EXPLORING PRESERVICE MIDDLE AND HIGH SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF DIRECTLY AND INVERSELY PROPORTIONAL RELATIONSHIPS

by

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(Under the Direction of Andrew G. Izsák)

ABSTRACT

This study used hands-on and missing-value word problems to examine preservice middle and high school teachers' knowledge resources when inferring directly and inversely proportional relationships between quantities. Additionally, the study examined preservice teachers' solution strategies and their difficulties when solving single and multiple proportion problems. An explanatory case study with multiple cases was used to make comparisons within and across cases. This study used the knowledge-in-pieces perspective in reporting preservice teachers' reasoning about ratios and proportional relationships. It appeared that the extent to which the preservice teachers were successful in coordinating the directly and inversely proportional relationships hinged on their attention to the specific features of the context. Although the preservice teachers accurately inferred the relationships between two covarying quantities as directly proportional or inversely proportional, their inferences were mainly based on attending to qualitative relationships—two quantities are increasing together—and the constancy of the rate of change. Thus, preservice teachers who relied heavily on the qualitative relationships and the constancy of the rate of change often judged nonproportional relationships that consisted of a constant difference or constant sum to be proportional, even after identifying correct nonproportional relationships. The results showed that the contexts of the hands-on problems facilitated the preservice teachers' coordination of the directly and inversely proportional relationships more than the contexts of the missing-value word problems.

INDEX WORDS: Preservice teachers, Teacher knowledge, Teacher education, Proportional reasoning, Ratio, Proportional relationships

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DEDICATION

I dedicate this study to

My mother, Gulbeyaz Arican

For believing in my potential

And

My wife, Esra and

My son, Suleyman Bera

For their love and patience

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CHAPTER 1

INTRODUCTION

Background

Understanding ratios, proportions, and proportional reasoning constitutes a main area of school mathematics that is critical for students to learn but difficult for teachers to teach (Lobato & Ellis, 2010). For instance, middle grade students need to understand proportionality well if they want to succeed in Grades 6-8 and in their following mathematical experiences (Lobato & Ellis, 2010). Therefore, in middle school, students need to develop skills that are essential for the development of proportionality. Two of those skills, as reflected in the National Council of Teachers of Mathematics' (NCTM; 2000) *Principles and Standards for School Mathematics*, are understanding and using ratios and proportions to represent quantitative relationships; and developing, analyzing, and explaining methods for solving problems involving proportions (Number and Operations Standards for Grades 6-8 section, para. 7). In addition to these two skills, in Grade 7, students should be able to "analyze proportional relationships and use them to solve real-world and mathematical problems (7.RP)" (CCSSM; Common Core State Standards Initiative, 2010, p. 48).

Researchers define *proportional reasoning* in a various ways. For example, Lamon (2007) defines it as follows:

> [*P*]*roportional reasoning* means supplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist

> of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities. (pp. 637-638)

The definition above is a very thorough way of defining proportional reasoning. Later in the same manuscript, she describes proportional reasoning briefly: “Proportional *reasoning* refers to detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships” (Lamon, 2007, p. 647). Lamon (2007) states that proportional reasoning advances as one studies fractions, and it is a sign of one’s rational number sense.

Karplus, Pulos, and Stage (1983a) describe proportional reasoning as a term that indicates reasoning in a “system of two variables between which there exists a linear functional relationship” (p. 219), and for them proportional reasoning leads one to reach conclusions about a condition or phenomenon that can be explained by a constant ratio. Lesh, Post, and Behr (1988) view proportional reasoning as a form of mathematical reasoning that entails “a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information” (p. 93). They also point out that proportional reasoning plays a key role in students’ mathematical development, viewing it as an important concept in children’s elementary school arithmetic and in higher mathematics.

According to Lamon (2007), *proportionality* is a much larger context than proportional reasoning. For Lamon (2007), “Proportionality is a mathematical construct referring to the condition or the underlying structure of a situation in which a special invariant (constant) relationship exists between two co-varying quantities (quantities that are linked and changing together)” (p. 638). She hypothesizes that proportions appear in the study of the natural expression of equivalent rational numbers and that one learns to reason proportionally by various experiences with rational numbers. For her, understanding proportionality develops later through interactions with mathematical and scientific systems, and these systems involve the invariance of a ratio or of a product.

As stated in the Common Core State Standards for Mathematics, to be able to reason proportionally, students should be able to “Decide whether two quantities are in a proportional relationship (7.RP.2a)” (CCSSM; Common Core State Standards Initiative, 2010, p. 48). There are two types of proportional relationships between quantities: *(directly) proportional relationships* and *inversely proportional relationships*. The word *directly* is written in parentheses because the concept of proportional relationship is usually understood to refer to directly proportional relationships (Beckmann, 2011). According to Beckmann and Izsák (2015), models of multiplication, division, and proportional relationships can be combined by the equation $M*N = P$, where M, N, and P stand for known constants. For Beckmann and Izsák (2015), a (directly) proportional relationship is “a collection of pairs of values for x and y” that either satisfy the equation $x * N = y$ or $M* x = y$ (p. 20). In both equations, N and M are known constants, and *x* and *y* are either unknown variable amounts or two co-varying values. In a directly proportional relationship, the unknown amounts (x and y) vary directly with each other. Varying directly implies that the values of these two quantities stay in a constant ratio.

On the other hand, an inversely proportional relationship is “a collection of pairs of values for x and y” that satisfy the equation $x * y = P$, where *x* and *y* are unknown quantities or two co-varying values, and P is a known constant (Beckmann & Izsák, 2015, p. 20). The term *inversely proportional* is used, because there is an inverse relationship between two quantities in that if the value of a quantity increases by a multiplicative constant, then the corresponding value of the second quantity decreases by the reciprocal of that constant. Hence, in an inversely proportional relationship, the unknown amounts (x and y) vary inversely with each other, and varying inversely implies that the product of the values of these two quantities remains constant.

Lamon (2007) states that $y = k \cdot x$ is the mathematical model for directly proportional relationships. In this model, the variables y and x represent the quantities that are in a proportional relationship, and the amount k represents the constant of proportionality. Since $y = k \cdot x$ necessitates $\frac{y}{x} = k$, in a proportional relationship, the quotient of the two co-varying quantities always remains constant. The mathematical model for an inversely proportional relationship is $y \cdot x = k$. Similarly, k represents the constant of proportionality. For Lamon (2007), the constant of proportionality plays a fundamental role in understanding proportionality. She describes the constant of proportionality as a *slippery character*, since its role depends on the situation where it is used. For example, she explains that in a graph it is the slope, in symbols it is a constant, in rate situations it is the constant rate, in reading maps it is the scale, in similar figures it is the scale factor, and it may mean the percentage if we think about sales tax.

Statement of the Problems

Two primary types of proportional relationship problems are used in mathematics education research: missing-value problems and comparison problems (Lamon, 2007). In missing-value type problems, a student is typically presented with three of the four values and asked to determine the fourth missing value (Lamon, 2007). However, in comparison problems, two ratios are compared to determine whether they are equal, or if one is larger or smaller (Lobato & Ellis, 2010). One of the problems of teaching and learning proportional relationships is that traditional proportion instruction places an emphasis on rule memorization and rote computations (Izsák & Jacobson, 2013). Hence, the most common textbook strategy for solving a missing-value problem is the cross-multiplication strategy (Fisher, 1988), which requires setting a proportion and cross-multiplying numbers within the proportion. This strategy can also be used with comparison problems to determine the equality of two ratios. As noted by Izsák and Jacobson (2013), "reasoning about proportional relationships involves much more than using cross-multiplication" (p. 2). For Izsák and Jacobson (2013), a strong understanding of proportional relationships involves "*understanding and using multiplicative relationships between two co-varying quantities and recognizing whether or not two co-varying quantities remain in the same constant ratio"* (p. 2). As observed by many researchers (e.g., Fisher, 1988; Riley, 2010), teachers can depend severely on using the cross-multiplication strategy when solving proportion problems.

A second problem is that, according to Izsák and Jacobson (2013), mathematics education research has overlooked teachers' proportional reasoning. In particular, only a few researchers (e.g., Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010) have studied teachers' proportional reasoning regarding inverse proportions. These researchers observed that when teachers were introduced to inverse proportion problems they struggled to solve those problems. For example, Fisher (1988) gave 20 secondary mathematics teachers the following inverse proportion problem:

> If it takes nine workers 5 hours to mow a certain lawn, how long would it take six workers to mow the same lawn? (p. 160).

As discussed by Fisher (1988), 12 out of 20 teachers solved this problem incorrectly, and nine of them approached the problem as if it were a direct proportion problem. Therefore, a third problem is that preservice and in-service teachers tend to judge nonproportional relationships to be proportional (Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010). In addition to these three problems, Lim (2009) observed that preservice teachers (PSTs) had greater difficulty with the conceptual understanding of the solution of a direct proportion problem than conceptual understanding of a nonproportional problem. They are likely to use additive strategies to solve proportion problems (Simon & Blume, 1994; Riley, 2010). They have difficulty creating suitable reciprocal multiplicative relationships for nonproportional problems (Izsák & Jacobson, 2013). Finally, they have difficulty understanding ratio-as-measure and the invariance of a ratio (Simon & Blume, 1994).

Purpose of the Study

My main goal in conducting this study was to explore how preservice middle and high school mathematics teachers infer directly and inversely proportional relationships in single and multiple proportion questions. Additionally, I was interested in understanding the types of strategies that PSTs use to solve single and multiple proportion questions, the ways they represent directly and inversely proportional relationships in the given questions, and the difficulties that they encounter while solving these questions. To accomplish these goals, I asked the following research questions.

Research Questions

1. How do preservice middle and high school mathematics teachers infer directly and inversely proportional relationships in single and multiple proportion problems; what types of knowledge resources do they use when inferring and explaining directly and inversely proportional relationships; and what kinds of difficulties do they encounter in the process of inferring, explaining, and expressing directly and inversely proportional relationships?

2. What types of solution strategies do preservice middle and high school mathematics teachers use to solve single and multiple proportion problems, and how do they express directly and inversely proportional relationships in those problems?

Significance of the Study

In earlier research, some of which I discussed previously, researchers investigated teachers' proportional reasoning mostly using missing-value word problems, which usually involved a single proportional or nonproportional relationship. Similarly, instruction on proportions traditionally uses missing-value word problems in teaching, and cross multiplication is the choice for a general solution strategy. Hence, preservice and in service teachers usually have some experiences with missing-value word problems. In this study, a combination of hands-on activities and real-world missing-value problems, which involved either single or multiple directly and inversely proportional relationships, were used. It was expected that the use of physical devices (e.g., plastic gears and mini number balance system) would provide hands-on experiences and generate a checking mechanism for PSTs, which would eventually help them have well developed understandings of directly and inversely proportional relationships. Because it is not easy to solve multiple proportion problems by simply forming a proportion and applying the cross-multiplication strategy, it was expected that teachers would avoid using cross multiplication and additive strategies in those problems. Therefore, the results of this study illuminate how PSTs reason about proportional relationships when they cannot rely on computation methods like cross multiplication. In addition, it was anticipated that the use of hands-on and multiple proportion problems would help reveal teachers' knowledge resources for inferring directly and inversely proportional relationships.

This study makes use of the knowledge-in-pieces epistemological perspective (diSessa, 1988, 1993, 2006) to analyze knowledge resources that teachers used to infer directly and inversely proportional relationships and multiplicative relationships. Most recently, Izsàk and Jacobson (under review) investigated preservice middle and secondary grades teachers' facility with multiplicative relationships and identification of directly and inversely proportional relationships by utilizing the knowledge-in-pieces perspective. However, the missing-value problems used by Izsàk and Jacobson (under review) involved either a single inversely proportional relationship or a constant difference relationship. Izsàk and Jacobson (under review) suggested that future research should involve more complex cognitive structures to analyze teachers' responses to the proportion problems. In order to examine complex cognitive structures Izsàk and Jacobson (under review) recommended using problem tasks that involve physical devices and other contexts with which teachers have less experience. Since this study uses hands-on and multiple proportion tasks to examine teachers' proportional reasoning, it extends and strengthens the knowledge-in-pieces perspective by applying core components of this perspective to understand the more complex cognitive structures used by teachers to identify directly and inversely proportional relationships and multiplicative relationships.

Thus, it was anticipated that this study would make four contributions to the current research base in mathematics education: First, very little research has been conducted on PSTs' proportional reasoning. In particular, only a few researchers (e.g., Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010) have studied teachers' proportional reasoning regarding inverse proportions, and even fewer researchers have studied multiple proportions (e.g., Vergnaud, 1983, 1988). Second, the use of hands-on tasks and real-world missing-value problems together precipitate the gathering of relevant information regarding PSTs' proportional reasoning. Third, this study builds a bridge between mathematics education and science education by making use of science concepts—velocity, gear ratio, and balance. Fourth, this study uses the knowledge-in-pieces perspective for analyzing PSTs' knowledge resources in detecting and explaining directly and inversely proportional relationships in problems tasks with more complex structures and with which teachers have less experience.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I elaborate on the key concepts and mathematics literature, some of which I already discussed in Chapter One, about proportional reasoning and proportional relationships. I first discuss key concepts that play important roles in studying proportions. Next, I present the types of problems that have been used in studying ratios and proportions and discuss strategies that students and teachers employ when solving proportion problems. I then summarize previous studies on students' and teachers' proportional reasoning. In the last section of the chapter, I explain how the term *multiplicative reasoning* is used in this study.

Discussion of the Key Concepts

Quantity

Researchers, such as Schwartz (1988), Shalin (1987), and Nesher (1988), "characterize quantities as ordered pairs of the form (*number*, *unit*)" while Steffe (1991b) "characterizes quantity as the outcome of unitizing or segmenting operations" (as cited in Thompson, 1994, p. 184). Although, for Thompson (1994), operations of unitizing and segmenting are essential for a person to create quantities, he states that he uses the term *quantity* more broadly than Steffe (1991b). Thompson (1990) defines *quantity* as a "quality of something that one has conceived as admitting some measurement process" (p. 5). Similarly, Lamon (2007) defines *quantity* as a "measurable quality of an object—whether that quality is actually quantified or not" (p. 630). As an example, one can compare the heights of two people without measuring by simply observing the difference when they stand beside each other. For Lamon (2006), linking quantities that are not quantified is an essential kind of reasoning, and this kind of reasoning is available to children when posed in contexts that they understand (as cited in Lamon, 2007). As an example, Lamon (2007) suggests that one way to help students develop this type of reasoning involves using a context in which teachers ask students how many cookies (the same, more, or less) their friends would get if they shared some cookies with some friends one day and if they shared fewer cookies with more friends the next day.

Two different kinds of quantity are described by researchers: *intensive quantity* and *extensive quantity*. Schwartz (1988) defines *intensive quantity* as “a type of quantity that is ordinarily not either counted or measured directly” (p. 42), and for him, intensive quantities can be recognized by the fact that their unit measures contains the word *per*. For Kaput and West (1994), *intensive quantity* is used as “a blanket term to cover all the types of quantities” (p. 239) that are described as rates, ratios, unit conversion factors, and scale conversion factors. As stated by Kaput and West (1994), intensive quantities can be used in two different ways: particular intensive quantity (or particular ratio) and rate intensive quantity (or rate-ratio). If we are using a particular ratio to describe some relationship between two quantities, then we are using a particular intensive quantity. For example, if we talk about a particular purchase of vegetables—let’s say if we pay 6 dollars for 3 pounds of vegetables—then we are talking about particular intensive quantity. On the other hand, because the statement 2 dollars per pound refers to the price of vegetables of any amount, then we are talking about a rate intensive quantity.

On the other hand, Thompson (1990) defines *extensive quantity* as “a quantity that may be measured directly or is a combination of directly measurable quantities” (p. 6). For Kaput and West (1994), two extensive quantities can be used to construct an intensive quantity. For example, if we pay 5 dollars for 3 pounds of vegetables, then the statement 5 dollars per 3 pounds corresponds to an intensive quantity, and 5 dollars and 3 pounds correspond to extensive quantities that can be used to construct the intensive quantity.

Ratio and Rate

As discussed by Thompson (1994), even though a conventional distinction is not made between ratio and rate, there is widespread confusion about existing distinctions. For Thompson (1994), because of the lack of conventional distinction between ratio and rate, these two terms are used frequently without definition. He sums up the most frequent distinctions as follows:

1) A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds), and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time; Vergnaud, 1983, 1988).
2) A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).
3) A ratio is a binary relation which involves ordered pairs of quantities. A rate is an intensive quantity—a relationship between one quantity and one unit of another quantity (Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, & Behr, 1988; Schwartz, 1988). (Thompson, 1994, p. 190)

According to Thompson (1994), each of the above distinctions seems to have some validity, although there is an evident controversy about those distinctions. For him, these distinctions have been based upon situations *per se* instead of mental operations. Thompson (1994) describes a ratio as being the "result of comparing two quantities multiplicatively" (p. 190) and rate as a "reflectively abstracted constant ratio" (p. 192). Similarly, Lobato and Ellis (2010) define the term *ratio* as a "multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit" (p. 12).

The most common definitions of ratio and rate are based on the nature of the quantities compared, so when two quantities that are compared have the same units, they become a ratio; but if they have different units, they become a rate. As stated in the draft *Ratio and Proportional Relationships Progression* (Common Core Standards Writing Team; 2011), some authors distinguish ratios from rates by looking at the nature of the quantities, but other authors use ratios to encompass both kinds of situations, and the CCSSM standards use ratio in this later sense. A colon is used in the notation of a ratio, as in $2\mathbf{:}\ 3$**,** and the quotient $\frac{2}{3}$ is called the value of the ratio $2\mathbf{:}\ 3$. If we let the ratio $2\mathbf{:}\ 3$ represents 2 cups of orange juice for every 3 cups of apple juice, then the associated rate is $\frac{2}{3}$ cups of orange juice for every 1 cup of apple juice, and the numerical term $\frac{2}{3}$ is called the *unit rate*.

If we multiply each measurement in a ratio by the same positive number, then we get an equivalent ratio, and equivalent ratios have the same unit rate. For example, if we multiply each measurement in $\frac{2\ cups\ orange\ juice}{3\ cups\ apple\ juice}$ by 2, then we get a ratio of 4: 6 and we still have a rate of $\frac{2}{3}$ cups of orange juice for every 1 cup of apple juice. Equivalent ratios play a critical role in exploring directly proportional relationships, since one has to consider equivalent ratios to solve problems that involve directly proportional relationships. The ratio of the reciprocal of the quantities is called an inverse of a ratio or simply inverse ratio. For example, the inverse ratio of $\frac{2\ cups\ orange\ juice}{3\ cups\ apple\ juice}$ is $\frac{3\ cups\ of\ apple\ juice}{2\ cups\ of\ orange\ juice}$. This concept is important, because to solve problems that involve inversely proportional quantities, we need to form a proportion in which the second ratio needs to be an inverse ratio.

Two other important concepts involving ratios are the concepts of within and between ratios. According to Freudenthal (1973, 1978), an internal (within) ratio is composed of "magnitudes" from the same system or measure space, while an external (between) ratio is composed of "magnitudes" from different systems or measure spaces (as cited in Lamon, 2007). For instance, if a car covers a distance of 60 miles in 1 hour, then it can cover 120 miles in 2 hours. In this example, 60 miles**:** 120 miles and 1 hour**:** 2 hours are internal (within) ratios, and 60 miles**:** 1 hour and 120 miles**:** 2 hours are external (between) ratios. As emphasized by Lamon (2007), one of the persistent issues in proportions is the confusion about the classification of students' strategies as either within or between. For her, the confusion results from different uses of the terms *within* and *between* in the earlier research, which originated in a science tradition. For example, Karplus et al. (1983a, 1983b) and Noelting (1980a, 1980b) used alternative definitions for within and between ratios (Lamon, 2007). They defined a system as "a set of interacting elements" (Lamon, 2007, p. 634). For them, an internal (within) ratio was "a comparison of elements within one scientific state or system" (Lamon, 2007, p. 634). In contrast, an external (between) ratio involved elements from different systems. According to this alternative view, in the example above, 60 miles and 1 hour define a system, and 120 miles and 2 hours define another system. Hence 60 miles**:** 1 hour and 120 miles**:** 2 hours become internal (within) ratios. Similarly, 60 miles**:** 120 miles and 1 hour**:** 2 hours become external (between) ratios.

As discussed by Lamon (2007), using terminologies *"*within or between *measure spaces"* or "within or between *systems*" can help us eliminate this confusion (p. 634). I give the following two examples to demonstrate these two terminologies:

Example 1: If 4 apples cost 1 dollar, then 12 apples cost 3 dollars.

In this example, our measure spaces are: number of apples (Measure space one or M1) and cost in dollars (Measure space two or M2). Therefore, the ratio $\frac{4\text{ apples}}{12\text{ apples}} = \frac{1}{3}$ is an example of a within measure space ratio. On the other hand, the ratio $\frac{\text{4 apples}}{\text{1 dollar}}$ is an example of a between measure space ratio. Here, the between measure space ratio becomes an intensive quantity.

Example 2: Mixture one was constituted using 2 cups of sugar and 6 cups of water.

Mixture two was constituted using 3 cups of sugar and 8 cups of water.

In this example, system one is mixture one and system two is mixture two. Hence, $\frac{\text{2 cups of sugar}}{\text{6 cups of water}}$ and $\frac{\text{3 cups of sugar}}{\text{8 cups of water}}$ are within-system ratios, and $\frac{\text{2 cups of sugar}}{\text{3 cups of sugar}}$ and $\frac{\text{6 cups of water}}{\text{8 cups of water}}$ are between-system ratios.

One can see the use of within or between measure spaces terminology in Vergnaud’s (1983, 1988) studies. In this study, I also use within or between measure spaces terminology to avoid confusion. Following Freudenthal (1973, 1978), Vergnaud’s (1983, 1988), and Lamon (2007), one may define *within* and *between measure space ratios* as follows: If a ratio consists of quantities that are taken from the same measure space, then this ratio is called a *within measure space ratio*. On the contrary, if a ratio consists of quantities that are taken from different measure spaces, then this ratio is called a *between measure space ratio*. As defined by Lamon (2007), “Measure spaces usually refer to different sets of objects, different types of quantities, or different units of measure” (p. 634). If students solve a problem by forming a proportion with two within measure space ratios, then their strategy is called a *within measure space strategy*. Similarly, if they form a proportion by using two between measure space ratios, then their strategy is called a *between measure space strategy*.

Covariation

As stated by Lamon (2007), “proportional relationships involve one of the simplest forms of co-variation” (p. 648). By covariation, Lamon (2007) implies that “two quantities are linked to each other in such a way that when one changes, the other one also changes in a precise way with the first quantity" (p. 648). As I discussed in Chapter One, if there is a directly proportional relationship between two quantities (e.g., x and y), we can write $y = k_1 \cdot x$ or $\frac{y}{x} = k_1$, where k_1 is called the constant of the proportionality. Similarly, if there is an inversely proportional relationship between x and y, we can write $y \cdot x = k_2$, and again, k_2 is called the constant of the proportionality. As one can understand from these two equations—$y = k_1 \cdot x$ and $y \cdot x = k_2$—directly and inversely proportional relationships require a multiplicative relationship between the two quantities. For example, in the directly proportional case, y is a constant multiple of x and, so when x changes (increases or decreases), y changes in proportion to x. Similarly in the inversely proportional case, y is a constant multiple of $\frac{1}{x}$, and x and y change in an opposite manner. For instance, when x increases, y decreases in proportion to x or, when x decreases, y increases in portion to x.

Proportion

According to the Common Core Standards Writing Team (2011), a proportion is "an equation stating that two ratios are equivalent" (p. 3). Similarly, Fisher (1988) defines the term *proportion* as a "statement of the equality of two ratios (i.e., a/b = c/d)" (p. 157), and Lobato and Ellis (2010) define it as a "relationship of equality between two ratios" (p. 12). Because a proportion is formed by two equivalent ratios, and because each of these two ratios represents the relationship between two quantities, in a proportion, even if we change the corresponding values of the quantities, the ratio of the two quantities remains constant.

According to the draft *Ratio and Proportional Relationships Progression* (Common Core Standards Writing Team, 2011), "The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades" (p. 2). Also, Lobato and Ellis (2010) state that elementary schools allow students to develop meanings for fractions and multiplication that have important foundations on which students build an understanding of ratios, proportions, and proportional reasoning. Hence, Vergnaud (1983, 1988) emphasizes that multiplication, division, fractions, ratios, and rational numbers are not mathematically independent of one another and places these concepts within a larger context that he calls the *multiplicative conceptual field*. For Vergnaud (1988), a *conceptual field* is "a set of situations, the mastering of which requires mastery of several concepts of different natures" (p. 141). Two main conceptual fields are *additive* and *multiplicative structures* (Vergnaud, 1983, 1988). Additive structures include a set of problems involving addition, subtraction, difference, interval, and translation, and multiplicative structures include a set of problems involving multiplication, division, fraction, ratio, and similarity (Vergnaud, 1983, 1988).

Vergnaud (1983, 1988) discusses three types of multiplicative structures: *isomorphism of measures, product of measures,* and *multiple proportion other than product*. The isomorphism of measures structure "consists of a simple direct proportion between two measure-spaces M_1 and M_2" (Vergnaud, 1988, p. 129). The product of measures structure "consists of the Cartesian composition of two measure-spaces, M_1 and M_2, into a third, M_3" (Vergnaud, 1988, p. 134). According to Vergnaud (1988), the problems in this structure are concerned with area, volume, Cartesian product, and work. Although this structure includes an inversely proportional relationship between quantities multiplied, Vergnaud (1983, 19888) did not focus on this inversely proportional relationship in detail. In the multiple proportion structure, "a measure-space M_3 is proportional to two different independent measure-spaces M_1 and M_2" (Vergnaud, 1988, p. 138). This type of proportional relationship is also called a *jointly proportional relationship*. For example: "The consumption of cereal in a scout camp is proportional to the number of persons and to the number of days" (Vergnaud, 1988, p. 138). As noted by Vergnaud

(1988), multiple proportion problems have not been researched widely, and most teachers are unaware of students' difficulties with these problems.

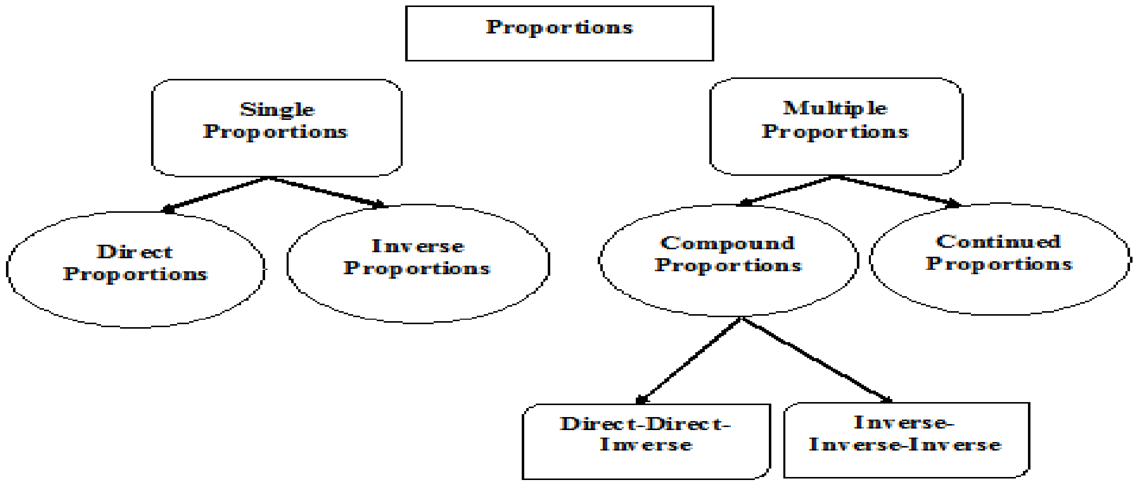


Figure 1. Classification of proportions.

Although there is not an agreed upon classification of proportions, in this study, I classify proportions into two main categories: single proportions and multiple proportions (Figure 1). In Figure 1, I classified multiple proportions for the case of the three measure spaces. Single proportions involve direct and inverse proportions, and multiple proportions involve compound and continued proportions. My classification uses Vergnaud's (1983, 1988) classification of multiplicative structures and extends it a little bit further. For instance, first, my definition of a single proportion structure combines isomorphism of measures and product of measures structures. Second, my definition of a multiple proportion structure involves two main types of proportions: compound and continued. A compound proportion structure involves direct-direct-inverse and inverse-inverse-inverse proportion structures. Following the multiplication statement provided by Beckmann and Izsák (2015), a direct-direct-inverse structure can be expressed by the multiplication statement, M*N = *k**P, and an inverse-inverse-inverse structure can be expressed by the multiplication statement, M*N*P = *k*. In these two statements, M, N, and P represent the quantities compared, and *k* represents the constant of the proportionality.

Vergnaud's (1983, 1988) multiple proportion structure is equivalent to a direct-direct-inverse proportion structure in my classification.

Single Proportions. Following Vergnaud (1983, 1988), if we only have two measure spaces (e.g., M1 and M2) that involve four quantities (e.g., a, b, c, and d), then the proportion that is formed by these four quantities has a single structure. In the direct proportion, quantities in two measure spaces, M1 and M2, are directly proportional to each other. Vergnaud (1983, 1988) called this structure *Isomorphism of Measures*. As discussed by Vergnaud (1983), this structure describes many situations in ordinary life – for example, fair sharing, uniform speed (speed and distance), and constant price (cost and objects bought). The following example is adapted from Vergnaud (1983):

Example: If the consumption of gas for a car is 6 liters per 100 km, then it needs 30 liters to travel 500 km. In this example, a = 100 km, b = 6 liters, c = 500 km, d = 30 liters, M1 = distance, and M2 = gas in liters.

Because the distance traveled and the amount of gas consumed are directly proportional, the direct proportion that is formed among the four quantities in this question is either *a:c* = *b:d* or *a:b* = *c:d*. Vergnaud (1983) demonstrated the simple direct proportion between M1 and M2 as in Figure 2.

M1	M2
a	b
c	d

Figure 2. A demonstration of a simple proportion structure.

Unlike the case of a direct proportion, in an inverse proportion, the quantities in two measure spaces are inversely proportional to each other. We can also see inversely proportional structures naturally in life. For example, if the distance is constant, then the speed and the time are inversely proportional; the number of people and the time spent completing a job are also inversely proportional if we take the amount of work to be constant and assume that each person works at the same rate, and the lengths of height and width of a rectangle are also inversely proportional if we take the area to be constant.

Example: If one can cover 120 miles in 2 hours driving at 60 mph, then he or she can cover the same distance in 3 hours driving at 40 mph.

In this example, M1 = speed, M2 = time, a = 60 mph, b = 2 hours, c = 40 mph, and d = 3 hours. Because there is an inversely proportional relationship between the speed and the time, the proportion that is formed among these four quantities is either a:c = d:b or a:d = c:b. Therefore, for this example, a×b= c×d, and they represent the distance traveled. Following Vergnaud (1983), I represent the inverse structure of this example in Figure 3.

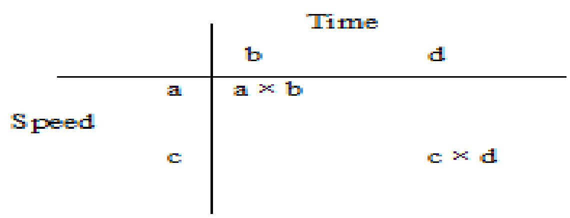


Figure 3. Representation of an inverse proportion structure.

Multiple Proportions. Multiple proportions are formed if we have more than two measure spaces. For simplicity, I use three measure spaces. Let our three measure spaces be *M1*, *M2*, and *M3*. Hence, in a multiple proportion structure, we can determine three relationships between three measure spaces: between M1 and M2, between M1 and M3, and between M2 and M3. There are two substructures of multiple proportions: a compound proportion structure and a connected proportions structure.

A compound proportion structure consists of a combination of direct and inverse proportions. For example, it involves direct-direct-inverse and inverse-inverse-inverse proportion structures for three quantities case. I demonstrate these two structures by using two different examples:

Example 1: If four workers paint two bedrooms in 6 hours, then two workers paint one bedroom in 6 hours.

Example 2: If 12 workers build an apartment in 24 days by working 8 hours per day, then 24 workers build the same apartment in 16 days working 6 hours per day.

In the first example, let M1 be the number of workers, M2 be the number of bedrooms, and M3 be the number of hours each worker works per day. There is a directly proportional relationship between quantities in M1 and M2 (assuming the number of hours each worker works per day to be constant). Similarly, there is a directly proportional relationship between quantities in M2 and M3 (assuming the number of workers to be constant). On the other hand, there is an inversely proportional relationship between quantities in M1 and M3 (assuming the number of bedrooms to be constant). I call this structure a *direct-direct-inverse proportion structure* (or *joint proportion structure*) and the relationship a *direct-direct-inverse proportional relationship*. The mathematical composition behind this structure can be described as follows: (number of workers) * (number of hours) = (number of bedrooms) * (man-hours per bedroom). In this example, man-hours per bedroom is the constant of the proportionality and can be calculated as $k = \frac{4 \cdot 6}{2} = 12$. In addition, if M1 consist of quantities a and b, M2 consist of quantities c and d, and M3 consist of quantities e and f; then, in the direct-direct-inverse proportion structure, we have $a \cdot d \cdot e = b \cdot c \cdot f$. For instance, in example one, we have a = 4, b = 2, c = 2, d = 1, e = 6, and f = 6. Here we have $a \cdot d \cdot e = 4 \cdot 1 \cdot 6 = 24$ and $b \cdot c \cdot f = 2 \cdot 2 \cdot 6 = 24$.

In the second example, let M1 be the number of workers, M2 be the number of days, and M3 be the number of hours workers work per day. The number of workers is inversely proportional to the number of days if we assume that the number of hours per day is constant. The number of workers is also inversely proportional to the number of hours per day if we assume that the number of days is constant. Furthermore, the number of days and the number of hours per day are inversely proportional if we take the number of workers as constant. I call this structure an *inverse-inverse-inverse proportion structure* and the relationship an *inverse-inverse-inverse proportional relationship*. The mathematical composition behind this structure can be described as follows: (number of workers) * (number of days) * (number of hours per day) = (man-hours). The constant of the proportionality, man-hours, can be found by multiplying the number of workers by the number of days and by the number of hours per day. In our example, we have $k = 12 \cdot 24 \cdot 8 = 2304 = 24 \cdot 16 \cdot 6$. If M1 consist of quantities k and l, M2 consist of quantities m and n, and M3 consist of quantities r and s; then, in the inverse-inverse-inverse proportion structure, we have $k \cdot m \cdot r = l \cdot n \cdot s$.

If there are four quantities that have a structure in which $a:b = b:c = c:d$ and so on, then we call this structure *continued proportions*. The following example illustrates this structure:

> *Example:* John, Mary, David, and Elizabeth want to share 30 cookies among themselves in the following ratios: John: Mary = 1: 2, Mary: David = 2: 4, and David: Elizabeth = 3: 6. How many cookies does each one get?

In this example, there is a continued proportion 1:2 = 2:4 = 3:6. This proportion implies that John gets $\frac{1}{2}$ times as many cookies as Mary, Mary gets $\frac{1}{2}$ times as many as David, and David gets $\frac{1}{2}$ times as many as Elizabeth. As a result, Elizabeth gets 8 times as many cookies as John. If x is the number of cookies John gets, then in total they have $x + 2x + 4x + 8x = 15x$ cookies. One can find that $x = 2$ and determine that John gets two cookies, Mary gets four cookies, David gets eight cookies, and Elizabeth gets 16 cookies.

Types of Problems Used in Studying Ratios and Proportions

Lobato and Ellis (2010) mentioned three types of proportion problems: missing-value problems, comparison problems, and transformation problems. The first two problem types are more frequently used in research than the last type. In a missing-value problem, three out of four quantities are usually given, and the goal is to determine the missing one. The following question is an example of a missing-value problem:

> *Example:* If a store sells two bags of apples for four dollars, then how much does the store charge for three bags of apples?

The problem above consists of a directly proportional relationship between the number of bags and the cost. One can also generate missing-value problems with inversely proportional relationships. To solve this problem, a student can generate the following proportion, $\frac{2}{4} = \frac{3}{x}$, and determine *x* by cross-multiplying or using other techniques. Following Vergnaud (1983), I illustrate two of these techniques in Figure 4. In this figure, M1 represents measure-space one, which is the bags of apples, and M2 represents measure-space two, which is the cost of the apples in dollars. In the first method, one can determine the constant relationship between the numbers of bags and the cost and, then, apply the same constant to get the missing-value. This method can be called a between measure space strategy, which I discussed previously. One can also determine the relationship between numbers of bags and use the same relationship between the numbers of dollars to get the missing value. This method can be called a within measure space strategy, which I also discussed previously.

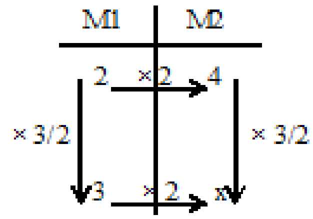


Figure 4. Representation of and strategies for solving a missing-value problem.

In a comparison problem, one needs to compare two given ratios by determining whether the first of the two ratios is greater than, less than, or equal to the second. Mixture problems can be given as examples for the comparison problems. For example, when comparing the tastiness of two mixtures, usually we compare two ratios that determine the flavors. The following example can be given to illustrate missing-value problems:

> *Example:* We have two pitchers of lemonades. The first one is made by mixing 2 cups of lemon juice with 5 cups of water, and the second one is made by mixing 3 cups of lemon juice with 7 cups of water. Which one of the lemonades tastes more lemony?

To solve this example, one can form two between measure space ratios: $\frac{\text{2 cups of lemon juice}}{\text{5 cups of water}}$ and $\frac{\text{3 cups of lemon juice}}{\text{7 cups of water}}$ and determine that $\frac{3}{7} > \frac{2}{5}$. One can conclude that the second mixture tastes more lemony.

The last problem type is transformation problems. Lobato and Ellis (2010) describe this problem type as follows: “Transformation problems give a ratio or two equivalent ratios and ask students either to change one or more quantities to change the ratio relationship or to determine how a given change in one or more quantities changes the relationship” (p. 5).

> *Example:* A particular brand of lemonade is made by mixing 2 cups of lemon concentrate with 3 cups of water, and a bottle of lemonade contains 20 cups of the mixture. Because the company had complaints from their costumers about the strong lemon taste in the

lemonade, they decided to reduce the lemon concentrate they used in the bottles to 5 cups. Can you determine the new relationship between the lemon concentrate and water?

Because the company used 20 cups of lemon-water mixture in a bottle of lemonade, then new bottles must have 20 cups of mixture also. We know that there are five cups of lemon concentrate used in this new mixture, so the remaining 15 cups must be water. Consequently, in the new mixture, the lemon-water ratio must be $\frac{5\ cups\ lemon}{15\ cups\ water} = \frac{1\ cups\ lemon}{3\ cups\ water}$. Thus, we can conclude that, in the new mixture, the company used one cup of lemon concentrate for every three cups of water.

Strategies Used to Solve Proportion Problems

Karplus et al. (1983b) developed lemonade puzzles to explore proportional and other types of reasoning of 60 sixth graders and 60 eight graders. They collected students' strategies in a strategy scale that involved four categories:

1. **Category I (incomplete or illogical strategy):** The responses in this category indicated that students did not know the answer, guessed, or used inappropriate quantitative operations.
2. **Category Q (qualitative strategy):** The responses in this category demonstrated that students compared given quantities qualitatively using words, such as *less* and *more*, or identical terms.
3. **Category A (additive strategy):** The responses in this category showed that students compared quantities by paying attention to the differences in the values of these quantities.

4. **Category P (proportional strategy):** The responses in this category revealed that students compared quantities by paying attention to the proportional relationships between quantities.

Tourniaire (1986) gave the following mixture problem and explained typical student answers for each of the four categories above:

> There are two mixtures of orange juice and water. One is made with 2 glasses of orange juice and 4 glasses of water. The other is made with 6 glasses of orange juice. How much water should be used to get the same taste? (p. 404)

For Tourniaire (1986), a strategy such as "6, because there are 6 glasses of orange juice" (p. 404) would fit in Category I. On the other hand, a strategy such as "10, because there is much more orange juice, so there should be much more water, too" (p. 404) would fit in Category Q. An additive strategy would look like "8, because there should be 2 more glasses of water than orange juice" (p. 404). Finally, a proportional strategy would look like "12, because there should be twice as much water as orange juice" (p. 404) or "12, because we used 3 times as much orange juice, so we need 3 times as much water" (p. 404).

To classify secondary teachers' solution strategies to solve two direct and two inverse proportion problems, Fisher (1988, pp. 161-162) used a list of nine strategies. She treated the first five of the following strategies as the incorrect strategies and the remaining four as the correct strategies:

1. **No answer.**
2. **Intuitive:** Guessing the answer or answering the question by just relying on feelings or intuition;

3. **Additive:** The subject incorrectly focuses on the additive differences between the given quantities and does not consider multiplicative relationships;
4. **Proportion attempt:** The subject understands that proportion was involved but cannot express the relationship;
5. **Incorrect other:** An incorrect strategy that cannot be placed in categories 1-4;
6. **Proportion formula:** A correct strategy in which the subject solves a problem by showing the equivalence of two ratios or by generating an equation that expresses the equality of two products followed by an explicit statement noticing the inverse relationship;
7. **Proportional reasoning:** The subject solves the problem by using a correct proportion strategy other than the proportion formula;
8. **Algebra:** The subject solves the problem by setting up an algebraic equation other than the proportion formula;
9. **Correct other:** A correct strategy that cannot be placed in categories 6-8.

While working with 24 sixth-grade students, Lamon (1993, p. 46) identified the following six strategies from their responses to a set of 40 ratio and proportion problems:

1. **Avoiding:** Students did not establish a genuine interaction with the problem;
2. **Visual or additive:** Students solved problems by using the trial-and-error method. They did not offer reasons for their responses, they employed visual judgments, or they used incorrect additive strategies;
3. **Pattern building:** Students used verbal or written patterns without considering numerical relationships;

4. **Pre-proportional reasoning:** Students used intuitive activities, such as charts, pictures, or models, to solve the problems. Some relative thinking was also involved in the solution processes;
5. **Qualitative proportional reasoning:** Students understood numerical relationships, used a ratio as a unit, and used relative thinking to solve the problems;
6. **Quantitative proportional reasoning:** Students used algebraic symbols to show proportions and understood functional and scalar relationships in those symbols.

In addition to the six strategies that I discussed above, for Lamon (2007), at an early age children use a *building-up strategy* to solve proportion problems. In the *building-up* strategy, students set up a ratio and, by addition, extend it to a second ratio. Following Lamon (2007, p.643), an example of this strategy looks like the following:

If two pencils cost 1$

1$ for 2 more makes 2$

1$ for 2 more makes 3$

For Lamon (2007), without additional information, this strategy would not be treated as a proportional reasoning strategy, since it does not take into account the constant ratio.

Lamon (2007) also discussed two intuitive strategies, *double counting* and *unitizing* strategies. A double counting strategy uses the *norming* process, described as “reinterpreting a situation in terms of some chosen unit” (Lamon, 2007, p. 644). In this strategy, students chose one of the ratios and used it to reinterpret the other ratio. For instance, adapting the example given by Lamon (2007, p. 644), suppose three pizzas were shared among five girls, and one pizza was shared between two boys. If we ask students to determine who gets more pizza, the girls or the boys, then they could reinterpret the 1 pizza**:**2 boys ratio as 3 pizzas**:**6 boys. Hence, they could realize that six boys share three pizzas. Therefore, students could arrive at the solution that the girls get more pizza, because they would be able to feed one more person.

As explained by Lamon (1996), "Unitizing is the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity" (p. 170). For instance, thinking of a case of cola as 24 cans, 2 (12-packs), 4 (six-packs), or 6 (four-packs) can be given as an example of unitizing (Lamon, 1996). As stated by Lamon (1996), "The ability to form and operate with increasingly complex unit structures appears to be an important mechanism by which more sophisticated reasoning develops" (p. 170). Therefore, the ability to correctly unitize given quantities plays an important role in the development of students' proportional reasoning. In the light of the three strategy categories, I further discuss additive and proportional reasoning strategies in the following paragraphs.

Additive strategies. As I explained above, a student who uses an additive strategy compares two quantities by paying attention to the additive differences in the values of these quantities. One of the additive strategies is a repeated addition strategy in which students understand multiplication as repeated addition. For example, a student who reasons additively can solve the following problem by repeatedly adding four three dollars (3 dollars + 3 dollars + 3 dollars + 3 dollars) instead of multiplying three dollars by 4.

> Jane wants to buy four plastic cars, each of which costs three dollars. How much money does she need to pay?

As mentioned by Tourniaire (1986), "repeated adders either do not master multiplication or fail to see its link with repeated addition" (p. 406). For Hart (1981), children use repeated addition to avoid multiplying a number by a fraction. Hence, for her, teachers should be aware of children's use of repeated addition. As mentioned by Lobato and Ellis (2010), there is a need to extend students' conception of multiplication beyond repeated addition if teachers want to develop their students' proportional reasoning.

Proportional reasoning strategies. A student who reasons proportionally can understand proportional relationships between the values of the quantities compared. As mentioned by Lamon (2007), "Proportional reasoners are able to differentiate between additive and multiplicative situations and to apply whichever transformation is appropriate" (p. 650). Students can determine proportional relationships between quantities in different ways. Doubling and halving strategies are two of the simplest ways of testing proportional relationships between quantities. Two other strategies to determine proportional relationships between quantities are within and between measure space ratio strategies. Since I have already explained these two strategies previously, I do not discuss them here.

Studies on Students' Proportional Reasoning

Noelting (1980a, 1980b) studied children's cognitive development. He explored the following two questions: (1) is cognitive development hierarchical? (2) what are the mechanisms involved in the processes of development? A sample of 321 children from ages 6 to 16 participated in the study. Children were chosen from mathematically advanced classrooms and from the same socio-economic level. A test, which had 25 items, was developed by the author. Each item involved a comparison between the relative orange tastes of two drinks. Items were categorized into stages according to their difficulty. Nine stages were formed. Noelting (1980a, 1980b) matched these stages with Piagetian operational levels to understand the cognitive development of students at each stage. As a result, he found that stage 0 corresponded to the *Symbolic* level. Stages IA, IB, and IC corresponded to *Lower*, *Middle*, and *Higher Intuitive* levels, respectively. Stages IIA and IIB corresponded to *Lower* and *Higher Concrete Operational* levels, respectively. IIIA and IIIB corresponded to *Lower* and *Higher Formal Operational* levels.

Noelting (1980a, 1980b) discussed that older students were more successful at attaining higher stages. The students who were in the lower stages usually used additive strategies, and the students who were in the higher stages usually used multiplicative reasoning strategies (i.e., within ratio, between ratio, and unit ratio). Students who used additive strategies focused on a single quantity, either the amount of water or the amount of orange juice, and this tendency resulted in their incorrect reasoning. Five of the items involved noninteger ratios, and as discussed by Noelting (1980a, 1980b), these items were more difficult for students. Therefore, these five items were placed in the higher stages.

Karplus et al. (1983a, 1983b) investigated adolescents' proportional reasoning. Karplus et al. (1983a) used four comparison problems, and Karplus et al. (1983b) used eight lemonade puzzles. The first study was conducted with 116 sixth graders and 137 eight graders. Four problems were administrated during one class period. Responses to those four problems were combined into seven categories: Category N (N: no response), Category I (I: incorrect), Category C (C: correct), Category E (E: error), Category Pw (Pw: proportional within), Category Pb (Pb: proportional between), and Category Po (Po: proportional other).

Karplus et al. (1983a, 1983b) and Noelting (1980a, 1980b) used a different approach to define between and within strategies than Vergnaud (1983, 1988). If a person compared quantities from two different measure spaces, then Vergnaud (1983, 1988) defined the strategy that the person used to compare quantities as a between strategy and the ratio as a between measure space ratio. On the other hand, if a person used quantities from the same measure space, then he defined the strategy as a within strategy and the ratio as a within measure space ratio.

According to Karplus et al. (1983a, 1983b) and Noelting (1980a, 1980b), the quantities compared in a between approach formed an extensive variable. For example, comparing distances 50 km and 60 km was a between strategy, and the ratio $\frac{5}{6}$ was a between ratio. Conversely, two quantities compared in a within approach formed an intensive variable. For instance, comparing 50 km and 2 hours was a within strategy, and the ratio $\frac{50 \text{ km}}{2 \text{ hours}}$ was a within ratio. Therefore, in the within strategy, two quantities formed a rate. For Vergnaud (1983, 1988), comparing distances 50 mph and 60 mph was a within measure space strategy, and $\frac{5}{6}$ was a within measure space ratio. Similarly, for him, comparing 50 km and 2 hours was a between measure space strategy, and the ratio $\frac{50 \text{ km}}{2 \text{ hours}}$ was a between measure space ratio.

As discussed by Karplus et al. (1983a), even though Vergnaud (1981) proposed the *between* strategy as more natural than the *within* strategy (here the terms *between* and *within strategies* were used in terms of Karplus et al.'s (1983a, 1983b) and Noelting's (1980a, 1980b) definitions), in Karplus et al. (1983a, 1983b), the *within* strategy seemed to be more natural than the *between* strategy. Karplus et al. (1983a) stated that, before the 14th century, the *between* strategy was the only accepted form of proportional computation. They also stated that the *within* strategy seemed more faithful to comparison and missing-value problems, since an intensive variable was defined by a within ratio. As a result, Karplus et al. (1983a) concluded that "…the relative frequencies with which the types of comparison and various strategies are used is affected greatly by the context and numerical content of the problem, and even the immediately preceding task" (p. 231). In addition, approximately one-fourth of the errors were because of incomplete calculations.

The second study explored the proportional and other types of reasoning of 60 sixth-graders and 60 eighth graders. Four of the lemonade puzzles had equal ratios, and the remaining four had unequal ratios. As discussed by Karplus et al. (1983b), the most difficult puzzle for students was the one which involved unequal nonintegral ratios. When unequal nonintegral ratios were presented, students generally preferred additive strategies. The subjects used *within* comparisons more than *between* comparisons, and this result for Karplus et al. (1983b) differed from what their previous research found. Hence, they concluded that this result was evidence of context and numerical relationships significantly influencing the types of comparison used by the subjects. Karplus et al. (1983a) mentioned that, in the Karplus et al. (1983b) study, "the reasoning approach actually used by most adolescents is determined by the presence of equal and/or integral ratios and not by the fact that sweetness of the concentrate is a constant intensive variable defined by the lemon/sugar ratio" (p. 222).

Tourniaire (1986) explored elementary school students' strategies in solving proportion problems. Sixty pupils from Grades 3, 4, and 5 participated in the study. Two interviews were conducted. Three problems were used in the first interviews, and four problems were used in the second interviews. As mentioned by Tourniaire (1986), since this study involved young students, small numbers and integer ratios were used. Also, all of the problems were missing-value type problems. In the first interview, all of the 60 pupils were interviewed, and in the second one, 30 pupils were interviewed. Two types of mixture problems were used: orange juice and paint. In order to classify students' strategies, Tourniaire (1986) used a strategy scale that was developed by Karplus et al. (1983b). The scale had four levels: incomplete, qualitative, additive, and proportional.

The success rate for the orange juice problem in the first interview was 60% and, in the second interview, 63%. However, for the paint problem, which was only used in the first interview, the success rate was 37%. Because students were forbidden to play with paints in their classrooms, they were less familiar with a paint context. Thus, as stated by Tourniaire (1986), familiarity with the use of ratios in a context made a difference in the children's abilities to answer the problem, such as the distinction between the orange juice problem and the paint problem. He also stated that the presence of a mixture appeared to increase the difficulty of the question. Also for Tourniaire (1986), students' responses indicated that young children have some idea of the concept of proportions. He also stated that the context of the problem influenced students' performance. In addition, he mentioned that very few additive strategies were presented and no qualitative strategies were found. However, a repeated addition strategy was usually used by the subjects, but it did not exist on the strategy scale.

Harel, Behr, Lesh, and Post (1994) studied the concept of taste constancy with tasks that required students to compare the "orangeness" of two glasses of orange juice that were taken from the same mixture. Since the glasses were filled from the same mixture, the compared ratios were always equal. As discussed by Harel et al. (1994), Noelting (1980a, 1980b) and Tourniaire (1986) had taken children's conceptions of taste constancy for granted. Therefore, Harel et al. (1994) aimed to show that children's conceptions of taste constancy cannot be taken for granted. Sixteen sixth grade children participated in this study. When students were shown two different sized glasses that were filled from the same orange juice box, eight of these children stated that the glasses would "taste different." Later, these eight students were interviewed to understand their responses. At the end of the interview processes, as stated by Harel et al. (1994), only one child changed his/her answer from "Not the same" to "The same," and the other remaining children did not change their answers. The student with the answer "they taste the same" was at the stage of understanding the quantity of taste as an intuitive internalized ratio. Students with answers “do not taste the same” believed that “the taste of a mixture depends on the volume occupied by the mixture” (Harel et al., 1994, p. 333).

