Moduli Spaces of Vector Bundles on Curves

by

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(Under the direction of Elham Izadi)

Abstract

In this work, we generalize Bertram's work on rank two vector bundles on a smooth irreducible projective curve to an irreducible singular curve $C$ with only one node as singularity. We resolve the indeterminancy of the rational map $\phi_L \colon \mathbb{P}(\operatorname{Ext}^1_C(L, \mathcal{O}_C)) \to \overline{\mathcal{SU}_C(2, L)}$ defined by $\phi_L([0 \to \mathcal{O}_C \to E \to L \to 0]) = E$ in the case $\deg L = 3, 4$ via a sequence of three blow-ups with smooth centers. Here $\overline{\mathcal{SU}_C(2, L)}$ is a compactification of the moduli space $\mathcal{SU}_C(2, L)$ of semi-stable vector bundles of rank 2 and determinant $L$ on $C$.

An additional part is on the base locus of the generalized theta divisor $\Theta_r$ on $\mathcal{SU}_C(r, L)$ for a smooth curve $C$. Among our results, we show, using results of Raynaud, that the base locus is always non-empty when $r \geq 2^g$ and $L = \mathcal{O}_C$.

Index words: Algebraic Geometry, Algebraic Curve, Vector Bundle, Moduli Space, Blow-up

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## Table of Contents

# Introduction

The moduli spaces of semi-stable vector bundles with trivial determinant on a smooth curve are one of the links between Mathematics and Physics which has been studied at the end of the twentieth century and at the beginning of the current one. There are several formulas about them which are known to the physicists, but that are still to be proved by mathematicians, and this is the main motivation for the following work. We want to understand those moduli spaces and their "cousins," which would be some similar and related moduli spaces. Also, we want to extend known constructions to the case of singular curves.

The moduli spaces of semi-stable vector bundles on a smooth curve were constructed in the sixties by several people, including Mumford, Narasimhan, Ramanan, and Seshadri. They are well understood in the case when the genus of the curve is either 0 or 1, thanks to work of Grothendieck and Atiyah. In the case of a singular curve, the moduli spaces were constructed in the seventies by Newstead and Seshadri, together with their natural compactifications using torsion-free sheaves.

The first part of this work deals with a construction of Bertram which uses extensions of line bundles to study rank-2 vector bundles of fixed determinant on a smooth curve. We generalize his construction to an irreducible curve $C$ with one node in the particular case when the degree of the fixed determinant is 3 or 4. The idea is to look at extensions of $L$ by $\mathcal{O}_C$, where $L$ is a generic line bundle of degree 3 or 4, and consider the 'forgetful' map which sends an extension to the vector bundle of rank 2 in the middle, forgetting the extension maps. This gives a rational map from $\mathbb{P}(\operatorname{Ext}^1_C(L, \mathcal{O}_C))$ to $\overline{\mathcal{SU}_C(2, L)}$, a compactification of the moduli space

$\overline{\mathcal{SU}_C(2,L)}$ of semi-stable vector bundles of rank 2 and determinant $L$. We resolve the indeterminancy of the map by a sequence of three blow-ups with smooth centers. A nice aspect of these blow-ups is that there exists at each stage a 'universal bundle' which gives the rational map in a natural way.

In the first chapter, we describe the rational map, and we relate coherent sheaves on the curve to coherent sheaves on its normalization, $N$, with some extra structure. The image of the rational map (on its domain of definition) is contained in the dense open subset $\mathcal{SU}_C(2,L)$ of $\overline{\mathcal{SU}_C(2,L)}$. In the second chapter, we blow-up the extension space at a point, and we describe the new rational map. The curve is naturally embedded in $\mathbb{P}(\operatorname{Ext}^1_C(L,\mathcal{O}_C))$, and we blow-up this space at the singular point of the curve. The image of this map contains all non-trivial extensions of $L\otimes(\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$ which are push-forwards of extensions from the normalization. In the third chapter, we blow-up the new space along a line. The image of the new rational map contains all non-trivial locally-free extensions of $L\otimes(\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$ of determinant $L$. Finally, in the fourth chapter, we blow-up the new space along the strict transform of the curve. The induced map is a morphism, and the image contains all non-trivial extensions of $L(-q)$ by $\mathcal{O}_C(q)$, where $q$ is any smooth point on the curve.

The second part of this work studies the base locus of the generalized theta divisor on the moduli space of vector bundles of fixed rank and determinant on a smooth curve. The generalized theta divisor is any divisor whose associated line bundle is the unique ample generator of the Picard group of the moduli space.

In the fifth chapter, we give a short survey of the known results, and we prove two new results. Among our results, we show that, if the degree of the vector bundle is a multiple of the rank, and the rank is at least $2^g$, then the base locus is non-empty, and we give a lower bound for its dimension. Moreover, we show that the base locus is non-empty for smaller ranks if the curve is a covering of another smooth curve.

The sixth and final chapter is a report on a conjecture introduced by Donagi and Tu which generalizes a previous conjecture known to the physicists. The conjecture expresses a duality between sections of multiples of the generalized theta divisors on different moduli spaces of vector bundles. After reformulating the conjecture, and supplying more evidence for it, we show how the conjecture would help in answering questions about the base locus of the generalized theta divisor.

The appendices contain technical results needed in the first part of this work that we did not want to include there to make the exposition more readable. The first appendix studies extension spaces of torsion-free sheaves of rank 1 on a nodal curve with one node, the second one studies similar extension spaces on the product of a smooth variety with the curve, and the third one contains a result that we need to prove the linearity of certain restrictions of our rational maps.

# Chapter 1

# Getting started

## 1.1 Background and notation

We extend Bertram's construction (see [Ber89], [Ber92]) to the case of an irreducible projective curve with one node, i.e., we use extensions to study the moduli spaces of (S-equivalence classes[1] of) semi-stable[2] vector bundles of rank 2. Let

| | |
|---|---|
| $C$ | an irreducible projective curve with one node $p$ as singularity |
| $\pi\colon N \to C$ | the normalization of $C$ |
| $p_1, p_2$ | the two points on $N$ which map to $p$ in $C$ |
| $g \geq 2$ | the arithmetic genus of $C$ |
| $L$ | a generic line bundle on $C$ |
| $\mathcal{SU}_C(2,L)$ | the moduli space of (S-equivalence classes of) semi-stable vector bundles of rank 2 and determinant $L$ on $C$ |
| $\overline{\mathcal{SU}_C(2,L)}$ | the compactification of $\mathcal{SU}_C(2,L)$ via torsion-free sheaves (see [Ses82] and [New78]) |
| $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ | the vector space of extensions of $L$ by $\mathcal{O}_C$ |
| $\mathbb{P}_L$ | the projectivization of $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ |
| $\langle S \rangle$ | the vector subspace of a vector space $V$ corresponding to a linear subspace $S \subseteq \mathbb{P}(V)$ |
| $T_x X$ | the tangent space to a variety $X$ at the point $x$ |

[1] Two vector bundles $E_1$ and $E_2$ of rank 2 are S-equivalent if there exists a line bundle $F$ such that $2\deg F = \deg E_1 = \deg E_2$, $F \subseteq E_1$ and either $F \subseteq E_2$ or $E_2/F \subseteq E_2$.

[2] A bundle $E$ of rank 2 is semi-stable if $2\deg F \leq \deg E$ for every line subbundle $F \subseteq E$.

The only exception to the table is that we shall use $T_pC$ to denote the projective tangent plane to the curve at $p$, which is a 2-dimensional linear subspace of $\mathbb{P}_L$, instead of the tangent plane itself.

Also, we shall abuse the notation at times. For example, a point $x$ in $\mathbb{P}_L$ corresponds to a line through the origin in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$. If one extension on such a line is $0 \to \mathcal{O}_C \to E_x \to L \to 0$, then all the non-trivial extensions on the line have the middle vector bundle isomorphic to $E_x$. Therefore, we shall sometimes refer to $E_x$ as the "extension in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ corresponding to $x$." In other words, even if there does not exist a unique extension determined by $x$, the middle vector bundles in the extensions corresponding to $x$ are all isomorphic, and we shall refer to any of them as the "extension corresponding to $x$."

Finally, for every extension space $\operatorname{Ext}^1(G, F)$ and every map $F \to F'$ [resp. $G' \to G$], there exists a linear homomorphism $\operatorname{Ext}^1(G, F) \to \operatorname{Ext}^1(G, F')$ [resp. $\operatorname{Ext}^1(G, F) \to \operatorname{Ext}^1(G', F)$] given by the pull-back

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

[resp. the push-forward

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F & \longrightarrow & E' & \longrightarrow & G' & \longrightarrow & 0 \end{array} \quad ].$$

We shall sometimes refer to these linear homomorphisms as "natural linear homomorhisms."

## 1.2 Description of the rational map

We study the rational map $\phi_L \colon \mathbb{P}_L \to \overline{\mathcal{SU}_C(2, L)}$ defined as follows. Let $x \in \mathbb{P}_L$. Then $x$ corresponds to a line through the origin in the vector space $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$.

Each point on the line corresponds to an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ and all the $E$'s (except for the one corresponding to the origin) are isomorphic. Define $\phi_L(x)$ to be the equivalence class of $E$ in $\overline{\mathcal{SU}_C(2,L)}$, that we shall denote by $E$.

Note that $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ is isomorphic to $H^0(C, L \otimes \omega_C)^*$ (see [Har77, III.6.7, III.7.6]). Therefore, the linear system $|L \otimes \omega_C|$ defines a rational map

$$\varphi_{L\otimes\omega_C} : C \longrightarrow |L \otimes \omega_C|^* \simeq \mathbb{P}_L.$$

Let $U_L \subseteq \mathbb{P}_L$ be the open locus of semi-stable extensions.

**Proposition 1.1.** (1) *If* $\deg L < 0$, *then* $U_L = \emptyset$.

(2) *If* $0 \leq \deg L \leq 2$, *then* $U_L = \mathbb{P}_L$.

(3) *If* $3 \leq \deg L \leq 4$, *then* $U_L = \mathbb{P}_L \setminus \varphi_{L\otimes\omega_C}(C)$.

*Remark.* The same is true in the case of a smooth curve (see [Ber92, section 3]).

*Proof.* (1) If $\deg L < 0$, then every extension $0 \to \mathcal{O}_C \to E \to L \to 0$ of $L$ by $\mathcal{O}_C$ is not semi-stable because $\mathcal{O}_C \subseteq E$ and $\deg \mathcal{O}_C = 0 > \deg L = \deg E$.

(2) This is equivalent to the following statement: If $0 \leq \deg L \leq 2$, then every non-trivial extension of $L$ by $\mathcal{O}_C$ is semi-stable. Let $0 \to \mathcal{O}_C \to E \to L \to 0$ be a non-trivial extension of $L$ by $\mathcal{O}_C$, and suppose that $E$ is not semi-stable. Then there exists a torsion-free quotient $F$ of $E$ of rank 1 such that $2 \deg F < \deg E \leq 2$ [3]. Therefore, $\deg F \leq 0$.

Case I: $\deg F < 0$. Then the composition $\mathcal{O}_C \hookrightarrow E \to F$ is the zero map, and the map $E \to F$ factors through $L$. But this is not possible because $\deg F < 0 \leq \deg L$ and every map from $L$ to $F$ is zero[4].

[3]This is a standard result about non-semi-stable torsion-free coherent sheaves (see, for example, [New78]).

[4]This is clear if $F$ is locally-free, and the same proof holds for a torsion-free sheaf (see [LeP97, 5.3.3]).

Case II: $\deg F = 0$. Consider the composition $\mathcal{O}_C \hookrightarrow E \to F$. If this is the zero map, we obtain a contradiction as in case I. If it is not the zero map, then it has to be an isomorphism[5], and the extension is trivial.

(3) Let $3 \leq \deg L \leq 4$, and let $0 \to \mathcal{O}_C \to E \to L \to 0$ be a non-trivial extension of $L$ by $\mathcal{O}_C$. If $E$ is not semi-stable, then there exists a torsion-free quotient $F$ of $E$ such that $2 \deg F < \deg E \leq 4$. Therefore, $\deg F \leq 1$. An argument similar to that of part (2) above shows that $\deg F$ cannot be $\leq 0$, and therefore $\deg F = 1$. Since $\deg F < \deg L$, the composition $\mathcal{O}_C \hookrightarrow E \to F$ cannot be the zero map (or $E \to F$ would factor through $L$), and this implies that $\mathcal{O}_C \hookrightarrow F$. If $F$ is locally-free, then $F = \mathcal{O}_C(q)$ for some $q \in C \setminus \{p\}$. If $F$ is not locally-free, then $F = \pi_* \mathcal{O}_N$ [6]. The rest of the proof follows from the following lemma. $\square$

**Lemma 1.2.** (1) *If $q$ is a smooth point of $C$ (i.e., $q \neq p$), then*

$$\varphi_{L\otimes\omega_C}(q) = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_q} \operatorname{Ext}^1_C(L, \mathcal{O}_C(q)))).$$

*Moreover,* $\varphi_{L\otimes\omega_C}(p) = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_p} \operatorname{Ext}^1_C(L, \pi_* \mathcal{O}_N)))$.

(2) *Let $0 \to \mathcal{O}_C \to E \to L \to 0$ be a non-trivial extension, and let $F$ be a torsion-free sheaf with $\deg F < \deg L$. If there exists a surjective map $E \to F$, then $E \in \ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_F} \operatorname{Ext}^1_C(L, F))$. The converse is true if $\mathcal{O}_C \subseteq F$ and $E \notin \ker \psi_G$ for every torsion-free sheaf $G$ such that $\mathcal{O}_C \subseteq G \subsetneq F$.*

*Remark.* The linear homomorphisms $\psi_q$, $\psi_p$, and $\psi_F$ are the natural push-forward maps defined by the inclusions of $\mathcal{O}_C$ into $\mathcal{O}_C(q)$, $\pi_* \mathcal{O}_N$, and $F$, respectively.

[5] Since the image is a subsheaf of the torsion-free sheaf $F$, if the map is non-zero, then the image has rank 1 and degree $\leq \deg \mathcal{O}_C = 0$; since this is also the degree of $F$, the map is surjective. It is clearly also injective, since $\mathcal{O}_C$ is torsion-free.

[6] A torsion-free non-locally-free coherent sheaf $F$ of rank 1 on $C$ is of the form $\pi_* \mathcal{F}$ for some line bundle $\mathcal{F}$ on $N$ (see [Ses82, VII.2, VII.10]). In our case, since $\deg F = 1$, $\deg \mathcal{F} = 0$, and $\mathcal{O}_C \hookrightarrow F$ implies that $\mathcal{O}_N = \pi^* \mathcal{O}_C \hookrightarrow \mathcal{F}$. Therefore, $\mathcal{F} = \mathcal{O}_N$.

*Proof.* (1) If $q \neq p$, then $\varphi_{L\otimes\omega_C}(q)$ is the hyperplane $H^0(C, L \otimes \omega_C(-q))$ in $H^0(C, L \otimes \omega_C)$. In particular, it corresponds to the kernel of the dual linear homomorphism $H^0(C, L\otimes\omega_C)^* \to H^0(C, L\otimes\omega_C(-q))^*$. This homomorphism is the natural linear homomorphism $\psi_q$ when we identify $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ [resp. $\operatorname{Ext}^1_C(L, \mathcal{O}_C(q))$] with $H^0(C, L \otimes \omega_C)^*$ [resp. $H^0(C, L \otimes \omega_C(-q))^*$].

If $q = p$, then $\varphi_{L\otimes\omega_C}(p)$ is the hyperplane of $H^0(C, L\otimes\omega_C)$ defined by the sections vanishing at $p$. Since the sheaf generated by the regular functions which vanish at $p$ is the sheaf $\pi_*(\mathcal{O}_N(-p_1 - p_2))$ and its dual is $\pi_*\mathcal{O}_N$, $\varphi_{L\otimes\omega_C}(p)$ corresponds to the kernel of the linear homomorphism $H^1(C, L^{-1}) \to H^1(C, L^{-1} \otimes \pi_*\mathcal{O}_N)$, where we identified $H^0(C, L\otimes\omega_C)^*$ with $H^1(C, L^{-1})$ (see [Har77,III.7.7]). If $G$ is any coherent sheaf, we can identify $\operatorname{Ext}^1_C(L, G)$ with $H^1(C, L^{-1}\otimes G)$ using [Har77, III.6.3, III.6.7], and the linear homomorphism above becomes $\psi_p$ as claimed.

(2) Let $f$ be the surjective map $E \to F$, and consider its composition with the inclusion $\mathcal{O}_C \hookrightarrow E$. This is a non-zero map $\mathcal{O}_C \hookrightarrow F$. Otherwise, $E \to F$ would factor through $L$ and this is not possible because $\deg F < \deg L$ and $F$ is torsion-free. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & F & \longrightarrow & \psi_F(E) & \longrightarrow & L & \longrightarrow & 0 \end{array},$$

and we need to prove that $\psi_F(E)$ splits. The surjective map $(-f + \mathrm{id})\colon E \oplus F \to F$ contains $\mathcal{O}_C$ in its kernel, and therefore it induces a surjective map from the quotient $\psi_F(E) = (E \oplus F)/\mathcal{O}_C$ to $F$. This surjective map, composed with the inclusion $F \hookrightarrow \psi_F(E)$ gives the identity on $F$, proving that $\psi_F(E)$ is the trivial extension.

For the converse, assume that $\mathcal{O}_C \subseteq F$, $E \in \ker\psi_F$, and $E \notin \ker\psi_G$ for every $\mathcal{O}_C \subseteq G \subsetneq F$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & F & \longrightarrow & F \oplus L & \longrightarrow & L & \longrightarrow & 0 \end{array}.$$

The composition $E \to F \oplus L \to F$, where the second map is the projection onto the first factor, must be surjective. Otherwise, its image would be a torsion-free sheaf $G$ such that $\mathcal{O}_C \subseteq G \subsetneq F$. Indeed, $\mathcal{O}_C$ is contained in the image because the composite map $\mathcal{O}_C \hookrightarrow E \hookrightarrow F \oplus L \to F$ is non-zero. □

From now on, we shall restrict ourselves to the first non-trivial case, i.e., when $\deg L$ is either 3 or 4.

**Lemma 1.3.** *If* $\deg L \geq 3$, *then* $\varphi_{L\otimes\omega_C}$ *is an embedding.*

*Remark.* Since $\varphi_{L\otimes\omega_C}$ is an isomorphism onto its image, we shall identify $C$ with $\varphi_{L\otimes\omega_C}(C) \subseteq \mathbb{P}_L$.

*Proof.* By [Bar87, 3.14-3.18], we need to prove that, for every $q, q' \in C$ not both equal to $p$,

$$H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) = H^1(C, L \otimes \omega_C) = 0,$$

where $\mathcal{I}_q$ [resp. $\mathcal{I}_{q'}$] is the ideal sheaf of the point $q$ [resp. $q'$] [7], and that[8]

$$H^1(C, L \otimes \omega_C \otimes \mathcal{I}_p^2) = H^1(C, L \otimes \omega_C) = 0.$$

Case I: $q, q' \neq p$. Then, by Serre duality,

$$H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) \simeq H^0(C, L^{-1}(q + q'))^*,$$

which is zero because $\deg(L^{-1}(q + q')) < 0$.

Case II: $q \neq p$ and $q' = p$. Since $\pi^*\omega_C \simeq \omega_N(p_1 + p_2)$ (see [Bar87, 3.7]), using the projection formula we obtain $\omega_C \otimes \mathcal{I}_p \simeq \pi_*\omega_N$, and

$$L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_p \simeq L(-q) \otimes \pi_*\omega_N \simeq \pi_*(\pi^*(L(-q)) \otimes \omega_N).$$

[7] If $q \neq p$, then $\mathcal{I}_q = \mathcal{O}_C(-q)$. If $q = p$, then $\mathcal{I}_p = \pi_*(\mathcal{O}_N(-p_1 - p_2))$.

[8] The case of $\mathcal{I}_p^2$ is a little different because $\mathcal{I}_q^2 = \mathcal{I}_q \otimes \mathcal{I}_q$ for a smooth point $q$, but $\mathcal{I}_p^2 \neq \mathcal{I}_p \otimes \mathcal{I}_p$ for the node $p$. We have that $\mathcal{I}_p^2 = \pi_*\omega_N(-2p_1 - 2p_2)$, and it is equal to $(\mathcal{I}_p \otimes \mathcal{I}_p)/\mathrm{Tors}$.

Therefore,

$$H^1(C, L \otimes \omega_C \otimes \mathcal{I}_q \otimes \mathcal{I}_{q'}) \simeq H^1(N, \pi^*(L(-q)) \otimes \omega_N) \simeq H^0(N, \pi^*(L^{-1}(q)))^*$$

(see [Har77, Ex. III.4.1]), which is zero because $\deg(\pi^*(L^{-1}(q))) < 0$.

Case III: $q = q' = p$. Then $\omega_C \otimes \mathcal{I}_p^2 \simeq \pi_*(\omega_N(-p_1 - p_2))$ and

$$L \otimes \omega_C \otimes \mathcal{I}_p^2 \simeq \pi_*(\pi^* L \otimes \omega_N(-p_1 - p_2)).$$

Therefore,

$$H^1(C, L \otimes \omega_C \otimes \mathcal{I}_p^2) \simeq H^1(N, \pi^* L \otimes \omega_N(-p_1 - p_2)) \simeq H^0(N, \pi^* L^{-1}(p_1 + p_2))^*,$$

which is zero because $\deg(\pi^* L^{-1}(p_1 + p_2)) < 0$. $\square$

*Remark.* Lemma 1.2 shows that a smooth point $q \in C \subseteq \mathbb{P}_L$ is the point $\mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_q} \operatorname{Ext}^1_C(L, \mathcal{O}_C(q))))$ and the singular point $p$ is the point $\mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_p} \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)))$.

**Lemma 1.4.** *The projective tangent plane to $C$ at $p$ is*

$$T_pC = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_{1,2}} \operatorname{Ext}^1_C(L, \pi_*(\mathcal{O}_N(p_1 + p_2))))).$$

*Proof.* It is easy to see that all the kernels involved in this proof have the right dimension (for a formal proof, see lemma A.1). The line between the singular point $p$ and a smooth point $q$ is given by $\mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N(q))))$, this being a 1-dimensional linear subspace of $\mathbb{P}_L$ which contains both $p$ and $q$. If we take the limit as $p \mapsto q$ along the branch corresponding to $p_i$ $(i = 1, 2)$, we see that the projective tangent line at $p$ to that branch is

$$X_{p_i} := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_C(L, \pi_*(\mathcal{O}_N(p_i))))) \quad (i = 1, 2).$$

Since $\mathbb{P}(\ker(\psi_{1,2}))$ is a 2-dimensional linear subspace of $\mathbb{P}_L$ which contains both $X_{p_1}$ and $X_{p_2}$, it is the projective tangent plane $T_pC$ to $C$ at $p$. $\square$

We end this section with an important way to describe the rational map $\phi_L$ [9].

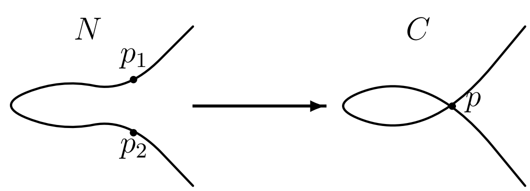
**Proposition 1.5.** *There exists a locally-free sheaf $\mathcal{E}_L$ on $\mathbb{P}_L \times C$ such that*

$$\mathcal{E}_L|_{\{x\}\times C} \simeq \phi_L(x)$$

*for every $x \in \mathbb{P}_L \setminus C$. Moreover, $\mathcal{E}_L$ is an extension in $\operatorname{Ext}^1_{\mathbb{P}_L\times C}(L, \mathcal{O}_{\mathbb{P}_L}(1))$, and for every $a \neq 0$ in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$, if we identify $\pi^*_{\mathbb{P}_L}\mathcal{O}_{\mathbb{P}_L}(1))|_{\{[a]\}\times C}$ with $\mathcal{O}_C$ using $a$, $\mathcal{E}_L$ restricts on $\{[a]\} \times C$ to the extension $a$ itself.*

*Proof.* The vector space of extensions $\operatorname{Ext}^1_{\mathbb{P}_L\times C}(L, \mathcal{O}_{\mathbb{P}_L}(1))$ is isomorphic to $\operatorname{Hom}(\operatorname{Ext}^1_C(L, \mathcal{O}_C), \operatorname{Ext}^1_C(L, \mathcal{O}_C))$ by lemma B.1. Let $\mathcal{E}_L$ be the extension corresponding to the identity homomorphism. Then, if $a \neq 0$ is an extension $0 \to \mathcal{O}_C \to E_a \to L \to 0$, $\mathcal{E}_L|_{\{[a]\}\times C}$ is $E_a$ by corollary B.2. $\square$

## 1.3 Relating sheaves on $C$ to sheaves on $N$




Every extension $0 \to \mathcal{O}_C \to E \to L \to 0$ of $L$ by $\mathcal{O}_C$ pulls-back to an extension $0 \to \mathcal{O}_N \to \pi^*E \to \pi^*L \to 0$ of $\pi^*L$ by $\mathcal{O}_N$ on $N$, because $L$ is locally-free. Therefore, if we denote by $\mathcal{L}$ the line bundle $\pi^*L$ on $N$, there exists a natural map $\pi^* \colon \operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$.

**Lemma 1.6.** *If we identify $\operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ with $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$ via the projection formula, the map $\pi^* \colon \operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ is the natural linear homomorphism $\psi_p \colon \operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$.*

[9] In the proposition, $\operatorname{Ext}^1_{\mathbb{P}_L\times C}(L, \mathcal{O}_{\mathbb{P}_L}(1))$ actually denotes $\operatorname{Ext}^1_{\mathbb{P}_L\times C}(\pi^*_C L, \pi^*_{\mathbb{P}_L}\mathcal{O}_{\mathbb{P}_L}(1))$. For an explanation of the notation, see appendix B.

*Proof.* Note that $\pi^*$ is linear. Indeed, the diagrams which define the operations on $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ pull-back to the diagrams which define the operations on $\operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ for the corresponding images. Hence, since[10] the dimension of $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ is equal to $g-1+\deg L$, which is equal to $\dim \operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)+1$, it is enough to show that $\pi^*$ has the same kernel as $\psi_p$. Since both kernels are 1-dimensional, it suffices to show that $\ker \psi_p$ is contained in $\ker \pi^*$.

We saw in lemma 1.2 that if an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ is in the kernel of $\psi_p$, then $E$ has $\pi_*\mathcal{O}_N$ as quotient, and therefore $\pi^*E$ has quotient $\mathcal{O}_N = (\pi^*(\pi_*\mathcal{O}_N))/\mathrm{Tors}$. Since $0 \to \mathcal{O}_C \to E \to L \to 0$ is in the kernel of $\pi^*$ if and only if $\pi^*E$ has $\mathcal{O}_N$ as quotient, the two kernels are the same. □

*Remark.* The linear homomorphism $\pi^*\colon \operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ gives a rational map $\pi^*\colon \mathbb{P}_L \to \mathbb{P}_{\mathcal{L}}$, where $\mathbb{P}_{\mathcal{L}}$ is $\mathbb{P}(\operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N))$. This map is just the projection from $p$.

The following lemma describes the fibers of $\pi^*$. For every vector bundle $\mathcal{G}$ of rank $r$ on $N$ and for every $r$-dimensional subspace $F(G)$ of $\mathcal{G}|_{p_1} \oplus \mathcal{G}|_{p_2}$, define a torsion-free sheaf $G$ on $C$ by

$$G := \ker(\pi_*\mathcal{G} \longrightarrow ((\mathcal{G}|_{p_1} \oplus \mathcal{G}|_{p_2})/F(G))).$$

Bhosle shows in [Bho92] that the correspondence $(\mathcal{G}, F(G)) \mapsto G$ is one-to-one if $G$ is locally-free. I.e., if $G$ is locally-free, then there exists a unique pair $(\mathcal{G}, F(G))$ which satisfies the equality above. In such a case, $\mathcal{G} = \pi^*G$.

**Lemma 1.7.** *Let $0 \to \mathcal{O}_N \to \mathcal{E} \to \mathcal{L} \to 0$ be a non-trivial extension of $\mathcal{L}$ by $\mathcal{O}_N$, and let $F(E)$ be a 2-dimensional vector subspace of $\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2}$. Then the torsion-free sheaf $E := \ker(\pi_*\mathcal{E} \to (\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2})/F(E))$ is in $\operatorname{Ext}^1_C(L, \mathcal{O}_C)$ if and only if $F(E) \cap (\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2}) = F(\mathcal{O}_C)$ and the image of $F(E)$ under the homomorphism*

[10] For calculations of dimensions of extension spaces, see appendix A.

*$\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2} \longrightarrow \mathcal{L}|_{p_1} \oplus \mathcal{L}|_{p_2}$ is contained in $F(L)$. There is a $\mathbb{P}^1 \setminus \{\mathrm{pt}\}$ of such $F(E)$'s, and the corresponding $E$'s are all the non-trivial extensions of $L$ by $\mathcal{O}_C$ such that $\pi^* E = \mathcal{E}$.*

*Proof.* We need to find, for a fixed $\mathcal{E} \in \operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$, which $F(E)$ give a commutative diagram as the following.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \pi_*\mathcal{O}_N & \longrightarrow & \dfrac{\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2}}{F(\mathcal{O}_C)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \pi_*\mathcal{E} & \longrightarrow & \dfrac{\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2}}{F(E)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \pi_*\mathcal{L} & \longrightarrow & \dfrac{\mathcal{L}|_{p_1} \oplus \mathcal{L}|_{p_2}}{F(L)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

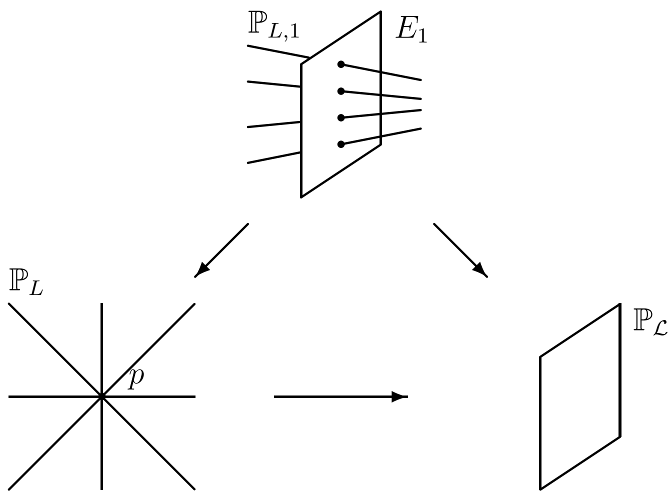
First of all, we need $F(E) \cap (\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2}) = F(\mathcal{O}_C)$ for $\mathcal{O}_C$ to be contained in $E$. Secondly, we want the bottom map in the following diagram to be well-defined:

$$\begin{array}{ccc}
\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2} & \xrightarrow{f} & \mathcal{L}|_{p_1} \oplus \mathcal{L}|_{p_2} \\
\downarrow & & \downarrow \\
\dfrac{\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2}}{F(E)} & \longrightarrow & \dfrac{\mathcal{L}|_{p_1} \oplus \mathcal{L}|_{p_2}}{F(L)}
\end{array}$$

i.e., we need $F(E) \subseteq f^{-1}(F(L))$.

In the projective space $\mathbb{P}(\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2}) \simeq \mathbb{P}^3$, such $F(E)$'s correspond to the lines contained in the plane corresponding to $f^{-1}(F(L))$ which pass through the point corresponding to $F(\mathcal{O}_C)$, except for the line corresponding to $\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2}$. Therefore, the $F(E)$'s that give an extension of $L$ by $\mathcal{O}_C$ for a fixed $\mathcal{E}$ are parametrized by $\mathbb{P}^1 \setminus \{\mathrm{pt}\}$. $\square$

Since the rational map $\pi^*\colon \mathbb{P}_L \to \mathbb{P}_{\mathcal{L}}$ is the projection from $p$, if we blow-up $\mathbb{P}_L$ at $p$, we obtain a well-defined morphism $\mathbb{P}_{L,1} := \mathcal{BL}_p\mathbb{P}_L \to \mathbb{P}_{\mathcal{L}}$.




Under this morphism, the exceptional divisor $E_1$ maps isomorphically onto $\mathbb{P}_{\mathcal{L}}$. This isomorphism corresponds to the isomorphism $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \simeq \operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ given by the projection formula. The preimage of a vector bundle $\mathcal{E} \in \mathbb{P}_{\mathcal{L}}$ in $\mathbb{P}_{L,1}$ is given by the $E$'s corresponding to the tuples $(\mathcal{E}, F(E))$ described in the lemma, plus a point in $E_1$ corresponding to the only extension $0 \to \pi_*\mathcal{O}_N \to E \to L \to 0$ of $L$ by $\pi_*\mathcal{O}_N$ such that $\pi^*E/\mathrm{Tors} = \mathcal{E}$.

# Chapter 2

# The first blow-up

## 2.1 The first blow-up

Since the indeterminancy locus of the rational map $\phi_L\colon \mathbb{P}_L \to \overline{\mathcal{SU}_C(2,L)}$ is the curve $C \subseteq \mathbb{P}_L$, to resolve the indeterminancy via a sequence of blow-ups with smooth centers, we need to begin the process with the blow-up of $\mathbb{P}_L$ at the singular point $p$ of $C$. By lemma 1.2, $p$ is $\mathbb{P}(\ker(\psi_p))$, where $\psi_p$ is the natural linear homomorphism $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$. Therefore, the exceptional divisor $E_1$ of

$$\mathbb{P}_{L,1} := \mathcal{BL}_p\mathbb{P}_L \xrightarrow{\varepsilon_1} \mathbb{P}_L$$

is canonically isomorphic to $\mathbb{P}(\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N))$ because we have the canonical isomorphisms

$$E_1 \simeq \mathbb{P}(\mathcal{N}_{\{p\}/\mathbb{P}_L}) \quad \text{and} \quad \mathcal{N}_{\{p\}/\mathbb{P}_L} \simeq T_p\mathbb{P}_L \simeq \frac{\operatorname{Ext}^1_C(L, \mathcal{O}_C)}{\ker \psi_p} \simeq \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N).$$

We saw in section 1.3 that there is a morphism $\mathbb{P}_{L,1} \to \mathbb{P}_{\mathcal{L}}$ which extends the rational map $\pi^*\colon \mathbb{P}_L \to \mathbb{P}_{\mathcal{L}}$, and that $E_1$ maps isomorphically onto $\mathbb{P}_{\mathcal{L}}$.

