

SIGN CHANGES OF ERROR TERMS
RELATED TO CERTAIN ARITHMETIC FUNCTIONS

by

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(Under the direction of Andrew Granville)

ABSTRACT

Let $H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x$. Motivated by a conjecture of P. Erdős, Y.-K. Lau developed a new method and proved $\#\{1 \leq n \leq T : H(n)H(n+1) < 0\} \gg T$.

We consider arithmetic functions $f(n) = \sum_{d|n} \frac{b_d}{d}$ whose summation can be expressed as $\sum_{n \leq x} f(n) = \alpha x + P(\log(x)) + E(x)$, where α is a real number, $P(x)$ is a polynomial and the error term $E(x)$ is of the form

$$E(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

for $\psi(x) = x - [x] - \frac{1}{2}$, and where $y(x)$, $k(x)$ and b_n satisfy some general conditions. We generalize Lau's method and prove results about the number of sign changes for these error terms. We illustrate our results with a list of well known arithmetic functions. In particular, we prove the following generalization of Lau's result:

Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a rational valued arithmetic function and suppose the sequence b_n satisfies $\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right)$ and $\sum_{n \leq x} b_n^4 \ll x \log^D x$, for some B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}, \quad \gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right), \quad E(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

Then, except when $\alpha = 0$, or $B = 0$ and α is rational, we have

$$\#\{1 \leq n \leq T : E(n)E(n+1) < 0\} \gg T \Leftrightarrow \#\{1 \leq n \leq T : \alpha E(n) < 0\} \gg T.$$

We also study the error term $\Delta(x) = \sum_{n \leq x} \tau(n) - x \log x - (2\gamma - 1)x$ and prove

$$\#\{1 \leq n \leq T : \Delta(n)\Delta(n+1) < 0\} > \sqrt{T} + O(1).$$

INDEX WORDS: sign changes, error term, Euler function, divisor function.

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NOTATION

The following notation and conventions are used systematically in the text.

We use Landau's notation $f(x) = O(g(x))$ and Vinogradov's $f(x) \ll g(x)$ to both mean that $|f(x)| \leq Cg(x)$, for some positive constant C , which may be absolute or depend upon various parameters, in which case these may be indicated in subscript. Also, $g(x) \gg f(x)$ means $f(x) \ll g(x)$ and we write $f(x) \asymp g(x)$ to indicate that $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold simultaneously. Moreover, $f(x) = o(g(x))$ means $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$, and $f(x) \sim g(x)$ means $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

In the opposite direction, we write $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ to respectively denote that $f(x_n) > Cg(x_n)$ and $f(x_n) < -Cg(x_n)$ holds for infinitely many x_n such that $x_n \rightarrow \infty$, with a certain positive constant C . Also, we write $f(x) = \Omega_\pm(g(x))$ to indicate that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold, and $f(x) = \Omega(g(x))$ means that $|f(x)| = \Omega_+(g(x))$.

Given a finite set A , we write $|A|$ and $\#(A)$ to both mean the cardinality of A .

The integer and fractional parts of the real number x are denoted by $[x]$ and $\{x\}$, respectively. Also, $\psi(x) = \{x\} - \frac{1}{2}$. We write $e(x)$ to denote $\exp(2\pi ix)$.

$a \mid b$ means a divides b , and $a \equiv b \pmod{m}$ means $m \mid (a - b)$. The greatest common divisor and the least common multiple of m and n are denoted by (m, n) and $[m, n]$, respectively.

Throughout this work, $s = \sigma + it$ is a complex variable, $\zeta(s)$ is the Riemann Zeta function, γ is the Euler constant, $\phi(n)$ is the Euler totient function, $\mu(n)$ is the Möbius function, $\tau(n)$ is the number of positive divisors of n , and $\sigma(n)$ is the sum of the positive divisors of n .

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CHAPTER 1

OVERVIEW

Formally, an arithmetic function is simply a sequence of real or complex values. In many arithmetic functions studied in Number Theory, their individual values fluctuate widely, however in many cases summation smooths out the fluctuation and it may be possible to find an asymptotic expression for the summation function. This asymptotic expansion consists of a main term, which includes an average order, and of an oscillating error term. Various authors have studied the properties of these error terms, namely its order of magnitude, Ω -estimates, mean value results, distributions and the number of sign changes. The object of our study concerns arithmetic functions for which the error term of the summation function involves $\psi(x) = \{x\} - \frac{1}{2}$. In this work, we will mention the results about error terms that exist in the literature for particular examples, but we will be mainly interested in finding the number of times the error term changes sign. There are two kinds of sign changes that we will consider:

1. If, given x and y , we have $f(x)f(y) < 0$, then we have (at least) one *sign change* (or *change of sign*) in the interval $[x, y]$.
2. If, given an integer n , we have $f(n)f(n+1) < 0$ then we have a *sign change on integers* (or *change of sign on integers*) at n .

1.1 INTRODUCTION

The most famous example of sign changes of error terms is related to the function $\pi(x)$. The prime number theorem states

$$\pi(x) = Li(x) + E(x),$$

where $E(x) = o(Li(x))$, and J. E. Littlewood [53] proved that $E(x)$ changes of sign infinitely often (for a full proof, see [28]), which disproved an old belief that $\pi(x) < Li(x)$, for all x (shared, among others by C. F. Gauss [21] and B. Riemann [75]). In fact, Littlewood proved that

$$E(x) = \Omega_{\pm} \left(\frac{\sqrt{x} \log \log \log x}{\log x} \right).$$

In two papers [83, 84], the first assuming the Riemann hypothesis and the other assuming it is false, S. Skewes proved that there must exist $x < 10^{10^{964}}$ such that $\pi(x) > Li(x)$. After improvements on this bound by R. S. Lehman [52] and H. J. J. te Riele [92], the best result is now 1.39×10^{316} , obtained by C. Bays and R. H. Hudson [4]. J. Kaczorowski [43] has worked on the number of sign changes of $E(x)$ and proved that $X_E(T) \gg \log T$, where $X_E(T)$ denote the number of sign changes of $E(x)$ in $[1, T]$. J. Kaczorowski [42, 43] also obtained similar results for the error terms $\sum_{n \leq x} \Lambda(n) - x$ and $\sum_{p \leq x} \log p - x$.

The motivation for our work was a paper by Y.-K. Lau [50], where he proves that the error term, $H(x)$, given by

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + H(x)$$

has a positive proportion of sign changes on integers. This result implies a conjecture stated by P. Erdős in 1967:

The error term, $R(x)$, of the summation of $\phi(n)$, has a positive proportion of changes of sign on integers

The main tool Lau used to prove his theorem, was that the error term $H(x)$ can be expressed as

$$H(x) = - \sum_{n \leq \frac{x}{\log^5 x}} \frac{\mu(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^{20} x}\right)$$

Throughout this work, we will be interested in arithmetic functions such that the asymptotic expansion of their summations have error terms of the form

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right), \quad (1.1)$$

where $y(x)$, $k(x)$ and the sequence b_n satisfy some general conditions. In chapter 2, we obtain some general results for the number of sign changes for functions $H(x)$ of the form (1.1).

We will study two classes of arithmetic functions for which Lau's result can be generalized. In chapter 3, we consider arithmetic functions $f(n)$, such that $f(n) = \sum_{d|n} \frac{b_d}{d}$ and the sequence b_n satisfies

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{for some } B \text{ real and } A > 1 \quad (1.2)$$

and

$$\sum_{n \leq x} b_n^4 \ll x \log^D x, \quad \text{for } D > 0. \quad (1.3)$$

We prove that these two conditions imply that the error term of $\sum_{n \leq x} f(n)$ is of the form (1.1). Notice that this class is closed for addition, i. e. if $f(n)$ and $g(n)$ are members of the class then also is $(f + g)(n)$.

In chapter 4, we consider arithmetic functions $f(n)$, such that $f(n) = \sum_{d|n} \frac{b_d}{d}$, the sequence b_n satisfies condition (1.3) and

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^\beta(s) g(s), \quad (1.4)$$

for some β real and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma > 1 - \lambda$, for some $\lambda > 0$. Again, these two conditions imply that the error term of $\sum_{n \leq x} f(n)$ is of the form (1.1).

Our main theorem is a generalization of Lau's result for some members of our first class of arithmetic functions.

Theorem 1.1. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a rational valued arithmetic function and suppose the sequence b_n satisfies*

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

Then, except when $\alpha = 0$, or $B = 0$ and α is rational, we have

$H(x)$ has a positive proportion of changes of sign on integers if and only if $\alpha H(x)$ has a positive proportion of negative values on integers.

Theorem 1.1 will be obtained in chapter 3 as a corollary of the more general theorem:

Theorem 1.2. *Assume the sequence b_n satisfies the hypothesis of theorem 1.1 and define α , γ_b and H as in theorem 1.1. If $\alpha \neq 0$ then*

1. *$\alpha H(x)$ has a positive proportion of positive values on integers;*
2. *If $H(x)$ has a positive proportion of changes of sign on integers then $\alpha H(x)$ has a positive proportion of negative values on integers;*

3. If $\alpha H(x)$ has a positive proportion of negative values on integers then $H(x)$ has a positive proportion of changes of sign on integers or a positive proportion of zeros.

For our second class of arithmetic functions, we obtain the following result:

Theorem 1.3. Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies

$$\sum_{n \leq x} b_n^4 \ll x \log^D x \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^\beta(s) g(s)$$

for some β real, $D > 0$, and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma > 1 - \lambda$, for some $\lambda > 0$. Let $\alpha = \zeta^\beta(2)g(2)$ and $H(x)$ be the error term of the asymptotic expansion of $\sum_{n \leq x} f(n)$. If $\alpha \neq 0$, we have

1. $\alpha H(x)$ has a positive proportion of positive values on integers;
2. If $H(x)$ has a positive proportion of changes of sign on integers then $\alpha H(x)$ has a positive proportion of negative values on integers;
3. If $\alpha H(x)$ has a positive proportion of negative values on integers then $H(x)$ has a positive proportion of changes of sign on integers or a positive proportion of zeros.

In chapters 3 and 4, we also exhibit examples for which our theorems can be applied as well as examples illustrating the necessity of some of our conditions.

In chapter 5, we study the error term $\Delta(x)$ of the summation of the divisor function $\tau(n)$. We apply the methods developed in chapter 2, but, unfortunately, we are not able to obtain a corresponding theorem about the number of sign changes on integers of $\Delta(x)$. Using a different method, we prove

Theorem 1.4. Let $N_\Delta(T)$ denote the number of sign changes on integers of $\Delta(t)$, in the interval $[T, 2T]$. Then, for sufficiently large T , $N_\Delta(T) > \sqrt{T}$. Moreover,

there exists a constant c_1 , and $t_1, t_2 \in [T, T + \sqrt{T}]$ such that $\Delta(t_1) \leq -c_1 T^{\frac{1}{4}}$ and $\Delta(t_2) \geq c_1 T^{\frac{1}{4}}$.

In the rest of this chapter, we give a review of the literature related to the error terms of some particular examples and explain the techniques we are going to use to prove the results in chapters 2, 3, 4 and 5.

1.2 AN ERDÖS'S CONJECTURE

As we noticed before, the proof of Erdős's conjecture about the sign changes on integers of the error term associated to the summation of $\phi(x)$ motivated our work. In this section, we state some results about this summation function and about the summation of $\frac{\phi(n)}{n}$.

P. G. L. Dirichlet [13] proved that

$$\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + R(x), \quad (1.5)$$

with $R(x) = O(x^{1+\epsilon})$, for any $\epsilon > 0$, and F. Mertens [57] (see [29, theorem 330]) obtained $R(x) = O(x \log x)$. A related problem which is easier to handle, is the study of the error term

$$H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x.$$

S. S. Pillai and S. D. Chowla [69] studied the relationship between $R(x)$ and $H(x)$ and obtained

$$\frac{R(x)}{x} = H(x) + O\left(\frac{1}{\log^4 x}\right). \quad (1.6)$$

So, Mertens result implies that $H(x) = O(\log x)$. Let $\psi(x) = x - [x] - \frac{1}{2}$. In 1932, S. Chowla [7] proved that

$$H(x) = - \sum_{n \leq \frac{x}{\log^5 x}} \frac{\mu(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^{20} x}\right) \quad (1.7)$$

From this it is immediate to obtain $H(x) = O(\log x)$. The best O -result is still the one obtained by A. Walfisz [104]¹:

$$H(x) = O\left((\log x)^{\frac{2}{3}} (\log \log x)^{\frac{4}{3}}\right). \quad (1.8)$$

On the other hand, Pillai and Chowla [69] proved that

$$R(x) = \Omega(x \log \log \log x), \quad (1.9)$$

and so

$$H(x) = \Omega(\log \log \log x). \quad (1.10)$$

We say that a real valued function g has N *changes of sign* in the interval $[1, T]$ if $[1, T]$ can be partitioned into $N + 1$ consecutive subintervals I_i , $i = 0, 1, \dots, N$, satisfying

- (i) For each $i \in \{0, 1, \dots, N\}$ and any $x, y \in I_i$, $g(x)g(y) \geq 0$;
- (ii) For each $i \in \{0, 1, \dots, N - 1\}$, there are $x_i \in I_i$ and $x_{i+1} \in I_{i+1}$, such that $g(x_i)g(x_{i+1}) < 0$

The number of sign changes of g in $[1, T]$ is denoted by $X_g(T)$.

By averaging H on certain adequately chosen arithmetical progressions $An + B$ ($n \leq x$) of very large moduli $A = A(x)$, P. Erdős and H. N. Shapiro [17] showed that (1.10) implies

$$H(x) = \Omega_{\pm}(\log \log \log x), \quad (1.11)$$

which implies that $H(x)$ changes sign infinitely often. In 1987, this result was improved to

$$H(x) = \Omega_{\pm}\left((\log \log x)^{\frac{1}{2}}\right),$$

¹The proof proposed by Saltykov [76] of $H(x) = O\left((\log x)^{\frac{2}{3}} (\log \log x)^{1+\epsilon}\right)$ is erroneous and once corrected only yields Walfisz result (see [68]).

by H. L. Montgomery [60] and independently by Pétermann [66, 67]. The natural question that follows is: How many changes of sign does $H(x)$ have in the interval $[1, T]$?

In 1986, Y.-F. S. Pétermann [63] studied this problem: First, using the following result of S. Chowla [7],

$$\int_1^T H^2(t) dt = \frac{1}{2\pi^2} T + O\left(\frac{T}{\log^4 T}\right), \quad (1.12)$$

Pétermann proved that

$$\#\{n \leq T : 0 < H(n) < \frac{6}{\pi^2}\} \geq \left(\frac{4}{3} - \frac{\pi^2}{18}\right) T$$

and, since $H(x)$ decreases linearly by $\frac{6}{\pi^2}$ on $[n, n+1)$, obtained

$$X_H(T) \geq \frac{8}{3} \left(1 - \frac{\pi^2}{24}\right) T + o(T). \quad (1.13)$$

Let D be the distribution function of $H(x)$ which is defined by

$$D(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \#\{1 \leq n \leq T : H(n) \leq u\}.$$

P. Erdős and H. N. Shapiro [18] proved that D is a well defined continuous function. This result allowed Pétermann [63] to refine (1.13). His argument goes as follows:

First, $0 < H(n) < \frac{6}{\pi^2}$ if and only if there is a change of sign from positive to negative on $[n, n+1)$. Also, between any two changes of sign from positive to negative, there must be a change of sign from negative to positive (notice that a change of sign from negative to positive always takes place at an integer). Hence

$$X_H(T) = 2 \left(D\left(\frac{6}{\pi^2}\right) - D(0)\right) T + o(T). \quad (1.14)$$

Although $H(x)$ changes signs often, the first values of $H(m)$, with m an integer, are all positive. J. J. Sylvester [90], misled by these initial values, conjectured that $R(n) > 0$ (and so $H(n) > 0$), for all integers n (see [90]). Notice that his table in

[89] goes until $n = 1000$, and it can be seen there that $R(820) < 0$. M. L. N. Sarma [77] rediscovered this counterexample in 1931. As we will see shortly, $H(n)$ and $R(n)$ have a positive proportion of negative values. So, it is natural to ask: How many changes of sign on the integers of the interval $[1, T]$, does $H(n)$ have?

We say that an arithmetic function $f(x)$ has a *sign change on integers* at $x = n$, if $f(n)f(n+1) < 0$. The number of sign changes on integers of $f(x)$ on the interval $[1, T]$ is defined as

$$N_f(T) = \#\{n \leq T, n \text{ integer} : f(n)f(n+1) < 0\}.$$

In a letter to J. Steinig in 1967, Erdős [15] conjectured that

$$N_R(T) \gg T$$

where $R(x)$ is defined in (1.5). Later, Erdős [16] proposed the weaker $N_R(T) = \Omega(T)$. Many authors studied $N_H(T)$, instead, and obtained results about $N_R(T)$ using (1.6).

Since, for any integer N , $\sum_{n \leq N} \frac{\phi(n)}{n}$ is a rational number and $\frac{6}{\pi^2}N$ is irrational then $H(N)$ is also irrational. So $H(N) \neq 0$ for all positive integer N . Figure 1.1 shows the number of sign changes on integers, for some initial values of T . From this data it seems $N_H(T) \approx \frac{T}{270}$. For $n \leq 10^7$, each occurrence of $H(n) < 0$ is isolated and occurs when n is even.

The first result about the number of changes of sign on integers of $H(n)$ was obtained by J. H. Proschan [71], who, using the result (1.11) of Erdős and Shapiro, proved

$$N_H(T) > IL(T),$$

where, $IL(T)$ is the smallest k such that the $4k$ -fold iterated logarithm of T (to a sufficiently large base) is less than 2. Y.-F. S. Pétermann [65] improved Proschan's result, to

$$N_H(T) \gg \log \log T,$$

T	10^3	2×10^3	3×10^3	4×10^3	5×10^3	10^4	2×10^4
$N_H(T)$	2	8	12	18	24	40	76
$\frac{T}{N_H(T)} \approx$	500	250	250	222	208	250	263
T	3×10^4	4×10^4	5×10^4	10^5	2×10^5	3×10^5	4×10^5
$N_H(T)$	120	158	196	384	754	1116	1484
$\frac{T}{N_H(T)} \approx$	250	253	255	260	265	269	270
T	5×10^5	10^6	2×10^6	3×10^6	4×10^6	5×10^6	10^7
$N_H(T)$	1868	3794	7496	11108	14804	18500	36808
$\frac{T}{N_H(T)} \approx$	268	264	267	270	270	270	272

Figure 1.1: Sign Changes on Integers of $H(x)$

still using the method of Erdős and Shapiro [17] and a refinement of the method that Pillai and Chowla used to obtain (1.9). Later, Pétermann [64] obtained

$$N_H(T) \gg \exp \left(C(\log T)^{\frac{3}{5}} (\log \log T)^{-\frac{1}{5}} \right),$$

for a constant $C > 0$, using the result

$$\int_0^T H(t) dt = O \left(T \exp \left(-C(\log T)^{\frac{3}{5}} (\log \log T)^{-\frac{1}{5}} \right) \right),$$

obtained by D. Suryanarayana [88]. Pétermann's idea was the following: Let

$$f(T) = \exp \left(-C(\log T)^{\frac{3}{5}} (\log \log T)^{-\frac{1}{5}} \right)$$

and suppose $H(x)$ stays negative on $[T, T + cTf(T)]$, for a positive constant c . Since $H(x)$ decreases linearly by $\frac{6}{\pi^2}$ on $[n, n+1)$, this implies that

$$\int_T^{T+cTf(T)} H(t) dt \leq -\frac{3c}{\pi^2} Tf(T) + O(1).$$

On the other hand, for $0 \leq c \leq \frac{1}{f(T)}$, we have by Suryanarayana's result,

$$\int_T^{T+cTf(T)} H(t) dt = O(Tf(T)),$$

where the implied constant is independent of c . It follows that $H(x)$ stays negative on intervals of length at most $aTf(T)$ in $(T, 2T)$, for some constant $a > 0$, for sufficiently large T . In the same paper, Pétermann proved that $D(u) \neq 0$ and $0 < D(u) < 1$ for any real u , which implies that there is a positive proportion of negative values of $H(n)$. Let $D(0) = \alpha \neq 0$. Since $H(n) \neq 0$ then, for sufficiently large T , the interval $(T, 2T)$ contains at least $\frac{\alpha}{2}T$ integers at which $H(x)$ is negative. This implies that there are at least $\frac{\alpha T}{2aTf(T)}$ changes of sign on integers of $H(x)$ in the interval.

The conjecture of Erdős was finally proved in 1999, by Y.-K. Lau [50]:

Theorem 1.5 (Lau, 1999). $N_R(T) \gg T$ and $N_H(T) \gg T$, where the implied constants are absolute.

The starting point for our work is Lau's result, so we will explain how he obtained $N_H(T) \gg T$. Using (1.7) and the expansion of $\psi(x)$ as a Fourier series,

$$\psi(u) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi ku)}{k}, \quad (1.15)$$

valid for non integer x , Lau was able to prove

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th, \quad (1.16)$$

for any fixed $1 \leq h \ll \log^4 T$. Lau's argument to prove (1.16), will be generalized in sections 2.2, 2.3 and 2.4. In the next section, we describe the main points of the argument and we also explain how it was generalized.

Formula (1.16) implies that there must be many cancellations in intervals $[t, t+h]$, for some t 's. So, Lau's idea was to prove that we cannot have many large subintervals of $[T, 2T]$, at which $H(x)$ doesn't change sign on integers. As before, if $H(x)$ is always negative in the interval $[t, t+h]$, then

$$\int_t^{t+h} H(u) du \leq -\frac{3}{\pi^2}(h-1).$$

Therefore, if n is an integer such that $H(m) < 0$ for all integer m in $[n, n + 2h]$, then we have

$$\int_n^{n+h} \left(\int_t^{t+h} H(u) du \right)^2 dt \gg h^3.$$

But, $D(0) \neq 0$, so there are cT integers in the interval $[T, 2T]$, at which $H(x)$ is negative. Divide the interval $[T, 2T]$ in subintervals of length h , and take one in each three of those subintervals that have some n for which $H(n) < 0$. This clever division allowed Lau to obtain $\frac{c}{3h}T$ intervals, separated by a distance of at least $2h$, each with $H(x)$ being negative on at least one integer. Let M be the number of the above intervals for which exists a n , with $H(m) < 0$ for all m in $[n, n + 2h]$, and let I be the set formed by those n 's. Then

$$\sum_{n \in I} \int_n^{n+h} \left(\int_t^{t+h} H(u) du \right)^2 dt \gg Mh^3.$$

Hence, by (1.16), $M \ll \frac{T}{h^2}$. If T is sufficiently large, we can take a fixed h suitably large, such that there are $\frac{c}{3h}T - M \geq \frac{c}{6h}T$ intervals of length $2h$ for which there are integers n and m with $H(n) < 0$ and $H(m) \geq 0$. But, as was noticed before, $H(m) \neq 0$, for any integer m , so $N_H(T) \gg T$. Since $D(u)$ is continuous, we can find a δ such that, at least half of the above intervals have integers n and m , with $H(n) < -\delta$ and $H(m) > \delta$. From (1.6), also $N_R(T) \gg T$.

Another example that was considered in the literature is $f(n) = \frac{\sigma(n)}{n}$. If we write

$$F(x) = \sum_{n \leq x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6}x + \frac{\log 2\pi x}{2} + \frac{\gamma}{2}$$

then $F(x)$ also have infinitely many changes of sign. Pétermann [63] used a mean square result of A. Walfisz [103] for $F(x) - \frac{\gamma + \log 2\pi}{2}$, from which it can be deduced that

$$\int_1^T F^2(t) dt = \frac{5\pi^2}{144}T(1 + o(1)), \quad (1.17)$$

to obtain

$$X_F(T) \geq \frac{8}{3} \left(1 - \frac{15}{4\pi^2} \right) T + o(T). \quad (1.18)$$

Later, Pétermann [64] also obtained $N_F(T) \gg T^{0.71\dots}$. As Y.-K. Lau remarked, his method can be applied to obtain $N_F(T) \gg T$. In figure 1.2, we present the initial values of $N_F(T)$, for small values of T .

T	10^3	2×10^3	3×10^3	4×10^3	5×10^3	10^4	2×10^4
$N_F(T)$	6	20	34	50	64	146	328
$\frac{T}{N_F(T)} \approx$	166.7	100	88.2	80	78.1	68.5	61
T	3×10^4	4×10^4	5×10^4	10^5	2×10^5	3×10^5	4×10^5
$N_F(T)$	510	684	872	1762	3526	5258	7046
$\frac{T}{N_F(T)} \approx$	58.8	58.5	57.3	56.8	56.7	57.1	56.8
T	5×10^5	6×10^5	7×10^5	8×10^5	9×10^5	10^6	
$N_F(T)$	8820	10600	12334	14098	15890	17686	
$\frac{T}{N_F(T)} \approx$	56.7	56.6	56.8	56.7	56.6	56.5	

Figure 1.2: Sign Changes on Integers of $F(x)$

1.3 NEW RESULTS

In order to generalize Lau's result we will study a class of arithmetic functions that have a behavior similar to $\frac{\phi(n)}{n}$, i. e. we consider functions $f(n)$ such that, writing

$$f(n) = \sum_{d|n} \frac{b_d}{d},$$

the sequence b_n satisfies conditions (1.2) and (1.3):

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right)$$

$$\sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some constants $A > 1$, $D > 0$ and real B , respectively.

When $b_n = \mu(n)$ or $b_n = 1$, both conditions are satisfied, so the corresponding functions $\frac{\phi(n)}{n}$ and $\frac{\sigma(n)}{n}$ belong to our class of arithmetic functions.

Condition (1.2) allow us to prove

$$\sum_{n \leq x} f(n) = \alpha x - \frac{B \log 2\pi x}{2} - \frac{\gamma_b}{2} + H(x),$$

where α and γ_b are constants, defined by

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$$

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right)$$

and, given $0 < C < A - 1$,

$$H(x) = - \sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right).$$

In sections 2.2, 2.3 and 2.4, we use condition (1.3) and its consequences (see lemma 2.2), to refine Lau's method, and prove

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}},$$

for any constant $h \ll \log^c T$, where $0 < c \leq 1$.

Here, we will outline the main topics of Lau's argument and our generalization.

Define

$$H_N(x) = - \sum_{d \leq N} \frac{b_d}{d} \psi\left(\frac{x}{d}\right). \quad (1.19)$$

Using Cauchy's inequality, one obtains

$$\left(\int_t^{t+h} H(u) du \right)^2 dt \leq 2 \left(\int_t^{t+h} H_N(u) du \right)^2 dt + 2 \left(\int_t^{t+h} (H(u) - H_N(u)) du \right)^2 dt.$$

Also, using Cauchy's inequality and interchanging the integrals,

$$\int_T^{2T} \left(\int_t^{t+h} (H(u) - H_N(u)) du \right)^2 dt \leq h^2 \int_T^{2T+h} (H(u) - H_N(u))^2 du$$

So, the idea is to estimate

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u) du \right)^2 dt \quad \text{and} \quad \int_T^{2T+h} (H(u) - H_N(u))^2 du.$$

Notice that

$$H(u) - H_N(u) = - \sum_{N < n \leq \frac{u}{\log^C u}} \frac{b_n}{n} \psi\left(\frac{u}{n}\right) + O\left(\frac{1}{\log^C u}\right) + O\left(\frac{1}{\log^{A-C-1} u}\right).$$

We begin the process of evaluating the integrals using the Fourier series (1.15) for the function $\psi(x)$. After some calculations, we obtain

$$\begin{aligned} & \int_T^{2T+h} (H(u) - H_N(u))^2 du \\ & \leq \frac{2}{\pi^2} \sum_{m,n=N+1}^X \frac{b_m b_n}{mn} \sum_{k,l=1}^{\infty} \frac{1}{kl} \int_{\eta(T,m,n)}^{2T+h} \sin\left(2\pi \frac{ku}{m}\right) \sin\left(2\pi \frac{lu}{n}\right) du \\ & \quad + O\left(\frac{T}{\log^{2C} T} + \frac{T}{(\log T)^{2(A-C-1)}}\right). \end{aligned}$$

In section 2.3, we clearly define X and $\eta(T, m, n)$, but here it's enough to know that $X \ll \frac{T}{\log^C T}$ and $\eta(T, m, n) \geq T$. After using a trigonometric identity and evaluating the integral, we are left with the estimation of three double sums

$$\begin{aligned} & \int_T^{2T+h} (H(u) - H_N(u))^2 du \ll T \sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \\ & \quad + \sum_{m,n \leq X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl(kn + lm)} \\ & \quad + \sum_{m,n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl|kn - lm|} \\ & \quad + O\left(\frac{T}{\log^{2C} T} + \frac{T}{(\log T)^{2(A-C-1)}}\right). \quad (1.20) \end{aligned}$$

In the case studied by Y.-K. Lau, $b_n = \mu(n)$, and so, $|b_m b_n|$ is 0 or 1, for any integers m and n . Lau uses lemma 7 and 8 of S. D. Chowla [7] - of which lemma 1.6 and 1.7

below, are generalizations, respectively - to estimate the two last sums and directly estimates the first, obtaining

$$\int_T^{2T+h} (H(u) - H_N(u))^2 \, du \ll \frac{T}{N} + \frac{T}{\log^4 T}.$$

So, if $h \leq \min\{N, \log^4 T\}$, then

$$h^2 \int_T^{2T+h} (H(u) - H_N(u))^2 \, du \ll Th.$$

In our general case, the estimation of the sums in (1.20) becomes much harder, in particular the third sum. We use a variety of elementary techniques (e. g. Hölder's inequality in the form $|\sum_i u_i v_i| \leq (\sum_j u_j^4)^{\frac{1}{4}} (\sum_k v_k^{\frac{4}{3}})^{\frac{3}{4}}$ and Cauchy's inequality), as well as (1.3) in order to obtain the following results,

Lemma 1.6. *Let $E = 4 + \frac{D}{2}$, then*

$$\sum_{m,n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \ll X (\log X)^E.$$

Lemma 1.7. *If $D > 0$ satisfies condition (1.3), then*

$$\sum_{m,n \leq X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl (kn + lm)} \ll X (\log X)^{1+\frac{D}{2}}.$$

Lemma 1.8. *For any $\delta > 0$*

$$\sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \ll \frac{1}{N^{1-\delta}}.$$

These results imply that

$$\begin{aligned} \int_T^{2T+h} (H(u) - H_N(u))^2 \, du &\ll \frac{T}{N^{1-\delta}} + \frac{T}{(\log T)^{C-E}} + \frac{T}{\log^{2C} T} \\ &\quad + \frac{T}{(\log T)^{2(A-C-1)}}. \end{aligned}$$

If A is large enough, we can take $C = E + 1$ and $2(A - C - 1) = c > 0$. Therefore, taking $h \leq \min(\log^c T, \log T)$ and $N > \log^2 T$, we obtain

$$h^2 \int_T^{2T+h} (H(u) - H_N(u))^2 du \ll Th.$$

Now, we evaluate the first integral

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u) du \right)^2 dt.$$

From (1.15) and a few calculations,

$$\begin{aligned} & \int_t^{t+h} H_N(u) du \\ &= -\frac{1}{2\pi^2} \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\cos\left(2\pi \frac{k(t+h)}{m}\right) - \cos\left(2\pi \frac{kt}{m}\right)}{k^2} \\ &= \frac{1}{4\pi^2} \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{1}{k^2} \left\{ \left[e\left(\frac{kh}{m}\right) - 1 \right] e\left(\frac{kt}{m}\right) \left[e\left(-k \frac{(2t+h)}{m}\right) - 1 \right] \right\}. \end{aligned}$$

Hence, as $|z|^2 = z\bar{z}$,

$$\begin{aligned} & \int_T^{2T} \left| \int_t^{t+h} H_N(u) du \right|^2 dt = \\ & \frac{1}{16\pi^4} \sum_{m,n \leq N} b_m b_n \sum_{k,l=1}^{\infty} \frac{(e(\frac{kh}{m}) - 1)(e(-\frac{lh}{n}) - 1)}{(kl)^2} \\ & \times \int_T^{2T} e\left(\frac{kt}{m}\right) e\left(-\frac{lt}{n}\right) \left(e\left(-k \frac{(2t+h)}{m}\right) - 1 \right) \left(e\left(l \frac{(2t+h)}{n}\right) - 1 \right) dt. \end{aligned}$$

After evaluating the integral and using $|e(t) - 1| \ll \min(1, |t|)$, we are left with

$$\begin{aligned} & \int_T^{2T} \left| \int_t^{t+h} H_N(u) du \right|^2 dt \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{(kl)^2 \left(\frac{l}{n} + \frac{k}{m}\right)} + \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{(kl)^2 \left|\frac{k}{m} - \frac{l}{n}\right|} \\ & + T \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right). \end{aligned} \tag{1.21}$$

Notice that the first term on the left is bounded by the second term. In the case $b_m = \mu(m)$, Lau used lemma 7 of [7], and obtained

$$\sum_{m,n \leq N} \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{(kl)^2 \left| \frac{k}{m} - \frac{l}{n} \right|} \ll N^3 \log N.$$

For the third term, Lau took $d = (m, n)$, $\alpha = \frac{m}{d}$, $\beta = \frac{n}{d}$ and $\gamma = \frac{k}{\alpha}$, which allowed him to transform the minimum factors into

$$\left(\min \left(1, \frac{h\gamma}{d} \right) \right)^2.$$

By considering the cases $d \leq h\gamma$ and $d > h\gamma$, Lau was able to bound the corresponding term in (1.21) by Th . Hence if N is a sufficiently small power of T , say $T^{\frac{1}{4}}$, then

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u) du \right)^2 dt \ll Th.$$

For general sequences b_n , we, again, need to develop a new method to estimate the sums of (1.21). As before, we generalize Lau's technique, using (1.3) and the lemmas above. We obtain a weaker result, but still sufficient for our purposes. With $N = T^{\frac{1}{4}}$ as above, we prove

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u) du \right)^2 dt \ll Th^{\frac{3}{2}}.$$

Hence

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}}.$$

Although this result is weaker than (1.16), it will still allow us to obtain a result about the number of sign changes on integers for this class of functions. More exactly, we prove

Theorem 1.2. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies both conditions*

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let $\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right)$,

$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$ and $H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}$. If $\alpha \neq 0$, then

1. $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$.
2. if $N_H(T) \gg T$, then $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$;
3. if $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$, then $N_H(T) \gg T$ or $z_H(T) \gg T$.

