Energy Minimization of Confined Plasmas in Tokamak Fusion Reactors

by

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(Under the direction of Robert Varley)

Abstract

The tokamak has been the preferred design for fusion test reactors since the 70's. While the tokamak holds promise to be one of the most efficient in the field of fusion research, it is still important to achieve minimal energy field configurations for the sake of demonstrating commercial potential. In 1986, John B. Taylor solved the problem of finding energy minimizing vector fields on the flat torus, given certain constraints. This paper will explore the fundamentals of fusion energy and give an overview of Taylor's solution.

Index words: Fusion, Torus, Tokamak, Plasma, Energy Minimization

ENERGY MINIMIZATION OF CONFINED PLASMAS IN TOKAMAK FUSION REACTORS

by

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Chapter 1

Introduction

1.1 A Brief History of Fusion Power

Fusion is the process of combining light nuclei in order to create heavier nuclei and energy. While this is a scientific definition, one does not need modern science to witness fusion power at work in nature. As we examine the night sky and all the stars scattered among its canvas, we are witnessing fusion power at work. It is fusion that powers the stars and keeps them burning over the billions of years of their lifespans. And it is for this reason that fusion is perhaps the most pure and generous source of energy we can draw upon. But before we can talk about harnessing fusion power on Earth, it would help us to know the conditions for it to take place in nature's source.

A star forms when a massive amount of hydrogen gas and dust, in what is known as a nebula, is pulled together by gravitational attraction. As the matter draws closer together, collisions begin to take place. Eventually, the gravitational attraction brings the matter so close together, that pressure is generated near the center of mass. This pressure causes heating and speeds up collisions. When temperatures are significantly high enough, hydrogen atoms ionize and form what is known as a plasma. This happens because, at high temperatures, the collisions are strong enough to knock electrons free. Simply put, a plasma is a ”soup” of free nuclei and electrons. When the plasma temperature and pressure rises enough such that the electromagnetic repulsion force between nuclei is less than the attracting strong force, then the nuclei *fuse* and form a helium nucleus. This fusing releases energy which generates an outward-pointing vector field of force - a field that can be pictured as vectors normal to the surface of a sphere. The outward force pushes against the inward force caused by the gravitational colapse. Over time, an equilibrium between the forces is reached and a new star becomes stable and begins its lifespan of billions of years.

The energy required to overcome the electromagnetic repulsion of hydrogen nuclei is high - so high in fact, that temperatures must be in the realm of millions of degrees Celcius before fusion reactions can take place. It is this high level of energy that is responsible for the fact that there are no small[1] stars in the universe. In order for the gravitational force (the weakest of the four fundamental forces) to create enough pressure to overcome the electromagnetic force, mass must be exceptionally high. Because of this fact of nature, we must exclude traditional means of star formation as a method of harnessing fusion power on earth. What then are we left with?

The primary barrier to overcome is the electromagnetic force. We must somehow create a *controlled*[2] environment with enough energy so that two hydrogen nuclei can collide with each other and bind through the strong force. Researchers have tried a couple promising methods of creating such an environment. One method is known as inertial confinement and it involves heating a pellet of hydrogen fuel with powerful lasers, causing an explosion of the outer shell, thus generating an inward force to start a fusion reaction at the core of the pellet. Although this method has shown promise, there is a different method that has been in the process of research since the 50's and it involves the use of externally generated electromagnetic fields.

Remember when we said that, as a star forms, hydrogen atoms ionize and form a plasma before fusion takes place. Well, one peculiar property of the plasma state of matter is that it has a charge. Because of the charged nature of a plasma, it has the ability to conduct a current and also to respond to a magnetic field. So if we could create a leak-proof "magnetic bottle" so to speak, one that encloses the plasma, and heat the plasma to sufficient temperatures, then theoretically we would have a controlled fusion reaction. This has, in fact, been what researchers in the field of

[1]Small is meant in terms of total mass, not volume. Black holes and neutron stars are small in terms of volume, but still extremely massive.

[2]A hydrogen bomb is powered by a fusion reaction, but it is certainly not controlled.

controlled fusion power have been doing for decades already. Since the 50's, plasma physicists have been successfully achieving fusion reactions in experimental reactors by use of magnetic fields to contain the plasmas. The Princeton Plasma Physics Laboratory and its TFTR(Tokamak Test Fusion Reactor), which ran from the 80's until decommissioning in 1997, achieved controlled fusion reactions and plasma temperatures over 500 million degrees Celcius.

In fusion reactor design, the Q-factor is defined to be a ratio of output energy to input energy. A Q-factor higher than 1 means that the output energy is higher than the input energy. In order for a fusion reactor to be commercially viable, it would have to have a Q-factor significantly higher than 1. Otherwise, there would be no point. The TFTR unfortunately was not able to demonstrate this, but the ITER project, which is currently under construction, will demonstrate a Q-factor of at least 10. Regardless of the Q-factor of the TFTR, it was a remarkable achievement in fusion energy research and it has contributed immensely to scientific knowledge.

Researchers knew that, in order to give fusion power a fighting chance in the commercial world, they would have to create a test reactor that could achieving a promising Q-factor. It is here that the ITER project steps up. ITER, or International Tokamak Experimental Reactor, is another test fusion reactor currently being constructed in Cadarache, France. ITER is a joint effort between several nations, all sharing the cost in various proportions, and it will be the largest test fusion reactor built to date. To give an idea of its size, the combined weight of the electromagnetic field coils alone is over 10,000 metric tons and the reactor is anticipated to generate vertical compression forces in the realm of 400MN.

Although TFTR is now decommisioned, the ”tokamak” design has remained promising in the field of controlled fusion power, leading the way to the current ITER project. The tokamak's geometry has a number of interesting properties, one being the bootstrap current present in a toroidal plasma. Although discussion of the bootstrap current is not within the scope of this thesis, a motivated reader may find it to a be good starting point for researching the history of popularity in the tokamak design. Suffice it to say, the geometry of a fusion reactor plays an integral role in its efficiency and viability.

1.2 The Tokamak and Toroidal Geometry

The word tokamak comes from a juxtaposition of Russian words that together mean toroidal magnetic chamber. A tokamak works by creating a toroidal magnetic field that confines hydrogen plasma so that it may be heated to sufficient temperatures for fusion reactions to take place. The field must be strong, as it is the pinch effect (the magnetic field compressing the plasma) that is responsible for much of the initial heating of the plasma. Other methods are used to induce plasma heating such as resistive heating (running a current through the plasma) and neutral beam injection, but the field configuration itself will be responsible for a significant portion of reactor efficiency. An inefficient field configuration can require more energy to sustain a reaction. While this is certainly of significance in the lab, it will be even more important when testing fusion energy for commercial purposes.

The standard euclidean metric on a solid torus in $\mathbb{R}^3$ can lead to some complicated mathematics when attempting to find minimal energy vector fields in said domain. By instead considering the domain of the solid flat torus[3], we can look at a simpler mathematical model for finding minimal energy vector fields. Any point in the flat torus can be specified using a 3-variable coordinate system, but unlike the standard torus, it is not embeddable in $\mathbb{R}^3$. We can visualize the flat torus as a cylinder with the top and bottom identified. Using the flat torus as our domain removes some of the mathematical complexity caused by the bending around the major axis in the standard torus.

[3]After this point, I will use the terms torus and flat torus to mean solid torus and solid flat torus respectively.

The Woltjer problem in plasma physics deals with finding minimal energy vector fields on the flat torus. We will discuss the Woltjer problem in this paper and examine some simplified cases.[4]

[4] See chapter 4.2 and chapter 5.

CHAPTER 2

ELECTROMAGNETISM BASICS

2.1 FUNDAMENTALS OF ELECTROMAGNETISM

An electromagnetic field is actually a composition of two separate fields. These two fields are known as the electric field, **E**, and the magnetic field, **B**. The etymology for the term ”electro-magnetic” involves the strong codependence these fields have on each other. A changing electric field will create a magnetic field and a changing magnetic field has the ability to induce an electric current, **J**, in a conductor. This relationship sets a basis by which many modern day electrical devices were designed. Consider the power supply that charges a laptop computer. Inside the power supply is a device called a transformer which steps the voltage down to a smaller AC voltage. This device uses the intimate relationship between electric currents and magnetic fields to achieve this. The power supply receives an input of 110VAC and has an output usually somewhere in the neighborhood of 16-20VDC. The sinusoidal wave form of the alternating input current creates a magnetic field in a coil of wire, known as the primary coil. This magnetic field induces a smaller AC current in an adjacent coil of wire, known as the secondary coil. The exact voltage of the output AC current is determined by the ratio of the number of windings between the primary and secondary coils. Then, the smaller AC output current passes through a rectifier to be converted to DC.

In the middle 1800’s, a man by the name of James Clerk Maxwell, through many years of experimental research and mathematical computation, wrote down exact mathematical relationships between electric and magnetic fields. He came up with four equations which are at the heart of classical electrodynamics. They are referred to as Maxwell's Equations. We list them below:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

The constants ε_0 and μ_0 are known respectively as the permittivity of free space and the permeability of free space.[1] It is equations 2.2 and 2.4 that give us the precise relationships between changing electric fields and changing magnetic fields, the relationships by which devices such as transformers operate.

Equation 2.3 is a powerful result as it tells us all magnetic fields are divergence-free. This peculiar property of magnetic fields leads to some eloquent mathematical formulations in their regard. In this paper, the vector fields we will be most interested in are those vector fields that are divergence free. After we introduce the Biot-Savart operator in the next section, we will use it to prove equation 2.3.