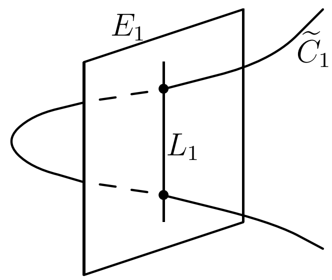
**Theorem 2.1.** $(a)$ *The composition* $\phi_L \circ \varepsilon_1\colon \mathbb{P}_{L,1} \longrightarrow \overline{\mathcal{SU}_C(2,L)}$ *extends to a rational map* $\phi_{L,1}$ *defined as follows: The image of a point* $x \in E_1$ *corresponding to an extension* $\mathcal{E}_x$ *in* $\operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ *is the torsion-free sheaf*

$$E_x := \ker(\pi_*\mathcal{E}_x \longrightarrow (\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2})/(\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2})).$$

*If* $E'_x$ *is the extension in* $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$ *corresponding to* $x$*, then* $E'_x$ *maps to* $E_x$ *under the natural homomorphism* $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \xrightarrow{\psi} \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$.

*(b) The indeterminancy locus of the rational map* $\phi_{L,1}\colon \mathbb{P}_{L,1} \to \overline{\mathcal{SU}_C(2,L)}$ *is the union of the strict transform* $\widetilde{C}_1$ *of* $C$ *and the line*[1]

$$L_1 := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N) \xrightarrow{\psi} \operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*,\pi_*\mathcal{O}_N))) \subseteq E_1.$$




We shall delay the proof of theorem 2.1 to the next section. Let us now understand the situation better. The strict transform $\widetilde{C}_1$ of $C$ is isomorphic to $N$ and intersects $E_1$ at the two points $p_1, p_2$ lying on $p$. The following lemma describes $L_1$.

**Lemma 2.2.** *The points on* $L_1$ *correspond to the directions in* $T_pC$*, the projective tangent plane to* $C$ *at* $p$*. In particular,* $L_1$ *is the line through* $p_1$ *and* $p_2$ *in* $E_1$*.*

*Proof.* It suffices to show that $L_1$ contains $p_1$ and $p_2$. Let us show first that $p_i = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N) \xrightarrow{\psi_i} \operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N(p_i)))) \subseteq \mathbb{P}(\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N)) \simeq E_1$ for $i=1,2$. We already saw in the proof of lemma 1.4 that the projective line tangent at $p$ to the branch corresponding to $p_i$ is

$$X_{p_i} = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L,\mathcal{O}_C) \to \operatorname{Ext}^1_C(L,\pi_*(\mathcal{O}_N(p_i)))))$$

for $i=1,2$. Therefore,

$$\langle p_i\rangle \simeq \frac{\langle X_{p_i}\rangle}{\langle p\rangle} \simeq \frac{\ker(\operatorname{Ext}^1_C(L,\mathcal{O}_C) \to \operatorname{Ext}^1_C(L,\pi_*(\mathcal{O}_N(p_i))))}{\ker(\operatorname{Ext}^1_C(L,\mathcal{O}_C) \to \operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N))}$$

is equal to $\ker\psi_i$ for $i=1,2$, as claimed.

[1]See lemma A.3 for a proof that $\dim(\ker\psi)=2$.

To prove that $p_1, p_2 \in L_1$, we need to show that $\ker\psi_i \subseteq \ker\psi$ for $i = 1, 2$. By the proof of lemma 1.2, a non-trivial extension $E$ in $\mathrm{Ext}\,^1_C(L, \pi_*\mathcal{O}_N)$ is in the kernel of $\psi_i$ if and only if there exists a surjective map $E \to \pi_*\mathcal{O}_N(p_i)$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L\otimes(\pi_*\mathcal{O}_N)^* & = & L\otimes(\pi_*\mathcal{O}_N)^* & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\mathcal{O}_N & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0. \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\mathcal{O}_N & \longrightarrow & \pi_*\mathcal{O}_N(p_i) & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $L \otimes (\pi_*\mathcal{O}_N)^* \subseteq E$, the extension is also in the kernel of $\psi$, i.e., there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*\mathcal{O}_N & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & || & & \uparrow & & \uparrow & & . \\
0 & \longrightarrow & \pi_*\mathcal{O}_N & \longrightarrow & \pi_*\mathcal{O}_N \oplus (L\otimes(\pi_*\mathcal{O}_N)^*) & \longrightarrow & L\otimes(\pi_*\mathcal{O}_N)^* & \longrightarrow & 0
\end{array}$$

□

To understand the image of $\phi_{L,1}|_{E_1}$, it is important to understand the image of the natural linear homomorphism $\psi$.

**Lemma 2.3.** *There exists a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Ext}\,^1_N(\mathcal{L}, \mathcal{O}_N) & \xrightarrow{\simeq} & \mathrm{Ext}\,^1_C(L, \pi_*\mathcal{O}_N) \\
\downarrow & & \downarrow \psi \\
\mathrm{Ext}\,^1_N(\mathcal{L}(-p_1-p_2), \mathcal{O}_N) & \xrightarrow{\pi_*} & \mathrm{Ext}\,^1_C(L\otimes(\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)
\end{array}, \tag{2.1}$$

*the linear homomorphism $\pi_*$ is an isomorphism onto its image, and*

$$\mathrm{Im}\,\psi \simeq \mathrm{Ext}\,^1_N(\mathcal{L}(-p_1-p_2), \mathcal{O}_N).$$

*Proof.* There is a natural commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^1_N(\mathcal{L},\mathcal{O}_N) & \xrightarrow{\pi_*} & \operatorname{Ext}^1_C(\pi_*\mathcal{L},\pi_*\mathcal{O}_N) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^1_N(\mathcal{L}(-p_1-p_2),\mathcal{O}_N) & \xrightarrow{\pi_*} & \operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*,\pi_*\mathcal{O}_N) \end{array},$$

where the vertical homomorphisms are pull-backs via the inclusions $\mathcal{L}(-p_1-p_2)\hookrightarrow\mathcal{L}$ and $L\otimes(\pi_*\mathcal{O}_N)^*=\pi_*(\mathcal{L}(-p_1-p_2))\hookrightarrow\pi_*\mathcal{L}$, respectively. Since the vertical homomorphism on the right factors through $\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N)$, to prove that (2.1) is commutative, it is enough to show that the isomorphism $\operatorname{Ext}^1_N(\mathcal{L},\mathcal{O}_N)\xrightarrow{\simeq}\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N)$ factors through $\operatorname{Ext}^1_C(\pi_*\mathcal{L},\pi_*\mathcal{O}_N)$.

Consider an extension $0\to\mathcal{O}_N\to\mathcal{E}\to\mathcal{L}\to 0$ of $\mathcal{L}$ by $\mathcal{O}_N$. Its image in $\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N)$ is the only extension $0\to\pi_*\mathcal{O}_N\to E\to L\to 0$ of $L$ by $\pi_*\mathcal{O}_N$ such that $\mathcal{E}=\pi^*E/\mathrm{Tors}$. There exists a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \pi_*\mathcal{O}_N & = & \pi_*\mathcal{O}_N & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & E & \longrightarrow & \pi_*\mathcal{E} & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & L & \longrightarrow & \pi_*\mathcal{L} & \longrightarrow & (\mathcal{L}|_{p_1}\oplus\mathcal{L}|_{p_2})/F(L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

and it is clear that $E$ is the pull-back of $\pi_*\mathcal{E}$ via the inclusion $L\hookrightarrow\pi_*\mathcal{L}$.

Since $\operatorname{Ext}^1_N(\mathcal{L},\mathcal{O}_N)\to\operatorname{Ext}^1_N(\mathcal{L}(-p_1-p_2),\mathcal{O}_N)$ is surjective[2], the fact that $\pi_*\colon\operatorname{Ext}^1_N(\mathcal{L}(-p_1-p_2),\mathcal{O}_N)\to\operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*,\pi_*\mathcal{O}_N)$ is an isomorphism onto its image follows from its injectivity, and $\dim\operatorname{Ext}^1_N(\mathcal{L}(-p_1-p_2),\mathcal{O}_N)=\dim\operatorname{Im}\pi_*$. Indeed, $\operatorname{Im}\pi_*=\operatorname{Im}\psi$, and[3] $\dim\operatorname{Im}\pi_*=\dim\operatorname{Im}\psi=\dim\operatorname{Ext}^1_C(L,\pi_*\mathcal{O}_N)-2$, which is equal to $\dim\operatorname{Ext}^1_N(\mathcal{L}(-p_1-p_2),\mathcal{O}_N)=\dim\operatorname{Ext}^1_N(\mathcal{L},\mathcal{O}_N)-2$. □

[2] Because $\operatorname{Ext}^2_N(\mathbb{C}_{p_1}\oplus\mathbb{C}_{p_2},\mathcal{O}_N)$ is zero, being $N$ a smooth curve.

[3] For the calculations of the dimensions of these extension spaces, see appendix A.

**Corollary 2.4.** *The image* $\phi_{L,1}(\mathbb{P}_{L,1} \setminus (L_1 \cup \widetilde{C}_1))$ *of* $\phi_{L,1}$ *in* $\overline{\mathcal{SU}_C(2,L)}$ *is given by*

$$\phi_L(\mathbb{P}_L \setminus C) \cup \{E \in \overline{\mathcal{SU}_C(2,L)} \mid E = \pi_*\mathcal{E} \text{ for some } \mathcal{E} \in \operatorname{Ext}^1_N(\mathcal{L}(-p_1 - p_2), \mathcal{O}_N)\}.$$

## 2.2 Proof of theorem 2.1

To prove theorem 2.1, we need the following result.

**Lemma 2.5.** *All non-trivial extensions*

$$0 \longrightarrow \pi_*\mathcal{O}_N \longrightarrow E \longrightarrow L \otimes (\pi_*\mathcal{O}_N)^* \longrightarrow 0$$

*in* $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ *are semi-stable.*

*Proof.* Assume that $E$ is not semi-stable. Then there exists a torsion-free quotient $F$ of $E$ of rank 1 and degree $\leq 1$ (see [LeP97, section 5.3]). Consider the composite map $\pi_*\mathcal{O}_N \hookrightarrow E \to F$. If it is the zero-map, then the morphism $E \to F$ factors through $L \otimes (\pi_*\mathcal{O}_N)^*$, and this is not possible since $\deg(L \otimes (\pi_*\mathcal{O}_N)^*) > 1 \geq \deg F$. If it is not the zero-map, then it is an inclusion because $\pi_*\mathcal{O}_N$ is torsion-free, and this implies that $\deg F = 1$ and $F \simeq \pi_*\mathcal{O}_N$. But this can happen only if the extension we started with is trivial. □

**Corollary 2.6.** *The natural map* $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)) \to \overline{\mathcal{SU}_C(2,L)}$ *defined by* $(0 \to \pi_*\mathcal{O}_N \to E \to L \otimes (\pi_*\mathcal{O}_N)^* \to 0) \mapsto E$ *is a morphism.*

*Remark.* Since this morphism is always generically injective and injective for $g > 3$ (or $g > 2$ for $\deg L = 3$) by lemma A.7, we shall usually think of $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$ as contained in $\overline{\mathcal{SU}_C(2,L)}$, i.e., if we say that a map to $\overline{\mathcal{SU}_C(2,L)}$ is given by a map to $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$, we are actually talking about the composition with the morphism in the corollary. For our purposes, the fact that for $g = 2$ or $3$ there might be a locus where this morphism is not injective is irrelevant, as it is irrelevant that there is a locus where $\phi_L$ is not injective. This shall become an issue when we study the fibers of the rational map, which is not done in this work.

*Proof* (*of theorem* 2.1). Let us start with the geometric idea of the proof. We already saw in section 1.3 that the space $\mathbb{P}_L \setminus \{p\}$ is isomorphic to the space of pairs $(\mathcal{E}, F(E))$, with $\mathcal{E} \in \mathbb{P}_{\mathcal{L}}$ and $F(E)$ satisfying the conditions in lemma 1.7, and the image of an extension $0 \to \mathcal{O}_C \to E \to L \to 0$ corresponding to a pair $(\mathcal{E}, F(E))$ via $\phi_L$ is exactly $E = \ker(\pi_*\mathcal{E} \to (\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2})/F(E))$. A line through the point $p$ in $\mathbb{P}_L$ corresponds to all such pairs for a fixed $\mathcal{E}$.

Let $x \in E_1$. It is clear from lemma 1.7 that, if $\phi_{L,1}(x)$ exists, then

$$\phi_{L,1}(x) = E_x := \ker(\pi_*\mathcal{E} \longrightarrow (\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2})/(\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2})),$$

since $\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2}$ is the natural limit of the planes $F(E)$'s for any fixed $\mathcal{E}_x$. The commutative diagram in the proof of lemma 1.7 changes into

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi_*\mathcal{O}_N & = & \pi_*\mathcal{O}_N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E_x & \longrightarrow & \pi_*\mathcal{E}_x & \longrightarrow & \dfrac{\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2}}{\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2}} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \longrightarrow & \pi_*(\mathcal{L}(-p_1 - p_2)) & \longrightarrow & \pi_*\mathcal{L} & \longrightarrow & \mathcal{L}|_{p_1} \oplus \mathcal{L}|_{p_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2.2}$$

showing that $E_x$ is the image under $\psi$ of the extension in $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$ corresponding to $\mathcal{E}_x$ [4].

We saw in proposition 1.5 that there exists a locally-free sheaf $\mathcal{E}_L$ on $\mathbb{P}_L \times C$ such that $\mathcal{E}_L|_{\{x\}\times C} \simeq \phi_L(x)$ for every $x \in \mathbb{P}_L \setminus C$. Let $\mathcal{E}_{L,1}$ be the torsion-free sheaf on

[4] Note that $\pi_*(\mathcal{L}(-p_1 - p_2)) \simeq \pi_*(\pi^* L \otimes \mathcal{O}_N(-p_1 - p_2)) \simeq L \otimes (\pi_*\mathcal{O}_N)^*$.

$\mathbb{P}_{L,1} \times C$ defined by[5]

$$\mathcal{E}_{L,1} := \ker((\varepsilon_1, 1)^* \mathcal{E}_L \longrightarrow \mathcal{O}_{E_1} \boxtimes \pi_* \mathcal{O}_N),$$

Note that the map is surjective because $\mathcal{E}_L|_{\{p\}\times C}$ is isomorphic to $\mathbb{P}(\ker \psi_p)$, which surjects onto $\pi_*\mathcal{O}_N$ by lemma 1.2. Moreover, the sheaf $\mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N$ is supported on $E_1 \times C$, and so $\mathcal{E}_{L,1}$ defines the same map as $\mathcal{E}_L$ on $\mathbb{P}_{L,1} \setminus E_1 = \mathbb{P}_L \setminus \{p\}$, i.e., $\mathcal{E}_{L,1}|_{\{x\}\times C} \simeq \mathcal{E}_L|_{\{\varepsilon_1(x)\}\times C}$ for every $x \in \mathbb{P}_{L,1} \setminus E_1$.

If $x \in E_1$, we have an exact sequence $\mathcal{E}_{L,1}|_{\{x\}\times C} \to (\varepsilon_1, 1)^*\mathcal{E}_L|_{\{x\}\times C} \to \pi_*\mathcal{O}_N \to 0$, which completes to an exact sequence $0 \to T \to \mathcal{E}_{L,1}|_{\{x\}\times C} \to \mathcal{E}_L|_{\{p\}\times C} \to \pi_*\mathcal{O}_N \to 0$ on $C$, where $T$ is the torsion sheaf $\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N, \mathcal{O}_{\{x\}\times C})$, that we shall show in section 2.3 to be isomorphic to $\pi_*\mathcal{O}_N$. Since $\mathcal{E}_L|_{\{p\}\times C}$ fits in the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L \otimes (\pi_*\mathcal{O}_N)^* & = & L \otimes (\pi_*\mathcal{O}_N)^* & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E}_L|_{\{p\}\times C} & \longrightarrow & L & \longrightarrow & 0, \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \pi_*\mathcal{O}_N & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

the kernel of $\mathcal{E}_L|_{\{p\}\times C} \to \pi_*\mathcal{O}_N$ is $L \otimes (\pi_*\mathcal{O}_N)^*$, and $\mathcal{E}_{L,1}|_{\{x\}\times C}$ is an extension of $L \otimes (\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$.

Therefore, by lemma 2.5, $\mathcal{E}_{L,1}|_{\{x\}\times C}$ is semi-stable if and only if it does not split as such extension. To conclude the proof of the theorem, we need to show that $\mathcal{E}_{L,1}|_{\{x\}\times C}$ (with $x \in E_1$) is the trivial extension $\pi_*\mathcal{O}_N \oplus (L \otimes (\pi_*\mathcal{O}_N)^*)$ if and only if

[5] For the notation for torsion-free sheaves on the product of a variety with $C$, see appendix B.

$x \in L_1$. We shall actually prove in proposition 2.9 that $\mathcal{E}_{L,1}|_{E_1\times C}$ induces the rational map $\phi_{L,1}|_{E_1}$. In particular, $\mathcal{E}_{L,1}|_{\{x\}\times C} \simeq E_x$ for every $x \in E_1$. Remember that $E_x$ fits in the diagram (2.2). It is not semi-stable if and only if $\pi_*\mathcal{L}(-p_1-p_2) \subseteq E_x \subseteq \pi_*\mathcal{E}_x$, i.e.[6], $\mathcal{L}(-p_1-p_2) \subseteq \mathcal{E}_x$. Since $\mathcal{E}_x$ does not split, and so $\mathcal{L} \nsubseteq \mathcal{E}_x$, there are only three possibilities:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(-p_1-p_2) & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{O}_N(p_1+p_2) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathcal{L}(-p_1) & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{O}_N(p_1) & \longrightarrow & 0, \\ 0 & \longrightarrow & \mathcal{L}(-p_2) & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{O}_N(p_2) & \longrightarrow & 0 \end{array}$$

and we know from Bertram's work[7] in [Ber89] and [Ber92] that the extensions in $\operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N)$ of this kind are exactly the ones on the secant line joining $p_1$ and $p_2$, which is $L_1$ under the identification $E_1 \simeq \mathbb{P}(\operatorname{Ext}^1_N(\mathcal{L}, \mathcal{O}_N))$. $\square$

## 2.3 Description of $\mathcal{E}_{L,1}$

In this section, we shall analyze $\mathcal{E}_{L,1}$ in more depth. We shall show that it is, in some sense, "canonical," by relating it to other universal sheaves, and we shall try to understand in more depth what sheaf it is by putting it into exact sequences. The main goal is to introduce the tools needed for the proof of proposition 2.9, that we shall prove in the next section, stating that $\mathcal{E}_{L,1}$ induces the rational map $\phi_{L,1}$.

Let us start by analyzing $\mathcal{E}_{L,1}$. We know that $\mathcal{E}_L$ fits into an exact sequence[8]

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_L}(1) \longrightarrow \mathcal{E}_L \longrightarrow L \longrightarrow 0$$

[6] If $\mathcal{L}(-p_1-p_2) \subseteq \mathcal{E}_x$, then $\pi_*\mathcal{L}(-p_1-p_2)$ maps to zero into $(\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2})/(\mathcal{O}_N|_{p_1} \oplus \mathcal{O}_N|_{p_2})$ which is isomorphic to $\mathcal{L}|_{p_1} \oplus \mathcal{L}|_{p_2}$, and so it is contained in $E_x$.

[7] This result is identical to lemma A.5.

[8] For the notation for torsion-free sheaves on the product of a variety with $C$, see appendix B.

on $\mathbb{P}_L \times C$, and we want to put $\mathcal{E}_{L,1}$ into a similar exact sequence on $\mathbb{P}_{L,1} \times C$. Using the definition of $\mathcal{E}_{L,1}$, we obtain the following commutative diagram on $\mathbb{P}_{L,1} \times C$:

$$
\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & \mathcal{E}_{L,1} & & \\
 & & \downarrow & \searrow f_1 & \\
0 \longrightarrow \varepsilon_1^* \mathcal{O}_{\mathbb{P}_L}(1) & \longrightarrow & (\varepsilon_1, 1)^* \mathcal{E}_L & \longrightarrow & L \longrightarrow 0 \\
 & \searrow g_1 & \downarrow & & \\
 & & \mathcal{O}_{E_1} \boxtimes \pi_* \mathcal{O}_N & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}
$$

Let us compute the image of $g_1$. First of all, since the support of $\mathcal{O}_{E_1} \boxtimes \pi_* \mathcal{O}_N$ is $E_1 \times C$, the support of $\operatorname{Im} g_1$ is contained in it. This means that we have the following commutative diagram on $E_1 \times C$, where the first row is exact because $\pi_C^* L$ is locally-free, and therefore $\mathcal{T}or_1^{\mathbb{P}_{L,1} \times C}(\pi_C^* L, \mathcal{O}_{E_1 \times C}) = 0$:

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{O}_{E_1 \times C} & \longrightarrow & \mathcal{E}_L|_{\{p\} \times C} \\
 & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & \operatorname{Im} g_1 & \longrightarrow & \pi_* \mathcal{O}_N \\
 & & \Big\downarrow & & \Big\downarrow \\
 & & 0 & & 0
\end{array}
$$

Since the map from $\mathcal{O}_{E_1 \times C}$ to $\pi_C^* \pi_* \mathcal{O}_N$ is injective[9], the map from $\mathcal{O}_{E_1 \times C}$ to $\operatorname{Im} g_1$ is both injective and surjective, i.e., $\operatorname{Im} g_1 \simeq \mathcal{O}_{E_1 \times C}$.

[9]This is clear since it is not the zero map, and the sheaves $\mathcal{O}_{E_1 \times C}$ and $\pi_C^* \pi_* \mathcal{O}_N$ do not contain torsion.

This gives us a commutative diagram on $\mathbb{P}_{L,1} \times C$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathcal{E}_{L,1} & \longrightarrow & \mathcal{B}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) & \longrightarrow & (\varepsilon_1, 1)^*\mathcal{E}_L & \longrightarrow & L & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E_1 \times C} & \longrightarrow & \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N & \longrightarrow & \mathcal{O}_{E_1 \times \{p\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.3}$$

where $\mathcal{A}_1$, $\mathcal{E}_{L,1}$, and $\mathcal{B}_1$ are defined by the exactness of the vertical exact sequences. In particular, $\mathcal{A}_1 \simeq \pi^*_{\mathbb{P}_{L,1}}(\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,1}}(-E_1))$.

This shows that $\mathcal{E}_{L,1}$ fits in the exact sequence

$$0 \longrightarrow \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,1}}(-E_1) \longrightarrow \mathcal{E}_{L,1} \longrightarrow \mathcal{B}_1 \longrightarrow 0$$

on $\mathbb{P}_{L,1} \times C$, which restricts to the exact sequence

$$0 \longrightarrow \mathcal{O}_{E_1}(1) \longrightarrow \mathcal{E}_{L,1}|_{E_1 \times C} \longrightarrow \mathcal{B}_1|_{E_1 \times C} \longrightarrow 0$$

on $E_1 \times C$. The restriction stays exact because $\mathcal{O}_{E_1}(1)$ is locally-free, and the map $\mathcal{O}_{E_1}(1) \to \mathcal{E}_{L,1}|_{E_1 \times C}$ is generically injective. Therefore, the image of any $\mathcal{T}or$ sheaf which would appear on the left is 0.

*Remark.* We shall use this fact several times when restricting diagrams or short exact sequences. When no comments are made about a sequence staying exact after a restriction, the reason shall be the same as here, i.e., the first sheaf is locally-free, and the first map is generically injective.

**Lemma 2.7.** *There exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_{E_1}(1) \boxtimes \pi_*\mathcal{O}_N \longrightarrow \mathcal{E}_{L,1}|_{E_1 \times C} \longrightarrow L \otimes (\pi_*\mathcal{O}_N)^* \longrightarrow 0$$

*on $E_1 \times C$.*

In the proof, we shall need the following result.

**Lemma 2.8.** *There are isomorphisms*

$$\begin{aligned}
\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1\times C},\mathcal{O}_{E_1\times C}) &\simeq \pi_{E_1}^*\mathcal{O}_{E_1}(1),\\
\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1\times \{p\}},\mathcal{O}_{E_1\times C}) &\simeq \mathcal{O}_{E_1}(1)\boxtimes \mathbb{C}_p,\\
\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1}\boxtimes \pi_*\mathcal{O}_N,\mathcal{O}_{E_1\times C}) &\simeq \mathcal{O}_{E_1}(1)\boxtimes \pi_*\mathcal{O}_N.
\end{aligned}$$

*Proof.* Consider the short exact sequence $0\to\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1)\to\mathcal{O}_{\mathbb{P}_{L,1}}\to\mathcal{O}_{E_1}\to 0$ on $\mathbb{P}_{L,1}$ and its pull-back to $\mathbb{P}_{L,1}\times C$. If we restrict it to $E_1\times C$ [resp. $E_1\times\{p\}$], we obtain the following exact sequence:

$$0\longrightarrow\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1\times C},\mathcal{O}_{E_1\times C})\longrightarrow\mathcal{O}_{E_1}(1)\xrightarrow{0}\mathcal{O}_{E_1\times C}\xrightarrow{\simeq}\mathcal{O}_{E_1\times C}\longrightarrow 0,$$

[resp. $0\longrightarrow\mathcal{T}or_1^{\mathbb{P}_{L,1}\times C}(\mathcal{O}_{E_1\times\{p\}},\mathcal{O}_{E_1\times C})\longrightarrow\mathcal{O}_{E_1}(1)\boxtimes\mathbb{C}_p\xrightarrow{0}\mathcal{O}_{E_1\times\{p\}}\xrightarrow{\simeq}\mathcal{O}_{E_1\times\{p\}}\longrightarrow 0$],

which proves the first two isomorphisms. The third one follows from the first two and the short exact sequence $0\to\mathcal{O}_{E_1\times C}\to\mathcal{O}_{E_1}\boxtimes\pi_*\mathcal{O}_N\to\mathcal{O}_{E_1\times\{p\}}\to 0$. $\square$

*Proof* (*of lemma* 2.7)*.* If we restrict the diagram (2.3) to $E_1\times C$, we obtain[10]

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{E_1}(1) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N \arrow[r] \arrow[d] & \mathcal{O}_{E_1\times\{p\}}(1) \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & \mathcal{O}_{E_1}(1) \arrow[r] \arrow[d, "0"'] & \mathcal{E}_{L,1}|_{E_1\times C} \arrow[r] \arrow[d] & \mathcal{B}_1|_{E_1\times C} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & \mathcal{O}_{E_1\times C} \arrow[r] \arrow[d, "\simeq"'] & \pi_C^*\mathcal{E}_L|_{\{p\}\times C} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & \mathcal{O}_{E_1\times C} \arrow[r] \arrow[d] & \pi_*\mathcal{O}_N \arrow[r] \arrow[d] & \mathcal{O}_{E_1\times\{p\}} \arrow[r] \arrow[d] & 0\\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

[10]For the first row, see lemma 2.8.

It follows from the commutativity of the diagram that

$$\ker(\pi_C^*\mathcal{E}_L|_{\{p\}\times C}\longrightarrow\pi_C^*(\pi_*\mathcal{O}_N)) \simeq \ker(\pi_C^*L\longrightarrow\mathcal{O}_{E_1\times\{p\}}) \simeq \pi_C^*(L\otimes(\pi_*\mathcal{O}_N)),$$

which implies our statement. $\square$

## 2.4 Relation between $\mathcal{E}_{L,1}$ and $\phi_{L,1}$

We shall prove in this section the following result.

**Proposition 2.9.** *The sheaf $\mathcal{E}_{L,1}$ on $\mathbb{P}_{L,1}\times C$ induces the rational map $\phi_{L,1}$.*

Let $\mathcal{E}'$ be the universal sheaf on $\mathbb{P}(\mathrm{Ext}\,^1_C(L,\pi_*\mathcal{O}_N))\times C \simeq E_1\times C$ corresponding to the identity under the isomorphism

$$\mathrm{Ext}\,^1_{E_1\times C}(L,\mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N) \simeq \mathrm{Hom}\,(\mathrm{Ext}\,^1_C(L,\pi_*\mathcal{O}_N),\mathrm{Ext}\,^1_C(L,\pi_*\mathcal{O}_N))$$

described in lemma B.1 and its following remark. In particular, $\mathcal{E}'|_{\{x\}\times C}\simeq E'_x$ for every $x\in E_1$, where $E'_x$ is the torsion-free sheaf which corresponds to the extension $x$. The main step in the proof of proposition 2.9 is that $\mathcal{E}'$ maps to $\mathcal{E}_{L,1}|_{E_1\times C}$ under the natural linear homomorphism

$$\mathrm{Ext}\,^1_{E_1\times C}(L,\mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N)\longrightarrow\mathrm{Ext}\,^1_{E_1\times C}(L\otimes(\pi_*\mathcal{O}_N)^*,\mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N),$$

induced by the short exact sequence $0\to L\otimes(\pi_*\mathcal{O}_N)^*\to L\to\mathbb{C}_p\to 0$ pulled-back from $C$ to $E_1\times C$.

*Proof* (*of proposition* 2.9). Let us pull-back the extension $\mathcal{E}_L$ to $\mathbb{P}_{L,1}\times C$, and then push it forward via the inclusion $\pi^*_{\mathbb{P}_{L,1}}(\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1))\hookrightarrow\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\boxtimes\pi_*\mathcal{O}_N$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) & \longrightarrow & (\varepsilon_1,1)^*\mathcal{E}_L & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\boxtimes\pi_*\mathcal{O}_N & \longrightarrow & \mathcal{E}'_0 & \longrightarrow & L & \longrightarrow & 0 \end{array} \tag{2.4}$$

The restriction of $\mathcal{E}'_0$ to $E_1\times C$ splits. Indeed, $\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)|_{E_1}\simeq\mathcal{O}_{E_1}$, and since $\mathrm{Ext}\,^1_{E_1\times C}(L,\pi_*\mathcal{O}_N)\simeq H^0(E_1,\mathcal{O}_{E_1})\otimes\mathrm{Ext}\,^1_C(L,\pi_*\mathcal{O}_N)$ by lemma B.1, we see that

$\mathcal{E}'_0|_{E_1\times C}$ splits as long as $\mathcal{E}'_0|_{\{x\}\times C}$ splits for some $x \in E_1$. Restricting the diagram (2.4) above to $\{x\} \times C$ for any $x \in E_1$, we see that $\mathcal{E}'_0|_{\{x\}\times C}$ is the trivial extension $\psi_p(\mathcal{E}_L|_{\{p\}\times C})$.

Therefore, there exists a surjective map $\mathcal{E}'_0 \to \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N$, and we can define $\mathcal{E}'_1$ to be its kernel:

$$0 \longrightarrow \mathcal{E}'_1 \longrightarrow \mathcal{E}'_0 \longrightarrow \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N \longrightarrow 0.$$

There exists a commutative diagram on $\mathbb{P}_{L,1} \times C$ similar to (2.3):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}'_1 & \longrightarrow & \mathcal{E}'_1 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \boxtimes \pi_*\mathcal{O}_N & \longrightarrow & \mathcal{E}'_0 & \longrightarrow & L & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N & = & \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with $\mathcal{A}'_1 \simeq (\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,1}}(-E_1)) \boxtimes \pi_*\mathcal{O}_N$. Moreover, we have the following commutative diagram on $\mathbb{P}_{L,1} \times C$ which relates $\mathcal{E}'_0$ and $\mathcal{E}'_1$ to $\mathcal{E}_L$ and $\mathcal{E}_{L,1}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{E}_{L,1} & \xrightarrow{i_1} & (\varepsilon_1, 1)^*\mathcal{E}_L & \longrightarrow & \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{E}'_1 & \xrightarrow{i'_1} & \mathcal{E}'_0 & \longrightarrow & \mathcal{O}_{E_1} \boxtimes \pi_*\mathcal{O}_N & \longrightarrow & 0\cdot \\
 & & \downarrow & & \downarrow & & & & \\
 & & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \boxtimes \mathbb{C}_p & = & \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \boxtimes \mathbb{C}_p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

When we restrict the first two rows of this diagram to $E_1 \times C$, and we look at the image of the restrictions of $i_1$ and $i'_1$ to $E_1 \times C$, we obtain the following diagram, where the first row is the exact sequence described in lemma 2.7.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{E_1}(1) \boxtimes \pi_*\mathcal{O}_N & \longrightarrow & \mathcal{E}_{L,1}|_{E_1\times C} & \xrightarrow{i_1|_{E_1\times C}} & L\otimes(\pi_*\mathcal{O}_N)^* & \longrightarrow & 0 \\
 & & || & & \downarrow & & \downarrow & & . \\
0 & \longrightarrow & \mathcal{O}_{E_1}(1) \boxtimes \pi_*\mathcal{O}_N & \longrightarrow & \mathcal{E}'_1|_{E_1\times C} & \xrightarrow{i'_1|_{E_1\times C}} & L & \longrightarrow & 0
\end{array}
$$

This shows that $\mathcal{E}_{L,1}|_{E_1\times C}$ is the pull-back of $\mathcal{E}'_1|_{E_1\times C}$ via the pull-back of the inclusion $L\otimes(\pi_*\mathcal{O}_N)^* \hookrightarrow L$ from $C$ to $E_1\times C$. We shall now show that $\mathcal{E}'_1|_{E_1\times C}$ is actually the canonical sheaf $\mathcal{E}' \in \operatorname{Ext}^1_{E_1\times C}(L, \mathcal{O}_{E_1}(1)\boxtimes \pi_*\mathcal{O}_N)$ described above. This is a summary of the steps used in the construction of $\mathcal{E}'_1|_{E_1\times C}$ [11]:

$$
\begin{array}{ccc}
\mathcal{E}_L & \in & \operatorname{Ext}^1_{\mathbb{P}_L\times C}(L, \mathcal{O}_{\mathbb{P}_L}(1)) \\
\downarrow & & \downarrow \\
(\varepsilon_1,1)^*\mathcal{E}_L & \in & \operatorname{Ext}^1_{\mathbb{P}_{L,1}\times C}(L, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)) \\
\downarrow & & \downarrow \\
\mathcal{E}'_0 & \in & \operatorname{Ext}^1_{\mathbb{P}_{L,1}\times C}(L, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\boxtimes\pi_*\mathcal{O}_N) \\
\downarrow & & \uparrow \\
\mathcal{E}'_1 & \in & \operatorname{Ext}^1_{\mathbb{P}_{L,1}\times C}(L, (\varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)\otimes\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1))\boxtimes\pi_*\mathcal{O}_N) \\
\downarrow & & \downarrow \\
\mathcal{E}'_1|_{E_1\times C} & \in & \operatorname{Ext}^1_{E_1\times C}(L, \mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N)
\end{array}
$$

Using the isomorphisms

$$\operatorname{Ext}^1_{Y\times C}(L, F\boxtimes G) \simeq H^0(Y,F)\otimes \operatorname{Ext}^1_C(L,G)$$

of lemma B.1, we can understand what extension $\mathcal{E}'_1|_{E_1\times C}$ is by tracking the corresponding elements in these spaces. Let $v_0,\ldots,v_n$ be a basis of $\operatorname{Ext}^1_C(L,\mathcal{O}_C)$, with $\operatorname{Span}\{v_0\} = \langle p\rangle$, and let $v_0^*,\ldots,v_n^*$ be the corresponding dual basis in $\operatorname{Ext}^1_C(L,\mathcal{O}_C)^* \simeq H^0(\mathbb{P}_L, \mathcal{O}_{\mathbb{P}_L}(1))$. Then $\mathcal{E}_L$ corresponds to the element

$$\sum_{i=0}^n v_i^*\otimes v_i \in H^0(\mathbb{P}_L,\mathcal{O}_{\mathbb{P}_L}(1))\otimes \operatorname{Ext}^1_C(L,\mathcal{O}_C).$$

[11] Note that one of the arrows goes in the other direction between the extension spaces.