As discussed by Harel et al. (1994), three factors affected children’s reasoning about the constancy of taste: the relative volumes of the mixtures to be tasted, the uniformity of the liquid to be tasted, that is, whether it is thought of as consisting of a single ingredient or more than one ingredient, and the numerical data of the problem. Students’ tendency to focus on a single quantity was a sign of their use of *univariate reasoning*, which was also discussed by Lobato and Ellis (2010). They mentioned that before students are able to reason with ratios, they reason with a single quantity. This type of reasoning is called *univariate reasoning*.

Studies on Teachers’ Proportional Reasoning

As discussed by Izsák and Jacobson (2013), a small number of studies on teachers’ reasoning about proportional relationships have reported that “teachers’ difficulties are similar to students’ difficulties” (p. 3). For example: Teachers tend to judge nonproportional relationships to be proportional (Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010); they have difficulty understanding ratio-as-measure and the invariance of a ratio and tend to use additive strategies to solve proportion problems (Riley, 2010; Simon & Blume, 1994); they have trouble creating suitable reciprocal multiplicative relationships for nonproportional problems (Izsák & Jacobson, 2013); and they have difficulty with the conceptual understanding of the solutions to direct proportion problems (Lim, 2009).

In order to identify secondary mathematics teachers’ strategies on proportion problems, Fisher (1988) interviewed 20 teachers. Four problems were designed of which there were two inverse proportion problems, a wheel problem and a work problem. She found out that 12 out of 20 teachers solved the work problem incorrectly, where nine of them attempted the problem as if it was a direct proportion problem. Similarly, six teachers solved the wheel problem incorrectly. However, none of the teachers attempted the wheel problem as a direct proportion problem. The teachers' scores were almost perfect on the remaining two direct proportion problems (Fisher, 1988).

Cramer, Post, and Currier (1993) designed a word problem in which there was a constant difference between the numbers of laps completed by two runners: Sue and Julie. They were running equally fast around a track. Sue started first, and when she had run nine laps, Julie had run three laps. Participants were asked to determine the number of laps Sue ran when Julie completed 15 laps. Although this problem did not involve a proportional relationship, as stated by Cramer et al. (1993), 32 out of 33 preservice elementary education teachers in a mathematics methods class approached this problem as if it were a direct proportion problem. In addition to this lap problem, Cramer et al. (1993) also provided a direct proportion problem to their participants, and they observed that all of the participants answered this problem correctly.

Simon and Blume (1994) focused on the development of the ratio-as-measure concept. For them, ratio-as-measure is "the ability to identify a ratio as the appropriate measure of a given attribute" (Simon & Blume, 1994, p. 184). A mathematics course on mathematics learning and teaching was designed to collect data. Twenty six prospective elementary teachers proceeded through a 5-week pre-student-teaching practicum and a 15-week student-teaching practicum. A pre-test problem, which involved understanding the "squareness" of three rectangles, was given to prospective elementary teachers. Nineteen out of the 26 teachers used an additive strategy to solve this problem. During the study, teachers were asked to compare the steepness of different ski ramps. As discussed by Simon and Blume (1994), teachers used two methods to indicate steepness: First, they compared the height**:** width ratios of two ski ramps. Second, they compared the difference between the heights and the widths. When teachers were given different sized ramps that had the same slopes, their reasoning seemed to focus neither on the invariance of the slopes nor the invariance of the ratios.

Lim (2009) studied the proportion concept with the assistance of 28 PSTs of Grades 4-8. The PSTs were given five tasks, each of which was either nonproportional or proportional. In one task, two similar candles, A and B, are burning at the same rate, but they were lit at different times. When Candle A had burned 20 mm, Candle B had burned 12 mm. The PSTs were asked to calculate how many millimeters of Candle A had burned given that Candle B had burned 30 mm. In another task, two candles, P and Q, are burning at different rates, and they were lit at the same time. When P had burned 16 mm, Candle Q had burned 10 mm. The PSTs were asked to calculate how many millimeters of Candle P had burned given that Candle Q had burned 35 mm. Lim (2009) reported that 23 out of 28 PSTs used the same strategy for both tasks. Seventeen out of the 23 used a proportional strategy, and of those 17, 13 set up a proportion and calculated the missing value, and the other four used a unit ratio strategy. Five out of the 23 used an additive strategy, and the remaining one used an incorrect strategy. One student solved the first task using a unit ratio strategy, and for the second task, he/she "wrote 'not enough information to determine how fast candle P is burning'" (Lim, 2009, p. 496). Only four of the 28 PSTs used a correct additive strategy for the first task and a correct proportional strategy for the second task. Lim (2009) stated that participants who correctly solved the first task appeared to have a *referential meaning* for the 18 mm difference and the 8 mm difference. He noted that 21 of the participants solved the first task correctly. He also stated that although more participants successfully completed the second task than the first task, they appeared to have more difficulty with the conceptual understanding of the solution for the second task. For example, Lim (2009) reported that none of the participants who used a proportional strategy for the second task explained the meaning of the ratio 16:10 or the ratio 35:10 in their written responses.

Riley (2010) used a lap problem in her study, and she mentioned that only 46% of the 80 preservice elementary teachers answered this problem correctly. She stated that participants who gave an incorrect answer usually set up a proportion and used the cross multiplication strategy as if the problem was a direct proportion problem. In addition to the lap problem, she also used a comparison problem, a missing-value problem, and three inverse proportion problems. She stated that 90% and 71% of the PSTs answered the missing-value problem and the comparison problem correctly respectively. Three inverse proportion problems were answered correctly by 48%, 39%, and 39% of the PSTs. Riley (2010) stated that teachers who gave incorrect responses to these three problems generally set up a proportion and then cross multiplied the given numbers.

In addition to these studies, Izsák and Jacobson (2013) also investigated the reasoning of four pairs of preservice middle school teachers and four pairs of preservice secondary teachers. Two nonproportional word problems, a dumpling problem and a team problem were used. The dumpling problem involved the constant difference of quantities, and the team problem involved the constant product of the quantities. As stated by Izsák and Jacobson (2013), in addition to these two problems, the preservice secondary teachers were also asked to sort six word problems. Only the responses from the three pairs of preservice middle school teachers for the dumpling problem were discussed. Even though all three pairs had a correct understanding of the constant difference between the numbers of dumplings, two pairs initially judged the relationship to be proportional. A common problem with these two pairs was that they had “trouble coordinating their meanings for ‘same rate’ and ‘constant ratio’ with the constant difference between the numbers of dumplings” (Izsák & Jacobson, 2013, p. 14). The last pair rejected the possibility of a proportional relationship quickly opting for a multiplicative comparison (Izsák & Jacobson, 2013). However, the secondary teachers quickly realized the nonproportional nature of the constant difference problems (Izsák & Jacobson, 2013). As discussed by Izsák and Jacobson (2013), none of the participants constructed a constant product in any of the constant product problems. Many of them, at first, thought that these problems could be solved using a direct proportion, but later, they realized the inverse relations in the problems. Most of the participants also had trouble creating suitable reciprocal multiplicative relationships; even participants who focused on multiplicative comparisons were unable to explain reciprocal multiplicative relationships. In addition, Izsák and Jacobson (2013) stated that participants who were able to solve these problems reasoned out the amount of time in which a single person could complete the job.

Multiplicative Reasoning

In this section, I explain how the term *multiplicative reasoning* is used in this dissertation. In her definition of the proportional reasoning, which I provided in Chapter One, Lamon states that proportional reasoning involves “the ability to discern a multiplicative relationship between two quantities” (2007, p. 638). Hence, proportional reasoning necessitates understanding multiplicative relationships between two covarying quantities. Following Vergnaud’s (1983, 1988) *multiplicative conceptual field* framework, I interpret the term *multiplicative* to include multiplication, division, fractions, ratios, and proportions. Vergnaud (1983, 1988) discusses three types of multiplicative structures: *isomorphism of measures, product of measures*, and *multiple proportion other than product*. This study contains *isomorphism of measures* and *multiple proportion other than product* structures and strengthens Vergnaud’s work by investigating constant product relationships. Therefore, by adapting Lamon’s (2007) definition of proportional reasoning and implementing it to the *multiplicative conceptual field* framework, I describe multiplicative reasoning to mean supplying reasons in support of claims made about the multiplicative structures presented in problem contexts that consists of the ability to understand multiplicative relationships and to form multiplicative comparisons between covarying quantities.

As I discussed in the previous chapter, multiplicative relationships can be expressed by the equation $M*N = P$, where M, N, and P stand for known constants (Beckmann & Izsák, 2015). According to Beckmann and Izsák (2015), in this equation the multiplier, M, is interpreted as the number of groups, the multiplicand, N, is interpreted as the number of units in each group, and the product, P, is interpreted as the number of units in M groups. In this equation, the *isomorphism of measures* structure can be expressed by either the equation $x * N = y$ or $M * x = y$, where x and y are either unknown variable amounts or two co-varying values. Based on the types of units of quantities that we are comparing, we can express multiplicative relationships between quantities either with the first equation or the second equation. The first equation can be used to express a multiplicative relationship between quantities with different types of units. For instance, the following question can be given as an example to illustrate this multiplicative relationship:

> *Example 1:* If Gear A, with a 2-cm radius, have 4 notches around, then how many notches around Gear B, with a 4-cm radius (assuming Gear A and Gear B can be meshed)?

In this question, the multiplicative relationship between the radii and number of notches can be expressed by either (R cm) * ($\frac{2}{1}$ notch/cm) = (N notches) or (N notches) * ($\frac{1}{2}$ cm/notch) = (R cm). In these two equations, multiplicands represent the unit rate. In the two equations the unit rates can be interpreted as “two notches per 1 cm” and “$\frac{1}{2}$ cm radius per one notch,” respectively. Substituting four for R in one of these two equations, one can determine the number of notches to be eight. In this question, “four notches in 2 cm” can be viewed as a composed unit (Lobato & Ellis, 2010) or “batch.” Hence, eight notches can be calculated iterating this batch two times (Figure 5). The perspective used to interpret this type of proportional relationships is called the *multiple batches perspective*. Similar interpretation also works for the second equation where we need to consider the unit rate, $\frac{1}{2}$ cm radius per one notch, as a batch. When two is substituted for x and four is substituted for y in the $x * \mathrm{N} = y$ equation, the division to obtain N, which is the unit rate, is called a *partitive* or *how many units in each group?* division (Beckmann & Izsák, 2015).

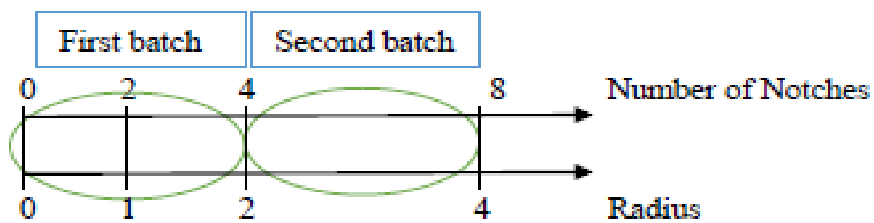


Figure 5. Expression of a directly proportional relationship from the multiple batches perspective.

On the other hand, the multiplicative relationship in the second equation—$\mathrm{M} * x = y$—can be obtained if x and y have the same type of units. The following example can be used to illustrate this situation:

Example 2: A certain brand of lemonade is made by mixing 3 ounces of water with 2 ounces of lemon concentrate. How many ounces of lemon concentrate are needed to be mixed with 12 ounces of water?

In this example, the multiplicative relationship between ounces of water and lemon concentrate can be expressed by either $\frac{2}{3} *$ (Ounces of Water) = (Ounces of Lemon Concentrate) or $\frac{3}{2} *$ (Ounces of Lemon Concentrate) = (Ounces of Water). In these two equations, the multipliers are numbers, which are unit-less. The multiplicative relationships in these equations can be interpreted as “The amount of lemon concentrate is two-thirds the amount of water” and “The amount of water is three-halves the amount of lemon concentrate,” respectively. Hence, the amount of lemon concentrate and water mixed in a fixed 2 to 3 ratio that means for same-sized part there are 2 parts of lemon concentrate and 3 parts of water. As seen in Figure 6, the number of parts is fixed but they can vary in size. For instance, the size of each part will be 1 ounce if we make a 5-ounce mixture. On the other hand, the size of each part will be 4 ounces if we make a 20-ounce mixture. According to Beckmann and Izsák (2015), the perspective used to interpret this type of proportional relationships is called the *variable parts perspective*. Substituting 12 for the ounces of water, one can calculate the amount of lemon concentrate to be 8 ounces. When 3 ounces is substituted for x and 2 ounces is substituted for y in the $M * x = y$ equation, the division to obtain M, which is a number, is called a *measurement* or *how many groups?* division (e.g., Greer, 1992).

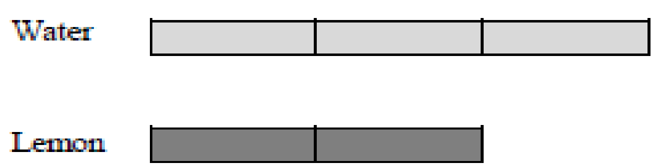


Figure 6. Expression of a directly proportional relationship from the variable parts perspective.

Based on the equation $M * N = P$, a constant product relationship satisfies the equation $x * y = P$, where x and y are unknown quantities or two co-varying values, and P is a known constant (Beckmann & Izsák, 2015). The gear context can also be used to illustrate a constant product relationship. The following example demonstrates a constant product relationship between the number of notches and number of revolutions:

> *Example 3:* If Gear A, with eight notches, revolves three times, then how many times does Gear B, with six notches, revolve (assuming Gear A and Gear B can be meshed)?

In this question, the product of the number of notches of a gear by its number of revolutions yield the total notches revolved around this gear. For instance, on Gear A, eight notches revolve in one revolution, so in three revolutions a total of 24 notches revolve. Hence, eight notches can be interpreted as the number of notches revolved in one revolution and the number of revolutions gives us how many groups of eight notches revolved. Because both gears are meshed, the same 24 notches revolve around both gears that means the product of the number of notches and number of revolutions remains constant (Figure 7). Therefore, in the question, the multiplication statement can be expressed by (R revolutions) * (N notches per revolution) = (P notches). Substituting six and 24 for N and P, respectively, one can make a measurement division and calculate the number of revolutions of Gear B as four.

Revolutions	Notches
1	24
2	12
3	8
4	6

Figure 7. Expression of a constant product relationship.

Vergnaud's (1983, 1988) *multiple proportion other than product* (direct-direct-inverse) structure can be expressed by the equation $x * y = z$, where x, y, and z are either unknown variable amounts or co-varying values. In this structure, *z* is directly proportional to *x* and *y* quantities, and *x* and *y* are inversely proportional. The following example illustrates this structure:

> *Example 4:* If four people frost 20 cupcakes in 12 minutes, then how long would it take for three people to frost 30 cupcakes (assuming all people work at the same pace)?

In this question, the number of cupcakes is directly proportional to the number of people and number of minutes, but the number of people and the number of minutes are inversely proportional. The multiplicative relationships among the three quantities can be expressed by the statement, (P people) * (T minutes) = (C cupcakes) * *k*, where *k* represents the constant of the proportionality. Substituting four, 12, and 20 for P, T, and C in the statement, one can calculate *k* as $\frac{48}{20}$ or simply $\frac{12}{5}$. For this particular example, $\frac{12}{5}$ represents the number of minutes needed to frost one cupcake by one person. Therefore, the multiplication statement becomes (P people) * (T minutes) = (C cupcakes) * $\frac{12}{5}$. Substituting three and 30 for P and C in the statement, one can calculate the number of minutes required for three people to frost 30 cupcakes as 24 minutes.

In the following chapter, I discuss the methodology that I used in this study. To do that, first, I explain the theoretical framework that informs this study. Next, I describe the research design and explain the participant selection criteria and the method for data collection and analysis. Later, I explain the problem tasks, give a rationale for inclusion of the tasks, and give advanced summaries of each task. Finally, I discuss the pilot study and its findings.

CHAPTER 3

METHODOLOGY

Theoretical Framework

The purpose of this study is to explore preservice middle and high school mathematics teachers' understanding of directly and inversely proportional relationships. More specifically, I examine the types of strategies they use to solve given problems, their ability to detect and represent directly and inversely proportional relationships, and the reasoning they engage in when solving these problems. I developed the conceptual and theoretical framework for this study drawing on a number of such frameworks reported in past research. In particular, I used Vergnaud's (1983, 1988) the *multiplicative conceptual field theory* to develop the categories of proportions (see Figure 1) and to explain multiplicative structures of the problems. I used the solution strategies framework described by Fisher (1988) and Lamon (1993) to classify and explain PSTs' solutions strategies. Finally, to analyze knowledge resources that PSTs used to infer directly and inversely proportional relationships and multiplicative relationships, I used core components of the *knowledge-in-pieces* epistemological perspective (diSessa, 1988, 1993, 2006). In the following paragraphs, I explain the knowledge-in-pieces perspective and describe its core components.

Knowledge-in-Pieces Perspective

Developed originally in the area of Newtonian mechanics (e.g., diSessa 1988, 1993), the knowledge-in-pieces epistemological perspective has its roots in science education research on conceptual change (Izsàk & Jacobson, under review). However, as discussed by Izsàk and

Jacobson (under review), this perspective has also been applied in several areas of mathematics, including whole-number multiplication, fractions, functions, and probability. The knowledge-in-pieces perspective acknowledges that elements of knowledge are "more diverse and smaller in grain size than those presented in textbooks" (Izsàk, 2005, pp. 361-362). According to this perspective, growth and change in one's knowledge entails various related processes, including the development of new knowledge elements as well as the coordination of diverse knowledge elements and the adjustment of conditions under which specific elements may be used productively (Izsàk, 2005). One of the central constructs that is used by diSessa (1988, 1993) to explain a particular example of knowledge structures is p-prims. For diSessa (1988), p-prims can be understood as "simple abstractions from common experiences that are taken as relatively primitive in the sense they generally need no explanation" (p. 52). An example given by diSessa (1988) is the statement that a person gets more result when he or she expends more effort.

diSessa and Sherin (1998) stated that previous research on conceptual change did not explain explicitly what constitutes a concept. For diSessa and Sherin (1998), although "concepts are at the core of our understanding…they are left as something of a black box" (p. 1161). Additionally, previous models of concept included only a small number of mental structures that were associated with a concept (diSessa & Sherin, 1998). However, diSessa and Sherin (1998) describe their model of concept as a knowledge system. As stated by diSessa (1988), students have a fragmented system of intuitive knowledge, and this system of intuitive knowledge reveals important educational problems. Furthermore, the traditional models of conceptual change are interested in whether or not a person has a concept, but diSessa and Sherin (1998) believe that instead of looking at people's possession of concepts, "it is necessary to describe specific ways in which a learner's concept system behaves like an expert's - and the ways and circumstances in which it behaves differently" (p. 1170). Hence, a decade later, within the knowledge-in-pieces perspective diSessa and Sherin (1998) proposed the *coordination classes* construct as a certain class of concepts. They defined a coordination class as a "systematic collection of strategies for reading a certain type of information out from the world" (p.1155). Therefore, for diSessa and Sherin (1998), coordination classes were "systematically connected ways of getting information from the world" (p. 1171).

As stated by diSessa and Sherin (1998), to highlight a coordination class perspective, they see *coordination* as a verb representing 'see' or 'determine information" (pp. 1171-1172). For instance, "when a person makes observations and uses prior knowledge to make inferences from those observations, the person has performed an act of coordination" (Thaden-Koch, 2003, p. 1). There are two mechanisms that make up a coordination class: readout strategies and causal net. Readout strategies "deal with the diversity of presentations of information to determine, for example, characteristic attributes of a concept exemplar in different situations" (diSessa & Sherin, 1998, p. 1171), or more simply, they are strategies for acquiring information about the physical world. diSessa and Sherin (1998) suggest two kinds of coordination that are central to readout: *integration* and *invariance*. Integration is frequently collecting, selecting, or combining "diverse observations to determine what we wish to see" (diSessa & Sherin, 1998, p. 1176), and invariance deals with "how observations in different circumstances can manage to determine the same information" (diSessa & Sherin, 1998, p. 1176). The second mechanism, the causal net, is "The general class of knowledge and reasoning strategies that determines when and how some observations are related to the information at issue" (diSessa & Sherin, 1998, p. 1176).

As stated by diSessa and Sherin (1998), read out strategies and the causal net are closely related. Therefore, they should "co-evolve" as learning happens, and features of one will have important effects on how the other operates and advances (diSessa & Sherin, 1998, p. 1177). The distinct changes in read out strategies and in the causal net determine the features of conceptual change (diSessa & Sherin, 1998). Following diSessa and Sherin (1998), in learning of a new coordination class, no new read out strategies may be necessary; however, the existing read out strategies can be arranged and used in other ways. Furthermore, at times new read out strategies may be necessary in acquiring information. On the other hand, according to diSessa and Sherin (1998), one may not have a prior causal net, so the construction of a new net from "whole cloth" may be required, or "an old causal net may need to be developed and reorganized to varying degrees" (p. 1177). According to diSessa and Sherin (1998), not all examples discussed as concepts are coordination classes. For example, p-prims may establish concepts; however, since by themselves p-prims are "too small and isolated," they cannot be counted as coordination classes (p. 1178). In this study, I did not utilize coordination classes construct to explain the PSTs' reasoning, but instead I investigated their coordination of the directly and inversely proportional relationships. Following diSessa and Sherin (1998), I use the term *coordination* in the sense of determining and integrating information within a problem context.

Applying Knowledge-in-Pieces Perspective to Proportional Relationships

Izsak and Jacobson (under review) stated that their participants attended to the following knowledge resources in inferring proportional relationships: "…associations with specific phrase like "same rate," facility with multiplicative relationships between numbers, and attention to multiplicative relationships within and between measure spaces" (p. 29). In addition to these knowledge resources, a PST might infer a proportional relationship attending to the qualitative relationships that illustrate coordinated increases or decreases in the values of covarying quantities. Izsak and Jacobson (under review) observed that attention to the multiplicative relationships helped the PSTs in correcting their initial incorrect inferences of the directly proportional relationships. Hence, in this study, I expect that PSTs who attend to multiplicative relationships between and within measure spaces when inferring directly and inversely proportional relationships are more likely to distinguish proportional relationships from nonproportional relationships. On the contrary, I expect that PSTs who just rely on the specific phrases and qualitative relationships may have difficulty distinguishing proportional relationships from nonproportional relationships. Besides determining PSTs' knowledge resources in inferring directly and inversely proportional relationships, the knowledge-in-pieces perspective is also effective in determining the range of knowledge resources PSTs use to reason about multiplicative relationships. Therefore, the knowledge-in-pieces perspective is utilized in determining these knowledge resources.

In this study, I expect that PSTs may use a variety of knowledge resources when inferring directly and inversely proportional relationships. I also expect that some PSTs may judge inversely proportional relationships as if they were directly proportional. Similarly, some PSTs may have prior knowledge resources that either help them to infer, for example, correct multiplicative relationships between two covarying quantities or prevent them from inferring correct multiplicative relationships. On the other hand, PSTs may not have prior knowledge on the topic and may need to construct a new knowledge in order to understand proportional relationships. I decided to use the knowledge-in-pieces perspective to analyze PSTs' responses for the following reasons: First, the knowledge-in-pieces perspective can be used effectively in identifying PSTs' knowledge resources in inferring directly and inversely proportional relationships between two covarying quantities and the reasoning behind their correct and incorrect inferences. Second, it is helpful in the sense of analyzing participants' "contextually sensitive" (Wagner, 2006, p. 7) intuitive knowledge and characterizing the evolution of this knowledge from naïve to expert. Third, the knowledge-in-pieces perspective offers effective tools for analyzing how PSTs infer directly and inversely proportional relationships, the strategies they use in inferring these relationships, and the difficulties they experience while working on the single and multiple proportion problems. Fourth, the knowledge-in-pieces perspective is effective in analyzing cognitive structures that PSTs use to identify directly and inversely proportional relationships. Finally, this study makes use of science concepts, and PSTs' comprehension of these concepts can be analyzed by employing the knowledge-in-pieces perspective.

Research Design

Multiple-Case Study Methodology

I designed this study to have two parts: a pilot study and a final study. In the pilot study, I tested my ideas about the mathematical tasks that I was planning to use, and I tried to understand PSTs' expertise in reasoning about proportional relationships. Consequently, I was able to refine my ideas about the mathematical tasks that I intended to use, and I was able to observe PSTs' reasoning about proportional relationships, which helped me revise my research questions, mathematical tasks, and criteria for participant selection. To explain my participants' knowledge resources when inferring directly and inversely proportional relationships, an explanatory multiple-case study methodology (e.g., Yin, 1993, 2009) was used to design this study. Yin (2009) offered a twofold definition of case study methodology. The first part of the definition involved the scope of a case study: "A case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (Yin, 2009, 18). The second part of the definition involved technical characteristics of a case study. For Yin (2009), the case study inquiry:

- copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result
- relies on multiple sources of evidence, with data needing to converge in a triangulation fashion, and as another result
- benefits from the prior development of theoretical propositions to guide data collection and analysis. (p. 18)

Yin (2009) defined a case as “a concrete entity, event, occurrence, action, but not an abstract topic such as a concept, argument, hypothesis, or theory” (p. x). As Stake (1978) points out, “The case need not be a person or enterprise. It can be whatever ‘bounded system’…is of interest” (p. 7). Because one of the purposes of this study was to explore PSTs’ reasoning, each individual participant constituted a case in this study. Since there was more than one case, a multiple-case study methodology best suited the scope of this study.

This study focused on a contemporary phenomenon—exploring PSTs’ understanding of and difficulties with directly and inversely proportional relationships. Interview videos, transcripts, and participants’ written responses were derived from the interview data and were used to explain the research phenomenon and the contextual conditions. Two research questions, which included a combination of sub questions, were designed to guide this study. The questions were intended to analyze how PSTs infer and express directly and inversely proportional relationships in the given problem contexts, the solution strategies that they use to solve single and multiple proportion problems, and the difficulties they encounter while solving these problems. While participants were working on the problem tasks, I tried not to interfere with the participants' ways of reasoning and dealing with the interview activities since my role was to facilitate these activities during the process of data collection. According to Yin (1993), a case study is a preferred method when (a) the researcher seeks to answer research questions that are designed to explain "how" and "why" a social phenomenon works, (b) the researcher has slight control over events, and (c) the study focuses on a contemporary phenomenon within an authentic situation (Yin, 2009). Thus, the case study appeared to be a proper methodological design for this study.

Because one of the goals of this study was to explore PSTs' understanding and reasoning about direct and inverse proportions and, as a result, to explain their comprehension of directly and inversely proportional relationships, I conceptualized this study as an explanatory case study methodology. Yin (1993) briefly described explanatory case study as follows: "An explanatory case study presents data bearing on cause-effect relationships—explaining which causes produced which effects" (p. 5). As highlighted by Mills, Durepos, and Wiebe (2010), besides exploring and describing phenomena, explanatory case studies can also be used to develop theory and explain causal relationships. For them, a key feature of an explanatory case study is that during the research process the researchers need to remain open to new discoveries.

One of the most important strengths of case study is that it allows researchers to explore a real-life phenomenon in depth (Yin, 2009). In addition to this strength, George and Bennett (2004) explained four other strengths of case studies: (a) case studies have potential for obtaining high conceptual validity; (b) they have well-built procedures for promoting new hypotheses; (c) by using case studies in the context of individual cases, researchers can intimately analyze the conjectured role of causal mechanisms; and (d) case studies can be used to address causal complexity. Meyer (2001) underlined that in contrast to grounded theory or survey research methods, there are nearly no clear-cut requirements instructing case study research. For Meyer (2001), this is both a strength and weakness of case study research. This is a strength because having no specific requirements allows researchers to tailor "the design and data collection procedures to the research questions" (Meyer, 2001, p. 330). Conversely, having no specific requirements may result in the development of poor case studies and this is one of the criticisms of the case study methodology. A second criticism is that case studies offer little basis for scientific generalization (Yin, 2009). Especially, the generalizability of the single-case studies is usually questioned. As an answer to this criticism Yin (2009) states that "…case studies, like experiments, are generalizable to theoretical propositions and not to populations or universes" (p. 15). Stake (1978) offers *naturalistic generalization* as an alternative to scientific generalization in qualitative research:

> Naturalistic generalizations develop within a person as a product of experience. They drive from the tacit knowledge of how things are, why they are, how people feel about them, and how these things are likely to be later or in other places with which this person is familiar. (Stake, 1978, p. 6)

According to Stake (1978), there is a necessity for generalization about a specific case or generalization to a similar case, and readers organize the foundation for naturalistic generalization as they identify important similarities to cases. A third criticism to the case studies is that "they take too long, and they result in massive, unreadable documents" (Yin, 2009, p. 15).

I used the following three tests to establish the validity of this case study: *construct validity*, *internal validity*, and *external validity* (e.g., Yin, 1993, 2009). Construct validity deals with "identifying correct operational measures for the concepts being studied" (Yin, 2009, p. 40). As explained by Yin (2009), internal validity, which is only used with explanatory case studies, is concerned with establishing a causal relationship. External validity deals with "defining the domain to which a study's findings can be generalized" (p. 40). Parallel to the case study tactics discussed in Yin (2009), I established each one of these three tests as follows: Construct validity was addressed by using sources of evidence (e.g., interview videos, transcripts, participants' written responses) that were derived from the interview data and by having key informants (the members of my advisory committee) review the draft of the case study report. Since cross-task analysis yielded information about the causal relationships, I used it to strengthen the internal validity of this study. Many qualitative researchers reject causation because they believe that causal explanation is "incompatible with an interpretivist or constructivist approach" (Maxwell, 2012, p. 655). For Maxwell (2012), the main reason for qualitative researchers' rejection of causal explanation is that since it has been used regularly in quantitative research, it has an association with positivism. Maxwell (2012) opposes the rejection of causality in qualitative research and offers a realist approach. To address external validity, I used literal replication logic in which each case was "carefully selected so that it…predicts similar results" (Yin, 2009, p. 54). Because, as opposed to purposefully selecting for diversity, I selected my participants from a small sample (PSTs who took or have been taking content courses on ratios and proportions), each case produced results that would be generalizable to the selected sample. Therefore, the participants represented the cases.

Participants and Selection Process

Since the purpose of this study was to understand PSTs' reasoning about directly and inversely proportional relationships, I recruited prospective middle and high school mathematics teachers from one large public university in the Southeast. The university offers separate programs leading to certification for secondary grades (6-12) and middle grades (4-8) mathematics teachers. PSTs, therefore, were selected from these two programs. Since the focus of this study involved challenging mathematical problems, PSTs with some college level experience on direct and inverse proportions were preferred. The secondary grades program includes one content course with a focus on multiplicative relationships, ratios, and proportions; the middle grades program includes two such content courses. Therefore, preservice teachers who attended or were attending one of these courses were selected. Although I was an observer in those classes, my relationship with those participants at the time did not extend beyond that role. Furthermore, participation in the study was voluntarily and each participant was given 10 dollars incentive for each hour of participation.

For the pilot study, in the spring semester of 2013, once Institutional Review Board (IRB) approval had been secured, I contacted a course instructor who had been teaching a secondary teacher education course. I asked the instructor to forward an electronic recruitment message to the students in his class. One female and two male preservice secondary grade teachers agreed to participate in the study. Later, in the fall semester of 2013, two preservice middle grade teachers, one female and one male, agreed to participate in the study. To maintain confidentiality, I used the following pseudonyms for the secondary grade PSTs: Sally, Robert, and Jason; for the middle grade PSTs, I used Abby and Michael. All participants were in the third year of their programs. Sally stated that she was majoring in mathematics education and was attending a university course in which she was studying proportional relationships. Robert stated that he was majoring in mathematics education with a minor in statistics. He pointed out that he had not previously attended any university courses with a focus on proportional relationships. However, he said he helped middle school students solve proportion problems that, according to him, were similar to the problems he had encountered in this study. Jason said that he was majoring in mathematics education with a concentration on mathematics, so he was pursuing a dual degree. He stated that he had taken two university courses that focused on proportional relationships. Abby was, at the time, taking a course that focused on proportional relationships. Michael stated that he had not taken any university courses on proportional relationships. Because Michael had conceptual difficulties understanding the topics discussed in the tasks, I excluded his case. Thus, only responses from four of the participants were analyzed.

For the final study, in the fall semester of 2014, I contacted two course instructors. One was teaching a course designed for middle grade PSTs, and the other one was teaching a course designed for secondary grade PSTs. I introduced my study to the PSTs in both courses and asked if they would like to participate in the study. I let them know that their participation in the study was voluntary, and that they would be given $10 of incentive per 1 hour of involvement. Six female secondary grade teachers, and one male and three female middle grade teachers agreed to participate in the study. Based on the information that they gave about their background on the direct and inverse proportions, three female secondary grade and two female middle grade PSTs were chosen among 10 teachers. To maintain confidentiality, I use the following pseudonyms for the secondary grade PSTs: Kathy, Susan, and Amy; for the middle grade PSTs, I use Carol and Helen. With the exception of Susan who was in the third year of her program, the remaining participants were in the fourth year of their programs. Amy and Kathy stated that they were majoring in mathematics education, and Susan said that she was majoring in mathematics education with a concentration on mathematics. They all had been attending a course with a focus on directly and inversely proportional relationships. Carol and Helen were both majoring in middle grades education with a concentration in mathematics, and they both stated that they took two courses in which they studied the directly and inversely proportional relationships.

During the interview process, I observed that Amy had difficulty understanding the topics we discussed. For that reason I excluded her case in this study. Thus, only responses from four of the participants were analyzed and discussed in detail.

Data Collection and Analysis

The data was collected through semi-structured clinical interviews (e.g., Bernard, 1994). This allowed me to maintain a consistent interview structure with each participant (see interview protocols, Appendices A and B), while also providing me the flexibility to probe or modify the follow up questions depending on the responses that I obtained from the participants. Semi-structured clinical interviews were helpful in capturing verbal and nonverbal components of the participants' explanations, gestures, and writing. During the interview process, I adopted a facilitator role, being careful not to create a teacher-student kind of relationship between my participants and myself. Furthermore, during the interviews, I tried not to intervene in their reasoning or direct their thinking in any particular way; however, to understand their reasoning in-depth, I asked for clarification, further explanation, or pointed out inconsistencies between their verbal and written statements. As a result, it is important to point out that all interview situations affect, to a certain extent, the responses obtained, and it is difficult to know to what extent the interview situation and questions may have influenced the results.

In order to best capture participants' reasoning processes, two video cameras were used during the pilot and final interviews. One focused on the participant's written work, and the other focused on the participant and the interviewer. I conducted all of the interviews with the participants, and one graduate student helped me operate the video cameras. In the pilot study, Robert, Jason, and Sally were interviewed for three hours each; Abby was interviewed for approximately 80 minutes. In the final study, Kathy was interviewed for approximately 4 hours, and Susan was interviewed for around four and a half hours. Carol and Helen were interviewed for approximately 5 hours each.

Roulston (2001) discussed *thematic analysis* as an approach to analyzing interview data. In this study, the interview data was analyzed using a thematic analysis approach. As stated by Mills et al. (2010):

> Thematic analysis is a systematic approach to the analysis of qualitative data that involves identifying themes or patterns of cultural meaning; coding and classifying data, usually textual, according to themes; and interpreting the resulting thematic structures by seeking commonalties, relationships, overarching patterns, theoretical constructs, or explanatory principles. (pp. 925-926)

According to Holstein and Gubrium (1997), thematic analysis is the most widely used approach in social sciences to analyze data (as cited in Roulston, 2001); however, it has not been well described (Mills et al., 2010). Boyatzis (1998) describes five purposes of thematic analysis: It can be used as

1. A way of *seeing*
2. A way of *making sense* out of seemingly unrelated material
3. A way of *analyzing* qualitative information
4. A way of *systematically observing* a person, an interaction, a group, a situation, an organization, or a culture
5. A way of *converting* qualitative information into quantitative data (p. 4)

The analytic strategy used in thematic analysis is *coding* (Mills et al., 2010). As stated by Saldaña (2012), a code is "most often a word or short phrase that symbolically assigns a summative, salient, essence-capturing, and/or evocative attribute for a portion of language-based or visual data" (p. 3). While coding the data, researchers also take notes, which constitute memos that are "informal analytic notes" (Charmaz, 2006, p. 72). The goal of the researcher in the process of data analysis is to identify themes. As described by Guest et al. (2011), thematic analysis concentrates on “identifying and describing both implicit and explicit ideas within the data, that is, themes” (p. 10). Two approaches to identifying themes are discussed by Mills et al. (2010): *deductive* and *inductive*. In the deductive approach “Researchers might use research questions, interview questions, or theory-derived categories as a start list of priori themes for coding data documents, an approach that can facilitate within- or cross-case comparisons” (Mills et al., 2010, p. 926), whereas in the inductive approach themes emerge from the data, and they are grounded in the data. As explained by Mills et al. (2010), the researcher builds a case analysis through a process of:

> [N]oticing patterns, attending to how participants label events, defining emergent themes, constantly comparing data against codes and categories, cycling back through documents to revise coding, recording interpretive insights in research memos, and developing data displays that reveal overarching patterns. (p. 926)

In this study, I used a mix of *deductive* and *inductive* approaches to identify themes.

I briefly present the steps that I followed to collect and analyze the final data. Although, I followed a slightly different approach in analyzing the pilot data, for the most part I engaged in these activities. First, I conducted semi-structured interviews for data collection. Second, the interviews were transcribed verbatim. Third, I open coded the interview transcripts line-by-line and wrote memos about these incidents. Fourth, I created a code file for each task in Microsoft Excel. Then, I counted the number of occurrences of each code and entered that number in the record for the task. Fifth, I wrote a descriptive analysis of each case. Sixth, I returned to the interview transcripts and recoded these to strengthen the reliability of the results and to reduce the number of codes. I then aggregated similar codes together to determine the connections among the codes and to identify relationships. Seventh, I determined the overall themes by considering the number of occurrences of each code, the connections among the codes, and the research questions at hand. Because I used my research questions, the codes, and their occurrences to determine the themes, deductive and inductive approaches were used in the analysis of the data. The last step of the data analysis was that I wrote cross-task analyses of each case based on the themes that I identified. In the following pages, I explain the steps that I followed in determining the themes for each case.

Thematic Analysis Process. To analyze the interview data thematically, I followed three stages. In Table 1, I present these three stages and the process of theme development for Kathy's case. In the first stage, I organized the open codes in each task according to their occurrences and determined the most frequently occurring codes throughout the interviews. I then sifted the open codes based on their frequencies, meaningfulness in answering the research questions, and connections with the other codes. Hence, I did not consider the open codes in the process of theme development if they were not meaningful in answering the research questions, did not have any connections with the other codes, and appeared less frequently.

Table 1
Theme Development Process for Kathy's Case

Stages
Stage Three: Themes
Theme 1: Attention to multiplicative and qualitative relationships when inferring directly and inversely proportional relationships. **Theme 2:** Proficiency in distinguishing directly and inversely proportional relationships from nonproportional relationships. **Theme 3:** The use of proportional reasoning strategies and reasoning within measure spaces when solving proportion questions.
Stage Two: Categorization of the open codes
Comprehension: Understanding the constant product relationship between quantities (9); Statement of a directly proportional relationship (4); Statement of an inversely proportional relationship (3); Distinguishing inversely and directly proportional graphs (2); Understanding starting points precluding proportional relationships in Graphs B and C (2); Not all linear graphs depict proportional relationships (1); Not all increase-decrease relationships are proportional (1); There is a need to have a multiplicative relationship between the pair of values to infer a proportional relationship (1);

Comprehension of the numerical multiplicative reciprocal relationships in an inversely proportional graph (1); None of the graphs express proportional relationships (1)
Knowledge Resources: Statement of an inverse qualitative relationship (15); Statement of a multiplicative relationship within measure spaces (10); Statement of a reciprocal multiplicative relationship (10); Inferring an inversely proportional relationship between quantities based on the inverse qualitative relationship (6); Statement of a qualitative relationship (4); Attention to the constancy of the quotients when inferring proportional relationships (3); Attention to the numerical reciprocal multiplicative relationship within two measure spaces in inferring inversely proportional relationship (3); Attention to the within measure space multiplicative factors being reciprocal of each other when inferring the inversely proportional relationship (3); Inferring a proportional relationship between quantities based on the qualitative relationship (2); Statement of a numerical multiplicative relationship between measure spaces (1); Attention to the unit ratios when inferring constant ratio relationships between quantities (1)
Difficulties: Expressing her difficulty solving a multiple proportion question (3); Difficulty writing inverse proportion formula (2); Expressing her difficulty solving the fence problem (1); Difficulty expressing verbal explanations into mathematical form (1); Difficulty relating constancy of the products with the existing of an inversely proportional relationship (1); Difficulty recognizing the reciprocal numerical relationship when the within measure multiplicative factor was not a whole number (1); Difficulty recognizing constancy of the products in the ratio table (1); Difficulty recognizing a constant product relationship between the number of people and number of minutes (1); Difficulty obtaining an expression to indicate relationships among the number of people, number of cupcakes, and time (1)
Solution Strategies & Expressions: Ratio table strategy (28); Multiplying within measure spaces (17); Algebra strategies (14); Multiplying between measure spaces (6); Double number line strategy (6); Scientific unit conversion method (4); Unit ratio strategy (2); Additive strategy (1); Double counting strategy (1) **&** Double number line (6); Table (6); Formula (4); Directly proportional graph (2); Inversely proportional graph (2)

Stage One: Organization of the Open Codes

The open codes that remained after sifting process are entered in this stage.

In the second stage, I identified first-hand relationships among the open codes that remained in Stage One and entered related codes under one of the four categories that I determined considering my research questions and research purposes: comprehension, knowledge resources, difficulties, and solution strategies and expressions. In the first category, I entered the open codes that suggested the PSTs' comprehension of the directly and inversely proportional relationships. In the second category, I entered the codes that demonstrated knowledge resources of the PSTs' when inferring directly and inversely proportional relationships. In the third category, I entered the codes that suggested difficulties the PSTs encountered in the process of determining, explaining, and expressing directly and inversely proportional relationships, and while solving the given questions. In the last category, I entered the codes that showcased the PSTs' solution strategies and expressions of the directly and inversely proportional relationships. I also entered the frequency of each code in parenthesis. Some of the codes were related with more than one category; however, I only entered a code within a single, best suited category. For instance, if a code was demonstrating a difficulty, I entered this code in the difficulty category, although it might have also suggested a comprehension issue. Because the first three categories were derived from Research Question One, the codes in these categories were related. I entered a code within a single category because that allowed me to present a relatively simple structure of the theme development process; however, I considered the relationships among the codes in four categories when identifying the themes.

In the last stage, I determined the themes that best reflected the overall ideas presented in these four categories. In my determination, I considered the relationships among the codes and their frequencies. For example, in Table 1, the knowledge resources category included codes that demonstrated Kathy's attention to the multiplicative and qualitative relationships that she constructed between two covarying quantities when inferring directly and inversely proportional relationships. Hence, the codes suggested her attention to the multiplicative and qualitative relationships when inferring directly and inversely proportional relationships. In the comprehension category, the codes including, "Not all linear graphs depict proportional relationships," "Not all increase-decrease relationships are proportional," and "None of the graphs express proportional relationships" demonstrated Kathy's ability to distinguish the directly and inversely proportional relationships from the nonproportional relationships.

Therefore, the codes in comprehension category suggested Kathy's proficiency in distinguishing the directly and inversely proportional relationships from the nonproportional relationships.

In the solution strategies category, the *ratio table strategy* was the most frequent code throughout the interviews, which appeared 28 times. In this strategy and in the remaining strategies, Kathy usually reasoned multiplicatively and preferred multiplying within measure spaces, which occurred 17 times. Considering Kathy's comprehension of directly and inversely proportional relationships, which was also suggested by the codes entered in the comprehension category, and the frequency of the additive strategies, which occurred only once, the codes suggested Kathy's preference for reasoning within measure spaces when solving the proportion questions. Because I did not detect a consistent difficulty throughout the tasks and the frequencies of the difficulties were relatively small in number, the codes did not suggest an overall theme for the difficulties Kathy encountered. On the other hand, I discussed some of the difficulties, which appear in Table 1, in the cross-task analysis.

I followed the same steps for the remaining three participants. In Table 2, I present the themes that I determined for Susan, Carol, and Helen. Common to all three participants, the open codes suggested the PSTs' difficulties in distinguishing directly and inversely proportional relationships from nonproportional relationships. They all preferred proportional reasoning strategies to solve the proportion questions. For example, Susan used ratio table and algebra strategies more often, which appeared 16 and 15 times, respectively. Whereas, Carol preferred ratio table and proportion formula strategies, which appeared 15 and 14 times, respectively. On the other hand, Helen used ratio table and unit ratio strategies more often than the other strategies, which appeared 20 and 15 times, respectively. The codes also indicated Carol and Helen's difficulties interpreting the cupcake order in terms of cupcakes in Task 2. The thematic analysis of the PSTs' responses demonstrated their attention to specific features of contexts of the mathematical tasks when inferring directly and inversely proportional relationships. For instance, Susan attended to the constancy of the rate of change, static points on graphs, and whether the values of points were swapped when inferring directly and inversely proportional relationships. On the other hand, Carol and Susan attended to the multiplicative and qualitative relationships when inferring directly and inversely proportional relationships.

Table 2
Themes for Susan, Carol, and Helen

Participants	Themes
Susan	**Theme 1:** Attention to the constancy of the rate of change when inferring directly proportional relationships. **Theme 2:** Attention to static points on graphs and values of points being swapped when inferring inversely proportional relationships. **Theme 3:** Difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships. **Theme 4:** The use of proportional reasoning strategies and reasoning within measure spaces when solving proportion questions.
Carol	**Theme 1:** Attention to unit rates, multiplicative relationships within measure spaces, and qualitative relationships when inferring directly proportional relationships. **Theme 2:** Attention to multiplicative and inverse qualitative relationships when inferring inversely proportional relationships. **Theme 3:** Difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships. **Theme 4**: The use of proportional reasoning strategies when solving proportion questions and difficulty interpreting the cupcake order in terms of cupcakes.
Helen	**Theme 1:** Attention to numerical multiplicative relationships between measure spaces and qualitative relationships when inferring directly proportional relationships. **Theme 2:** Attention to inverse qualitative relationships and the context of balancing when inferring inversely proportional relationships. **Theme 3:** Difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships. **Theme 4:** The use of proportional reasoning strategies when solving proportion questions and difficulty interpreting the cupcake order in terms of cupcakes.

Problem Tasks

A total of nine mathematical tasks were used in this study (see Table 3). Some combinations of the first eight tasks were used in the pilot study. The final study involved the Gear, Bakery, Balance, Speed, Fence, and Scout Camping tasks in which the Fence and Scout Camping tasks were used as extras. The interview protocols for the pilot and final study are in Appendix A and B, respectively. For the purposes of this study, I developed the Gear, Painter, Apartment, and Balance System tasks. I adapted Bakery, Fence, Cookie Factory, and Speed tasks from Dr. Sybilla Beckman's textbook, *Mathematics for Elementary Teachers* (2011), and adapted the Scout Camping task from Vergnaud's (1983) study. The tasks involved either single or multiple proportional relationships and all of the problems were missing-value type problems. In all of the tasks, participants were required to detect and explain either directly or inversely proportional relationships.

The Gear task involved single directly and inversely proportional relationships that were investigated in two separate parts, respectively. The Bakery, Painter, Fence, Apartment and Speed tasks involved multiple proportional relationships. The Bakery task included two parts in the pilot study and three parts in the final study. The Painter task also included two parts. The Cookie Factory and the Balance System tasks involved single inversely proportional relationships. In Table 3, I briefly describe these nine tasks and report the names of the participants who worked on these tasks during the pilot study. In the final study, all participants worked on the same, Gear, Bakery, Balance, Speed, Fence, and Scout Camping tasks.

Table 3
Descriptions of the Mathematical Tasks

Name of the task	Brief descriptions	Participants
Gear	A directly proportional relationship between the size of a gear and the number of notches it possessed and an inversely proportional relationship between the number of revolutions that a gear made and its size are involved in this task. For example, in one of the questions, participants calculated the number of notches of a gear with a 2-cm radius, given that the second gear had a 3-cm radius and 12 notches. In another question, they calculated the number of revolutions of a gear with a 3-cm radius, given that the second gear had a 4-cm radius and revolved six times.	Robert, Sally, and Jason
Bakery	In this task, participants explored one inversely and two directly proportional relationships among the number of people, the number of cupcakes, and the number of minutes. The task involved single and multiple proportion questions. For example, in one of the questions, participants calculated how many cupcakes could be frosted by two people in T minutes, considering that three people frosted N cupcakes in T minutes.	Robert, Sally, and Jason
Balance	In this task, I provided participants with a mini-number balance system with which they balanced the system through hanging weights on hooks that were placed on both sides of the system. They explored an inversely proportional relationship between the distance (how far from the center a weight hung) and the number of weights that were hung.	Abby and Michael
Speed	In this task, one inversely and two directly proportional relationships among the distance, speed, and time are investigated. The participants worked on questions similar to this one: If you covered the distance between two markers in 90 seconds driving at 60 mph and if you want to cover the same distance in 60 seconds, then what must your speed be?	Sally, Jason, Abby, and Michael
Fence	This task involved identifying one inversely and two directly proportional relationships among the number of workers, the number of days, and the number of fences painted. The participants worked on questions similar to the following one: If three people take two days to paint five fences, how long will it take two people to paint one fence?	Robert, Sally, and Jason
Apartment	This task involved an inverse-inverse-inverse relationship among the number of workers, the number of hours they work each day, and the number of days required to build an apartment. For example, participants needed to calculate the number of days it would take for 12 workers to build an apartment when each worker worked 8 hours per day, given that eight workers built the same apartment in 24 days working 6 hours per day.	Robert, Sally, and Jason

Painter	The participants explored a directly proportional relationship between the number of painters and number of bedrooms and an inversely proportional relationship between the number of painters and number of hours. For example, the participants calculated the number of bedrooms painted by eight painters in 6 hours, given that four painters painted three bedrooms in 6 hours. Similarly, they calculated the number of hours needed by eight painters to paint three bedrooms, given that four painters painted the same three bedrooms in 6 hours.	Robert
Cookie Factory	This task involved an inversely proportional relationship between the number of assembly lines used to make boxes of cookies and the time required to fill a truck with those boxes. The participants worked on questions similar to the following one: If four assembly lines could make enough boxes of cookies to fill a truck in 10 hours, how many hours are needed to fill the same truck if eight assembly lines were used?	Robert
Scout Camp	This task involved three inversely proportional relationships among the number of people, the amount of cereal each person eats per day, and the number of days they stayed in the camp. Participants worked on questions to calculate the number of people, the amount of cereal each person ate per day, or the number of days they stayed in the camp.	Only final study participants

Rationale for Inclusion of the Tasks. When developing the interview protocols, I considered mathematical tasks to represent real-life examples. For example, I included the Gear, Balance, Speed and Bakery tasks because I expected them to be helpful to research participants in making connections with real-life conditions. In the Gear and Balance tasks, plastic gears and a mini-number balance system were provided, respectively, to explore directly and inversely proportional relationships. Therefore, these tasks provided hands-on experiences with direct and inverse proportions. In addition, the use of physical materials provided the following two advantages:

- Because the plastic gears and balance system had features that were suitable for forming directly and inversely proportional relationships, they helped me investigating participants' reasoning about these two relationships.
- As emphasized by Hart (1981), "children who are presented with practical problems (e.g., gears), which needed a ratio for solution, do improve and abandon the addition

strategy" (p. 100). Therefore, it is expected that hands-on tasks would also help PSTs to abandon the additive strategies.

The use of hands-on tasks and real-world problems together, precipitated the gathering of relevant information regarding the research questions. The tasks contained a variety of proportion categories, which I discussed in Chapter Two, and the use of these different categories was effective in understanding the participants' reasoning for different situations. The multiple proportion problems could not be easily solved by cross-multiplication or additive strategies. Hence, multiple proportion problems were effective in determining the solution strategies of the PSTs' when they could not rely on the cross-multiplication and rote computations. These problems also helped me determine the participants' conceptual understanding and difficulties. Furthermore, the mathematical tasks used were appropriate for exposing the participants' reasoning because using those tasks I was able to generate conversations around the research topics. The interactive nature of the tasks was also helpful in building rapport between the participants and myself. Additionally, the interactive nature of the interviews was helpful for exposing participants' knowledge resources, coordination of these resources, development of cognition, and construction of meanings.

Advance Summaries of the Tasks. These advance summaries include the information about what mathematics I targeted with the tasks and possible solution methods.