The following example shows that we cannot eliminate $z_H(T) \gg T$ from part 3:

Example 1.1. Consider the sequence $b_1 = 0$, $b_2 = 4$, $b_3 = 6$, $b_4 = b_5 = 0$, $b_6 = -24$ and $b_n = 0$ for $n > 6$. Then $\alpha = 1$, $B = 0$ and $\gamma_b = 0$. We will show later, in section 3.4, that if $b_n = 0$ for $n > N$ then $f(n) := \sum_{d|n} \frac{b_d}{d}$ is periodic (see proposition 1.11). In this example, $f(1) = f(5) = f(6) = 0$, $f(2) = f(3) = f(4) = 2$ and $f(n)$ has period 6. So, $H(n)$ is also periodic with period 6 and $H(1) = -1$, $H(2) = 0$, $H(3) = 1$, $H(4) = 2$, $H(5) = 1$ and $H(6) = 0$. Hence, $\#\{1 \leq n \leq T : \alpha H(n) < 0\} = \frac{T}{6} + O(1)$, $N_H(T) = 0$ and $z_H(T) = \frac{T}{3} + O(1)$.

Part 3 is not as good as we would desire, but if other assumptions about the constants B and α are made, we can prove that $z_H(T)$ is very small. In order to do that, we use a corollary of A. Baker's result on algebraic numbers [2, theorem 1],

Proposition 1.9 (Baker, 1967). Let $\alpha_1, \dots, \alpha_n$ and $\beta_0, \dots, \beta_n$ denote nonzero algebraic numbers. Suppose that $\kappa > n + 1$, and let d and H denote, respectively, the maximum of the degrees and heights of $\beta_0, \dots, \beta_n$. Then

$$|\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n| > C e^{-(\log H)^\kappa},$$

for some effectively computable number $C > 0$.

Hence, if $\alpha_1, \dots, \alpha_n$ and $\beta_0, \dots, \beta_n$ are nonzero algebraic numbers, then

$$\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n \neq 0$$

From this we obtain

Theorem 1.10. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a rational valued arithmetic function and suppose the sequence b_n satisfies $\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right)$, for some real B and $A > 1$. Let r be a real number and*

$$H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2},$$

where $\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right)$ and $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$. Then

1. If $B = 0$ and α is irrational then $\#\{n \leq T, n \text{ integer} : H(n) = r\} \leq 1$;
2. If B is a nonzero algebraic number then $\#\{n \leq T, n \text{ integer} : H(n) = r\} \leq 2$;
3. If B is transcendental then there exists a constant C that depends on r and on the function $f(n)$, such that

$$\#\{n \leq T, n \text{ integer} : H(n) = r\} \ll (\log T)^C.$$

Using theorem 1.10 and parts 2 and 3 of theorem 1.2, we obtain theorem 1.1.

In the next few sections of chapter 3 we study examples of arithmetic functions for which our theorems can be applied. In section 3.4, we consider sequences b_n with only a finite number of nonzero terms. These sequences, trivially, satisfy both conditions (1.2) and (1.3). We prove

Proposition 1.11. *Let b_n be a sequence of real numbers such that $b_n = 0$ for $n > N$, for some integer N . Then the sequence $f(n) = \sum_{d|n} \frac{b_d}{d}$ is periodic with period, say q , dividing $[1, 2, \dots, N]$ and $f(i) = f((i, q))$, for any integer i .*

Reciprocally, if there exists q for which $f(i) = f((i, q))$, for all integers i , then $b_n = 0$ whenever $n \nmid q$.

Moreover, using the above notation, we have $\alpha = \frac{1}{q} \sum_{n \leq q} f(n)$ and $\gamma_b = f(q)$.

In section 3.5, we study the function $\frac{n}{\phi(n)}$. Since

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)},$$

then $b_n = \frac{n\mu^2(n)}{\phi(n)}$. So, both conditions (1.2) and (1.3) are valid for this sequence (and we can take A as large as we want). We have

$$\sum_{n \leq x} \frac{n}{\phi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}x - \frac{\log x}{2} - \frac{\log 2\pi + \gamma + \sum_p \frac{\log p}{p(p-1)}}{2} + H(x)$$

and we can apply the theorems above to the error term $H(x)$, so that $N_H(T) \gg T$ if and only if $\#\{1 \leq n \leq T : H(n) < 0\} \gg T$.

In 1900, E. Landau [49] proved that

$$E_0(x) := \sum_{n \leq x} \frac{1}{\phi(n)} - \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\log x + \gamma - \sum_p \frac{\log p}{p^2 - p + 1} \right) = O\left(\frac{\log x}{x}\right),$$

which was improved by R. Sitaramachandrarao [80] in 1982:

$$E_0(x) = O\left(\frac{(\log x)^{\frac{2}{3}}}{x}\right).$$

Later, Sitaramachandrarao [81, Lemma 2.4] proved that

$$E_1(x) = xE_0(x) + O(1),$$

where

$$E_1(x) = \sum_{n \leq x} \frac{n}{\phi(n)} - \frac{\zeta(2)\zeta(3)}{\zeta(6)}x + \frac{\log x}{2}.$$

Clearly, we also have

$$H(x) = xE_0(x) + O(1).$$

Therefore, we can apply our results to $E_0(x)$ and obtain $N_{E_0}(T) \gg T$ if and only if $\#\{1 \leq n \leq T : \alpha H(n) < -\delta\} \gg T$, for some $\delta > 0$.

One of the conditions for theorem 1.2 is that $\alpha \neq 0$. In section 3.6, we prove the following result:

Proposition 1.12. *If b_n is a completely multiplicative sequence satisfying condition (1.3), then $\alpha \neq 0$.*

We end that section with a couple of examples of multiplicative and strongly multiplicative sequences for which $\alpha = 0$.

In section 3.7, we use the method developed in section 2.3 in order to obtain the mean square of $H(x)$. We prove the following theorem:

Theorem 1.13. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies both conditions*

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 7 + \frac{3D}{4}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

Let $g(n) = \sum_{d|n} b_d$. Then,

$$\int_1^x H^2(u) \, du = \frac{x}{2\pi^2} \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} + O\left(\frac{x}{\log^L x}\right),$$

where $L > 0$.

The methods developed to prove theorem 1.2, also allow us to obtain $X_H(T) \gg T$. This is done in section 3.8. But to generalize (1.14), we would need to prove the existence of distribution functions associated with the error terms H . This problem was not studied in our work.

1.4 METHODS OF COMPLEX ANALYSIS

For many sequences b_n satisfying conditions (1.2) and (1.3), the corresponding arithmetic functions $f(n) = \sum_{d|n} \frac{b_d}{d}$ don't have interesting expressions. For example,

1. $b_n = \Lambda(n)$, where Λ is the Von Mangoldt's function, i. e.

$$b_n = \begin{cases} \log p, & \text{if } n = p^\nu \text{ for some } \nu \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

- 2.

$$b_n = \begin{cases} 1, & \text{if } n \text{ is a prime number} \\ 0, & \text{otherwise} \end{cases}$$

3. $b_n = (-1)^{\Omega(n)}$, i. e. b_n is the Liouville's function.

On the other hand, the sequences b_n corresponding to the arithmetic functions

$$\frac{n}{\sigma(n)}, \quad \frac{\sigma(n)}{\phi(n)}, \quad \frac{\phi(n)}{\sigma(n)}$$

or, more generally,

$$\left(\frac{\phi(n)}{n}\right)^r, \quad \left(\frac{\sigma(n)}{n}\right)^r, \quad \left(\frac{\phi(n)}{\sigma(n)}\right)^r \quad (1.22)$$

for any real $r \neq 0$, don't have simple expressions, and, in general, don't satisfy condition (1.2). In order to obtain a refinement of theorem 1.2 that can be applied to the above functions, we use methods of complex analysis and Dirichlet series expansions. First, we state some results about their summation functions. We start with the Euler function. Following U. Balakrishnan and Y.-F. S. Pétermann [3], we will write, for real $r \neq 0$,

$$\sum_{n \leq x} \left(\frac{\phi(n)}{n}\right)^r = \begin{cases} \alpha_r x + \sum_{k=0}^{[-r]} a_k (\log x)^{-r-k} + H(x, r), & \text{if } r < 0 \\ \alpha_r x + H(x, r) & \text{if } r > 0 \end{cases} \quad (1.23)$$

where α_r and a_k are constants. S. D. Chowla [6], proved that $H(x, r) = O(\log^r x)$ and

$$\alpha_r = \prod_p \left(1 - \frac{1}{p} \left(1 - \left(1 - \frac{1}{p} \right)^r \right) \right),$$

for any positive integer r . In 1969, I. I. Il'yasov [36] generalized Walfisz's estimate (1.8), obtaining

$$H(x, r) = O \left((\log x)^{\frac{2}{3}} (\log \log x)^{\frac{4}{3}} \right),$$

for $0 < r \leq 1$. Ten years later, A. Sivaramasarma [82] obtained

$$H(x, r) = O \left((\log x)^{r-\frac{1}{3}} (\log \log x)^{\frac{4}{3}} \right),$$

for $r \geq 1$. The best result to date about the order of $H(x, r)$ is due to U. Balakrishnan and Y.-F. S. Pétermann [3, theorem 4]. They proved that

$$H(x, r) = O \left((\log x)^{\frac{2|r|}{3}} (\log \log x)^{\frac{4|r|}{3}} \right),$$

for every real $r \neq 0$.

Now, we consider the sum of divisors function. We have

$$\sum_{n \leq x} \left(\frac{\sigma(n)}{n} \right)^r = \begin{cases} \beta_r x + E(x, r), & \text{if } r < 0 \\ \beta_r x + \sum_{k=0}^{[r]} b_k (\log x)^{r-k} + E(x, r), & \text{if } r > 0 \end{cases} \quad (1.24)$$

where β_r and b_k are constants.

Instead of the error term $E(x, 2)$, R. A. Smith [85], considered a related function, say $F(x)$, that can be expressed as $F(x) = E(x, 2) + C \log x$ and proved that

$$F(x) = O \left((\log x)^{\frac{5}{3}} \right).$$

Balakrishnan and Pétermann [3] also considered the error terms $E(x, r)$ and obtained

$$E(x, r) = O \left((\log x)^{\frac{2|r|}{3}} (\log \log x)^{\frac{4|r|}{3}} \right).$$

In the same paper, they considered Ω -estimates for both $H(x, r)$ and $E(x, r)$ and proved the following results

$$H(x, r) = \begin{cases} \Omega_{\pm} \left((\log \log x)^{|r|} \right) & \text{if } r < 0 \\ \Omega_{\pm} \left((\log \log x)^{\frac{|r|}{2}} \right) & \text{if } r > 0 \end{cases}$$

and

$$E(x, r) = \begin{cases} \Omega_{\pm} \left((\log \log x)^{|r|} \right) & \text{if } r > 0 \\ \Omega_{\pm} \left((\log \log x)^{\frac{|r|}{2}} \right) & \text{if } r < 0 \end{cases}$$

Using the asymptotic expansions (1.23) and (1.24), Balakrishnan and Pétermann obtained similar expressions for the sums of $\phi^r(n)$ and $\sigma^r(n)$, for $r > 0$:

$$\sum_{n \leq x} \phi^r(n) = \frac{\alpha_r}{r+1} x^{r+1} + G(x, r)$$

and

$$\sum_{n \leq x} \sigma^r(n) = \frac{\beta_r}{r+1} x^{r+1} + x^r \sum_{k=0}^{[r]} c_k (\log x)^{r-k} + F(x, r),$$

where c_k are constants. They also proved that $G(x, r) = x^r H(x, r)(1 + o(1))$, and $F(x, r) = x^r E(x, r)(1 + o(1))$. Therefore, O - and $\Omega_{\pm}$ -results for $G(x, r)$ and $F(x, r)$ can be derived from the corresponding results for $H(x, r)$ and $E(x, r)$.

The arithmetic functions in (1.22) have Dirichlet series expansions of the form $\zeta(s)\zeta^{\beta}(s+1)g(s+1)$, where β only depends on r , and $g(s)$ is absolutely convergent for $\sigma > 1 - \lambda$, for some $\lambda > 0$.

We use the following result to obtain theorem 1.3.

Proposition 1.14 (Balakrishnan & Pétermann [3], 1996). *Let $f(n)$ be a complex valued arithmetic function satisfying*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s)\zeta^{\beta}(s+1)g(s+1), \quad (1.25)$$

for a complex number β , and $g(s)$ having a Dirichlet series expansion

$$g(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

which is absolutely convergent in the half plane $\sigma > 1 - \lambda$ for some $\lambda > 0$. Let β_0 be the real part of β . If

$$\zeta^\beta(s)g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

then there is a real number b , $0 < b < 1/2$, and constants B_j , such that

$$\sum_{n \leq x} f(n) = \begin{cases} \zeta^\beta(2)g(2)x - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1), & \text{if } \beta_0 < 0 \\ \zeta^\beta(2)g(2)x + \sum_{j=0}^{[\beta_0]} B_j (\log x)^{\beta-j} - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1), & \text{if } \beta_0 > 0, \end{cases}$$

where $y(x) = x \exp(-(\log x)^b)$.

We are only interested in the cases when $f(n)$ and β are real. For the proof of this theorem, Balakrishnan and Pétermann used Hankel's and Perron's formulæ, and bounds on the zeta function in the critical strip. Below, we will explain the easier case when β is an integer, using the residue theorem instead of Hankel's.

Notice that, in proposition 1.14,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b_n}{n^{s+1}},$$

which implies that $f(n) = \sum_{d|n} \frac{b_d}{d}$. Reciprocally, we prove

Lemma 1.15. *Given a sequence of real numbers b_n , let $f(n) = \sum_{d|n} \frac{b_d}{d}$. Then*

$$\sum_{n \leq x} \frac{f(n)}{n^s} = \zeta(s) \sum_{n \leq x} \frac{b_n}{n^{s+1}} - \sum_{n \leq x} \frac{b_n}{n^{s+1}} \sum_{m > \frac{x}{n}} \frac{1}{m^s}$$

for any $s = \sigma + it$ with $\sigma > 1$. Define

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad B(s) := \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

whenever the series in question exist. If the sequence b_n satisfies the condition (1.3), i. e.

$$\sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for $D > 0$, then

$$F(s) = \zeta(s)B(s+1),$$

for $\sigma > 1$.

Now, we will explain how to get proposition 1.14 when β is an integer. Let

$$G(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$$

be a Dirichlet series with abscissa of absolute convergence σ_a . Perron's effective formula (see [91, theorem II 2.2]) states,

$$\sum_{n \leq x} \alpha_n = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} G(s) \frac{x^s}{s} ds + O(x^{c-1}),$$

where $x \geq 1$ and $c > \max(0, \sigma_a)$. The integral on the right can be computed if we complete the segment from $c - ix$ to $c + ix$ into a closed contour and use the residue theorem of complex analysis.

Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a sequence of real numbers satisfying (1.25). Then

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n \leq x} \frac{b_n}{n} \left[\frac{x}{n} \right] \\ &= x \sum_{n \leq x} \frac{b_n}{n^2} - \frac{1}{2} \sum_{n \leq x} \frac{b_n}{n} - \sum_{n \leq x} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) \end{aligned}$$

Using Perron's formula, we evaluate $\sum_{n \leq x} \frac{b_n}{n}$ and $\sum_{n \leq x} \frac{b_n}{n^2}$. We have

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^\beta(s+1)g(s+1),$$

where $g(s)$ is absolutely convergent in the half plane $\sigma > 1 - \lambda$ for some positive λ .

So

$$\sum_{n \leq x} \frac{b_n}{n} = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} \zeta^\beta(s+1)g(s+1)\frac{x^s}{s} ds + O(x^{c-1}),$$

for any $0 < c < 1$. It is known, that $\zeta(s+1)$ has no zero in the region $\sigma \geq -\frac{c_1}{\log^{1-\epsilon}(|t|)}$, for any $\epsilon < \frac{1}{3}$ (see [38, Chapter 6])². There is also no zero of $\zeta(s+1)$ in the region $|s| < 1$ (see [91, section II 5.2]).

Take $0 < \eta < \min(\lambda, 1)$, $\epsilon < \frac{1}{3}$ and

$$G(s) = \zeta^\beta(s+1)g(s+1)\frac{x^s}{s}.$$

In order to form a closed contour, say L , take the vertical line connecting the points $c - ix$ and $c + ix$, then join horizontally the points $c + ix$ and $c - ix$ with the curve $\sigma = -\frac{c_1}{\log^{1-\epsilon}(|t|)}$, and join vertically the point $-\eta$ with that curve. Notice that, inside L , $\zeta(s+1)$ has no zero and has one simple pole at $s = 0$. Moreover, $g(s+1)$ is absolutely convergent inside L . Therefore, $G(s)$ has a pole at $s = 0$ of order $\beta + 1$, if $\beta \geq 0$, and is analytic, if $\beta < 0$. By the residue theorem

$$\frac{1}{2\pi i} \int_{c-ix}^{c+ix} G(s) ds = \Phi(x) - \frac{1}{2\pi i} \int_{L \setminus [c-ix, c+ix]} G(s) ds,$$

where, when $\beta \geq 0$, $\Phi(x)$ is the residue of $G(s)$ at $s = 0$, and, when $\beta < 0$, $\Phi(x) = 0$.

Since, close to $s = 0$, we have $\zeta(s+1) = \frac{1}{s} + \gamma + O(|s|)$, then

$$\Phi(x) = \sum_{j=0}^{\beta} A_j \log^j x,$$

²N. M. Korobov [48] and I. M. Vinogradov [99] obtained independently an upper bound for $\zeta(s)$ in a region just to the left of $\sigma = 1$ which implies that $\zeta(s)$ has no zero in the region $\sigma > 1 - \frac{C}{\log^{\frac{2}{3}} |t| (\log \log |t|)^{\frac{1}{3}}}$

for $\beta \geq 0$, where A_j are computable constants. Since $\zeta(s+1) = O(\log |t|)$, inside L (see [91, theorem II 3.7]), then

$$\begin{aligned} \frac{1}{2\pi i} \int_{L \setminus [c-ix, c+ix]} G(s) ds &= O\left(x^{-\frac{c_1}{\log^{1-\epsilon} x}} \log x\right) + O(x^{c-1}) \\ &= O(\exp(-c_2 \log^\epsilon x)) \end{aligned}$$

Therefore

$$\sum_{n \leq x} \frac{b_n}{n} = \begin{cases} O(\exp(-c_2 \log^\epsilon x)), & \text{if } \beta < 0 \\ \sum_{j=0}^{\beta} A_j \log^j x + O(\exp(-c_2 \log^\epsilon x)), & \text{if } \beta > 0 \end{cases}$$

Now, we evaluate $\sum_{n \leq x} \frac{b_n}{n^2}$. We have

$$\sum_{n \leq x} \frac{b_n}{n^2} = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} \zeta^\beta(s+2)g(s+2) \frac{x^s}{s} ds + O(x^{c-1}),$$

for any $0 < c < 1$. Take the vertical line $[c-ix, c+ix]$, connect horizontally its end points to the curve $\sigma = -1 - \frac{c_1}{\log^{1-\epsilon}(|t|)}$, and connect vertically the point $-1 - \eta$ to the same curve to form a closed contour L_1 . Inside L_1 , $\zeta^\beta(s+2)g(s+2) \frac{x^s}{s}$ has a simple pole at $s = 0$ and a pole of order β , if $\beta \geq 1$, at the point $s = -1$. Therefore,

$$\sum_{n \leq x} \frac{b_n}{n^2} = \begin{cases} \zeta^\beta(2)g(2) + O(\exp(-c_2 \log^\epsilon x)), & \text{if } \beta < 1 \\ \zeta^\beta(2)g(2) + \frac{1}{x} \sum_{j=0}^{\beta-1} C_j \log^j x + O(\exp(-c_2 \log^\epsilon x)), & \text{if } \beta > 1 \end{cases}$$

After joining everything together, we obtain

$$\sum_{n \leq x} f(n) = \begin{cases} \zeta^\beta(2)g(2)x - \sum_{n \leq x} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O(\exp(-c_2 \log^\epsilon x)), & \text{if } \beta < 0 \\ \zeta^\beta(2)g(2)x + \sum_{j=0}^{\beta} D_j \log^j x - \sum_{n \leq x} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) \\ \quad + O(\exp(-c_2 \log^\epsilon x)), & \text{if } \beta > 0 \end{cases}$$

Now, we just need to estimate $\sum_{y < n \leq x} \frac{b_n}{n} \psi\left(\frac{x}{n}\right)$. Here, we follow Balakrishnan and Pétermann [3, Lemma 2.5]), who proved that

$$\sum_{y < n \leq x} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) = \begin{cases} o(1), & \text{if } \beta < 1 \\ \sum_{j=0}^{\beta-1} E_j (\log x)^{\beta-1-j} + o(1), & \text{if } \beta > 1 \end{cases}$$

and so proposition 1.14 is obtained, for any integer β .

Define

$$H(x) = \begin{cases} \sum_{n \leq x} f(n) - \zeta^\beta(2)g(2)x, & \text{if } \beta < 0 \\ \sum_{n \leq x} f(n) - \zeta^\beta(2)g(2)x - \sum_{j=0}^{[\beta]} B_j (\log x)^{\beta-j}, & \text{if } \beta > 0 \end{cases}$$

From proposition 1.14, there is an increasing function $k(x)$, with

$$\lim_{x \rightarrow \infty} k(x) = \infty,$$

such that

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right).$$

The methods developed in chapter 2, allow us to obtain

$$\int_T^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \ll Th^{\frac{3}{2}},$$

for any constant $h \ll \min(\log T, k^2(T))$, and using this result we prove theorem 1.3, which we state here again:

Theorem 1.3. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a arithmetic function and suppose the sequence b_n satisfies conditions (1.3) and (1.4), i. e.*

$$\sum_{n \leq x} b_n^4 \ll x \log^D x \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^\beta(s)g(s)$$

for some β real, $D > 0$, and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma > 1 - \lambda$, for some $\lambda > 0$. Let $\alpha = \zeta^\beta(2)g(2)$ and

$$H(x) = \begin{cases} \sum_{n \leq x} f(n) - \alpha x, & \text{if } \beta < 0 \\ \sum_{n \leq x} f(n) - \alpha x - \sum_{j=0}^{[\beta]} B_j (\log x)^{\beta-j}, & \text{if } \beta > 0 \end{cases}$$

where the constants B_j are defined by proposition 1.14. If $\alpha \neq 0$ then

1. $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$;
2. if $N_H(T) \gg T$, then $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$;
3. if $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$, then $N_H(T) \gg T$ or $z_H(T) \gg T$.

We end chapter 4 proving that the list of examples in (1.22) satisfy the conditions of theorem 1.3.

1.5 THE DIVISOR FUNCTION $\tau(n)$

Let $\tau(n)$ denote the number of divisors of n . It was proved by P. G. L. Dirichlet [12] that

$$D(x) := \sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x), \quad (1.26)$$

where

$$\Delta(x) = -2 \sum_{d \leq \sqrt{x}} \psi\left(\frac{x}{d}\right) + O(1)$$

So $\Delta(x) = O(\sqrt{x})$. The *Dirichlet's divisor problem* consists of determining as precisely as possible the maximum order of $\Delta(x)$. G. H. Hardy conjectured that $\Delta(x) = O(x^{\frac{1}{4}+\epsilon})$, for any $\epsilon > 0$. Using only elementary methods, G. F. Voronoï [100] proved that $\Delta(x) = O\left(x^{\frac{1}{3}} \log x\right)$.

In 1904, Voronoï [101] obtained the following explicit formula

$$\Delta(x) = \frac{1}{4} + \frac{\delta(x)}{2} + \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(4\pi\sqrt{nx} - \frac{\pi}{4}) - \frac{3}{32\pi^2\sqrt{2}} x^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi\sqrt{nx} - \frac{\pi}{4}) + O\left(x^{-\frac{3}{4}}\right), \quad (1.27)$$

where $\delta(x) = \tau(x)$ if x is an integer and 0 otherwise. Next, we will give an idea of how a truncated version of the above formula can be obtained (see [94, chapter XII] for details).

The generating function of $\tau(n)$ is $\zeta^2(s)$, i. e.

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

for $\sigma > 1$. Let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with abscissa of convergence σ_c . Then, Perron's formula (see [91, theorem II.2.1]) states,

$$\sum_{n < x} a_n + \frac{a_x}{2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds,$$

where $x > 0$, $a_x = 0$ if x is not integer and $c > \max(0, \sigma_c)$. Therefore, when x is not an integer,

$$D(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^2(s) \frac{x^s}{s} ds,$$

for $c > 1$. Considering the integral round the rectangle $c - iT$, $c + iT$, $-a + iT$, $-a - iT$, where $a > 0$, we find a double pole at $s = 1$ and a simple pole at $s = 0$. The residue at $s = 0$ is $\zeta^2(0) = \frac{1}{4}$. Since, close to $s = 1$, we have,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|).$$

Then the residue at $s = 1$ is $x \log x + (2\gamma - 1)x$. Moreover, using bounds on $\zeta(s)$, can be proved that

$$\int_{-a+iT}^{c+iT} \zeta^2(s) \frac{x^s}{s} ds = O\left(\frac{T^{2a}}{x^a}\right) + O\left(\frac{x^c}{T}\right).$$

A similar result holds for the integral of $\zeta^2(s) \frac{x^s}{s}$, when s goes from $-a-iT$ to $c-iT$.

Therefore,

$$\begin{aligned} D(x) := \sum_{n \leq x} \tau(n) &= x \log x + (2\gamma - 1)x + \frac{1}{4} + \frac{1}{2\pi i} \int_{-a-iT}^{-a+iT} \zeta^2(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{T^{2a}}{x^a}\right) + O\left(\frac{x^c}{T}\right) \end{aligned}$$

Using (1.26) and the functional equation of $\zeta(s)$, we obtain

$$\begin{aligned} \Delta(x) &= \frac{1}{4} + \frac{1}{2\pi i} \int_{-a-iT}^{-a+iT} \zeta^2(s) \frac{x^s}{s} ds + O\left(\frac{T^{2a}}{x^a}\right) + O\left(\frac{x^c}{T}\right) \\ &= \frac{1}{4} + \frac{1}{2\pi i} \int_{-a-iT}^{-a+iT} \chi^2(s) \zeta^2(1-s) \frac{x^s}{s} ds + O\left(\frac{T^{2a}}{x^a}\right) + O\left(\frac{x^c}{T}\right) \\ &= \frac{1}{4} + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \tau(n) \int_{-a-iT}^{-a+iT} \frac{\chi^2(s) x^s}{n^{1-s} s} ds + O\left(\frac{T^{2a}}{x^a}\right) + O\left(\frac{x^c}{T}\right) \end{aligned}$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}s\pi\right) \Gamma(1-s).$$

After making the change of variable $w = 1 - s$, and using the well known relation $\Gamma(w) = (w-1)\Gamma(w-1)$, one can apply Bessel functions (see [93, (7.9.8) and (7.9.11)]) and obtain

$$\Delta(x) = \frac{1}{4} - \frac{2}{\pi} x^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{\frac{1}{2}}} \left(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2} Y_1(4\pi\sqrt{nx}) \right) + O\left(\frac{T^{2a}}{x^a}\right) + O\left(\frac{x^c}{T}\right).$$

Take $N \ll x$, $T = \sqrt{Nx}$, $a = \epsilon > 0$ and $c = 1 + \epsilon$. Using the asymptotic formulæ for Bessel functions (see [105]), a truncated form of (1.27) is obtained

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(x^{\frac{1+\epsilon}{2}} N^{-\frac{1}{2}}\right). \quad (1.28)$$

Define α as the least number such that $\Delta(x) < x^{\alpha+\epsilon}$, for every positive ϵ . The following bounds for α have been obtained:

$$\begin{aligned} \frac{33}{100} &= 0.330000\dots \text{ van der Corput [97]} \\ \frac{27}{82} &= 0.329268\dots \text{ van der Corput [98]} \\ \frac{15}{46} &= 0.326086\dots \text{ Chih [5], Richert [74]} \\ \frac{12}{37} &= 0.324324\dots \text{ Kolesnik [44]} \\ \frac{346}{1067} &= 0.324273\dots \text{ Kolesnik [45]} \\ \frac{35}{108} &= 0.324074\dots \text{ Kolesnik [46]} \\ \frac{139}{429} &= 0.324009\dots \text{ Kolesnik [47]} \\ \frac{7}{22} &= 0.318181\dots \text{ Iwaniec and Mozzochi [40], Heath-Brown and Huxley [32]} \\ \frac{23}{73} &= 0.315068\dots \text{ M. N. Huxley [34]} \end{aligned}$$

Recently, M. N. Huxley [35] obtained the bound $\frac{131}{416} = 0.314904$. Hence

$$\Delta(x) = O\left(x^{\frac{131}{416}+\epsilon}\right).$$

On the other hand, Hardy [25] proved³ that

$$\Delta(x) = \begin{cases} \Omega_+ \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} \log \log x \right) \\ \Omega_- \left(x^{\frac{1}{4}} \right), \end{cases} \quad (1.29)$$

The Ω_- -result for $\Delta(x)$ has been gradually improved, culminating in the work of K. Corrádi and I. Kátai [9], who, in 1967, showed that for a positive constant c ,

$$\Delta(x) = \Omega_- \left(x^{\frac{1}{4}} \exp \left(c (\log \log x)^{\frac{1}{4}} (\log \log \log x)^{-\frac{3}{4}} \right) \right)$$

³Hardy stated

$$\Delta(x) = \begin{cases} \Omega_+ \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} \log \log x \right) \\ \Omega_- \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} \log \log x \right), \end{cases}$$

but the Ω_- -result cannot be obtained as it is by Hardy's method (see [29, pp. 326])

The first improvement on the Ω_+ -result had to wait until 1981, when J. L. Hafner [22] obtained

$$\Delta(x) = \Omega_+ \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} (\log \log x)^{\frac{1}{4}(3+\log 4)} \exp \left(-c(\log \log \log x)^{\frac{1}{2}} \right) \right)$$

Recently, K. Soundararajan [86], refined Hafner's argument and obtained

$$\Delta(x) = \Omega \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} (\log \log x)^{\frac{3}{4}(2^{\frac{3}{4}}-1)} (\log \log \log x)^{-\frac{5}{8}} \right). \quad (1.30)$$

Starting with Voronoï's formula, we may model $\pi x^{-\frac{1}{4}} \sqrt{2} \Delta(x)$ by a random trigonometric series $\sum_{n=1}^{\infty} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(X_n)$, where the X_n are independent random variables uniformly distributed on $[0, 2\pi]$. Using estimates, obtained by H. L. Montgomery and A. M. Odlyzko [61], for the probability of large values attained by this trigonometric series, Soundararajan also provided a heuristic justification that the Ω -result in (1.30) is the best possible up to $(\log \log x)^{o(1)}$.

These results seem to support Hardy's conjecture, which is believed to be extremely difficult. Therefore, it is natural to consider the mean values of $\Delta(x)$.