We have the following diagram[12]:

$$
\begin{array}{ccccc}
\mathcal{E}_L & \longleftrightarrow & \displaystyle\sum_{i=0}^{n} v_i^* \otimes v_i & \in & H^0(\mathbb{P}_L, \mathcal{O}_{\mathbb{P}_L}(1)) \otimes \operatorname{Ext}^1_C(L, \mathcal{O}_C) \\
\downarrow & & \downarrow & & \downarrow \\
(\varepsilon_1, 1)^*\mathcal{E}_L & \longleftrightarrow & \displaystyle\sum_{i=0}^{n} v_i^* \otimes v_i & \in & H^0(\mathbb{P}_{L,1}, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)) \otimes \operatorname{Ext}^1_C(L, \mathcal{O}_C) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}'_0 & \longleftrightarrow & \displaystyle\sum_{i=1}^{n} v_i^* \otimes \psi_p(v_i) & \in & H^0(\mathbb{P}_{L,1}, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1)) \otimes \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \\
\downarrow & & \downarrow & & \uparrow \\
 & & & & H^0(\mathbb{P}_{L,1}, \varepsilon_1^*\mathcal{O}_{\mathbb{P}_L}(1) \otimes \mathcal{O}_{\mathbb{P}_{L,1}}(-E_1)) \\
\mathcal{E}'_1 & \longleftrightarrow & \displaystyle\sum_{i=1}^{n} v_i^* \otimes \psi_p(v_i) & \in & \otimes \\
 & & & & \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}'_1|_{E_1 \times C} & \longleftrightarrow & \displaystyle\sum_{i=1}^{n} \psi_p(v_i)^* \otimes \psi_p(v_i) & \in & H^0(E_1, \mathcal{O}_{E_1}(1)) \otimes \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)
\end{array}
$$

This proves our claim that $\mathcal{E}'_1|_{E_1\times C}$ is $\mathcal{E}'$. Therefore, since $\mathcal{E}_{L,1}|_{E_1\times C}$ is the pull-back of $\mathcal{E}'_1|_{E_1\times C}$ via $L \otimes (\pi_*\mathcal{O}_N)^* \hookrightarrow L$, and the pull-back corresponds to composing a homomorphism in $\operatorname{Hom}(\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N), \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N))$ with $\psi$ to obtain an element in $\operatorname{Hom}(\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N), \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$, $\mathcal{E}_{L,1}|_{E_1\times C}$ corresponds to $\psi$ itself. This proves that, for any $a \in \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$, $a \neq 0$, $\mathcal{E}_{L,1}|_{\{[a]\}\times C}$ is $\psi(a)$ as extensions of $L \otimes (\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$.

This proves our proposition, since $\mathcal{E}_{L,1}$ defines $\phi_{L,1}$ outside of $E_1$. $\square$

[12] Note that $\psi_p(v_0) = 0$.

# Chapter 3

# The second blow-up

## 3.1 The second blow-up

To resolve the indeterminancy of $\phi_{L,1}$, we first blow-up $\mathbb{P}_{L,1}$ along the line $L_1 \subseteq E_1$. Let

$$\mathbb{P}_{L,2} := \mathcal{BL}_{L_1}\mathbb{P}_{L,1} \xrightarrow{\varepsilon_2} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_1} \mathbb{P}_L,$$

and let $E_2 \subseteq \mathbb{P}_{L,2}$ be the exceptional divisor.

**Theorem 3.1.** $(a)$ *The composition* $\phi_{L,1} \circ \varepsilon_2 \colon \mathbb{P}_{L,2} \longrightarrow \overline{\mathcal{SU}_C(2,L)}$ *extends to a rational map* $\phi_{L,2}$ *satisfying the following: For each* $l \in L_1$*, the rational map*

$$\phi_{L,2}|_{E_2|_l} \colon E_2|_l \longrightarrow \overline{\mathcal{SU}_C(2,L)}$$

*is the projectivization of a linear homomorphism* $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_l \to H'$*, where*

$$H' = \overline{\{\det E \simeq L\}} \subseteq \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N).$$

*This linear homomorphism is an isomorphism if* $l \neq p_1, p_2$*, and it maps* $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_{p_i}$ $(i = 1, 2)$ *surjectively onto* $\operatorname{Im}\psi \subseteq H'$*.*

$(b)$ *In particular, the indeterminancy locus of* $\phi_{L,2} \colon \mathbb{P}_{L,2} \to \overline{\mathcal{SU}_C(2,L)}$ *is the strict transform* $\widetilde{C}_2$ *of* $\widetilde{C}_1$*.*

**Corollary 3.2.** *The image* $\phi_{L,2}(\mathbb{P}_{L,2} \setminus \widetilde{C}_2)$ *of* $\phi_{L,2}$ *in* $\overline{\mathcal{SU}_C(2,L)}$ *is given by*[1]

$$\phi_L(\mathbb{P}_L \setminus C) \cup \mathbb{P}(H').$$

[1] Remember that by $\mathbb{P}(H')$ we actually mean its image into $\overline{\mathcal{SU}_C(2,L)}$ by the morphism described in corollary 2.6.

We shall delay the proof of the theorem until section 3.3, after we study the exceptional divisor in this section, and another important situation in section 3.2. Note that the strict transform $\widetilde{C}_2$ of $\widetilde{C}_1$ is isomorphic to $\widetilde{C}_1$ and $N$, and it intersects $E_2$ at two points $\widetilde{p}_1$ and $\widetilde{p}_2$ lying over $p_1$ and $p_2$, respectively.

The first step in the proof of theorem 3.1 is the analysis of the exceptional divisor $E_2$, which is canonically isomorphic to $\mathbb{P}(\mathcal{N}_{L_1/\mathbb{P}_{L,1}})$. It is a projective bundle over $L_1$. Since $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}$ is the normal bundle to $L_1$ in $\mathbb{P}_{L,1}$, it contains the normal bundle to $L_1$ in $E_1$, and we have the following exact sequence of vector bundles on $L_1$:

$$0 \longrightarrow \mathcal{N}_{L_1/E_1} \longrightarrow \mathcal{N}_{L_1/\mathbb{P}_{L,1}} \longrightarrow \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1} \longrightarrow 0.$$

For each $l \in L_1$, if we let $X_l$ be the projective line in $\mathbb{P}_L$ which passes through $p$ and corresponds to $l$, we obtain the following canonical isomorphisms:

$$\begin{array}{ccccc} \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_l \simeq \dfrac{T_l\mathbb{P}_{L,1}}{T_l E_1} \xrightarrow{\simeq} T_p X_l \simeq & \dfrac{\langle X_l \rangle}{\langle p \rangle} & \simeq & \langle l \rangle \\ & \cap & & \cap \\ & \dfrac{\operatorname{Ext}^1_C(L, \mathcal{O}_C)}{\langle p \rangle} & \simeq & \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \end{array}$$

and

$$\mathcal{N}_{L_1/E_1}|_l \simeq \frac{T_l E_1}{T_l L_1} \simeq \frac{\dfrac{\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)}{\langle l \rangle}}{\dfrac{\langle L_1 \rangle}{\langle l \rangle}} \simeq \frac{\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)}{\langle L_1 \rangle}.$$

Since $\langle L_1 \rangle = \ker \psi$, we have that, for every $l \in L_1$, $\mathcal{N}_{L_1/E_1}|_l$ is canonically $\operatorname{Im} \psi$. In particular, it is independent from $l$, and for every $l \in L_1$ we have the following exact sequence

$$0 \longrightarrow \operatorname{Im} \psi \longrightarrow \mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_l \longrightarrow \langle l \rangle \longrightarrow 0.$$

The geometric reason for the independence of $\mathcal{N}_{L_1/E_1}|_l$ from $l$ is simple. We saw that the rational map $\phi_{L,1}|_{E_1} \colon E_1 \to \overline{\mathcal{SU}_C(2,L)}$ is given by the rational map

$$E_1 \simeq \mathbb{P}(\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)) \xrightarrow{\mathbb{P}(\psi)} \mathbb{P}(\operatorname{Im} \psi) \subseteq \mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)),$$

which is the projection from $L_1$. It is constant on each plane containing $L_1$. Blowing-up $E_1$ at the line $L_1$, we obtain a morphism

$$\mathcal{BL}_{L_1}E_1 \longrightarrow \mathbb{P}(\operatorname{Im}\psi).$$

For each $l \in L_1$, a point of $\mathbb{P}(\mathcal{N}_{L_1/E_1}|_l)$ corresponds to a plane through $L_1$, and it maps to the same point as the plane. The projectivization of each fiber of $\mathcal{N}_{L_1/E_1}$ maps isomorphically onto $\mathbb{P}(\operatorname{Im}\psi)$.

**Lemma 3.3.** *The short exact sequence*

$$0 \longrightarrow \mathcal{N}_{L_1/E_1} \longrightarrow \mathcal{N}_{L_1/\mathbb{P}_{L,1}} \longrightarrow \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1} \longrightarrow 0$$

*splits. If* $N = \dim \mathbb{P}_L$, *then* $\mathcal{N}_{L_1/E_1} \simeq \mathcal{O}_{L_1}(1)^{\oplus N-2}$, *and* $\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1} \simeq \mathcal{O}_{L_1}(-1)$. *Moreover,* $\mathbb{P}(\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1})$ *is canonically isomorphic to* $\widetilde{T_pC} \cap E_2$, *where* $\widetilde{T_pC}$ *is the strict transform of* $T_pC$ *in* $\mathbb{P}_{L,2}$ *via* $\varepsilon_2 \circ \varepsilon_1$.

We shall denote $\mathbb{P}(\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1})$ by $L_2$. It is isomorphic to $L_1$ via $\varepsilon_2|_{L_2}$, and it corresponds to a section of the projective bundle $E_2 \to L_1$.

*Proof.* Claim 1: $\mathcal{N}_{L_1/E_1} \simeq \mathcal{O}_{L_1}(1)^{\oplus N-2}$. There is a short exact sequence

$$0 \longrightarrow \mathcal{I}_{L_1}/\mathcal{I}_{L_1}^2 \longrightarrow \Omega_{E_1}|_{L_1} \longrightarrow \Omega_{L_1} \longrightarrow 0,$$

which together with the standard short exact sequence for $\Omega_{\mathbb{P}^n}$ (see [Har77, II.8.13]) gives the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{L_1}/\mathcal{I}_{L_1}^2 & \longrightarrow & \Omega_{E_1}|_{L_1} & \longrightarrow & \Omega_{L_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f & \longrightarrow & \mathcal{O}_{E_1}(-1)^{\oplus N}|_{L_1} & \xrightarrow{f} & \mathcal{O}_{L_1}(-1)^{\oplus 2} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker g & \longrightarrow & \mathcal{O}_{E_1}|_{L_1} & \xrightarrow{g} & \mathcal{O}_{L_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $g$ is an isomorphism, $\mathcal{I}_{L_1}/\mathcal{I}_{L_1}^2 \simeq \ker f \simeq \mathcal{O}_{L_1}(-1)^{\oplus N-2}$, and so

$$\mathcal{N}_{L_1/E_1} \simeq \mathcal{O}_{L_1}(1)^{\oplus N-2}.$$

Claim 2: $\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1} \simeq \mathcal{O}_{L_1}(-1)$. It is a standard fact about blow-ups (see [Har77, II.8.24(c)]) that $\mathcal{N}_{E_1/\mathbb{P}_{L,1}} \simeq \mathcal{O}_{E_1}(-1)$. The claim follows by restricting this isomorphism to $L_1$.

Claim 3: The following short exact sequence splits:

$$0 \longrightarrow \mathcal{N}_{L_1/E_1} \longrightarrow \mathcal{N}_{L_1/\mathbb{P}_{L,1}} \longrightarrow \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1} \longrightarrow 0.$$

This follows from the fact that every extension of $\mathcal{O}_{L_1}(-1)$ by $\mathcal{O}_{L_1}(1)^{\oplus N-2}$ splits. Indeed,

$$\begin{aligned} \operatorname{Ext}^1_{L_1}(\mathcal{O}_{L_1}(-1), \mathcal{O}_{L_1}(1)^{\oplus N-2}) &\simeq H^1(L_1, \mathcal{O}_{L_1}(1) \otimes \mathcal{O}_{L_1}(1)^{\oplus N-2}) \\ &\simeq H^1(L_1, \mathcal{O}_{L_1}(2)^{\oplus N-2}) \\ &\simeq \oplus^{N-2} H^1(L_1, \mathcal{O}_{L_1}(2)) \quad = \quad 0. \end{aligned}$$

The last statement of the lemma follows from the description of $\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1}$ given at the beginning of the section. ☐

We conclude this section with the following important geometric fact.

**Lemma 3.4.** *The two points $\widetilde{p}_1$ and $\widetilde{p}_2$ where the strict transform $\widetilde{C}_2$ of $C$ intersects the exceptional divisor $E_2$ do not lie in the strict transform $\widetilde{T_pC}$ of $T_pC$.*

This lemma is important because one of the main steps in the proof of theorem 3.1 is that the rational map $\phi_{L,2}$ is defined everywhere on $\widetilde{T_pC}$, and this would not be possible if $\widetilde{C}_2$ intersected it.

*Proof.* Let $i \in \{1, 2\}$. We need to show that, in $\mathbb{P}_{L,1}$, the tangent direction of $\widetilde{C}_1$ at $p_i$ is not the same as the tangent direction of the strict transform $\widetilde{X}_{p_i}$ of $X_{p_i}$

at $p_i$, where $X_{p_i} = \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \pi_*(\mathcal{O}_N(p_i)))))$ [2] is the projective line in $\mathbb{P}_L$ tangent at $p$ to the branch of $C$ corresponding to $p_i$.

There exists local coordinates at $p$ such that $X_{p_i}$ is parametrized by the map $t \mapsto (t, 0, \ldots, 0)$, and the branch of $C$ corresponding to $p_i$ is locally parametrized by $t \mapsto (t, \alpha_2 t^2 + O(t^3), \ldots, \alpha_n t^2 + O(t^3))$ for some $\alpha_1, \ldots, \alpha_n$. After blowing-up $\mathbb{P}_L$ at $p$, and choosing the local coordinates $u_1, \ldots, u_n$ at $p_i$ in such a way that $(u_1, \ldots, u_n) \mapsto (u_1, u_1 u_2, \ldots, u_1 u_n)$, $\widetilde{X}_{p_i}$ is parametrized by $t \mapsto (t, 0, \ldots, 0)$, and $\widetilde{C}_1$ is parametrized by $t \mapsto (t, \alpha_2 t + O(t^2), \ldots, \alpha_n t + O(t^2))$.

Therefore, to prove that the tangent direction to $\widetilde{C}_1$ at $p_i$ is not the same as the tangent direction of $\widetilde{X}_{p_i}$, it suffices to show that one of the $\alpha_j$'s is not zero. Remember that $|L \otimes \omega_C|$ is the linear system which defines the embedding of $C$ in $\mathbb{P}_L$. If we compose this embedding with the normalization $\pi : N \to C$, the composition is the morphism associated to the linear system $W \subseteq |\pi^*(L \otimes \omega_C)|$ defined by

$$W = \mathbb{P}(\{s \in H^0(N, \pi^*(L \otimes \omega_C)) \mid s(p_1) = s(p_2)\}).$$

In terms of this linear system, the condition that one of the $\alpha_j$'s is not zero becomes the condition that there exists an effective divisor $D$ in the linear system $W$ such that $D \geq 2p_i + p_{3-i}$ but $D \not\geq 3p_i + p_{3-i}$. Since

$$\{D \in W \mid D \geq mp_i + p_{3-i}\} = \{D \in |\pi^*(L \otimes \omega_C)| \mid D \geq mp_i + p_{3-i}\}$$

for every $m > 0$, because $D \geq p_i$ and $D \in W$ implies that $D \geq p_i + p_{3-i}$, to prove our lemma we need to prove that

$$h^0(N, \pi^*(L \otimes \omega_C) \otimes \mathcal{O}_N(-2p_i - p_{3-i})) = h^0(N, \pi^*(L \otimes \omega_C) \otimes \mathcal{O}_N(-3p_i - p_{3-i})) + 1.$$

Since $\pi^*\omega_C \simeq \omega_N(p_1 + p_2)$, $\pi^*(L \otimes \omega_C) \otimes \mathcal{O}_N(-mp_i - p_{3-i}) \simeq \pi^* L \otimes \omega_N((1-m)p_i)$ for every integer $m$. In our case, $m$ is 2 or 3, and $m - 1$ is less than $\deg L$. Therefore,

[2] See the proof of lemma 1.4.

$h^1(N, \pi^*(L \otimes \omega_C) \otimes \mathcal{O}_N(-mp_i - p_{3-i})) = h^0(N, \pi^* L^{-1} \otimes \mathcal{O}_N((m-1)p_i)) = 0$, and the result follows from the Riemann-Roch theorem. $\square$

## 3.2 Limits along lines in $T_pC$ through $p$

Before we proceed to the proof of theorem 3.1, it is important to study the following situation: Let $T_pC$ be the projective tangent plane to $C$ at $p$, and let $X$ be a projective line in $T_pC$ passing through $p$. As we saw in lemma 2.2, such lines are parametrized by $L_1$. Any such line $X_l$ ($l \in L_1$) intersects $C$ at $p$ (and possibly at other points, but always a finite number), and there exists a rational map

$$\phi_L|_{X_l} \colon X_l \longrightarrow \overline{\mathcal{SU}_C(2, L)},$$

which extends uniquely to a morphism $\psi_l$ defined on the whole $X_l$. We are interested in finding $\psi_l(p)$. If $M_l$ is the extension of $\mathbb{C}_p$ by $\pi_*\mathcal{O}_N$ corresponding to a point $l \in L_1 \simeq \mathbb{P}(\mathrm{Ext}^1_C(\mathbb{C}_p, \pi_*\mathcal{O}_N))$ [3], then[4]

$$X_l = \mathbb{P}(\ker(\mathrm{Ext}^1_C(L, \mathcal{O}_C) \longrightarrow \mathrm{Ext}^1_C(L, M_l))).$$

The following lemma describes the $M_l$'s.

**Lemma 3.5.** *If $l \neq p_1, p_2$, then $M_l$ is a line bundle of degree 2. If $l = p_i$ ($i = 1, 2$), then $M_{p_i}$ is $\pi_*\mathcal{O}_N(p_i)$.*

*Proof.* All of the extensions of $\mathbb{C}_p$ by $\pi_*\mathcal{O}_N$ (which are sometimes called "elementary transformations" of $\pi_*\mathcal{O}_N$) are of the form

$$M_l = \ker(\pi_*\mathcal{O}_N(p_1 + p_2) \longrightarrow (\mathcal{O}_N(p_1 + p_2)|_{p_1} \oplus \mathcal{O}_N(p_1 + p_2)|_{p_2})/F(M_l)),$$

where $F(M_l)$ is a one-dimensional vector subspace of $\mathcal{O}_N(p_1+p_2)|_{p_1} \oplus \mathcal{O}_N(p_1+p_2)|_{p_2}$, and $M_l$ is locally-free if and only if $F(M_l)$ projects isomorphically onto $\mathcal{O}_N(p_1+p_2)|_{p_1}$

[3] For this isomorphism, see lemma A.3.

[4] The right side contains $\{p\} = \mathbb{P}(\ker(\mathrm{Ext}^1_C(L, \mathcal{O}_C) \to \mathrm{Ext}^1_C(L, \pi_*\mathcal{O}_N)))$ and is contained in $T_pC = \mathbb{P}(\ker(\mathrm{Ext}^1_C(L, \mathcal{O}_C) \to \mathrm{Ext}^1_C(L, \pi_*\mathcal{O}_N(p_1 + p_2))))$.

and $\mathcal{O}_N(p_1+p_2)|_{p_2}$ (see [Bho92, 1.8]). It is clear that this happens for all $F(M_l)'s$ except for the two vector subspaces $F(M_{p_i}) = \mathcal{O}_N(p_1+p_2)|_{p_i}$ $(i=1,2)$ for which $M_{p_i} = \ker(\pi_*\mathcal{O}_N(p_1+p_2) \to \mathcal{O}_N(p_1+p_2)|_{p_{3-i}}) = \pi_*\mathcal{O}_N(p_i)$. □

We are now ready to find $\psi_l(p)$.

**Lemma 3.6.** *Let $l \in L_1$. Then $\psi_l(p)$ is the unique (up to isomorphisms) torsion-free sheaf $E_l$ which can be written both as an extension*

$$0 \longrightarrow \pi_*\mathcal{O}_N \longrightarrow E_l \longrightarrow L \otimes (\pi_*\mathcal{O}_N)^* \longrightarrow 0$$

*and an extension*

$$0 \longrightarrow L \otimes M_l^* \longrightarrow E_l \longrightarrow M_l \longrightarrow 0.$$

*In particular, it is locally-free if and only if $l \neq p_1, p_2$.*

*Proof.* Since for every $x \in X_l \setminus \{p\}$, $\psi_l(x)$ maps onto $M_l$ by lemma 1.2, the same is true for $\psi_l(p)$. Indeed, it cannot surject onto something of smaller degree, or it would not be semi-stable. Since $\psi_l(p)$ is in $\overline{\mathcal{SU}_C(2,L)}$, the kernel of $\psi_l(p) \to M_l$ must then be $L \otimes M_l^*$, and we have a short exact sequence $0 \to L \otimes M_l^* \to \psi_l(p) \to M_l \to 0$.

Since $\psi_l(x)$ surjects onto $L$ for every $x \in X_l \setminus \{p\}$, $\psi_l(p)$ surjects onto some torsion-free sheaf $F \subseteq L$, which must have $\deg F \geq 2$ because $\psi_l(p)$ is semi-stable. Moreover, $F \neq L$ because in that case $\psi_l(p)$ would be an extension of $L$ by $\mathcal{O}_C$, but we know that the limit in $\mathbb{P}_L$ is $\mathcal{E}_L|_{\{p\}\times C}$, which is not semi-stable. Since $\psi_l(p)$ is an extension of $M_l$ by $L \otimes M_l^*$, and every map from $M_l$ to $F$ is zero[5], the composite map $L \otimes M_l^* \to \psi_l(p) \to F$ is non-zero, and therefore $L \otimes M_l^* \subseteq F \subseteq L$, which implies that $F \simeq L \otimes (\pi_*\mathcal{O}_N)^*$.

Therefore, $\psi_l(p)$ is both an extension of $M_l$ by $L \otimes M_l^*$ and of $L \otimes (\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$, as claimed. There is only one such torsion-free sheaf (up to isomorphisms),

[5]Since they are both torsion-free shaves of degree 2, there exists a map between them only if they are isomorphic, and this does not happen for a generic $L$.

because such sheaves are all in the kernel of the natural linear homomorphism $\mathrm{Ext}^1_C(M_l, L\otimes M_l^*) \to \mathrm{Ext}^1_C(M_l, L\otimes(\pi_*\mathcal{O}_N)^*)$ by lemma 1.2 [6], and the kernel is one-dimensional, being isomorphic to $\mathrm{Hom}_C(M_l, \mathbb{C}_p)$. $\square$

We now give another description of $E_l$ for $l \neq p_1, p_2$. Pulling-back the short exact sequence $0 \to L\otimes M_l^{-1} \to E_l \to M_l \to 0$ to $N$, we obtain the short exact sequence $0 \to \mathcal{L}(-p_1-p_2) \to \mathcal{E}_l \to \mathcal{O}_N(p_1+p_2) \to 0$, where $\mathcal{E}_l := \pi^* E_l$. This follows from the fact that $\pi^* M_l \simeq \mathcal{O}_N(p_1+p_2)$. Indeed, $M_l$ is locally-free because $l \neq p_1, p_2$, and $M_l \subseteq \pi_*(\mathcal{O}_N(p_1+p_2))$. Therefore, $\pi^* M_l$ is contained in $\mathcal{O}_N(p_1+p_2)$, which is $\pi^*\pi_*(\mathcal{O}_N(p_1+p_2))/\mathrm{Tors}$. Since they have the same degree, they are isomorphic. Moreover, since $E_l$ surjects onto $L\otimes(\pi_*\mathcal{O}_N)^* \simeq \pi_*(\mathcal{L}(-p_1-p_2))$, $\mathcal{E}_l$ surjects onto $\mathcal{L}(-p_1-p_2) \simeq \pi^*\pi_*(\mathcal{L}(-p_1-p_2))/\mathrm{Tors}$, and the sequence above splits, i.e.,

$$\mathcal{E}_l \simeq \mathcal{O}_N(p_1+p_2) \oplus \mathcal{L}(-p_1-p_2).$$

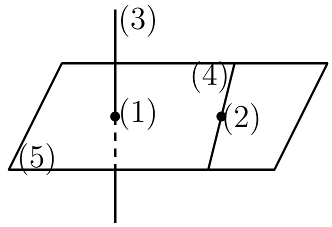
This proves that, for all $l \in L_1$, $l \neq p_1, p_2$, $\mathcal{E}_l = \pi^* E_l$ is the same vector bundle $\mathcal{O}_N(p_1+p_2) \oplus \mathcal{L}(-p_1-p_2)$. To simplify the notation, we shall denote $\mathcal{O}_N(p_1+p_2)$ by $\mathcal{O}'$ and $\mathcal{L}(-p_1-p_2)$ by $\mathcal{L}'$. Let $F(E_l)$ be the two dimensional vector subspace of $\mathcal{E}_l|_{p_1} \oplus \mathcal{E}_l|_{p_2}$ such that $E_l = \ker(\pi_*\mathcal{E}_l \to (\mathcal{E}_l|_{p_1} \oplus \mathcal{E}_l|_{p_2})/F(E_l))$. Consider the inclusion $\mathcal{L}' \subseteq \mathcal{E}_l$. Since $E_l \in \mathrm{Ext}^1_C(M_l, L\otimes M_l^{-1})$, we have that

$$F(E_l) \cap (\mathcal{L}'|_{p_1} \oplus \mathcal{L}'|_{p_2}) = F(L\otimes M_l^{-1}), \quad F(E_l) \subseteq \mathrm{Span}\,\{\mathcal{L}'|_{p_1} \oplus \mathcal{L}'|_{p_2}, F(M_l)\}.$$

Looking at the inclusion $\mathcal{O}' \subseteq \mathcal{E}$, we see that, since $E_l$ surjects onto $L\otimes(\pi_*\mathcal{O}_N)^*$, $F(E_l) \cap (\mathcal{O}'|_{p_1} \oplus \mathcal{O}'|_{p_2}) = \{0\}$. There is an $\mathbb{A}^1 \setminus \{0\}$ of such $F(E_l)$'s: Let us show that, as expected, they all give isomorphic $E_l$'s.

[6]Lemma 1.2 is about extensions of $L$ by $\mathcal{O}_C$, but a similar statement is true about extensions of $M_l$ by $L\otimes M_l^*$.

In this picture of $\mathbb{P}(\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2})$,




(1) $\mathbb{P}(F(M_l))$
(2) $\mathbb{P}(F(L \otimes M_l^{-1}))$
(3) $\mathbb{P}(\mathcal{O}'|_{p_1} \oplus \mathcal{O}'|_{p_2})$
(4) $\mathbb{P}(\mathcal{L}'|_{p_1} \oplus \mathcal{L}'|_{p_2})$
(5) $\mathbb{P}(\mathcal{L}'|_{p_1} \oplus \mathcal{L}'|_{p_2} \oplus F(M_l))$

the lines corresponding to the $F(E_l)$'s are the lines in the plane (5) going through (2), except for the line (4) and the line joining (2) and (1).

Let us fix a basis of $\mathcal{E}|_{p_1} \oplus \mathcal{E}|_{p_2} \simeq \mathbb{C}^4$ such that the standard directions correspond respectively to $\mathcal{O}'|_{p_1}$, $\mathcal{O}'|_{p_2}$, $\mathcal{L}'|_{p_1}$, $\mathcal{L}'|_{p_2}$, and $F(M_l)$, $F(L \otimes M_l^{-1})$ are generated by $(1, l, 0, 0)$ and $(0, 0, l, 1)$ respectively. Then $F(E_l)_m = \text{Span}\,\{(0, 0, l, 1), (1, l, m, 0)\}$ for some $m \neq 0$. The corresponding $E$'s are all isomorphic because the automorphism $\phi_\lambda : \mathcal{O}' \oplus \mathcal{L}' \to \mathcal{O}' \oplus \mathcal{L}'$ given by $(a, b) \mapsto (\lambda a, b)$ identifies $F(E_l)_m$ with $F(E_l)_{m/\lambda}$.

We shall prove in proposition 3.16 that the points of $L_2 = \widetilde{T_pC} \cap E_2 \simeq L_1$ map to these vector bundles $E_l$. The following lemma describes their geometry.

**Lemma 3.7.** *The torsion-free shaves $E_l$ ($l \in L_1$) form a conic in a quadric $Q$ in*

$$\mathbb{P}^3 \simeq \mathbb{P}(\ker(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N(p_1 + p_2)))).$$

*Proof.* For a proof that the dimension of the kernel above is really four, see lemma A.6. For every $l \in L_1$, let

$$\begin{aligned} X_{l,1} &:= \mathbb{P}(\ker(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, M_l))), \\ X_{l,2} &:= \mathbb{P}(\ker(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes M_l^*, \pi_*\mathcal{O}_N))). \end{aligned}$$

These are all lines by lemma A.6. For each $l \in L_1$, $X_{l,1}$ and $X_{l',2}$ span the plane $\mathbb{P}(\ker(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \to \operatorname{Ext}^1_C(L \otimes M_{l'}^*, M_l)))$.

The union of all these lines is a quadric $Q$. We shall show in lemma 3.9 that $H' := \overline{\{\det E \simeq L\}}$ is a hyperplane in $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$. Similarly, $\mathbb{P}(H') \cap \mathbb{P}^3$

is a hyperplane in this $\mathbb{P}^3$, and the intersection of $\mathbb{P}(H')$ with $Q$ is the conic in the lemma, since we know that each $E_l$ is contained in both.

For another proof of this lemma, using lemma B.5, see section 3.5. $\square$

### 3.3 Proof of theorem 3.1

As in the proof of theorem 2.1, we first give a geometric idea of the proof. We saw in section 3.1 that the normal bundle $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}$ splits as $\mathcal{N}_{L_1/E_1} \oplus \mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_{L_1}$, and that $\mathcal{N}_{L_1/E_1}|_l$ is canonically isomorphic to $\operatorname{Im}\psi$ for every $l \in L_1$. Moreover, for each $l \in L_1$, there is a natural point in $\mathcal{N}_{E_1/\mathbb{P}_{L,1}}|_l$, the point determined by the direction of the strict transform $\widetilde{X}_l$ of the line $X_l$ described in section 3.2.

If $l \neq p_1, p_2$, the image of the corresponding point in $E_2|_l$ is the vector bundle $E_l$ described in lemma 3.6, and the statement in the theorem that $\phi_{L,2}|_{E_2|_l}$ maps $E_2|_l$ isomorphically onto $\mathbb{P}(H')$ shall follow from the linearity of the map.

When $l$ is $p_1$ or $p_2$, it is clear that $\phi_{L,2}$ cannot be defined at the two points $\widetilde{p}_1$ and $\widetilde{p}_2$ where $\widetilde{C}_2$ intersects $E_2$. We shall see that the all the torsion-free sheaves which are in the image of $E_2$ are extensions of $L \otimes (\pi_* \mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$, and we saw in lemma 2.5 that the only such extension which is not semi-stable is the trivial extension. Therefore, even in this case, the statement in the theorem that $\phi_{L,2}|_{E_2|_l}$ maps $E_2|_l$ onto $\mathbb{P}(\operatorname{Im}\psi)$ shall follow from the linearity of the map.

As for the first blow-up, let us construct a torsion-free sheaf $\mathcal{E}_{L,2}$ on $\mathbb{P}_{L,2} \times C$ such that $\mathcal{E}_{L,2}|_{\{x\}\times C} \simeq \phi_{L,2}(x)$ for every $x \in \mathbb{P}_{L,2} \setminus \widetilde{C}_2$. We can construct $\mathcal{E}_{L,2}$, starting with the torsion-free sheaf $\mathcal{E}_{L,1}$ corresponding to the rational map $\phi_{L,1}$, as follows[7]

$$\mathcal{E}_{L,2} := \ker((\varepsilon_2, 1)^* \mathcal{E}_{L,1} \longrightarrow \varepsilon_2^* \mathcal{O}_{L_1}(1) \boxtimes \pi_* \mathcal{O}_N).$$

[7]For the existence of the map in the definition of $\mathcal{E}_{L,2}$, for a proof of its surjectivity, and for an in more depth analysis of the sheaf, see section 3.4.