Another interesting property of electric and magnetic fields is that they exert a force on charged particles and this force can be quantified exactly. The force is given by the Lorentz Force Law as follows:

$$\mathbf{F} = Q(\mathbf{E} + (\mathbf{v} \times \mathbf{B}))$$

[1]Quantitatively, $\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{C^2}{Nm^2}$ and $\mu_0 = 4\pi \times 10^{-7}\ \frac{N}{A^2}$

It should be noted that this equation is axiomatic and verified by experimental evidence.

2.2 The Biot Savart Operator and Ampere's Law

Since the most fundamental aspect of controlled thermonuclear fusion is the magnetic field, I find it pedagogically consistent to begin with the Biot-Savart Operator. The Biot-Savart Law describes the magnetic field $\mathbf{B}$ at a position $\mathbf{y}$ induced by a steady line current $\mathbf{J}$ at a position $\mathbf{x}$. The Biot-Savart Operator is given by[2]:

$$\mathbf{B}(\mathbf{y}) = \mathbf{BS}(\mathbf{J}(\mathbf{x})) = \frac{\mu_0}{4\pi}\int_{\Omega} \mathbf{J}(\mathbf{x}) \times \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} d\mathbf{x}.$$

where Ω is the domain which the current flows through (for instance a wire, which can be modelled as a long cylinder). Since the magnetic force obeys an inverse square law, the $\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}$ term follows naturally. [3] (From this point on, we will reference $\mathbf{B}(\mathbf{y})$ as $\mathbf{B}$ and $\mathbf{J}(\mathbf{x})$ as $\mathbf{J}$ where the position dependence is assumed to be understood.)

Ampere's Law describes the magnetic field induced by a current. It can be thought of as an inverse of the Biot-Savart operator. Through some basic vector algebra, we can derive Ampere's Law from the Biot-Savart Operator:

We begin by taking the curl of both sides of the Biot-Savart Law:

$$\nabla \times \mathbf{B} = \nabla \times \frac{\mu_0}{4\pi}\int_{\Omega} \mathbf{J} \times \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} d\mathbf{x}.$$

[2] I have explicitly indicated $\mathbf{y}$-dependence on $\mathbf{B}$ and $\mathbf{x}$-dependence on $\mathbf{J}$ in the definition as this will be important to note when we derive Ampere's Law.

[3] If we turn the numerator into a unit vector, the denominator becomes squared rather than cubed.

$$= \frac{\mu_0}{4\pi}\int_\Omega \nabla \times \left(\mathbf{J} \times \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}\right) d\mathbf{x}.$$

$$= \frac{\mu_0}{4\pi}\int_\Omega \left[\left(\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \cdot \nabla\right)\mathbf{J} - (\mathbf{J}\cdot\nabla)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} + \mathbf{J}\left(\nabla \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}\right) - \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}(\nabla\cdot\mathbf{J})\right] d\mathbf{x}$$

The first and fourth terms in the integral will be zero since nabla is operating on **y**, and **J** is only dependent on **x**. The third term will integrate to zero leaving only the second term to make a contribution. The second term simplifies[4] into $4\pi\delta^3(\mathbf{x})$. Thus we are now left with:[5]

$$\frac{\mu_0}{4\pi}\int_\Omega \mathbf{J}\cdot 4\pi\delta^3(\mathbf{x})\, d\mathbf{x}$$

$$= \frac{\mu_0}{4\pi}(4\pi\mathbf{J}) = \mu_0\mathbf{J}$$

We have now derived Ampere's Law and we can write it as:[6]

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J}$$

We will find both the Biot-Savart Law and Ampere's Law essential as we develop more focused concepts for which to describe the flow of plasma in a magnetic field. One of these concepts, helicity, we will define shortly. It will put into quantitative measure the degree to which magnetic field lines wrap around each other. But first we will prove one of Maxwell's Equations, equation 2.3, using the Biot Savart Operator. We start with the operator and take its divergence:

$$\nabla\cdot\mathbf{B} = \nabla\cdot\frac{\mu_0}{4\pi}\int_\Omega \mathbf{J}\times\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3}d\mathbf{x}$$

[4] See Griffiths' Introduction to Electrodynamics p. 223-224 for a more detailed explanation as the simplification is somewhat cumbersome.

[5] The δ represents the dirac-delta function, which is not really a function in a true sense. It is defined by the following two properties: 1) $\delta(x-a) = 0$, when $x \neq a$, and 2) $\int_\Omega \delta(x-a)dx = 1$ if $a \in \Omega$.

[6] This is essentially equation 2.4 without the time-dependent term.

$$= \frac{\mu_0}{4\pi} \int_{\Omega} \nabla \cdot \left(\mathbf{J} \times \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \right) d\mathbf{x}$$

$$= \frac{\mu_0}{4\pi} \int_{\Omega} \left[\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \right) \right] d\mathbf{x}$$

Both terms in our integral will be zero since we have the curl of $\mathbf{J}$ with respect to $\mathbf{y}$ and the the curl of a position vector. Thus we obtain:[7]

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

2.3 Flux, Energy, and Helicity

2.3.1 Flux

Qualitatively, flux is a measure of the vector field lines that pass through some given surface. Let Ω denote the flat solid torus and picture a circular cross section, Σ, which plasma is passing through, such that $\partial\Sigma \in \partial\Omega$, and for which V models the flow of the plasma.[8] If the flow through this cross-section is high, we say the flux is high. We define the flux through a surface Σ to be:

$$F_{\Sigma}(V) = \oint_{\Sigma} V \cdot n dA \tag{2.5}$$

[7] Up until this point, I have followed a convention to use bold letters when indicating vector quantities as to avoid confusion. But from here on, quantities which are vector quantities should be clear through context and will be printed in standard typeset.

[8] V may be thought of as analogous to the current density we discussed when talking about Biot-Savart.

Moreover, in the solid flat torus (and in the standard torus), the flux value is independent of the cross-sectional surface chosen. Thus we can drop the Σ subscript on the $F(V)$ and just say:

$$F(V) = \oint_{\Sigma} V \cdot n dA \tag{2.6}$$

where Σ is any cross-sectional surface in the solid flat torus.

Mathematically, flux would be an extra variable to consider in the case of plasma confinement. But as it turns out, flux that moves with a plasma through a closed contour is constant.[9] This fact will be important later as we establish a relationship between Energy, Helicity, and Flux.

2.3.2 Energy

We may assign, to each vector field V, a scalar quantity we shall call Energy. Energy is given by:

$$\int_{\Omega} |V|^2 d\tau$$

If we let $\mathbf{VF}(\Omega)$ denote the space of vector fields on Ω, then we can define the L^2 inner product on that space as:

$$< V, W >= \int_{\Omega} V \cdot W d\tau$$

Now we can write energy simply as:

$$E(V) = \langle V, V \rangle \tag{2.7}$$

[9] See p. 124 of Goldston's and Rutherford's Introduction to Plasma Physics.

Since we stated earlier that V can be thought of as modeling the flow of a confined plasma, one can think of the energy definition as "kinetic energy" associated with a vector field. This analogy should help us better understand the real world scenarios we are attempting to model.

2.3.3 Helicity

A vector potential for a vector field, B, is defined as a vector field, A, such that its curl is equal to B, i.e. $\nabla \times A = B$. Using this definition of vector potential, magnetic helicity is defined classically as follows:

$$H(B) = \int_{\Omega} A \cdot B d\tau$$

where A is the vector potential for B.

To migrate this definition to one we will use for arbitrary divergence free vector fields, we note that, for any divergence free vector field, V, $BS(V)$ is a vector potential for V. Thus we may now define helicity for a divergence free vector field as:

$$H(V) = \int_{\Omega} V \cdot BS(V) d\tau$$

or equivalently,[10]

$$H(V) = \langle V, BS(V) \rangle \tag{2.8}$$

Due to the toroidal geometry of most modern test fusion reactors, helicity becomes an important quantity in determining the level of energy within the reactor. During the first fraction of a second when plasma is injected in a reactor, the plasma is highly turbulent. After this time however, the plasma "relaxes" and helicity is, for our purposes, a conserved quantity. Thus it makes sense for

[10] This definition could also apply to vector fields with non-zero divergence; however, the scope of this paper is primarily focused on those vector fields that are divergence free.

us to consider helicity a constraint for calculating energy.

CHAPTER 3

MATHEMATICAL PRELIMINARIES

In looking for energy minimizing vector fields, it will help us to be more familiar with Bessel's equation and the decomposition of vector fields. Rather than incorporating these things in the middle of the discussion of finding energy minimizing vector fields, I found it more organizationally sound to place them in a preliminary fashion. A brief presentation of Bessel's equation and its solution is given in the first section of this chapter. Then, we will give an overview of The Hodge Decomposition for Vector Fields and state the theorem. As much as I would like to discuss some of the proof for the Hodge Theorem, my lack of background in algebraic topology hinders me. Moreover, it it is quite lengthy. A motivated reader can examine the proof in Cantarella, DeTurck, and Gluck's *Vector Calculus and the Topology of Domains in 3-Space*.

