Tautological rings of Prym varieties

by

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(Under the direction of Elham Izadi)

Abstract

The ring of algebraic cycles modulo algebraic equivalence on an abelian variety is an interesting and mysterious object. When the abelian variety is the Jacobian of a smooth curve, Arnaud Beauville defined a certain subring, called the tautological ring, which has become of great interest to a number of mathematicians. Recently Ben Moonen defined the small and the big tautological rings for Jacobians modulo rational equivalence, both of which surject onto the tautological ring of Beauville.

In this thesis, the notions of tautological rings of Beauville and Moonen are generalized to pairs, consisting of an abelian variety and a subvariety. The tautological ring modulo algebraic equivalence is then studied for the pairs: Prym variety P of a double cover $\tilde{C} \to C$ and Abel-Prym curve $\psi(\tilde{C})$. Generators and certain relations, called "Polishchuk relations", for the tautological ring of the pair $(P, \psi(\tilde{C}))$ are determined. Given a complete linear system g^r_d on C, Beauville constructed and studied two subvarieties V_0 and V_1 of P, called *special subvarieties*. He showed that V_0 and V_1 have the same class in the cohomology ring of P. In this thesis it is shown that in many cases V_0 and V_1 are, in fact, algebraically equivalent. The class of the union of V_0 and V_1 turns out to belong to the tautological ring and is expressed in terms of its generators.

INDEX WORDS: Abelian varieties, Prym varieties, algebraic cycles, algebraic curves, Brill-Noether theory.

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Contents

Chapter 1

Introduction

1.1 Tautological rings

For a non-singular variety V, we let $\mathrm{CH}(V)$ denote the Chow ring of V modulo rational equivalence with $\mathbb{Q}$-coefficients. The quotient $\mathrm{CH}(V)/\!\sim_{\mathrm{alg}}$ modulo algebraic equivalence is denoted by $A(V)$. We work over $\mathbb{C}$, the field of complex numbers.

Let X be an abelian variety. Besides the intersection product, the ring $\mathrm{CH}(X)$ is endowed with Pontryagin product defined by

$$x_1 * x_2 = m_*(p_1^* x_1 \cdot p_2^* x_2),$$

where $m\colon X \times X \to X$ is the addition morphism and $p_j\colon X \times X \to X$ is the projection onto the j^{th} factor, [BL, p.530]. If $V, W \subset X$ are subvarieties, set-theoretically $[V] * [W] = \{v + w\colon v \in V, w \in W\}$. More precisely, $[V] * [W] = \deg(m_{|V \times W})[V + W]$ if the addition map $m\colon V \times W \to V + W$ is generically finite, and $[V] * [W] = 0$ otherwise. Moreover, the

Chow ring of X is bi-graded,

$$\mathrm{CH}(X) = \bigoplus_{p,s} \mathrm{CH}^p(X)_{(s)}.$$

The p-grading is by codimension. The Beauville grading (s) is characterized by: $x \in \mathrm{CH}^p(X)_{(s)}$ if and only if $k^*x = k^{2p-s}x$ for all $k \in \mathbb{Z}$, where k also denotes the endomorphism of X given by $x \mapsto kx$, see [Be86]. The (s)-component of a cycle Z is denoted by $Z_{(s)}$. There is a Fourier transform

$$\mathcal{F}_X \colon \mathrm{CH}(X) \to \mathrm{CH}(X),$$

which has been defined by Beauville in relation to the Fourier-Mukai transform, see Section 2.4. The operations $*, \mathcal{F}_X$ and the bi-grading descend to $A(X)$.

When X is the Jacobian J of a smooth curve C of genus g, we may fix a point $o \in C$ and embed C in J via the Abel map $\varphi \colon x \mapsto \mathcal{O}_C(x - o)$. The *small tautological ring* $\mathrm{taut}(C)$ of J is defined to be the smallest $\mathbb{Q}$-subalgebra of $\mathrm{CH}(J)$ under the intersection product, which contains the class of the image of C under φ, and is stable under the operations $*, \mathcal{F}_J, k^*$ and k_* for all $k \in \mathbb{Z}$. The *big tautological ring* $\mathrm{Taut}(C)$ is defined in the same way, except it is required to contain the image of $\varphi_* \colon \mathrm{CH}(C) \to \mathrm{CH}(J)$, see [Mo, Def.3.2, p.487]. The tautological ring for Jacobians was originally defined and studied by Beauville in [Be04] as a $\mathbb{Q}$-subalgebra $\mathscr{T}(C)$ of $A(J)$ under the intersection product. In [Be04] it was shown that $\mathscr{T}(C)$ is generated by the classes $w^1, \ldots, w^{g-1}$, where $w^{g-d} = (1/d!)C^{*d}$. The generators for the tautological rings $\mathrm{taut}(C)$ and $\mathrm{Taut}(C)$ of the Jacobian J have also been determined in [Po07, Thm.0.2, p.461] and [Mo, Thm.3.6, p.489], respectively. The rings $\mathrm{taut}(C)$ and $\mathrm{Taut}(C)$ have the same image, namely $\mathscr{T}(C)$, in $A(J)$.

The following definition generalizes the notions of various tautological rings for a Jacobian. The original idea of considering pairs is due to R. Varley.

Definition 1.1.1. *Let X be an abelian variety and let $V \subset X$ be a subvariety. The* small *and the* big tautological rings $\mathrm{taut}(X,V)$ *and* $\mathrm{Taut}(X,V)$*, respectively, of the pair* (X,V) *are the smallest subrings of* $\mathrm{CH}(X)$ *under the intersection product, which contain* $[V]$ *and* $\mathrm{CH}(V)$*, respectively, and are stable under the operations* $*, \mathcal{F}_X, k^*$ *and* k_* *for all* $k \in \mathbb{Z}$*. The image of* $\mathrm{Taut}(X,V)$ *in* $A(X)$ *is called the* tautological ring *of* (X,V) *and is denoted by* $\mathscr{T}(X,V)$*.*

Our definition of the tautological rings:

$$\mathrm{taut}(J,\varphi(C)), \mathrm{Taut}(J,\varphi(C)) \text{ and } \mathscr{T}(J,\varphi(C))$$

coincides with the previous definitions denoted in [Mo] by $\mathrm{taut}(C), \mathrm{Taut}(C)$ and $\mathscr{T}(C)$, respectively.

Let $\pi\colon \tilde{C} \to C$ be a degree 2 morphism, which is either étale or ramified at two points, and let $\tilde{J}$ be the Jacobian of $\tilde{C}$. The norm map $\mathrm{Nm}\colon \tilde{J} \to J$ takes a linear equivalence class $[\sum_j \tilde{p}_j] \in \tilde{J}$ of degree zero divisors on $\tilde{C}$ to the linear equivalence class $[\sum_j \pi(\tilde{p}_j)] \in J$. By [Mu74], the connected component of the identity in $\ker(\mathrm{Nm}\colon \tilde{J} \to J)$ is a principally polarized abelian variety (P,ξ), called the Prym variety of $\tilde{C} \to C$. If we fix $\tilde{o} \in \tilde{C}$, there is a morphism $\psi\colon \tilde{C} \to P$, called the *Abel-Prym map*, which is obtained by composing $\tilde{C} \to \tilde{J}, \tilde{x} \mapsto \mathcal{O}_{\tilde{C}}(\tilde{x} - \tilde{o})$ with $1 - \iota\colon \tilde{J} \to \tilde{J}$, where ι is the involution induced by the involution on $\tilde{C}$ exchanging the sheets of the cover. If $\tilde{C}$ is non-hyperelliptic, then ψ is a closed embedding in the étale case, and identifies the ramification points in the ramified case, [BL, Prop.12.5.2, p.378]. If $\tilde{C}$ is hyperelliptic, ψ has degree two onto its image.

The outline of the contents of this thesis is as follows. In Chapter 2, we collect the foundational material that will be used in the sequel. In Chapter 3 we study the generators and relations for the ring $\mathscr{T}(P,\psi(\tilde{C}))$ under the intersection product. Consider the classes $\zeta_n = \mathcal{F}_P\big([\psi(\tilde{C})]_{(n-1)}\big) \in A(P)$ for $1 \le n \le \dim P$. In Section 3.2 we show the following.

Theorem 1.1.2. *The tautological ring $\mathscr{T}(P, \psi(\tilde{C}))$ is generated as a $\mathbb{Q}$-subalgebra of $A(P)$ under the intersection product by the cycles ζ_n, where $1 \le n \le \dim P - 1$ and n is odd.*

In [Po05], a collection of relations, which we call *Polishchuk relations*, among the generators of $\mathscr{T}(\tilde{C})$ is obtained and studied. In Section 3.3 we show that Polishchuk relations on $\tilde{J}$ restrict to trivial relations on P, see Proposition 3.3.1. Also, using Polishchuk's methods we find relations among the generators of $\mathscr{T}(P, \psi(\tilde{C}))$, which resemble those for Jacobians. To state the precise result, we recall some notation from [Po05]. Let $[1, r] := \{1, 2, 3, \dots, r\}$. Assume that we are given integers $k_j > 1$ for $j = 1, \dots, r$. Given a subset $I = \{i_1, i_2, \dots, i_s\} \subset [1.r]$, define the numbers:

$$b(I) := \frac{(k_{i_1} + \ldots + k_{i_s})!}{k_{i_1}! \cdots k_{i_s}!} \qquad \text{and} \qquad d(I) := k_{i_1} + \ldots + k_{i_s} - s + 1.$$

Theorem 1.1.3. *For each integer $r \ge 1$, odd integers $k_1, \dots, k_r > 1$, and each d with $0 \le d \le r - 1$ we have the relation,*

$$\sum_{\mathscr{P}_m} \binom{m-1}{d+m-r} b(I_1) \cdots b(I_m) \zeta_1^{[p-d-m+r-\sum_{i=1}^r k_i]} \zeta_{d(I_1)} \cdots \zeta_{d(I_m)} = 0$$

in $A^{p-d}(P)$, where the sum is taken over all unordered partitions $\mathscr{P}_m = \{I_1, \dots, I_m\}$ of $[1, r]$ into m disjoint nonempty parts such that $r - d \le m \le p - d + r - \sum_{i=1}^r k_i$.

In Chapters 4 and 5 we study the special subvarieties V_0 and V_1 of P associated to a complete g^r_d on C (see Section 4.2 for the definition of V_0, V_1). The Brill-Noether variety $W^r_d(C)$ parametrizes invertible sheaves L on C with $\deg L = d$ and $h^0(L) \ge r + 1$, see [ACGH, p.153]. The expected dimension of $W^r_d(C)$ is the Brill-Noether number $\rho(g, r, d) = g - (r + 1)(g - d + r)$. In Section 4.6 we prove the following.

Theorem 1.1.4. *Assume that $W^r_d(C)$ is reduced and of dimension $\rho(g, r, d)$. If $\rho(g, r, d) > 0$, then V_0 and V_1 are algebraically equivalent.*

Note that the inequalities $d < 2g$ and $0 < 2r < d$ are needed for the special subvarieties to be defined and homologically equivalent, respectively (see Section 4.2 for details). As a consequence of Theorem 1.1.2, the tautological ring $\mathscr{T}(P, \psi(\tilde{C}))$ is spanned as a $\mathbb{Q}$-vector space by cycles of the form

$$[\psi(\tilde{C})]_{(n_1)} * [\psi(\tilde{C})]_{(n_2)} * \cdots * [\psi(\tilde{C})]_{(n_r)},$$

where $n_i, r \geq 0$ are varying integers. In Section 5.2 we show that the class $[V]$ of $V := V_0 \cup V_1$ belongs to $\mathscr{T}(P, \psi(\tilde{C})) \subset A(P)$ and express $[V]$ in terms of the generators of $\mathscr{T}(P, \psi(\tilde{C}))$:

Theorem 1.1.5. *If $0 < 2r < d < 2g$, then the component of the class $[V]$ in $A^{g-r-1}(P)_{(t)}$ is given by the formula $[V]_{(t)} = c_{t,r,d}\big([\psi(\tilde{C})]^{*r}\big)_{(t)}$, where $c_{t,r,d}$ are certain rational numbers defined in Section 5.2.*

Remark 1.1.6. The pairs: a Jacobian with an Abel curve and a Prym variety with an Abel-Prym curve are special cases of a pair $(X, \tau(C))$, where $X \subset J$ is a Prym-Tyurin variety defined by the endomorphism σ of J satisfying certain properties and $\tau \colon C \to X$ is the composition of an Abel map $C \to J$ and $(1 - \sigma) \colon J \to X$, see [BL, p.369] for details. Moreover, every principally polarized abelian variety is a Prym-Tyurin variety, although not in a unique way, [BL, Cor.12.2.4, p.371].

1.2 Notation and conventions

We work over the field $\mathbb{C}$ of complex numbers and the Chow rings are considered with $\mathbb{Q}$-coefficients. When we write "point", we mean a closed point. The word "scheme" will mean a noetherian separated scheme over $\mathbb{C}$. By a node on a curve we mean an ordinary double point (locally analytically given by $xy = 0$) and by a cusp on a curve we mean an ordinary cusp (locally analytically given by $y^2 = x^3$). By an (étale) double cover we mean a finite (étale) morphism $f\colon X \to Y$ of schemes that has degree 2 on each irreducible component of Y. $\pi\colon \tilde{C} \to C$ denotes a connected double cover, étale or ramified at exactly two points, of a smooth curve C of genus g. (P, ξ) denotes the principally polarized Prym variety of dimension $p \geq 2$ associated to $\pi\colon \tilde{C} \to C$, where $\xi \in A^1(P)$ is the class of a theta divisor. If π is étale, $p = g-1$, and, if π is ramified at two points, $p = g$. $\tilde{J}$ and J denote the Jacobians of $\tilde{C}$ and C. We fix $\tilde{o} \in \tilde{C}$ and let $\tilde{\varphi}\colon \tilde{C} \to \tilde{J}$ be the Abel map $x \mapsto \mathcal{O}_{\tilde{C}}(x - \tilde{o})$. ι denotes the involution on $\tilde{C}$ such that $\tilde{C}/\iota = C$ and also the induced involution on $\tilde{J}$. We let $u\colon \tilde{J} \to P$ be the restriction of $(1-\iota)$ on the image and $\psi := u \circ \tilde{\varphi}\colon \tilde{C} \to P$ be the Abel-Prym map. We take $o = \pi(\tilde{o})$ and let $\varphi\colon C \to J$ be the Abel map $x \mapsto \mathcal{O}_C(x - o)$. We take the Fourier transform $\mathcal{F}_P\colon A(P) \to A(P)$ to be $\mathcal{F}_P(x) = p_{2*}(p_1^* x \cdot e^{\ell})$, where $\ell := p_1^*\xi + p_2^*\xi - m^*\xi$ is the class of the Poincaré line bundle on $P \times P$, $m\colon P \times P \to P$ is the addition morphism and p_1, p_2 are the projections from $P \times P$ onto P. If X is a scheme, we say that a generic (resp., general) element $x \in X$ has property $\mathscr{P}$, if $\mathscr{P}$ holds on the complement of a countable union of closed subschemes of X (resp., on a dense Zariski open subset of X). The Brill-Noether number is denoted by $\rho(g, r, d) = g - (r+1)(g - d + r)$.

Chapter 2

Equivalence relations on algebraic cycles

2.1 Rational, algebraic, and homological equivalence

Let X be an irreducible projective variety of dimension n. Given an integer $0 \leq k \leq n$, consider the group $\mathcal{Z}_k(X)$ of k-cycles on X, which are by definition finite formal sums

$$m_1 Z_1 + m_2 Z_2 + \cdots + m_r Z_r,$$

where $m_i \in \mathbb{Z}$ and Z_i is a k-dimensional subvariety of X. A k-cycle $Z = \sum_i m_i Z_i$ is called *effective* if $m_i \geq 0$ for all i. By definition, two k-cycles Z and Z' on X are *rationally equivalent* if there is an effective $(k+1)$-cycle $V = \sum_i n_i V_i$ on $X \times \mathbb{P}^1$, projecting dominantly to $\mathbb{P}^1$ (i.e., each V_i dominates $\mathbb{P}^1$), and an effective k-cycle W on X such that

$$V_{|X \times \{b\}} = Z + W \text{ and } V_{|X \times \{b'\}} = Z' + W$$

for some points $b, b' \in \mathbb{P}^1$. We shall denote the group of k-cycles modulo rational equivalence by $\mathrm{CH}_k(X)$ or by $\mathrm{CH}^{n-k}(X)$ (the upper index is the codimension of the cycle). If X is smooth, then $\mathrm{CH}(X) := \bigoplus_l \mathrm{CH}^l(X)$ has the structure of a graded (by codimension l) commutative ring with 1 (= the fundamental class $[X]$ of X) under the intersection product, which is called the *Chow ring of* X, [Fu, 8.3, p.140] (note that $\mathrm{CH}^l(X)$ is denoted by $A^l(X)$ in [Fu]).

A coarser equivalence relation on $\mathcal{Z}_k(X)$, which is also of great interest, is *algebraic equivalence*. The definition of algebraic equivalence is analogous to the definition of rational equivalence, except instead of $\mathbb{P}^1$ we are allowed to use any smooth connected curve C. The group of k-cycles modulo algebraic equivalence will be denoted by $A_k(X)$ (or by $A^{n-k}(X)$, if the emphasis is on codimension). When X is smooth, $A(X) := \bigoplus_l A^l(X)$ also has the structure of a graded commutative ring with 1 (= the fundamental class $[X]$ of X), which is called the *Chow ring of* X *modulo algebraic equivalence*.

There is yet a coarser equivalence relation on $\mathcal{Z}_k(X)$ called *homological equivalence*, which comes from topology. One way to define this equivalence relation is to consider the singular homology groups $H_k(X)$. There is a group homomorphism $cl\colon \mathcal{Z}_k(X) \to H_{2k}(X)$ called the *cycle class map*, [Fu, p.372]. A cycle Z is *homologically trivial* if $cl(Z) = 0$. Two cycles are *homologically equivalent* if their difference is homologically trivial.

The three equivalence relations on $\mathcal{Z}_k(X)$ defined above are related as follows. Let $\mathrm{Rat}_k(X), \mathrm{Alg}_k(X)$, and $\mathrm{Hom}_k(X)$ be the subgroups of $\mathcal{Z}_k(X)$ consisting of cycles that are rationally, algebraically, and homologically equivalent to zero, respectively. We have inclusions of groups

$$\mathrm{Rat}_k(X) \subset \mathrm{Alg}_k(X) \subset \mathrm{Hom}_k(X).$$

As pointed out in [Bl76b], $\mathrm{Alg}_k(X)/\mathrm{Rat}_k(X)$ should be thought of as the "continuous" part of $\mathrm{CH}_k(X)$. However, $A_k(X) = \mathcal{Z}_k(X)/\mathrm{Alg}_k(X)$ need not be finitely generated, as we'll see.

There are group homomorphisms $\mathrm{CH}_k(X) \to A_k(X) \to H_{2k}(X)$. However, the following examples show that the three equivalence relations can be very different. If X is a generic quintic threefold in $\mathbb{P}^4$ or a generic abelian threefold $\big(\mathrm{Hom}_1(X)/\mathrm{Alg}_1(X)\big) \otimes \mathbb{Q}$ is infinite dimensional, [Cl, No]. If X is a non-singular projective variety of dimension ≥ 2 with $h^0(X, \Omega_X^q) > 0$ for some $q \geq 2$, then $\mathrm{Alg}_0(X)/\mathrm{Rat}_0(X)$ is larger that the group of rational points on any abelian variety (for precise statements see [Mu68] for the case $\dim X = 2$ and [Ro] for the general case). On the other hand, there are examples of Fano threefolds for which $\mathrm{Alg}_1(X)/\mathrm{Rat}_1(X)$ is isomorphic to an abelian variety, see [Bl76b, BlMur]. Also, there are algebraic varieties, for example flag varieties, for which all three equivalence relations coincide.

Cycles of dimension $n-1$ (equivalently, of codimension 1) on X are of special interest and they are called *divisors*. Algebraic and homological equivalence coincide for divisors and are in general strictly coarser than rational equivalence, which for divisors is usually called linear equivalence, [Fu, 19.3.1, p.385].

2.2 Facts from intersection theory

We shall repeatedly use the following facts from intersection theory. For simplicity we assume that X and Y are projective varieties, although the results stated below hold in a more general setting.

1. *Pull-back*: If $f\colon X \to Y$ is either a flat morphism or an l.c.i. morphism, then there is a group homomorphism $f^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$, called the pull-back. If X and Y are non-singular, f^* is a ring homomorphism. If f is flat, then for a subvariety $V \subset Y$, $f^*([W]) = [f^{-1}(W)]$, [Fu, 1.7, p.18; 6.6, p.112].

2. *Proper push-forward*: If $f\colon X \to Y$ is a proper morphism, then for every k there is a group homomorphism $f_*\colon \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$, called *push-forward*. If $V \subset X$ is a

subvariety, put $W = f(V)$ and let $\deg(V/W)$ be the degree of $f_{|V}\colon V \to W$ or 0 if $f_{|V}$ is not finite, then $f_*([V]) = \deg(V/W)[W]$, [Fu, 1.4, p.11]. Note that f_* is not a ring homomorphism for the intersection product.

3. *Projection formula*: For any $\alpha \in \mathrm{CH}(Y), \beta \in \mathrm{CH}(X)$: $f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta$, whenever all the operations involved are defined.

4. *Flat base change of a proper morphism*: If $g\colon S \to Y$ is flat and $f\colon X \to Y$ is proper, form the fiber-product diagram:

$$\begin{tikzcd} X_S \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S \arrow[r, "g"] & Y, \end{tikzcd}$$

then for any $\alpha \in \mathrm{CH}(X)$: $g^* f_* \alpha = (f')_*(g')^*\alpha$.

2.3 Basics on cycles on abelian varieties

Let (X, θ) be a principally polarized abelian variety (ppav) of dimension g. The dual abelian variety is denoted by $X^t := \mathrm{Pic}^0(X)$ and we let $\lambda_\theta\colon X \to X^t$ be the isomorphism induced by $\theta \in \mathrm{CH}^1(X)$. There are two ring structures on $\mathrm{CH}(X)$: intersection product and Pontryagin product. As we have seen, pull-back is a ring homomorphism for the intersection product, but push-forward in general is not a ring homomorphism for this operation. On the other hand, push-forward by a homomorphism of abelian varieties is a ring homomorphism for the Pontryagin product $*$, but pull-back is not in general a ring homomorphism for $*$, [La, p.9; MvdG]. The relationship between the two products has been studied in [Po08] and [Bl76a, Lem.1.1, p.218].