Moreover, the sheaf $\varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N$ is supported on $E_2\times C$, and so $\mathcal{E}_{L,2}$ defines the same map as $\mathcal{E}_{L,1}$ on $\mathbb{P}_{L,2}\setminus E_2=\mathbb{P}_{L,1}\setminus L_1$. The situation is very similar to the one in the first blow-up, and it is easy to see that, for every $l\in L_1$ and every $x\in E_2|_l$, we have an exact sequence

$$0\longrightarrow\pi_*\mathcal{O}_N\longrightarrow\mathcal{E}_{L,2}|_{\{x\}\times C}\longrightarrow\mathcal{E}_{L,1}|_{\{l\}\times C}\longrightarrow\pi_*\mathcal{O}_N\longrightarrow 0.$$

Since $\mathcal{E}_{L,1}|_{\{l\}\times C}\simeq\pi_*\mathcal{O}_N\oplus(L\otimes(\pi_*\mathcal{O}_N)^*)$ for every $l\in L_1$, we obtain that, for every $x\in E_2$, $\mathcal{E}_{L,2}|_{\{x\}\times C}$ is an extension of the following type:

$$0\longrightarrow\pi_*\mathcal{O}_N\longrightarrow\mathcal{E}_{L,2}|_{\{x\}\times C}\longrightarrow L\otimes(\pi_*\mathcal{O}_N)^*\longrightarrow 0.$$

We shall prove in proposition 3.14 that the sheaf $\mathcal{E}_{L,2}$ induces the rational map $\phi_{L,2}$. In particular, for every $l\in L_1$, the restriction of $\phi_{L,2}$ to $E_2|_l$ is a rational map

$$E_2|_l\longrightarrow\mathbb{P}(\operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*,\pi_*\mathcal{O}_N)),$$

that we want to prove to be linear, and to be a morphism for $l\neq p_1,p_2$.

**Lemma 3.8.** *For every $l\in L_1$, the rational map*

$$\phi_{L,2}|_{E_2|_l}\colon E_2|_l\longrightarrow\mathbb{P}(\operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*,\pi_*\mathcal{O}_N))$$

*is linear.*

*Proof.* We give two different proofs.

First proof: We already saw in section 3.1 that, for every $l\in L_1$, we have the following commutative diagram:

$$\begin{array}{ccc} E_2|_l & \xrightarrow{\phi_{L,2}|_{E_2|_l}} & \mathbb{P}(\operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*,\pi_*\mathcal{O}_N)) \\ \bigcup & & \bigcup \\ \mathbb{P}(\mathcal{N}_{L_1/E_1}|_l) & \xrightarrow{\simeq} & \mathbb{P}(\operatorname{Im}\psi) \end{array}.$$

Since the morphism $\mathbb{P}(\mathcal{N}_{L_1/E_1}|_l)\to\mathbb{P}(\operatorname{Im}\psi)$ is a linear isomorphism, the map itself is linear by proposition C.1.

Second Proof: It follows from lemma 3.11 using lemma B.5. ☐

Let us now show that $H'$ is a hyperplane in $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$.

**Lemma 3.9.** *The closure of* $\{E \in \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \mid \det E \simeq L\}$ *in* $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ *is a vector subspace of codimension* 1.

*Proof.* It is enough to show that the closure of

$$\mathbb{P}\left(\{E \in \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \mid \det E \simeq L\}\right)$$

in $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$ is a linear hyperplane. Let $E$ be a vector bundle in $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$: What are the possible values for $\det E$?

Consider the lines $X_{l,1}$ and $X_{l,2}$ that we defined in lemma 3.7. By lemma 1.2 [8], every vector bundle in $X_{l,1}$ surjects onto $M_l$. Therefore, if $E$ is a non-trivial extension in $X_{l,1}$, there exists a commutative diagram[9]

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker f \arrow[r, equal] \arrow[d] & \ker f \arrow[d] & \\
0 \arrow[r] & \pi_*\mathcal{O}_N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & L \otimes (\pi_*\mathcal{O}_N)^* \arrow[r] \arrow[d, "f"] & 0. \\
0 \arrow[r] & \pi_*\mathcal{O}_N \arrow[r] & M_l \arrow[r] \arrow[d] & \mathbb{C}_p \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Hence, the vector bundles in $X_{l,1}$ are of the form $0 \to L \otimes M^*_{l'} \to E \to M_l \to 0$, and the determinant of the locally-free such $E$'s is of the form $L \otimes M_l \otimes M^*_{l'}$ for $l, l' \in L_1 \setminus \{p_1, p_2\}$. Similarly, all of the vector bundles $E$ in $X_{l,2}$ are of the form

[8] Lemma 1.2 is about extensions of $L$ by $\mathcal{O}_C$, but a similar statement is true about extensions of $L \otimes (\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$.

[9] This diagram also illustrates the isomorphism $X_{l,1} \simeq \mathbb{P}(\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \mathbb{C}_p))$ described in the proof of lemma A.6.

$0 \to L \otimes M_l^* \to E \to M_{l'} \to 0$, and so their determinant is of the same form as above. Consider the rational map

$$\det : \mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)) \longrightarrow \overline{\{L' \in \operatorname{Pic}^3(C) \mid \pi^*L' \simeq \pi^*L\}} \simeq \mathbb{P}^1.$$

It is defined on the locus of locally-free sheaves, and it extends to the locus of the extensions which are not push-forwards of extension from $N$ (see [Bho92, 4.7]). Since it is an isomorphism on each line $X_{l,i}$ with $l \neq p_1, p_2$ and $i \in \{1, 2\}$, it is a linear map by proposition C.2. $\square$

We conclude this section with the following proposition, which completes the proof of theorem 3.1.

**Proposition 3.10.** *For every $l \in L_1$, $l \neq p_1, p_2$, the rational map*

$$E_2|_l \to \mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$$

*is an isomorphism onto its image $\mathbb{P}(H')$. If $l = p_i$ $(i = 1, 2)$, then it maps $E_2|_l$ onto $\mathbb{P}(\operatorname{Im}\psi)$. In particular, if $l \neq p_1, p_2$, then $\phi_2|_{E_2|_l}$ is a morphism.*

*Proof.* We already saw in lemma 3.8 that the map is linear for every $l \in L_1$. Let $x_l$ be the point described at the beginning of the section, i.e., the point on the intersection of the strict transform of the line $X_l$ with $E_2$. We shall prove in proposition 3.16 that, if $l \neq p_1, p_2$, $\mathcal{E}_{L,2}|_{\{x_l\}\times C}$ is isomorphic to the vector bundle $E_l$ of lemma 3.6. Therefore, in this case, the image of $E_2|_l$ contains $\mathbb{P}(\operatorname{Im}\psi)$ and $E_l$. Since $E_l \notin \mathbb{P}(\operatorname{Im}\psi)$, the image of $E_2|_l$ is the hyperplane $H'$.

If $l = p_i$ $(i = 1, 2)$, then the image is just $\mathbb{P}(\operatorname{Im}\psi)$. Indeed, we already know that the map cannot be defined everywhere on $E_2|_{p_i}$ $(i = 1, 2)$ because it contains a point on the strict transform $\widetilde{C}_2$ of $C$ which is contained in the locus of indeterminancy of $\phi_{L,2}$. Therefore, it cannot be an isomorphism. Being a linear map, its image is contained in a hyperplane, which has to be $\mathbb{P}(\operatorname{Im}\psi)$. $\square$

## 3.4 Description of $\mathcal{E}_{L,2}$

We saw in section 2.3 that $\mathcal{E}_{L,1}|_{E_1\times C}$ is an extension

$$0 \longrightarrow \mathcal{O}_{E_1}(1) \boxtimes \pi_*\mathcal{O}_N \longrightarrow \mathcal{E}_{L,1}|_{E_1\times C} \longrightarrow L \otimes (\pi_*\mathcal{O}_N)^* \longrightarrow 0,$$

and that it is the image of $\mathcal{E}'$ [10] under the natural linear homomorphism

$$\operatorname{Ext}^1_{E_1\times C}(L, \mathcal{O}_{E_1}(1) \boxtimes \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_{E_1\times C}(L \otimes (\pi_*\mathcal{O}_N)^*, \mathcal{O}_{E_1}(1) \boxtimes \pi_*\mathcal{O}_N).$$

In particular $\mathcal{E}_{L,1}|_{L_1\times C}$ is the image of the universal sheaf corresponding to the line $L_1$. This implies that $\mathcal{E}_{L,1}|_{L_1\times C}$ splits as

$$(\mathcal{O}_{L_1}(1) \boxtimes \pi_*\mathcal{O}_N) \oplus (L \otimes (\pi_*\mathcal{O}_N)^*),$$

and we have the surjective map $\mathcal{E}_{L,1} \to \mathcal{O}_{L_1}(1) \boxtimes \pi_*\mathcal{O}_N \to 0$ whose pull-back to $\mathbb{P}_{L,2} \times C$ appears in the definition of $\mathcal{E}_{L,2}$ in Section 3.3.

Using the definition of $\mathcal{E}_{L,2}$ and the short exact sequence with $\mathcal{E}_{L,1}$ that we saw in section 2.3, we obtain the following commutative diagram on $\mathbb{P}_{L,2} \times C$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{E}_{L,2} \arrow[d] \arrow[dr, "f_2"] & & \\
0 \arrow[r] & (\varepsilon_2, 1)^*\mathcal{A}_1 \arrow[r] \arrow[dr, "g_2"] & (\varepsilon_2, 1)^*\mathcal{E}_{L,1} \arrow[r] \arrow[d] & (\varepsilon_2, 1)^*\mathcal{B}_1 \arrow[r] & 0. \\
 & & \varepsilon_2^*\mathcal{O}_{L_1}(1) \boxtimes \pi_*\mathcal{O}_N \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Let us compute the image of $g_2$. Since the support of the target sheaf $\varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N$ is $E_2 \times C$, the support of $\operatorname{Im} g_2$ is contained in it. This means that we have the

[10] Remember that $\mathcal{E}'$ was the universal sheaf corresponding to $\mathbb{P}(\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)) \simeq E_1$.

following commutative diagram on $E_2 \times C$:

$$\begin{array}{ccccc}
0 & \longrightarrow & \varepsilon_2^*\mathcal{O}_{L_1}(1) & \longrightarrow & (\varepsilon_2,1)^*\mathcal{E}_{L,1}|_{L_1\times C} \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Im} g_2 & \longrightarrow & \varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes \pi_*\mathcal{O}_N. \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

The situation is similar to the one we saw for $\operatorname{Im} g_1$ in section 2.3: the map from $\pi_{E_2}^*\varepsilon_2^*\mathcal{O}_{L_1}(1)$ to $\varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N$ is injective, and so the map from $\pi_{E_2}^*\varepsilon_2^*\mathcal{O}_{L_1}(1)$ to $\operatorname{Im} g_2$ is injective and surjective, i.e.,

$$\operatorname{Im} g_2 \simeq \pi_{E_2}^*\varepsilon_2^*\mathcal{O}_{L_1}(1).$$

We have then the following commutative diagram on $\mathbb{P}_{L,2}\times C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \mathcal{E}_{L,2} & \longrightarrow & \mathcal{B}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\varepsilon_2,1)^*\mathcal{A}_1 & \longrightarrow & (\varepsilon_2,1)^*\mathcal{E}_{L,1} & \longrightarrow & (\varepsilon_2,1)^*\mathcal{B}_1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_2^*\mathcal{O}_{L_1}(1) & \longrightarrow & \varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N & \longrightarrow & \varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.1}$$

where $\mathcal{A}_2$, $\mathcal{E}_{L,2}$, and $\mathcal{B}_2$ are defined by the vertical exact sequences.

Now we want to show that, for every $l\in L_1$, $l\neq p_1,p_2$, $\mathcal{E}_{L,2}|_{E_2|_l\times C}$ is the universal bundle associated to $H'$ when we identify $E_2|_l$ with $H'$. First of all, let us show the following

**Lemma 3.11.** *For every $l\in L_1$, $l\neq p_1,p_2$, there exists a short exact sequence*

$$0\longrightarrow\mathcal{O}_{E_2|_l}(1)\boxtimes\pi_*\mathcal{O}_N\longrightarrow\mathcal{E}_{L,2}|_{E_2|_l\times C}\longrightarrow L\otimes(\pi_*\mathcal{O}_N)^*\longrightarrow 0$$

*on $E_2|_l\times C$.*

As for the proof of lemma 2.7, we shall need to restrict the diagram (3.1) to $E_2 \times C$. Let us first calculate some torsion sheaves.

**Lemma 3.12.** *There are isomorphisms*

$$\begin{aligned}
\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{E_2\times C}) &\simeq \pi_{E_2}^*\mathcal{O}_{E_2}(-E_2),\\
\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{E_2\times C}) &\simeq \mathcal{O}_{E_2}(-E_2)\boxtimes\mathbb{C}_p,\\
\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2}\boxtimes\pi_*\mathcal{O}_N, \mathcal{O}_{E_2\times C}) &\simeq \mathcal{O}_{E_2}(-E_2)\boxtimes\pi_*\mathcal{O}_N,
\end{aligned}$$

*where $\mathcal{O}_{E_2}(-E_2)$ denotes $\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{E_2}$. Moreover, if $F$ is a locally-free sheaf on $E_2$, then $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\pi_{E_2}^*F\otimes -, \mathcal{O}_{E_2\times C})$ is $\pi_{E_2}^*F\otimes\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(-, \mathcal{O}_{E_2\times C})$.*

*Proof.* The first part of this proof is the exact copy of the proof of lemma 2.8: We just need to replace $\mathbb{P}_{L,1}$, $E_1$, and $\mathcal{O}_{E_1}(1)$ with $\mathbb{P}_{L,2}$, $E_2$, and $\mathcal{O}_{E_2}(-E_2)$, respectively. The last statement follows from the fact that if we tensor any exact sequence with a locally-free sheaf, it stays exact. If we tensor any of the exact sequences that we used to calculate the $\mathcal{T}or_1$ sheaves with $\pi_{E_2}^*F$, we see that $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\pi_{E_2}^*F\otimes -, \mathcal{O}_{E_2\times C})$ is $\pi_{E_2}^*F\otimes\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(-, \mathcal{O}_{E_2\times C})$. $\square$

*Proof* (*of lemma* 3.11). The restriction of diagram (3.1) to $E_2\times C$ is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F(-E_2) \arrow[r] \arrow[d, "\simeq"'] & F(-E_2)\boxtimes\pi_*\mathcal{O}_N \arrow[r] \arrow[d] & F(-E_2)\boxtimes\mathbb{C}_p \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & F(-E_2) \arrow[r] \arrow[d, "0"'] & \mathcal{E}_{L,2}|_{E_2\times C} \arrow[r] \arrow[d] & \mathcal{B}_2|_{E_2\times C} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & F \arrow[r] \arrow[d, "\simeq"'] & (\varepsilon_2,1)^*\mathcal{E}_{L,1}|_{L_1\times C} \arrow[r] \arrow[d] & (\varepsilon_2,1)^*\mathcal{B}_1|_{L_1\times C} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & F \arrow[r] \arrow[d] & F\boxtimes\pi_*\mathcal{O}_N \arrow[r] \arrow[d] & F\boxtimes\mathbb{C}_p \arrow[r] \arrow[d] & 0\\
 & 0 & 0 & 0 &
\end{tikzcd},$$

where we denoted the locally-free sheaf $\varepsilon_2^*\mathcal{O}_{L_1}(1)$ [resp. $\varepsilon_2^*\mathcal{O}_{L_1}(1)\otimes\mathcal{O}_{E_2}(-E_2)$] on $E_2$ by $F$ [resp. $F(-E_2)$] to simplify the notation. This shows that $\mathcal{B}_2|_{E_2\times C}$ has torsion.

Also, for every $l$, $\mathcal{B}_2|_{E_2|_l\times C}$ has torsion, and $\mathcal{B}_2|_{E_2|_l\times C}/\mathrm{Tors}$ is isomorphic to $\ker((\varepsilon_2,1)^*\mathcal{B}_1|_{\{l\}\times C}\longrightarrow\mathcal{O}_{E_2|_l\times\{p\}})$. It is clear that this kernel is just the pull-back via $(\varepsilon_2,1)$ of $\mathcal{B}_1|_{\{l\}\times C}$ modulo torsion, which is $\pi_C^*(L\otimes(\pi_*\mathcal{O}_N)^*)$. From the diagram, it is clear that the kernel of the map $\mathcal{E}_{L,2}|_{E_2|_l\times C}\longrightarrow\pi_C^*(L\otimes(\pi_*\mathcal{O}_N)^*)$ is the same as the kernel of the map $\mathcal{E}_{L,2}|_{E_2|_l\times C}\longrightarrow(\varepsilon_2,1)^*\mathcal{E}_{L,1}|_{\{l\}\times C}$, which is $\mathcal{O}_{E_2|_l}(1)\boxtimes\pi_*\mathcal{O}_N$. □

We conclude the section with the following result, which follows directly from the proof above, and that we shall need in section 4.4.

**Lemma 3.13.** *There exists a short exact sequence*

$$0\longrightarrow(\varepsilon_2^*\mathcal{O}_{L_1}(1)\otimes\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{E_2})\boxtimes\pi_*\mathcal{O}_N\longrightarrow\mathcal{E}_{L,2}|_{E_2\times C}\longrightarrow L\otimes(\pi_*\mathcal{O}_N)^*\longrightarrow 0$$

*on* $E_2\times C$.

*Proof.* This follows directly from the diagram in the proof of lemma 3.11 by looking at the middle column and observing that the kernel of the map $(\varepsilon_2,1)^*\mathcal{E}_{L,1}|_{L_1\times C}\to\varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N$ on $E_2\times C$ is[11] $\pi_C^*(L\otimes(\pi_*\mathcal{O}_N)^*)$. □

## 3.5 Relation between $\mathcal{E}_{L,2}$ and $\phi_{L,2}$

The main goal of this section is to prove the following result.

**Proposition 3.14.** *The sheaf $\mathcal{E}_{L,2}$ on $\mathbb{P}_{L,2}\times C$ induces the rational map $\phi_{L,2}$.*

For the proof, we need the following lemma.

**Lemma 3.15.** *If $Y\subseteq\mathbb{P}_{L,2}$ is a smooth subvariety such that*

$$\mathrm{codim}(Y,\mathbb{P}_{L,2})=\mathrm{codim}(Y\cap E_2,E_2),$$

*then* $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C},\mathcal{O}_{Y\times C})=0$ *and* $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}},\mathcal{O}_{Y\times C})=0$.

[11]Remember that $\mathcal{E}_{L,1}|_{L_1\times C}$ splits.

*Proof.* Consider the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}_{L,2}}(-E_2) \xrightarrow{f} \mathcal{O}_{\mathbb{P}_{L,2}} \xrightarrow{g} \mathcal{O}_{E_2} \to 0$ on $\mathbb{P}_{L,2}$ and its pull-back to $\mathbb{P}_{L,2} \times C$. If we tensor it with $\mathcal{O}_{Y\times C}$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C}) \longrightarrow \mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_Y \xrightarrow{f|_{Y\times C}} \mathcal{O}_{Y\times C} \xrightarrow{g|_{Y\times C}} \mathcal{O}_{(Y\cap E_2)\times C} \longrightarrow 0$$

on $Y \times C$, where the zero on the left occurs because $\mathcal{O}_{\mathbb{P}_{L,2}\times C}$ is locally-free.

Since the codimension of $Y \cap E_2$ in $Y$ is 1, $g$ is zero on the dense open subset $(Y \setminus (Y \cap E_2)) \times C$, and $f$ is an isomorphism on it. Therefore, $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C})$ is supported on $(Y \cap E_2) \times C$, and it must be zero, being a subsheaf of $\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{Y\times C}$, which is a locally-free sheaf on a bigger dimensional variety.

Consider now the short exact sequence $0 \to (\pi_*\mathcal{O}_N)^* \to \mathcal{O}_C \to \mathbb{C}_p \to 0$ on $C$ and its pull-back $0 \to \mathcal{O}_{E_2} \boxtimes (\pi_*\mathcal{O}_N)^* \to \mathcal{O}_{E_2\times C} \to \mathcal{O}_{E_2\times\{p\}} \to 0$ to $E_2 \times C$. If we tensor this exact sequence with $\mathcal{O}_{Y\times C}$ over $\mathcal{O}_{\mathbb{P}_{L,2}\times C}$, we obtain the exact sequence

$$0 \longrightarrow T \longrightarrow \mathcal{O}_{Y\cap E_2} \boxtimes (\pi_*\mathcal{O}_N)^* \longrightarrow \mathcal{O}_{(Y\cap E_2)\times C} \longrightarrow \mathcal{O}_{(Y\cap E_2)\times\{p\}} \longrightarrow 0$$

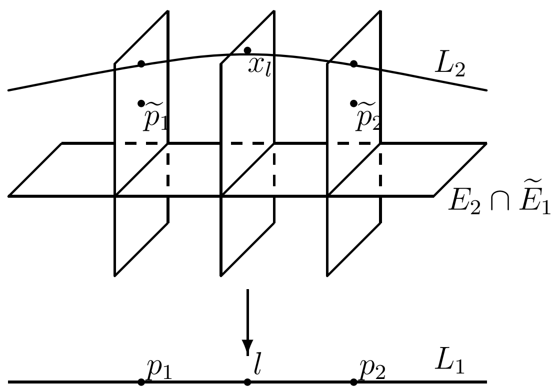
on $(Y \cap E_2) \times C$, where $T = \mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{Y\times C})$ and the zero on the left is $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{Y\times C})$. Just as above[12], $\mathcal{O}_{Y\cap E_2} \boxtimes (\pi_*\mathcal{O}_N)^* \to \mathcal{O}_{(Y\cap E_2)\times C}$ is an isomorphism on the dense open subset $(Y \cap E_2) \times (C \setminus \{p\})$, whose complement has codimension 1, and therefore $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{Y\times C})$ must be zero, being supported on $(Y \cap E_2) \times \{p\}$ and contained in the torsion-free sheaf $\mathcal{O}_{Y\cap E_2} \boxtimes (\pi_*\mathcal{O}_N)^*$, which is supported on a bigger dimensional variety. □

*Proof (of proposition* 3.14*).* It is clear that $\mathcal{E}_{L,2}$ defines $\phi_{L,2}$ on $\mathbb{P}_{L,2} \setminus E_2$. On $E_2$, we shall divide the proof in two part. We shall first show that $\mathcal{E}_{L,2}$ defines the rational map $\phi_{L,2}$ on $\widetilde{E}_1 \cap E_2 \simeq \mathbb{P}(\mathcal{N}_{L_1/E_1}) \subseteq E_2$, where $\widetilde{E}_1$ is the strict transform of $E_1$, and then we shall prove in proposition 3.16 that, if $l \neq p_1, p_2$, then $\mathcal{E}_{L,2}|_{\{x_l\}\times C} \simeq E_l$, where $E_l$ is the vector bundle of lemma 3.6.

[12]For another proof of $T$ being 0, note that the map $\mathcal{O}_{Y\cap E_2} \boxtimes (\pi_*\mathcal{O}_N)^* \to \mathcal{O}_{(Y\cap E_2)\times C}$ is injective because it is the pull-back of the injective map $(\pi_*\mathcal{O}_N)^* \to \mathcal{O}_C$ via the flat morphism $(Y \cap E_2) \times C \to C$.

Since $\phi_{L,2}$ agrees with the rational map defined by $\mathcal{E}_{L,2}$ on a dense open subset, we have that $\phi_{L,2}(x)$ is $\mathcal{E}_{L,2}|_{\{x\}\times C}$ whenever this is semi-stable. In particular, the proposition will follow from the fact that $\mathcal{E}_{L,2}$ is semi-stable for every $x \in E_2$ except for $x = \widetilde{p}_i$, $i = 1, 2$, which are the only two points on $E_2$ where we know that $\phi_{L,2}$ cannot be defined.

Here is a picture of $E_2$:




To prove that $\mathcal{E}_{L,2}$ defines the rational map $\phi_{L,2}$ on $\widetilde{E}_1 \cap E_2 \simeq \mathbb{P}(\mathcal{N}_{L_1/E_1}) \subseteq E_2$, restrict the commutative diagram (3.1) to $\widetilde{E}_1 \times C$ to obtain:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}_2|_{\widetilde{E}_1\times C} & \longrightarrow & \mathcal{E}_{L,2}|_{\widetilde{E}_1\times C} & \longrightarrow & \mathcal{B}_2|_{\widetilde{E}_1\times C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_{E_1}(1) & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,1}|_{E_1\times C} & \longrightarrow & (\sigma,1)^*\mathcal{B}_1|_{E_1\times C} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_{L_1}(1) & \longrightarrow & \sigma^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N & \longrightarrow & \sigma^*\mathcal{O}_{L_1}(1)\boxtimes\mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\sigma\colon \widetilde{E}_1 \to E_1$ is the restriction of $\varepsilon_2$ to $\widetilde{E}_1$. The vertical columns are exact because $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\varepsilon_2^*\mathcal{O}_{L_1}(1), \mathcal{O}_{\widetilde{E}_1\times C}) = \mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\varepsilon_2^*\mathcal{O}_{L_1}(1)\boxtimes\mathbb{C}_p, \mathcal{O}_{\widetilde{E}_1\times C}) = 0$. This is true because $\varepsilon_2^*\mathcal{O}_{L_1}(1) \simeq \varepsilon_2^*\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1) \otimes \mathcal{O}_{E_2\times C}$, and since $\varepsilon_2^*\mathcal{O}_{\mathbb{P}_{L,1}}(-E_1)$

is a locally-free sheaf, it is enough to show that $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times C}, \mathcal{O}_{\widetilde{E}_1\times C})$ and $\mathcal{T}or_1^{\mathbb{P}_{L,2}\times C}(\mathcal{O}_{E_2\times\{p\}}, \mathcal{O}_{\widetilde{E}_1\times C})$ are both 0, which was proved in lemma 3.15.

Since $(\sigma,1)^*\mathcal{E}_{L,1}|_{E_1\times C}$ is an extension of $\pi_C^*(L\otimes(\pi_*\mathcal{O}_N)^*)$ by $\sigma^*\mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N$, there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{A}'_2 & \longrightarrow & \mathcal{E}_{L,2}|_{\widetilde{E}_1\times C} & \longrightarrow & L\otimes(\pi_*\mathcal{O}_N)^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_{E_1}(1)\boxtimes\pi_*\mathcal{O}_N & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,1}|_{E_1\times C} & \longrightarrow & L\otimes(\pi_*\mathcal{O}_N)^* & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \sigma^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N & = & \sigma^*\mathcal{O}_{L_1}(1)\boxtimes\pi_*\mathcal{O}_N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{A}'_2 \simeq (\sigma^*\mathcal{O}_{E_1}(1)\otimes\mathcal{O}_{\widetilde{E}_1}(-(\widetilde{E}_1\cap E_2)))\boxtimes\pi_*\mathcal{O}_N$.

Using the isomorphisms $\mathrm{Ext}^1_{Y\times C}(L, F\boxtimes G)\simeq H^0(Y,F)\otimes\mathrm{Ext}^1_C(L,G)$ of lemma B.1 as we did in the proof of proposition 2.9, we have the following diagram[13]

$$\begin{array}{ccccc}
\mathcal{E}_{L,1}|_{E_1\times C} & \longleftrightarrow & \sum_{i=1}^n w_i^*\otimes\psi(w_i) & \in & H^0(E_1,\mathcal{O}_{E_1}(1))\otimes V \\
\downarrow & & \downarrow & & \downarrow \\
(\sigma,1)^*\mathcal{E}_{L,1}|_{E_1\times C} & \longleftrightarrow & \sum_{i=1}^n w_i^*\otimes\psi(w_i) & \in & H^0(\widetilde{E}_1,\sigma^*\mathcal{O}_{E_1}(1))\otimes V \\
\downarrow & & \downarrow & & \uparrow \\
\mathcal{E}_{L,2}|_{\widetilde{E}_1\times C} & \longleftrightarrow & \sum_{i=3}^n w_i^*\otimes\psi(w_i) & \in & H^0(\widetilde{E}_1,\mathcal{A}'_2)\otimes V \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_{L,2}|_{(\widetilde{E}_1\cap E_2|_l)\times C} & \longleftrightarrow & \sum_{i=3}^n \psi(w_i)^*\otimes\psi(w_i) & \in & H^0(\widetilde{E}_1\cap E_2|_l,\mathcal{O}_{\widetilde{E}_1\cap E_2|_l}(1))\otimes V
\end{array},$$

[13]Note that $\psi(w_1)=\psi(w_2)=0$.

where $w_1, \dots, w_n$ is a basis of $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)$ such that $\operatorname{Span}\{w_1, w_2\} = \ker\psi$, $w_1^*, \dots, w_n^*$ is the corresponding dual basis of $\operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N)^* \simeq H^0(E_1, \mathcal{O}_{E_1}(1))$, and we denoted $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ by $V$ to simplify the diagram.

This proves that $\mathcal{E}_{L,2}|_{(\widetilde{E}_1 \cap E_2|_l)\times C}$ corresponds to the inclusion when we identify the vector space $\operatorname{Ext}^1_{(\widetilde{E}_1 \cap E_2|_l)\times C}(L \otimes (\pi_*\mathcal{O}_N)^*, \mathcal{O}_{\widetilde{E}_1 \cap E_2|_l}(1) \boxtimes \pi_*\mathcal{O}_N)$ with the vector space $\operatorname{Hom}(\operatorname{Im}\psi, \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$ as described in lemma B.1. In particular, for every $a \in \operatorname{Im}\psi$, $a \neq 0$, $[a] \in \mathbb{P}(\operatorname{Im}\psi) \simeq \mathbb{P}(\mathcal{N}_{L_1/E_1}|_l) \simeq \widetilde{E}_1 \cap E_2|_l$, and $\mathcal{E}_{L,2}|_{\{[a]\}\times C} \simeq a$ as extensions of $L \otimes (\pi_*\mathcal{O}_N)^*$ by $\pi_*\mathcal{O}_N$. The proposition now follows from proposition 3.16. □

**Proposition 3.16.** *For every $l \in L_1$, $l \neq p_1, p_2$, $\mathcal{E}_{L,2}|_{\widetilde{X}_l \times C}$ induces the morphism $\psi_l : X_l \to \overline{\mathcal{SU}_C(2, L)}$ in a neighborhood of $p$. In particular, $\mathcal{E}_{L,2}|_{\{x_l\}\times C} \simeq E_l$, and $\phi_{L,2}(x_l) = E_l$, where $\{x_l\} = E_2 \cap \widetilde{X}_l$, and $E_l$ is the vector bundle of lemma* 3.6*.*

For the proof, we need the following lemma.

**Lemma 3.17.** *The kernel of the natural push-forward* $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, M_l)$ *equals the kernel of the natural pull-back* $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L \otimes M_l^{-1}, \mathcal{O}_C)$*.*

*Proof.* We already saw in lemma 1.2 that if an extension is in the kernel of $\psi_{M_l} : \operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, M_l)$, then the middle vector bundle surjects onto $M_l$ or $\pi_*\mathcal{O}_N$ (or it splits). By lemma A.1, the dimension of this kernel is 2, and since the subspace of vector bundles surjecting onto $\pi_*\mathcal{O}_N$ is 1-dimensional, we can choose a basis $\{E_1, E_2\}$ of the kernel given by two extensions which surject onto $M_l$.

Since the dimension of the kernel of $\psi_{M_l^{-1}} : \operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L \otimes M_l^{-1}, \mathcal{O}_C)$ is also 2 by lemma A.2, it is enough to show that $E_1$ and $E_2$ become trivial when pulled-back via $L \otimes M_l^{-1} \hookrightarrow L$, i.e., that there exists a commutative diagram $(i = 1, 2)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & E_i & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_C \oplus (L \otimes M_l^{-1}) & \longrightarrow & L \otimes M_l^{-1} & \longrightarrow & 0 \end{array}.$$

For each $i \in \{1,2\}$, since the kernel of the surjective map $E_i \to M_l$ is $L \otimes M_l^{-1}$, and the composition of $L \otimes M_l^{-1} \hookrightarrow E_i$ with $E_i \to L$ is the inclusion $L \otimes M_l^{-1} \hookrightarrow L$, it follows that there exists a map $L \otimes M_l^{-1} \hookrightarrow \psi_{M_l^{-1}}(E_i)$ which gives the splitting. $\square$

*Proof* (*of proposition* 3.16). To simplify the notation throughout this proof, we shall identify $X_l$ with its strict transforms in $\mathbb{P}_{L,1}$ and $\mathbb{P}_{L,2}$, and therefore identify the points $x_l \in E_2 \subseteq \mathbb{P}_{L,2}$, $l \in L_1 \subseteq E_1 \subseteq \mathbb{P}_{L,1}$, and $p \in C \subseteq \mathbb{P}_L$. Then $\mathcal{E}_l := \mathcal{E}_L|_{X_l \times C}$ and $\mathcal{E}_{l,i} := \mathcal{E}_{L,i}|_{\tilde{X}_l \times C}$ $(i = 1,2)$ are sheaves on $X_l \times C$ which define the same rational map $X_l \to \overline{\mathcal{SU}_C(2,L)}$ on $X_l \setminus \{p\}$.

Since $\mathcal{E}_l|_{\{p\}\times C}$ and $\mathcal{E}_{l,1}|_{\{p\}\times C}$ are not semi-stable, the rational maps defined by $\mathcal{E}_l$ and $\mathcal{E}_{l,1}$ are not morphisms. We shall prove that $\mathcal{E}_{l,2}$ induces a morphism in a neighborhood of $p$, i.e., that $\mathcal{E}_{l,2}|_{\{p\}\times C}$ is semi-stable (and, in particular, it is isomorphic to $E_l$).

