COHOMOLOGY AND SUPPORT VARIETIES

FOR LIE SUPERALGEBRAS

by

Irfan Bagci

(Under the direction of Daniel K. Nakano)

Abstract

Let $\mathfrak{g}$ be a Lie superalgebra over the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{t}$ be a Lie subsuperalgebra of $\mathfrak{g}$. In this thesis, we determine necessary conditions to identify the relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ with a ring of invariants.

In this work we calculate the cohomology rings of the Lie superalgebras $W(n)$ and $\bar{S}(n)$ relative to degree zero components $W(n)_0$ and $\bar{S}(n)_0$, respectively. The crucial point is that these rings are finitely generated polynomial rings. Finite generation of these cohomology rings allows one to define support varieties for finite dimensional $W(n)$ (resp. $\bar{S}(n)$)-modules which are completely reducible over $W(n)_0$ (resp. over $\bar{S}(n)_0$). We calculate the support varieties of all simple modules in these categories. Remarkably our computations coincide with the prior notion of atypicality for Cartan type Lie superalgebras due to Serganova. We also present new results on the realizability of support varieties which hold for both classical and Cartan type Lie superalgebras.

Index words: Lie superalgebra, cohomology, support varieties, rank varieties

COHOMOLOGY AND SUPPORT VARIETIES FOR LIE SUPERALGEBRAS

by

IRFAN BAGCI

B.S., Abant Izzet Baysal University, 1999

M.S., Abant Izzet Baysal University, 2002

M.A., The University of Georgia, 2006

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2009

COHOMOLOGY AND SUPPORT VARIETIES
FOR LIE SUPERALGEBRAS

by

IRFAN BAGCI

Approved:

Major Professor: Daniel K. Nakano

Committee: Brian D. Boe
Mitchell Rothstein

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2009

Dedication

This dissertation is dedicated to my family.

Acknowledgments

I would like to express my gratitude to my Ph.D. advisor Daniel Nakano. His understanding, encouragement, support, and advice have been invaluable throughout my graduate studies. I would also like to thank Jonathan Kujawa for his help, support, and advice. My appreciation also goes to Edward Azoff, Brian Boe, Jon Carlson, Ben Jones, and Robert Varley for their help during my studies at UGA.

Table of Contents

Chapter 1

Introduction

1.1 History

Representation theory is concerned with realizing a group or an algebra as a collection of matrices. In this way, one can understand the way the group or algebra acts linearly on a vector space, where the action respects the operations in the group or algebra. In the process, one is able to understand more completely the structure of the group or algebra. Representation theory has found applications in many areas, particularly where symmetry arises. These areas include physics, chemistry and mathematics itself.

Suppose that $\mathfrak{g}$ is a finite dimensional Lie superalgebra over the field $\mathbb{C}$ of complex numbers. If V is a superspace on which there is an action of $\mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and respects the bracket in $\mathfrak{g}$, then V is called a $\mathfrak{g}$-module. One often discusses representations of $\mathfrak{g}$ via the equivalent language of $\mathfrak{g}$-modules.

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called classical if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$. In [BKN1] Boe, Kujawa and Nakano initiated a study of the representation theory of classical Lie superalgebras via a cohomological approach. Let us first summarize what is known for the classical Lie superalgebras as given in [BKN1, BKN2]. The first fundamental result is that the relative cohomology ring $\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is a finitely generated commutative ring. Note that this result crucially depends on the reductivity of $\mathfrak{g}_{\bar{0}}$. By applying invariant theory results in [LR] and [DK], it was shown under mild conditions that a natural "detecting" subalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ of $\mathfrak{g}$ arises such that the restriction map in cohomology induces an isomorphism

$$R := \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W,$$

where W is a finite pseudoreflection group. The vector space dimension of the degree $\bar{1}$ part of the detecting subalgebra and the Krull dimension of R both coincide with the combinatorial notion of the defect of $\mathfrak{g}$ previously introduced by Kac and Wakimoto [KW]. The fact that R is finitely generated can be employed to define the cohomological support varieties $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ for any finite dimensional $\mathfrak{g}$-module M. The variety $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ can be identified as a certain subvariety of $\mathfrak{e}_{\bar{1}}$ using a rank variety description [BKN1, Theorem 6.3.2]. For the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ the support varieties of all finite dimensional simple modules were computed in [BKN2]. A remarkable consequence of this calculation is that the dimensions of the support varieties of a given simple module (over $\mathfrak{g}$ or $\mathfrak{e}$) coincide with the combinatorially defined degree of atypicality of the highest weight as defined by Kac and Serganova.

Duflo and Serganova introduce associated varieties for finite dimensional modules of a Lie superalgebra in [DS]. It remains unclear what connection, if any, exists between their work and the cohomological support varieties considered in [BKN1, BKN2] and here.

1.2 Outline of the dissertation

Chapter 2 is concerned with preliminaries. In Section 2.1 we recall the notions of Lie superalgebra, classical Lie superalgebra, and PBW Theorem. In Sections 2.2 and 2.3 we review basic properties of representation theory for Lie superalgebras, we also record some facts about the representation theory of rank one Lie superalgebras. In Section 2.4 we recall Kac classification of simple Lie superalgebras over the complex numbers and we briefly define these algebras. In the remaining sections of this chapter we discuss the representation theory and atypicality for Lie superalgebras $W(n)$ and $\bar{S}(n)$.

Chapter 3 is concerned with cohomology for Lie superalgebras. In this chapter we review the relative cohomology for Lie superalgebras and record the properties we are going to need in the rest. After that we give necessary conditions to identify relative cohomology ring with a ring of invariants and discuss when the ring of invariants will be finitely generated. We also introduce the notion of a detecting subalgebra for a Lie superalgebra.

Chapter 4 is concerned with varieties for Lie superalgebras. Here we associate varieties which are called support varieties to modules for Lie superalgebras. These varieties are affine conical varieties. We prove a theorem which is called realization theorem in the theory of support varieties. This theorem basically says that we can realize any conical subvariety as variety of some module.

The real substance of this work appears in Chapter 5. In this chapter we give the first application of the abstract theory defined in Chapters 3 and 4. Here we first identify the relative cohomology ring for the simple Cartan Lie superalgebra $\mathfrak{g} = W(n)$ relative to degree zero component $W(n)_0$ in the $\mathbb{Z}$-grading. This has been done by using detailed representation theory of $\mathfrak{g}_0 = W(n)_0$ which is isomorphic to $\mathfrak{gl}(n)$ as a Lie algebra. After identification of the relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ with a ring of invariants the next step is to compute the ring of invariants. This has been done by applying invariant theoretic results due to Luna and Richardson. In particular we show that $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ is a polynomial ring in $n-1$ variables. We also prove that for any finite dimensional $\mathfrak{g}$-module M, $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M)$ is a finitely generated $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ module. By using this finite generation result we define support varieties for finite dimensional modules. With the aid of results of Serganova on representations of $\mathfrak{g}$ we compute support varieties of all finite dimensional $\mathfrak{g}$ modules which are completely reducible over $\mathfrak{g}_0$. We also show that the realization theorem proven in Chapter 4 holds for $\mathfrak{g}$.

In Chapter 6 we give a second application. Here following the work we have done for $W(n)$ we compute the relative cohomology ring for the Lie superalgebra $\mathfrak{g} = \bar{S}(n)$ relative to the degree zero component $\mathfrak{g} = \bar{S}(n)_0$ which is isomorphic to $\mathfrak{gl}(n)$ as a Lie algebra. In particular we show that $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ is a polynomial ring with $n-2$ variables. By using finite generation theorem we defined support varieties for finite dimensional $\mathfrak{g}$-modules. By using the fact that a simple module of $W(n)$ is typical if and only if its superdmension is zero and results of Serganova on representations of $\mathfrak{g}$, we were able to compute support varieties of all finite dimensional simple modules which are completely reducible over $\mathfrak{g}_0$. We also show that the realization theorem proven in Chapter 4 holds for $\mathfrak{g}$.

In Chapter 7 we discuss the connections between combinatorial notions defect and atypicality defined for only basic classical Lie superalgebras introduced by Kac and Serganova and our cohomology and support variety results for Lie superalgebras. Our cohomological calculations for $W(n)$ and $\bar{S}(n)$ show that we can extend the notions of defect and atypicality to $W(n)$ and $\bar{S}(n)$. New definitions of defect and atypicality agree with the earlier definitions for basic classical Lie superalgebras and also the computations we have done show that the Kac-Wakimoto conjecture holds for $W(n)$ and $\bar{S}(n)$.

Finally, in Chapter 8, we discuss the connections between the representation theory of Lie superalgebras and our support variety constructions. We present a conjecture relating representation type for a Lie superalgebra to our support variety construction.

Chapter 2

Preliminaries

In this chapter we survey some necessary definitions and set some notation which will be used throughout. Most of the arguments and details can be found in [Kac, Sch].

2.1 Lie Superalgebras

In this section, we introduce Lie superalgebras and the properties of them we will need in the subsequent chapters.

Lie superalgebras are a topic of interest in physics in the context of supersymmetry [CNS]. Physicists call them $\mathbb{Z}_2$-graded Lie algebras but they are not Lie algebras. Lie superalgebras occur in several cohomology theories, for example in deformation theory [CNS, MM].

In this thesis we will work over the field $\mathbb{C}$ of complex numbers.

Definition 2.1.1. A *superspace* is a $\mathbb{Z}_2$-graded vector space and, given a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and a homogeneous vector $v \in V$, we write $\overline{v} \in \mathbb{Z}_2$ for the *parity* (or *degree*) of v. Elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are called *even* (resp. *odd*).

Example 2.1.2. If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $W = W_{\bar{0}} \oplus W_{\bar{1}}$ are two superspaces, then the space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is naturally $\mathbb{Z}_2$-graded by $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)_r$ $(r \in \mathbb{Z}_2)$ if $f(V_s) \subseteq W_{s+r}$ for all $s \in \mathbb{Z}_2$.

Definition 2.1.3. A *superalgebra* is a $\mathbb{Z}_2$-graded, unital, associative algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ which satisfies $A_r A_s \subseteq A_{r+s}$ for all $r, s \in \mathbb{Z}_2$.

Example 2.1.4. Let $\Lambda(n)$ be the exterior algebra in n variables $\xi_1, \ldots, \xi_n$. Then $\Lambda(n)$ becomes a superalgebra if one sets $\overline{\xi_i} = \overline{1}$, for $i = 1, \ldots, n$.

Definition 2.1.5. A *Lie superalgebra* is a finite dimensional superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

(1) $[\mathfrak{g}_r, \mathfrak{g}_s] \subseteq \mathfrak{g}_{r+s}$,

(2) $[x,y] = -(-1)^{\overline{x}\,\overline{y}}[y,x]$,

(3) $(-1)^{\overline{x}\,\overline{z}}[x,[y,z]] + (-1)^{\overline{x}\,\overline{y}}[y,[z,x]] + (-1)^{\overline{y}\,\overline{z}}[z,[x,y]] = 0$,

for all $r, s \in \mathbb{Z}_2$ and homogeneous $x, y, z \in \mathfrak{g}$.

Observe that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action.

Example 2.1.6. Given any superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ one can define the super commutator on homogeneous elements by

$$[a,b] = ab - (-1)^{\overline{a}\,\overline{b}} ba$$

and then extending by linearity to all elements. The algebra A together with the super commutator then becomes a Lie superalgebra.

Example 2.1.7. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$ graded vector space. Then the associative algebra $\mathfrak{gl}(V)$ of endomorphisms of V has a natural $\mathbb{Z}_2$-grading as follows:

$$\mathfrak{gl}(V)_{\bar{0}} = \{f \in \mathfrak{gl}(V) \mid f(V_{\bar{r}}) \subseteq V_{\bar{r}}, \bar{r} \in \mathbb{Z}_2\},$$

$$\mathfrak{gl}(V)_{\bar{1}} = \{f \in \mathfrak{gl}(V) \mid f(V_{\bar{r}}) \subseteq V_{\overline{r+1}}, \bar{r} \in \mathbb{Z}_2\}.$$

The algebra $\mathfrak{gl}(V)$ becomes a Lie superalgebra with the super commutator bracket.

Definition 2.1.8. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called *classical* if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

Example 2.1.9. Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ be the set of all $(m+n) \times (m+n)$ matrices over the complex numbers. If we write $E_{i,j}$ $(1 \leq i, j \leq m+n)$ for the matrix unit with a one in the (i,j) position, then the $\mathbb{Z}_2$-grading is obtained by setting $\overline{E_{i,j}} = \bar{0}$ if $1 \le i,j \le m$ or $m+1 \le i,j \le m+n$, and $\overline{E_{i,j}} = \bar{1}$, otherwise. That is, let

$$\mathfrak{g}_{\bar{0}} = \left(\begin{array}{c|c} A_{m\times m} & O \\ \hline O & B_{n\times n} \end{array}\right) \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \left(\begin{array}{c|c} O & C_{m\times n} \\ \hline D_{n\times m} & O \end{array}\right).$$

The bracket is given by super commutator,

$$[X,Y] = XY - (-1)^{\overline{X}\ \overline{Y}} YX,$$

for homogeneous $X, Y \in \mathfrak{gl}(m|n)$. Then $\mathfrak{gl}(m|n)$ is a classical Lie superalgebra with $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \times \mathfrak{gl}(n)$.

A superspace $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ is called a *subalgebra* of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ if $\mathfrak{a}_{\bar{0}} = \mathfrak{a} \cap \mathfrak{g}_{\bar{0}}$, $\mathfrak{a}_{\bar{1}} = \mathfrak{a} \cap \mathfrak{g}_{\bar{1}}$ and $\mathfrak{a}$ is closed under the bracket. A subalgebra I of a Lie superalgebra $\mathfrak{g}$ is an *ideal* if $[I, \mathfrak{g}] \subseteq I$. Homomorphisms (isomorphisms, automorphisms) of Lie superalgebras are always assumed to be consistent with the $\mathbb{Z}_2$-grading, i.e., they are homogeneous linear mappings of degree 0.

Definition 2.1.10. A *universal enveloping superalgebra* of the Lie superalgebra $\mathfrak{g}$ is a superalgebra $U(\mathfrak{g})$ together with a linear map $i : \mathfrak{g} \to U(\mathfrak{g})$ satisfying

$$i([x,y]) = xy - (-1)^{\overline{x}\ \overline{y}} yx \tag{2.1}$$

and such that for any other superalgebra A and any linear map $i' : \mathfrak{g} \to A$ satisfying (2.1), there is a unique homomorphism of superalgebras $f : U(\mathfrak{g}) \to A$ such that $f \circ i = i'$ and $f(1) = 1$.

We can construct $U(\mathfrak{g})$ as follows: Let $T(\mathfrak{g})$ be the tensor superalgebra over the space $\mathfrak{g}$ with the induced $\mathbb{Z}_2$-grading, and I be the ideal of $T(\mathfrak{g})$ generated by elements of the form $a \otimes b - (-1)^{\bar{a}\,\bar{b}} b \otimes a - [a,b]$. Set $U(\mathfrak{g}) = T(\mathfrak{g})/I$. One can construct a basis for $U(\mathfrak{g})$ using the following result.

Theorem 2.1.11. ***(PBW Theorem)*** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, $a_1, \ldots, a_m$ be an ordered basis of $\mathfrak{g}_{\bar{0}}$ and $b_1, \ldots, b_n$ be a basis of $\mathfrak{g}_{\bar{1}}$. Then the vectors of the form*

$$a_1^{i_1} \ldots a_m^{i_m} b_1^{j_1} \ldots b_n^{j_n} \quad i_k \in \mathbb{Z}_+ \ \forall \ 1 \leq k \leq m, \ 0 \leq j_l \leq 1 \ \ \forall \ \ 1 \leq l \leq n$$

form a basis for $U(\mathfrak{g})$.

2.2 Representation theory of Lie superalgebras

Definition 2.2.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$ graded vector spcae. A *representation* ρ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ in V is a Lie superalgebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. The space V is called a $\mathfrak{g}$-module.

We have that V is a $U(\mathfrak{g})$-module if and only if V is a $\mathfrak{g}$-module. A *simple* $\mathfrak{g}$-module is a $\mathfrak{g}$-module $V \neq 0$ whose only submodules are V and 0.

Definition 2.2.2. Let M be a finite dimensional $\mathfrak{g}$-module. The *superdimension* of M is defined to be $\operatorname{sdim} M = \dim M_{\bar{0}} - \dim M_{\bar{1}}$.

Example 2.2.3. If $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra , we have

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined by

$$\operatorname{ad}_x(y) = [x, y].$$

One can easily check that ad is a Lie superalgebra homomorphism and is, therefore, a representation of $\mathfrak{g}$, called the *adjoint representation.*

The restriction of ad to $\mathfrak{g}_{\bar{0}}$ is a representation of $\mathfrak{g}_{\bar{0}}$ in the vector space $\mathfrak{g}$. Since the subspaces $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ are invariant under this representation the adjoint representation of the Lie superalgebra induces a representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ in the odd space $\mathfrak{g}_{\bar{1}}$.

Definition 2.2.4. A module M is called *finitely semisimple* if it is isomorphic to a direct sum of finite dimensional simple submodules.

Lemma 2.2.5. ***(Schur's Lemma)*** *Let $\mathfrak{g}$ be a Lie superalgebra and let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite dimensional simple $\mathfrak{g}$-module. Then*

$$\operatorname{Hom}_{\mathfrak{g}}(V,V)_{\bar{0}} = \mathbb{C}.Id, \quad \operatorname{Hom}_{\mathfrak{g}}(V,V)_{\bar{1}} = \mathbb{C}.u$$

where either $u = 0$ or else $u^2 = Id$.

2.3 Rank one Lie superalgebras

The definition of rank varieties depends on restricting to rank one Lie superalgebras. Here we summarize some basic results on finite dimensional representations of these Lie superalgebras. As a background source we refer the reader to [BKN1, Section 5.2].

Let $\mathfrak{s}$ be a Lie superalgebra generated by a single odd dimensional vector, x. There are the following two possibilities for $\mathfrak{s}$.

Case I. We have $[x,x] = 0$. Then $\dim(\mathfrak{s}_{\bar{0}}) = 0$, $\dim(\mathfrak{s}_{\bar{1}}) = 1$, and $\mathfrak{s}$ is a one dimensional abelian Lie superalgebra concentrated in degree $\bar{1}$.

Case II. We have $h := [x,x] \neq 0$. Then $\dim(\mathfrak{s}_{\bar{0}}) = 1$, $\dim(\mathfrak{s}_{\bar{1}}) = 1$, and $\mathfrak{s}$ is isomorphic to the Lie superalgebra $\mathfrak{q}(1)$ (see [Kac]).

Proposition 2.3.1. *The following statements about the category of finite dimensional $\mathfrak{s}$-modules hold.*

(a) If $\mathfrak{s}$ is as in Case I, then

- *(i) the trivial module $L(0) := \mathbb{C}$ is the only simple module.*
- *(ii) the projective cover of $L(0)$ is $P(0) := U(\mathfrak{s})$.*
- *(iii) the module $P(0)$ is self dual, hence injective.*
- *(iv) the set $\{P(0), L(0)\}$ is a complete set of indecomposable modules.*

(b) Let $\mathfrak{s}$ be as in Case II. Given $\lambda \in \mathbb{C}$, let $P(\lambda) := U(\mathfrak{s}) \otimes_{U(\mathfrak{s}_{\bar{0}})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ denotes $\mathbb{C}$ viewed as an $\mathfrak{s}_{\bar{0}}$-module concentrated in degree $\bar{0}$ and h acts by the scalar λ. Let $L(\lambda)$ denote the head of $P(\lambda)$.

(i) The set $\{L(\lambda) \mid \lambda \in \mathbb{C}\}$ *is a complete set of simple* $\mathfrak{s}$*-modules.*

(ii) For all $\lambda \in \mathbb{C}$*,* $P(\lambda)$ *is the projective cover of* $L(\lambda)$*.*

(iii) If $\lambda \neq 0$*, then* $L(\lambda) = P(\lambda)$*.*

(iv) For all $\lambda \in \mathbb{C}$*,* $P(\lambda)$ *is dual to some* $P(\mu)$*, hence injective.*

(v) The set $\{L(\lambda) \mid \lambda \in \mathbb{C}\} \cup \{P(0)\}$ *is a complete set of indecomposable modules.*

We remark that in both cases the modules $P(\lambda)$ *always satisfy*

$$\dim(P(\lambda)_{\bar{0}}) = \dim(P(\lambda)_{\bar{1}}) = 1.$$

In particular, $P(\lambda)$ *is always two dimensional.*

2.4 Simple Lie superalgebras

A Lie superalgebra $\mathfrak{g}$ is said to be *simple* if its only ideals are $\{0\}$ and $\mathfrak{g}$. Simple Lie superalgebras were classified in the 1970s by V. Kac. These superalgebras can be divided into three groups.

1. *Basic classical Lie superalgebras*, i.e., Lie superalgebras which can be determined by a Cartan matrix. These superalgebras have an invariant form and Cartan involution. There are two families of such algebras $\mathfrak{sl}(m|n)$ (factored by center when $m = n$) and $\mathfrak{osp}(m|n)$. The Lie superalgebra $\mathfrak{osp}(4|2)$ has a one-parameter deformation, called $D(\alpha)$. There are also two exceptional Lie superalgebras G_3 and F_4.

2. *Strange Lie superalgebras* $Q(n)$ and $P(n)$, the former consists of operators commuting with an odd nondegenerate operator, the latter consists of operators preserving a nondegenerate odd symmetric form.

3. *Cartan Lie superalgebras* $W(n), S(n), \tilde{S}(n)$ and $H(n)$, i.e., superalgebras of vector fields on a supermanifold of pure odd dimension and its simple subalgebras.

2.5 Representation theory and atypicality for $W(n)$ and $\bar{S}(n)$

In this section we briefly define Lie superalgebras $W(n)$ and $\bar{S}(n)$. For detailed definitions we refer the reader to Chapters 5 and 6, respectively.