In what follows we shall consider the rings $\mathrm{CH}(X)$ *and* $A(X)$ *with* $\mathbb{Q}$*-coefficients, that is, tensored by* $\mathbb{Q}$*, without changing the notation.*

2.4 The Fourier transform

A brief historical account is as follows. It appears that the Fourier transform on the cohomology ring was defined for the first time in the paper [Li] of Lieberman. In [Muk], Mukai gave a more general definition on the derived category of an abelian variety. In [Be83, Be86], Beauville introduced and studied the analogous notion on the Chow ring. Since its appearance, the Fourier transform in all of its incarnations has been a powerful tool in the study of abelian varieties and was generalized to other settings as well. In what follows, we shall be working with the Fourier transform on the Chow ring $\mathrm{CH}(X)$ and the Chow ring $A(X)$ modulo algebraic equivalence. The definitions and properties below are stated for $\mathrm{CH}(X)$ but they hold verbatim for $A(X)$.

Let $\ell := p_1^*\theta + p_2^*\theta - m^*\theta$ be the first Chern class of the Poincaré sheaf on $X \times X$, where $m\colon X \times X \to X$ is the addition morphism and $p_1, p_2\colon X \times X \to X$ are the projections. The Fourier transform $\mathcal{F}_X\colon \mathrm{CH}(X) \to \mathrm{CH}(X)$ on the Chow ring of X is defined by the formula

$$\mathcal{F}_X(z) := p_{2,*}(e^\ell \cdot p_1^* z).$$

Note that usually the Fourier transform takes objects on X to objects on the dual abelian variety X^t. More precisely, let us denote the "usual" Fourier transform by $\mathscr{F}_X\colon \mathrm{CH}(X) \to \mathrm{CH}(X^t)$, whose definition is

$$\mathscr{F}_X(z) := p_{X^t,*}\big(e^{l_X} \cdot p_X^* z\big),$$

where p_X, p_{X^t} are the projections from $X \times X^t$ and $l_X := (1_X \times \lambda_\theta)^*\ell$ is the first Chern class of the Poincaré bundle on $X \times X^t$. Since λ_θ is flat and p_{X^t} is proper, then by Section 2.2, item 4, and the Cartesian diagram

$$\begin{tikzcd} X \times X \arrow[r, "1_X \times \lambda_\theta"] \arrow[d, "p_2"'] & X \times X^t \arrow[d, "p_{X^t}"] \\ X \arrow[r, "\lambda_\theta"] & X^t, \end{tikzcd}$$

we obtain

$$\begin{aligned} \big((\lambda_\theta)^* \circ \mathscr{F}_X\big)(z) &= (\lambda_\theta)^* p_{X^t,*}\big(e^{l_X} \cdot p_X^* z\big) \\ &= p_{2,*}(1_X \times \lambda_\theta)^*\big(e^{l_X} \cdot p_X^* z\big) \\ &= p_{2,*}(e^{\ell} \cdot p_1^* z) = \mathcal{F}_X(z). \end{aligned}$$

Therefore, the two definitions are related by the formula $\mathcal{F}_X = (\lambda_\theta)^* \circ \mathscr{F}_X$.

Let Y be a ppav of dimension h and let $f\colon X \to Y$ be a homomorphism of abelian varieties. The dual homomorphism $f^t\colon Y^t \to X^t$ is defined by $L \mapsto f^*L$. The main properties of the Fourier transform that we shall use are the following:

(P1) $\mathcal{F}_X \circ \mathcal{F}_X = (-1)^g(-1)^*$;

(P2) $\mathcal{F}_X(x * y) = \mathcal{F}_X(x) \cdot \mathcal{F}_X(y)$ and $\mathcal{F}_X(x \cdot y) = (-1)^g \mathcal{F}_X(x) * \mathcal{F}_X(y)$;

(P3) $\mathcal{F}_X(x) = e^\theta \cdot \big((\bar{x} \cdot e^\theta) * e^{-\theta}\big)$, where $\bar{x} = (-1)^* x$.

(P4) $\mathscr{F}_Y \circ f_* = (f^t)^* \circ \mathscr{F}_X$ and $\mathscr{F}_X \circ f^* = (-1)^{g+h}(f^t)_* \circ \mathscr{F}_Y$;

Proof. For proofs of (P1) and (P2) we refer to [Be83, Prop.3, p.243] or [BL, Ch.16]. The property (P3) is taken from [Be04, 2.3iv, p.684] and the proof is as follows. Let $\omega\colon X \times X \to X \times X$ be the automorphism $(x, y) \mapsto (-x, x + y)$, then $p_1 \circ \omega = -p_1, p_2 \circ \omega = m$, and $m \circ \omega = p_2$. Therefore, substituting the definition of ℓ and using the fact that $(-1)^*\theta = \theta$

we obtain:

$$\begin{aligned}\mathcal{F}_X(x) &= p_{2,*}(e^{\ell}\cdot p_1^* x) = e^{\theta}p_{2,*}(p_1^*(e^{\theta}\cdot x)\cdot e^{-m^*\theta})\\ &= e^{\theta}p_{2,*}\omega_*\omega^*(p_1^*(e^{\theta}\cdot x)\cdot e^{-m^*\theta}) = e^{\theta}\cdot m_*((-p_1)^*(e^{\theta}\cdot x)\cdot e^{-p_2^*\theta})\\ &= e^{\theta}m_*(p_1^*(\bar{x}\cdot e^{\theta})\cdot p_2^*(e^{-\theta})) = e^{\theta}\cdot((\bar{x}\cdot e^{\theta}) * e^{-\theta}).\end{aligned}$$

A more general version of (P3) is proven in [Po08, Lem.1.4, p.707].

The identities in (P4) were originally stated for an isogeny [Be83, Prop.3(iii), p.243], but they hold more generally for any homomorphism of abelian varieties, see also [MvdG] for the proof of $\mathscr{F}_Y \circ f_* = (f^t)^* \circ \mathscr{F}_X$ in the setting of abelian schemes. The proof of (P4) is as follows. Let p_Y, p_{Y^t} and q_X, q_{Y^t} be the projections from $Y \times Y^t$ and $X \times Y^t$ onto the indicated factors. As above, we let l_X and l_Y be the first Chern classes of Poincaré bundles on $X \times X^t$ and $Y \times Y^t$, respectively. On the one hand,

$$\begin{aligned}\mathscr{F}_Y(f_*z) &= p_{Y^t,*}\big(e^{l_Y}\cdot p_Y^* f_*(z)\big) = p_{Y^t,*}\big(e^{l_Y}\cdot (f\times 1_{Y^t})_* q_X^* z\big)\\ &= p_{Y^t,*}(f\times 1_{Y^t})_*\big((f\times 1_{Y^t})^* e^{l_Y}\cdot q_X^* z\big) = q_{Y^t,*}\big((1_X\times f^t)^* e^{l_X}\cdot q_X^* z\big),\end{aligned}$$

where for the last equality we used the identity $(f \times 1_{Y^t})^* l_Y = (1_X \times f^t)^* l_X$, which follows from the universal property of the Poincaré bundle [BL, Prop.2.5.2, p.38].

On the other hand, using the Cartesian diagram

$$\begin{tikzcd} X \times Y^t \arrow[r, "1_X \times f^t"] \arrow[d, "q_{Y^t}"'] & X \times X^t \arrow[d, "p_{X^t}"] \\ Y^t \arrow[r, "f^t"] & X^t, \end{tikzcd}$$

we may compute:

$$
\begin{aligned}
(f^t)^* \circ \mathscr{F}_X(z) &= (f^t)^* p_{X^t,*}\big(e^{l_X} \cdot p_X^*(z)\big) \\
&= q_{Y^t,*}(1_X \times f^t)^*\big(e^{l_X} \cdot p_X^*(z)\big) \\
&= q_{Y^t,*}\big((1_X \times f^t)^* e^{l_X} \cdot q_X^*(z)\big).
\end{aligned}
$$

This shows the first identity: $\mathscr{F}_Y \circ f_* = (f^t)^* \circ \mathscr{F}_X$.

To prove $\mathscr{F}_X \circ f^* = (-1)^{g+h}(f^t)_* \circ \mathscr{F}_Y$, we apply the first identity to f^t and use (P1) and the fact that $(f^t)^t = f$ under the canonical identifications of $(X^t)^t$ with X and $(Y^t)^t$ with Y:

$$
\begin{aligned}
\mathscr{F}_X \circ f^*(w) &= \mathscr{F}_X \circ f^* \circ (-1)^h (-1)_Y^* \circ \mathscr{F}_{Y^t} \circ \mathscr{F}_Y(w) \\
&= (-1)^h \mathscr{F}_X \circ ((-1)_Y \circ f)^* \circ \mathscr{F}_{Y^t} \circ \mathscr{F}_Y(w) \\
&= (-1)^h \mathscr{F}_X \circ \mathscr{F}_{X^t} \circ ((-1)_Y \circ f)^t_* \circ \mathscr{F}_Y(w) \\
&= (-1)^{g+h} (-1)_{X^t}^* \circ (f^t)_* \circ (-1)_{Y^t,*} \circ \mathscr{F}_Y(w) \\
&= (-1)^{g+h} (-1)_{X^t}^* \circ (-1)_{X^t,*} \circ (f^t)_* \circ \mathscr{F}_Y(w) \\
&= (-1)^{g+h} (f^t)_* \circ \mathscr{F}_Y(w).
\end{aligned}
$$

□

2.5 The Beauville grading

The Chow ring of X is graded by codimension, $\mathrm{CH}(X) = \bigoplus_{a=0}^{g} \mathrm{CH}^a(X)$, where $\mathrm{CH}^a(X)$ is the group of codimension a cycles on X modulo rational equivalence. There is a second grading, which is due to Beauville [Be86], and is defined as follows. For $k \in \mathbb{Z}$, let $k\colon X \to X$ also denote the morphism $x \mapsto kx$. For any $0 \le a \le g$, let $\mathrm{CH}^a(X)_{(s)}$ be the subgroup of

$\mathrm{CH}^a(X)$ consisting of classes x with the property $k^*x = k^{2a-s}x$ for all $k \in \mathbb{Z}\backslash\{0\}$. In other words, $\mathrm{CH}^a(X)_{(s)}$ is the simultaneous eigenspace of weight $2a - s$ for the operators k^*. In [Be86], it is shown that $\mathrm{CH}^a(X) = \bigoplus_{s=a-g}^{a} \mathrm{CH}^a(X)_{(s)}$. Note that there is an alternative characterization: $x \in \mathrm{CH}^a(X)_{(s)}$ if and only if $k_*x = k^{2g-2a+s}x$ for all $k \in \mathbb{Z}\backslash\{0\}$.

Moreover, in [Be86, p.255], it was conjectured that $\mathrm{CH}^a(X)_{(s)} = 0$ for all $s < 0$ and all a. This has been verified by Beauville and Bloch for $a \in \{0, 1, g-2, g-1, g\}$, [Be86, Prop.8, p.255; Bl74a, Thm.4.7, p.227].

Note that $\mathrm{CH}^0(X) = \mathrm{CH}^0(X)_{(0)}$. Also, $\mathrm{CH}^1(X)_{(0)} = \mathrm{Pic}^s(X)_{\mathbb{Q}}$ and $\mathrm{CH}^1(X)_{(1)} = \mathrm{Pic}^0(X)_{\mathbb{Q}}$, where $\mathrm{Pic}^s(X)_{\mathbb{Q}}$ is the subgroup of symmetric invertible sheaves, tensored with $\mathbb{Q}$. Thus, the Beauville decomposition of $\mathrm{CH}^1(X)$ can be reinterpreted as the fact that any $\mathbb{Q}$-divisor D on X can be written as a sum of a symmetric $\mathbb{Q}$-divisor and an algebraically trivial $\mathbb{Q}$-divisor:

$$D = \frac{D + (-1)^*D}{2} + \frac{D - (-1)^*D}{2}$$

and the summands are unique up to linear equivalence.

Also, $A^1(X) = A^1(X)_{(0)} \simeq \mathrm{Pic}^s(X)_{\mathbb{Q}} \simeq \mathrm{NS}(X)_{\mathbb{Q}}$, where $\mathrm{NS}(X)_{\mathbb{Q}}$ is the Néron-Severi group of X consisting of $\mathbb{Q}$-divisors on X modulo algebraic equivalence.

Furthermore, the Beauville grading is preserved by the following operations on cycles on abelian varieties:

(P5) $\mathcal{F}_X \mathrm{CH}^a(X)_{(s)} = \mathrm{CH}^{g-a+s}(X)_{(s)}$;

(P6) if $x \in \mathrm{CH}^a(X)_{(s)}$ and $y \in \mathrm{CH}^b(X)_{(t)}$, then $x \cdot y \in \mathrm{CH}^{a+b}(X)_{(s+t)}$ and $x * y \in \mathrm{CH}^{a+b-g}(X)_{(s+t)}$;

(P7) if $f\colon X \to Y$ is a homomorphism of abelian varieties, then $f^*\mathrm{CH}^a(Y)_{(s)} \subset \mathrm{CH}^a(X)_{(s)}$ and $f_*\mathrm{CH}^a(X)_{(s)} \subset \mathrm{CH}^{a+c}(Y)_{(s)}$, where $c = \dim Y - \dim X$.

Proof. See [Be86, Prop.2, p.648] or [BL, 16.5, p.534]. $\square$

2.6 Examples

Put $\ell_x := \ell_{|\{x\}\times X}$, let $[X]$ be the fundamental class of X, and for a point $x \in X$, let $[x] \in \mathrm{CH}^g(X)$ be its class. We shall use the following additional properties of $\mathcal{F}_X$:

1. $\mathcal{F}_X([X]) = (-1)^g[0]$ and $\mathcal{F}_X([0]) = [X]$;

2. $\mathcal{F}_X(\ell_x) = (-1)^g \sum_{k=1}^g \frac{1}{k}([0]-[x])^{*k}$ and $\mathcal{F}_X([x]) = e^{\ell_x}$,

whose proofs follow immediately from [BL, Cor.16.3.3, p.529; Prop.16.3.6, p.531] using the fact that $\mathcal{F}_X = (\lambda_\theta)^* \circ \mathscr{F}_X$ (F in [BL] is our $\mathscr{F}_X$).

Note that since any two points on X are algebraically equivalent (X is assumed to be connected), then $A^g(X) \simeq A^g(X)_{(0)} \simeq \mathbb{Q}$. In particular, $A^g(X)_{(s)} = 0$ for all $s \geq 1$. By Fourier dualty $A^m(X)_{(m)} = 0$ for all $m \geq 1$. Also, $A^0(X)_{(0)} = \mathbb{Q}\cdot[X]$ and $A^1(X) \simeq A^1(X)_{(0)} = \mathbb{Q}\cdot\theta$.

Example 1. Let E be an elliptic curve with a marked point 0. We may take $\theta = [0]$. We have $\mathrm{CH}^1(E)_{(1)} = \mathrm{Pic}^0(E)_{\mathbb{Q}}$, $\mathrm{CH}^1(E)_{(0)} = \mathbb{Q}\cdot[0]$, and $\mathrm{CH}^0(E) = \mathbb{Q}\cdot[E]$. The Fourier transform exchanges $\mathrm{CH}^0(E)$ and $\mathrm{CH}^1(E)_{(0)}$ by sending $[E]$ to $-[0]$ and sends $\mathrm{CH}^1(E)_{(1)}$ to itself by the rule $\ell_x \mapsto [x]-[0]$.

Modulo algebraic equivalence, $A^1(E)_{(1)} = 0$ and $\mathcal{F}_X$ exchanges $A^0(E) = \mathbb{Q}\cdot[E]$ and $A^1(E)_{(0)} = \mathbb{Q}\cdot[0]$ as above.

Example 2. Let (S,θ) be a principally polarized abelian surface. Then $\mathrm{CH}^0(S) = \mathbb{Q}\cdot[S]$; $\mathrm{CH}^1(S)_{(1)} = \mathrm{Pic}^0(S)_{\mathbb{Q}}$, $\mathrm{CH}^1(S)_{(0)} = \mathbb{Q}\cdot\theta$. The Fourier transform induces isomorphisms $\mathrm{CH}^0(S) \to \mathrm{CH}^2(S)_{(0)}$ and $\mathrm{CH}^1(S)_{(1)} \to \mathrm{CH}^2(S)_{(1)}$ and automorphisms of $\mathrm{CH}^1(S)_{(0)}$ and $\mathrm{CH}^2(S)_{(2)}$. Since the geometric genus $p_g(S) = 1$, by a theorem of Mumford [Mu68], $\mathrm{CH}^2(S)$ is "infinite dimensional", i.e., is larger than the set of closed points of any abelian variety. By Fourier duality, $\mathrm{CH}^2(S)_{(1)} \simeq \mathrm{CH}^1(S)_{(1)} \simeq \mathrm{Pic}^0(S)_{\mathbb{Q}}$ and $\mathrm{CH}^2(S)_{(0)} \simeq \mathrm{CH}^0(S)_{(0)} \simeq \mathbb{Q}$

are “finite dimensional”, we see that $\mathrm{CH}^2(S)_{(2)}$ is the part that makes $\mathrm{CH}^2(S)$ “infinite dimensional”.

Modulo algebraic equivalence, the only non-zero groups are $A^0(S), A^1(S)_{(0)}$, and $A^2(S)_{(0)}$ each of which is isomorphic to $\mathbb{Q}$.

Example 3. Let X be a principally polarized abelian threefold. If X is generic, then by a theorem of Nori [No] (see also [Ba]), $A^2(X)$ is infinite dimensional as a vector space over $\mathbb{Q}$. Since $A^2(X)_{(2)} = 0$ and $A^2(X)_{(0)} \simeq \mathbb{Q}$, then we conclude that $A^2(X)_{(1)}$ is the part that makes $A^2(X)$ infinite dimensional, see also [CoPi]. The Fourier transform induces an automorphism of $A^2(X)_{(1)}$.

Chapter 3

Tautological rings modulo algebraic equivalence

Throughout this section $\pi\colon \tilde{C} \to C$ is a connected double cover, which is either étale or ramified at two points. Also, we shall use the following additional notation. The translation map $x \mapsto x + a$ on an abelian variety is denoted by τ_a. Let $i\colon P \hookrightarrow \tilde{J}$ be the inclusion of the Prym variety P into the Jacobian $\tilde{J}$ of $\tilde{C}$. Let Θ_J, $\Theta_{\tilde{J}}$, and Ξ be theta divisors on J, $\tilde{J}$, and P, respectively, such that $\Theta_{\tilde{J}}$ restricts to 2Ξ on P. Let $\mathscr{L}_P$ be the Poincaré sheaf on $P \times P$ normalized by the conditions $\mathscr{L}_P|_{\{b\}\times P} \simeq \mathscr{L}_P|_{P\times\{b\}} \simeq \mathcal{O}_P(\Xi - \tau_b^*\Xi)$ for all $b \in P$. Let $\mathscr{L}_{\tilde{J}}$ and $\mathscr{L}_J$ be Poincaré sheaves on $\tilde{J} \times \tilde{J}$ and $J \times J$, respectively, normalized analogously.

3.1 The Fourier transform of an Abel-Prym curve

In this section we compute the Fourier transform $\mathcal{F}_P[\psi(\tilde{C})]$ of an Abel-Prym curve in $A(P)$ and obtain a formula analogous to the one in [Be04, Prop.3.3, p.685]. The formula for $\mathcal{F}_P[\psi(\tilde{C})]$ can also be deduced from [Na, Prop.3.1, p.226], where the K-theoretic Fourier transforms of certain bundles on $\tilde{J}$ are related to their counterparts on P.

Recall that g and $\tilde{g}$ denote the genera of C and $\tilde{C}$, respectively, and consider the classes

$$\tilde{w}^{\tilde{g}-d} := \frac{1}{d!}\tilde{C}^{*d} \qquad \text{and} \qquad N^k(\tilde{w}) := \frac{1}{k!}\sum_{i=1}^{\tilde{g}} \lambda_i^k,$$

where $\lambda_1, \ldots, \lambda_{\tilde{g}}$ are the roots of the equation $\lambda^{\tilde{g}} - \lambda^{\tilde{g}-1}\tilde{w}^1 + \ldots + (-1)^{\tilde{g}}\tilde{w}^{\tilde{g}} = 0$. The classes $N^k(\tilde{w})$ are Newton polynomials in the classes $\tilde{w}^i$ and are used to express the Chern character of a vector bundle in terms of its Chern classes, [Fu, Ex.15.1.2, p.284]. Recall from the introduction that $i\colon P \hookrightarrow \tilde{J}$ is the inclusion and $u\colon \tilde{J} \to P$ is the morphism $(1-\iota)\colon \tilde{J} \to \tilde{J}$ restricted to P on the image.

Lemma 3.1.1. *The Fourier transforms on P and $\tilde{J}$ are related by the formulas*

$$\mathcal{F}_P \circ u_* = i^* \circ \mathcal{F}_{\tilde{J}} \text{ and } \mathcal{F}_P \circ i^* = (-1)^{\tilde{g}+p} u_* \circ \mathcal{F}_{\tilde{J}},$$

where $\tilde{g} = \dim \tilde{J}$ and $p = \dim P$.