Gear. The development of the Gear tasks was influenced by the concept of Gear (Speed) Ratio, an essential concept in engineering. Gear (Speed) Ratio is defined as the ratio of the radii of two gears, the ratio of the number of notches of two gears, or the inverse ratio of the velocity of each gear. This definition assumes that the two gears are meshed into each other, meaning that they have the same sized notches. Hence, if the movement ratio is represented by R, the radii by *r*, the number of notches by *n*, and the angular velocity by *v*, the Gear (Speed) Ratio is expressed by $R = \frac{r_1}{r_2} = \frac{n_1}{n_2} = \frac{v_2}{v_1}$.

In the first part of this task, I provided the participants with two plastic gears to investigate the directly proportional relationship between the number of notches of a gear and the size of its radius. For example, one of the problems involved determining the number of notches of Gear B, with a 6-cm radius, given that Gear A had a 3-cm radius and 12 notches. Because Gears A and B had circumferences 6π and 12π, respectively, and notches evenly placed, Gear B had 2 times as many notches as Gear A. Hence, the answer was 24 notches. After realizing this directly proportional relationship, as seen in the equation above, one could set up either $\frac{r_1}{r_2} = \frac{n_1}{n_2}$ or $\frac{r_1}{n_1} = \frac{r_2}{n_2}$ as the proportion and could determine the missing value. For this study, I followed the conceptual framework given by Vergnaud (1983, 1988) to explain multiplicative relationships. If we designate radii as measure space one (M1), then the number of notches become measure space two (M2). Hence, $\frac{r_1}{r_2}$ and $\frac{n_1}{n_2}$ become within measure space ratios, and $\frac{r_1}{n_1}$ and $\frac{r_2}{n_2}$ become between measure space ratios. I expected participants to realize that the $\frac{3\ cm}{6\ cm} = \frac{1}{2}$ within measure space ratio was equal to the $\frac{12\ notches}{x\ nothes} = \frac{12}{x}$ within measure space ratio, and to realize that the $\frac{12\ notches}{3\ cm} = \frac{4\ notches}{1\ cm}$ between measure space ratio, which was the unit rate between the number of notches of a gear and the size of its radius, was equal to the $\frac{x\ notches}{6\ cm}$ between measure space ratio. Thinking either way would help participants to determine the number of notches of Gear B. Furthermore, I expected participants to realize that the number of notches and radii were covarying in a fixed $\frac{4\ notches}{1\ cm}$ ratio. Additionally, I expected that the participants could express this relationship with a directly proportional graph and see that the slope would be equal to the value of this fixed ratio.

In the second part of this task, the goal was to explore the inversely proportional relationship between the number of revolutions a gear makes and size of its radius**.** Because the number of notches a gear has was proportionally related to the size of its radius, I expected the participants to also understand the inversely proportional relationship between the number of revolutions and number of notches. For instance, one of the problems involved determining the number of revolutions of Gear K with four notches, given that Gear F had 8 notches and revolved three times. I expected my participants to realize the inverse relation between the size of a gear and the number of revolutions it makes. They should see that for every one revolution of Gear F, eight notches of it moved; hence, in three revolutions a total of 24 notches of it moved. Gear K was meshed with Gear F, so I expected that the participants should see the same 24 notches as revolving on Gear K. Because four notches moved on Gear K in one full rotation, Gear K needed to make six revolutions. One could solve this problem forming either $\frac{r_1}{r_2} = \frac{v_2}{v_1}$ or $\frac{n_1}{n_2} = \frac{v_2}{v_1}$ proportions and calculating the missing value. Thus, I presumed that the participants could realize that the $\frac{8\,notches}{4\,notches} = \frac{2}{1}$ within measure space ratio was equal to the inverse of the $\frac{3\,revolutions}{x\,revolutions} = \frac{3}{x}$ within measure space ratio. The between measure space ratios were not equal in this task, so there was not a single constant rate. For this reason, I expected to see more of the reasoning within measure spaces. I anticipated possible difficulties for recognition of the inversely proportional relationship because, as I discussed earlier, teachers tend to judge nonproportional relationships to be proportional (Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010). I also anticipated that the participants could have difficulty recognizing the product of the number of revolutions and number of notches being constant and the reciprocal multiplicative relationship between the number of revolutions and number of notches. In addition, I expected that participants could express the inversely proportional relationship with an inversely proportional graph.

Bakery. This task involved three parts. In the first part, the goal was to explore the directly proportional relationship between the number of people and the number of cupcakes they frosted in a fixed amount of time, as well as the directly proportional relationship between the number of cupcakes and time considering the number of people as constant. In the second part, the goal was to investigate the inversely proportional relationship between the number of people and the time required for that many people to frost a fixed amount of cupcakes. The last part involved exploration of multiple proportion problems. Each part also involved two small sections. In the first section, I used a combination of letters and numbers, and in the second section, I used just numbers. So, if a participant could not answer the questions in the first section, then I continued with the questions in the second section.

For example, in the first part of this task, I asked participants to calculate the number of cupcakes frosted by two people in T minutes, given that three people frosted N cupcakes in T minutes. I expected my participants to realize that the number of cupcakes and number of people was covarying in a fixed $\frac{N\ cupcakes}{3\ people}$ ratio. For this task, M1 was the number of people, and M2 was the number of cupcakes they frosted in T minutes. I imagined my participants to realize that the $\frac{2\ people}{3\ people} = \frac{2}{3}$ within measure space ratio was equal to the $\frac{x\ cupcakes}{N\ cupcakes} = \frac{x}{N}$ within measure space ratio, and to realize that the $\frac{N\ cupcakes}{3\ people}$ between measure space ratio was equal to the $\frac{x\ cupcakes}{2\ people}$ between measure space ratio. Also the participants needed to realize that the value of the between measure space ratio would become the slope of the directly proportional relationship between the number of cupcakes and the number of people.

In the second part, the participants worked on problems including the following: At a bakery, three people can frost a total of N cupcakes in T minutes. How long would it take for six people to frost N cupcakes? Because this problem involved fixing the number of cupcakes and doubling the number of people, the time required to frost N cupcakes would be halved. Therefore, using this information and other similar information, I expected my participants to realize the inversely proportional relationship between the time required to frost N cupcakes and number of people. For this task, M1 was the number of people, and M2 was the time needed to frost N cupcakes. I supposed my participants to realize that the $\frac{3\ people}{6\ people} = \frac{1}{2}$ within measure space ratio was equal to the inverse of the $\frac{T\ minutes}{x\ minutes} = \frac{T}{x}$ within measure space ratio. The between measure space ratios were not equal in this task, so there was not a single constant rate. For this reason, I expected to see more of the reasoning within measure spaces.

In the third part, the participants worked on the multiple proportion problems including the following: At a bakery, three people can frost a total of N cupcakes in T minutes. How long would it take for one person to frost 2N cupcakes? I anticipated that the participants might experience some difficulty solving this problem because three quantities were covarying together and the value of none of them was fixed. I expected the participants to reason within measure spaces because it was not easy to reason between measure spaces for this type of problems. They needed to solve the problem by fixing the value of one of the three quantities at a time and identifying the relationship between the remaining two quantities.

Speed. The first problem of this task necessitated the calculation of the time needed to cover a distance driving at 50 mph, considering that driving at 60 mph another car covered the same distance in 90 seconds. Because the faster car could cover the same distance in fewer seconds, there was an inversely proportional relationship between speed and time. I expected that one could solve this problem by first calculating the distance covered in 90 seconds. Since driving at 60 mph necessitated covering one mile every minute, then in 90 seconds the fast car could cover one and half miles. Since the slower car could cover 50 miles in one hour, it could cover one and a half miles in $\frac{1.5\,miles}{50miles/hour} = \frac{3}{100}$ of an hour, which was equal to 108 seconds. However, an easier method to solve this problem could be as follows: If one considers M1 as speed and M2 as time, one could form a $\frac{60\,miles/hour}{50\,miles/hour} = \frac{6}{5}$ within measure space ratio and could infer that this ratio is equal to the inverse of $\frac{90\,seconds}{x\,seconds} = \frac{90}{x}$ within measure space ratio. Also I expected that some participants may have known the distance formula—$D = S \times T$—and that they might use it to solve this problem. In addition, I presumed some difficulties with unit conversions.

Balance. The mini number balance system was a simple version of an equal-arm beam balance scale. In this study, I used it to explore inversely proportional relationships. If we hang some weights (A weight was represented with a blue rectangular rod) on one side of the balance system, then we need to hang some other weights on the other side to balance the system. I expected that participants could balance the system by hanging the same number of weights on both sides. However, for this study participants were required to explore multiple ways of balancing the system. The main idea in a balance system was that there was an inversely proportional relationship between the distance from the center and the numbers of weights hung. For example, if we hang two weights at 10 cm, then we need to hang five weights at 4 cm, or 10 weights at 2 cm, or 20 weights at 1 cm to balance the system. Thus, I expected that the participants could generate the balance formula as follows: $D1 \times W1 = D2 \times W2$, where $D1$ was the distance from the center in the first direction; *D2* was the distance from the center in the second direction; *W1* was the number of weights hung in the first direction; and *W2* was the number of weights hung in the second direction.

Fence. This task involved a multiple proportional relationship. In the first problem, participants had to calculate the number of days required for two people to paint one fence, considering that three people painted five fences in two days. For this task, the participants needed to realize two proportional relationships: between the number of people and the number of fences and between the number of days and the number of fences. And they also had to understand one inversely proportional relationship: between the number of people and the number of days. After determining these relationships, one could solve this problem by fixing one quantity at a time. For example, using the information that three people can paint five fences in two days, a participant could determine that one person can paint the same five fences in six days because having a third of the number of people requires tripling the number of days. For the second step, the participant could determine that two people can paint five fences in three days because doubling the number of people necessitates halving the number of days. For the last step, the participant could conclude that two people can paint one fence in three-fifths of a day because having a fifth of the number of fences requires a fifth the number of days. I demonstrated this solution method in Figure 8. However, one could solve this problem by following a different order. I expected that this task could be difficult for the participants because it involved multiple relationships, and it was not easy for someone to simply set up a proportion and to use cross-multiplication strategy. Hence, the participants needed to identify and comprehend all the existing relationships in the multiple proportion problems, and this feature of the multiple proportion problems allowed me to better understand the participants' reasoning and their knowledge resources when inferring the relationships.

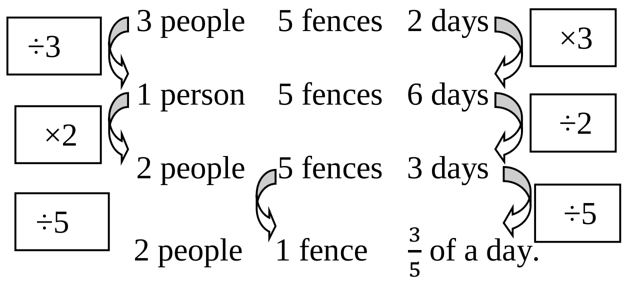


Figure 8. The three steps to solve the Fence problem.

Apartment. This task also involved multiple proportional relationships. For the first problem, participants needed to calculate the number of days required by 12 workers to build an apartment if each worker works 8 hours per day, considering that eight workers can build the same apartment in 24 days each working 6 hours per day. In this task, the participants were required to realize three inversely proportional relationships between the number of workers, the number of hours they work each day, and the number of days it took them to build the apartment. After determining these inverse relationships, one could solve this problem by fixing one quantity at a time. For example, using the information that by working 6 hours per day eight workers could build the apartment in 24 days, a participant could determine the number of hours each worker works as 6 × 24 = 144 hours. Since there were eight workers, they worked 8 × 144 = 1152 hours in total. Here 1152 hours became the constant of the proportionality. The problem required calculating the number of days for 12 workers, where each of them was working 8 hours per day. So they could work 12 × 8 = 96 hours a day. For the last step, the participant could divide the total number of hours—1152—by 96 hours and could get 12 days. This is one of the easiest methods to solve this problem; however, this solution may not involve consideration of the proportionality. Another solution, which has a higher possibility for involving proportionality, is to follow a method that is very much similar to the method I discussed in the Fence task. I demonstrated this solution method in Figure 9. However, one could solve this problem by following a different order.

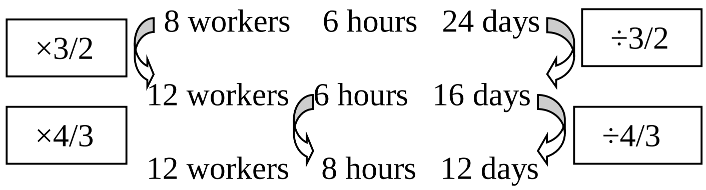


Figure 9. The steps to solve the Apartment problem.

Painter. In the first part of this task, the initial problem was about determining the number of bedrooms painted by eight painters in 6 hours, given that four painters painted three bedrooms in 6 hours. Reasoning within measure spaces, I expected the participants to recognize that doubling the number of painters required doubling the number of bedrooms they painted in 6 hours. So, the answer was six bedrooms. Hence, this task involved determining a fixed $\frac{3\ bedrooms}{4\ painters}$ ratio relationship between the number of painters and number of bedrooms they painted in 6 hours. For this task, M1 was the number of painters, and M2 was the number of bedrooms they painted in 6 hours. I expected that my participants could realize the $\frac{4\ painters}{8\ painters} = \frac{1}{2}$ within measure space ratio as equal to the $\frac{3\ bedrooms}{x\ bedrooms} = \frac{3}{x}$ within measure space ratio. And I further anticipated that they could realize that the $\frac{3\ bedrooms}{4\ painters}$ between measure space ratio was equal to the $\frac{x\ bedrooms}{8\ painters}$ between measure space ratio.

In the second part of this task, the participants were required to determine the number of hours needed by eight painters to paint three bedrooms, given that four painters painted the same three bedrooms in 6 hours. I expected the participants to recognize the reciprocal multiplicative relationship, doubling the number of painters would halve the number of hours to paint three bedrooms. So, the answer was 3 hours. As a result, this task involved determining an inversely proportional relationship between the number of painters and number of hours. Designating M1 as the number of painters and M2 as the number of hours, I expected that my participants could recognize that the $\frac{4\,painters}{8\,painters} = \frac{1}{2}$ within measure space ratio was equal to the inverse of the $\frac{6\,hours}{x\,hours} = \frac{6}{x}$ within measure space ratio. The between measure space ratios were not equal, so there was not a constant rate.

Cookie Factory. Knowing that four assembly lines can make enough boxes of cookies to fill a truck in 10 hours, the first problem involved determining the number of hours required to fill the same truck if eight assembly lines were used. Because the assembly lines were doubled, the time needed to fill the truck would be halved, and so the answer was 5 hours. Hence, I expected the participants to realize the inversely proportional relationship between the number of assembly lines and the time it took them to make boxes of cookies to fill a truck. For this problem, I presumed the participants could designate M1 as the number of assembly lines and M2 as the time needed to fill the truck. Later, they could determine the $\frac{4\,assembly\ lines}{8\,assembly\ lines} = \frac{1}{2}$ within measure space ratio, and could show that this ratio was equal to the inverse of the second $\frac{10\,hours}{x\,hours} = \frac{10}{x}$ within measure space ratio. Because the between measure space ratios were not equal, there was not a constant rate between the number of assembly lines and the time.

Scout Camp. This task involved inverse-inverse-inverse multiplicative relationships among the number of people, number of cereal, and number of days. For instance, the first question was about determining the amount of cereal 20 people needed to eat, so the cereal they brought with them would last for 16 days, given that it would take 20 people 12 days if each one of them consumes half of a pound of cereal every day. Similar to the previous multiple proportion tasks, I expected that the participants could solve this problem by fixing the value of one of the three quantities at a time and working on identifying the relationship between the remaining two quantities. I also expected them to calculate the total amount of cereal to be 120 pounds by multiplying 20 people with 12 days and one-half pound per person per day; however, I did not think that they could recognize 120 pounds as representing the constant of proportionality.

Multiplicative Structures Presented in the Mathematical Tasks. The mathematical tasks used to collect data were appropriate for compiling relevant information from the PSTs about their inferences of the directly and inversely proportional relationships. Because the mathematical tasks used in this study involved comparing quantities with different units, it was expected that the PSTs might have difficulty expressing and stating multiplicative relationships between measure spaces. For instance, in Task 1A, there was a directly proportional relationship between the size of a gear and its number of notches. The PSTs were provided two gears, which were thought to be meshed, and they were asked to calculate either the number of notches or the radius of the gears. In one of the questions, the PSTs needed to calculate the number of notches of Gear B, with a 2-cm radius, given that Gear A had a 3-cm radius and 12 notches. The mathematical structure of the relationship between the number of notches and radius for this particular question can be expressed with the multiplication statement (3 cm) * (4 notches/cm) = 12 notches. This mathematical statement can be best explained with a multiple batch perspective. Assuming 4 notches per 1 cm as a batch, which is a unit rate, an expert can iterate this batch two times to calculate the number of notches on Gear B as eight. In general, the multiplicative relationship between the number of notches and radii can be expressed with the statement (R cm) * ($\frac{N}{R}$ notches/cm) = N notches (Figure 10). Because it was not meaningful to state the multiplicative relationship by saying either "12 notches is four times 3 cm" or "3 cm is one-fourth of 12 notches," I expected that when comparing quantities between measures, the PSTs either would state the unit rate or focus on the numerical multiplicative relationship—12 is 4 times the 3.

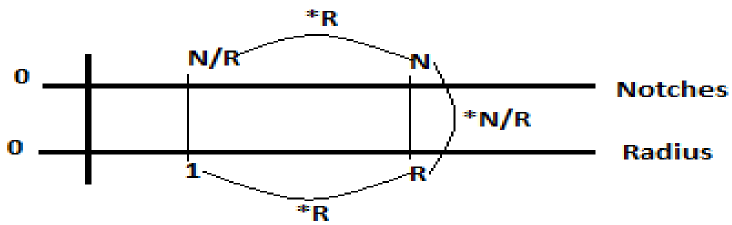


Figure 10. Expression of the multiplicative relationship between the number of notches and radii.

Because the mathematical tasks in this study involved comparing quantities with different units rather than the same units, I expected that it would be difficult for the PSTs to express multiplicative relationships between measure spaces. For example, the following information and question can be given as an illustration of two quantities with the same units. Three ounces of rose oil and 2 ounces of chamomile oil are mixed to make a fragrance. How much chamomile oil was needed if 12 ounces of rose oil were used in the fragrance? In this question, the mathematical structure of the relationship can be expressed with $(\frac{2}{3})$ * (Rose oil) = Chamomile oil. A PST could state the multiplicative relationship between measure spaces by saying either "The chamomile oil is two-thirds the rose oil" or "the rose oil is three-halves the chamomile oil," which are relatively easier to determine than the multiplicative relationships between measure spaces with different units that I provided in the preceding paragraph. As in the gear example above, quantities in the separate within measure spaces had the same units. Therefore, I expected a tendency from the PSTs to make multiplicative comparisons within measure spaces—3 cm is three-halves the 2 cm, so 12 notches needed to be three-halves the eight notches—rather than making multiplicative comparisons between measure spaces.

In the inverse proportion tasks, there was a constant product relationship between the two covarying quantities. For example, in Task 1B, the product of the number of revolutions and number of notches yielded the total number of notches revolved on a gear, which was constant. Hence, the multiplicative structure of the relationship could be expressed with the following statement: (number of revolutions) * (number of notches per revolution) = total notches. In the Balance task, because the PSTs needed to balance the system on two sides, they were required to have the same value, which was determined by multiplying the number of weights and the distance from the center of the system, on both sides. Therefore, the contexts of the Gear and Balance tasks were appropriate for facilitating the PSTs' inferences of constant product relationships. In the Bakery task, there was an inversely proportional relationship between the number of people and the number of minutes. It was expected that an expert might infer the cupcakes order as the product of the number of people and the cupcakes frosted by each person. Because the cupcake order consisted of a fixed number of cupcakes, this expert could recognize that as the number of people increases, the time to complete the order decreases by the multiplicative inverse of the increases in the number of people. On the other hand, a novice might have difficulty inferring the cupcakes order as the product of the number of people and the cupcakes frosted by each person.

Because more than two quantities were involved in the multiple proportion tasks, identifying multiplicative relationships were expected to be more difficult than single proportion tasks. For example, in the Bakery task, there was a direct-direct-inverse proportional relationships structure. The multiplicative relationships among the number of people, the number of minutes, and the number of cupcakes can be expressed with the following multiplication statement: (number of people) * (number of minutes) = (number of cupcakes) * (number of person-minutes per cupcake), where "number of person-minutes per cupcake" represents the constant of proportionality. To infer the multiplicative relationship between any two quantities, PSTs needed to fix the value of the third quantity as constant. In the Scout Camp task, there was an inverse-inverse-inverse proportional relationships structure. Hence, the product of the number of people, the number of days, and the amount of cereal eaten by a person in a single day yielded the total cereal, which was constant. Therefore, the multiplicative relationships could be expressed by the following multiplication statement: (number of people) * (number of days) * (pounds of cereal eaten per person per day) = total pounds of cereal.

Pilot Study

In the following pages, I present the data analysis of one middle grades teacher, Abby, and three secondary grades teachers, Sally, Jason, and Robert. Each case analysis begins with a brief summary, and a cross-task analysis of the participants' responses follows it. The data analysis is concluded with a discussion of the findings. I provided a brief summary of the pilot study at the end of this chapter in order to discuss the transition from the pilot to final study.

Abby's Case

Summary. Abby correctly inferred the directly and inversely proportional relationships between quantities in the Balance and Speed tasks. Although, in some instances, there was evidence that Abby coordinated multiplicative relationships within two separate measure spaces, her inference of the directly and inversely proportional relationships was mainly based on attending to the qualitative relationships, not to the multiplicative relationships between quantities. To solve the problems in the Balance task, she used the balance formula that she generated. In the Speed task, by reasoning within measure spaces, Abby used the unit ratio and ratio table strategies. She expressed the relationships with tables, balance and distance formulas, and directly and inversely proportional graphs.

Cross-Task Analysis. In the Balance task, I provided Abby with a mini number balance system by which she balanced the system through hanging weights on hooks that were placed on both directions of the system. Her goal was to explore the relationship between the distance (how far from center a weight hung) and the number of weights hung. Although Abby did not necessarily anticipate an inversely proportional relationship, she seemed to have a sense of proportionality about the way the balance system was working, and noted her suspicion by stating the following idea:

Abby: I am trying to think how exactly this [pointing at the balance] works but I think ummm maybe this balance is set, maybe it is set to stay in a certain proportion. So, only things that like did that proportion will make it balanced.

After working a while to figure out the way the system was working, she inferred an inversely proportional relationship between the distance and the number of weights. Her inference of the inversely proportional relationship was based on attending to inverse nature of increments and decrements in the values of corresponding quantities.

Int: Can you tell me what kind of relation is this?
Abby: I think it is a proportional relationship [sounded hesitant]. Well it is a, I think, it is an inversely proportional relationship because a proportional relationship would be uhmm every time the distance increases by a certain amount uhmmm the amount of weight would increase by certain amount. This one decreasing while the other increasing, I think it would be inversely proportional because it is opposite.

According to the exchange, Abby's correct inference of the inversely proportional relationship was based on attending to the inverse qualitative relationship, which she described by saying, "This one decreasing while the other [is] increasing." The following exchange demonstrated Abby's recognition of a covariation between the number of weights and distance:

Int: Here you talked about relation between distance and weight. Can you tell me more about this relation?

Abby: Yeah so here I said as the distance from zero increases the weight that you need to put on decreases. So there always the reason why this always works is because they are always changing together.

In the exchange, Abby's statement, "…they are always changing together" indicated her understanding of the covariation.

Abby successfully explained the reciprocal multiplicative relationships within measure spaces if the number of weights and distances involved halving and doubling. For example, when told that six weights were hung at $7\frac{1}{2}$ cm in one direction of the system and asked where to hang three weights to balance the system in the other direction, Abby stated that because the number of weights on one side was double the other side, for her, the distance needed to be doubled so that the system could be balanced:

Abby: I think that since this weight [pointed at six weights] is double this weight [pointed at three weights] then, or three is one half [of] six, so this weight is half of this weight. Then my distance should be double because like I said before, the closer ummm that the distance is to zero the more weights that you need to put on. And so since this is exactly half of this. I think that new distance is going to be exactly double seven and a half which is fifteen.

I should have noted that the problem above was an inverse proportion problem and very easy since it only involved doubling and halving; however, there was not enough evidence if she could infer reciprocal multiplicative relationships within measure spaces in an inverse proportion problem for fractional values. During the interviews, Abby did not explain multiplicative relationships between measure spaces, and that suggested her preference for reasoning about multiplicative relationships within measure spaces.

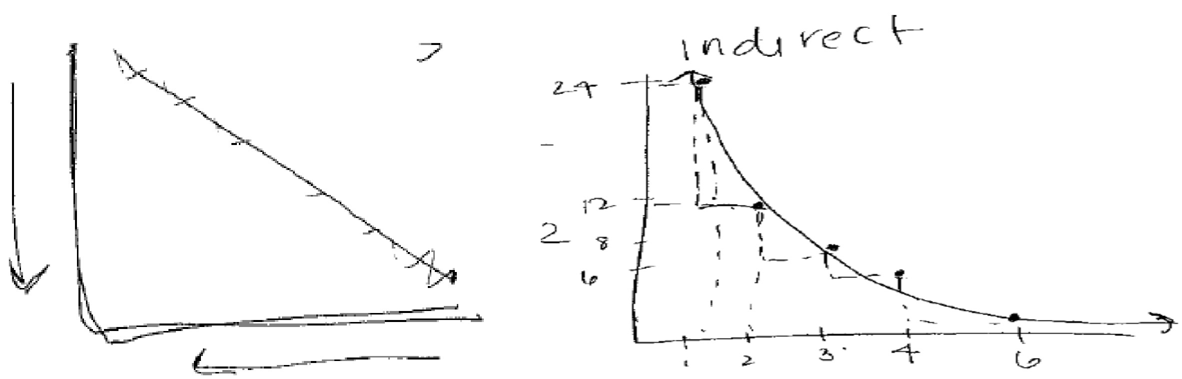
When I asked Abby to draw a directly proportional graph and an inversely proportional graph, she successfully drew a directly proportional graph, but initially thought the graph in Figure 11 a, which she drew, was also representing a direct proportion.

Abby: Yeah. Uhm so for direct as one increasing the other is increasing so like something like that. Uhm or as one is decreasing the other is decreasing so it would be like this…I think nope that's wrong. Uhm okay for indirect this would be indirect because uhm like here uhm I am just going to pretend like I am plotting these points. So one and 24 and 2 and 12, and 3 and [inaudible], and four and six.

The explanation demonstrated that Abby's attention to the qualitative relationships directed her to incorrect conclusions—a direct proportion and inverse proportion—about the relationship in Figure 11 a. A few exchanges later, by marking the values, which she had in her ratio table, she obtained the correct inversely proportional graph (Figure 11 b), in which the line was curved instead of straight. When I asked which one—Figure 11 a or 11 b—was showing an inversely proportional relationship, she said that the graph in Figure 11 b was showing an inversely proportional relationship and explained:

Abby: Because even then it's like so next would be like six and four uhmmm because they're not because they're not changing at the same rate.
Int: In that case?
Abby: In this case they're still change like a straight line means they're changing at the same rate.

The exchange showed that for Abby, a straight line necessitated the change of the values of quantities "at the same rate." Because she detected that the values in her ratio table and in the curved graph were not changing at the same rate, she concluded that the graph with a curved line was representing the inversely proportional relationship.



(a) (b)

Figure 11. Abby's determination of a graph with an inversely proportional relationship.

In the Speed task, Abby inferred one inversely and two directly proportional relationships among the distance, speed, and time by fixing a quantity at a time and describing appropriate qualitative relationships between the remaining two quantities. She usually solved the given questions reasoning within measure spaces that involved coordinated multiplication and division. For instance, when I asked Abby her reason for using the same and opposite operations in two separate problems, she explained:

Abby: It is because as the amount of miles that you travel increases, the amount of time that you move uhmm the amount of time that you take increases too because you are traveling at the same rate at you traveling at the same miles per hour. So, as long as your speed does not change, the more miles you travel the more time is going to take. And they are moving at the same time as long as the miles per hour stay the same.

Int: So what kind of relation is this you are talking about?

Abby: So this is a proportional relationship. The amount of miles and the amount of seconds is the proportional relationship if the miles per hour stay the same. But here if you are looking at the seconds and the miles per hours, [it is] inversely proportional. Because uhmmm at the amount miles that you are traveling stay the same then as the seconds as the amount of time you take increases the miles per hour that you are driving decreases if you are traveling the same distance. So it depends on what uhmmm what variables you look at or what two variables you are comparing. So this makes more sense.

This exchange demonstrated that Abby inferred the directly proportional relationship between the distance and the time and the inversely proportional relationship between the time and the speed by considering the speed and distance as constants one at a time and by qualitatively describing the coordinated increases and decreases in the values of the remaining two quantities. The exchange also demonstrated how Abby successfully coordinated the need for taking value of a quantity as constant with the presence of a directly or inversely proportional relationship between the other two quantities. For example, she said, “The amount of miles and the amount of seconds is the proportional relationship if the miles per hour stay the same.”

Sally's Case

Summary. Sally usually inferred the directly and inversely proportional relationships in the given questions by attending to and explaining the qualitative relationships between quantities. She also inferred the multiplicative relationships within measure spaces. In the direct proportion questions, by making multiplicative comparisons between measure spaces, she inferred the constancy of the quotients. On the other hand, in the inverse proportion questions, she did not recognize the constancy of the products. She expressed the relationships among quantities with proportions, ratio tables, directly and inversely proportional graphs, equations, and formulas. She seemed to be comfortable while she was working on the tasks, so she solved problems in the absence of numbers and set up direct and inverse proportions and used other proportional reasoning strategies to solve given problems.

Cross-Task Analysis. Similar to Abby, Sally's main strategy for inferring relationships between quantities was to describe qualitative relationships. If the problem tasks involved more than two quantities such as in the Fence, Apartment, and Speed tasks, she fixed one quantity at a time and described the relationship between the remaining two. As a strategy, before attempting to solve any problems, she described the qualitative relationships among quantities and solved the problems later. Her qualitative descriptions generally embodied some type of causal relationships (e.g., *radius increases so revolutions decreases, more workers so less time*) and resembled *p-prims* like knowledge pieces since the effect-reaction relationships defined in those descriptions were self-evident. The following exchange from the Bakery II task showed how she inferred an inverse relationship between the number of people and the time to frost 50 cupcakes:

Int: If two people frost 50 cupcakes in 12 minutes, then how long would it take for four people to frost 50 cupcakes?

Sally: Okay this is going to be an inverse relationship because we still know two people can frost 50 cupcakes in 12 minutes. So it is not going to take them, if this was like a regular

> proportion, then four people to frost 50 cupcakes it would take, if we multiply two by 2 to get four people so multiply the time by 2 and we get 24 minutes. But that doesn't make any sense because it will actually take them less time to frost cupcakes because more people there. And then they work at the same pace and so it is going to take less time.

The exchange suggested that Sally's inference of the inverse relationship was based on her coordination of the inverse qualitative relationship—"…more people…so it is going to take less time"—in the given question with the knowledge of what would look like the qualitative relationship if it was a "regular" proportional relationship by which she implied the directly proportional relationship.

Sally reasoned multiplicatively while solving the questions and seemed to be aware of the consequences of addition and multiplication. For example, in the Gear I task, when asked how many notches would Gear 2, which had a 4-cm radius and 16 notches, have if Gear 1 with a 3-cm radius had 18 notches instead of 12. She explained that multiplying 12 notches by $\frac{3}{2}$ would yield 18 notches, so, she said that she also need to multiply 16 notches by $\frac{3}{2}$ to obtain the number of notches on Gear 2. When asked how she knew multiplying 16 notches by $\frac{3}{2}$ would work, she explained:

Sally: Well honestly, I just tried to figure out what you could multiply by this number [pointed at 12] to get this number [pointed at 18]…First, I thought you add six but that is not a definition of proportionality…It should be like constant factor, so it should be something like you are multiplying by. So, I found that if you multiply 12 by three halves that will give you 18. So, to check it is proportional or not, then we can go over here this number [pointed at 16] and multiply by the same amount and that will give us 24, which corresponds to the 18. These two [pointed at 12 and 16] are corresponding and these two [pointed at 18 and 24] are corresponding. And then I think we can say it is proportional.

Sally's explanation suggested that for her, the idea of proportionality necessitated multiplying values of two separate within measure space quantities by the same multiplicative factor. Her correct explanation also suggested that she had facility with fractions as multiplicative operators. It appeared that Sally's consideration of multiplication in her reasoning to explain proportionality was an example for her coordination of the proportionality with the necessity of the existence of multiplicative relationships between two covarying quantities.

Sally successfully identified appropriate multiplicative relationships within measure spaces for whole numbers and proper and improper fractions. For instance, in the Gear I task, one of the questions involved calculation of the number of notches around Gear 2, which had a 4-cm radius, given that Gear 1, which had a 3-cm radius, had 18 notches. She explained the multiplicative relationship within measure spaces by saying, “I know that Gear 1 is always going to have three-fourths the amount of little notches that Gear 2 has. And so if I know how many notches Gear 2 has, I can multiply this by $\frac{3}{4}$ and get the amounts of notches that Gear 1 has.” Sally also made multiplicative comparisons between measure spaces in the direct proportion questions to infer constant ratio relationships between two covarying quantities. For instance, in the Fence task, from the information—two people paint five fences in three days—she inferred the constant ratio between the number of people and the number of fences as $\frac{2}{5}$ (Figure 12). Fixing the number of people, she also explained a constant ratio relationship between the number of fences and number of days:

Sally: So people over fences is going to be constant. So, the amount of people in this was just two and the amount of fences was five, so two-fifths. It also equals to same ratio, which is, four over 10 is also equal to two-fifths. So if you are just changing these [pointed at the number of fences and number of people] and keeping days constant and then that will work. But uhmmm yeah like this one [pointed at 6 people/5 fences ≠ 1 person/5 fences] it won't work with because you are changing the amount of days. And so, if you…so, this does…this actually does work. So, day one over fence one equal day two over fence two. So, I don't have any example of this though. But if you increase the amount of days you have then you increase and if you keep the amount of people the same. So, like whatever this is, this does not have any equation has to be constant. So, in this case this is people. So, you keep people constant then your day to fence ratio is going to be same for all your examples.

Figure 12 and the statement above demonstrated Sally’s inference of a constant ratio relationship between the number of people and number of fences. She clearly stated this relationship by saying, “So people over fences is going to be constant.” Sally’s statement, “So, the amount of people in this was just two and the amount of fences was five, so two-fifths. It also equals to same ratio, which is, four over 10 is also equal to two-fifths,” suggested that she was attending to the numerical multiplicative relationships between measure spaces in inferring the constant ratio relationship.

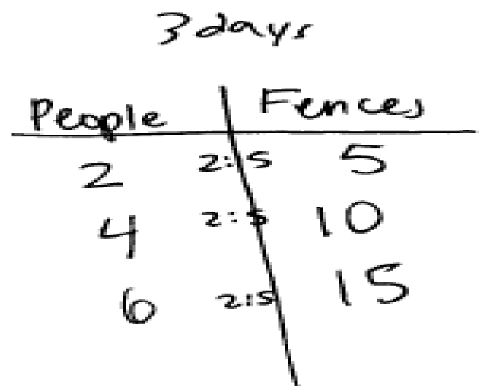


Figure 12. Sally’s ratio table for the number of people and number of fences relationship.

In the Gear II task, when asked to draw a graph to express the relationship between the number of revolutions and the radius, Sally marked some values that she obtained earlier and drew the graph in Figure 13, but she could not decide whether the graph needed to be straight or curved (Figures 14 a and b). She then explained:

Sally: Between points 1 and 3, I am going to figure out the slope. If it is a linear relationship, the slope should be constant between all these points.

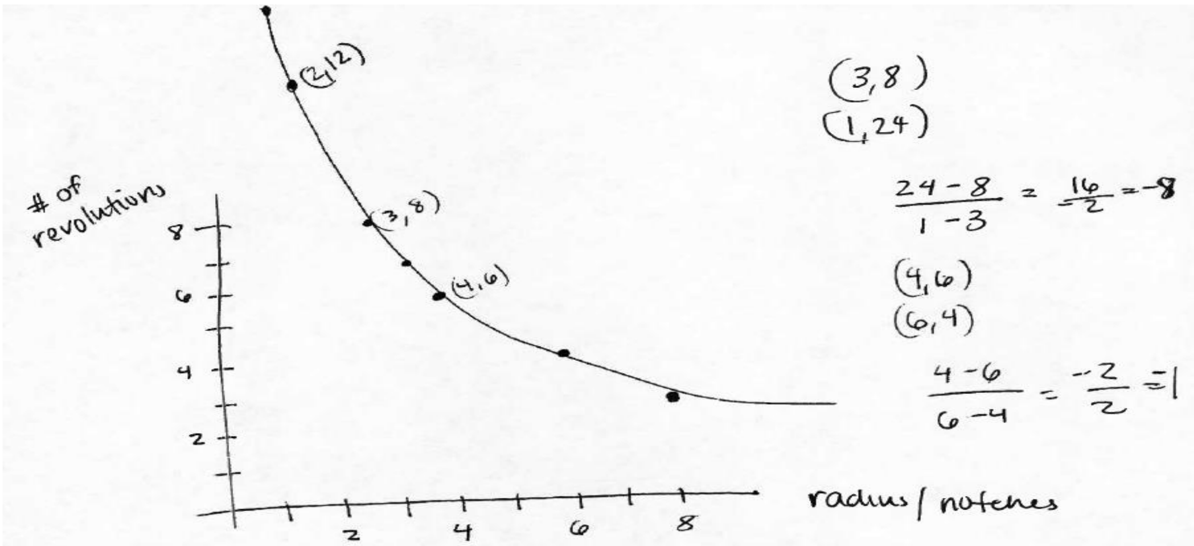
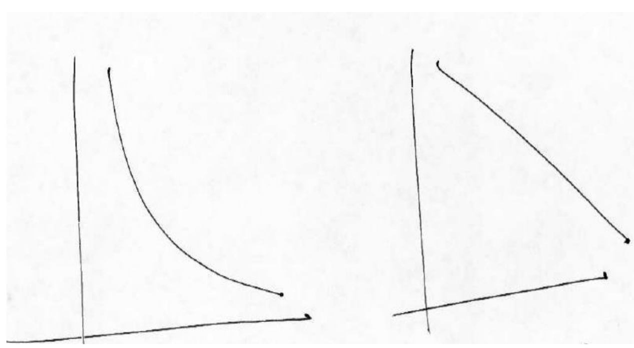


Figure 13. Sally's inversely proportional graph for the number of notches and number of revolutions relationship.

Sally calculated the slopes for some of the values and determined that the slope was changing (Figure 13). Hence, she decided Figure 14 b was the graph of the inverse relationship. Similar to Abby, Sally also initially assumed the graph in Figure 14 b as an inversely proportional graph and like Abby, by attending to the constancy of the rate of change, she obtained the correct inversely proportional graph. Abby and Sally's initial erroneous assumptions suggested constraints in their understanding of proportional and nonproportional relationships.



(a) (b)

Figure 14. Sally's attempts to draw an inversely proportional graph.

Jason's Case

Summary. Jason used both proportion formula and proportional reasoning strategies. His solution strategies and reasoning demonstrated his understanding of directly proportional relationships; however, he experienced conceptual difficulties in understanding and explaining inversely proportional relationships. For example, in the Gear II task, he incorrectly inferred a directly proportional relationship between the size of a gear and the number of revolutions it made. Jason had trouble with multiple proportion questions that involved the time concept. For instance, in the Speed task, he incorrectly read out an inversely proportional relationship between the distance traveled and the number of minutes. Jason used ratio tables, graphs, equations, or formulas to represent the directly and inversely proportional relationships.

Cross-Task Analysis. Similar to Abby and Sally, to decide the relationships between two covarying quantities, Jason usually increased (or decreased) the value of a quantity and observed the corresponding change in the other quantity. If this second quantity also increased, then he said there was a directly proportional relationship between those two quantities. On the contrary, if the second quantity decreased, then he said these two quantities were inversely proportional. Jason also made multiplicative comparisons within measure spaces if the covarying quantities involved doubling and halving and used those multiplicative comparisons to infer the relationships or to solve questions.

In the Bakery II task, Jason explored the relationship between the number of people and number of minutes needed to frost 50 cupcakes. He stated that the multiplicative relationship within two measure spaces by saying:

Jason: We have twice as many people working so it should not take longer, it should take half the time not the double the time. Six minutes. How to explain it...It is because we do not have a proportional relationship. We have an inversely proportional relationship.

Jason's statement, "We have twice as many people working so it should not take longer, it should take half the time not the double the time" provided evidence for his attending to reciprocal multiplicative relationships within measure spaces. Jason's response suggested that he used this reciprocal multiplicative relationships as a knowledge resource to infer an inversely proportional relationship between the number of people and number of minutes.

If the problems involved more than two quantities, Jason fixed one quantity at a time to discuss the relationship between the other two quantities. For example, in the Bakery II task, he expressed the relationships among the number of people, number of cupcakes, and number of minutes with the equation: $p * m = c$. Although this equation was not complete, it was important because Jason formed this equation by fixing a quantity at a time and exploring the directly and inversely proportional relationships between the other two quantities. When asked to replace some values into this equation for checking, Jason realized that the equation was not working for the values he replaced. His reasoning was understandable, but he needed to recognize that $\frac{p*m}{c}$ was equal to some constant k, so the equation was going to be $p * m = k * c$.

In the Gear II task, Jason initially endorsed a directly proportional relationship between the number of revolutions and radius. To determine the number of revolutions of a gear, with a 3-cm radius, given that another gear, with a 4-cm radius, made six revolutions, he set up the direct proportion $\frac{3\ cm}{4\ cm} = \frac{x\ revolutions}{6\ revolutions}$ and cross-multiplied to obtain an incorrect answer, $\frac{9}{2}$ revolutions (Figure 15). To explain his solution, Jason used a concept of "distance travelled," which appeared to be a piece of knowledge that the participants usually coordinated with the gear concept. He calculated that the gear with a 4-cm radius travelled a distance of 48π and the gear with a 3-cm radius travelled a distance of $x * 6\pi$, but he could not find a way to coordinate the concept of distance traveled with the directly proportional relationship. Hence, he could not explain that the gears had to travel the same distance since they were meshed. His difficulty understanding the gears traveling the same distance suggested constraints in his coordination of constant product relationships. At that moment, I asked him whether the gears traveled the same distances or different distances. He reacted to my question by saying, “Ohhh I see what you are saying.” As can be understood from his reaction, my question directed him to think about the possibility of the gears traveling the same distances. It would have been good if I had waited for him to finalize his response, instead of directing him with my question. Although he explained that the gears required traveling the same distances because they were “linked,” it is a question whether he could have been obtained the same conclusion if I had not helped him with my question. On the other hand, his initial incorrect inference can be given as an example for his judgment of an inversely proportional relationship to be directly proportional.

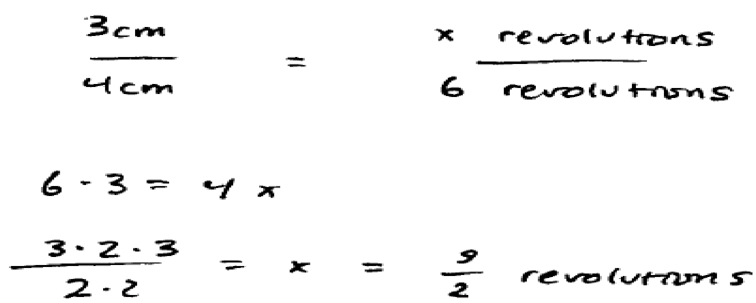


Figure 15. Jason’s incorrect direct proportion strategy.

In the Speed task, when he was calculating the speed of a car that was driving two miles in 100 seconds, Jason assumed the relationship between the distance and time to be inversely proportional. As a result of assuming an inversely proportional relationship, he used opposite operations within two separate measure spaces and obtained an incorrect answer (Figure 16). The following exchange shows our discussion:

Int: Twenty miles per hour your speed?
Jason: Yeah.

Int: How did you derive that conclusion?
Jason: So, you know I can go two miles in 100 seconds, so again using inverse proportionality five-thirds time two miles in 60 seconds because six-fifths I mean six-tenths time 100 is 60. So, ten-fifths I mean ten-sixths which is equal to five-thirds time two equals ten-thirds.

These exchanges suggested constraints in Jason's understanding of the directly and inversely proportional relationships. In addition, his incorrect judgement of the relationship showed that Jason might not have had well-developed strategies for inferring directly and inversely proportional relationships and for successfully distinguishing these relationships from each other. As seen in Figure 16, he also incorrectly multiplied $\frac{10}{3}$ by 6 instead of 60. When reminded if it was okay to drive more miles in fewer seconds, Jason recognized his incorrect assumption of inversely proportional and said, "Oh I used inverse proportion. Yeah okay I used, I assumed an inversely proportional relationship and that is not the case." He then corrected his answer by assuming a directly proportional relationship.

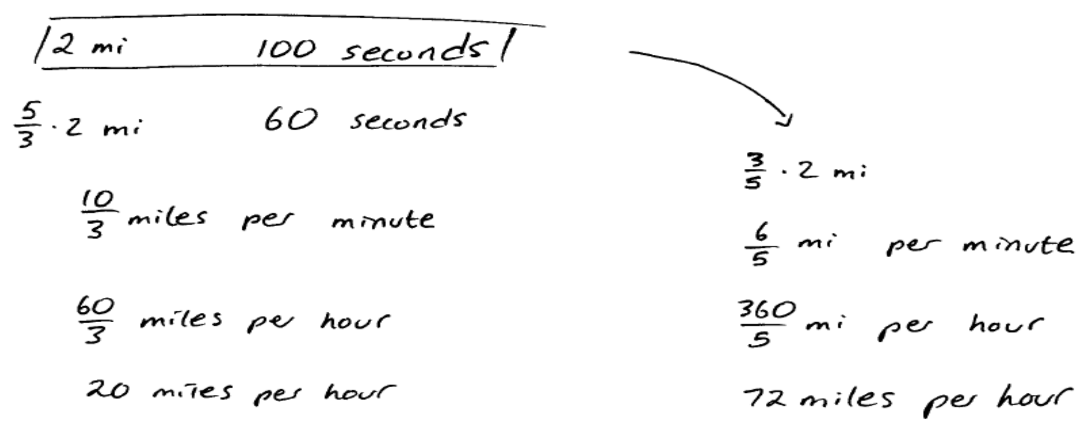


Figure 16. Jason's response to the Speed problem.

Robert's Case

Summary. Robert solved the given questions generating algebraic formulas or equations, or using the proportion formula strategy. In his solution strategies, he appeared to depend on the numerical relationships and preferred to express relationships among quantities with algebraic equations and formulas. He was comfortable with mathematical calculations, but he had difficulty explaining the meanings of his calculations and the meanings of numerical values. In each task, he generated the mathematical expressions by trying out numbers, and he checked the accurateness of his expressions by replacing the given numbers. He did not explain directly and inversely proportional relationships between quantities; instead, he described qualitative relationships among them. Therefore, his reasoning and solution strategies indicated that he had a limited coordination of directly and inversely proportional relationships. He represented directly and inversely proportional relationships with graphs, equations, or formulas.

Cross-Task Analysis. Similar to the other three participants, Robert compared the given quantities qualitatively and decided the relationships to be inverse or linear. In a few instances, he described multiplicative relationships between quantities. Therefore, comparing quantities qualitatively was his main strategy for inferring relationships. For example, the following exchanges in the Gear II task showed how Robert described the relationship between the size of a gear and its number of revolutions:

Int: Do you think is there a relationship between the radius and the number of revolutions the gear makes?
Robert: It is related because like we said this is the largest radius and this is the smallest radius so let’s say the greater the radius the fewer amount of rotations. So if this one is greater than this, this one [pointed at the small gear] had to make more rotations than this one [pointed at the big gear] because its radius is greater.
Int: What happens if we make this one too big you know? If we make the second gear too large or too big then what happens if we return this one [second gear] six times again?
Robert: This [pointed at the first gear] has to turn more.

In the exchanges above, Robert stated the inverse qualitative relationship by saying, “…the greater the radius the fewer amount of rotations.”

In the Cookie Factory task, when asked to calculate the number of hours needed by eight assembly lines to fill a truck with boxes of cookies, given that four assembly lines filled the same truck in 10 hours, Robert inaccurately stated a “linear relationship” between the number of assembly lines and number of hours. Hence, he solved the problem incorrectly and obtained 20 hours as his answer. To explain his answer, he made an incorrect multiplicative comparison within measure spaces and stated that it would take eight assembly lines twice the time of four assembly lines to fill the same truck. Some exchanges later, he described the correct inverse qualitative relationship between the number of assembly lines and the number of hours by saying, “As the assembly lines increase hours decrease and vice versa.” When reminded of the contradiction between his description of the relationship and his answer, Robert divided 10 hours by 2 instead of multiplying and obtained the correct answer, 5 hours. Thus, it appeared that my reminder created a disequilibrium between the computation that Robert made and the qualitative relationship he had described. This was the only case he incorrectly inferred a relationship between given quantities. It was possible that he might have been attracted by the possibility of a linear relationship between given quantities. Therefore, this was an example of a preservice teachers’ incorrect judgment of an inversely proportional relationship to be a directly proportional relationship.

In the Gear I and II tasks, when asked how he obtained the units of his answers, Robert had difficulty to explain how he obtained those units. For example, when asked to calculate the radius of a gear with 11 notches, given that another gear, with a 3-cm radius, had 12 notches, Robert divided 11 notches by four notches and obtained 2.75 cm as the radius of the gear:

Int: If we had another gear with 11 notches, what would be the radius of that new gear?

Robert: So, you would keep this thing [circled the 4 to 1 ratio] the same and then you just take 11 notches divided by four notches because you are using the same standard. And that will give you 2.75 cm. Because you are using the same standard as the last, so the ratio will be the same.

Earlier in the Gear I task, simplifying the 12-notches-to-3-cm relationship, Robert obtained a 4 notches**:**1 cm ratio and interpreted it as “four notches for every one cm.” Robert’s explanation in the exchange suggested that he was searching for the same constant ratio relationship. Hence, he divided 11 notches by 4 notches and got 2.75 cm as the radius of the gear. When I asked how he got the unit of his division to be centimeters, he seemed confused and could not give an answer. When dividing, he seemed to focus on the numbers rather than referent units. Therefore, his difficulty answering my question suggested that he might not know the unit of this 4 notches**:**1 cm ratio could be written as notches per cm (or notch/cm). Similarly, in the Gear II task, Robert used the idea of the “total number of notches moved” on one gear as a result of some number of revolutions to solve the questions. For instance, he calculated the total number of notches moved on Gear 2, with 16 notches, after three revolutions as 48 notches. He then divided 48 notches by the number of notches around Gear 1, which had 12 notches, and obtained the number of revolutions as four. When asked how he obtained *revolutions* as the unit, he had difficulty to explain. Hence, he used statements such as “12 notches is one rotation” or “12 notches is equivalent to one rotation,” but he did not realize that the statements could be written as 12 notches/rotation. Therefore, he should have realized that the accurate within measure space ratio could be written as $\frac{48\ (\text{notches})}{12\ (\text{notches/rotation})} = 4$ rotations. As I explained in Chapter Two, the division operation he made in this question is called *measurement division*. In this question, 48 notches was the constant product that obtained by multiplying the number of notches by the number of revolutions. Robert’s difficulty interpreting the meaning of his division suggested that he calculated the correct answer by just focusing on the numbers and without knowing the significance of the operations he made.

Robert generally searched for the numerical relationships between quantities and generated algebraic expressions, equations, or formulas to solve the given problems. He also set up proportions and used additive strategies. Hence, his solution strategies can be classified in Fisher's (1988) algebra and proportion formula strategies. Although Robert preferred using equations and formulas, he had trouble explaining the meanings of his equations and formulas. For example, in the Bakery II task, when asked to calculate the time needed by *n* people to frost 50 cupcakes, given that two people frosted 50 cupcakes in 12 minutes, as a result of trying out the values of the given quantities, Robert obtained the correct formula $\frac{24}{n}$ to indicate the time needed to frost 50 cupcakes by *n* people. When asked the meaning of 24 in $\frac{24}{n}$, he could not explain that it represented the time required for one person to frost 50 cupcakes. Thus, Robert's difficulty interpreting the meaning of units in his solutions, and his difficulty understanding the meaning of his formulas, suggested constraints in his coordination of the directly and inversely proportional relationships in the given tasks.