In [101], Voronoï also proved

$$\begin{aligned} \int_2^T \Delta(x) dx &= \frac{1}{4}T + \left(2\pi^2\sqrt{2}\right)^{-1} T^{\frac{3}{4}} \sum_{n=1}^{\infty} \tau(n) n^{-\frac{5}{4}} \sin \left(4\pi\sqrt{nT} - \frac{\pi}{4}\right) \\ &\quad + \frac{15}{64\pi^3\sqrt{2}} T^{\frac{1}{4}} \sum_{n=1}^{\infty} \tau(n) n^{-\frac{7}{4}} \cos \left(4\pi\sqrt{nT} - \frac{\pi}{4}\right) + O(1). \end{aligned} \quad (1.31)$$

Notice that

$$\Delta(x^2) = \frac{\sqrt{x}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(4\pi x\sqrt{n} - \frac{\pi}{4}) + O(x^{1+\epsilon} N^{-\frac{1}{2}}),$$

which is easier to work with. Starting with this formula, a variation of (1.16) for the function $\Delta(x)$ is obtained

Theorem 1.16. *Let $\epsilon > 0$. For T sufficiently large and $r \ll T^{\frac{1}{2}-2\epsilon}$,*

$$\int_T^{2T} \left(\int_{t-\frac{r}{\sqrt{T}}}^{t+\frac{r}{\sqrt{T}}} \Delta(u^2) du \right)^2 dt = \frac{3\zeta^4(\frac{3}{2})}{2\pi^2\zeta(3)} Tr^2 + O \left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon} \right) + O \left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon} \right).$$

From this, we were unable to get a new version of theorem 1.2. Going a step further, we prove

Theorem 1.17. *Let $\epsilon > 0$. Let T sufficiently large and $r, X \ll T^{\frac{1}{2}-\epsilon}$. For $t \in [T, 2T]$ and any $h > 0$, define*

$$A_{t,h} = \int_{t-h}^{t+h} \Delta(u^2) du.$$

Then,

$$\begin{aligned} \int_T^{2T} \left(A_{t, \frac{r}{\sqrt{T}}} + A_{t+X, \frac{r}{\sqrt{T}}} \right)^2 dt &= \frac{3\zeta^4(\frac{3}{2})}{\pi^2\zeta(3)} T r^2 + \frac{3}{2\pi^2} T r^2 \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \cos(4\pi X \sqrt{n}) \\ &\quad + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right) + O(T^{1-\epsilon}) + O\left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon}\right) \end{aligned}$$

But, again, this result was not enough to prove that there are many cancellations in small intervals, which would allow us to obtain results about the number of sign changes.

Higher power moments of $\Delta(x)$ were considered by previous authors. The mean of $\Delta^2(x)$ was obtained by H. Cr  mer [10], in 1922,

$$\int_2^T \Delta^2(t) dt = \frac{\zeta^4(\frac{3}{2})}{6\pi^2\zeta(3)} T^{\frac{3}{2}} + R(T), \quad (1.32)$$

where $R(T) = O\left(T^{\frac{5}{4}+\epsilon}\right)$. The error term above have been improved to $O(T \log^5 T)$ by K. C. Tong [95] and to $O(T \log^4 T)$ by E. Preissmann [70]. As Heath-Brown mentioned (For details, see [94, pag. 327]), a result of the form $R(T) \ll F(T)$ implies $\Delta(T) \ll (F(T) \log T)^{\frac{1}{3}}$, but Y.-K. Lau and K.-M. Tsang [51] proved

$$\int_2^T R(t) dt = -\frac{1}{8\pi^2} T^2 \log^2 T + O(T^2 \log T)$$

which implies $R(T) = \Omega_-(T \log^2 T)$.

Using Cr  mer's result (1.32) and Vorono  's formula (1.31), we were able to prove that there is a positive proportion of negative values for $\Delta(x)$, more exactly

Theorem 1.18. *There are positive constants c_1, c_2 and c_3 , such that, for T sufficiently large,*

$$\#\{1 \leq n \leq T : c_1 T^{\frac{1}{4}} < \Delta(n) < c_2 T^{\frac{1}{4}}\} > c_3 T$$

and

$$\#\{1 \leq n \leq T : -c_2 T^{\frac{1}{4}} < \Delta(n) < -c_1 T^{\frac{1}{4}}\} > c_3 T.$$

D. R. Heath-Brown and K. Tsang had already proved that there is a positive proportion of large values of $|\Delta(x)|$:

Theorem (Heath-Brown & Tsang [33], 1994). *Let $\delta > 0$ be any given small quantity. Then for any $T \geq T_0(\delta)$, there are at least $c_1 \delta \sqrt{T} \log^5 T$ disjoint intervals of length $c_2 \delta \sqrt{T} \log^{-5} T$ in $[T, 2T]$, such that $|\Delta(x)| > (c_3 - \delta)x^{\frac{1}{4}}$ whenever x lies in any of these subintervals. The positive constants c_1, c_2 and c_3 can be computable.*

A few years earlier, K.-M. Tsang obtained asymptotic expansions for the third and fourth moments of $\Delta(x)$:

Theorem (Tsang [96], 1992).

$$\begin{aligned} \int_2^T \Delta^3(t) dt &= \frac{3c_1}{28\pi^3} T^{\frac{7}{4}} + O\left(T^{\frac{47}{28}+\epsilon}\right), \\ \int_2^T \Delta^4(t) dt &= \frac{3c_2}{64\pi^4} T^2 + O\left(T^{\frac{45}{23}+\epsilon}\right), \end{aligned}$$

where

$$\begin{aligned} c_1 &= \sum_{\alpha, \beta=1}^{\infty} (\alpha\beta(\alpha+\beta))^{-\frac{3}{2}} \sum_{h=1}^{\infty} \frac{\mu^2(h)}{h^{\frac{9}{4}}} \tau(\alpha^2 h) \tau(\beta^2 h) \tau((\alpha+\beta)^2 h) \\ c_2 &= \sum_{\substack{k, l, m, n=1 \\ \sqrt{k}+\sqrt{l}=\sqrt{m}+\sqrt{n}}}^{\infty} \frac{\tau(k)\tau(l)\tau(m)\tau(n)}{(klmn)^{\frac{3}{4}}} \end{aligned}$$

For larger powers of $\Delta(x)$, A. Ivić [37], showed that

$$\int_2^T \Delta(t)^A dt \ll T^{1+\frac{A}{4}+\epsilon},$$

for any $0 \leq A \leq \frac{35}{4}$ and any $\epsilon > 0$. In an important paper, D. R. Heath-Brown [31] prove the existence of a distribution function for $x^{-\frac{1}{4}}\Delta(x)$ and extended Ivić's result for $A \leq \frac{28}{3}$, using the estimate of Iwaniec and Mozzochi [40] for $\Delta(x)$. Using Huxley's estimate, Ivić's result is easily extended to $A \leq \frac{184}{19}$.

The question about the number of sign changes was also considered for this function. In 1969, J. Steinig [87], proved a general result about the number of sign changes of error terms associated with the coefficients of Dirichlet series that satisfy a certain functional equation. In particular, Steinig obtained $X_\Delta(T) > 4\sqrt{T} - A$, where A is a constant independent of T . By a different method, A. Ivić and H. J. J. te Riele [39] proved that $\Delta(x)$ changes sign in $[x, x + c\sqrt{x}]$, for x sufficiently large. Later, D. R. Heath-Brown and K.-M. Tsang obtained

Theorem (Heath-Brown & Tsang [33], 1994). *For any real-valued function $f(t)$ satisfying $|f(t)| \leq c_1 t^{\frac{1}{4}}$, the function $\Delta(t) + f(t)$ changes sign at least once in the interval $[T, T + c_2\sqrt{T}]$, for large T . In particular, there exists $t_1, t_2 \in [T, T + c_1\sqrt{T}]$ such that $\Delta(t_1) \leq -c_2 t_1^{\frac{1}{4}}$ and $\Delta(t_2) \geq c_2 t_2^{\frac{1}{4}}$.*

Since $\Delta(x)$ varies at most $\log x$ in the interval $[n, x]$, where $n = [x]$, this result implies that $N_H(T) \gg \sqrt{T}$.

In order to obtain an explicit lower bound for the number of sign changes on integers of $\Delta(x)$, we prove that,

$$\begin{aligned} & \int_{t_k-h}^{t_k+h} \int_{t_{k-1}-h}^{t_{k-1}+h} \cdots \int_{t_2-h}^{t_2+h} \int_{t_1-h}^{t_1+h} \Delta(t_0^2) dt_0 dt_1 \cdots dt_{k-2} dt_{k-1} \\ &= \frac{1}{2^k \pi^{k+1} \sqrt{2}} \sqrt{t_k} \sum_{n=1}^{t_k^2} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h \sqrt{n}) \cos\left(4\pi t_k \sqrt{n} - \frac{\pi}{4}\right) + O(h^{k+1} t_k^\epsilon) \end{aligned}$$

for any $\epsilon > 0$, $k, h \geq 1$ and t_k sufficiently large. Since $\zeta^2(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$, the sum on the right is smaller than $\zeta^2\left(\frac{3}{4} + \frac{k}{2}\right)$. We choose k and h such that

$$\sin^k(4\pi h) > \zeta^2\left(\frac{3}{4} + \frac{k}{2}\right) - 1$$

For these values of k and h , the multiple integrations of $\Delta(x^2)$ changes sign whenever the first term of the sum changes sign. From this we obtain theorem 1.4:

Theorem 1.4. *Let $N_{\Delta}(T)$ be the number of sign changes on integers of $\Delta(t)$, for $T \leq t \leq 2T$. Then, for sufficiently large T , $N_{\Delta}(T) > \sqrt{T}$. Moreover, there exists a constant c_1 , and $t_1, t_2 \in [T, T + \sqrt{T}]$ such that $\Delta(t_1) \leq -c_1 T^{\frac{1}{4}}$ and $\Delta(t_2) \geq c_1 T^{\frac{1}{4}}$.*

1.6 MORE FUNCTIONS

Let $r(n)$ denote the number of representations of n as the sum of two squares. It is well known that

$$r(n) = 4 \sum_{d|n} \chi(n), \tag{1.33}$$

where $\chi(n)$ is the non principal of modulus 4, i. e.

$$\chi(n) = \begin{cases} 0 & \text{if } 2 \mid n \\ 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv -1 \pmod{4} \end{cases}$$

Using formula (1.33) or by counting the number of lattice points in a circle, it can be proved that

$$\sum_{n \leq x} r(n) = \pi x + P(x),$$

where $P(x) = O(\sqrt{x})$ (see, for example, [29]). The above estimation of $P(x)$ was obtained by C. F. Gauss [20], and Sierpinski [79] obtained $P(x) = O(x^{\frac{1}{3}})$. In [25],

G. H. Hardy investigated the function $\Delta(x)$ together with the function $P(x)$ and obtained the Ω -result

$$P(x) = \Omega_+ \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} \right).$$

In the same journal, Hardy [26] obtained average results for $\Delta(x)$ and $P(x)$. Since then, many results concerning $\Delta(x)$ have been associated with similar results for $P(x)$ (e. g. [22, 33, 34, 35, 86, 96]). In particular,

$$P(x) = O \left(x^{\frac{131}{416} + \epsilon} \right)$$

and

$$P(x) = \Omega \left(x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} (\log \log x)^{\frac{3}{4}(2^{\frac{1}{3}} - 1)} (\log \log \log x)^{-\frac{5}{8}} \right).$$

The conclusions of theorem 1.18 remain valid if we replace $\Delta(x)$ by $P(x)$. This can be proved using the result of G. H. Hardy and E. Landau [27]

$$\int_2^T P(t) dt = O(T^{\frac{3}{4}})$$

and the result of H. Cr  mer [10]

$$\int_2^T P(t) dt = \frac{1}{3\pi^2} \sum_{n=1}^{\infty} \left(\frac{r^2(n)}{n^{\frac{3}{2}}} \right) T^{\frac{3}{2}} + O \left(T^{\frac{5}{4} + \epsilon} \right)$$

Notice that the above implies that the mean value of $P(x)$ is 0 while the mean value of $\Delta(x)$ is $\frac{1}{4}$, by Vorono  's result.

There is also a Vorono   type formula for $P(x)$ (see [38, equation (13.74)])

$$P(x) = \frac{x^{\frac{1}{4}}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{\frac{3}{4}}} \cos(2\pi\sqrt{nx} + \frac{\pi}{4}) + O(x^{\epsilon}).$$

Using this formula, we can obtain

Theorem. *Let $N_P(T)$ denote the number of sign changes on integers of $P(t)$, for $T \leq t \leq 2T$. Then, for sufficiently large T , $N_P(T) \gg \sqrt{T}$. Moreover, there exists positive constants c_1 and c_2 , and $t_1, t_2 \in [T, T + c_2\sqrt{T}]$ such that $P(t_1) \leq -c_1T^{\frac{1}{4}}$ and $P(t_2) \geq c_1T^{\frac{1}{4}}$.*

For any integer n , let $\tau_k(n)$ be the number of ways of expressing n as a product of k factors. Consider the error term

$$\Delta_k(x) = \sum_{n \leq x} \tau_k(n) - x P_k(\log x),$$

where P_k is a polynomial of degree $k - 1$. From [94, equation (12.4.6)], we have

$$\Delta_3(t) = \frac{1}{\pi\sqrt{3}} t^{\frac{1}{3}} \sum_{n \leq T^2} \frac{\tau_3(n)}{n^{\frac{2}{3}}} \cos\left(6\pi(nt)^{\frac{1}{3}}\right) + O(T^\epsilon),$$

for any $\epsilon > 0$, T sufficiently large and $T \leq t \leq 2T$. Since $\zeta^3(s) = \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^s}$, for $s > 1$, we can use the method developed to prove theorem 1.4 and obtain $N_{\Delta_3}(T) \gg T^{\frac{1}{3}}$.

The reader may have noticed that many of the papers mentioned above refer to the behavior of the zeta function on the critical line or to the function $E(T)$. This function $E(T)$ denotes the error term in the asymptotic formula for the mean square of the Riemann zeta-function on the critical line, i. e.

$$E(T) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt - T \log\left(\frac{T}{2\pi}\right) - (2\gamma - 1)T$$

It have been proved that, if $E(t) \ll t^\alpha$ then $|\zeta(\frac{1}{2} + it)| \ll t^{\frac{\alpha}{2}}$ (see [94, Notes for Chapter 7]). In 1949, F. V. Atkinson [1] found a Voronoï's type formula for $E(T)$ which, for $N \asymp T$, may be written as

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T),$$

where

$$\begin{aligned} \Sigma_1(T) &= \left(\frac{2T}{\pi}\right) \sum_{n \leq N} (-1)^n \frac{\tau(n)}{n^{\frac{3}{4}}} e(T, n) \cos(f(T, n)), \\ \Sigma_2(T) &= -2 \sum_{n \leq N'} \frac{\tau(n)}{n^{\frac{1}{2}}} \left(\log \frac{T}{2\pi n}\right)^{-1} \cos(g(T, n)), \end{aligned}$$

and $N' = \frac{T}{2\pi} + \frac{N}{2} - \left(\frac{N^2}{4} + \frac{NT}{2\pi}\right)^{\frac{1}{2}}$. The functions $e(T, n)$, $f(T, n)$ and $g(T, n)$ are defined below

$$\begin{aligned} e(T, n) &= \left(1 + \frac{\pi n}{2T}\right)^{-\frac{1}{4}} \left[\left(\frac{2T}{\pi n}\right)^{\frac{1}{2}} \operatorname{arc sinh} \left(\frac{\pi n}{2T}\right)^{\frac{1}{2}} \right]^{-1} \\ f(T, n) &= 2T \operatorname{arc sinh} \left(\frac{\pi n}{2T}\right)^{\frac{1}{2}} + (2\pi nT + \pi^2 n^2)^{\frac{1}{2}} - \frac{\pi}{4}, \\ g(T, n) &= T \log \left(\frac{T}{2\pi n}\right) - T + \frac{\pi}{4}. \end{aligned}$$

D. R. Heath-Brown [30] used the Voronoï's type formula above to prove the following theorem:

Theorem.

$$\int_2^T E^2(t) dt = \left(\frac{2}{3\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^{\frac{3}{2}}} \right) T^{\frac{3}{2}} + O\left(T^{\frac{5}{4}} \log^5 T\right).$$

As for $\Delta(x)$, the error term above has been improved to $O(T \log^5 T)$ by T. Meurman [59]. J. L. Hafner and A. Ivić [23] proved that the average of $E(T)$ is π :

Theorem.

$$\int_2^T E(T) dt = \pi T + \frac{1}{2} \left(\frac{2T}{\pi}\right)^{\frac{3}{4}} \sum_{n=1}^{\infty} (-1)^n \frac{\tau(n)}{n^{\frac{5}{4}}} \sin \left(4\pi \sqrt{\frac{nT}{2\pi}} - \frac{\pi}{4}\right) + O\left(T^{\frac{2}{3}} \log T\right)$$

With the two results above we can prove a version of theorem 1.18 for $E(t)$.

In the paper mentioned above Hafner and Ivić obtained also Ω -results for $E(T)$:

$$\begin{aligned} E(t) &= \Omega_+ \left\{ (T \log T)^{\frac{1}{4}} (\log \log T)^{\frac{3+\log 4}{4}} \exp \left(-c_1 \sqrt{\log \log \log T} \right) \right\} \\ E(t) &= \Omega_- \left\{ T^{\frac{1}{4}} \exp \left(c_2 \frac{(\log \log T)^{\frac{1}{4}}}{(\log \log \log T)^{\frac{3}{4}}} \right) \right\} \end{aligned}$$

Many authors (e.g. [55, 62, 58]) found analogues of Atkinson's formula to other error terms, e. g. error terms of mean square of $\zeta(\sigma + it)$, for $\frac{1}{2} < \sigma < \frac{3}{4}$ or similar

error terms associated with Dirichlet L-functions. With these formulas, they obtained O-, Ω - and mean square results for those error terms (see K. Matsumoto's survey [56]). It seems plausible that results about sign changes, can be obtained for these or more general error terms, but these problems were not considered in our work.

CHAPTER 2

ERROR TERMS

The main tool Y.-K. Lau used to obtain theorem 1.5, was what he called his Main Lemma, where he proves that if

$$H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x$$

then

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th,$$

for sufficiently large T and any $1 \leq h \ll \log^4 T$. As we explained in the previous chapter, Lau's argument depends essentially on the formula (1.7):

$$H(x) = - \sum_{n \leq \frac{x}{\log^5 x}} \frac{\mu(n)}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^{20} x}\right).$$

Many arithmetic functions have summations with error terms that can be expressed with a similar formula. In this chapter, we obtain a generalization of Lau's Main Lemma for functions similar to $H(x)$, above, satisfying some general conditions:

Main Lemma. *Suppose $H(x)$ is a function that can be expressed as*

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where each b_n is a real number and

1. for some $D > 0$, we have $y(x) \ll \frac{x}{(\log x)^{5+\frac{D}{2}}}$ and $\sum_{n \leq x} b_n^4 \ll x \log^D x$;

2. $k(x)$ is an increasing function, satisfying $\lim_{x \rightarrow \infty} k(x) = \infty$.

Then, for all large T and fixed $h \leq \min(\log T, k^2(T))$, we have

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}}. \quad (2.1)$$

In section 2.5, we use the Main Lemma to prove the following general theorem, from which theorems 1.2 and 1.3 are corollaries.

Theorem 2.1. *Suppose $H(x)$ is a function that can be expressed as*

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where b_n , $y(x)$ and $k(x)$ satisfy the hypothesis of the Main Lemma, for some $D > 0$.

Suppose also that $H(x) = H([x]) - \alpha\{x\} + \theta(x)$, where $\alpha \neq 0$ and $\theta(x) = o(1)$ and let $\prec \in \{<, =, \leq\}$. If

$$\#\{1 \leq n \leq T : \alpha H(n) \prec 0\} \gg T$$

then exists a positive constant c_0 and $c_0 T$ disjoint subintervals of $[1, T]$, with each of them having at least two integers, n and m , such that $\alpha H(n) \prec 0$ and $\alpha H(m) > 0$.

Moreover,

1. $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$.
2. if $N_H(T) \gg T$, then $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$;
3. if $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$, then $N_H(T) \gg T$ or $z_H(T) \gg T$.

2.1 PRELIMINARY RESULTS

Before we prove the Main Lemma we need a technical result for sequences b_n satisfying condition (1.3):

Lemma 2.2. *Let b_n be a sequence of real numbers satisfying (1.3), i. e.*

$$B_4(N) := \sum_{n \leq N} b_n^4 \ll N \log^D N,$$

for some $D > 0$. Then

1.

$$B_2(N) := \sum_{n \leq N} b_n^2 \ll N \log^{\frac{D}{2}} N, \quad (2.2)$$

2.

$$B_1(N) := \sum_{n \leq N} |b_n| \ll N \log^{\frac{D}{4}} N \quad (2.3)$$

3.

$$\sum_{n \leq N} \frac{b_n^2}{n} \ll (\log N)^{1+\frac{D}{2}} \quad (2.4)$$

4.

$$\sum_{n \leq N} \frac{|b_n|}{n} \ll (\log N)^{1+\frac{D}{4}} \quad (2.5)$$

5. For any $\delta > 0$,

$$\sum_{n > N} \frac{b_n^4}{n^2} \tau(n) \ll \frac{1}{N^{1-\delta}}, \quad (2.6)$$

where $\tau(n)$ is the number of divisors of n .

Remark. *If $H(x)$ can be expressed in the form (1.1), then (2.5) implies*

$$|H(x)| \leq \sum_{n \leq y(x)} \frac{|b_n|}{n} + O\left(\frac{1}{k(x)}\right) \ll (\log x)^{1+\frac{D}{4}} \quad (2.7)$$

Proof: The inequality (2.2) follows at once from Cauchy's inequality:

$$\sum_{n \leq N} b_n^2 \leq \left(\sum_{n \leq N} 1 \right)^{\frac{1}{2}} \left(\sum_{n \leq N} b_n^4 \right)^{\frac{1}{2}} \ll N \log^{\frac{D}{2}} N$$

Using the Cauchy Inequality again, we obtain (2.3).

To prove (2.4) we use partial summation. Let $B_2(t) = \sum_{n \leq t} b_n^2$, then

$$\begin{aligned} \sum_{n \leq N} \frac{b_n^2}{n} &= \int_{1^-}^N \frac{dB_2(t)}{t} \\ &= O\left(\log^{\frac{D}{2}} N\right) + \int_{1^-}^N \frac{B_2(t)}{t^2} dt \\ &= O\left(\log^{1+\frac{D}{2}} N\right) \end{aligned}$$

Using $B_1(t)$ instead of $B_2(t)$, we get (2.5). Now, we prove the last statement in the lemma. Take any $\delta > 0$. Theorem 315 of Hardy & Wright [29] tells us that $\tau(n) = O(n^\epsilon)$ for any $\epsilon > 0$. So, take $\epsilon < \frac{\delta}{2}$. Then

$$\sum_{n > N} \frac{b_n^4}{n^2} \tau(n) \ll \sum_{n > N} \frac{b_n^4}{n^{2-\epsilon}}$$

By partial summation,

$$\begin{aligned} \sum_{n > N} \frac{b_n^4}{n^{2-\epsilon}} &= \int_N^\infty \frac{dB_4(t)}{t^{2-\epsilon}} \\ &= \left[\frac{B_4(t)}{t^{2-\epsilon}} \right]_N^\infty + (2-\epsilon) \int_N^\infty \frac{B_4(t)}{t^{3-\epsilon}} dt \\ &= O\left(\frac{1}{N^{1-2\epsilon}}\right) + O\left(\int_N^\infty \frac{t \log^D t}{t^{3-\epsilon}} dt\right) \end{aligned}$$

Since $t^\epsilon \log^D t < t^\delta$, for sufficiently large t , then $\frac{t(t^\epsilon \log^D t)}{t^3} < \frac{1}{t^{2-\delta}}$. Hence,

$$\sum_{n > N} \frac{b_n^4}{n^2} \tau(n) \ll \frac{1}{N^{1-\delta}}$$

for any $\delta > 0$. □

2.2 THE MAIN LEMMA

We are going to prove the Main Lemma, assuming a technical result, which will be proved in the next two sections.

For any integer N , define

$$H_N(x) = - \sum_{d \leq N} \frac{b_d}{d} \psi\left(\frac{x}{d}\right) \quad (2.8)$$

Lemma 2.3. *Suppose $H(x)$ is a function that can be expressed as*

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where each b_n is a real number and

1. for some $D > 0$, we have $y(x) \ll \frac{x}{(\log x)^{5+\frac{D}{2}}}$ and $\sum_{n \leq x} b_n^4 \ll x \log^D x$;
2. $k(x)$ is an increasing function, satisfying $\lim_{x \rightarrow \infty} k(x) = \infty$.

Take $\delta > 0$ and $D > 0$ satisfying condition 1, above. Let $E = 4 + \frac{D}{2}$. Then,

a) For all large T , any $Y \ll T$ and $N \leq y(T)$, we have

$$\int_T^{T+Y} (H(u) - H_N(u))^2 du \ll \frac{Y}{N^{1-\delta}} + \frac{Y}{k^2(T)} + y(T+Y) (\log T)^E;$$

b) For all large T , $N \leq y(T)$ and fixed $1 \leq h \leq \min(\log T, k^2(T))$, we have

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u) du \right)^2 dt \ll Th^{\frac{3}{2}} + N^3 (\log N)^E.$$

Proof of the Main Lemma: Assume the two results of lemma 2.3 and take $N = T^{\frac{1}{4}}$. An application of Cauchy's inequality in the form $(a+b)^2 \leq 2(a^2+b^2)$ gives us

$$\left(\int_t^{t+h} H(u) du \right)^2 dt \leq 2 \left(\int_t^{t+h} H_N(u) du \right)^2 dt + 2 \left(\int_t^{t+h} (H(u) - H_N(u)) du \right)^2 dt$$

Since $N = T^{\frac{1}{4}}$ then, for sufficiently large T , $N^3 \log^E N \ll T$. So, using part b) of lemma 2.3, we have

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u) \, du \right)^2 \, dt \ll Th^{\frac{3}{2}} + N^3 \log^E N \ll Th^{\frac{3}{2}}.$$

Also, using Cauchy's inequality and interchanging the integrals,

$$\begin{aligned} \int_T^{2T} \left(\int_t^{t+h} (H(u) - H_N(u)) \, du \right)^2 \, dt &\leq h \int_T^{2T} \left(\int_t^{t+h} (H(u) - H_N(u))^2 \, du \right) \, dt \\ &\leq h \int_T^{2T+h} \left(\int_{\max(u-h, T)}^{\min(u, 2T)} (H(u) - H_N(u))^2 \, dt \right) \, du \\ &\leq h^2 \int_T^{2T+h} (H(u) - H_N(u))^2 \, du \\ &\ll h^2 \left(\frac{T+h}{N^{1-\delta}} + \frac{T+h}{k^2(T)} + y(2T+h) (\log T)^E \right) \\ &\ll T + Th \ll Th^{\frac{3}{2}} \end{aligned}$$

since $y(2T+h) \ll \frac{T}{(\log T)^{E+1}}$ and $h \leq \min(\log T, k^2(T))$. Hence

$$\int_T^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \ll Th^{\frac{3}{2}}.$$

□

2.3 STEP I

In this section, we will prove part a) of lemma 2.3 using the three technical lemmas 1.6, 1.7 and 1.8, stated in section 1.3. We finish this section with the proof of these three lemmas.

Using expression (1.1) and Cauchy's inequality in the form $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$\int_T^{T+Y} (H(u) - H_N(u))^2 du \leq 2 \int_T^{T+Y} \left(\sum_{m=N+1}^{y(u)} \frac{b_m}{m} \psi\left(\frac{u}{m}\right) \right)^2 du + O\left(\frac{Y}{k^2(T)}\right).$$

Let $\eta(T, m, n) = \max(T, y^{-1}(m), y^{-1}(n))$, then

$$2 \int_T^{T+Y} \left(\sum_{m=N+1}^{y(u)} \frac{b_m}{m} \psi\left(\frac{u}{m}\right) \right)^2 du = 2 \sum_{m,n=N+1}^{y(T+Y)} \frac{b_m b_n}{mn} \int_{\eta(T,m,n)}^{T+Y} \psi\left(\frac{u}{m}\right) \psi\left(\frac{u}{n}\right) du.$$

The Fourier series of $\psi(u) = u - [u] - \frac{1}{2}$, when u is not an integer, is given by

$$\psi(u) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k u)}{k}, \quad (2.9)$$

so we obtain

$$\frac{2}{\pi^2} \sum_{m,n=N+1}^{y(T+Y)} \frac{b_m b_n}{mn} \sum_{k,l=1}^{\infty} \frac{1}{kl} \int_{\eta(T,m,n)}^{T+Y} \sin\left(2\pi \frac{ku}{m}\right) \sin\left(2\pi \frac{lu}{n}\right) du$$

which is equal to

$$\frac{1}{\pi^2} \sum_{m,n=N+1}^{y(T+Y)} \frac{b_m b_n}{mn} \sum_{k,l=1}^{\infty} \frac{1}{kl} \int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} + \frac{l}{n}\right)\right) - \cos\left(2\pi u \left(\frac{k}{m} - \frac{l}{n}\right)\right) du.$$

Next, we estimate the integral above. For the first term we get

$$\int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} + \frac{l}{n}\right)\right) du \ll \frac{1}{\left(\frac{k}{m} + \frac{l}{n}\right)}.$$

If $\frac{k}{m} = \frac{l}{n}$, then

$$\int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} - \frac{l}{n}\right)\right) du \leq Y,$$

otherwise

$$\int_{\eta(T,m,n)}^{T+Y} \cos\left(2\pi u \left(\frac{k}{m} - \frac{l}{n}\right)\right) du \ll \frac{1}{\left|\frac{k}{m} - \frac{l}{n}\right|}.$$

Part a) of lemma 2.3 will follow from the next three lemmas.

Lemma 1.6. *Let $E = 4 + \frac{D}{2}$ as in lemma 2.3. Then*

$$\sum_{m,n \leq X} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \ll X (\log X)^E$$

Lemma 1.7. *If $D > 0$ satisfies condition (1.3), then*

$$\sum_{m,n \leq X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl (kn + lm)} \ll X (\log X)^{1+\frac{D}{2}}$$

Lemma 1.8. *For any $\delta > 0$*

$$\sum_{N < m,n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \ll \frac{1}{N^{1-\delta}}$$

To finish the proof of part a) of lemma 2.3 we just need to take $X = y(T + Y)$ in the previous lemmas. Hence

$$\int_T^{T+Y} (H(u) - H_N(u))^2 du \ll \frac{Y}{N^{1-\delta}} + y(T + Y) (\log T)^E + \frac{Y}{k^2(T)}$$

□

Proof of lemma 1.7: Since the arithmetical mean is greater or equal then the geometrical mean, we have

$$\sum_{m,n \leq X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl(kn + lm)} \leq 2 \sum_{m,n \leq X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl \sqrt{knlm}}$$

$$\ll \left(\sum_{m \leq X} \frac{|b_m|}{\sqrt{m}} \right)^2$$

$$\ll \left(\sum_{m \leq X} 1 \right) \left(\sum_{M \leq X} \frac{b_M^2}{M} \right)$$

$$\stackrel{(2.4)}{\ll} X (\log X)^{1+\frac{D}{2}}$$

□

Proof of lemma 1.8: For the second sum, take $d = (m, n)$, $m = d\alpha$ and $n = d\beta$.