Let $\Lambda(n)$ denote the exterior algebra on n odd generators $\xi_1, \ldots, \xi_n$. Then $W(n)$ is the Lie superalgebra of derivations of $\Lambda(n)$. An element $D \in W(n)$ can be written in the form $\sum_{i=1}^{n} f_i \partial_i$, where $f_i \in \Lambda(n)$ and ∂_i is the derivation defined by

$$\partial_i(\xi_j) = \delta_{ij}.$$

The $\mathbb{Z}$-grading on $\Lambda(n)$ induces a $\mathbb{Z}$-grading on $W(n)$ where ∂_i has degree -1, so that

$$W(n) = W(n)_{-1} \oplus W(n)_0 \oplus \cdots \oplus W(n)_{n-1}$$

and $W(n)_0$ is isomorphic to $\mathfrak{gl}(n)$.

The superalgebra $S(n)$ is the subalgebra of $W(n)$ consisting of all elements $D \in W(n)$ such that $\mathrm{div}(D) = 0$, where

$$\mathrm{div}(\sum_{i=1}^{n} f_i \partial_i) = \sum_{i=1}^{n} \partial_i(f_i).$$

The superalgebra $S(n)$ has a $\mathbb{Z}$-grading induced by the grading of $W(n)$

$$S(n) = S(n)_{-1} \oplus S(n)_0 \oplus \cdots \oplus S(n)_{n-2}$$

and $S(n)_0$ is isomorphic to $\mathfrak{sl}(n)$.

Let $\mathcal{E} = \Sigma_{i=1}^{n} \xi_i \partial_i$. In order to keep track of the $\mathbb{Z}$-grading we will attach $\mathcal{E}$ to $S(n)$ and consider the subalgebra $\bar{S}(n) = S(n) \oplus \mathbb{C}\mathcal{E}$ of $W(n)$.

Let $\mathfrak{g} = W(n)$ or $\bar{S}(n)$ with a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_s.$$

Moreover, let $\mathfrak{g}^+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ and $\mathfrak{g}^- = \mathfrak{g}_{-1}$ so that $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^+.$$

Throughout $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ will denote the category of $\mathfrak{g}$-modules which are finitely semisimple as $\mathfrak{g}_0$-module.

2.6 Kac modules and Finite Dimensional Simple modules

Let $\mathfrak{g} = W(n)$ or $\bar{S}(n)$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincides with a Cartan subalgebra of $\mathfrak{g}_0$. Fix a maximal torus $\mathfrak{h} \subseteq \mathfrak{g}_0$ and a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$. Let $X_0^+ \subset \mathfrak{h}^*$ denote the parametrizing set of highest weights for the simple finite dimensional $\mathfrak{g}_0$-modules with respect to the pair $(\mathfrak{h}, \mathfrak{b}_0)$ and let $L_0(\lambda)$ denote the simple finite dimensional $\mathfrak{g}_0$-module with highest weight $\lambda \in X_0^+$. We view $L_0(\lambda)$ as a $\mathfrak{g}_0$-module concentrated in degree $\bar{0}$.

The *Kac module* $K(\lambda)$ is the induced representation of $\mathfrak{g}$,

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}^+)} L_0(\lambda),$$

where $L_0(\lambda)$ is viewed as a $\mathfrak{g}_0 \oplus \mathfrak{g}^+$ via inflation through the canonical quotient map $\mathfrak{g}_0 \oplus \mathfrak{g}^+ \to \mathfrak{g}_0$. By the PBW theorem for Lie superalgebras the module $K(\lambda)$ is a finite dimensional indecomposable object in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$. With respect to the choice of Borel subalgebra $\mathfrak{b}_0 \oplus \mathfrak{g}^+ \subseteq \mathfrak{g}$ one has a dominance order on weights. With respect to this ordering $K(\lambda)$ has highest weight λ and, therefore, a unique simple quotient which we denote by $L(\lambda)$. Conversely, every finite dimensional simple module appears as the head of some Kac module (cf. [Ser, Theorem 3.1]).

From our discussion above one observes that the set

$$\{L(\lambda) \mid \lambda \in X_0^+\}$$

is a complete irredundant collection of simple finite dimensional $\mathfrak{g}$-modules.

2.7 Root Decomposition

Let $\mathfrak{g} = W(n)$ or $\bar{S}(n)$. Recall from the previous section that we fixed a maximal torus $\mathfrak{h} \subseteq \mathfrak{g}_0 \subseteq \mathfrak{g}$. With respect to this choice we have a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Many properties of root decompositions for semisimple Lie algebras do not hold in our case. For example, a root can have multiplicity bigger than one, and $\alpha \in \Phi$ does not imply that

$-\alpha \in \Phi$. Still any root space $\mathfrak{g}_\alpha$ is concentrated in either degree $\bar{0}$ or degree $\bar{1}$ and in this way one can define a natural parity function on roots.

Let us describe the roots. If $\mathfrak{g} = W(n)$, choose the standard basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathfrak{h}^*$ where $\varepsilon_i(\xi_j \partial_j) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. Then the root system of $\mathfrak{g}$ is the set

$$\Phi = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < \cdots < i_k \leq n,\ 1 \leq j \leq n\}.$$

If $\mathfrak{g} = \bar{S}(n)$, then the root system is the set

$$\Phi = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < \cdots < i_k < n,\ 1 \leq j \leq n\}.$$

In either case the set of simple roots for $\mathfrak{g}$ is

$$\triangle = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n\}.$$

2.8 Typical and Atypical Weights

We consider Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{b}_0$. Among such subalgebras we distinguish $\mathfrak{b}_{\max} = \mathfrak{b}_0 \oplus \mathfrak{g}^+$ and $\mathfrak{b}_{\min} = \mathfrak{b}_0 \oplus \mathfrak{g}^-$. Let $\mathfrak{b}$ denote either $\mathfrak{b}_{\max}$ or $\mathfrak{b}_{\min}$. Then $\lambda \in \mathfrak{h}^*$ defines a one dimensional representation of $\mathfrak{b}$ which we denote by $\mathbb{C}_\lambda$. The induced module $M^{\mathfrak{b}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ has a unique proper maximal submodule. We denote the unique irreducible quotient by $L^{\mathfrak{b}}(\lambda)$. In particular, if $\lambda \in X_0^+$, then $L(\lambda) \cong L^{\mathfrak{b}_{\max}}(\lambda)$.

Denote by λ' the weight such that $L^{\mathfrak{b}_{\min}}(\lambda') \cong L(\lambda)$. Let $\Phi(\mathfrak{g}_{-1})$ be the set of roots which lie in $\mathfrak{g}_{-1}$. By Serganova [Ser, Lemma 5.1],

$$\lambda' = \lambda + \sum_{\alpha \in \Phi(\mathfrak{g}_{-1})} \alpha \tag{2.2}$$

for a Zariski open set of $\lambda \in \mathfrak{h}^*$. Following Serganova, we call $\lambda \in \mathfrak{h}^*$ *typical* if (2.2) holds for λ and otherwise λ is *atypical*. Serganova determines a necessary and sufficient combinatorial condition for λ to be typical. Namely, by [Ser, Lemma 5.3] one has that the set of atypical weights Ω for $\mathfrak{g} = W(n)$ is

$$\Omega = \{a\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n \in \mathfrak{h}^* \mid a \in \mathbb{C},\ 1 \leq i \leq n\}.$$

If $\mathfrak{g} = \bar{S}(n)$, then

$$\Omega = \{a\varepsilon_1 + \ldots. + a\varepsilon_{i-1} + b\varepsilon_i + (a+1)\varepsilon_{i+1} + \cdots + (a+1)\varepsilon_n \mid a, b \in \mathbb{C}, 1 \leq i \leq n\}.$$

Chapter 3

Cohomology

One of our goals is to develop an algebro-geometric theory which unifies the representation and cohomology theories of Lie superalgebras. For this purpose, it is important to use the relative cohomology rather than ordinary Lie superalgebra cohomology. The latter is usually non-zero in only finitely many degrees [Fu] and this will not capture much information about the representation theory for finite dimensional Lie superalgebras. Relative cohomology for Lie algebras was first defined by Hochschild [Hoc] and the super case is considered in Fuks [Fu]. The main theme here is that once one accounts for the $\mathbb{Z}_2$-grading, results from the purely even case hold here as well.

3.1 Relative Cohomology

In this section we outline basic definitions and results for relative cohomology. Let R be a superalgebra and S be a subsuperalgebra. In particular we asume $S_r = R_r \cap S$ for $r \in \mathbb{Z}_2$. Let

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \dots$$

be a sequence of R-modules. We say this sequence is (R,S)-*exact* if it is exact as a sequence of R-modules and if, when viewed as a sequence of S-modules, $\mathrm{Ker} f_i$ is a direct summand of M_i for all i.

An R-module P is (R,S)-*projective* if given any (R,S)-exact sequence

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0,$$

and R-module homomorphism $h : P \to M_3$ there is an R-homomorphism $\tilde{h} : P \to M_2$ satisfying $g \circ \tilde{h} = h$.

In particular, note that if P is a projective module, then it is automatically (R,S)-projective. Also since g is assumed to be even, if h is homogeneous, then we can choose $\tilde{h}$ to be homogeneous of the same degree as h. More generally, if we write $h = h_{\overline{0}} + h_{\overline{1}}$ where $h_r \in \mathrm{Hom}_R(P, M_3)_r \ \ (r = \overline{0}, \overline{1})$, then we can lift each h_r and $\tilde{h}_{\overline{0}} + \tilde{h}_{\overline{1}}$ is a lift of h.

An (R,S)-*projective resolution* of an R-module is an (R,S)-exact sequence

$$\cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \to 0,$$

where each P_i is an (R,S)-projective module. We remind the reader that implicit in the definition is the fact that maps δ_i are all assumed to be even.

The following lemma is proven as in [Kum].

Lemma 3.1.1. *Let R be a superalgebra and S be a subsuperalgebra of R.*

(a) If M is any S-module, then $R \otimes_S M$ is an (R,S)-projective R-module. The $\mathbb{Z}_2$-grading on $R \otimes_S M$ is given in the usual way by:

$$(R \otimes_S M)_i = \bigoplus_{\substack{k+l=i \\ k,l \in \mathbb{Z}_2}} R_k \otimes_S M_l.$$

(b) Any R-module admits an (R,S)-projective resolution, namely,

$$\cdots \xrightarrow{\delta_2} R \otimes_S \mathrm{Ker}\,\delta_0 \xrightarrow{\delta_1} R \otimes_S M \xrightarrow{\delta_0} M \to 0. \tag{3.1}$$

Here δ_i is the "multiplication" map $R \otimes_S N \to N$ given by $r \otimes n \mapsto rn$ for any R-module N.

Note that since the multiplication map is even, its kernel is an R-submodule of the domain and, hence, we can recursively define the above sequence as indicated. Given an R-module M with (R,S)-projective resolution $P_\bullet \to M$, apply the functor $\mathrm{Hom}_R(-, N)$ and set

$$\mathrm{Ext}^i_{(R,S)}(M,N) = \mathrm{H}^i(\mathrm{Hom}_R(P_\bullet, N)).$$

One can show that $\mathrm{Ext}^{\bullet}_{(R,S)}(M,N)$ can be defined with the dually defined (R,S)-injective resolutions, and it is functorial in both arguments and well defined. It is of interest to track the $\mathbb{Z}_2$-grading. As we remarked earlier, $\mathrm{Hom}_R(P_i, N)$ is naturally $\mathbb{Z}_2$-graded and since δ_i was assumed to be even the induced homomorphism $\mathrm{Hom}_R(P_{i-1}, N) \to \mathrm{Hom}_R(P_i, N)$ is also even. Consequently, $\mathrm{Ext}^{i}_{(R,S)}(M,N)$ inherits a $\mathbb{Z}_2$-grading.

3.2 Relative Cohomology for Lie superalgebras

Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{t} \subseteq \mathfrak{g}$ be a Lie subsuperalgebra, and M be a $\mathfrak{g}$-module. For $p \geq 0$ set

$$C^p(\mathfrak{g}; M) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^p_s(\mathfrak{g}), M),$$

where $\Lambda^p_s(\mathfrak{g})$ is the super wedge product. That is, $\Lambda^p_s(\mathfrak{g})$ is the p-fold tensor product of $\mathfrak{g}$ modulo the $\mathfrak{g}$-submodule generated by elements of the form

$$x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_p + (-1)^{\overline{x_k}\ \overline{x_{k+1}}} x_1 \otimes \cdots \otimes x_{k+1} \otimes x_k \otimes \cdots \otimes x_p$$

for homogeneous $x_1, \ldots, x_p \in \mathfrak{g}$. Therefore, x_k, x_{k+1} skew commute unless both are odd in which case they commute.

Let $d^p : C^p(\mathfrak{g}; M) \to C^{p+1}(\mathfrak{g}; M)$ be given by the formula:

$$\begin{aligned} d^p(\phi)(x_1 \wedge \cdots \wedge x_{p+1}) = & \sum_{i<j} (-1)^{\sigma_{i,j}(x_1,\ldots,x_p)} \phi([x_i, x_j]\ \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{p+1}) \\ & + \sum_i (-1)^{\gamma(x_1,\ldots,x_p,\phi)} x_i \phi(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{p+1}), \end{aligned} \tag{3.2}$$

where $x_1, \ldots, x_{p+1}$ and ϕ are assumed to be homogeneous, and

$$\begin{aligned} \sigma_{i,j}(x_1, \ldots, x_p) &:= i + j + \overline{x_i}(\overline{x_1} + \cdots + \overline{x_{i-1}}) + \overline{x_j}(\overline{x_1} + \cdots + \overline{x_{j-1}} + \overline{x_i}), \\ \gamma_i(x_1, \ldots, x_p, \phi) &:= i + 1 + \overline{x_i}(\overline{x_1} + \cdots + \overline{x_{i-1}} + \overline{\phi}). \end{aligned}$$

Ordinary Lie superalgebra cohomology is then defined as

$$\mathrm{H}^p(\mathfrak{g}; M) = \mathrm{Ker}\ d^p / \mathrm{Im}\ d^{p-1}.$$

The relative version of the above construction is given as follows. Define

$$C^p(\mathfrak{g}, \mathfrak{t}; M) = \mathrm{Hom}_{\mathfrak{t}}(\Lambda^p_s(\mathfrak{g}/\mathfrak{t}), M).$$

Then the map d^p induces a well defined map $d^p : C^p(\mathfrak{g}, \mathfrak{t}; M) \to C^{p+1}(\mathfrak{g}, \mathfrak{t}; M)$ and we define

$$\mathrm{H}^p(\mathfrak{g}, \mathfrak{t}; M) = \mathrm{Ker}\ d^p/\mathrm{Im}\ d^{p-1}.$$

3.3 Relating Cohomology Theories

Let R be a superalgebra and S a subsuperalgebra. Given R-modules M and N one can define cohomology with respect to the pair (R, S). In particular, if $\mathfrak{t}$ is a Lie subsuperalgebra of $\mathfrak{g}$ then one can define cohomology for the pair $(U(\mathfrak{g}), U(\mathfrak{t}))$. The following proposition relates the relative cohomology with the cohomology theories of $(U(\mathfrak{g}), U(\mathfrak{t}))$ and $\mathcal{C}_{(\mathfrak{g},\mathfrak{t})}$, where $\mathcal{C}_{(\mathfrak{g},\mathfrak{t})}$ denotes the category of $\mathfrak{g}$-modules which are finitely semisimple as $\mathfrak{t}$-modules.

Proposition 3.3.1. *Let $\mathfrak{t}$ be a Lie subsuperalgebra of $\mathfrak{g}$, and M, N be $\mathfrak{g}$-modules in $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{t})}$ and assume that $\mathfrak{g}$ is finitely semisimple as a $\mathfrak{t}$-module under the adjoint action. Then,*

(a) $\mathrm{Ext}^{\bullet}_{(U(\mathfrak{g}),U(\mathfrak{t}))}(M, N) \cong \mathrm{Ext}^{\bullet}_{(U(\mathfrak{g}),U(\mathfrak{t}))}(\mathbb{C}, \mathrm{Hom}_{\mathbb{C}}(M, N)) \cong \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{t}; \mathrm{Hom}_{\mathbb{C}}(M, N))$;

(b) $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(M, N) \cong \mathrm{Ext}^{\bullet}_{(U(\mathfrak{g}),U(\mathfrak{t}))}(M, N)$.

Proof. (a) The proof given in [Kum], also see [BKN1, Lemma 2.3.1], can be used to prove the statement. (b) According to [BKN1, Proposition 2.4.1], if L is a finite-dimensional simple $\mathfrak{t}$-module and $\hat{L} = U(\mathfrak{g}) \otimes_{U(\mathfrak{t})} L$ then $\hat{L}$ is $(U(\mathfrak{g}), U(\mathfrak{t}))$-projective and a projective module in the category $\mathcal{C}$. Now one can apply [BKN1, Corollary 2.4.2] with $\mathcal{D} = \mathcal{C}$.

□

3.4 Relating cohomology rings to invariants

In this section we give necessary conditions to identify the relative cohomology ring $\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ with a ring of invariants.

Theorem 3.4.1. *Let* $\mathfrak{t}$ *be a Lie subsuperalgebra of* $\mathfrak{g}$ *and* $\mathfrak{u} \subseteq \mathfrak{g}$ *be a* $\mathfrak{t}$*-module. Then* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ *is isomorphic to the ring of invariants* $\Lambda_s^\bullet(\mathfrak{u}^*)^{\mathfrak{t}}$ *if*

(a) $\mathrm{Hom}_{\mathfrak{t}}(\Lambda_s^p(\mathfrak{g}/\mathfrak{t}), \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{t}}(\Lambda_s^p(\mathfrak{u}), \mathbb{C})$ *for all* $p \geq 0$*, and*

(b) $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{t}$.

Proof. By (a) one has

$$\begin{aligned} C^p(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) &= \mathrm{Hom}_{\mathfrak{t}}(\Lambda_s^p(\mathfrak{g}/\mathfrak{t}), \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathfrak{t}}(\Lambda_s^p(\mathfrak{u}), \mathbb{C}) \\ &\cong \Lambda_s^p(\mathfrak{u}^*)^{\mathfrak{t}}. \end{aligned}$$

Now observe that in this case the differential d^p in (3.2) is identically zero. Namely, in the first sum of (3.2) each $[x_i, x_j]$ is zero in the quotient $\mathfrak{g}/\mathfrak{t}$ since $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{t}$ and the terms in the second sum of (3.2) are zero since here M is the trivial module. Consequently, the cohomology is simply the cochains themselves. Thus

$$\mathrm{H}^p(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) \cong \Lambda_s^p((\mathfrak{g}/\mathfrak{t})^*)^{\mathfrak{t}} \cong \Lambda_s^p(\mathfrak{u}^*)^{\mathfrak{t}}.$$

□

Example 3.4.2. Suppose that $\mathfrak{t} = \mathfrak{g}_{\bar{0}}$. Since super wedge product is symmetric product on odd spaces and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}}$ we have the following important isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong S(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}}.$$

Corollary 3.4.3. *Maintain the hypotheses of Theorem 3.4.1 and assume in addition that* $\mathfrak{t}$ *is a reductive Lie algebra. Let* G *be the connected reductive algebraic group such that* $\mathrm{Lie}(G) = \mathfrak{t}$. *Let* M *be a finite dimensional* $\mathfrak{g}$*-module. Then,*

(a) The superalgebra $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ *is finitely generated as a ring.*

(b) $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; M)$ *is finitely generated as an* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$*-module.*

Proof. (a) Since $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C}) \cong \Lambda_s(\mathfrak{u}^*)^{\mathfrak{t}} = \Lambda_s(\mathfrak{u}^*)^G$ and G is reductive this statement follows from the classical invariant theory result of Hilbert [PV, Theorem 3.6].

(b) Observe that

$$\mathrm{Hom}_{\mathbb{C}}(\Lambda_s^\bullet(\mathfrak{g}/\mathfrak{t}), M) \cong \Lambda_s^\bullet((\mathfrak{g}/\mathfrak{t})^*) \otimes M$$

is finitely generated as a $\Lambda_s^\bullet((\mathfrak{g}/\mathfrak{t})^*)$-module since M is finite dimensional. One can invoke [PV, Theorem 3.25] to see that

$$\mathrm{Hom}_{\mathbb{C}}(\Lambda_s^\bullet(\mathfrak{g}/\mathfrak{t}), M)^G = \mathrm{Hom}_G(\Lambda_s^\bullet(\mathfrak{g}/\mathfrak{t}), M) = C^\bullet(\mathfrak{g},\mathfrak{t};M)$$

is finitely generated as a $\Lambda_s^\bullet((\mathfrak{g}/\mathfrak{t})^*)^G \cong \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-module. One can now use the finite generation of $C^\bullet(\mathfrak{g},\mathfrak{t};M)$ over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ to show that $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M)$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$. Given $r \in \mathrm{H}^p(\mathfrak{g},\mathfrak{t};\mathbb{C})$ and $x \in C^\bullet(\mathfrak{g},\mathfrak{t};M)$, we have $d(rx) = d(r)x + (-1)^p r d(x) = (-1)^p r d(x)$. The second equality follows from the fact that the differentials for $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ are identically zero. Hence, $d : C^\bullet(\mathfrak{g},\mathfrak{t};M) \to C^\bullet(\mathfrak{g},\mathfrak{t};M)$ is a graded $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-module homomorphism. Since $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ is finitely generated, any subquotient of a finitely generated graded module is finitely generated.

□

There are lots of examples of rings of invariants which are not finitely generated (see [Bl]). We make the following assumption for the relative cohomology ring $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$:

Hypothesis 3.4.4. *Given a Lie superalgebra* $\mathfrak{g}$ *and a subalgebra* $\mathfrak{t}$*, assume that* $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ *is a finitely generated* $\mathbb{C}$*-algebra.*

3.5 Detecting subalgebras

In [BKN1, BKN2] by applying invariant theory results in [LR] and [DK] the authors showed that under suitable conditions a classical Lie superalgebra $\mathfrak{g}$ admits a subalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$

such that the restriction map in cohomology induces an isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W,$$

where W is a finite pseudoreflection group. In this section we will give an abstract definition of a detecting subalgebra and in Chapters 5 and 6, explicit detecting subalgebras for $W(n)$ and $\bar{S}(n)$ will be constructed.