Because the Fence task involved multiple relationships, it was difficult to think about setting up proportions. Therefore, he seemed to have difficulty solving the questions in this task. For example, when asked to calculate the number of days needed for two people to paint one fence, given that three people painted five fences in two days, first, he identified that one person could paint $\frac{5}{3}$ fences in two days. He then divided five-thirds by 2 to decide how many fences one person could paint in one day and with a calculator he achieved .8333. He had trouble with converting this result to its fractional form that indicated problems with fractions. A few exchanges later, he realized that .8333 was equal to five-sixths. He then determined that two people could paint $\frac{10}{6} = \frac{5}{3}$ of a fence in one day. After he obtained this result, he said "but that is too much." He seemed confused, so when asked the time needed for one person to paint five fences, he calculated that one person could paint five fences in six days. As an answer to the time required for two people to paint one fence, he said that "I want to say it is three-fifths of a day, but I do not know why." Using the information "$\frac{5}{6}$ fence = 1 day," which he already identified, he determined that it would take six-fifths of a day for one person to paint one fence. He explained that he multiplied both sides of the equation by $\frac{6}{5}$. Next, he divided $\frac{6}{5}$ by 2 and determined that two people could paint one fence in three-fifths of a day (Figure 17). My follow-up questions and the conversation between us seemed to provide clarity to Robert; however, he did not appear to understand the meanings of the operations that he used in his solution.

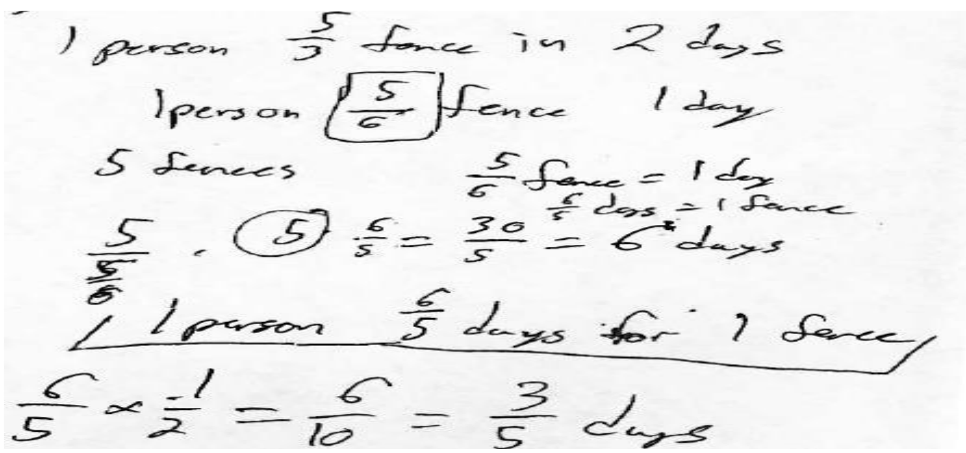


Figure 17. Robert's response to the Fence problem.

When the numbers were not presented, Robert appeared to have difficulty solving the given questions. If the questions involved a single directly proportional relationship, then he easily obtained a formula or equation to express numerical relationships. For example, in the Bakery I task, given that two people can frost 50 cupcakes in 12 minutes, Robert represented the number of cupcakes frosted by *n* people in 12 minutes by the formula x = 25n. On the contrary, if the problems involved a single inversely proportional relationship, Robert had to spend some time to figure out the correct formula. For example, in the Bakery II task, he had to try out three formulas to obtain the correct formula $\frac{50}{25} \times \frac{12}{\mathrm{n}} = min$. By this formula, he was able to calculate the time required for *n* people to frost 50 cupcakes. Similarly, in the Cookie Factory task, he had to spend some time to obtain the formula $\frac{\text{4 lines}}{\text{x lines}} \times 10 = y\ hours$ by which he calculated the time or the number of assembly lines. Thus, it appeared that Robert had difficulties when the numbers were not presented and in solving multiple proportion problems because he did not seem to coordinate the directly and inversely proportional relationships and did not consider proportionality in his reasoning. Coordinating proportionality requires performing correct operations in complex situations, even if the numbers were not presented.

Discussion of the Pilot Study Findings

In the following pages, I discuss the pilot study findings around each research question by making cross-case comparisons.

Research Question 1: *How do preservice middle and high school mathematics teachers infer directly and inversely proportional relationships in single and multiple proportion problems; what types of knowledge resources do they use when inferring and explaining directly and inversely proportional relationships; and what kinds of difficulties do they encounter in the process of inferring, explaining, and expressing directly and inversely proportional relationships?*

Before attempting to solve problems, all four of the PSTs initially decided the relationships between quantities by paying attention to the qualitative relationships (e.g., coordinated increments/and or decrements of the values in each related quantities). For example, if the values of two quantities increased (or decreased) together, then they inferred a directly proportional relationship. On the other hand, if the value of a quantity increased and the value of the related quantity decreased, then they inferred an inversely proportional relationship.

Furthermore, it appeared that they usually expected a clear dichotomy, if a relationship is not directly proportional, then it is inversely proportional. None recognized that if a relationship is not directly proportional, then it does not have to be inversely proportional.

If the given questions involved more than two quantities, such as in the multiple proportion tasks, then they usually fixed one quantity at a time to explain the relationship between the other two quantities. In the multiple proportion tasks, I also observed that PSTs were able to coordinate the need for taking the value of a quantity as constant with the presence of a directly or inversely proportional relationship between the other two quantities, and this coordination seemed to be very important. The PSTs' qualitative comparisons usually involved causal relationships (e.g., *x increases so y increases* or *x increases so y decreases*), and they used these qualitative comparisons to infer the directly and inversely proportional relationships. In addition, looking at the PSTs' responses, I can say that deciding relationships before attempting to solve the given questions increased participants' successes of getting correct answers.

Based on the inaccurate dichotomy that they expected, all four PSTs used a similar strategy in inferring an inversely proportional relationship between two quantities. This strategy involved coordination of the inverse qualitative relationship—Quantity A increases and Quantity B decreases—in the given question with the knowledge of what the qualitative relationship would look like if it was a directly proportional relationship. Therefore, by coordinating these two knowledge pieces, they were able to demonstrate a contradiction between directly and inversely proportional relationships, and by using this contradiction, they rationalized their inferences of inversely proportional relationships. Besides comparing quantities qualitatively, which was participants' main strategy to infer the directly and inversely proportional relationships, PSTs also compared quantities multiplicatively and used those multiplicative relationships to infer directly and inversely proportional relationships and to solve questions. They usually made comparisons within measure spaces. In their study, Izsàk and Jacobson (2013) pointed out how PSTs' formation of multiplicative relationships between quantities played an important role in their inferences about directly proportional relationships. Similarly, in this study, I observed that PSTs' formation of multiplicative relationships significantly affected their inferences of the relationships and their abilities to meaningfully distinguish directly and inversely proportional relationships from each other.

Each PST had some difficulties explaining what directly and inversely proportional relationships implied and distinguishing these two relationships from one another. Because Robert depended on algebraic equations and formulas to solve given problems, I observed that he had more difficulties than other participants in explaining and making sense of his solutions. For instance, he had difficulty using correct units, explaining the meaning of the units, and unit conversions. Since Robert searched for numerical relationships, when the numbers were not presented, he had trouble generating equations. He also had difficulty with fractions and fraction operations and with solving multiple proportion questions. Abby initially had two meanings of the rate concept and endorsed a single rate for the inversely proportional relationship between the number of weights and distance. Jason incorrectly judged an inversely proportional relationship to be directly proportional. This result was consistent with the findings from previous studies, which reported that teachers tend to judge nonproportional relationships to be proportional (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010).

In the Speed task, Sally and Jason incorrectly inferred an inversely proportional relationship between the distance and time. It seemed that inclusion of time might have inclined Sally and Jason to make incorrect inferences about the directly proportional relationships. Their incorrect inferences can be given as evidence for their difficulties in inferring and distinguishing directly and inversely proportional relationships in the multiple proportion tasks. In addition, Abby and Sally's initial expectations of the graph of an inversely proportional relationships to be straight demonstrated their difficulties in distinguishing inverse additive relationships from inverse multiplicative relationships.

Research Question 2: *What types of solution strategies do preservice middle and high school mathematics teachers use to solve single and multiple proportion problems, and how do they express directly and inversely proportional relationships?*

Except Robert, the remaining three PSTs generally considered proportionality in their responses and mainly used a proportion formula and/or proportional reasoning strategies to solve the given questions. Robert usually generated algebraic equations and/or formulas to solve the questions. However, he also used the proportion formula strategy in which he set up direct and inverse proportions and cross multiplied values to get the missing one. I observed that Abby, Sally, and Jason generated equations or formulas as well to solve problems, but they usually generated these equations or formulas if I asked them to do so or as a second approach. In the multiple proportion questions, Abby, Sally, and Jason generally used a ratio table strategy, which involved coordinated multiplications or divisions. In addition, I observed that Jason and Robert used correct additive strategies in the Gear I and Gear II tasks, respectively; however, they used those strategies as backups. Hence, it appeared to me that the use of hands-on tasks, in which physical devices were provided, and the multiple proportion questions prevented the use of additive and incorrect strategies and precipitated the use of the proportion formula and proportional reasoning strategies.

The PSTs usually expressed the directly and inversely proportional relationships with ratio tables, direct and inverse proportions, formulas, equations, or graphs. I observed that none of the participants knew what a graph of an inversely proportional relationship would look like. Jason and Robert obtained an inversely proportional graph by marking points without making a prediction about what it would look like. On the other hand, Abby and Sally initially thought that the graph of an inversely proportional relationship could be straight (Figures 11 a and 13 b, respectively). All participants correctly identified the slopes of the directly proportional graphs and explained their meanings. They all explained that, in a directly proportional graph, the slope was constant and there was not a single slope in an inversely proportional graph. Robert also added that the slope was positive in a linear graph, and it was negative in an inverse graph, but he also stated that the graph of a directly proportional relationship could be curved.

A Brief Summary of the Pilot Study

The pilot study yielded information about the quality of the research questions, theoretical approach, research design (e.g., data collection, participant selection, data analysis), and the interview process (e.g., quality of the interview tasks and follow-up questions). In the following pages, I discuss how I used this information to improve the quality of the final study.

Research Questions. I revised my initial research questions according to the pilot study results and to the change in my theoretical approach. For instance, based on my initial theoretical approach, the social constructivist theory, I was mainly interested in PSTs' comprehension of the directly and inversely proportional relationships and their constructions of the knowledge of these two relationships. Later, based on the feedback that I received from my advisory committee on my pilot study, I decided that employing the *knowledge in pieces* perspective to make sense of my interview data would be a better option. Therefore, I revised my research questions considering this new theoretical approach and the initial findings.

Theoretical Approach. As I stated in the above paragraph, I initially planned to employ a social constructivist theory in analyzing my pilot data. Later, my advisory committee suggested that the social constructivist approach may not be compatible with my research purposes and the research questions at hand. Hence, I decided to employ the *knowledge-in-pieces* perspective to interpret the interview data, because, as I discussed earlier, the *knowledge-in-pieces* perspective offer effective tools (e.g., read out strategies and the causal net) for analyzing the knowledge resources of PSTs in inferring directly and inversely proportional relationships. Furthermore, the *knowledge-in-pieces* perspective is effective in analyzing PSTs' responses to the problems with complex cognitive structures. In addition, PSTs' comprehension of science concepts (e.g., gear ratio, velocity, and balance) can be interpreted by employing a *knowledge-in-pieces* perspective.

Research Design. The use of case study methodology helped me analyze the pilot interview data closely and report findings in-depth. Therefore, I decided to continue using the case study methodology to report my final findings. Semi-structured interviews also helped me generate reflective conversations between the participants and me. These reflective conversations assisted me in understanding the PSTs' reasoning, and they also appeared to help the PSTs understand directly and inversely proportional relationships and to realize their mistakes. In addition, the pilot study showed the importance of participants having some college level experiences with directly and inversely proportional relationships. Except Robert, I recruited the PSTs from courses that treated proportional relationships because Izsàk and Jacobson (under review) observed that the PSTs usually enter these courses with a few knowledge resources, and they usually use computation methods like cross-multiplication. As in Robert's case, they may also focus on the numerical relationships rather than direct and inverse covariation between quantities. Because the mathematical tasks that I used required proportional reasoning skills to understand directly and inversely proportional relationships between quantities, and the multiple proportion tasks were challenging and could not solved using rote computation methods, for the final study, I decided to recruit participants with some college level experience on proportions.

Interview Process. The analysis of the pilot tasks revealed that the Gear and Speed tasks yielded richer information than the other ones. Therefore, I decided to continue using these tasks in the final study. Because the pilot study demonstrated that the PSTs had a dichotomy—a relationship is either directly proportional or inversely proportional—in the Gear task, I decided to provide the PSTs with three graphs (see Appendix B Task 1B) that involved nonpoportional relationships to investigate their ability to distinguish proportional relationships from nonproportional relationships. The Bakery and Painter tasks were similar in nature; however, because the problems in the Bakery tasks could be converted to multiple proportion problems, I decided to continue with the Bakery tasks. The Fence problem was a little difficult, because it involved multiple relationships, and the Apartment task, which also involved multiple relationships, was solved easily without considering proportionality. Hence, I decided to use the Fence task as an extra in the final study. On the other hand, I decided to use the Scout Camp task adapted from Vergnaud (1983), which I did not use in the pilot study, as an extra instead of the Apartment task. The reason for including the Scout camp task was that unlike the Apartment task, it involved three quantities with different referent units (the number of people, amount of cupcakes, and the number of days). In addition, because the Balance task offered hands-on experience about inversely proportional relationships, I also decided to use it in the final study.

Therefore, I decided to use the Gear, Bakery, Balance, Speed, Fence, and Scout tasks in the final study.

After deciding the mathematical tasks for the final study, I revised the pilot interview protocol to develop the final interview protocol. While revising the pilot interview protocol, I considered the pilot findings, my research questions, and the feedback that I obtained from my advisory committee. Thus, I developed a first draft of my final interview protocol. Later, I shared this final protocol with two doctoral students and with my advisory committee for their feedback. Next, I revised this first draft according to their feedback and generated the last form of the final interview protocol.

In the following chapter, I report the findings of the final study and discuss the PSTs' reasoning about directly and inversely proportional relationships.

CHAPTER 4

RESULTS

Research Problems, Purposes, and Questions

It is my contention that in mathematics education literature, the concept of inversely proportional relationships is largely overlooked because of its complex nature and the fact that directly proportional relationships occur more frequently in school mathematics. Hence, issues such as PSTs' reasoning about and comprehension of inversely proportional relationships is not well-explored. Similarly, the concept of multiple proportions has been explored by only a few researchers such as Vergnaud (1983, 1988). Thus, my main goal in conducting these case studies was to investigate how preservice middle and high school mathematics teachers infer directly and inversely proportional relationships in the given mathematical tasks and distinguish them from each other and from nonproportional relationships. Additionally, I was interested in understanding the types of strategies that PSTs used to solve single and multiple proportion problems, their ability to represent directly and inversely proportional relationships in the given problems, and the difficulties that they encountered while solving these problems. To achieve these goals, I was guided by the following research questions:

1. How do preservice middle and high school mathematics teachers infer directly and inversely proportional relationships in single and multiple proportion problems; what types of knowledge resources do they use when inferring and explaining directly and inversely proportional relationships; and what kinds of difficulties do they encounter in

the process of inferring, explaining, and expressing directly and inversely proportional relationships?

2. What types of solution strategies do preservice middle and high school mathematics teachers use to solve single and multiple proportion problems, and how do they express directly and inversely proportional relationships in those problems?

Analysis and Findings

In the following pages, I present the analysis of four cases. The case analysis begins with a brief summary of the cross-case findings. This is followed by a cross-task analysis of the PSTs' responses. In the cross-task analysis, the analysis of the PSTs' responses is organized around the themes that I provided in Chapter Three. To interpret the PSTs' responses, I use the *knowledge-in-pieces* perspective and *multiplicative conceptual field* framework. The knowledge-in-pieces perspective is employed to understand the knowledge resources of the PSTs and their coordination of directly and inversely proportional relationships. On the other hand, the *multiplicative conceptual field* framework is employed to understand multiplicative structures presented in the problems. Because the PSTs worked on various questions, I only provide their strategies on some of those questions. I selected those strategies based on the following three criteria: (a) Did the participant present a different perspective in her solution than the remaining participants? (b) Did the strategy exhibit a significant understanding or constraint in the participant's reasoning? (c) Did the strategy involve a new way of expressing the relationships contained in the questions? There are no deletions in the transcripts that I provide. I show pauses with ellipses and describe actions within square brackets.

Summary of the Cross-Case Findings

Kathy was the more proficient of the two secondary grade PSTs in proportional reasoning. She successfully inferred the constant ratio relationships between two covarying quantities by making multiplicative and qualitative comparisons. She easily determined the multiplicative relationships within measure spaces. In addition, if the values of quantities in two separate measure spaces involved doubling and halving situations, she was able to state the reciprocal multiplicative relationships. Otherwise, she stated the numerical reciprocal multiplicative relationships. Kathy was the only participant among the four PSTs who inferred appropriately that relationships illustrated in Graphs B and C (see Appendix B Task 1B) are nonproportional. Kathy's attention to the numerical multiplicative relationships within the separate measure spaces allowed her to infer the relationships in those graphs as nonproportional. On the other hand, although Susan was able to form multiplicative relationships within measure spaces and reciprocal multiplicative relationships, she attended to the constancy of the rate of change when inferring constant ratio relationships and attended to the static points on graphs and values of points being swapped to infer constant product relationships. Hence, she had difficulty distinguishing the nonproportional relationships that were depicted in Graphs B and C from directly and inversely proportional relationships. Both Kathy and Susan recognized the constancy of the products in Tasks 1B and 3 but not in the remaining inverse proportion tasks. They both had difficulty explaining multiplicative relationships between measure spaces. Hence, they preferred reasoning within measure spaces when solving the given tasks.

The two middle grade PSTs, Carol and Helen, did not differ much in their reasoning. They were successful in determining multiplicative relationships within measure spaces, reciprocal multiplicative relationships, and qualitative relationships between two covarying quantities. Carol inferred directly proportional relationships between quantities by attending to unit rates, multiplicative relationships within measure spaces, and qualitative relationships—two quantities are increasing (or decreasing) together. On the other hand, Helen's main knowledge resource for inferring given relationships was attending to qualitative relationships and constancy of the rate of change. They both attended to inverse qualitative relationships—one quantity is increasing and other quantity is decreasing—when inferring inversely proportional relationships, and both PSTs' responses to inverse proportion questions indicated their difficulties coordinating constant product relationships. Because of their attention to qualitative relationships and constancy of the rate of change when inferring proportional relationships, Carol and Helen had trouble distinguishing proportional relationships from nonproportional relationships depicted in Graphs B and C in Task 1B. With the exception of Task 1A, in which they reasoned between measure spaces, they usually preferred reasoning within measure spaces when solving single and multiple proportion questions. They also used a variety of proportional reasoning and other strategies to solve the proportion questions. In Task 2B, they both tended to interpret the cupcake order in terms of minutes rather than cupcakes. Hence, they inappropriately shared the number of minutes among the number of people. Carol and Helen used graphs, formulas, tables, pictures, or some combination to express the directly and inversely proportional relationships in the given tasks.

Case One: Kathy

Summary

Kathy was successful in inferring directly and inversely proportional relationships in the given tasks. She attended to multiplicative relationships (e.g., the constancy of the quotients, unit ratio relationships, and numerical multiplicative relationships between two separate measure spaces) and to qualitative relationships—two quantities are increasing (or decreasing) together—when inferring directly proportional relationships between quantities. On the other hand, she attended to numerical reciprocal multiplicative relationships, context of balancing, and inverse qualitative relationships—one quantity is increasing and other quantity is decreasing—when inferring inversely proportional relationships. Kathy recognized the constancy of the products in Tasks 1B and 3, but she did not recognize them in the remaining inverse proportion tasks. Therefore, the contexts of the hands-on tasks were effective in facilitating Kathy's recognition of the constant product relationships. She successfully distinguished directly and inversely proportional relationships from the nonproportional relationships that consisted of a quadratic growth, constant difference, or a constant sum. Kathy generally used proportional reasoning strategies and preferred reasoning within measure spaces when solving single and multiple proportion questions. She expressed directly and inversely proportional relationships with graphs, double number lines, formulas, tables, or some combination.

Cross-Task Analysis

In Chapter Three, I determined three themes for Kathy's case among the codes that I provided in Table 1. In the following pages, I elaborate on these three themes to explain Kathy's reasoning across tasks. In the first theme, I discuss how Kathy inferred directly proportional relationships in Tasks 1A, 2, 4, and 5, and inversely proportional relationships in Tasks 1B, 3, and 4 by attending to multiplicative and qualitative relationships. In the second theme, I investigate how Kathy distinguished directly and inversely proportional relationships in Task 1B from nonproportional relationships. In the last theme, I discuss selected proportional reasoning strategies that Kathy used across tasks, and her preference for reasoning within measure spaces.

Theme 1: Attention to multiplicative and qualitative relationships when inferring directly and inversely proportional relationships.

In Task 1A, when asked to discuss the relationship between the number of notches and radii, Kathy used the given information about two meshed gears, X and Y, to discuss the relationship. Gear X had a radius of r_1 cm and n_1 notches, and Gear Y had a radius of r_2 cm and n_2 notches. The following exchanges demonstrate how Kathy inferred a constant ratio relationship between the radii and number of notches:

Kathy: Well yeah so you, I mean you would know that, like if this, like if we are just focusing on this relationship [drew a rectangle around r_1 and n_1] with this gear [pointed Gear X] like we know that the radius and the notches are, we know they are related because they are on the same circle, or gear, yeah on the same gear. And so just given that we increase the radius, size, like we also know decrease whatever we are doing, we are multiplying this radius [pointed r_1] by a number. We know that this radius [pointed r_2] depends on this radius here [pointed r_1] and then this [pointed n_1]…this number of notches [pointed n_2] depends on the number of notches here [n_1]. So, I think that if we know these two [moved her pen across r_1 and n_1] and then given whatever other one [made an imaginary circle around r_2 and n_2], we know like we find the relationship here [moved her pen across r_1 and n_1] and use it to find the same relationship here [moved her pen across r_2 and n_2].

Int: So, they have the same relationship here [pointed to r_2 and n_2], like this relationship [pointed to r_1 and n_1] and this relationship [pointed to r_2 and n_2] are the same?

Kathy: From like radii to notches.

Int: Yeah radii to notches.

Kathy: Yeah yeah yeah. Well it could be like, it [moved her pen across r_2 and n_2] is going to be a multiple of whatever we are given here [moved her pen across r_1 and n_1] yeah they are the same.

Although Kathy used the term *relationship*, it was not clear that she was referring to the ratio of the radii and number of notches until she stated the phrase "radii to notches." There was an indication of a ratio relationship in the "radii to notches" phrase, and her gestures were supporting this idea, because she was moving her pen across r_1 and n_1, and again across r_2 and n_2. Her statement "…we find the relationship here [moved her pen across r_1 and n_1] and use it to find the same relationship here [moved her pen across r_2 and n_2]" provided evidence for her suggestion of a constant ratio relationship. Although Kathy's final statement "…it is going to be a multiple of whatever we are given here yeah they are the same" can be interpreted as $\frac{r_2}{n_2} = \frac{r_{1*k}}{n_1}$, which precludes a constant ratio relationship, the phrase "yeah they are the same" suggests that she was considering the equivalence of the two ratios. Hence, it is my conjecture that by this statement Kathy implied $\frac{r_2}{n_2} = \frac{r_{1*k}}{n_{1}*k}$. Therefore, these exchanges suggested that she might have been attending to the numerical multiplicative relationship between the separate measure spaces of the radii and number of notches. Thus, these data provided initial evidence for Kathy's recognition of a constant ratio relationship between the radii and number of notches.

In Task 2A, when asked to determine the relationship between the number of people and number of cupcakes and between the number of cupcakes and time, Kathy inferred directly proportional relationships between those quantities. She used the information three people frosting 60 cupcakes in 12 minutes, which I provided her earlier, to explain that there was a 1-person-to-20-cupcakes and a 5-cupcakes-to-1-minute constant ratio relationship.

Kathy: This [pointed at people and cupcakes] is proportional yeah and so is this [pointed at cupcakes and time].
Int: So, the same?
Kathy: [Nodded]
Int: How did you determine these to be proportional?
Kathy: I guess because we can…we can get them all down to like this base case like where we know that for every one person they frost 20 cupcakes, and so then from there I can tell you any number of people. I mean, I could do the same thing here [pointed at cupcakes and time]. It would give me the same result but it is just like if I wanted to know how many cupcakes were made in 1 minute, then I would divide by…both these [pointed at 60 cupcakes and 12 minutes] by 12, and there were five cupcakes in 1 minute, and then I could get any number of cupcakes and any number of minutes.

Kathy's phrases "…for every one person they frost 20 cupcakes" and "…there were five cupcakes in 1 minute" illustrated her formation of unit ratios and suggested that she was considering the two unit ratios as batches. Kathy's statement "…we can get them all down to like this base case…" provided evidence for her recognition of a constant ratio relationship between the number of people and number of cupcakes and between the number of cupcakes and number of minutes, because she indicated that any people and cupcakes and cupcakes and minutes relationship could be simplified to the 1-person-to-20-cupcakes and 5-cupcakes-to-1-minute constant ratio relationships, respectively. Therefore, these exchanges demonstrated Kathy's attention to unit ratio relationships when justifying her inference of the directly proportional relationships.

In Task 4, Kathy successfully calculated the speed of a car, given that it covered 2 miles in 100 seconds, to be 72 mph using a scientific unit conversion method (Figure 18) which, she said, she used in chemistry and physics. When asked if there was a relationship between the distance and time, Kathy correctly inferred a proportional relationship by attending to the relationships between numbers in her ratio table and constancy of the quotients.

Int: So, what is the relation between the distance and the time? You talked about the relation between speed and the seconds to be inversely proportional, how about the relation between the distance you covered and the time it takes to cover?
Kathy: It should be proportional I think. Yeah this should be proportional.
Int: Please tell me why it is proportional.
Kathy: I mean, so you are going at a constant speed, okay so then miles and seconds [drawing a ratio table], miles is here and seconds and then we know this relationship 2 and 100 and we know 1 is 50 and 3 is 150 and 4 is 200 and so on.
Int: So, then you think it is proportional?
Kathy: Yeah because all these are all these have to same like ratio this 1 over 50, and 2 over 100 is going to be 1 over 50…3 over 150 is 1 over 50, it keeps going.

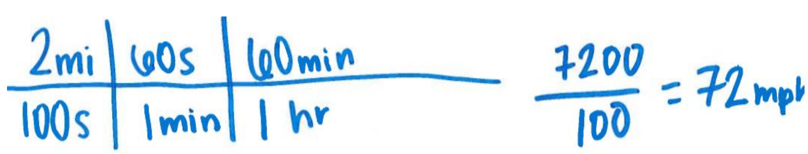


Figure 18. Kathy's scientific unit conversion method.

Kathy drew the ratio table in Figure 19 and showed that there was a constant $\frac{1}{50}$ ratio relationship between the distance and time. The exchanges above also showed her explicit statement of this constant ratio relationship. Therefore, the exchanges and Figure 19 confirmed that Kathy's causal net was sufficient to see that driving at a constant speed was yielding a constant ratio relationship between the distance and time. Because Kathy did not consider the referent units in her explanation of the constant ratio relationship between miles and seconds and did not show the multiplicative relationship by an expression, she appeared to attend to the relationships between numbers and constancy of the quotients when justifying her inference of the proportional relationship.

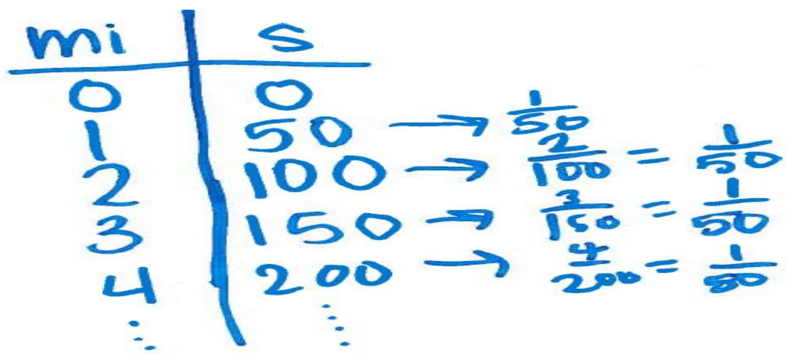


Figure 19. Kathy's ratio table for expressing a constant ratio relationship between the distance and time.

Some exchanges later, by taking 30 minutes as constant, Kathy expressed the constant ratio relationship between distance and speed with a ratio table based on an example—a car covering 10 miles in 30 minutes driving 20 miles per hour—that she generated (Figure 20). Kathy explained that she generated the ratio table based on the fact that the value of the speed was double the value of the distance:

Kathy: Uhmm my miles per hour you just double it because you are keeping 30 minutes constant. So, if I travel this [pointed at 15 miles] match in half an hour then the whole hour I would travel 30 miles ohh no no that is 30 miles per hour yeah yeah yeah that still makes sense. In 10 minutes, I will drive, no in 30 minutes I drive 10 miles and I am going 20 miles per hour. In 30 minutes I drive 15 miles...

Int: Then your speed is?

Kathy: 30 miles per hour.

Kathy's attention to the value of speed always being double of the value of distance and her phrase "my miles per hour you just double it" illustrated that she was attending to numerical multiplicative relationships between measure spaces. Hence, her reasoning in this task extended my understanding about what she was capable of. When I asked Kathy to generate a formula to express relationships among the distance, speed, and time, she easily determined the correct formula from the fact that the unit of the speed was written as miles per hour and tested the accurateness of the formula on the values in her ratio table. Kathy calculated *T* to be $\frac{1}{2}$ hour using the speed formula, but she did not explain that her constant $\frac{1}{2}$ ratio was also representing $\frac{1}{2}$ hour (see Figure 20). Kathy's absence of attaching an appropriate referent unit to constant $\frac{1}{2}$ ratio suggested her attention to the numerical multiplicative relationship. Therefore, Kathy inferred constant ratio relationships in Task 4 by attending to numerical multiplicative relationships between measure spaces.

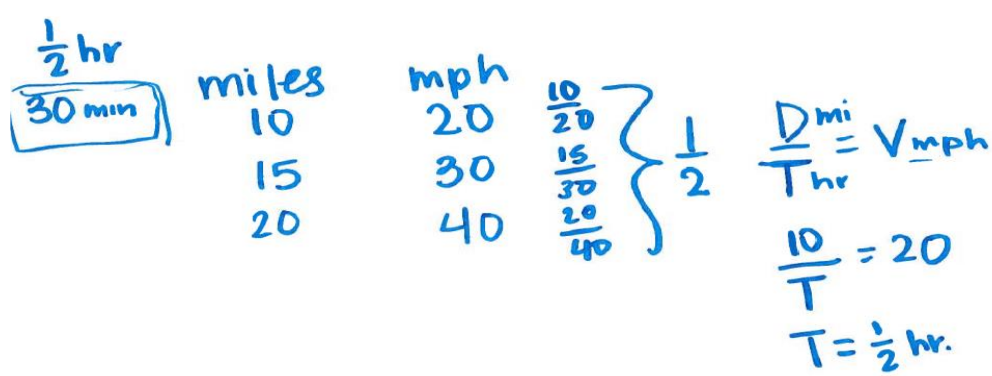


Figure 20. Kathy's expression of a constant ratio relationship between the distance and speed.

In Task 5, Kathy worked on calculating the number of days needed by two people to paint one fence, given that three people painted five fences in two days. Kathy inferred a directly proportional relationship between the number of fences and number of days based on the qualitative relationship that she constructed. She stated this qualitative relationship by saying, "…there is less fences it is going to take less time, so this is a proportional…" This was the only instance throughout the interviews that Kathy attended to a qualitative relationship when inferring a directly proportional relationship. Thus, Kathy's reasoning in Tasks 1A, 2A, 4, and 5 illustrated that she attended to the unit ratio relationships, constancy of the quotients, numerical multiplicative relationships between separate measure spaces, and qualitative relationships when inferring directly proportional relationships.

In the following pages, I discuss how Kathy inferred inversely proportional relationships in Tasks 1B, 3, and 4. In Task 1B, Kathy recognized that the product of the number of notches of a gear by the number of revolutions it made was giving the total number of notches moving in some number of revolutions. For instance, in one of the questions, Kathy needed to calculate the number of notches of Gear K, given that it completed two-thirds of a revolution, when Gear F, with eight notches, revolved three times. Kathy explained:

Kathy: No that is okay…let's see revolves three times and has eight notches, so that means like in total it goes through 24 notches right? Because it revolves three times with eight notches so this is [pointed at Gear F], it goes through 24 notches in this three revolutions…and then K revolves two-thirds of a time and goes through the exact same number of notches.

Int: How do you know it goes through the same number of notches?

Kathy: Because I guess like the reason why I am thinking that is because like in these three revolutions it [Gear K] goes through…goes through two-thirds of a revolution but because we know that F has eight notches we know that in total in three revolutions it is going to go through 24, so that we can say that. Okay I know, that [pointed at Gear K] is going to go through 24 notches in two-thirds of a revolution.

For Kathy, Gear F was "going through" a total of 24 notches in three revolutions, which she calculated by multiplying three revolutions and eight notches. By stating, "…K revolves two-thirds of a time and goes through the exact same number of notches," Kathy inferred that both gears were rotating the same total number of notches. Hence, for Kathy, Gear K needed to "go through" 24 notches in two-thirds of a revolution. A few exchanges later, she incorrectly multiplied 24 notches by two-thirds and obtained an answer, 16 notches, but she immediately recognized that this answer was not correct:

Kathy: Uhmm so I guess 24 notches in two-thirds how many notches in one whole is what we want to know. So, I guess we know it goes 24 notches in two-thirds of a revolution and we want to know how many in one. Ohh so it is three-halves not two-thirds I am backwards that is why. So, to get to here [pointed at 1 revolution] it is three-halves so two [inaudible]…

Kathy's search for the number of notches in one revolution was consistent with partitive division. Hence, by multiplying 24 notches by $\frac{3}{2}$, she calculated the correct answer as 36 notches. Kathy's explanation in this task provided evidence for her consideration of the constant total notches when explaining the inverse relationship between the number revolutions and number of notches. She used the idea of two gears "going through the same number of notches" to solve the remaining questions in this task. Thus, Kathy's idea of two gears "going through the same number of notches" can be given as an early sign of her understanding of the constancy in the situation.

Some exchanges later, Kathy worked on a new question. In this question, she needed to calculate the number of revolutions of Gear L, with eight notches, given that Gear M had 14 notches and revolved four times. She used the same idea—two gears "go through the same number of notches"—and calculated the answer to be seven revolutions. When asked if she could use a ratio table strategy to solve the same question, Kathy said she did not have familiarity with this strategy. Hence, I described for her what a ratio table looks like, and she was able to generate one for the relationship between the number of notches and number of revolutions depicted in the question (Figure 21). She recognized that the product of all rows (notches and revolutions) was equal to 56:

Kathy: Okay so well that has to be 56, I mean this is 56 here. I just know, I just kind of know that like all of these, like these two [pointed at notches and revolutions] have to multiply to give me 56 like every single time. So, I am saying what times two is 56 and that is, I do not know, 28. And then $\frac{56}{3}$, I do not what that is.

Int: You can leave like that. Knowing that 56, you said 56 is the?

Kathy: is the product of notches and revolutions.

Here, Kathy explicitly stated that 56 was the product of notches and revolutions. In the exchange, Kathy used multiplication and attended to the multiplicative relationships between the number of notches and number of revolutions to discuss the constant product relationship. Therefore, the exchange and Figure 21 suggested Kathy's coordination of the constant product relationship between the number of notches and number of revolutions.

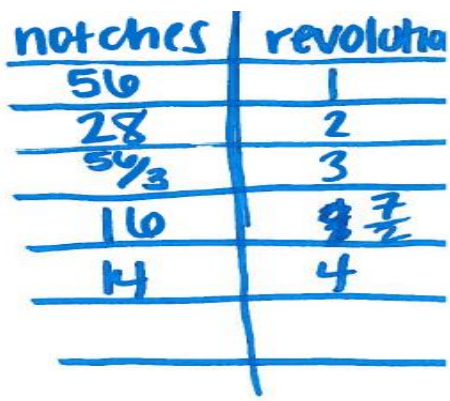


Figure 21. Kathy's ratio table for expressing a constant product relationship in Task 1B.

Some exchanges later, when asked to determine the number of revolutions of Gear K, with 12 notches, given that Gear F, with eight notches, revolved *p* times, Kathy calculated the number of revolutions to be $\frac{2}{3}$p (Figure 22). In her calculation, she multiplied eight notches by *p* revolutions to find the total notches moved on Gear F and then divided the product, 8*p*, by the number of notches of Gear K, which was 12. Kathy then explained an inversely proportional relationship between the number of notches and number of revolutions based on the numerical reciprocal multiplicative relationship that she constructed.

Kathy: I mean, I can say, this [pointed at 12] is three-halves of eight, so this [pointed at $\frac{2}{3}p$] is going to be two-thirds of p because it is inverse proportions.
Int: How do you know, like you said an inverse proportion?
Kathy: Yeah.
Int: How do you know it is inverse?
Kathy: I just by the definition of what an inverse proportion is. I do not really, we do not talk about this long time but it is just like…like you know like notches and revolutions are going to be inversely proportional because here you multiplying by three-halves and then you multiplying by two-thirds here. It is like you are multiplying by the reciprocal so yeah. You are multiplying one relationship like notches by the reciprocal of revolution, and then vice versa.

Kathy's initial statement "…this [pointed at 12] is three-halves of 8, so this [pointed at $\frac{2}{3}p$] is going to be two-thirds of p because it is inverse proportions" demonstrated how she attributed her description of the numerical reciprocal relationship between the number of notches and number of revolutions to the inversely proportional relationship. She then attributed the inversely proportional relationship to the within measure space multiplicative factors being reciprocal of each other by saying, "…notches and revolutions are going to be inversely proportional because here you multiplying by three-halves and then you multiplying by two-thirds here." Thus, these exchanges demonstrated Kathy's consideration of the constant product relationship between the number of notches and number of revolutions when explaining her solution and showed her attention to a relationship between the inversely proportional relationship and numerical reciprocal multiplicative relationship that she constructed.

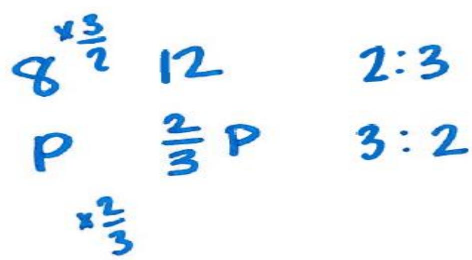


Figure 22. Kathy's explanation of a numerical reciprocal multiplicative relationship.

In Task 3, Kathy investigated the relationship between the number of weights hanging on a balance and the distance of those weights from the center of the balance. By hanging some number of weights on one side and by experimenting on the other side, Kathy obtained the balance formula, W1*D1 = W2*D2. When I asked how she got this formula, Kathy explained, "I was just observing what I saw, what happened, so I just kind of made a conjecture from there." Hence, Kathy's determination of the formula was based on her experimentation with the balance. Some exchanges later, I asked if she could generate a ratio table from the values of quantities that she needed to balance the system on one side, given that on the other side eight weights were hung at a 3-cm distance from the center. Considering the balance formula, Kathy multiplied 8 by 3 and got 24 and explained that she needed the products being equal to 24 on the other side. She generated a ratio table (Figure 23) and explained that all products were equal to 24.

Kathy: Okay. So we know that the multiplication of weights and the length is going to be 24. And so if we wanted to do balancing it out, all the things we can do, add on here. Okay so it will be three and eight and eight and three, and then four and six, and six and four.

The phrase "…the multiplication of weights and the length is going to be 24" and Figure 23 clearly demonstrated Kathy's attention to a constant product between the number of weights and distance from the center of the balance. Kathy's reasoning in this task suggested that the context of balancing seemed to facilitate her observation of the constant product relationship.

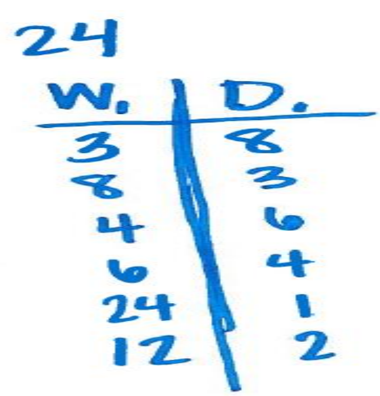


Figure 23. Kathy's ratio table for expressing a constant product relationship in Task 3.

In Task 4, Kathy needed to calculate the speed of a car that covered a certain distance in 60 seconds, given that another car covered the same distance in 90 seconds driving at 60 mph. She used a ratio table strategy and reasoned between measure spaces (Figure 24). In this strategy, she determined two-thirds as the multiplicative factor between 90 seconds and 60 mph. She then multiplied 60 seconds by the multiplicative reciprocal of two-thirds and calculated the speed of the car to be 90 mph. Kathy then explained:

Kathy: Okay so what I am thinking is that if there, so if you want to cover the same distance in less time then you have to go faster. So, I am going to take this is being inversely proportional. So, if it takes 90 seconds drive 60 mph and this would be two-third, two-third? Yeah. And then 60 seconds we need three-half so 90, 90 mph.

Kathy's explanation above suggested that her inference of the inversely proportional relationship between the speed and time was based on the inverse qualitative relationship—"…so if you want to cover the same distance in less time then you have to go faster." Therefore, this was an example to demonstrate Kathy's use of a qualitative compensation to infer an inversely proportional relationship. Although multiplying between measure spaces in this question worked, it would not work with all numbers. It seemed that the specific repetition of the numbers allowed Kathy to use this strategy.

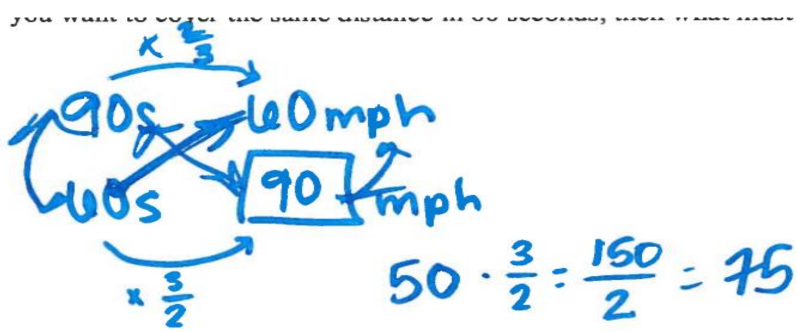


Figure 24. Kathy's ratio table strategy in Task 4.

Thus, Kathy attended to the numerical reciprocal multiplicative relationships in Task 1B, context of balancing in Task 3, and inverse qualitative relationships in Task 4 when inferring inversely proportional relationships. Because Kathy only recognized the constancy of the products in Tasks 1B and 3, the contexts of Gear and Balance tasks seemed to facilitate her coordination of the constant product relationships.

Theme 2: Proficiency in distinguishing directly and inversely proportional relationships from nonproportional relationships.

In Task 1B, I asked Kathy to compare the relationships in her directly and inversely proportional graphs (Figure 25) that she drew to show relationships between the number of notches and radii and between the number of notches and revolutions, respectively, with three graphs (see Appendix B Task 1B) that expressed nonproportional relationships. Graph A depicted quadratic growth that can be expressed with $y = x^2$, Graph B depicted a constant difference that can be expressed with $y = x + 2$, and Graph C depicted a constant sum that can be expressed with $y = -x + 5$. Kathy quickly inferred the nonproportional relationships in Graphs B and C.

Kathy: Okay I do not think…I do not think B and C are showing the same thing that this [pointed at the directly proportional graph] is showing just because I mean they are linear, like you want to say they are the same but they are not because…I mean…this [pointed at Graph B] has like a starting value and just like this is, like this one [pointed at Graph C]. These [pointed at Graphs B and C] are not proportional because I mean when you…you know what I am saying, like when you multiply…so like what we are looking, can I draw on this [pointed at Graph B]?

Int: Yeah you can draw on it.

Kathy: Yeah and then 3 and 5 [drawing on Graph B] okay, so but like 2 is with 0 like do you see what I am saying you cannot like…from like…like 1 on we can talk about it. But it just like because we are starting at (0, 2), it is like how do you multiply…like in proportionality like we were doing here how do you find something that multiply is by that [pointed at (0, 2)] to get you to 1 and 3 or 2 and 4, just it does not happen. In this [pointed at Graph C], the same idea, that here it is like 5…this is 0 and 1 goes with 4 and 2 goes with 3, like all, I mean you cannot like I am just starting with 0 and 5 and I want to know then I want to calculate like 2 and 3 that I, how do I get there from 0 and 5. It does not, that [pointed at Graph C] is not proportional.

In the exchange above, it appeared that Kathy understood the need to have a multiplicative relationship between the pair of values of two covarying quantities to infer a directly proportional relationship. Comparing Graph B with her directly proportional graph (Figure 25), Kathy explained that because Graph B had a starting point (0, 2), there was not a multiplicative factor to get (1, 3) and (2, 4) from (0, 2). Kathy's statement "…how do you find something that multiply is by that [pointed at (0, 2)] to get you to 1 and 3 or 2 and 4, just it does not happen" suggested that she was attending to the numerical multiplicative relationships within the separate values of quantities in the x and y-axis. On the other hand, her statement "…from like…like 1 on we can talk about it" suggested that she might be mistakenly thinking about a proportional relationship on the remaining part of the graph that starts at (1, 3). Because she did not explain further what her reason was for a possible proportional relationship on some parts of the graph, I do not have enough evidence for what she was thinking.

In the exchange, Kathy also compared Graph C with the directly proportional graph and explained that Graph C had a starting point (0, 5) and there was not a multiplicative factor to get (2, 3) from (0, 5). Kathy's reasoning was not entirely clear, but she may have been saying that because multiplying zero by any number would result in zero, there was not a multiplicative way to obtain the remaining points from the starting points (0, 2) and (0, 5). I expected Kathy to compare Graph C with her inversely proportional graph because in both graphs there was an inverse qualitative relationship—one quantity was increasing and other quantity was decreasing. It is possible that she might be inclined by the fact that Graphs B and C were expressing linear relationships. In her directly proportional graph in Figure 25, the origin, (0, 0) could not be multiplied to obtain the values of the remaining points. Therefore, as I will discuss in the following pages, Kathy explained that she could neglect the origin. Nevertheless, Kathy's reasoning was sufficient enough to see that for her the starting points of Graphs B and C precluded directly proportional relationships.

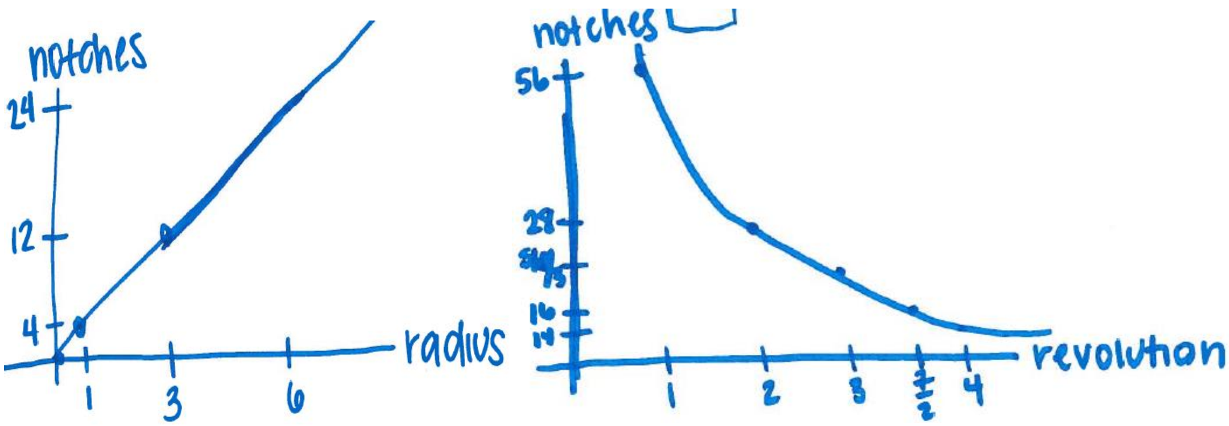


Figure 25. Kathy's directly and inversely proportional graphs in Task 1.

The following exchange also demonstrated some of the ways Kathy distinguished directly proportional relationship from nonproportional relationships:

Int: In this case [I pointed to the directly proportional graph she drew in the previous task] how do you go from one [point] to [the] other? Like you were saying something.

Kathy: Right, so because like we are not given any sort of like starting value you know it is like 0 and 0, so it is like that is pretty clear but then from there [pointed the directly proportional graph] like we can see 1 and 4 coincides and 3 and 12 coincides. So, it is like we know the…this thing…the like 1 and 4 and 3 and 12 are proportional…because like we can kind of like neglect this [pointed at the origin] when it starts, they both started at 0 but like when you are given something that it does not start at 0 you have to…you have to see like…if one of them is non zero like only way to get to 0 is to multiply by 0 so it is like, it does not really make sense to call this [pointed at Graph C] the same.

In the exchange, without referring to the units, Kathy explained that 1 cm and 4 notches and 3 cm and 12 notches were proportional, but she did not explicitly state how she inferred 1 cm and 4 notches and 3 cm and 12 notches to be proportional. It is possible that she might have been attending to both points being on the directly proportional line to which she referred by saying, "…1 and 4 coincides and 3 and 12 coincides." Although what Kathy implied by her statement "…when you are given something that it does not start at 0 you have to…you have to see like if one of them is non zero like only way to get to 0 is to multiply by 0" seemed to be not clear, I offer the following interpretation of this statement. By the first 0, she appeared to imply the origin. For Kathy, if a graph had a starting point other than the origin, such as (0, 5), the only way to obtain the zero value of this starting point from any remaining points on Graph C was to multiply within separate measure spaces of x and y by 0. Hence, this idea did not make sense to Kathy because multiplying by 0 would not yield the starting point that she intended to get. Therefore, Kathy's reasoning suggested that her attention to the multiplicative relationships within separate measure spaces allowed her to understand that the starting points of Graphs B and C precluded proportional relationships on these graphs.

On the other hand, Kathy initially assumed an inversely proportional relationship in Graph A.

Kathy: Like, I can actually call it [pointed at Graph A] maybe, call it an inverse proportion graph but I am trying to kind of figure it out. Because I mean here [pointed at the inversely proportional graph that she drew] neither of them [pointed at two end points of her inversely proportional line], you know, are coming…are going to come and contact with the x and y axes but I guess it is okay can we neglect that, too. Like we can neglect that with proportions like where it hits the zero whatever…I do not really know, like I am looking at it there, it just looks like then it really look graph of x squared, I mean the positive side. I am just trying to think like…

In a directly proportional graph, the line of the graph passing through the origin was contradicting Kathy's idea of multiplying the values of two separate quantities by the same number to get the proportional pair of values. Hence, Kathy appeared to have difficulty making sense of the origin from the idea of multiplying within separate measure spaces of x and y, and so she stated in the current and preceding exchanges that we could "neglect" the origin. Therefore, for Kathy, whether a graph "hits" (0, 0), by which she implied that the line of a graph passes through the origin, seemed to be a key feature. It is possible that Kathy's initial incorrect assumption might be related to Graph A having a curved line similar to the inversely proportional line that did not intercept the axes at $x = 0$ and $y = 0$; however, as it appears in

Kathy's statement, she was not sure about the accuracy of her assumption. She successfully explained that Graph A was showing a $y = x^2$ relationship. When reminded, in her inversely proportional graph, the number of revolutions was increasing as the number of notches was decreasing and asked if that was also the case in Graph A, Kathy said both quantities were increasing in Graph A. Therefore, she explained that calling Graph A an inverse proportion graph was not meaningful:

Kathy: So, I do not think that does not make sense. No it does not make sense…because like if like one is increasing by whatever number then other quantity needs to be like multiply by the reciprocal of that number so it will be decreasing. I guess whatever the number it depends, like the quantities are varying but like in the opposite way. So that is wrong.

The exchange showed that for Kathy, because of the numerical reciprocal multiplicative relationship that existed between inversely proportional quantities, when the value of a quantity increased multiplicatively by a number, the value of the inversely related quantity would decrease by the multiplicative reciprocal of that number. She noted this inverse covariation by saying, "…the quantities are varying but like in the opposite way." Therefore, the exchange indicated that Kathy was able to explain why her initial assumption of a proportional relationship in Graph A was wrong by attending to the numerical reciprocal multiplicative relationship within the separate measure spaces.

Thus, all of these data in Task 1B showed that Kathy's attention to the numerical multiplicative relationships within separate measure spaces helped her determining nonproportional relationships in Graphs A, B, and C. Because distinguishing directly and inversely proportional relationships from nonproportional relationships require an expert's skills and knowledge, Kathy's reasoning in this task demonstrated her proficiency in proportional reasoning.

Theme 3: The use of proportional reasoning strategies and reasoning within measure spaces when solving proportion questions.

In the previous pages, I presented various ways that Kathy inferred whether relationships were directly proportional, inversely proportional, or neither. Henceforth, I will explain the strategies that she used to solve given multiple and single proportion questions. Kathy used a variety of proportional reasoning strategies (e.g., Fisher, 1988) for solving proportions and was able to use additional strategies that I suggested. In Task 1A, I asked Kathy to calculate the number of notches around Gear B, with a 6-cm radius, given that Gear A, with a 3-cm radius, had 12 notches. Kathy recognized that the radius of Gear B was double the radius of Gear A, so she said that the circumference of Gear B was also doubling the circumference of Gear A. Hence, she was attending to multiplicative relationships within measure spaces.