Proof. By definition, the composition

$$P \xrightarrow{\lambda_\xi} P^t \xrightarrow{u^t} \tilde{J}^t \xrightarrow{\lambda_{\tilde{\theta}}^{-1}} \tilde{J}$$

is the inclusion $i\colon P \hookrightarrow \tilde{J}$. Therefore, using (P4) we obtain:

$$\mathcal{F}_P \circ u_* = \lambda_\xi^* \circ \mathscr{F}_P \circ u_* = \lambda_\xi^* \circ (u^t)^* \circ \mathscr{F}_{\tilde{J}} = \lambda_\xi^* \circ (u^t)^* \circ (\lambda_{\tilde{\theta}}^{-1})^* \mathcal{F}_{\tilde{J}} = i^* \circ \mathcal{F}_{\tilde{J}}.$$

On the other hand, $i^t = (\lambda_{\tilde{\theta}}^{-1} \circ u^t \circ \lambda_\xi)^t = (\lambda_\xi)^t \circ u \circ (\lambda_{\tilde{\theta}}^{-1})^t = \lambda_\xi \circ u \circ \lambda_{\tilde{\theta}}^{-1}$, and therefore, $\lambda_\xi^{-1} \circ i^t \circ \lambda_{\tilde{\theta}} = u$. Thus, using (P4) and the identity $(\lambda_\xi)^* = (\lambda_\xi^{-1})_*$ (remember that λ_ξ is an isomorphism), we obtain the other part of the lemma:

$$\begin{aligned}\mathcal{F}_P \circ i^* = (\lambda_\xi)^* \circ \mathscr{F}_P \circ i^* &= (-1)^{\tilde{g}+p}(\lambda_\xi^{-1})_* \circ (i^t)_* \circ \mathscr{F}_{\tilde{J}} \\ &= (-1)^{\tilde{g}+p}(\lambda_\xi^{-1})_* \circ (i^t)_* \circ (\lambda_{\tilde{\theta}})_* \circ \mathcal{F}_{\tilde{J}} = (-1)^{\tilde{g}+p} u_* \circ \mathcal{F}_{\tilde{J}}.\end{aligned}$$

□

Proposition 3.1.2. *The Fourier transform of the class of the Abel-Prym curve $\psi(\tilde{C})$ on a Prym variety P of dimension p is given by the formula:*

$$\mathcal{F}_P[\psi(\tilde{C})] = -\big(i^*N^1(\tilde{w}) + \ldots + i^*N^p(\tilde{w})\big).$$

Proof. By [Be04, Prop.3.3], $\mathcal{F}_{\tilde{J}}[\tilde{\varphi}(\tilde{C})] = -(N^1(\tilde{w}) + \ldots + N^{\tilde{g}}(\tilde{w}))$. Therefore, by Lemma 3.1.1 we get $\mathcal{F}_P[\psi(\tilde{C})] = (\mathcal{F}_P \circ u_*)[\tilde{\varphi}(\tilde{C})] = (i^* \circ \mathcal{F}_{\tilde{J}})[\tilde{\varphi}(\tilde{C})] = -(i^*N^1(\tilde{w}) + \ldots + i^*N^p(\tilde{w}))$. □

3.2 Generators

Let $[\psi(\tilde{C})] = [\psi(\tilde{C})]_{(0)} + \cdots + [\psi(\tilde{C})]_{(p-1)}$ be the decomposition of $[\psi(\tilde{C})]$ into homogeneous components for the Beauville grading, i.e., $[\psi(\tilde{C})]_{(n)} \in A^{p-1}(P)_{(n)}$.

Definition 3.2.1. *For each $1 \leq n \leq p = \dim P$ define the cycle $\zeta_n = \mathcal{F}_P\big([\psi(\tilde{C})]_{(n-1)}\big)$.*

Note that by [Be04, Cor.3.4, p.686], $N^n(\tilde{w}) \in A^n(\tilde{J})_{(n-1)}$, and therefore, by Proposition 3.1.2, $\zeta_n = -i^*N^n(\tilde{w}) \in A^n(P)_{(n-1)}$. Furthermore, since $\mathcal{F}_P \circ \mathcal{F}_P = (-1)^p(-1)^*$, then $\mathcal{F}_P(\zeta_n) = (-1)^{p-n+1}[\psi(\tilde{C})]_{(n-1)}$. The main theorem in [Be04] gives a set of generators for the tautological ring of the Jacobian of a smooth connected curve. In the remainder of this section we shall show how to extend the argument in [Be04, Sec.4] in order to obtain a set of generators for $\mathscr{T}(P, \psi(\tilde{C}))$.

Consider the $\mathbb{Q}$-vector subspace $\mathscr{T}'$ of $A(P)$ spanned by the cycles of the form

$$\zeta_{n_1} \cdot \zeta_{n_2} \cdots \zeta_{n_r},$$

where $1 \leq n_i \leq p$ and $r \geq 1$ are integers. The image $\mathcal{F}_P(\mathscr{T}')$ of $\mathscr{T}'$ under $\mathcal{F}_P$ is spanned as a $\mathbb{Q}$-vector space by the elements of the form

$$\mathcal{F}_P(\zeta_{n_1} \cdot \zeta_{n_2} \cdots \zeta_{n_r}) = \pm[\psi(\tilde{C})]_{(n_1-1)} * \cdots * [\psi(\tilde{C})]_{(n_r-1)}.$$

We have the following lemma, whose proof is taken from [Be04, Lem.4.2].

Lemma 3.2.2. *$\mathcal{F}_P(\mathscr{T}')$ is spanned by the classes $(k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})])$, where r and $k_1, \ldots, k_r$ are positive integers.*

Proof. Since $k_*[\psi(\tilde{C})] = \sum_{n=0}^{p-1} k^{2+n}[\psi(\tilde{C})]_{(n)}$ then

$$(k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})]) = (k_1 \cdots k_r)^2 \sum_{n_1,\ldots,n_r} k_1^{n_1} \cdots k_r^{n_r} [\psi(\tilde{C})]_{(n_1)} * \cdots * [\psi(\tilde{C})]_{(n_r)},$$

where the sum is taken over $\mathbf{n} = (n_1, \ldots, n_r) \in [0, p-1]^r$. So, we see that $(k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})])$ belongs to $\mathcal{F}_P(\mathscr{T}')$. We claim that we can choose p^r r-tuples, $\mathbf{k} = (k_1, \ldots, k_r)$, which make the $p^r \times p^r$ matrix $(a_{\mathbf{k},\mathbf{n}})$ with entries $a_{\mathbf{k},\mathbf{n}} := k_1^{n_1} \cdots k_r^{n_r}$ invertible ($\mathbf{n}$ runs through $[0, p-1]^r$). For each $1 \leq \ell \leq p^r$, let $\mathbf{k}_\ell := (\ell, \ell^p, \ldots, \ell^{p^{r-1}})$, then $\det(a_{\mathbf{k}_\ell,\mathbf{n}})$ is a non-zero Vandermonde determinant. Hence, the matrix $(a_{\mathbf{k}_\ell,\mathbf{n}})$ is invertible. This shows that each cycle $[\psi(\tilde{C})]_{(n_1)} * \cdots * [\psi(\tilde{C})]_{(n_r)}$ can be expressed as a $\mathbb{Q}$-linear combination of cycles of the form $(k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})])$. $\square$

Theorem 3.2.3. *The tautological ring $\mathscr{T}(P, \psi(\tilde{C}))$ is generated as a $\mathbb{Q}$-subalgebra of $A(P)$ under the intersection product by the cycles ζ_n, where $1 \leq n \leq p-1$ and n is odd.*

Proof. First, we note that $\zeta_p \in A^p(P)_{(p-1)} = \{0\}$ (recall from Section 2.6 that $A^p(P)_{(s)} = 0$

for all $s > 0$ and $p > 1$ by assumption), hence $\zeta_p = 0$. Also, since $\psi(\tilde{C})$ has symmetric translates, $[\psi(\tilde{C})]_{(n)} = 0$ for all odd n, and therefore, $\zeta_n = 0$ for all even n. The rest of the proof proceeds as in [Be04, Sec.4]. First, we note that $\zeta_n \in \mathscr{T}(P, \psi(\tilde{C}))$ for all $1 \leq n \leq p$, and therefore, $\mathscr{T}' \subset \mathscr{T}(P, \psi(\tilde{C}))$. We shall show that $\mathscr{T}' = \mathscr{T}(P, \psi(\tilde{C}))$. By definition, $\mathscr{T}'$ is generated as a $\mathbb{Q}$-algebra under the intersection product by elements which are homogeneous for the Beauville grading, and therefore, $\mathscr{T}'$ is stable under the intersection product and the operations k_*, k^*. Thus, to prove the equality $\mathscr{T}' = \mathscr{T}(P, \psi(\tilde{C}))$, we must check that $\mathscr{T}'$ contains $[\psi(\tilde{C})]$ and is stable under $\mathcal{F}_P$ and $*$.

Assume that $\mathscr{T}'$ is stable under $\mathcal{F}_P$. Then $(-1)^{p-n+1}[\psi(\tilde{C})]_{(n-1)} = \mathcal{F}_P(\zeta_n)$ implies that $\mathscr{T}'$ contains $[\psi(\tilde{C})] = \sum_n [\psi(\tilde{C})]_{(n)}$. Also, given $x_i \in \mathscr{T}'$, let $\hat{x}_i = (-1)^p(-1)^*\mathcal{F}_P(x_i) \in \mathscr{T}'$ for $i = 1, 2$. Using [P1, Sec.2.4; Lem.3.1.1] we obtain:

$$\mathcal{F}_P(\hat{x}_i) = (-1)^p(-1)_*\mathcal{F}_P \circ \mathcal{F}_P(x_i) = (-1)^p(-1)_*(-1)^*(-1)^p x_i = x_i,$$

and therefore, $x_1 * x_2 = \mathcal{F}_P(\hat{x}_1) * \mathcal{F}_P(\hat{x}_2) = (-1)^p\mathcal{F}_P(\hat{x}_1 \cdot \hat{x}_2) \in \mathscr{T}'$ by [P2, Sec.2.4]. This shows that it suffices to prove that $\mathscr{T}'$ is invariant under $\mathcal{F}_P$, which we do next.

Since $\mathscr{T}'$ is stable under the intersection product, $\mathcal{F}_P(\mathscr{T}')$ is stable under Pontryagin product. Furthermore, by [P3, Sec.2.4], $\mathcal{F}_P(x) = e^{\xi} \cdot \big((\bar{x} \cdot e^{\xi}) * e^{-\xi}\big)$, where $\bar{x} = (-1)^*x$. Thus, to prove the inclusion $\mathcal{F}_P(\mathscr{T}') \subset \mathscr{T}'$ it is enough to check that $\mathcal{F}_P(\mathscr{T}')$ is invariant under intersection with ξ.

By Lemma 3.2.2 it suffices to show that $\xi \cdot \big((k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})])\big)$ belongs to $\mathcal{F}_P(\mathscr{T}')$. Consider the composition

$$v\colon\ \tilde{C}^r \xrightarrow{\vec{\psi}} P^r \xrightarrow{\vec{k}} P^r \xrightarrow{m} P\ ,$$

where $\vec{\psi} = (\psi, \ldots, \psi)$, $\vec{k} = (k_1, \ldots, k_r)$ with $k_i\colon x \mapsto k_i x$, and $m\colon P^r \to P$ is the addition morphism. Note that

$$v_*[\tilde{C}^r] = (k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})]).$$

Therefore,

$$\xi \cdot \big((k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})])\big) = \xi \cdot v_*[\tilde{C}^r] = v_*\big(v^*(\xi) \cdot [\tilde{C}^r]\big) = v_* v^* \xi.$$

Hence, it suffices to show that $v_* v^* \xi \in \mathcal{F}_P(\mathscr{T}')$.

By the theorem of the cube [Mu70, p.55], we have

$$m^*\xi = \sum_i p_i^*\xi - \sum_{i<j} p_{ij}^* c_1(\mathscr{L}_P),$$

where p_i, p_{ij} are projections from P^r onto the i^{th}, i^{th} and j^{th} factors, respectively. This, together with $k_i^*\xi = k_i^2\xi$ and $(k_i, k_j)^* c_1(\mathscr{L}_P) = k_i k_j c_1(\mathscr{L}_P)$, implies that

$$v^*\xi = \sum_i k_i^2 q_i^* \psi^* \xi - \sum_{i<j} k_i k_j q_{ij}^* (\psi, \psi)^* c_1(\mathscr{L}_P), \tag{3.2.1}$$

where q_i, q_{ij} are projections of $\tilde{C}^r$ onto the i^{th}, i^{th} and j^{th} factors, respectively.

Next, we compute $(\psi, \psi)^* c_1(\mathscr{L}_P)$. The result [BL, Prop.12.3.4, p.374] in our notation states that $2\Theta_{\tilde{J}} \sim_{\text{alg}} \text{Nm}^*(\Theta_J) + u^*\Xi$. Consequently, the following identity holds in $A(\tilde{J} \times \tilde{J})$

$$(u, u)^* c_1(\mathscr{L}_P) = 2c_1(\mathscr{L}_{\tilde{J}}) - (\text{Nm}, \text{Nm})^* c_1(\mathscr{L}_J). \tag{3.2.2}$$

Let $\Delta_{\tilde{C}}$ and Δ_C be the diagonals in $\tilde{C} \times \tilde{C}$ and $C \times C$, respectively. Applying the See-saw theorem [Mu70, p.54], we see that

$$(\tilde{\varphi}, \tilde{\varphi})^* \mathscr{L}_{\tilde{J}} \simeq \mathcal{O}_{\tilde{C}^2}(\Delta_{\tilde{C}} - \tilde{C} \times \tilde{o} - \tilde{o} \times \tilde{C}) \tag{3.2.3}$$

and likewise $(\varphi,\varphi)^*\mathscr{L}_J \simeq \mathcal{O}_{C^2}(\Delta_C - C\times o - o\times C)$. Using the commutative diagram

$$\begin{array}{ccc} \tilde{C}\times\tilde{C} & \xrightarrow{(\tilde{\varphi},\tilde{\varphi})} & \tilde{J}\times\tilde{J} \\ {\scriptstyle(\pi,\pi)}\downarrow & & \downarrow{\scriptstyle(\mathrm{Nm},\mathrm{Nm})} \\ C\times C & \xrightarrow{(\varphi,\varphi)} & J\times J, \end{array}$$

we obtain

$$(\mathrm{Nm}\circ\tilde{\varphi},\mathrm{Nm}\circ\tilde{\varphi})^*\mathscr{L}_J \simeq (\pi,\pi)^*\mathcal{O}_{C^2}(\Delta_C - C\times o - o\times C). \tag{3.2.4}$$

Since $\psi = u\circ\tilde{\varphi}$, then from (3.2.2), (3.2.3) and (3.2.4) we obtain:

$$\begin{aligned} (\psi,\psi)^*c_1(\mathscr{L}_P) &= (\tilde{\varphi},\tilde{\varphi})^*(u,u)^*c_1(\mathscr{L}_P) \\ &= 2(\tilde{\varphi},\tilde{\varphi})^*c_1(\mathscr{L}_{\tilde{J}}) - (\tilde{\varphi},\tilde{\varphi})^*(\mathrm{Nm},\mathrm{Nm})^*c_1(\mathscr{L}_J) \\ &= 2(\tilde{\varphi},\tilde{\varphi})^*c_1(\mathscr{L}_{\tilde{J}}) - (\mathrm{Nm}\circ\tilde{\varphi},\mathrm{Nm}\circ\tilde{\varphi})^*c_1(\mathscr{L}_J) \\ &\sim_{\mathrm{alg}} 2(\Delta_{\tilde{C}} - \tilde{C}\times\tilde{o} - \tilde{o}\times\tilde{C}) - (\pi,\pi)^*(\Delta_C - C\times o - o\times C). \end{aligned}$$

Furthermore, $(\pi,\pi)^*\Delta_C = \Delta_{\tilde{C}} + (1,\iota)^*\Delta_{\tilde{C}}$, where $1\colon \tilde{C}\to\tilde{C}$ denotes the identity morphism, and therefore,

$$(\psi,\psi)^*c_1(\mathscr{L}_P) \sim_{\mathrm{alg}} \Delta_{\tilde{C}} - (1,\iota)^*\Delta_{\tilde{C}}. \tag{3.2.5}$$

Substituting (3.2.5) into the identity (3.2.1), we see that $v^*\xi$ is algebraically equivalent to a linear combination of divisors of the form $q_i^*\tilde{o}$ and $q_{ij}^*\beta^*\Delta_{\tilde{C}}$, where β is one of the morphisms $(1,1)$ or $(1,\iota)$. The cycles $v_*q_{ij}^*\Delta_{\tilde{C}}$ and $v_*q_{ij}^*(1,\iota)^*\Delta_{\tilde{C}}$ are proportional to cycles of the form $(l_{1*}[\psi(\tilde{C})]) * \cdots * (l_{(r-1)*}[\psi(\tilde{C})])$, where $(l_1,\dots,l_{r-1})$ is

$$(k_1,\dots,\widehat{k_i},\dots,\widehat{k_j},\dots,k_r,k_i+k_j) \text{ and } (k_1,\dots,\widehat{k_i},\dots,\widehat{k_j},\dots,k_r,k_i-k_j),$$

respectively, and the symbol $\widehat{k_j}$ means that k_j is omitted from the list. Since $v_*q_i^*\tilde{o}$ is proportional to the cycle $(k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})])$ with $k_{i*}[\psi(\tilde{C})]$ omitted, we see that $v_* v^* \xi$ belongs to $\mathcal{F}_P(\mathscr{T}')$. $\square$

Modulo homological equivalence, the tautological ring of the Jacobian J is the subalgebra of $H^*(J,\mathbb{Q})$ generated by the class θ of the theta divisor on J. If C is generic of genus ≥ 3, the Ceresa cycle $C-(-1)^*C$ is not zero on J modulo algebraic equivalence, [Ce]. In particular, this implies that C is not proportional to a power of θ in $\mathscr{T}(J,\varphi(C))$. Indeed, any power of θ is symmetric, i.e. is stable under $(-1)^*$, which is not the case for any non-zero scalar multiple of C. Hence, at least for a generic Jacobian, $\mathscr{T}(J,\varphi(C))$ is not generated by θ. As in the case of J, the tautological ring of the pair $(P,\psi(\tilde{C}))$ modulo homological equivalence is the subalgebra of $H^*(P,\mathbb{Q})$ generated by ξ, the class of the principal polarization of P. On the other hand, in contrast with Jacobians, for any Prym variety (P,ξ) the Ceresa-type cycle $[\psi(\tilde{C})]-(-1)^*[\psi(\tilde{C})]$ is zero in $\mathscr{T}(P,\psi(\tilde{C}))$, because the Abel-Prym curve has symmetric translates. Nevertheless, if P is generic and of dimension $p \geq 5$, then by the proof of [Fa, Thm.4.5, p.117], the class $([\psi(\tilde{C})]^{*r})_{(2)}$ is non-zero in $A(P)$ for $1 \leq r \leq p-3$. Also, $A^p(P)_{(1)}=0$, because $A^p(P) \simeq \mathbb{Q} \simeq A^p(P)_{(0)}$. Since $\mathcal{F}_P\colon A^p(P)_{(1)} \to A^1(P)_{(1)}$ is an isomorphism, then $A^1(P)_{(1)}=0$, and therefore, $A^1(P)=A^1(P)_{(0)}$. This shows that $[\psi(\tilde{C})]_{(2)}$ is not in the subring generated by ξ, because $\xi=\xi_{(0)}$, and thus, $\xi^{p-1} \in A^{p-1}(P)_{(0)}$ by [P6, Sec.2.5]. *Therefore, if P is generic and of dimension ≥ 5, the tautological ring $\mathscr{T}(P,\psi(\tilde{C}))$ is not generated by ξ.*

The Torelli theorem [ACGH, p.245] for principally polarized Jacobians implies that every principally polarized Jacobian has a unique tautological ring in the sense of [Be04]. It is well known that the Torelli theorem does not hold for Prym varieties, which means that a given principally polarized abelian variety may have a structure of a Prym variety in multiple ways. As a consequence, given a Prym variety P, we may have choices for an Abel-Prym curve, which a priori may give different tautological rings of pairs $(P,\psi(\tilde{C}))$. There are four explicit counter-examples to the Prym-Torelli problem: covers of hyperelliptic curves, Donagi's tetragonal construction, Verra's construction with plane sextics, and the recent construction of Izadi and Lange using ramified covers, [Mu74, p.346; Do; IL; Ve]. We do not know whether the tautological ring of the pair $(P, \psi(\tilde{C}))$ is always independent of the choice of an Abel-Prym curve. One approach to answer this question is to see whether the various Abel-Prym curves are algebraically equivalent on the Prym variety. As explained below, the answer to this question is known in the case of general (resp., generic) Pryms of dimension $2 \leq p \leq 4$ (resp., $p \geq 6$).

Let $\mathscr{R}_g$ and $\mathscr{A}_{g-1}$ be the moduli spaces of étale double covers of smooth curves of genus g and principally polarized abelian varieties of dimension $g-1$, respectively. The Prym map $\mathscr{P}\colon \mathscr{R}_g \to \mathscr{A}_{g-1}$ associates to an étale double cover the corresponding Prym variety. We know that the general fiber of the Prym map is connected, whenever $2 \leq g-1 \leq 4$, see [Do92, §6; Iz95]. Therefore, on general Pryms of dimension $2 \leq p \leq 4$, the Abel-Prym curves are algebraically equivalent. According to [DS, Thm.2.1, p.34], the Prym map $\mathscr{R}_6 \to \mathscr{A}_5$ has degree 27. We do not know whether the 27 Abel-Prym curves on a generic Prym of dimension 5 are algebraically equivalent or not. Finally, we remark that due to the generic injectivity of the Prym map for curves of genus $g \geq 7$ [De, Ka, FS, We87], a *generic* Prym variety P of dimension ≥ 6 has a unique Abel-Prym curve $\psi(\tilde{C})$, and therefore, the tautological ring of the pair $\mathscr{T}(P, \psi(\tilde{C}))$ depends only on P.

3.3 Polishchuk relations and the $\mathfrak{sl}_2$ action

By [Ku], any polarization (the class of an ample symmetric divisor) on an abelian variety X gives rise to an $\mathfrak{sl}_2 := \mathfrak{sl}_2(\mathbb{Q})$ action on $\mathrm{CH}(X)$, which descends to an $\mathfrak{sl}_2$ action on $A(X)$. Let $\mathfrak{sl}_2 = \mathbb{Q}\cdot f + \mathbb{Q}\cdot h + \mathbb{Q}\cdot e$ with $[e,f]=h; [e,h]=-2e; [f,h]=2f$. We normalize the $\mathfrak{sl}_2$

action on $\mathrm{CH}(\tilde{J})$ as in [Mo]:

$$e(\alpha) = -\tilde{\theta} \cdot \alpha, \quad f(\alpha) = -[\tilde{\varphi}(\tilde{C})]_{(0)} * \alpha, \quad h_{|\mathrm{CH}^i(\tilde{J})_{(j)}} = (2i - j - \tilde{g})\mathrm{id}.$$

Analogously, the $\mathfrak{sl}_2$ action on $\mathrm{CH}(P)$ is given by:

$$e(\alpha) = -\xi \cdot \alpha, \quad f(\alpha) = -[\psi(\tilde{C})]_{(0)} * \alpha, \quad h_{|\mathrm{CH}^i(P)_{(j)}} = (2i - j - p)\mathrm{id}.$$

When $\pi\colon \tilde{C} \to C$ is étale, $\tilde{g} = 2g-1$ and $\dim P = g-1$, and, when $\pi\colon \tilde{C} \to C$ is ramified at two points, $\tilde{g} = 2g$ and $\dim P = g$. Following [Po05 and Mo], let

$$p_n := \mathcal{F}_{\tilde{J}}\big([\tilde{\varphi}(\tilde{C})]_{(n-1)}\big).$$

By [CvG, Thm.1.3(3); Po05, Cor.0.2, p.877], $p_n = 0$ for $n \geq \tilde{g}/2+1$, i.e., for $n \geq g+1$. Thus, $\mathscr{T}(\tilde{C}) := \mathscr{T}(\tilde{J}, \tilde{\varphi}(\tilde{C}))$ is generated by $p_1, \ldots, p_g$. In what follows we shall use the notation $p_1^{[d]} := p_1^d/d!$ for $d \geq 0$ and $p_1^{[d]} := 0$ for $d < 0$, following [Po05], and the analogous notation $\zeta_1^{[d]} := \zeta_1^d/d!$ for $d \geq 0$ and $\zeta_1^{[d]} := 0$ for $d < 0$.