Definition 3.5.1. Let $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ be a subalgebra of $\mathfrak{g}$ such that

(a) $\mathfrak{e}$ is a classical Lie superalgebra, and

(b) the inclusion map $\mathfrak{e} \hookrightarrow \mathfrak{g}$ induces an isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W$$

for some group W such that $(-)^W$ is exact.

A subalgebra with these properties will be called a *detecting subalgebra* for the pair $(\mathfrak{g}, \mathfrak{t})$.

The subsuperalgebra $\mathfrak{e}$ can be viewed as an analogue of a Sylow subgroup with

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W$$

looking like a theorem involving transfer for finite groups.

Example 3.5.2. Let $\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and $\mathfrak{t} = \mathfrak{g}_{\bar{0}}$. As in [BKN1, Section 8.10], we can take $\mathfrak{e}_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$ to be the subspace spanned by the distinguished basis

$$x_s := E_{m+1-s,m+s} + E_{m+s,m+1-s} \quad \text{for} \quad s = 1, \ldots, r.$$

Let $\mathfrak{e}_{\bar{0}}$ be the stabilizer of $\mathfrak{e}_{\bar{1}}$ in $\mathfrak{g}_{\bar{0}}$. Then $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ is a detecting subalgebra of $\mathfrak{g}$.

For more examples of detecting subalgebras in the case of simple classical Lie superalgebras one can see [BKN1] and for the Lie superalgebras $W(n)$ and $\bar{S}(n)$ one can see Chapters 5 and 6, below.

Since a pair $(\mathfrak{g}, \mathfrak{t})$ may not have a detecting subalgebra we make the following technical assumption about the pair $(\mathfrak{g}, \mathfrak{t})$.

Hypothesis 3.5.3. *The pair* $(\mathfrak{g}, \mathfrak{t})$ *has a detecting subalgebra.*

Chapter 4

Varieties

One of the most useful innovations in representation theory is the support variety, which is defined using cohomological operations. Support varieties were first defined for modules over finite groups by Carlson [Ca1] in the early 1980's and have been a key to learning more about the representation theory of finite groups. The theory of support varieties was later extended to the Frobenius kernel of an algebraic group scheme G (denoted G_1) by Friedlander and Parshall [FPa]. This work was refined and generalized to all infinitesimal group schemes by Bendel, Friedlander and Suslin [SFB]. Later the work of [SFB] was generalized to all finite group schemes by Friedlander and Pevtsova by introducing π-points [FPe]. Support varieties for Lie superalgebras were defined by Boe, Kujawa and Nakano in [BKN1].

4.1 Support Varieties

Suppose that the Hypotheses 3.4.4 and 3.5.3 hold for the pair $(\mathfrak{g}, \mathfrak{t})$. That is, $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ is finitely generated and there exists a classical subalgebra $\mathfrak{e}$ of $\mathfrak{g}$ such that restriction map induces an isomorphism $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W$ for some group W with $(-)^W$ exact.

Let M and N be $\mathfrak{g}$-modules such that $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathrm{Hom}_{\mathbb{C}}(M, N)))$ is finitely generated as an $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$-module.

Let

$$I_{(\mathfrak{g},\mathfrak{t})}(M, N) = \mathrm{Ann}_{\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})}(\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathrm{Hom}_{\mathbb{C}}(M, N)))$$

be the annihilator ideal of this module. We define the *relative support variety* of the pair (M, N) to be

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M, N) = \mathrm{MaxSpec}(\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})/I_{(\mathfrak{g},\mathfrak{t})}(M, N)),$$

the maximal ideal spectrum of the quotient of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ by $I_{(\mathfrak{g},\mathfrak{t})}(M,N)$. For short when $M=N$, write

$$I_{(\mathfrak{g},\mathfrak{t})}(M) = I_{(\mathfrak{g},\mathfrak{t})}(M,M),$$

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M) = \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M,M).$$

We call $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M)$ the *support variety* of M. Let us remark that, just as for finite groups, $I_{(\mathfrak{g},\mathfrak{t})}(M)$ is precisely the annihilator ideal of the identity element of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M^*\otimes M)$ viewed as a ring under the Yoneda product.

Since the detecting subalgebra $\mathfrak{e}$ is classical $\mathrm{H}^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};\mathbb{C}) \cong S(\mathfrak{e}_{\bar{1}}^*)^{\mathfrak{e}_{\bar{0}}}$ is finitely generated and one can define support varieties for the detecting subalgebra $\mathfrak{e}$. The canonical restriction map

$$\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};\mathbb{C})$$

induces a map of varieties

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(\mathbb{C}).$$

By the isomorphism $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};\mathbb{C})^W$ one then has

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})/W \cong \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(\mathbb{C}).$$

In particular, $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W$ and $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M)$ can naturally be viewed as affine subvarieties of $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(\mathbb{C})$. Since the defining ideals are graded these are conical varieties.

Furthermore, res^* restricts to give a map,

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M).$$

Since $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ is stable under the action of W we have the following embedding induced by res^*,

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W \cong \mathrm{res}^*(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)) \hookrightarrow \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M). \tag{4.1}$$

4.2 Rank varieties

Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ be a subalgebra of $\mathfrak{g}$. Let M be an $\mathfrak{a}$-module. Given an element $x \in \mathfrak{a}$, let $\langle x \rangle$ denote the Lie subsuperalgebra generated by x. Define the *rank variety* of M to be

$$\mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M) = \{x \in \mathfrak{a}_{\bar{1}} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

We record some basic properties of rank varieties in the following proposition. For the proofs we refer the reader to [BKN1, Proposition 6.3.1].

Proposition 4.2.1. *Let M and N be finite dimensional $\mathfrak{a}$-modules. Then*

(a) $\mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M \otimes N) = \mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M) \cap \mathcal{V}_{\mathfrak{a}}^{\text{rank}}(N)$.

(b) $\mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M^*) = \mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M)$.

(c) $\mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M^* \otimes M) = \mathcal{V}_{\mathfrak{a}}^{\text{rank}}(M)$.

Hypothesis 4.2.2. *We assume that the detecting subalgebra $\mathfrak{e}$ has a rank variety description; i.e., for any $\mathfrak{e}$-module M, $\mathcal{V}_{\mathfrak{e}}^{\text{rank}}(M) \cong \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$.*

The following theorem taken together with Proposition 4.2.1 shows that support varieties for finite dimensional $\mathfrak{e}$-modules satisfy the desirable properties of a support variety theory (cf.[BKN1, Theorem 6.4.2]).

Theorem 4.2.3. *Suppose that the detecting subalgebra $\mathfrak{e}$ has rank variety description, i.e, Hypothesis 4.2.2 holds. Let M, N, M_1, M_2 and M_3 be finite dimensional $\mathfrak{e}$-modules . Then,*

(a) M is projective if and only if $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) = \{0\}$.

(b) $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \oplus N) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \cup \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)$.

(c) If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence, then

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_i) \subseteq \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_j) \cup \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_k),$$

where $\{i, j, k\} = \{1, 2, 3\}$.

4.3 Invariants and ideals

As a matter of notation, if J is an ideal of some commutative ring A, then let $\mathcal{Z}(J)$ be the variety defined by J. That is,

$$\mathcal{Z}(J) = \{\mathfrak{m} \in \mathrm{MaxSpec}(A) \mid J \subseteq \mathfrak{m}\}.$$

In particular, for $a \in A$ let $\mathcal{Z}(a)$ denote the variety defined by the ideal (a).

Lemma 4.3.1. *Let S be a finitely generated, commutative, graded algebra where each graded summand is finite dimensional. Furthermore assume some group Γ acts semisimply on S and the action repects the grading on S. Let $i : S^\Gamma \to S$ be the canonical embedding. If J is a graded ideal of S, then the ideal $i^{-1}(J) = J^\Gamma$. In addition, if I, J are both graded ideals of S, then one has*

$$(I + J)^\Gamma = I^\Gamma + J^\Gamma. \tag{4.2}$$

Proof. One first notes that one has the inclusion $I^\Gamma + J^\Gamma \subseteq (I + J)^\Gamma$. However, as graded Γ-modules one has

$$(I \oplus J) / (I \cap J) \cong I + J.$$

By using the fact that taking fixed points under Γ is exact (because the action of Γ is semisimple) one has

$$(I \oplus J)^\Gamma / (I \cap J)^\Gamma \cong (I + J)^\Gamma.$$

However, $(I \cap J)^\Gamma = I^\Gamma \cap J^\Gamma$ and $(I \oplus J)^\Gamma = I^\Gamma \oplus J^\Gamma$. Thus one has

$$(I \oplus J)^\Gamma / \left(I^\Gamma \cap J^\Gamma\right) \cong \left(I^\Gamma + J^\Gamma\right). \tag{4.3}$$

On the other hand, considering I^Γ and J^Γ as Γ-modules one has

$$\left(I^\Gamma \oplus J^\Gamma\right) / \left(I^\Gamma \cap J^\Gamma\right) \cong I^\Gamma + J^\Gamma. \tag{4.4}$$

Using (4.3) and (4.4) to compare dimensions of the graded summands of (4.2), one sees that the earlier inclusion must, in fact, be an equality.

□

Throughout the remainder of this chapter we assume the pair $(\mathfrak{g}, \mathfrak{t})$ satisfies the Hypotheses 3.4.4, 3.5.3, and 4.2.2.

4.4 Tensor Products

One of the fundamental results in the theory of support varieties for finite group schemes is that the support variety of the tensor product of two modules is the intersection of the two modules' support varieties. Lacking equality in (4.1) we are limited to the following analogue.

Lemma 4.4.1. *Let M and N be $\mathfrak{e}$-modules for which $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; M^* \otimes M)$, $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; N^* \otimes N)$, and $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; M^* \otimes N^* \otimes M \otimes N)$ are finitely generated $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$-modules. Then,*

$$\mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \otimes N)\right) = \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right) \cap \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)\right).$$

Proof. If M is a finite dimensional $\mathfrak{g}$-module and $I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ is the ideal which defines $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$, then $\mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right)$ is the variety defined by the ideal $\mathrm{res}^{-1}\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right)$. Recall that

$$\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) \xrightarrow{\cong} \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W \subseteq \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C}).$$

If we identify $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ with its image under this map, one has $\mathrm{res}^{-1}(J) = J^W$ for any ideal J in $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$. Furthermore, given an ideal I we write $\sqrt{I}$ for the radical of the ideal.

By the tensor product property of $\mathfrak{e}$ support varieties one has

$$\mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \otimes N)\right) = \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \cap \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)\right).$$

Applying the earlier remarks, at the level of ideals the above equality becomes

$$\sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M\otimes N)^W} = \sqrt{\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)+I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)\right)^W}.$$

However by (4.2) one has

$$\sqrt{\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)+I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)\right)^W} = \sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)^W+I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)^W}.$$

As the latter ideal defines the variety $\operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right) \cap \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)\right)$, this yields the desired result. □

4.5 Carlson modules

To prove realizability one needs to introduce a family of modules for which one can explicitly calculate their support varieties. Let $n>0$ and let $\zeta \in \mathrm{H}^n(\mathfrak{g},\mathfrak{t};\mathbb{C})$. We can consider ζ to be a $\mathfrak{g}$-homomorphism from the nth syzygy of the trivial module, $\Omega^n(\mathbb{C})$, to $\mathbb{C}$. Set

$$L_\zeta = \operatorname{Ker}(\zeta : \Omega^n(\mathbb{C}) \to \mathbb{C}) \subseteq \Omega^n(\mathbb{C}).$$

These modules are often referred to as "Carlson modules." As in the theory of support varieties for finite group schemes the importance of the module L_ζ is that one can explicitly realize its support as the zero locus of ζ in $\operatorname{MaxSpec}(\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C}))$.

The first step is to compute the support variety of L_ζ over the detecting subalgebra.

Lemma 4.5.1. *Given $\zeta \in \mathrm{H}^n(\mathfrak{g},\mathfrak{t};\mathbb{C})$ and L_ζ as above, then*

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_{\operatorname{res}(\zeta)}) = \mathcal{Z}(\operatorname{res}(\zeta)).$$

Proof. Given $\zeta \in \mathrm{H}^n(\mathfrak{g},\mathfrak{t};\mathbb{C})$, we compute the $\mathfrak{e}$ support variety of L_ζ as follows. Construct the short exact sequence of $\mathfrak{g}$-modules,

$$0 \to L_\zeta \to \Omega^n(\mathbb{C}) \xrightarrow{\zeta} \mathbb{C} \to 0.$$

Upon restriction to $\mathfrak{e}$ one obtains the short exact sequence,

$$0 \to L_\zeta \downarrow_{\mathfrak{e}} \to \Omega^n(\mathbb{C}) \downarrow_{\mathfrak{e}} \xrightarrow{\operatorname{res}(\zeta)} \mathbb{C} \to 0.$$

By using the graded version of Schanuel's Lemma, $L_\zeta \downarrow_{\mathfrak{e}} \cong L_{\mathrm{res}(\zeta)} \oplus P$ and $\Omega^n(\mathbb{C}) \downarrow_{\mathfrak{e}} \cong \Omega^n_{\mathfrak{e}}(\mathbb{C}) \oplus P$, where $\Omega^n_{\mathfrak{e}}(\mathbb{C})$ denotes $\Omega^n(\mathbb{C})$ for the trivial $\mathfrak{e}$-module, $L_{\mathrm{res}(\zeta)}$ is the Carlson $\mathfrak{e}$-module for $\mathrm{res}(\zeta) \in \mathrm{H}^n(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$, and P is some projective $\mathfrak{e}$-module. Therefore, we have the following short exact sequence of $\mathfrak{e}$-modules:

$$0 \to L_{\mathrm{res}(\zeta)} \oplus P \to \Omega^n_{\mathfrak{e}}(\mathbb{C}) \oplus P \to \mathbb{C} \to 0.$$

By the rank variety description of $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_{\mathrm{res}(\zeta)})$ one has that $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_{\mathrm{res}(\zeta)}) = \mathcal{Z}(\mathrm{res}(\zeta))$. Therefore, since $L_\zeta \downarrow_{\mathfrak{e}} \cong L_{\mathrm{res}(\zeta)} \oplus P$, one has that

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_{\mathrm{res}(\zeta)}) = \mathcal{Z}(\mathrm{res}(\zeta)).$$

□

We should warn the reader that it may be that L_ζ is infinite dimensional and, hence, $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; L_\zeta^* \otimes L_\zeta)$ is no longer necessarily finitely generated as an $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$-module. Let us mention that since as $\mathfrak{e}$-modules $L_\zeta \cong L_{\mathrm{res}(\zeta)} \oplus P$ and $L_{\mathrm{res}(\zeta)}$ is finite dimensional (since the projective indecomposible $\mathfrak{e}$-modules are finite dimensional by [BKN2, Proposition 5.2.2]), this complication did not arise in Lemma 4.5.1. Similarily, when $\mathfrak{g}$ is classical L_ζ is necessarily finite dimensional. However, if one wishes to consider support varieties for $\mathfrak{g}$ whose Carlson modules are not necessarily finite dimensional, then the issue can no longer be ignored. To circumvent this difficulty one can instead choose to work with relative support varieties as we now demonstrate.

Proposition 4.5.2. *Let* $\zeta_1, \dots, \zeta_s \in \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ *be homogeneous elements with corresponding Carlson modules* $L_{\zeta_1}, \dots, L_{\zeta_s}$*. Then,*

(a) $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; L_{\zeta_1}^* \otimes \cdots \otimes L_{\zeta_s}^*)$ *is finitely generated over* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$*.*

(b) $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}, \mathbb{C}) \subseteq \cap_{i=1}^s \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_i}, \mathbb{C})$

Proof. (a) We will prove this by induction on s. For $s = 1$, consider the short exact sequence

$$0 \to L_\zeta \xrightarrow{\alpha} \Omega^n(\mathbb{C}) \xrightarrow{\zeta} \mathbb{C} \to 0. \tag{4.5}$$

This induces a long exact sequence of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-modules:

$$\cdots \xrightarrow{d} \mathrm{H}^r(\mathfrak{g},\mathfrak{t};\mathbb{C}) \xrightarrow{\zeta_*} \mathrm{H}^r(\mathfrak{g},\mathfrak{t};\Omega^n(\mathbb{C})^*) \xrightarrow{\alpha_*} \mathrm{H}^r(\mathfrak{g},\mathfrak{t};L_\zeta^*) \xrightarrow{d} \mathrm{H}^{r+1}(\mathfrak{g},\mathfrak{t};\mathbb{C}) \to \cdots,$$

where α_* and ζ_* are the maps induced by α and ζ, respectively, and d denotes the connecting morphism in the long exact sequence.

For $r \geq 0$, set

$$A_r = \mathrm{H}^r(\mathfrak{g},\mathfrak{t};\Omega^n(\mathbb{C})^*)/\operatorname{Ker}(\alpha_*),$$
$$B_r = \operatorname{Im}(d) \subseteq \mathrm{H}^{r+1}(\mathfrak{g},\mathfrak{t};\mathbb{C}).$$

Let $A_\bullet = \oplus_r A_r$ and $B_\bullet = \oplus_r B_r$. Note that

$$\alpha_*^\bullet : \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\Omega^n(\mathbb{C})^*) \to \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};L_\zeta^*)$$
$$d^\bullet : \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};L_\zeta^*) \xrightarrow{d} \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$$

are $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-module homomorphisms, where $\alpha_*^\bullet$ and $d^\bullet$ are the maps obtained by taking the direct sum of the maps α_* and d, respectively. Hence, $A_\bullet$ and $B_\bullet$ are $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-modules and from the long exact sequence given above one has the short exact seqence of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-modules,

$$0 \to A_\bullet \xrightarrow{\bar{\alpha}_*^\bullet} \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};L_\zeta^*) \xrightarrow{\bar{d}^\bullet} B_\bullet \to 0, \tag{4.6}$$

where $\bar{\alpha}_*^\bullet$ and $\bar{d}^\bullet$ are the maps induced by $\alpha_*^\bullet$ and $d^\bullet$, respectively.

However, for all $r \geq 0$, $\mathrm{H}^r(\mathfrak{g},\mathfrak{t};\Omega^n(\mathbb{C})^*) \cong \mathrm{H}^{n+r}(\mathfrak{g},\mathfrak{t};\mathbb{C})$ by degree shifting. Taking the direct sum of these maps yields an $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-module isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\Omega^n(\mathbb{C})^*) \cong \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C}).$$

Therefore $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\Omega^n(\mathbb{C})^*)$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$. Since $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ is a Noetherian ring it follows that the quotient module $A_\bullet$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$. Similarily, since $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$, the submodule $B_\bullet$ is also finitely generated. Finally, using (4.6) and the fact that $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ is Noetherian one has that $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};L_\zeta^*)$ must be finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$.

For the inductive step we claim that if M is a module for which $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M^*)$ is a finitely generated $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$-module and ζ is a homogeneous element of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$, then $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M^*\otimes L_\zeta^*)$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$. The argument parallels the base case considered above. Namely, consider the short exact sequence obtained by tensoring (4.5) with M :

$$0 \to M\otimes L_\zeta \to M\otimes \Omega^n(\mathbb{C}) \to M\otimes \mathbb{C} \to 0.$$

Note that by assumption (i) $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M^*)$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$, and (ii) $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M^*\otimes\Omega^n(\mathbb{C})^*)$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ by applying a dimension shift argument (as in the $s=1$ case). Applying the long exact sequence in cohomology and arguing as in the base case shows that $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};M^*\otimes L_\zeta^*)$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$.

(b) The statement clearly holds for $s=1$. Now assume that the statement holds for $s-1$ factors. For a fixed $i=1,2,\dots,s$, set $N = L_{\zeta_1}\otimes\cdots\otimes\widehat{L_{\zeta_i}}\otimes\cdots\otimes L_{\zeta_s}$. Consider the following short exact sequence given by ζ_i,

$$0\to L_{\zeta_i}\to\Omega^n(\mathbb{C})\to\mathbb{C}\to 0.$$

By tensoring by N we obtain

$$0\to L_{\zeta_i}\otimes N\to\Omega^n(\mathbb{C})\otimes N\to N\to 0.$$

Therefore, by induction

$$\begin{aligned}\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_s},\mathbb{C}) &\subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(\Omega^n(\mathbb{C})\otimes N,\mathbb{C})\cup\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(N,\mathbb{C})\\ &= \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(N,\mathbb{C})\\ &\subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_1},\mathbb{C})\cap\cdots\cap\widehat{\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_i},\mathbb{C})}\cap\cdots\cap\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_s},\mathbb{C}).\end{aligned}$$

Since i is arbitrary we conclude that

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_s},\mathbb{C})\subseteq\cap_{i=1}^s\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_{\zeta_i},\mathbb{C}).$$

□

4.6 Support Varieties for Carlson modules

We are now prepared to compute (relative) support varieties for the Carlson modules.

Proposition 4.6.1. *Let* $\zeta \in \mathrm{H}^n(\mathfrak{g},\mathfrak{t};\mathbb{C})$ *and* L_ζ *be given as above. Then,*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta,\mathbb{C}) = \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right) = \mathcal{Z}(\zeta). \tag{4.7}$$

If L_ζ *is a finite dimensional* $\mathfrak{g}$*-module, then one has*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta) = \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right) = \mathcal{Z}(\zeta). \tag{4.8}$$

Proof. We first prove (4.8). Since

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta),$$

one has that

$$\mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta).$$

However, by Lemma 4.5.1 the variety $\mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right)$ is defined by the ideal $\mathrm{res}^{-1}\left((\mathrm{res}(\zeta))\right) = (\zeta)$, where the equality of ideals follows from the explicit description of the map res . Therefore, one has

$$\mathcal{Z}(\zeta) = \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta). \tag{4.9}$$

On the other hand, one can use the proof given in [Ca2, Proposition 6.13] to show that ζ^2 annihilates $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};L_\zeta^*\otimes L_\zeta)$. Let $I_{(\mathfrak{g},\mathfrak{t})}(L_\zeta)$ denote the annihilator of $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ acting on this group (i.e., the ideal which defines the support variety). So we have $\zeta^2 \in I_{(\mathfrak{g},\mathfrak{t})}(L_\zeta)$. This implies that $I_{(\mathfrak{g},\mathfrak{t})}(L_\zeta)$ contains the ideal generated by ζ^2. This in turn implies that the radical of the ideal $I_{(\mathfrak{g},\mathfrak{t})}(L_\zeta)$ contains the ideal generated by ζ. Thus the variety defined by the ideal $\sqrt{I_{(\mathfrak{g},\mathfrak{t})}(L_\zeta)}$ is contained in $\mathcal{Z}(\zeta)$, that is,

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta) \subseteq \mathcal{Z}(\zeta). \tag{4.10}$$

Combining equations (4.9) and (4.10) one has (4.8).