Kathy: Okay, so I guess my first thought is that because the radius of the Gear B is double the radius of Gear A that means that the circumference is also going to be doubled. So, if it is twelve notches around then Gear B be 24 notches around.

When asked how she could express her verbal solution mathematically, by reasoning between measure spaces, Kathy explained:

Kathy: Okay so if it is I guess that circumference, so like A and B we know circumference $2\pi r$ so that is 6π and circumference here is 12π. And then this [Gear A] has 12 notches. I am just looking at how these two numbers [pointed at 6π and 12] are related, and so this is well I mean not including the π, we know what that [pointed at π] is. It is like times 2, that would be 24. That is how I am thinking of it.

Kathy compared 6π and 12 multiplicatively, and following the same type of reasoning she determined the number of notches of Gear B to be 24 notches. Kathy's statement above showed her initial understanding of a numerical multiplicative relationship between the circumference of a gear and number of notches, a between measure space comparison.

Some exchanges later, when asked to calculate the number of notches around Gear B again if Gear A had 7 notches instead of 12 notches, Kathy immediately determined the amount to be 14 notches and pointed out that in the first question, she should have considered how 6π and 12π were related instead of considering how 6π and 12 notches were related.

Kathy: Okay then I will take 14…because I guess I am still looking at…I guess, I should be looking at how these two [circled 6π and 12π]…
Int: In that case you looked at that two [pointed 6π and 12 notches] right?
Kathy: Yeah and that was incorrect because I guess they [pointed 12π and 12 notches] are confuses me…
Int: Why do you think that is..?
Kathy: Because they have both…I mean these two [pointed 12π and 12 notches] has 12 so I was looking, I guess I was just thinking about like that but I mean if this, like if the radius doubles and the number of notches would also double. So if the radius goes, like if the radius goes from 3 to 6, then their notches would go from 7 to 14 because [inaudible multiplied 3 cm and 7 notches by 2] how I am thinking about it.

Although Kathy multiplicatively compared quantities both between and within measure spaces, these exchanges suggested her preference of comparing quantities within measure spaces over between measure spaces. While the data did not provide any evidence, her preference might have based on the easiness of comparing quantities within measure spaces because they had the same referent units. As I discussed earlier, Kathy usually attended to the numerical multiplicative relationships when determining the constant ratio relationships. Her claim of comparing quantities between measure spaces as incorrect in this task together with her avoidence of the referent units in determining the constant ratio relationships suggested possible constraints in her reasoning about comparing quantities between measure spaces. Therefore, she focused on within measure spaces and by comparing radii multiplicatively calculated the number of notches around Gear B to be 14 notches (Figure 26). When asked what she would obtain if she multiplied between measure spaces, Kathy multiplied 3 and 6 by $\frac{7}{3}$ and got the same answer, 14 notches,

(Figure 26). When reminded that she thought the multiplying between measure spaces to be incorrect, Kathy explained:

Kathy: I guess, I was just dealing up here with such pretty whole numbers [pointed at 3 cm to 6 cm and 7 notches to 14 notches]. I did not really consider like that the idea of multiplying by an improper fraction. I see how it works I just did not put that together first yeah.

The statement above sugested that Kathy's preference of multiplying within measure spaces in this question was based on her ease with multiplying by whole numbers. Hence, it appeared that Kathy's solution methods depended on the numbers provided. On the other hand, her ability to multiply by improper fractions was a sign of her competence with dealing difficult numbers.

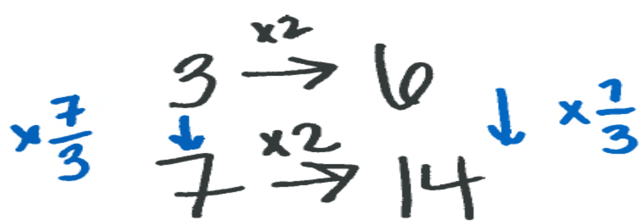


Figure 26. Kathy's multiplication operations within and between measure spaces.

In Task 1B, Kathy solved inverse proportion questions. She immediately recognized that the relationship between the number of notches and revolutions was different than the relationship between the radii and number of notches. Hence, when asked to calculate the number of revolutions of Gear K, with four notches, given that Gear F, with eight notches, revolved three times, she stated "I want to say 6" and noted that she did not know how to write it down mathematically. Some exchanges later, when asked how many revolutions Gear K would make if it had six notches instead of four notches. She incorrectly stated that Gear K rotated one and a fourth times when Gear F rotated one time:

Kathy: So, when this one, so when Gear F rotates one time, Gear K rotates one and a fourth times.
Int: Okay, how did you get that?
Kathy: Okay because huhahh I found you! Yess! I think I did it, we will see how that goes. Let's see if I can actually explain it. Okay so…maybe that one is a third let me think…So, it is

> like this [pointed at Gear F] rotates one time goes through eight notches and then this one [pointed at Gear K] has to go through six notches. So, when it [Gear K] goes to full six, it still has two more notches leftover before this one [pointed at Gear F] is like out of full circle, revolution…One and, I do not…if it was one and a fourth or one and a third let me think. (Three) because like is it of this [pointed at six notches on Gear K], if it was of this one [pointed at Gear K] and it would be 2 out of 6 but if it was of this one [pointed at Gear F] it is 2 out of 8.

This exchange showed Kathy's struggle to decide between one and a fourth or one and a third as the number of revolutions of Gear K. By reasoning additively, she explained that once Gear F completed a full revolution, Gear K was completing a full revolution and there were two notches "left over" on Gear K. She then could not decide whether to compare these two notches with eight nothes or six notches. Comparing two notches with eight and six notches multiplicatively was yielding one-fourth and one-third, respectively. The mathematical statement in this question can be written as (X revolutions) * (6 not/rev) = 8 notches. Although this situation can be modeled by measurement division, whether Kathy was thinking about division at this point was not clear. What was clear was her difficulty with referent units, she did not know whether the compare the two notches to Gear K or to Gear F. Kathy's confusion might have arisen from reliance on the additive reasoning by which she found that there were "two notches left over" on Gear K. This exchange happened before Kathy recognized a constant product relationship between the number of notches and revolutions.

Some exchanges later, I introduced two new gears, Gears L and M. Gear L had eight notches, and Gear M had 14 notches and revolved four times. Considering her idea of the constancy of the total notches moved on both gears in some revolutions, Kathy calculated the total notches moved on Gear M in four revolutions as 56 notches and divided 56 by 8 and got the correct answer of seven revolutions. In Task 1A, Kathy used a double number line to express the relationship between the radii and number of notches. When asked if she could use a double number line to solve this question, Kathy drew two sperate double number lines (Figure 27), one for Gear M and one for Gear L. In both number lines, she matched the number of notches with the number of revolutions and, by multiplying within measure spaces, she calculated the total notches moving as 56 notches. Kathy explained her reasoning by stating:

Kathy: Okay, so double number line. I can try see how that works. Uhhmm so okay so we want to know…okay this is Gear M. So, we know that 14 notches are in one revolution. We want to know how many notches are in four revolutions because we have to have that before we can even think about L. So, to get here [pointed at four revolutions] we multiply by 4, and the same thing here [pointed at 14 notches] to get 56. So, then for the L, uhmm okay let me see…so we know that we have okay so now we have eight notches make one revolution, and we know that in total we are looking for 56 notches because that is how many notches M goes through in four revolutions. So, we multiply this number [pointed at eight notches] by 7 to get 56. That is how those numbers are related and then we have to do the same thing here [pointed at one revolution], seven revolutions.

Because the relationship between the number of notches and revolutions was an inversely proportional relationship and the fact that a number line cannot be used to express an inversely proportional relationship, I expected Kathy to have difficulty solving this question using a double number line. My purpose for suggesting Kathy to use a double number line to solve this question was to understand her proficiency with proportional relationships. Because Kathy inferred 56 notches to be the total notches moved on both gears by multiplying 14 notches per revolution and four revolutions, these data suggested her coordination of the constant product relationship between the number of notches and number of revolutions. Therefore, Kathy's coordination of the constant product relationship seemed to facilitate her understanding of using two seperate double number lines to solve this question. Thus, Kathy's reasoning in this task demonstrated her proficiency in reasoning within measure spaces about proportional relationships.

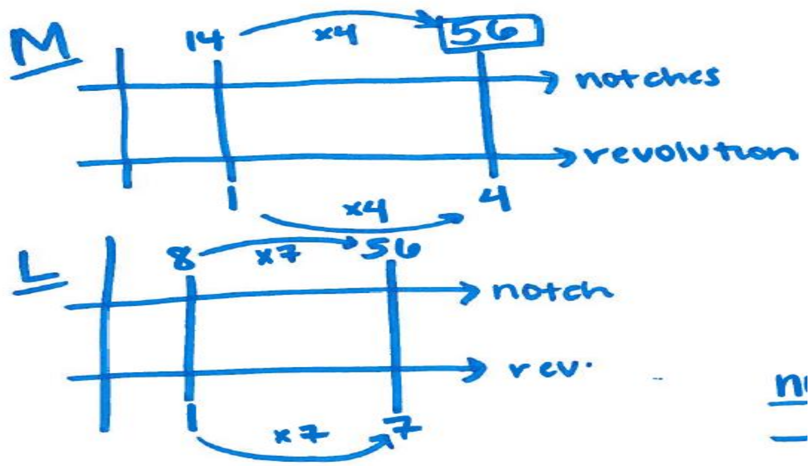


Figure 27. Kathy's two double number lines strategy.

When Kathy was working on Task 2B, I asked her if she could use a double number line to calculate the number of people needed to frost 60 cupcakes in 9 minutes, given that three people frosted 60 cupcakes in 12 minutes. She explained:

Kathy: Not for that though because you cannot I mean double number line increase together or decrease together, these [pointed at people and time] go opposite so we cannot use that.

Kathy's statement above provided evidence for her attention to the qualitative relationships in explaining why a double number cannot be used to express an inverse relationship. Some exchanges later, I reminded Kathy how she used the double number line strategy to solve an inverse proportion question in the Gear task. She explained:

Kathy: Well if you want something proportional because these like, these number lines [pointed at the double number lines for the number of notches moved and revolutions] show proportional relationships between two things.

Kathy's statement showed her understanding of using double number lines when showing proportional relationships. For her, there was a proportional relationship between the number of notches moved and revolutions. Because the context of the Gear task facilitated Kathy's understanding of the total notches as the product of number of revolutions and number of notches per revolution, she was able to use double number lines in that task by thinking of two pairs of increasing quantities. In the current question, there was also a constant product relationship between the number of people and number of minutes, but Kathy did not see 36 as the number of minutes to frost a total of 60 cupcakes. Therefore, Kathy stated that she could not use double number line in this question. In this task, Kathy might have had difficulty working with time. In addition, in the Gear task, the unit of the product—total notches moved—was the same as the unit of the number of notches; however, in the Bakery task the product was presented in terms of cupcakes but could also be interpreted in terms of person-minutes.

Kathy worked on the multiple proportion questions in Task 2C, and in those questions she preferred reasoning within measure spaces. For instance, when asked to calculate the number of cupcakes frosted by two people in $\frac{T}{2}$ minutes, given that three people frosted N cupcakes in T minutes, she used a ratio table strategy, which she said involved two steps (Figure 28):

Kathy: So, [thinking]…okay so I would probably take two steps again and do like three people in half the time frost half the cupcakes, N over 2.
Int: Okay.
Kathy: And then I guess multiplying, we want to know how many.
Int: Yeah two people.
Kathy: Two people so we need to multiply that [pointed three people] by two-thirds, less people less cupcakes…[pause over 30 seconds] I think [she wrote $\frac{2N}{6}$] yeah.

In the first step, Kathy determined that 3 people could frost $\frac{N}{2}$ cupcakes in $\frac{T}{2}$ minutes. She then multiplied three people by two-thirds to get two people and so multiplied $\frac{N}{2}$ cupcakes by the same two-thirds and got the correct answer $\frac{2N}{6}$ cupcakes. The exchange showed that Kathy attended to the numerical multiplicative relationships within measure spaces and the qualitative relationship "…less people less cupcakes…" to solve the question. Kathy attended to these two features throughout the interview to solve the multiple proportion questions.

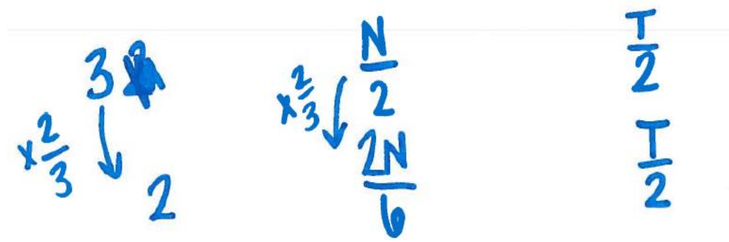


Figure 28. Kathy's ratio table strategy involving two steps.

In Task 4, I asked Kathy to calculate the speed of a car that covered a certain distance in 50 seconds, given that another car covered the same distance in 90 seconds driving at 60 mph. She used a ratio strategy and, reasoning between measure spaces incorrectly, determined the speed to be 75 mph (see Figure 24). Kathy immediately recognized the inaccuracy of her answer and said:

Kathy: 50 seconds, you multiply 50 by $\frac{3}{2}$, and you get hahh wait a minute.
Int: What happened?
Kathy: 50 over, 50 seconds and one and a half, what is it…[she calculated $\frac{150}{2} = 75$ mph]. No it does not make sense hold on a second.

Kathy then multiplied within measure spaces and calculated the correct answer of 108 mph. When asked why 75 mph did not make sense, she explained:

Kathy: Yeah it does not make sense because it should be faster because it less time.
Int: It is less time than?
Kathy: 60.

In the preceding question, Kathy calculated the speed of the same car as 90 mph to cover the same distance in 60 seconds. Because she was driving the same distance in less time, 50 seconds, she expected to obtain a speed of more than 90 mph. Hence, her understating of the qualitative compensation between the speed and time helped her to recognize the mistake in her solution. After obtaining 108 mph, she stated "I guess, I should have been doing it this way" that suggested her inclination towards reasoning within measure spaces. When I reminded her that she reasoned between measure spaces in the preceding question, Kathy said that reasoning between measure spaces worked for that particular question, because the numbers (60 mph and 60 seconds, and 90 seconds and 90 mph) in two opposite corners of her strategy were the same (see Figure 24). She concluded that it did not matter going from within measure space values or between measure space values for that specific question; however, she noted that reasoning within measure spaces would always work. Therefore, for Kathy, the specific repetition of the numbers allowed her to calculate the correct answer in the preceding question.

Case Two: Susan

Summary

Susan correctly inferred directly and inversely proportional relationships in the given tasks. Although she was successful in forming multiplicative relationships within measure spaces and reciprocal multiplicative relationships, her main knowledge resource for inferring directly proportional relationships was attention to the constancy of the rate of change. On the other hand, Susan identified relationships as inversely proportional by attending to the static points on graphs and whether the values of points were swapped (e.g., (x, y) and (y, x)). Because of her preference for attending to the constancy of the rate of change and static points when inferring directly and inversely proportional relationships, Susan had difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships that consisted of a constant difference or constant sum. Susan used proportion formula, algebra strategies (e.g., equations, formulas), and other proportional reasoning strategies (e.g., ratio table, double number lines) and preferred reasoning within measure spaces in solving multiple and single proportion questions. She expressed directly and inversely proportional relationships with graphs, double number lines, formulas, ratio tables, or with some combination.

Cross-Task Analysis

In Chapter Three, I determined four themes, which I provided in Table 2, for Susan's case based on the thematic analysis. In the following pages, I elaborate on these four themes to explain Susan's reasoning across tasks. In the first theme, I discuss Susan's attention to the constancy of the rate of change in Tasks 1A and 2A when inferring directly proportional relationships between two covarying quantities. In the second theme, I explain Susan's focus on static points on graphs and values of points being swapped in Tasks 1B and 3 when inferring inversely proportional relationships. In the third theme, I investigate Susan's difficulty distinguishing directly and inversely proportional relationships from the nonproportional relationships. Finally, in the last theme, I conclude the cross-task analysis with a discussion of select proportional reasoning strategies that Susan used to solve the given questions across tasks.

Theme 1: Attention to the constancy of the rate of change when inferring directly proportional relationships.

Susan correctly inferred directly proportional relationships in the given tasks. Susan's responses to the direct proportion questions demonstrated that her inference was mainly based on her attention to the constancy of the rate of change. Although Susan successfully obtained multiplicative relationships within measure spaces, she did not recognize multiplicative relationships between measure spaces. She even obtained the multiplicative relationships within measure spaces in the presence of proper fractions. For example, in Task 1A, Susan worked on a question in which she needed to calculate the number of notches of Gear B, with a 2-cm radius, given that Gear A had a 3-cm radius and 12 notches. She accurately obtained two-thirds as the multiplicative relationship within measure spaces:

Susan: Okay, so it would have…, so B is two-thirds the size of A.
Int: Okay. So, two-thirds because.

Susan: Because…the circumference 4π, so 6π simplifies to two-thirds.
Int: Two-thirds okay.
Susan: So, then it would have [multiplied 12 notches by two-thirds] eight notches.

Susan's statement, "…so B is two-thirds the size of A," exemplified her determination of a multiplicative relationship within measure spaces. She determined this multiplicative relationship by comparing the circumferences of the two gears multiplicatively. Because she expected to have the same multiplicative relationship between the number of notches of the two gears, these exchanges suggested her coordination of multiplicative relationships within measure spaces.

On only one occasion did Susan seem to attend to a multiplicative relationship between measure spaces. In Task 1A, using the 3-cm-to-12-notches relationship, she drew a linear graph (Figure 29) to express the relationship between the number of notches and radii. Some exchanges later, Susan generated a ratio table between the values of the number of notches and radii (Figure 29). When asked if there was a pattern or something important to talk about in her table, Susan explained:

Susan: Umm, it's always increasing by 4, the number of notches it increases by 4 each time the centimeters increases by 1.

Susan's statement demonstrated that she was attending to the constancy of the rate of changes within two separate measure spaces. When asked, Susan inferred a proportional relationship between the number of notches and radii in her ratio table.

Susan: It is a proportion.
Int: How do you know that is proportional?
Susan: Because there is a constant rate of change.
Int: Constant rate of change?
Susan: The notches equals 4 times the centimeters, the amount of centimeters.
Int: Always, do you mean?
Susan: Yeah.

These exchanges showed that for Susan, the existence of a constant rate of change suggested a proportional relationship between the number of notches and radii. She calculated the rate of change as four (Figure 29) and explained it as follows: "The notches equals 4 times the centimeters, the amount of centimeters." Although her definition of the constant rate of change suggested an understanding of a multiplicative relationship between measure spaces, there was no other evidence in her interviews that would allow one to claim that she coordinated multiplicative relationships between measure spaces. Therefore, in this occasion, Susan seemed to be expressing an association between the number of notches and radii based on the entrees in her ratio table. Thus, these exchanges and Figure 29 revealed that Susan's inference of the proportional relationship was based on attending to the constancy of the rate of change.

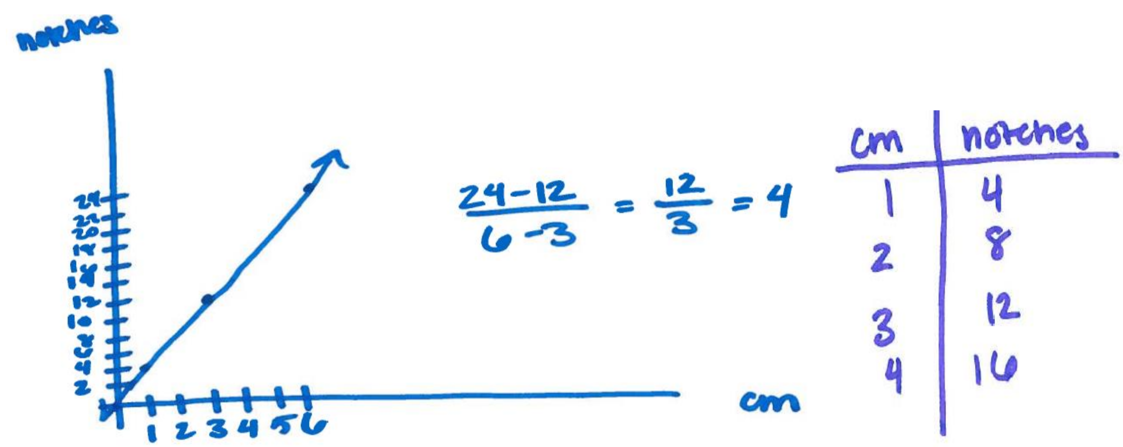


Figure 29. Susan's directly proportional graph and ratio table.

In Task 2A, when asked to talk about the relationship between the number of people and number of cupcakes, Susan used the information confirming that three people frost 12 cupcakes in T minutes to draw a linear graph (Figure 30). She then inferred the relationship as a proportional relationship.

Int: What is the relation…, what do you think the relation is?
Susan: Between cupcakes and people?
Int: Yeah.
Susan: They are proportional.
Int: The reason is you have the graph or something…what was your main idea to graph it? Like, when I ask you to identify the relationship between these two, like the number of cupcakes and people, you said I can graph it. What was the reason for graphing to identify the relationship?

Susan: So, I could show there was a linear relationship. So, that the…the ratio…, there is a constant ratio between the people and the cupcakes.

According to these exchanges, Susan's reason for drawing the graph was to show that it was linear and that there was a constant ratio relationship between the number of people and cupcakes. These data suggested a consistency between Susan's reasoning in Task 1A and Task 2A, because in both tasks she drew linear graphs first and inferred relationships based on those graphs. Her strategy of drawing a graph to infer a relationship suggested her possible coordination of a directly proportional relationship with the linearity of its graph. These exchanges also included the first instance of Susan's mentioning the term *constant ratio relationship*. When asked what she meant by the constant ratio, Susan explained:

Susan: Uhh, so we have 0, 0; 1, 4; 2, 8; 3, 12, and umm so this [pointed at 12 cupcakes] is three times the amount of the one person; this [pointed at eight cupcakes] is two times the amount of whatever is made by the first person.
Int: So the ratio is here.
Susan: Oh, sorry.
Int: What was the ratio, like you mean these two, like 1 over 2 or the other 3…like you said in that case, 3 and 12 something?
Susan: I don't know. It increases by four every time. Whatever you do to the… like if you look at the original, the 1 to 4. Whatever you do from the 1 to get to the…any amount of people that is you do to the amount of cupcakes.
Int: Same thing.
Susan: Same four and that'll give you the answer.

These exchanges suggested that Susan was attending to a multiplicative relationship—"…so this [pointed at 12 cupcakes] is three times the amount of the one person, this [pointed at eight cupcakes] is two times the amount of whatever is made by the first person"—and the constancy of the increments within measure spaces—"It increases by four every time." Although she used the term *constant ratio relationship,* there was not any indication that she was attending to the relationships between measure spaces. Hence, these data suggested that she might have used the term *constant ratio relationship* to indicate the constancy of the rate of change. Her ratio table in

Figure 30 confirmed my conjecture about her reasoning because she was attending to the constancy of the increments within measure spaces. Therefore, similar to Task 1A, these exchanges and Figure 30 implied that Susan's inference of the proportional relationship between the number of people and cupcakes was based on her attention to the constancy of the rate of change.

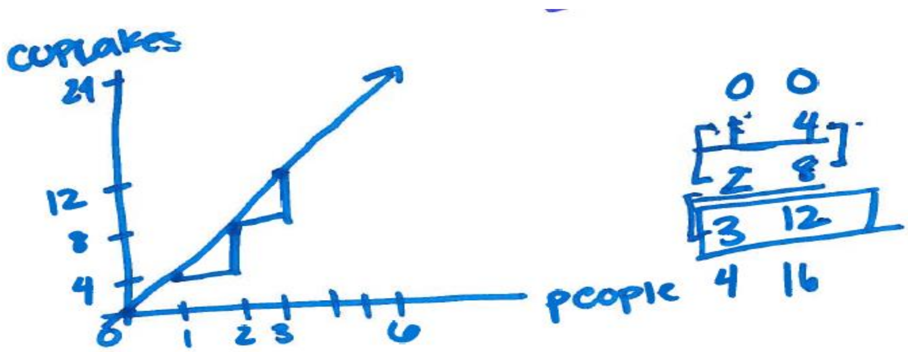


Figure 30. Susan's expression of the proportional relationship between the number of people and number of cupcakes.

Theme 2: Attention to static points on graphs and values of points being swapped when inferring inversely proportional relationships.

Susan successfully inferred inversely proportional relationships in the given inverse proportion questions. She recognized the constancy of the products in Tasks 1B and 3 but did not recognize them in the remaining inverse proportion questions. Although Susan was successful at determining the multiplicative reciprocal relationships between quantities, she mainly attended to the numbers provided or points in graphs when justifying the inversely proportional relationships. For example, in Task 1B, when asked the number of revolutions of Gear K, with four notches, given that Gear F, with eight notches, revolved three times, Susan correctly determined the number of revolutions to be six.

Susan: Umm, so the distance that F has to travel is 2 times as far, so if it [pointed at Gear F] goes around 3 times, K is going to go around 6 times…6 times.

Int: You said distance F traveled, what do you mean by the distance?
Susan: The circumference of F is going to be 2 times as big as circumference of K.

In the exchange, by the term *distance,* Susan implied the lengths of the circumferences of Gears F and K. Because Gear F's circumference was double the circumference of Gear K, for Susan Gear F was traveling two times the circumference of Gear K. A few exchanges later, Susan stated the reciprocal multiplicative relationship between the sizes of Gears F and K and their number of revolutions:

Susan: Yeah so, so this, Gear F goes around three times and it's two times as big; K only needs half the time to go a full revolution, so it's going to go twice as many revolutions as F. So, it's going to go six revolutions.

This statement clarified what Susan implied by the distance traveled by two gears. Without this statement, Susan's explanation, "…so the distance that F has to travel is 2 times as far," could be considered incorrect because both gears traveled the same distance. For Susan, Gear K was completing a full revolution in half the time Gear F completed a full revolution. For that reason, Gear K was making twice the number of revolutions Gear F made. In these data, the gear context seemed to facilitate Susan's coordination of the size of a gear with its circumference and thereby with its notches, and her determination of the correct reciprocal multiplicative relationship between the size of a gear and its revolutions. Therefore, these data were an illustration of the influence of the gear context on a PST's reasoning.

Although Susan stated an accurate reciprocal multiplicative relationship between the sizes of Gears F and K and their number of revolutions, she seemed to have difficulty expressing her solution mathematically. Susan stated her difficulty as follows:

Susan: And since there's eight notches around that means that the circumference makes up eight notches. So that, it's…I don't know I don't know what to write for it.

Based on Susan's response above, I asked what the distance traveled by Gear F was in three revolutions. In response, Susan calculated the distance traveled by Gear F in three revolutions as 24 notches. My question appeared to trigger something in Susan's understanding, because she then recognized that Gear K was traveling the same 24 notches in six revolutions.

Susan: It goes around the same amount of notches.
Int: What do you mean by the same amount?
S: Because four goes around 6 times, that's 24 notches as well.
Int: Can you tell me about that one? It's interesting.
Susan: Umm, because since this [pointed at Gear K] is 2 times as small, the amount of revolutions are going to be 2 times as many. So, that's why I multiplied it by the 6.

Susan's last statement—"…since this [pointed at Gear K] is 2 times as small, the amount of revolutions are going to be 2 times as many"—illustrated her attention to the reciprocal multiplicative relationships between the number of notches and number of revolutions when explaining constancy of the total notches traveled in both gears. Although the question that I asked seemed to facilitate her understanding of the constancy of the total notches traveled in some number of revolutions, Susan was able to reason about the necessity of both gears traveling the same distance. She explained:

Susan: Since, yes, to make one full revolution it has to hit all four notches, it's hitting all four notches six times. So, you get 24 notches, which is the same as the Gear F.

According to Susan, each of the four notches on Gear K was hit by six times by Gear F. Hence, a total of 24 notches were hit on Gear K and this number was the same as total notches revolving around Gear F. This explanation demonstrated that Susan's causal net was sufficient to see that the same number of notches was revolving around both gears. Because the total number of notches revolved on a gear was the product of number of revolutions and notches per one revolution, these data provided evidence for Susan's rudimentary understanding of a constant product relationship between the number of notches and number of revolutions. Therefore, she usually solved the inverse questions in this task by using a "total notches traveled" strategy. In this strategy, Susan calculated the total number of notches revolved on a gear by multiplying its number of notches by the number of revolutions and equated this value with the product of the number of notches and revolutions in the other gear.

Susan obtained the reciprocal multiplicative relationship between the size of a gear and its revolutions even in the presence of proper fractions. For instance, in one of the questions, using the "total notches traveled" strategy, she calculated the number of revolutions of Gear K, with 12 notches, to be $\frac{2}{3}p$ revolutions, given that Gear F, with eight notches, made p revolutions. She explained the reciprocal multiplicative relationship as follows:

Susan: Since Gear F is two-thirds the size of Gear K, it's going to…Gear K's going to make two-thirds the amount of revolutions as Gear F.

Some exchanges later, when asked to draw the graph of the relationship between the number of notches and revolutions using the 8-notches-to-3-revolutions relationship, Susan drew an inversely proportional graph (Figure 31) and declared an inverse relationship between the number of notches and revolutions.

Susan: These are inversely related.
Int: Okay, how do you know that are inversely…?
Susan: Because when there is 24 notches, there is only one revolution, but when there is 24 revolutions, there is one notch and you can find the inverse relationship between them.
Int: When you are saying inverse relationship, what do you imply with inverse?
Susan: I mean, like 24 over 1 that is the inverse of 1 over 24.
Int: Okay, because of that reason, you think that is the inverse…
Susan: Yeah.
Int: inversely…inverse relationship?
Susan: Yes and you can find that throughout the entire graph.
Int: Do you mean you will have an inverse of some…one point appears here [I pointed at the inversely proportional graph]?
Susan: Every point on the graph has an inverse somewhere else on the graph.

Susan wrote that $\frac{24\,notches}{1\,revolution}$ and $\frac{8\,notches}{3\,revolutions}$ ratios were the inverses of $\frac{1\,notch}{24\,revolutions}$ and $\frac{3\,notches}{8\,revolutions}$ ratios, respectively (see Figure 31). Her statement "…24 over 1 that is the inverse of 1 over 24" suggested that she was attending to $\frac{24}{1}$ and $\frac{1}{24}$ being reciprocal of each other when explaining the inverse relationship. She stated that every point on the inversely proportional graph had an inverse somewhere else on the graph. It appeared that for Susan, the points (1 notch, 24 revolutions) and (24 notches, 1 revolution) were the inverses of each other because the values of the quantities were swapped. Therefore, when inferring two points as inverses of each other, Susan seemed to attend to the pair of values of two points being swapped rather than the quotients that they formed.

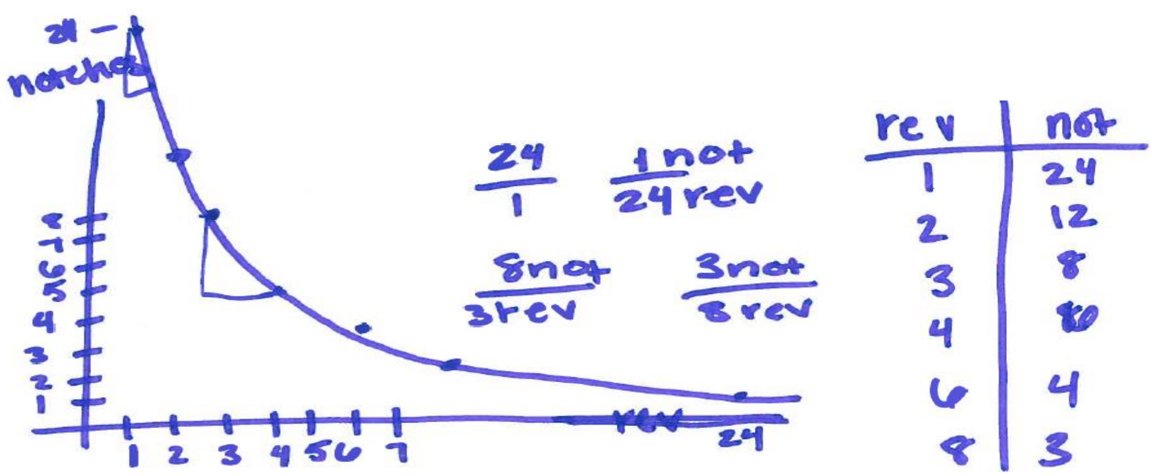


Figure 31. Susan's inversely proportional graph and ratio table.

A few exchanges later, I asked Susan if she could express the relationship between the number of notches and revolutions with a ratio table. She generated the ratio table in Figure 31 and explained:

Susan: The amount of notches and the amount of revolutions…every time we multiply together, they equal 24.

Susan explicitly stated the constant product relationship between the number of notches and number of revolutions. Susan appeared to attend to the numerical relationships between numbers in her ratio table when determining the constant product relationship. Because Susan also explained 24 as the total notches traveled in both gears, the context of the Gear task seemed to facilitate her understanding of this constant product relationship.

In Task 3, Susan determined the balance formula, W1*D1 = W2*D2, playing with the plastic weights and hanging them at different distances. She described the inverse qualitative relationship between the number of weights and distance by stating, “The amount of weights is increasing as you are decreasing the distance.” She generated the ratio table in Figure 32 for a particular number of weights and distance relationship and, when asked, endorsed an inversely proportional relationship between the number of weights and distance:

Susan: Yeah. So, they are inversely proportional.
Int: Why do you think that is, they are inversely [proportional]?
…….
Susan: Because the 2 times the 6 equals 12, the 6 times the 2 equals 12, 4 times 3 equals 12. They are always…the distance times the amount of weights like for that distance always multiply to 12.

In the particular number of weights and distance relationship that Susan decided, the product of the number of weights and the distance was always equal to 12. Susan’s reasoning when endorsing an inversely proportional relationship in this task was similar to that on Task 1B because she appeared to attend to swapping pairs of values.

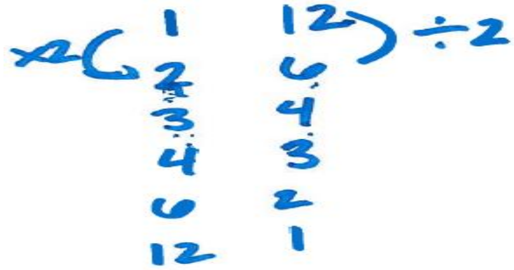


Figure 32. Susan’s ratio table for expressing constancy of the products in Task 3.

Besides attending to the static points and swapping pair of numbers, in Task 2B, Susan inferred an inversely proportional relationship between the number of people and number of minutes by attending to the numerical multiplicative relationships within measure spaces. When I asked Susan to calculate the number of minutes need by six people to frost N cupcakes given that three people frosted N cupcakes in T minutes, she reasoned within measure spaces and explained:

Susan: Umm, if I think if my thinking is right, the amount of people is inversely proportional to the amount of time.
Int: Time? Why do you think that is inversely proportional?
Susan: Because here to go from the original 3 people to the 6 people we multiplied by 2 and then the time we divided by 2, so you're doing like the opposite umm operation.
Int: How about for 3 people and 2 people?
Susan: So, going from 3 people to 2 people you multiplied it by 2 over 3 so then you multiplied the time by 3 over 2.

These exchanges demonstrated Susan's attention to the numerical multiplicative relationships within measure spaces when justifying her inference of an inversely proportional relationship. Therefore, all these data in Tasks 1B, 2B, and 3 confirmed Susan's attention to the static points on graphs and values of points being swapped and numerical multiplicative relationships within measure spaces when inferring inversely proportional relationships between quantities.

In summary, although in some instances, Susan accurately stated reciprocal multiplicative relationships and recognized the constancy of products in Tasks 1B and 3, her inference of inversely proportional relationships appeared to be mainly based on her attention to static points on graphs and values of points being swapped. While Susan's conjecture of pair of values being swapped in an inversely proportional was correct, it was not sufficient to distinguish proportional relationships from nonproportional relationships. She should have attributed the reason for inferring the inversely proportional relationships to the reciprocal multiplicative relationships between two quantities and constancy of the products. Except Tasks 1B and 3, Susan did not recognize the constancy of the products in the remaining inverse proportion tasks; however, after I asked her if the products of the inversely proportional quantities were constant in those tasks, she realized the constancy of those products. Therefore, the contexts of Tasks 1B and 3 seemed to facilitate Susan's recognition of constant product relationships between quantities more than the remaining inverse proportion tasks.

Theme 3: Difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships.

Susan had difficulty distinguishing directly and inversely proportional relationships from the nonproportional relationships that consisted of constant difference or constant sum. When working on Task 1B, I provided Susan with three graphs (Figure 33) and asked her to determine the relationships in them. She inferred a linear relationship in Graph B:

Susan: That [pointed at Graph B] is linear.
Int: Is that also proportional like the one here [I pointed at the directly proportional graph in Figure 29]?
Susan: Yes.
Int: Okay, please tell me how do you know this is also proportional?
Susan: So, slope between the two points is always the same distance, because that [pointed at x] is always changing 1 and that [pointed at y] is always changing 1.

As discussed earlier, Susan attended to the constancy of the rate of change when inferring the directly proportional relationships. These exchanges also showed Susan's attention to the constancy of the rate of change when inferring a proportional relationship in Graph B. She emphasized the constancy of the change by stating, "…slope between the two points is always the same distance."

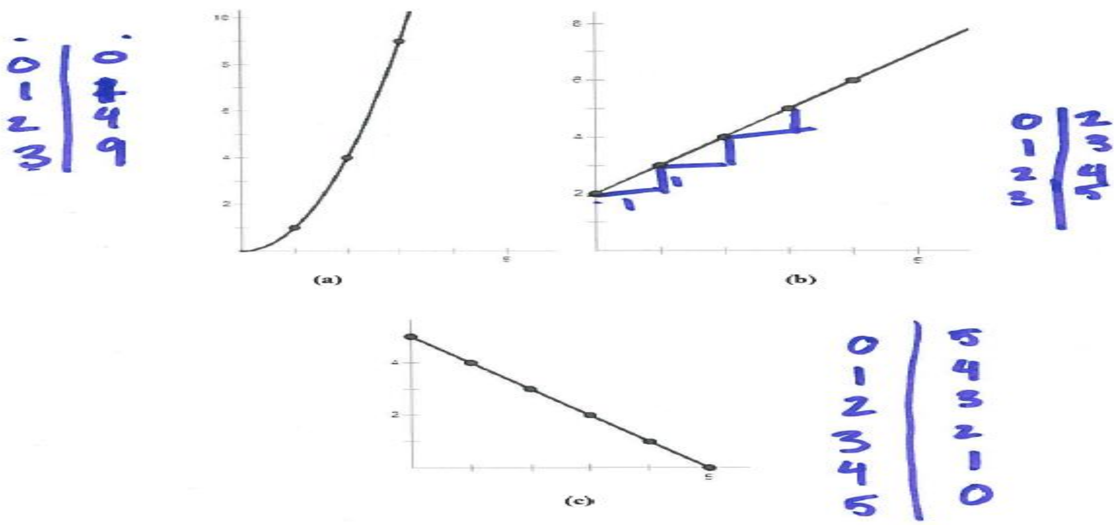


Figure 33. Susan's determination of the relationships in Graphs A, B and C.

For the relationship in Graph C, Susan generated the ratio table in Figure 33 and stated that Graph C looked like an inversely proportional relationship graph.

Int: Okay, how about [Graph] C?
Susan: So, it looks like it is inversely proportional.
Int: Why do you think it looks like…it is inversely proportional?
Susan: Because we have…for 0 x there is 5 y, and for 5 x there is 0 y, yeah.
Int: Do you mean the pairs (0, 5) and (5, 0)?
Susan: The pairs have the opposite…

For Susan, Graph C looked like an inversely proportional relationship graph, because for each point on Graph C there was an opposite point, which she characterized as swapping the values of x and y. For example, for Susan, the two pairs (0, 5) and (5, 0) were opposite because x and y values were swapped. These exchanges provided evidence for Susan's attention to the static points on Graph C when inferring an inversely proportional relationship. Therefore, her reasoning when inferring an inversely proportional relationship for Graph C supported the consistent pattern in her reasoning when inferring the constant product relationship between the number of notches and revolutions, which I discussed in the previous category. Susan then explained that in the inversely proportional relationship between the number of notches and revolutions, the products were all equal to 24 and noted that in Graph C the x and y values added up to five.

Susan: The one with 24s. In this one they all add up to five, but in that one they all multiplied or the product was 24.
Int: Can you talk about that? You said these are, here products were 24 but here they add up [to] 5.
Susan: They add up to five.
Int: Add up to five, yeah. That do you think that makes them same or different or? But you said they are same right inversely proportional?
Susan: I yeah I said that. Umm…
Int: But how that difference makes…you know changes like, product and…addition.
Susan: So, I think it kind of just depends on what you are relating. So, here we were multiplying notches to revolutions…I don't know.

Although I tried to ask Susan if she attached any significance to the difference between constant sum and constant product, her response "…I think it kind of just depends on what you are relating" suggested that for Susan obtaining the constant sum or constant product did not have significantly different implications for inferring inversely proportional relationships. Nevertheless, that she did not seem confident about her response indicated a possible confusion about constant sum and constant product relationships. Therefore, even though Susan recognized that the x and y values in Graph C were adding up to five, she did not see that the values adding up to five precluded an inversely proportional relationship between x and y.

A few exchanges later, Susan generated a ratio table (Figure 33) and identified Graph A as representing the $y = x^2$ relationship. She then stated that because x and y were not increasing at a constant rate, there was not a proportional relationship.

Susan: It's the graph of x-squared.
Int: How about the relationship between x and y? Is that a proportional, inversely proportional, or you know neither kind of?
Susan: I want to say no but I do not know why.
Int: No…do you mean like…?
Susan: They are not, it is not proportional. I do not know.
Int: From what aspect of the relationship you want to believe that they are…this is not proportional?

Susan: Because it is not increasing at a constant rate and it is not the 3…9 is the 3 squared, but it does not have anything to do with the 4, other than they are both the squared of that [pointed x].

Although Susan stated that she did not know why she believed Graph A was not proportional, in her causal net, initially the lack of a constant rate was the reason for Graph A being a nonproportional graph. Therefore, Susan's attention to the constancy of the rate of change facilitated her in identifying the relationship in Graph A as nonproportional, even though she was hesitant in her response.

In summary, Susan's attention to the constancy of the rate of change and relationships between static points on graphs precluded her from determining nonproportional relationships exhibited in Graphs B and C. Susan's incorrect inferences suggested her difficulty differentiating proportional relationships from nonproportional relationships.

Theme 4: The use of proportional reasoning strategies and reasoning within measure spaces when solving proportion questions.

In the previous pages, I presented several ways that Susan inferred whether relationships were directly proportional, inversely proportional, or neither. Hereafter, I will discuss some of the proportional strategies that she used to solve given multiple and single proportion questions. In Task 1B, when asked if she could calculate the number of notches of Gear K, which revolved eight times, given that Gear F, with *n* notches, revolved six times using a strategy other than the "total notches traveled" strategy, Susan used a double number line incorrectly and determined the number of notches to be $\frac{4}{3}n$ (Figure 34). While explaining her solution, Susan immediately recognized that she would not have obtained the same result if she had used her original "total notches traveled" strategy:

Susan: Okay…so, since I knew that for every six revolutions there were *n* notches…we were trying to find out how many *n* notches in eight revolutions…You can since they are occurring at the same time, you can manipulate it to get to eight. You can manipulate six into eight and by doing that you also manipulate *n* to get to whatever would be co-occurring with the eight revolutions.

Int: You obtained four-thirds…?

Susan: Four-thirds *n*.

Int: How about if you…?

Susan: Which I would not have done…

Int: Sorry?

Susan: I would not have gotten that if I did it my way, the same way I was doing, previously.

Int: What…what would you get?

Susan: Well the way I was going to do it…I would have had six revolutions times *n* notches, and then I would have divided that by the eight revolutions, and I would have gotten three-fourths *n* notches.

Susan incorrectly stated that "…for every six revolutions there were *n* notches." The correct statement should resemble "for every one revolution there were *n* notches." Susan's following statement, "…how many *n* notches in eight revolutions," and her usage of the term *co-occurring* suggested that she might have considered the number of notches and revolutions to be varying directly. Because there was a directly proportional relationship between the total number of notches and number of revolutions, Susan seemed to confuse the number notches on Gear K with the total number of notches. Kathy, on the other hand, was able to use two double number lines to solve a similar inverse proportion question by comparing the total notches moved and number of revolutions.

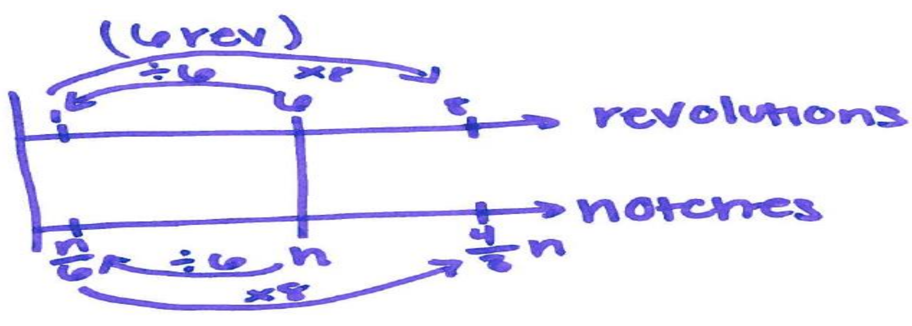


Figure 34. Susan's incorrect double number line strategy.

A few exchanges later, Susan explained that one of these two results was incorrect but could not decide which one was. Her difficulty determining the incorrect result also supported my claim of possible constraints in her understanding of the constant product relationship between the number of notches and number of revolutions. When asked how she interpreted $\frac{4}{3}n$ and $\frac{3}{4}n$ notches, Susan explained:

Susan: So, the three-fourths *n* notches means that K has three-fourths the amount of notches as F. Is that what you were asking?
Int: I was asking…like that one has six revolutions and that one has eight revolutions. Does that say anything about the notches?
Susan: So, okay…so, eight revolutions…means that K is going to have less distance to travel than…F would. So, that means there is going to be fewer notches if they are…
Int: Fewer notches, okay, on Gear F or K?
Susan: There is going to be fewer notches on Gear K.
Int: But how about if you if you look at your answers, which one has fewer notches?
Susan: This one [pointed at $\frac{3}{4}n$ nothes].

As I explained earlier, by the term *distance,* Susan meant the circumferences of Gears F and K. Because Gear K traveled eight times, by attending to inverse qualitative relationship, Susan decided that Gear K had "fewer notches" than Gear F. Hence, without careful attention, Susan's usage of the term *distance* to imply the circumference of a gear (or the number of notches) might create confusion. Therefore, these exchanges showed that Susan's comparison of the inverse qualitative relationship between the number of revolutions and number of notches facilitated her decision of Gear K having less notches than Gear F. Because $\frac{3}{4}n$ notches was less than $\frac{4}{3}n$ notches, Susan decided that the number of notches of Gear K was $\frac{3}{4}n$ notches. When asked why the double number line did not work, she was unable to explain. Thus, unlike Kathy, she could not adapt her methods for using double number lines to reason about the inversely proportional relationships between the number of notches and number of revolutions.

Overall, in the Bakery task, Susan preferred a ratio table strategy to solve the given multiple proportion questions. In this strategy, she fixed the value of a quantity as constant and then used mathematical operations within separate measure spaces for the remaining two quantities. For example, in Task 2B, she successfully calculated the time required by one person to frost 2N cupcakes, given that three people frosted N cupcakes in T minutes, to be 6T minutes (Figure 35). In her ratio table strategy, first she fixed N cupcakes as a constant, and by reasoning within measure spaces, she determined that one person could frost N cupcakes in 3T minutes. Next, she fixed one person as a constant, and by multiplying within measure spaces, she calculated that one person could frost 2N cupcakes in 6T minutes.

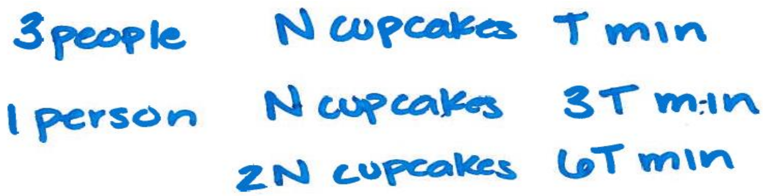


Figure 35. Susan's ratio table strategy for solving a multiple proportion question.

Some exchanges later, when asked if she could use any other method to solve the same question, Susan said she could use two different double number lines (Figure 36):

Susan: I mean you could use a double number line…but I would use two different double number lines.
Int: How do you…can you show it to me, like how do you…?
Susan: Okay. So, we have…I left cupcakes the same. So, we have people and time, so…ohh…see I do not know…they are inversely proportional.
Int: What happened? You said they are inversely proportional. Cannot you use double number lines with that, do you mean that?
Susan: Yeah…I do not know because…because the three people is occurring at the same time that the T minutes is. But I divided this [pointed at three people] by 3 to get to one, so I had to multiply this [pointed at T minutes] by 3 to get to 3T.

Susan's idea of using two number lines seemed to be resting on the two steps in her ratio table strategy. Following the same strategy, she fixed the number of cupcakes and expressed the relationship between the number of people and number of minutes with one double number line. She then fixed the number of people and expressed the relationship between the number of cupcakes and number of minutes. These exchanges suggested that because Susan needed to divide three people by 3 to get one person, and needed to multiply T minutes to get 3T minutes as the time required for one person to frost N cupcakes, she realized the inversely proportional relationship between the number of people and time precluded the use of a double number line. Assuming the correctness of the first double number line, she was able to express the relationship between the number of cupcakes and number of minutes on the second double number line. In Task 1B, Susan was unable to explain why the double number line did not work for the gear question, but in this task, she attributed the inappropriateness of using double number line to the inversely proportional relationship between the number of people and number of minutes. Similar to Kathy, Susan also did not see 3T minutes as the constant product of the number of people and number of minutes. She should have seen 3T as the total "person-minutes" to frost N cupcakes. Thus, these data suggested that although Susan was expert at using double number lines to express directly proportional relationships, her difficulty recognizing 3T minutes as the total "person-minutes" to frost N cupcakes precluded her from using double number lines with inverse proportions.

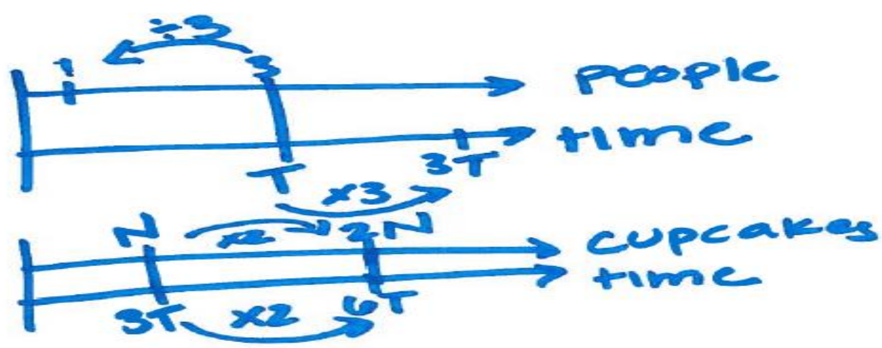


Figure 36. Susan's two double number lines strategy.

In Task 4, Susan preferred reasoning within measure spaces and using a scientific unit conversion strategy. For example, in one of the questions, she needed to calculate the speed of a car that covered a certain distance in 60 seconds, given that another car covered the same distance in 90 seconds at 60 mph. When asked what her initial impression was, Susan inferred the speed to be 90 miles an hour:

Susan: It is going to be 90 miles an hour.
Int: How do you know that it is 90 mph? How did you get that quick?
Susan: I do not know. Because he is driving 60 miles an hour, so he is driving a mile a minute. So, he drove a mile and a half because he drove for 90 seconds. So, if we want to drive a mile and a half in 60 seconds…you would have to travel a mile and a half in a minute.
Int: Then how do you know your speed is 90?
Susan: Because 90 miles in an hour divided by 60 seconds is 1.5.

In Tasks 1B and 3, Susan inferred inversely proportional relationships based on numbers swapping. In this task, the numbers were also swapped, but she did not see that. Instead, Susan seemed to be focusing on the numerical relationship between values of the speed and time because she incorrectly stated the relationship among speed, time, and distance by saying, “Because 90 miles in an hour divided by 60 seconds is 1.5.” Later, dividing 90 seconds by 60 mph, she wrote 1.5 (Figure 37) without its units that also suggested her attention to the relationship between numbers. To further investigate this possibility, I asked her if she was getting “miles” from the division of “seconds” by “miles per hour,” and Susan stated, “I don’t know if it makes sense that I write 1.5 miles.” Her response was a reflection of her confusion about the accurateness of her distance expression. Later, when she was reasoning out her answer, she incorrectly explained that multiplying 1.5 miles by 60 seconds yielded 90 mph (Figure 37). Susan’s incorrect expressions indicated that her difficulty with using appropriate referent units and expressing multiplicative relationship among quantities in the multiple proportion problems.