The relations in the rings $\mathscr{T}(\tilde{C}), \mathrm{taut}(\tilde{C})$, and $\mathrm{Taut}(\tilde{C})$ have been studied recently in [CvG, He, vdGK, Po05, Po07, Mo]. In [Po05], Polishchuk developed a powerful tool for producing relations among the generators of $\mathscr{T}(\tilde{C})$. Consider the operator $\mathscr{D}$, which acts on $A(\tilde{J})$ as $-f \in \mathfrak{sl}_2$, that is, $\mathscr{D}\colon \alpha \mapsto [\tilde{\varphi}(\tilde{C})]_{(0)} * \alpha$. Note that $[\tilde{\varphi}(\tilde{C})]_{(0)} * \alpha = \mathcal{F}_{\tilde{J}}(p_1 \cdot \mathcal{F}_{\tilde{J}}^{-1}(\alpha))$, and S and U in [Po05] are our $\mathcal{F}_{\tilde{J}}$ and $\mathscr{D}$, respectively. It turns out that $\mathscr{D}$ is a differential operator with respect to the intersection product and can be described as follows. Let $R = \mathbb{Q}[x_1, x_2, \ldots]$ be the polynomial ring in infinitely many indeterminates with $\deg x_n = n$

and consider the differential operator

$$D := -\tilde{g}\partial_1 + \frac{1}{2}\sum_{m,n\geq 1}\binom{m+n}{n}x_{m+n-1}\partial_m\partial_n,$$

acting on R, where $\partial_k = \partial_{x_k}$ is the partial derivative with respect to x_k. Let $\kappa\colon R \to \mathscr{T}(\tilde{C})$ be the surjective map $x_n \mapsto p_n$ for $n \geq 1$. By [Po05], $\kappa \circ D = \mathscr{D} \circ \kappa$, which describes the action of $\mathscr{D}$ explicitly.

Polishchuk showed that the polynomials of the form $F(x_1, x_2, \ldots) = D^d(x_1^{m_1} \cdots x_k^{m_k})$ with $d \geq 0$, $\sum_{i=1}^k im_i = \tilde{g}$, and $m_1 < \tilde{g}$ give the relations $F(p_1, p_2, \ldots) = 0$ in $\mathscr{T}(\tilde{C})$, see [Po05, Thm.0.1(i); Mo, 2.1, p.476]. We shall call these relations *Polishchuk's relations.* Note that with the above assumption on the m_i's, we have $p_1^{m_1} \cdots p_k^{m_k} \in A^{\tilde{g}}(\tilde{J})_{(s)}$ with $s > 0$ because $m_1 < \tilde{g}$. Since $A^{\tilde{g}}(\tilde{J})_{(s)} = 0$ for all $s > 0$, then $p_1^{m_1} \cdots p_k^{m_k} = 0$. Following [Mo], we shall call the relations of the form $p_1^{m_1} \cdots p_k^{m_k} = 0$ with $\sum_{i=1}^k im_i = \tilde{g}$ and $m_1 < \tilde{g}$ *trivial relations.* Thus, Polishchuk started with trivial relations and after applying powers of $\mathscr{D}$ obtained new relations many of which were no longer trivial. Polishchuk conjectured that for a generic curve, this procedure gives a complete set of relations among the generators for $\mathscr{T}(\tilde{C})$, [Po05, p.879].

Next, we observe that Polishchuk's relations "descend" to relations among the tautological cycles on P. Indeed, by Proposition 3.1.2 and Theorem 3.2.3, the restriction homomorphism $i^*\colon A(\tilde{J}) \to A(P)$ induces a surjective ring homomorphism (under the intersection product): $i^*\colon \mathscr{T}(\tilde{J}) \twoheadrightarrow \mathscr{T}(P, \psi(\tilde{C}))$ such that $p_n \mapsto \zeta_n$ for all n. Thus, polynomial relations among the p_n's restrict to analogous polynomial relations among the ζ_n's. However, we have the following proposition.

Proposition 3.3.1. *All Polishchuk relations among the generators of $\mathscr{T}(\tilde{C})$ become trivial after restriction to P.*

Proof. By [Po05, Thm.0.1, p.876 and 2.3(ii), p.885], any Polishchuk relation can be written as

$$\mathscr{D}^d(p_1^{[\tilde{g}-\sum_{i=1}^k n_i]} p_{n_1}\cdots p_{n_k}) = 0, \tag{3.3.1}$$

where $n_i > 1$ for all i (Polishchuk's U and g are our $\mathscr{D}$ and $\tilde{g}$, respectively). For the above relation to be non-trivial we must have $\sum_{i=1}^k n_i \leq \tilde{g}$ and $d \leq k-1$. As a consequence, $k \leq \tilde{g}/2$, and therefore, $\tilde{g} - d \geq \tilde{g}/2 + 1$. Since $\mathscr{D}(A^l(P)) \subset A^{l-1}(P)$, then the left hand side of (3.3.1) is homogeneous of degree $\tilde{g} - d$. Since $\tilde{g} - d \geq \tilde{g}/2 + 1 > \dim P$, then the above relation becomes $0 = 0$ after pull-back by i^*. □

By analogy with [Po05], let us consider the operator

$$\mathscr{D}_P \colon A(P) \to A(P), \qquad \alpha \mapsto [\psi(\tilde{C})]_{(0)} * \alpha,$$

which can also be written as $\mathscr{D}_P(\alpha) = \mathcal{F}_P(\zeta_1 \cdot \mathcal{F}_P^{-1}(\alpha))$. Let $R_P := \mathbb{Q}[x_1, x_3, x_5, \ldots]$ be the polynomial ring in infinitely many indeterminates with odd indices and $\deg x_n = n$, and let $\kappa_P \colon R_P \to \mathscr{T}(P, \psi(\tilde{C}))$ be the natural homomorphism $x_n \mapsto \zeta_n$. In what follows we shall compute, similarly to [Po05], a differential operator D_P on R_P such that $\kappa_P \circ D_P = \mathscr{D}_P \circ \kappa_P$.

Given integers $k_1, \ldots, k_r$, define the cycle

$$\varpi(k_1, \ldots, k_r) := (k_{1*}[\psi(\tilde{C})]) * \cdots * (k_{r*}[\psi(\tilde{C})]).$$

Note that $\varpi(k_1, \ldots, k_r)$ is related to Polishchuk's $w(k_1, \ldots, k_r) \in A(\tilde{J})$, [Po05, p.881], by the formula $u_*\big(w(k_1, \ldots, k_r)\big) = \varpi(k_1, \ldots, k_r)$. By analogy with [Po05, Lem.2.2], we have:

Lemma 3.3.2. *Given integers $r \geq 1$ and $k_1, \ldots, k_r$, the following identity holds in $A(P)$:*

$$\xi \cdot \varpi(k_1, \ldots, k_r) = 2p \sum_{i=1}^r k_i^2 \varpi(k_1, \ldots, \widehat{k_i}, \ldots, k_r)$$
$$- \sum_{i<j} k_i k_j \bigg(\varpi(k_i + k_j, \ldots, \widehat{k_i}, \ldots, \widehat{k_j}, \ldots, k_r) - \varpi(k_i - k_j, \ldots, \widehat{k_i}, \ldots, \widehat{k_j}, \ldots, k_r) \bigg).$$

Proof. Using the notation of the proof of Theorem 3.2.3 we have:

$$\begin{aligned} v^*\xi &= \sum_{i=1}^{r} k_i^2 q_i^* \psi^* \xi - \sum_{1 \le i < j \le r} k_i k_j q_{ij}^* (\psi, \psi)^* c_1(\mathscr{L}_P) \\ &= 2p \sum_{i=1}^{r} k_i^2 q_i^* [\tilde{o}] - \sum_{1 \le i < j \le r} k_i k_j q_{ij}^* (\Delta_{\tilde{C}} - (1, \iota)^* \Delta_{\tilde{C}}). \end{aligned}$$

Since $\xi \cdot \varpi(k_1, \ldots, k_r) = v_* v^* \xi$, the result of the lemma follows from the formulas:

$$\begin{aligned} v_* q_{ij}^* \Delta_{\tilde{C}} &= \varpi(k_i + k_j, \ldots, \widehat{k}_i, \ldots, \widehat{k}_j, \ldots, k_r) \\ v_* q_{ij}^* (1, \iota)^* \Delta_{\tilde{C}} &= \varpi(k_i - k_j, \ldots, \widehat{k}_i, \ldots, \widehat{k}_j, \ldots, k_r). \end{aligned}$$

□

Let $Q(t) := \sum_{i=1}^{p-1} \zeta_i t^i$, where t is an indeterminate. We have the following lemma.

Lemma 3.3.3. *There is an identity of polynomials in $t_1, \ldots, t_r$:*

$$\begin{aligned} \mathscr{D}_P(Q(t_1) \cdots Q(t_r)) &= -2p \sum_{i=1}^{r} t_i Q(t_1) \cdots \widehat{Q(t_i)} \cdots Q(t_r) \\ &+ \sum_{1 \le i < j \le r} \big((t_i + t_j) Q(t_i + t_j) - (t_i - t_j) Q(t_i - t_j)\big) Q(t_1) \cdots \widehat{Q(t_i)} \cdots \widehat{Q(t_j)} \cdots Q(t_r). \end{aligned}$$

Proof. First, for any integers $k_1, \ldots, k_r$, using Proposition 3.1.2, we obtain the formula:

$$\mathcal{F}_P(\varpi(k_1, \ldots, k_r)) = k_1 \cdots k_r Q(k_1) \cdots Q(k_r).$$

Using this formula together with Lemma 3.3.2 and the fact that $\mathscr{D}_P(\alpha) = \mathcal{F}_P(\zeta_1 \cdot \mathcal{F}_P^{-1}(\alpha)) = -\mathcal{F}_P(\xi \cdot \mathcal{F}_P^{-1}(\alpha))$ we may check that the identity of the lemma holds if $t_i = k_i$ for all $i = 1, \ldots, r$. Since $k_1, \ldots, k_r$ are arbitrary integers, the identity of polynomials follows. □

Proposition 3.3.4. *For any integer $n \geq 0$ and any **odd** integers $k_1, \ldots, k_r \geq 3$:*

$$\mathscr{D}_P(\zeta_1^{[n]}\zeta_{k_1}\cdots\zeta_{k_r}) = 2\big(-p+n-1+\sum_{i=1}^{r}(k_i+1)\big)\zeta_1^{[n-1]}\zeta_{k_1}\cdots\zeta_{k_r}$$
$$+2\sum_{1\leq i<j\leq r}\binom{k_i+k_j}{k_i}\zeta_1^{[n]}\zeta_{k_i+k_j-1}\zeta_{k_1}\cdots\widehat{\zeta_{k_i}}\cdots\widehat{\zeta_{k_j}}\cdots\zeta_{k_r}.$$

Proof. First, note that for any integers $n_1, \ldots, n_r$, not necessarily odd, $\mathscr{D}_P(\zeta_1^{[n]}\zeta_{n_1}\cdots\zeta_{n_r})$ is equal to $1/n!$ times the coefficient of $t_1^{n_1}\cdots t_r^{n_r}t^n$ in $\mathscr{D}_P\big(Q(t_1)\cdots Q(t_r)Q(t)^n\big)$. The coefficient in question can be computed:

$$\Big(-2pn+2n(n-1)+2n\sum_{i=1}^{r}(n_i+1)\Big)\zeta_1^{n-1}\zeta_{n_1}\cdots\zeta_{n_r}$$
$$+\sum_{1\leq i<j\leq r}\big(1+(-1)^{n_j-1}\big)\binom{n_i+n_j}{n_i}\zeta_1^{n}\zeta_{n_i+n_j-1}\zeta_{n_1}\cdots\widehat{\zeta_{n_i}}\cdots\widehat{\zeta_{n_j}}\cdots\zeta_{n_r},$$

If we take $n_i = k_i$ odd for all $i = 1, \ldots, r$, then the above expression can be rewritten as

$$2n\big(-p+n-1+\sum_{i=1}^{r}(k_i+1)\big)\zeta_1^{n-1}\zeta_{k_1}\cdots\zeta_{k_r} \tag{3.3.2}$$
$$+2\sum_{1\leq i<j\leq r}\binom{k_i+k_j}{k_i}\zeta_1^{n}\zeta_{k_i+k_j-1}\zeta_{k_1}\cdots\widehat{\zeta_{k_i}}\cdots\widehat{\zeta_{k_j}}\cdots\zeta_{k_r},$$

and the result follows after multiplying by $1/n!$. □

Theorem-Definition 3.3.5. *The operator $\mathscr{D}_P$ acts on $\mathscr{T}(P,\psi(\tilde{C}))$ in the same way as the operator*

$$D_P := -2p\partial_1 + \sum_{\substack{m,n\geq 1\\ \text{odd}}}\binom{m+n}{m}x_{m+n-1}\partial_m\partial_n$$

acts on R_P (the summation is over all odd *integers $m, n \geq 1$). More precisely, $\kappa_P \circ D_P = \mathscr{D}_P \circ \kappa_P$.*

Proof. By looking at the Expression 3.3.2 in the proof of Proposition 3.3.4, we see that $\mathscr{D}_P$

acts as the differential operator

$$2(-p-1)\partial_1 + 2\partial_1 x_1 \partial_1 + 2\sum_{\substack{m\geq 3\\ \text{odd}}}(m+1)x_m\partial_m\partial_1 + \sum_{\substack{m,n\geq 3\\ \text{odd}}}\binom{m+n}{m}x_{m+n-1}\partial_m\partial_n.$$

The desired result follows by rewriting the above operator using the identities:

$$\sum_{\substack{m,n\geq 1\\ \text{odd}}}\binom{m+n}{m}x_{m+n-1}\partial_m\partial_n = 2x_1\partial_1^2 + 2\sum_{\substack{m\geq 3\\ \text{odd}}}(m+1)x_m\partial_m\partial_1 + \sum_{\substack{m,n\geq 3\\ \text{odd}}}\binom{m+n}{m}x_{m+n-1}\partial_m\partial_n$$

and $\partial_1 x_1 \partial_1 - x_1\partial_1^2 = \partial_1$. $\square$

Thus, starting with trivial relations in $\mathscr{T}(P, \psi(\tilde{C}))$, we may apply $\mathscr{D}_P$ to obtain new relations. Next, we shall write these relations explicitly, as was done in [Po05] in the case of Jacobians. Given an integer $n \geq 0$ and a number or an operator x, let us use the notation $\binom{x}{n} := x(x-1)\cdots(x-n+1)/n!$. We set $\binom{x}{0} = 1$ and if $n < 0$, $\binom{x}{n} = 0$. Define the following differential operators:

$$\Delta_P := \frac{1}{2}\sum_{\substack{m,n\geq 3\\ \text{odd}}}\binom{m+n}{m}x_{m+n-1}\partial_m\partial_n$$

$$H_P := -p - 1 + x_1\partial_1 + \sum_{\substack{n\geq 3\\ \text{odd}}}(n+1)x_n\partial_n,$$

then we may check that $D_P = 2(\partial_1 H_P + \Delta_P)$. For an operator T, we let $T^{[n]} := T^n/n!$ be the n^{th} divided power.

Lemma 3.3.6. *For every $n \geq 0$ we have the equality of differential operators acting on R_P:*

$$D_P^{[n]} = 2^n \sum_{i=0}^{n} \partial_1^{n-i}\Delta_P^{[i]}\binom{H_P - i}{n-i}.$$

Proof. As in [Po05, Lem.2.7, p.884], we may check this by induction on n using the commutation relations:

$$[\partial_1, \Delta_P] = 0, [H_P, \partial_1] = -\partial_1, [H_P, \Delta_P] = -2\Delta_P.$$

$\square$

Definition 3.3.1. *Given an integer $r \geq 1$, integers $k_1, \ldots, k_r > 1$, and a subset $I = \{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, r\}$ define the numbers*

$$b(I) := \frac{(k_{i_1} + \ldots + k_{i_s})!}{k_{i_1}! \cdots k_{i_s}!} \qquad \textit{and} \qquad d(I) := k_{i_1} + \ldots + k_{i_s} - s + 1.$$

Theorem 3.3.7. *For each integer $r \geq 1$, odd integers $k_1, \ldots, k_r > 1$, and each d with $0 \leq d \leq r-1$, we have the relation,*

$$\sum_{\mathscr{P}_m} \binom{m-1}{d+m-r} b(I_1) \cdots b(I_m) \zeta_1^{[p-d-m+r-\sum_{i=1}^r k_i]} \zeta_{d(I_1)} \cdots \zeta_{d(I_m)} = 0$$

in $A^{p-d}(P)$, where the sum is taken over all unordered partitions $\mathscr{P}_m = \{I_1, \ldots, I_m\}$ of $[1, r] = \{1, 2, \ldots, r\}$ into m disjoint nonempty subsets such that $r - d \leq m \leq p - d + r - \sum_{i=1}^r k_i$.

Proof. Using [Po05, Lem.2.8, p.884], we may check that for each $j \geq 0$,

$$\Delta_P^{[j]}(x_1^{[n]} x_{k_1} \cdots x_{k_r}) = \sum_{\mathscr{P}_{r-j}} b(I_1) \cdots b(I_{r-j}) x_1^{[n]} x_{d(I_1)} \cdots x_{d(I_{r-j})},$$

where the sum is taken over all unordered partitions $\mathscr{P}_{r-j} = \{I_1, \ldots, I_{r-j}\}$ of $[1, r]$ into $r-j$ disjoint nonempty parts. Since $\zeta_1^{[p-\sum_{i=1}^r k_i]} \zeta_{k_1} \cdots \zeta_{k_r} = 0$ in $A(P)$, we obtain the relation $\mathscr{D}_P^{[d]}(\zeta_1^{[p-\sum_{i=1}^r k_i]} \zeta_{k_1} \cdots \zeta_{k_r}) = 0$. Using Theorem 3.3.5 and Lemma 3.3.6, this relation can be rewritten as

$$2^n \sum_{j=0}^{d} \binom{r-1-j}{n-j} \zeta_1^{[p-d+j-\sum_{i=1}^r k_i]} \zeta_{k_1} \cdots \zeta_{k_r} \sum_{\mathscr{P}_{r-j}} b(I_1) \cdots b(I_{r-j}) \zeta_{d(I_1)} \cdots \zeta_{d(I_{r-j})} = 0,$$

which differs from the relation in the statement of the theorem by a factor of 2^n (after the substitution $m = r - j$ and rearrangement of the summands). $\square$

Corollary 3.3.8. *If $p \geq 3$, then $\zeta_n = 0$ for all $n \geq \frac{2}{3}p + 1$.*

Proof. If $d = r - 1$, the integers $k_1, \ldots, k_r \geq 3$ are odd and $\sum_{i=1}^r k_i = p$, then by Theorem 3.3.7, $\zeta_{p-r+1} = 0$. If p is even, then we may write p as one of the following: $6l - 2, 6l$, or $6l + 2$ for some $l \geq 1$, and we obtain $\zeta_{p+1} = \zeta_{p-1} = \ldots = \zeta_{p-(2l-3)} = 0$. If p is odd, then we may write p as one of the following: $6l - 3, 6l - 1$, or $6l + 1$ for some $l \geq 1$, and therefore, $\zeta_p = \zeta_{p-2} = \ldots = \zeta_{p-(2l-2)} = 0$. Since $\zeta_n = 0$ for all n even, the result follows. $\square$

It is interesting to determine the kernels

$$\ker\big(\kappa_P \colon R_P \twoheadrightarrow \mathscr{T}(P, \psi(\tilde{C}))\big) \text{ and } \ker\big(i^* \colon \mathscr{T}(\tilde{J}) \twoheadrightarrow \mathscr{T}(P, \psi(\tilde{C}))\big).$$

We don't have complete answers to these questions at this point, but we offer the following observations and conjectures. From the proof of [Fa, Thm.4.5, p.117], we know that for a generic Prym variety P of dimension $p \geq 5$, the cycle $([\psi(\tilde{C})]^{*r})_{(2)}$ is non-zero in $A(P)$ for $1 \leq r \leq p - 3$, and therefore, by Fourier duality, $\zeta_1^j \zeta_3 \neq 0$ in $A(P)$ for $0 \leq j \leq p - 4$. Note, however, that $\zeta_1^j \zeta_3 = 0$ for $j > p - 4$. Since $\zeta_1 = -\xi \in A^1(P)_{(0)}$, then $\zeta_1^j \neq 0$ in $A(P)$ for $1 \leq j \leq p$. These results seem to be the only non-vanishing results for tautological classes on Pryms known thus far. On the other hand, by analogy with Polishchuk's conjecture for $\mathscr{T}(\tilde{C})$, [Po05, p.879], we may conjecture that for a generic Prym the set of relations given in Theorem 3.3.7 is complete.

Examples: *For $p \leq 5$ there are no non-trivial Polishchuk relations.*

$p = 2$: $\mathscr{T}(P, \psi(\tilde{C})) \simeq \mathbb{Q}[\zeta_1]/(\zeta_1^3)$.

$p = 3$: $\mathscr{T}(P, \psi(\tilde{C})) \simeq \mathbb{Q}[\zeta_1]/(\zeta_1^4)$.

$p = 4$: $\mathbb{Q}[x_1, x_3]/(x_1^5, x_3^2, x_1x_3) \twoheadrightarrow \mathscr{T}(P, \psi(\tilde{C}))$ with $x_n \mapsto \zeta_n$, and I do not know if ζ_3 is non-zero on a generic Prym of dimension 4.