3.1 BESSEL'S EQUATION

Bessel's equation is a second order differential equation of the following form:

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0, \ (v \geq 0) \tag{3.1}$$

More precisely, we would call this Bessel's equation of order v. Let $p(x), q(x) \in \mathbb{R}[x]$. If p and q are analytic at x_0 then every solution of $y'' + p(x)y' + g(x)y = 0$ is analytic at some x_0. Thus we can write $y(x)$ as:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Using the method of Frobenius:

If p has a simple pole and q has a pole of order 2, then we can rewrite Equation 3.1 as:

$$x^2y'' + x(xp(x))y' + (x^2q(x))y = 0 \tag{3.2}$$

Then we know $xp(x)$ and $x^2q(x)$ are analytic. Thus we can expect a solution:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n.$$

Now we find a Frobenius Solution:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

Putting this back into Bessel's Equation:

$$\sum_{k=0}^{\infty} \left(\left((k+r)^2 - v^2\right) a_k + a_{k-2}\right) x^{k+r} = 0 \ \ (a_0 \neq 0).$$

Now we set each coefficient of each power of x equal to 0:

$$k = 0: \ (r^2 - v^2)a_0 = 0 \tag{3.3}$$

$$k = 1: \ \left((r+1)^2 - v^2\right) a_1 = 0 \tag{3.4}$$

$$k \geq 2: \ \left((r+k)^2 - v^2\right) a_k + a_{k-2} = 0 \tag{3.5}$$

From equation 3.3, $r = \pm v$. From equation 3.5, $a_k = -\frac{1}{k(k+2r)} a_{k-2}$ $(k \geq 2)$. From equation 3.4, $a_1 = 0$ and thus $a_1 = a_3 = a_5 = \cdots = 0$.

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (v_k)(v+k-1)\cdots(v+1)} = \frac{(-1)^k v! a_0}{2^{2k} k! (v+k)!} \ \ (v \in \mathbb{Z}).$$

If $v \notin \mathbb{Z}$:

$$a_{2k} = \frac{(-1)^k \gamma(v+1) a_0}{2^{2k} k! \Gamma(v+k+1)}$$

and thus we have

$$y(x) = a_0 2^v \Gamma(v+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k+v}.$$

This function is called Bessel's function of the first kind of order v and is usually written:

$$y(x) = J_v(x)$$

There is also a Bessel function of the 2nd kind, usually denoted Y_v, but this particular function has a singularity and is of no use to us here.

3.2 The Hodge Decomposition Theorem for Vector Fields

Let Ω be a compact domain in $\mathbb{R}^3$ having a smooth boundary $\partial\Omega$. Define $\mathbf{VF}(\Omega)$ to be the infinite dimensional vector space of all $\mathbf{C}^\infty$ vector fields in Ω. Then the Hodge Theorem tells us that we can decompose $\mathbf{VF}(\Omega)$ into five mutually orthogonal subspaces. We measure orthogonality by means of the L^2 inner product:

$$< V, W >= \int_\Omega V \cdot W d\tau$$

We will not prove the Hodge Decomposition Theorem for Vector fields as it is lengthy and beyond the scope of this paper. However, we will discuss briefly the decomposition and its organization.

Now rather than just outright stating all five subspaces that compose $\mathbf{VF}(\Omega)$, I find it pedagogically consistent to first note that $\mathbf{VF}(\Omega)$ can be written in terms of two larger subspaces; that is, knots and gradients. We define knots, **K**, and gradients, **G**, as follows:

$$\mathbf{K} = \{V \in \mathbf{VF}(\Omega) : \nabla \cdot V = 0, V \cdot n = 0\}$$
$$\mathbf{G} = \{V \in \mathbf{VF}(\Omega) : V = \nabla \varphi\}$$

We see that a knot is any vector field that is divergence free and tangent to the boundary while a gradient is simply a vector field that is the gradient of some scalar. A powerful result of this decomposition is that the space of all smooth vector fields can be broken into orthogonal components that lie in these two sets. This decomposition is non-trivial and requires finding solutions of the Laplace and Poisson Equations with Dirichlet and Neumann Boundary Conditions. [1] However, once the decomposition has been made, it can be shown relatively easy that the two spaces are orthogonal. We can check this by simply taking their L^2 inner product:

Let $V_K \in \mathbf{K}$ and $V_G \in \mathbf{G}$. Then:

$$< V_K, V_G >= \int_\Omega V_K \cdot V_G d\tau = \int_\Omega V_K \cdot \nabla \varphi d\tau$$

for some φ. Now, using vector identity B.5, we obtain:

$$\int_\Omega V_K \cdot \nabla \varphi d\tau = \int_\Omega (\nabla \cdot (\varphi V_K) - \varphi(\nabla \cdot V_K))\, d\tau$$

The second term in the integral is zero since V_K must be divergence free by definition. Thus we have only the first term to consider. Applying the divergence theorem, we have:

$$\int_\Omega \nabla \cdot (\varphi V_K) d\tau = \oint_{\partial\Omega} \varphi V_K d\sigma$$

And since V_K is tangent to the boundary by definition, we have that our integral is zero.

[1] See Vector Calculus and the Topology of Domains in 3-Space by Cantarella, DeTurck, and Gluck.

We may now write:

$$\mathbf{VF}(\Omega) = \mathbf{K} \oplus \mathbf{G} \tag{3.6}$$

Now we have two primary categories for which our vector fields can be decomposed. Each of these categories has its own decomposition as well. More specifically, we can decompose **K** into the two orthogonal subspaces of fluxless knots, **FK**, and harmonic knots, **HK**. We define **FK** and **HK** as follows:

$$\mathbf{FK} = \{V \in \mathbf{K} : \text{all interior fluxes are zero}\}$$
$$\mathbf{HK} = \{V \in \mathbf{K} : \nabla \times V = 0\}$$

We can write:

$$\mathbf{K} = \mathbf{FK} \oplus \mathbf{HK} \tag{3.7}$$

We can also decompose **G** into three mutually orthogonal subspaces. These three subspaces are curly gradients, **CG**, harmonic gradients, **HG**, and grounded gradients, **GG**. We define **CG**, **HG**, **GG** as follows:

$$\mathbf{CG} = \{V \in \mathbf{G} : \nabla \cdot V = 0, \text{all boundary fluxes are zero}\}$$
$$\mathbf{HG} = \{V \in \mathbf{G} : \nabla \cdot V = 0, \varphi \text{ is locally constant on } \partial\Omega\}$$
$$\mathbf{GG} = \{V \in \mathbf{G} : \varphi|_{\partial\Omega} = 0\}$$

We can write:

$$\mathbf{G} = \mathbf{CG} \oplus \mathbf{HG} \oplus \mathbf{GG} \tag{3.8}$$

Now that we have decomposed both knots and gradients into their mutually respective subspaces, we can combine equations 3.6, 3.7, and 3.8 to obtain:[2]

$$\mathbf{VF}(\Omega) = \mathbf{FK} \oplus \mathbf{HK} \oplus \mathbf{CG} \oplus \mathbf{HG} \oplus \mathbf{GG}$$

On a last note of preliminaries, we'd like to define a special kind of vector field called a curl eigenfield. We will denote the space of curl eigenfields as **CE** where:[3]

$$\mathbf{CE} = \{V \in \mathbf{VF}(\Omega) : \exists \lambda \text{ s.t. } \nabla \times V = \lambda V\}$$

[2]It should be noted that what we have not proved anything here, but rather only stated a result that we get from the Hodge Decomposition Theorem for Vector Fields. The actual proofs of these statements are not trivial and are beyond the scope of this paper.

[3]For a much more thorough treatment of the Hodge Decomposition Theorem for Vector Fields, please see "Vector Calculus and the Topology of Domains in 3-Space" by Cantarella, DeTurck, and Gluck

CHAPTER 4

THE WOLTJER PROBLEM

4.1 PERIODS OF THE TORUS

We now take our discussion to focus on properties of periodicity in the flat torus. We will assume that our flat torus has an aspect ratio bounded below by unity. First, let us imagine a loop going around the long way[1] of our torus. Let us call this loop L. If we imagine another loop wrapping the short way around the torus, we can call this loop M. Moreover, M bounds a surface inside the torus that we shall call Σ_M. The importance of Σ_M will become apparent later because we'll be interested in the flux through this surface. Now suppose we are given a vector field V and we wish to follow our respective loops L and M around the torus through our vector field V. Then we define quantities called longitudinal and meridional periods respectively as $p_L(V)$ and $p_M(V)$ where:

$$p_L(V) = \int_L V \cdot ds \tag{4.1}$$

$$p_M(V) = \int_M V \cdot ds \tag{4.2}$$

The relationship of these periods to energy, flux, and helicity will become more apparent as we prove a few theorems that follow.

[1]Since the aspect ratio must be greater than unity, the long way will always be the circumference generated by the major radius.

4.2 Introduction to the Woltjer Problem

The Woltjer Problem involves finding minimal energy fluid knots on a domain, given a fixed value of helicity. It can be stated by the following pair of equations:

$$E(V) = < V, V > \tag{4.3}$$

$$H(V) = C \tag{4.4}$$

Recall we stated in Chapter 2 that the flux of a plasma will be conserved. Considering also the flux of the plasma, we can simplify the Woltjer problem by adding the constraint that flux be held fixed. This special case of the Woltjer problem is known as the Taylor problem. Moreover, we are looking for solutions that are fluid knots, that is, vector fields that are smooth, divergence free, and tangent to the boundary. We will see in Chapter 5 that solutions to this simplified case of the Woltjer problem must necessarily be curl eigenfields and this will reduce the energy minimizing problem to finding solutions of the equation $\nabla \times V = \lambda V$. But before we proceed, we will establish the relationship between energy, helicity, and flux. We start this process by looking at some vector field relations.