There exists a short exact sequence $0 \to \mathcal{O}_{X_l}(1) \to \mathcal{E}_l \to L \to 0$ on $X_l \times C$. If we apply the functor $\operatorname{Hom}_{X_l \times C}(L \otimes M_l^{-1}, -)$, we obtain

$$\operatorname{Hom}_{X_l\times C}(L \otimes M_l^{-1}, \mathcal{E}_l) \hookrightarrow \operatorname{Hom}_{X_l\times C}(L \otimes M_l^{-1}, L) \to \operatorname{Ext}^1_{X_l\times C}(L \otimes M_l^{-1}, \mathcal{O}_{X_l}(1)).$$

The natural inclusion $L \otimes M_l^{-1} \hookrightarrow L$ gives an element of $\operatorname{Hom}_{X_l\times C}(L \otimes M_l^{-1}, L)$ which maps to zero in $\operatorname{Ext}^1_{X_l\times C}(L \otimes M_l^{-1}, \mathcal{O}_{X_l}(1))$. Indeed, its image would be the extension $\mathcal{E}'_l$ on $X_l \times C$ defined by:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{X_l}(1) \arrow[r] \arrow[d, equal] & \mathcal{E}_l \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_l}(1) \arrow[r] & \mathcal{E}'_l \arrow[r] & L \otimes M_l^{-1} \arrow[r] & 0
\end{tikzcd},$$

which splits because $\mathcal{E}_l = v_1^* \otimes v_1 + v_2^* \otimes v_2 \in H^0(X_l, \mathcal{O}_{X_l}(1)) \otimes \operatorname{Ext}^1_C(L, \mathcal{O}_C)$, $\operatorname{Span}\{v_1, v_2\} = \langle X_l \rangle$, and it maps to $0 \in H^0(X_l, \mathcal{O}_{X_l}(1)) \otimes \operatorname{Ext}^1_C(L \otimes M_l^{-1}, \mathcal{O}_C)$ because $v_1$ and $v_2$ map to zero under the pull-back via $L \otimes M_l^{-1} \hookrightarrow L$ by lemma 3.17. Therefore, there exists an inclusion $\pi_C^*(L \otimes M_l^{-1}) \hookrightarrow \mathcal{E}_l$, and a commutative diagram on $X_l \times C$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L\otimes M_l^{-1} & = & L\otimes M_l^{-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_l}(1) & \longrightarrow & \mathcal{E}_l & \longrightarrow & L & \longrightarrow & 0, \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_l}(1) & \longrightarrow & G & \longrightarrow & S_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $G$ and $S_p$ are defined by the exactness of the last two columns. In particular, $S_p$ is the pull-back from the second factor of a skyscreaper sheaf of degree 2 supported at $p$. This skyscreaper sheaf is $L/(L\otimes M_l^{-1}) \simeq M_l/\mathcal{O}_C$, and it fits into a short exact sequence $0 \to \pi_*\mathcal{O}_N/\mathcal{O}_C \to M_l/\mathcal{O}_C \to M_l/\pi_*\mathcal{O}_N \to 0$.

The idea of the proof is to show that $\mathcal{E}_{l,2} \in \mathrm{Ext}^1_{X_l\times C}(\mathcal{O}_{X_l}(-1)\boxtimes M_l, L\otimes M_l^{-1})$, and that $\mathcal{E}_{l,2}|_{\{p\}\times C}$ is $E_l$.

Since $\mathcal{E}_{l,1}$ can be defined as the kernel of $\mathcal{E}_l \to \mathcal{O}_{\{p\}}\boxtimes\pi_*\mathcal{O}_N$ [14], we have the following commutative diagram on $X_l \times C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_l} & \longrightarrow & \mathcal{E}_{l,1} & \longrightarrow & \mathcal{B}_{l,1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X_l}(1) & \longrightarrow & \mathcal{E}_l & \longrightarrow & L & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\{p\}\times C} & \longrightarrow & \mathcal{O}_{\{p\}}\boxtimes\pi_*\mathcal{O}_N & \longrightarrow & \mathcal{O}_{\{p\}\times\{p\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

[14]This is just the restriction of the short exact sequence defining $\mathcal{E}_{L,1}$ to $X_l\times C$. It stays exact because of lemma 3.15, since $\mathrm{codim}\,(X_l, \mathbb{P}_{L,2}) = \mathrm{codim}\,(\{p\}, E_2)$.

Since the composition $\pi_C^*(L \otimes M_l^{-1}) \hookrightarrow \mathcal{E}_l \to \mathcal{O}_{\{p\}} \boxtimes \pi_*\mathcal{O}_N$ is the zero map, the line bundle $\pi_C^*(L \otimes M_l^{-1})$ is contained in $\mathcal{E}_{l,1}$, and the following diagram is commutative.

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & L \otimes M_l^{-1} \arrow[d] \arrow[r, equal] & L \otimes M_l^{-1} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{X_l} \arrow[r] \arrow[d, equal] & \mathcal{E}_{l,1} \arrow[r] \arrow[d] & \mathcal{B}_{l,1} \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O}_{X_l} \arrow[r] & G_1 \arrow[r] \arrow[d] & S_{p,1} \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

where $G_1$ and $S_{p,1}$ are defined by the exactness of the last two columns. In particular, $S_{p,1}$ is an extension of $\mathcal{O}_{X_l}(-1) \boxtimes (M_l/\pi_*\mathcal{O}_N)$ by $\mathcal{O}_{X_l} \boxtimes (\pi_*\mathcal{O}_N/\mathcal{O}_C)$ because there exists a commutative diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{X_l} \boxtimes \dfrac{\pi_*\mathcal{O}_N}{\mathcal{O}_C} \arrow[r] \arrow[d, equal] & S_{p,1} \arrow[r] \arrow[d] & \mathcal{O}_{X_l}(-1) \boxtimes \dfrac{M_l}{\pi_*\mathcal{O}_N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_l} \boxtimes \dfrac{\pi_*\mathcal{O}_N}{\mathcal{O}_C} \arrow[r] & S_p \arrow[r] \arrow[d] & \mathcal{O}_{X_l} \boxtimes \dfrac{M_l}{\pi_*\mathcal{O}_N} \arrow[r] \arrow[d] & 0^{\cdot} \\
& & \mathcal{O}_{\{p\}\times\{p\}} \arrow[d] \arrow[r, equal] & \mathcal{O}_{\{p\}\times\{p\}} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

Moreover, $S_{p,1}$ is the image of $S_p$ under the pull-back linear homomorphism via the inclusion $\mathcal{O}_{X_l}(-1) \boxtimes (M_l/\pi_*\mathcal{O}_N) \hookrightarrow \mathcal{O}_{X_l} \boxtimes (M_l/\pi_*\mathcal{O}_N)$.

Similarly, we obtain three commutative diagrams describing $\mathcal{E}_{l,2}$, which can be defined as the kernel of $\mathcal{E}_{l,1} \to \mathcal{O}_{\{p\}} \boxtimes \pi_* \mathcal{O}_N$ [15]:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{X_l}(-1) \arrow[r] \arrow[d] & \mathcal{E}_{l,2} \arrow[r] \arrow[d] & \mathcal{B}_{l,2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_l} \arrow[r] \arrow[d] & \mathcal{E}_{l,1} \arrow[r] \arrow[d] & \mathcal{B}_{l,1} \arrow[r] \arrow[d] & 0; \\
0 \arrow[r] & \mathcal{O}_{\{p\}\times C} \arrow[r] \arrow[d] & \mathcal{O}_{\{p\}} \boxtimes \pi_* \mathcal{O}_N \arrow[r] \arrow[d] & \mathcal{O}_{\{p\}\times\{p\}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & L \otimes M_l^{-1} \arrow[r, equal] \arrow[d] & L \otimes M_l^{-1} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{X_l}(-1) \arrow[r] \arrow[d, equal] & \mathcal{E}_{l,2} \arrow[r] \arrow[d] & \mathcal{B}_{l,2} \arrow[r] \arrow[d] & 0; \\
0 \arrow[r] & \mathcal{O}_{X_l}(-1) \arrow[r] & G_2 \arrow[r] \arrow[d] & S_{p,2} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

[15] As for $\mathcal{E}_{l,1}$, this is just the restriction of the short exact sequence defining $\mathcal{E}_{L,2}$ which stays exact when restricted to $X_l \times C$ by lemma 3.15.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{X_l}(-1)\boxtimes\dfrac{\pi_*\mathcal{O}_N}{\mathcal{O}_C} & \longrightarrow & S_{p,2} & \longrightarrow & \mathcal{O}_{X_l}(-1)\boxtimes\dfrac{M_l}{\pi_*\mathcal{O}_N} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{X_l}\boxtimes\dfrac{\pi_*\mathcal{O}_N}{\mathcal{O}_C} & \longrightarrow & S_{p,1} & \longrightarrow & \mathcal{O}_{X_l}(-1)\boxtimes\dfrac{M_l}{\pi_*\mathcal{O}_N} & \longrightarrow & 0^{\cdot} \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\{p\}\times\{p\}} & = & \mathcal{O}_{\{p\}\times\{p\}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

In particular, $S_{p,1}$ is the push-forward of $S_{p,2}$ via the natural homomorphism induced by the inclusion $\mathcal{O}_{X_l}(-1)\boxtimes(\pi_*\mathcal{O}_N/\mathcal{O}_C)\hookrightarrow\mathcal{O}_{X_l}\boxtimes(\pi_*\mathcal{O}_N/\mathcal{O}_C)$. Under the natural isomorphism $\mathrm{Ext}\,^1_{X_l\times C}(\mathcal{O}_{X_l}(-1)\boxtimes -,\mathcal{O}_{X_l}(-1)\boxtimes -)\simeq\mathrm{Ext}\,^1_{X_l\times C}(\mathcal{O}_{X_l}\boxtimes -,\mathcal{O}_{X_l}\boxtimes -)$, $S_{p,2}$ maps to $S_p$, since they both map to $S_{p,1}$ under the natural linear homomorphisms described above.

Indeed, those homomorphisms are injective, because, using lemma B.3, they become the injective linear homomorphisms

$$\mathrm{Hom}\,_{X_l}(\mathcal{O}_{X_l},\mathcal{O}_{X_l})\longrightarrow\mathrm{Hom}\,_{X_l}(\mathcal{O}_{X_l}(-1),\mathcal{O}_{X_l})$$

and

$$\mathrm{Hom}\,_{X_l}(\mathcal{O}_{X_l}(-1),\mathcal{O}_{X_l}(-1))\longrightarrow\mathrm{Hom}\,_{X_l}(\mathcal{O}_{X_l}(-1),\mathcal{O}_{X_l})$$

tensored by the identity on $\mathrm{Ext}\,^1_C(M_l/\pi_*\mathcal{O}_N,\pi_*\mathcal{O}_N/\mathcal{O}_C)$. Therefore,

$$S_{p,2}=\pi^*_{X_l}\mathcal{O}_{X_l}(-1)\otimes S_p=\mathcal{O}_{X_l}(-1)\boxtimes\frac{M_l}{\mathcal{O}_C},$$

and there exists a short exact sequence $0\to L\otimes M_l^{-1}\to\mathcal{E}_{L,2}\to G_2\to 0$ on $X_l\times C$, where $G_2$ is an extension of $\mathcal{O}_{X_l}(-1)\boxtimes(M_l/\mathcal{O}_C)$ by $\mathcal{O}_{X_l}(-1)$.

As observed above, we would like to show that $G_2$ is $\mathcal{O}_{X_l}(-1) \boxtimes M_l$. Consider $G_2 \otimes \pi_{X_l}^* \mathcal{O}_{X_l}(1)$. It is an element of

$$\begin{aligned} \operatorname{Ext}^1_{X_l \times C}\left(\mathcal{O}_{X_l} \boxtimes \frac{M_l}{\mathcal{O}_C}, \mathcal{O}_{X_l \times C}\right) &\simeq H^1\left(X_l \times C, \left(\frac{M_l}{\mathcal{O}_C}\right) \otimes \omega_{X_l \times C}\right)^* \\ &\simeq H^1(X_l, \omega_{X_l})^* \otimes H^0\left(C, \frac{M_l}{\mathcal{O}_C} \otimes \omega_C\right)^* \\ &\simeq H^0(X_l, \mathcal{O}_{X_l}) \otimes \operatorname{Ext}^1_C\left(\frac{M_l}{\mathcal{O}_C}, \mathcal{O}_C\right), \end{aligned}$$

and it is therefore a pull-back from $C$ of an extension in $\operatorname{Ext}^1_C(M_l/\mathcal{O}_C, \mathcal{O}_C)$. Since $(G_2 \otimes \pi_{X_l}^* \mathcal{O}_{X_l}(1))|_{\{x\} \times C}$ is isomorphic to $M_l$ for every $x \neq p$, $G_2$ is $\mathcal{O}_{X_l}(-1) \boxtimes M_l$ as claimed. Therefore, there exists a short exact sequence

$$0 \longrightarrow L \otimes M_l^{-1} \longrightarrow \mathcal{E}_{l,2} \longrightarrow \mathcal{O}_{X_l}(-1) \boxtimes M_l \longrightarrow 0.$$

The proposition is now proved, because $\mathcal{E}_{l,2}|_{\{p\} \times C}$ cannot split, being an extension of $M_l$ by $L \otimes M_l^{-1}$ and an extension of $L \otimes (\pi_* \mathcal{O}_N)^*$ by $\pi_* \mathcal{O}_N$, and therefore it is semi-stable. By continuity, it must be isomorphic to $E_l$, that we proved to be the limit of $\psi_l(x)$ as $x \mapsto p$. $\square$

We conclude this chapter by giving another proof of lemma 3.7. Let us first show that there exists a short exact sequence

$$0 \longrightarrow \mathcal{O}_{L_2}(2) \boxtimes \pi_* \mathcal{O}_N \longrightarrow \mathcal{E}_{L,2}|_{L_2 \times C} \longrightarrow L \otimes (\pi_* \mathcal{O}_N)^* \longrightarrow 0$$

on $L_2 \times C$. This follows by restricting the short exact sequence of lemma 3.13 to $L_2 \times C$, and observing that $(\varepsilon_2^* \mathcal{O}_{L_1}(1) \otimes \mathcal{O}_{E_2}(-E_2))|_{L_2} \simeq \mathcal{O}_{L_2}(2)$. Now, by lemma B.5, it suffices to show that the image of the induced rational map is contained in the $\mathbb{P}^3$ defined in lemma 3.7, and this is clear because $L_2 \subseteq \widetilde{T_p C}$, and $T_p C$ is $\mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \pi_*(\mathcal{O}_N(p_1 + p_2)))))$.

## Chapter 4

## The third blow-up

### 4.1 The third blow-up

To resolve the indeterminancy of $\phi_{L,2}$, we now blow-up $\mathbb{P}_{L,2}$ along $\widetilde{C}_2$. Let

$$\mathbb{P}_{L,3} := \mathcal{BL}_{\widetilde{C}_2}\mathbb{P}_{L,2} \xrightarrow{\varepsilon_3} \mathbb{P}_{L,2} \xrightarrow{\varepsilon_2} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_2} \mathbb{P}_L,$$

and let $E_3 \subseteq \mathbb{P}_{L,3}$ be the exceptional divisor.

**Theorem 4.1.** *The composition $\phi_{L,2} \circ \varepsilon_3 \colon \mathbb{P}_{L,3} \longrightarrow \overline{\mathcal{SU}_C(2,L)}$ extends to a morphism $\phi_{L,3}$ such that for each $q \in \widetilde{C}_2$, $q \neq \widetilde{p}_1, \widetilde{p}_2$, the restriction of $\phi_{L,3}$ to $E_3|_q$ maps $E_3|_q$ isomorphically onto*[1] *$\mathbb{P}(\mathrm{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$, and for $i = 1, 2$, its restriction to $E_3|_{\widetilde{p}_i}$ sends $E_3|_{\widetilde{p}_i}$ isomorphically onto $\mathbb{P}(H')$.*

**Corollary 4.2.** *The image of $\phi_{L,2}$ in $\overline{\mathcal{SU}_C(2,L)}$ is given by*[2]

$$\phi_L(\mathbb{P}_L \setminus C) \cup \mathbb{P}(H') \bigcup_{q \in C, q \neq p} \mathbb{P}(\mathrm{Ext}^1_C(L(-q), \mathcal{O}_C(q))).$$

We shall prove theorem 4.1 in the next section, after we study the exceptional divisor $E_3$ in this section. We know that $E_3$ is canonically isomorphic to $\mathcal{N}_{\widetilde{C}_2/\mathbb{P}_{L,2}}$.

Let $q \neq \widetilde{p}_1, \widetilde{p}_2$. Then we have the canonical isomorphisms

$$\mathcal{N}_{\widetilde{C}_2/\mathbb{P}_{L,2}}|_q \simeq \frac{T_q\mathbb{P}_{L,2}}{T_q\widetilde{C}_2} \simeq \frac{T_q\mathbb{P}_L}{T_qC} \simeq \frac{\dfrac{\mathrm{Ext}^1_C(L, \mathcal{O}_C)}{\langle q \rangle}}{T_qC},$$

[1]We identify here a point $q$ on $\widetilde{C}_2$, $q \neq \widetilde{p}_1, \widetilde{p}_2$, with its image $q$ on $C$.

[2]Note that by $\mathbb{P}(H')$ [resp. $\mathbb{P}(\mathrm{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$] we actually mean its image into $\overline{\mathcal{SU}_C(2,L)}$ by the morphism described in corollary 2.6 [resp. 4.4].

and so, to prove that $E_3|_q \xrightarrow{\simeq} \mathbb{P}(\mathrm{Ext}\,^1_C(L(-q),\mathcal{O}_C(q)))$, it is necessary to prove that

$$T_qC \simeq \frac{\ker(\mathrm{Ext}\,^1_C(L,\mathcal{O}_C) \longrightarrow \mathrm{Ext}\,^1_C(L(-q),\mathcal{O}_C(q)))}{\langle q\rangle}.$$

Since the secant line joining two smooth points $q, q'$ of $C$ is[3]

$$\mathbb{P}(\ker(\mathrm{Ext}\,^1_C(L,\mathcal{O}_C) \longrightarrow \mathrm{Ext}\,^1_C(L,\mathcal{O}_C(q+q')))),$$

and $\mathrm{Ext}\,^1_C(L,\mathcal{O}_C(q+q')) \simeq \mathrm{Ext}\,^1_C(L(-q'),\mathcal{O}_C(q))$, this follows when taking the limit as $q' \to q$.

Let now $q = \widetilde{p}_i$ with $i \in \{1,2\}$. Then

$$\mathcal{N}_{\widetilde{C}_2/\mathbb{P}_{L,2}}|_{\widetilde{p}_i} \simeq \frac{T_{\widetilde{p}_i}\mathbb{P}_{L,2}}{T_{\widetilde{p}_i}\widetilde{C}_2} \simeq T_{\widetilde{p}_i}E_2.$$

This contains the canonical hyperplane $T_{\widetilde{p}_i}(E_2|_{p_i})$ that maps isomorphically to $\mathrm{Im}\,\psi$. Indeed, using lemma 3.3 and the fact that $E_2 \simeq \mathbb{P}(\mathcal{N}_{L_1/\mathbb{P}_{L,1}})$,

$$T_{\widetilde{p}_i}(E_2|_{p_i}) \simeq \frac{\mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_{p_i}}{\langle\widetilde{p}_i\rangle} \simeq \frac{\mathcal{N}_{L_1/E_1}|_{p_i} \oplus \mathcal{O}_{L_1}(-1)|_{p_i}}{\mathcal{O}_{L_1}(-1)|_{p_i}} \simeq \mathcal{N}_{L_1/E_1}|_{p_i},$$

which we already saw to be canonically isomorphic to $\mathrm{Im}\,\psi$ in section 3.1. We shall show in proposition 4.9 that the morphism $\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i})) \to \overline{\mathcal{SU}_C(2,L)}$ factors through this canonical isomorphism, i.e., there exists a commutative diagram

$$\begin{array}{cccc} E_3|_{\widetilde{p}_i} \simeq & \mathbb{P}(T_{\widetilde{p}_i}E_2) & \longrightarrow & \phi_{L,3}(E_3|_{\widetilde{p}_i}) \subseteq \overline{\mathcal{SU}_C(2,L)} \\ & \cup & & \uparrow \\ & \mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i})) & \xrightarrow{\simeq} & \mathbb{P}(\mathrm{Im}\,\psi) \end{array}. \tag{4.1}$$

We shall then show that the top map factors through an isomorphism $E_3|_{\widetilde{p}_i} \xrightarrow{\simeq} \mathbb{P}(H')$.

## 4.2 Proof of theorem 4.1

As for the other blow-ups, the strategy is to construct a universal sheaf $\mathcal{E}_{L,3}$ on $\mathbb{P}_{L,3} \times C$, and then prove that $\mathcal{E}_{L,3}$ induces the expected rational map. In this case,

[3] See lemma A.5.

we also want to prove that $\mathcal{E}_{L,3}$ induces a morphism, i.e., that $\mathcal{E}_{L,3}|_{\{x\}\times C}$ is semi-stable for every $x \in \mathbb{P}_{L,3}$. The definition of $\mathcal{E}_{L,3}$ is not as evident as in the other two blow-ups, and we postpone it to section 4.3.

By construction, $\mathcal{E}_{L,3}$ shall agree with $\mathcal{E}_{L,2}$ on $\mathbb{P}_{L,3} \setminus E_3 = \mathbb{P}_{L,2} \setminus \widetilde{C}_2$, and we shall show in propositions 4.7 and 4.8 that, if $q \in \widetilde{C}_2$, $q \neq \widetilde{p}_1, \widetilde{p}_2$, then $\mathcal{E}_{L,3}|_{E_3|_q \times C}$ induces the isomorphism of $E_3|_q$ with $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$ described in section 4.1. To prove that this induces a morphism from $E_3|_q$ to $\overline{\mathcal{SU}_C(2,L)}$, we need to prove the following result.

**Lemma 4.3.** *All non-trivial extensions*

$$0 \longrightarrow \mathcal{O}_C(q) \longrightarrow E \longrightarrow L(-q) \longrightarrow 0$$

*in* $\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q))$ *are semi-stable.*

*Proof.* This proof is identical to the one of lemma 2.5 when we change $\pi_*\mathcal{O}_N$ into $\mathcal{O}_C(q)$ and $L \otimes (\pi_*\mathcal{O}_N)^*$ into $L(-q)$. □

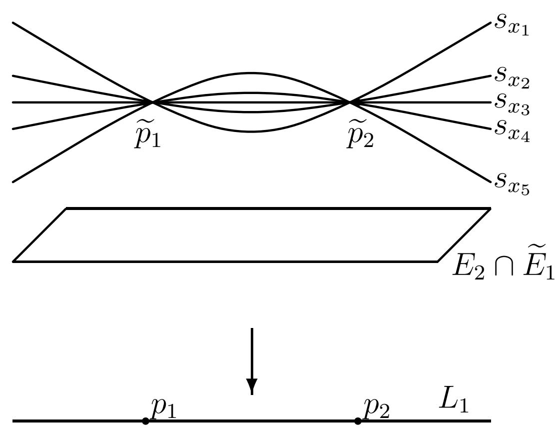
**Corollary 4.4.** *The natural map* $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q))) \to \overline{\mathcal{SU}_C(2,L)}$ *defined by* $(0 \to \mathcal{O}_C(q) \to E \to L(-q) \to 0) \mapsto E$ *is a morphism.*

Note that this morphism is always generically injective and actually injective for $g > 3$ (or $g > 2$ for $\deg L = 3$) by lemma A.7.

We now want to give the idea of the proof that $\mathcal{E}_{L,3}|_{E_3|_{\widetilde{p}_i} \times C}$ $(i = 1, 2)$ induces an isomorphism of $E_3|_{\widetilde{p}_i}$ with $\mathbb{P}(H')$. We saw in chapter 3 that each fiber of $E_2$ over $L_1$, except for the two special fibers over $p_1$ and $p_2$, maps isomorphically onto $\mathbb{P}(H')$, while those two special fibers map onto $\mathbb{P}(\operatorname{Im}\psi)$, which is a hyperplane in $\mathbb{P}(H')$.

Therefore, for each point $x \in \mathbb{P}(H') \setminus \mathbb{P}(\operatorname{Im}\psi)$, there exists a section $s_x$ of $E_2 \to L_1$ defined as follows: If $q \neq p_1, p_2$, $s_x(q)$ is the unique point of $E_2|_q$ that maps to $x$ in $\mathbb{P}(H')$. This defines a section on $L_1 \setminus \{p_1, p_2\}$, which can be completed to a section of $E_2 \to L_1$ by taking its closure. Note that its closure must satisfy $s_x(p_i) = \widetilde{p}_i$ for $i = 1, 2$, because $\widetilde{p}_i$ is the only point on $E_2|_{p_i}$ that does not map to $\mathbb{P}(\operatorname{Im}\psi)$, and $x \notin \mathbb{P}(\operatorname{Im}\psi)$.

We shall show that each such section has a different tangent direction at $\widetilde{p}_1$ and $\widetilde{p}_2$, so that when we blow-up these two points in $E_2$, the strict transforms of the $s_x(L_1)$'s will not intersect and will define the map $E_3|_{\widetilde{p}_i} \to \mathbb{P}(H')$ by $y \mapsto x$ with $\{y\} = \widetilde{s_x(L_1)} \cap E_3$.




From the definition of $\mathcal{E}_{L,3}$, it will be clear that, as in the case of the first two blow-ups, $\phi_{L,3}(E_3|_{\widetilde{p}_i}) \subseteq \mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$, and therefore, using diagram (4.1), we can prove the following linearity result.

**Lemma 4.5.** *For $i = 1, 2$, the rational map*

$$\phi_{L,3}|_{E_3|_{\widetilde{p}_i}} \colon E_3|_{\widetilde{p}_i} \longrightarrow \mathbb{P}(\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$$

*is linear.*

*Proof.* The proof is the same as the one of lemma 3.8. □

For each $i \in \{1, 2\}$, since the map is linear, and we know it to send the hyperplane $\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))$ isomorphically onto $\mathbb{P}(\operatorname{Im}\psi)$, to prove that it maps $E_3|_{\widetilde{p}_i}$ isomorphically onto $\mathbb{P}(H')$, it suffices to show that there exists a point $y \in E_3|_{\widetilde{p}_i}$ that maps to some point $x \in \mathbb{P}(H') \setminus \mathbb{P}(\operatorname{Im}\psi)$. We shall prove in proposition 4.10 that, for every $l \in L_1$, $l \neq p_1, p_2$, the point $y_{l,i}$ defined as the intersection of the strict transform of $s_{E_l}(L_1)$ with $E_3|_{\widetilde{p}_i}$ maps to $E_l$ for $i = 1, 2$, and this completes the proof of theorem 4.1. $\square$

## 4.3 Definition of $\mathcal{E}_{L,3}$

We shall define $\mathcal{E}_{L,3}$ as the kernel of a map $(\varepsilon_3, 1)^*\mathcal{E}_{L,2} \to (\varepsilon_3, 1)^*(\mathcal{A}_2|_{\widetilde{C}_2 \times C} \otimes \mathcal{F}_2)$, with $\mathcal{F}_2$ a sheaf on $\widetilde{C}_2 \times C$ such that $\mathcal{F}_2|_{\{q\}\times C} \simeq \mathcal{O}_C(q)$ if $q \neq \widetilde{p}_1, \widetilde{p}_2$, and $\mathcal{F}_2|_{\{\widetilde{p}_i\}\times C} \simeq \pi_*\mathcal{O}_N$ for $i = 1, 2$. The map corresponds to a map $\mathcal{E}_L \to \pi^*_{\mathbb{P}_L}\mathcal{O}_{\mathbb{P}_L}(1)|_{C\times C} \otimes \mathcal{F}$, where $\mathcal{F} := \mathcal{I}^*_\Delta$, $\mathcal{I}_\Delta$ being the ideal sheaf of the diagonal $\Delta$ in $C \times C$.

Before we define the map, let us study $\mathcal{F}$ in more detail.

**Lemma 4.6.** *There exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_{C\times C} \longrightarrow \mathcal{F} \longrightarrow \omega_\Delta^{-1} \longrightarrow 0.$$

*Moreover,* $\mathcal{F}|_{\{q\}\times C} \simeq \mathcal{O}_C(q)$ *if* $q \neq p$, *and* $\mathcal{F}|_{\{p\}\times C} \simeq \pi_*\mathcal{O}_N$.

*Proof.* Starting with the short exact sequence $0 \to \mathcal{I}_\Delta \to \mathcal{O}_{C\times C} \to \mathcal{O}_\Delta \to 0$, and applying the functor $\mathcal{H}om_{C\times C}(-, \mathcal{O}_{C\times C})$, we obtain the short exact sequence[4]

$$0 \longrightarrow \mathcal{O}_{C\times C} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta, \mathcal{O}_{C\times C}) \longrightarrow 0.$$

Moreover, $\mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta, \mathcal{O}_{C\times C}) \simeq \mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta, \omega_{C\times C}) \otimes \omega^{-1}_{C\times C} \simeq \omega_\Delta \otimes \omega^{-1}_{C\times C} \simeq \omega_\Delta^{-1}$ since $\omega_\Delta \simeq \mathcal{E}xt^1_{C\times C}(\mathcal{O}_\Delta, \omega_{C\times C})$ (see [Eis95, 21.15]), and $\omega_{C\times C}|_\Delta \simeq \omega_\Delta^{\otimes 2}$.

Now, for any $q \in C$, restricting the short exact sequence to $\{q\} \times C$, we obtain a short exact sequence $0 \to \mathcal{O}_C \to \mathcal{F}|_{\{q\}\times C} \to \mathbb{C}_q \to 0$, which we need to prove does not split.

It is enough to show that $\mathcal{F}|_{\{q\}\times C}$ is a torsion-free sheaf, and it is enough to study the situation in a neighborhood of $\{q\} \times \{q\}$. This is a nice local calculation,

[4] The sequence starts with $\mathcal{H}om_{C\times C}(\mathcal{O}_\Delta, \mathcal{O}_{C\times C})$ which is zero because $\mathcal{O}_{C\times C}$ is torsion-free, and ends with $\mathcal{E}xt^1_{C\times C}(\mathcal{O}_{C\times C}, \mathcal{O}_{C\times C})$ which is also zero (see [Har77, III.6.7]).

which we shall not write up, though. We take this opportunity to use Macaulay 2 (see [M2]). If $q \neq p$, then it is enough to study the following situation[5]. Let

$$R = \mathbb{C}[x_1, x_2], I = (x_1 - x_2), S = \frac{R}{(x_1)},$$

and show that $\mathrm{Hom}_R(I, R) \otimes_R S \simeq \mathrm{Hom}_S(I \otimes_R S, S)$, where $\mathrm{Hom}_R(I, R) \otimes_R S$ corresponds to $\mathcal{I}_\Delta^*|_{\{q\}\times C}$, and $\mathrm{Hom}_S(I \otimes_R S, S)$ corresponds to $(\mathcal{I}_\Delta|_{\{q\}\times C})^* \simeq \mathcal{O}_C(q)$. Here is the Macaulay 2 code:

```
i1 : R = QQ[x1,x2]
I = ideal (x1 - x2)
M = module I
J = ideal (x1)
S = R / J
N = Hom (M, R^1)
P = N ** S
Q = M ** S
R = Hom (Q, S^1)
P == R
```

The last line of the output will be “true” and it checks that the two modules are the same. If $q = p$, then we need to prove that $\mathrm{Hom}_R(I, R) \otimes_R S \simeq \mathrm{Hom}_S(I \otimes_R S, S)$ for

$$R = \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{(x_1 y_1, x_2 y_2)}, I = (x_1 - y_1, x_2 - y_2), S = \frac{R}{(x_1, y_1)}.$$

The following Macaulay 2 code proves it[6]:

```
i1 : R = QQ[x1,x2,y1,y2] / (x1 * y1, x2 * y2)
I = ideal (x1 - x2, y1 - y2)
M = module I
J = ideal (x1, y1)
S = R / J
N = Hom (M, R^1)
P = N ** S
Q = M ** S
R = Hom (Q, S^1)
C = resolution P
D = resolution R
C.dd
D.dd
```

The last part of the code output shows that the two sheaves have the same resolution, and in particular that $\mathcal{F}|_{\{p\}\times C}$ is torsion-free. □

[5] This is enough because the completion of an analytic neighborhood of $\{q\} \times \{q\}$ is the same as the completion $\mathbb{C}[[x, y]]$ of $\mathbb{C}[x, y]$.

[6] Note that this time we cannot just type “P==R” as before, because the two modules do not live in the same space, one is a quotient of $S^2$, and one is a sub-module. Therefore, we look at their resolutions.

We know that $\mathcal{E}_L$ is the extension of $\pi_C^* L$ by $\pi_{\mathbb{P}_L}^* \mathcal{O}_{\mathbb{P}_L}(1)$ that corresponds to the identity in $\mathrm{Hom}\,(\mathrm{Ext}\,^1_C(L, \mathcal{O}_C), \mathrm{Ext}\,^1_C(L, \mathcal{O}_C))$. Since $\pi_{\mathbb{P}_L}^* \mathcal{O}_{\mathbb{P}_L}(1)|_{C\times C} \simeq \pi_1^*(L \otimes \omega_C)$, where $\pi_1$ is the first projection $C \times C \to C$, we obtain the following short exact sequence on $C \times C$:

$$0 \longrightarrow \pi_1^*(L \otimes \omega_C) \longrightarrow \mathcal{E}_L|_{C\times C} \longrightarrow \pi_2^* L \longrightarrow 0.$$

The map $\pi_1^*(L \otimes \omega_C) \hookrightarrow \pi_1^*(L \otimes \omega_C) \otimes \mathcal{F}$ extends to a map $\mathcal{E}_L|_{C\times C} \to \pi_1^*(L \otimes \omega_C) \otimes \mathcal{F}$ if $\mathcal{E}_L|_{C\times C}$ is in the kernel of the natural linear homomorphism

$$\mathrm{Ext}\,^1_{C\times C}(\pi_2^* L, \pi_1^*(L \otimes \omega_C)) \longrightarrow \mathrm{Ext}\,^1_{C\times C}(\pi_2^* L, \pi_1^*(L \otimes \omega_C) \otimes \mathcal{F}),$$

i.e., if $\mathcal{E}_L|_{C\times C}$ is in the image of the natural linear homomorphism

$$\mathrm{Hom}_{\,C\times C}(\pi_2^* L, \pi_1^*(L \otimes \omega_C) \otimes \omega_\Delta^{-1}) \longrightarrow \mathrm{Ext}\,^1_{C\times C}(\pi_2^* L, \pi_1^*(L \otimes \omega_C)).$$

Let us prove that this is the case. Since $\pi_1^*(L \otimes \omega_C) \otimes \omega_\Delta^{-1}$ is isomorphic to $L$ on $\Delta \simeq C$, $\mathrm{Hom}_{\,C\times C}(\pi_2^* L, \pi_1^*(L \otimes \omega_C) \otimes \omega_\Delta^{-1}) \simeq H^0(\Delta, \mathcal{O}_\Delta) \simeq \mathbb{C}$. Moreover,

$$\begin{aligned} \mathrm{Ext}\,^1_{C\times C}(\pi_2^* L, \pi_1^*(L \otimes \omega_C)) &\simeq H^1(C \times C, (L \otimes \omega_C) \boxtimes L^{-1}) \\ &\simeq H^0(C, L \otimes \omega_C) \otimes H^1(C, L^{-1}) \\ &\simeq \mathrm{Ext}\,^1_C(L, \mathcal{O}_C)^* \otimes \mathrm{Ext}\,^1_C(L, \mathcal{O}_C), \end{aligned}$$

which has the canonical identity element corresponding to $\mathcal{E}_L|_{C\times C}$. Clearly, the constant section 1 of $\mathcal{O}_\Delta$ maps to the identity, and our claim is proved, i.e., there exists a map $\mathcal{E}_L|_{C\times C} \to \pi_{\mathbb{P}_L}^* \mathcal{O}_{\mathbb{P}_L}(1)|_{C\times C} \otimes \mathcal{F}$ as claimed at the beginning of the section.