$p = 5$: $\mathbb{Q}[x_1, x_3]/(x_1^6, x_3^2, x_1^2x_3) \twoheadrightarrow \mathscr{T}(P, \psi(\tilde{C}))$, and this is an isomorphism for a generic Prym of dimension 5, because $\zeta_1\zeta_3 \neq 0$ on such a Prym.

*For $6 \leq p \leq 10$, the **non-trivial** Polishchuk relations are:*

$p = 6$: $20\zeta_5 = 0$ $(d = 1, k_1 = k_2 = 3)$. Furthermore, if P is generic, $\mathscr{T}(P, \psi(\tilde{C})) \simeq \mathbb{Q}[x_1, x_3]/(x_1^7, x_3^2, x_1^3x_3)$.

$p = 7$: $20\zeta_1\zeta_5 + \zeta_3^2 = 0$ $(d = 1, k_1 = k_2 = 3)$.

$p = 8$: $10\zeta_1^2\zeta_5 + \zeta_1\zeta_3^2 = 0$ $(d = 1, k_1 = k_2 = 3)$, $56\zeta_7 = 0$ $(d = 1, k_1 = 3, k_2 = 5)$.

$p = 9$: $\frac{10}{3}\zeta_1^3\zeta_5 + \frac{1}{2}\zeta_1^2\zeta_3^2 = 0$ $(d = 1, k_1 = k_2 = 3)$, $56\zeta_1\zeta_7 + \zeta_3\zeta_5 = 0$ $(d = 1, k_1 = 3, k_2 = 5)$, $60\zeta_3\zeta_5 = 0$ $(d = 1, k_1 = k_2 = k_3 = 3)$, $1680\zeta_7 = 0$ $(d = 2, k_1 = k_2 = k_3 = 3)$.

$p = 10$: $\frac{5}{6}\zeta_1^4\zeta_5 + \frac{1}{6}\zeta_1^3\zeta_3^2 = 0$ $(d = 1, k_1 = k_2 = 3)$, $28\zeta_1^2\zeta_7 + \zeta_1\zeta_3\zeta_5 = 0$ $(d = 1, k_1 = 3, k_2 = 5)$, $120\zeta_9 = 0$ $(d = 1, k_1 = 3, k_2 = 7)$, $60\zeta_1\zeta_3\zeta_5 + 2\zeta_3^3 = 0$ $(d = 1, k_1 = k_2 = k_3 = 3)$, $1680\zeta_1\zeta_7 + 60\zeta_3\zeta_5 = 0$ $(d = 2, k_1 = k_2 = k_3 = 3)$.

Remark 3.3.9. *If $\tilde{C}$ has a base-point-free g_d^r, then by [He, vdGK], for every $N \geq d - 2r + 1$, the following relation holds in $A(\tilde{J})$:*

$$\sum_{a_1 + \cdots + a_r = N} (a_1 + 1)! \cdots (a_r + 1)! [\tilde{\varphi}(\tilde{C})]_{(a_1)} * \cdots * [\tilde{\varphi}(\tilde{C})]_{(a_r)} = 0.$$

Applying u_* this yields the following relation in $A(P)$ for every $N \geq d - 2r + 1$:

$$\sum_{a_1 + \cdots + a_r = N} (a_1 + 1)! \cdots (a_r + 1)! [\psi(\tilde{C})]_{(a_1)} * \cdots * [\psi(\tilde{C})]_{(a_r)} = 0.$$

Chapter 4

Algebraic equivalence of special subvarieties

In this chapter we assume that the double cover $\pi\colon \tilde{C} \to C$ is connected and étale. In this section we let r, d, g be integers such that $0 < 2r < d < 2g$. Also, S will denote a smooth connected but not necessarily complete curve. For a morphism $X \to S$, the fiber over s is denoted by X_s, and for a sheaf $\mathcal{F}$ on X, $\mathcal{F}_s := \mathcal{F}_{|X_s}$ is the restriction. If X is an integral projective scheme, we let Pic^0_X be the connected component of the identity in the Picard scheme of X. If X is a smooth curve, we let X_d denote the d^{th} symmetric product of X. An integral curve with n ordinary double points (resp., n ordinary cusps) and no other singularities will be called n-nodal (resp., n-cuspidal).

B will denote a smooth connected curve of genus $g-1$. B_{pq} will denote the 1-nodal curve obtained by gluing distinct points p and q on B. Also, B_{pp} will denote the 1-cuspidal curve, whose normalization is B, and such that the point $p \in B$ maps to the cusp of B_{pp}. For simplicity, both normalization morphisms $B \to B_{pq}$ and $B \to B_{pp}$ will be denoted by ν and the distinction will be clear from the context.

4.1 Étale double covers and square trivial invertible sheaves

Let us recall our convention that "scheme" means (here) a noetherian separated scheme over $\mathbb{C}$. Also, by an étale double cover we mean a finite étale morphism $f\colon X \to Y$ of schemes of degree 2 on each irreducible component of Y.

The exercise [Ha, Ex.2.7(b), p.306] asks to show that for a smooth curve Y there is a one-to-one correspondence between étale covers $f\colon X \to Y$ of degree two and points of order two in $\mathrm{Pic}(Y)$, i.e., invertible sheaves $\mathcal{L}$ on Y such that $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_Y$. This correspondence is valid in a more general setting, as we show next.

Let us recall from [AK70, p.124] the definition of the discriminant of a flat morphism $f\colon X \to Y$ of schemes. Since f is flat, $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module. The trace

$$\mathrm{Tr}\colon \underline{\mathrm{End}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X) \to \mathcal{O}_Y$$

is a homomorphism of $\mathcal{O}_Y$-modules, which can be described locally as follows. If V is an open affine set in Y such that $f_*\mathcal{O}_X$ is free over V, then Tr sends $g_V \in \mathrm{End}_{\mathcal{O}_Y(V)}(f_*\mathcal{O}_X(V))$ to the trace

$$\mathrm{tr}\big(g_V\colon f_*\mathcal{O}_X(V) \to f_*\mathcal{O}_X(V)\big) \in \mathcal{O}_Y(V).$$

Since $f_*\mathcal{O}_X$ is locally free then there is a natural isomorphism $\underline{\mathrm{End}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X) = (f_*\mathcal{O}_X)^\vee \otimes f_*\mathcal{O}_X$, [Ha, Ex.II.5.1b, p.123], which gives a natural map $f_*\mathcal{O}_X \to \underline{\mathrm{End}}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$. The $\mathcal{O}_Y$-module homomorphism

$$\mathrm{Tr}_{X/Y}\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$$

obtained by composing the above natural map with Tr is also called the trace.

To $\mathrm{Tr}_{X/Y}$ we may associate a natural map $u\colon f_*\mathcal{O}_X \to (f_*\mathcal{O}_X)^\vee$. For $V \subset Y$ open and

$a \in f_*\mathcal{O}_X(V)$ define

$$u_V(a)\colon f_*\mathcal{O}_X(V) \to \mathcal{O}_Y(V)$$

by $b \mapsto \mathrm{Tr}_{X/Y}(ab)$. This gives a homomorphism

$$f_*\mathcal{O}_X(V) \to \mathrm{Hom}_{\mathcal{O}_Y(V)}\big(f_*\mathcal{O}_X(V), \mathcal{O}_Y(V)\big), a \mapsto u_V(a),$$

which is the local description of u. This induces the *discriminant*

$$d_{X/Y} := \det(u) \in \mathrm{Hom}\big(\det f_*\mathcal{O}_X, \det(f_*\mathcal{O}_X)^\vee\big),$$

which is the map on top exterior powers induced by u. The image of

$$d_{X/Y} \otimes \mathrm{id}\colon (\det f_*\mathcal{O}_X)^{\otimes 2} \to \mathcal{O}_Y$$

is called the *discriminant ideal* and is denoted by $D_{X/Y}$. The main result we shall use is: $D_{X/Y} = \mathcal{O}_Y$ if and only if f is étale, [AK70, p.124]. In this case, $d_{X/Y} \otimes \mathrm{id}$ is an isomorphism, [Ha, Ex.II.7.1, p.169], so

$$(\det f_*\mathcal{O}_X)^{\otimes 2} \simeq \mathcal{O}_Y.$$

Proposition 4.1.1. *Let Y be a scheme. There is a one-to-one correspondence between étale double covers of Y and locally free sheaves $\mathcal{L}$ on Y with $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_Y$.*

Proof. Let $\mathcal{L}$ be an invertible sheaf on Y such that $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_Y$. Let us fix an isomorphism $\varphi\colon \mathcal{L}^{\otimes 2} \to \mathcal{O}_Y$. Define an $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus \mathcal{L}$ by

$$\langle a, b\rangle \cdot \langle a', b'\rangle = \langle aa' + \varphi(bb'), ab' + a'b\rangle.$$

Let $X := \mathbf{Spec}(\mathcal{O}_Y \oplus \mathcal{L})$ with the natural morphism $f\colon X \to Y$, [Ha, Ex.II.5.17, p.128].

Since $f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{L}$, then $(\det f_*\mathcal{O}_X)^{\otimes 2} \simeq \mathcal{L}^{\otimes 2} \simeq \mathcal{O}_Y$. This implies that the discriminant ideal $D_{X/Y}$ coincides with $\mathcal{O}_Y$, and therefore, f is étale.

Conversely, let $f\colon X \to Y$ be an étale double cover. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{f^\sharp} f_*\mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow 0, \tag{4.1.1}$$

where $\mathcal{L} := \operatorname{coker}(f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X)$. Since f is étale and $\mathcal{L} \simeq \det f_*\mathcal{O}_X$ [Ha, Ex.II.5.16d, p.128], then by the discussion preceding the proposition: $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_Y$.

The trace homomorphism

$$\mathrm{Tr}_{X/Y}\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$$

provides a splitting of the sequence (4.1.1), and therefore, $f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{L}$. This shows that the two constructions are inverses of each other. $\square$

4.2 Special subvarieties

Assume that C has a complete g^r_d, which is viewed as a subvariety $G_d \subset C_d$ isomorphic to $\mathbb{P}^r$. Let us recall the definition of the special subvarieties V_0 and V_1 of P associated to G_d, cf. [Be82]. First, we assume that G_d contains a reduced divisor. Consider the following commutative diagram

$$\begin{tikzcd} \tilde{C}_d \arrow[r, "\tilde{\varphi}_d"] \arrow[d, "\pi_d"'] & \tilde{J} \arrow[d, "\mathrm{Nm}"] \\ C_d \arrow[r, "\varphi_d"] & J, \end{tikzcd}$$

where the horizontal maps are abelian sum mappings $\varphi_d\colon D \mapsto \mathcal{O}_C(D - do)$, $\tilde{\varphi}_d\colon \tilde{D} \mapsto \mathcal{O}_{\tilde{C}}(\tilde{D} - d\tilde{o})$ and π_d is induced by $\pi\colon \tilde{C} \to C$. The variety $\tilde{\varphi}_d(\pi_d^{-1}(G_d))$ is contained in the kernel of the norm map $\mathrm{Nm}\colon \tilde{J} \to J$ and has two connected components V_0 and V_1, [Be82, p.365; We81, p.98]. After a translation of one of the components V_0 or V_1, we assume that both of them are contained in P and, by definition, V_0 and V_1 are called *special subvarieties* of P. The union $V_0 \cup V_1$ is denoted by V. In the case where G_d has a base-point of multiplicity ≥ 2, V is non-reduced and we may define it as a cycle $[V] := \tilde{\varphi}_{d*}(\pi_d^*[G_d])$ with multiplicities, where $[G_d] \in \mathrm{CH}^{d-r}(C_d)$, [Be82, p.359]. In this case V_0 and V_1 can also be defined as cycles so that $[V] = [V_0] + [V_1]$.

By Clifford's theorem, the inequalities $0 < d < 2g$ imply that $d > 2r$, except when the g_d^r is the canonical system or a multiple of a g_2^1. In these two cases the special subvarieties are not even homologically equivalent, see [Be82, Rem.3, p.362 and p.366]. However, if $2r < d$, then the subvarieties V_0 and V_1 are homologically equivalent, i.e., V_0 and V_1 have the same cohomology class

$$2^{d-2r-1} \cdot \frac{\xi^{g-r-1}}{(g-r-1)!}$$

in $H^{2(g-r-1)}(P, \mathbb{Z})$, see [Be82, Prop.1, p.360 and Thm.1, p.364].

Special subvarieties have been used in the study of threefolds and intersections of three quadrics in $\mathbb{P}^{2n+4}$, [We81; Be82, Sec.4,5] and also in the Prym-Torelli problem [Be82, Sec.3; SV02; Na; NL].

4.3 Compactified Jacobians and autoduality

We shall use the results of [AK80, EGK] in the sequel and we recall the part of the theory that we need. Let $\mathcal{C} \to S$ be a flat family of integral curves whose fiber over s is denoted by $\mathcal{C}_s$. We assume that the family $\mathcal{C}/S$ has a section σ whose image lies in the smooth locus of the morphism $\mathcal{C} \to S$. Given an integer n and an S-scheme T, a *torsion-free rank one sheaf of degree* n on $\mathcal{C}_T := \mathcal{C} \times_S T$ is a T-flat coherent $\mathcal{O}_{\mathcal{C}_T}$-module $\mathcal{F}$ such that $\mathcal{F}_t := \mathcal{F}_{|\mathcal{C}_t}$ is a torsion free rank one sheaf on the fiber $\mathcal{C}_t$ over $t \in T$ and $\chi(\mathcal{F}_t) - \chi(\mathcal{O}_{\mathcal{C}_t}) = n$ for every $t \in T$. There is a projective S-scheme $\bar{J}^n_{\mathcal{C}/S}$, called the *compactified Jacobian of* $\mathcal{C}/S$, that parametrizes torsion-free rank one sheaves of degree n on the fibers of $\mathcal{C}/S$, see [EGK, p.594] and references therein. More precisely, $\bar{J}^n_{\mathcal{C}/S}$ is the fine moduli scheme representing the following functor of S-schemes:

$$T \mapsto \{\text{torsion-free rank one sheaves of degree } n \text{ on } C \times_S T\}/\sim,$$

where $\mathcal{F} \sim \mathcal{G}$ if and only if there exists an invertible sheaf $\mathcal{L}$ on T such that $\mathcal{F} \simeq \mathcal{G} \otimes \pi_T^* \mathcal{L}$ and $\pi_T \colon \mathcal{C} \times_S T \to T$ is the projection. A sheaf $\mathcal{P}_d$ on $\mathcal{C} \times_S \bar{J}^d_{\mathcal{C}/S}$ is called a *Poincaré sheaf*, if $\mathcal{P}_d$ satisfies the following universal property. For any S-scheme T and any T-flat torsion-free rank one sheaf $\mathcal{F}$ of degree n on $\mathcal{C} \times_S T$, there is a unique morphism $f \colon T \to \bar{J}^d_{\mathcal{C}/S}$ such that $\mathcal{F}$ is equivalent to the pull-back of $\mathcal{P}_d$ by the induced morphism $1 \times f \colon \mathcal{C} \times_S T \to \mathcal{C} \times_S \bar{J}^d_{\mathcal{C}/S}$. More precisely, there is an invertible sheaf $\mathcal{L}$ on T such that $(1 \times f)^* \mathcal{P}_d \simeq \mathcal{F} \otimes \pi_T^* \mathcal{L}$. Let us note that $\mathcal{P}_d$ exists because all the fibers of $\mathcal{C} \to S$ are integral, [AK80]. Furthermore, $\mathcal{P}_d$ is uniquely determined up to tensoring with a pull-back of an invertible sheaf on $\bar{J}^d_{\mathcal{C}/S}$.

There is also an open subscheme $J^n_{\mathcal{C}/S} \subset \bar{J}^n_{\mathcal{C}/S}$, called the (*generalized*) *Jacobian of* $\mathcal{C}/S$, parametrizing those sheaves that are invertible. The schemes $J^n_{\mathcal{C}/S}$ and $\bar{J}^n_{\mathcal{C}/S}$ are flat over S by [AIK, p.8] and their fibers over $s \in S$ are denoted by $J^n_{\mathcal{C}_s}$ and $\bar{J}^n_{\mathcal{C}_s}$, respectively. The section $\sigma \colon S \to \mathcal{C}$ gives an invertible sheaf $\mathcal{N}$ of degree one on $\mathcal{C}$, which determines an *Abel map* $A_{\mathcal{N}} \colon \mathcal{C} \to \bar{J}^0_{\mathcal{C}/S}$, see [EGK, p.595]. On the fiber $\mathcal{C}_s$, the Abel map is given by $x \mapsto \mathfrak{m}_x \otimes \mathcal{N}_s$, where $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_{\mathcal{C}_s,x}$. If the geometric fibers of $\mathcal{C} \to S$ have double points at worst, then by [EGK, Thm.2.1] the Abel map induces an isomorphism of group-schemes:

$$A^*_{\mathcal{N}} \colon \mathrm{Pic}^0_{\bar{J}^0_{\mathcal{C}/S}} \to J^0_{\mathcal{C}/S}.$$

4.4 Presentation schemes

In Section 4.6 we shall use the construction of [AK], called the *presentation scheme*, which we recall next. Let X be an integral curve with a unique double point and let $\nu\colon X' \to X$ be the normalization. Given an integer n, the presentation scheme P^n_X parametrizes injective morphisms $h\colon L \hookrightarrow \nu_* M$, called presentations, such that $L \in \bar{J}^n_X$, $M \in J^n_{X'}$. By [AK], the presentation scheme P^n_X fits into a diagram

$$\begin{tikzcd} P^n_X \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & \bar{J}^n_X \\ J^n_{X'}, & \end{tikzcd}$$

where κ and λ send $(h\colon L \hookrightarrow \nu_* M)$ to L and M, respectively. Note that in [AK] the degree of L is $\chi(L)$, not $\chi(L) - \chi(\mathcal{O}_X)$, as it is for us.

When $X = B_{pq}$ is 1-nodal, $\lambda\colon P^n_{B_{pq}} \to J^n_B$ is a $\mathbb{P}^1$-bundle, which has two distinguished sections s_p and s_q. The morphism $\kappa\colon P^n_{B_{pq}} \to \bar{J}^n_{B_{pq}}$ identifies the images $\mathrm{Im}(s_p)$ and $\mathrm{Im}(s_q)$ with a shift and is an isomorphism outside of $\mathrm{Im}(s_p) \cup \mathrm{Im}(s_q)$. More precisely, if $M \in J^n_B$, then κ identifies $s_p(M)$ with $s_q(M \otimes \mathcal{O}_B(q-p))$. Furthermore, the common image of $\kappa \circ s_p$ and $\kappa \circ s_q$ coincides with $\partial \bar{J}^n_{B_{pq}}$, the locus of non-invertible sheaves.

When $X = B_{pp}$ is 1-cuspidal, let $p' \subset B$ be the fiber over the cusp of B_{pp}. There is an embedding $s_{p'}\colon J^n_B \times p' \to P^n_{B_{pp}}$ such that $\lambda \circ s_{p'}\colon J^n_B \times p' \to J^n_B$ is the first projection. The scheme $P^n_{B_{pp}}$ is non-reduced along $\mathrm{Im}(s_{p'})$, although $\bar{J}^n_{B_{pp}}$ itself is reduced. The morphism κ is bijective but is not an isomorphism: it maps the non-reduced locus $\mathrm{Im}(s_{p'})$ to the reduced subscheme $\partial \bar{J}^n_{B_{pp}} \subset \bar{J}^n_{B_{pp}}$ and is an isomorphism outside of $\mathrm{Im}(s_{p'})$.

4.5 Brill-Noether varieties and their families

In classical algebraic geometry, to a smooth curve C and integers $r \geq 0, d \geq 1$, one associates the Brill-Noether scheme $W_d^r(C)$, which parametrizes invertible sheaves L on C with $\deg L = d$ and $h^0(L) \geq r + 1$, [ACGH, p.153]. $W_d^r(C)$ can be expressed as a degeneracy locus of a morphism of locally free sheaves on $\mathrm{Pic}^d(C)$, [ACGH, p.177]. The expected dimension of $W_d^r(C)$ is the Brill-Noether number $\rho(g, r, d) = g - (r+1)(g - d + r)$. The basic results of Brill-Noether theory which we shall use are:

1. *Existence*: if $\rho(g, r, d) \geq 0$, then $W_d^r(C) \neq \emptyset$.

2. *Connectedness*: if $\rho(g, r, d) \geq 1$, then $W_d^r(C)$ is connected.

3. *Dimension*: if C is a general curve, then $W_d^r(C)$ is reduced and of dimension $\rho(g, r, d)$, when $\rho(g, r, d) \geq 0$ and is empty otherwise.

These fundamental results are due to many mathematicians: [FL, EH, GH, KL, Ke], see also [ACGH].

In [AC, p.6 or ACGH, p.177] we may find a construction of the family of Brill-Noether varieties associated to a family of smooth curves of genus g. In fact, this construction works in a more general setting. We describe this next. Let $\mathcal{C} \to S$ be a family of integral curves of arithmetic genus g having a section whose image lies in the smooth locus of each fiber. We shall construct a proper S-subscheme $\mathcal{W}_d^r$ of $\bar{J}^d_{\mathcal{C}/S}$, extending the classical construction for families of smooth curves.