4.3 Vector Field Relations

Since solutions to Taylor's problem will be knots that are also curl eigenfields, we will proceed to establish a property of such vector fields.[2]

Proposition: Suppose $V \in \mathbf{K} \cap \mathbf{CE}$. Then $p_M(V) = \lambda F(V)$.

[2] For a more detailed reference of these propositions, refer to "The Taylor Problem in Plasma Physics" by Dr. Jason Parsley.

Proof: Using Stokes Theorem, we obtain that:

$$p_M(V) = \oint_M V \cdot ds = \int_{\Sigma_M} (\nabla \times V) \cdot ndA$$

Since $V \in \mathbf{CE}$, we can write:

$$\int_{\Sigma_M} (\nabla \times V) \cdot ndA = \lambda \int_{\Sigma_M} V \cdot ndA = \lambda F(V)$$

QED

Proposition: $E(V) = \lambda H(V) + p_L(V)F(V)$

We have two cases to consider here: the case where $\lambda = 0$ and then when $\lambda \neq 0$. If $\lambda = 0$ then we essentially want to show that $E(V) = p_L(V)F(V)$. We start out by letting $V = c\hat{Z}$. Then we have:

$$E(V) = \int_\Omega V \cdot V d\tau = \int_0^l \int_0^{2\pi} \int_0^a c^2 r dr d\varphi dz = \pi a^2 l c^2$$

Now we calculate flux:

$$F(V) = \int_\Omega V \cdot ndA = \int_0^{2\pi} \int_0^a c r dr d\varphi = \pi a^2 c$$

Finally, period is given by:

$$p_L(V) = \int_L V \cdot ds = \int_0^l c dz = cl$$

Thus we have that $E(V) = p_L(V)F(V)$ when $\lambda = 0$.

To show the case for $\lambda \neq 0$, we first need to introduce the modified Biot-Savart operator. The modified Biot Savart operator, $BS'(V)$, is defined by restricting $BS(V)$ to Ω and then subtracting some gradient, V_G, so that $\nabla \cdot BS'(V) = 0$ and $BS'(V) \cdot n = 0$. Thus we can say:

$$BS'(V) = BS(V) - V_G$$

Moreover, from the Hodge Decomposition Theorem, we also know that:

$$< V, V_G >= 0$$

because of the inner product structure of $VF(\Omega)$. Now we can establish a relationship between helicity and the modified Biot-Savart operator:

$$H(V) =< V, BS(V) >=< V, BS'(V) - V_G >=< V, BS'(V) >$$

For the next step, we need to call on the fact that we can write:[3]

$$BS'(V) = \frac{1}{\lambda}V - \frac{p_L(V)}{\lambda l}\hat{z} \tag{4.5}$$

Accepting this fact, our result follows smoothly by starting with our definition of helicity:

$$H(V) = \int_\Omega V \cdot BS'(V) d\tau = \int_\Omega V \cdot \left(\frac{1}{\lambda}V - \frac{p_L(V)}{\lambda l}\hat{z} \right) d\tau$$

$$= \frac{1}{\lambda}\int_\Omega V \cdot V d\tau - \frac{1}{\lambda}\int_\Omega V \cdot \frac{p_L(V)}{l}\hat{z} = \frac{1}{\lambda}E(V) - \frac{1}{\lambda}p_L(V)\int_\Omega V \cdot \frac{1}{l}\hat{z} d\tau$$
$$= \frac{1}{\lambda}E(V) - \frac{1}{\lambda}p_L(V)\int_\Omega V \cdot d\mathbf{a} = \frac{1}{\lambda}E(V) - \frac{1}{\lambda}p_L(V)F(V)$$

Now multiplying through by λ we obtain:

$$\lambda H(V) = E(V) - p_L(V)F(V)$$

[3] This is a nontrivial result and too involved to include justification of here. Please reference section 7 of "The Taylor Problem in Plasma Physics" by Dr. Jason Parsley for a detailed treatment.

QED

For our next proposition, recall from section 2 that we stated solutions to the Taylor problem must necessarily be curl eigenfields (we will show this in Chapter 5).

Proposition $H(V) = 0$ if and only if Taylor solutions correspond to eigenfields with eigenvalue zero.

$\Longrightarrow$ Suppose $\lambda = 0$. Since Taylor solutions correspond to vector fields that are curl eigenfields, we have $\nabla \times V = 0$. This implies $V \in \mathbf{HK}$. Moreover zero curl for V implies our vector potential A is zero. In electromagnetism, helicity is defined classically as:

$$H(V) = \int_{\Omega} A \cdot B d\tau$$

Since A is zero, then we have that $H(V) = 0$.

$\Longleftarrow$ We established that $E(V) = \lambda H(V) + p_L(V)F(V)$. If $H(V) = 0$, then we have $E(V) = p_L(V)F(V)$. Recall that we let $V = c\hat{Z}$, and then we showed that $E(V) = \pi a^2 lc^2$ and $F(V) = \pi a^2 c$. So from that, we have $E(V) = lcF(V)$. Choose $W \in \mathbf{K}$ such that $F(W) = F(V)$. By the Hodge Decomposition Theorem, we can write W as the direct sum of two fields lying in distinct orthogonal subspaces. Namely, we can say $W = W_{FK} + W_{HK}$ where $W_{FK} \in \mathbf{FK}$ and $W_{HK} \in \mathbf{HK}$. We know $F(W_{FK}) = 0$ by definition of $\mathbf{FK}$. Thus we conclude that $F(W_{HK}) = F(W) = F(V)$ which implies $W_{HK} = V$. Since E differs from F only by a scalar factor of lc, we may now state that $E(V) = E(W_{HK}) \leq E(W) = E(W_{HK}) + E(W_{FK})$. Thus any choice of W will only yield a higher energy field.

QED

CHAPTER 5

AN OVERVIEW OF TAYLOR'S SOLUTION

5.1 LOOKING FOR CURL EIGENFIELDS

Let Ω be the domain on which our vector field V is defined. Taylor's problem asks us to find fluid knots of minimal energy, given that helicity and flux are fixed. In section 9 of **Influence of Geometry and Topology on Helicity** by Cantarella, DeTurck, Gluck, and Teytel, it is shown that if a vector field V is divergence free and tangent to our boundary, $\partial\Omega$, then the following equation holds:

$$\nabla \times \mathbf{BS}(V) = V$$

We will use this result to show that solutions to the Taylor Problem must necessarily be curl eigenfields. We start with a Lagrange multipliers method from calculus of variations, given the equation we obtained in the previous chapter:

$$E(V) = \lambda H(V) + p_L(V)F(V)$$

We will also need to know a couple properties of the modified Biot Savart operator. These will be given here without proof:

1. BS' is linear: $BS'(c_1V_1 + c_2V_2) = c_1BS'(V_1) + c_2BS'(V_2)$

2. BS' is self-adjoint: $< BS'(V), W >=< V, BS'(W) >$

Accepting these two properties above and using methods from calculus of varations, we can now derive the following:

Proposition: If V is a solution to $E(V) = \lambda H(V) + p_L(V)F(V)$, then $\nabla \times V = \lambda V$

We begin by considering small perturbations in our vector field, V, starting at time zero and represented by a quantity, W. Recall that the value of flux is independent of the surface we choose so long as $\partial\Sigma \in \partial\Omega$. So we are free to choose for any V, a Σ_V such that $\nabla \times p_L(V)F(V) = 0$. If we differentiate $E(V+tW) = \lambda H(V+tW) + p_L(V+tW)F(V+tW)$ with respect to time, we have $\frac{d}{dt}E(V+tW) = \lambda \frac{d}{dt}H(V+tW) + \frac{d}{dt}(p_L(V+tW)F(V+tW))$. Rewriting, using our inner product definitions, we have $\frac{d}{dt} < V+tW, V+tW >= \lambda \frac{d}{dt} < BS'(V+tW), V+tW > + \frac{d}{dt}(p_L(V+tW)F(V+tW))$. Expanding this out, it becomes $\frac{d}{dt}(< V,V > +2t < V,W > +t^2 < W,W >) = \frac{d}{dt}(< BS'(V),V > +2t < BS'(V),W > +t^2 < W,W >) + \frac{d}{dt}(p_L(V+tW)F(V+tW))$. Now applying our time operator to the energy and helicity terms, we obtain $2 < V,W > +2t < W,W >= 2 < BS'(V),W > +2t < W,W > + \frac{d}{dt}(p_L(V+tW)F(V+tW))$. If we let $t = 0$, then we have $2 < V,W >= 2\lambda < BS'(V),W > + \frac{d}{dt}\mid_{t=0} (p_L(V+tW)F(V+tW))$. Now, taking the curl of both sides, we have $\nabla \times (2 < V,W >) = \nabla \times (2\lambda < BS'(V),W >) + \nabla \times (\frac{d}{dt}\mid_{t=0} (p_L(V+tW)F(V+tW)))$. The curl of the last term will be zero, and then we are left with $\nabla \times (2 < V,W >) = \nabla \times (2\lambda < BS'(V),W >)$. Since this expression is true for any W, then $\nabla \times V = \nabla \times \lambda BS'(V)$. But we have shown for the modified Biot Savart operator that $\nabla \times BS'(V) = V$. Thus $\nabla \times V = \lambda V$. QED.

To solve our energy minimization problem, we now only need look for vector fields that are curl eigenfields.

5.2 The Lundquist Solution

We will begin by solving our Lagrange multiplier problem for energy minimization for the simplest case. Given a standard basis of cylindrical coordinates, we start by holding φ and z contant. Thus what we will be looking for is an energy minimizing vector field with dependence only on r. The solution that we obtain by doing this is known as the Lundquist solution.