Since $kn = lm$, then $\alpha|k$ and $\beta|l$. Hence $k = \alpha\gamma$, say, and $l = \beta\gamma$. Since

$$\sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} = \frac{1}{\alpha\beta} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^2} = \frac{\pi^2}{6} \frac{(m, n)^2}{mn} \quad (2.10)$$

Then,

$$\begin{aligned} \sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} &= \frac{\pi^2}{6} \sum_{N < m, n \leq X} \frac{|b_m b_n| (m, n)^2}{m^2 n^2} \\ &= \frac{\pi^2}{6} \sum_{d \leq X} d^2 \sum_{\substack{N < m, n \leq X \\ (m, n) = d}} \frac{|b_m b_n|}{m^2 n^2} \\ &\leq \frac{\pi^2}{6} \sum_{d \leq X} \left(d \sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \end{aligned}$$

In order to obtain the result stated in the lemma we have to prove that

$$\sum_{d \leq X} \left(d \sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \ll \frac{1}{N^{1-\delta}} \quad (2.11)$$

The next step will be to estimate the inner sum using Hölder inequality, in the

$$\begin{aligned} \text{form } \left| \sum_i u_i v_i \right| &\leq \left(\sum_j u_j^4 \right)^{\frac{1}{4}} \left(\sum_k v_k^{\frac{4}{3}} \right)^{\frac{3}{4}} \\ &\leq \left(\sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \leq \left(\sum_{\substack{N < m \leq X \\ d|m}} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \left(\sum_{\substack{N < M \leq X \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}} \end{aligned}$$

Take $\beta = \frac{M}{d}$, then

$$\left(\sum_{\substack{N < M \leq X \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}} = \frac{1}{d^3} \left(\sum_{\substack{N < \beta \leq \frac{X}{d}}} \frac{1}{\beta^2} \right)^{\frac{3}{2}} \ll \frac{1}{d^3} \min \left\{ 1, \frac{d^3}{N^3} \right\}^{\frac{1}{2}} \quad (2.12)$$

We complete the proof of lemma 1.8 using Cauchy's inequality and the estimate (2.6). Take any $\delta > 0$, then

$$\begin{aligned}
\sum_{N < m, n \leq X} \frac{|b_m b_n|}{mn} \sum_{\substack{k, l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} &\ll \sum_{d \leq X} \frac{1}{d} \min \left\{ 1, \frac{d^3}{N^3} \right\}^{\frac{1}{2}} \left(\sum_{\substack{N < m \leq X \\ d|m}} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \\
&\ll \left(\sum_{d \leq X} \frac{1}{d^2} \min \left\{ 1, \frac{d^3}{N^3} \right\} \right)^{\frac{1}{2}} \left(\sum_{D \leq X} \left(\sum_{\substack{N < m \leq X \\ D|m}} \frac{b_m^4}{m^2} \right) \right)^{\frac{1}{2}} \\
&\ll \left[\left(\frac{1}{N^3} \sum_{d \leq N} d + \sum_{d > N} \frac{1}{d^2} \right) \sum_{N < m \leq X} \left(\frac{b_m^4}{m^2} \sum_{D|m} 1 \right) \right]^{\frac{1}{2}} \\
&\leq \left(\frac{1}{N} \sum_{N < m \leq X} \frac{b_m^4}{m^2} \tau(m) \right)^{\frac{1}{2}} \\
&\ll \frac{1}{N^{1-\delta}}.
\end{aligned}$$

□

Proof of lemma 1.6: This lemma is a generalization of *Hilfssatz 6* in [102] of A. Walfisz. Notice first that

$$\sum_{m, n \leq X} \left(|b_m b_n| \sum_{\substack{k, l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \right) \leq 2 \sum_{m \leq n \leq X} \left(|b_m b_n| \sum_{\substack{k, l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} \right).$$

Like in [102] we begin by separating the interior sum into four terms.

$$\sum_{\substack{k, l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl |kn - lm|} = \left(\sum_{\substack{k, l=1 \\ lm \leq \frac{kn}{2}}}^{\infty} + \sum_{\substack{k, l=1 \\ lm \geq 2kn}}^{\infty} + \sum_{\substack{k, l=1 \\ \frac{kn}{2} < lm < kn}}^{\infty} + \sum_{\substack{k, l=1 \\ kn < lm < 2kn}}^{\infty} \right) \left(\frac{1}{kl |kn - lm|} \right)$$

For the first term, we just need to use the estimate (2.3) and the fact that $\sum_{l \leq x} \frac{1}{l} \ll \log x$.

$$\begin{aligned}
\sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ lm \leq \frac{kn}{2}}}^{\infty} \frac{1}{kl |kn - lm|} &\leq 2 \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ lm \leq \frac{kn}{2}}}^{\infty} \frac{1}{k^2 l n} \\
&= 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{2m}} \frac{1}{l} \\
&\ll \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \left(\frac{\log k}{k^2} + \frac{\log X}{k^2} \right) \\
&\ll \log X \sum_{n \leq X} \frac{|b_n|}{n} \sum_{m \leq n} |b_m| \\
&\ll (\log X)^{1+\frac{D}{4}} \sum_{n \leq X} |b_n| \\
&\ll X (\log X)^{1+\frac{D}{2}}
\end{aligned}$$

The second term is treated similarly, except here we also use (2.5).

$$\begin{aligned}
\sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ kn \leq \frac{lm}{2}}}^{\infty} \frac{1}{kl |kn - lm|} &\leq 2 \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ kn \leq \frac{lm}{2}}}^{\infty} \frac{1}{k l^2 m} \\
&= 2 \sum_{m \leq n \leq X} |b_n| \frac{|b_m|}{m} \sum_{l=1}^{\infty} \frac{1}{l^2} \sum_{k \leq \frac{lm}{2n}} \frac{1}{k} \\
&\ll \sum_{m \leq n \leq X} |b_n| \frac{|b_m|}{m} \sum_{l=\frac{2n}{m}}^{\infty} \frac{\log l}{l^2} \\
&\ll \sum_{n \leq X} |b_n| \sum_{m \leq n} \frac{|b_m|}{m} \\
&\ll (\log X)^{1+\frac{D}{4}} \sum_{n \leq X} |b_n| \\
&\ll X (\log X)^{1+\frac{D}{2}}
\end{aligned}$$

The estimation of the third term is more complicated and we will have to use a different approach. In this case, we have $\frac{1}{l} < \frac{2m}{kn}$, so

$$\begin{aligned}
& \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ \frac{kn}{2} < lm < kn}}^{\infty} \frac{1}{kl |kn - lm|} \\
& < 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{2m} < l < \frac{kn}{m}}} \frac{m}{kn - lm} \\
& < 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{m} - 1} \left(\frac{1}{\frac{kn}{m} - l} \right) \\
& + 2 \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{m} - 1 < l < \frac{kn}{m}}} \frac{m}{kn - lm}.
\end{aligned}$$

Now,

$$\sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{m} - 1} \left(\frac{1}{\frac{kn}{m} - l} \right) = \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l \leq \frac{kn}{m} - 1} \frac{1}{l}$$

$$\ll X (\log X)^{1+\frac{D}{2}}$$

as in the first term. If there exists an integer l with $\frac{kn}{m} - 1 < l < \frac{kn}{m}$, then $m \nmid kn$.

In this case, $kn - lm = m \left\{ \frac{kn}{m} \right\}$ and $m < n$. So, we have to estimate

$$\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k=1 \\ m \nmid kn}}^{\infty} \frac{1}{k^2 \left\{ \frac{kn}{m} \right\}} \tag{2.13}$$

Notice that the fractional part of $\frac{kn}{m}$ is at least $\frac{1}{m}$. So, when $k \geq m$, we can estimate (2.13), using (2.3).

$$\begin{aligned}
\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=m}^{\infty} \frac{m}{k^2} &\ll \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \\
&\ll \sum_{n \leq X} \frac{|b_n|}{n} \sum_{m \leq n} |b_m| \\
&\ll (\log X)^{\frac{D}{4}} \sum_{n \leq X} |b_n| \\
&\ll X (\log X)^{\frac{D}{2}}
\end{aligned}$$

We are left with the estimation of

$$\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k < m \\ m \nmid kn}} \frac{1}{k^2 \left\{ \frac{kn}{m} \right\}}.$$

Since $m \nmid kn$, then, given k and n , we can take $a_{k,n}$, such that $1 \leq a_{k,n} < m$ and $a_{k,n} \equiv kn \pmod{m}$. Then,

$$\begin{aligned}
\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k < m \\ m \nmid kn}} \frac{1}{k^2 \left\{ \frac{kn}{m} \right\}} &\leq \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k < m} \frac{m}{k^2 a_{k,n}} \\
&\leq \sum_{a, k \leq X} \frac{1}{ak^2} \sum_{\max(a, k) < m \leq X} m |b_m| \sum_{\substack{m < n \leq X \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} \quad (2.14)
\end{aligned}$$

We need to estimate the inner sums. In order to do that, we will partition the interval $[1, X]$ in intervals of the form $[M, 2M)$ and apply Cauchy's inequality. Take $1 \leq P \leq Q \leq X$, then,

$$\sum_{P \leq m < 2P} m |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} \ll \frac{P}{Q} \sum_{P \leq m < 2P} |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} |b_n|.$$

Next, we apply Cauchy's inequality twice, first to the first sum on the right and afterwards to the second sum.

$$\begin{aligned}
\left(\sum_{P \leq m < 2P} \left(|b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod m}} |b_n| \right) \right)^2 &\leq \left(\sum_{P \leq m < 2P} b_m^2 \right) \left(\sum_{P \leq m < 2P} \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod m}} |b_n| \right)^2 \\
&\stackrel{(2.2)}{\ll} P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left(\sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod m}} b_n^2 \sum_{\substack{Q \leq N < 2Q \\ kN \equiv a \pmod m}} 1 \right) \\
&\ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left(\left(1 + \frac{Q}{(k, m)} \right) \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod m}} b_n^2 \right)
\end{aligned}$$

Since $m \leq 2P \leq 2Q$, we have $\frac{Q}{m} \geq \frac{1}{2}$. Using also $(k, m) \leq k$, we obtain $1 + \frac{Q}{(k, m)} \leq 3 \frac{Qk}{m}$. Therefore,

$$\begin{aligned}
\left(\sum_{P \leq m < 2P} \left(|b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod m}} |b_n| \right) \right)^2 &\ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left(3 \frac{Qk}{m} \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod m}} b_n^2 \right) \\
&\ll P \frac{3kQ}{P} \log^{\frac{D}{2}} P \sum_{Q \leq n < 2Q} \left(b_n^2 \sum_{\substack{P \leq m < 2P \\ m | kn - a}} 1 \right) \\
&\ll kQ (\log P)^{\frac{D}{2}} \sum_{Q \leq n < 2Q} b_n^2 \tau(kn - a)
\end{aligned}$$

By a theorem of S. Ramanujan [72], we have $\sum_{n \leq X} \tau^2(n) \sim X \log^3 X$ (for a proof see [106]). So, by another application of Cauchy inequality and condition (1.3), we obtain

$$\begin{aligned} \left(\sum_{Q \leq n < 2Q} b_n^2 \tau(kn - a) \right)^2 &\leq \left(\sum_{Q \leq n < 2Q} b_n^4 \right) \left(\sum_{Q \leq n < 2Q} \tau^2(kn - a) \right) \\ &\ll Q \log^D Q \sum_{kQ - a \leq n < 2kQ - a} \tau^2(n) \\ &\ll kQ^2 \log^{D+3} X. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{P \leq m < 2P} \left(m |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} \right) &\ll \frac{P}{Q} \left(kQ (\log P)^{\frac{D}{2}} \sum_{Q \leq n < 2Q} b_n^2 \tau(kn - a) \right)^{\frac{1}{2}} \\ &\ll \frac{P}{Q} \left(kQ (\log X)^{\frac{D}{2}} (kQ^2 \log^{D+3} X)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\ll P k^{\frac{3}{4}} (\log X)^{1 + \frac{D}{2}} \end{aligned}$$

The number of pairs of intervals of the form $([P, 2P], [Q, 2Q])$ that we have to consider is at most $\ll \log^2 X$, hence

$$\begin{aligned} \sum_{a, k \leq X} \frac{1}{ak^2} \sum_{a < m \leq X} m |b_m| \sum_{\substack{m < n \leq X \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n} &\ll \sum_{a, k \leq X} \frac{k^{\frac{3}{4}}}{ak^2} \sum_{P, Q} P (\log X)^{1 + \frac{D}{2}} \\ &\ll X (\log X)^{4 + \frac{D}{2}} \end{aligned}$$

We still have to estimate the fourth term

$$\sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ kn < lm < 2kn}}^{\infty} \frac{1}{kl |kn - lm|}$$

We will use the method we developed in order to estimate the third term. In this case, we have $\frac{1}{l} < \frac{m}{kn}$, so

$$\begin{aligned}
& \sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ kn < lm < 2kn}}^{\infty} \frac{1}{kl |kn - lm|} \\
& < \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{m} < l < \frac{2kn}{m}}} \left(\frac{m}{lm - kn} \right) \\
& < \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{m} + 1 \leq l < \frac{2kn}{m}}} \left(\frac{1}{l - \frac{kn}{m}} \right) \\
& + \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{m} < l < \frac{kn}{m} + 1}} \left(\frac{m}{lm - kn} \right)
\end{aligned}$$

The first case is again easy to estimate,

$$\begin{aligned}
& \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\substack{\frac{kn}{m} + 1 \leq l < \frac{2kn}{m}}} \left(\frac{1}{l - \frac{kn}{m}} \right) \\
& = \sum_{m \leq n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l < \frac{kn}{m}} \frac{1}{l}
\end{aligned}$$

$$\ll X (\log X)^{1 + \frac{D}{2}}$$

If $\frac{kn}{m} < l < \frac{kn}{m} + 1$, then $l = \frac{kn}{m} + 1 - \left\{ \frac{kn}{m} \right\}$ and again $m \nmid kn$ (so $m < n$).

Hence we need to estimate

$$\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k=1 \\ m \nmid kn}}^{\infty} \frac{1}{k^2 (1 - \left\{ \frac{kn}{m} \right\})}.$$

Since $1 - \left\{ \frac{kn}{m} \right\}$ is at least $\frac{1}{m}$ we again obtain, for $k \geq m$,

$$\begin{aligned}
\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k=m}^{\infty} \frac{m}{k^2} & \ll \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \\
& \ll X (\log X)^{\frac{D}{2}},
\end{aligned}$$

where we used inequalities (2.3) and (2.5). Therefore, we are left with the estimation of

$$\sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k < m \\ m \nmid kn}} \frac{1}{k^2 (1 - \{\frac{kn}{m}\})}.$$

Given k and n , take $a_{k,n}$ such that $1 \leq a_{k,n} < m$ and $a_{k,n} \equiv -kn \pmod{m}$, then

$$\begin{aligned} & \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k < m \\ m \nmid kn}} \frac{1}{k^2 (1 - \{\frac{kn}{m}\})} \\ & \leq \sum_{m < n \leq X} |b_m| \frac{|b_n|}{n} \sum_{k < m} \frac{m}{k^2 a_{k,n}} \\ & = \sum_{a, k \leq X} \frac{1}{ak^2} \sum_{\max(a, k) < m \leq X} m |b_m| \sum_{\substack{m < n \\ kn \equiv a \pmod{m}}} \frac{|b_n|}{n}. \end{aligned}$$

So, we are in the same situation as in (2.14). Hence

$$\sum_{m \leq n \leq X} |b_m b_n| \sum_{\substack{k, l=1 \\ kn < lm < 2kn}}^{\infty} \frac{1}{kl |kn - lm|} \ll X (\log X)^{4 + \frac{D}{2}}$$

This completes the proof of Lemma 1.6. $\square$

2.4 STEP II

In this section we will finalize the proof of the Main Lemma by proving part b) of lemma 2.3.

From equation (2.9), we can get

$$\begin{aligned} \int_0^t \psi(u) du &= -\frac{1}{\pi} \int_0^t \left(\sum_{k=1}^{\infty} \frac{\sin(2\pi ku)}{k} \right) du \\ &= -\frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{1}{k} \int_0^t \sin(2\pi ku) du \right) \\ &= \frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{1}{k} \left[\frac{\cos(2\pi ku)}{2\pi k} \right]_0^t \right) \\ &= \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \left(\frac{\cos(2\pi kt)}{k^2} - \frac{1}{12} \right) \end{aligned}$$

where we used Lebesgue dominated convergence theorem. Using the definition of H_N stated in (2.8), we obtain

$$\begin{aligned}
\int_t^{t+h} H_N(u) du &= - \sum_{m \leq N} \frac{b_m}{m} \int_t^{t+h} \psi\left(\frac{u}{m}\right) du \\
&= - \sum_{m \leq N} \frac{b_m}{m} \int_{\frac{t}{m}}^{\frac{t+h}{m}} m \psi(v) dv \\
&= - \frac{1}{2\pi^2} \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\cos\left(2\pi \frac{k(t+h)}{m}\right) - \cos\left(2\pi \frac{kt}{m}\right)}{k^2}
\end{aligned}$$

As usual, let us write $e(t)$ for $e^{2\pi it}$. Then

$$\begin{aligned}
\cos\left(2\pi k \frac{t+h}{m}\right) - \cos\left(2\pi k \frac{t}{m}\right) &= \frac{e\left(\frac{k(t+h)}{m}\right) + e\left(-\frac{k(t+h)}{m}\right) - e\left(\frac{kt}{m}\right) - e\left(-\frac{kt}{m}\right)}{2} \\
&= \frac{e\left(\frac{kt}{m}\right) \left(e\left(\frac{kh}{m}\right) - 1\right)}{2} - \frac{e\left(-\frac{k(t+h)}{m}\right) \left(e\left(\frac{kh}{m}\right) - 1\right)}{2} \\
&= \frac{e\left(\frac{kh}{m}\right) - 1}{2} e\left(\frac{kt}{m}\right) \left[1 - e\left(-k \frac{(2t+h)}{m}\right)\right]
\end{aligned}$$

So,

$$\int_t^{t+h} H_N(u) du = \frac{1}{4\pi^2} \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) e\left(\frac{kt}{m}\right) \left(e\left(-k \frac{(2t+h)}{m}\right) - 1\right)}{k^2}$$

Therefore, using $|z|^2 = z\bar{z}$,

$$\begin{aligned}
&16\pi^4 \int_T^{2T} \left| \int_t^{t+h} H_N(u) du \right|^2 dt \\
&= \int_T^{2T} \left| \sum_{m \leq N} b_m \sum_{k=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) e\left(\frac{kt}{m}\right) \left(e\left(-k \frac{(2t+h)}{m}\right) - 1\right)}{k^2} \right|^2 dt \\
&= \sum_{m, n \leq N} b_m b_n \sum_{k, l=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right) - 1\right) \left(e\left(-\frac{lh}{n}\right) - 1\right)}{(kl)^2} \\
&\quad \times \int_T^{2T} e\left(\frac{kt}{m}\right) e\left(-\frac{lt}{n}\right) \left(e\left(-k \frac{(2t+h)}{m}\right) - 1\right) \left(e\left(l \frac{(2t+h)}{n}\right) - 1\right) dt
\end{aligned}$$

After multiplying the terms inside the integral above, we obtain the following four terms that we will estimate below:

$$\begin{aligned} & \int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt + e\left(\frac{lh}{n} - \frac{kh}{m}\right) \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt \\ & - e\left(-\frac{kh}{m}\right) \int_T^{2T} e\left(-\frac{lt}{n} - \frac{kt}{m}\right) dt - e\left(\frac{lh}{n}\right) \int_T^{2T} e\left(\frac{lt}{n} + \frac{kt}{m}\right) dt \end{aligned}$$

Notice that,

$$\begin{aligned} \left| \int_T^{2T} e^{2\pi i r t} dt \right| &= \frac{1}{|2\pi i r|} |e^{4\pi i r T} - e^{4\pi i r T}| \\ &\leq \frac{1}{\pi |r|} \end{aligned}$$

for any $r \neq 0$. We begin with the last term and use $|e(t) - 1| \leq 2$. Then

$$\begin{aligned} & \left| \sum_{m,n \leq N} b_m b_n \sum_{k,l=1}^{\infty} \frac{(e(\frac{kh}{m}) - 1)(e(-\frac{lh}{n}) - 1)e(\frac{lh}{n})}{(kl)^2} \int_T^{2T} e\left(\frac{lt}{n} + \frac{kt}{m}\right) dt \right| \\ & \leq 4 \sum_{m,n \leq N} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{(kl)^2} \left| \int_T^{2T} e\left(\frac{lt}{n} + \frac{kt}{m}\right) dt \right| \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{(kl)^2 \left(\frac{l}{n} + \frac{k}{m}\right)} \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl} \left(\frac{m}{k}\right) \left(\frac{n}{l}\right) \frac{1}{(lm + kn)} \\ & \ll N^3 (\log N)^{1+\frac{D}{2}}. \end{aligned}$$

where we used lemma 1.7 in the last step. The third term is treated similarly and we obtain

$$\left| \sum_{m,n \leq N} b_m b_n \sum_{k,l=1}^{\infty} \frac{(e(\frac{kh}{m}) - 1)(e(-\frac{lh}{n}) - 1)e(-\frac{kh}{m})}{(kl)^2} \int_T^{2T} e\left(-\frac{lt}{n} - \frac{kt}{m}\right) dt \right| \ll N^3 (\log N)^{1+\frac{D}{2}}.$$

Now, if $kn = lm$, then

$$\int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt + e\left(\frac{lh}{n} - \frac{kh}{m}\right) \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt = 2T.$$

If $kn \neq lm$ then,

$$\int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt + e\left(\frac{lh}{n} - \frac{kh}{m}\right) \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt \ll \frac{1}{\left|\frac{k}{m} - \frac{l}{n}\right|}$$

Let's study first the case when $kn \neq lm$.

$$\begin{aligned} & \left| \sum_{m,n \leq N} b_m b_n \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{(e(\frac{kh}{m}) - 1)(e(-\frac{lh}{n}) - 1)}{(kl)^2} \int_T^{2T} e\left(\frac{kt}{m} - \frac{lt}{n}\right) dt \right. \\ & \quad \left. + \sum_{m,n \leq N} b_m b_n \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{(e(\frac{kh}{m}) - 1)(e(-\frac{lh}{n}) - 1)e(\frac{lh}{n} - \frac{kh}{m})}{(kl)^2} \int_T^{2T} e\left(\frac{lt}{n} - \frac{kt}{m}\right) dt \right| \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{(kl)^2 \left|\frac{k}{m} - \frac{l}{n}\right|} \\ & \ll \sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \left(\frac{m}{k}\right) \left(\frac{n}{l}\right) \frac{1}{kl |kn - lm|} \\ & \ll N^3 \log^E N, \end{aligned}$$

by Lemma 1.6. If $kn = ml$, we will use $|e(t) - 1| \leq \min(2, 6\pi|t|)$ instead. The expression obtained has some similarities with lemma 1.8. We are going to use the same argument to prove:

$$\sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right) \ll h^{\frac{3}{2}} \quad (2.15)$$

As in Lemma 1.8, take $d = (m, n)$, $\alpha = \frac{m}{d}$, $\beta = \frac{n}{d}$ and $\gamma = \frac{k}{\alpha}$. So $l = \beta\gamma$. Then,

$$\sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right) = \frac{1}{\alpha^2 \beta^2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{d}\right) \right)^2$$

If $d \leq h$, we obtain

$$\sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{d}\right) \right)^2 = \frac{\pi^4}{90},$$

and if $h < d \leq N$, then

$$\begin{aligned} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{d}\right) \right)^2 &= \left(\frac{h}{d}\right)^2 \sum_{\gamma \leq \frac{d}{h}} \frac{1}{\gamma^2} + \sum_{\gamma > \frac{d}{h}} \frac{1}{\gamma^4} \\ &\ll \left(\frac{h}{d}\right)^2 + \left(\frac{h}{d}\right)^3 \ll \left(\frac{h}{d}\right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{m,n \leq N} |b_m b_n| \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{(kl)^2} \min\left(1, \frac{kh}{m}\right) \min\left(1, \frac{lh}{n}\right) \\ &= \sum_{m,n \leq N} |b_m b_n| \frac{(m,n)^4}{m^2 n^2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left(\min\left(1, \frac{h\gamma}{(m,n)}\right) \right)^2 \\ &\leq \sum_{d \leq N} d^4 \sum_{\substack{m,n \leq N \\ d|(m,n)}} \frac{|b_m b_n|}{m^2 n^2} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^4} \left[\min\left(1, \frac{h\gamma}{d}\right) \right]^2 \\ &\ll \sum_{d \leq h} \left(d^2 \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2} \right)^2 + h^2 \sum_{h < d \leq N} \left(d \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \end{aligned}$$

The second sum on the right above, will be estimated using inequality (2.11), which states

$$\sum_{d \leq X} \left(d \sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \ll \frac{1}{N^{1-\delta}},$$

for, since $d|m$ we have $m > h$, so,

$$h^2 \sum_{h < d \leq N} \left(d \sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \ll h^{1+\delta}.$$

To estimate the first term we begin with Hölder inequality:

$$\sum_{d \leq h} d^4 \left(\sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \leq \sum_{d \leq h} d^4 \left(\sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|^4}{m^2} \right)^{\frac{1}{2}} \left(\sum_{\substack{M \leq N \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}}$$

The third sum is $O\left(\frac{1}{d^3}\right)$ (similar to (2.12)). Then

$$\begin{aligned} \sum_{d \leq h} d^4 \left(\sum_{\substack{m \leq N \\ d|m}} \frac{|b_m|^4}{m^2} \right)^{\frac{1}{2}} \left(\sum_{\substack{M \leq N \\ d|M}} \frac{1}{M^2} \right)^{\frac{3}{2}} &\ll \sum_{d \leq h} d \left(\sum_{\substack{d \leq m \leq N \\ d|m}} \frac{|b_m|^4}{m^2} \right)^{\frac{1}{2}} \\ &\ll \left(\sum_{d \leq h} d^2 \right)^{\frac{1}{2}} \left[\sum_{D \leq h} \left(\sum_{\substack{D \leq m \leq N \\ D|m}} \frac{|b_m|^4}{m^2} \right) \right]^{\frac{1}{2}} \\ &\ll h^{\frac{3}{2}} \left(\sum_{m \leq N} \frac{|b_m|^4}{m^2} \sum_{\substack{D \leq h \\ D|m}} 1 \right)^{\frac{1}{2}} \\ &\ll h^{\frac{3}{2}} \left(\sum_{m \leq N} \frac{|b_m|^4}{m^2} \tau(m) \right)^{\frac{1}{2}}. \end{aligned}$$

But equation (2.6) implies that $\sum_{m > N} \frac{|b_m|^4}{m^2} \tau(m) \rightarrow 0$ as $N \rightarrow \infty$. In particular,

we have $\sum_{m \leq N} \frac{|b_m|^4}{m^2} \tau(m) = O(1)$, where the implied constant doesn't depend on N .

Therefore, we obtain inequality (2.15) and part b) of lemma 2.3, follows. $\square$

2.5 A GENERAL THEOREM

In this section, we prove theorem 2.1, from which the main theorems 1.2 and 1.3, will be deduced.

Theorem 2.1. *Suppose $H(x)$ is a function that can be expressed as*

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where each b_n is a real number and

1. for some $D > 0$, we have $y(x) \ll \frac{x}{(\log x)^{5+\frac{D}{2}}}$ and $\sum_{n \leq x} b_n^4 \ll x \log^D x$;
2. $k(x)$ is an increasing function, satisfying $\lim_{x \rightarrow \infty} k(x) = \infty$.

Suppose also that $H(x) = H([x]) - \alpha\{x\} + \theta(x)$, where $\alpha \neq 0$ and $\theta(x) = o(1)$. Let $\prec \in \{<, =, \leq\}$. If $\#\{1 \leq n \leq T : \alpha H(n) \prec 0\} \gg T$ then there exists a positive constant c_0 and $c_0 T$ disjoint subintervals of $[1, T]$, with each of them having at least two integers, m and n , such that $\alpha H(m) > 0$ and $\alpha H(n) \prec 0$.

Moreover

1. $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$.
2. if $N_H(T) \gg T$, then $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$;
3. if $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$, then $N_H(T) \gg T$ or $z_H(T) \gg T$.

Proof: From the Main Lemma, we have

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}}.$$

for all large T and $h \leq \min(\log T, k^2(T))$.

Assume $\#\{1 \leq n \leq T : \alpha H(n) \prec 0\} \gg T$ and let $c > 0$ be a constant such that, if T be a sufficiently large number, then

$$\#\{1 \leq n \leq 2T : \alpha H(n) \prec 0\} > cT.$$

Take T large. Divide the interval $[1, 2T]$ into subintervals of length h , where h is a sufficiently large constant (we will see later how large it must be) satisfying $h \leq \log T$. Then at least $\frac{cT}{h}$ of those subintervals must have at least one integer n with $\alpha H(n) < 0$. Let $\mathcal{C}$ be the set of the subintervals which satisfy that property. Write

$$\mathcal{C} = \{J_r \mid 1 \leq r \leq R\}$$

where the subintervals are indexed by its position in the interval $[1, 2T]$ and where $R > \frac{cT}{h}$. Define $K_s = J_{3s-2}$, for $1 \leq s \leq \frac{R}{3}$, and let $\mathcal{D}$ be the set of these subintervals. We have $\#(\mathcal{D}) > \frac{cT}{3h}$. Notice that any two members of $\mathcal{D}$ are separated by a distance of at least $2h$.

Let M be the number of sets K in $\mathcal{D}$ for which there exists an integer $n \in K$ such that $\alpha H(n) < 0$ and $\alpha H(m) \leq 0$ for every integer $m \in (n, n + 2h)$, and let $\mathcal{S}$ be the set of the corresponding values of n .

Lemma 2.4.

$$M \leq c_1 \frac{T}{h^{\frac{3}{2}}}$$

for some absolute constant c_1 .

Proof: Since $H(x) = H([x]) - \alpha\{x\} + \theta(x)$, then

$$\alpha H(x) - \alpha H([x]) = -\alpha^2\{x\} + \alpha\theta(x).$$

So, if x is sufficiently large and not an integer then

$$-\frac{5}{4}\alpha^2\{x\} < \alpha H(x) - \alpha H([x]) < -\frac{3}{4}\alpha^2\{x\}. \quad (2.16)$$

Let n_1 be the smallest integer such that any non integer $x > n_1$ satisfies condition (2.16). If $\#\{n \in \mathcal{S} : n \geq n_1\} = 0$ then $M \leq n_1$, so, the lemma is clearly true for sufficiently large T . Otherwise, $\#\{n \in \mathcal{S} : n \geq n_1\} \geq M - n_1 \gg M$.

Take $n \in \mathcal{S}$ with $n \geq n_1$ and $t \in [n, n+h]$. Then for any integer $m \in [t, t+h]$, $\alpha H(m) \leq 0$. Moreover,

$$\int_t^{t+h} H(u) \, du = \int_t^{[t]+1} H(u) \, du + \sum_{j=1}^{h-1} \int_{[t]+j}^{[t]+j+1} H(u) \, du + \int_{[t]+h}^{t+h} H(u) \, du.$$

Now, for any $1 \leq j < h$,

$$\begin{aligned} \int_{[t]+j}^{[t]+j+1} H(u) \, du &= \int_{[t]+j}^{[t]+j+1} (H(u) - H([t] + j) + H([t] + j)) \, du \\ &= \int_{[t]+j}^{[t]+j+1} (H(u) - H([t] + j)) \, du + \int_{[t]+j}^{[t]+j+1} H([t] + j) \, du. \end{aligned}$$

Therefore, by (2.16),

$$\begin{aligned} \int_{[t]+j}^{[t]+j+1} \alpha H(u) \, du &< \int_0^1 \left(-\frac{3}{4} \alpha^2 x \right) \, dx + \alpha H([t] + j) \\ &< -\frac{3}{8} \alpha^2 \end{aligned}$$

because $\alpha H([t] + j) \leq 0$. Since $[t] \geq n$, we also have

$$\int_t^{[t]+1} \alpha H(u) \, du < \int_{\{t\}}^1 \left(-\frac{3}{4} \alpha^2 x \right) \, dx + \alpha H([t]) (1 - \{t\}) < 0$$

and

$$\int_{[t]+h}^{t+h} \alpha H(u) \, du < \int_0^{\{t\}} \left(-\frac{3}{4} \alpha^2 x \right) \, dx + \alpha H([t] + h) \{t\} \leq 0.$$

Hence,

$$\int_t^{t+h} \alpha H(u) \, du \leq -\frac{3}{8} \alpha^2 (h-1)$$

and so

$$\left| \int_t^{t+h} H(u) \, du \right| \geq \frac{3}{8} |\alpha| (h-1).$$

To finalize the proof of this lemma we are going to use condition (2.1):

$$\int_T^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \ll Th^{\frac{3}{2}}.$$

Take an integer $r = r(T)$ such that $2^r > (\log T)^{3+\frac{D}{2}}$ then, using (2.1) and (2.7), we obtain

$$\begin{aligned}
\int_0^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt &= \int_0^{\frac{T}{2^r}} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \\
&\quad + \sum_{j=0}^r \int_{\frac{T}{2^j}}^{\frac{T}{2^{j-1}}} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \\
&\ll \frac{T}{2^r} h^2 (\log T)^{2+\frac{D}{2}} + h^{\frac{3}{2}} \sum_{j=0}^r \frac{T}{2^j} \\
&\ll Th^{\frac{3}{2}}
\end{aligned}$$

since $h \leq \log T$. On the other hand,

$$\begin{aligned}
\int_0^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt &\geq \sum_{n \in \mathcal{S}} \int_n^{n+h} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \\
&\geq \sum_{\substack{n \in \mathcal{S} \\ n \geq n_1}} \int_n^{n+h} \left(\frac{3}{8} |\alpha| (h-1) \right)^2 \, dt \\
&\gg Mh^3
\end{aligned}$$

Hence $M \leq c_1 \frac{T}{h^{\frac{3}{2}}}$ for some absolute constant c_1 . □

If h is a suitably large integer such that $c_1 \frac{T}{h^{\frac{3}{2}}} < \frac{c}{6h} T$ (its enough to pick $h > \left(\frac{6c_1}{c} \right)^2$), then there are at least $\frac{c}{6h} T$ intervals K in $\mathcal{D}$ such that $\alpha H(n) \prec 0$ for some integer $n \in K$ and $\alpha H(m) > 0$ for some integer m lying in $(n, n+2h)$. Hence, take $c_0 = \frac{c}{6h}$ and, for each of the above $\frac{c}{6h} T$ intervals, take $I = K \cup (n, n+2h)$.

Now, we prove part 1. Suppose $\#\{1 \leq n \leq T : \alpha H(n) \leq 0\} \gg T$. Take T sufficiently large and take the order relation ' $\prec$ ' to be ' $\leq$ '. Therefore, we have $c_0 T$ integers m in the interval $[1, 2T]$, for which $\alpha H(m)$ is positive. So, in this case,

$$\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T.$$

If we don't have $\#\{1 \leq n \leq T : \alpha H(n) \leq 0\} \gg T$, then, we must have $\alpha H(n) > 0$ for almost all $n \in [1, T]$, i. e.

$$\#\{1 \leq n \leq T : \alpha H(n) > 0\} = T(1 + o(1)).$$

Next, we prove part 3. Take ' $\prec$ ' to be ' $<$ '. Then, there exists a positive constant c_0 and c_0T disjoint subintervals of $[1, T]$, with each of them having at least two integers, m and n , such that $H(m) > 0$ and $H(n) < 0$. Therefore, in each of those intervals we have at least one l with either $H(l) = 0$ or $H(l)H(l+1) < 0$. Whence, $z_H(T) > \frac{c_0}{2}T$ or $N_H(T) > \frac{c_0}{2}T$.