Let π_1, π_2 be the projections of $\mathcal{C} \times_S \bar{J}^d_{\mathcal{C}/S}$ onto the first and the second factor, respectively. The Poincaré sheaf $\mathcal{P}_d$ is flat over $\mathcal{C}$ relative to the projection π_1, and therefore, we may apply the theory of cohomology and base-change, [Mu70, p.46]. This theory implies that every point in $\bar{J}^d_{\mathcal{C}/S}$ has an affine open neighborhood $U = \mathrm{Spec}\, A$ such that there exists a complex $K^\bullet$ of free $\mathcal{O}_U$-modules with the following property. Given any affine scheme $V = \mathrm{Spec}\, B$ and a morphism of affine schemes $V \to U$, there is an isomorphism of functors on the category of A-algebras B:

$$H^i\big(\pi_2^{-1}(U) \times_U V, \mathcal{P}_{d|\pi_2^{-1}(U)} \otimes_{\mathcal{O}_U} \mathcal{O}_V\big) \simeq H^i\big(\Gamma(V, K^\bullet \otimes_{\mathcal{O}_U} \mathcal{O}_V)\big), \text{ for all } i \geq 0. \tag{4.5.1}$$

Let m and n be the ranks of K^0 and K^1, respectively, and let us identify $\Gamma(U, K^0)$ and $\Gamma(U, K^1)$ with $\Gamma(U, \mathcal{O}_U^m)$ and $\Gamma(U, \mathcal{O}_U^n)$, respectively. By (4.5.1), there is an exact sequence

$$\Gamma(U, \mathcal{O}_U^m) \xrightarrow{A} \Gamma(U, \mathcal{O}_U^n) \longrightarrow H^1(\pi_2^{-1}(U), \mathcal{P}_{d|\pi_2^{-1}(U)}) \longrightarrow 0,$$

where A is an $n \times m$ matrix with coefficients in $\Gamma(U, \mathcal{O}_U)$. Define $\overline{W}^r_{d,U}$ to be the subscheme of U whose ideal I_U is generated by $(m-r) \times (m-r)$ minors of A. The ideal I_U is also the $(g-d+r)^{th}$ Fitting ideal associated to the above exact sequence. The formation of Fitting ideals is compatible with base-change and is independent of the choice of presentation (see [AC, p.6] and references therein). Consequently, there exists a subscheme $\mathcal{W}^r_d$ of $\bar{J}^d_{\mathcal{C}/S}$ such that for every affine open set U as above

$$\mathcal{W}^r_d \cap U = \overline{W}^r_{d,U}.$$

The fiber of $\mathcal{W}^r_d$ over s is denoted by $\overline{W}^r_d(\mathcal{C}_s)$. As a set, $\overline{W}^r_d(\mathcal{C}_s)$ consists of $F \in \bar{J}^d_{\mathcal{C}_s}$ with $h^0(\mathcal{C}_s, F) \geq r+1$.

The sets of locally free and non-locally free elements of $\overline{W}^r_d(\mathcal{C}_s)$ will be denoted by $W^r_d(\mathcal{C}_s)$ and $\partial\overline{W}^r_d(\mathcal{C}_s)$, respectively. If $\mathcal{C}_s$ is smooth then $\overline{W}^r_d(\mathcal{C}_s) = W^r_d(\mathcal{C}_s)$ is the classical Brill-Noether scheme as in [ACGH, p.153].

In Section 4.6 we shall make repeated use of the following two observations.

Proposition 4.5.1 (Flatness criterion). *If all fibers of $\mathcal{W}^r_d(\mathcal{C}) \to S$ are reduced and have the expected dimension $\rho(g, r, d)$, then $\mathcal{W}^r_d(\mathcal{C})$ is flat over S.*

Proof. Since S is a smooth connected curve by our ongoing assumption, then by [Ha, Prop.9.7, p.257] it suffices to check that no component of $\mathcal{W}_d^r(\mathcal{C})$ is contained in a fiber $\overline{W}_d^r(\mathcal{C}_s)$ for some $s \in S$. Since $\mathcal{W}_d^r(\mathcal{C})$ is determinantal, then each irreducible component of $\mathcal{W}_d^r(\mathcal{C})$ has dimension $\geq \rho(g,r,d)+1$ at every point. But $\dim \overline{W}_d^r(\mathcal{C}_s) = \rho(g,r,d)$ for all $s \in S$ by assumption, and therefore, no component of $\mathcal{W}_d^r$ can be contained in a fiber (this argument is adapted from [HM, p.267]). $\square$

Proposition 4.5.2 (Principle of connectedness)**.** *Let $X \to S$ be a flat family of projective schemes. If there exists a point $s_0 \in S$ such that X_{s_0} is connected and reduced, then X_s is connected for each point $s \in S$.*

Proof. The hypotheses of the proposition imply that $h^0(\mathcal{O}_{X_{s_0}}) = 1$. By upper semicontinuity of $s \mapsto h^0(\mathcal{O}_{X_s})$, there is a Zariski open subset $U \subset S$ such that for all points $s \in U$, X_s is connected. By [Ha, Ex.11.4, p.281], X_s is connected for each point $s \in S$. $\square$

4.6 Main result

The following proposition is due to E. Izadi.

Proposition 4.6.1. *Let C be a smooth curve of genus g, which has a complete g_d^r with $0 < 2r < d < 2g$, and let $\tilde{C} \to C$ be a connected étale double cover. If $W_d^r(C)$ is connected and either $W_{d-1}^r(C)$ or $W_{d+1}^{r+1}(C)$ is nonempty, then the special subvarieties V_0 and V_1 associated to the g_d^r are algebraically equivalent.*

Proof. The cover $\tilde{C} \to C$ determines a point of order two in J_C^0. Translating by $\mathcal{O}_C(do)$ and going to the dual abelian variety of J_C^d, gives a point of order two in $\mathrm{Pic}^0(J_C^d)$, which determines an étale double cover $\widetilde{J}_C^d \to J_C^d$. Assume that $W_{d-1}^r(C)$ is nonempty. Take $L \in W_{d-1}^r(C)$ and let $C \to W_d^r(C)$ be the embedding $x \mapsto L(x)$. We may check that there is a commutative diagram

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[d] & \widetilde{W^r_d}(C) \arrow[r] \arrow[d] & \widetilde{J^d_C} \arrow[d] \\ C \arrow[r] & W^r_d(C) \arrow[r] & J^d_C, \end{tikzcd}$$

where the two squares are Cartesian. The double cover $\widetilde{J^d_C} \to J^d_C$ restricts to the connected double cover $\tilde{C} \to C$, and therefore, the intermediate double cover $\widetilde{W^r_d}(C) \to W^r_d(C)$ is non-trivial. Since $W^r_d(C)$ is connected, this shows that $\widetilde{W^r_d}(C)$ is also connected. The special subvarieties V_0 and V_1 are members of a family of cycles over $\widetilde{W^r_d}(C)$. Since $\widetilde{W^r_d}(C)$ is connected, V_0 and V_1 are algebraically equivalent. The case when $W^{r+1}_{d+1}(C)$ is nonempty is analogous (embed C in $W^r_d(C)$ by $x \mapsto L(-x)$ for a fixed $L \in W^{r+1}_{d+1}(C)$). $\square$

If $\rho(g,r,d) < \min\{r+1, g-d+r\}$ and C is general in the sense of Brill-Noether theory, both $W^r_{d-1}(C)$ and $W^{r+1}_{d+1}(C)$ are empty, because $\rho(g,r,d-1)$ and $\rho(g,r+1,d+1)$ are negative, [ACGH, Thm.1.5, p.214]. Nevertheless, in what follows we shall show that V_0 and V_1 are algebraically equivalent whenever $\rho(g,r,d) > 0$ and $W^r_d(C)$ is reduced and of the expected dimension.

The *Wirtinger cover* is the étale double cover $\tilde{B}_{pq} \to B_{pq}$, where the curve $\tilde{B}_{pq}$ is obtained from two copies of B by identifying p and q on one copy with q and p, respectively, on the other copy. Using [Be77, 6.1] we may find a smooth connected curve S such that $\tilde{C} \to C$ and $\tilde{B}_{pq} \to B_{pq}$ vary in a family of double covers

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[rr, "\tau"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & S & \end{tikzcd}$$

(S in [Be77] is not the same as our S) with the following properties: the fibers of $\mathcal{C} \to S$ are integral and at worst nodal, $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are flat over S, τ is étale, and there is a section σ of

$\mathcal{C} \to S$, which induces a degree one invertible sheaf $\mathcal{N}$ on $\mathcal{C}$, as in Section 4.3. Assume that $\tilde{B}_{pq} \to B_{pq}$ lies over the special point $s_0 \in S$. By Section 4.5 there is an associated family $\mathcal{W}^r_d(\mathcal{C}) \to S$.

The étale double cover $\tau\colon \tilde{\mathcal{C}} \to \mathcal{C}$ induces a morphism $\mathcal{O}_{\mathcal{C}} \hookrightarrow \tau_*\mathcal{O}_{\tilde{\mathcal{C}}}$, whose cokernel is an invertible sheaf $\mathcal{L}$ with the property $\mathcal{L}^2 \simeq \mathcal{O}_{\mathcal{C}}$. Let $\mathcal{M} := (A^*_{\mathcal{N}})^{-1}(\mathcal{L})$, then $\mathcal{M}^2$ is isomorphic to the pull-back of an invertible sheaf on S. After replacing S by a Zariski open subset containing s_0, we may assume that $\mathcal{M}^2$ is trivial. The relative spectrum $\widetilde{J}^0_{\mathcal{C}/S} := \mathbf{Spec}(\mathcal{O}_{\bar{J}^0_{\mathcal{C}/S}} \oplus \mathcal{M})$ over $\bar{J}^0_{\mathcal{C}/S}$ gives an étale double cover $\widetilde{J}^0_{\mathcal{C}/S} \to \bar{J}^0_{\mathcal{C}/S}$. Pulling back via the isomorphism $\bar{J}^d_{\mathcal{C}/S} \to \bar{J}^0_{\mathcal{C}/S}$ of tensoring with $\mathcal{N}^{-d}$, we obtain an étale double cover $\widetilde{J}^d_{\mathcal{C}/S} \to \bar{J}^d_{\mathcal{C}/S}$, whose restriction to $\mathcal{W}^r_d(\mathcal{C})$ is denoted by $\widetilde{\mathcal{W}}^r_d(\mathcal{C}) \to \mathcal{W}^r_d(\mathcal{C})$. Over the special point $s_0 \in S$, we use the notation $\widetilde{J}^d_{B_{pq}} \to \bar{J}^d_{B_{pq}}$ and $\widetilde{W}^r_d(B_{pq}) \to \overline{W}^r_d(B_{pq})$ for the induced double covers. We have:

Proposition 4.6.2. *If $0 < \rho(g,r,d) < \min\{r+1, g-d+r\}$, then a general 1-nodal curve B_{pq} of arithmetic genus g satisfies: (1) $\overline{W}^r_d(B_{pq})$ is connected, reduced, and has the expected dimension $\rho(g,r,d)$; (2) $\widetilde{W}^r_d(B_{pq})$ is connected.*

Assuming the above proposition, we prove the main result of this section:

Theorem 4.6.3. *Let C be a smooth curve of genus g, which has a complete g^r_d, and such that $W^r_d(C)$ is reduced and of dimension $\rho(g,r,d)$. Let $\tilde{C} \to C$ be a connected étale double cover and let V_0, V_1 be the special subvarieties associated to the g^r_d. If $\rho(g,r,d) > 0$, then V_0 and V_1 are algebraically equivalent.*

Proof. The idea of the proof is taken from [IT]. If $\rho(g,r,d) \geq \min\{r+1, g-d+r\}$, then either $W^r_{d-1}(C)$ or $W^{r+1}_{d+1}(C)$ is nonempty [Ke, KL], and Proposition 4.6.1 applies. Thus, we may assume that $0 < \rho(g,r,d) < \min\{r+1, g-d+r\}$.

Let B_{pq} be a 1-nodal curve satisfying the conclusions of Proposition 4.6.2. We shall degenerate $\widetilde{W}^r_d(C)$ to $\widetilde{W}^r_d(B_{pq})$ and apply the principle of connectedness from Section 4.5.

Consider the flat family $\tau\colon \tilde{\mathcal{C}} \to \mathcal{C}$ over S and the associated family $\widetilde{\mathcal{W}}^r_d(\mathcal{C}) \to \mathcal{W}^r_d(\mathcal{C})$, as described before Proposition 4.6.2. The schemes $W^r_d(C)$ and $\overline{W}^r_d(B_{pq})$ are reduced, have dimension $\rho(g,r,d)$, and appear as fibers of $w_{\mathcal{C}}\colon \mathcal{W}^r_d(\mathcal{C}) \to S$. After shrinking S, if necessary, we may assume that all fibers of $w_{\mathcal{C}}$ are reduced and have dimension $\rho(g,r,d)$, hence $w_{\mathcal{C}}$ is flat by Proposition 4.5.1. This implies that the morphism $\widetilde{\mathcal{W}}^r_d(\mathcal{C}) \to S$ is also flat and all of its fibers are reduced. Since $\widetilde{W}^r_d(B_{pq})$ is connected, then by Proposition 4.5.2, so is $\widetilde{W}^r_d(C)$. As in the proof of Proposition 4.6.1, this implies that V_0 and V_1 are algebraically equivalent. $\square$

Before giving the proof of Proposition 4.6.2, we need two preliminary observations. First, let us show that $\widetilde{J}^d_{B_{pq}} \to \bar{J}^d_{B_{pq}}$ has a description analogous to the Wirtinger cover $\tilde{B}_{pq} \to B_{pq}$. Using the notation from Section 4.4, we see that the morphism $\kappa\colon P^d_{B_{pq}} \to \bar{J}^d_{B_{pq}}$ is the normalization. Let $\alpha\colon B_{pq} \to \bar{J}^d_{B_{pq}}$ be the composition of the Abel map (Section 4.3) followed by the morphism $\bar{J}^0_{B_{pq}} \to \bar{J}^d_{B_{pq}}$ of tensoring by $\mathcal{N}^d_{s_0}$. By the universal property of normalization, the composition $\alpha \circ \nu\colon B \to \bar{J}^d_{B_{pq}}$ induces a morphism $\tilde{\alpha}\colon B \to P^d_{B_{pq}}$, such that the diagram

$$\begin{tikzcd} B \arrow[r, "\tilde{\alpha}"] \arrow[d, "\nu"'] & P^d_{B_{pq}} \arrow[d, "\kappa"] \\ B_{pq} \arrow[r, "\alpha"'] & \bar{J}^d_{B_{pq}} \end{tikzcd}$$

commutes. Applying Pic^0, we get a commutative diagram:

$$\begin{tikzcd} \mathrm{Pic}^0_B & \mathrm{Pic}^0_{P^d_{B_{pq}}} \arrow[l, "\tilde{\alpha}^*"'] \\ \mathrm{Pic}^0_{B_{pq}} \arrow[u, "\nu^*"] & \mathrm{Pic}^0_{\bar{J}^d_{B_{pq}}} \arrow[l, "\alpha^*"] \arrow[u, "\kappa^*"'] \end{tikzcd} .$$

The composition $\lambda \circ \tilde{\alpha}\colon B \to P^d_{B_{pq}} \to J^d_B$ is an Abel map (see Section 4.4 for the definition of λ), which implies that $\tilde{\alpha}^*$ is injective. By the construction preceding Proposition 4.6.2, the invertible sheaves $\mathcal{L}_{s_0}$ and $\mathcal{M}_{s_0}$ that determine $\tilde{B}_{pq} \to B_{pq}$ and $\widetilde{J}^d_{B_{pq}} \to \bar{J}^d_{B_{pq}}$, respectively, satisfy $\alpha^*(\mathcal{M}_{s_0}) = \mathcal{L}_{s_0}$. Also, $\mathcal{L}_{s_0}$ induces the trivial double cover of B, i.e., $\nu^*(\mathcal{L}_{s_0}) = \mathcal{O}_B$. Therefore, using the commutativity of the above diagram and the injectivity of $\tilde{\alpha}^*$, we see that $\kappa^*(\mathcal{M}_{s_0}) = \mathcal{O}_{P^d_{B_{pq}}}$. This shows that there is a Cartesian square

$$\begin{tikzcd} P^d_{B_{pq}} \coprod P^d_{B_{pq}} \arrow[r] \arrow[d] & \widetilde{J}^d_{B_{pq}} \arrow[d] \\ P^d_{B_{pq}} \arrow[r, "\kappa"] & \bar{J}^d_{B_{pq}}, \end{tikzcd}$$

where the left vertical arrow is the trivial double cover. We conclude that $\widetilde{J}^d_{B_{pq}}$ is obtained from $P^d_{B_{pq}} \coprod P^d_{B_{pq}}$ by gluing $\mathrm{Im}(s_p)$ and $\mathrm{Im}(s_q)$ on one copy of $P^d_{B_{pq}}$ to $\mathrm{Im}(s_q)$ and $\mathrm{Im}(s_p)$, respectively, on the other copy (the gluing is with a shift, as in the case of $\bar{J}^d_{B_{pq}}$ described in Section 4.4).

Next, consider the induced double cover $\widetilde{W}^r_d(B_{pq}) \to \overline{W}^r_d(B_{pq})$. Let

$$W(p,q) := \kappa^{-1}\big(\overline{W}^r_d(B_{pq})\big) \qquad W(p) := \mathrm{Im}(s_p) \cap W(p,q) \qquad W(q) := \mathrm{Im}(s_q) \cap W(p,q),$$

where the preimage and the images are scheme-theoretic. From the above description of $\widetilde{J}^d_{B_{pq}}$ we see that $\widetilde{W}^r_d(B_{pq})$ is obtained from two copies of $W(p,q)$ by gluing $W(p)$ and $W(q)$ on one copy to $W(q)$ and $W(p)$, respectively, on the other copy. Therefore, to show that $\widetilde{W}^r_d(B_{pq})$ is connected, it suffices to prove that $W(p,q)$ is connected and $W(p)$, $W(q)$ are nonempty. Non-emptiness of $W(p)$ and $W(q)$ can be seen as follows. Since $\dim W^r_{d-1}(B) \geq \rho(g-1, r, d-1) = \rho(g,r,d) - 1 \geq 0$ (the last inequality holds by our ongoing assumption $\rho(g,r,d) > 0$) and $\partial\overline{W}^r_d(B_{pq}) = \{\nu_* M \mid M \in W^r_{d-1}(B)\}$, then $\partial\overline{W}^r_d(B_{pq})$ is nonempty, [Ke, KL]. It follows from Section 4.4 that κ maps each of $W(p)$ and $W(q)$ onto $\partial\overline{W}^r_d(B_{pq})$, which implies that $W(p)$ and $W(q)$ are nonempty. Connectedness of $W(p,q)$ (for a general 1-nodal curve B_{pq}) is much harder to show and it is the main point of the proof of Proposition 4.6.2

below. The idea is to let p and q come together and to consider the analogous locus $W(p,p)$ for the 1-cuspidal curve B_{pp}. Although the scheme $W(p,p)$ turns out to be connected, it is also non-reduced. Hence, we may have $h^0(\mathcal{O}_{W(p,p)}) > 1$, and therefore, we cannot apply the principle of connectedness directly to conclude that $W(p,q)$ is connected. Furthermore, it is not clear a priori that the specialization $W(p,q) \rightsquigarrow W(p,p)$ is flat. To overcome these difficulties, we shall use the determinantal loci $Y(x,y)$, which are introduced in the next paragraph.

Let us introduce two last bits of notation, state two lemmas (whose proofs are at the end of the section), and give the proof of Proposition 4.6.2. First, by [EH], if X is an integral curve with double points at worst, there is a scheme $\overline{G}^r_d(X)$ parametrizing pairs (L,V) such that $L \in \overline{W}^r_d(X)$ and $V \subset H^0(X,L)$ is a subspace of dimension $r+1$. There is a forgetful morphism $\overline{G}^r_d(X) \to \overline{W}^r_d(X)$ and a subscheme $G^r_d(X) \subset \overline{G}^r_d(X)$ pararmetrizing pairs (L,V) with L locally free. Second, given points $x, y \in B$ (not necessarily distinct), define

$$Y(x,y) := \{M \in J^d_B \mid h^0(M) \geq r+1 \text{ and } h^0(M(-x-y)) \geq r\}$$

as a subscheme of J^d_B with its natural structure of a determinantal locus (see Appendix to this section for details), whose expected dimension can be computed using [Fu, Ch.14.3, p.249] and is equal to $\rho(g,r,d)$.

Lemma 4.6.4. *If $\rho(g,r,d) < \min\{r+1, g+r-d\}$, then for a general 1-cuspidal curve B_{pp} of arithmetic genus g, the scheme $Y(p,p)$ is connected, reduced, and of dimension $\rho(g,r,d)$.*

Lemma 4.6.5. *If both $W^{r+1}_d(B)$ and $W^r_{d-2}(B)$ are empty, then for any distinct points $p, q \in B$, the underlying topological spaces of the schemes $Y(p,q)$ and $W(p,q)$ are homeomorphic.*

Proof of Prop.4.6.2. Say that an integral curve X of arithmetic genus g has property $\mathcal{P}$, if $\overline{W}^r_d(X)$ is connected, reduced, and of dimension $\rho(g,r,d)$. Note that property $\mathcal{P}$ is open and depends on the integers r, d. By [EH, Thm.4.5 and Sec.9], there exists a rational g-cuspidal curve X for which $\overline{G}^r_d(X)$ is connected, reduced, and of the expected dimension, which implies that $\mathcal{P}$ holds for X. Since a 1-nodal curve can be specialized to a g-cuspidal rational curve, then $\mathcal{P}$ holds for a general 1-nodal curve. This proves part (1) of the proposition.

Connectedness of $\widetilde{W}^r_d$ is an open property of 1-nodal curves. Therefore, to prove part (2) of the proposition, it suffices to exhibit a single 1-nodal curve B_{pq} such that $\widetilde{W}^r_d(B_{pq})$ is connected. This will be done using 1-cuspidal curves. By Lemma 4.6.4, there exists a 1-cuspidal curve B_{pp} such that $Y(p,p)$ is connected, reduced, and of dimension $\rho(g,r,d)$. Moreover, by the proof of Lemma 4.6.4, we may assume that both $W^{r+1}_d(B)$ and $W^r_{d-2}(B)$ are empty. Let us show that there exists a point $q \in B\backslash\{p\}$ such that $Y(p,q)$ is connected. Take a smooth, connected but not necessarily complete, curve T parametrizing divisors $\{p + q_t \,|\, t \in T\}$ on B such that $q_t \neq p$ for all $t \in T\backslash\{t_0\}$ and $q_{t_0} = p$. Consider a family $\mathscr{Y} \to T$, such that the fiber over a point $t \in T$ is $Y(p, q_t)$. Since $Y(p,p)$ is connected, reduced, and of the expected dimension $\rho(g,r,d)$, we may replace T with a Zariski open neighborhood of t_0, if necessary, to ensure that $\mathscr{Y} \to T$ is flat. By the principle of connectedness, this implies that $Y(p, q_t)$ is connected for each point $t \in T$.