In the general Taylor Problem, we would have the following:

$$V(r,\varphi,z) = V_r(r,\varphi,z)\hat{r} + V_\varphi(r,\varphi,z)\hat{\varphi} + V_z(r,\varphi,z)\hat{z}$$

But since we are letting V depend only on r, we can write the above as:

$$V(r) = V_r(r)\hat{r} + V_\varphi(r)\hat{\varphi} + V_z(r)\hat{z}$$

The equation we wish to solve is:

$$\nabla \times V = \nabla \times \left[V_r(r)\hat{r} + V_\varphi(r)\hat{\varphi} + V_z(r)\hat{z}\right] = \lambda \left[V_r(r)\hat{r} + V_\varphi(r)\hat{\varphi} + V_z(r)\hat{z}\right]$$

We start by working with just the left hand side and simplifying:

$$\nabla \times V = \left[\frac{1}{r}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right]\hat{r} + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\hat{\varphi} + \frac{1}{r}\left[\frac{\partial (rV_\varphi)}{\partial r} - \frac{\partial V_r}{\partial \varphi}\right]\hat{z}$$

Every partial derivative that is not with respect to r is going to be zero and thus we obtain:

$$\nabla \times V = -\frac{\partial V_z}{\partial r}\hat{\varphi} + \frac{1}{r}\frac{\partial (rV_\varphi)}{\partial r}\hat{z} = -\frac{\partial V_z}{\partial r}\hat{\varphi} + \left(\frac{\partial V_\varphi}{\partial r} + \frac{V_\varphi}{r}\right)\hat{z}$$

Now we have that:

$$-\frac{\partial V_z}{\partial r}\hat{\varphi}+\left(\frac{\partial V_\varphi}{\partial r}+\frac{V_\varphi}{r}\right)\hat{z}=\lambda\left[V_r(r)\hat{r}+V_\varphi(r)\hat{\varphi}+V_z(r)\hat{z}\right]$$

and thus we obtain the following three equations:

$$\begin{aligned}V_r&=0\\-\frac{\partial V_z}{\partial r}&=\lambda V_\varphi\\\frac{\partial V_\varphi}{\partial r}+\frac{V_\varphi}{r}&=\lambda V_z\end{aligned}$$

From our first equation, we get:

$$V_\varphi=-\frac{1}{\lambda}\frac{\partial V_z}{\partial r}$$

We plug this back into our second equation and obtain:

$$-\frac{1}{\lambda}\frac{\partial^2 V_z}{\partial r^2}-\frac{1}{r\lambda}\frac{\partial V_z}{\partial r}-\lambda V_z=0$$

Or likewise:

$$r^2\frac{\partial^2 V_z}{\partial r^2}+r\frac{\partial V_z}{\partial r}+r^2\lambda^2 V_z=0$$

This equation may not appear to exactly take on the form of Bessel's equation, but it does however take on the form of what is commonly known as the parametric Bessel equation. The parametric Bessel equation is given by $x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0$. It can be shown fairly easily that the parametric Bessel equation implies the Bessel equation. If we simply make the substitution that $x'=\lambda x$, then we obtain:

$$\frac{x'^2}{\lambda^2}\frac{d^2y}{dx^2}+\frac{x'}{\lambda}\frac{dy}{dx}+(x'^2-\nu^2)y=\frac{x'^2}{\lambda^2}\frac{d^2y}{dx'^2}\lambda^2+\frac{x'}{\lambda}\frac{dy}{dx}\lambda+(x'^2-\nu^2)y=x'^2\frac{d^2y}{dx'^2}+x'\frac{dy}{dx'}+(x'^2-\nu^2)y=0$$

And this is the original Bessel equation.

Now we know the solution to this equation must be:

$$V = J_1(\lambda r)\,\hat{\varphi} + J_0(\lambda r)\,\hat{z}$$

We now have the Lundquist solution for energy minimization sans λ. To find λ all we need do is apply our boundary conditions and use the known values of zeros for the Bessel function. Since energy must be zero at the boundary, then if we assume a normalized radius ($R = 1$), the value of the Bessel function must also be zero when R=1. Thus we want to find a λ such that $J_0(\lambda) = 0$. The first value of λ for which this occurs can be determined numerically and it is approximately 2.404.

5.3 A More General Solution

Making certain assumptions in order to simplify the mathematics, we have obtained a basic solution to energy minimization of confined plasmas. While this solution has many merits, it is difficult to resist the temptation to go a step further. Recall the small angle approximation for the pendulum problem from elementary mechanics. You'll remember that for small angles, we just assume $\sin\theta \approx \theta$. This makes the mathematics simple, but deep down we may crave something more intricate and exact. To get precise answers for the pendulum problem, we must consider the general case which introduces the use of eliptic integrals to find solutions. But we can't deny that the small angle approximation provided us with a level of understanding. For the Lundquist solution, using cylindrical coordinates, we hold both φ and z constant. To get a much better approximation, we must consider fields where φ and z are not held constant. In order to do this, we'd set up our equation as before only now we consider dependency on all three variables:

$$\left[\frac{1}{r}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right]\hat{r} + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\hat{\varphi} + \frac{1}{r}\left[\frac{\partial (rV_\varphi)}{\partial r} - \frac{\partial V_r}{\partial \varphi}\right]\hat{z} = \lambda\left[V_r(r,\varphi,z)\hat{r} + V_\varphi(r,\varphi,z)\hat{\varphi} + V_z(r,\varphi,z)\hat{z}\right]$$

From this we obtain the following three coupled partial differential equations:

$$\begin{aligned} \frac{1}{r}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z} &= \lambda V_r \\ \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} &= \lambda V_\varphi \\ \frac{1}{r}\left[\frac{\partial (rV_\varphi)}{\partial r} - \frac{\partial V_r}{\partial \varphi}\right] &= \lambda V_z \end{aligned}$$

Since the solution to the Taylor problem must lie in **CE**, we know that it must be a combination of the Lundquist solution, V_0, and the arbitrary curl eigenfields $V^i_{m,k}$. We know this because it was shown[1] that these are all the curl eigenfields that are both tangent to the boundary and divergence free. For reference we state what these are.

$$\begin{aligned} V_0 &= J_1(\lambda r)\hat{\varphi} + J_0(\lambda r)\hat{z} \\ V^{\pm 1}_{m,k} &= \left(-\frac{m}{r}J_m(nr) \mp \frac{mk}{\lambda r}J_m(nr) \pm \frac{nk}{\lambda}J_{m+1}(nr)\right) sin(m\varphi \pm kz)\hat{r} \\ &+ \left(-\frac{m}{r}J_m(nr) \mp \frac{mk}{\lambda r}J_m(nr) + nJ_{m+1}(nr)\right) cos(m\varphi \pm kz)\hat{\varphi} \\ &+ \frac{n^2}{\lambda}J_m(nr)cos(m\varphi \pm kz)\hat{z} \\ V^{\pm 2}_{m,k} &= \left(-\frac{m}{r}J_m(nr) \pm \frac{mk}{\lambda r}J_m(nr) \mp \frac{nk}{\lambda}J_{m+1}(nr)\right) cos(m\varphi \mp kz)\hat{r} \\ &+ \left(-\frac{m}{r}J_m(nr) \pm \frac{mk}{\lambda r}J_m(nr) + nJ_{m+1}(nr)\right) sin(m\varphi \mp kz)\hat{\varphi} \\ &+ \frac{n^2}{\lambda}J_m(nr)sin(m\varphi \mp kz)\hat{z} \end{aligned}$$

where $m \in \mathbb{Z}^+ \cup \{0\}$, $k \in \frac{2\pi}{l}(\mathbb{Z}^+ \cup \{0\})$, and $\lambda = \sqrt{k^2+n^2}$.

[1] See "The Spectrum of the Curl Operator on the Flat Torus" by Cantarella, DuTurck, Gluck.

Proposition: $\forall i$ and for fields $V^i_{m,k}$, $p_L(V^i_{m,k}) = 0$ and $F(V^i_{m,k}) = 0$.

Proof:

$$p_L(V^i_{m,k}) = \int_0^l \frac{n^2}{\lambda} J_m(na) cos(m\varphi_0 + kz) dz$$

We consider two cases here: one for $k \neq 0$ and one for $k = 0$. If $k \neq 0$, then note we are integrating over exactly one period(that is, we are going from 0 to l. Since cosine is a periodic function and there is no other z dependence in our integral, then the value of the integral must be 0. If $k = 0$, then our cosine term is constant and we obtain:

$$p_L(V_{m,0}) = \lambda l J_m(\lambda a) cos(m\varphi_0)$$

We know that $V^{\pm i}_{m,k}$ is tangent to the boundary when:[2]

$$\frac{m}{a} J_m(na) = \pm \left(-\frac{mk}{\lambda a} J_m(na) + \frac{nk}{\lambda} J_{m+1}(na) \right)$$

Since $k = 0$, and we know that the above equation must hold for $V_{m,0}$ to be tangent to the boundary, then this implies the right hand side is zero and thus $J_m(\lambda a) = 0$.