This map is surjective because its restriction to $\{q\}\times C$, $q \neq p$, [resp. to $\{p\}\times C$] is the surjective map $\mathcal{E}_L|_{\{q\}\times C} \to \mathcal{O}_C(q)$ [resp. $\mathcal{E}_L|_{\{p\}\times C} \to \pi_* \mathcal{O}_N$] that makes $\mathcal{E}_L|_{\{q\}\times C}$ [resp. $\mathcal{E}_L|_{\{p\}\times C}$] not semi-stable.

There exists a commutative diagram

$$\begin{array}{ccccc} \mathcal{E}_{L,1}|_{\widetilde{C}_1\times C} & \longrightarrow & \mathcal{A}_1|_{\widetilde{C}_1\times C} \otimes \mathcal{F}_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ (\sigma, 1)^*(\mathcal{E}_L|_{C\times C}) & \xrightarrow{g} & \sigma^* \mathcal{O}_{\mathbb{P}_L}(1)|_{\widetilde{C}_1\times C} \otimes (\sigma, 1)^* \mathcal{F} & \longrightarrow & 0 \end{array},$$

where $\sigma : \widetilde{C}_1 \to C$ is the restriction of $\varepsilon_1$ to $\widetilde{C}_1$, and $\mathcal{F}_1$ is defined by the first row being exact. Since the restriction of the short exact sequence defining $\mathcal{E}_{L,1}$ to $\widetilde{C}_1 \times C$ stays exact by lemma 3.15 [7] the cokernel of the vertical map on the left is $\mathcal{O}_{\{p_1,p_2\}} \boxtimes \pi_*\mathcal{O}_N$. Since the fiber of $\ker g$ at $\{p_i\} \times C$ has degree 2 for $i = 1, 2$, it maps to zero into $\pi_*\mathcal{O}_N$, and we obtain the following commutative diagram on $\widetilde{C}_1 \times C$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker g & \longrightarrow & \mathcal{E}_{L,1}|_{\widetilde{C}_1\times C} & \longrightarrow & \mathcal{A}_1|_{\widetilde{C}_1\times C} \otimes \mathcal{F}_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker g & \longrightarrow & (\sigma,1)^*(\mathcal{E}_L|_{C\times C}) & \xrightarrow{g} & \sigma^*\mathcal{O}_{\mathbb{P}_L}(1)|_{\widetilde{C}_1} \otimes (\sigma,1)^*\mathcal{F} & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\{p_1,p_2\}} \boxtimes \pi_*\mathcal{O}_N & = & \mathcal{O}_{\{p_1,p_2\}} \boxtimes \pi_*\mathcal{O}_N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

If we restrict to $\{q\} \times C$, for $q \in \widetilde{C}_1$, $q \neq p_1, p_2$, then $\mathcal{F}_1|_{\{q\}\times C} \simeq \mathcal{F}|_{\{q\}\times C} \simeq \mathcal{O}_C(q)$. Let $i \in \{1, 2\}$. If we restrict the right column to $\{p_i\} \times C$, we obtain

$$0 \longrightarrow T \longrightarrow \mathcal{F}_1|_{\{p\}\times C} \longrightarrow \mathcal{F}|_{\{p\}\times C} \xrightarrow{\simeq} \pi_C\mathcal{O}_N \longrightarrow 0,$$

where $T = \mathcal{T}or_1^{\widetilde{C}_1\times C}(\mathcal{O}_{\{p_1,p_2\}} \boxtimes \pi_*\mathcal{O}_N, \mathcal{O}_{\{p_i\}\times C})$.

To calculate this sheaf, consider $0 \to \mathcal{O}_{\widetilde{C}_1}(-p_i) \to \mathcal{O}_{\widetilde{C}_1} \to \mathcal{O}_{\{p_i\}} \to 0$ on $\widetilde{C}_1$ and its pull-back to $\widetilde{C}_1 \times C$. We want to tensor it with $\mathcal{O}_{\{p_1,p_2\}} \boxtimes \pi_*\mathcal{O}_N$, and we do it in two steps. We first tensor it with $\pi_C^*(\pi_*\mathcal{O}_N)$ to obtain

$$0 \longrightarrow \mathcal{O}_{\widetilde{C}_1}(-p_i) \boxtimes \pi_*\mathcal{O}_N \longrightarrow \pi_*\mathcal{O}_N \longrightarrow \mathcal{O}_{\{p_i\}} \boxtimes \pi_*\mathcal{O}_N \longrightarrow 0,$$

where the map $\mathcal{O}_{\widetilde{C}_1}(-p_i) \boxtimes \pi_*\mathcal{O}_N \to \pi_*\mathcal{O}_N$ is injective because it is an isomorphism on the dense open subset $(\widetilde{C}_1 \setminus \{p_i\}) \times C$ whose complement has codimension 1, and

[7] Lemma 3.15 is about a subvariety $Y \subseteq \mathbb{P}_{L,2}$, but the same statement about $Y \subseteq \mathbb{P}_{L,1}$ with $\mathbb{P}_{L,2}$ [resp. $E_2$] replaced by $\mathbb{P}_{L,1}$ [resp. $E_1$] is also true, with the same proof.

therefore the image of any torsion sheaf appearing on the left will be zero, being a subsheaf of a torsion-free sheaf supported on a codimension 1 subvariety. Then we tensor the short exact sequence with $\pi^*_{\widetilde{C}_1}\mathcal{O}_{\{p_1,p_2\}}$ to obtain

$$0\longrightarrow T\longrightarrow\mathcal{O}_{\widetilde{C}_1}(-p_i)|_{\{p_1,p_2\}}\boxtimes\pi_*\mathcal{O}_N\longrightarrow\mathcal{O}_{\{p_1,p_2\}}\boxtimes\pi_*\mathcal{O}_N\longrightarrow\mathcal{O}_{\{p_i\}}\boxtimes\pi_*\mathcal{O}_N\longrightarrow 0,$$

from which it is clear that $T\simeq\mathcal{O}_{\{p_i\}}\boxtimes\pi_*\mathcal{O}_N$, and therefore $\mathcal{F}_1|_{\{p_i\}\times C}\simeq\pi_*\mathcal{O}_N$.

An identical process defines a sheaf $\mathcal{F}_2$ on $\widetilde{C}_2\times C$ such that

$$\begin{array}{ccccc} \mathcal{E}_{L,2}|_{\widetilde{C}_2\times C} & \longrightarrow & \mathcal{A}_2|_{\widetilde{C}_2\times C}\otimes\mathcal{F}_2 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{E}_{L,1}|_{\widetilde{C}_1\times C} & \longrightarrow & \mathcal{A}_1|_{\widetilde{C}_1\times C}\otimes\mathcal{F}_1 & \longrightarrow & 0 \end{array},$$

where we identify $\widetilde{C}_2$ and $\widetilde{C}_1$ via the isomorphism $\varepsilon_2|_{\widetilde{C}_2}$. Since the cokernel of the vertical maps is again $\mathcal{O}_{\{\widetilde{p}_1,\widetilde{p}_2\}}\boxtimes\pi_*\mathcal{O}_N$, the exact same proof as above shows that $\mathcal{F}_2|_{\{q\}\times C}\simeq\mathcal{O}_C(q)$ if $q\in\widetilde{C}_2$, $q\neq\widetilde{p}_1,\widetilde{p}_2$, and $\mathcal{F}_2|_{\{\widetilde{p}_i\}\times C}\simeq\pi_*\mathcal{O}_N$ for $i=1,2$.

We define $\mathcal{E}_{L,3}$ to be the kernel of the map $(\varepsilon_3,1)^*\mathcal{E}_{L,2}\to(\varepsilon_3,1)^*(\mathcal{A}_2|_{\widetilde{C}_2\times C}\otimes\mathcal{F}_2)$.

## 4.4 Relation between $\mathcal{E}_{L,3}$ and $\phi_{L,3}$

**Proposition 4.7.** *For every $q\in\widetilde{C}_2$, $q\neq\widetilde{p}_1,\widetilde{p}_2$, $\phi_{L,3}|_{E_3|_q}$ is a morphism, and it maps $E_3|_q$ isomorphically to $\mathbb{P}(\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$.*

*Proof.* We proved in section 4.1 that $E_3|_q\simeq\mathbb{P}(\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$, and therefore we need to show that under this identification, $\phi_{L,3}$ is the identity map. The proposition follows from proposition 4.8. □

**Proposition 4.8.** *The restriction of $\mathcal{E}_{L,3}$ to $E_3|_q\times C$ is the extension of $\pi^*_C L(-q)$ by $\mathcal{O}_{E_3|_q}(1)\boxtimes\mathcal{O}_C(q)$ that corresponds to the identity under the identification of this extension space with* $\mathrm{Hom}\,(\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q)),\mathrm{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$.

*Proof.* This proof is very similar to the proof of proposition 2.9. Let $H\subseteq\mathbb{P}_L$ be a linear hyperplane that contains $q$, does not contain $p$, and is transverse to the curve

$C$ at $q$ (i.e., $H$ does not contain $T_qC$). Then $H$ is isomorphic to its strict transform in $\mathbb{P}_{L,2}$, which we shall still denote by $H$. It is clear that $\mathcal{E}_{L,2}|_{H\times C}$ is $\mathcal{E}_L|_{H\times C}$, and therefore it is an extension $0 \to \mathcal{O}_H(1) \to \mathcal{E}_{L,2}|_{H\times C} \to L \to 0$ on $H \times C$.

Let $\sigma : \widetilde{H} \to H$ be the blow-up of $H$ at $q$, and let $E' \subseteq \widetilde{H}$ be the exceptional divisor. Then there exists a commutative diagram on $\widetilde{H} \times C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) \otimes \mathcal{O}_{\widetilde{H}}(-E') & \longrightarrow & \mathcal{E}_{L,3}|_{\widetilde{H}\times C} & \longrightarrow & \mathcal{B}_3|_{\widetilde{H}\times C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} & \longrightarrow & L & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{E'\times C} & \longrightarrow & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) & \longrightarrow & \mathcal{O}_{E'\times\{q\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{B}_3$ is defined exactly in the same way we defined $\mathcal{B}_1$ and $\mathcal{B}_2$, and the columns are exact because of lemma 3.15 [8].

Let $\mathcal{E}'_H$ be the push-forward of $(\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C}$ via $\sigma^*\mathcal{O}_H(1) \hookrightarrow \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) & \longrightarrow & (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \longrightarrow & \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q) & \longrightarrow & \mathcal{E}'_H & \longrightarrow & L & \longrightarrow & 0
\end{array} . \tag{4.2}$$

Then the restriction of $\mathcal{E}'_H$ to $E' \times C$ splits. Indeed, $\sigma^*\mathcal{O}_H(1)|_{E'} \simeq \mathcal{O}_{E'}$, and since $\operatorname{Ext}^1_{E'\times C}(L, \mathcal{O}_C(q)) \simeq H^0(E', \mathcal{O}_{E'}) \otimes \operatorname{Ext}^1_C(L, \mathcal{O}_C(q))$ by lemma B.1, we see that $\mathcal{E}'_H|_{E'\times C}$ splits as long as $\mathcal{E}'_H|_{\{x\}\times C}$ splits for some $x \in E'$. Resctricting the diagram (4.2) above to $\{x\} \times C$ for any $x \in E'$, we see that $\mathcal{E}'_H|_{\{x\}\times C}$ is the trivial extension $\psi_q(\mathcal{E}_L|_{\{q\}\times C})$.

[8] In lemma 3.15, replace $Y$ with $\widetilde{H}$, and $E_2 \subseteq \mathbb{P}_{L,2}$ with $E_3 \subseteq \mathbb{P}_{L,3}$.

Therefore, there exists a surjective map $\mathcal{E}'_H \to \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q)$, and we can define $\mathcal{E}'_{H,1}$ to be its kernel:

$$0 \longrightarrow \mathcal{E}'_{H,1} \longrightarrow \mathcal{E}'_H \longrightarrow \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \longrightarrow 0.$$

Then there exists the following commutative diagram on $\widetilde{H} \times C$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & (\sigma^*\mathcal{O}_H(1) \otimes \mathcal{O}_{\widetilde{H}}(-E')) \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d] & \mathcal{E}'_{H,1} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d] & \mathcal{E}'_H \arrow[r] \arrow[d] & L \arrow[r] & 0. \\
 & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[r, equal] \arrow[d] & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Moreover, we have the following commutative diagram on $\widetilde{H} \times C$ that relates $\mathcal{E}'_H$ and $\mathcal{E}'_{H,1}$ to $\mathcal{E}_{L,2}|_{H\times C}$ and $\mathcal{E}_{L,3}|_{\widetilde{H}\times C}$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E}_{L,3}|_{\widetilde{H}\times C} \arrow[r, "i_1"] \arrow[d] & (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} \arrow[r] \arrow[d] & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}'_{H,1} \arrow[r, "i'_1"] \arrow[d] & \mathcal{E}'_H \arrow[r] \arrow[d] & \mathcal{O}_{E'} \boxtimes \mathcal{O}_C(q) \arrow[r] & 0\cdot \\
 & \sigma^*\mathcal{O}_H(1) \boxtimes \mathbb{C}_q \arrow[r, equal] \arrow[d] & \sigma^*\mathcal{O}_H(1) \boxtimes \mathbb{C}_q \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

When we restrict the first two rows of this diagram to $E' \times C$, and we look at the image of the restrictions of $i_1$ and $i'_1$ to $E' \times C$, we obtain the following diagram, where the first row shows that the restriction of $\mathcal{E}_{L,3}$ to $E_3|_q \times C \simeq E' \times C$ is an extension of $\pi_C^* L(-q)$ by $\mathcal{O}_{E_3|_q}(1) \boxtimes \mathcal{O}_C(q)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) & \longrightarrow & \mathcal{E}_{L,3}|_{E'\times C} & \xrightarrow{i_1|_{E'\times C}} & L(-q) & \longrightarrow & 0 \\ & & || & & \big\downarrow & & \big\downarrow & & . \\ 0 & \longrightarrow & \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q) & \longrightarrow & \mathcal{E}'_{H,1}|_{E'\times C} & \xrightarrow{i'_1|_{E'\times C}} & L & \longrightarrow & 0 \end{array}$$

This shows that $\mathcal{E}_{L,3}|_{E'\times C}$ is the pull-back of $\mathcal{E}'_{H,1}|_{E'\times C}$ via the pull-back of the inclusion $L(-q) \hookrightarrow L$ from $C$ to $E' \times C$. Here is a summary of how to construct $\mathcal{E}_{L,3}|_{E'\times C}$ [9]:

$$\begin{array}{ccc} \mathcal{E}_{L,2}|_{H\times C} & \in & \mathrm{Ext}\,^1_{H\times C}(L, \mathcal{O}_H(1)) \\ \downarrow & & \downarrow \\ (\sigma,1)^*\mathcal{E}_{L,2}|_{H\times C} & \in & \mathrm{Ext}\,^1_{\widetilde{H}\times C}(L, \sigma^*\mathcal{O}_H(1)) \\ \downarrow & & \downarrow \\ \mathcal{E}'_H & \in & \mathrm{Ext}\,^1_{\widetilde{H}\times C}(L, \sigma^*\mathcal{O}_H(1) \boxtimes \mathcal{O}_C(q)) \\ \downarrow & & \uparrow \\ \mathcal{E}'_{H,1} & \in & \mathrm{Ext}\,^1_{\widetilde{H}\times C}(L, (\sigma^*\mathcal{O}_H(1) \otimes \mathcal{O}_{\widetilde{H}}(-E')) \boxtimes \mathcal{O}_C(q)) \\ \downarrow & & \downarrow \\ \mathcal{E}'_{H,1}|_{E'\times C} & \in & \mathrm{Ext}\,^1_{E'\times C}(L, \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q)) \\ \downarrow & & \downarrow \\ \mathcal{E}_{L,3}|_{E'\times C} & \in & \mathrm{Ext}\,^1_{E'\times C}(L(-q), \mathcal{O}_{E'}(1) \boxtimes \mathcal{O}_C(q)) \end{array}$$

Using the isomorphisms

$$\mathrm{Ext}\,^1_{Y\times C}(L, F \boxtimes G) \simeq H^0(Y, F) \otimes \mathrm{Ext}\,^1_C(L, G)$$

of lemma B.1, we can understand what extension $\mathcal{E}_{L,3}|_{E'\times C}$ is by tracking the corresponding elements in these spaces.

Let $v_0, \ldots, v_n$ be a basis of $\mathrm{Ext}\,^1_C(L, \mathcal{O}_C)$ with $\mathrm{Span}\,\{v_1, \ldots, v_n\} = \langle H \rangle$, and $\mathrm{Span}\,\{v_0, v_1\} = \langle T_q C \rangle$. Let $v_0^*, \ldots, v_n^*$ be the corresponding dual basis in

[9] Note that one of the arrows goes in the other direction between the extension spaces.

$\operatorname{Ext}^1_C(L,\mathcal{O}_C)^*$, with $v_1^*,\dots,v_n^*$ a basis of $\langle H\rangle^* \simeq H^0(H,\mathcal{O}_H(1))$. Then $\mathcal{E}_{L,2}|_{H\times C}$ corresponds to the element

$$\sum_{i=1}^n v_i^*\otimes v_i \in H^0(H,\mathcal{O}_H(1))\otimes \operatorname{Ext}^1_C(L,\mathcal{O}_C).$$

We have the following diagram, where $\psi_{-q}$ is the natural linear homomorphism $\operatorname{Ext}^1_C(L,\mathcal{O}_C(q))\to \operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q))$ [10]:

$$\begin{array}{ccccc}
\mathcal{E}_{L,2}|_{H\times C} & \longleftrightarrow & \sum_{i=1}^n v_i^*\otimes v_i & \in & H^0(H,\mathcal{O}_H(1))\otimes \operatorname{Ext}^1_C(L,\mathcal{O}_C)\\
\downarrow & & \downarrow & & \downarrow\\
(\sigma,1)^*\mathcal{E}_{L,2} & \longleftrightarrow & \sum_{i=1}^n v_i^*\otimes v_i & \in & H^0(\widetilde{H},\sigma^*\mathcal{O}_H(1))\otimes \operatorname{Ext}^1_C(L,\mathcal{O}_C)\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{E}'_H & \longleftrightarrow & \sum_{i=2}^n v_i^*\otimes \psi_q(v_i) & \in & H^0(\widetilde{H},\sigma^*\mathcal{O}_H(1))\otimes \operatorname{Ext}^1_C(L,\mathcal{O}_C(q))\\
\downarrow & & \downarrow & & \uparrow\\
 & & & & H^0(\widetilde{H},\sigma^*\mathcal{O}_H(1)\otimes\mathcal{O}_{\widetilde{H}}(-E'))\\
\mathcal{E}'_{H,1} & \longleftrightarrow & \sum_{i=2}^n v_i^*\otimes \psi_q(v_i) & \in & \otimes\\
 & & & & \operatorname{Ext}^1_C(L,\mathcal{O}_C(q))\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{E}'_{H,1}|_{E'\times C} & \longleftrightarrow & \sum_{i=2}^n w_i^*\otimes \psi_q(v_i) & \in & H^0(E',\mathcal{O}_{E'}(1))\otimes \operatorname{Ext}^1_C(L,\mathcal{O}_C(q))\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{E}_{L,3}|_{E'\times C} & \longleftrightarrow & \sum_{i=2}^n w_i^*\otimes w_i & \in & H^0(E',\mathcal{O}_{E'}(1))\otimes \operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q))
\end{array},$$

where, for $2\le i\le n$, $w_i=\psi_{-q}(\psi_q(v_i))$. Therefore, $\mathcal{E}_{L,3}$ corresponds to the identity in $\operatorname{Hom}(\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q)),\operatorname{Ext}^1_C(L(-q),\mathcal{O}_C(q)))$, as claimed. $\square$

We now prove that $\phi_{L,3}|_{\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))}$ factors through the canonical isomorphism $\mathbb{P}(T_{\widetilde{p}_i}(E_2|_{p_i}))\xrightarrow{\simeq}\mathbb{P}(\operatorname{Im}\psi)$ described in section 4.1.

[10] Note that $\psi_q(v_1)=0$ and $\ker(\psi_{-q}\circ\psi_q)=\operatorname{Span}\{v_0,v_1\}$.

**Proposition 4.9.** *For $i \in \{1,2\}$, the extension*

$$\mathcal{E}_{L,3}|_{\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})\times C} \in \operatorname{Ext}^1_{\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})\times C}(L\otimes(\pi_*\mathcal{O}_N)^*, \mathcal{O}_{\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})}(1)\boxtimes\pi_*\mathcal{O}_N)$$

*corresponds to the inclusion in* $\operatorname{Hom}(\operatorname{Im}\psi, \operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N))$ *under the canonical identification* $\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}} \simeq \operatorname{Im}\psi$.

*Proof.* Let $i \in \{1,2\}$. We already saw in chapter 3 that $\mathcal{E}_{L,2}|_{E_2|_{p_i}\times C}$ induces the linear map given by projection from $\widetilde{p}_i$, i.e., it corresponds to a linear homomorphism $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_{p_i} \to \operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ of kernel $\langle\widetilde{p}_i\rangle$ and image $\operatorname{Im}\psi$ (lemma B.1).

Therefore, we can find a basis $w_1,\dots,w_n$ of $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_{p_i}$ and a basis $v_0,\dots,v_n$ of $\operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ such that $\langle\widetilde{p}_i\rangle = \operatorname{Span}\{w_1\}$, $\operatorname{Im}\psi = \operatorname{Span}\{v_2,\dots,v_n\}$, and $\operatorname{Span}\{w_1,w_j\}$ maps to $\operatorname{Span}\{v_j\}$ for every $2\le j\le n$ under the homomorphism corresponding to $\mathcal{E}_{L,2}|_{E_2|_{p_i}\times C}$. In particular, this sheaf corresponds to $\sum_{j=2}^n w_j^*\otimes v_j$ in $H^0(E_2|_{p_i}, \mathcal{O}_{E_2|_{p_i}}(1))\otimes\operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$.

To simplify the notation, let us denote $E_2|_{p_i}$ by $X$ and its blow-up at $\widetilde{p}_i$ by $\sigma:\widetilde{X}\to X$, with $E'$ the exceptional divisor. Then there exists a short exact sequence

$$0\longrightarrow\mathcal{E}_{L,3}|_{\widetilde{X}\times C}\longrightarrow(\sigma,1)^*\mathcal{E}_{L,2}|_{X\times C}\longrightarrow\mathcal{O}_{E'}\boxtimes\pi_*\mathcal{O}_N\longrightarrow 0,$$

obtained by restricting the short exact sequence defining $\mathcal{E}_{L,3}$ to $\widetilde{X}\times C$. It stays exact because of lemma 3.15.

There exists the following commutative diagram on $\widetilde{X}\times C$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{A}\boxtimes\pi_*\mathcal{O}_N \arrow[r] \arrow[d] & \mathcal{E}_{L,3}|_{\widetilde{X}\times C} \arrow[r] \arrow[d] & L\otimes(\pi_*\mathcal{O}_N)^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \sigma^*\mathcal{O}_X(1)\boxtimes\pi_*\mathcal{O}_N \arrow[r] \arrow[d] & (\sigma,1)^*\mathcal{E}_{L,2}|_{X\times C} \arrow[r] \arrow[d] & L\otimes(\pi_*\mathcal{O}_N)^* \arrow[r] & 0, \\
 & \mathcal{O}_{E'}\boxtimes\pi_*\mathcal{O}_N \arrow[r, equal] \arrow[d] & \mathcal{O}_{E'}\boxtimes\pi_*\mathcal{O}_N \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\mathcal{A} = \sigma^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\widetilde{X}}(-E')$. If we restrict the first row to $E' \times C$, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{O}_{E'}(1) \boxtimes \pi_*\mathcal{O}_N \longrightarrow \mathcal{E}_{L,3}|_{E'\times C} \longrightarrow L \otimes (\pi_*\mathcal{O}_N)^* \longrightarrow 0.$$

Remember that $E'$ is $\mathbb{P}(\mathcal{N}_{\{\widetilde{p}_i\}/E_2|_{p_i}})$. The following diagram, where we denoted $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ by $V$ to simplify the notation, illustrates the steps we took in finding $\mathcal{E}_{L,3}|_{E'\times C}$:

$$\begin{array}{ccccc}
\mathcal{E}_{L,2}|_{X\times C} & \longleftrightarrow & \sum_{j=2}^n w_j^* \otimes v_j & \in & H^0(X, \mathcal{O}_X(1)) \otimes V \\
\downarrow & & \downarrow & & \downarrow \\
(\sigma,1)^*\mathcal{E}_{L,2}|_{X\times C} & \longleftrightarrow & \sum_{j=2}^n w_j^* \otimes v_j & \in & H^0(\widetilde{X}, \sigma^*\mathcal{O}_X(1)) \otimes V \\
\downarrow & & \downarrow & & \uparrow \\
\mathcal{E}_{L,3}|_{\widetilde{X}\times C} & \longleftrightarrow & \sum_{j=2}^n w_j^* \otimes v_j & \in & H^0(\widetilde{X}, \mathcal{A}) \otimes V \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_{L,3}|_{E'\times C} & \longleftrightarrow & \sum_{j=2}^n v_j^* \otimes v_j & \in & H^0(E', \mathcal{O}_{E'}(1)) \otimes V
\end{array}.$$

Therefore, $\mathcal{E}_{L,3}|_{E'\times C}$ corresponds to the inclusion $\operatorname{Im}\psi \hookrightarrow \operatorname{Ext}^1_C(L\otimes(\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$, as claimed. □

Let $l \in L_1$, $l \neq p_1, p_2$, and let $Y_l := s_{E_l}(L_1)$, where $s_{E_l}$ is the section of $E_2 \to L_1$ defined in section 4.2. Remember that we denoted by $y_{l,i}$ the only point of intersection of the strict transform $\widetilde{Y}_l$ of $Y_l$ with $E_3|_{\widetilde{p}_i}$ $(i = 1, 2)$.

**Proposition 4.10.** *The restriction of $\mathcal{E}_{L,3}$ to $\widetilde{Y}_l \times C$ is a non-zero element of $\operatorname{Ext}^1_{\widetilde{Y}_l\times C}(L \otimes (\pi_*\mathcal{O}_N), \pi_*\mathcal{O}_N) \simeq H^0(\widetilde{Y}_l, \mathcal{O}_{\widetilde{Y}_l}) \otimes \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N), \pi_*\mathcal{O}_N)$, where $\widetilde{Y}_l$ is the strict transform of $Y_l$ in $\mathbb{P}_{L,3}$. In particular $\mathcal{E}_{L,3}|_{\{y_{l,i}\}\times C} \simeq E_l$ for $i = 1, 2$.*

For the proof, we need the following result.

**Lemma 4.11.** *The restriction of $\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)$ to $L_2 \simeq \mathbb{P}^1$ is $\mathcal{O}_{L_2}(1)$.*

*Proof.* Remember that $L_2 = \widetilde{T_pC} \cap E_2$. It is isomorphic to $L_1 = \widetilde{T_pC} \cap E_1$ via $\varepsilon_2$, and it is therefore the exceptional divisor of the blow-up of $T_pC$ at $p$. Therefore,

$$\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{L_2} = (\mathcal{O}_{\mathbb{P}_{L,2}}(-E_2)|_{\widetilde{T_pC}})|_{L_2} = \mathcal{O}_{\widetilde{T_pC}}(-L_2)|_{L_2} = \mathcal{O}_{L_2}(1).$$

□

*Proof* (*of proposition* 4.10). We saw in lemma 3.13 that there exists a short exact sequence $0 \to (\varepsilon_2^* \mathcal{O}_{L_1}(1) \otimes \mathcal{O}_{E_2}(-E_2)) \boxtimes \pi_* \mathcal{O}_N \to \mathcal{E}_{L,2}|_{E_2 \times C} \to L \otimes (\pi_* \mathcal{O}_N)^* \to 0$ on $E_2 \times C$. If we restrict it to $Y_l \times C$, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_l}(2) \boxtimes \pi_* \mathcal{O}_N \longrightarrow \mathcal{E}_{L,2}|_{Y_l \times C} \longrightarrow L \otimes (\pi_* \mathcal{O}_N)^* \longrightarrow 0,$$

because $\mathcal{O}_{E_2}(-E_2)|_{Y_l} = \mathcal{O}_{Y_l}(1)$ since $Y_l$ and $L_2$ are in the same linear system[11]. Therefore, there exists the following commutative diagram on $\widetilde{Y}_l \times C \simeq Y_l \times C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\widetilde{Y}_l} \boxtimes \pi_* \mathcal{O}_N & \longrightarrow & \mathcal{E}_{L,3}|_{\widetilde{Y}_l \times C} & \longrightarrow & L \otimes (\pi_* \mathcal{O}_N)^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{Y_l}(2) \boxtimes \pi_* \mathcal{O}_N & \longrightarrow & \mathcal{E}_{L,2}|_{Y_l \times C} & \longrightarrow & L \otimes (\pi_* \mathcal{O}_N)^* & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\{\widetilde{p}_1, \widetilde{p}_2\}} \boxtimes \pi_* \mathcal{O}_N & = & \mathcal{O}_{\{\widetilde{p}_1, \widetilde{p}_2\}} \boxtimes \pi_* \mathcal{O}_N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and $\mathcal{E}_{L,3}|_{\widetilde{Y}_l \times C}$ is an element of $\operatorname{Ext}^1_{\widetilde{Y}_l \times C}(L \otimes (\pi_* \mathcal{O}_N), \pi_* \mathcal{O}_N)$. Since this is isomorphic to $H^0(\widetilde{Y}_l, \mathcal{O}_{\widetilde{Y}_l}) \otimes \operatorname{Ext}^1_C(L \otimes (\pi_* \mathcal{O}_N), \pi_* \mathcal{O}_N)$ by lemma B.1, $\mathcal{E}_{L,3}|_{\widetilde{Y}_l \times C}$ is non-zero because we know that $\mathcal{E}_{L,3}|_{\{y\} \times C}$ does not split for every $y \in \widetilde{Y}_l$, $y \neq y_{l,1}, y_{l,2}$. □

[11] Each section of $E_2 \to L_1$ corresponds to a line subbundle of $\mathcal{N}_{L_1/\mathbb{P}_{L,1}}$, which we saw to be isomorphic to $\mathcal{O} \oplus \mathcal{O}_{L_1}(-1)$, with $\mathcal{O} := \mathcal{O}_{L_1}(1) \oplus \cdots \oplus \mathcal{O}_{L_1}(1)$. Both $Y_l$ and $L_2$ are images of sections of $E_2 \to L_1$ of the form $\mathcal{O}_{L_1}(-1) \xrightarrow{(\alpha, \mathrm{id})} \mathcal{O} \oplus \mathcal{O}_{L_1}(-1)$, since they both do not intersect $\mathbb{P}(\mathcal{O}) = \mathbb{P}(\mathcal{N}_{L_1/E_1})$.

## 4.5 Where to go next

As mentioned in chapter 2, the next step would be to study the fibers of the morphism $\phi_{L,3}$, and then to use the morphism to study the moduli space $\overline{\mathcal{SU}_C(2,L)}$. The most interesting case would probably be when $g = 2$, since then $\dim \overline{\mathcal{SU}_C(2,L)} = 3$, and $\dim \mathbb{P}_L = \deg L$.

Next, since our rational maps factor through projective spaces like $\mathbb{P}(H')$ and $\mathbb{P}(\operatorname{Ext}^1_C(L(-q), \mathcal{O}_C(q)))$ when restricted to fibers of the exceptional divisors, we can use our construction to study other different compactifications of $\mathcal{SU}_C(2,L)$. We are particularly interested in the following two other important compactifications of this moduli space: The one introduced by Bhosle in [Bho92], which we used in this work, and another one introduced by Gieseker in [Gie84]. Bhosle's construction uses generalized parabolic bundles, and Gieseker's uses sheaves on semi-stable models of the curve.

Finally, we would like to find a good way to generalize this construction to higher degree. The indeterminancy locus is not as easy to calculate after the first blow-up, even when $\deg L = 5$.

# Chapter 5

## The base locus of the generalized theta divisor

### 5.1 A brief survey

Let $C$ be a smooth irreducible projective curve of genus $g \geq 2$. For a positive integer $r$ and an integer $d$, let $\mathcal{U}_C(r,d)$ be the moduli space of (S-equivalence classes of) semi-stable[1] vector bundles of rank $r$ and degree $d$. If $L$ is a line bundle on $C$, let $\mathcal{SU}_C(r,L)$ be the moduli space of (S-equivalence classes of) semi-stable vector bundles of rank $r$ and determinant $L$. If $L_1, L_2$ are two line bundles of degree $d$, then[2]

$$\mathcal{SU}_C(r,L_1) \simeq \mathcal{SU}_C(r,L_2).$$

Therefore, we shall denote this moduli space also by $\mathcal{SU}_C(r,d)$ when we do not want to emphasize which line bundle has been fixed as determinant.