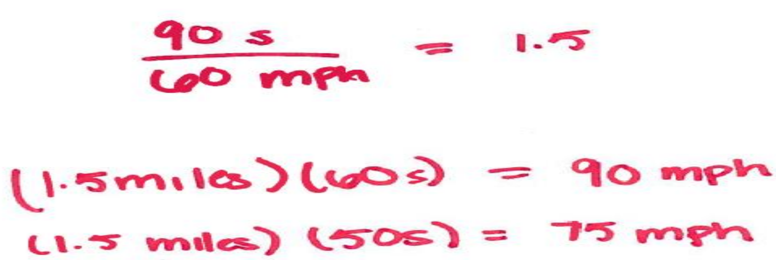


Figure 37. Susan’s initial distance formula.

Some exchanges later, when asked to calculate the speed of a car that covered a distance in 50 seconds, given that another car covered the same distance in 90 seconds driving at 60 mph, Susan multiplied 1.5 miles by 50 seconds and, as in Kathy's case, incorrectly calculated the speed to be 75 mph (Figure 37). Because she earlier calculated that the same distance was covered in 60 seconds at 90 mph, Susan immediately recognized that her answer of 75 mph did not make sense.

Susan: 50 seconds, then you get 75 miles an hour, which does not make sense because if it took less time they should be traveling at a faster speed.
Int: Faster speed, right.
Susan: So, this should be greater than 90.

In Susan's causal net, because 50 seconds was less than 60 seconds, the speed of the car needed to be more than 90 mph. Therefore, her coordination of the inverse qualitative relationship appeared to facilitate the detection of the mistake in her calculation. When asked what the relationship between the speed and time was, Susan used her knowledge of the inverse qualitative relationship to infer an inversely proportional relationship.

A few exchanges later, I told Susan that she could use any method to solve this question. She incorrectly set up a $\frac{90\ mph}{60\ s} = \frac{X\ mph}{50\ s}$ direct proportion, which would have yielded the same incorrect 75 mph answer, but she did not calculate the result. She indicated her difficulty with the question by stating, "I do not know why this is so hard for me to do." Some exchanges later, Susan decided to use a ratio table strategy to solve this question. She first tried out this strategy on the previous question in which she calculated the speed of a car that covered a distance in 60 seconds, given that another car covered the same distance in 90 seconds at 60 mph. She showed that she was multiplying one side by 1.5 and dividing the other side by 1.5. Hence, she decided that she needed to multiply 60 seconds by $\frac{5}{6}$ to get 50 seconds and divide 90 mph by $\frac{5}{6}$, allowing her to correctly determine the speed of the car to be 108 mph. These data suggested that Susan was better able to coordinate quantities when making within measure space comparisons than when making between measure space comparisons.

When asked if the $\frac{90\,mph}{60\,s} = \frac{X\,mph}{50\,s}$ proportion would have helped her to calculate the same correct answer, Susan said the proportion would not have helped her to calculate the same answer and explained the reason:

Susan: Because it is…it is not a proportional relationship, so.
Int: How about if it was a proportional [relationship], then do you think it will…it would work?
Susan: If it was proportional, then the ratio between the two [pointed at the speed and time] would be the same throughout.

In Susan's causal net, because the relationship between the speed and time was not a proportional relationship, the direct proportion that she set up would not have yielded the correct answer. Her statement, "If it was proportional, then the ratio between the two [pointed at the speed and time] would be the same throughout" provided evidence of her coordination of a directly proportional relationship with the constancy of the quotients between measure spaces. Some exchanges later, Susan explained that the product of the time and speed was constant, reasoning that there was an inversely proportional relationship between these two quantities:

Susan: So, like…if we multiplied these like 50, 50 seconds times 108 miles per hour, that should be the same as multiplying 60 seconds times 90 miles per hour because the inverse proportion.

Although Susan's statement "…50 seconds times 108 miles per hour, that should be the same as multiplying 60 seconds times 90 miles per hour because the inverse proportion" suggested her attention to constancy of the products, she did not see that the product of time and speed yielded the distance. Therefore, this example was evidence of Susan's difficulty with referent units and coordinating the relationships among distance, speed, and time. When asked what the result of multiplying 60 seconds by 90 mph was, Susan used a "scientific unit conversion" strategy (Figure 38) and successfully converted 60 mph to $\frac{1}{60}$ miles per second. She multiplied $\frac{1}{60}$ miles per second by 90 seconds and obtained 1.5 miles. She realized that the product of the time and speed yielded the distance. Thus, my question seemed to facilitate Susan's determination of the correct distance formula.

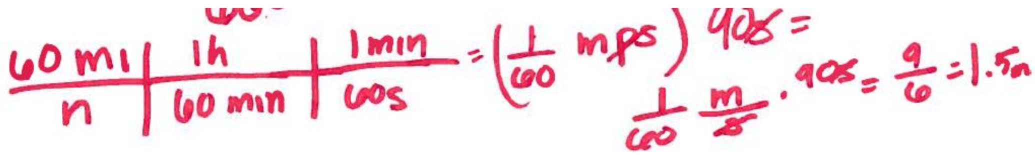


Figure 38. Susan's unit conversion strategy.

Case Three: Carol

Summary

Carol correctly inferred directly and inversely proportional relationships in the given tasks. She was successful in determining multiplicative relationships within measure spaces, multiplicative reciprocal relationships, and qualitative relationships between two covarying quantities. Carol inferred directly proportional relationships between quantities by attending to the unit rate, multiplicative relationships within measure spaces, and qualitative relationships—two quantities are increasing (or decreasing) together. On the other hand, she usually attended to the inverse qualitative relationships—one quantity is increasing and other quantity is decreasing—when inferring inversely proportional relationships. She recognized the constancy of the products in Tasks 1B and 3, but did not recognize similar relationships in the remaining inverse proportion tasks. Hence, the Tasks 1B and 3 appeared to facilitate Carol's recognition of the constant product relationships more than the remaining inverse proportion tasks. Because Carol focused on the constancy of the rate of change and qualitative relationships that she constructed between quantities, she had difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships that consisted of a constant difference or a constant sum. In Task 2, she tended to interpret the cupcake order in terms of minutes rather than cupcakes. Carol used a variety of proportional reasoning strategies to solve single and multiple proportion questions. She expressed directly and inversely proportional relationships with graphs, formulas, tables, pictures, or some combination.

Cross-Task Analysis

In Chapter Three, I determined four themes, which I provided in Table 2, for Carol's case based on the thematic analysis. In the following pages, I elaborate on these four themes to explain Carol's reasoning across tasks. In the first theme, I discuss Carol's attention to unit rates and qualitative relationships in Task 1A and to the multiplicative relationships within measure spaces and equivalence of the between measure space ratios in Task 2A when inferring directly proportional relationships between two covarying quantities. In the second theme, I discuss Carol's focus on the multiplicative and inverse qualitative relationships when inferring inversely proportional relationships in Tasks 1B and 3. In the third theme, I discuss Carol's difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships. In the last theme, I conclude the cross-task analysis with a discussion of select proportional reasoning strategies that Carol used to solve the given questions across tasks.

Theme 1: Attention to unit rates, multiplicative relationships within measure spaces, and qualitative relationships when inferring directly proportional relationships.

In Task 1A, Carol worked on questions that involved a directly proportional relationship between the number of notches and radii. For instance, in one of the questions, I asked Carol to calculate the number of notches of Gear B, with a radius of $\frac{3}{4}$ cm, given that Gear A had a radius of 3 cm and 12 notches. By setting up a $\frac{3\ cm}{12\ notches} = \frac{\frac{3}{4}\ cm}{x\ notches}$ proportion and cross-multiplying the values, Carol successfully determined that Gear B had three notches. Unlike the other three PSTs, Carol usually preferred using proportions to solve questions and to explain her solutions in Tasks 1A and 1B. When asked how she made sense of that answer, Carol drew a strip diagram that had a size of 1 cm and divided it evenly into four parts, with each part having a size of $\frac{1}{4}$ cm (Figure 39). She explained:

Carol: Yeah, well, if you have, say it is the same as 1 centimeter and you have four notches, okay. We're looking at it as, since, you're…this is three-fourths of 1 centimeter. You're going to want to break your centimeter up by fourths because you're looking at that's how many parts give you a whole. So, if you have your 1 centimeter here and you get four notches. Let's just say 1, 2, 3, 4. Okay, well, this works out nice because we have four notches for 1 centimeter and we have four parts of our whole. That's how we're going to break up our centimeters. So, we have it like, here is, we have…we're looking just at 1, 2, 3 parts of our whole because the numerator, that's what it tells us 1, 2, 3. So, we're looking at three parts of our whole. So, then, if this is our different…we're going to look at three parts of the four. So, now we have three of our four notches is to three-fourths of a centimeter.

Int: So, you made a match here between each point showing one match for each kind.

Carol: It's like saying how much would…then you basically break it down into saying instead of 1 centimeter you have four notches, for one notch you have one-fourth of a centimeter. So, that's another unit rate you can look at instead of per 1 centimeter, it's per one notch.

Carol's unit rate statements—"…it is the same as 1 centimeter and you have four notches..." and "…1 centimeter you have four notches, for one notch you have one-fourth of a centimeter"—suggested her attention to the unit ratio relationship between the radius and the number of notches. By the first statement, Carol implied that 3-cm-to-12-notches relationship was equivalent to the 1-cm-to-4-notches relationship. The exchange showed that because Carol knew that there were four notches per 1 cm radius, she partitioned the whole strip into four equal parts. For Carol, the whole strip represented a 1-cm radius, and the four small boxes below the strip represented four notches. She verbally explained the association between four parts and the whole strip saying, "…this is three-fourth of 1 centimeter." Hence, this statement made it clear why she shaded three parts of the whole strip. Carol also stated the association between the four boxes—represented the notches—and the strip diagram—represented 1-cm radius—verbally saying, "…we have three of our four notches is to $\frac{3}{4}$ of a centimeter…" Therefore, for Carol, each part in her strip diagram represented one-fourth of 1 cm as shown by her annotation (see Figure 39). Because three parts formed $\frac{3}{4}$ of a centimeter, she matched each of the three parts with one box and determined the number of notches to be three. This example showed that Susan's determination of the unit rate between the radius and the number of notches facilitated her making sense of the correct answer.

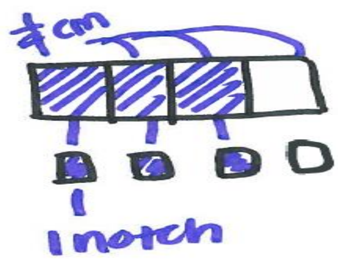


Figure 39. Carol's strip diagram strategy.

Some exchanges later, similar to the previous example, setting up a $\frac{3\ cm}{10\ notches} = \frac{\frac{6}{5}\ cm}{x\ notches}$ proportion and cross-multiplying the values, Carol calculated the number of notches of Gear B, with a $\frac{6}{5}$ cm radius, to be four notches, given that Gear A had a 3-cm radius and 10 notches. When asked if she could use another strategy, Carol said she could make a table (Figure 40):

Carol: So, you can do…let's do it for every, since we're working with this, let's do it for every fifth of a centimeter. So, for one notch gives you…let's do it per notch instead. So, you have notches here and then centimeters, you know for 10 notches you have 3 centimeters. And then you know for 1 notch you have three-tenths of a centimeter. So, for 2 notches you're going to have 0.6 because for 1 notch is 0.3. If you add another notch for 0.3 centimeters, you just add 0.6. So, then you have for 3 notches, it's going to be 0.9 because you added another 0.3 and then for 4 notches, it's going to be 1.2 and so on until you get 3 centimeters for 10. And you would hopefully notice that six-fifth, you're looking for six-fifth, and since you're going to be working with decimals, I would say put that in decimal form first, which is 1.2 centimeters. And then once you hit this 1.2 you would see, oh, okay, for 1.2 centimeters, I would have 4 notches.

Carol attended to the repeated addition of batches as indicated by her double counting approach. First, knowing that there were 10 notches for a 3-cm radius, she determined a 1-notch-to-0.3-cm radius relationship as one batch, which also was the unit rate. By repeatedly adding this batch, she calculated a 1.2-cm-radius-to-4-notches relationship. Therefore, this was another example demonstrating Carol's attention to the unit rate in making sense of her response to a given question. When asked if she would prefer addition, Carol stated, "…for me it's not meaningful just because I know that multiplication is just a bunch of addition." Carol's statement suggested a close association between addition and multiplication when concatenating batches based on a unit rate. At the same time, she suggested that if she had been given big numbers such as 100 notches, she would have needed to multiply the 10 notches by 10 and 3 cm by 10. These data suggested that Carol's preference for repeated addition or multiplication was influenced by the numbers involved.

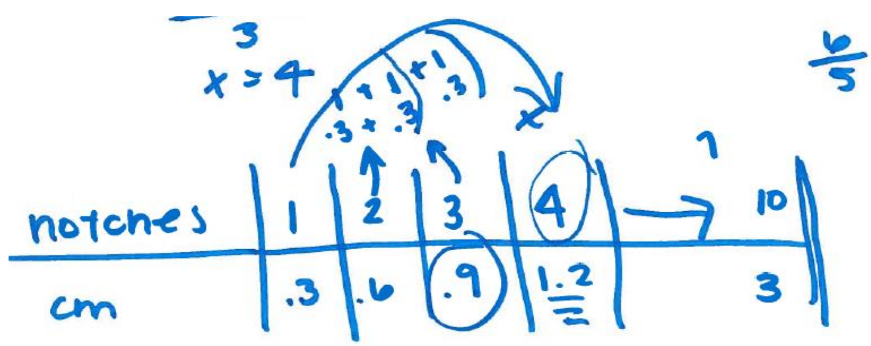


Figure 40. Carol's ratio table strategy.

A few exchanges later, when the second interviewer asked how she was making sense of her table from the meaning of proportional relationships, Carol explained:

Carol: It just shows that the proportions…all of these relationships, all these ratios [pointed at 1 notch and 0.3 cm] are equivalent. The proportions are equal, they stay consistent throughout the table. So, but if this was 3 and this was 0.8, it wouldn't be proportional. If this was anything but 0.9 for 3 notches it would not be proportional.

Int: It is because?

Carol: Because of the unit rate we found from the original ratio we were given.

In the exchange, it appeared that Carol used the terms *ratio* and *proportion* interchangeably to mean the same thing. Hence, this suggested confusion of the two terms on her part. She also used these two terms interchangeably in some other instances. For instance, some exchanges earlier, she called the $\frac{3\ cm}{12\ notches}$ between measure space ratio a proportion, saying "…if this [pointed at 3-cm/12 notches ratio] is the proportion that we're using." Because a proportion is formed by two equivalent ratios, she might have confused the terms *ratio* and *proportion*. Carol's explanation "So, but if this was 3 and this was 0.8, it wouldn't be proportional. If this was anything but 0.9 for 3 notches it would not be proportional" provided evidence for her understanding of a constant ratio relationship. Thus, these data showed that Carol's attention to the unit rate assured the correct conclusion that the number of notches and radii were in a constant ratio.

Some exchanges later, when asked if there was a relationship between the number of notches and the radii in the given questions, Carol described the qualitative relationship:

Carol: The greater the radius, or the greater the size of the gear, the more notches you're going to have.

When asked if there was a special name for the qualitative relationship that she described, Carol stated, "It was a proportional relationship" and explained:

Carol: The size of the gear and the radius, yeah. And then it's…I know what you're asking, I just can't remember the words. Hold on, let me think. It's a…we did this so much last semester in Dr. Anna's (pseudonym) class…I can't…I don't know what it's called now. Oh gosh, it's the proportional because it is like…if it was like the radius, the bigger the size of the gear, the less notches you have it would be an inversely proportional relationship. But with this one it's just a proportional relationship then, because they're both growing in size. They are both getting bigger as you get bigger. The bigger radius, the more notches you're going to have…the greater number of notches you're going to have. So, if your radius gets smaller, that means the size of your circle is getting smaller. That means you're going to have less notches.

Although Carol explained that the relationship between the number of notches and radii was proportional, she seemed to be searching for another term. Hence, she tried to remember what students called it in Dr. Anna's class. Because the relationship was a directly proportional relationship, she might have been trying to remember the term *directly proportional*. For Carol, because the radii and number of notches were both increasing, there was a proportional relationship. She also explained that if the relationship could be characterized like this—the bigger the size of the gear, the less notches—then it would be an inversely proportional relationship. Therefore, these data demonstrated Carol's attention to qualitative relationships when inferring a directly proportional relationship.

In Task 2A, Carol investigated a directly proportional relationship between the number of people and number of cupcakes frosted in a fixed time. For example, in one of the questions, I asked Carol to calculate the number of cupcakes frosted by four people in T minutes, given that three people frosted N cupcakes in T minutes. She drew a table (Figure 41) and suggested a proportional relationship between the number of people and number of cupcakes:

Carol: Yes, like okay, these [pointed at people and cupcakes] are going to be proportional like they're like…
Int: What do you mean by…?
Carol: If this [pointed at people] doubles, then this one [pointed at cupcakes] will double.
Int: Which are the proportional, [can] you show…?
Carol: The number of people plus the number of cupcakes. So if…if 300 people worked, then 300N would be made. If one person worked 1/3 of N person…of N would be made. So, these [pointed at people and cupcakes] will be at the same ratio, so if…

As these exchanges and Figure 41 demonstrated, Carol multiplied within separate measure spaces to show that the number of people and cupcakes remained in a constant ratio. In the exchanges, she slipped and said 300 people would make 300N cupcakes, but some exchanges later, she corrected her mistake by saying 100N cupcakes. Her last statement, "So, these [pointed at people and cupcakes] will be at the same ratio," provided evidence for her coordination of a constant ratio relationship between the number of people and number of cupcakes. Therefore, these exchanges showed Carol's attention to the multiplicative relationships within measure spaces when inferring a proportional relationship. When asked what she meant by the term *ratio*, Carol explained:

Carol: I just know that as this [pointed at people] increases, like multiplicatively, this [pointed at cupcakes] will increase. It's not adding. You don't add three [pointed at people] and then add three [pointed at cupcakes]. You don't add one [pointed at people] and then add one [pointed at cupcakes]. It's like you have to, because like if it was from 3 to 300 people, you wouldn't add 297 cupcakes because they could be making 10 cupcakes per person. So, it's whatever you multiply your original ratio…the ratio goes from this [pointed at people] to this [pointed at cupcakes], not this [pointed at people] to this [pointed at people]. So, it is three people for N, so six people for 2N, four people for 4/3N, this is your original thing, 3 to N. So…

Int: You said the ratio do not…doesn't go from that [I pointed at people] to that one [I pointed at people], what was the reason for you going from that [people] to that [cupcakes] instead of that one [people]?

Carol: Because it's…you're going to com…I don't…I would, I compare two different quantities…

Int: You compare…?

Carol: So, if you have three people making N cupcakes, so you have 3 to N. If you have one, alright? It is going to be 1/3N. If you have a 100, it's going to be…or 300 it's going to be 100N. How'd you get from 3 to 100…300, 3 multiplied by 100 and N multiplied by 100.

Int: So, do you obtain the, these ratios to be equal or different, kind of?

Carol: They're equivalent…so they're proportional in that way…and then yeah, so…

Carol described the increments in the number of people and cupcakes multiplicatively and as a simultaneous action. This suggested her understanding of a covariation between the number of people and number of cupcakes. Her comparison of the multiplication with addition implied that Carol was aware of the consequences of addition and multiplication. When describing the ratio, she explained that she would compare values between measure spaces. She did not give a clear reason for that, but her explanation suggested that by comparing the values of the two between measure spaces, she was able to show that all ratios were equal to the original ratio, which she referred to as 3 people to N cupcakes. Her explanation suggested this reason because by multiplying and dividing within measure spaces, she showed that 3 people to N cupcakes, 1 person to $\frac{1}{3}$N cupcakes, and 300 people to 100N cupcakes, stayed at the same constant ratio (Figure 41). In her last statement, she provided the equivalence of these ratios as the reason for inferring a proportional relationship between the number of people and cupcakes. Therefore, these data showed Carol's attention to the multiplicative relationships within measure spaces and equivalence of the between measure space ratios when inferring the constant ratio relationship.

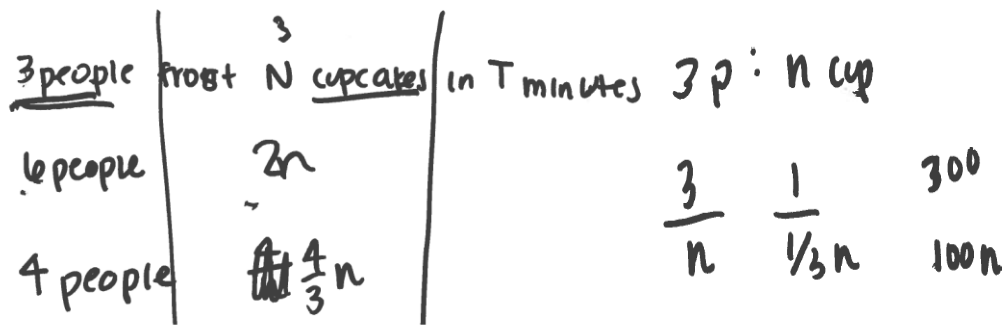


Figure 41. Carol's ratio table for expressing the number of people and number of cupcakes relationship.

Carol's reasoning on Task 1A and 2A provided evidence for her coordination of the constant ratio relationships. Her knowledge resources in coordinating the constant ratio relationships were that she attended to unit rates and qualitative relationships in Task 1A and that she attended to the multiplicative relationships within measure spaces and equivalence of the between measure space ratios in Task 2A.

Theme 2: Attention to multiplicative and inverse qualitative relationships when inferring inversely proportional relationships.

In Task 1B, Carol investigated an inversely proportional relationship between the number of notches and number of revolutions. Carol recognized that the product of the number of notches of a gear by the number of revolutions it made gave the total distance it revolved. For instance, in the first question, she needed to calculate the number of revolutions of Gear K, with four notches, given that Gear F, with eight notches, revolved three times. She drew the two gears as in Figure 42 and explained as follows:

Carol: Each notch, since they're the same…since it's like there's same spacing, it is just smaller gear than this one [pointed Gear F]. They're going to, these notches are going to hit twice. So, if there's three rotations for eight, then for four notches there's going to be six rotations.

As is clear in the statement above and Figure 42, Carol's explanation suggested that her understanding that Gear K revolved twice the number of revolutions Gear F had completed was based on the notion that each notch on Gear K would be hit twice when Gear F completed a full revolution. In Task 1A, Carol found an association between the circumference of a gear and its number of notches and explained the directly proportional relationship between the number of notches and radius based on this association. Hence, when asked how she could use the same idea in this question, she took the radii of Gears F and K as 4 cm and 2 cm, respectively, and calculated the circumferences as 8π and 4π, respectively. She then explained her idea that both gears would travel the same total length:

Carol: [Gear F] is 8π centimeters, okay. We want to know how many turns will be made if we completed three full turns. So, we're going to be turning three full turns so it's going to be 24, a total length of 24 centimeters turned. Like, do you see what I'm trying to say when I say that? Like, if you had a string and you traced, it's going to…
Int: One string is…one turn is the…
Carol: is 8π.
Int: And then you say…
Carol: So three turns gives you 24π centimeters.
Int: So, the length of the string.
Carol: Okay, so if, well I don't know why…hold on umm, because this is, I'm thinking that this is…these [pointed at Gears F and K] are going to turn the same distance.
Int: How do you know they turn the same distance?
Carol: I don't know, I just know that if they turn the same distance, they're going to have a different number of turns. And so if this…if you want the both turned 24 centimeters, 24π centimeters, this one [pointed at Gear F] took three turns and so this circumference is 4π, and you want to turn a complete 24π centimeters, you would, in each turn, and this one [pointed at Gear K] has 4π instead of 8π. You would do 24π divided by 4π and that would give you six turns.

These exchanges showed Carol's successful coordination of the circumferences of the gears with the "distances" they traveled in some number of revolutions. By multiplying the number of revolutions of Gear F by the length of its circumference, she calculated the distance traveled by Gear F as 24π. Carol explicitly stated that both gears "turn the same distance," although she did not give a clear reason to explain why they were turning the same distance. Carol's multiplication operation in calculating the distances traveled by the gears could be expressed with the following multiplication statement: (number of revolutions) * (length of the circumference) = distance. Because Carol calculated the circumference of Gear K as 4π and divided 24π by 4π to obtain six revolutions, she seemed to attend to the multiplicative relationships between quantities. The division that Carol made in this question is called measurement division. Therefore, Carol's mathematical operations and her comprehension of two meshed gears traveling the same distance provided evidence for her understanding of using constant product to explain the reciprocal relationship between the size of a gear and its number of revolutions.

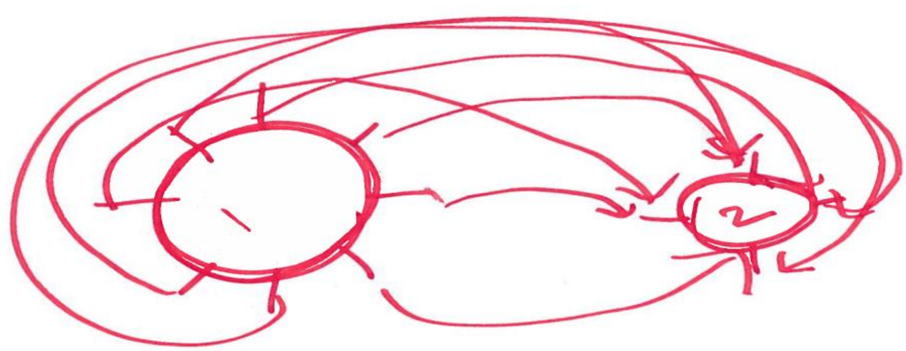


Figure 42. Carol's drawing depicting the number of notches and revolutions relationship.

Some exchanges later, when asked to calculate the number revolutions of Gear L, with eight notches, given that Gear M with 14 notches revolved four times, Carol calculated the total notches being touched on Gear M to be 56 following similar reasoning. She divided 56 by 8 and found the number of revolutions of Gear L to be seven. Because in the question 14 [notches] divided by 7 [revolutions] was equal to 8 [notches] divided by 4 [revolutions], by attending to the relationship between numbers, Carol observed a numerical inversely proportional relationship between the values. During a later exchange, based on the idea of the numerical inversely proportional relationship, Carol set the inverse proportion and calculated the number of revolutions of Gear Z, with n_2 notches, as $r_2 = \frac{n_1 r_1}{n_2}$ (Figure 43), given that Gear T, with n_1 notches, revolved r_1 times. When asked what $n_1 r_1$was, Carol explained as follows:

Carol: n1r1 is the number of notches times the number of rotations from Gear T. And then n2 is the number of notches for Gear Z and so through this, through that, realizing that relationship and then setting up like the proportions and cross-multiplying and dividing to find x, I just don't know why it works.

The exchange showed Carol's explicit multiplicative statement of $n_1 r_1$ as a product of the number of notches and revolutions; however, she accepted that she did not know why setting up a proportion and cross-multiplying worked. She did not notice that the inverse proportion was expressing the equality of the product of the number of notches and revolutions in both gears ($n_1 r_1 = n_2 r_2$). Therefore, Carol's recognition of the inverse proportion by attending to the numbers and her inability to notice the inverse proportion expressing the equality of the product of the number of notches and revolutions provided evidence for her difficulty understanding why setting the inverse proportion and cross-multiplying worked. When asked what 56 meant, Carol explained in the following manner:

Int: What was 56 in your head? I'm asking what that means.
Carol: 56 is the number of notches times the number of rotations. Or it could be the number of total notches touched, okay? So, like saying, okay, oh that's why it makes sense. Okay, so it's saying for one full rotation, n1 notches will be touched.
Int: Yes.
Carol: Okay, so if you have x, r1 rotations, then that times the notches will give you how many total notches were touched throughout those rotations.

These exchanges showed Carol's explicit statement of 56 as the number of notches per rotation times the number of rotations and suggested her understanding of a constant product relationship. From the context of gears, Carol made sense of 56 by explaining it as "the number of total notches touched" on both gears. Hence, for Carol, $n_1 r_1$ represented the total number of notches that were touched on Gear T in r_1 rotations. When asked why she divided $n_1 r_1$ by n_2, Carol explained in the following manner:

Carol: By n2 because you want to know how many times it took Gear Z to make…how many times it took Gear Z to touch that many notches.

Carol's explanation suggested that she divided $n_1 r_1$by n_2 to calculate how many times Gear Z touched $n_1 r_1$ notches. For Carol, because n_2 notches would be touched in one rotation of Gear Z, dividing the total $n_1 r_1$ notches by n_2 yielded the number of revolutions of Gear Z. Therefore, in this example, Carol used measurement division to calculate the number of revolutions of Gear Z. Although these data suggested Carol's understanding of a constant product relationship between the number of notches and number of revolutions, her difficulty recognizing the equality of two such products suggested she did not fully understand the significance of what she had done by setting the inverse proportion and cross-multiplying.

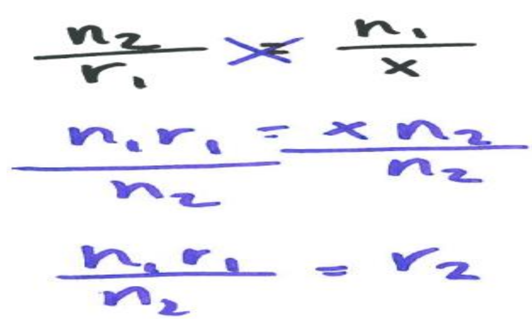


Figure 43. Carol's inverse proportion and proportion formula strategy in Task 1B.

In Task 3, Carol investigated an inversely proportional relationship between the number of weights hung and the distance from the center of a balance. By playing with plastic weights and hanging them in different places, Carol recognized that the number of weights times the distance on one side of the balance was equal to the number of weights times the distance on the other side. Hence, in addition to the context of gears, the context of balancing also seemed to facilitate Carol's recognition of a constant product relationship. When asked to describe relationship between the number of weights and distance, Carol inferred an inversely proportional relationship:

Carol: So, if you are closer, so if your distance is less, then your weight will be more. So, as your distance decreases your weight increases, so it is like the inverse, like proportional.
Int: You said proportional, inverse or?
Carol: Yeah, it's inverse.
Int: But how do you know it's proportional?
Carol: Because like this like we had a distance of 8, but our weight was 1.
Int: Okay.
Carol: Okay. And then we had a distance of 1 but our weight was 8.

These exchanges suggested that Carol's inference was based on the inverse qualitative relationship—"…as your distance decreases your weight increases…"—that she constructed between the number of weights and distance. When asked to explain how she knew that the relationship was proportional, similar to Susan, Carol attended to pair of values of two quantities being swapped to justify proportionality. In Tasks 2B and 4, Carol also worked on the inverse proportion questions and usually inferred inversely proportional relationships by attending to the inverse qualitative relationships—one quantity is increasing and other quantity is decreasing. She did not recognize the constancy of the products in those tasks. Therefore, the contexts of the Tasks 1B and 3 facilitated Carol's understanding of the constant product relationships more than the contexts of the Tasks 2B and 4.

Theme 3: Difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships.

In Task 1A, when asked what kind of graph she would have if she plotted the directly proportional relationship between the number of notches and radii, Carol stated, “A linear one.” She drew the linear graph in Figure 44 and explained:

Carol: Every…for every, like the length of centimeters you have, the amount of notches is going to fall on this line. It’s never going to fall off this line as long as it’s a 1 centimeter to 4 notches ratio.

Carol’s explanation seemed to be consistent with her coordination of the constant ratio relationship between the number of notches and radii, which I discussed earlier. She suggested a constant 1-centimeter-to-4-notches relationship for each and every notch-to-radii relationship that falls on the directly proportional line. Carol’s explanation, consistent with previous data in Task 1A, focused on to the unit rate when determining a linear relationship between the number of notches and radii.

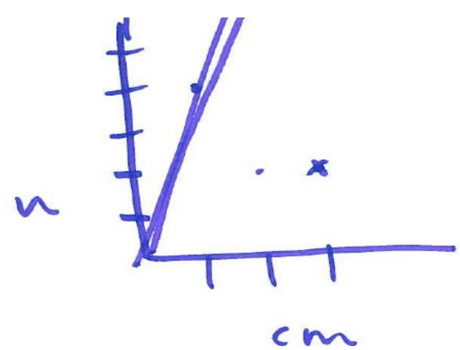


Figure 44. Carol’s linear graph for expressing the number of notches and radii relationship.

Although I planned to provide three graphs (see Appendix B Task 1B) that consisted of quadratic growth, constant difference, and constant sum in Task 1B, I decided instead to provide them at this point because I expected to receive an explanation from Carol similar to the one provided above when determining the relationships in those three graphs. When asked to compare the relationships in the three graphs with her linear graph (see Figure 44), Carol immediately stated that the relationships in Graph B and her linear graph were the same:

Carol: It would be B, yes.
Int: Is [it] the same relationship kind of, between x and y here [I pointed at Graph B] with the number of notches and radius?
Carol: I think so because, although it's not the same slope, it is the same like linear function. It's still y = x + b.
Int: Is that also…you said this [I pointed at her linear graph] is proportional relationship, right? Is that [I pointed at Graph B] also proportional like that one?
Carol: Yes. The only thing that's throwing me off is the 0 for 2.
Int: Sorry?
Carol: The only thing that throws me off is for 0 of x, you have 2 of y and then for 1 of x you have 3 and 4 of x. So like you can't…like, for some like this, this here, this 0 to 2 is not an equal proportion from 1 to 3. So, that's the only thing that's throwing me off saying that they're not exactly alike because if they're exactly alike, they're going to have…, well…

Carol pointed out that Graph B and her graph had different slopes but that they expressed "the same" linear relationship, which she claimed to be y = x + b. When asked, she claimed that the relationship in Graph B was also proportional; however, she did not seem confident in her claim. The reason for Carol's lack of confidence in her response seemed to be related to Graph B's starting point (0, 2). Although Carol explained that the ratio 0 to 2 was not equal to 1 to 3, her responses suggested that she did not have a clear explanation why the starting point (0, 2) prevented a constant ratio relationship between x and y quantities. Therefore, she did not see the significance of this starting point for inferring a proportional relationship.

Because Carol's final statement above included the phrase "if they're exactly alike," the second interviewer asked what she meant by this phrase. Carol responded to this question as follows:

Carol: Well, I don't know. Like, if you're starting from the origin I guess…but I guess it doesn't matter where you start from as long as you know for every 1 it's going to be 3…No because if it was 0, it would be 0 because it is, [sighed] I do not know.
Int 2: Is that matter or it doesn't matter?
Carol: I guess it doesn't matter.

Int 2: It doesn't.

Carol: No, because you cannot get divide…the only thing you can divide 1 by to get 0 is 0 or something, I don't know but…or multiply 1. So, I guess it doesn't matter where you start from as long as you see that, like, for every value on the x you're going to have a value for y that lands on this line. And this one [pointed at Graph A] is not like that [pointed at her directly proportional graph] because it's like exponential.

Carol's response "So, I guess it doesn't matter where you start from as long as you see that, like, for every value on the x you're going to have a value for y that lands on this line" suggested that she did not see the significance of Graph B's starting at (0, 2). Carol's response also demonstrated her attention to linearity when determining a proportional relationship. Furthermore, her statement "…the only thing you can divide 1 by to get 0 is 0 or something, I don't know but…or multiply 1" implied her difficulty with dividing zero. Therefore, these exchanges showed that attention to linearity and the presence of "the same" functional relationship, which she claimed to be $y = x + b$, were the two main reasons for Carol's inference of a proportional relationship in Graph B.

As is clear in her final statement above—"And this one [pointed at Graph A] is not like that [pointed at her directly proportional graph] because it's like exponential"—Carol inferred a nonproportional relationship for Graph A, but she misinterpreted the quadratic growth in Graph A to be exponential. Similar to Carol, in the pilot study, Robert misinterpreted a hyperbolic growth in an inversely proportional graph as exponential. When asked, Carol continued explaining her reason for inferring a nonproportional relationship for Graph A as follows:

Carol: [Nodded] and then, well, I guess it would be because you plotted this line, but these [pointing at x and y] aren't going to be proportional rates, like there's…because you see it's like a curve. It's like an exponential curve and then this one [pointed at Graph C] has a negative slope, so it's going to decrease. But this one [pointed at Graph B] is the most similar to this [pointed at her directly proportional graph], because it has a positive increase. So, this [pointed at Graph C] is like inverse proportionality because as x grows bigger, y grows smaller.

For Carol, because the line in Graph A was curved, which she incorrectly claimed to be an exponential curve, the x and y values were not increasing at a constant rate. She stated her observation of the absence of a constant rate by saying, "But these aren't going to be proportional rates." Therefore, she seemed to infer the nonproportional relationship from the absence of a constant rate. Carol's further explanation supports this interpretation:

Carol: I mean it…the only reason I say it is not directly proportional is because of…for every 1, it's a 1 and then you go over 1 and it's a 4. And then you go over another 1 and it's a 9. It's not the same unit rate for all the things. So, like if these were connected as a slope like this, not all the *y*s would fall. You see what I'm saying? It's like an exponential…I don't know the word is for it.

In her explanation, Carol explicitly stated that there was not a constant rate of change in Graph A—"It's not the same unit rate for all the things"—and for her that was the reason that Graph A did not depict a proportional relationship. As seen in the preceding exchange, Carol inferred an inversely proportional relationship in Graph C based on an inverse qualitative relationship—"…as x grows bigger, y grows smaller." Therefore, Carol's inference of the inversely proportional relationship between x and y was consistent with her definition of what an inversely proportional relationship was.

When asked for clarifications of her responses regarding the relationships in Graphs A, B, and C, Carol explained as follows:

Int: And these are…this is [I pointed at Graph C] you said inversely proportional and this is [I pointed at Graph B]?
Carol: Proportional.
Int 2: Because the slope is constant?
Carol: Yeah, on this one [pointed at Graph B].
Int 2: Is slope was also constant in this one [pointed at Graph C]?
Carol: It is constant, but it is inverse because as your x increases, your y decreases.
Int 2: Hang on, just one more time, just for clarification, you say that for proportionality the slope should be constant and both variables should increase, and if slope is constant but if one is decreasing the other increases then it is not a proportional.
Carol: No it is still proportional it is just inversely proportional.

Carol’s main points of focus in determining proportionality was the constancy of the rate of change and linearity of the graphs. It was clear from the data that after determining the constancy of the rate of change, she decided that the relationship was either directly proportional or inversely proportional based on the qualitative relationships between the two covarying quantities. It appeared that for Carol, the term *inverse* meant “opposite,” and so the inversely proportional relationship in Graph C was in some sense the opposite of the proportional relationship depicted in Graph B. Carol did not indicate that Graphs B and C showed a constant difference and a constant sum between x and y, respectively. In Task 1B, when asked to draw the graph of the relationship between the number of notches and revolutions, Carol drew the graph in Figure 45, where the line was almost straight. Although she explained that the line did not intersect the x- and y-axis at zero and that there was not a constant slope, her drawing was consistent with her incorrect inference of an inversely proportional relationship in Graph C. Thus, these data suggested that Carol attempted to integrate certain features of the context (i.e., constancy of the rate of change and linearity of the graphs) with her understanding of inverse but did not recognize that relationships in Graphs B and C were nonproportional.

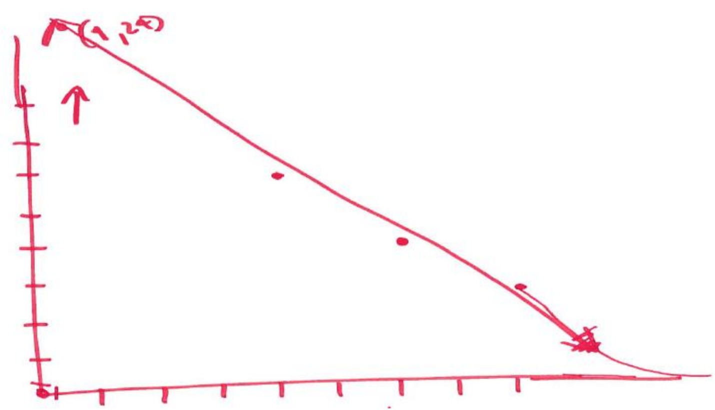


Figure 45. Carol’s inversely proportional graph.

Theme 4: The use of proportional reasoning strategies when solving proportion questions and difficulty interpreting the cupcake order in terms of cupcakes.

In the previous pages, I presented several ways that Carol inferred whether relationships were directly proportional, inversely proportional, or neither. Hereafter, I will explain some of the strategies that Carol used to solve the given multiple and single proportion questions and the difficulties that she encountered while solving those questions. For instance, in Task 2B, Carol investigated an inversely proportional relationship between the number of people and number of minutes required to frost a fixed number of cupcakes. When asked how long it would take for four people to frost N cupcakes, given that three people frosted N cupcakes in T minutes, Carol used a ratio table strategy and by reasoning within measure spaces, she found the correct answer to be $\frac{3}{4}T$ minutes. She offered the following explanation for her solution:

Carol: For four people? Three people, four people, T time…alright…how'd you get from 3 to 4…multiply it by four-thirds, yeah. So, to get you do the opposite so T times three-fourths and you would get three-fourths of the time because like I said, the more people you have the less time it will take and if the time is like divvied up, so like you're…here's like one person, here's one person, here's one person, so then you have, okay, T here, so a third of T, a third of T, if you divvied up the time like that and then if you add another, hold on, trying to do it in my head, but now I don't. Then, you still have like a third of the T, this [wrote a third of T over the three people she drew] is like a third of T. But now that you have of the T and we don't know T. So now that you have 4 people, these thirds are going to be split up by like 4, so you have like 1, 2, 3, and then here's 4. So, you have four total people and then you have just the thirds of the time split up because this was like your original thing. So, then you have three ways split up over four people. Does that make sense? That's the way I saw it in my head. I don't think that's imagined to be correct, like I don't know. Never mind, it's just how I looked at it. But I know that, just based on this, if it is four-thirds here, it's going to be three-fourths there.

Although Carol correctly solved the question by reasoning within measure spaces and forming an inverse qualitative relationship that "the more people you have the less time it will take," her attention to the inverse qualitative relationship and to the numbers indicated that she did not necessarily understand why $\frac{3}{4}$T minutes made sense. Hence, she had difficulty coordinating the cupcake order with the number of people. She wanted to make a pictorial representation of the question to explain the solution (Figure 46) but tended to interpret the cupcake order in terms of minutes rather than cupcakes. She distributed T minutes evenly among the three people, which suggested her misunderstanding that people worked sequentially instead of concurrently. Because one additional person joined the original three people, Carol then explained that she needed to split up $\frac{1}{3}$T minutes over four people and that for her there were "three ways split up over four people." Carol's phrase "three ways split up over four people" indicated that she was reasoning in a way similar to sharing three objects among four people. Therefore, her reasoning suggested that she had been attending to the numbers and looking for a way to get $\frac{3}{4}$T minutes. Ultimately, she admitted that this was the way she saw the question in her head and stated that she did not think it was a correct solution.

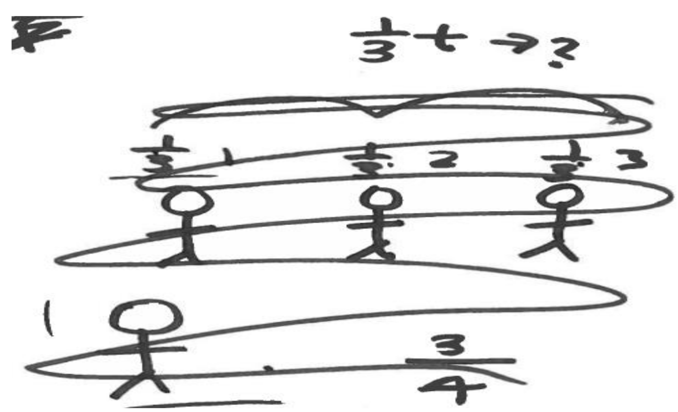


Figure 46. Carol's pictorial representation of the number of people and time relationship.

When asked how she divvied up T minutes by three people, Carol explained as follows:

Carol: I do not know. I don't think…I think that's the one way to look at it. It doesn't make sense anymore. It made sense in my head and after I drew it, I don't know, it didn't make sense. Because the more people you have, if you're like dividing up your time by the number of people that you have, then it will be 1/4 T with four people and so it does not, it does not, like…never mind, just….

Although Carol recognized that her pictorial representation did not make sense, her explanation of dividing up T minutes by four people suggested that she was still thinking of the cupcake order in terms of minutes. To understand whether she was thinking people working sequentially or concurrently, I asked the following question:

Int: Do you think they work…do you think those people working in this bakery…work like…say we three working at this bakery. Like we make some number of cupcakes in T minutes like say 12 minutes, okay? So, do you think you work 4 minutes, Rachel (pseudonym) works 4 minutes, and I work 4 minutes?
Carol: No, we all work 12 minutes.

Carol's response suggested that she seemed to understand that all of us were working 12 minutes; however, it was not clear whether she thought three of us were working concurrently or sequentially. She then explained:

Carol: But like if somebody's working, we're all going to work the same amount of time, the total time that it took. Like you're not…I'm not going to work harder, I'm working 4 minutes and now it's your turn to work 4 minutes, we're all going to work the 12 minutes, so.

Carol's statement, "…I'm working 4 minutes and now it's your turn to work 4 minutes. We're all going to work the 12 minutes…" can be interpreted as a rejection for a sequential working order because she was focusing on 12 minutes.

Because Carol focused on sharing the number of minutes among the number of people, she had difficulty making sense of the correct answer. Therefore, I reoriented her attention toward sharing cupcakes by asking how many cupcakes each of us would frost in 12 minutes, she correctly stated that it was one-third of N cupcakes. Henceforth, she decided to use N cupcakes to explain the solution. She drew four people again and correctly explained that four of us were frosting four-thirds of N cupcakes in 12 minutes (Figure 47). Carol then correctly calculated the time required for four people to frost N cupcakes as 9 minutes and explained:

Carol: Then instead of it being four-thirds of N, it will be three-fourths of 12 because it's that inverse proportion. If we're keeping N the same, it's going to take less time to frost the same amount of minutes. And if each person is frosting a third of N, then instead of having four-thirds of N, we have three-fourths of the time that it took because of that inverse relationship that they have. So, it would be three-fourths of 12 and that is 9.

Carol's response showed that she did not seem to see that $\frac{1}{4}$N cupcakes was three-fourths of $\frac{1}{3}$N cupcakes, so the time needed to frost $\frac{1}{4}$N cupcakes was three-fourths the time required to frost $\frac{1}{3}$N cupcakes. In addition, she also slipped when saying "If we're keeping N the same, it's going to take less time to frost the same amount of minutes." Whereas she needed to say that it's going to take less time to frost the same amount of cupcakes. All these data in Task 2B did not suggest a clear explanation for Carol's inappropriate approach of distributing the number of minutes evenly among the people but suggested her possible confusion. Sharing N cupcakes among four people would hold with her representation; however, sharing number of minutes among four people did not work that way. Hence, her previous experience with sharing some number of objects evenly among the people and the involvement of the time concept might have led to her confusion. In addition, Carol did not seem to see the constant product relationship between the number of people and number of minutes, and that appeared to affect her making sense of the correct answer.

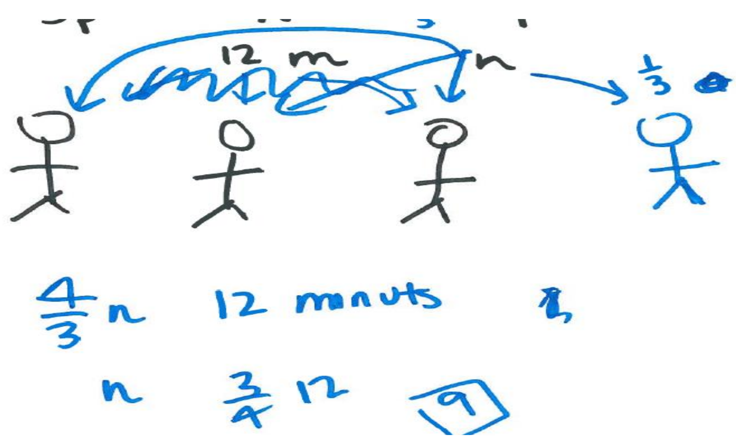


Figure 47. Carol's second attempt to explain the number of people and time relationship.

In Task 4, Carol investigated relationships among the speed of a car, the distance it covered, and the time to travel that distance. The first question of the task concerned calculating the speed of a car that covered a certain distance in 60 seconds, given that another car covered the same distance in 90 seconds at 60 mph. Carol incorrectly set up a direct proportion and by cross-multiplying the values, she calculated 40 miles per hour to be the speed of the first car (Figure 48). She immediately recognized that the answer she found was not correct:

Carol: Yeah, I cross multiply by, but it's not going to equals 90x and x equals 40 but that doesn't make sense because if you are traveling the same distance in the shorter amount of time, your speed will be…will increase, not decrease. So, that's not right, that's why I x'd it out…let me think, hold on…So, if you traveled X distance in 90 seconds at 60 mph and you want to travel X in 60 seconds and you want to know how much…how fast that would take. Gosh, I don't know…It would be I think it's…well if your time is decreasing, your speed is increasing because your distance's staying the same, right?

Carol's determination of the inverse qualitative relationship, which she stated by saying "if your time is decreasing, your speed is increasing," seemed to help her understand that her claim of the speed of the car as 40 mph was incorrect. In her explanation, Carol considered the distance to be fixed when determining the inverse qualitative relationship, and this provided evidence for her proficiency in determining qualitative relationships if more than two quantities were presented. Although Carol recognized that setting up a direct proportion was an inappropriate strategy, she seemed to have difficulty finding a more appropriate strategy to solve this question.

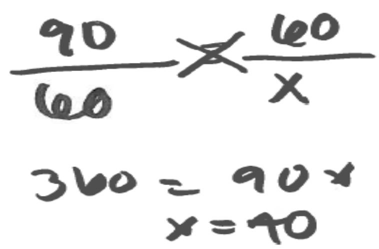


Figure 48. Carol's incorrect proportion formula strategy in Task 4.

A few exchanges later, by multiplicatively comparing quantities in within separate measure spaces, Carol correctly calculated the speed to be 90 miles per hour. She then explained:

Carol: Or this [pointed at 90 seconds] would decrease or times…this [pointed at 60 seconds] is two-thirds of 90, yeah. So, times two-thirds…so, then you're going to…if you're multiplying this [pointed at 90 seconds] times two-thirds, you're doing the inverse, so you're going to multiply this [pointed at 60 mph] by three-halves. And so 60 times 3, 180 divided by 2 is 90.
Int: 90…?
Carol: 90 seconds…90 mph.

This exchange demonstrated how Carol coordinated the numerical multiplicative relationships within separate measure spaces of speed and seconds with the inverse qualitative relationship. When she discussed her solution, Carol suggested working with within measure spaces instead of between measure spaces:

Carol: You can, but I don't. I suggest keeping your units together.
Int: Okay. You said you can if you do that kind of relationship multiplicative?
Carol: Well, I mean that you're like I think it only works because you [have] 60 miles per hour in 60 seconds.

By saying "I suggest keeping your units together," Carol appeared to imply comparing quantities with the same units (i.e., seconds with seconds and mph with mph), and so suggested reasoning within measure spaces. For Carol, because the given numbers were the same, 60 mph and 60 seconds, it was okay to multiply between measure spaces for this specific question. Perhaps, because Carol did not recognize the constancy of the products, she might have assumed that reasoning between measure spaces would be incorrect except for some specific situations. Therefore, this exchange demonstrated how Carol used specific features of the problem situation to choose one solution strategy over another.

Case Four: Helen

Summary

Helen accurately inferred directly and inversely proportional relationships in the given tasks. Although she was successful in making multiplicative comparisons between and within measure spaces, her main knowledge resources for inferring directly and inversely proportional relationships was that she attended to qualitative relationships and constancy of rate of change. Therefore, she had difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships that consisted of a constant difference or a constant sum. Similar to Carol in Task 2, she tended to interpret the cupcake order in terms of minutes rather than cupcakes. With the exception of Task 1A, she usually preferred reasoning within measure spaces when solving single and multiple proportion questions. She also reasoned in a variety of ways about proportional relationships. Helen expressed directly and inversely proportional relationships using graphs, double number lines, formulas, tables, pictures, or with some combination.