Now, let us fix a point $q \in B\backslash\{p\}$ such that $Y(p,q)$ is connected and consider the 1-nodal curve B_{pq}. Since both $W^{r+1}_d(B)$ and $W^r_{d-2}(B)$ are empty, then by Lemma 4.6.5, the schemes $W(p,q)$ and $Y(p,q)$ have homeomorphic underlying topological spaces, hence $W(p,q)$ is connected. As described before the proof, $\widetilde{W}^r_d(B_{pq})$ is obtained by gluing two copies of $W(p,q)$, which implies that $\widetilde{W}^r_d(B_{pq})$ is connected. □

Proof of Lemma 4.6.4. Say that a smooth curve B of genus $g-1$ has property $\mathcal{E}$, if $W^{r+1}_d(B) = W^r_{d-2}(B) = \emptyset$. Note that $\mathcal{E}$ is an open property and depends on the integers r, d. Since $\rho(g,r,d) < \min\{r+1, g+r-d\}$, then both $\rho(g-1, r+1, d)$ and $\rho(g-1, r, d-2)$ are negative. Therefore, by [ACGH, Thm.1.5, p.214], $\mathcal{E}$ holds for a general smooth curve of genus $g-1$. By [EH, Thm.4.5 and Sec.9], there exists a rational g-cuspidal curve X for which

$\overline{G}^r_d(X)$ is connected, reduced, and of the expected dimension. It follows that for a general 1-cuspidal curve B_{pp} of arithmetic genus g the scheme $\overline{G}^r_d(B_{pp})$ is connected, reduced, and of dimension $\rho(g,r,d)$.

For the remainder of the proof let us fix one such curve B_{pp} such that its normalization B has property $\mathcal{E}$. In particular, $W^{r+1}_d(B) = \emptyset$, which implies that $\overline{W}^r_d(B_{pp}) = \overline{G}^r_d(B_{pp})$. By Section 4.4, there is a diagram

$$\begin{array}{ccc} P^d_{B_{pp}} & \xrightarrow{\kappa} & \bar{J}^d_{B_{pp}} \\ {\scriptstyle\lambda}\downarrow & & \\ J^d_B, & & \end{array}$$

where κ is proper, birational and bijective (hence, a homeomorphism on the underlying topological spaces). Let $W(p,p)$ be the scheme-theoretic preimage $\kappa^{-1}(\overline{W}^r_d(B_{pp}))$, i.e., $W(p,p)$ is the fiber product of $P^d_{B_{pp}}$ and $\overline{W}^r_d(B_{pp})$ over $\bar{J}^d_{B_{pp}}$. Since κ is birational and bijective and $\overline{W}^r_d(B_{pp})$ is connected, reduced, and of dimension $\rho(g,r,d)$, then $W(p,p)$ is connected, generically reduced, and of dimension $\rho(g,r,d)$. Note that $W(p,p)$ is non-reduced along $\kappa^{-1}(\partial\overline{W}^r_d(B_{pp}))$. The scheme $Y(p,p)$ is a determinantal locus whose expected dimension is $\rho(g,r,d)$, and therefore, $\dim Y(p,p) \geq \rho(g,r,d)$, if $Y(p,p)$ is nonempty. In the paragraph below, we shall show that $Y(p,p)$ is the set-theoretic image of $W(p,p)$ under λ. This implies that $Y(p,p)$ is connected and of dimension $\rho(g,r,d)$. Moreover, being a determinantal locus of the expected dimension, $Y(p,p)$ is Cohen-Macaulay, [Fu, Thm.14.3c, p.250], and therefore, has no embedded components. Thus, to show that $Y(p,p)$ is reduced, it remains to prove that $Y(p,p)$ is generically reduced. In [EH, Sec.4], it is shown that the G^r_d of a cuspidal curve X is closely related to a certain determinantal locus in a Grassmann bundle over the scheme of linear series of the normalization of X. In the case of 1-cuspidal curves this determinantal locus is the scheme $Y(p,p)$. In particular, using [EH, Thm.4.1, p.388 and Remark on p.389], it is easy to see that the morphism $G^r_d(B_{pp}) \to Y(p,p)$, given by $L \mapsto \nu^*L$, is birational. Since $G^r_d(B_{pp})$ is reduced, this shows that $Y(p,p)$ is generically reduced.

To complete the proof, let us check the set-theoretic equality $\lambda(W(p,p)) = Y(p,p)$. Let sk be the sky-scraper sheaf on B_{pp} supported at the cusp with fiber $\mathbb{C}$. If $L \in \overline{W}^r_d(B_{pp})$ and L is invertible, then $\kappa^{-1}(L)$ is a single reduced point $(L \hookrightarrow \nu_*\nu^*L) \in W(p,p)$, whose image in J^d_B is ν^*L. Consider the short exact sequence $0 \to L \to \nu_*\nu^*L \to \mathrm{sk} \to 0$ and the associated long exact sequence

$$0 \longrightarrow H^0(L) \longrightarrow H^0(\nu^*L) \xrightarrow{\beta} \mathbb{C} \longrightarrow \cdots$$

Since $W^{r+1}_d(B) = \emptyset$, then $h^0(\nu^*L) = r+1$ and β is the zero map. Therefore, the linear system $|L|$ pulls-back to the *complete* linear system $|\nu^*L|$, which implies that $h^0(\nu^*(L)(-2p)) = r$. Hence, $\nu^*L \in Y(p,p)$. If $L \in \partial\overline{W}^r_d(B_{pp})$, let $M := (\nu^*(L)/torsion) \otimes \mathcal{O}_B(p)$. In this case we have $L \simeq \nu_*(M(-p))$, [Al, Lem.1.5], and there is a natural presentation $\nu_*(M(-p)) \hookrightarrow \nu_*M$. The fiber $\kappa^{-1}(L)$ is the point $(\nu_*(M(-p)) \hookrightarrow \nu_*M) \in W(p,p)$ with multiplicity 2 (see Section 4.4), whose image in J^d_B is M. Since $h^0(L) \geq r+1$ and $W^{r+1}_d(B) = \emptyset$, the inclusion $L \simeq \nu_*(M(-p)) \hookrightarrow \nu_*M$ induces an isomorphism $H^0(M(-p)) \simeq H^0(M)$. Therefore, $h^0(M) = r+1$ and $h^0(M(-2p)) \geq r$, hence $M \in Y(p,p)$. $\square$

Proof of Lemma 4.6.5. Let sk be the sky-scraper sheaf on B_{pq} supported at the node with fiber $\mathbb{C}$. By Section 4.4, the presentation scheme $P^d_{B_{pq}}$ fits into the diagram

$$\begin{tikzcd} P^d_{B_{pq}} \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & \bar{J}^d_{B_{pq}} \\ J^d_B, & \end{tikzcd}$$

where λ is a $\mathbb{P}^1$-bundle and κ is the normalization morphism.

To prove the lemma, we shall show that λ restricts to a bijective morphism $\bar{\lambda}\colon W(p,q) \to Y(p,q)$. Since $\bar{\lambda}$ is also proper, this will imply that $\bar{\lambda}$ induces a homeomorphism on the underlying topological spaces.

First, let us check that $\lambda(W(p,q)) = Y(p,q)$. Recall from Section 4.4 that λ sends a presentation $(L \hookrightarrow \nu_* M)$ to $M \in J_B^d$. If $L \in \overline{W}^r_d(B_{pq})$ and L is invertible, then $\kappa^{-1}(L)$ is a single reduced point $(L \hookrightarrow \nu_*\nu^* L) \in W(p,q)$. Consider the short exact sequence $0 \to L \to \nu_*\nu^*L \to \mathrm{sk} \to 0$ and the associated long exact sequence

$$0 \longrightarrow H^0(L) \longrightarrow H^0(\nu^*L) \xrightarrow{\beta} \mathbb{C} \longrightarrow \cdots$$

Since $W_d^{r+1}(B) = \emptyset$, then $h^0(\nu^*L) = r+1$ and β is the zero map. Therefore, the linear system $|L|$ pulls-back to the *complete* linear system $|\nu^*L|$, which implies that $h^0(\nu^*(L)(-p-q)) = r$, see also [Ca09, Rem.2.2.1, p.1397]. Hence, $\nu^*L \in Y(p,q)$. If $L \in \partial\overline{W}^r_d(B_{pq})$, then $\kappa^{-1}(L)$ consists of two reduced points: $(\nu_*M(-p) \hookrightarrow \nu_*M)$ and $(\nu_*M'(-q) \hookrightarrow \nu_*M')$, where $M = (\nu^*(L)/torsion)\otimes\mathcal{O}_B(p)$ and $M' = M\otimes\mathcal{O}_B(q-p)$, see Section 4.4. Note that $L \simeq \nu_*M(-p) \simeq \nu_*M'(-q)$, which implies that $h^0(M), h^0(M') \geq r+1$ and $h^0(M(-p-q)), h^0(M'(-p-q)) \geq r$. Hence, $M, M' \in Y(p,q)$. This shows that $\lambda(W(p,q)) = Y(p,q)$.

Second, let us show that $\bar{\lambda}$ is bijective. Since $W_d^{r+1}(B)$ and $W_{d-2}^r(B)$ are empty, then for each $M \in Y(p,q)$, we have $h^0(M) = r+1$ and $h^0(M(-p-q)) = r$. As a consequence, there is a *unique* morphism $h\colon \nu_*M \twoheadrightarrow \mathrm{sk}$, which induces the zero map on global sections (if p and q are not base points of M, this also follows from [Ca09, Lem.5.1.3(2), p.1420]). The sheaf $L_M := \ker(h)$ has $h^0(L_M) = r+1$, and therefore, $L_M \in \overline{W}^r_d(B_{pq})$. From the description of $\bar{\lambda}$ given above, we see that the assignment $M \mapsto (L_M \hookrightarrow \nu_*M)$ is a set-theoretic inverse of $\bar{\lambda}$, which shows that $\bar{\lambda}$ is bijective. $\square$

Remark 4.6.6. The special subvarieties are not always algebraically equivalent, as is shown in the following example. In the case of a trigonal curve C, the special subvarieties V_0 and V_1 associated to a g^1_3 on C are related by $V_1 = (-1)^*V_0$ (after an appropriate translation) and P is isomorphic to the Jacobian of V_0, which has a g^1_4, see [Be82, p.360 and p.366; Re]. Since $\rho(g,1,4) = 6-g$, then all curves of genus at most 6 admit a g^1_4. It follows from [Ce] and [Re] that V_0 and V_1 are not algebraically equivalent on a generic "trigonal" Prym of dimension $3 \leq p = g-1 \leq 6$. Note that $\rho(g,1,3) = 4-g \leq 0$, whenever $p \geq 3$.

4.7 Appendix

Fix integers g, r, d such that $0 < 2r < d < 2g$ and let B be a smooth curve of genus $g-1$. For any two points (not necessarily distinct) $x, y \in B$, we have considered the locus

$$Y(x,y) = \{M \in J_B^d \,|\, h^0(M) \geq r+1 \text{ and } h^0(M(-x-y)) \geq r\}.$$

The purpose of this appendix is to show that $Y(x,y)$ carries a natural scheme structure of a determinantal locus in J_B^d. Such loci are sometimes also referred to as Schubert loci or Schubert varieties. A reference for their theory is [Fu, Ch.14]. We shall use the following notation:

- $\mathscr{L}$:= a Poincaré sheaf on $B \times J_B^d$,
- D_m := an effective divisor of degree $m \geq 2(g-1) - d + r - 1$,
- $\Gamma := D_m \times J_B^d$ considered as a divisor on $B \times J_B^d$,
- $\mathscr{L}(\Gamma) := \mathscr{L} \otimes \mathcal{O}_{B \times J_B^d}(\Gamma)$,
- $\phi\colon B \times J_B^d \to J_B^d$ is the second projection.

When there is no danger of confusion, given a divisor D on B, we let $\mathscr{L}(D) := \mathscr{L} \otimes$(pull-back of $\mathcal{O}_B(D)$) and $\mathscr{L}(\Gamma + D) := \mathscr{L}(D) \otimes \mathcal{O}_{B \times J_B^d}(\Gamma)$. Our assumption $m \geq 2(g-1) - d + r - 1$ and the Riemann-Roch formula imply that for any $1 \leq j \leq r-1$ and any points $p_1, \ldots, p_{r-1} \in B$, the sheaf $\phi_* \mathscr{L}(\Gamma - p_1 - \cdots - p_j)$ is locally free of rank $d + m - j - g + 2$ on J_B^d and $R^1 \phi_* \mathscr{L}(\Gamma - p_1 - \cdots - p_j) = 0$.

Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \phi_*\mathscr{L} \arrow[r] & \phi_*\mathscr{L}(\Gamma) \arrow[r, "\sigma"] & \phi_*(\mathscr{L}(\Gamma)/\mathscr{L}) \arrow[r] & R^1\phi_*\mathscr{L} \arrow[r] & 0 \\
0 \arrow[r] & \phi_*\mathscr{L}(-x-y) \arrow[r] \arrow[u] & \phi_*\mathscr{L}(\Gamma-x-y) \arrow[r, "\bar{\sigma}"] \arrow[u, "\alpha"] & \phi_*\left(\frac{\mathscr{L}(\Gamma-x-y)}{\mathscr{L}(-x-y)}\right) \arrow[r] \arrow[u, "\beta"] & R^1\phi_*\mathscr{L}(-x-y) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

It is easy to check that α is injective and β is an isomorphism. Hence, α induces an injection from $\ker(\bar{\sigma})$ to $\ker(\sigma)$.

Put $E := \phi_*\mathscr{L}(\Gamma)$ and $F := \phi_*(\mathscr{L}(\Gamma)/\mathscr{L})$. To set up a determinantal locus as in [Fu, p.243] we shall construct a flag $A_1 \subset A_2 \subset \ldots \subset A_r \subset A_{r+1} \subset E$. Put $A_{r+1} := E$ and $A_r := \phi_*\mathscr{L}(\Gamma - x - y)$. Fix points $p_1, \ldots, p_{r-1} \in B$ and for $1 \leq i \leq r-1$, put $A_{r-i} := \phi_*\mathscr{L}(\Gamma - x - y - p_1 - \cdots - p_i)$. Thus, we get a flag

$$A_1 = \phi_*\mathscr{L}(\Gamma - x - y - p_1 - \cdots - p_{r-1}) \subset \ldots \subset A_{r-1} = \phi_*\mathscr{L}(\Gamma - x - y - p_1) \subset A_r = \phi_*\mathscr{L}(\Gamma - x - y)$$

of locally free subsheaves of E. Let $a_i := \mathrm{rk}(A_i)$. Using Riemann-Roch, we may compute: $a_{r+1} = \mathrm{rk}(E) = d + m - g + 2$ (recall that m is the degree of the fixed divisor D_m on B), $a_r = a_{r+1} - 2$, and $a_i = a_{r+1} - 2 - (r - i)$ for $1 \leq i \leq r-1$. Note that the rank of F is m. For $M \in J^d_B$, let $A_i(M)$ be the fiber of A_i over M and let $\sigma(M)\colon E(M) \to F(M)$ be the morphism on the fibers induced by $\sigma\colon E \to F$. Consider the determinantal locus

$$\Omega(\underline{A}; \sigma) = \{M \in J^d_B \mid \dim(\ker(\sigma(M)) \cap A_i(M)) \geq i,\ 1 \leq i \leq r+1\}.$$

For $1 \leq i \leq r-1$, the condition $\dim(\ker(\sigma(M)) \cap A_i(M)) \geq i$ is a consequence of $\dim(\ker(\sigma(M)) \cap A_r(M)) \geq r$, because the ranks of the A_i's go down by one starting with $i = r$. Let $\sigma_i\colon A_i \to F$ be composition of the inclusion $A_i \hookrightarrow E$ followed by $\sigma\colon E \to F$.

Using the calculation on the bottom of p.178 in [ACGH] for the 4th equality below, we have:

$$\begin{aligned}
\Omega(\underline{A};\sigma) &= \{M \in J_B^d \,|\, \dim(\ker(\sigma(M)) \cap A_i(M)) \geq i, \;\text{ for } i \in \{r, r+1\}\} \\
&= \{M \in J_B^d \,|\, \dim(\ker(\sigma_i(M))) \geq i, \;\text{ for } i \in \{r, r+1\}\} \\
&= \{M \in J_B^d \,|\, \operatorname{rank}(\sigma_{r+1}(M)) \leq d + m - (g-1) - r \text{ and } \operatorname{rank}(\sigma_r(M)) \leq d + m - g - r\} \\
&= \{M \in J_B^d \,|\, h^0(M) \geq r+1 \text{ and } h^0(M(-x-y)) \geq r\} \\
&= Y(x, y).
\end{aligned}$$

This defines the scheme structure of a determinantal locus on $Y(x,y)$. By [Fu, Thm.14.3, p.249], the expected codimension of $Y(x,y) = \Omega(\underline{A};\sigma)$ in J_B^d is $\sum_{i=1}^{r+1} m - a_i + i$. For $1 \leq i \leq r$,

$$\begin{aligned}
m - a_i + i = m - (n - 2 - (r - i)) + i &= m - n + r + 2 \\
&= m - (d + m - g + 2) + r + 2 = g - d + r,
\end{aligned}$$

and therefore,

$$\begin{aligned}
\sum_{i=1}^{r+1} m - a_i + i = \sum_{i=1}^{r} (m - a_i + i) &+ (m - a_{r+1} + r + 1) \\
&= r(g - d + r) + m - n + r + 1 = r(g - d + r) + m - (d + m - g + 2) + r + 1 \\
&= (r+1)(g - d + r) - 1.
\end{aligned}$$

Hence, the expected dimension of $Y(x,y) = g - 1 - ((r+1)(g-d+r) - 1) = \rho(g, r, d)$.

Chapter 5

Classes of special subvarieties

5.1 Beauville-Poincaré formulas

Let (J, Θ) be a principally polarized Jacobian of a smooth curve C of genus g. Let $[C]$ and $[\Theta]$ be the classes of the Abel curve and the theta divisor, respectively. The following classical formula in $H^*(J, \mathbb{Z})$ is due to Poincaré:

$$[\Theta]^{\cdot p}\frac{1}{p!} = \frac{1}{(g-p)!}[C]^{*(g-p)}$$

for $1 \leq p \leq g$.

For quite a while it was not known whether the above formula holds in $A(J)$. In [Co75, Cor.4, p.97], it has been shown that the Poincaré formula does hold in $A(J)$, if C is hyperelliptic. This also follows from the result of Colombo and van Geemen [CvG, Thm.1.3(3)], as observed in [Be04, p.687].

On the other hand, if C is a generic curve of genus $g \geq 3$, then C is non-hyperelliptic. In fact, the result of Ceresa [Ce] implies that for such a curve $[C]_{(1)} \neq 0$ in $A(J)$. Therefore, $[C]$ is not contained in the subring of $A(J)$ generated by the class of the theta divisor, because

$[\Theta]^{\cdot p} \in A^p(J)_{(0)}$ for all $0 \leq p \leq g$. In particular, Poincaré formulas do not hold for generic curves of genus $g \geq 3$ even modulo algebraic equivalence.

Nevertheless, Beauville has found certain formulas in $\mathrm{CH}(J)$, which generalize Poincaré formulas in cohomology, [Be83]. These formulas hold verbatim in $A(X)$. Let L be an ample symmetric invertible sheaf on X (e.g., $L = \mathcal{O}_X(\Theta)$), $\nu := h^0(L) = \frac{L^{\cdot g}}{g!}$, and $c_L := \frac{L^{\cdot(g-1)}}{\nu(g-1)!}$.

Theorem 5.1.1 (Beauville-Poincaré formula)**.** *For all* $0 \leq p \leq g$*,*

$$\frac{L^{\cdot p}}{p!} = \nu \frac{c_L^{*(g-p)}}{(g-p)!} \in \mathrm{CH}^p(X).$$

Proof. See either [Be83, Cor.2, p.249] or [BL, 16.5.6, p.537]. □

We shall use is the following corollary.

Corollary 5.1.2 (Beauville, 1983)**.** *For* $p, q \geq 0$ *we have the following identity in* $\mathrm{CH}^{p+q-g}(X)$*:*

$$\frac{L^{\cdot p}}{p!} * \frac{L^{\cdot q}}{q!} = \nu \binom{2g-p-q}{g-p} \frac{L^{\cdot(p+q-g)}}{(p+q-g)!}.$$

Proof. As in [BL, p.538], we use Beauville-Poincaré formula twice:

$$\begin{aligned}
\frac{L^{\cdot p}}{p!} * \frac{L^{\cdot q}}{q!} &= \nu^2 \frac{c_L^{*(g-p)}}{(g-p)!} * \frac{c_L^{*(g-q)}}{(g-q)!} \\
&= \nu^2 \binom{2g-p-q}{g-p} \frac{c_L^{*(2g-p-q)}}{(2g-p-q)!} \\
&= \nu \binom{2g-p-q}{g-p} \frac{L^{\cdot(p+q-g)}}{(p+q-g)!}.
\end{aligned}$$

□

5.2 Classes of special subvarieties in $\mathscr{T}(P, \psi(\tilde{C}))$

In this section we assume that $\pi\colon \tilde{C} \to C$ is étale and recall that in this case $\dim P = p = g - 1$. Let us fix two integers r and d such that $0 < 2r < d < 2g$. Throughout this section the letters n and m will denote r-tuples of non-negative integers $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_r)$, respectively. As in Section 4.2, $\tilde{\varphi}_d\colon \tilde{C}_d \to \tilde{J}$, $\varphi_d\colon C_d \to J$ are abelian sum mappings and $\pi_d\colon \tilde{C}_d \to C_d$ is the map induced by $\pi\colon \tilde{C} \to C$. Also, let $|n| := \sum_{j=1}^r n_j$ and $\mu_n := \prod_{j=1}^r \frac{(-1)^{n_j - 1}}{n_j}$.

For any pair n, m of r-tuples with $1 \le n_1 \le \ldots \le n_r$, $\sum_{j=1}^r n_j \le d$, and $0 \le m_j \le n_j/2$ for all j, define the following numbers:

1. $\nu_{n,m}$: For each $\ell \ge 1$ let $q(\ell)$ be the number of n_j's that are equal to ℓ and suppose that $n_{j_1} = n_{j_2} = \ldots = n_{j_{q(\ell)}} = \ell$. If $q(\ell)$ is not zero, let $p(\ell, n, m)$ be the number of permutations of the ordered $q(\ell)$-tuple $(m_{j_1}, m_{j_2}, \ldots, m_{j_{q(\ell)}})$, and otherwise let $p(\ell, n, m) = 1$. Then we set

$$\nu_{n,m} := \prod_{\ell=1}^{d-r+1} \frac{1}{p(\ell, n, m)}.$$

Note that if $r = 1$, then $\nu_{n,m} = 1$ for all n, m. If $r = 2$, then $\nu_{n,m} = 1/2$ if $n_1 = n_2$ and $m_1 \neq m_2$, and $\nu_{n,m} = 1$ otherwise.