Also, if $d_1 \equiv d_2 \pmod{r}$, then[3]

$$\mathcal{SU}_C(r,d_1) \simeq \mathcal{SU}_C(r,d_2) \quad \text{and} \quad \mathcal{U}_C(r,d_1) \simeq \mathcal{U}_C(r,d_2).$$

Therefore, for each $r$, it is enough to consider degrees $d$ such that $0 \leq d < r$.

[1] A vector bundle $E$ of rank $r$ and degree $d$ is called semi-stable if, for every subbundle $F \subseteq E$, $r \deg F \leq d \mathrm{rk} F$.

[2] The isomorphism is $E \mapsto E \otimes L$, where $L$ is a line bundle such that $L^r \simeq L_1^{-1} \otimes L_2$.

[3] The isomorphisms are both given by $E \mapsto E \otimes L$, where $L$ is an appropriate line bundle of degree $(d_2 - d_1)/r$.

Finally, if $d_1 + d_2 = r$, $0 < d_1, d_2 < r$, then $-d_1 \equiv d_2 \pmod r$, and

$$\begin{array}{rcl} \mathcal{SU}_C(r, d_1) & \xrightarrow{\simeq} & \mathcal{SU}_C(r, -d_1) \simeq \mathcal{SU}_C(r, d_2) \\ \mathcal{U}_C(r, d_1) & \xrightarrow{\simeq} & \mathcal{U}_C(r, -d_1) \simeq \mathcal{U}_C(r, d_2) \\ E & \mapsto & E^* \end{array}$$

and it is enough to consider degrees $d$ such that $0 \leq d \leq r/2$.

The Picard group of $\mathcal{SU}_C(r, d)$ is generated by one ample line bundle (see [DreNar89]), which we shall denote by $\mathcal{L}_{r,d}$. If $\Theta_{r,d}$ is a divisor on $\mathcal{SU}_C(r, d)$ such that $\mathcal{L}_{r,d} = \mathcal{O}_{\mathcal{SU}_C(r,d)}(\Theta_{r,d})$, then $\Theta_{r,d}$ is called a generalized theta divisor. We are interested in the base locus of its linear system.

If $d = 0$, a point in the base locus corresponds to a semi-stable vector bundle $E$ with integral slope[4] $\mu(E) \leq g - 1$ such that $H^0(E \otimes L) \neq 0$ for every line bundle $L$ of degree 0 [5]. This was proved by Beauville in [Bea88] for $r = 2$ and by Beauville, Narasimhan and Ramanan in [BeaNarRam89] in general. Raynaud studied such bundles in [Ray82], and Beauville summarizes his results as follows in [Bea95].

**Theorem (Raynaud).** *(a) For $r = 2$, the linear system $|\Theta_{2,0}|$ has no base points.*

*(b) For $r = 3$, $|\Theta_{3,0}|$ has no base points if $g = 2$, or if $g \geq 3$ and $C$ is generic.*

*(c) Let $n$ be an integer $\geq 2$ dividing $g$. For $r = n^g$, the linear system $|\Theta_{r,0}|$ has base points.*

Using the bundles constructed by Raynaud, we improve the result in part (c) of his theorem.

**Theorem 5.1.** *If $r \geq 2^g$, then $|\Theta_{r,0}|$ has base points. Moreover, the dimension of its base locus is at least $(r - 2^g)^2(g - 1) + 1$* [6].

[4] The slope of a vector bundle $E$ is defined as $\mu(E) = \deg E/\mathrm{rk}E$.

[5] In particular, then, $\mu(E) \geq 0$.

[6] Note that $n^2(g - 1) + 1$ is the dimension of $\mathcal{U}_C(n, 0)$.

We also prove that, for some covering curves, the base locus of $|\Theta_{r,0}|$ is non-empty for ranks smaller than $2^g$.

**Proposition 5.2.** *Let $\widetilde{C} \to C$ be an $n:1$ covering of smooth irreducible projective curves of genus $\widetilde{g}$ and $g$, respectively. If $s$ is a positive integer such that*[7]

$$\left\lceil \frac{g}{n} \right\rceil \geq \left\lceil \frac{g}{s} \right\rceil + 1,$$

*then there exists a vector bundle in the base locus of $|\widetilde{\Theta}_{s^g,0}|$, where $\widetilde{\Theta}_{s^g,0}$ is a generalized theta divisor in $\mathcal{SU}_{\widetilde{C}}(s^g,0)$.*

*Remark.* Note that the condition $\lceil g/n \rceil \geq \lceil g/s \rceil + 1$ implies that $\lceil g/n \rceil \geq 2$. Therefore, $g/n > 1$, i.e., $n < g$.

**Corollary 5.3.** *For every covering as above with $1 < n < g$, there exists an $s$ satisfying the condition above such that $s^g < 2^{\widetilde{g}}$, unless $g = 4$ and $n = 2$.*

About stability of the vector bundles in the base locus, note that all of the vector bundles constructed in the proof of theorem 5.1 are semi-stable and not stable, but we prove the following[8]

**Proposition 5.4.** *If $r_0 = \min\{r \in \mathbb{N} \mid |\Theta_{r,0}|$ has base points$\}$, then every vector bundle in the base locus of $|\Theta_{r_0,0}|$ is stable.*

To our knowledge, no stable bundles have been proved to exist in the base locus of $|\Theta_{r,0}|$ if $r > r_0$.

If $d = 1$, we have the following result proved by Brivio and Verra in [BriVer99] for $r = 2$ and [BriVer02] in general.

**Theorem (Brivio,Verra).** *The line bundle $\mathcal{L}_{r,1}$ is very ample for every positive $r$.*

[7]For a real number $m$, we denote by $\lceil m \rceil$ the smallest integer $\geq m$.

[8]This is probably a known fact to the experts in the area, but we could not find it in the literature.

In particular, the linear systems $|\Theta_{r,1}|$ are base-point-free.

To our knowledge, nothing is known about the base locus of the linear systems $|\Theta_{r,d}|$ if $2 \leq d \leq r/2$.

## 5.2 Proof of theorem 5.1

In this section, we shall set $d = 0$, and drop the degree from the notation of the generalized theta divisor explained above. Also, since all of the moduli spaces of the form $\mathcal{SU}_C(r, L)$ for some line bundle $L$ of degree 0 are isomorphic, we shall assume that $L = \mathcal{O}_C$.

The proof of theorem 5.1 can be divided in two parts. We first use the bundles constructed by Raynaud in [Ray82] to show that the base locus of $|\Theta_r|$ is always non-empty if $r = 2^g$, and then we prove the following lemma.

**Lemma 5.5.** *If $E$ is a vector bundle in the base locus of $|\Theta_r|$, then, for every $F \in \mathcal{U}_C(n, 0)$, the vector bundles of the form $(E \oplus F) \otimes L$ (with $L \in Pic^0(C)$) which have trivial determinant are in the base locus of $|\Theta_{r+n}|$. Moreover, the space of all such bundles has dimension $n^2(g-1)+1$.*

As we saw above, Beauville says in [Bea95] that the linear system $|\Theta_r|$has base points for $r = n^g$ where $n$ is an integer dividing $g$. What Raynaud actually constructs is, for every $n$, a semi-stable vector bundle of rank $n^g$ and slope $g/n$ such that $H^0(E \otimes L) \neq 0$ for every line bundle $L$ of degree 0. Then, for $g, n \geq 2$, we have $g/n \leq g - 1$.

If $g$ is even, we are then done because $n = 2$ divides $g$, and we obtain base points for $r = n^g = 2^g$. If $g$ is odd, it is actually possible to construct, using Raynaud's bundle, another vector bundle with integral slope and the same property. This is actually proved by Raynaud's himself[9]:

[9]For our purposes we can say that a semi-stable vector bundle has the property $(*)$ if $0 \leq \mu(E) \leq g-1$ and $H^0(E \otimes L) \neq 0$ for every line bundle $L$ of degree 0.

**Proposition ([Ray82, 1.7.2]).** *Let $r_0$ be an integer. Every semi-stable vector bundle on $C$ of rank $r \leq r_0$ satisfies $(*)$ if and only if every stable vector bundle of slope $g-1$ and rank $\leq r_0$ does.*

We shall prove this stronger statement.

**Proposition 5.6.** *If $E$ is a semi-stable vector bundle of slope $\mu(E) \leq g-1$ such that $H^0(E \otimes L) \neq 0$ for every line bundle $L$ of degree 0, then there exists a semi-stable vector bundle $E'$ of rank $\leq rkE$ and slope $\lceil \mu(E) \rceil (\leq g-1)$ with the same property.*

*Proof.* If $\mu(E)$ is an integer, we can take $E' = E$. If $\mu(E) < \lceil \mu(E) \rceil$, let $E_1$ be an elementary transformation of $E$, i.e., $0 \to E \to E_1 \to \mathbb{C}_p \to 0$, for some $p \in C$. Clearly, $\mathrm{rk} E_1 = \mathrm{rk} E$, and $H^0(E_1 \otimes L) \supseteq H^0(E \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^0(C)$.

Assume that $E_1$ is not semi-stable. Let $F_1 \subseteq E_1$ be a subbundle of maximal slope. Then both $F_1$ and $E_1/F_1$ are semi-stable vector bundles. Moreover, the short exact sequence $0 \to F_1 \to E_1 \to E_1/F_1 \to 0$ induces, for every $L \in \mathrm{Pic}^0(C)$, a long exact sequence of cohomology

$$0 \longrightarrow H^0(F_1 \otimes L) \longrightarrow H^0(E_1 \otimes L) \longrightarrow H^0((E_1/F_1) \otimes L) \longrightarrow \cdots$$

which implies, by semi-continuity, that either $H^0(F_1 \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^0(C)$ or $H^0((E_1/F_1) \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^0(C)$. Let $G$ be the vector bundle (either $F_1$ or $E_1/F_1$) for which this is true.

Let $F = E \cap F_1$. Since $\mu(F_1) > \mu(E_1) > \mu(E)$, and $E$ is semi-stable, $F \neq F_1$, and we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & F_1 & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & E/F & = & E_1/F_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $E$ is semi-stable, $\mu(F) \leq \mu(E) < \lceil \mu(E) \rceil$, and therefore $\mu(F_1) \leq \lceil \mu(E) \rceil$. Also, $\mu(E) \leq \mu(E/F) = \mu(E_1/F_1) < \mu(E_1) \leq \lceil \mu(E) \rceil$.

To summarize, we proved that, if $E_1$ is not semi-stable, then there exists a vector bundle $G$ of rank $\mathrm{rk}G < \mathrm{rk}E_1$ with $\mu(E) \leq \mu(G) \leq \lceil \mu(E) \rceil$, and $H^0(G \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^0(C)$. If $E_1$ is semi-stable, then we produced a vector bundle of rank $\mathrm{rk}E$ with $\mu(E) < \mu(E_1) \leq \lceil \mu(E) \rceil$ with the same property. If $E_1$ is semi-stable, let $E_1' = E_1$. Otherwise, let $E_1' = G$.

We can now repeat the process, starting with $E_1'$ as $E$, and we can produce semi-stable vector bundles $E_i'$ such that $\mu(E_i') \leq \lceil \mu(E) \rceil$, and $H^0(E_i' \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^0(C)$. The process ends (i.e., $\mu(E_i') = \lceil \mu(E) \rceil$ for some $i$) after a finite number of steps, because at each step either the rank decreases or the degree increases, and $\mathrm{rk}E_i'$ [resp. $\deg E_i'$] is bounded below by 1 [resp. above by $\mathrm{rk}E_i' \cdot \lceil \mu(E) \rceil$]. $\square$

We shall now prove lemma 5.5.

*Proof* (*of lemma* 5.5). Let $E$ be a vector bundle in the base locus of $|\Theta_r|$. There are two special cases for which the proof is simpler: when $n = 1$ and when $E$ is stable. We shall explain how these two cases work as we go through the proof.

Consider the set of extensions $0 \to E \otimes L_F \to W_F \to F \to 0$, where $F \in \mathcal{U}_C(n, 0)$, and $L_F$ is a line bundle such that $L_F^r \simeq \det F^*$ [10]. Let us show that $W_F$ is a vector bundle of rank $r+n$ that is in the base locus of $|\Theta_{r+n}|$.

First of all, $W_F$ is semi-stable (see [LeP97, 5.3.5]). Also, note that, for a fixed $F$, all extensions of $F$ by $E \otimes L_F$ are $S$-equivalent to $(E \otimes L_F) \oplus F$, so it suffices to consider $W_F = (E \otimes L_F) \oplus F$. Moreover, $W_F$ contains $E \otimes L_F$, and so, for every $L \in \mathrm{Pic}^{g-1}C$, $H^0(E \otimes L_F \otimes L) \neq 0 \Rightarrow H^0(W_F \otimes L) \neq 0$, since $\deg(L_F \otimes L) = g - 1$, and $E$ is in the base locus of $|\Theta_r|$. And $W_F$ has also trivial determinant; indeed,

$$\det W_F \simeq \det(E \otimes L_F) \otimes \det F \simeq \det E \otimes L_F^r \otimes \det F \simeq \det E \simeq \mathcal{O}_C.$$

The idea for the rest of the proof is that at most a finite number of $W_F$'s have the same S-equivalence class, and so the dimension of the vector bundles in the base locus of $|\Theta_{r+n}|$ that we constructed is $\dim \mathcal{U}(n, 0) = n^2(g-1) + 1$.

Let $\varphi$: $\mathrm{Pic}^0 C \times \mathcal{U}(n, 0) \to \mathcal{U}(r+n, 0)$ be the morphism $(L, F) \mapsto (E \otimes L) \oplus F$. There is a natural embedding $\mathcal{SU}_C(r+n, \mathcal{O}_C) \subseteq \mathcal{U}(r+n, 0)$, and we claim that[11]

**Claim.** *Let*

$$B = \varphi(\mathrm{Pic}^0 C \times \mathcal{U}(n, 0)) \cap \mathcal{SU}_C(r+n, \mathcal{O}_C).$$

*Then* $\dim B = n^2(g-1) + 1$, *and* $B$ *is contained in the base locus of* $|\Theta_{r+n}|$.

*Proof.* It is clear that $B$ is contained in the base locus of $|\Theta_{r+n}|$. The preimage of $\mathcal{SU}_C(r+n, \mathcal{O}_C)$ under $\varphi$ is $A = \{(L, F) \in \mathrm{Pic}^0 C \times \mathcal{U}(n, 0) \mid L^r \simeq \det F^*\}$, and $B = \varphi(A)$.

[10] If $n = 1$, the proof is a little bit simpler; we do not need to use $L_F$, since $\det F^*$ is just $F^{-1}$. We can just use the set of extensions $0 \to E \otimes F^{-1} \to W_F \to F^r \to 0$, with $F$ a line bundle of degree 0.

[11] For $n = 1$, we use the map $\varphi : \mathrm{Pic}^0 C \to \mathcal{SU}_C(r+1, \mathcal{O}_C)$ given by $F \mapsto (E \otimes F^{-1}) \oplus F^r$. If we then let $B = \varphi(\mathrm{Pic}^0 C) \subseteq \mathcal{SU}_C(r+1, \mathcal{O}_C)$, the claim is that $B$ is contained in the base locus of $|\Theta_{r+1}|$ and $\dim B = g$.

The claim now is that $\dim A = \dim \mathcal{U}(n,0)$ and the general fiber of $\varphi$ is finite. We shall assume in what follows that $F$ is a stable vector bundle; this does not effect the computation since a generic element of $\mathcal{U}(n,0)$ is stable.

Consider the second projection $A \to \mathcal{U}(n,0)$: it is surjective because for every stable $F \in \mathcal{U}(n,0)$ there exists $L \in \mathrm{Pic}^0 C$ such that $L^r \simeq \det F^*$. Moreover, there are only a finite number of them, and this proves the claim about the dimension.

Let $\oplus_{i=1}^k \mathrm{Gr}(E)_i$ be the associated grading of $E$ (see [LeP97, section 5.3]). Since $F$ is a stable bundle, the associated grading for $\varphi(L,F)$ is

$$\bigoplus_{i=1}^{k} (\mathrm{Gr}(E)_i \otimes L) \oplus F.$$

Let $H \in B$; it is of the form $\varphi(L,F) = (E \otimes L) \oplus F$ for some $(L,F) \in A$. We shall prove that there are only a finite number of $(L', F') \in A$ such that $(E \otimes L) \oplus F$ is S-equivalent to $(E \otimes L') \oplus F'$, i.e., there exists a permutation $\sigma \in S_{k+1}$ such that, for every $i \in \{1, \ldots, k+1\}$, $\mathrm{Gr}(\varphi(L,F))_i \simeq \mathrm{Gr}(\varphi(L',F'))_{\sigma(i)}$ [12].

There are two cases[13].

Case I: $k \geq 2$. Then there exist at least one $i \in \{1, \ldots, k\}$ such that $\sigma(i)$ is also in $\{1, \ldots, k\}$. Then, setting $\mathrm{Gr}_i = \mathrm{Gr}(E)_i$ for $i = 1, \ldots, k$,

$$\begin{aligned}
\mathrm{Gr}_i \otimes L \simeq \mathrm{Gr}_{\sigma(i)} \otimes L' \quad &\Rightarrow \quad \det(\mathrm{Gr}_i \otimes L) \simeq \det(\mathrm{Gr}_{\sigma(i)} \otimes L') \\
&\Rightarrow \quad \det \mathrm{Gr}_i \otimes L^n \simeq \det \mathrm{Gr}_{\sigma(i)} \otimes (L')^n \\
&\Rightarrow \quad (L')^n \simeq \det \mathrm{Gr}_i \otimes L^n \otimes \det \mathrm{Gr}_{\sigma(i)}^{-1},
\end{aligned}$$

where $n = \mathrm{rk}\mathrm{Gr}_i = \mathrm{rk}\mathrm{Gr}_{\sigma(i)}$, and there exist only a finite number of $L'$ with this property. We shall show now that $F'$ is uniquely determined. There are two subcases.

Subcase I: $\sigma(k+1) = k+1$. Then $F' \simeq F$.

[12] For the case $n = 1$, the associated grading for $H$ is $\bigoplus_{i=1}^k (\mathrm{Gr}(E)_i \otimes F^{-1}) \oplus F^r$.

[13] When $E$ is stable, we only have the second case. See the note after the proof.

Subcase II: $\sigma(j) = k+1$ for some $j \neq k+1$. Then $F' \simeq \mathrm{Gr}_j \otimes L$ [14].

Case II: $k = 1$, i.e., $E$ is stable. If $\sigma = \mathrm{id}_{S_2}$, we use the same technique as before. If $\sigma(1) = 2$ and $\sigma(2) = 1$, then $F'$ is uniquely determined by $E \otimes L \simeq F'$, and

$$\begin{aligned} E \otimes L' \simeq F \quad &\Rightarrow \quad \det(E \otimes L') \simeq \det F \\ &\Rightarrow \quad \det E \otimes (L')^r \simeq \det F \\ &\Rightarrow \quad (L')^r \simeq \det E^{-1} \otimes \det F; \end{aligned}$$

again there are only a finite number of $L'$ that satisfy this property[15]. $\square$

Note that we could assume at the beginning of the proof that $E$ is stable. Indeed, assume for a second that we prove that, for $E$ stable in the base locus of $|\Theta_r|$, we can costruct a family of semi-stable vector bundles in the base locus of $|\Theta_{r+n}|$ of dimension $n^2(g-1)+1$. As we shall prove in proposition 5.4, if the base locus of $|\Theta_r|$ is not empty, there exist a stable vector bundle in the base locus of $|\Theta_{r'}|$ for some $r' \leq r$, and doing the construction using the stable bundle, we would find a family of vector bundles in the base locus of $|\Theta_{r+n}|$ of dimension $(r+n-r')^2(g-1)+1 \geq n^2(g-1)+1$, which is what we want to prove.

We wrote the proof for the general case because many of the concrete examples that have been provided so far of vector bundles in the base locus are examples of vector bundles that are not known to be stable. The construction can be used for those vector bundles to construct some other concrete vector bundles of higher rank in the base locus of the generalized theta divisor.

We shall now end the section with the proof of proposition 5.4.

[14] For $n = 1$: Case I: $k \geq 2$. Then there exist at least one $i \in \{1, \ldots, k\}$ such that $\sigma(i) \in \{1, \ldots, k\}$. Then $\mathrm{Gr}_i \otimes F^{-1} \simeq \mathrm{Gr}_{\sigma(i)} \otimes (F')^{-1} \Rightarrow (F')^n \simeq \det \mathrm{Gr}_i^{-1} \otimes F^n \otimes \det \mathrm{Gr}_{\sigma(i)}$, where $n = \mathrm{rk}\mathrm{Gr}_i = \mathrm{rk}\mathrm{Gr}_{\sigma(i)}$, and there exist only a finite number of $F'$ with this property.

[15] For $n = 1$: Case II: $k = 1$, i.e., $E$ is stable. Since $E$ is in the base locus of $|\Theta_r|$, it cannot be a line bundle; hence, $\sigma = \mathrm{id}$. Then $E \otimes F^{-1} \simeq E \otimes (F')^{-1}$ implies $(F')^r \simeq F^r$, and there exist again only a finite number of $F'$ with this property.

*Proof.* Let $E$ be a vector bundle in the base locus of $|\Theta_{r_0}|$; $E$ is in the same equivalence class of its associated grading $\oplus_{i=1}^k \mathrm{Gr}_i$. Since $H^0(E \otimes L) \neq 0$ if and only if there exists an $i$ such that $H^0(\mathrm{Gr}_i \otimes L) \neq 0$, we see by semicontinuity that there exists $i$ such that $\mathrm{Gr}_i$ is in the base locus of $|\Theta_{\mathrm{rk}\mathrm{Gr}_i}|$. By the minimality of $r_0$, $k = 1$, and $E$ is stable. □

## 5.3 Filtrations of the Base Locus

We introduce here subspaces of $|\Theta_r|$ that we shall need to prove proposition 5.2.

**Definition.** For every $n \in \mathbb{N}$, let

$$|\Theta_r|_n = \{E \in \mathcal{SU}_C(r, \mathcal{O}_C) \mid \forall\ F \in U_C(n, n(g-1))\ H^0(E \otimes F) \neq 0\}.$$

Note that $|\Theta_r|_1$ is the base locus of $|\Theta_r|$.

**Lemma 5.7.** *If $m|n$, then $|\Theta_r|_n \subseteq |\Theta_r|_m$. In particular, every $|\Theta_r|_n$ is contained in the base locus of the generalized theta divisor.*

*Remark.* For every $n \in \mathbb{N}$, we have a filtration of $|\Theta_r|_1$:

$$|\Theta_r|_1 \supseteq |\Theta_r|_n \supseteq |\Theta_r|_{2n} \supseteq |\Theta_r|_{3n} \supseteq \cdots$$

*Proof.* Suppose that $n = lm$, and let $E \in |\Theta_r|_n$. Then, if $F \in \mathcal{U}_C(m, m(g-1))$, $\oplus^l F \in \mathcal{U}_C(n, n(g-1))$, and $H^0\left(E \otimes \left(\oplus^l F\right)\right) \neq 0 \Rightarrow H^0(E \otimes F) \neq 0$. □

The main result is the following.

**Proposition 5.8.** *Let $\widetilde{C} \to C$ be an $n:1$ covering of smooth irreducible projective curves of genus $\widetilde{g}$ and $g \geq 2$, respectively. If $E \in |\Theta_r|_n$, then $\pi^* E \in |\widetilde{\Theta}_r|_1$.*

*Remarks.* (1) It will be clear from the proof that the assumption $E \in |\Theta_r|_n$ is strong. What is really needed is that $H^0(C, E \otimes F) \neq 0$ for every $F$ in the image of the rational map $\pi_* : \mathrm{Pic}^{\widetilde{g}-1}\widetilde{C} \to \mathcal{U}_C(n, n(g-1))$.

(2) From [BeaNarRam89], we know that there exists and $n:1$ covering for which $\pi_*$ is dominant, and in that case we really need $E \in |\Theta_r|_n$.

*Proof.* This is clear since we know that $H^0(\widetilde{C}, \pi^* E \otimes L) \neq 0 \Leftrightarrow H^0(C, E \otimes \pi_* L) \neq 0$ by the projection formula. If $L$ is a line bundle of degree $\widetilde{g} - 1$, then $\pi_* L$ is a vector bundle of rank $n$ and degree $n(g-1)$. If $\pi_* L$ is semi-stable, then $H^0(C, E \otimes \pi_* L) \neq 0$ because $E \in |\Theta_r|_n$. If $\pi_* L$ is not semi-stable, then it has a subbundle $F$ of bigger slope. This means that $\chi(E \otimes F) > 0$, and so $H^0(C, E \otimes \pi_* L) \supseteq H^0(C, E \otimes F) \neq 0$. $\square$

## 5.4 Proof of proposition 5.2

*Proof* (*of proposition* 5.2). Since Raynaud showed in [Ray82] the existence of a semi-stable vector bundle of rank $s^g$ and slope $g/s$ with has sections when tensored by any $L \in \mathrm{Pic}^0(C)$, by proposition 5.6, there exists a semi-stable vector bundle $E'$ of rank $s^g$ and slope $\lceil g/s \rceil$ such that $H^0(E' \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^0(C)$. By tensoring with a suitable line bundle of degree $-\lceil g/s \rceil$, we can construct a vector bundle $E \in \mathcal{SU}_C(s^g)$ such that $H^0(E \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^{\lceil g/s \rceil}(C)$.

Let $F \in \mathcal{U}_C(n, n(g-1))$, and let $L$ be a line subbundle of $F$ of maximal degree. Mukai and Sakai proved in [MukSak85] that $\mu(F/L) - \mu(L) \leq g$. Since

$$\frac{n(g-1) - \deg L}{n-1} - \deg L \leq g \Longleftrightarrow -n - n \deg L \leq -g \Longleftrightarrow \deg L \geq \frac{g}{n} - 1,$$

this implies that every such $F$ has a line subbundle $L$ of degree $\geq \lceil g/s \rceil$, and therefore $H^0(E \otimes F) \supseteq H^0(E \otimes L) \neq 0$. Since this is true for every $F \in \mathcal{U}_C(n, n(g-1))$, $E \in |\Theta_{s^g}|_n$. The result follows from proposition 5.8. $\square$

Here is a table of values of $s$ for small $g$:

| | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ |
|---|---|---|---|---|---|---|---|---|
| $g=3$ | 3 | | | | | | | |
| $g=4$ | 4 | 4 | | | | | | |
| $g=5$ | 3 | 5 | 5 | | | | | |
| $g=6$ | 3 | 6 | 6 | 6 | | | | |
| $g=7$ | 3 | 4 | 7 | 7 | 7 | | | |
| $g=8$ | 3 | 4 | 8 | 8 | 8 | 8 | | |
| $g=9$ | 3 | 5 | 5 | 9 | 9 | 9 | 9 | |
| $g=10$ | 3 | 4 | 5 | 10 | 10 | 10 | 10 | 10 |

For example, if $g=3, n=2$, $|\Theta_{27}|_2$ in non-empty, and so $|\widetilde{\Theta}_{27}|_1$ is non-empty, while $\widetilde{g} \geq 5$, and $2^{\widetilde{g}} \geq 32$.

For every $n$, if $g=n^2+1$, then $|\Theta_{(n+1)^g}|_n$ is non-empty, and so $|\widetilde{\Theta}_{(n+1)^g}|_1$ is non-empty, while $\widetilde{g} \geq n^3+1$, and $2^{\widetilde{g}} \geq 2^{n^3+1} \geq (n+1)^g$; indeed,

$$2^{n^3+1} = 2 \cdot (2^n)^{n^2} \geq (n+1) \cdot (n+1)^{n^2} = (n+1)^g.$$

For every $n$, if $g=n+1$, then $|\Theta_{(n+1)^g}|_n$ is non-empty, and so $|\widetilde{\Theta}_{(n+1)^g}|_1$ is non-empty, while $\widetilde{g} \geq n^2+1$, and $2^{\widetilde{g}} \geq 2^{n^2+1} \geq (n+1)^g$; indeed,

$$2^{n^2+1} = 2 \cdot (2^n)^n \geq (n+1) \cdot (n+1)^n = (n+1)^g.$$

# Chapter 6

# The strange duality conjecture

## 6.1 Formulation of the conjecture

Let $r$ be a positive integer, let $d$ be an integer such that $0 \leq d \leq r/2$, and let $h$ be their greatest common divisor. Then, for every $F \in \mathcal{U}_C(r/h, (r(g-1)-d)/h)$, it is possible to define a divisor $\Theta_F$ in $\mathcal{SU}_C(r,d)$ [resp. $\mathcal{U}_C(r,d)$] as the closure of

$$\{E \in \mathcal{SU}_C(r,d) \text{ [resp. } \mathcal{U}_C(r,d)] \mid E \text{ is stable and } H^0(E \otimes F) \neq 0\}.$$

The strange duality conjecture, inspired by physicists in the case $d = 0$ (see [Bea95, section 8]) and extended by Donagi and Tu to every degree in [DonTu94], states that, if $h = \gcd(r,d)$, then for every $F \in \mathcal{U}_C(r/h, d/h)$ and every positive integer $k$,

$$H^0(\mathcal{SU}_C(r, (\det F)^{\otimes h}), k\Theta_{r,d})^* \simeq H^0(\mathcal{U}_C(rk/h, (r(g-1)-d)k/h), h\Theta_F).$$

In particular, if $k = 1$, the strange duality conjecture would give a commutative diagram

$$\begin{array}{ccc} \mathcal{SU}_C(r,d) & \longrightarrow & |\Theta_{r,d}|^* \\ & \searrow D & \downarrow \simeq \\ & & |h\Theta_F| \end{array}$$

where $D(E) = \Theta_E$, and the base locus of $|\Theta_{r,d}|$ would be isomorphic to

$$\{E \in \mathcal{SU}_C(r,d) \mid H^0(E \otimes F) \neq 0 \ \forall F \in \mathcal{U}_C(r/h, (r(g-1)-d)/h)\}.$$

## 6.2 Improvements on the evidence

We want to contribute here to the evidence in favor of the strange duality conjecture, to add to the evidence in [Bea95] and [DonTu94].

**Lemma 6.1.** *For every $F \in \mathcal{U}_C(r/h, d/h)$,*

$$\dim H^0(\mathcal{SU}_C(r, (\det F)^{\otimes h}), h\Theta_{r,d})^* = \dim H^0(\mathcal{U}_C(r, r(g-1)-d), h\Theta_F).$$

*I.e., the two vector spaces in the strange duality conjecture have the same dimension when $k = h$.*

*Proof.* Let $L \in \mathrm{Pic}^{g-1}(C)$, and consider the isomorphism

$$\begin{array}{ccc} \mathcal{U}_C(r, r(g-1)-d) & \xrightarrow{\simeq} & \mathcal{U}_C(r, d) \\ E & \mapsto & E^* \otimes L \end{array}.$$

Now, $E \in \Theta_F \subseteq \mathcal{U}_C(r, r(g-1)-d)$ if and only if $H^0(E \otimes F) \neq 0$. Since $\chi(E \otimes F) = 0$, this is equivalent to

$$H^0((E \otimes F)^* \otimes \omega_C)^* \simeq H^1(E \otimes F) \neq 0.$$

Therefore, under the isomorphism above, $\Theta_F$ gets identified with $\Theta_{F^* \otimes \omega_C \otimes L^{-1}}$. Indeed, as we saw above

$$H^0(E \otimes F) \neq 0 \Longleftrightarrow H^0((E^* \otimes L) \otimes (F^* \otimes \omega_C \otimes L^{-1})) \neq 0.$$

Therefore, since $\deg(F^* \otimes \omega_C \otimes L^{-1}) = -\deg F + \mathrm{rk}F(g-1) = (r(g-1)-d)/h$, and $\mathrm{rk}(F^* \otimes \omega_C \otimes L^{-1}) = \mathrm{rk}F = r/h$, we obtain

$$\dim H^0(\mathcal{U}_C(r, r(g-1)-d), h\Theta_F) = \dim H^0(\mathcal{U}_C(r, d), h\Theta_{F^* \otimes \omega_C \otimes L^{-1}}),$$

and this is equal to

$$\dim H^0(\mathcal{SU}_C(r, d), h\Theta_{r,d})$$

by [DonTu94, Theorem 1]. □

### 6.3 If the conjecture is true...

In the case $d = 0$, we proved in chapter 5 that, for every $k < g$, if $s$ is a positive integer such that

$$\left\lceil \frac{g}{k} \right\rceil \geq \left\lceil \frac{g}{s} \right\rceil + 1,$$

then there exists a vector bundle $E \in \mathcal{SU}_C(s^g, 0)$ satisfying

$$H^0(E \otimes F) \neq 0 \quad \forall F \in \mathcal{U}_C(k, k(g-1)).$$

If the strange duality conjecture is true, then a corollary would be that, for every $k < g$, the base locus of $|k\Theta_{r,0}|$ is non-empty for $r = s^g$ with $s$ as above[1].

In the case $d = 1$, Brivio and Verra's theorem mentioned in chapter 5 would imply that for every $E \in \mathcal{U}_C(r, 1)$ there exists an $F \in \mathcal{U}_C(r, r(g-1) - 1)$ such that $H^0(E \otimes F) = 0$.

### 6.4 Where to go next

Our main goal is to relate the two parts of this work by using calculations on nodal curves to prove statements about a general smooth curve. This has already been done succesfully in many other situations, and we hope to use the techniques developed in Part I to prove some of the open problems mentioned in Part II.

For example, we would like to prove the following conjecture.

**Conjecture.** *If the base locus of $|\Theta_{r,d}|$ is empty for a curve of genus $g$, then it is also empty for a generic curve of any genus $\geq g$.*

The proof would use an irreducible nodal curve of arithmetic genus $g + 1$ with the normalization equal to the curve of genus $g$ with the empty base locus of $|\Theta_r|$. We would like to show that the base locus for the nodal curve is also empty.

[1] There is a table at the end of chapter 5 with values of $s$ for small $g$.

When $d = 0$, we would also like to produce more examples of smooth curves when the base locus of $|\Theta_{r,0}|$ is non-empty for some $r < 2^g$, where $g$ is the genus of the curve, and find a better upper bound for

$$r_0(g) := \min\{r \in \mathbb{Z}^+ \mid |\Theta_{r,0}| \text{ is non-empty for all curves of genus } g\}.$$

This would give examples of morphisms of $\mathcal{SU}_C(r, 0)$ into projective space. Note that the conjecture above, together with lemma 5.5 would imply that $r_0(g)$ is non-decreasing as a function of $g$.