Cross-Task Analysis

In Chapter Three, I determined four themes, which I provided in Table 2, for Helen's case based on the thematic analysis. In the following pages, I discuss these four themes to explain Helen's reasoning across tasks. In the first theme, I explain Helen's attention to numerical multiplicative relationships between measure spaces and qualitative relationships when inferring directly proportional relationships in Tasks 1A and 4. In the second theme, I analyze Helen's attention to inverse qualitative relationships and the context of balancing when inferring inversely proportional relationships in Tasks 1B and 3. In the third theme, I discuss Carol's difficulty distinguishing the directly and inversely proportional relationships from the nonproportional relationships in Task 1B. In the last theme, I conclude the cross-task analysis with a discussion of selected proportional reasoning strategies that Carol used to solve the given questions across tasks.

Theme 1: Attention to numerical multiplicative relationships between measure spaces and qualitative relationships when inferring directly proportional relationships.

In Task 1A, Helen explored a directly proportional relationship between the number of notches and radii. She usually made multiplicative comparisons between measure spaces to solve the given questions. For example, in one of the questions, I asked Helen to calculate the size of Gear D with 21 notches, given that Gear E with a 4-cm radius had 14 notches. She explained:

Helen: Okay. I can just do the same as I did before. So, I have 4 over 14 which is two-sevenths. So, this 4 cm is two-sevenths of 14 notches, and so I have to ask myself what x amount of cm is two-sevenths of 21. So, I have 21 times two-sevenths which is 42 over 7 which is 6 cm, yeah for D.

Although Helen considered the referent units in her statement, "...4 cm is two-sevenths of 14 notches," her reasoning and multiplication statement (Figure 49) suggested that she might have been attending to a numerical multiplicative relationship between the radii and number of notches. Helen's statement, "…so I have to ask myself what x amount of cm is two-sevenths of 21," showed that she was searching for the same multiplicative relationship (radius is two-sevenths of number of notches) between measure spaces for Gear D. Therefore, she multiplied 21 notches by $\frac{2}{7}$ and found that the radius of Gear D was 6 cm. In the multiplication statement (see Figure 49), Helen used $\frac{2}{7}$ without its referent unit—cm/notch—and that was the reason for claiming she was attending to a numerical multiplicative relationship. From the meaning of multiplication, the correct statement needed to be (21 notches) * ($\frac{2}{7}$ cm/notch) = 6 cm. Although Helen did not infer a directly proportional relationship between the number of notches and radii, the multiplication statement in Figure 49 suggested that she expected to have the same constant $\frac{2}{7}$ ratio relationship between 21 notches and 6 cm. Thus, these data can be given as a sign of Helen's expectation of a constant ratio relationship between the number of notches and radii.

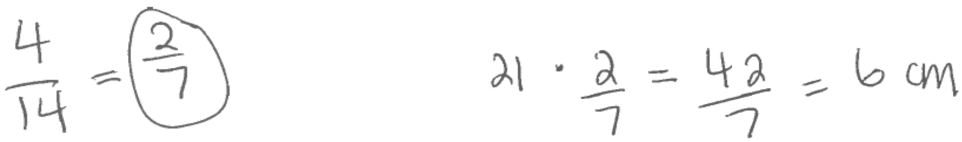


Figure 49. Helen's multiplication statement for explaining the radii and number of notches relationship.

Some exchanges later, I asked Helen to calculate the number of notches on Gear B with a 6-cm radius, given that Gear A had a 3-cm radius and *m* notches. In her solution, Helen made multiplicative comparisons within measure spaces rather than making multiplicative comparisons between measure spaces and explained:

Helen: Well, I think since well I would say that if this is [pointing out 3 cm and *m* notches] unit rate so I would have basically 1*m* like there is a certain amount of notches for 3 cm. So, since I do not know what this [pointed at the number of notches of Gear B] is and I have 6-cm, I double this [pointed 3-cm] so I would just say that 2*m* will give me 6 cm so we are looking for.

In the previous questions of this task, Helen was given numbers that she used to identify the numerical multiplicative relationships between measure spaces. In the current question, Helen identified the multiplicative relationship within measure spaces and correctly determined the number of notches to be 2*m*. Helen's inclination towards reasoning within measure spaces might have been based on the number of notches being represented by the letter *m* and the fact that doubling the numbers of centimeters and notches allowed working with the same units. When reminded that she talked about unit rates in the previous questions and asked what the unit rate was in this question, Helen explained:

Helen: Actually I do not know because if we have 1*m*.
Int: Okay.
Helen: I was just saying it is like 1 but that does not make sense.
Int: One? One notch?
Helen: We do not know it. I am…just like *m*, I am just saying if we have 1*m*, if it is *m* and then we know that for every *m* there is 3 cm that since we already know that three times…like doubling three is six. We doubled what he had here [pointed at 3 cm and m notches] so

that's why we would say there is 2*m* because that would be as just we have 3 m plus 3 m is 6 and 2m. Like this [pointed at 3 and *m*] is be 6 and 2m. Like I was saying that is how I think about it.

In the exchanges, Helen did not recognize that the unit rate between the radii and number of notches could be stated as either there is $\frac{m}{3}$ notches per 1 cm radius or $\frac{3}{m}$ cm radius per 1 notch. Her difficulty seemed to be related with the involvement of the letter *m*. On the other hand, by the phrase "for every *m* there is 3 cm," Helen characterized a fixed ratio relationship between the number of notches and radii, and she used a multiple batches approach (Figure 50) to explain her solution. In her approach, she seemed to consider the 3-cm-to-*m*-notches relationships as a batch and, by adding two batches, determined a 6-cm-to-2*m*-notches relationship. Therefore, Helen's reasoning in this task demonstrated a possible discomfort about forming multiplicative relationships between measure spaces in the absence of numbers and suggested constraints in the extent to which she could coordinate constant ratio relationships between two covarying quantities.

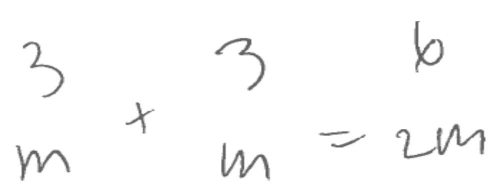


Figure 50. Helen's repeated addition of the batches in Task 1A.

In Task 4, Helen explored relationships among the speed of a car, the distance it covered, and the time to cover that distance. For example, in one of the questions, I asked Helen to calculate the time required for a car, which was traveling at a constant rate, to cover 16 miles, given that it traveled 40 miles in 50 minutes. She used a ratio table strategy, but incorrectly inferred an inverse relationship between the distance and time.

Helen: So, I can just do the table. So, if I was traveling 1 mile it would take me more, like it would take me longer to travel 1 mile at the same rate.

Int: Yeah at the same rate, sorry I forgot to mention that. You are driving at the same rate…same speed.
Helen: So, that means it would take 2,000 minutes for 1 mile.
Int: Okay.
Helen: Right? Because 4 times 5 is 20 yeah that is right yeah that time I have two zeros that is 2,000 yeah yeah that is right. Okay so then to get to 16 miles, that [pointed at 1 mile] would times by 16, so that means I would divide this [pointed at 2,000 minutes] by 16. So it would be 2,000/16, which is [simplifying, obtained 125 minutes]. So, 16 miles ohhh that does not make sense.

It was possible that because earlier Helen studied an inversely proportional relationship between speed and time, she might have confused the distance and speed concepts because the units of both concepts involve the term *mile* (mile and miles per hour). Similarly, in the pilot study, Sally and Jason incorrectly stated an inversely proportional relationship between the distance and time on the same task. Hence, the involvement of the time concept appeared to affect Helen's incorrect inference. When she determined the time it would take to cover 16 miles as 125 minutes, Helen recognized the mistake and said, "…ohhh that does not make sense." When asked why 125 minutes did not make sense to her, Helen explained:

Helen: Because if I traveled less miles...
Int: Less miles comparing 40?
Helen: Ohh wait I just realized okay these two problems were different.
Int: Which ones?
Helen: This one [pointed at part a] and this one [pointed at part c]. So, I was doing the same thing that I did here [pointed at part a] with this but that is not the same because for this one I am traveling a certain distance in 90 seconds at 60 miles in 1 hour. So, if I want to cover the same distance…in less seconds, I have to travel faster so it is inversely proportional. But this one [pointed at part c] if I travel 40 miles in 50 minutes…then to travel 1 mile it is, if I am just traveling wait if I am just going to travel just 1 mile it is going to take less time.

In these exchanges, Helen recognized that while driving at the same rate, covering 1 mile must have taken less time than covering 40 miles. These exchanges supported my conjecture above for Helen's incorrect inference of an inversely proportional relationship between the distance and time, because she admitted her confusion of the directly proportional relationship in the current question with the inversely proportional relationship in the previous question. For Helen, because in the previous question she was covering "the same distance in less seconds," there was an inversely proportional relationship between the speed and time. She then stated "…if I am just going to travel just 1 mile it is going to take less time." Therefore, these statements indicated that Helen's inferences of directly and inversely proportional relationships were based on her attention to qualitative relationships.

In her second attempt to solve the question, Helen reasoned between measure spaces. Hence, she set up the proportion in Figure 51, cross-multiplied, and correctly determined the time to be 20 minutes.

Helen: The same speed. Okay so basically what I did here was I said how many…so the speed here miles per minute, I am trying to…same miles per minutes, in this [pointed at 40 miles/50 minutes = 16 miles/? min] I want, I need it to be the same.
Int: Why do you want those equal to that?
Helen: Because you want the same speed. And if the speed is miles per minutes this speed [pointed at 16 miles/? min] has to be the same miles per minute.
Int: Do you mean this is the speed [I pointed at 40 miles/50 minutes]?
Helen: This one [pointed at 40 miles/50 minutes], it is the speed.
Int: Okay.
Helen: So this speed [pointed at 16 miles/? min] you want the speed equal each other.
Int: Then you interpret those two as speeds.
Helen: Right so basically I am looking at this 40 to 50 equals 16 to 20 [she wrote 40/50 =16/20]. So if I reduce this [pointed at 40/50] to four-fifths this [pointed at 16/20] also equals four-fifths. These ratios are the same.

These exchanges demonstrated that Helen coordinated the multiplicative relationships among the speed, distance, and time, because she explained that speed was expressed by the ratio between the distance and time. She seemed to deduce this expression from the fact that the unit of speed was written as "miles per minute." Because the cars were driving at a constant rate, Helen showed that both $\frac{40\, miles}{50\, min}$ and $\frac{16\, miles}{20\, min}$ were equal and could be simplified to $\frac{4}{5}$. Therefore, by equating the rates, Helen explained a constant ratio relationship between the distance and time.

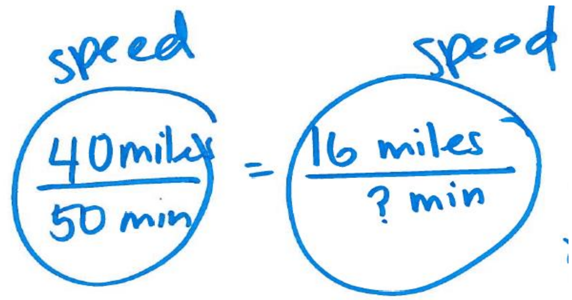


Figure 51. Helen's expression of the constancy of the between measure space ratios.

Thus, in Tasks 1A and 4, Helen made multiplicative comparisons between measure spaces and attended to the qualitative relationship—the less distance in less seconds—when inferring directly proportional relationships. She was comfortable identifying numerical multiplicative relationships between measure spaces when numbers were presented, but forming multiplicative relationships between measure spaces in the absence of numbers may have required more effort.

Theme 2: Attention to inverse qualitative relationships and the context of balancing when inferring inversely proportional relationships.

In Task 1B, Helen explored an inversely proportional relationship between the number of notches and number of revolutions. To solve the given questions, she usually used a ratio table strategy that involved multiplying and dividing the values within separate measure spaces simultaneously. For example, when asked to calculate the number of revolutions of Gear L, with eight notches, given that Gear M, with 14 notches, revolved four times, Helen generated the ratio table in Figure 52 and explained:

Helen: Okay so I just do 14 notches and then 4 revolutions [generating a ratio table]…So, for one, for one notch I would divided by 14, so I do 14 divided by…ohh four divided by, ohhh since I divided by 14 to get one notch, I would do four times 14 to get the number of revolutions. So, for one notch for these new L and M, I would have 56 revolutions.

Int: You are dividing and multiplying because?

Helen: Because this is an inverse relationship. So, to get 1 to 8 I multiply by 8. So, that means I have to divide by 8 here [pointed at 56 revolutions]. So, it will be seven revolutions.

Int: Does that make sense looking at the number of notches?
Helen: No, does it? Ohh yeah it does make sense because…since I have 14 notches and four revolutions that means if I have smaller number of notches I am going to need more revolutions.

According to Helen's explanation, she was multiplying a quantity on one side of the table by a number and dividing the covarying quantity on the other side by the same number, because for her there was an inverse relationship between the number of notches and revolutions. These data suggested that her determination of the inverse relationship was based on the inverse qualitative relationship that she described by saying, "…if I have smaller number of notches I am going to need more revolutions." I reminded Helen of how she used the unit rate concept to explain the relationship between the number of notches and the radii, and I asked if there was something similar in this task. I expected Helen to recognize that the product of the number of notches and revolutions was always equal to 56. She did not express the constancy of the products but explained:

Helen: Well it was different than the other one because whatever you do to the one you have to do opposite to the other because of it is an inverse relationship. And like same with this since I multiply it here I divide it here. And so for this type of problems with like revolution and relationships that like that is what we have to do this, do the opposite.

Helen's explanation provided evidence for how she distinguished the relationship between the number of notches and number of revolutions from the relationship between the number of notches and radii. For Helen, an inverse relationship was the reason for multiplying and dividing the values of two separate within measure space quantities by the same number. Although I provided Helen with opportunities to identify a constant product relationship, she did not express the constant product between the number of notches and number of revolutions in her ratio table.

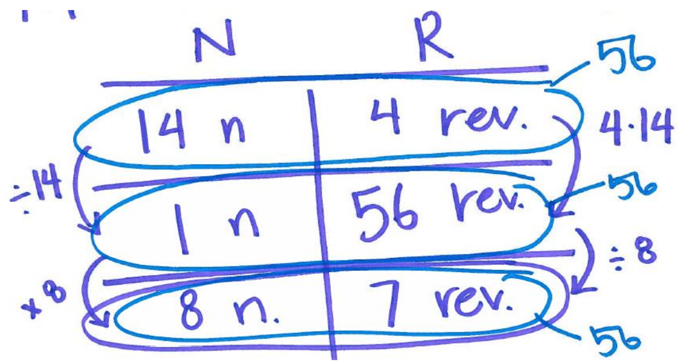


Figure 52. Helen's ratio table for expressing the number of notches and revolutions relationship.

In Task 3, Helen explored an inversely proportional relationship between the number of weights hung on a balance and the distance from the center of the balance. To clarify what the task was about, I explained that W1 number of weights was hung at D1 distance on one side of the balance, and W2 number of weights was hung on the second side. The first question was about determining D2, the distance in the second side, in terms of D1, W1, and W2. By hanging different variations of weights, Helen recognized that the product of the number of weights and distance was equal on both sides. Therefore, she expressed the relationship between the number of weights and distance with W1*D1 = W2*D2.

Helen: Well I think, well I was, I think that W1 or yeah W1 times the distance, distance one would have to equal, would has to equal W2 times the distance in order to balance [she wrote W1D1=W2D2]. So in order to find out what D2 is, I would just do W1 times D1 divided by W2.

Int: How do you know that [I pointed at W1D1] one was equal to this two [I pointed at W2D2] to balance?

Helen: Well if they are balanced like they have to be the same on both sides in order to balance.

Int: Same what?

Helen: The same value or the same weights essentially because if this is like one out here [pointed at number 10] like there has to be more, they just have to equal to same, like these two numbers [pointed at W1 and D1] multiplied by each other equal to same.

The exchanges provided evidence that Helen's recognition of the constant product relationship between the number of weights and distance was based on her understanding of balancing—

“Well if they are balanced like they have to be the same on both sides in order to balance.” It appeared that Helen determined this constant product relationship by experimenting on the balance system, but it was possible that she might have had past instruction on balancing and that might have helped her determining the constant product relationship. Hence, the context of balancing seemed to be helpful in Helen’s determination of the constant product relationship.

Some exchanges later, I asked Helen if she could generate a ratio table from the values of quantities that she needed to use to balance the system on one side, given that on the other side six weights were hung on a 4-cm distance from the center. Using the balance formula, Helen multiplied 6 by 4 and got 24 and explained that she needed combinations of 24 on the other side.

Helen: Okay so I guess the first way is try about it is that I have 4 cm and then six weights, so based on this W1D1=W2D2, I started off by thinking okay well I multiply this and that is 4 times 6 is 24 and…so, what are the like what are the…possible combinations that I can come up with. So, I can do…I can do 4 and 6, I can do…four weights on 6 cm away. So, I can try that first [she hung six weights at 4 cm], that works. And then I could do…three weights, so three weights on 8 cm.

Int: Do you want to talk about anything?

Helen: Well all these [circled pairs of weights and distances] values here like if I multiply these together they have to equal 24 for to balance.

Int: What is that 24?

Helen: Twenty four is the amount of distance and weight of the first side. Yeah like I showed here that this is four weights on the distance of six from the center, so that is why it has to be 24.

Helen searched for “possible combinations” of 24, and by circling each pair (Figure 53), she showed that the products were all equal to 24. When asked what 24 was, she said, “Twenty four is the amount of distance and weight of the first side.” As I discussed in the previous paragraph, Helen considered the product of the number of weights and distance to be a value or an amount. Therefore, these exchanges provided evidence for Helen’s explicit attention to the constancy of the products of number of weights and distance. These data suggested that Helen’s recognition of the constant product relationship was based on her attention to the numbers or, as I discussed in the preceding paragraph, she might have recalled this constant product relationship from a past instruction on balancing.

Figure 53. Helen's ratio table for expressing the number of weights and distance relationship.

Some exchanges later, when asked if it was a coincidence that all products were equal to 24, Helen explained:

Helen: No, it is not a coincidence because they need to equal to 24 to balance out.
Int: To balance out. I mean this is…is this related to the balance issue or for any inversely proportional relationship do you need that you know products being equal? Like in the revolution task, I remember you did not discuss that kind of thing.
Helen: Because the balance.
Int: The balance issue.
Helen: Yeah if I would go back to that [revolution task] and look at the balance issue then maybe it would. I would make...
Int: The revolution issue do you mean?
Helen: Right if I compare the revolution issue to balance maybe it would, it probably would.

For Helen, to balance the two sides of the system, she needed all products equal to 24. When asked if the products were equal because of the balancing issue or applied to any inversely proportional relationship product, Helen stated that it was because of the balancing. Hence, Helen's response suggested that she might not have a well-developed coordination between inversely proportional relationships and the necessity of products being constant. Helen then returned to Task 1B to investigate whether the relationship between the number of notches and number of revolutions could be explained by the idea of balancing.

Helen: So, it would be well if I, let's…I can compare it to the how I have it here. So, that they linked, like I could just say like L1 times ohh not L1, N1 times R1 equals N2 times R2.
Int: Which is?
Helen: Number of notches and the number of revolutions. So, if I have eight notches and R1, I do not know. That was equal to 14 notches and four revolutions. [Multiplied 14 by 4] So, I would have R1 or 8R1 equals 14 times four and then 56 divided by [8]…seven. Okay, so basically this [pointed at 14*4] is equal to 56 and I need this [pointed at 8*R1] to equal to 56 as well. So, what can I multiply eight by to bala…basically yes like balance, it is 56. So, what times eight would give me 56, it will be 8 times 7. So 8 times 7 is equal to 14 times 4.
Int: In your ratio table you also obtained seven revolutions for that one right?
Helen: Right. So, these all multiply and equal, this [pointed at 14 notches and four revolutions] equal 56 here, this [pointed at one notch and 56 revolutions] equal to 56, the product is 56, and then the product of these [pointed at eight notches and seven revolutions] is 56 here.
Int: You could not recognize that one that time because…the reason was?
Helen: The balance, I think the balance helped me to connect.

Helen's immediate determination of the equation N1*R1=N2*R2 suggested that she might have been following the same multiplicative structure presented in her balance formula because she did not interpret her equation in terms of equal groups of notches for each rotation. Multiplying the values of number of notches and number of revolutions that she presented in her ratio table (see Figure 52), Helen determined that all products were equal to 56. For Helen, the products were all equal to 56 in both sides of her equation N1*R1=N2*R2 and that was supporting her claim the products being equal because of balancing. Helen's reasoning demonstrated that she used the equal signs in her balance formula and in equation N1*R1=N2*R2 to represent the concept of balancing, because she explained:

Helen: I think it is just because of what I was presented with like balance, so I used the equal sign as the balance. So, when I see that these two quantities would have to equal for to balance. So, that is how I came up with. I used it like resembled this balance.

I do not have enough evidence about why Helen did not recognize the constancy of the products when she was working on Task 1B. Although Helen recognized the constant product relationship in Task 3, and when asked, she recognized the constant product relationship in Task 1B, her reasoning indicated that she did not have a well-coordinated understanding of the constant product relationships—for instance, she did not explain total notches and the product of notches per revolution and the number of revolutions. My main reason for this claim is that Helen's recognition of the constant product relationship in Task 3 was context-dependent, and she seemed to attend to the numbers rather than the reciprocal multiplicative relationships between quantities. Hence, she could not recognize the constancy of the products in the remaining inverse proportion tasks. My second reason is that the products were equal because of the inversely proportional relationship between quantities, but Helen did not see that and rather attributed the reason to the context of balancing.

Theme 3: Difficulty distinguishing directly and inversely proportional relationships from nonproportional relationships.

In Task 1A, Helen asserted a directly proportional relationship between the number of notches and radii. When asked how she knew that the relationship was proportional, Helen drew the linear graph in Figure 54 and explained:

Helen: Well I just think that the points, like wherever I just see this, like if I looked it up on graph like this [drawing a graph], I will see that for every…if this was centimeters and then notches, I will see that for every 3 centimeters I am up some notch. So, if I have, like 6 centimeters I still have more notches and it will keep going and it would be like…that for one, for every 1 centimeters goes up, it will increase in the other. So, that is way it is proportional.

Int: You said it will increase in the other?

Helen: So, like if centimeters increases the number of notches will increase as well.

These exchanges showed that Helen's assertion of the proportional relationship was based on her attention to the rate of change—"…for every 3 centimeters I am up some notch…for every 1 centimeters goes up, it will increase in the other" and to the qualitative relationship—"…if centimeters increases the number of notches will increase as well."

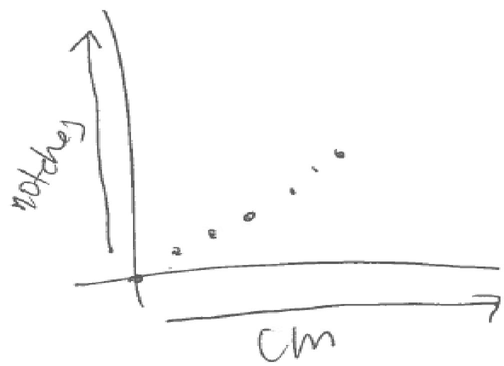


Figure 54. Helen's linear graph for expressing the number of notches and radii relationship.

Some exchanges later, I asked if any relationship with an increase-increase situation, such as the one Helen described in the previous exchange, was proportional. To demonstrate such a situation, I gave a track example in which she and I were running at a constant speed around a circular track, and she started running after I ran 100 meters. She recognized that the distance between us was constant:

Helen: If we have a constant speed that means we have a constant distance in between.

Although Helen recognized that the distance between us was constant, when I asked if the relationship between the distances she and I ran was the same as the relationship in her directly proportional graph, she stated that these two relationships were the same and explained:

Int: Okay. Is that the same kind of relationship [I referred to the directly proportional relationship] we are talking here or...?
Helen: I think it is because one does like if you go, if you are 400, like what I say you are 400 meters from the starting point and I started 100 meters behind you.
Int: Okay.
Helen: Then I be 300 meters behind you, and then as you increase distance I increase distance but we are still in the same like we still have the same…we still have the same distance apart. It is always going to be 100 meters apart.

Helen's statement, "…as you increase distance I increase distance but we are still in the same… distance apart," suggested her understanding of the constant difference of our running distances.

Although these exchanges did not support a clear explanation for why Helen thought the relationship between the distances we ran was as same with the relationship in her directly proportional graph, her definition of the qualitative relationship—"as you increase distance I increase distance"— implied her attention to the qualitative relationships. Hence, Helen did not appear to understand that because she always remained 100 meters behind me it was not possible to conclude that the relationship between our running distances was the same with the relationship in her directly proportional graph. It is possible that she might thinking about the constancy of the rate of change in both relationships when inferring the same relationship in both of them. Therefore, the data suggested Helen's lack of differentiation between proportional and nonproportional relationships.

In Task 1B, when asked to discuss the relationship between the number of notches and number of revolutions, Helen explicitly stated an inversely proportional relationship. She said:

Helen: I think it is an inverse relationship. So, whenever one goes up the other goes down. So, the number of notches…is inversely proportional to the number of revolutions.

Helen's statement of the inversely proportional relationship was based on the inverse qualitative relationship that she constructed between the number of notches and revolutions because she said, "…it is an inverse relationship so whenever one goes up the other goes down." Helen's usage of the terms *inverse relationship* and *inversely proportional* in the same sentence to describe the inversely proportional relationship suggested that she was using both terms to mean the same thing. Because not all inverse relationships are inversely proportional, Helen's usage of these two terms interchangeably supported my conjecture about her lack of differentiation between proportional and nonproportional relationships. Hence, when reminded, Helen explained:

Helen: Because proportional like they are also related. So, that is why I say inversely proportional.
Int: Do you mean they are the, they mean the same thing for you?
Helen: Yeah what I say inverse, I think so. I think I am using it at the same way. The inverse of something is the same as saying it is inversely proportional, because they are related.

In these exchanges, it was not clear what Helen meant by the phrase, “they are related,” and why she believed that this implied that inverse and inversely proportional mean the same thing. My interpretation for her use of the term is that she might be using the term *proportion* to mean *related* and might be suggesting the existence of an increase-decrease kind of covariation between the quantities.

A few exchanges later, Helen drew an inversely proportional graph to express the relationship between the number of notches and revolutions and explained:

Helen: Okay so, for eight notches I had three revolutions, for four notches I had six revolutions. It is going to end up looking like this [drawing an inversely proportional graph] because when I, as I decrease the number of notches like it is going to keep going like that and if I, as I, it is going to slowly like approach zero.

Helen’s explanation demonstrated her comprehension of the inversely proportional line approaching both the x- and y-axis at zero. Although knowing this feature of the inversely proportional graphs was important to coordinate inversely proportional relationships depicted in graphs, it was not sufficient for Helen to distinguish proportional relationships from nonpoportional relationships. For instance, some exchanges later, I provided her with three graphs (see Appendix B Task 1B) that expressed nonproportional relationships, and I asked if she could identify the relationships on those graphs. She stated that there were constant rates in Graph B and C (Figure 55), so the relationships in these graphs were similar to the directly proportional relationship that she discussed in the previous task.

Helen: Okay, so I would just say that this [pointed at Graph B] one is…the rate is constant so as I go over, this is…okay so for…I would start out at some point [pointing at y = 2]. I do not like if it was distance from home or something. In each time I go, let’s say, or each time one minute goes by I move up one foot or like I can, I do not know just something like that. So, like this rate right here, like I am going to go over and this is like we are up and over, up and over, and up, that is going to be a constant rate. And same with this one [pointed at Graph C]. I start off maybe some distance from home and I go down certain amount and I also this is [pointed at the small rectangle that she drew to express rate of change] going to be constant stay the same every time. But this one [pointed at Graph A] is like increasing exponentially so no.

Int: So these two relationships are?

Helen: These [pointed at Graphs B and C] are similar.

This exchange suggested that Helen inferred the relationships in Graphs B and C to be similar based on the existence of constancy of the rate of changes. On the other hand, for her, the relationship in Graph A was different than the relationships in Graphs B and C, because there was not a constant rate but it was "increasing exponentially." Hence, as in Carol's case, Helen also misinterpreted the quadratic growth in Graph A as exponential.

It was not clear from the preceding exchanges what Helen meant when she said, "These [pointed at Graphs B and C] are similar." When asked, Helen explained that the relationships in these two graphs were similar to the directly proportional relationship that she discussed in the previous task. When I pointed out that the x and y values in Graph B were both increasing but the x and y values in Graph C were not like that, Helen explained:

Helen: Yeah that is true. But it is I think it is even though they are like different graphs, they still have the same relationship, like...

Int: Proportional?

Helen: Yeah like they still proportional, they still increases at a constant rate but this one's [pointed at Graph A] rate is not constant like if you try to find, if you found like slope of that with tangent or whatever…it would not be a, it does not increase at a constant… because like here 1 over 1, 2 [over] 3. If it was constant the line would be like that [drew a straight line].

This exchange provided evidence that for Helen x and y values increasing at a constant rate suggested a proportional relationship. Although I pointed out the differences in the relationships in Graphs B and C, she still inferred that these two graphs showed proportional relationships. It seemed that for Helen, straightness of the line of a graph was evidence for the existence of a constant rate. As I discussed earlier, Helen usually attended to the qualitative relationships between two covarying quantities when inferring directly and inversely proportional relationships. Therefore, it seemed that Helen's attention to qualitative relationships together with the existence of constant rate of change created difficulties for her distinguishing proportional relationships from nonproportional relationships.

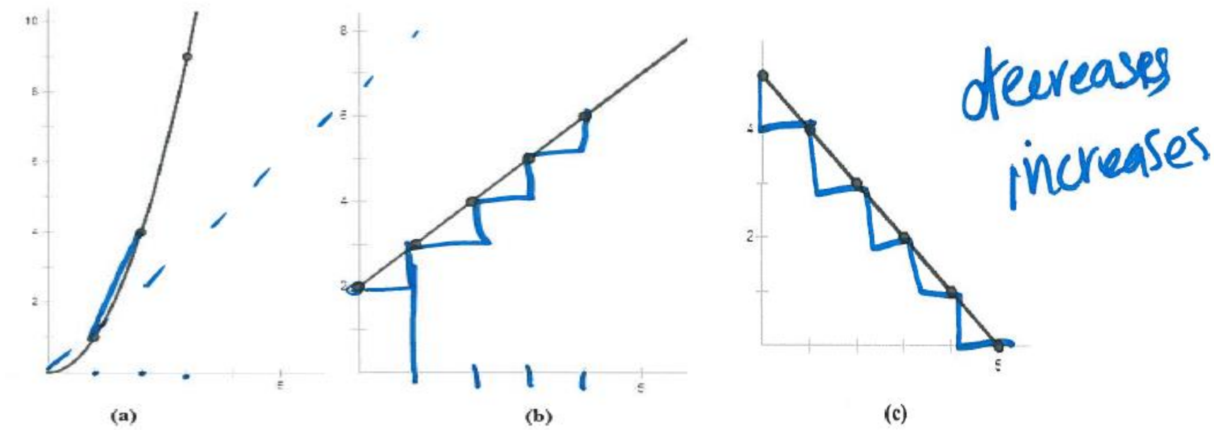


Figure 55. Helen's descriptions of the relationships in Graphs A, B and C.

Theme 4: The use of proportional reasoning strategies when solving proportion questions and difficulty interpreting the cupcake order in terms of cupcakes.

In the previous pages, I presented several ways that Helen inferred whether relationships were directly proportional, inversely proportional, or neither. Henceforth, I will explain some of the proportional reasoning strategies that Helen used to solve the given multiple and single proportion questions and the difficulties that she encountered while solving those questions. In Task 1A, when asked to calculate the number of notches on Gear Y that had a radius of r_2, given that Gear X had n_1 notches and r_1 radius, Helen set up a proportion, $\frac{r_1}{n_1} = \frac{r_2}{x}$, and found that n_2 was equal to $\frac{r_2 n_1}{r_1}$. She explained as follows:

Helen: So, I guess I would just set up a proportion like I did before. I have certain amount of cm for every amount of notches I have [wrote r1/n1], and then I have the same thing here certain amount of cm for x amounts of notches [wrote r2/x]. So say, so r1 times x, which is x is the amount of notches for Y…is equal to r2 times n1, and so x would equal r2 times n1 over r1.

Helen's statement—"I have certain amount of cm for every amount of notches I have, and then I have the same thing here certain amount of cm for x amounts of notches"—can be provided as her definition of the constant ratio relationship between the radii and number of notches. Hence, this statement provided evidence for Helen's expectation of a constant ratio relationship between the radii and number of notches. When asked if she could use other strategies to solve the question, Helen explained:

Helen: If I would set up something similar to this [wrote r1 --> n1 notches and r2 --> x notches], r1 has n1 notches and then r2 is what but there is no way for me to get to like there is no way for me to, I guess I cannot necessarily it is not necessarily 2r that is the whole. So, it is 2r then I could do something but that is r2. Would that make any kind of sense?
Int: Yeah I understand. So, if you were given something 2r or…
Helen: If it was 2r then I just double it and say this one [pointed at the notches on Gear Y] would be like 2n.
Int: But here you think you cannot or go or get something you said?
Helen: I do not think so because I do not really know…I do not know anything about r. I just know that for every *r* there is *n* amount of notches. So, for every r2 there is going to be a relationship between this one [pointed at r1] but I think that is expressed with this [pointed at $\frac{r2n1}{r1}$].

For Helen, there was "no way" for her to get r_2 from r_1. Although she expected a constant ratio relationship between the radii and number of notches and expressed this relationship correctly with the $\frac{r_1}{n_1} = \frac{r_2}{x}$ proportion, she could not recognize that r_2 was $\frac{r_2}{r_1}$ times r_1 and that the same relationship would hold for n_2, which was $\frac{r_2}{r_1}$ times n_1. Therefore, these exchanges provided evidence of Helen's difficulty identifying multiplicative relationships within and between measure spaces in the absence of numbers.

In Task 1B, when asked to calculate the number of revolutions of Gear K, with six notches, given that Gear F, with eight notches, completed three revolutions, Helen decided to use a double number line. As can be seen in Figure 56, she used opposite mathematical operations in the top and bottom number lines and obtained the correct answer—four revolutions. Hence, when reminded that they were using the same operations in the top and bottom number line of the double number line in one of her classes, which I also observed, and asked if it was okay to use opposite operations in the double number lines, Helen explained:

Helen: I think we did the same operation depending on the relationship.
Int: Did you that for the...
Helen: We did it for…we did…okay so usually depending on the relationship, like since this is inverse, we know that you have to do the opposite. Because we did both of in the Dr. Betty's (pseudonym) class.
Int: You did both of them?
Helen: Yeah because we, first we like learned how to do it with like the proportional or directly proportional things like that but now that one we start moving with this [pointing 3 revolutions] like inverse relationships we also did.

In a double number line, a pair of quantities from separate measure spaces generates a batch, and one can iterate this batch by multiplying by a factor to get multiple batches. Hence, the quantities that generate a batch covary directly. Because, in an inverse proportion, quantities covary inversely, using double number line was not an appropriate strategy to express an inversely proportional relationship; however, Helen just showed how to use a double number line to solve an inverse proportion question without attending to the operations used. On the other hand, the product of the inversely proportional quantities remains constant, and there is a directly proportional relationship between this product and the inversely proportional quantities. As in Kathy’s case, Helen could calculate 24 notches as the constant product and use two separate double number lines or one triple number line to express the relationship between the number of notches and number of revolutions. Nevertheless, Helen was able to use a double number line to support her reasoning in this task.

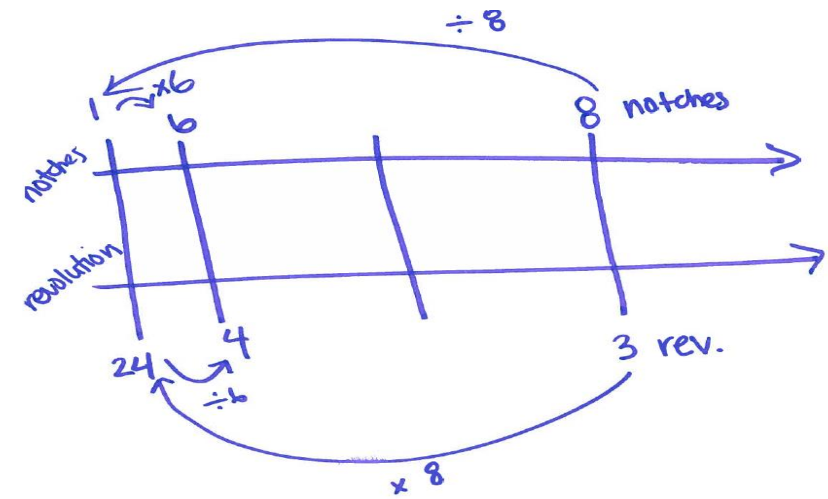


Figure 56. Helen's double number line strategy.

Some exchanges later, when asked to calculate the number of notches of Gear K that revolved two-third times, given that Gear F, with eight notches, revolved three times, Helen again used a double number line. She determined a 1-notch-to-24-revolutions relationship on the double number line and tried to figure out "how many two-third revolutions were in 24 revolutions." Helen's question statement can be expressed mathematically with (X notches) * ($\frac{2}{3}$ revolutions per notch) = 24 revolutions. Hence, she was searching for a measurement division. Although Helen spent a great deal of time switching between the double number line and the ratio table strategies, she struggled to obtain the correct answer, 36 notches. Thus, these data suggested Helen's difficulty determining multiplicative relationships when the values of two covarying quantities involved nonwhole number multiplicative relationships within and between measure spaces.

In Task 2B, when asked to calculate the time required for six people to frost N cupcakes, given that three people frosted N cupcakes in T minutes, Helen used a ratio table strategy and reasoned within measure spaces correctly to determine that the answer was $\frac{1}{2}$T minutes. A few exchanges later, I asked Helen if she could use another strategy to solve the same question and suggested that she could create drawings or diagrams. Helen represented N cupcakes with a rectangular diagram (Figure 57 a) in which she shared N cupcakes among three people. The following statement showed Helen's difficulty coordinating the number of minutes with the cupcake order and number of people:

Helen: I will just say like this, I have this [drew a rectangular diagram] and I will just say that like this is N cupcakes here and it takes three people T minutes. Uhmm so basically I would just I guess divide this [diagram] by 3 and just say that this is for person one, and this is for person two, and this is for person three. But uhmm noo this is stuck, I mean I am just like trying to picture but I do not know I think. Like I just know if I add more people and the time to be half, like if I double the amount of people it [time] is going to be half but I do not know.
Int: In this picture you divided N cupcakes by…
Helen: Three people.
Int: Is this [diagram] helping you to get your answer or?
Helen: [negative nodding]
Int: Ohh okay. Why?
Helen: Because I have to do something with the time but like, I understand that if each person have the third to do, it is going to take more time than versus like if we have the same number of N and then we do one, two, three like that each person is going to get the sixth so it will take…I guess I do not really know but this makes sense. I just know we have half the time [wrote 1/2 T and circled it].

Helen knew that her diagrams needed to have information about the minutes, but she stated her difficulty expressing the minutes in her diagrams. Helen's reasoning indicated her tendency to interpret the cupcake order in terms of minutes rather than cupcakes. She distributed the minutes evenly among the three people, which suggested her misunderstanding that people worked sequentially instead of concurrently. In addition, she seemed to conflate fixing number of people and sharing N cupcakes among T minutes and fixing T minutes and sharing N cupcakes among three people.

When I asked why she distributed minutes evenly among the three people, Helen explained:

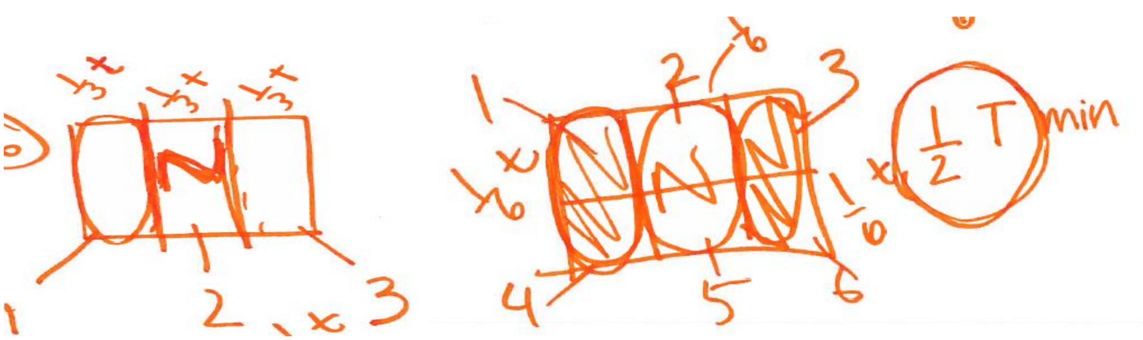
Int: Why do you think that is a third of the time?

Helen: Because one-third plus one-third plus one-third…one-third T sorry. So three-thirds T which is T.

Helen did not recognize that each person was doing his or her part in T minutes rather than one-third T minutes. It was possible that Helen might have been considering one-third T minutes as the time required to frost one-third N cupcakes by three people rather than the time needed by one person to frost one-third N cupcakes. The following exchange suggested that Helen's inclination towards the first explanation:

Int: You said one person takes one-third of the time.
Helen: Yeah
Int: To frost one-third of N?
Helen: Yeah. Ohh maybe I am right or wrong like I know that or one like I was just dividing up the number of cupcakes evenly and for this number of cupcakes it is the third of time like...

Helen's explanation—"I was just dividing up the number of cupcakes evenly and for this number of cupcakes it is the third of time"—demonstrated that she was considering one-third T minutes as the time needed to frost one-third N cupcakes. Because the total work for cupcake order was the product of the number of people and number of minutes, it could be expressed in units of "person-minutes." Helen's tendency interpreting the cupcake order in terms of minutes rather than cupcakes suggested a confusion in her side between the number of minutes and "person-minutes."



(a) (b)

Figure 57. Helen's diagrams expressing the number of cupcakes and time relationship.

A few exchanges later, Helen drew a new diagram (Figure 57 b) to express N cupcakes and divided it into six even parts. She then explained that one person was doing one-sixth of the work and by circling two parts, she stated that two people were doing their parts in one-sixth T minutes.

Helen: That is what it is [she divided 1/3 by 2 and got 1/6]. It is two people do one-sixth like two people complete the cupcakes in one-sixths of the time, and so the same thing two people complete the one-third of the cupcakes so you divided amounts by 2. I just I do not know I am just kind of think like, this is another one-sixths of the time, and then two people do this, they complete this in one-sixth of the time. So one-sixth plus one-sixth plus one-sixth equals to three-sixth which is equal one-halves the time. That is the only way I can see the picture it but I really do not know.

It seemed that because the number of people was doubled, Helen halved one-third T minutes and used similar inappropriate reasoning to arrive at the correct answer. When asked if she was assuming that people worked separately, she said, “No, they work together but it still has the same amount of time.” Although the statement suggested her understanding that the people worked concurrently rather than sequentially, it was not clear why she tended to distribute the minutes evenly by each individual. Her distribution strategy would be appropriate if she had distributed the cupcake order (or “person-minutes”) among the people rather than distributing the minutes among the people. Helen’s explanations of the diagrams were not accurate, and she did not seem completely confident about her explanations. Because, in the particular problem, the cupcake order was discussed in minutes rather than total number of cupcakes, it was not surprising to see Helen’s difficulty coordinating the cupcakes order as a product of number of people and number of cupcakes per person. In this task, although Helen successfully constructed the cupcake order as equal-sized groups (e.g., Izsàk & Jacobson, under review) of cupcakes, she did not see how to use her equal groups in determining the number of minutes needed by each person to frost his/her own part (Figure 57). That the cupcake order was described by two different units might also affected Helen’s difficulty interpreting the cupcake order in terms of cupcakes.

In the following chapter, I provide a discussion of the findings and conclude the study with a discussion of implications and suggestions for future research.

CHAPTER 5

DISCUSSION and IMPLICATIONS

Discussion

The findings of this study confirmed some findings reported in existing mathematics education research literature and added detailed information about the knowledge resources that PSTs used when determining directly and inversely proportional relationships, their difficulties determining these relationships, and the types of strategies they used to solve the given problems. This study contributes to the mathematics education literature by elucidating the research questions discussed. In the following paragraphs, I provide a discussion of the findings for each research question.

Research Question 1: *How do pre-service middle and high school mathematics teachers infer directly and inversely proportional relationships in single and multiple proportion problems; what types of knowledge resources do they use when inferring and explaining directly and inversely proportional relationships; and what kinds of difficulties do they encounter in the process of inferring, explaining, and expressing directly and inversely proportional relationships?*

The PSTs' responses to the proportion problems suggested their initial tendencies to infer relationships in the given tasks either as proportional relationships, which some of them referred to as linear relationships, or as inverse relationships based on attending to qualitative relationships between two covarying quantities. For instance, if the values of two quantities increased (or decreased) together, they usually inferred a proportional relationship. On the other hand, if the value of a quantity increased and the value of the related quantity decreased, they inferred an inverse relationship. The PSTs' qualitative comparisons usually involved causal relationships (e.g., *x* increases so *y* increases, or *x* increases so *y* decreases). Except for Kathy, who explicitly stated that not all relationships are proportional, the remaining PSTs usually inferred linear relationships as directly proportional and inverse relationships as inversely proportional. Hence, they tended to use the terms *inverse* and *inversely proportional*, and *linear* and *directly proportional* interchangeably. It appeared that these PSTs usually expected a dichotomy: If a relationship is not directly proportional, then it is inversely proportional. They seemed not to recognize that if a relationship is not directly proportional, then it does not have to be inversely proportional. Therefore, the PSTs' reasoning suggested a possible difficulty differentiating proportional relationships from nonproportional relationships.

Following the preceding paragraph, in the final study, with the exception of Kathy, none of the participants recognized nonproportional relationships in Graphs B and C (see Appendix B). Kathy was the only participant who attended to the multiplicative relationships when determining the relationships in Graphs B and C as nonproportional. The remaining participants usually attended to qualitative relationships (i.e., the value on the x-axis increases, so the value on the y-axis increases), constancy of the rate of change (or the inconstancy of the rate of change), shapes of graphs (i.e., whether the line of the graph was straight or curved), points (i.e., the values of the points being swapped), or some combination. Thus, the findings of this study suggested that the extent to which the PSTs were successful in distinguishing proportional and nonproportional relationships from each other hinged on their attention to the multiplicative relationships between quantities. This result was consistent with the findings of the Izsàk and Jacobson (2013) study in which they explained that "…teachers' capacities to form multiplicative relationships played a key role in their capacities to judge when relationships between two quantities were and were not proportional" (p. 1).

Besides attending to qualitative relationships, the PSTs also attended to multiplicative relationships within and between measure spaces when determining directly and inversely proportional relationships. Because the mathematical tasks used in this study included quantities with different units, identifying multiplicative relationships within measure spaces seemed easier for the PSTs than identifying multiplicative relationships between measure spaces. For this reason, the PSTs usually formed multiplicative relationships within measure spaces. On the other hand, when they made multiplicative comparisons between measure spaces, they usually appeared to attend to numerical multiplicative relationships. The PSTs' reasoning indicated that identifying constant ratio relationships was easier for them than identifying constant product relationships. Although they recognized the constancy of the products and reasoned about the constant product relationships in Tasks 1 and 3, none of the PSTs recognized the constancy of the products in the remaining inverse proportion tasks. Therefore, the contexts of the Gear and Balance tasks seemed to facilitate the PSTs' inferences of the constant product relationships more than the contexts of the remaining tasks. This result is very important, because it shows the effectiveness of using hands-on tasks in teaching directly and inversely proportional relationships, and thus encourages educators to use hands-on tasks in their teaching.

Because the multiple proportion tasks (i.e., Bakery, Speed, Fence, and Scout Camp) involved more than two quantities, the PSTs usually fixed one quantity at a time to explain the relationship between the other two quantities. Hence, the PSTs were generally able to coordinate the need for taking the value of a quantity as constant with the presence of a directly or inversely proportional relationship between the other two quantities, and this coordination seemed important. On the other hand, none of the PSTs was able to express the multiplicative relationships among the quantities in the Bakery and Fence tasks, and most of them had difficulty expressing the multiplicative relationships in the Speed task. In the Scout Camp task, although they usually calculated the total pounds of cereal by multiplying the given three quantities, they did not recognize that the product was constant. Therefore, the PSTs' difficulty expressing multiplicative relationships in the multiple proportion tasks suggested possible constraints in their coordination of the multiplicative relationships when more than two quantities are present.

As explained earlier, I observed that the extent to which the PSTs were successful in coordinating directly and inversely proportional relationships hinged on their attention to specific features of the contexts. These specific features comprised the knowledge resources of the PSTs' for determining directly and inversely proportional relationships. As discussed by Izsak and Jacobson (under review), attention to some of these features including qualitative relationships (two quantities are increasing together or one quantity is increasing and the other is decreasing), constancy of the rate of change, and textual features (two cars driving at the same rate or all workers working at the same pace, numbers, points, graphs) are "…associated with an expert perspective on proportional relationships, but they do not reliably discriminate between relationships that are and are not proportional" (p. 9). Based on the PSTs' responses to the proportion problems, the following five main knowledge resources of the PSTs' for determining directly and inversely proportional relationships were observed: (a) attention to qualitative relationships; (b) attention to multiplicative relationships between and within measure spaces; (c) facility with multiplicative relationships between numbers; (d) attention to the constancy of the rate of change and shape of the graphs (i.e., the line is straight or curved); and (e) attention to the static points on graphs and values of points being swapped. In their study, in addition to the associations with a particular phrase such as “same rate” and attention to equal groups, Izsàk and Jacobson (under review) mentioned the second and third knowledge resources for their participants’ inferences of proportional relationships (p. 29). In comparison to Izsàk and Jacobson (under review), the knowledge resources observed in this study were more diverse, and that could be attributed to the inclusion of the physical devices and multiple proportion tasks.

The PSTs usually attended to some combination of these five knowledge resources when determining relationships between quantities. For instance, while Susan made multiplicative comparisons within measure spaces, she attended to qualitative relationships, constancy of the rate of change, and static points on graphs and values of points being swapped when determining constant ratio and constant product relationships. It appeared that some of these five knowledge resources might have influenced the PSTs in inferring relationships more than others. For example, although Susan, Carol, and Helen made multiplicative comparisons within and between quantities, they tended to judge nonproportional relationships to be proportional. Their incorrect judgments were consistent with findings reported in existing mathematics education research (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010). This result supported my conjecture above: Some of these features might have influenced the PSTs’ reasoning when determining relationships more than others. Another indicator of my claim is that although Helen recognized a constant difference relationship between running distances in Task 1B, she inferred a proportional relationship based on the constancy of the rate of change. Similarly, Susan recognized the constant sum relationship in Graph C, but she inferred an inversely proportional relationship based on the constancy of the rate of change. Izsàk and Jacobson (under review) also documented a similar difficulty to the one Helen encountered and noted that "…the cues on which the preservice teachers relied for anticipating a proportional relationship were not always well integrated with the quantitative relationships they were forming" (p. 12). In this study, I used the PSTs' incorrect inferences to determine their expertise in coordinating directly and inversely proportional relationships.

I observed that each PST had some difficulties reasoning about directly and inversely proportional relationships. The extent of their difficulties differed based on their ability to coordinate directly and inversely proportional relationships. For instance, because Kathy was the most proficient of the four PSTs in reasoning about proportions, she had less difficulty than the remaining three PSTs. Kathy was the only PST who successfully inferred the relationships depicted in Graphs B and C in Task 1B as nonproportional. On the other hand, Susan, Carol, and Helen's incorrect inferences suggested difficulties in their coordination of directly and inversely proportional relationships. In addition, in Task 4, Helen incorrectly endorsed an inversely proportional relationship between the distance and time. As I discussed earlier, in the pilot study, Sally and Jason also incorrectly endorsed an inversely proportional relationship between the distance and time. It seemed that inclusion of time might have influenced the PSTs to make incorrect judgments about directly proportional relationships. I should have noted that PSTs' incorrect judgment of directly proportional relationships to be inversely proportional is a new finding.

Although the PSTs were comfortable in expressing multiplicative relationships between quantities with algebraic equations or formulas in the single proportion tasks, they all had difficulty generating algebraic equations or formulas in the multiple proportion tasks. Because the multiple proportion tasks involved relationships among at least three quantities, the PSTs' difficulties generating algebraic equations or formulas to express multiplicative relationships suggested their difficulties coordinating multiplicative relationships when more than two quantities were presented. In addition, although the PSTs were comfortable with describing multiplicative relationships within measure spaces, they described multiplicative relationships between measure spaces in fewer instances and when they did, they had difficulty stating those relationships. Because, in this study, quantities in two separate measure spaces had different referent units, it appeared that describing multiplicative relationships between measure spaces was more difficult than describing multiplicative relationships within measure spaces. Therefore, this feature of the tasks might have affected the PSTs' preferences of describing multiplicative relationships within measure spaces. Izsàk and Jacobson (under review) noted a similar difficulty for the PSTs who participated in their study.