Now,

$$F(V_{m,k}) = \frac{p_M(V_{m,k})}{\lambda} = \frac{1}{\lambda} \int_0^{2\pi} aV(a, \varphi, z_0) \cdot \hat{\varphi} d\varphi$$

$$= \frac{a}{\lambda} \left(-\frac{m}{r} J_m(na) - \frac{mk}{\lambda a} J_m(na) + nJ_{m+1}(na) \right) \int_0^{2\pi} cos(m\varphi + kz_0) d\varphi$$

The periodicity of the cosine implies the integral is zero for $m \neq 0$. If $m = 0$, then we have

$$F(V_{m,k}) = \frac{2\pi na}{\lambda} J_1(na) cos(kz_0)$$

[2]See: Parsley, The Taylor Problem in Plasma Physiscs, Section 5.

But we know that $J_1(\lambda a) = 0$ and thus $\forall m,k$ we have $F(V_{m,k}) = 0$.

From this point, we have a few more steps to take before we arrive at our goal. We have already stated that our solution must be a combination of V_0 and $V_{m,k}$. So let us write the Taylor solution as $W = \alpha V_0 + \beta V_{m,k}$ where α, β, m, and k are specific values we must determine.

The first thing to note is that the ratio of helicity to the square of flux determines the values of our constants α and β. To give a sketch of why this is true, we recall that $F(V_{m,k}) = 0$ and so we can write:

$$F(W) = \alpha F(V_0)$$

Also:

$$H(W) = \alpha^2 H(V_0) + 2\alpha\beta < BS(V_0), V_{m,k} > +\beta^2 H(V_{m,k})$$

From these two equations, we can obtain values for α and β.

$$\alpha = \frac{F(W)}{F(V_0)}$$

$$\beta = \frac{-2\alpha\beta < BS(V_0), V_{m,k} > \pm\sqrt{(2\alpha\beta < BS(V_0), V_{m,k} >)^2 - 4H(V_{m,k})(\alpha^2 H(V_0) - H(W))}}{2(H(V_{m,k}))}$$

We also want the eigenvalue that will yield us the lowest energy level out of all $\lambda_{m,k}$'s. It turns out that this occurs for $m = 1$ and $k \approx \frac{1.25}{a}$.[3]

While the importance of these exact constants can primarily be seen in experimental physics, it is nice to know that we can indeed quantify our results mathematically. This quantification, although moderately tedious for our simplified scenarios, becomes almost impossible when more real-world factors are introduced.

[3] See: Parsley, The Taylor Problem in Plasma Physiscs, Section 8.

CHAPTER 6

A GLANCE AT THE TRANSITION TO THE STANDARD TORUS

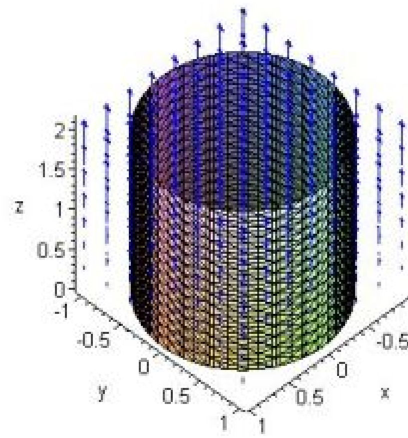
While the Taylor solution provides us with simple geometric model for plasma flows, it should be clear that its simplification omits several factors that must be considered in real-world applications. The most obvious omission would be the bend or curvature of the standard torus.

The flat torus serves as an almost perfect model for high-aspect ratio curved tori because, for any given small cross section, the degree of curvature is minimal. However for low-aspect ratio tori, the degree of curvature is high, even for a small cross section, and thus we cannot expect vector fields to behave the same as they do in a flat torus. All modern tokamaks are modelled best by very low-aspect ratio tori. And in fact, the cross sections of these reactors are more like a D-shape rather than a circle. These factors must be taken into account when considering the most efficient field configuration of a fusion reactor.

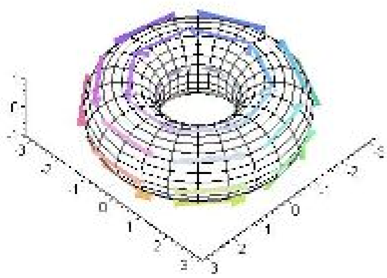
Let us define the projected position of a particle, q, inside the torus, with respect to its major axis to be as follows:

$$P_q = R - rcos(\varphi)$$

If we imagine a uniform vector field, say V_1, under cylindrical coordinates, flowing in the $\hat{z}$ direction in the flat torus, we note that the field lines all travel at the same rate no matter what their distance from the boundary. See image:



Now consider bending this flat torus and connecting the identified ends as to form a standard torus, as shown:



If we retain the same vector field under our new geometry, although the linear velocity remains the same among all field lines, the difference in angular velocity, which now must be accounted for, causes a metric separation of all particles that flow in different projected positions on the major axis. In other words, two particles that, in the flat torus, would have remained next to each other forever, now drift apart at a constant rate proportional to the difference in their projected positions on the major axis. We can calculate this rate, Λ_q as follows:

$$\Lambda_q = k(P_{q_1} - P_{q_2})$$

where k is the proportionality constant. We note that k will remain fixed as long as the major radius of the torus remains fixed.

As we can see in the above image of the standard torus, the vector field is flowing in the $\hat{\varphi}$ direction. Although this image does not give reference to the rate of flow of the field lines, imagine a new vector field that would appear visually the same as the one above. But instead of the linear velocity of all particles in the field being the same, we now have equal angular velocity. We will call this new vector field V_2 and we note that there is some W such that $V_2 = V_1 + W$. As a field quantity, W will be dependent on Λ_q. And to consider the transition from flat to curved torus, W must be taken into account.

We already stated heuristically, that as the aspect ratio of the curved torus is made smaller, the role of W will become more significant. To take a brief look at this fact mathematically, we can write:

$$|W| = \int_{\Omega} \Lambda_q d\tau$$

When the aspect ratio increases, the projected positions of particles become smaller. Thus we have $\Lambda_q - > 0$ for every particle, q. So we see that $|W| - > 0$ uniformly and thus W is identically zero.

Appendix A

Visualizations and Coding

In this section, I have placed some visualizations of vector fields and tori that were created using Maple 11. The Maple code used to generate these images precedes them.

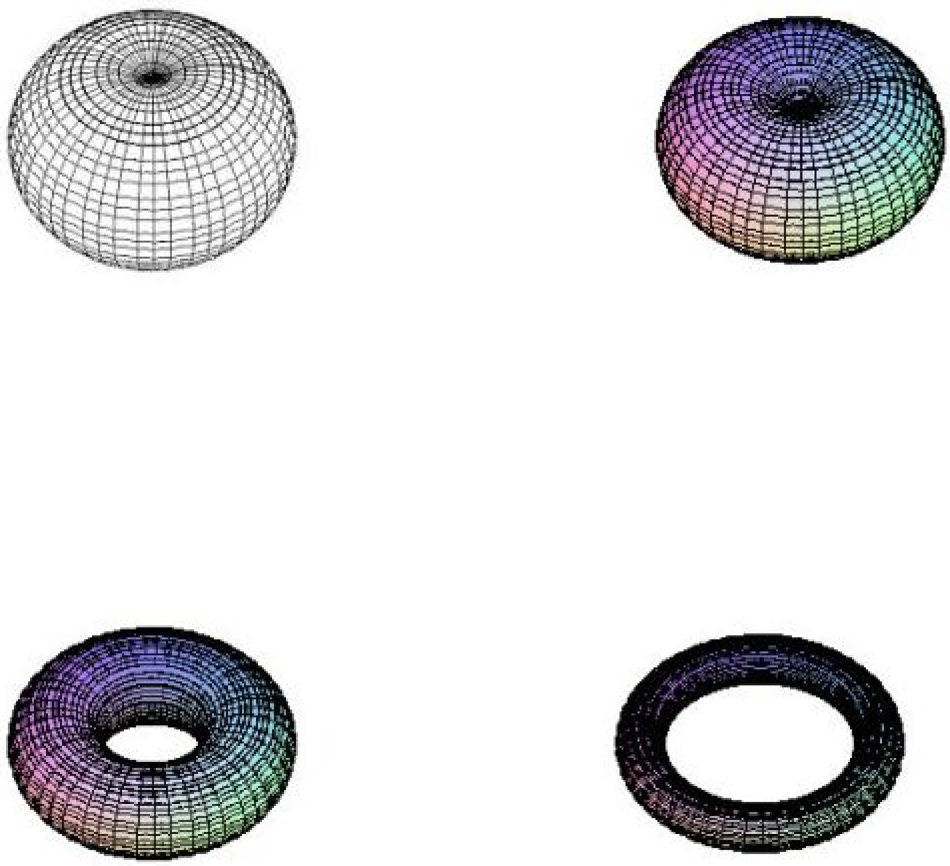
A.1 The Aspect Ratio of a Torus

The aspect ratio of a torus is defined to be the ratio of the major radius (the radius from the geometric center to the center of any circle in the cross-section of the interior) to the minor radius (the radius of any circle in the cross section of the interior). The aspect ratio of a proper torus has a lower bound of 1. We can visualize this by imaging a scenario where the major radius is actually smaller than the minor radius (as seen in the first picture). The object would no longer be a torus and, in fact, would be topologically equivallent to a sphere. We also note that as the aspect ratio approaches zero, the object atains the shape of a perfect sphere.

The second image is interesting in that it is actually tangent to itself at its geometric center. The solid torus of aspect ratio 1 has the property that any longitudinal loop on the boundary is homotopic to a point. This cannot be said about a torus with aspect ratio greater than 1.