To prove (4.7) one argues much as above. Namely, one has

$$\operatorname{res}^* : \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta, \mathbb{C}) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta, \mathbb{C}).$$

However, recall that $L_\zeta \cong L_{\operatorname{res}(\zeta)} \oplus P$ as $\mathfrak{e}$-modules where P is a projective $\mathfrak{e}$-module. Also note that by definition $L_{\operatorname{res}(\zeta)}$ can be assumed to lie within the principal block of $\mathfrak{e}$. However, by [BKN2, Proposition 5.2.2] the trivial module is the only simple module in the principal block of $\mathfrak{e}$. Taken together with [Ben2, Proposition 5.7.1] these observations imply that $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta, \mathbb{C})$. Then Lemma 4.5.1 implies that the variety $\operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right)$ is defined by the ideal $\operatorname{res}^{-1}((\operatorname{res}(\zeta))) = (\zeta)$. Therefore, one has

$$\mathcal{Z}(\zeta) = \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta, \mathbb{C})\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta, \mathbb{C}). \tag{4.11}$$

On the other hand, one can once again use the proof given in [Ca2, Proposition 6.13] to show that ζ^2 annihilates $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t}; L_\zeta^* \otimes L_\zeta)$. As for finite groups (cf. [Ben2, Section 5.7]), this implies ζ^2 annihilates $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t}; L_\zeta^*)$.and so is an element of the ideal which defines $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(L_\zeta, \mathbb{C})$. Just as before this implies

$$\mathcal{V}_{\mathfrak{g}}(L_\zeta, \mathbb{C}) \subseteq \mathcal{Z}(\zeta). \tag{4.12}$$

Combining equations (4.11) and (4.12) one has (4.7). □

4.7 Realization Theorem

One important property in the theory of support varieties is the realizability of any conical variety as the support variety of some module in the category. Carlson [Ca1] first proved this for finite groups in the 1980s. Friedlander and Parshall [FPa] later used Carlson's proof to establish realizability for restricted Lie algebras. For arbitrary finite group schemes the finite generation of cohomology due to Friedlander and Suslin [FS] allowed one to define support varieties. In this generality the realizability of supports was established using Friedlander and Pevtsova's method [FPe] of concretely describing support varieties through π-points.

In the classical Lie superalgebra setting the realizability of supports was established for the detecting subalgebra $\mathfrak{e}$ in [BKN1, Theorem 6.4.3]. The main tool to establish this theorem is the tensor product theorem [BKN1, Proposition 6.3.1].

We are now ready to prove the realization theorem.

Theorem 4.7.1. *Suppose that $\mathfrak{g}$ satisfies the Hypotheses 3.4.4, 3.5.3 and 4.2.2. Let X be a conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(\mathbb{C})$. Then there exists a $\mathfrak{g}$-module M such that*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M,\mathbb{C}) = X.$$

If the Carlson modules are finite dimensional for $\mathfrak{g}$ then there exists a finite dimensional $\mathfrak{g}$-module M such that

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M) = X.$$

Proof. First, express X as the zero locus of homogeneous elements $\zeta_1, \dots, \zeta_s \in \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$. That is, fix homogeneous elements $\zeta_1, \dots, \zeta_s \in \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{t};\mathbb{C})$ such that

$$X = \mathcal{Z}(\zeta_1) \cap \cdots \cap \mathcal{Z}(\zeta_s).$$

Let $M = L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}$. If Carlson modules are finite dimensional for $\mathfrak{g}$ then one can combine (4.8), Lemma 4.4.1, and the fact that $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(N_1 \otimes N_2) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(N_1) \cap \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(N_2)$ for any two modules N_1, N_2 (cf. [BKN2, (4.6.4)]) to obtain

$$\begin{aligned}
X &= \cap_{i=1}^s \mathcal{Z}(\zeta_i) \\
&= \cap_{i=1}^s \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}\left(L_{\zeta_i}\right) \\
&= \cap_{i=1}^s \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}\left(L_{\zeta_i}\right)\right) \\
&= \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right) \\
&\subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}\left(M\right) \\
&\subseteq \cap_{i=1}^s \mathcal{V}_{(\mathfrak{g},\mathfrak{t})}\left(L_{\zeta_i}\right) \\
&= \cap_{i=1}^s \mathcal{Z}(\zeta_i) \\
&= X.
\end{aligned}$$

It then follows that $\mathcal{V}_{(\mathfrak{g},\mathfrak{t})}(M) = X$.

The relative case is argued similarly using instead (4.7) and Proposition 4.5.2(b). $\square$

Chapter 5

Cohomology and support varieties for $W(n)$

5.1 $W(n)$

We begin by recalling the definition of the simple Lie superalgebras of type $W(n)$. As a background source we refer the reader to [Kac, Sch, Ser].

Assume that $n \geq 2$. The Lie superalgebra $W(n)$ may be described as follows. Let $\Lambda(n)$ be the exterior algebra of the vector space $V = \mathbb{C}^n$. The algebra $\Lambda(n) = \oplus_{k=0}^{n} \Lambda^k(n)$ is an associative superalgebra of dimension 2^n with a $\mathbb{Z}$-grading given by total degree. The $\mathbb{Z}_2$-grading is inherited from the $\mathbb{Z}$-grading by setting $\Lambda(n)_{\bar{0}} = \oplus_k \Lambda^{2k}(n)$ and $\Lambda(n)_{\bar{1}} = \oplus_k \Lambda^{2k+1}(n)$.

A (homogeneous) *superderivation* of $\Lambda(n)$ is a linear map $D : \Lambda(n) \to \Lambda(n)$ which satisfies $D(xy) = D(x)y + (-1)^{\overline{D}\,\overline{x}} x D(y)$ for all homogenous $x, y \in \Lambda(n)$. Set $W(n)$ to be the vector space of all superderivations of $\Lambda(n)$. Then $W(n)$ is a Lie superalgebra via the supercommutator bracket. Furthermore, $W(n)$ inherits a $\mathbb{Z}$-grading,

$$W(n) = W(n)_{-1} \oplus W(n)_0 \oplus \cdots \oplus W(n)_{n-1},$$

from $\Lambda(n)$ by setting $W(n)_k$ to be the superderivations which increase the degree of a homogeneous element by k. The $\mathbb{Z}_2$-grading on $W(n)$ is inherited from the $\mathbb{Z}$-grading by setting $W(n)_{\bar{0}} = \oplus_k W(n)_{2k}$ and $W(n)_{\bar{1}} = \oplus_k W(n)_{2k+1}$. One can verify that $[W(n)_k, W(n)_l] \subseteq W(n)_{k+l}$ for all $k, l \in \mathbb{Z}$. Most importantly this implies $W(n)_0$ is a Lie algebra and $W(n)_k$ $(k = -1, \ldots, n-1)$ is a $W(n)_0$-module under the adjoint action.

Every element of $W(n)$ restricts to a linear map $V \to \Lambda(n)$. Conversely, every element of $W(n)$ arises in this way and so one has an isomorphism of vector spaces

$$W(n) \cong \Lambda(n) \otimes V^*.$$

This identification will be useful for computations. Fix an ordered basis $\{\xi_1, \ldots, \xi_n\}$ for V. For each ordered subset $I = \{i_1, \ldots, i_s\}$ of $N = \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_s$, let $\xi_I = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_s}$. The set of all such ξ_I forms a basis for $\Lambda(n)$. For $1 \leq i \leq n$ let ∂_i be the element of $W(n)$ such that $\partial_i(\xi_j) = \delta_{ij}$. An explicit basis for $\Lambda(n) \otimes V^*$ is then given by the set of all $\xi_I \otimes \partial_i$, where here we identify ∂_i with its restriction to V. We shall write $\xi_I\partial_i$ instead of $\xi_I \otimes \partial_i$. We use the isomorphism above to identify $W(n)$ and $\Lambda(n) \otimes V^*$.

In particular, one has $W(n)_0 \cong V \otimes V^* \cong \mathfrak{gl}(n)$ and the element $\xi_i\partial_j$ corresponds to the matrix unit $E_{i,j}$ (i.e. the matrix with a one in the (i,j) position and zeros elsewhere). Also $W(n)_{-1} \cong V^*$ as a $W(n)_0$-module. In general the basis elements $\xi_I\partial_i$ belonging to $W(n)_k$ are those with $|I| = k+1$. Thus $\dim_{\mathbb{C}} W(n)_k = n\binom{n}{k+1}$ and $\dim_{\mathbb{C}} W(n) = n2^n$.

We use the following notational conventions throughout this chapter. Set $\mathfrak{g} = W(n)$ with $\mathfrak{g}_i = W(n)_i$, $i \in \mathbb{Z}$, and $\mathfrak{g}_{\bar{\imath}} = W(n)_{\bar{\imath}}$, $\bar{\imath} \in \mathbb{Z}_2$. Moreover, let $\mathfrak{g}^+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ and $\mathfrak{g}^- = \mathfrak{g}_{-1}$, so that $\mathfrak{g}$ has the lopsided triangular decomposition

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^+.$$

Throughout this chapter all $\mathfrak{g}$-modules will be assumed to be objects in the category $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$.

5.2 Cohomology in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$

The goal of this section is to compute the cohomology ring $R = \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$. The main result is Theorem 5.3.1 which shows that R can be identified with a ring of invariants. Consequently, one sees that R is finitely generated and $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M)$ is a finitely generated R-module for any finite dimensional $\mathfrak{g}$-module M.

We begin by showing that the calculation of $\mathfrak{g}_0$-invariants on $\Lambda_s^\bullet((\mathfrak{g}/\mathfrak{g}_0)^*)$ reduces to looking at $\mathfrak{g}_0$-invariants on $\Lambda_s^\bullet\left(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*\right)$. This will be accomplished by using information from the representation theory of $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$.

Theorem 5.2.1. *Let $\mathfrak{g} = W(n)$ and let $p \geq 0$. Then, $\Lambda_s^p((\mathfrak{g}/\mathfrak{g}_0)^*)^{\mathfrak{g}_0} \cong \Lambda_s^p(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*)^{\mathfrak{g}_0}$*

Proof. First observe that $\mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ as $\mathfrak{g}_0$-modules. We then have

$$\begin{aligned}
&\Lambda_s^p(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \oplus \cdots \oplus \mathfrak{g}_{n-1}^*)^{\mathfrak{g}_0} \\
&\qquad \cong \bigoplus \left(\Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}^*) \otimes \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-1}}(\mathfrak{g}_{n-1}^*)\right)^{\mathfrak{g}_0} \\
&\qquad \cong \bigoplus \operatorname{Hom}_{\mathfrak{g}_0}\left(\mathbb{C}, \Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}^*) \otimes \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-1}}(\mathfrak{g}_{n-1}^*)\right) \\
&\qquad \cong \bigoplus \operatorname{Hom}_{\mathfrak{g}_0}\left(\Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}), \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-1}}(\mathfrak{g}_{n-1}^*)\right)
\end{aligned}$$

where the direct sums are taken over all nonnegative integers $i_{-1}, i_1, \ldots, i_{n-1}$ such that $i_{-1} + i_1 + \cdots + i_{n-1} = p$.

Recall that $\mathfrak{g}_{-1}$ is isomorphic to the dual of the natural $\mathfrak{g}_0$-module and, since $\mathfrak{g}_{-1}$ is concentrated in degree $\bar{1}$, one has $\Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}) \cong S^{i_{-1}}(\mathfrak{g}_{-1})$ as $\mathfrak{g}_0$-modules. It is well known that symmetric powers of the dual of the natural module are simple (cf. [Jan, II 2.16]) and so $\Lambda_s^{i_{-1}}(\mathfrak{g}_{-1})$ is a simple $\mathfrak{g}_0$-module with highest weight

$$\mu = (\mu_1, \ldots, \mu_n) = (0, \ldots, 0, -i_{-1}).$$

Since $\mathfrak{g}_k^* \cong \Lambda^{k+1}(V^*) \otimes V \quad (-1 \leq k \leq n-1)$, where V is the natural $\mathfrak{g}_0$-module, and since $\Lambda^{k+1}(V^*)$ (resp. V) is a simple $\mathfrak{g}_0$-module with highest weight $(0, \ldots, 0, -1, \ldots, -1)$ (resp. $(1, 0, \ldots, 0)$), the highest weight of $\mathfrak{g}_k^*$ will be the sum of these two weights, $(1, 0, \ldots, 0, -1, \ldots, -1)$. Therefore the largest possible weight that could occur in the weight space decomposition of the $\mathfrak{g}_0$-module

$$A := (\mathfrak{g}_1^*)^{\otimes i_1} \otimes (\mathfrak{g}_2^*)^{\otimes i_2} \otimes \cdots \otimes (\mathfrak{g}_{n-1}^*)^{\otimes i_{n-1}}$$

is

$$\begin{aligned}
\lambda &= (\lambda_1, \ldots, \lambda_n) \\
&= (i_1, 0, \ldots, 0, -i_1, -i_1) + (i_2, 0, \ldots, 0, -i_2, -i_2, -i_2) + \cdots + (0, -i_{n-1}, \ldots ., -i_{n-1}) \\
&= (\Sigma_{t=1}^{n-1} i_t, -i_{n-1}, -i_{n-2} - i_{n-1}, \ldots, -\Sigma_{t=2}^{n-1} i_t, -\Sigma_{t=1}^{n-1} i_t, -\Sigma_{t=1}^{n-1} i_t).
\end{aligned}$$

If the Hom-space

$$\mathrm{Hom}_{\mathfrak{g}_0}(\Lambda_s^{i-1}(\mathfrak{g}_{-1}), \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-1}}(\mathfrak{g}_{n-1}^*)) \tag{5.1}$$

is non-zero, then the weight μ occurs in the weight space decomposition of the $\mathfrak{g}_0$-module A. However, if μ occurs in the weight space decomposition of this $\mathfrak{g}_0$-module, then it has to be less than or equal to λ in the dominance order; that is, $\Sigma_{i=1}^k \mu_i \leq \Sigma_{i=1}^k \lambda_i$ for all $k > 0$. By considering this inequality when $k = n-1$ one obtains

$$0 \leq -i_2 - 2i_3 - \cdots - (n-2)i_{n-1}.$$

Thus the Hom space in (5.1) is nonzero only when $i_2 = i_3 = \cdots = i_{n-1} = 0$. This gives the stated result. □

5.3 Calculation of the cohomology

The previous theorem can be used to show that the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ can be identified with a ring of invariants. Recall that $G_0 \cong \mathrm{GL}(n)$ denotes the connected reductive group such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$ and the adjoint action of G_0 on $\mathfrak{g}$ differentiates to the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$.

Theorem 5.3.1. *Let $\mathfrak{g} = W(n)$. Then,*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^*)^{\mathfrak{g}_0} = S((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^*)^{G_0}.$$

Proof. By Theorem 5.2.1 one has

$$\begin{aligned} C^p(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) &= \mathrm{Hom}_{\mathfrak{g}_0}(\Lambda_s^p(\mathfrak{g}/\mathfrak{g}_0), \mathbb{C}) \\ &\cong \Lambda_s^p\left((\mathfrak{g}/\mathfrak{g}_0)^*\right)^{\mathfrak{g}_0} \\ &\cong \Lambda_s^p\left(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*\right)^{\mathfrak{g}_0} \\ &\cong \mathrm{Hom}_{\mathfrak{g}_0}(\Lambda_s^p(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1), \mathbb{C}). \end{aligned}$$

Now observe that in this case the differential d^p in (3.2) is identically zero. Namely, in the first sum of (3.2) each $[x_i, x_j]$ is zero in the quotient $\mathfrak{g}/\mathfrak{g}_0$ since the bracket preserves the

$\mathbb{Z}$-grading and the terms in the second sum of (3.2) are zero since here M is the trivial module.

As a consequence the cohomology can be identified with the cochains. It remains to observe that since $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is concentrated in degree $\bar{1}$, one has

$$C^p(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \Lambda_s^p \left((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^* \right)^{\mathfrak{g}_0} \cong S^p \left((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^* \right)^{\mathfrak{g}_0} = S^p \left((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^* \right)^{G_0}.$$

□

5.4 Finite Generation Results

Let M be a finite dimensional $\mathfrak{g}$-module. By using the Yoneda product $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M)$ is a module for the cohomology ring R. A result from invariant theory shows that this module is finitely generated over R.

Theorem 5.4.1. *Let M be a finite dimensional $\mathfrak{g}$-module. Then,*

(a) The superalgebra $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ is a finitely generated commutative ring;

(b) The cohomology $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M)$ is finitely generated as an $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$-module.

Proof. Since $\mathfrak{g}_0$ is a reductive Lie algebra the result follows from Corollary 3.4.3 and Theorem 5.3.1. □

5.5 Invariant Theory Calculations

Recall from Theorem 5.3.1 that

$$R = \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S \left((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^* \right)^{\mathfrak{g}_0}. \tag{5.2}$$

Thus to compute R it suffices to compute the invariant ring on the right hand side of (5.2). To do so we use a result of Luna and Richardson [LR]. First we require certain preliminaries.

If G is an algebraic group which acts on a variety X, then we write $g.x$ for the action of $g \in G$ on the element $x \in X$. Set

$$\mathrm{Stab}_G(x) = \{g \in G \mid g.x = x\},$$

the *stabilizer* of x. An element $x \in X$ is *semisimple* if the orbit $G.x$ is closed in X. An element $x \in X$ is said to be *regular* if the dimension of the orbit $G.x$ is of maximal possible dimension among all orbits. Equivalently, x is regular if the dimension of $\mathrm{Stab}_G(x)$ is of minimal dimension among all stabilizer subgroups.

The group G_0 acts on $\mathfrak{g}$ by the adjoint action and its action preserves the $\mathbb{Z}$-grading of $\mathfrak{g}$. Let

$$\beta : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_0$$

be the G_0-equivariant map given by $\beta(x, y) = [x, y]$ for all $x \in \mathfrak{g}_{-1}$ and $y \in \mathfrak{g}_1$.

Fix $T \subseteq G_0$ to be the maximal torus consisting of all diagonal matrices. Then $\mathfrak{h} = \mathrm{Lie}\,(T)$, the Cartan subalgebra we fixed in Section 2.8.

The following lemma summarizes some well known results about the adjoint action of G_0 on $\mathfrak{g}_0$. See, for example, [CM, Hum].

Lemma 5.5.1. *Let $h \in \mathfrak{h}$. Then,*

(a) *The element h is regular if and only if $\mathrm{Stab}_{G_0}(h) = T$;*

(b) *The element h is regular if and only if all the eigenvalues of h are pairwise distinct elements of $\mathbb{C}$;*

(c) *An element $x \in \mathfrak{g}_0$ is semisimple if and only if it is G_0-conjugate to an element of $\mathfrak{h}$;*

(d) *An element of $x \in \mathfrak{g}_0$ is semisimple and regular if and only if it is G_0-conjugate to a regular element of $\mathfrak{h}$;*

(e) *The semisimple regular elements of $\mathfrak{g}_0$ form a dense open set in $\mathfrak{g}_0$.*

Lemma 5.5.2. *Let G be an algebraic group acting on the varieties X and Y. Let $f : X \to Y$ be a G-equivariant map. Then the following statements hold true.*

(a) If $y \in Y$ and $x \in f^{-1}(y)$, then

$$\mathrm{Stab}_G(x) \subseteq \mathrm{Stab}_G(y).$$

(b) If $y \in Y$, then

$$G.f^{-1}(y) = f^{-1}(G.y).$$

In particular, if $x \in f^{-1}(y)$, then $G.x \subseteq f^{-1}(G.y)$.

Proof. To prove (a), let $g \in \mathrm{Stab}_G(x)$. Then $g.y = g.f(x) = f(g.x) = f(x) = y$, so $g \in \mathrm{Stab}_G(y)$.

To prove (b), let $g.x \in G.f^{-1}(y)$ for some $g \in G$ and $x \in f^{-1}(y)$. Then $f(g.x) = g.f(x) = g.y$, so $g.x \in f^{-1}(G.y)$. On the other hand, if $z \in f^{-1}(G.y)$, then $f(z) = g.y$ for some $g \in G$. That is, $g^{-1}.f(z) = f(g^{-1}.z) = y$. So $g^{-1}.z \in f^{-1}(y)$ and, hence, $z \in G.f^{-1}(y)$. □

We saw in Section 5.1 that $\mathfrak{g}_{-1} \cong V^*$ as a $\mathfrak{g}_0$-module and has basis $\{\partial_i\}$, $\mathfrak{g}_0$ has basis $\{\xi_i\partial_j\}$, and $\mathfrak{g}_1 = \Lambda^2(V) \otimes V^*$ with basis $\{\xi_i\xi_j\partial_k \mid i < j\}$), where $1 \leq i, j, k \leq n$. Recall that the isomorphism $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$ is given by $\xi_i\partial_j \leftrightarrow E_{i,j}$, where $E_{i,j}$ is the (i,j) matrix unit. In particular, $\mathfrak{h}$ is spanned by the set $\{\xi_i\partial_i \mid 1 \leq i \leq n\}$.

Lemma 5.5.3. *Let $h \in \mathfrak{h}$ be a semisimple regular element and write*

$$h = \sum_{i=1}^{n} c_i\xi_i\partial_i,$$

with $c_i \in \mathbb{C}$. One then has the following.

(a) If $c_1, \ldots, c_n$ are all nonzero, then $\beta^{-1}(h) = \emptyset$.