In addition to these difficulties, in the Bakery task, there was an inversely proportional relationship between the number of people and number of minutes. Although both Carol and Helen correctly explained that the workers made equal numbers of cupcakes in a fixed amount of time, they had difficulty coordinating the cupcake order with the number of people. Hence, when Carol and Helen drew a pictorial representation of the problem (see Figures 46 and 57), they tended to interpret the cupcake order in terms of minutes rather than cupcakes, and so they had difficulty explaining their solutions. Their inappropriate reasoning suggested that Carol and Helen might not have coordinated the cupcakes order as the product of the number of people and the cupcakes frosted by each person. In their study, Izsàk and Jacobson (under review) also noted a similar difficulty that their participants were having.

Research Question 2: *What types of solution strategies do preservice middle and high school mathematics teachers use to solve single and multiple proportion problems, and how do they express directly and inversely proportional relationships?*

Based on the analysis of the PSTs' responses to the proportion tasks in the pilot and final studies, I entered the solution strategies of the PSTs' in Table 4. In the final study, I encouraged the PSTs to use different strategies by asking them if they could use a different strategy than they normally used; however, I did not do that in the pilot study. Hence, as it appears in Table 4, in the final study, the PSTs used many more strategies than they did in the pilot study. Table 4 suggests that many of the PSTs' strategies could be classified within Fisher's (1988) proportion formula, proportional reasoning (i.e., ratio table, unit ratio, and double number line), and algebra strategies. Additive, computation (e.g., unit conversion method), and intuitive (e.g., double counting strategy) strategies were also observed but occurred in fewer instances.

In Table 4, the most used strategy appeared to be the ratio table strategy. Three different variations of this strategy were observed. In the most common usage, the PSTs entered the given information side by side, without necessarily having rows and columns, and either multiplied or divided within and/or between measure spaces (e.g., see Figures 26 and 35). In the second type of usage, they entered the information and separated the values of the quantities from different measure spaces by rows and columns, and again either multiplied or divided within and/or between measure spaces (e.g., see Figures 21 and 23). In the last type of usage, which was only used by Sally in the Bakery II task, the information was entered in a parenthesis rather than in a table (Figure 58). The ratio table strategy usually yielded the correct results if the PSTs inferred the correct relationships between quantities. As it appears in Table 4, some of the PSTs obtained incorrect results using the ratio table strategy because they inferred incorrect relationships between the quantities compared. Because the multiple proportion tasks involved three quantities, the ratio table strategy allowed the PSTs to fix one quantity at a time and make multiplicative operations on the remaining two quantities. Therefore, the PSTs usually preferred the ratio table strategy in the multiple proportion tasks (i.e., Bakery, Speed, Fence, Apartment, and Scout Camp).

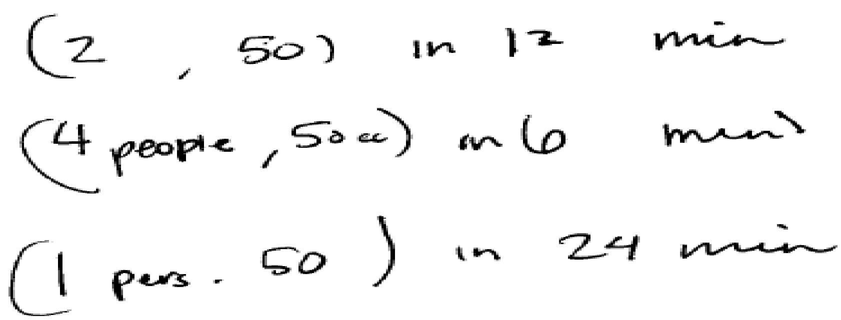


Figure 58. Sally's ratio table strategy in the Bakery II task.

Table 4
Preservice Teachers' Solution Strategies

PILOT STUDY				
	Abby	**Sally**	**Jason**	**Robert**
Gear I		Proportion Formula Strategy Ratio Table Strategy	Proportion Formula Strategy Additive Strategy	Unit Ratio Strategy Proportion Formula Strategy
Gear II		Proportion Formula Strategy	*Proportion Formula Strategy Algebra Strategy	Algebra Strategy Additive Strategy
Bakery I		Ratio Table Strategy Proportion Formula Strategy	Ratio Table Strategy	Unit Ratio Strategy Algebra Strategy
Bakery II		Ratio Table Strategy Algebra Strategy	Ratio Table Strategy Algebra Strategy	Unit Ratio Strategy Algebra Strategy
Painter I				Ratio Table Strategy Unit Ratio Strategy
Painter II				Unit Ratio Strategy Algebra Strategy

Fence		Ratio Table Strategy	Ratio Table Strategy	Ratio Table Strategy
Apartment		Ratio Table Strategy	Ratio Table Strategy	Algebra Strategy
Cookie Factory				*Ratio Table Strategy Algebra Strategy
Speed	Unit Ratio Strategy Ratio Table Strategy	*Ratio Table Strategy	*Ratio Table Strategy	
Balance	Algebra Strategy Ratio Table Strategy			

FINAL STUDY

	Kathy	**Susan**	**Carol**	**Helen**
Gear I	Ratio Table Strategy Double Number Line Strategy Unit Ratio Strategy	*Proportional Reasoning Strategy Proportion Formula Strategy Ratio Table Strategy	Proportion Formula Strategy Unit Ratio Strategy Strip Diagram Strategy Ratio Table Strategy Double Counting Strategy	Proportion Formula Strategy Ratio Table Strategy Double Counting Strategy Unit Ratio Strategy
Gear II	Additive Strategy Algebra Strategy Double Number Line Strategy Ratio Table Strategy	Algebra Strategy Ratio Table Strategy *Double Number Line Strategy	*Proportion Formula Strategy Algebra Strategy	Ratio Table Strategy *Double Number Line Strategy
Bakery I	Ratio Table Strategy Double Number Line Strategy	Ratio Table Strategy Proportion Formula Strategy Unit Ratio Strategy Double Counting Strategy	Ratio Table Strategy	Ratio Table Strategy Unit Ratio Strategy
Bakery II	Ratio Table Strategy	Ratio Table Strategy Unit Ratio Strategy	Ratio Table Strategy *Pictorial Representation Strategy	Ratio Table Strategy *Pictorial Representation Strategy
Bakery III	Ratio Table Strategy	Ratio Table Strategy *Double Number Line Strategy Unit Ratio Strategy	Ratio Table Strategy Unit Ratio Strategy	Ratio Table Strategy
Balance	Algebra Strategy Ratio Table Strategy	Algebra Strategy Ratio Table Strategy	Algebra Strategy Ratio Table Strategy	Algebra Strategy Ratio Table Strategy

Speed	*Ratio Table Strategy Unit Conversion Strategy Double Counting Strategy	*Algebra Strategy *Proportion Formula Strategy Ratio Table Strategy Double Number Line Strategy Unit Conversion Strategy Unit Ratio Strategy	*Proportion Formula Strategy Ratio Table Strategy Additive Strategy Unit Ratio Strategy Double Number Line Strategy	Unit Conversion Strategy *Ratio Table Strategy Proportion Formula Strategy
Fence	Ratio Table Strategy			Ratio Table Strategy
Scout Camp	Ratio Table Strategy	Algebra Strategy	Pictorial Representation Strategy	Pictorial Representation Strategy Ratio Table Strategy

Note. * indicates that the PST obtained an incorrect answer.

The second most used strategy appeared to be the algebra strategy. Following Fisher's (1988) strategies framework, I classified a strategy as an algebra strategy if the PSTs solved a problem setting up an algebraic equation other than a proportion formula. Except Robert, who heavily relied on this strategy, all other PSTs seemed to consider proportionality in their explanations of the algebraic expressions. Robert usually generated the formulas and equations by focusing on the relationships between numbers and so, when asked, he could not explain the meanings of his formulas and equations. In the pilot study, all three secondary grade teachers generated the $time = \frac{24}{n}$ formula in the Bakery task to calculate the time needed by n people to frost 50 cupcakes, but none of the final study participants used the algebra strategy in this task. On the other hand, in the Gear II task, the PSTs generally considered the equality of the total number of notches revolved on two meshed gears after some number of revolutions. In this task, the PSTs' ideas of total number of notches revolved on a gear could be expressed by the equation (number of revolutions) * (number of notches per revolution) = total number of notches. Similarly, in the Balance task, the PSTs generated the balance formula, W1*D1 = W2*D2, and used it to solve the given questions. Therefore, it appeared that the contexts of the Gear and

Balance tasks facilitated the PSTs' understanding of the algebraic relationships between quantities.

After the ratio table and algebra strategies, the PSTs preferred the proportion formula strategy. In this strategy, the PSTs set up a direct or an inverse proportion, showing the equivalence of two ratios, and they calculated the missing value by multiplying (or dividing) within or between measure spaces or by cross-multiplying values within the proportion (see Figures 43, 48, 51, and 59). The common mistake of the PSTs with this strategy was that because some of them tended to judge inversely proportional relationships to be directly proportional (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Lim, 2009; Riley, 2010), they set up a direct proportion to solve an inverse proportion problem. For instance, in the pilot study, Jason set up a direct proportion in the Gear II task to solve an inverse proportion problem (see Figure 15). In the final study, Carol set up a direct proportion in the Speed task to solve an inverse proportion problem (see Figure 48). Nevertheless, overall, the PSTs tended to use the incorrect proportion formula strategy in a few instances. This might have happened because of the inclusion of hands-on and multiple proportion tasks and the PSTs' prior experiences with direct and inverse proportions.

The PSTs also used the unit ratio, double number line, additive, unit conversion, pictorial, and double counting strategies. In the unit ratio strategy, the PSTs usually inferred a unit ratio relationship between two quantities, and they used this relationship to calculate a missing value. The PSTs usually used this strategy within the other strategies. For instance, in Figure 56, Helen used this strategy within a double number line strategy, and in Figure 59, Carol used it within a proportion formula strategy. In the pilot study, Robert frequently stated the unit ratio in an equation (e.g., 1 cm = 4 notches, 1 person = 25 cupcakes) and used this equation to calculate the missing-value. The double number line strategy was used by only final study participants. The PSTs usually used the double number line strategy to solve the direct proportion questions. In the inverse proportion questions, the product of the inversely proportional quantities was directly proportional to each inversely proportional quantity. Hence, inverse proportion questions could also be solved by the double number line strategy if the intention was to express a directly proportional relationship. For example, Kathy used two double number lines to solve an inverse proportion question in the Gear II task; however, Susan and Helen used this strategy inappropriately in the same task. Kathy's correct usage demonstrated her coordination of the constant product relationship in the Gear II task, and Susan and Helen's inappropriate usages showed limitations in their coordination of the constant product relationships.

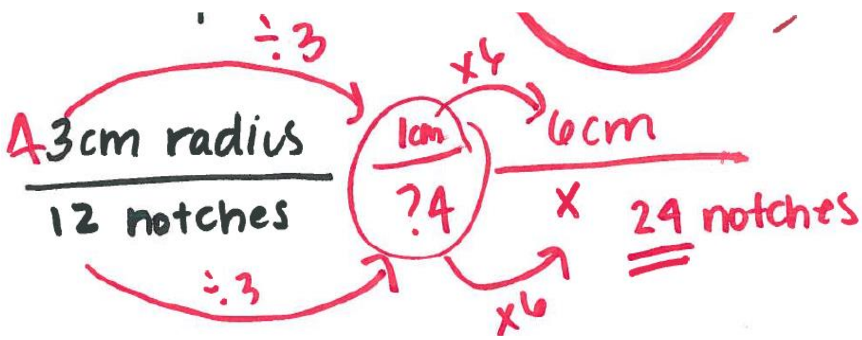


Figure 59. Carol's unit ratio strategy in Task 1A.

The PSTs also used correct additive strategies to solve problems. Because the additive strategies were used in a small number of instances, Table 4 suggests that the PSTs appeared to prefer reasoning multiplicatively rather than additively, and this appeared to be supported by the inclusion of hands-on and multiple proportion tasks. This result also supports the idea of using hands-on and multiple proportion tasks in teaching directly and inversely proportional relationships, and it contradicts the results obtained by Simon and Blume (1994) and Riley (2010) who stated that PSTs are likely to use additive strategies to solve proportion problems. In the final study, Kathy, Susan, and Helen used a “unit conversion” strategy to solve the questions in the Speed task. This strategy involved conversion of the units and did not involve reasoning proportionally. Hence, it seemed to be a computation method of which the PSTs made use. In addition to these strategies, Carol and Helen used pictorial drawings to solve the inverse proportion problems in the Bakery II and Scout Camp tasks. The double counting strategy, which was referred by Lamon (2007) as an intuitive strategy, appeared in a couple instances.

The PSTs used ratio tables, direct and inverse proportions, algebraic formulas and equations, graphs, double number lines, or some combination of these to express the directly and inversely proportional relationships. It appeared that none of the PSTs knew what a graph of an inversely proportional relationship would look like, so they usually obtained an inversely proportional graph by marking given points. For example, in the pilot study, Abby and Sally initially thought that the graph of an inversely proportional relationship could be linear (see Figures 11 a and 13 b). I observed a similar tendency in the final study from Susan, Carol, and Helen, all of whom incorrectly inferred an inversely proportional relationship in Graph C based on the constancy of the rate of change and straightness of the line. Additionally, Carol drew her inversely proportional graph as almost straight (see Figure 45). Finally, the PSTs easily recognized the constancy of the quotients and products when they entered the given data into a ratio table. Otherwise, they had difficulty recognizing these constant relationships, especially the constant product relationships.

Implications

In this study, I investigated how preservice middle and high school mathematics teachers reason about proportional relationships. This is a critical topic, because existing mathematics education research documents numerous difficulties that students and teachers have with this topic, some of which I discussed in Chapters One and Two. In the studies that I discussed, researchers generally used word problems with a single proportional or a nonproportional relationship to investigate how students or teachers reason about proportional relationships. On the other hand, in this study, the PSTs solved problems about proportional relationships that were presented through word problems and physical devices (e.g., plastic gears, mini number balance system), and the problems involved either single or multiple directly and inversely proportional relationships. Results of this study illuminate how PSTs reason about proportional relationships when they cannot rely on computation methods like cross-multiplication.

As I discussed in Chapter One, this study makes four contributions to the current research base in mathematics education: First, very little research has been conducted on PSTs' proportional reasoning. In particular, only a few researchers (e.g., Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010) have studied teachers' proportional reasoning regarding inverse proportions. Although these studies have included inversely proportional relationships, they have not focused on teachers' reasoning about such relationships to the extent that I have in this study. Additionally, multiple proportions were studied by only a very small number of researchers (e.g., Vergnaud, 1983, 1988). Because it was not easy to solve multiple proportion problems by simply forming a proportion and applying the cross-multiplication strategy, the PSTs avoided using cross-multiplication and additive strategies in those problems. Hence, the use of multiple proportion problems appeared to incline the PSTs to use cognitively more demanding strategies such as ratio table and algebra strategies. Therefore, this study benefits university educators and teachers by illuminating strategies that PSTs use when solving single and multiple proportion problems and difficulties that they encounter when solving those problems. By paying more attention to these solutions strategies and difficulties, university educators and teachers may contribute to the development of their students' proportional reasoning.

Second, the use of hands-on tasks and real-world missing-value problems together precipitate the gathering of relevant information for revealing teachers' knowledge resources for determining directly and inversely proportional relationships. One of the main findings of this study is that Gear and Balance tasks facilitated the PSTs' coordination of inversely proportional relationships more than the missing-value word problems (i.e., Bakery, Speed, Fence, and Scout Camp). In this study, the PSTs easily recognized the constancy of the products in the Gear and Balance tasks, but they usually had difficulty recognizing constant products in the remaining word problems. In the Balance task, the PSTs made experiments to balance the system on both sides, and so they empirically determined the constant product in the balance. Similarly, in the Gear task, the context facilitated the PSTs in determining the constant product—the total notches moved—by coordinating the number of groups (where a group corresponded to one rotation) and the size of groups (where the size was the number of notches). Thus, among hands-on activities, there may be important differences in how students reason about multiplicative relationships. Therefore, this result encourages educators to use hands-on activities such as the ones used in this study in teaching inversely proportional relationships.

Third, this study builds a bridge between mathematics education and science education by making use of science concepts—velocity, gear ratio, and balance. The contexts of the Gear, Balance, and Speed tasks were effective in facilitating the PSTs' proportional reasoning because they provided connections with real-life conditions. Hence, these tasks facilitated the PSTs' coordination of directly and inversely proportional relationships by enabling them to use their life experiences in making sense of the problems discussed. Some forms of proportional relationships are usually involved in the science concepts, but the existing research on proportional reasoning reveals that researchers have rarely utilized these science concepts in investigating students' or teachers' proportional reasoning. Therefore, considering the effectiveness of these concepts in understanding the PSTs' reasoning about proportions, the results of this study may inspire mathematics educators to use science concepts in their investigations.

Fourth, this study uses the knowledge-in-pieces perspective for analyzing PSTs' knowledge resources in determining and explaining directly and inversely proportional relationships in problem tasks with more complex structures and with which teachers have less experience. In this study, it appeared that the context of the tasks significantly affected the PSTs' correct inferences of the directly and inversely proportional relationships. The PSTs usually attended to specific features of the contexts when determining relationships between quantities, and some features (e.g., constancy of the rate of change, linearity of the graphs, and attention to the static points) influenced the PSTs in inferring relationships more than the others (e.g., multiplicative relationships between and within measure spaces). The knowledge resources that I entered in the first parenthesis were relevant for inferring directly or inversely proportional relationships but were not sufficient to distinguish proportional relationships from nonproportional relationships. Hence, the PSTs' difficulty distinguishing proportional relationships from nonproportional relationships suggested that these knowledge resources provided a foundation for reasoning about proportional relationships, but they needed further development. Therefore, mathematics educators should aware of the influences of these specific features on students' correct and incorrect inferences of the proportional relationships and pay more attention to the features that direct students to make incorrect inferences. It was not clear in this study why some of these specific features influenced the PSTs in inferring certain relationships more than the others. It is possible that the PSTs' previous experiences with these specific features might have inclined them to pay more attention to them when determining relationships.

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APPENDICES

Appendix A

Preservice Middle School Mathematics Teachers' Proportional Reasoning Pilot Project

Interview Protocol

Interviewer: MuhammetArican.

Discussion of goals: The goal for the interview is to explore in-depth PSTs' understanding of proportional and inversely proportional relationships. In particular, I am looking for answers to the following questions: How do preservice middle and high school mathematics teachers identify proportional relationships and inversely proportional relationships?, how do preservice middle and high school mathematics teachers compare and contrast proportional relationships with inversely proportional relationships?, what strategies do they use to compare and contrast the two relationships?, and what types of reasoning do they engage in when solving tasks that involve proportional and inversely proportional relationships? I believe investigating problems with directly proportional relationships and inversely proportional relationships is critical because research indicates that in-service and preservice mathematics teachers have problems understanding directly proportional and inversely proportional relationships especially the latter. Also these kinds of problems are examples of multi-step problems and it is important for middle and high school mathematics teachers to learn to apply their knowledge in multi-step situations.

Materials: Each participant will be given a printout of the mathematical tasks, papers, pencils, plastic gears with different sizes, and colored pencils in case they need.

Introduction: In the first two tasks participants are going to explore the relationships on given plastic gears so I will introduce the gears they will be exploring and explain assumptions such as the notches have the same size in each gear so they can be meshed one another, meshed gears have different sizes, etc… Later I will ask them to work on the tasks and I will let them know that I am very interested in their thinking. Also I will tell them that this interview is not a test so they may skip questions and they may discontinue the interview at any time without explanation. As the participants discuss their thinking, I will ask follow up questions.

Interview Tasks

Note: In tasks 1 and 2, Gear 1 and Gear 2 are connected to each other so if one turns around then the other gear also will turn around. Also in tasks 1 and 2 we are assuming all the notches are the same size, so that they can be meshed to one another, and equally placed around the gears (Gear 1 and Gear 2).

Task 1

Assume you are given two gears, Gear 1 and Gear 2. Gear 1 has a radius r1 = 3 cm and Gear 2 has a radius r2 = 4 cm. If Gear 1 has n1 = 12 notches, then how many notches, n2, does Gear 2 have?

Possible follow-up questions:

- Can you explain to me what you are thinking?
- If Gear 1 had 18 notches, then how many notches does Gear 2 have?
- Numbers of notches of given three gears are as follows. Can you obtain what should be the radius of each gear? (Assume that the sizes of notches are the same with notches in Gear 1 and Gear 2, and equally placed around the gears.)

  A gear with 8 notches

A gear with 11 notches

A gear with 18 notches

- Assume you replaced Gear 1 with following gears, radius of each given. Can you obtain the number of notches of each gear? (Assume that the sizes of notches are the same with notches in Gear 1, and equally placed around the gears.)

  A gear with a radius 5 cm

  A gear with a radius $\frac{1}{2}$ cm

  A gear with a radius $\frac{3}{4}$ cm

  A gear with a radius $\frac{7}{2} = 3\frac{1}{2}$ cm

- Now assume if we had two different gears, Gear 3 and Gear 4, with radii 4 cm and 6 cm, respectively. If we were able to place 14 notches around the Gear 3, then how many of these notches can we place around Gear 4? (Assume that sizes of the notches are the same in Gear 3 and Gear 4 again.)
- Considering all the parts of Task 1, do you think is there a relationship between the radius of a gear and the number of notches it has? If there is can you represent this relationship with anything that you think will be proper? Can you explain me what do you think about this relationship?
- Let think if the question was given in the following format: Gear 1 has a radius r1 and n1 number of notches and Gear 2 has a radius r2 and n2 number of notches then how would you solve describe the relationship?
- Can you write this relationship in an algebraic form? (If does not understand, then explain that for example as an equation or another form.)

Task 2

Assume you are given the gears in task 1 again. If Gear 2 revolves R2 = 6 times, then how many times does Gear 1 revolve?

Possible follow-up questions:

- Can you explain to me what you are thinking?
- If Gear 2 revolves 3 times, then how many times does Gear 1 revolve?
- If Gear 2 revolves 1 time, then how many times does Gear 1 revolve?
- If Gear 2 completes $\frac{3}{4}$ of a revolution, then how much revolutions does Gear 1 complete?
- If Gear 2 revolves $\frac{15}{2} = 7\frac{1}{2}$ times, then how many times does Gear 1 revolve?
- If we replaced Gear 1 with each of the following gears with given radius, can you obtain the number (or the parts) of the revolutions each gear makes still assuming that Gear 2 revolves 6 times? (Assume that the sizes of notches are the same with notches in Gear 1 and 2 and are equally placed around the gears.)

  A gear with radius 8 cm

  A gear with radius $\frac{1}{2}$ cm

  A gear with radius $\frac{3}{4}$ cm

  A gear with radius $\frac{7}{2} = 3\frac{1}{2}$ cm
- Now assume you replace Gear 1 and Gear 2 with two new gears, Gear 5 and Gear 6, with 15 and 21 notches respectively. If Gear 5 revolves 8 times, then how many revolutions does Gear 6 make? (Assume that Gear 5 and Gear 6 have the same size notches but they are different in size then the notches in Gear 1 and Gear 2.)

- Is there a relationship between the radius of Gear 1 and the number of revolutions it makes, still assuming that Gear 2 revolves 6 times? If there is, can you represent this relationship with a table, graph, or anything you think proper?
- Is there a relationship between the number of notches a gear has and the number of revolutions it makes? If there is, can you represent this relationship with anything that you think will be proper?
- If the question was given in the following format: Gear 1 has a radius r1, has n1 notches and makes R1 revolutions and Gear 2 has a radius r2, has n2 number of notches, and makes R2 revolutions, then how would you represent the relationships between those quantities?
- Can you represent these relationships in an algebraic form? (If they do not understand what I mean, then I will tell them if they can represent the relations in an equation or another form.)

Task 3

At a bakery, 2 people can frost a total of 50 cupcakes in 12 minutes. How many cupcakes can 4 people frost in 12 minutes? (Assume that all people work at the same steady pace.)

Possible Follow-up Questions:

- Can you explain what you are thinking?
- How many cupcakes can 1 person frost in 12 minutes?
- How many cupcakes can 3 people frost in 12 minutes?
- How many cupcakes can N people frost in 12 minutes?
- How many people will be needed to frost 350 cupcakes in 12 minutes?

- How about if we increase the number of people by adding six more people, then do we need to increase the number of cupcakes by subtracting the same number?
- How many people will be needed to frost M cupcakes in 12 minutes?
- Is there a relationship between the number of people and the number of cupcakes?
- How do you know it is directly proportional?
- Can you represent this relationship with a graph and a table?

Task 4

At a bakery, 2 people can frost a total of 50 cupcakes in 12 minutes. How long will it take for 4 people to frost 50 cupcakes? (Assume that all people work at the same steady pace.)

Possible Follow-up Questions:

- Can you explain what you are thinking?
- How long will it take for 1 person to frost 50 cupcakes?
- How long will it take for 6 people to frost 50 cupcakes?
- How about if we increase the number of people by adding six more people, then do we need to decrease the time by subtracting the same number?
- How long will it take for N people to frost 50 cupcakes?
- If we know that 50 cupcakes were frost in 8/3 minutes, then how many people frosted that many cupcakes? (Assuming that 2 people can frost a total of 50 cupcakes in 12 minutes.)
- How many people do we need to frost 50 cupcakes in ¾ of a minute? (Assuming that 2 people can frost a total of 50 cupcakes in 12 minutes.)
- How long would it take for 4 people to frost 75 cupcakes? (Assuming that 2 people can frost a total of 50 cupcakes in 12 minutes.)

- Is there a relationship between the number of people and number of minutes when frosting 50 cupcakes? How do you know it is inversely proportional?
- Can you represent this relationship with a graph and a table?

Task 5

If 4 painters can paint 3 bedrooms in 6 hours then how many bedrooms can 8 painters paint in the same 6 hours? (Assume that all painters work at the same steady pace.)

Possible Follow-up Questions:

- Can you explain what you are thinking?
- How many bedrooms or parts of a bedroom can a single painter paint in 6 hours?
- How many bedrooms or parts of a bedroom can two painters paint in 6 hours?
- How many bedrooms or parts of a bedroom can 6 painters paint in 6 hours?
- How many painters will be needed to paint 9/2 bedrooms in 6 hours?
- How many bedrooms or parts of bedrooms can be painted by N painters in 6 hours?
- Is there a relationship between the number of painters and the number of bedrooms painted in 6 hours? If there is, can you explain what that relation is?
- Can you represent this relationship with anything that you think will be proper?

Task 6

If 4 painters can paint 3 bedrooms in 6 hours, then how many hours would it take for 8 painters to paint the same 3 bedrooms? (Assume that all painters work at the same steady pace.)

Possible Follow-up Questions:

- Can you explain what you are thinking?
- How many hours must 1painter spend to paint the same 3 bedrooms?
- How many hours must 2 painters spend to paint the same 3 bedrooms?

- How many painters do we need to paint 3 bedrooms in 1 hour?
- How many painters do we need to paint 3 bedrooms in ¾ of an hour?
- How many painters do we need to paint 3 bedrooms in 4/3 hours?
- Now if 4 painters can paint 3 bedrooms in 6 hours then how many painters do we need to paint 4 bedrooms in 4 hours?
- Is there a relationship between the number of painters and the number of hours spent for painting 3 bedrooms? If there is, can you explain what that relation is?
- Can you represent this relationship with a table, graph, or anything you think appropriate?

Task 7: Direct-Direct-Inverse Proportional Relationship Problems

If 3 people take 2 days to paint 5 fences, how long will it take 2 people to paint 1 fence? (Assume that the fences are all the same size and the painters work at the same steady rate.)

Possible Follow-up Questions:

- Can you explain what you are thinking?
- How many days would it take for 1 person to paint 5 fences?
- How many days would it take for 1 person to paint 1 fence?
- How many days would it take for 2 painters to paint 1 fence?
- Assuming that 3 people take 2 days to paint 5 fences, how many fences can 6 painters paint in 3 days?
- Assuming that 3 people take 2 days to paint 5 fences, how many painters do we need to paint 10 fences in 3 days?
- Are there any relationships between the given three quantities? If there are, then can you explain what those relations are?

- Can you represent these relationships with anything that you think may proper?

Task 8: Inverse-Inverse-Inverse Proportional Relationship Problems

If 8 workers can build an apartment in 24 days by working 6 hours each day, then how many days does it take for 12 workers to build the same apartment if each works 8 hours every day? (Assume that all workers work at the same steady pace.)

Possible Follow-up Questions:

- Can you explain what you are thinking?
- How many days does it take 1 worker to build the same apartment if he/she works 6 hours each day?
- How many days does it take 1 worker to build the same apartment if he/she works 8 hours each day?
- How many days does it take 12 workers to build the same apartment if each work 8 hours every day?
- Assuming that 8 workers can build an apartment in 24 days by working 6 hours each day, how many workers do we need to build the same apartment in 18 days assuming each worker work 4 hours a day?
- Still we assume that 8 workers can build an apartment in 24 days by working 6 hours each day. Now if we have 12 workers, so how many hours a day should each worker need to work to build the same apartment in 16 days?
- Are there any relationships between the given three quantities? If there are, then can you explain what those relations are?
- Can you represent these relationships with anything that you think may proper?

Task 9

In a cookie factory, 4 assembly lines make enough boxes of cookies to fill a truck in 10 hours. How long will it take to fill the truck if 8 assembly lines are used? (Assume that all assembly lines work at the same steady rate.)

Possible Follow-up questions:

- How long will it take to fill a truck if 2 assembly lines are used?
- How long will it take to fill a truck if 1 assembly line is used?
- How long will it take to fill a truck if 6 assembly lines are used?
- How many assembly lines do we need to fill the truck in $\frac{2}{3}$ of an hour?
- How many assembly lines do we need to fill the truck in $\frac{5}{4}$ hours?
- Is there a relationship between the number of assembly lines and the number of hours to fill the truck? If there is, can you represent this relationship with anything that you think will be proper?

Task 10

If you covered the distance between two markers in 90 seconds driving at 60 mph. How long would it take you to cover the same distance driving at 50 mph?

Possible Follow-up questions:

- Is there another way to solve this problem?
- How long would it take someone to cover the same distance if she/he walks 1 mile per hour? (Assume she/he walks in a steady pace.)
- How long would it take someone to cover the same distance driving at 80mph?
- If you cover 2 miles in 100 seconds, then what is the speed of your car in mile per hour?
- If you trveled 40 miles 50 minutes, then how long would it take you to travel 15 miles?

- Are there relationships between speed of a car, the distance it traveled, and the amount of time traveled? Can you tell me what kinds of relationships are these?

Task 11 (Used only with middle grade PSTs)

Note: I will introduce the mini number balance system and explain the goal, which is to figure a balance between weights that can be hung at both directions. I will tell them they will need to assume they are only allowed to hang weights on one place at a time in both directions. Also, I will explain that each weight weighs one gram.

Question: If you hang 3 grams on number 4 at one direction, then can you show me different ways of forming the balance in the system?

Possible follow-up questions:

- Can you explain to me what you are thinking?
- Can you tell me more about why do you think hanging this way works?
- Similarly, if you had hung 4 grams at number 8, then can you show me the ways of forming the balance in the system?
- Can you think the numbers on the both directions in another way? What can they also represent here?
- Can you find an equation that can be used to explain the ways of balancing the system?
- Is there a relation between weights and the distance where the weights hung?

  Can you draw a graph of this relationship?

Appendix B

EXPLORING PRESERVICE MIDDLE AND HIGH SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF DIRECTLY AND INVERSELY PROPORTIONAL RELATIONSHIPS

Interview Protocol 1

Interviewer: Muhammet Arican

Discussion of goals: The goal for the first half of Interview 1 is to explore how preservice middle school mathematics teachers infer the directly proportional relationship between the size of a gear and the number of notches around it. In the second half, I will explore how preservice middle school mathematics teachers infer the inversely proportional relationship between the size of a gear and the number of revolutions it makes and the inversely proportional relationship between the number of notches the gear has and its number of revolutions. I will provide PSTs with plastic gears and give them some problems that will help me explore their reasoning. During the first interview, I will focus on the knowledge resources that they use to detect the directly and inversely proportional relationships. Additionally, the types of strategies that they use to solve given proportion problems, the ways they represent the directly and inversely proportional relationships, and the types of reasoning they engage in when solving these problems will be explored.

Materials: Each participant will be given printouts of the problems, paper, pencils, and plastic gears.

Introduction: I will let the participants know that I am most interested in their thinking processes. Also, I will tell them that this interview is not a test, so they may skip questions and may discontinue the interview at any time without explanation. I will ask them to work on the tasks and to think aloud while they are attempting to solve the problems. I will let them know that they can play with the plastic gears to solve the problems and encourage them to take their time. I will also let them know that I may ask them to use a new method to solve given problems but this doesn't mean their first way was wrong. Before moving to the tasks, I will ask them if they have any questions for me. As the participants discuss their thinking, I will ask them follow-up questions.

Interview Tasks and Follow-up Questions

Task 1A

Note: In Tasks 1A and 1B, two gears are meshed with each other, so if one gear rotates, then the other gear also rotates. Because the two gears are meshed, they have the same-sized notches, and the notches are equally placed around the gears.

Problem: Two gears, Gear A and Gear B, are meshed with each other as seen in picture below. Gear A has a 3-cm radius, and Gear B has a 6-cm radius. If Gear A has 12 notches, then how many notches does Gear B have?

Possible follow-up problems and questions:

Note: I will start with easy problems and gradually increase the difficulty of the questions. I do not plan to ask each one of the following problems. Depending on the interviewee's responses, I may ask three or four of these problems.

- Please tell me what you are thinking.

- How else might you solve this problem? Please tell me methods that could be used to solve this problem. (I will suggest using a drawing if students are stuck trying to calculate.)
- If Gear A had seven notches, then how many notches would Gear B have?
- If Gear B had a 2-cm radius, then how many notches would Gear B have?
- If Gear B had a $\frac{3}{4}$-cm radius, then how many notches would Gear B have?
- What if Gear A had 3-cm radius and 10 notches, and Gear B had a $\frac{6}{5}$-cm radius, then how many notches would Gear B have?
- What if Gear A had 5-cm radius and four notches, and Gear B had six notches, then what would be the size of Gear B?
- Now, assume we have two different gears, Gear D and Gear E that are also meshed and have 21 and 14 notches, respectively. If Gear D has a radius of 6 cm, then what is the radius of Gear E? (We are assuming that the sizes of the notches of Gear D and Gear E are the same.)
- If Gear E had 8 notches, then what would be the size of Gear E?

Note: I will continue with the following problems.

- Gear A and Gear B with 3-cm and 6-cm radius, respectively, are meshed with each other again. If Gear A has m notches, then please calculate the number of notches of Gear B in terms of m.
- What if Gear B had a 2-cm radius, then how many notches would it have?

If an interviewee cannot answer the problem above, then I will ask the following problem.

- What if Gear B had a 1-cm radius, then how many notches would it have?

- Now, consider what if Gear A had a r_1-cm radius and six notches, and Gear B had eight notches, then what would be the value of Gear B's radius (r_2) in terms of r_1?

Note: Although the following problem might be a little difficult for PSTs in a middle grades program, I will use this problem, since it may help me understand PSTs' abilities to solve problems when the numbers are not provided. This problem may also help me detect their reasoning for more complex situations.

- Gear X with a radius r_1-cm and Gear Y with a radius r_2-cm are meshed with each other. If Gear X has n_1 notches, then how many notches (n_2) does Gear Y have?
- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that could be used to solve this problem. (I will suggest using a drawing if students are stuck trying to calculate.)

Note: I will ask the following questions to each of the interviewees.

- Considering all of these problems, would you talk about any patterns or relationships, if any, between the size of a gear and the number of its notches?
- How did you determine your description of the relationship?
- Please represent the relationship you claim graphically.
- How else might you express this relationship?

Task 1B

Problem: Gear F and Gear K are two meshed gears where Gear F has 8 notches and Gear K has 4 notches. If Gear F revolves 3 times, then how many times does Gear K revolve?

Possible follow-up problems and questions:

Note: Depending on the interviewees' responses, I may ask three or four of the following problems.

- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that could be used to solve this problem. (I will suggest using a drawing if students are stuck trying to calculate.)
- What if Gear F completed $\frac{3}{2}$ revolutions, then how much revolutions would Gear K complete?
- What if Gear K had six notches, then how much revolutions would Gear K complete when Gear F completed three revolutions?
- How many notches should Gear K have, so when Gear F completes three revolutions, it can complete $\frac{2}{3}$ of a revolution? (We are assuming that Gear F and Gear K are meshed, and Gear F has eight notches.)
- Now, assume that you are given a new pair of gears (Gear L and Gear M) that are also meshed and have eight and 14 notches, respectively. If Gear M revolves four times, then how many revolutions does Gear M make?
- How many notches should Gear L have, so when Gear M completes four revolutions, it can complete $\frac{7}{2}$ revolutions?

Note: I will continue with the following problems.

- You are given Gear F and Gear K with eight and four notches, respectively. If Gear F made *p* revolutions, then determine the number of revolutions that Gear K made in terms of *p*.

- What if Gear K had 12 notches, then how many revolutions would it complete? (We are assuming that Gear F made *p* revolutions.)

If an interviewee cannot answer the problem above, then I will ask the following problem.

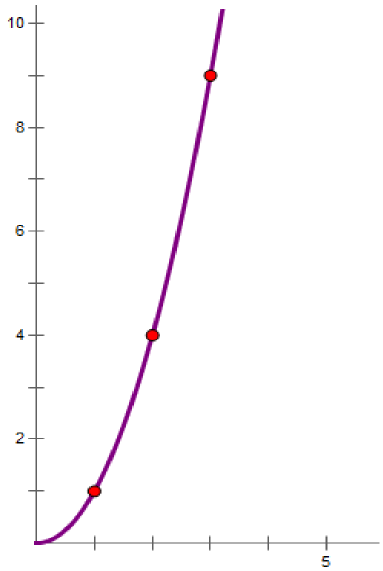
- What if Gear K had one notch, then how many revolutions would it complete? (We are assuming that Gear F made *p* revolutions.)
- When Gear F with *n* notches completed six revolutions, Gear K completed eight revolutions. Please determine the number of notches of Gear K in terms of *n*.

Note: Although the following problem might be a little difficult for PSTs in a middle grades program, I will use this problem, since it may help me understand PSTs' abilities to solve problems when the numbers are not provided. This problem may also help me detect their reasoning for more complex situations.

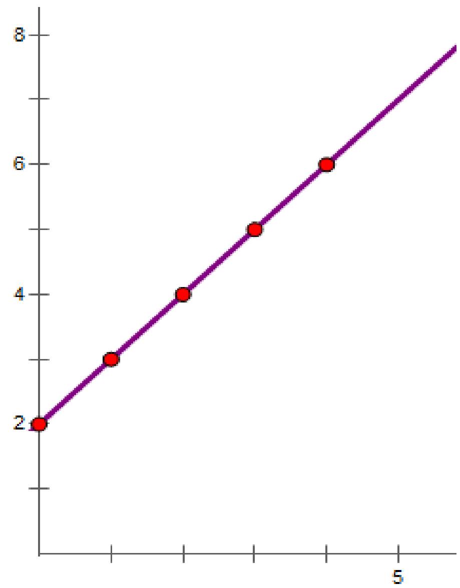
- Gear T has n_1 notches, and Gear Z has n_2 notches. If Gear T revolves R_1times, then how many times (R_2) does Gear Z revolve?
- Please tell me what you are thinking.

Note: I will ask the following questions to each interviewee.

- Considering all of these problems, would you talk about any patterns or relationships, if any, between the number of notches on a gear and the number of its revolutions?
- How did you determine your description of the relationship?
- Please represent the relationship you claim graphically.
- Please compare this relationship with the relationship that you identified in the previous task (Task 1A). Please explain similarities and differences.
- Please describe the relationships in the following three graphs and compare those relationships with each other.



(a)



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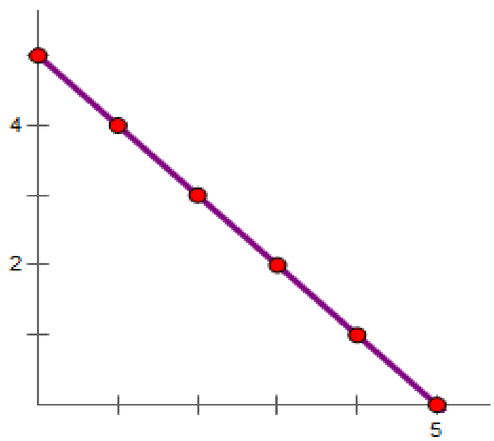

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Interview Protocol 2

Interviewer: Muhammet Arican

Discussion of goals: The goal for the second interview is to explore how preservice middle school mathematics teachers infer the directly proportional relationship between the number of people and the number of cupcakes they frost in some fixed time, the directly proportional relationship between the number of cupcakes frosted by some fixed number of people and the time needed to frost that many cupcakes, and an inversely proportional relationship between the number of people and the time they needed to frost some fixed number of cupcakes. During the second interview, I will focus on the knowledge resources that they use to detect the directly and inversely proportional relationships. Additionally, the types of strategies that they use to solve given proportion problems, the ways they represent the inversely proportional relationships, and the types of reasoning they engage in when solving these problems will be explored.

Materials: Each participant will be given printouts of the problems, paper, and pencils.

Introduction: I will let the participants know that I am most interested in their thinking processes. Also, I will tell them that this interview is not a test, so they may skip questions and may discontinue the interview at any time without explanation. I will ask them to work on the tasks and to think aloud while they are attempting to solve the problems. I will let them know that they can play with the plastic gears to solve the problems and encourage them to take their time. Before moving to the tasks, I will ask them if they have any questions for me. As the participants discuss their thinking, I will ask them follow-up questions.

Interview Tasks and Follow-up Questions

Task 2A

Problem: At a bakery, 3 people can frost a total of N cupcakes in T minutes. How many cupcakes can 6 people frost in the same T minutes? (Assume that all people work at the same steady pace.)

Possible follow-up problems and questions:

- Please tell me what you are thinking.

- How else might you solve this problem? Please tell me methods that could be used to solve this problem. (I will suggest using a drawing if students are stuck trying to calculate.)
- How many cupcakes can four people frost in T minutes?
- How many cupcakes can M people frost in T minutes?
- How many people are needed to frost $\frac{2N}{3}$ cupcakes in T minutes?
- How long would it take for three people to frost $\frac{3N}{2}$ cupcakes?

Note: If an interviewee cannot answer the initial problem or has difficulty with the follow up questions, then I will skip them and continue with the next problem and follow up questions. If he/she can answer it, then I will not introduce the next problem and continue with the last part of this task.

- At a bakery, three people can frost a total of 60 cupcakes in 12 minutes. How many cupcakes can one person frost in 12 minutes? (Assume that all people work at the same steady pace.)
- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that could be used to solve this problem.
- How many cupcakes can six people frost in 12 minutes?
- How many people are needed to frost 80 cupcakes in 12 minutes?
- How long would it take for three people to frost 90 cupcakes?

Note: I will ask the following questions to each interviewee.

- Considering all of these problems, would you talk about any patterns or relationships, if any, between the number of people and the number of cupcakes? What are they?

- How did you determine your description of the relationship?
- Please represent the relationship you claim with a drawing.
- Considering all of these problems, would you talk about any patterns or relationships, if any, between the number of cupcakes and the time? What are they?
- How did you determine your description of the relationship?
- Please represent the relationship you claim with a drawing.

Task 2B

Problem: At a bakery, 3 people can frost a total of N cupcakes in T minutes. How long would it take for 6 people to frost N cupcakes? (Assume that all people work at the same steady pace.)

Possible follow-up questions:

- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that could be used to solve this problem.
- How long would it take for two people to frost N cupcakes?
- How many people are needed to frost N cupcakes in $\frac{3T}{4}$ minutes?

Note: If an interviewee cannot answer the initial problem or has difficulty with the follow up questions, then I will skip them and continue with the next problem and follow up questions. If he/she can answer it, then I will not introduce the next problem and continue with the last part of this task.

- At a bakery, three people can frost a total of 60 cupcakes in 12 minutes. How long would it take for six people to frost 60 cupcakes? (Assume that all people work at the same steady pace.)
- How long would it take for one person to frost 60 cupcakes?

- How many people are needed to frost 60 cupcakes in 9 minutes?

Note: I will ask the following questions to each interviewee.

- Considering all of these problems, would you talk about any patterns or relationships, if any, between the number of people and the time? What are they?
- How did you determine your description of the relationship?
- Please represent the relationship you claim graphically.
- Please compare this relationship with the relationships in the previous task. Please explain the similarities and the differences.

Task 2C

Problem: At a bakery, 3 people can frost a total of N cupcakes in T minutes. How long would it take for 1 person to frost 2N cupcakes? (Assume that all people work at the same steady pace.)

- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that could be used to solve this problem.
- How many cupcakes can two people frost in $\frac{T}{2}$ minutes?
- How many people are needed to frost $\frac{2N}{3}$ cupcakes in $\frac{T}{3}$ minutes?

Note: If an interviewee cannot answer the initial problem or has difficulty with the follow up questions, then I will skip them and continue with the next problem and follow up questions.

- At a bakery, three people can frost a total of 60 cupcakes in 12 minutes. How long would it take for one person to frost 120 cupcakes? (Assume that all people work at the same steady pace.)
- Please tell me what you are thinking.

- How else might you solve this problem? Please tell me the ways that you think could be used to solve this problem.
- How many cupcakes can two people frost in 6 minutes? (Assuming that three people can frost a total of 60 cupcakes in 12 minutes, and they all work at the same steady pace.)
- How many people are needed to frost 40 cupcakes in 4 minutes? (Assuming that three people can frost a total of 60 cupcakes in 12 minutes.)
- Considering parts A, B, and C, please formulate an equation to express relationships among the number of people, the number of cupcakes, and the number of minutes?

Interview Protocol 3

Interviewer: Muhammet Arican

Discussion of goals: I will provide each preservice teacher with a mini number balance system, which is a simple version of an equal-arm beam balance scale. In this interview, the goal is to explore how PSTs infer an inversely proportional relationship between the number of weights hung and the distance (how far from the center a weight is hung). I will tell the PSTs to hang some number of weights on a number in one direction of the system and ask them to balance the system in the other direction. In addition, I will explain that they are allowed to hang weights on only one number to balance the system. I will also explore the knowledge resources that PSTs use to detect the inversely proportional relationship, the types of strategies that they use to solve given proportion problems, the ways they represent the inversely proportional relationship, and the types of reasoning they engage in when solving these problems.

Materials: Each participant will be given printouts of the problems, paper, pencils, and a mini number balance system.

Introduction: I will let the participants know that I am most interested in their thinking processes. Also, I will tell them that this interview is not a test, so they may skip questions and may discontinue the interview at any time without explanation. I will ask them to work on the tasks and to think aloud when they are attempting to solve the problems. I will let them know that they can play with the plastic gears to solve the problems and encourage them to take their time. Before moving to the tasks, I will ask them if they have any questions for me. As the participants discuss their thinking, I will ask them follow-up questions.

Interview Tasks and Follow-up Questions

Task 3

Problem: You are given that W_1 number of weights were hung on a number that has a D_1 distance from the center on one side of the balance system. To balance the system on the other side, you want to hang W_2 number of weights. What would be the distance (D_2) in terms of D_1, W_1, and W_2, so you could balance the system?

Possible follow-up problems and questions:

- Please tell me what you are thinking.

If the interviwee obtain $D_1 * W_1 = D_2 * W_2$ equation, then I will ask the following question:

- How did you obtain this equation? Why do you think the product of D_1 and W_1 is equal to the product of D_2 and W_2?
- How else might you express this equation?

Note: If the interviewee cannot answer the problem, I will skip it and continue with the following problems.

- If you are given that $W_1 = 6$ weights, $D_1 = a\ cm$, and $W_2 = 8$ weights, then what would be the value of D_2 in terms of a?

- Please tell me how do you make sense of D_2 in terms of given number of weights and a?
- If you are given that $D_1 = 4\ cm$, $D_2 = 3\ cm$, and $W_1 = m$ weights, then what would be the value of W_2 in terms of m?
- Please tell me how do you make sense of W_2 in terms of given distances and m?

Note: If the interviewee cannot answer the problems above, then I will skip these problems and continue with the following problems.

- If you are given that $D_1 = 4\ cm$, $D_2 = 3\ cm$, and $W_2 = 8$ weights, then what would be the value of W_1?
- How did you obtain 24?
- Please tell me more about why you think hanging them this way works.

Note: I will ask the following questions to each interviewee.

- If you hang six weights on number eight on one side, then please show me different ways of balancing the system on the other side?
- Please express these different ways of balancing the system with a ratio table.
- Do you recognize anything significant in your table? Please explain what you recognize as significant.
- Would you talk about any patterns or relationships, if any, between the number of weights and the place where the weights were hung? What are they?
- How did you determine your description of the relationship?
- Please represent the relationship you claim graphically.
- Do you recognize anything significant in your graph? Please explain what you recognize as significant.

Interview Protocol 4

Interviewer: Muhammet Arican

Discussion of goals: Tasks 4, 5, and 6 involve multiple proportional relationships. The goal for Task 4 is to explore how preservice middle school mathematics teachers infer the directly or inversely proportional relationships among the speed, distance, and time. In Task 5, PSTs' determination of two directly proportional relationships and one inversely proportional relationship will be studied. In Task 6, PSTs' determination of three inversely proportional relationships will be explored. Because these tasks involve multiple proportional relationships, I expect difficulties from PSTs in solving these problems. Since it is difficult to set up proportions and use the cross-multiplications strategy to solve the problems in these tasks, by using these tasks, I expect to detect PSTs' reasoning for complex situations. During the interview, I will focus on the knowledge resources that they use to detect directly and inversely proportional relationships. Additionally, the types of strategies that they use to solve given proportion problems, the ways they represent the directly and inversely proportional relationships, and the types of reasoning they engage in when solving these problems will be explored.

Materials: Each participant will be given printouts of the problems, paper, and pencils.

Introduction: I will let the participants know that I am most interested in their thinking processes. Also, I will tell them that this interview is not a test, so they may skip questions and may discontinue the interview at any time without explanation. I will ask them to work on the tasks and to think aloud when they are attempting to solve the problems. I will let them know that they can play with the plastic gears to solve the problems and encourage them to take their time. Before moving to the tasks, I will ask them if they have any questions for me. As the participants discuss their thinking, I will ask them follow-up questions.

Interview Tasks and Follow-up Questions

Task 4

If you covered the distance between two markers in 90 seconds driving at 60 mph and if you want to cover the same distance in 60 seconds, then what must be your speed?

Possible follow-up questions:

- Please tell me what you are thinking.
- Please calculate how much miles you covered in 90 seconds driving at 60 mph.
- How long would it take someone to cover the same distance if she/he walks 1 mile per hour? (Assume she/he walks at a steady pace.)

Note: I will ask the following questions to each interviewee.

- If you cover 2 miles in 100 seconds, then what is the speed of your car in miles per hour?
- If you traveled 40 miles in 50 minutes, then how long would it take you to travel 16 miles?
- You covered the distance (*D*) between two markers in 90 seconds driving at V mph. If you want to cover the same distance in 60 seconds, then what must your speed be in terms of *V*?
- How else might you solve this problem? Please tell me methods that could be used to solve this problem.
- Please calculate how many miles you covered in 90 seconds driving at V mph in terms of *V*.
- If you reduce your speed to $\frac{2V}{3}$ mph, then how long would it take you to cover the same distance (D)?

- Would you talk about any patterns or relationships, if any, among the speed of a car, the distance it travels, and the amount of time traveled? What are they?
- Please generate a formula to express these relationships.

Extra Tasks:

Task 5: Direct-Direct-Inverse Proportional Relationships Problem

If three people take two days to paint five fences, how long will it take two people to paint one fence? (Assume that the fences are all the same size, and the painters work at the same rate.)

Possible follow-up questions:

- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that could be used to solve this problem.
- Assuming that three painters painted five fences in two days, how many fences can six painters paint in three days?
- Assuming that three painters painted five fences in two days, how many painters do we need to paint 10 fences in three days?
- Would you talk about any patterns or relationships, if any, among the number of painters, the number of fences painted, and the number of days? What are they?

Task 6: Inverse-Inverse-Inverse Proportional Relationships Problem

A total of 20 people went on a scout camping trip. If each person consumes $\frac{1}{2}$ of a pound of cereal every day, then the cereal will last for 12 days. On the first day, they decided to extend the length of their stay from 12 days to 16 days. How much cereal should each person eat so that the cereal will last for 16 days?

Possible follow-up questions:

- Please tell me what you are thinking.
- How else might you solve this problem? Please tell me methods that you could be used to solve this problem.
- If each of the 20 people consumed $\frac{1}{4}$ of a pound of cereal every day, how many days would the cereal last?
- If each of the 20 people consumed $\frac{3}{2}$ of a pound of cereal very day, how many days would the cereal last?
- If 10 more students joined the scout group the day before they went camping, how much cereal could each person eat so that the cereal would last for 12 days?
- Would you talk about any patterns or relationships, if any, among the number of people, cereal consumption per person, and the length of their stay?