If $N_H(T) \gg T$ then, for sufficiently large T ,

$$\#\{1 \leq n \leq T : H(n)H(n+1) < 0\} > c_2T,$$

for some positive constant c_2 . Therefore, between 1 and T , there are more than c_2T integers n such that $\alpha H(n) < 0$, i.e. $\#\{1 \leq n \leq T : \alpha H(n) < 0\} > c_2T$ for all large T . This proves part 2. $\square$

CHAPTER 3

A CLASS OF ARITHMETIC FUNCTIONS

In this chapter, we consider arithmetic functions $f(n)$ such that $f(n) = \sum_{d|n} \frac{b_d}{d}$ and the sequence b_n satisfies conditions (1.2) and (1.3), i. e. for some $B, A > 1$ and $D > 0$,

$$B(x) := \sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right),$$

and

$$\sum_{n \leq x} b_n^4 \ll x \log^D x,$$

We will determine the asymptotic expansion of $\sum_{n \leq x} f(n)$ and will apply the Main Lemma to its error term, $H(x)$. The Main Lemma will enable us to prove theorem 1.2. If the arithmetic functions are rational valued (as they are in most of the cases studied), we prove a result about how frequently can $H(x)$ take any given value, using a theorem of A. Baker. This result, together with theorem 1.2 enable us to prove theorem 1.1, which generalizes Lau's theorem. In sections 3.4, 3.5 and 3.6, we examine some examples for which our conditions are satisfied and so the conclusions of the theorems are true. In section 3.7, we obtain mean square results for $H(x)$, and, in section 3.8, we generalize Pétermann's results (1.13) and (1.18) and prove $X_H(T) \gg T$.

3.1 PRELIMINARY RESULTS

In this section, we will prove some elementary results about this class of arithmetic functions.

Using condition (1.2), we immediately obtain the following lemma:

Lemma 3.1. *Let b_n be a sequence of real numbers satisfying (1.2), for some constants B and $A > 1$, then there exist constants γ_b and α such that*

$$\sum_{n \leq x} \frac{b_n}{n} = B \log x + \gamma_b + O\left(\frac{1}{\log^{A-1} x}\right) \quad (3.1)$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^2} = \alpha, \quad (3.2)$$

$$\sum_{n > x} \frac{b_n}{n^2} = \frac{B}{x} + O\left(\frac{1}{x \log^{A-1} x}\right) \quad (3.3)$$

Proof: Let $R(x) = \sum_{n \leq x} b_n - Bx$. Using partial summation, we get

$$\begin{aligned} \sum_{n \leq x} \frac{b_n}{n} &= \int_{1-}^x \frac{dB(t)}{t} \\ &= \left[\frac{B(t)}{t} \right]_{1-}^x + \int_{1-}^x \frac{B(t)}{t^2} dt \\ &= B + \frac{R(x)}{x} + \int_{1-}^x \frac{Bt + R(t)}{t^2} dt \end{aligned}$$

Since $A > 1$ and using condition (1.2), we obtain

$$\begin{aligned} \int_x^{\infty} \frac{R(t)}{t^2} dt &\ll \int_x^{\infty} \frac{dt}{t \log^A t} \\ &\ll \frac{1}{\log^{A-1} x}, \end{aligned}$$

which converges to zero as $x \rightarrow \infty$. Therefore, $\lim_{x \rightarrow \infty} \int_{1-}^x \frac{R(t)}{t^2} dt$ exists. Let's define

$$\gamma_b = \int_{1-}^{\infty} \frac{R(t)}{t^2} dt + B.$$

Hence,

$$\begin{aligned}\sum_{n \leq x} \frac{b_n}{n} &= B + B \log x + \int_{1^-}^{\infty} \frac{R(t)}{t^2} dt - \int_x^{\infty} \frac{R(t)}{t^2} dt + O\left(\frac{1}{\log^A x}\right) \\ &= B \log x + \gamma_b + O\left(\frac{1}{\log^{A-1} x}\right)\end{aligned}$$

Next, we prove (3.3),

$$\begin{aligned}\sum_{n > x} \frac{b_n}{n^2} &= \int_x^{\infty} \frac{dB(t)}{t^2} \\ &= \left[\frac{B(t)}{t^2} \right]_x^{\infty} + 2 \int_x^{\infty} \frac{B(t)}{t^3} dt \\ &= -\frac{B}{x} + O\left(\frac{1}{x \log^A x}\right) + 2 \int_x^{\infty} \frac{B}{t^2} + O\left(\frac{1}{t^2 \log^A t}\right) dt \\ &= \frac{B}{x} + O\left(\frac{1}{x \log^A x}\right)\end{aligned}$$

The last result also implies

$$\sum_{n=1}^{\infty} \frac{b_n}{n^2} = \alpha < \infty$$

□

Notice that, if $b_n = \mu(n)$, then $\alpha = \zeta^{-1}(2) = \frac{6}{\pi^2}$, $B = 0$, $\gamma_b = 0$ and A can be any real number greater than 1. If $b_n = 1$, then $\alpha = \frac{\pi^2}{6}$, $B = 1$, $\gamma_b = \gamma$ and, again, A can be any real number greater than 1.

We will be interested in functions $f(n)$ that can be written as

$$f(n) = \sum_{d|n} \frac{b_d}{d},$$

where b_n satisfy conditions (1.2) and (1.3). If $b_n = \mu(n)$, then $f(n) = \frac{\phi(n)}{n}$, which is the function studied by Y.-K. Lau. When $b_n = 1$, then $f(n) = \frac{\sigma(n)}{n}$. If the first

condition is satisfied, the summation function of $f(n)$ will have an explicit main term and an error term that depends on the function $\psi(x) = x - [x] - \frac{1}{2}$.

Lemma 3.2. *Let b_n be a sequence of real numbers as in lemma 3.1, and let $f(n) = \sum_{d|n} \frac{b_d}{d}$. Then*

$$\sum_{n \leq x} f(n) = \alpha x - \frac{B \log 2\pi x}{2} - \frac{\gamma_b}{2} + H(x),$$

where, for $x > 1$,

$$H(x) = - \sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right), \quad (3.4)$$

for any $0 < C < A - 1$.

Proof: We have

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n \leq x} \sum_{d|n} \frac{b_d}{d} \\ &= \sum_{d \leq x} \frac{b_d}{d} \sum_{\substack{n \leq x \\ d|n}} 1 \\ &= \sum_{d \leq x} \frac{b_d}{d} \sum_{m \leq \frac{x}{d}} 1 \\ &= \sum_{m \leq x} \sum_{d \leq \frac{x}{m}} \frac{b_d}{d} \end{aligned}$$

Next, we will separate the double sum above in two parts. Let $0 < C < A - 1$ and $y = \log^C x$. Then

$$\begin{aligned} \sum_{m \leq x} \sum_{d \leq \frac{x}{m}} \frac{b_d}{d} &= \sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} + \sum_{y < n \leq x} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} \\ &= \sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} + \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \sum_{y < n \leq \frac{x}{d}} 1 \end{aligned} \quad (3.5)$$

In order to evaluate the first term on the right, we start with an application of formula (3.1). We obtain

$$\begin{aligned} \sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} &= \sum_{n \leq [y]} \left(B \log x - B \log n + \gamma_b + O \left(\frac{1}{\log^{A-1} \left(\frac{x}{n} \right)} \right) \right) \\ &= B[y] \log x - B \sum_{n \leq [y]} \log n + \gamma_b[y] + O \left(\frac{y}{\log^{A-1} \left(\frac{x}{y} \right)} \right) \end{aligned}$$

Now, by Stirling formula,

$$\sum_{n \leq [y]} \log n = [y] \log[y] - [y] + \frac{\log[y]}{2} + \frac{\log 2\pi}{2} + O \left(\frac{1}{y} \right)$$

Notice that, $\log[y] = \log \left(y \left(1 - \frac{\{y\}}{y} \right) \right) = \log y + O \left(\frac{1}{y} \right)$. Hence,

$$\sum_{n \leq y} \sum_{d \leq \frac{x}{n}} \frac{b_d}{d} = B[y] (\log x - \log[y] + 1) + \gamma_b[y] - \frac{B(\log 2\pi y)}{2} + O \left(\frac{y}{\log^{A-1} x} \right) + O \left(\frac{1}{y} \right)$$

For the second term, we get

$$\begin{aligned} \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \sum_{y < n \leq \frac{x}{d}} 1 &= \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \left(\left[\frac{x}{d} \right] - [y] \right) \\ &= x \sum_{d \leq \frac{x}{y}} \frac{b_d}{d^2} - \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \psi \left(\frac{x}{d} \right) - \frac{1}{2} \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} - [y] \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \\ &= x \sum_{d=1}^{\infty} \frac{b_d}{d^2} - x \sum_{d > \frac{x}{y}} \frac{b_d}{d^2} - \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \psi \left(\frac{x}{d} \right) - \left(\frac{1}{2} + [y] \right) \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \end{aligned}$$

By lemma 3.1, we have

$$\begin{aligned} x \sum_{d=1}^{\infty} \frac{b_d}{d^2} &= \alpha x, \\ x \sum_{d > \frac{x}{y}} \frac{b_d}{d^2} &= By + O \left(\frac{y}{\log^A \frac{x}{y}} \right) = By + O \left(\frac{y}{\log^A x} \right) \end{aligned}$$

and

$$\sum_{d \leq \frac{x}{y}} \frac{b_d}{d} = B(\log x - \log y) + \gamma_b + O \left(\frac{1}{\log^{A-1} x} \right).$$

Using the three results above, we evaluate the second double sum on the right of (3.5), obtaining

$$\begin{aligned} \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \sum_{y < n \leq \frac{x}{d}} 1 &= \alpha x - By - \sum_{d \leq \frac{x}{y}} \frac{b_d}{d} \psi\left(\frac{x}{d}\right) - \frac{\gamma_b + B \log x}{2} - B[y](\log x - \log y) \\ &\quad + \frac{B \log y}{2} - \gamma_b[y] + O\left(\frac{y}{\log^{A-1} x}\right) \end{aligned}$$

Notice also that

$$\begin{aligned} B[y](\log y - \log[y]) &= B[y] \left(-\log \left(1 - \frac{\{y\}}{y} \right) \right) \\ &= B\{y\} + O\left(\frac{1}{y}\right) \end{aligned}$$

Hence, joining everything together, we obtain

$$\begin{aligned} H(x) &= \sum_{n \leq x} f(n) - \left(\alpha x - \frac{B \log 2\pi x}{2} - \frac{\gamma_b}{2} \right) \\ &= - \sum_{d \leq \frac{x}{\log^C x}} \frac{b_d}{d} \psi\left(\frac{x}{d}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right) + O\left(\frac{1}{\log^C x}\right) \end{aligned}$$

□

3.2 GENERALIZATION OF LAU'S THEOREM

In this section, we will prove theorem 1.2. We begin with some definitions and notation.

The *distribution function* of $H(x)$ is given by

$$D(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \# \{1 \leq n \leq T : H(n) \leq u\}$$

whenever the limit exists. We say that $D(u)$ is symmetric if $D(u) + D(-u^-) = 1$.

Let's also define

$$\begin{aligned} Z_f(T) &= \#\{x \leq T : f(x) = 0\} \\ z_f(T) &= \#\{n \leq T, n \text{ integer} : f(n) = 0\}. \end{aligned}$$

We state theorem 1.2 again:

Theorem 1.2. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies both conditions*

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

If $\alpha \neq 0$, then

1. $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$.
2. if $N_H(T) \gg T$, then $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$;
3. if $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$, then $N_H(T) \gg T$ or $z_H(T) \gg T$.

Remark. *The result of Y.-K. Lau uses the fact that, when*

$$H(x) = \sum_{n \leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x,$$

the distribution function $D(u)$ exists and is continuous (proved by P. Erdős and H. Shapiro [18]) and $D(0) > 0$ (obtained by Y.-F. Pétermann [64]). Since continuity of $D(u)$ implies $z_H(T) = o(T)$ then part 3 generalizes Lau's result.

Remark. In many applications, $f(n)$ is rational for all n which in certain cases enables us to guarantee that $z_H(T)$ is very small (see theorem 1.10, below), and so we obtain $N_H(T) \gg T$ in part 3. However, we cannot eliminate $z_H(T) \gg T$ from part 3 in general, as example 1.1 demonstrates.

Proof: We just have to show that $H(x)$ satisfies the conditions of theorem 2.1. From lemma 3.2, for any x

$$\begin{aligned} H(x) - H([x]) &= -\alpha\{x\} - \frac{B}{2} (\log 2\pi[x] - \log 2\pi x) \\ &= -\alpha\{x\} - \frac{B}{2} \log \left(1 - \frac{\{x\}}{x}\right) \\ &= -\alpha\{x\} + \frac{B}{2} \frac{\{x\}}{x} + O\left(\frac{1}{x^2}\right) \end{aligned}$$

In lemma 3.2, we also obtained

$$H(x) = - \sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right),$$

for any $0 < C < A - 1$. Take $C = 5 + \frac{D}{2}$, $y(x) = \frac{x}{\log^C x}$ and

$$k(x) = \min(\log^C x, \log^{A-C-1} x).$$

Since $A > 6 + \frac{D}{2}$, then $C < A - 1$ and $A - C - 1 > 0$. Theorem 1.2 now follows from theorem 2.1. □

In theorem 2.1, we proved above that if a function $H(x)$ satisfies condition (2.1) and decreases in all intervals of the form $(n, n + 1)$, where n is an integer, then

$$\#\{1 \leq n \leq T : H(n) \leq 0\} \gg T \quad \text{implies} \quad \#\{1 \leq n \leq T : H(n) > 0\} \gg T.$$

Unfortunately, the converse is not true in general, as the following example shows.

Example 3.1. Consider the arithmetic function $f(n) = \delta$ for all integer n , where δ is a real number. Then $b_1 = \delta$ and $b_n = 0$ for all $n > 1$. Clearly, the sequence b_n satisfies conditions (1.2) and (1.3), and we have $B = 0$, $\alpha = \delta$, $\gamma_b = \delta$ and $H(x) = \delta \left(\frac{1}{2} - \{x\} \right)$. In this case, $\alpha H(n) = \frac{\delta^2}{2}$, for all n . Hence, if $\delta \neq 0$,

$$\#\{1 \leq n \leq T : \alpha H(n) > 0\} = T \quad \text{and} \quad \#\{1 \leq n \leq T : \alpha H(n) \leq 0\} = 0$$

We will prove that when $H(x)$ has very few zeros then

$$\#\{1 \leq n \leq T : H(n) \leq 0\} \gg T \quad \text{implies} \quad \#\{1 \leq n \leq T : H(n) > 0\} \gg T,$$

and from this we obtain $N_H(T) \gg T$. More exactly, we prove

Theorem 3.3. Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies both conditions

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

If $Z_H(T) = o(T)$, then $N_H(T) \gg T$.

Proof: As in theorem 1.2, take $C = 5 + \frac{D}{2}$, $y(x) = \frac{x}{\log^C x}$ and

$$k(x) = \min(\log^C x, \log^{A-C-1} x).$$

Then, the conditions of theorem 2.1 are satisfied, so, conditions (2.1), (2.7) and (2.16), are still valid, i. e., for T and x sufficiently large, and $h \leq \min(\log T, k^2(T))$, we have

$$\int_T^{2T} \left(\int_t^{t+h} H(u) du \right)^2 dt \ll Th^{\frac{3}{2}},$$

$$|H(x)| \ll (\log x)^{1+\frac{D}{4}},$$

$$-\frac{5}{4}\alpha^2\{x\} < \alpha H(x) - \alpha H([x]) < -\frac{3}{4}\alpha^2\{x\}.$$

From part 1 of theorem 1.2, we know that $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$. As in the proof of theorem 2.1, we take T and h large, and divide the interval $[1, 2T]$ into subintervals of length h . Then we take those subintervals which have at least one element n , with $\alpha H(n) > 0$ and are separated by a distance of at least $2h$. The idea is to prove that we cannot have too many of the above subintervals satisfying the condition:

there exists an integer $n \in J$ such that $\alpha H(n) > 0$ and $\alpha H(m) \geq 0$ for
all integer $m \in (n, n + 2h)$

Let M be the number of sets satisfying the above condition and $\mathcal{L}$ be the set of the corresponding values of n . We are going to prove that

$$M \leq C_3 \frac{T}{h^{\frac{3}{2}}}$$

for some absolute constant C_3 . Let n_1 be the smallest integer such that any non integer $x > n_1$ satisfies condition (2.16). First we prove

$$\left| \int_t^{t+h} H(u) du \right| \geq \frac{1}{8} |\alpha| (h - 1),$$

where $t \in [n, n + h]$, $n > n_1$ and $n \in \mathcal{L}$. Since $Z_H(T) = o(T)$, then the intervals of the form $[t, t + h]$ where $H(u) = 0$, for some u , will be excluded. Notice that we still have, for some constant $c > 0$, $\frac{cT}{h}$ intervals, each one with at least one n for which $\alpha H(n) > 0$ and separated by a distance of at least $2h$. Now, in each of those intervals $H(u) \neq 0$, for every u , so $\alpha H(m) > \frac{3}{4} \alpha^2$.

As in the proof of lemma 2.4,

$$\int_t^{t+h} H(u) du = \int_t^{[t]+1} H(u) du + \sum_{j=1}^{h-1} \int_{[t]+j}^{[t]+j+1} H(u) du + \int_{[t]+h}^{t+h} H(u) du$$

and, for any $1 \leq j < h$,

$$\int_{[t]+j}^{[t]+j+1} H(u) du = \int_{[t]+j}^{[t]+j+1} (H(u) - H([t] + j)) du + \int_{[t]+j}^{[t]+j+1} H([t] + j) du.$$

Therefore, using the fact that $-\frac{5}{4}\alpha^2\{x\} < \alpha H(x) - \alpha H([x]) < -\frac{3}{4}\alpha^2\{x\}$, for any $x > n_1$, we obtain

$$\begin{aligned} \int_{[t]+j}^{[t]+j+1} \alpha H(u) \, du &> \int_0^1 \left(-\frac{5}{4}\alpha^2 x \right) \, dx + \alpha H([t] + j) \\ &> -\frac{5}{8}\alpha^2 + \frac{3}{4}\alpha^2 \\ &> \frac{1}{8}\alpha^2 \end{aligned}$$

We also have

$$\int_t^{[t]+1} \alpha H(u) \, du > -\frac{5}{8}\alpha^2 (1 - \{t\}^2) + \alpha H([t]) (1 - \{t\})$$

and

$$\int_{[t]+h}^{t+h} \alpha H(u) \, du > -\frac{5}{8}\alpha^2 \{t\}^2 + \alpha H([t] + h) \{t\}.$$

which implies,

$$\int_t^{[t]+1} \alpha H(u) \, du + \int_{[t]+h}^{t+h} \alpha H(u) \, du > 0.$$

Hence,

$$\int_t^{t+h} \alpha H(u) \, du > \frac{1}{8}\alpha^2(h-1)$$

and so

$$\left| \int_t^{t+h} H(u) \, du \right| > \frac{1}{8}|\alpha|(h-1).$$

Taking an integer $r = r(T)$ such that $2^r > (\log T)^{3+\frac{D}{2}}$ and using (2.1) and (2.7), we obtain

$$\begin{aligned}
\int_0^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt &= \int_0^{\frac{T}{2^r}} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \\
&\quad + \sum_{j=0}^r \int_{\frac{T}{2^j}}^{\frac{T}{2^{j-1}}} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \\
&\ll \frac{T}{2^r} h^2 (\log T)^{2+\frac{D}{2}} + h^{\frac{3}{2}} \sum_{j=0}^r \frac{T}{2^j} \\
&\ll Th^{\frac{3}{2}}
\end{aligned}$$

since $h \leq \log T$. On the other hand,

$$\begin{aligned}
\int_0^{2T} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt &\geq \sum_{n \in \mathcal{L}} \int_n^{n+h} \left(\int_t^{t+h} H(u) \, du \right)^2 \, dt \\
&\geq \sum_{\substack{n \in \mathcal{L} \\ n \geq n_1}} \int_n^{n+h} \left(\frac{3}{8} |\alpha|(h-1) \right)^2 \, dt \\
&\gg Mh^3
\end{aligned}$$

Hence $M \leq C_3 \frac{T}{h^{\frac{3}{2}}}$. Therefore, for a suitable large h , there are at least $\frac{c}{2h}T$ intervals K (separated by a distance of at least $2h$) such that $\alpha H(n) > 0$ for some integer $n \in K$ and $\alpha H(m) < 0$ for some integer m lying in $(n, n+2h)$. Hence

$$\#\{1 \leq n \leq T : \alpha H(n) < 0\} \geq \frac{c}{2h}T$$

To obtain $N_H(T) \gg T$, we just have to notice that for each n and m as above, there are no integer $n < k < m$ for which $H(k) = 0$. $\square$

3.3 RATIONAL ARITHMETIC FUNCTIONS

In some particular cases, with rational valued arithmetic functions $f(n)$, we can obtain a better result than part 3 of theorem 1.2. In this section, we prove such a result. We will need the following consequence of [2, theorem 1] by A. Baker.

Proposition 3.4. *Let $\alpha_1, \dots, \alpha_n$ and $\beta_0, \dots, \beta_n$ denote nonzero algebraic numbers. Then*

$$\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n \neq 0$$

Using the result above, we will prove that, in certain conditions, the error term of the summation of a rational valued arithmetic function, cannot take any given value very often.

Theorem 1.10. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a rational valued arithmetic function and suppose the sequence b_n satisfies $\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right)$, for some real B and $A > 1$. Let r be a real number and*

$$H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2},$$

where $\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right)$ and $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$. Then

1. If $B = 0$ and α is irrational then $\#\{n \leq T, n \text{ integer} : H(n) = r\} \leq 1$;
2. If B is a nonzero algebraic number then $\#\{n \leq T, n \text{ integer} : H(n) = r\} \leq 2$;
3. If B is transcendental then there exists a constant C that depends on r and on the function $f(n)$, such that

$$\#\{n \leq T, n \text{ integer} : H(n) = r\} < (\log T)^C.$$

Proof: Suppose that $B = 0$ and α is irrational. Suppose also that there are two integers, say $M \neq N$, such that $H(M) = H(N)$. Then

$$\sum_{n \leq M} f(n) - \alpha M + \frac{\gamma_b}{2} = \sum_{n \leq N} f(n) - \alpha N + \frac{\gamma_b}{2}.$$

But this implies that α is rational, a contradiction.

Next, suppose B is a nonzero algebraic number and that there are $M > N > Q$ integers, satisfying $H(M) = H(N) = H(Q)$. We have

$$\sum_{n \leq M} f(n) - \alpha M + \frac{B \log 2\pi M}{2} + \frac{\gamma_b}{2} = \sum_{n \leq N} f(n) - \alpha N + \frac{B \log 2\pi N}{2} + \frac{\gamma_b}{2}$$

which implies

$$\alpha = \frac{B}{M - N} \log \left(\frac{M}{N} \right) + \frac{1}{M - N} \sum_{N < n \leq M} f(n).$$

Similarly,

$$\alpha = \frac{B}{M - Q} \log \left(\frac{M}{Q} \right) + \frac{1}{M - Q} \sum_{Q < n \leq M} f(n).$$

Subtracting the second from the first, we obtain

$$B \log \left(\frac{\left(\frac{M}{N} \right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q} \right)^{\frac{1}{M-Q}}} \right) = \frac{1}{M - Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M - N} \sum_{N < n \leq M} f(n).$$

We are going to prove that

$$\left(\frac{M}{N} \right)^{\frac{1}{M-N}} \neq \left(\frac{M}{Q} \right)^{\frac{1}{M-Q}}. \quad (3.6)$$

Since B is a nonzero algebraic number, the above implies that

$$\frac{1}{M - Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M - N} \sum_{N < n \leq M} f(n) \neq 0.$$

Since the values of $f(n)$ are rational, for any integer n , proposition 3.4 implies

$$B \log \left(\frac{\left(\frac{M}{N} \right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q} \right)^{\frac{1}{M-Q}}} \right) \neq \frac{1}{M - Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M - N} \sum_{N < n \leq M} f(n),$$

and so we get a contradiction, which implies $\#\{n \leq T, n \text{ integer} : H(n) = r\} \leq 2$, for any real r .

In fact, instead of proving (3.6), we are going to prove that

$$M^{N-Q}Q^{M-N} < N^{M-Q}, \quad (3.7)$$

for any positive integers $M > N > Q$. Clearly, this implies (3.6). The inequality (3.7) is just a particular case of the geometric mean-analytic mean inequality

$$\left(\prod_{i=1}^n u_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^n u_i, \quad (3.8)$$

where equality only happens if $u_1 = u_2 = \dots = u_n$. In fact, taking $n = M - Q$, $u_i = M$ for $1 \leq i \leq N - Q$ and $u_i = Q$ for $N - Q < i \leq M - Q$, we obtain

$$(M^{N-Q}Q^{M-N})^{\frac{1}{M-Q}} < \frac{1}{M-Q} ((N-Q)M + (M-N)Q) = N$$

Hence, we obtain (3.7) and part 2 of the lemma.

Finally we prove part 3. Suppose r is a real number such that

$$\#\{n \leq T : H(n) = r\} \geq 4.$$

Let $Q < N < M$ be the three smallest positive integers in the above set. Then

$$B \log \left(\frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}} \right) = \frac{1}{M-Q} \sum_{Q < n \leq M} f(n) - \frac{1}{M-N} \sum_{N < n \leq M} f(n).$$

Suppose L is such that $H(L) = r$. Then $L > N > Q$, and as in part 2.

$$B \log \left(\frac{\left(\frac{L}{N}\right)^{\frac{1}{L-N}}}{\left(\frac{L}{Q}\right)^{\frac{1}{L-Q}}} \right) = \frac{1}{L-Q} \sum_{Q < n \leq L} f(n) - \frac{1}{L-N} \sum_{N < n \leq L} f(n).$$

The two expressions above are nonzero. After we cross multiply them, we obtain

$$\log \left(\frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}} \right) = r_1 \log \left(\frac{\left(\frac{L}{N}\right)^{\frac{1}{L-N}}}{\left(\frac{L}{Q}\right)^{\frac{1}{L-Q}}} \right),$$

for some rational r_1 . Therefore, there are four rational numbers r_2, r_3, r_4 and r_5 , such that

$$L^{r_2} = M^{r_3} N^{r_4} Q^{r_5}.$$

Now, any prime dividing L must divide MNQ . Notice that, if p is a prime, k is an integer and $p^k \leq x$ then $k \leq \frac{\log x}{\log p}$. Therefore, the number of integers smaller than x , which have all prime divisors smaller than M is smaller than $(\log x)^{\pi(M)}$. This finishes our proof. $\square$

As a corollary of theorem 1.2 and theorem 1.10 we obtain our main theorem 1.1:

Theorem 1.1. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a rational valued arithmetic function and suppose the sequence b_n satisfies*

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right)$$

and

$$H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

Then, except when $\alpha = 0$, or $B = 0$ and α is rational, we have

$$N_H(T) \gg T \quad \text{if and only if} \quad \#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$$

3.4 PERIODIC SEQUENCES

In this section, we study a family of sequences that trivially satisfy conditions (1.2) and (1.3).

If the terms of the sequence b_n are zero except for finite number of terms than this sequence clearly satisfies conditions (1.2) (with $B = 0$) and (1.3). In this case the sequence $f(n)$ have a simple structure:

Proposition 1.11. *Let b_n be a sequence of real numbers such that $b_n = 0$ for $n > N_0$, for some integer N_0 . Then the sequence $f(n) = \sum_{d|n} \frac{b_d}{d}$ is periodic with period, say q , dividing $[1, 2, \dots, N_0]$ and $f(i) = f((i, q))$, for any integer i .*

Reciprocally, if there exists q satisfying $f(i) = f((i, q))$ for all integers i , then $b_n = 0$ for $n \nmid q$.

Moreover, in this case, $\alpha = \frac{1}{q} \sum_{n \leq q} f(n)$ and $\gamma_b = f(q)$.

Proof: Let $L = [1, 2, \dots, N_0]$ and i be a positive integer. Suppose $d \mid L + i$. If $d \leq N_0$, then $d \mid L$, so $d \mid i$. If $d > N_0$ then $b_d = 0$. Hence

$$f(L + i) = \sum_{d|L+i} \frac{b_d}{d} = \sum_{d|i} \frac{b_d}{d} = f(i).$$

Let q be the period and $g = (i, q)$. Then $(\frac{i}{g}, \frac{q}{g}) = 1$. Take a prime $p \equiv \frac{i}{g} \pmod{\frac{q}{g}}$, with $p > N_0$. Then $pg \equiv i \pmod{q}$ and

$$f(i) = f(pg) = \sum_{d|pg} \frac{b_d}{d} = \sum_{d|g} \frac{b_d}{d} = f(g),$$

because, if d is a divisor of pg and $p \mid d$, then $b_d = 0$. Next, suppose $f(i) = f((i, q))$, then,

$$\begin{aligned} \frac{b_n}{n} &= \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) \\ &= \sum_{\substack{d|n \\ i=(d,q)}} \mu\left(\frac{n}{d}\right) f(i) \\ &= \sum_{i|(n,q)} f(i) \sum_{\substack{d|n \\ i=(d,q)}} \mu\left(\frac{n}{d}\right) \end{aligned}$$

Let $e = (n, q)$. Then there exists integers N and Q , such that $(N, Q) = 1$, $n = Ne$ and $q = Qe$. Write $d = ir$, then

$$\begin{aligned} \frac{b_n}{n} &= \sum_{i|e} f(i) \sum_{\substack{r|\frac{e}{i}N \\ (r, \frac{e}{i}Q)=1}} \mu\left(\frac{e}{i} \frac{N}{r}\right) \\ &= \sum_{i|e} f(i) \sum_{\substack{r|Q \\ (r, \frac{e}{i})=1}} \mu\left(\frac{e}{i} \frac{N}{r}\right) \end{aligned}$$

since $(r, \frac{e}{i}Q) = 1$ implies $(r, \frac{e}{i}) = 1$. Now, if $\left(\frac{e}{i}, \frac{N}{r}\right) > 1$ then $\mu\left(\frac{e}{i} \frac{N}{r}\right) = 0$, on the other hand, if $\left(\frac{e}{i}, \frac{N}{r}\right) = 1$ then $\left(\frac{e}{i}, N\right) = 1$, because $(r, \frac{e}{i}) = 1$. Therefore

$$\begin{aligned} \frac{b_n}{n} &= \sum_{\substack{i|e \\ (\frac{e}{i}, N)=1}} f(i) \sum_{r|N} \mu\left(\frac{e}{i}\right) \mu\left(\frac{N}{r}\right) \\ &= \sum_{\substack{i|e \\ (\frac{e}{i}, N)=1}} f(i) \mu\left(\frac{e}{i}\right) \sum_{r|N} \mu(r) \end{aligned}$$

and the inner sum above will be equal to 1 if $N = 1$ and 0 otherwise. Hence, if $n \mid q$ then

$$b_n = n \sum_{d|n} f(d) \mu\left(\frac{n}{d}\right)$$

and $b_n = 0$ otherwise.

Since $B = 0$, then

$$\gamma_b = \sum_{n=1}^{\infty} \frac{b_n}{n} = \sum_{n|q} \frac{b_n}{n} = f(q)$$

Finally,

$$\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2} = \sum_{d|q} \frac{b_d}{d^2} = \frac{1}{q} \sum_{d|q} \frac{b_d}{d} \frac{q}{d} = \frac{1}{q} \sum_{d|q} \frac{b_d}{d} \sum_{m \leq \frac{q}{d}} 1 = \frac{1}{q} \sum_{n \leq q} \sum_{d|n} \frac{b_d}{d} = \frac{1}{q} \sum_{n \leq q} f(n).$$

□

3.5 AN ERROR TERM OF LANDAU

We finish this chapter with the well known arithmetic function $\frac{n}{\phi(n)}$. Using this function and a result of R. Sitamarachandrarao [81] we can extend our results to the error term associated to $\sum_{n \leq x} \frac{1}{\phi(n)}$ which was studied by E. Landau [49] in the end of the XIX century.

Notice that

$$\frac{n}{\phi(n)} = \prod_{p|n} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p|n} \left(1 + \frac{1}{p-1}\right) = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}$$

Let $b_n = \frac{\mu^2(n)n}{\phi(n)}$. Then $f(n) = \sum_{d|n} \frac{b_d}{d}$. In [81], R. Sitaramachandrarao proved that

$$\sum_{n \leq x} \frac{\mu^2(n)n}{\phi(n)} = x + O\left(x^{\frac{1}{2}}\right),$$

so condition (1.2) is satisfied for any A and with $B = 1$. Since, by Merten's theorem

$$\prod_{p|n} \left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{p \leq n} \left(1 - \frac{1}{p}\right)^{-1} \sim e^{\gamma \log n},$$

then

$$\sum_{n \leq x} b_n^4 = \sum_{n \leq x} \mu^2(n) \frac{n^4}{\phi^4(n)} = \sum_{n \leq x} O(\log^4 n) = O(x \log^4 x),$$

and condition (1.3) is satisfied for $D \geq 4$.