2. $\lambda_{n,m}$: This is the number

$$\lambda_{n,m} := 2^{d-|n|} \cdot \mu_n \cdot \nu_{n,m} \cdot \binom{d}{|n|}\binom{n_1}{m_1} \cdots \binom{n_r}{m_r}.$$

3. $d_{n,m}$: Let $e_1, \ldots, e_k$ count the number of repeats in the sequence of pairs $(n_1, m_1), \ldots, (n_r, m_r)$. For example, if the sequence of pairs is

$$(1,2), (1,2), (2,5), (2,3), (2,3), (2,3), (7,5), (3,3), (3,3), (3,3),$$

then the associated sequence of repeats is $2, 1, 3, 1, 3$. Let us define

$$d_{n,m} := e_1!e_2! \cdots e_k!.$$

Theorem 5.2.1. *Let $0 < 2r < d < 2g$ and let $V = V_0 \cup V_1$ be the union of special subvarieties of P associated to a complete and base point free g^r_d on C. The component of the class $[V]$ in $A^{p-r}(P)_{(t)}$ is given by the formula*

$$[V]_{(t)} = c_{t,r,d}\big([\psi(\tilde{C})]^{*r}\big)_{(t)}$$

where

$$c_{t,r,d} := 2^{-2r-t} \sum_{n} \sum_{m \leq \frac{n}{2}} \frac{\lambda_{n,m}}{d_{n,m}} \prod_{j=1}^{r} (n_j - 2m_j)^{t+2},$$

the outer sum is taken over the choices of r-tuples $n = (n_1, \ldots, n_r)$ of integers with $1 \leq n_1 \leq \ldots \leq n_r$ and $\sum_{j=1}^{r} n_j \leq d$, the inner sum is taken over the choices of r-tuples $m = (m_1, \ldots, m_r)$ of integers with $0 \leq m_j \leq \frac{n_j}{2}$ for all j.

Proof. Let G_d denote the complete and base-point-free g^r_d on C, considered as a subvariety of C_d isomorphic to $\mathbb{P}^r$. Given an r-tuple n of positive integers, consider the generalized diagonal

$$\delta_n = \Big\{ n_1x_1 + n_2x_2 + \cdots + n_rx_r \,|\, x_1, \ldots, x_r \in C \Big\}$$

in the $|n|$-fold symmetric product of C. Let $D \in G_d$ be a fixed effective divisor, whose support consists of d distinct points. In [He, Thm.3, p.888] we may find the following formula for the class $[G_d] \in \mathrm{CH}^{d-r}(C_d)$:

$$[G_d] = \sum_{n, \underline{o}_s} \mu_n[\delta_n + o_1 + \cdots + o_s],$$

where $s = s(n) := d - |n| \geq 0$ and the sum is taken over all r-tuples n with $1 \leq n_1 \leq \ldots \leq n_r$ and the choices of (unordered) sums $\underline{o}_s := o_1 + \cdots + o_s$ of pairwise distinct points in the support of the fixed divisor D.

In order to compute $\pi_d^*[G_d]$, for each pair n, m of r-tuples with $m_j \leq n_j$ for all j, we introduce the *modified generalized diagonals*

$$\tilde{\delta}_{n,m} := \Big\{ \sum_{j=1}^{r} m_j \iota(\tilde{x}_j) + (n_j - m_j)\tilde{x}_j \mid \tilde{x}_1, \ldots, \tilde{x}_r \in \tilde{C} \Big\},$$

considered as subvarieties of the $|n|$-fold symmetric product of $\tilde{C}$. We may check that

$$\pi_d^*[\delta_n + o_1 + \cdots + o_s] = \sum_{m, \underline{\tilde{u}}_s} \nu_{n,m} \binom{n_1}{m_1} \cdots \binom{n_r}{m_r} \big[\tilde{\delta}_{n,m} + \tilde{u}_1 + \cdots + \tilde{u}_s\big],$$

where the sum is taken over all r-tuples m with $0 \leq m_j \leq \frac{n_j}{2}$ for all j and all (unordered) sums $\underline{\tilde{u}}_s = \tilde{u}_1 + \cdots + \tilde{u}_s$ with $\tilde{u}_j \in \pi^{-1}(o_j) = \{\tilde{o}_j, \iota(\tilde{o}_j)\}$. To explain the number $\nu_{n,m}$, note that two ordered indexing pairs n, m and n, m' with $m \neq m'$ may label the same generalized diagonal. The number $\nu_{n,m}$ is the adjustment for this redundancy. Passing to $A(\tilde{C}_d)$, the formula for the pull-back of a generalized diagonal becomes

$$\pi_d^*[\delta_n + o_1 + \cdots + o_s] = \sum_{m \leq \frac{n}{2}} 2^{d-|n|} \nu_{n,m} \binom{n_1}{m_1} \cdots \binom{n_r}{m_r} \big[\tilde{\delta}_{n,m} + (d - |n|)\tilde{o}\big]$$

and we get:

$$\pi_d^*[G_d] = \sum_{n} \sum_{m \leq \frac{n}{2}} \lambda_{n,m} \big[\tilde{\delta}_{n,m} + (d - |n|)\tilde{o}\big]. \tag{5.2.1}$$

The abelian sum mapping $\tilde{\varphi}_d$ maps the subvariety $\tilde{\delta}_{n,m} + (d - |n|)\tilde{o}$ of $\tilde{C}_d$ bijectively onto a translate of the variety $(m_1\iota + n_1 - m_1)(\tilde{C}) + \cdots + (m_r\iota + n_r - m_r)(\tilde{C})$, where $\tilde{C} \subset \tilde{J}$ also denotes the Abel curve and $(m_j\iota + n_j - m_j)$ is viewed as an endomorphism of $\tilde{J}$. The number $d_{n,m}$ computes the degree of the addition morphism

$$(m_1\iota + n_1 - m_1)(\tilde{C}) \times \cdots \times (m_r\iota + n_r - m_r)(\tilde{C}) \to \sum_{j=1}^{r}(m_j\iota + n_j - m_j)(\tilde{C}),$$

and therefore, by the definition of Pontryagin product we have:

$$(m_1\iota + n_1 - m_1)_*[\tilde{C}] * \cdots * (m_r\iota + n_r - m_r)_*[\tilde{C}] = d_{n,m}\Big[\sum_{j=1}^{r}(m_j\iota + n_j - m_j)(\tilde{C})\Big],$$

which implies the formula:

$$u_*\tilde{\varphi}_{d*}[\tilde{\delta}_{n,m} + (d - |n|)\tilde{o}] = \frac{1}{d_{n,m}}(n_1 - 2m_1)_*[\psi(\tilde{C})] * \cdots * (n_r - 2m_r)_*[\psi(\tilde{C})]. \tag{5.2.2}$$

Since the composition of $P \hookrightarrow \tilde{J}$ with $u\colon \tilde{J} \to P$ is multiplication by 2, then by (5.2.1) and (5.2.2) we have the following identity in $A(P)$:

$$2_*[V] = \sum_{n}\sum_{m \leq \frac{n}{2}} \frac{\lambda_{n,m}}{d_{n,m}}(n_1 - 2m_1)_*[\psi(\tilde{C})] * \cdots * (n_r - 2m_r)_*[\psi(\tilde{C})].$$

We may extract the formulas for the homogeneous components of $[V]$ by recalling that $k_*x = k^{2(p-l)+t}x$ for $x \in A^l(P)_{(t)}$. □

In general, there is no canonical way of distinguishing V_0 from V_1, and consequently, we may not extract the formulas for $[V_0]$ and $[V_1]$ from the formula for $[V]$ in a direct way. In particular, we do not know in general if $[V_i] \in \mathscr{T}(P, \psi(\tilde{C}))$. However, when V_0 and V_1 are algebraically equivalent, $[V_0] = [V_1]$ in $A(P)$ and we easily obtain a formula for their classes: $[V_0] = [V_1] = \frac{1}{2}[V] \in \mathscr{T}(P, \psi(\tilde{C}))$.

5.3 Examples

First, we note that even when $c_{t,r,d} \neq 0$, the cycle $([\psi(\tilde{C})]^{*r})_{(t)}$ may itself be zero, e.g., when t is odd and $r = 1$, see the proof of Theorem 3.2.3.

Example 1. When G_d is a g_d^1 (the case $r = 1$), the formula in Theorem 5.2.1 reads

$$[V]_{(t)} = \sum_{n=1}^{d} \sum_{m \leq \frac{n}{2}} \frac{(-1)^{n-1} 2^{d-n-t-2}}{n} \binom{d}{n} \binom{n}{m} (n-2m)^{2+t} [\psi(\tilde{C})]_{(t)}. \tag{5.3.1}$$

In particular,

$$[V]_{(0)} = 2^{d-3} [\psi(\tilde{C})]_{(0)} = 2^{d-2} \cdot \frac{\xi^{g-2}}{(g-2)!},$$

which corresponds to the singular cohomology class of V and agrees with the formula in [Be82, Thm.1, p.364]. When $t = 2s \geq 0$ is even, using the package `ekhad` from [PWZ] for the software system `maple` and the resource [OEIS], it appears that

$$c_{2s,1,d} = \frac{(4^{s+1} - 1) B_{2s+2}}{s+1} \cdot 2^{d-2},$$

where B_m is the m^{th} Bernoulli number, defined by $\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$. In particular, the coefficient $c_{t,1,d}$ is non-zero when t is even. When t is odd, a closed formula for $c_{t,1,d}$ can also be found using the package `ekhad`, but the formula that we obtained was very bulky, and therefore, we do not include it here. In any case, when t is odd, the class $[\psi(\tilde{C})]_{(t)}$ is zero.

Example 2. When G_d is a g^3_7, we may check that

$$\begin{aligned}2_*[V] = \ & \tfrac{821}{6}Z^{*3} - 84Z^{*2} * 2_*Z + \tfrac{89}{6}Z^{*2} * 3_*Z - \tfrac{7}{4}Z^{*2} * 4_*Z + \tfrac{1}{10}Z^{*2} * 5_*Z \\ & + \tfrac{89}{8}Z * (2_*Z)^{*2} - \tfrac{7}{3}Z * 2_*Z * 3_*Z + \tfrac{1}{8}Z * 2_*Z * 4_*Z + \tfrac{1}{18}Z * (3_*Z)^{*2} \\ & - \tfrac{7}{24}(2_*Z)^{*3} + \tfrac{1}{24}(2_*Z)^{*2} * 3_*Z,\end{aligned}$$

where $Z = [\psi(\tilde{C})]$. Using the identification $[\psi(\tilde{C})]_{(0)} = 2 \cdot \frac{\xi^{g-2}}{(g-2)!}$ and Corollary 5.1.2, an elementary calculation shows that $[V]_{(0)} = 2 \cdot \frac{\xi^{g-4}}{(g-4)!}$, which agrees with the formula in [Be82].

By the proof of [Fa, Thm.4.5, p.117], we know that for a generic Prym variety P of dimension $p \geq 5$, the cycle $([\psi(\tilde{C})]^{*r})_{(2)}$ is non-zero in $A(P)$ for $1 \leq r \leq p-3$. Consequently, we may deduce the following non-vanishing results.

From the first example above we know that $c_{2,1,d} = -2^{d-4}$, which implies that $[V]_{(2)} \neq 0$, where V is associated to a g^1_d on a generic curve C of genus $g \geq 6$.

When $r = 2$, we have verified on a computer that $c_{2,2,d} = 2^{d-7}$ for $0 \leq d \leq 100$. Therefore, $[V]_{(2)} \neq 0$ at least for $4 \leq d \leq 100$, where V is associated to a g^2_d on a generic curve C of genus $g \geq 6$.

When $r = 3$, the coefficients $c_{2,r,d}$ are not always integers and do not seem to follow an obvious pattern, but they still appear to be non-zero, which has been checked on a computer for $3 \leq d \leq 50$.

Bibliography

[Al] Alexeev, V.: *Compactified Jacobians and Torelli map.* Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1241–1265.

[AIK] Altman, A., Iarrobino, A., Kleiman, S.: *Irreducibility of the compactified Jacobian.* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math.), pp. 1–12. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[AC] Arbarello, E., Cornalba, M.: *Su una congettura di Petri.* Comment. Math. Helv. **56** (1981), no. 1, 1–38.

[AK70] Altman, A., Kleiman, S.: *Introduction to Grothendieck duality theory.* LNM **146**, Springer-Verlag, Berlin-New York, 1970.

[AK80] Altman, A., Kleiman, S.: *Compactifying the Picard scheme.* Adv. Math. **35** (1980), 50-112.

[AK90] Altman, A., Kleiman, S.: *The presentation functor and the compactified Jacobian.* The Grothendieck Festschrift, Vol. I, 15–32, Progr. Math., **86**, Birkhäuser Boston, Boston, MA, 1990.

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P. A., Harris, J.: *Geometry of Algebraic Curves, Vol. I.* Grund. der Math. Wis., **267**. Springer-Verlag, New York, 1985.

[Ba] Bardelli, F.: *Curves of genus three on a general abelian threefold and the nonfinite generation of the Griffiths group.* Arithmetic of complex manifolds (Erlangen, 1988), 10–26, LNM **1399**, Springer, Berlin, 1989.

[Be77] Beauville, A.: *Prym varieties and the Schottky problem.* Invent. Math. **41** (1977), no. 2, 149–196.

[Be82] Beauville, A.: *Sous-variétés spéciales des variétés de Prym.* Compositio Math. **45** (1982), no. 3, 357–383.

[Be83] Beauville, A.: *Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne.* Algebraic geometry (Tokyo/Kyoto, 1982), 238–260, Lecture Notes in Math., **1016**, Springer, Berlin, 1983.

[Be86] Beauville, A.: *Sur l'anneau de Chow d'une variété abélienne.* Math. Ann. **273** (1986), no. 4, 647–651.

[Be04] Beauville, A.: *Algebraic cycles on Jacobian varieties.* Compositio Math. **140** (2004), no. 3, 683–688.

[BL] Birkenhake, C., Lange, H.: *Complex Abelian Varieties.* 2nd ed. Grund. der Math. Wis., **302**. Springer-Verlag, Berlin, 2004.

[Bl76a] Bloch, S.: *Some elementary theorems about algebraic cycles on Abelian varieties.* Invent. Math. **37** (1976), no. 3, 215–228.

[Bl76b] Bloch, S.: *An example in the theory of algebraic cycles.* Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston), LNM **551**, Springer, (1976) 1–29.

[BlMur] Bloch, S., Murre, J. P.: *On the Chow group of certain types of Fano threefolds.* Compositio Math. **39** (1979), no. 1, 47–105.

[Ce] Ceresa, G.: *C is not algebraically equivalent to C^- in its Jacobian.* Ann. of Math. (2) **117** (1983), no. 2, 285–291.

[Cl] Clemens, H.: *Homological equivalence, modulo algebraic equivalence, is not finitely generated.* Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, 19–38.

[Co] Collino, A.: *Poincaré's formulas and hyperelliptic curves.* Atti Accad. Sci. Torino **109** (1975), 89–101.

[CvG] Colombo, E., van Geemen, B.: *Note on curves in a Jacobian.* Compositio Math. **88** (1993), no. 3, 333–353.

[CoPi] Colombo, E., Pirola, G. P.: *New cycles in the Griffiths group of the generic abelian threefold.* Amer. J. Math. **116** (1994), no. 3, 637–667.

[De] Debarre, O.: *Sur le problème de Torelli pour les variétés de Prym.* Amer. J. Math. **111** (1989), no. 1, 111-134.

[Do81] Donagi, R.: *The tetragonal construction.* Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 2, 181–185.

[Do92] Donagi, R.: *The fibers of the Prym map.* Curves, Jacobians, and abelian varieties (Amherst, MA, 1990), 55–125, Contemp. Math., **136**, Amer. Math. Soc., Providence, RI, 1992.

[DS] Donagi, R., Smith, R. C.: *The structure of the Prym map.* Acta Math. **146** (1981), no. 1-2, 25–102.

[EGK] Esteves, E., Gagné, M., Kleiman, S.: *Autoduality of the compactified Jacobian.* J. London Math. Soc. (2) **65** (2002), no. 3, 591–610.

[EH] Eisenbud, D., Harris, J.: *Divisors on general curves and cuspidal rational curves.* Invent. Math. **74** (1983), no. 3, 371–418.

[Fa] Fakhruddin, N.: *Algebraic cycles on generic Abelian varieties.* Compositio Math. **100** (1996), no. 1, 101–119.

[FS] Friedman, R., Smith, R.: *The generic Torelli theorem for the Prym map.* Invent. Math. **67** (1982), no. 3, 473–490.

[Fu] Fulton, W.: *Intersection theory.* 2^{nd} ed. Ergeb. der Math. und ihrer Grenz., **2**. Springer-Verlag, 1998.

[FL] Fulton, W., Lazarsfeld, R: *On the connectedness of degeneracy loci and special divisors.* Acta Math. **146** (1981), no. 3-4, 271–283.

[GH] Griffiths, P., Harris, J.: *On the variety of special linear systems on a general algebraic curve.* Duke Math. J. **47** (1980), no. 1, 233–272.

[HM] Harris, J., Morrison, I.: *Moduli of curves.* Graduate Texts in Mathematics, **187**. Springer-Verlag, New York, 1998.

[Ha] Hartshorne, R.: *Algebraic geometry.* Graduate Texts in Mathematics, No. **52**. Springer-Verlag, New York-Heidelberg, 1977.

[He] Herbaut, F.: *Algebraic cycles on the Jacobian of a curve with a linear system of given dimension.* Compositio Math. **143** (2007), no. 4, 883–899.

[Iz95] Izadi, E.: *The geometric structure of $\mathscr{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors.* J. reine angew. Math. **462** (1995) 93–158.

[Iz01] Izadi, E.: *Subvarieties of abelian varieties.* Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), 207–214, NATO Sci. Ser. II Math. Phys. Chem., **36**, Kluwer Acad. Publ., Dordrecht, 2001.

[IL] Izadi, E., Lange, H.: *Counter-examples of high Clifford index to Prym-Torelli.* arXiv: 1001.3610v1, 2010.

[IT] Izadi, E., Tamás, Cs.: *The primitive cohomology of the theta divisor of an abelian fivefold.* Unpublished manuscript.

[Ka] Kanev, V. I.: *A global Torelli theorem for Prym varieties at a general point.* Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 2, 244–268, 431.

[Ku] Künnemann, K.: *A Lefschetz decomposition for Chow motives of abelian schemes.* Invent. Math. **113** (1993), no. 1, 85–102.

[La] Lang, S.: *Abelian varieties.* Springer-Verlag, New York-Berlin, 1983.

[Li] Lieberman, D.: *Numerical and homological equivalence of algebraic cycles on Hodge manifolds.* Amer. J. Math., **90**, (1968) 366–374.

[Ke] Kempf, G.: *Schubert methods with an application to algebraic curves.* Mathematisch Centrum, Amsterdam, 1971.

[KL] Kleiman, S., Laksov, D.: *On the existence of special divisors.* Amer. J. Math. **94** (1972), 431–436.

[Mo] Moonen, B.: *Relations between tautological cycles on Jacobians.* Comment. Math. Helv. **84** (2009), no. 3, 471–502.

[MvdG] Moonen, B., van der Geer, G.: *Abelian Varieties.* Book draft.

[Muk] Mukai, S.: *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves.* Nagoya Math. J. **81** (1981), 153–175.

[Mu68] Mumford, D.: *Rational equivalence of* 0*-cycles on surfaces.* J. Math. Kyoto Univ. **9** (1968), 195–204.

[Mu70] Mumford, D.: *Abelian Varieties.* Tata Inst. of Fund. Res. Studies in Math., No. 5. Oxford University Press, London 1970.

[Mu74] Mumford, D.: *Prym varieties. I.* Contributions to analysis, pp. 325–350. Academic Press, New York, 1974.

[Na] Naranjo, J.C.: *Fourier transform and Prym varieties.* J. Reine Angew. Math. **560** (2003), 221–230.

[NL] Lahoz, M., Naranjo, J.C.: *Theta-duality on Prym varieties and a Torelli theorem.* arXiv: 1010.0398v1, 2010.

[No] Nori, M.: *Cycles on the generic abelian threefold.* Proc. Indian Acad. Sci. Math. Sci. **99** (1989), no. 3, 191–196.

[OEIS] The On-Line Encyclopedia of Integer Sequences.

[PWZ] Petkovšek, M., Wilf, H.S., Zeilberger, D.: $\mathbf{A} = \mathbf{B}$. A K Peters, Ltd., Wellesley, MA, 1996.

[Po05] Polishchuk, A.: *Universal algebraic equivalences between tautological cycles on Jacobians of curves.* Math. Z. **251** (2005), no. 4, 875–897.

[Po07] Polishchuk, A.: *Lie symmetries of the Chow group of a Jacobian and the tautological subring.* J. Algebraic Geom. **16** (2007), no. 3, 459–476.

[Po08] Polishchuk, A.: *Fourier-stable subrings in the Chow rings of abelian varieties.* Math. Res. Lett. **15** (2008), no. 4, 705–714.

[Re] Recillas, S.: *Jacobians of curves with g^1_4's are the Prym's of trigonal curves.* Bol. Soc. Mat. Mex. (2) **19** (1974), no. 1, 9–13.

[Ro] Roitman, A.A.: *Rational equivalence of* 0*-dimensional cycles.* Math. U.S.S.R.-Sbornik **18** (1972), 571–588.

[SV02] Smith, R., Varley, R.: *A Torelli theorem for special divisor varieties X associated to doubly covered curves $\tilde{C}/C$.* Int. J. of Math., Vol. **13** (2002), No. 1, 67–91.

[Ve] Verra, A.: *The Prym map has degree two on plane sextics.* The Fano Conference, 735–759, Turin, 2004.

[vdGK] van der Geer, G., Kouvidakis, A.: *Cycle relations on Jacobian varieties.* With an appendix by Don Zagier. Compos. Math. **143** (2007), no. 4, 900–908.

[We81] Welters, G. E.: *Abel-Jacobi isogenies for certain types of Fano threefolds.* Mathematical Centre Tracts, **141**. Mathematisch Centrum, Amsterdam, 1981.

[We87] Welters, G. E.: *Recovering the curve data from a general Prym variety.* Amer. J. Math. **109** (1987), no. 1, 165–182.