## Appendix A

## Dimensions of extension spaces

Let $C$ be an irreducible projective curve with one node $p$ as singularity, and let $N$ be its normalization.

*Remark.* Let $F$ and $G$ be torsion-free coherent sheaves on $C$ of rank 1.

(1) If $F$ and $G$ are line bundles on $C$, then

$$\operatorname{Ext}^1_C(F, G) \simeq H^0(C, \omega_C \otimes F \otimes G^{-1}).$$

(2) If $F$ is locally-free and $G = \pi_* \mathcal{G}$ for some line bundle $\mathcal{G}$ on $N$, then

$$\operatorname{Ext}^1_C(F, \pi_* \mathcal{G}) \simeq H^1(N, \pi^* F^{-1} \otimes \mathcal{G}) \simeq \operatorname{Ext}^1_N(\pi^* F, \mathcal{G}).$$

(3) If $F = \pi_* \mathcal{F}$ for some line bundle $\mathcal{F}$ on $N$, and $G$ is locally-free, then

$$\operatorname{Ext}^1_C(\pi_* \mathcal{F}, G) \simeq H^0(N, \mathcal{F} \otimes \pi^* G^{-1} \otimes \pi^* \omega_C)^*.$$

To compute the dimension of these cohomology spaces we use the Riemann-Roch theorem on $C$ or $N$.

*Proof.* The isomorphisms follow from the projection formula (see [Har77, ex. II.5.1(d)]) and standard results on extension spaces (see [Har77, III.6.3, III.6.7, III.7.6]). □

The following lemma gives a formula for the dimensions needed in lemma 1.4.

**Lemma A.1.** *Let $L$ be a line bundle on $C$, and let $F$ be a torsion-free coherent sheaf of rank 1 on $C$ containing $\mathcal{O}_C$. If $\deg F < \deg L$, then*

$$\dim(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_F} \operatorname{Ext}^1_C(L, F))) = \deg F.$$

*Proof.* There exists a short exact sequence $0 \to \mathcal{O}_C \to F \to S \to 0$, where $S$ is a skyscreaper sheaf with degree equal to $\deg F$. Consider the long exact sequence associated to the functor $\operatorname{Hom}_C(L, -)$:

$$0 \longrightarrow \operatorname{Hom}_C(L, S) \longrightarrow \operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_F} \operatorname{Ext}^1_C(L, F) \longrightarrow \operatorname{Ext}^1_C(L, S) \longrightarrow 0,$$

where we started the sequence with $\operatorname{Hom}_C(L, F)$ which is 0 because $\deg L > \deg F$, and we ended the sequence with $\operatorname{Ext}^2_C(L, \mathcal{O}_C) \simeq H^2(C, L^{-1})$ which is also 0 because $L^{-1}$ is locally-free and we are on a curve.

Therefore, the dimension of the kernel of $\psi_F$ is the dimension of $\operatorname{Hom}_C(L, S)$, which is equal to $\deg F$, being $S$ a skyscreaper sheaf of that degree. □

A similar lemma gives the dimension of the kernel of pull-back homomorphisms.

**Lemma A.2.** *Let $L$ be a line bundle on $C$, and let $F$ be a torsion-free coherent sheaf of rank 1 on $C$ containing $\mathcal{O}_C$. If $\deg F < \deg L$, then*

$$\dim(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \stackrel{\psi_{F^*}}{\to} \operatorname{Ext}^1_C(L \otimes F^*, \mathcal{O}_C))) = \deg F.$$

*Proof.* There exists a short exact sequence $0 \to L \otimes F^* \to L \to S \to 0$, where $S$ is a skyscreaper sheaf with degree equal to $\deg F$. Consider the long exact sequence associated to the functor $\operatorname{Hom}_C(-, \mathcal{O}_C)$:

$$0 \longrightarrow \operatorname{Ext}^1_C(S, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_C(L, \mathcal{O}_C) \xrightarrow{\psi_{F^*}} \operatorname{Ext}^1_C(L \otimes F^*, \mathcal{O}_C) \longrightarrow 0,$$

where the sequence starts with $\operatorname{Hom}_C(L \otimes F^*, \mathcal{O}_C) = 0$ (because $\deg(L \otimes F^*) > 0$), and ends with $\operatorname{Ext}^2_C(S, \mathcal{O}_C) = 0$.

Therefore, $\dim(\ker \psi_{F^*})$ equals the dimension of $\operatorname{Ext}^1_C(S, \mathcal{O}_C) \simeq H^0(C, S \otimes \omega_C)$, which is equal to $\deg F$, being $S \otimes \omega_C$ a skyscreaper sheaf of that degree. □

The following lemma studies the natural linear homomorphism $\psi$ that appears in theorems 2.1, 3.1, and 4.1.

**Lemma A.3.** *Let* $\psi\colon \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \to \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N)$ *be the natural pull-back linear homomorphism defined by the inclusion* $L \otimes (\pi_*\mathcal{O}_N)^* \hookrightarrow L$*. Then*

$$\dim(\ker\psi) = \dim(\operatorname{coker}\psi) = 2 \quad \textit{and} \quad \ker\psi \simeq \operatorname{Ext}^1_C(\mathbb{C}_p, \pi_*\mathcal{O}_N).$$

*Proof.* Consider the short exact sequence $0 \to L \otimes (\pi_*\mathcal{O}_N)^* \to L \to \mathbb{C}_p \to 0$. Apply the functor $\operatorname{Hom}_C(-, \pi_*\mathcal{O}_N)$, and and consider the long exact sequence of cohomology:

$$0 \longrightarrow \operatorname{Hom}_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Hom}_C(L, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^1_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L, \pi_*\mathcal{O}_N) \xrightarrow{\psi} \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^2_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^2_C(L, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^2_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \cdots$$

Since $\deg L \otimes (\pi_*\mathcal{O}_N)^* \geq \deg \pi_*\mathcal{O}_N$, we have that $\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) = 0$. Moreover, $\operatorname{Ext}^2_C(L, \pi_*\mathcal{O}_N) \simeq H^2(C, L^{-1} \otimes \pi_*\mathcal{O}_N) \simeq H^2(N, \pi^*L^{-1})$ is also 0, and therefore,

$$\ker\psi \simeq \operatorname{Ext}^1_C(\mathbb{C}_p, \pi_*\mathcal{O}_N), \quad \operatorname{coker}\psi \simeq \operatorname{Ext}^2_C(\mathbb{C}_p, \pi_*\mathcal{O}_N).$$

To calculate the dimensions of these spaces, start with the short exact sequence $0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_N \to \mathbb{C}_p \to 0$, and consider the long exact sequence of cohomology for the functor $\operatorname{Hom}_C(\mathbb{C}_p, -)$:

$$0 \longrightarrow \operatorname{Hom}_C(\mathbb{C}_p, \mathcal{O}_C) \longrightarrow \operatorname{Hom}_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Hom}_C(\mathbb{C}_p, \mathbb{C}_p) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^1_C(\mathbb{C}_p, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(\mathbb{C}_p, \mathbb{C}_p) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^2_C(\mathbb{C}_p, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^2_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^2_C(\mathbb{C}_p, \mathbb{C}_p) \longrightarrow \cdots$$

As soon as $i \geq 2$, we get an isomorphism $\operatorname{Ext}^i_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \simeq \operatorname{Ext}^i_C(\mathbb{C}_p, \mathbb{C}_p)$, because $\operatorname{Ext}^i_C(\mathbb{C}_p, \mathcal{O}_C) = 0$. Moreover, $\operatorname{Hom}_C(\mathbb{C}_p, \mathcal{O}_C) = \operatorname{Hom}_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) = 0$ because $\mathcal{O}_C$ and $\pi_*\mathcal{O}_N$ are torsion-free, and $\operatorname{Hom}_C(\mathbb{C}_p, \mathbb{C}_p) \simeq \operatorname{Ext}^1_C(\mathbb{C}_p, \mathcal{O}_C) \simeq \mathbb{C}$. Therefore,

$$\forall\ i \geq 1, \quad \operatorname{Ext}^i_C(\mathbb{C}_p, \pi_*\mathcal{O}_N) \simeq \operatorname{Ext}^i_C(\mathbb{C}_p, \mathbb{C}_p).$$

The lemma follows from lemma A.4. $\square$

**Lemma A.4.** *For all $i \geq 1$,*

$$\dim \operatorname{Ext}^i_C(\mathbb{C}_p, \mathbb{C}_p) = 2.$$

*Proof.* Since the completion of an analytic neighborhood of $p$ is the same as the completion $\mathbb{C}[[x,y]]/(xy)$ of $\mathbb{C}[x,y]/(xy)$, it is enough to check this for the curve $\operatorname{Spec}(\mathbb{C}[x,y]/(xy))$, and we shall do this with the help of Macaulay 2.

In the following code, $N$ will be the module corresponding to the quotient of $R = \mathbb{C}[x,y]/(xy)$ by the ideal $(x,y)$. It corresponds to $\mathbb{C}_p$, and it is clear from the output that $\operatorname{Ext}^i(N,N)$ is two dimensional for every $i$.

```
i1 : R = QQ[x,y] / (x*y)
I = ideal (x, y)
M = module I
f = map(R^1, M)
N = cokernel f
Ext^1(N,N)
Ext^2(N,N)
Ext^3(N,N)
Ext^4(N,N)
```

$\square$

We prove here a known result that we need in the proof of theorems 3.1 and 4.1.

**Lemma A.5.** *The secant line joining two smooth points $q, q' \in C \subseteq \mathbb{P}_L$ is*

$$\mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_C(L, \mathcal{O}_C(q+q')))).$$

*Proof.* Let $X := \mathbb{P}(\ker(\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \mathcal{O}_C(q+q'))))$. It is enough to show that $q, q' \in X$ and that $X$ is a line. The first statement is obvious since the natural linear homomorphism $\operatorname{Ext}^1_C(L, \mathcal{O}_C) \to \operatorname{Ext}^1_C(L, \mathcal{O}_C(q+q'))$ factors through $\operatorname{Ext}^1_C(L, \mathcal{O}_C(q))$ and $\operatorname{Ext}^1_C(L, \mathcal{O}_C(q'))$, respectively. The second statement follows from lemma A.1. $\square$

The following lemma is needed in the proof of lemma 3.7.

**Lemma A.6.** $(a)$ *The kernel of the natural linear homomorphism*

$$\operatorname{Ext}^1_C(L \otimes (\pi_* \mathcal{O}_N)^*, \pi_* \mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes (\pi_* \mathcal{O}_N)^*, \pi_* \mathcal{O}_N(p_1 + p_2))$$

*has dimension* 4.

(*b*) *For every* $l \in L_1$*, the kernels of the natural linear homomorphisms*

$$\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, M_l)$$

$$\text{and} \quad \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes M_l^*, \pi_*\mathcal{O}_N)$$

*have dimension* 2.

*Proof.* (a) Consider the short exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_N \longrightarrow \pi_*\mathcal{O}_N(p_1 + p_2) \longrightarrow \mathbb{C}_p \oplus \mathbb{C}_p \longrightarrow 0.$$

We know from the long exact sequence of cohomology associated to the functor $\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, -)$ that the kernel of

$$\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N(p_1 + p_2))$$

is the image of

$$\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \mathbb{C}_p \oplus \mathbb{C}_p) \longrightarrow \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N),$$

and this is equal to the four-dimensional vector space $\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \mathbb{C}_p \oplus \mathbb{C}_p)$ itself because $\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N(p_1 + p_2))$ is zero for a generic $L$.

(b) The first part is very similar to part (a). Starting with the short exact sequence $0 \to \pi_*\mathcal{O}_N \to M_l \to \mathbb{C}_p \to 0$, and considering the long exact sequence of cohomology associated to the functor $\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, -)$, we see that the kernel of $\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \to \operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, M_l)$ is the two-dimensional vector subspace $\operatorname{Hom}_C(L \otimes (\pi_*\mathcal{O}_N)^*, \mathbb{C}_p)$.

For the second part, consider the short exact sequence

$$0 \longrightarrow L \otimes M_l^* \longrightarrow L \otimes (\pi_*\mathcal{O}_N)^* \longrightarrow \mathbb{C}_p \longrightarrow 0.$$

The long exact sequence of cohomology associated to the functor $\operatorname{Hom}_C(-, \pi_*\mathcal{O}_N)$ shows that the kernel of

$$\operatorname{Ext}^1_C(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N) \longrightarrow \operatorname{Ext}^1_C(L \otimes M_l^*, \pi_*\mathcal{O}_N)$$

is the two-dimensional subspace $\operatorname{Ext}^1_C(\mathbb{C}_p, \pi_*\mathcal{O}_N)$ because $\operatorname{Hom}_C(L \otimes M_l^*, \pi_*\mathcal{O}_N)$ is zero for a generic $L$. □

The following lemma is very important, because we factor certain restrictions of our rational maps through projective spaces of the form $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes F^*, F))$, with $F$ as in the lemma. The lemma shows that the rational map is defined exactly where the map to $\mathbb{P}(\operatorname{Ext}^1_C(L \otimes F^*, F))$ is defined.

**Lemma A.7.** *Let $F$ be a torsion-free sheaf of rank 1 on $C$ such that $\mathcal{O}_C \subseteq F$, $\deg F = 1$. Then the natural rational map*

$$\mathbb{P}(\operatorname{Ext}^1_C(L \otimes F^*, F)) \longrightarrow \overline{\mathcal{SU}_C(2, L)}$$

*defined by*

$$(0 \to F \to E \to L \otimes F^* \to 0) \mapsto E$$

*is a morphism*[1]*, and it is generically injective. Moreover, it is injective for $g > 3$ if $\deg L = 4$ and for $g > 2$ if $\deg L = 3$.*

*Proof.* The proof that the rational map is a morphism is an exact copy of the proof of lemma 2.5 with $L \otimes (\pi_*\mathcal{O}_N)^*$ and $\pi_*\mathcal{O}_N$ replaced by $L \otimes F^*$ and $F$.

It is clear that the morphism is injective at a point $E$ if $h^0(C, E) = 1$, because in that case there can be only one way to write $E$ as an extension of $L \otimes F^*$ by $F$. Since $\chi(E) = \deg E + 2(1 - g) = 5 - 2g \leq 1$, and we know that $h^0(C, E) \geq 1$ since $\mathcal{O}_C \subseteq F \subseteq E$, it is clear that a generic $E \in \operatorname{Ext}^1_C(L \otimes F^*, F)$ has $h^0(C, E) = 1$.

To prove the last statement, consider the short exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow L \otimes F^* \longrightarrow 0,$$

[1] We are assuming that $\deg L$ is 3 or 4.

and the corresponding long exact sequence

$$0 \longrightarrow H^0(C,F) \longrightarrow H^0(C,E) \longrightarrow H^0(C, L \otimes F^*) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^1(C,F) \longrightarrow H^1(C,E) \longrightarrow H^1(C, L \otimes F^*) \longrightarrow 0.$$

Since $\chi(L \otimes F^*) = \deg L - 1 + 1 - g = \deg L - g$ and $L$ is generic, $h^0(C, L \otimes F^*) = 0$ if $g > \deg L - 1$. In this case, $h^0(C,E) = h^0(C,F) = 1$. $\square$

# Appendix B

## Analysis of extensions spaces on products of varieties

Throughout this appendix, $C$ is an irreducible projective curve with only one node $p$ as singularity.

If a coherent sheaf on $X \times C$, where $X$ is an irreducible projective variety, is of the type $\pi_X^* F \otimes \pi_C^* G$ for some coherent sheaves $F$ on $X$ and $G$ on $C$, we shall sometimes denote it by $F \boxtimes G$. If it is a pull-back of a coherent sheaf $F$ from one of the factors, we shall sometimes denote it by $F$ itself, if it is clear from the context what is the base space for the sheaf. For example, $\pi_C^* L$ on $\mathbb{P}_L \times C$ is sometimes denoted just by $L$ if we are writing an extension on $\mathbb{P}_L \times C$.

**Lemma B.1.** *Let $Y$ be a smooth projective variety, let $F_Y$ [resp. $F_C$] be a vector bundle on $Y$ [resp. $C$], and let $G$ be a torsion-free sheaf on $C$. If $H^0(C, F_C^* \otimes G) = 0$, then*

$$\operatorname{Ext}^1_{Y\times C}(F_C, F_Y \boxtimes G) \simeq H^0(Y, F_Y) \otimes \operatorname{Ext}^1_C(F_C, G).$$

*If $Y = \mathbb{P}(V)$ for some vector space $V$, and $F_Y \simeq \mathcal{O}_{\mathbb{P}(V)}(1)$, this is also naturally isomorphic to*

$$\operatorname{Hom}(V, \operatorname{Ext}^1_C(F_C, G)).$$

*Proof.* Since $F_C$ is locally-free, $\operatorname{Ext}^1_{Y\times C}(F_C, F_Y \boxtimes G) \simeq H^1(Y \times C, F_Y \boxtimes (F_C^* \otimes G))$. By the Künneth Formula (see [Kem93, Proposition 9.2.4]), this is isomorphic to $H^0(Y, F_Y) \otimes H^1(C, F_C^* \otimes G)$ (remember that $H^0(C, F_C^* \otimes G) = 0$), which is isomorphic to $H^0(Y, F_Y) \otimes \operatorname{Ext}^1_C(F_C, G)$, as claimed.

If $Y = \mathbb{P}(V)$ and $F_Y \simeq \mathcal{O}_{\mathbb{P}(V)}(1)$, $H^0(Y, F_Y)$ is canonically isomorphic to $V^*$. $\square$

*Remark.* The isomorphisms of lemma B.1 can be described as follows:

$$(1)\quad H^0(\mathbb{P}(V), F_V) \otimes \operatorname{Ext}^1_C(F_C, G) \xrightarrow{\simeq} \operatorname{Ext}^1_{\mathbb{P}(V)\times C}(F_C, F_V \boxtimes G).$$

If we start with an element of the form $s \otimes a$, with $s$ a section $\mathcal{O}_{\mathbb{P}(V)} \hookrightarrow F_V$ and $a$ an extension $0 \to G \to E_a \to F_C \to 0$, we first pull $a$ back to $\mathbb{P}(V) \times C$, and then we use the section $\pi^*_{\mathbb{P}(V)} s : \pi^*_C G \hookrightarrow F_V \boxtimes G$ to push $\pi^*_C E_a$ forward to get an extension $\mathcal{E}_{s\otimes a}$ in $\operatorname{Ext}^1_{\mathbb{P}(V)\times C}(F_C, F_V \boxtimes G)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & E_a & \longrightarrow & F_C & \longrightarrow & 0, \\
 & & {\scriptstyle \pi^*_{\mathbb{P}(V)} s}\big\downarrow & & \big\downarrow & & \| & & . \\
0 & \longrightarrow & F_V \boxtimes G & \longrightarrow & \mathcal{E}_{s\otimes a} & \longrightarrow & F_C & \longrightarrow & 0
\end{array}$$

$$(2)\quad H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \otimes \operatorname{Ext}^1_C(F_C, G) \xrightarrow{\simeq} \operatorname{Hom}(V, \operatorname{Ext}^1_C(F_C, G)).$$

An element of the form $s\otimes a$ goes to the linear homomorphism mapping $b$ to[1] $s(b)\cdot a$.

$$(3)\quad \operatorname{Ext}^1_{\mathbb{P}(V)\times C}(F_C, \mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes G) \xrightarrow{\simeq} \operatorname{Hom}(V, \operatorname{Ext}^1_C(F_C, G)).$$

An extension $a$ of the form $0 \to \mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes G \to \mathcal{E}_a \to F_C \to 0$ goes to the linear homomorpshim that sends $v \in V$ to the extension obtained by restricting $a$ to $\{[v]\} \times C$, and identifying $(\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes G)|_{\{[v]\}\times C}$ with $G$ using $v$:

$$0 \longrightarrow G \longrightarrow \mathcal{E}_a|_{\{[v]\}\times C} \longrightarrow F_C \longrightarrow 0.$$

In particular,

**Corollary B.2.** *If* $V = \operatorname{Ext}^1_C(F_C, G)$*, and* $\mathcal{E}_V \in \operatorname{Ext}^1_{\mathbb{P}(V)\times C}(F_C, \mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes G)$ *is the element that corresponds to the identity in* $\operatorname{Hom}(\operatorname{Ext}^1_C(F_C, G), \operatorname{Ext}^1_C(F_C, G))$*, then, for every non-zero extension* $a \in \operatorname{Ext}^1_C(F_C, G)$*,* $\mathcal{E}_V|_{\{[a]\}\times C}$ *is* $a$ *when we identify* $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes G|_{\{[v]\}\times C}$ *with* $G$ *using* $a$.

Here is a generalization of lemma B.1.

[1] Remember that $s \in H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \simeq V^*$.

**Lemma B.3.** *Let $Y$ be a smooth projective variety, let $F_Y, G_Y$ be locally-free sheaves on $Y$, and let $F_C, G_C$ be quasi-coherent sheaves on $C$. Then, for every $i \geq 0$,*

$$\operatorname{Ext}^i_{Y\times C}(F_Y \boxtimes F_C, G_Y \boxtimes G_C) \simeq \bigoplus_{j+k=i} \operatorname{Ext}^j_Y(F_Y, G_Y) \otimes \operatorname{Ext}^k_C(F_C, G_C).$$

*Proof.* The vector space $\operatorname{Ext}^i_{Y\times C}(F_Y \boxtimes F_C, G_Y \boxtimes G_C)$ can be calculated as $E^{p+q}_\infty(Y \times C)$ using the spectral sequence $E^{p,q}_2(Y \times C) \Rightarrow E^{p+q}_\infty(Y \times C)$, where $E^{p,q}_2(Y \times C) = H^p(Y \times C, \mathcal{E}xt^q_{Y\times C}(F_Y \boxtimes F_C, G_Y \boxtimes G_C))$ (see [AltKle70, IV.2.4]). Moreover

$$\mathcal{E}xt^q_{Y\times C}(F_Y \boxtimes F_C, G_Y \boxtimes G_C) \simeq \pi_Y^* F_Y^* \otimes \pi_Y^* G_Y \otimes \mathcal{E}xt^q_{Y\times C}(\pi_C^* F_C, \pi_C^* G_C)$$

(see [Har77, III.6.7]), and, by lemma B.4, this is $(F_Y^* \otimes G_Y) \boxtimes \mathcal{E}xt^q_C(F_C, G_C)$. Therefore, by the Künneth formula (see [Kem93, Proposition 9.2.4]),

$$H^p(Y\times C, (F_Y^*\otimes G_Y)\boxtimes \mathcal{E}xt^q_C(F_C, G_C)) \simeq \bigoplus_{j+l=p} H^j(Y, F_Y^*\otimes G_Y)\otimes H^l(C, \mathcal{E}xt^q_C(F_C, G_C)).$$

Thus, the original spectral sequence $E^{p,q}_2(Y \times C) \Rightarrow E^{p+q}_\infty(Y \times C)$ is induced by the spectral sequence

$$E^{l,q}_2(C) = H^l(C, \mathcal{E}xt^q_C(F_C, G_C)) \Longrightarrow \operatorname{Ext}^{l+q}_C(F_C, G_C) = E^{l+q}_\infty(C).$$

Therefore, since $H^j(Y, F_Y^* \otimes G_Y) \simeq \operatorname{Ext}^j_Y(F_Y, G_Y)$, we obtain an isomorphism

$$E^i_\infty(Y \times C) \simeq \bigoplus_{j+k=i} \operatorname{Ext}^j_Y(F_Y, G_Y) \otimes E^k_\infty(C),$$

as claimed. ∎

**Lemma B.4.** *For any two quasi-coherent sheaves $F$ and $G$ on $C$ and any smooth projective variety $Y$,*

$$\mathcal{E}xt^i_{Y\times C}(\pi_C^* F, \pi_C^* G) \simeq \pi_C^* \mathcal{E}xt^i_C(F, G).$$

*Proof.* Let $R_\bullet\colon \cdots \to F_2 \to F_1 \to F_0 \to F \to 0$ be a locally-free resolution of $F$ on $C$. Then its pull-back $\pi_C^* R_\bullet\colon \cdots \to \pi_C^* F_2 \to \pi_C^* F_1 \to \pi_C^* F_0 \to \pi_C^* F \to 0$ to $Y \times C$ is a locally-free resolution of $\pi_C^* F$. Since

$$\mathcal{E}xt^i_{Y\times C}(\pi_C^* F, \pi_C^* G) = \mathcal{H}^i(\mathcal{H}om_{Y\times C}(\pi_C^* R_\bullet, \pi_C^* G))$$

(see [Har77, III.6.5]), and, for $i \geq 0$,

$$\begin{aligned}
\mathcal{H}om_{Y\times C}(\pi_C^* F_i, \pi_C^* G) &\simeq \mathcal{H}om_{Y\times C}(\mathcal{O}_{Y\times C}, \pi_C^* G) \otimes \pi_C^* F_i^* \\
&\simeq \pi_C^* G \otimes \pi_C^* F_i^* \simeq \pi_C^*(G \otimes F_i^*) \\
&\simeq \pi_C^*(\mathcal{H}om_C(\mathcal{O}_C, G) \otimes F_i^*) \\
&\simeq \pi_C^*(\mathcal{H}om_C(F_i, G))
\end{aligned}$$

because $F_i$ is locally-free, then $\mathcal{E}xt^i_{Y\times C}(\pi_C^* F, \pi_C^* G) \simeq \mathcal{H}^i(\pi_C^*(\mathcal{H}om_C(R_\bullet, G)))$. Since $\pi_C$ is flat, $\mathcal{H}^i(\pi_C^*(\mathcal{H}om_C(R_\bullet, G))) \simeq \pi_C^*(\mathcal{H}^i(\mathcal{H}om_C(R_\bullet, G)))$, and the lemma follows because $\mathcal{E}xt^i_C(F, G) = \mathcal{H}^i(\mathcal{H}om_C(R, G))$. $\square$

**Lemma B.5.** *If $\mathcal{E}$ is an extension of $F$ by $\mathcal{O}_{\mathbb{P}^N}(n) \boxtimes G$ on $\mathbb{P}^N \times C$, then $\mathcal{E}$ induces a rational map $\mathbb{P}^N \to \mathbb{P}(\operatorname{Ext}^1_C(F, G))$ given by sections of $\mathcal{O}_{\mathbb{P}^N}(n)$.*

*Proof.* We saw in lemmas B.1 and B.3 that

$$\operatorname{Ext}^1_{\mathbb{P}^N\times C}(F, \mathcal{O}_{\mathbb{P}^N}(n) \boxtimes G) \simeq H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \otimes \operatorname{Ext}^1_C(F, G).$$

Let us first assume that $\mathcal{E}$ corresponds to a non-zero element of the form $s \otimes a$ in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \otimes \operatorname{Ext}^1_C(F, G)$. Then the induced rational map will map to the point $[a]$, and it will send a point $x \in \mathbb{P}^N$ to the extension $\phi_{s\otimes a}(x)$ defined by restricting $\mathcal{E}$ to $\{x\} \times C$ and identifying $\mathcal{O}_{\mathbb{P}^N}(n)|_{\{x\}\times C}$ with $\mathcal{O}_C$. Note that it does not matter which identification we choose, because the extensions we obtain are all multiples of $a$, and they give a well-defined element in $\mathbb{P}(\operatorname{Ext}^1_C(F, G))$.

Note that $\mathcal{E}|_{\{x\}\times C}$ splits if and only if $s(x) = 0$. Indeed, since $\mathcal{E}$ corresponds to $s \otimes a$, it is defined by pushing $\pi_C^* a$ forward via $\pi_{\mathbb{P}^N}^* s : \mathcal{O}_{\mathbb{P}^N} \boxtimes G \to \mathcal{O}_{\mathbb{P}^N}(n) \boxtimes G$. If

$s(x) \neq 0$, then the restriction of $\mathcal{E}$ to $\{x\} \times C$ is a multiple of $a$, and if $s(x) = 0$, then the restriction of $\mathcal{E}$ to $\{x\} \times C$ is the push-forward of $a$ via the zero map $G \xrightarrow{0} G$, which splits. Indeed, the push-forward is $(E_a \oplus G)/\sim$, where $\sim$ identifies $G \subseteq E_a$ with 0, and therefore we obtain $F \oplus G$.

Therefore, if $\mathcal{E}$ corresponds to an element of the form $s \otimes a$, then the induced rational map is exactly given by $s$:

$$\begin{array}{rcl} \mathbb{P}^N & \to & \mathbb{P}^0 = \{[a]\} \subseteq \mathbb{P}(\operatorname{Ext}^1_C(F,G)) \\ x & \mapsto & s(x) \end{array}$$

In general, if $\mathcal{E}$ corresponds to an element of the form $\sum_{i=0}^m s_i \otimes a_i$, with $\{a_0, \ldots, a_m\}$ a basis of $\operatorname{Ext}^1_C(F,G)$ and $s_0, \ldots, s_m \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N(n)})$, the rational map is given by:

$$\begin{array}{rcl} \mathbb{P}^N & \to & \mathbb{P}(\operatorname{Ext}^1_C(F,G)) \\ x & \mapsto & (s_0(x) : \cdots : s_m(x)). \end{array}$$

$\square$

# Appendix C

## Results on maps of projective spaces

This is probably a well-known result on rational maps between projective spaces that we need in the proofs of linearity in chapters 3 and **??**.

**Proposition C.1.** *Let $\phi : \mathbb{P}^m \to \mathbb{P}^n$ be a rational map given by sections of $\mathcal{O}_{\mathbb{P}^m}(d)$ with no common factor. Let $U_\phi$ be the open set where $\phi$ is regular, and let $H$ be a hyperplane in $\mathbb{P}^m$. If $H \subseteq U_\phi$, then $\phi|_H$ is also given by sections of $\mathcal{O}_H(d)$ with no common factor.*

Before we prove the proposition, let us point out the importance of the hypothesis $H \subseteq U_\phi$. Let $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map of degree 2 defined by $\phi(x_0, x_1, x_2) = (x_0^2, x_0x_1, x_2^2)$. It is not defined only at the point $(0, 1, 0)$.

If we restrict $\phi$ to the line $L_1$ defined by $x_1 = 0$, we get the degree 2 morphism from $L_1$ to itself given by $\phi|_{\{x_1=0\}}(x_0, 0, x_2) = (x_0^2, 0, x_2^2)$.

On the other hand, if we restrict $\phi$ to the line $L_2$ defined by $x_2 = 0$, which contains the point $(0, 1, 0)$, then we get the degree 1 morphism from $L_2$ to itself[1] given by $\phi|_{\{x_2=0\}}(x_0, x_1, 0) = (x_0, x_1, 0)$.

The difference between $L_1$ and $L_2$ is that when we restrict the rational map to $L_2$, the degree 2 sections that define $\phi$ all have the common factor $x_0$. When we simplify the common factor, the degree drops. But the presence of the common factor implies that the original rational map is not defined on the zero locus of the common factor in $L_2$.

[1] Note that the rational map restricted to $L_2$ actually extends to the whole line.

*Proof* (*of proposition C*.1). Let $s_0, \ldots, s_n \in H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$ be such that

$$\phi(x) = (s_0(x), \ldots, s_n(x)),$$

and let $a_0, \ldots, a_m \in k$ be such that $H$ has equation $a_0x_0 + \cdots + a_mx_m = 0$. We shall assume that $a_m \neq 0$, and identify $H$ with $\mathbb{P}^{m-1}$ by sending $(x_0, \ldots, x_m)$ to $(x_0, \ldots, x_{m-1})$. This is clearly an isomorphism with the inverse sending the point $(x_0, \ldots, x_{m-1})$ to $(x_0, \ldots, x_{m-1}, -(a_0/a_m)x_0 - \cdots - (a_{m-1}/a_m)x_{m-1})$.

Then $\phi|_H \colon \mathbb{P}^{m-1} \to \mathbb{P}^n$ is given by $x \mapsto (\overline{s}_0(x), \ldots, \overline{s}_n(x))$, where, for $i = 0, \ldots, n$,

$$\overline{s}_i(x_0, \ldots, x_{m-1}) = s_i\left(x_0, \ldots, x_{m-1}, -\frac{a_0}{a_m}x_0 - \cdots - \frac{a_{m-1}}{a_m}x_{m-1}\right).$$

Since the $s_i$'s are homogeneous polynomials of degree $d$ in $x_0, \ldots, x_m$, the $\overline{s}_i$'s are homogeneous polynomials of degree $d$ in $x_0, \ldots, x_{m-1}$. All is left to prove is that the $\overline{s}_i$'s do not have a common factor. But this is clear, because if they had a common factor $f$, $\phi$ would not be defined on the non-empty locus $\{f = 0\} \subseteq H$, contradicting the fact that $H \subseteq U_\phi$. $\square$

A very similar proof actually proves a stronger statement:

**Proposition C.2.** *Let $\phi : \mathbb{P}^m \to \mathbb{P}^n$ be a rational map given by sections of $\mathcal{O}_{\mathbb{P}^m}(d)$ with no common factor. Let $U_\phi$ be the open set where $\phi$ is regular, and let $L$ be a linear subspace in $\mathbb{P}^m$ of dimension $\geq 1$. If $L \subseteq U_\phi$, then $\phi|_L$ is given by sections of $\mathcal{O}_L(d)$ with no common factor.*

As a corollary, we obtain a stronger statement.

**Corollary C.3.** *Let $\phi : \mathbb{P}^m \to \mathbb{P}^n$ be a rational map given by sections of $\mathcal{O}_{\mathbb{P}^m}(d)$ with no common factor. Let $U_\phi$ be the open set where $\phi$ is regular, and let $L$ be a linear subspace of $\mathbb{P}^m$. If $L$ contains a line $L_1$ such that $L_1 \subseteq U_\phi$, then $\phi|_L$ is given by sections of $\mathcal{O}_L(d)$ with no common factor.*

*Proof.* If $\phi|_L$ were given by sections of $\mathcal{O}_L(d')$ with $d' < d$, then $\phi|_{L'}$ would be given by sections of $\mathcal{O}_{L'}(d'')$ with $d'' \leq d' < d$, contradicting Proposition C.2. $\square$

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