In this case, $\alpha = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$ and $\gamma_b = \gamma + \sum_p \frac{\log p}{p(p-1)}$. Notice that our error term is different from the one studied by R. Sitaramachandrarao. In his article

$$E_1(x) = \sum_{n \leq x} \frac{n}{\phi(n)} - \frac{\zeta(2)\zeta(3)}{\zeta(6)}x + \frac{\log x}{2}$$

here

$$H(x) = \sum_{n \leq x} \frac{n}{\phi(n)} - \frac{\zeta(2)\zeta(3)}{\zeta(6)}x + \frac{\log x}{2} + \frac{\log 2\pi + \gamma + \sum_p \frac{\log p}{p(p-1)}}{2}$$

Since $B = 1$ we can apply theorem 1.10, and so $z(T) \leq 2$. Therefore, if

$$\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T,$$

T	10^3	2×10^3	3×10^3	4×10^3	5×10^3	10^4	2×10^4
$N_H(T)$	38	84	132	180	224	444	920
$\frac{T}{N_H(T)} \approx$	26.3	23.8	22.7	22.2	22.3	22.5	21.7
T	3×10^4	4×10^4	5×10^4	10^5	2×10^5	3×10^5	4×10^5
$N_H(T)$	1386	1862	2332	4666	9352	14064	18758
$\frac{T}{N_H(T)} \approx$	21.6	21.5	21.4	21.4	21.4	21.3	21.3
T	5×10^5	6×10^5	7×10^5	8×10^5	9×10^5	10^6	
$N_H(T)$	23410	28132	32834	37512	42220	46976	
$\frac{T}{N_H(T)} \approx$	21.4	21.3	21.3	21.3	21.3	21.3	

Figure 3.1: Sign Changes on Integers of Landau's error term

then $N_H(T) \gg T$. In this particular example, we seem to have $N_H(T) > \frac{T}{21.3} + o(T)$, as can be seen in figure 3.1.

3.6 MULTIPLICATIVE SEQUENCES

In the main theorems we assumed that $\alpha \neq 0$. In this section we will prove that this must happen whenever the sequence b_n is *completely multiplicative*, i. e. for any positive integers n and m , $b_{nm} = b_n b_m$. We will also give examples of *multiplicative* sequences (i. e. $b_1 = 1$ and $b_{nm} = b_n b_m$, whenever $(n, m) = 1$), for which $\alpha = 0$.

Proposition 1.12. *If b_n is a completely multiplicative sequence satisfying condition (1.3), then $\alpha \neq 0$.*

Proof: Using (2.3), it is plain that

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n^2} < \infty \quad \text{and} \quad \sum_{n>x} \frac{|b_n|}{n^2} = o(1).$$

Therefore, for any prime p ,

$$\sum_{m=1}^{\infty} \frac{b_{p^m}}{p^{2m}} \leq \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} < \infty.$$

Hence,

$$\prod_{p \leq x} \left(1 + \frac{b_p}{p^2} + \frac{b_{p^2}}{p^4} + \cdots \right) < \infty.$$

Since

$$\left| \sum_{n=1}^{\infty} \frac{b_n}{n^2} - \prod_{p \leq x} \left(1 + \frac{b_p}{p^2} + \frac{b_{p^2}}{p^4} + \cdots \right) \right| \leq \sum_{n > x} \frac{|b_n|}{n^2},$$

then, taking $x \rightarrow \infty$ we obtain

$$\alpha = \prod_p \left(1 + \frac{b_p}{p^2} + \frac{b_{p^2}}{p^4} + \cdots \right).$$

Since b_n is completely multiplicative, we can write the Euler product as

$$\alpha = \prod_p \left(1 - \frac{b_p}{p^2} \right)^{-1}.$$

Next, we apply logarithm to the left side and obtain

$$- \sum_p \log \left(1 - \frac{b_p}{p^2} \right)$$

which is

$$\sum_p \sum_{m=1}^{\infty} \frac{b_p^m}{mp^{2m}}.$$

Notice that

$$\sum_p \sum_{m=1}^{\infty} \frac{|b_p^m|}{mp^{2m}} \leq \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} < \infty$$

so

$$\sum_p \sum_{m=1}^{\infty} \frac{b_p^m}{mp^{2m}} > -\infty$$

which implies $\alpha \neq 0$. □

Example 3.2. Consider the multiplicative sequence defined by $b_1 = 1$, $b_2 = 1$, $b_3 = -9$, $b_6 = -9$ and $b_n = 0$ for $n \notin \{1, 2, 3, 6\}$. Then $\alpha = 0$.

Even when the sequence b_n is *strongly multiplicative* (i. e. is multiplicative and $b_{p^\nu} = b_p$ for any prime p and any $\nu \geq 1$), we cannot expect α to be nonzero, as the following example illustrates:

Example 3.3. Define $b_{2n} = -3$ and $b_{2n-1} = 1$. This sequence satisfies both conditions (1.2) and (1.3), but since

$$\sum_{n \text{ even}} \frac{1}{n^2} = \frac{\pi^2}{24} \quad \text{and} \quad \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8},$$

then $\alpha = 0$.

3.7 MEAN SQUARE OF $H(x)$

In this section, we generalize the mean square results (1.12) and (1.17), for our class of functions $H(x)$. We are going to prove

Theorem 1.13. Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies both conditions

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 7 + \frac{3D}{4}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

Let $g(n) = \sum_{d|n} b_d$. Then,

$$\int_1^x H^2(u) \, du = \frac{x}{2\pi^2} \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} + O\left(\frac{x}{\log^L x}\right),$$

where $L > 0$.

Proof: Notice that

$$\int_1^x H^2(u) \, du = \int_2^x H^2(u) \, du + O(1).$$

Suppose $x > 1$. From (3.4), we have

$$H(x) = - \sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right)$$

Take $E = 4 + \frac{D}{2}$ and $C = E + 1$. Then

$$\begin{aligned} H^2(x) &= \left(\sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{u}{n}\right) \right)^2 + \left(\sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) \right) O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{2C} x}\right) \\ &\quad + \left(\sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) \right) O\left(\frac{1}{\log^{A-C-1} x}\right) + O\left(\frac{1}{(\log x)^{2(A-C-1)}}\right) \end{aligned}$$

Using (2.7), we obtain

$$H^2(x) = \left(\sum_{n \leq \frac{x}{\log^C x}} \frac{b_n}{n} \psi\left(\frac{u}{n}\right) \right)^2 + O\left(\frac{1}{\log^K x}\right)$$

where $K = A - 7 - \frac{3D}{4}$. Let $y(x) = \frac{x}{\log^C x}$ and $\eta(m, n) = \max(2, y^{-1}(m), y^{-1}(n))$.

Then

$$\int_2^x \left(\sum_{n=1}^{y(u)} \frac{b_n}{n} \psi\left(\frac{u}{n}\right) \right)^2 \, du = \sum_{m, n=1}^{y(x)} \frac{b_m b_n}{mn} \int_{\eta(m, n)}^x \psi\left(\frac{u}{m}\right) \psi\left(\frac{u}{n}\right) \, du.$$

As in section 2.3 we use the Fourier series (2.9) to evaluate the integral. Since,

$$\psi(u) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k u)}{k},$$

we obtain

$$\frac{1}{\pi^2} \sum_{m, n=1}^{y(x)} \frac{b_m b_n}{mn} \sum_{k, l=1}^{\infty} \frac{1}{kl} \int_{\eta(m, n)}^x \sin\left(2\pi \frac{k u}{m}\right) \sin\left(2\pi \frac{l u}{n}\right) \, du$$

which is equal to

$$\frac{1}{2\pi^2} \sum_{m,n=1}^{y(x)} \frac{b_m b_n}{mn} \sum_{k,l=1}^{\infty} \frac{1}{kl} \int_{\eta(m,n)}^x \cos \left(2\pi u \left(\frac{k}{m} + \frac{l}{n} \right) \right) - \cos \left(2\pi u \left(\frac{k}{m} - \frac{l}{n} \right) \right) du.$$

Now,

$$\int_{\eta(m,n)}^x \cos \left(2\pi u \left(\frac{k}{m} + \frac{l}{n} \right) \right) du \ll \frac{1}{\left(\frac{k}{m} + \frac{l}{n} \right)}$$

Since $y(x) = \frac{x}{\log^C x}$ then, using lemma 1.7, we obtain

$$\sum_{m,n=1}^{y(x)} \frac{|b_m b_n|}{mn} \sum_{k,l=1}^{\infty} \frac{1}{kl \left(\frac{k}{m} + \frac{l}{n} \right)} \ll \frac{T}{\log T}$$

If $\frac{k}{m} \neq \frac{l}{n}$ then

$$\int_{\eta(m,n)}^x \cos \left(2\pi u \left(\frac{k}{m} - \frac{l}{n} \right) \right) du \ll \frac{1}{\left| \frac{k}{m} - \frac{l}{n} \right|}.$$

so, using lemma 1.6,

$$\sum_{m,n=1}^{y(x)} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn \neq lm}}^{\infty} \frac{1}{kl \left| \frac{k}{m} - \frac{l}{n} \right|} \ll \frac{T}{\log T},$$

If $kn = lm$ then

$$\int_{\eta(m,n)}^x \cos \left(2\pi u \left(\frac{k}{m} - \frac{l}{n} \right) \right) du = x - \eta(m,n)$$

Next, take $d = (m, n)$, $m = d\alpha$, $n = d\beta$, $k = \alpha\gamma$ and $l = \beta\gamma$. We have

$$y(x \log^C x) = \frac{x \log^C x}{(\log(x) + C \log \log x)^C} \gg x$$

so, $y^{-1}(x) \ll x \log^C x$. Therefore, using (2.10),

$$\begin{aligned} \sum_{m,n=1}^{y(x)} \frac{b_m b_n}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{\eta(m,n)}{kl} &\ll \sum_{m,n=1}^{y(x)} \frac{|b_m b_n| (m,n)^2 \eta(m,n)}{m^2 n^2} \\ &\ll \sum_{d=1}^{y(x)} d^2 \sum_{\substack{d \leq m \leq y(x) \\ d|m}} \frac{|b_m|}{m^2} \sum_{\substack{m \leq n \leq y(x) \\ (m,n)=d}} \frac{|b_n| n \log^C n}{n^2} \end{aligned}$$

We are going to estimate the inner sum using Hölder inequality, in the form

$$|\sum_i u_i v_i| \leq \left(\sum_j u_j^4 \right)^{\frac{1}{4}} \left(\sum_k v_k^{\frac{4}{3}} \right)^{\frac{3}{4}}. \text{ Take } \delta > 0, \text{ then, using (2.6),}$$

$$\begin{aligned} \sum_{\substack{m \leq n \leq y(x) \\ d|n}} \frac{|b_n| \log^C n}{n} &\leq \left(\sum_{\substack{m \leq n \leq y(x) \\ d|n}} \frac{b_n^4}{n^2} \right)^{\frac{1}{4}} \left(\sum_{\substack{m \leq n \leq y(x) \\ d|n}} \frac{\log^{\frac{4}{3}C} n}{n^{\frac{2}{3}}} \right)^{\frac{3}{4}} \\ &\ll \frac{x^{\frac{1}{4}+\delta}}{m^{\frac{1}{4}-\delta} \sqrt{d}} \end{aligned}$$

Once again, we use Hölder inequality,

$$\begin{aligned} \sum_{\substack{d \leq m \leq y(x) \\ d|m}} \frac{|b_m|}{m^{\frac{9}{4}-\delta}} &\leq \left(\sum_{\substack{d \leq m \leq y(x) \\ d|m}} \frac{b_m^4}{m^2} \right)^{\frac{1}{4}} \left(\sum_{\substack{d \leq m \leq y(x) \\ d|m}} \frac{1}{m^{\frac{7}{3}-\frac{4\delta}{3}}} \right)^{\frac{3}{4}} \\ &\ll \frac{1}{d^{\frac{1}{4}-\delta} d^{\frac{7}{4}-\delta} d^{1-\delta}} \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{m,n=1}^{y(x)} \frac{b_m b_n}{mn} \sum_{k,l=1}^{\infty} \frac{\eta(m,n)}{kl} &\ll x^{\frac{1}{4}+\delta} \sum_{d=1}^{y(x)} \frac{d^2}{d^{\frac{1}{4}-\delta} d^{\frac{7}{4}-\delta} d^{1-\delta} \sqrt{d}} \\ &\ll x^{\frac{1}{4}+\delta} \end{aligned}$$

Take $L = \min(K, 1)$. Joining everything together, we get

$$\int_1^x H^2(u) du = \frac{x}{2\pi^2} \sum_{m,n=1}^{y(x)} \frac{b_m b_n}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} + O\left(\frac{x}{\log^L x}\right)$$

From lemma 1.8, we have

$$\sum_{m,n > y(x)} \frac{|b_m b_n|}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} \ll \frac{1}{(y(x))^{1-\delta}}$$

Then, using (2.10),

$$\begin{aligned} \int_1^x H^2(u) \, du &= \frac{x}{2\pi^2} \sum_{m,n=1}^{\infty} \frac{b_m b_n}{mn} \sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} + O\left(\frac{x}{\log^L x}\right) \\ &= \frac{x}{12} \sum_{m,n=1}^{\infty} \frac{b_m b_n (m,n)^2}{m^2 n^2} + O\left(\frac{x}{\log^L x}\right) \end{aligned}$$

We still have to prove that

$$\frac{6}{\pi^2} \sum_{m,n=1}^{\infty} \frac{b_m b_n (m,n)^2}{m^2 n^2} = \sum_{k=1}^{\infty} \frac{g_k^2}{k^2}, \quad (3.9)$$

where $g(k) = \sum_{n|k} b_n$. We follow S. Chowla [7, lemma 5]. Notice that

$$\begin{aligned} g_k^2 &= \left(\sum_{n|k} b_n \right)^2 \\ &= \sum_{\substack{m|k \\ n|k}} b_n b_m \end{aligned}$$

Now, take $\nu = [m, n]$. Notice that $\nu = \frac{mn}{(m, n)}$, therefore

$$\begin{aligned} g_k^2 &= \sum_{\nu|k} \sum_{\substack{mn \\ (m,n)=\nu}} b_m b_n \\ &= \sum_{\nu|k} h_{\nu} \end{aligned}$$

where $h_{\nu} = \sum_{\substack{mn \\ (m,n)=\nu}} b_m b_n$. Hence

$$\sum_{k=1}^{\infty} \frac{g_k^2}{k^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\nu=1}^{\infty} \frac{h_{\nu}}{\nu^2}$$

We also have

$$\begin{aligned} \sum_{\nu=1}^{\infty} \frac{h_{\nu}}{\nu^2} &= \sum_{\nu=1}^{\infty} \frac{\left(\sum_{\substack{mn \\ (m,n)=\nu}} b_m b_n \right)}{\nu^2} \\ &= \sum_{m,n=1}^{\infty} \frac{b_m b_n (m,n)^2}{m^2 n^2} \end{aligned}$$

So, we proved (3.9). Whence

$$\int_1^x H^2(u) \, du = \frac{x}{2\pi^2} \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} + O\left(\frac{x}{\log^L x}\right)$$

□

In the case studied by Y.-K. Lau, $b_n = \mu(n)$, so $g_n = 1$ if $n = 1$ and $g_n = 0$ for all $n > 1$. Therefore, we obtain (1.12). Suppose $b_n = 1$ for all n . The corresponding error term is

$$F(x) = \sum_{n \leq x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6}x + \frac{\log 2\pi x}{2} + \frac{\gamma}{2}$$

In this case, $g_n = \tau(n)$. Since

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$$

then, we obtain formula (1.17)

$$\begin{aligned} \int_1^x F^2(u) \, du &= \frac{\zeta^4(2)}{2\pi^2\zeta(4)}x(1 + o(1)) \\ &= \frac{5\pi^2}{144}x(1 + o(1)) \end{aligned}$$

which was proved by A. Walfisz [103].

In general, the generation function of g_n^2 is not easy to determine. In the next lemma, we give estimates for the sum

$$\sum_{n=1}^{\infty} \frac{g^2(n)}{n^2}$$

that only depend on the sequence b_n .

Lemma 3.5. *Suppose $g(n) = \sum_{d|n} b_d$ and suppose the sequence b_n satisfies the condition $\sum_{n \leq x} b_n^4 \ll x \log^D x$, for some $D > 0$. Then*

$$\zeta(2) \left(\sum_{n=1}^{\infty} \frac{b_n}{n^2} \right)^2 \leq \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} \leq \zeta^3(2) \left(\sum_{n=1}^{\infty} \frac{b_n^4 \tau(n)}{n^2} \right)^{\frac{1}{2}}$$

Proof: We have

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n=1}^{\infty} \frac{b_n}{n^2}$$

Therefore, using Cauchy's inequality

$$\left(\sum_{n=1}^{\infty} \frac{b_n}{n^2} \right)^2 \leq \left(\frac{6}{\pi^2} \right)^2 \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hence

$$\sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} \geq \zeta(2) \left(\sum_{n=1}^{\infty} \frac{b_n}{n^2} \right)^2$$

On the other hand, by (3.9),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{g_n^2}{n^2} &= \frac{\pi^2}{6} \sum_{m,n=1}^{\infty} \frac{b_m b_n (m,n)^2}{m^2 n^2} \\ &\leq \frac{\pi^2}{6} \sum_{d=1}^{\infty} d^2 \left(\sum_{\substack{m=1 \\ d|m}}^{\infty} \frac{|b_m|}{m^2} \right)^2 \end{aligned}$$

The infinite series in the right is absolutely convergent, since we proved in (2.11) that, for any $X > N$,

$$\sum_{d \leq X} d^2 \left(\sum_{\substack{N < m \leq X \\ d|m}} \frac{|b_m|}{m^2} \right)^2 \ll \frac{1}{N^{1-\delta}}$$

Now, using Hölder inequality,

$$\begin{aligned} \left(\sum_{\substack{m=1 \\ d|m}}^{\infty} \frac{|b_m|}{m^2} \right)^2 &\leq \left(\sum_{\substack{m=1 \\ d|m}}^{\infty} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \left(\sum_{\substack{M=1 \\ d|M}}^{\infty} \frac{1}{M^2} \right)^{\frac{3}{2}} \\ &\leq \left(\sum_{\substack{m=1 \\ d|m}}^{\infty} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \frac{\zeta^{\frac{3}{2}}(2)}{d^3} \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{g_n^2}{n^2} &\leq \zeta^{\frac{5}{2}}(2) \sum_{d=1}^{\infty} \frac{1}{d} \left(\sum_{\substack{m=1 \\ d|m}}^{\infty} \frac{b_m^4}{m^2} \right)^{\frac{1}{2}} \\
&\leq \zeta^{\frac{5}{2}}(2) \left(\sum_{d=1}^{\infty} \frac{1}{d^2} \right)^{\frac{1}{2}} \left(\sum_{D=1}^{\infty} \left(\sum_{\substack{m=1 \\ D|m}}^{\infty} \frac{b_m^4}{m^2} \right) \right)^{\frac{1}{2}} \\
&\leq \zeta^3(2) \left(\sum_{m=1}^{\infty} \frac{b_m^4}{m^2} \sum_{D|m} 1 \right)^{\frac{1}{2}} \\
&\leq \zeta^3(2) \left(\sum_{m=1}^{\infty} \frac{b_m^4 \tau(m)}{m^2} \right)^{\frac{1}{2}}
\end{aligned}$$

□

3.8 ON $X_H(T)$

In this section, we prove that, under our usual conditions (1.2) and (1.3), we have a positive proportion of sign changes for the error terms considered, i. e. $X_H(T) \gg T$. We also show how to use the results proved in the previous section to obtain a lower bound for $X_H(T)$.

Theorem 3.6. *Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be an arithmetic function and suppose the sequence b_n satisfies both conditions*

$$\sum_{n \leq x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \quad \text{and} \quad \sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for some B real, $D > 0$ and $A > 6 + \frac{D}{2}$, respectively. Let $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{n^2}$,

$$\gamma_b = \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{b_n}{n} - B \log x \right) \quad \text{and} \quad H(x) = \sum_{n \leq x} f(n) - \alpha x + \frac{B \log 2\pi x}{2} + \frac{\gamma_b}{2}.$$

If $\alpha \neq 0$ then $X_H(T) \gg T$.

Proof: Suppose $z_H(T) \gg T$, and take the order relation ' $\prec$ ' to be '=' in theorem 2.1. Then there is a fixed constant h and cT disjoint intervals, with $c > 0$, with each of them having at least two integers, m and n , such that $\alpha H(m) > 0$ and $H(n) = 0$. Notice that, we have (2.16), i. e.

$$-\frac{5}{4}\alpha^2\{x\} < \alpha H(x) - \alpha H([x]) < -\frac{3}{4}\alpha^2\{x\}$$

so, if $H(n) = 0$, then, for any $x \in (n, n+1)$, $\alpha H(x) < 0$. Therefore, $X_H(T) \gg T$.

If $Z_H(T) = o(T)$ then by theorem 3.3, $N_H(T) \gg T$. Now, if n is an integer such that $H(n)H(n+1) < 0$ then there is a change of sign in $[n, n+1]$. Therefore, $X_H(T) \gg T$.

The last case we need to consider is $Z_H(T) - z_H(T) \gg T$. If $H(x) = 0$ for a non integer x then we have one change of sign in the interval $[n, n+1]$, where $n = [x]$. Therefore, $X_H(T) \gg T$. $\square$

In 1986, Y.-F. S. Pétermann proved the following general theorem

Theorem (Pétermann [63], 1986). *Let $H : [1, \infty) \rightarrow \mathbb{R}$ be such that for each $n \geq 1$, $H(x) = H([x]) - \alpha\{x\} + \theta(x)$, where $\alpha \neq 0$ is a constant, and $\theta(x) = o(1)$. Suppose further that there is a constant $K > 0$ such that*

$$\int_1^x H^2(u) \, du = Kx(1 + o(1))$$

Then, for T sufficiently large,

$$X_H(T) \geq \frac{8}{3} \left(1 - \frac{3K}{\alpha^2}\right) T + o(T)$$

For the class of arithmetic functions studied in this chapter, we have

$$K = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2}$$

from theorem 1.13. So, if

$$G(2) := \sum_{n=1}^{\infty} \frac{g^2(n)}{n^2} < \frac{2\alpha^2\pi^2}{3}$$

we can explicitly find a constant c , that depends on $G(2)$, such $X_H(T) \geq cT$.

CHAPTER 4

MORE ARITHMETIC FUNCTIONS

Given a sequence of real numbers b_n , and a complex number s , we define

$$B(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}.$$

In this chapter, we consider arithmetic functions $f(n)$, such that $f(n) = \sum_{d|n} \frac{b_d}{d}$ and the sequence b_n satisfies condition (1.3)

$$\sum_{n \leq x} b_n^4 \ll x \log^D x$$

and condition (1.4)

$$B(s) = \zeta^\beta(s)g(s),$$

for some β real, $D > 0$, and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma > 1 - \lambda$, for some $\lambda > 0$.

We will obtain new versions of the results from the previous chapter that can be applied to the examples mentioned in section 1.4. Using proposition 1.14, we prove theorem 1.3. Throughout this chapter, $s = \sigma + it$ will denote a complex number.

4.1 A NEW VERSION OF THE MAIN THEOREM

In this section, we prove that the conditions of theorem 2.1 are valid for the error terms associated with the arithmetic functions defined above and using this result we prove theorem 1.3.

We begin this section by proving an interesting connection between the function $f(n)$ and the sequence b_n . Let $\zeta(s)$ be the Riemann Zeta-function, that is

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \text{if } s = \sigma + it \text{ and } \sigma > 1.$$

Lema 1.15. *Given a sequence of real numbers b_n , let $f(n) = \sum_{d|n} \frac{b_d}{d}$. Then*

$$\sum_{n \leq x} \frac{f(n)}{n^s} = \zeta(s) \sum_{n \leq x} \frac{b_n}{n^{s+1}} - \sum_{n \leq x} \frac{b_n}{n^{s+1}} \sum_{m > \frac{x}{n}} \frac{1}{m^s}$$

for any $s = \sigma + it$ with $\sigma > 1$. Define

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad B(s) := \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

whenever the series in question exist. If the sequence b_n satisfies the condition (1.3),

i. e.

$$\sum_{n \leq x} b_n^4 \ll x \log^D x,$$

for $D > 0$, then

$$F(s) = \zeta(s)B(s+1),$$

for $\sigma > 1$.

Proof: The first part is easy:

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{n^s} &= \sum_{n \leq x} \frac{1}{n^s} \sum_{d|n} \frac{b_d}{d} \\ &= \sum_{d \leq x} \frac{b_d}{d^{s+1}} \sum_{m \leq \frac{x}{d}} \frac{1}{m^s} \\ &= \sum_{d \leq x} \frac{b_d}{d^{s+1}} \left(\sum_{m=1}^{\infty} \frac{1}{m^s} - \sum_{m > \frac{x}{d}} \frac{1}{m^s} \right) \end{aligned}$$

To prove the second part of the lemma, we just have to notice that under condition (1.3), and, for $\sigma > 1$, $\lim_{x \rightarrow \infty} \sum_{n=1}^x \frac{|b_n|}{n^\sigma}$ exists and

$$\begin{aligned} \sum_{d \leq x} \frac{b_d}{d^{s+1}} \sum_{m > \frac{x}{d}} \frac{1}{m^s} &\leq \sum_{d \leq x} \frac{|b_d|}{d^{\sigma+1}} \sum_{m > \frac{x}{d}} \frac{1}{m^\sigma} \\ &\ll \frac{1}{x^{\sigma-1}} \sum_{d \leq x} \frac{|b_d|}{d^{\sigma+1}} d^{\sigma-1} \\ &\ll \frac{1}{x^{\sigma-1}} \end{aligned}$$

So, taking $x \rightarrow \infty$ we obtain the stated result. $\square$

U. Balakrishnan and Y.-F. S. Pétermann [3] proved that:

Proposition 1.14. *Let $f(n)$ be a complex valued arithmetic function satisfying*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \zeta^\beta(s+1) g(s+1),$$

for a complex number β , and $g(s)$ having a Dirichlet series expansion

$$g(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

which is absolutely convergent in the half plane $\sigma > 1 - \lambda$ for some $\lambda > 0$. Let β_0 be the real part of β . If

$$\zeta^\beta(s) g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

then there is a real number b , $0 < b < 1/2$, and constants B_j , such that, taking $y(x) = x \exp(-(\log x)^b)$,

$$\sum_{n \leq x} f(n) = \begin{cases} \zeta^\beta(2) g(2) x - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1), & \text{if } \beta_0 < 0 \\ \zeta^\beta(2) g(2) x + \sum_{j=0}^{[\beta_0]} B_j (\log x)^{\beta-j} - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + o(1), & \text{if } \beta_0 > 0, \end{cases}$$

Remark. The constants b and B_j are all computable.

Remark. The proposition is valid for complex β and complex functions $f(n)$, but we will be only interested in the real version.

We are in conditions to prove theorem 1.3:

Theorem 1.3. Let $f(n) = \sum_{d|n} \frac{b_d}{d}$ be a arithmetic function and suppose the sequence b_n satisfies conditions (1.3) and (1.4), i. e.

$$\sum_{n \leq x} b_n^4 \ll x \log^D x \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^\beta(s) g(s)$$

for some β real, $D > 0$, and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma > 1 - \lambda$, for some $\lambda > 0$. Let $\alpha = \zeta^\beta(2)g(2)$ and

$$H(x) = \begin{cases} \sum_{n \leq x} f(n) - \alpha x, & \text{if } \beta < 0 \\ \sum_{n \leq x} f(n) - \alpha x - \sum_{j=0}^{[\beta]} B_j (\log x)^{\beta-j}, & \text{if } \beta > 0 \end{cases}$$

where the constants B_j are defined by proposition 1.14. If $\alpha \neq 0$ then

1. $\#\{1 \leq n \leq T : \alpha H(n) > 0\} \gg T$;
2. if $N_H(T) \gg T$, then $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$;
3. if $\#\{1 \leq n \leq T : \alpha H(n) < 0\} \gg T$, then $N_H(T) \gg T$ or $z_H(T) \gg T$.

Proof: We are going to use theorem 2.1. Notice that, for any $c > 0$

$$\begin{aligned} \log^c[x] &= \left(\log x + \log \left(1 - \frac{\{x\}}{x} \right) \right)^c \\ &= \log^c x - c \frac{\{x\}}{x} \log^{c-1} x + O\left(\frac{1}{x}\right) \end{aligned}$$

So, $H(x) = H([x]) - \alpha\{x\} + o(1)$. From proposition 1.14, there is an increasing function $k(x)$, with $\lim_{x \rightarrow \infty} k(x) = \infty$, such that

$$H(x) = - \sum_{n \leq y(x)} \frac{b_n}{n} \psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where $y(x) = x \exp(-(\log x)^b)$, for some $0 < b < 1/2$. Hence, the result follows from theorem 2.1. $\square$

$$4.2 \quad f(n) = \left(\frac{\phi(n)}{n}\right)^r$$

Given $r \neq 0$ real, let $f(n) = \left(\frac{\phi(n)}{n}\right)^r$. We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left(\frac{\phi(n)}{n}\right)^r}{n^s} &= \prod_p \left(1 + \frac{(1-p^{-1})^r}{p^s} + \frac{(1-p^{-1})^r}{p^{2s}} + \dots\right) \\ &= \zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right) \left(1 + \frac{(1-p^{-1})^r}{p^s} + \frac{(1-p^{-1})^r}{p^{2s}} + \dots\right) \\ &= \zeta(s) \prod_p \left(1 + \frac{(1-p^{-1})^r - 1}{p^s}\right) \end{aligned}$$

Consider the binomial theorem

$$\left(1 + \frac{1}{p}\right)^r = \sum_{k=1}^{\infty} \binom{r}{k} \frac{1}{p^k}.$$

Where we have infinitely many terms except when r is a nonnegative integer. The series converges because

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k} = O(k^{-r-1})^1$$

¹the last equality is justified by the following limit.

$$\frac{1}{z!} = \lim_{n \rightarrow \infty} \frac{\binom{n+z}{n}}{n^z}.$$

Euler used the fact that this limit always exists to define factorial of a number in general.

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\left(\frac{\phi(n)}{n}\right)^r}{n^s} &= \zeta(s) \prod_p \left(1 + \frac{\left(\frac{-r}{p} + \frac{\binom{r}{2}}{p^2} - \frac{\binom{r}{3}}{p^3} + \dots \right)}{p^s} \right) \\
&= \zeta(s) \zeta^{-r}(s+1) \prod_p \left(1 - \frac{1}{p^{s+1}} \right)^{-r} \left(1 + \frac{\left(-r + \frac{\binom{r}{2}}{p} - \frac{\binom{r}{3}}{p^2} + \dots \right)}{p^{s+1}} \right) \\
&= \zeta(s) \zeta^{-r}(s+1) \prod_p \left(1 + \frac{\binom{-r}{2} - r^2}{p^{2(s+1)}} + \frac{\binom{r}{2}}{p^{s+2}} + \dots \right) \\
&= \zeta(s) \zeta^{-r}(s+1) g_r(s+1),
\end{aligned}$$

with $g_r(s)$ having an Euler product absolutely convergent for $\sigma > \frac{1}{2}$. So, condition (1.4) is satisfied.

Remark. If $r > 0$ then the main term of $\sum_{n \leq x} \left(\frac{\phi(n)}{n}\right)^r$ is $\zeta^{-r}(2) g_r(2) x$, with no logarithmic terms, where $g_r(s)$ is defined above. When $r = 1$, $g_1(s) = 1$ and we recover the case studied by Y.-K. Lau.

Next, we prove condition (1.3). Since $f(n) = \sum_{d|n} \frac{b_d}{d}$, we have

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^{-r}(s) g_r(s)$$

by lemma 1.15.

Lemma 4.1.

$$\sum_{n=1}^{\infty} \frac{b_n^4}{n^s} = \zeta^{r^4}(s) h(s),$$

where $h(s)$ is absolutely convergent for $\sigma > \frac{1}{2}$.