(b) If $c_1 = 0$ and $x \in \beta^{-1}(h)$, then

$$x = a_1\partial_1 + \sum_{l=2}^{n} \frac{c_l}{a_1}\xi_1\xi_l\partial_l + \sum_{\substack{r,s,t \\ 1<r<s}} b_{r,s,t}\xi_r\xi_s\partial_t, \tag{5.3}$$

where $a_1, b_{r,s,t} \in \mathbb{C}$ and $a_1 \neq 0$.

Proof. We only sketch the calculation here. First, let $x \in \beta^{-1}(h)$ and write $x \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ in our preferred basis:

$$x = \sum_i a_i \partial_i + \sum_{\substack{i,j,k \\ i<j}} b_{i,j,k} \xi_i \xi_j \partial_k. \tag{5.4}$$

By a direct calculation of $\beta(x)$, one sees that if $a_t \neq 0$ for some $1 \leq t \leq n$, then necessarily $c_t = 0$. Therefore, if all the coefficients of h are nonzero, then there is no x such that $\beta(x) = h$. This proves part (a). Now say $c_1 = 0$. Since h is regular, $c_2, \ldots, c_n$ are all nonzero by Lemma 5.5.1(b). But then by the proof of part (a) one has $a_2 = \cdots = a_n = 0$. This observation simplifies the calculation of $\beta(x)$. Doing so and using that the image is equal to h, one obtains (5.3). □

Note that if $h \in \mathfrak{h}$ has any entry equal to zero then up to G_0-conjugacy (indeed up to Weyl group conjugacy), one can assume it is in the upper left corner. Thus the general case is easily deduced from part (b) of the lemma.

Proposition 5.5.4. *Let $h \in \mathfrak{h}$ be a semisimple regular element as in part (b) of the previous lemma. Let $x_0 \in \beta^{-1}(h)$ be chosen so that all the coefficents $b_{r,s,t}$ are zero in* (5.3)*. Then $x_0 \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is a semisimple element.*

Proof. First one computes $\mathrm{Stab}_{G_0}(x)$ for any $x \in \beta^{-1}(h)$. By Lemma 5.5.2 and Lemma 5.5.1(a) one has $\mathrm{Stab}_{G_0}(x) \subseteq \mathrm{Stab}_{G_0}(h) = T$. By part (b) of the previous lemma x is a linear combination of distinct weight vectors. From this one sees that $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$ fixes x if $t_1 = 1$. This is also sufficient in the case of x_0. Otherwise there will be additional contraints on t and the stabilizer will be a proper, smaller dimensional subgroup of

$$T_{n-1} := \{t = \mathrm{diag}(t_1, \ldots, t_n) \in T \mid t_1 = 1\}. \tag{5.5}$$

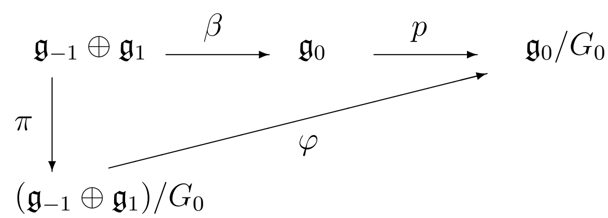
Thus x_0 has maximal stabilizer dimension, hence, minimal orbit dimension in the closed set $\beta^{-1}(G.h)$. By the well known fact on the closure of orbits, it follows that $G.x_0$ must be closed. □

Let G be a reductive algebraic group acting on an affine variety X. Let $\pi : X \to X/G$ be the canonical quotient map. An element $\zeta \in X/G$ is said to be *principal* if there is an open neighborhood U such that $\zeta \in U \subseteq X/G$ and for any semisimple $x, y \in \pi^{-1}(U)$, the groups $\mathrm{Stab}_G(x)$ and $\mathrm{Stab}_G(y)$ are conjugate in G [LR, Definition 3.2, Remark 3.3]. Let $(X/G)_{\mathrm{pr}}$ denote the set of principal elements of X/G. By [LR, Lemma 3.4] $(X/G)_{\mathrm{pr}}$ is a nonempty, dense, open subset of X/G.

Let

$$\pi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0 \qquad \text{and} \qquad p : \mathfrak{g}_0 \to \mathfrak{g}_0/G_0$$

denote the canonical quotient morphisms. Let $\varphi : (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0 \to \mathfrak{g}_0/G_0$ be the morphism induced by the map $p \circ \beta : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_0/G_0$. That is, the following diagram commutes.



We observe that it follows from Lemma 5.5.1 that the set $(\mathfrak{g}_0/G_0)_{\mathrm{pr}}$ is precisely the image under p of the semisimple regular elements of $\mathfrak{h}$.

Proposition 5.5.5. *Let $x_0 \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ be as in Proposition 5.5.4. Then $\pi(x_0)$ is a principal element of $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0$.*

Proof. Let $U := \varphi^{-1}((\mathfrak{g}_0/G_0)_{\mathrm{pr}})$. By definition, $\beta(x_0)$ is semisimple and regular, so $p(\beta(x_0)) = \varphi(\pi(x_0))$ is principal in $\mathfrak{g}_0/G_0$. That is, $\pi(x_0) \in U$. Therefore, U is a nonempty open neighborhood of $\pi(x_0)$ in $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0$. Let $\zeta \in U$ and let $y \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ be a semisimple element in $\pi^{-1}(\zeta)$. Then $\varphi(\pi(y)) = p(\beta(y))$, so $\beta(y) \in p^{-1}(\eta)$ for some principal $\eta \in \mathfrak{g}_0/G_0$. But then $\eta = p(h)$ for some semisimple regular $h \in \mathfrak{h}$. However, since h is semisimple and regular, it follows that $p^{-1}(\eta) = G_0.h$. That is, up to G_0-conjugacy one can assume $\beta(y)$ is a semisimple regular element of $\mathfrak{h}$. However, this implies that y is of the form given in

Lemma 5.5.3 and the stabilizer of such elements was computed in the proof of Proposition 5.5.4. By that calculation and the fact that y is semisimple one sees that the stabilizer of y is T_{n-1}. Therefore, all semisimple elements in the fibers of U have stabilizer conjugate to T_{n-1} and so x_0 is principal. □

The stage is now set to apply the results of Luna and Richardson [LR, Corollary 4.4] to calculate R. To do so requires certain preliminary calculations. Let $x_0 \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ be the semisimple element fixed in the previous section.

Let

$$H = \mathrm{Stab}_{G_0}(x_0) = T_{n-1}, \tag{5.6}$$

where the last equality is by the calculations made in the proof of Lemma 5.5.4. Let

$$N = \mathrm{Norm}_{G_0}(H) = \left\{ g \in G_0 \mid gHg^{-1} = H \right\}. \tag{5.7}$$

Let us first calculate the group N.

Lemma 5.5.6. *Let $N = \mathrm{Norm}_{G_0}(H)$. Recall that T is the torus of G_0. Let Σ_n be the permutation matrices of G_0 and let Σ_{n-1} be the permutation matrices which normalize T_{n-1}. Then,*

$$N = T\Sigma_{n-1}.$$

Proof. The first step is to prove that $N \subseteq \mathrm{Norm}_{G_0}(T)$. Fix a semisimple regular element $t_0 \in T_{n-1}$ (for the action of G_0 on itself by conjugation). Then $T = \mathrm{Stab}_{G_0}(t_0)$. Let $n \in N$. We claim that nTn^{-1} fixes t_0, hence $nTn^{-1} = T$, hence $n \in \mathrm{Norm}_{G_0}(T)$. Let $t \in T$ and consider

$$(ntn^{-1})t_0(ntn^{-1})^{-1} = ntn^{-1}t_0nt^{-1}n^{-1}.$$

However, since $t_0 \in T_{n-1}$ and $n^{-1} \in \mathrm{Norm}_{G_0}(T_{n-1})$, one has that $n^{-1}t_0n \in T_{n-1} \subseteq T$; since $t \in T$ and T fixes T pointwise under conjugation, one has $tn^{-1}t_0nt^{-1} = n^{-1}t_0n$. Thus,

$$ntn^{-1}t_0nt^{-1}n^{-1} = nn^{-1}t_0nn^{-1} = t_0.$$

Therefore, $ntn^{-1} \in \mathrm{Stab}_{G_0}(t_0) = T$. That is, as discussed above, $n \in \mathrm{Norm}_{G_0}(T) = T\Sigma_n$.

One can now verify that T fixes H pointwise, and that the elements of Σ_n which stabilize H are precisely Σ_{n-1}. □

We next need to calculate $\mathfrak{f}_{\bar{1}} := (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^H$.

Lemma 5.5.7. *The subvariety* $\mathfrak{f}_{\bar{1}} = (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^H$ *is the* $\mathbb{C}$*-span of the vectors*

$$\{\partial_1, \xi_1\xi_i\partial_i \mid i = 2, \ldots, n\}.$$

Proof. Since $H = T_{n-1}$, $\mathfrak{f}_{\bar{1}}$ is simply the span of all weight zero vectors with respect to this torus. Using the fixed basis of weight vectors for $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ established in Section 5.1 it can be seen that $\mathfrak{f}_{\bar{1}}$ is spanned by the given vectors. □

5.6 Explicit Description of the cohomology

We can now give an explicit description of the cohomology ring R. Let $Y_i \in \mathfrak{f}_{\bar{1}}^*$ be given by $Y_i(\xi_1\xi_j\partial_j) = \delta_{i,j}$ $(i, j = 2, \ldots, n)$ and $Y_i(\partial_1) = 0$. Let $\partial_1^* \in \mathfrak{f}_{\bar{1}}^*$ be given by $\partial_1^*(\xi_1\xi_j\partial_j) = 0$ for all $j = 2, \ldots, n$ and $\partial_1^*(\partial_1) = 1$.

Theorem 5.6.1. *Restriction of functions defines an isomorphism,*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S(\mathfrak{f}_{\bar{1}}^*)^N = \mathbb{C}[Y_2\partial_1^*, \ldots, Y_n\partial_1^*]^{\Sigma_{n-1}},$$

where Σ_{n-1} *acts on* $Y_2\partial_1^*, \ldots, Y_n\partial_1^*$ *by permutations. In particular,* R *is a polynomial ring in* $n-1$ *variables of degree* $2, 4, \ldots, 2n-2$.

Proof. The first isomorphism follows from (5.2) and [LR, Corollary 4.4]. Namely, $x_0 \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is a semisimple element with $\pi(x_0)$ a principal element of $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0$ so it follows by [LR, Corollary 4.4] that restriction of functions defines an isomorphism between $S((\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^*)^{G_0}$ and $S(\mathfrak{f}_{\bar{1}}^*)^N$. Since T is a normal subgroup of N, one can first compute that $S^\bullet(\mathfrak{f}_{\bar{1}}^*)^T = \mathbb{C}[Y_2\partial_1^*, \ldots, Y_n\partial_1^*]$ and check that Σ_{n-1} acts on this ring by permuting the variables. By a well known result on invariants under a symmetric group, it follows that R is a polynomial ring generated by elementary symmetric polynomials in the $Y_i\partial_1^*$, where degree of $Y_i\partial_1^*$ is two. □

Let $\mathfrak{f}_{\bar{1}} = (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)^H \subset \mathfrak{g}_{\bar{1}}$ be the subspace calculated in Lemma 5.5.7 and let

$$\mathfrak{f}_{\bar{0}} = \mathrm{Lie}(N) = \mathrm{Lie}(T) = \mathfrak{h} \subset \mathfrak{g}_{\bar{0}}.$$

One can verify by direct calculation that

$$[\mathfrak{f}_r, \mathfrak{f}_s] \subseteq \mathfrak{f}_{r+s} \tag{5.8}$$

for all $r, s \in \mathbb{Z}_2$. Thus $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ is a Lie subsuperalgebra of $\mathfrak{g}$. By [LR, Lemma 2.5] $\mathfrak{f}$ is unique up to conjugacy in the sense that if one chooses another semisimple $x \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ such that $\pi(x)$ is principal, then following the aforementioned construction leads to a subsuperalgebra which is G_0-conjuate to $\mathfrak{f}$.

Applying the definition of relative cohomology in Section 3.2 one can calculate $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})$ as follows. First, note that the $\mathbb{Z}_2$-grading implies that the differentials defining $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})$ are identically zero (cf. the proof of [BKN1, Theorem 2.5.2]). Thus the cohomology is given by the cochains; that is,

$$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong S\left(\mathfrak{f}_{\bar{1}}^*\right)^{\mathfrak{f}_{\bar{0}}}.$$

Furthermore, note that the elements of $S(\mathfrak{f}_{\bar{1}}^*)$ which are invariant under $\mathfrak{f}_{\bar{0}}$ are simply those of weight zero with respect to the torus T. Therefore, recalling that $S(\mathfrak{f}_{\bar{1}}^*) \cong \mathbb{C}[\partial_1^*, Y_2, \ldots, Y_n]$, one has

$$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong S\left(\mathfrak{f}_{\bar{1}}^*\right)^T = \mathbb{C}[Y_2\partial_1^*, \ldots, Y_n\partial_1^*].$$

The following theorem relates the $\mathfrak{g}$ and $\mathfrak{f}$ cohomology rings via the natural restriction map.

Theorem 5.6.2. *The inclusion map* $\mathfrak{f} \hookrightarrow \mathfrak{g}$ *induces a restriction map* $\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})$ *so that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C}) & \xrightarrow{\text{res}} & \mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar 0};\mathbb{C}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{C}[Y_2\partial_1^*,\ldots,Y_n\partial_1^*]^{\Sigma_{n-1}} & \xrightarrow{\subseteq} & \mathbb{C}[Y_2\partial_1^*,\ldots,Y_n\partial_1^*] \end{array} \tag{5.9}$$

That is, the restriction map induced by inclusion gives the following graded algebra isomorphism,

$$\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C}) \xrightarrow{\cong} \mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar 0};\mathbb{C})^{\Sigma_{n-1}} \cong \mathbb{C}[Y_2\partial_1^*,\ldots,Y_n\partial_1^*]^{\Sigma_{n-1}}.$$

In particular, both $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C})$ *and* $\mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar 0};\mathbb{C})$ *are isomorphic to graded polynomial rings in* $n-1$ *variables.*

Proof. The isomorphisms are a reinterpretation of Theorem 5.6.1 in terms of cohomology, and the commutativity of the diagram can be checked directly. □

5.7 Detecting Subalgebra for $W(n)$

Recall that $H = T_{n-1} = \{\mathrm{diag}(t_1,\ldots,t_n) \in T \mid t_1 = 1\}$. Let $\mathfrak{e}_{\bar 0} = \mathrm{Lie}(H) \subset \mathfrak{g}_0$ and $\mathfrak{e}_{\bar 1} = \mathfrak{f}_{\bar 1}$. Set

$$\mathfrak{e} = \mathfrak{e}_{\bar 0} \oplus \mathfrak{e}_{\bar 1}. \tag{5.10}$$

As with $\mathfrak{f}$ one can verify that $\mathfrak{e}$ is a Lie subsuperalgebra of $\mathfrak{g}$. In fact one has

$$[\mathfrak{e}_{\bar 0},\mathfrak{e}_{\bar 0}] = [\mathfrak{e}_{\bar 0},\mathfrak{e}_{\bar 1}] = 0. \tag{5.11}$$

One can also verify that the differentials defining $\mathrm{H}^\bullet(\mathfrak{e},\mathfrak{e}_{\bar 0};\mathbb{C})$ are identically zero and so the cohomology ring is again given by the cochains. In this case, however, $\mathfrak{e}_{\bar 0}$ acts trivially on $\mathfrak{e}_{\bar 1}$ and so one has

$$\mathrm{H}^\bullet(\mathfrak{e},\mathfrak{e}_{\bar 0};\mathbb{C}) \cong S(\mathfrak{e}_{\bar 1}^*) = S\left(\mathfrak{f}_{\bar 1}^*\right) \cong \mathbb{C}[\partial_1^*, Y_2,\ldots,Y_n].$$

Furthermore, the inclusion $\mathfrak{e} \hookrightarrow \mathfrak{f}$ defines a restriction map, res, so that the following diagram commutes,

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar 0};\mathbb{C}) & \xrightarrow{\text{res}} & \mathrm{H}^\bullet(\mathfrak{e},\mathfrak{e}_{\bar 0};\mathbb{C}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{C}[\partial_1^*Y_2,\ldots,\partial_1^*Y_n] & \xrightarrow{\subseteq} & \mathbb{C}[\partial_1^*, Y_2,\ldots,Y_n] \end{array} \tag{5.12}$$

Since $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^T$ and $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^{\Sigma_{n-1}}$ we have the following isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^{T\Sigma_{n-1}}.$$

As one can see from the discussion above the Lie subsuperalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ satisfies all the assumptions of a detecting subalgebra; i.e., it is classical and the restriction map $\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$ induces the isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^{T\Sigma_{n-1}}$$

with the group $T\Sigma_{n-1}$ being exact.

5.8 Support Varieties

The inclusion $\mathfrak{f} \hookrightarrow \mathfrak{g}$ induces a restriction map on cohomology which, in turn, induces maps of support varieties. That is, given modules M and N in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ one has $M \in \mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$ by restriction to $\mathfrak{f}$ and one has maps of varieties

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M, N) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M, N),$$
$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M).$$

Viewing the support varieties as subvarieties of $\mathbb{A}^{n-1}$ and using Theorem 5.6.2 one can explicitly describe this map as the quotient by the action of Σ_{n-1} on $\mathbb{A}^{n-1}$ by permutation of coordinates. Therefore one has

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)/\Sigma_{n-1} \cong \mathrm{res}^* \left(\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M). \tag{5.13}$$

We conjecture that the inclusion in (5.13) is in fact an equality for all finite dimensional $\mathfrak{g}$-modules $M \in \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$.

By using the pair $(\mathfrak{e}, \mathfrak{e}_{\bar{0}})$ and the setup of Section 4.1 one can define the support variety $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ for any finite dimensional module $M \in \mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$. Since $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$ is a polynomial ring in n variables, one can naturally identify the support variety of M with the conical affine subvariety of the affine n-space

$$\operatorname{MaxSpec}\left(\mathrm{H}^{\bullet}(\mathfrak{e},\mathfrak{e}_{\bar{0}};\mathbb{C})\right)=\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})\cong\mathbb{A}^{n}$$

defined by the ideal $I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$.

Since by (5.11) the structure of $\mathfrak{e}$ is of the type considered in [BKN1, Sections 5, 6], [BKN1, Theorem 6.3.2] implies that one has a canonical isomorphism

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\cong\mathcal{V}_{\mathfrak{e}}^{\mathrm{rank}}(M) \tag{5.14}$$

for any finite dimensional $\mathfrak{e}$-module M which is an object of $\mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$. We identify the rank and support varieties of $\mathfrak{e}$ via this isomorphism.

5.9 Relating $\mathfrak{e}$ and $\mathfrak{f}$ Support Varieties

We now wish to relate the support varieties of $\mathfrak{e}$- and $\mathfrak{f}$-modules. Note that if $M\in\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$, then via restriction it is an object in $\mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$. Therefore, whenever M is finite dimensional one has an induced map of varieties,

$$\mathrm{res}^{*}:\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\to\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M).$$

The present task is to better understand this map.

As a consequence of (5.12) one has that the map

$$\mathrm{res}^{*}:\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})\to\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C}).$$

is given by the canonical quotient map

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})\to\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})/T.$$

That is, for $M\in\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{0})}$ one has

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/T\cong\mathrm{res}^{*}\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right)\subseteq\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C}). \tag{5.15}$$

In fact one has the following theorem.

Theorem 5.9.1. *Let M be a finite dimensional object in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$, then*

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/T \cong \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)\right) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M). \tag{5.16}$$

Proof. The first isomorphism is (5.15). It remains to show that the map res^* is surjective.

To do so first requires a better understanding of the relationship between $\mathfrak{e}$ and $\mathfrak{f}$ cohomology with coefficents in a finite dimensional $\mathfrak{f}$-module U. Recall the definition of the cochains for relative cohomology in Section 3.2 and that $\mathfrak{e}_{\bar{1}} = \mathfrak{f}_{\bar{1}}$. If U is a finite dimensional module in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$, then the torus T acts on the cochains $C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U) = \operatorname{Hom}_{\mathfrak{f}_{\bar{0}}}(\Lambda_s^\bullet(\mathfrak{f}_{\bar{1}}),U)$ by $(t.\varphi)(x) = t\varphi(t^{-1}x)$ for all $t \in T$, $\varphi \in C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)$, and $x \in \Lambda_s^\bullet(\mathfrak{f}_{\bar{1}})$. If N is a T-module and $\lambda \in X(T)$ is a weight, then write N_λ for the λ weight space of N. Since T acts semisimply on $C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)$ one has

$$C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U) = C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)^T \oplus \bigoplus_{\substack{\lambda\in X(T)\\ \lambda\neq 0}} C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)_\lambda$$

as T-modules. Observe that the action of T commutes with the differential in the definition of relative cohomology. Thus one has

$$\begin{aligned}\mathrm{H}(\mathfrak{e},\mathfrak{e}_{\bar{0}};U) &= \mathrm{H}(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)^T \oplus \bigoplus_{\substack{\lambda\in X(T)\\ \lambda\neq 0}} \mathrm{H}(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)_\lambda \\ &\cong \mathrm{H}(\mathfrak{f},\mathfrak{f}_{\bar{0}};U) \oplus \bigoplus_{\substack{\lambda\in X(T)\\ \lambda\neq 0}} \mathrm{H}(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)_\lambda,\end{aligned}$$

where the isomorphism follows from the equality $C^\bullet(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)^T = C^\bullet(\mathfrak{f},\mathfrak{f}_{\bar{0}};U)$ and the exactness of taking T invariants. In particular, one has

$$\operatorname{res}: \mathrm{H}(\mathfrak{f},\mathfrak{f}_{\bar{0}};U) \xrightarrow{\cong} \mathrm{H}(\mathfrak{e},\mathfrak{e}_{\bar{0}};U)^T \subseteq \mathrm{H}(\mathfrak{e},\mathfrak{e}_{\bar{0}};U). \tag{5.17}$$

We are now prepared to prove the theorem. Let $(\mathfrak{a},\mathfrak{a}_{\bar{0}})$ denote either $(\mathfrak{f},\mathfrak{f}_{\bar{0}})$ or $(\mathfrak{e},\mathfrak{e}_{\bar{0}})$. Note that, just as for finite groups, an equivalent characterization of $I_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M)$ is the ideal of elements in $\mathrm{H}^\bullet(\mathfrak{a},\mathfrak{a}_{\bar 0};\mathbb{C})$ which annihilate the element $1_{\mathfrak{a},M} \in \mathrm{Ext}^0_{\mathcal{C}_{(\mathfrak{a},\mathfrak{a}_{\bar 0})}}(M,M)$ corresponding to the identity morphism. Note, too, that $\mathrm{res}(1_{\mathfrak{f},M}) = 1_{\mathfrak{e},M}$ for any $\mathfrak{f}$-module M and that $\mathrm{res}(x.z) = \mathrm{res}(x).\mathrm{res}(z)$ for any $x \in \mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar 0};\mathbb{C})$ and $z \in \mathrm{Ext}^\bullet_{\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar 0})}}(M,M)$.