Below are some examples of tori (although the first image is not actually a torus) with aspect ratios in increasing order.

```
> reset:
> with(plots):with(plottools):
> c[1]:=0.5;c[2]:=1;c[3]:=2;c[4]:=5;
                              c[1] := 0.5

                              c[2] := 1

                              c[3] := 2

                              c[4] := 5

> plot3d([(1+(1/c[1])*cos(v))*cos(u),(1+(1/c[1])*cos(v))*sin(u)
,(1/c[1])*sin(v)],v=-10..10,u=-10..10,scaling=constrained,
numpoints=2500,shading=NONE);plot3d([(1+(1/c[2])*cos(v))*cos(u),
(1+(1/c[2])*cos(v))*sin(u),(1/c[2])*sin(v)],v=-10..10,u=-10..10,
scaling=constrained,numpoints=2500);plot3d
([(1+(1/c[3])*cos(v))*cos(u),(1+(1/c[3])*cos(v))*sin(u),
(1/c[3])*sin(v)],v=-10..10,u=-10..10,scaling=constrained,
numpoints=2500);plot3d([(1+(1/c[4])*cos(v))*cos(u),(1+(1/c[4])
*cos(v))*sin(u),(1/c[4])*sin(v)],v=-10..10,u=-10..10,
scaling=constrained,numpoints=2500);
>
```

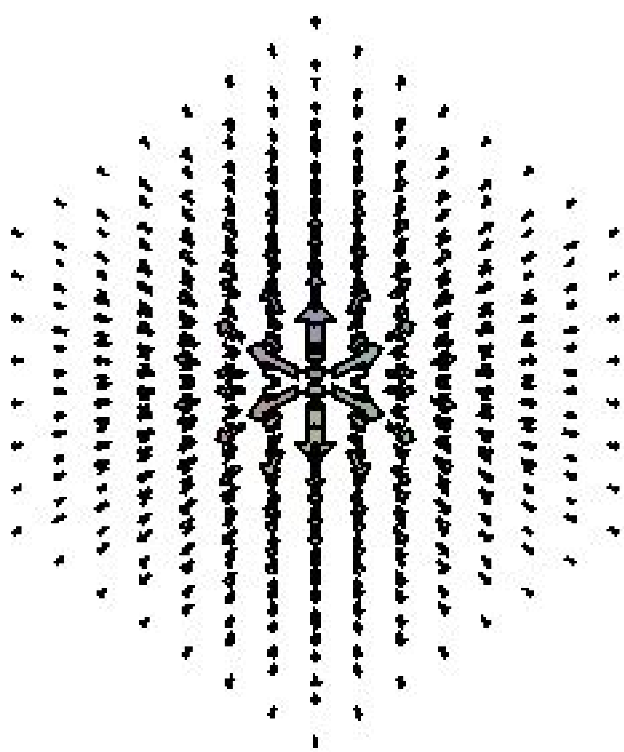


A.2 VECTOR FIELDS

A vector field can be visualized as a set of arrows indicating a direction of flow. Since a plasma in a tokamak fusion reactor behaves as a fluid confined to the interior of a solid torus, we can model them effectively using vector fields.

Below is an example of one of the most basic vector fields in cartesian coordinates. This vector field is symmetric about all axes and points outward from the origin. But most important to note is that it actually gets ”weaker” as it goes further out from the origin (this is represented by the arrows becoming smaller). As it happens, this particular vector field can represent an electric field of a point charge or a gravitational field from a point (or any field that disperses its intensity at a rate of an inverse square).

```
> F[1]:=vector([x/(x^2+y^2+z^2),y/(x^2+y^2+z^2),z/(x^2+y^2+z^2)]);

                 [      x              y              z       ]
        F[1] :=  [------------, ------------, ------------]
                 [ 2     2     2   2     2     2   2     2     2]
                 [x   + y   + z   x   + y   + z   x   + y   + z ]

> fieldplot3d(F[1],x=-1..1,y=-1..1,z=-1..1,thickness=2,scaling
=constrained,arrows=THICK);
```



If we pictured the above vector field as, say, the gravitational field of the earth on a large enough scale that the earth could be considered a point, we would note that these arrows are normal to the surface of the earth.

But since we are dealing with tori primarily, let us define a vector field that is normal to the surface of a torus - particularly, the vector field represented by the gradient.

```
> a:=0.5;b:=Pi/2;c:=1;
```

$$a := 0.5$$

$$b := \frac{\pi}{2}$$

$$c := 1$$

```
> F[2]:=(x,y,z)->(c-sqrt(x^2+y^2))^2+z^2-a^2;
```

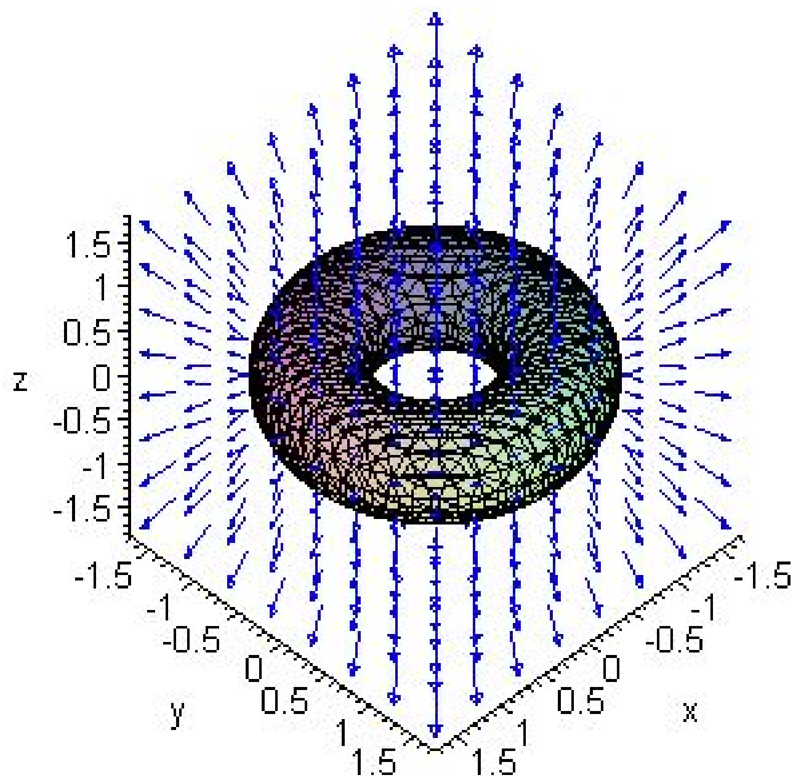
$$F[2] := (x, y, z) \to (c - \mathrm{sqrt}(x^2 + y^2))^2 + z^2 - a^2$$

```
> gradF2:=VectorCalculus[Gradient](F[2](x,y,z),[x,y,z]);
```

$$gradF2 := -\frac{2\,(1 - (x^2 + y^2)^{1/2})\,x}{(x^2 + y^2)^{1/2}}\,\overline{e}_x - \frac{2\,(1 - (x^2 + y^2)^{1/2})\,y}{(x^2 + y^2)^{1/2}}\,\overline{e}_y + 2\,z\,\overline{e}_z$$

```
> VectorCalculus[Gradient](gradF2);

> A[2]:=implicitplot3d(F[2](x,y,z),x=-b..b,y=-b..b,z=-b..b,
scaling=constrained,axes=framed,numpoints=5000):
```

```
> B[2]:=fieldplot3d(gradF2,x=-b..b,y=-b..b,z=-b..b,scaling=constrained
,axes=framed,color=blue,thickness=1,arrows=SLIM):
> display([A[2],B[2]]);
```



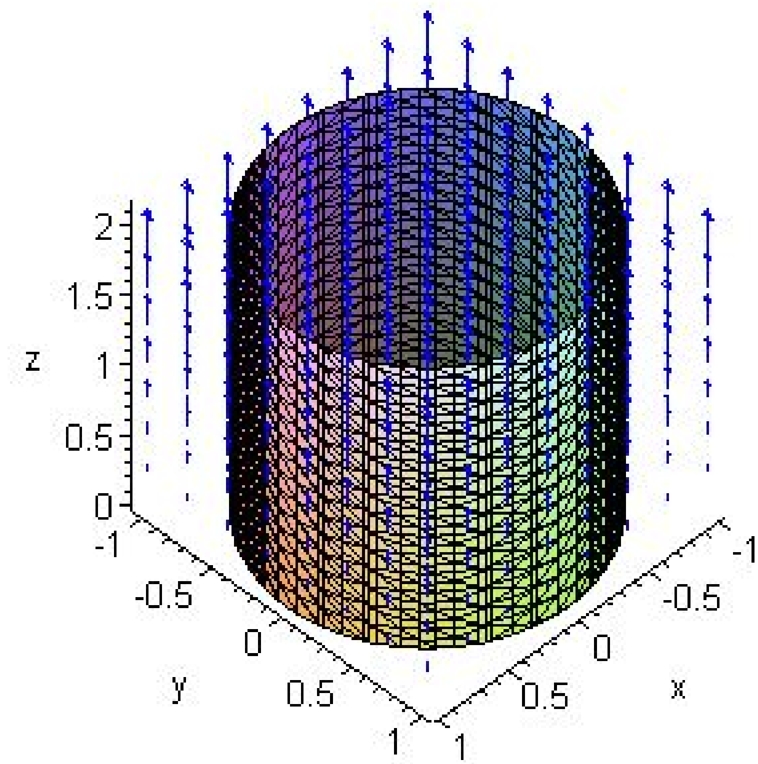
A.3 Vector Fields Inside the Standard and Flat Tori

Now that we have seen examples of different aspect ratio tori as well as some basic vector fields, we might wish to picture some vector fields inside the torus. In a tokamak, the plasma flows on the inside of a magnetic bottle whose interior is similar to that of a solid torus (A tokamak is not a perfect torus however).