Proof: We have

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \zeta^{-r}(s) g_r(s) = \prod_p \left(1 + \frac{p((1-p^{-1})^r - 1)}{p^s} \right).$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n^4}{n^s} &= \prod_p \left(1 + \frac{(p((1-p^{-1})^r - 1))^4}{p^s} \right) \\ &= \prod_p \left(1 + \frac{\left(-r + \frac{\binom{r}{2}}{p} - \frac{\binom{r}{3}}{p^2} + \dots \right)^4}{p^s} \right) \\ &= \prod_p \left(1 + \frac{r^4}{p^s} + \frac{d_1}{p^{s+1}} + \frac{d_2}{p^{s+2}} + \dots \right) \end{aligned}$$

where the d_j only depend on j , for each j . Since we want to find an Euler product that is convergent for $\sigma > \frac{1}{2}$, we just have to get rid of the term $\frac{r^4}{p^s}$. This can be done by multiplying the above by $\zeta^{-r^4}(s)$. Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n^4}{n^s} &= \zeta^{r^4}(s) \prod_p \left(1 - \frac{1}{p^s} \right)^{r^4} \left(1 + \frac{r^4}{p^s} + \frac{d_1}{p^{s+1}} + \frac{d_2}{p^{s+2}} + \dots \right) \\ &= \zeta^{r^4}(s) \prod_p \left(1 - \frac{r^4}{p^s} + \frac{\binom{r^4}{2}}{p^{2s}} - \dots \right) \left(1 + \frac{r^4}{p^s} + \frac{d_1}{p^{s+1}} + \frac{d_2}{p^{s+2}} + \dots \right) \\ &= \zeta^{r^4}(s) h_r(s). \end{aligned}$$

□

Therefore,

$$\begin{aligned}
\sum_{n \leq x} b_n^4 &\leq \sum_{n=1}^{\infty} b_n^4 \left(\frac{x}{n}\right)^{1+\frac{c}{\log x}} \\
&= x^{1+\frac{c}{\log x}} \sum_{n=1}^{\infty} \frac{b_n^4}{n^{1+\frac{c}{\log x}}} \\
&= e^c x \zeta^{r^4} \left(1 + \frac{c}{\log x}\right) h_r \left(1 + \frac{c}{\log x}\right) \\
&\ll x (\log x)^{r^4}
\end{aligned}$$

where we used $\zeta(s) \sim \frac{1}{s-1}$ as $s \rightarrow 1$. Hence, the sequence b_n satisfies condition (1.3), with $D = r^4$. So, theorem 1.3 is valid for the arithmetic functions of the form $f(n) = \left(\frac{\phi(n)}{n}\right)^r$.

$$4.3 \quad f(n) = \left(\frac{\sigma(n)}{n}\right)^r$$

Let $r \neq 0$ be a real number, here we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\left(\frac{\sigma(n)}{n}\right)^r}{n^s} &= \prod_p \left(1 + \frac{(1+p^{-1})^r}{p^s} + \frac{(1+p^{-1}+p^{-2})^r}{p^{2s}} + \dots\right) \\
&= \zeta(s) \prod_p \left(1 + \frac{(1+p^{-1})^r - 1}{p^s} + \frac{(1+p^{-1}+p^{-2})^r - (1+p^{-1})^r}{p^{2s}} + \dots\right) \\
&= \zeta(s) \zeta^r(s+1) G_r(s+1)
\end{aligned}$$

where

$$\begin{aligned}
G_r(s+1) &= \prod_p \left(1 - \frac{r}{p^{s+1}} + \frac{\binom{r}{2}}{p^{2(s+1)}} - \dots\right) \\
&\quad \times \prod_p \left(1 + \frac{(1+p^{-1})^r - 1}{p^s} + \frac{(1+p^{-1}+p^{-2})^r - (1+p^{-1})^r}{p^{2s}} + \dots\right) \\
&= \prod_p \left(1 + \frac{c_1}{p^{2(s+1)}} + \frac{c_2}{p^{s+2}} + \dots\right)
\end{aligned}$$

so $G_r(s)$ have an Euler product absolutely convergent for $\sigma > \frac{1}{2}$. Hence, condition (1.4) is satisfied.

Remark. When $r < 0$, the main term of $\sum_{n \leq x} \left(\frac{\sigma(n)}{n} \right)^r$ is $\zeta^r(2)G_r(2)x$, with no logarithmic terms, where $G_r(s)$ is defined above.

As before, writing $\left(\frac{\sigma(n)}{n} \right)^r = \sum_{d|n} \frac{b_d}{d}$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n}{n^s} &= \zeta^r(s)G_r(s) \\ &= \prod_p \left(1 + \frac{(1+p^{-1})^r - 1}{p^{s-1}} + \frac{(1+p^{-1}+p^{-2})^r - (1+p^{-1})^r}{p^{2(s-1)}} + \dots \right) \\ &= \prod_p \left(1 + \frac{r + \frac{\binom{r}{2}}{p} + \dots}{p^s} + \frac{\binom{r}{2} (1+p^{-1})^{r-1} + \dots}{p^{2s}} + \dots \right) \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n^4}{n^s} &= \prod_p \left(1 + \frac{\left(r + \frac{\binom{r}{2}}{p} + \dots \right)^4}{p^s} + \frac{\left(\binom{r}{2} (1+p^{-1})^{r-1} + \dots \right)^4}{p^{2s}} + \dots \right) \\ &= \prod_p \left(1 + \frac{r^4}{p^s} + \left(\frac{c_1}{p^{s+1}} + \frac{c_2}{p^{s+2}} + \dots \right) + \left(\frac{d_1}{p^{2s}} + \frac{d_2}{p^{2s+1}} + \dots \right) + \dots \right) \\ &= \zeta^{r^4}(s)H_r(s), \end{aligned}$$

where $H_r(s)$ has an Euler product that is convergent for $\sigma > \frac{1}{2}$. Condition (1.3) follows as in the previous section. So, theorem 1.3 is valid for the arithmetic functions of the form $f(n) = \left(\frac{\sigma(n)}{n} \right)^r$.

$$4.4 \quad f(n) = \left(\frac{\sigma(n)}{\phi(n)} \right)^r$$

Notice that

$$\begin{aligned} \frac{\sigma(n)}{\phi(n)} &= \prod_{p^k \parallel n} \frac{1 + p^{-1} + p^{-2} + \cdots + p^{-k}}{1 - p^{-1}} \\ &= \prod_{p^k \parallel n} (1 + p^{-1} + p^{-2} + \cdots + p^{-k}) (1 + p^{-1} + p^{-2} + \cdots) \\ &= \prod_{p^k \parallel n} (1 + 2p^{-1} + \cdots + kp^{-(k-1)} + (k+1)p^{-k} + (k+1)p^{-(k+1)} + \cdots) \end{aligned}$$

Fix r . Then $\sum_{n=1}^{\infty} \frac{\left(\frac{\sigma(n)}{\phi(n)} \right)^r}{n^s}$ is equal to the Euler product

$$\prod_p \left(1 + \frac{(1 + 2p^{-1} + 2p^{-2} + \cdots)^r}{p^s} + \frac{(1 + 2p^{-1} + 3p^{-2} + 3p^{-3} + \cdots)^r}{p^{2s}} + \cdots \right).$$

We will analyze the case $r = 1$. The result will also follow for the other values of r as in the two previous examples. With $r = 1$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left(\frac{\sigma(n)}{\phi(n)} \right)^r}{n^s} &= \zeta(s) \prod_p \left(1 + \frac{2p^{-1} + 2p^{-2} + \cdots}{p^s} + \frac{p^{-2} + p^{-3} + \cdots}{p^{2s}} + \cdots \right) \\ &= \zeta(s) \zeta^2(s+1) \gamma_1(s+1) \end{aligned}$$

where $\gamma_1(s)$ has an Euler product that is convergent for $\sigma > \frac{1}{2}$. Taking b_n such that

$$\frac{\sigma(n)}{\phi(n)} = \sum_{d|n} \frac{b_d}{d}, \text{ we can write}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n}{n^s} &= \prod_p \left(1 + \frac{2p^{-1} + 2p^{-2} + \cdots}{p^{s-1}} + \frac{p^{-2} + p^{-3} + \cdots}{p^{2(s-1)}} + \cdots \right) \\ &= \prod_p \left(1 + \frac{2 + 2p^{-1} + \cdots}{p^s} + \frac{1 + p^{-1} + \cdots}{p^{2s}} + \cdots \right) \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n^4}{n^s} &= \prod_p \left(1 + \frac{(2 + 2p^{-1} + \cdots)^4}{p^s} + \frac{(1 + p^{-1} + \cdots)^4}{p^{2s}} + \cdots \right) \\ &= \zeta^{16}(s) \theta_1(s) \end{aligned}$$

where $\theta_1(s)$ has an Euler product that is convergent for $\sigma > \frac{1}{2}$. Again we can obtain condition (1.3) and so the theorem 1.3 is also valid in this case. For general r , we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\phi(n)}{\sigma(n)}\right)^r}{n^s} = \zeta(s)\zeta^{2r}(s+1)\gamma_r(s+1),$$

$$\sum_{n=1}^{\infty} \frac{b_n^4}{n^s} = \zeta^{16r^4}(s)\theta_r(s),$$

As in the previous sections, condition (1.3) is valid. Hence, we can apply theorem 1.3 for the arithmetic functions $f(n) = \left(\frac{\sigma(n)}{\phi(n)}\right)^r$, where $r \neq 0$.

$$4.5 \quad f(n) = \left(\frac{\phi_m(n)}{n}\right)^r$$

We define $\phi_m(n)$ to be the number of distinct groups of m consecutive integers all prime to and smaller than n . Notice that, if $p \mid n$ is a prime number and $m \geq p$ then any set of m consecutive integers has one member divisible by p , so $\phi_m(n) = 0$. We can write

$$\phi_m(n) := \begin{cases} n \prod_{p \mid n} \left(1 - \frac{m}{p}\right) & \text{if } m \text{ is smaller than every prime factor of } n \\ 0 & \text{otherwise} \end{cases}$$

These functions were studied as earlier as 1869 by V. Schemmel [78]. Let

$$C_m(s) = \prod_{p \leq m} \left(1 + \frac{(1 - mp^{-1})^r}{p^s} + \frac{(1 - mp^{-1})^r}{p^{2s}} + \dots\right)^{-1}$$

Therefore, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\left(\frac{\phi_m(n)}{n}\right)^r}{n^s} &= \prod_{p>m} \left(1 + \frac{(1 - mp^{-1})^r}{p^s} + \frac{(1 - mp^{-1})^r}{p^{2s}} + \dots\right) \\
&= C_m(s) \zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right) \left(1 + \frac{(1 - mp^{-1})^r}{p^s} + \frac{(1 - mp^{-1})^r}{p^{2s}} + \dots\right) \\
&= C_m(s) \zeta(s) \prod_p \left(1 + \frac{(1 - mp^{-1})^r - 1}{p^s}\right) \\
&= \zeta(s) \zeta^{-mr}(s+1) \nu_r(s+1) \\
\sum_{n=1}^{\infty} \frac{b_n}{n^s} &= \zeta^{-mr}(s) \nu_r(s)
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{b_n^4}{n^s} = \zeta^{(mr)^4}(s) \xi_r(s),$$

where $\nu_r(s)$ and $\xi_r(s)$ have Euler products that are convergent for $\sigma > \frac{1}{2}$. Therefore, theorem 1.3 can also be applied to these family of arithmetic functions.

CHAPTER 5

THE DIVISOR FUNCTION

Let $\Delta(x) = \sum_{n \leq x} \tau(n) - x \log x - (2\gamma - 1)x$, where $\tau(n)$ is the number of divisors of n . There is a very extensive literature about the error term $\Delta(x)$. Many properties have been studied, namely, its maximum order, Ω -estimates and estimates on its moments. In this chapter, we study some other properties of $\Delta(x)$. For example, we will prove $\Delta(x)$ has a positive proportion of pairs (x_1, x_2) , such that $\Delta(x_1) < -cx^{\frac{1}{4}}$ and $\Delta(x_2) > cx^{\frac{1}{4}}$, for some constant $c > 0$; we will obtain an explicit result about the number of sign changes; and we also obtain a version of Lau's main lemma, for the function $\Delta(x)$.

5.1 PRELIMINARY RESULTS

Let $\tau(n)$ denote the number of divisors of n . It was proved by Dirichlet [12] that

$$D(x) := \sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x).$$

Our present interest in the arithmetic function $\Delta(x)$ is about the number of its sign changes. First we will obtain an approximation to $D(x)$ that isolates the oscillating term of $\Delta(x)$.

Lemma 5.1.

$$D(x) = x \log x + (2\gamma - 1)x - 2 \sum_{d \leq \sqrt{x}} \psi\left(\frac{x}{d}\right) - \frac{1}{6} + 2(\{\sqrt{x}\} - \{\sqrt{x}\}^2) + O\left(\frac{1}{\sqrt{x}}\right).$$

Proof: Following Dirichlet we have

$$\begin{aligned}
 D(x) &= \sum_{n \leq x} \tau(n) \\
 &= \sum_{n \leq x} \sum_{d|n} 1 \\
 &= \sum_{d \leq x} \sum_{m \leq \frac{x}{d}} 1 \\
 &= \sum_{d \leq \sqrt{x}} \sum_{m \leq \sqrt{x}} 1 + \sum_{d \leq \sqrt{x}} \sum_{\sqrt{x} < m \leq \frac{x}{d}} 1 + \sum_{d > \sqrt{x}} \sum_{m \leq \frac{x}{d}} 1.
 \end{aligned}$$

But

$$\sum_{d > \sqrt{x}} \sum_{m \leq \frac{x}{d}} 1 = \sum_{m \leq \sqrt{x}} \sum_{\sqrt{x} < d \leq \frac{x}{m}} 1,$$

and

$$\begin{aligned}
 \sum_{d \leq \sqrt{x}} \sum_{\sqrt{x} < m \leq \frac{x}{d}} 1 &= \sum_{d \leq \sqrt{x}} \left(\left[\frac{x}{d} \right] - [\sqrt{x}] \right) \\
 &= \sum_{d \leq \sqrt{x}} \left[\frac{x}{d} \right] - [\sqrt{x}]^2.
 \end{aligned}$$

So,

$$\begin{aligned}
 D(x) &= 2 \sum_{d \leq \sqrt{x}} \left[\frac{x}{d} \right] - [\sqrt{x}]^2 \\
 &= 2 \left(\sum_{d \leq \sqrt{x}} \frac{x}{d} - \sum_{d \leq \sqrt{x}} \psi \left(\frac{x}{d} \right) - \frac{1}{2} \sum_{d \leq \sqrt{x}} 1 \right) - [\sqrt{x}]^2
 \end{aligned}$$

We have the following estimate of the partial sums of the harmonic series (e.g. [91]), for $n \geq 1$,

$$\sum_{m \leq n} \frac{1}{m} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right).$$

In our case, we obtain

$$2 \sum_{d \leq \sqrt{x}} \frac{x}{d} = 2x \left(\log [\sqrt{x}] + \gamma + \frac{1}{2[\sqrt{x}]} - \frac{1}{12[\sqrt{x}]^2} + O\left(\frac{1}{x^2}\right) \right)$$

Now,

$$\begin{aligned}
 \log [\sqrt{x}] &= \log (\sqrt{x} - \{\sqrt{x}\}) \\
 &= \log (\sqrt{x}) + \log \left(1 - \frac{\{\sqrt{x}\}}{\sqrt{x}}\right) \\
 &= \log (\sqrt{x}) - \frac{\{\sqrt{x}\}}{\sqrt{x}} - \frac{\{\sqrt{x}\}^2}{2x} + O\left(x^{-\frac{3}{2}}\right)
 \end{aligned}$$

and

$$\frac{1}{[\sqrt{x}]} = \frac{1}{\sqrt{x}} \left(1 + \frac{\{\sqrt{x}\}}{\sqrt{x}} + O\left(\frac{1}{x}\right)\right).$$

Therefore

$$2 \sum_{d \leq \sqrt{x}} \frac{x}{d} = x \log x - 2 \{\sqrt{x}\} \sqrt{x} - \{\sqrt{x}\}^2 + 2\gamma x + \sqrt{x} + \{\sqrt{x}\} - \frac{1}{6} + O\left(\frac{1}{\sqrt{x}}\right).$$

Notice that

$$[\sqrt{x}]^2 = (\sqrt{x} - \{\sqrt{x}\})^2 = x - 2 \{\sqrt{x}\} \sqrt{x} + \{\sqrt{x}\}^2.$$

Hence, after joining everything together we get:

$$D(x) = x \log x + (2\gamma - 1)x - 2 \sum_{d \leq \sqrt{x}} \psi\left(\frac{x}{d}\right) - \frac{1}{6} + 2(\{\sqrt{x}\} - \{\sqrt{x}\}^2) + O\left(\frac{1}{\sqrt{x}}\right).$$

□

From this lemma we can see that the major oscillating term of $\Delta(x)$ is related to $\psi(x/n)$ and since this function seems to have a random behavior when n is close to the square root of x , it makes sense to predict that the maximum order of $\Delta(x)$ is $x^{\frac{1}{4}+\epsilon}$ as was conjectured by G. H. Hardy. The Ω -results show us that $|\Delta(x)|$ can be larger than $x^{\frac{1}{4}}$ and as D. R. Heath-Brown and K. Tsang [33] proved (see pp. 37), $|\Delta(x)|$ is this large very often. Since, for any $\delta > 0$, $\Delta(n+1) - \Delta(n)$ is smaller than n^δ , for n sufficiently large, it makes sense to expect that the number of sign changes of $\Delta(x)$ between 1 and T is about $T^{\frac{3}{4}-\epsilon}$.

5.2 POSITIVE AND NEGATIVE VALUES OF $\Delta(x)$

In this section, we prove that the inequalities $\Delta(x) < -cx^{\frac{1}{4}}$ and $\Delta(x) > cx^{\frac{1}{4}}$ occur very often.

In chapters 3 and 4, we prove that the error terms $H(x)$ studied, satisfy condition (2.16). We are going to prove a corresponding result for $\Delta(x)$:

Lemma 5.2. *For any $x > 1$,*

$$-\{x\} \log x - 2\gamma\{x\} < \Delta(x) - \Delta([x]) < -\{x\} \log x - (2\gamma - 1)\{x\} \quad (5.1)$$

Proof: Using the definition (1.26) of $\Delta(x)$, we obtain

$$\Delta(x) - \Delta([x]) = -x \log x + [x] \log [x] - (2\gamma - 1)\{x\}$$

As before, $\log [x] = \log x - \frac{\{x\}}{x} - \frac{\{x\}^2}{2x^2} - \dots$. Therefore

$$\begin{aligned} \Delta(x) - \Delta([x]) &= -\{x\} \log x - 2\gamma\{x\} + \frac{\{x\}^2}{x} - \frac{\{x\}^2}{2x} + \frac{\{x\}^3}{2x^2} - \frac{\{x\}^3}{3x^2} + \dots \\ &= -\{x\} \log x - 2\gamma\{x\} + \sum_{n=1}^{\infty} \frac{\{x\}^{n+1}}{n(n+1)x^n} \end{aligned}$$

Since $\{x\}^n \leq \{x\}$, for $n > 1$, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, then

$$\sum_{n=1}^{\infty} \frac{\{x\}^{n+1}}{n(n+1)x^n} < \frac{\{x\}}{x}$$

for $x > 1$. □

In [26], G. H. Hardy proved that

$$\int_2^T |\Delta(t)| dt = O(T^{\frac{5}{4}+\epsilon}),$$

for any $\epsilon > 0$. Next, we prove that the order of the above integral is exactly $T^{\frac{5}{4}}$.

Lemma 5.3.

$$\int_2^T |\Delta(t)| dt = cT^{\frac{5}{4}}(1 + o(1)),$$

where $c \neq 0$.

Proof: D. R. Heath-Brown [31, theorem 2] proved that, for any $k \in [0, 9]$, the mean value

$$X^{-1-\frac{k}{4}} \int_0^X |\Delta(x)|^k dx$$

converges to a finite limit as X tends to infinity, which implies relation above for a finite c . K.-M. Tsang [96, corollary 3] proved that, for any $k \in [0, 9]$,

$$\int_0^X |\Delta(x)|^k dx \asymp X^{1+\frac{k}{4}}.$$

So, $c \neq 0$. □

In the same direction, H. Crámer proved

Proposition 5.4 (H. Crámer [10], 1922).

$$\int_2^T \Delta^2(t) dt = \frac{\zeta^4(\frac{3}{2})}{6\pi^2\zeta(3)} T^{\frac{3}{2}} + O\left(T^{\frac{5}{4}+\epsilon}\right)$$

The two results above allow us to prove that $\Delta(n) < 0$, for a positive proportion of the values of n .

Theorem 1.18. *There are positive constants c_1, c_2 and c_3 , such that, for T sufficiently large,*

$$\#\{1 \leq n \leq T : c_1 T^{\frac{1}{4}} < \Delta(n) < c_2 T^{\frac{1}{4}}\} > c_3 T$$

and

$$\#\{1 \leq n \leq T : -c_2 T^{\frac{1}{4}} < \Delta(n) < -c_1 T^{\frac{1}{4}}\} > c_3 T.$$

In particular

$$\#\{1 \leq n \leq T : \Delta(n) < 0\} \gg T \quad \text{and} \quad \#\{1 \leq n \leq T : \Delta(n) > 0\} \gg T$$

Proof: From Voronoï's result (1.31), we have

$$\int_2^T \Delta(x) dx = \frac{1}{4}T + \left(2\sqrt{2}\pi^2\right)^{-1} T^{\frac{3}{4}} \sum_{n=1}^{\infty} \tau(n) n^{-\frac{5}{4}} \sin\left(4\pi\sqrt{nT} - \frac{\pi}{4}\right) + O\left(T^{\frac{1}{4}}\right),$$

so, from lemma 5.3, we obtain

$$\int_1^T \Delta(t) dt = \left(\frac{c}{2} + o(1)\right) T^{\frac{5}{4}} \quad \text{and} \quad \int_1^T |\Delta(t)| dt = \left(\frac{c}{2} + o(1)\right) T^{\frac{5}{4}}. \quad (5.2)$$

Let $c_4 = \frac{c}{6}$. We have,

$$\int_1^T \Delta(t) dt \leq c_4 T^{\frac{5}{4}}_{0 < \Delta(t) \leq c_4 T^{\frac{1}{4}}}$$

and

$$\int_1^T |\Delta(t)| dt \leq c_4 T^{\frac{5}{4}}_{-c_4 T^{\frac{1}{4}} \leq \Delta(t) < 0}$$

For any $\rho > 0$, let

$$A_\rho = \int_1^T |\Delta(t)| dt_{|\Delta(t)| \geq \rho T^{\frac{1}{4}}}$$

and let $c_5 > \frac{\zeta^4(\frac{3}{2})}{6\pi^2\zeta(3)}$. Then, by proposition 1.32, for T sufficiently large,

$$\begin{aligned} c_5 T^{\frac{3}{2}} &> \int_1^T \Delta(t)^2 dt \\ &\geq \int_1^T \Delta(t)^2 dt_{|\Delta(t)| \geq \rho T^{\frac{1}{4}}} \\ &\geq \rho T^{\frac{1}{4}} A_\rho. \end{aligned}$$

Therefore, $A_\rho < \frac{c_5}{\rho} T^{\frac{5}{4}}$. Take ρ such that $\rho > c_4$ and $\frac{c_5}{\rho} < c_4$, i. e. take

$$\rho > \max\left(\frac{c_5}{c_4}, c_4\right),$$

then

$$\int_1^T |\Delta(t)| dt < c_4 T^{\frac{5}{4}}_{|\Delta(t)| \geq \rho T^{\frac{1}{4}}}$$

Hence, using (5.2),

$$\int_1^T \Delta(t) dt > c_4 T^{\frac{5}{4}}$$

and

$$\int_1^T |\Delta(t)| dt > c_4 T^{\frac{5}{4}}$$

On the other hand,

$$\int_1^T \Delta(t) dt < \rho T^{\frac{1}{4}} \int_1^T 1 dt$$

Since $|\Delta(t) - \Delta([t])| = \{t\} \log t + O(1)$, by (5.1), then, for any $c_1 < c_4$, $c_2 > \rho$ and T sufficiently large,

$$\begin{aligned} \#\{1 \leq n \leq T : c_1 T^{\frac{1}{4}} < \Delta(n) < c_2 T^{\frac{1}{4}}\} &\geq \int_1^T 1 dt \\ &> \frac{1}{\rho T^{\frac{1}{4}}} \int_1^T \Delta(t) dt \\ &> \frac{c_4}{\rho} T. \end{aligned}$$

The first result follows with $c_3 = \frac{c_4}{\rho}$. Similarly,

$$\#\{1 \leq n \leq T : -c_2 T^{\frac{1}{4}} < \Delta(n) < -c_1 T^{\frac{1}{4}}\} > c_3 T$$

□

5.3 AVERAGE RESULTS

In this section, we will use Voronoï's result (1.28), in order to obtain a new version of Lau's Main Lemma, for the error term $\Delta(x)$.

Theorem 1.16. *Let $\epsilon > 0$. For T sufficiently large and $1 \leq r \ll T^{\frac{1}{2}-2\epsilon}$,*

$$\int_T^{2T} \left(\int_{t-\frac{r}{\sqrt{T}}}^{t+\frac{r}{\sqrt{T}}} \Delta(u^2) du \right)^2 dt = \frac{3\zeta^4(\frac{3}{2})}{2\pi^2\zeta(3)} Tr^2 + O\left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon}\right) + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right). \quad (5.3)$$

Proof: Instead of Voronoï's explicit formula (1.27), we will use the truncated form (1.28) that we state here again:

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(x^{\frac{1}{2}+\epsilon} N^{-\frac{1}{2}}\right).$$

where $1 \leq N \ll x$. In order to simplify the notation, write $h = \frac{r}{\sqrt{T}}$. Let $T \leq t \leq 2T$, $t - h \leq u \leq t + h$ and $N = T^2$, then $\sqrt{u} = \sqrt{t} + O\left(\frac{h}{\sqrt{t}}\right)$ and

$$\Delta(u^2) = \frac{\sqrt{t}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos\left(4\pi u\sqrt{n} - \frac{\pi}{4}\right) + O(T^\epsilon).$$

Now, we can calculate the inner integral of (5.3).

$$\begin{aligned} & \int_{t-h}^{t+h} \Delta(u^2) du \\ &= \frac{\sqrt{t}}{4\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \left(\sin\left(4\pi(t+h)\sqrt{n} - \frac{\pi}{4}\right) - \sin\left(4\pi(t-h)\sqrt{n} - \frac{\pi}{4}\right) \right) + O(hT^\epsilon) \\ &= \frac{\sqrt{t}}{2\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h\sqrt{n}) \cos\left(4\pi t\sqrt{n} - \frac{\pi}{4}\right) + O(hT^\epsilon). \end{aligned} \quad (5.4)$$

In the last equality we used $\sin(a+b) - \sin(a-b) = 2\sin b \cos a$. Taking the square,

$$\begin{aligned} & \left(\int_{t-h}^{t+h} \Delta(u^2) du \right)^2 = \\ & \frac{t}{8\pi^4} \sum_{m,n \leq N} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h\sqrt{m}) \sin(4\pi h\sqrt{n}) \cos\left(4\pi t\sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi t\sqrt{n} - \frac{\pi}{4}\right) \\ & + \frac{\sqrt{t}}{\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h\sqrt{n}) \cos\left(4\pi t\sqrt{n} - \frac{\pi}{4}\right) \times O(hT^\epsilon) + O(h^2T^{2\epsilon}). \end{aligned}$$

Next, we will get an upper bound for the second term above. Since, for $a < \frac{\pi}{4}$ we have $\sin a = a + O(a^3)$, then we will separate the sum in consideration into two

sums, in the first we take those small values of n for which we can use the Taylor approximation of the function $\sin x$ with a small error.

$$\begin{aligned} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h \sqrt{n}) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) &= O\left(h \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)}{n^{\frac{3}{4}}}\right) + O\left(\sum_{n > \frac{1}{256h^2}} \frac{\tau(n)}{n^{\frac{5}{4}}}\right) \\ &= O\left(h^{\frac{1}{2}-2\epsilon}\right) \end{aligned}$$

since, for the given ϵ , we have $\tau(n) = O(n^\epsilon)$. Hence

$$\frac{\sqrt{t}}{2\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h \sqrt{n}) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) \times O(hT^\epsilon) = O\left(h^{\frac{3}{2}-2\epsilon} T^{\frac{1}{2}+\epsilon}\right). \quad (5.5)$$

Therefore

$$\begin{aligned} \int_T^{2T} \left(\int_{t-h}^{t+h} \Delta(u^2) du \right)^2 dt &= \frac{1}{8\pi^4} \sum_{m,n \leq N} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h \sqrt{m}) \sin(4\pi h \sqrt{n}) \\ &\times \int_T^{2T} t \cos\left(4\pi t \sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) dt + O\left(h^{\frac{3}{2}-2\epsilon} T^{\frac{3}{2}+\epsilon}\right) + O(h^2 T^{1+2\epsilon}) \end{aligned} \quad (5.6)$$

Next, we analyze the integral of the right hand side. We have

$$\begin{aligned} &2 \int_T^{2T} t \cos\left(4\pi t \sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) dt \\ &= \int_T^{2T} t [\sin(4\pi t(\sqrt{m} + \sqrt{n})) + \cos(4\pi t(\sqrt{m} - \sqrt{n}))] dt \\ &= \int_T^{2T} t \sin(4\pi t(\sqrt{m} + \sqrt{n})) dt + \int_T^{2T} t \cos(4\pi t(\sqrt{m} - \sqrt{n})) dt \end{aligned} \quad (5.7)$$

We will use integration by parts in both terms. We begin with the second term. If $m \neq n$, then

$$\begin{aligned} \int_T^{2T} t \cos(4\pi t(\sqrt{m} - \sqrt{n})) dt &= \frac{T}{2\pi(\sqrt{m} - \sqrt{n})} \sin(8\pi T(\sqrt{m} - \sqrt{n})) \\ &\quad - \frac{T}{4\pi(\sqrt{m} - \sqrt{n})} \sin(4\pi T(\sqrt{m} - \sqrt{n})) \\ &\quad + O\left(\frac{1}{(\sqrt{m} - \sqrt{n})^2}\right) \\ &\ll \frac{T}{|\sqrt{m} - \sqrt{n}|} + \frac{1}{(\sqrt{m} - \sqrt{n})^2}. \end{aligned}$$

If $m = n$, then

$$\int_T^{2T} t \cos(4\pi t(\sqrt{m} - \sqrt{n})) dt = \frac{3}{2}T^2.$$

Now, we evaluate the first term of (5.7).

$$\begin{aligned} \int_T^{2T} t \sin(4\pi t(\sqrt{m} + \sqrt{n})) dt &= \frac{T}{4\pi(\sqrt{m} + \sqrt{n})} \cos(4\pi T(\sqrt{m} + \sqrt{n})) \\ &\quad - \frac{T}{2\pi(\sqrt{m} + \sqrt{n})} \cos(8\pi T(\sqrt{m} + \sqrt{n})) + O(1) \\ &\ll \frac{T}{\sqrt{m} + \sqrt{n}} \end{aligned}$$

since $m, n \leq N = T^2$. When $m \neq n$,

$$\int_T^{2T} t \sin(4\pi t(\sqrt{m} + \sqrt{n})) dt \ll \frac{T}{|\sqrt{m} - \sqrt{n}|}$$

and if $m = n$,

$$\int_T^{2T} t \sin(4\pi t(\sqrt{m} + \sqrt{n})) dt \ll \frac{T}{\sqrt{n}}$$

As before, we will separate the sum in (5.6) in two sums. When n is small we use the Taylor approximation of the function $\sin x$. When $m = n$ and n is small, we obtain

$$\begin{aligned} \frac{1}{8\pi^4} \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{(n)^{\frac{5}{2}}} (4\pi h\sqrt{n})^2 \left(\frac{3}{4}T^2\right) + O\left(T^2 \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{(n)^{\frac{5}{2}}} (4\pi h\sqrt{n})^6\right) \\ + \frac{1}{8\pi^4} \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{(n)^{\frac{5}{2}}} (4\pi h\sqrt{n})^2 O\left(\frac{T}{\sqrt{n}}\right) \end{aligned}$$

which is equal to

$$\frac{3}{2\pi^2} T^2 h^2 \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{n^{\frac{3}{2}}} + O(h^{3-4\epsilon} T^2) + O(Th^2), \quad (5.8)$$

where we used $\sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{n^2} = O(1)$ for the third term. Joining all the other terms

and writing $r = h\sqrt{T}$, the equation (5.6) becomes

$$\begin{aligned} \int_T^{2T} \left(\int_{t-\frac{r}{\sqrt{T}}}^{t+\frac{r}{\sqrt{T}}} \Delta(u^2) du \right)^2 dt &= \frac{3}{2\pi^2} T r^2 \sum_{n \leq \frac{T}{256r^2}} \frac{\tau(n)^2}{n^{\frac{3}{2}}} + S_1 + S_2 + S_3 + S_4 + S_5 \\ &\quad + O\left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon}\right) + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right) + O(r^2 T^{2\epsilon}) \quad (5.9) \end{aligned}$$

Where

$$\begin{aligned}
S_1 &= \frac{3}{32\pi^4} T^2 \sum_{\frac{T}{256r^2} < n \leq N} \frac{\tau(n)^2}{n^{\frac{5}{2}}} \sin^2 \left(4\pi \frac{r}{\sqrt{T}} \sqrt{n} \right) \\
S_2 &= O \left(r^2 \sum_{\substack{m, n \leq \frac{T}{256r^2} \\ m \neq n}} \frac{\tau(m)\tau(n)}{(mn)^{\frac{3}{4}} |\sqrt{m} - \sqrt{n}|} \right) \\
S_3 &= O \left(T \sum_{\substack{\frac{T}{256r^2} < m, n \leq N \\ m \neq n}} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}} |\sqrt{m} - \sqrt{n}|} \right) \\
S_4 &= O \left(\sum_{\substack{m, n \leq N \\ m \neq n}} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}} |\sqrt{m} - \sqrt{n}|^2} \right) \\
S_5 &= O \left(T \sum_{\frac{T}{256r^2} < n \leq N} \frac{\tau(n)^2}{n^3} \right)
\end{aligned}$$

The sum in the first term (5.9) can be expressed as an infinite sum plus an error term and the infinite sum can be evaluated using the following result due to Ramanujan