Since the ideal $\mathrm{res}^{-1}\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar 0})}(M)\right)$ defines the variety $\mathrm{res}^*\left(V_{(\mathfrak{e},\mathfrak{e}_{\bar 0})}(M)\right)$, it suffices to prove

$$\mathrm{res}^{-1}\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar 0})}(M)\right) = I_{(\mathfrak{f},\mathfrak{f}_{\bar 0})}(M).$$

Let $x \in I_{(\mathfrak{f},\mathfrak{f}_{\bar 0})}(M)$. That is, $x.1_{\mathfrak{f},M} = 0$ and so

$$0 = \mathrm{res}(x.1_{\mathfrak{f},M}) = \mathrm{res}(x).\mathrm{res}(1_{\mathfrak{f},M}) = \mathrm{res}(x).1_{\mathfrak{e},M}.$$

That is, $\mathrm{res}(x) \in I_{(\mathfrak{e},\mathfrak{e}_{\bar 0})}(M)$ and so $x \in \mathrm{res}^{-1}\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar 0})}(M)\right)$.

Conversely, let $x \in \mathrm{res}^{-1}\left(I_{(\mathfrak{e},\mathfrak{e}_{\bar 0})}(M)\right)$. Then

$$0 = \mathrm{res}(x).1_{\mathfrak{e},M} = \mathrm{res}(x).\mathrm{res}(1_{\mathfrak{f},M}) = \mathrm{res}\left(x.1_{\mathfrak{f},M}\right).$$

However by (5.17) (applied to the case $U = M^* \otimes M$) one has that res is injective and so $0 = x.1_{\mathfrak{f},M}$. That is, $x \in I_{(\mathfrak{f},\mathfrak{f}_{\bar 0})}(M)$. This proves (5.16). □

Note that one outcome of the above proof is that for any finite dimensional module M in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar 0})}$ one has

$$\mathrm{H}^p(\mathfrak{f},\mathfrak{f}_{\bar 0};M) \cong \mathrm{H}^p(\mathfrak{e},\mathfrak{e}_{\bar 0};M)^T$$

for all $p \geq 0$.

5.10 Properties of $\mathfrak{f}$ Support Varieties

We record some basic properties of support varieties for $\mathfrak{f}$-modules which follow from the rank variety description of $\mathfrak{e}$ support varieties and the isomophism given in Theorem 5.9.1. The situation is reminiscent of the connection between support varieties for the Frobenius kernels G_r and G_rT considered in [Nak]. Other properties of rank varieties can be found in [BKN1, Theorem 6.4.2].

Theorem 5.10.1. *Let M, N, M_1, M_2 and M_3 be finite dimensional $\mathfrak{f}$-modules in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar 0})}$. Then,* *(a)* $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M \otimes N) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) \cap \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(N)$*;*

(b) $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M^*) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)$;

(c) $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M^* \otimes M) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)$;

(d) If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence, then

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M_i) \subseteq \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M_j) \cup \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M_k),$$

where $\{i,j,k\} = \{1,2,3\}$.

Proof. To prove part (a), one first notes that as a consequence of the rank variety description one has by [BKN1, Proposition 6.3.1] that

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \otimes N) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \cap \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N).$$

Then the above equality translates into the equality

$$\sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \otimes N)} = \sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) + I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)}.$$

Taking invariants with respect to T and applying (4.2), one obtains

$$\begin{aligned}
\sqrt{I_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M \otimes N)} &= \sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \otimes N)^T} \\
&= \left(\sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M \otimes N)}\right)^T \\
&= \left(\sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) + I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)}\right)^T \\
&= \sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)^T + I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N)^T} \\
&= \sqrt{I_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) + I_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(N)}.
\end{aligned}$$

This proves the desired equality of varieties.

Part (b) is proven by a similar but easier argument and part (c) follows from parts (a) and (b).

Finally, to prove part (d) one observes that the rank variety description implies (cf. [BKN1, Theorem 6.4.2(d)]) that one has

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_i) \subseteq \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_j) \cup \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_k),$$

where $\{i,j,k\} = \{1,2,3\}$. One then argues as in part (a) using instead that

$$\left(\sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_j)I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_k)}\right)^T = \sqrt{I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_j)^T I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M_k)^T} = \sqrt{I_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M_j)I_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M_k)}.$$

□

Another important property of support varieties is their ability to detect projectivity. This is illustrated by the following theorem.

Theorem 5.10.2. *Let M be a finite dimensional module in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$. Then the following are equivalent:*

(a) The module M is projective in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$;

(b) The module M is projective in $\mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$;

(c) The variety $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) = \{0\}$.

Proof. If M is a projective $\mathfrak{f}$-module, then it remains so upon restriction to $\mathfrak{e}$, hence one has that (a) implies (b).

To prove (b) implies (a) it suffices to show

$$\operatorname{Ext}^i_{\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}}(M,L) = \mathrm{H}^i(\mathfrak{f},\mathfrak{f}_{\bar{0}}; M^* \otimes L) = 0$$

for all objects L in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$ and $i > 0$. Since $\mathfrak{e}$ is an ideal in $\mathfrak{f}$ one can consider the Lyndon-Hochschild-Serre spectral sequence for the pairs $(\mathfrak{e},\mathfrak{e}_{\bar{0}}) \subseteq (\mathfrak{f},\mathfrak{f}_{\bar{0}})$:

$$E_2^{i,j} = \mathrm{H}^i\left(\mathfrak{f}/\mathfrak{e}, \mathfrak{f}_{\bar{0}}/\mathfrak{e}_{\bar{0}}; \mathrm{H}^j(\mathfrak{e},\mathfrak{e}_{\bar{0}}; M^* \otimes L)\right) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{f},\mathfrak{f}_{\bar{0}}; M^* \otimes L).$$

By assumption M is a projective object in $\mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$ and so $\mathrm{H}^j(\mathfrak{e},\mathfrak{e}_{\bar{0}}; M^* \otimes L) = 0$ for $j > 0$ and the spectral sequence collapses. That is, for $i \geq 0$ one has

$$\mathrm{H}^i\left(\mathfrak{f}/\mathfrak{e}, \mathfrak{f}_{\bar{0}}/\mathfrak{e}_{\bar{0}}; (M^* \otimes L)^{\mathfrak{e}}\right) \cong \mathrm{H}^i(\mathfrak{f},\mathfrak{f}_{\bar{0}}; M^* \otimes L).$$

Since the objects of $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$ are finitely semisimple as $\mathfrak{f}_{\bar{0}}$-modules and since $\mathfrak{f}/\mathfrak{e} = \mathfrak{f}_{\bar{0}}/\mathfrak{e}_{\bar{0}}$ is a one dimensional subtorus of $\mathfrak{f}_{\bar{0}}$ one has $\mathrm{H}^i(\mathfrak{f}/\mathfrak{e}, \mathfrak{f}_{\bar{0}}/\mathfrak{e}_{\bar{0}}; (M^* \otimes L)^{\mathfrak{e}}) = 0$ for $i > 0$. It follows that $\mathrm{H}^i(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; M^* \otimes L) = 0$ and so M is projective in $\mathcal{C}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$.

The equivalence of (b) and (c) follows from [BKN1, Theorem 6.4.2(b)]. □

Note that it is *not* true that if $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) = \{0\}$, then M is projective as a $\mathfrak{f}$-module. One can find examples of $\mathfrak{f}$-modules, M, so that $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \neq \{0\}$, but by (5.16)

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) \cong \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/T = \{0\}.$$

On the other hand, by the previous theorem M is not projective as an $\mathfrak{f}$-module since $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \neq \{0\}$.

5.11 Calculation of Support Varieties

Recall that for $\lambda \in X_0^+$ we constructed the Kac module $K(\lambda)$ in Section 2.6. The following result shows that the $\mathfrak{g}$ and $\mathfrak{f}$ support varieties are zero for all Kac modules.

Proposition 5.11.1. *Let $\lambda \in X_0^+$ and N be a finite dimensional module in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$. Then,*

(a) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda), N) = \{0\}$;

(b) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda)) = \{0\}$;

(c) $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)) = \{0\}$.

Proof. We present a modified version of the argument used to prove [BKN2, Theorem 3.2.1]. First observe that part (b) follows immediately from part (a). Also, as in the proof of [BKN2, Corollary 3.3.1], for part (a) it suffices to prove that for n sufficiently large, $\mathrm{Ext}^n_{\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda), N) = 0$.

By Frobenius reciprocity, for all n we have

$$\mathrm{Ext}^n_{\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda), N) \cong \mathrm{Ext}^n_{\mathcal{C}_{(\mathfrak{g}_0 \oplus \mathfrak{g}^+,\mathfrak{g}_0)}}(L_0(\lambda), N).$$

Since $\mathfrak{g}^+$ is an ideal in $\mathfrak{g}_0 \oplus \mathfrak{g}^+$ one can apply the Lyndon-Hochschild-Serre spectral sequence to $(\mathfrak{g}^+, \{0\}) \subseteq (\mathfrak{g}_0 \oplus \mathfrak{g}^+, \mathfrak{g}_0)$:

$$E_2^{i,j} = \operatorname{Ext}^i_{\mathcal{C}_{(\mathfrak{g}_0,\mathfrak{g}_0)}}(L_0(\lambda), \operatorname{Ext}^j_{\mathcal{C}_{(\mathfrak{g}^+,\{0\})}}(\mathbb{C}, N)) \Rightarrow \operatorname{Ext}^{i+j}_{\mathcal{C}_{(\mathfrak{g}_0\oplus\mathfrak{g}^+,\mathfrak{g}_0)}}(L_0(\lambda), N).$$

Since $\mathcal{C}_{(\mathfrak{g}_0,\mathfrak{g}_0)}$ consists of $\mathfrak{g}_0$-modules which are finitely semisimple over $\mathfrak{g}_0$, this spectral sequence is zero for $i > 0$. That is, it collapses at the E_2-page and yields

$$\operatorname{Hom}_{\mathfrak{g}_0}(L_0(\lambda), \operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g}^+,\{0\})}}(\mathbb{C}, N)) \cong \operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g}_0\oplus\mathfrak{g}^+,\mathfrak{g}_0)}}(L_0(\lambda), N). \tag{5.18}$$

According to the definition of relative cohomology, $\operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g}^+,\{0\})}}(\mathbb{C}, N)$ is a subquotient of $\Lambda^n_s((\mathfrak{g}^+)^*) \otimes N$. But $\Lambda^n_s((\mathfrak{g}^+)^*)$ is positively graded by degree and N is finite dimensional so for sufficiently large n (depending on λ), $\Lambda^n_s((\mathfrak{g}^+)^*) \otimes N$ contains no composition factors of the form $L_0(\lambda)$. Thus $\operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda), N) = 0$ for $n \gg 0$.

Finally, to prove part (c) it suffices to observe that the map $\operatorname{res}^* : \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda)) = \{0\}$ is finite-to-one. Since $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda))$ is a conical variety it follows that it must be equal to $\{0\}$. □

Recall that Serganova [Ser, Lemma 5.3] proved that the set of atypical weights for $\mathfrak{g}$ is

$$\Omega = \{a\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n \mid a \in \mathbb{C},\ 1 \le i \le n\}.$$

Moreover, she determined the characters of the simple $\mathfrak{g}$-modules by determining composition series for the Kac modules. Serganova's abridged results for finite dimensional simple modules are presented in the following theorem.

Theorem 5.11.2. *[Ser, Theorem 6.3, Corollary 7.6] Let $\lambda \in X_0^+$.*

(a) If $\lambda \notin \Omega$ then $K(\lambda) \cong L(\lambda)$.

(b) Let $\lambda \in \Omega$.

(i) If $\lambda = a\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n$ with $a \neq 0, 1$, then there is the following exact sequence:

$$0 \to L(\lambda - \varepsilon_i) \to K(\lambda) \to L(\lambda) \to 0. \tag{5.19}$$

(ii) The structure of $K(0)$ and $K(\varepsilon_1 + \cdots + \varepsilon_n)$ is described by the exact sequences

$$0 \to L(-\varepsilon_n) \to K(0) \to L(0) \to 0, \tag{5.20}$$

$$0 \to L(0) \to K(\varepsilon_1 + \cdots + \varepsilon_n) \to L(\varepsilon_1 + \cdots + \varepsilon_n) \to 0. \tag{5.21}$$

From the above theorem one has an alternative characterization of typical/atypical for $\lambda \in X_0^+$: namely, λ is typical if and only if $K(\lambda)$ is simple.

The following theorem presents the computation of support varieties of simple $\mathfrak{g}$-modules. Our results demonstrate that $L(\lambda)$ is typical if and only if the support variety of $L(\lambda)$ is zero.

Theorem 5.11.3. *Let $\lambda \in X_0^+$ and let $L(\lambda)$ be finite dimensional simple $\mathfrak{g}$-module with highest weight λ.*

(a) If $\lambda \notin \Omega$ then $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = \{0\}$;

(b) If $\lambda \in \Omega$ then $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C})$ and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)) = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C})$.

Proof. Part (a) is immediate from Proposition 5.11.1(c) and Theorem 5.11.2(a).

To prove part (b), first observe that it suffices to prove $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C})$. Namely, one will then have

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}) = \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C})\right) = \operatorname{res}^*\left(\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda))\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}),$$

which implies the result for $\mathfrak{g}$. Furthermore, observe from Section 2.8 that

$$\Omega \cap X_0^+ = \{a\varepsilon_1 + \cdots + \varepsilon_n \mid a = 1, 2, 3, \ldots\} \cup \{b\varepsilon_n \mid \ b = 0, -1, \ldots\}.$$

We will repeatedly use two facts about support varieties of $\mathfrak{f}$-modules: the support variety of a module in a short exact sequence is contained in the union of support variety of the other two modules by Theorem 5.10.1(d), and the support variety of a Kac module is zero by Proposition 5.11.1(c).

Note that $L(0) \cong \mathbb{C}$ and set $\mathcal{V} := \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C})$. From (5.20) it follows that $\mathcal{V} = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(-\varepsilon_n))$. One can now use the remarks from the previous paragraph and (5.19) to recursively prove that $\mathcal{V} = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(-b\varepsilon_n))$ for $b = 2, 3, \dots$. Similarly, we have $\mathcal{V} = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\varepsilon_1 + \cdots + \varepsilon_n))$ from (5.21). Applying (5.19) recursively shows that $\mathcal{V} = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(a\varepsilon_1 + \cdots + \varepsilon_n))$ for all $a = 2, 3, \dots$. Note that all elements of $\Omega \cap X_0^+$ were considered above and thus the theorem is proven. □

5.12 Realization of Support varieties

The goal of this section is to show that a realization theorem holds for $\mathfrak{g}$. Recall that $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ denotes the category of $\mathfrak{g}$-modules which are finitely semisimple as $\mathfrak{g}_0$-modules and L_ζ denotes the Carlson module corresponding to the homogeneous element $\zeta \in \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$. We also remind the reader that Carlson modules may not be finite dimensional which is why we are going to work with relative support varieties.

Let us summarize what we know about the pair $(\mathfrak{g}, \mathfrak{g}_0)$:

(1) $\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M^* \otimes N)$ for $\mathfrak{g}$-modules $M, N \in \mathcal{C}$. Since $\mathfrak{g}$ is finitely semisimple as a $\mathfrak{g}_0$-module under the adjoint action the isomorphism follows from Proposition 3.3.1.

(2) $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ is a polynomial ring with $n-1$ variables (Theorem 5.6.1). Thus Hypothesis 3.4.4 holds for $\mathfrak{g}$.

(3) $\mathfrak{g}$ has a detecting subalgebra $\mathfrak{e}$ (Section 5.7) and $\mathfrak{e}$ support varieties have rank variety descriptions (Section 5.8); i.e., Hypotheses 3.5.3 and 4.2.2 hold for $\mathfrak{g}$.

(4) $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_{\mathrm{res}(\zeta)}) = \mathcal{Z}(\mathrm{res}(\zeta))$. This follows from Lemma 4.5.1.

(5) $\mathrm{Ext}^\bullet_{\mathcal{C}}(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}, \mathbb{C})$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$. This follows from (1) and Proposition 4.5.2.

(6) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_s}, \mathbb{C}) \subseteq \cap_{i=1}^s \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L_{\zeta_i}, \mathbb{C})$ This follows form (1) and Proposition 4.5.2.

(7) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L_\zeta,\mathbb{C}) = \operatorname{res}^*\big(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\big) = \mathcal{Z}(\zeta)$ This follows from Proposition 4.6.1.

We are now ready to prove the realization theorem for $\mathfrak{g}$.

Theorem 5.12.1. *Let X be a conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C})$. Then there exists a $\mathfrak{g}$-module $M \in \mathcal{C}$ such that*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M,\mathbb{C}) = X.$$

Proof. As we have seen from the discussion above the Hypotheses 3.4.4, 3.5.3 and 4.2.2 hold for $\mathfrak{g}$. Since the assumptions hold the result follows from Theorem 4.7.1. □

Chapter 6

Cohomology and support varieties for $\bar{S}(n)$

6.1 $S(n)$

Since $W(n)_0$ is canonically isomorphic to $\mathfrak{gl}(n)$, $W(n)$ has a natural structure of a $\mathfrak{gl}(n)$-module. The action on $W(n)$ of the center of $\mathfrak{gl}(n)$ is well-known so we may restrict our attention to the $\mathfrak{sl}(n)$-module structure of $W(n)$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}(n)$. Choose a set of simple roots of $\mathfrak{sl}(n)$ with respect to $\mathfrak{h}$ and let $\omega_1, \ldots, \omega_{n-1}$ be the fundamental weights. Since $W(n)_k \cong \Lambda^{k+1}(V) \otimes V^*$, the representation of $\mathfrak{sl}(n)$ in $W(n)_k$ is equivalent to $L(\omega_{k+1}) \otimes L(\omega_{n-1})$, $-1 \leq k \leq n-1$. On the other hand we have $L(\omega_{k+1}) \otimes L(\omega_{n-1}) \cong L(\omega_{k+1} + \omega_{n-1}) \oplus L(\omega_k)$ if $0 \leq k \leq n-2$.

Let $-1 \leq k \leq n-2$. Then the subspace of $W(n)_k \cong \Lambda^{k+1}(V) \otimes V^*$ corresponding to $L(\omega_{k+1}+\omega_{n-1})$ is denoted by $S(n)_k$. It is easy to see that $S(n) = \bigoplus_{k=-1}^{n-2} S(n)_k$ is a subalgebra of $W(n)$ and $\mathbb{Z}$-grading induces the $\mathbb{Z}_2$-grading. Note also that $S(n)_0 \cong \mathfrak{sl}(n)$.

6.2 Basis for $S(n)$

We now describe basis elements for $S(n)$. Let $N = \{1, \ldots, n\}$ and let I be an ordered subset of N. A spanning set for each $S(n)_k$ can be defined as follows and contains two distinct types of elements. Those elements of type (I, k) are all those of the form $\xi_I \partial_i$ with $i \notin I$ and $|I| = k+1$. Those of type (II, k) are of the form $\xi_A h_{ij}$ where $i, j \notin A$ and $|A| = k$. Here by definition $h_{ij} = \xi_i \partial_i - \xi_j \partial_j$.

The type I elements are all linearly independent, and their span $S(n)_k^I$ is independent of the span $S(n)_k^{II}$ of the type II elements. The type II elements are not independent however, since $h_{ij} + h_{jk} = h_{ik}$. We reduce the set of type II elements to a basis for $S(n)_k^{II}$ as follows. For each A with $|A| = k$, order the complement $B = N - A$ in the natural way as a subset of N and let i be the first element of B. Select those element of the form $\xi_A h_{ij}$ where $i < j \in B$. These are easily seen to be independent and span $S(n)_k^{II}$.

The calculations used to justify the last statement are essentially the same as those showing that the standard basis for the Lie algebra $\mathfrak{sl}(n)$ is indeed a basis. This is not an accident as the restriction of the isomorphism $W(n)_0 \cong \mathfrak{gl}(n)$ carries $S(n)_0$ onto $\mathfrak{sl}(n)$. Under this the type (I,0) basis elements $\xi_i\partial_j$ correspond to the off-diagonal matrix units E_{ij} and the type (II,0) basis elements h_{2j} to the diagonal elements $E_{22} - E_{jj}$.

6.3 $\bar{S}(n)$

Let $\mathcal{E} = \Sigma_{i=1}^n \xi_i \partial_i$. In order to keep track of the $\mathbb{Z}$-grading we will attach $\mathcal{E}$ to $S(n)$ and consider the subalgebra $\bar{S}(n) = S(n) \oplus \mathbb{C}\mathcal{E}$ of $W(n)$. The Lie superalgebra $\bar{S}(n)$ admits a $\mathbb{Z}$-grading and in this grading $\bar{S}(n)_0 \cong \mathfrak{gl}(n)$ and $\bar{S}(n)_k = S(n)_k$ for $\neq 0$. We use the following notational convention throughout this chapter. Set $\mathfrak{g} = \bar{S}(n)$ with $\mathfrak{g}_i = \bar{S}(n)_i$, $i \in \mathbb{Z}$, and $\mathfrak{g}_{\bar{\imath}} = \bar{S}(n)_{\bar{\imath}}$, $\bar{\imath} \in \mathbb{Z}_2$. Moreover, let $\mathfrak{g}^+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-2}$ and $\mathfrak{g}^- = \mathfrak{g}_{-1}$, so that Lie superalgebra has a lopsided triangular decomposition

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^+.$$

A $\mathfrak{g}$-module will be assumed to be an object in the category $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$.