The easiest model for vector flows in a torus is to start with the flat torus. The flat torus can be visualized as a cylinder with the top and bottom identified. The primary difference from a flat torus and a standard torus is the metric. If we imagine a standard torus with an aspect ratio that we allow to become larger and larger, the metric approaches that of a flat torus. So flat tori can be quite effective for modelling behavior of flows in large aspect ratio tori. Unfortunately, modern tokamaks have low aspect ratios and the flat torus has its limitations as a model.

Below we show a vector field pointed in the $\hat{z}$ direction of a flat torus.

```
a[0]:=1;b[0]:=1;
                              a[0] := 1


                              b[0] := 1

> F[3]:=(x,y,z)->(x/a[0])^2+(y/b[0])^2-1;

                                      2      2
                                     x      y
                F[3] := (x, y, z) -> ----- + ----- - 1
                                        2      2
                                     a[0]   b[0]

> x[0]:=1;y[0]:=1;z[0]:=2;

                              x[0] := 1


                              y[0] := 1


                              z[0] := 2

> C[1]:=implicitplot3d(F[3](x,y,z),x=-x[0]..x[0],y=-y[0]..y[0],z=0..z[0]
,scaling=constrained,numpoints=2500,axes=framed):
> V[1]:=vector([0,0,z]);

                           V[1] := [0, 0, z]

> C[2]:=fieldplot3d(V[1],x=-x[0]..x[0],y=-y[0]..y[0],z=0..z[0],
scaling=constrained,axes=framed,color=blue,thickness=1,arrows=SLIM):
> display([C[1],C[2]]);
```

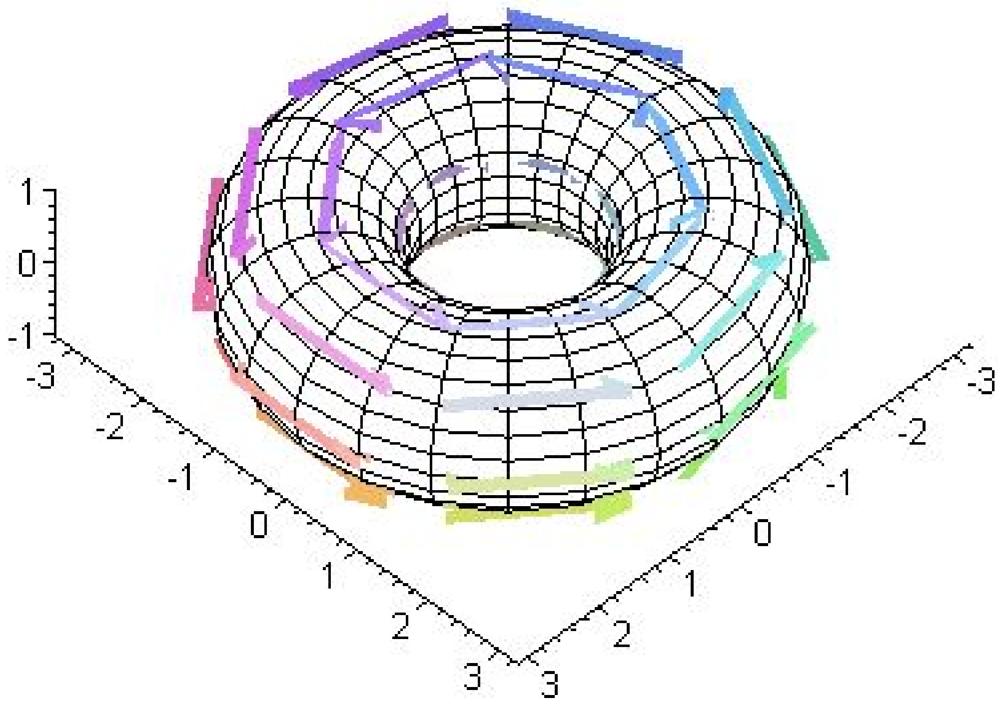


We might wish to map this particular vector field to its equivalent in the standard torus. Unfortunately, it doesn't have an exact equivalent due to the non-uniform change in the metric. We see above that all the field lines are flowing in a uniform direction at a uniform speed. However, an analagous vector field in the standard torus would have field lines flowing faster and faster as they moved further away from the geometric center.

Below we see an analagous vector field in a standard torus.

```
> R:=2;
                                        R := 2
> addcoords(tokamak,[r,u,v],[(R+r*cos(u))*cos(v),(R+r*cos(u))
*sin(v),r*sin(u)]);
> F:=vector([(R+cos(u))*cos(v),(R+cos(u))*sin(v),sin(u)]);
```

```
F := [(2 + cos(u)) cos(v), (2 + cos(u)) sin(v), sin(u)]

> G:=vector([0,0,1]);

                    G := [0, 0, 1]

> A:=plot3d(F,u=0..2*Pi,v=0..2*Pi,scaling=constrained,shading=NONE):
> B:=fieldplot3d(G,r=0..1,u=0..2*Pi,v=0..2*Pi,scaling=constrained,
axes=framed,coords=tokamak,thickness=5, arrows=THIN):
> display([A,B]);
```

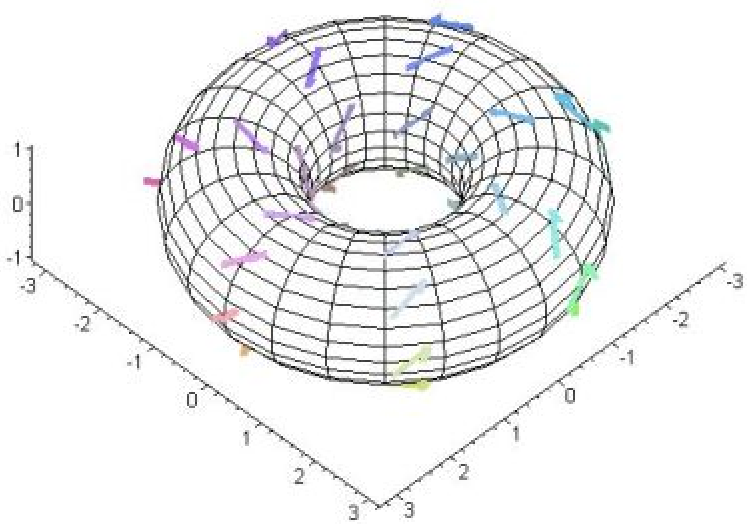


The above vector field has zero helicity, but plasma flows in actual tokamaks do wrap around each other with non-zero helicity. Below we can see a vector field flowing in both the $\hat{u}$ as well as the $\hat{v}$ direction.

```
> H:=vector([0,1,1]);
```

$$H := [0, 1, 1]$$

```
> C:=fieldplot3d(H,r=0..1,u=0..2*Pi,v=0..2*Pi,scaling=constrained
,axes=framed,coords=tokamak,thickness=5, arrows=THIN):
> display([A,C]);
```



A.4 EXTRAS (ENERGY AND HELICITY)

```
> with(LinearAlgebra);
```

Energy of a vector field V is defined to be the L1 inner product of V with itself over a given domain. Our domain in this case is a torus where dV is given by:

```
dV:=(R-r*cos(u))*r*dr*du*dv;
                    dV := (2 - r cos(u)) r dr du dv
```

We will enter a few vector fields here including the two we plotted above and compare the energy levels of these fields.

```
> V[1]:=<0,0,1>;V[2]:=<0,1,1>;V[3]:=<r,u,v>;
```

```
                    [0]
                    [ ]
          V[1] :=   [0]
                    [ ]
                    [1]


                    [0]
                    [ ]
          V[2] :=   [1]
                    [ ]
                    [1]


                    [r]
                    [ ]
          V[3] :=   [u]
                    [ ]
                    [v]
```

```
> for i from 1 to 3 do E[i]:=int(int(int(V[i].V[i]*(R-r*cos(u))
*r,r=0..1),u=0..2*Pi),v=0..2*Pi) end do;
```

```
                          2
          E[1] := 4 Pi


                          2
          E[2] := 8 Pi


                      2           4
  E[3] := -2/3 Pi  + 32/3 Pi
```

```
> for i from 1 to 3 do E[i]:=evalf(E[i]) end do;
```

```
        E[1] := 39.47841762
```

```
E[2] := 78.95683523

E[3] := 1032.450569
```

We can see that the first vector field has half the energy level of the second. This difference in energy is simply due to the non-zero helicity of the second field. Although it is practically impossible to reduce the helicity to zero in a fusion reactor, we can see that efforts to reduce it as much as possible are beneficial.

APPENDIX B

VECTOR IDENTITIES

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \tag{B.1}$$

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \tag{B.2}$$

$$\nabla(fg) = f(\nabla g) + g(\nabla f) \tag{B.3}$$

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A \tag{B.4}$$

$$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot (\nabla f) \tag{B.5}$$

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \tag{B.6}$$

$$\nabla \times (fA) = f(\nabla \times A) - A \times (\nabla f) \tag{B.7}$$

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A) \tag{B.8}$$

$$\nabla \cdot (\nabla \times A) = 0 \tag{B.9}$$

$$\nabla \times (\nabla f) = 0 \tag{B.10}$$

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A \tag{B.11}$$

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