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s},$$

which is theorem 304 in Hardy & Wright's book [29]. Since, for the given ϵ , we have $\tau(n) = O(n^\epsilon)$, then

$$\begin{aligned}
\frac{3}{2\pi^2} T r^2 \sum_{n \leq \frac{T}{256r^2}} \frac{\tau(n)^2}{n^{\frac{3}{2}}} &= \frac{3}{2\pi^2} T r^2 \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{\frac{3}{2}}} - \frac{3}{2\pi^2} T r^2 \sum_{n > \frac{T}{256r^2}} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \\
&= \frac{3\zeta^4(\frac{3}{2})}{2\zeta(3)\pi^2} T r^2 + O \left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon} \right)
\end{aligned}$$

The term S_1 has the same upper bound as the error term above,

$$\frac{3}{32\pi^4} T^2 \sum_{\frac{T}{256r^2} < n \leq N} \frac{\tau(n)^2}{n^{\frac{5}{2}}} \sin^2 \left(4\pi \frac{r}{\sqrt{T}} \sqrt{n} \right) = O(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}), \quad (5.10)$$

For the next two terms we have to be more careful. Our first step will be to get rid of the square roots in the denominator. We have

$$r^2 \sum_{\substack{m, n \leq \frac{T}{256r^2} \\ m \neq n}} \frac{\tau(m)\tau(n)}{(mn)^{\frac{3}{4}} |\sqrt{m} - \sqrt{n}|} \ll r^2 \sum_{n \leq \frac{T}{256r^2}} \frac{\tau(n)}{n^{\frac{3}{4}}} \sum_{n < m \leq \frac{T}{256r^2}} \frac{\tau(m)\sqrt{m}}{m^{\frac{3}{4}}(m-n)}.$$

We estimate the inside sum after making the change of variable, $k = m - n$.

$$\begin{aligned} \sum_{n < m \leq \frac{T}{256r^2}} \frac{\tau(m)}{m^{\frac{1}{4}}(m-n)} &\ll T^\epsilon \sum_{k \leq \frac{T}{256r^2}} \frac{1}{k(k+n)^{\frac{1}{4}}} \\ &\ll T^\epsilon \left(\sum_{k < n} \frac{1}{k(k+n)^{\frac{1}{4}}} + \sum_{n \leq k \leq \frac{T}{256r^2}} \frac{1}{k(k+n)^{\frac{1}{4}}} \right) \\ &\ll T^\epsilon \frac{\log n}{n^{\frac{1}{4}}} \end{aligned}$$

Hence,

$$\begin{aligned} S_2 &\ll T^{2\epsilon} r^2 \log T \sum_{n \leq \frac{T}{256r^2}} \frac{1}{n} \\ &\ll T^{2\epsilon} r^2 \log^2 T \\ &\ll T^{3\epsilon} r^2 \end{aligned} \tag{5.11}$$

Analogously,

$$\begin{aligned} S_3 &\ll \sum_{\frac{T}{256r^2} < n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sum_{n < m \leq N} \frac{\tau(m)\sqrt{m}}{m^{\frac{5}{4}}(m-n)} \\ &\ll T^{1+4\epsilon} \sum_{\frac{T}{256r^2} < n \leq N} \frac{1}{n^{\frac{5}{4}}} \sum_{k \leq N} \frac{1}{k(k+n)^{\frac{3}{4}}} \\ &\ll T^{1+4\epsilon} \sum_{\frac{T}{256r^2} < n \leq N} \frac{1}{n^{\frac{5}{4}}} \left(\frac{\log N}{n^{\frac{3}{4}}} + \frac{1}{n^{\frac{3}{4}}} \right) \\ &\ll T^{1+4\epsilon} (\log T) \left(\frac{r^2}{T} \right) \\ &\ll T^{5\epsilon} r^2. \end{aligned} \tag{5.12}$$

For S_4 we don't need to cancel T , so we can be more relaxed in our estimate. We, again, make the change of variable $k = m - n$.

$$\begin{aligned}
\sum_{\substack{m, n \leq N \\ m \neq n}} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}} |\sqrt{m} - \sqrt{n}|^2} &\ll T^{4\epsilon} \sum_{n \leq N} \frac{1}{n^{\frac{5}{4}}} \sum_{n < m \leq N} \frac{1}{m^{\frac{1}{4}} (m - n)^2} \\
&\ll T^{4\epsilon} \sum_{n \leq N} \frac{1}{n^{\frac{5}{4}}} \sum_{k \leq N} \frac{1}{k^2 (k + n)^{\frac{1}{4}}} \\
&\ll T^{4\epsilon}.
\end{aligned} \tag{5.13}$$

Since $\tau(n) = O(n^\epsilon)$,

$$\begin{aligned}
S_5 &= O \left(T \sum_{\frac{T}{256r^2} < n \leq N} \frac{\tau(n)^2}{n^3} \right) \\
&\ll \frac{r^{4-4\epsilon}}{T^{1-2\epsilon}}
\end{aligned} \tag{5.14}$$

Whence,

$$\begin{aligned}
\int_T^{2T} \left(\int_{t-\frac{r}{\sqrt{T}}}^{t+\frac{r}{\sqrt{T}}} \Delta(u^2) du \right)^2 dt &= \frac{3\zeta^4(\frac{3}{2})}{2\zeta(3)\pi^2} Tr^2 + O \left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon} \right) + O \left(T^{5\epsilon} r^2 \right) \\
&\quad + O \left(\frac{r^{4-4\epsilon}}{T^{1-2\epsilon}} \right) + O \left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon} \right) \\
&= \frac{3\zeta^4(\frac{3}{2})}{2\zeta(3)\pi^2} Tr^2 + O \left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon} \right) + O \left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon} \right)
\end{aligned}$$

□

Notice that, even if $\Delta(u^2)$ was always very large, say $\Delta(u^2) = cT^{\frac{1}{2}}(1 + o(1))$, for all $u \in [T, 2T]$, we would obtain

$$\int_T^{2T} \left(\int_{t-\frac{r}{\sqrt{T}}}^{t+\frac{r}{\sqrt{T}}} \Delta(u^2) du \right)^2 dt = 4c^2 Tr^2(1 + o(1))$$

which doesn't contradict theorem 1.16, if c is small enough. So, from theorem 1.16, we cannot assure that $\Delta(x)$ must have many changes of sign. Our next step is to

explore if we can find a constant, say X , such that, for some pairs of intervals separated by a distance $X - 2h$, say

$$(I_1, I_2) = ([t - h, t + h], [t + X - h, t + X + h]),$$

we have enough cancellations in the sum of the integrals

$$\int_{I_1} \Delta(u^2) du \quad \text{and} \quad \int_{I_2} \Delta(u^2) du.$$

If exists X satisfying the above condition then we must have many changes of signs for $\Delta(x)$.

Theorem 1.17. *Let $\epsilon > 0$, T sufficiently large and $1 \leq r, X \ll T^{\frac{1}{2}-\epsilon}$. For $t \in [T, 2T]$ and any $h > 0$, define*

$$A_{t,h} = \int_{t-h}^{t+h} \Delta(u^2) du.$$

Then,

$$\begin{aligned} \int_T^{2T} \left(A_{t, \frac{r}{\sqrt{T}}} + A_{t+X, \frac{r}{\sqrt{T}}} \right)^2 dt &= \frac{3\zeta^4(\frac{3}{2})}{\pi^2\zeta(3)} Tr^2 + \frac{3}{2\pi^2} Tr^2 \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \cos(4\pi X \sqrt{n}) \\ &\quad + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right) + O(T^{1-\epsilon}) + O\left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon}\right) \end{aligned} \quad (5.15)$$

Moreover, for any X ,

$$\int_T^{2T} \left(A_{t, \frac{r}{\sqrt{T}}} + A_{t+X, \frac{r}{\sqrt{T}}} \right)^2 dt \asymp Tr^2$$

Proof: First, we prove the second statement. The second term of (5.15) is of the size of the first only if X is such that $\cos(4\pi X \sqrt{n})$ is almost always close to -1 . But, if for some n and X , $\cos(4\pi X \sqrt{n})$ is close to -1 , then $\cos(4\pi X \sqrt{4n})$ is closed to 1. Therefore, we have

$$\left| \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \cos(4\pi X \sqrt{n}) \right| < c,$$

for some $c < \frac{\zeta^4(\frac{3}{2})}{\zeta(3)}$. Hence, for any X , we cannot cancel the main term of (5.15), i.

e. Tr^2 is the exact average order of

$$\int_T^{2T} \left(A_{t, \frac{r}{\sqrt{T}}} + A_{t+X, \frac{r}{\sqrt{T}}} \right)^2 dt$$

Now, we evaluate the integral on the left of (5.15). Notice that

$$\begin{aligned} \int_T^{2T} A_{t+X, \frac{r}{\sqrt{T}}}^2 dt &= \int_T^{2T} A_{t, \frac{r}{\sqrt{T}}}^2 dt + O(Xr^2) \\ &= \frac{3\zeta^4(\frac{3}{2})}{2\zeta(3)\pi^2} Tr^2 + O\left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon}\right) + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right) \end{aligned}$$

so we only need to calculate

$$\int_T^{2T} A_{t, \frac{r}{\sqrt{T}}} A_{t+X, \frac{r}{\sqrt{T}}} dt.$$

Let $h = \frac{r}{\sqrt{T}}$ and $N = T^2$. Using (5.4) and (5.5), we have

$$\begin{aligned} A_{t,h} A_{t+X,h} &= \left(\int_{t-h}^{t+h} \Delta(u^2) du \right) \left(\int_{t+X-h}^{t+X+h} \Delta(u^2) du \right) \\ &= \left(\frac{\sqrt{t}}{2\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h\sqrt{n}) \cos\left(4\pi t\sqrt{n} - \frac{\pi}{4}\right) \right) \\ &\quad \times \left(\frac{\sqrt{t+X}}{2\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h\sqrt{n}) \cos\left(4\pi(t+X)\sqrt{n} - \frac{\pi}{4}\right) \right) \\ &\quad + O\left(h^{\frac{3}{2}-2\epsilon} t^{\frac{1}{2}+\epsilon}\right) + O(h^2 t^{2\epsilon}) \\ &= \frac{\sqrt{t(t+X)}}{8\pi^4} \sum_{m,n \leq N} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h\sqrt{m}) \sin(4\pi h\sqrt{n}) \\ &\quad \times \cos\left(4\pi t\sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi(t+X)\sqrt{n} - \frac{\pi}{4}\right) dt \\ &\quad + O\left(h^{\frac{3}{2}-2\epsilon} t^{\frac{1}{2}+\epsilon}\right) + O(h^2 t^{2\epsilon}) \end{aligned}$$

Next, we the integral,

$$\begin{aligned}
\int_T^{2T} A_{t,h} A_{t+X,h} dt &= \frac{1}{8\pi^4} \sum_{m,n \leq N} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h\sqrt{m}) \sin(4\pi h\sqrt{n}) \\
&\times \int_T^{2T} \sqrt{t(t+X)} \cos\left(4\pi t\sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi(t+X)\sqrt{n} - \frac{\pi}{4}\right) dt \\
&+ O\left(h^{\frac{3}{2}-2\epsilon} T^{\frac{3}{2}+\epsilon}\right) + O(h^2 T^{1+2\epsilon})
\end{aligned} \tag{5.16}$$

Using another trigonometric identity,

$$\begin{aligned}
2 \int_T^{2T} \sqrt{t(t+X)} \cos\left(4\pi t\sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi(t+X)\sqrt{n} - \frac{\pi}{4}\right) dt \\
= \int_T^{2T} \sqrt{t(t+X)} \sin\left(4\pi t(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}\right) \\
+ \int_T^{2T} \sqrt{t(t+X)} \cos\left(4\pi t(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}\right) dt
\end{aligned} \tag{5.17}$$

Now, $\sqrt{t(t+X)} = t \left(1 + \frac{X}{2t} + O\left(\frac{X^2}{t^2}\right)\right)$. We have,

$$\int_T^{2T} \frac{X^2}{t} \cos\left(4\pi t\sqrt{m} - \frac{\pi}{4}\right) \cos\left(4\pi(t+X)\sqrt{n} - \frac{\pi}{4}\right) dt \ll X^2 \log T$$

Therefore,

$$\frac{1}{8\pi^4} \sum_{m,n \leq N} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h\sqrt{m}) \sin(4\pi h\sqrt{n}) X^2 \log T = O(T^{1-\epsilon})$$

We will use integration by parts in the next term,

$$\begin{aligned}
&\int_T^{2T} t \sin\left(4\pi t(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}\right) dt \\
&= \frac{T}{4\pi(\sqrt{m} + \sqrt{n})} \cos\left(4\pi T(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}\right) \\
&\quad - \frac{T}{2\pi(\sqrt{m} + \sqrt{n})} \cos\left(8\pi T(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}\right) + O(1) \\
&\ll \frac{T}{\sqrt{m} + \sqrt{n}}
\end{aligned}$$

We also have,

$$\begin{aligned}
& \int_T^{2T} X \sin(4\pi t(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}) \, dt \\
&= \frac{X}{4\pi(\sqrt{m} + \sqrt{n})} \cos(4\pi T(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}) \\
&\quad - \frac{X}{4\pi(\sqrt{m} + \sqrt{n})} \cos(8\pi T(\sqrt{m} + \sqrt{n}) + 4\pi X\sqrt{n}) \\
&\ll \frac{T}{\sqrt{m} + \sqrt{n}}
\end{aligned}$$

since $X \ll T^{\frac{1}{2}-\epsilon}$. So, using $r = h\sqrt{T}$, (5.14) and the third part of (5.8),

$$\frac{1}{8\pi^4} \sum_{m,n \leq N} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h\sqrt{m}) \sin(4\pi h\sqrt{n}) O\left(\frac{T}{\sqrt{m} + \sqrt{n}}\right) \ll r^2$$

For the second term of (5.17), we have to distinguish two cases. If $m \neq n$, then

$$\begin{aligned}
& \int_T^{2T} t \cos(4\pi t(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}) \, dt \\
&= \frac{T}{2\pi(\sqrt{m} - \sqrt{n})} \sin(8\pi T(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}) \\
&\quad - \frac{T}{4\pi(\sqrt{m} - \sqrt{n})} \sin(4\pi T(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}) + O\left(\frac{1}{(\sqrt{m} - \sqrt{n})^2}\right) \\
&\ll \frac{T}{|\sqrt{m} - \sqrt{n}|} + \frac{1}{(\sqrt{m} - \sqrt{n})^2}
\end{aligned}$$

and

$$\int_T^{2T} X \cos(4\pi t(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}) \, dt \ll \frac{X}{|\sqrt{m} - \sqrt{n}|}.$$

From (5.11), (5.12) and (5.13), we obtain

$$\begin{aligned}
& \frac{1}{8\pi^4} \sum_{\substack{m,n \leq N \\ m \neq n}} \frac{\tau(m)\tau(n)}{(mn)^{\frac{5}{4}}} \sin(4\pi h\sqrt{m}) \sin(4\pi h\sqrt{n}) O\left(\frac{T}{|\sqrt{m} - \sqrt{n}|} + \frac{1}{(\sqrt{m} - \sqrt{n})^2}\right) \\
&\ll S_2 + S_3 + S_4 \ll T^{5\epsilon} r^2
\end{aligned}$$

If $m = n$, then

$$\int_T^{2T} t \cos(4\pi t(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}) \, dt = \frac{3}{2} T^2 \cos(4\pi X\sqrt{n})$$

and

$$\int_T^{2T} X \cos(4\pi t(\sqrt{m} - \sqrt{n}) - 4\pi X\sqrt{n}) dt = TX \cos(4\pi X\sqrt{n}).$$

As in the proof of theorem 1.16, we will separate each of the sums

$$\frac{3}{32\pi^4} T^2 \sum_{n \leq N} \frac{\tau(n)^2}{n^{\frac{5}{2}}} \sin(4\pi h\sqrt{n})^2 \cos(4\pi X\sqrt{n})$$

and

$$\frac{1}{16\pi^4} TX \sum_{n \leq N} \frac{\tau(n)^2}{n^{\frac{5}{2}}} \sin(4\pi h\sqrt{n})^2 \cos(4\pi X\sqrt{n})$$

into two sums, one with $n \leq \frac{1}{256h^2}$ for which we can use $\sin x = x + O(x^3)$, and the other with $\frac{1}{256h^2} < n \leq N$. Take again $r = h\sqrt{T}$. Notice that

$$\frac{1}{16\pi^4} TX \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{n^{\frac{5}{2}}} (4\pi h\sqrt{n})^2 \cos(4\pi X\sqrt{n}) \ll Xr^2$$

and

$$\frac{1}{16\pi^4} T \left(X + \frac{3}{2}T \right) \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{n^{\frac{5}{2}}} (4\pi h\sqrt{n})^6 \cos(4\pi X\sqrt{n}) \ll T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}$$

We estimate the terms with large n as we did in (5.10),

$$\frac{1}{16\pi^4} T \left(X + \frac{3}{2}T \right) \sum_{\frac{T}{256r^2} < n \leq N} \frac{\tau(n)^2}{n^{\frac{5}{2}}} \sin(4\pi h\sqrt{n})^2 \cos(4\pi X\sqrt{n}) \ll T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}$$

So, we are left with

$$\begin{aligned} & \frac{3}{32\pi^4} T^2 \sum_{n \leq \frac{1}{256h^2}} \frac{\tau(n)^2}{n^{\frac{5}{2}}} (4\pi h\sqrt{n})^2 \cos(4\pi X\sqrt{n}) \\ &= \frac{3}{2\pi^2} Tr^2 \sum_{n \leq \frac{T}{256r^2}} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \cos(4\pi X\sqrt{n}) \\ &= \frac{3}{2\pi^2} Tr^2 \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \cos(4\pi X\sqrt{n}) + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right) \end{aligned}$$

Hence, joining everything together, we obtain

$$\begin{aligned} & \int_T^{2T} A_{t,h} A_{t+X,h} dt = \\ & \frac{3}{2\pi^2} T r^2 \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{\frac{3}{2}}} \cos(4\pi X \sqrt{n}) + O\left(T^{\frac{1}{2}+2\epsilon} r^{3-4\epsilon}\right) + O(T^{1-\epsilon}) + O\left(T^{\frac{3}{4}+2\epsilon} r^{\frac{3}{2}-2\epsilon}\right) \end{aligned}$$

□

5.4 CHANGES OF SIGN

In this section, we use a different approach to the problem of finding the number of sign changes of $\Delta(x)$, for $1 \leq x \leq T$. We need a technical lemma before we continue.

Lemma 5.5. *Let $\epsilon > 0$ and let T be sufficiently large. Take $h \leq T^\epsilon$, $k \geq 1$ an integer and $N = T^2$. Take also $t = t_k \geq T$, then*

$$\begin{aligned} F(t, k) &:= \int_{t_k-h}^{t_k+h} \int_{t_{k-1}-h}^{t_{k-1}+h} \cdots \int_{t_2-h}^{t_2+h} \int_{t_1-h}^{t_1+h} \Delta(t_0^2) dt_0 dt_1 \cdots dt_{k-2} dt_{k-1} \\ &= \frac{1}{2^k \pi^{k+1} \sqrt{2}} \sqrt{t_k} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}+\frac{k}{2}}} \sin^k(4\pi h \sqrt{n}) \cos\left(4\pi t_k \sqrt{n} - \frac{\pi}{4}\right) + O(h^{k+1} T^\epsilon). \end{aligned} \quad (5.18)$$

Proof: We will use again the truncated form of Voronoï's result (1.28):

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi \sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(4\pi \sqrt{nx} - \frac{\pi}{4}) + O\left(x^{\frac{1+\epsilon}{2}} N^{-\frac{1}{2}}\right),$$

where $1 \leq N \ll x$. Let $h < T^\epsilon$. Given $k \geq 1$ and $t \geq T$, let $t_k = t$ and for any $1 \leq i \leq k$, take $t_{k-i} \in [t_{k-i+1} - h, t_{k-i+1} + h]$. We are going to prove, by induction, that for every $1 \leq i \leq k$, we have

$$\sqrt{t_{k-i}} = \sqrt{t_k} + O\left(\frac{h}{\sqrt{t_k}}\right). \quad (5.19)$$

Since $t_k - h \leq t_{k-1} \leq t_k + h$ and $\sqrt{t_k \pm h} = \sqrt{t_k} + O\left(\frac{h}{\sqrt{t_k}}\right)$, then

$$\sqrt{t_{k-1}} = \sqrt{t_k} + O\left(\frac{h}{\sqrt{t_k}}\right).$$

Now, suppose (5.19) is valid for some $1 \leq i < k$, i. e. $\sqrt{t_{k-i}} = \sqrt{t_k} + O\left(\frac{h}{\sqrt{t_k}}\right)$. We are going to prove that (5.19) is also valid for $i+1$. We have

$$\sqrt{t_{k-(i+1)}} = \sqrt{t_{k-i}} + O\left(\frac{h}{\sqrt{t_{k-i}}}\right)$$

and

$$\begin{aligned} \frac{h}{\sqrt{t_{k-i}}} &= \frac{h}{\sqrt{t_k} + O\left(\frac{h}{\sqrt{t_k}}\right)} \\ &= \frac{h}{\sqrt{t_k}} \left(\frac{1}{1 + O\left(\frac{h}{t_k}\right)} \right) \\ &= \frac{h}{\sqrt{t_k}} \left(1 + O\left(\frac{h}{t_k}\right) \right) \\ &= O\left(\frac{h}{\sqrt{t_k}}\right) \end{aligned}$$

Therefore, $\sqrt{t_{k-(i+1)}} = \sqrt{t_k} + O\left(\frac{h}{\sqrt{t_k}}\right)$. Hence, (5.19) is valid for any $k \geq 1$ and any $1 \leq i \leq k$.

Take $t_0, t_1 \geq T$, $N = T^2$. Then

$$O\left(\frac{h}{\sqrt{t_1}}\right) \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos\left(4\pi t_0 \sqrt{n} - \frac{\pi}{4}\right) = O(hT^\epsilon). \quad (5.20)$$

and, for $k \geq 1$, $t_0, \dots, t_k \geq T$ and $1 \leq j \leq k$,

$$O\left(\frac{h}{\sqrt{t_k}}\right) \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{j}{2}}} \sin^j(4\pi h \sqrt{n}) \cos\left(4\pi t_j \sqrt{n} - \frac{\pi}{4}\right) = o(1), \quad (5.21)$$

since $h \leq T^\epsilon$. We are going to prove (5.18) by induction. Let $N = T^2$ and suppose that $k = 1$. Take $t_1 \geq T$ and $t_0 \in [t_1 - h, t_1 + h]$, then

$$\begin{aligned} \Delta(t_0^2) &= \frac{\sqrt{t_0}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(4\pi t_0 \sqrt{n} - \frac{\pi}{4}) + O(T^\epsilon) \\ &= \frac{\sqrt{t_1} + O\left(\frac{h}{\sqrt{t_1}}\right)}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(4\pi t_0 \sqrt{n} - \frac{\pi}{4}) + O(T^\epsilon) \\ &= \frac{\sqrt{t_1}}{\pi\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4}}} \cos(4\pi t_0 \sqrt{n} - \frac{\pi}{4}) + O(hT^\epsilon) \end{aligned}$$

where we used (5.20) in the last step. Therefore,

$$\begin{aligned}
& \int_{t_1-h}^{t_1+h} \Delta(t_0^2) dt_0 \\
&= \frac{\sqrt{t_1}}{4\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \left(\sin \left(4\pi(t_1+h)\sqrt{n} - \frac{\pi}{4} \right) - \sin \left(4\pi(t_1-h)\sqrt{n} - \frac{\pi}{4} \right) \right) + O(h^2 T^\epsilon) \\
&= \frac{\sqrt{t_1}}{2\pi^2\sqrt{2}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{5}{4}}} \sin(4\pi h\sqrt{n}) \cos \left(4\pi t_1\sqrt{n} - \frac{\pi}{4} \right) + O(h^2 T^\epsilon).
\end{aligned}$$

Suppose we have (5.18), for some $k \geq 1$, i. e.

$$\begin{aligned}
& \int_{t_k-h}^{t_k+h} \int_{t_{k-1}-h}^{t_{k-1}+h} \cdots \int_{t_2-h}^{t_2+h} \int_{t_1-h}^{t_1+h} \Delta(t_0^2) dt_0 dt_1 \cdots dt_{k-2} dt_{k-1} = \\
& \quad \frac{1}{2^k \pi^{k+1} \sqrt{2}} \sqrt{t_k} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h\sqrt{n}) \cos \left(4\pi t_k\sqrt{n} - \frac{\pi}{4} \right) + O(h^{k+1} T^\epsilon)
\end{aligned}$$

We are going to prove that the above implies

$$\begin{aligned}
& \int_{t_{k+1}-h}^{t_{k+1}+h} \left(\int_{t_k-h}^{t_k+h} \int_{t_{k-1}-h}^{t_{k-1}+h} \cdots \int_{t_2-h}^{t_2+h} \int_{t_1-h}^{t_1+h} \Delta(t_0^2) dt_0 dt_1 \cdots dt_{k-2} dt_{k-1} \right) dt_k = \\
& \quad \frac{1}{2^{k+1} \pi^{k+2} \sqrt{2}} \sqrt{t_{k+1}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k+1}{2}}} \sin^{k+1}(4\pi h\sqrt{n}) \cos \left(4\pi t_{k+1}\sqrt{n} - \frac{\pi}{4} \right) + O(h^{k+2} T^\epsilon)
\end{aligned}$$

Now, using (5.19) and (5.20),

$$\begin{aligned}
& \int_{t_{k+1}-h}^{t_{k+1}+h} \left(\sqrt{t_k} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h\sqrt{n}) \cos \left(4\pi t_k\sqrt{n} - \frac{\pi}{4} \right) \right) dt_k \\
&= \int_{t_{k+1}-h}^{t_{k+1}+h} \left(\sqrt{t_{k+1}} \sum_{n \leq N} \left(\frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h\sqrt{n}) \cos \left(4\pi t_k\sqrt{n} - \frac{\pi}{4} \right) \right) + o(1) \right) dt_k
\end{aligned}$$

Since,

$$\begin{aligned}
& \int_{t_{k+1}-h}^{t_{k+1}+h} \cos \left(4\pi t_k\sqrt{n} - \frac{\pi}{4} \right) dt_k \\
&= \frac{1}{4\pi\sqrt{n}} \left(\sin \left(4\pi(t_{k+1}+h)\sqrt{n} - \frac{\pi}{4} \right) - \sin \left(4\pi(t_{k+1}-h)\sqrt{n} - \frac{\pi}{4} \right) \right) \\
&= \frac{1}{2\pi\sqrt{n}} \sin(4\pi h\sqrt{n}) \cos \left(4\pi t_{k+1}\sqrt{n} - \frac{\pi}{4} \right)
\end{aligned}$$

we obtain

$$\int_{t_{k+1}-h}^{t_{k+1}+h} \left(\int_{t_k-h}^{t_k+h} \int_{t_{k-1}-h}^{t_{k-1}+h} \cdots \int_{t_2-h}^{t_2+h} \int_{t_1-h}^{t_1+h} \Delta(t_0^2) dt_0 dt_1 \cdots dt_{k-2} dt_{k-1} \right) dt_k =$$

$$\frac{1}{2^{k+1} \pi^{k+2} \sqrt{2}} \sqrt{t_{k+1}} \sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k+1}{2}}} \sin^{k+1}(4\pi h \sqrt{n}) \cos\left(4\pi t_{k+1} \sqrt{n} - \frac{\pi}{4}\right) + O(h^{k+2} T^\epsilon)$$

Hence, by induction, (5.18) is true for all $k \geq 1$. $\square$

We are in conditions of obtaining a different proof that $N_\Delta(T) \gg T^{\frac{1}{2}}$.

Theorem 1.4. *Let $N_\Delta(T)$ denote the number of sign changes of $\Delta(t)$, in the interval $[T, 2T]$. Then, for sufficiently large T , $N_\Delta(T) > \sqrt{T}$. Moreover, there exists a constant c_1 , and $t_1, t_2 \in [T, T + \sqrt{T}]$ such that $\Delta(t_1) \leq -c_1 T^{\frac{1}{4}}$ and $\Delta(t_2) \geq c_1 T^{\frac{1}{4}}$.*

Proof: For h, k, N as in the previous lemma and $t \geq T$, define $g(t, k)$ by

$$F(t, k) = \frac{1}{2^k \pi^{k+1} \sqrt{2}} \sqrt{t} \left(\sum_{n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h \sqrt{n}) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) + g(t, k) \right).$$

Notice that, $g(t, k) \ll h^{k+1} T^{-\frac{1}{2} + \epsilon}$. It is well known [29, theorem 289] that

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}, \quad \text{for } s > 1.$$

Take $\delta > 0$ small and, for fixed h and k (we will later take explicit values for h and k), take T sufficiently large so that, for any $t \geq T$, $|g(t, k)| < \delta$. Then, for any real $t \geq T$,

$$\left| \sum_{2 \leq n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h \sqrt{n}) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) + g(t, k) \right| \leq \zeta^2\left(\frac{3}{4} + \frac{k}{2}\right) - 1 + |g(t, k)|$$

$$< \zeta^2\left(\frac{3}{4} + \frac{k}{2}\right) - 1 + \delta$$

Since,

$$\frac{2^k \pi^{k+1} \sqrt{2}}{\sqrt{t}} F(t, k) =$$

$$\sin^k(4\pi h) \cos\left(4\pi t - \frac{\pi}{4}\right) + \sum_{2 \leq n \leq N} \frac{\tau(n)}{n^{\frac{3}{4} + \frac{k}{2}}} \sin^k(4\pi h \sqrt{n}) \cos\left(4\pi t \sqrt{n} - \frac{\pi}{4}\right) + g(t, k)$$

we want to find h and k such that $\sin^k(4\pi h) > \zeta^2\left(\frac{3}{4} + \frac{k}{2}\right) - 1 + \delta$ and hk to take the smallest possible value. This happens for $h = 0.084901\dots$ and $k = 4$. As we will see below, there is no lost if we take $h = \frac{1}{11}$ instead. Notice that, $\zeta^2\left(\frac{11}{4}\right) - 1 = 0.588089\dots$ and $\sin^4\left(\frac{4}{11}\pi\right) = 0.684641\dots$. Take $x_0 = [T] + 1 + \frac{1}{16}$ and $x_i = x_0 + \frac{i}{4}$, for any $i \geq 1$. Then

$$\cos\left(4\pi x_i - \frac{\pi}{4}\right) = (-1)^i.$$

Therefore, $F(x_i, 4) > 0$ if i is even and $F(x_i, 4) < 0$ if i is odd, i. e. $F(t, 4)$ changes sign for $t = t_k \in (x_i, x_{i+1})$. This implies that $\Delta(t_0^2)$ changes sign when

$$t_0 \in (x_i - hk, x_{i+1} + hk) = \left(x_i - \frac{4}{11}, x_i + \frac{27}{44}\right),$$

and so, $\Delta(t)$ changes sign in $\left(x_i^2 - \frac{8}{11}x_i + \frac{16}{121}, x_i^2 + \frac{27}{22}x_i + \frac{729}{1936}\right)$. Take only the x_i such that $i \equiv 0 \pmod{4}$, in this way, the intervals will be disjoint, since

$$\begin{aligned} (x_i + 1)^2 - \frac{8}{11}(x_i + 1) + \frac{16}{121} &= x_i^2 + \frac{14}{11}x_i + \frac{49}{121} \\ &> x_i^2 + \frac{27}{22}x_i + \frac{729}{1936} \end{aligned}$$

We proved that there are $T + O(1)$ changes of sign in the interval $(T^2, 2T^2)$. Therefore $X_\Delta(T) > \sqrt{T} + O(1)$.

Let $c = \frac{1}{16\pi^5\sqrt{2}}$. Now, we prove the second part of the theorem take c_1 sufficiently small (it's enough to take $c_1 < 0.09655c$) and T sufficiently large such that $|g(t, 4)| + \frac{c_1}{c} < \delta$, for all $t \geq T$. Take $\alpha = \sin^4\left(\frac{4}{11}\pi\right)$, then

$$c\sqrt{t} \left(\alpha \cos\left(4\pi t - \frac{\pi}{4}\right) + \sum_{2 \leq n \leq N} \frac{\tau(n)}{n^{\frac{11}{4}}} \sin^4\left(\frac{\pi}{2}\sqrt{n}\right) \cos\left(4\pi t\sqrt{n} - \frac{\pi}{4}\right) + g(t, 4) \pm \frac{c_1^2}{c} \right)$$

changes sign depending only on $\cos\left(4\pi t - \frac{\pi}{4}\right)$ as $F(t, 4)$ above. Hence, $\Delta(t) \pm c_1 T^{\frac{1}{4}}$ also changes signs in

$$\left(x_i^2 - \frac{8}{11}x_i + \frac{16}{121}, x_i^2 + \frac{27}{22}x_i + \frac{729}{1936}\right),$$

for any i . Therefore, for every $0 \leq j \leq \sqrt{T}$, exists $t_1, t_2 \in [T + j\sqrt{T}, T + (j+1)\sqrt{T}]$ such that $\Delta(t_1) \leq -c_1 T^{\frac{1}{4}}$ and $\Delta(t_2) \geq c_1 T^{\frac{1}{4}}$. Since $\Delta(t)$ changes at most $\log t$ in intervals of the form $[n, n+1)$ (see (5.1)), we have $N_\Delta(T) > \sqrt{T} + O(1)$ $\square$

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