6.4 Cohomology in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$

We begin by showing that the calculation of $\mathfrak{g}_0$-invariants on $\Lambda_s^\bullet((\mathfrak{g}/\mathfrak{g}_0)^*)$ reduces to looking at the $\mathfrak{g}_0$-invariants on $\Lambda_s^\bullet(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*)$. This will be accomplished by using detailed information about the representation theory of $\mathfrak{g}_0$.

Theorem 6.4.1. *Let $\mathfrak{g} = \bar{S}(n)$ and let $p \geq 0$. Then, $\Lambda_s^p((\mathfrak{g}/\mathfrak{g}_0)^*)^{\mathfrak{g}_0} \cong S^p(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*)^{\mathfrak{g}_0}$.*

Proof. First observe that $\mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-2}$ as $\mathfrak{g}_0$-modules. We then have

$$\begin{aligned}
&\Lambda_s^p(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \oplus \cdots \oplus \mathfrak{g}_{n-2}^*)^{\mathfrak{g}_0} \\
&\qquad \cong \bigoplus \left(\Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}^*) \otimes \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-2}}(\mathfrak{g}_{n-2}^*)\right)^{\mathfrak{g}_0} \\
&\qquad \cong \bigoplus \operatorname{Hom}_{\mathfrak{g}_0}\left(\mathbb{C}, \Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}^*) \otimes \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-2}}(\mathfrak{g}_{n-2}^*)\right) \\
&\qquad \cong \bigoplus \operatorname{Hom}_{\mathfrak{g}_0}\left(\Lambda_s^{i_{-1}}(\mathfrak{g}_{-1}), \Lambda_s^{i_1}(\mathfrak{g}_1^*) \otimes \Lambda_s^{i_2}(\mathfrak{g}_2^*) \otimes \cdots \otimes \Lambda_s^{i_{n-2}}(\mathfrak{g}_{n-2}^*)\right)
\end{aligned}$$

where the direct sums are taken over all nonnegative integers $i_{-1}, i_1, \ldots, i_{n-2}$ such that $i_{-1} + i_1 + \cdots + i_{n-2} = p$. Now one can argue as in the proof of the Theorem 5.2.1. □

The previous theorem can be used to show that the cohomology ring $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ can be identified with a ring of invariants. Let $G_0 \cong GL(n)$ be the connected reductive group such that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$ and the adjoint action of G_0 on $\mathfrak{g}$ differentiates to the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$.

Theorem 6.4.2. *Let* $\mathfrak{g} = \bar{S}(n)$. *Then,*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*)^{\mathfrak{g}_0} = S(\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^*)^{G_0}.$$

Proof. This follows from Theorem 6.4.1 and the argument used in the proof of the Theorem 5.3.1. □

Theorem 6.4.3. *Let M be a finite dimesional* $\mathfrak{g} = \bar{S}(n)$*-module.*

(a) The superalgebra $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ *is a finitely generated commutative ring.*

(b) The cohomology $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M)$ *is finitely generated as an* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$*-module.*

Proof. Follows from Theorem 6.4.2 and Theorem 3.4.3. □

6.5 Invariant theory Calculations

Fix the same notation as in the previous section. The Lie superalgebra $\mathfrak{g} = \bar{S}(n)$ admits a $\mathbb{Z}$-grading and $\mathfrak{g}_0 \cong \mathfrak{sl}(n) \oplus \mathbb{C} \cong \mathfrak{gl}(n)$ as a Lie algebra. In light of [BKN1], we are interested in the natural problem of computing the relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_0)$.

By Theorem 6.4.1, we can reduce the calculation via the following isomorphism:

$$R = \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S^{\bullet}(W^*)^{\mathfrak{g}_0}, \tag{6.1}$$

where $W := \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Thus to compute R, it suffices to compute the invariant ring on the right hand side of (6.1). To do so, we use the technology of [LR]. The first step is to obtain an element $x_0 \in W$ which is semisimple and principal in the sense of [LR].

The group G_0 acts on $\mathfrak{g}$ by the adjoint action and its action preserves the $\mathbb{Z}$-grading of $\mathfrak{g}$. Let $\beta : W \to \mathfrak{g}_0$ be given by $\beta(x, y) = [x, y]$ where $x \in \mathfrak{g}_{-1}$ and $y \in \mathfrak{g}_1$. Note that this map is G_0-equivariant.

Fix $T \subseteq G_0$ to be the maximal torus consisting set of all diagonal matrices and $\mathfrak{h} \subseteq \mathfrak{g}_0$ to be $\mathfrak{h} = \mathrm{Lie}\,(T)$, the Cartan subalgebra of $\mathfrak{g}_0$.

Recall that $\mathfrak{g}_{-1} = V^*$ as a $\mathfrak{g}_0$-module and has basis $\{\partial_i \mid 1 \leq i \leq n\}$, $\mathfrak{g}_0$ has basis $\{\xi_i\partial_j \mid 1 \leq i, j \leq n\}$ and $\mathfrak{g}_1$ has basis $\{\xi_i\xi_j\partial_k \mid 1 \leq i \neq k, j \neq k \leq n,\ i < j\} \cup \{\xi_1 h_{2j} \mid 3 \leq j \leq n\} \cup \{\xi_i h_{1k} \mid 2 \leq i \neq k \leq n\}$.

Lemma 6.5.1. *Let $h \in \mathfrak{h}$ be a semisimple regular element and write*

$$h = \sum_{i=1}^{n} c_i \xi_i \partial_i$$

Then we have the following.

(a) If all the entries of h are nonzero, then $\beta^{-1}(h) = \emptyset$.

(b) If $h = \mathrm{diag}(c_1, c_1 + c_3 + \ldots + c_n, c_3, \ldots, c_n)$ and $x \in \beta^{-1}(h)$, then

$$x = a_1\partial_1 + \sum_{i=3}^{n} \frac{c_i}{a_1}(\xi_1\xi_2\partial_2 - \xi_1\xi_i\partial_i) + \sum_{\substack{r,s,t \\ r\neq s\neq t \\ 1<r<s}} b_{r,s,t}\xi_r\xi_s\partial_t, \tag{6.2}$$

where $a_1, b_{r,s,t} \in \mathbb{C}$, and $a_1 \neq 0$.

Proof. We only sketch the calculation here. First, let $x \in \beta^{-1}(h)$, and write $x \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ in our preferred basis:

$$x = \sum_i a_i\partial_i + \sum_{l=3}^{n} b_{1,l}\xi_1 h_{2l} + \sum_{\substack{i,k \\ 2\leq i\neq k\leq n}} d_{i,k}\xi_i h_{1k} + \sum_{\substack{i,j,k \\ 1<i<j}} e_{i,j,k}\xi_i\xi_j\partial_k.$$

By a direct calculation of $\beta(x)$, one sees that if $a_t \neq 0$ for some $1 \leq t \leq n$, then necessarily $c_t = 0$. Therefore, if all the entries of h are nonzero, then there is no x such that $\beta(x) = h$. This proves part (a). Now let $c_1 = 0$. By our earlier remark, $a_2 = \cdots = a_n = 0$. This observation simplifies the calculation of $\beta(x)$. Doing so and using that the image is equal to h, one obtains (6.2).

□

Note that if $h \in \mathfrak{h}$ has any entry equal to zero then up to G_0-conjugacy (indeed up to Weyl group conjugacy), one can assume it is in the upper left corner. Thus the general case is easily deduced from part (b) of the lemma.

Proposition 6.5.2. *Let $h \in \mathfrak{h}$ be a semisimple regular element as in part (b) of the previous lemma. Let $x_0 \in \beta^{-1}(h)$ be chosen so that all the coefficents $b_{r,s,t}$ are zero in (6.2). Then $x_0 \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$ is a semisimple element.*

Proof. First one computes $\mathrm{Stab}_{G_0}(x)$ for any $x \in \beta^{-1}(h)$. By Lemma 5.5.2 and Lemma 5.5.1(a) one has $\mathrm{Stab}_{G_0}(x) \subseteq \mathrm{Stab}_{G_0}(h) = T$. By part (b) of the previous lemma x is a linear combination of distinct weight vectors. From this one sees that $t = \mathrm{diag}(t_1, t_2, \ldots, t_n) \in T$ fixes x if $t_1 = t_2 = 1$. This is also sufficient in the case of x_0. Otherwise there will be additional contraints on t and the stabilizer will be a proper, smaller dimensional subgroup of

$$T_{n-2} := \{t = \mathrm{diag}(t_1, t_2 \ldots, t_n) \in T \mid t_1 = t_2 = 1\}. \tag{6.3}$$

Thus, x_0 has maximal stabilizer dimension, hence minimal orbit dimension in the closed set $\beta^{-1}(G.y)$. But by the well known fact on the closure of orbits, it follows that $G.x_0$ must be closed.

□

Let

$$\pi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0 \qquad \text{and} \qquad p : \mathfrak{g}_0 \to \mathfrak{g}_0/G_0$$

denote the canonical quotient morphisms. Let $\varphi : (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0 \to \mathfrak{g}_0/G_0$ be the morphism induced by the map $p \circ \beta : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_0/G_0$. That is, the following diagram commutes.

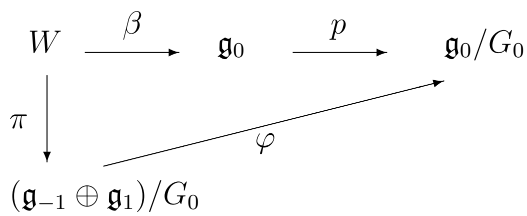


Figure 6.1:

Proposition 6.5.3. *Let* $x_0 \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ *be as in Proposition 6.5.2. Then* $\pi(x_0)$ *is a principal element of* $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0$.

Proof. Let $U := \varphi^{-1}((\mathfrak{g}_0/G_0)_{\text{pr}})$. By definition, $\beta(x_0)$ is semisimple and regular, so $p(\beta(x_0)) = \varphi(\pi(x_0))$ is principal in $\mathfrak{g}_0/G_0$. That is, $\pi(x_0) \in U$. Therefore, U is a nonempty open neighborhood of $\pi(x_0)$ in $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_1)/G_0$. Let $\zeta \in U$ and let $y \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ be a semisimple element in $\pi^{-1}(\zeta)$. Then $\varphi(\pi(y)) = p(\beta(y))$, so $\beta(y) \in p^{-1}(\eta)$ for some principal $\eta \in \mathfrak{g}_0/G_0$. But then $\eta = p(h)$ for some semisimple regular $h \in \mathfrak{h}$. However since h is semisimple and regular, it follows that $p^{-1}(\eta) = G_0.h$. That is, up to G_0-conjugacy one can assume $\beta(y)$ is a semisimple regular element of $\mathfrak{h}$. However, this implies that y is of the form given in Lemma 6.5.1 and the stabilizer of such elements was computed in the proof of Proposition 6.5.2. By that calculation and the fact that y is semisimple one sees that the stabilizer of y is T_{n-2}. Therefore all semisimple elements in the fibers of U have stabilizer conjugate to T_{n-2} and so $\pi(x_0)$ is principal. □

Now that we've proved that the element $x_0 \in W$ considered above satisfies the requirements of the Luna-Richardson theorem [LR] (i.e., that x_0 is semisimple and $\pi(x_0)$ is principal), we can calculate R using their theory. Namely,

Let

$$H = \text{Stab}_{G_0}(x_0) = T_{n-2} = \{t = (t_1, t_2, \ldots, t_n) \in T \mid t_1 = t_2 = 1\}, \tag{6.4}$$

and let

$$N = \mathrm{Norm}_{G_0}(H). \tag{6.5}$$

We first need to calculate the group N.

Lemma 6.5.4. *Let $N = \mathrm{Norm}_{G_0}(H)$. Recall that T is the torus of G_0. Let Σ_n be the permutation matrices of G_0 and let Σ_{n-2} be the permutation matrices which normalize T_{n-2}. Then,*

$$N = T\Sigma_{n-2}.$$

Proof. The first step is to prove that $N \subseteq \mathrm{Norm}_{G_0}(T)$. Fix a semisimple regular element $t_0 \in T_{n-2}$ (for the action of G_0 on itself by conjugation). Then $T = \mathrm{Stab}_{G_0}(t_0)$. Let $n \in N$. We claim that nTn^{-1} fixes t_0, hence $nTn^{-1} = T$, hence $n \in \mathrm{Norm}_{G_0}(T)$. Let $t \in T$ and consider

$$(ntn^{-1})t_0(ntn^{-1})^{-1} = ntn^{-1}t_0nt^{-1}n^{-1}.$$

However, since $t_0 \in T_{n-2}$ and $n^{-1} \in \mathrm{Norm}_{G_0}(T_{n-2})$, one has that $n^{-1}t_0n \in T_{n-2} \subseteq T$; since $t \in T$ and T fixes T pointwise under conjugation, one has $tn^{-1}t_0nt^{-1} = n^{-1}t_0n$. Thus,

$$ntn^{-1}t_0nt^{-1}n^{-1} = nn^{-1}t_0nn^{-1} = t_0.$$

Therefore, $ntn^{-1} \in \mathrm{Stab}_{G_0}(t_0) = T$. That is, as discussed above, $n \in \mathrm{Norm}_{G_0}(T) = T\Sigma_{n-1}$.

One can now verify that T fixes H pointwise, and that the elements of Σ_{n-1} which stabilize H are precisely Σ_{n-2}. □

We next need to calculate $\mathfrak{f}_{\bar{1}} = W^H$.

Lemma 6.5.5. *The subvariety $\mathfrak{f}_{\bar{1}} = W^H$ is the $\mathbb{C}$-span of the vectors*

$$\{\partial_1, \xi_1\xi_2\partial_2 - \xi_1\xi_i\partial_i \mid i = 3, \ldots, n\}.$$

Proof. Since $H = T_{n-2}$, $\mathfrak{f}_{\bar{1}}$ is simply all weight zero vectors with respect to this torus. Using our choice of weight basis for W, it is straightforward to see that $\mathfrak{f}_{\bar{1}}$ is spanned by the given vectors. □

6.6 Explicit Description of the cohomology

We are now give an explicit description of the cohomology ring R. Let $Z_k \in \mathfrak{f}_{\bar{1}}^*$ be given by $Z_k(\xi_1\xi_2\partial_2 - \xi_1\xi_i\partial_i) = \delta_{i,k}$ $(i,k = 3,\dots,n)$ and $Z_k(\partial_1) = 0$. Let ∂_1^* be given by $\partial_1^*(\xi_1\xi_2\partial_2 - \xi_1\xi_i\partial_i) = 0$ for all $i = 3,\dots,n$ and $\partial_1^*(\partial_1) = 1$.

Theorem 6.6.1. *Restriction of functions defines an isomorphism,*

$$\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C}) \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N = \mathbb{C}[Z_3\partial_1^*,\dots,Z_n\partial_1^*]^{\Sigma_{n-2}},$$

where Σ_{n-2} acts on $Z_3\partial_1^*,\dots,Z_n\partial_1^*$ by permuations. Therefore, R is a polynomial ring in $n-2$ variables of degree $2,4,\dots,2n-4$.

Proof. The first isomorphism is the Luna-Richardson Theorem of [LR]. Since T is a normal subgroup of N, one can first compute that $S^\bullet(\mathfrak{f}_{\bar{1}}^*)^T = \mathbb{C}[Z_3\partial_1^*,\dots,Z_n\partial_1^*]$. It's straightforward to check that Σ_{n-2} acts by permuting the variables $Z_3\partial_1^*,\dots,Z_n\partial_1^*$. This gives the stated result.

□

Let $\mathfrak{f}_{\bar{1}} = (\mathfrak{g}_{-1}\oplus\mathfrak{g}_1)^H \subset \mathfrak{g}_{\bar{1}}$ be the subspace calculated in Lemma 6.5.5 and let

$$\mathfrak{f}_{\bar{0}} = \mathrm{Lie}(N) = \mathrm{Lie}(T) = \mathfrak{h} \subset \mathfrak{g}_{\bar{0}}.$$

One can verify by direct calculation that

$$[\mathfrak{f}_r,\mathfrak{f}_s] \subseteq \mathfrak{f}_{r+s} \tag{6.6}$$

for all $r,s \in \mathbb{Z}_2$. Thus $\mathfrak{f} = \mathfrak{f}_{\bar{0}}\oplus\mathfrak{f}_{\bar{1}}$ is a Lie subsuperalgebra of $\mathfrak{g}$. By [LR, Lemma 2.5] $\mathfrak{f}$ is unique up to conjugacy in the sense that if one chooses another semisimple $x \in \mathfrak{g}_{-1}\oplus\mathfrak{g}_1$ such that $\pi(x)$ is principal, then following the aforementioned construction leads to a subsuperalgebra which is G_0-conjugate to $\mathfrak{f}$.

Applying the definition of relative cohomology in Section 3.2 one can calculate $\mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar{0}};\mathbb{C})$ as follows. First, note that the $\mathbb{Z}_2$-grading implies that the differentials defining $\mathrm{H}^\bullet(\mathfrak{f},\mathfrak{f}_{\bar{0}};\mathbb{C})$

are identically zero (cf. the proof of [BKN1, Theorem 2.5.2]). Thus the cohomology is given by the cochains; that is,

$$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong S\left(\mathfrak{f}_{\bar{1}}^*\right)^{\mathfrak{f}_{\bar{0}}}.$$

Furthermore, note that the elements of $S(\mathfrak{f}_{\bar{1}}^*)$ which are invariant under $\mathfrak{f}_{\bar{0}}$ are simply those of weight zero with respect to the torus T. Therefore, recalling that $S(\mathfrak{f}_{\bar{1}}^*) \cong \mathbb{C}[\partial_1^*, Z_3, \ldots, Z_n]$, one has

$$\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong S\left(\mathfrak{f}_{\bar{1}}^*\right)^T = \mathbb{C}[Z_3\partial_1^*, \ldots, Z_n\partial_1^*].$$

The following theorem relates the $\mathfrak{g}$ and $\mathfrak{f}$ cohomology rings via the natural restriction map.

Theorem 6.6.2. *The inclusion map* $\mathfrak{f} \hookrightarrow \mathfrak{g}$ *induces a restriction map* $\mathrm{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})$ *so that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) & \xrightarrow{\mathrm{res}} & \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathbb{C}[Z_3\partial_1^*, \ldots, Z_n\partial_1^*]^{\Sigma_{n-2}} & \xrightarrow{\subseteq} & \mathbb{C}[Z_3\partial_1^*, \ldots, Z_n\partial_1^*] \end{array} \tag{6.7}$$

That is, the restriction map induced by inclusion gives the following graded algebra isomorphism,

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \xrightarrow{\cong} \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^{\Sigma_{n-2}} \cong \mathbb{C}[Z_3\partial_1^*, \ldots, Z_n\partial_1^*]^{\Sigma_{n-2}}.$$

In particular, both $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ *and* $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})$ *are isomorphic to graded polynomial rings in* $n-2$ *variables.*

Proof. The isomorphisms are a reinterpretation of Theorem 6.6.1 in terms of cohomology and the commutativity of the diagram can be checked directly. □

6.7 Detecting subalagebra for $\bar{S}(n)$

Recall that $H = T_{n-2} = \{\mathrm{diag}(t_1, t_2, \ldots, t_n) \in T \mid t_1 = t_2 = 1\}$. Let $\mathfrak{e}_{\bar{0}} = \mathrm{Lie}(H) \subset \mathfrak{g}_0$ and $\mathfrak{e}_{\bar{1}} = \mathfrak{f}_{\bar{1}}$. Set

$$\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}. \tag{6.8}$$

As with $\mathfrak{f}$ one can verify that $\mathfrak{e}$ is a Lie subsuperalgebra of $\mathfrak{g}$. In fact one has

$$[\mathfrak{e}_{\bar{0}}, \mathfrak{e}_{\bar{0}}] = [\mathfrak{e}_{\bar{0}}, \mathfrak{e}_{\bar{1}}] = 0. \tag{6.9}$$

One can also verify that the differentials defining $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$ are identically zero and so the cohomology ring is again given by the cochains. In this case, however, $\mathfrak{e}_{\bar{0}}$ acts trivially on $\mathfrak{e}_{\bar{1}}$ and so one has

$$\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C}) \cong S(\mathfrak{e}_{\bar{1}}^*) = S\left(\mathfrak{f}_{\bar{1}}^*\right) \cong \mathbb{C}[\partial_1^*, Z_3, \ldots, Z_n].$$

Furthermore, the inclusion $\mathfrak{e} \hookrightarrow \mathfrak{f}$ defines a restriction map, res, so that the following diagram commutes,

$$\begin{array}{ccc} \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) & \xrightarrow{\text{res}} & \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{C}[\partial_1^* Z_3, \ldots, \partial_1^* Z_n] & \xrightarrow{\subseteq} & \mathbb{C}[\partial_1^*, Z_3, \ldots, Z_n] \end{array} \tag{6.10}$$

Since $\mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^T$ and $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^{\Sigma_{n-2}}$ we have the following isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^{T\Sigma_{n-2}}.$$

As one can see from the discussion above the Lie subsuperalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ satisfies all the assumptions of a detecting subalgebra; i.e., it is classical and the restriction map $\text{res} : \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$ induces the isomorphism

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^{T\Sigma_{n-2}}$$

with the group $T\Sigma_{n-2}$ being exact.

6.8 Support Varieties for $\bar{S}(n)$

By using the pair $(\mathfrak{e}, \mathfrak{e}_{\bar{0}})$ and the setup of Section 4.1 one can define the support variety $\mathcal{V}_{(\mathfrak{e}, \mathfrak{e}_{\bar{0}})}(M)$ for any finite dimensional module $M \in \mathcal{C}_{(\mathfrak{e}, \mathfrak{e}_{\bar{0}})}$. Since $\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$ is a polynomial ring in $n-1$ variables, one can naturally identify the support variety of M with the conical affine subvariety of the affine $(n-1)$-space

$$\operatorname{MaxSpec}\left(\mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})\right) = \mathcal{V}_{(\mathfrak{e}, \mathfrak{e}_{\bar{0}})}(\mathbb{C}) \cong \mathbb{A}^{n-1}$$

defined by the ideal $I_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$.

Since by (6.9) the structure of $\mathfrak{e}$ is of the type considered in [BKN1, Sections 5, 6], [BKN1, Theorem 6.3.2] implies that one has a canonical isomorphism

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \cong \mathcal{V}_{\mathfrak{e}}^{\mathrm{rank}}(M) \tag{6.11}$$

for any finite dimensional $\mathfrak{e}$-module M which is an object of $\mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$. We identify the rank and support varieties of $\mathfrak{e}$ via this isomorphism.

The inclusion $\mathfrak{e} \hookrightarrow \mathfrak{g}$ induces a restriction map on cohomology which, in turn, induces maps of support varieties. That is, given modules M and N in $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ one has $M \in \mathcal{C}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}$ by restriction to $\mathfrak{e}$ and one has maps of varieties

$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M,N) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M,N),$$
$$\mathrm{res}^* : \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M) \to \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M).$$

Serganova [Ser, Lemma 5.3] proved that the set of atypical weights for $\mathfrak{g}$ is

$$\Omega = \{a\varepsilon_1 + \cdots + a\varepsilon_{i-1} + b\varepsilon_i + (a+1)\varepsilon_{i+1} + \cdots + (a+1)\varepsilon_n \mid a, b \in \mathbb{C}, 1 \le i \le n\}.$$

Let $\sigma = \varepsilon_1 + \cdots + \varepsilon_n$. For each $\lambda \in \Omega, \lambda \neq a\sigma$ there exists a unique $\overline{\lambda} = \lambda - a\sigma$ such that $\overline{\lambda}$ is atypical for $W(n)$. Since $\dim L(a\sigma) = 1$, we have

$$L(\lambda) \cong L(\overline{\lambda}) \otimes L(a\sigma).$$

Moreover, she determined the characters of the simple $\mathfrak{g}$-modules by determining composition series for the Kac modules. Serganova's abridged results for finite dimensional simple modules are presented in the following theorem

Theorem 6.8.1. *[Ser, Theorem 6.3, Theorem 8.6] Let $\lambda \in X_0^+$.*

(a) If $\lambda \notin \Omega$ then $K(\lambda) \cong L(\lambda)$.

(b) Let $\lambda \in \Omega$ and $L'(\overline{\lambda})$ denote the irreducible $W(n)$-module with highest weight $\overline{\lambda}$ restricted to $\bar{S}(n)$. If $\lambda \neq a\sigma - \varepsilon_n$, then $L'(\overline{\lambda}) = L(\overline{\lambda})$.

We remark that from the theorem above one can deduce immediately that for $\lambda \in X_0^+$: $\lambda \notin \Omega$ (i.e., λ is typical) if and only if $K(\lambda)$ is simple.

The following theorem presents the computation of support varieties of simple $\mathfrak{g}$-modules. Our results demonstrate that $L(\lambda)$ is typical if and only if the support variety of $L(\lambda)$ is zero.

Theorem 6.8.2. *Let $\lambda \in X_0^+$, $K(\lambda)$ be the associated Kac module and let $L(\lambda)$ be the finite dimensional simple $\mathfrak{g}$-module with highest weight λ. Then*

(a) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K(\lambda)) = \{0\}$.

(b) If $\lambda \notin \Omega$ then $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)) = \{0\}$.

(c) If $\lambda \in \Omega$ then $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)) = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C})$. In this case the support variety has dimension $n-2$.

Proof. For part (a) it suffices to prove that for n sufficiently large, $\operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda), K(\lambda)) = 0$.

By Frobenius reciprocity, for all n we have

$$\operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda), K(\lambda)) \cong \operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g}_0\oplus\mathfrak{g}^+,\mathfrak{g}_0)}}(L_0(\lambda), K(\lambda)).$$

Since $\mathfrak{g}^+$ is an ideal in $\mathfrak{g}_0 \oplus \mathfrak{g}^+$ one can apply the Lyndon-Hochschild-Serre spectral sequence to $(\mathfrak{g}^+, \{0\}) \subseteq (\mathfrak{g}_0 \oplus \mathfrak{g}^+, \mathfrak{g}_0)$:

$$E_2^{i,j} = \operatorname{Ext}^i_{\mathcal{C}_{(\mathfrak{g}_0,\mathfrak{g}_0)}}(L_0(\lambda), \operatorname{Ext}^j_{\mathcal{C}_{(\mathfrak{g}^+,\{0\})}}(\mathbb{C}, K(\lambda))) \Rightarrow \operatorname{Ext}^{i+j}_{\mathcal{C}_{(\mathfrak{g}_0\oplus\mathfrak{g}^+,\mathfrak{g}_0)}}(L_0(\lambda), K(\lambda)).$$

Since $\mathcal{C}_{(\mathfrak{g}_0,\mathfrak{g}_0)}$ consists of $\mathfrak{g}_0$-modules which are finitely semisimple over $\mathfrak{g}_0$, this spectral sequence is zero for $i > 0$. That is, it collapses at the E_2 page and yields

$$\operatorname{Hom}_{\mathfrak{g}_0}(L_0(\lambda), \operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g}^+,\{0\})}}(\mathbb{C}, K(\lambda)) \cong \operatorname{Ext}^n_{\mathcal{C}_{(\mathfrak{g}_0\oplus\mathfrak{g}^+,\mathfrak{g}_0)}}(L_0(\lambda), K(\lambda)). \tag{6.12}$$

Now one can argue as in the proof of the Proposition 5.11.1.

Part (b) can be deduced from Part (a) and Theorem 6.8.1(a).

For part (c), first observe that

$$\Omega \cap X_0^+ = \{a\varepsilon_1 + a\varepsilon_2 \cdots + a\varepsilon_{n-1} + b\varepsilon_n \mid a, b \in \mathbb{Z}, b \leq a\}.$$

Moreover, it suffices to show that if $\lambda \in \Omega \cap X_0^+$ then $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})$. Namely, one will then have

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}) = \text{res}^* \left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})\right) = \text{res}^* \left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda))\right) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda)) \subseteq \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}),$$

which implies the result for $\mathfrak{g}$.

There are two cases we need to consider: $\lambda = a\sigma$ and $\lambda \neq a\sigma$.

If $\lambda = a\sigma$, since $\dim L(a\sigma) = 1$, $\text{sdim}\, L(\lambda) \neq 0$, then by [BKN1, Corollary 6.4.1] $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})$.

Now if $\lambda \neq a\sigma$, then there exists an atypical weight $\bar{\lambda}$ for $W(n)$ such that $L(\lambda) = L(\bar{\lambda}) \otimes L(a\sigma)$. One can also easily observe from Theorem 5.11.2 that a simple finite dimensional module for $W(n)$ is atypical if and only if its superdimension is zero. Since $\bar{\lambda}$ is atypical for $W(n)$, $\text{sdim}\, L(\bar{\lambda}) \neq 0$ and thus $\text{sdim}\, L(\lambda) = \text{sdim}\, L(\bar{\lambda}) \neq 0$. Now again from [BKN1, Corollary 6.4.1] it follows that $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(\mathbb{C})$.

□

6.9 Realization of Support varieties

The goal of this section is to show that a realization theorem holds for $\mathfrak{g}$. Recall that $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ denotes the category of $\mathfrak{g}$-modules which are finitely semisimple as $\mathfrak{g}_0$-module and L_ζ denotes the Carlson module corresponding to the homogeneous element $\zeta \in \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$. We also remind the reader that Carlson modules may not be finite dimensional which is why we are going to work with relative support varieties.

Let us summarize what we know about the pair $(\mathfrak{g}, \mathfrak{g}_0)$:

(1) $\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; M^* \otimes N)$ for $\mathfrak{g}$-modules $M, N \in \mathcal{C}$. Since $\mathfrak{g}$ is finitely semisimple as a $\mathfrak{g}_0$-module under the adjoint action the isomorphism follows from Proposition 3.3.1.

(2) $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C})$ is a polynomial ring with $n-2$ variables (Theorem 6.6.1). Thus Hypothesis 3.4.4 holds for $\mathfrak{g}$.

(3) $\mathfrak{g}$ has a detecting subalgebra $\mathfrak{e}$ (Section 6.7) and $\mathfrak{e}$ support varieties have rank variety descriptions (Section 6.8), i.e., Hypotheses 3.5.3 and 4.2.2 hold for $\mathfrak{g}$.

(4) $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta) = \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_{\mathrm{res}(\zeta)}) = \mathcal{Z}(\mathrm{res}(\zeta))$. This follows from Lemma 4.5.1.

(5) $\mathrm{Ext}^\bullet_{\mathcal{C}}(L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_s},\mathbb{C})$ is finitely generated over $\mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C})$. This follows from (1) and Proposition 4.5.2.

(6) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_s},\mathbb{C}) \subseteq \cap_{i=1}^{s}\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L_{\zeta_i},\mathbb{C})$. This follows from (1) and Proposition 4.5.2.

(7) $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(L_\zeta,\mathbb{C}) = \mathrm{res}^*\left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L_\zeta)\right) = \mathcal{Z}(\zeta)$. This follows from Proposition 4.6.1.

We are now ready to prove the realization theorem for $\mathfrak{g}$.

Theorem 6.9.1. *Let X be a conical subvariety of $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C})$. Then there exists a $\mathfrak{g}$-module $M\in\mathcal{C}$ such that*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(M,\mathbb{C}) = X.$$

Proof. As we have seen from the discussion above the Hypotheses 3.4.4, 3.5.3 and 4.2.2 hold for $\mathfrak{g}$. Since the assumptions hold the result follows from Theorem 4.7.1. □

Chapter 7

Defect, Atypicality and superdimension

7.1 Defect

Let $\mathfrak{g}$ be a classical Lie superalgebra and $\mathfrak{h}$ be a maximal torus contained in $\mathfrak{g}_{\bar{0}}$. Let Φ be the set of roots with respect to $\mathfrak{h}$. We have $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ where $\Phi_{\bar{0}}$ (resp. $\Phi_{\bar{1}}$) is the set of even roots (resp. odd roots). The positive roots will be denoted by Φ^+ and negative roots by Φ^-.

If $\mathfrak{g}$ is a basic classical Lie superalgebra, with the non-degenerate bilinear form denoted by $(-,-)$, Kac-Wakimoto [KW] defined the *defect* of $\mathfrak{g}$, denoted by $\mathrm{def}(\mathfrak{g})$, to be the dimension of a maximal isotropic subspace in the $\mathbb{R}$-span of Φ.

The defects for various simple basic classical Lie superalgebras are given as follows

Example 7.1.1. $\mathrm{def}(\mathfrak{sl}(m|n)) = \min(m,n)$, $\mathrm{def}(\mathfrak{sl}(n|n)) = n$, $\mathrm{def}(\mathfrak{osp}(2m+1|2n)) = \mathrm{def}(\mathfrak{osp}(2m|2n)) = \min(m,n)$, and the exceptional Lie superalgebras $D(2,1;\alpha)$, $G(3)$, and $F(4)$ all have defect 1.

7.2 Atypicality

Let $\mathfrak{g}$ be a basic classical Lie superalgebra as above. Let $\lambda \in \mathfrak{h}^*$ be a weight. The *atypicality* of λ, denoted by $\mathrm{atyp}(\lambda)$, is the maximal number of mutually orthogonal positive isotropic roots $\alpha \in \Phi^+$ such that $(\lambda + \rho, \alpha) = 0$, where $\rho = \frac{1}{2}(\Sigma_{\alpha \in \Phi_{\bar{0}}^+} \alpha - \Sigma_{\alpha \in \Phi_{\bar{1}}^+} \alpha)$.

Note that $\mathrm{atyp}(\lambda) \leq \mathrm{def}(\mathfrak{g})$. Let $L(\lambda)$ be a simple finite dimensional $\mathfrak{g}$-module with highest weight λ. The atypicality of $L(\lambda)$, is defined to be $\mathrm{atyp}(\lambda)$.

Kac and Wakimoto give the following conjecture relating the superdimension of simple finite-dimensional $\mathfrak{g}$-modules with the atypicality of the module [KW].

Conjecture 7.2.1 (Kac-Wakimoto). *Let $\mathfrak{g}$ be a simple basic classical Lie superalgebra and $L(\lambda)$ be a finite dimensional simple $\mathfrak{g}$-module. Then* $\operatorname{sdim} L(\lambda) = 0$ *if and only if* $\operatorname{atyp}(L(\lambda)) < \operatorname{def}(\mathfrak{g})$.

As suggested in [BKN1] one can extend the definitions of defect and atypicality as follows: If $\mathfrak{g} = W(n)$ or $S(n)$ (resp. $\mathfrak{g}$ is classical) define the *defect* of $\mathfrak{g}$ to be the Krull dimension of $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ (resp. $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$). In particular, this new definition would allow us to define the notion of defect for simple classical Lie superalgebras with no non-degenerate bilinear form, $W(n)$ and $\bar{S}(n)$. These cohomological and combinatorial definitions do not agree for $\mathfrak{sl}(n|n)$. Namely, $\operatorname{def}(\mathfrak{sl}(n|n)) = n$, whereas Krull dimension of $\mathrm{H}^\bullet(\mathfrak{sl}(n|n), \mathfrak{sl}(n|n)_{\bar{0}}; \mathbb{C}) = n - 1$.

Define the atypicality of a simple module $L(\lambda)$ with highest weight λ to be $\dim(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda))/W)$. These new definitions extend the Kac-Wakimoto Conjecture to simple basic classical Lie superalgebras and Cartan type Lie superalgebras $W(n)$ and $\bar{S}(n)$.

Chapter 8

Representation Type

8.1

Indecomposable modules of a finite dimensional algebra provide a complete description of all the modules of the algebra. Therefore, classification of the indecomposable modules for a fixed finite dimensional algebra A is a central theme in the representation theory of such algebras. The algebra A will fall into one of three classes depending on the classifiability of its indecomposable modules. If there are only finitely many isomorphism classes of indecomposable A-modules, then we say that A has *finite representation type*, and if there are infinitely many such isomorphism classes, then A has *infinite representation type*. If A has infinite representation type, it can be further classified as having *tame representation type* or *wild representation type*. In the former case all but finitely many indecomposable A-modules of a given dimension can be parametrized by essentially one parameter of base field. If A is wild, then the classification of the indecomposables is harder than bringing two matrices simultaneously into Jordan from, a problem which is generally considered hopeless.

8.2

Germoni [Ger] investigated the representation type for the Lie superalgebra $\mathfrak{sl}(m|n)$. He proved that if $m, n \geq 2$ then $\mathfrak{sl}(m|n)$ has wild representation type. Germoni also conjectured that this should hold for blocks of atypicality greater than or equal to two. Later, Shomron [Sho] proved that each block of the Lie superalgebra $W(n)$ has wild representation type for $n \geq 3$. Both cases are based on studying the Ext^1 quiver.

Recently Farnsteiner [Far, Theorem 3.1] showed that if the dimension of the support variety of some simple module in a block for a finite group scheme has dimension at least three, then the block has wild representation type. The proof chiefly depends upon using the finite group scheme analogue of the above realizability result to construct sufficiently many indecomposable modules in the block. With these results in mind we present the following conjecture relating the representation type of Lie superalgebras with our construction of support varieties for both the classical and Cartan type Lie superalgebras.

8.3

There is a slight difference in the way that the support varieties and detecting subalgebras are defined in the two cases. We will fix a common notation which allows us to treat both cases more or less simultaneously. Let $\mathfrak{g}$ be a classical Lie superalgebra with a polar and stable action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ as in [BKN1] or let $\mathfrak{g} = W(n)$ or $\bar{S}(n)$. Let $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ if $\mathfrak{g}$ is classical and $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ if $\mathfrak{g} = W(n)$ or $\bar{S}(n)$. Let

$$\mathrm{H}^\bullet = \begin{cases} \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C}), & \text{if } \mathfrak{g} \text{ is classical;} \\ \mathrm{H}^\bullet(\mathfrak{g},\mathfrak{g}_0;\mathbb{C}), & \text{if } \mathfrak{g} = W(n) \text{ or } \mathfrak{g} = \bar{S}(n). \end{cases}$$

If M and N are objects in $\mathcal{C}$ for which $\operatorname{Ext}^\bullet_{\mathcal{C}}(M,N)$ (resp. $\operatorname{Ext}^\bullet_{\mathcal{C}}(M,M)$) is a finitely generated $\mathrm{H}^\bullet$-module, then write $\mathcal{V}_{\mathfrak{g}}(M,N)$ for the corresponding relative support variety (resp. $\mathcal{V}_{\mathfrak{g}}(M)$ for the corresponding support variety).

Conjecture 8.3.1. *Let $\mathcal{B}$ be a block of $\mathcal{C}$. If there exists a simple module S in $\mathcal{B}$ with* $\dim \mathcal{V}_{\mathfrak{g}}(S) \geq 3$, *then $\mathcal{B}$ has wild representation type.*

In light of Conjecture 8.3.1 and Germoni's conjecture on the representation type of the blocks of $\mathfrak{sl}(m|n)$, it is worthwhile to note that by the calculations in [BKN2] one has that if $\mathcal{B}$ is a block of $\mathfrak{gl}(m|n)$ of atypicality k, then

$$\mathcal{V}_{\mathfrak{g}}(S) \cong \mathbb{A}^k$$

for all simple modules S in $\mathcal{B}$.

Bibliography

[BAKN] Irfan Bagci, Jonathan R. Kujawa, and Daniel K. Nakano, *Cohomology and support varieties for Lie superalgebras of type $W(n)$*, Int. Math. Res. Notices, vol. 2008, Article Id rnn115.

[Ben1] D. J. Benson, *Representations and Cohomology. I*, second ed., Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1998.

[Ben2] D. J. Benson, *Representations and Cohomology. II*, second ed., Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, Cambridge, 1998.

[Bl] J. Blowers, *The Cohomology Rings of Certain Finite Permutation Representations,* Proc. of American Math. Soc. **45** (1974), no. 2, 157–163.

[BKN1] B. D. Boe, J. R. Kujawa, and D. K. Nakano, *Cohomology and support varieties for Lie superalgebras*, arXiv:math.RT/0609363, (2006).

[BKN2] B. D. Boe, J. R. Kujawa, and D. K. Nakano, *Cohomology and support varieties for Lie superalgebras II*, Proc. London Math. Soc., **98** (2009), 19–44.

[Ca1] J.F. Carlson, *The varieties and the cohomology ring of a module*, J. Algebra **85** (1983), 104–143.

[Ca2] J.F. Carlson, *The variety of an indecomposable module is connected*, Invent. Math., **77** (1984), 291–299.

[Ca3] J.F. Carlson, *modules and group algebras*, Notes by Ruedi Suter, Lectures in Mathematics ETH Zurich. Birkhuser Verlag, Basel, 1996.

[CM] D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[CNS] L. Corwin, Y. Ne'eman, and S. Sternberg, *Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry)*, Rev. Mod. Phys. **47** (1975), 573–604.

[DK] J. Dadok and V. Kac, *Polar representations*, J. Algebra **92** (1985), no. 2, 504–524.

[DS] M. Duflo and V. Serganova, *On associated variety for Lie superalgebras*, arXiv:math.RT/0507198, (2005).

[Ev] Leonard Evens, *The Cohomology of Groups*, Oxford Science Publications, 1991.

[Far] R. Farnsteiner, *Tameness and complexity of finite group schemes*, Bull. London Math. Soc. **39** (2007), no. 1, 63–70.

[FPa] E.M. Friedlander, B.J. Parshall, *Support varieties for restricted Lie algebras*, Invent. Math. **86** (1986), 553–562.

[FPe] E.M. Friedlander, J. Pevtsova, *Representation theoretic support spaces for finite group schemes*, American J. Math. **127** (2005), 379–420.

[FS] E.M. Friedlander, A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270.

[Fu] D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, 1986

[Ger] J. Germoni, *Indecomposable representations of special linear Lie superalgebras*, J. Algebra **209** (1998), 367–401.

[Hoc] G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc., **82** (1956), 246–269.

[Hum] J. E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, Mathematical Surveys and Monographs, 43. American Mathematical Society, Providence, RI, 1995.

[Jan] J.C. Jantzen, *Representations of Algebraic Groups*, Second Edition, American Mathematical Society, Providence R.I., 2003.

[Kac] V. G. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96.

[KW] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, 415–456.

[Kum] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics, vol. 204, 2002.

[LR] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496.

[MM] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Ann. Math. **81** (1965), 211–264.

[Nak] D. K. Nakano, *Varieties for G_rT-modules*, Group representations: cohomology, group actions and topology (Seattle, WA, 1996), Proc. Sympos. Pure Math. **63**, Amer. Math. Soc., (1998), 441–452.

[PV] V. L. Popov and E. B. Vinberg, *Invariant Theory*, Algebraic Geometry IV (A. N. Parshin and I. R.Shafarevich, eds.), Encylopedia of Mathematical Sciences, vol. 55, Springer-Verlag, Berlin, 1994.

[Pr] A.A. Premet, *The theorem on restriction of invariants and nilpotent elements in W_n*, Math. USSR Sbornik **73** (1992), 135–159.

[R] U. Ray, *Automorphic Forms and Lie Superalgebras*, Algebra and Applications (Springer), vol.5, 2006.

[Sch] M. Scheunert, *The Theory of Lie Superalgebras*, Lec. Notes Math (Springer-Verlag) **716**, 1976.

[Ser] V. Serganova, *On representations of Cartan type Lie superalgebras*, Amer. Math. Soc. Transl. **213** (2005), no. 2, 223–239.

[Sho] N. Shomron, *Blocks of Lie superalgebras of type $W(n)$*, J. Algebra **251** (2002), 739–750.

[SFB] A. Suslin, E. Friedlander and C. Bendel, *Support varieties for infinitesimal group schemes*, J. Amer. Math. Soc. **10** (1997), 729–759.