Cohomological n-Equivalence of Differential Graded Algebras

by

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(Under the Direction of Robert Varley)

Abstract

For $n \geq 0$, we study a formalism for approximating rational homotopy types of differential graded algebras with cohomological n-equivalences, algebra morphisms inducing isomorphisms on cohomology through degree n and a monomorphism in degree $n+1$. We show that localizing with respect to cohomological n-equivalences has many fundamental properties in common with localizing with respect to quasi-isomorphisms. Moreover, we show that cohomological n-equivalences "detect" Massey products in degree $n+1$.

Index words: Differential graded algebra, Localization, Rational homotopy theory, Massey product

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Chapter 1

Introduction

In this first chapter, we briefly motivate and outline our study of cohomological n-equivalences.

1.1 Background in Rational Homotopy Theory

In 1953, Serre published [Ser53], which studied "homotopy mod $\mathfrak{C}$", where $\mathfrak{C}$ is a class of commutative groups satisfying certain closure axioms. In particular, when $\mathfrak{C}$ is is taken to be the class of torsion abelian groups, one obtains a notion of rational homotopy equivalence via the following result of Serre:

Theorem 1 (Whitehead-Serre)**.** *For a continuous map $f : X \to Y$ between simply connected spaces, the following are equivalent:*

1. $\pi_*(f) : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is an isomorphism*
2. $H_*(f; \mathbb{Q}) : H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$ *is an isomorphism.*
3. $H^*(f; \mathbb{Q}) : H^*(X; \mathbb{Q}) \to H^*(Y; \mathbb{Q})$ *is an isomorphism.*

Definition 1. A map $f : X \to Y$ satisfying the above equivalent conditions is called a *rational homotopy equivalence*. Two spaces X and Y have the same *rational homotopy type* if they can be connected by a string of rational equivalences (i.e. not necessarily composable sequence of rational homotopy equivalences).

In [Qui67, Qui69], Quillen describes a categorical formalism for homotopy theory by axiomatizing the classes of weak equivalences, fibrations, and cofibrations. In particular, the category of topological spaces with the above notion of rational equivalence fell under this formalism (under suitable connectivity assumptions). This framework importantly provided a way of verifying that various homotopy categories (the categories obtained by localizing weak equivalences) are equivalent. In particular, the rational homotopy category of topological spaces was shown to be equivalent to several other homotopy categories coming from algebraic categories, including differential graded Lie algebras and differential graded cocommutative coalgebras. This allows for a purely algebraic formulation of rational homotopy theory. By taking linear duals for differential graded coalgebras of finite type, one could also extend the results to the category of differential commutative graded algebras, where the notion of rational equivalence is given by quasi-isomorphisms, i.e. maps that induce isomorphisms on cohomology.

An alternative construction by Sullivan, [Sul77, DGMS75, Sul73] provided a direct functor from spaces to differential graded commutative algebras via the functor of "polynomial de Rham forms" of a space. This approach did not require the same connectivity conditions as [Qui69] nor did it require the differential graded algebras to be of finite type. It is shown in [BG76] that this induces an equivalence on the homotopy categories of spaces, as per Quillen's homotopical algebra. Moreover, Sullivan also emphasized the use of *minimal models* of differential graded algebras, which provide representatives of rational homotopy types that are unique up to isomorphism and enable many computations. In particular, the minimal model of the algebra of polynomial de Rham forms on a space was shown to correspond precisely to the Postnikov tower of the space. He also defined *formal* differential graded algebras as those having the same rational homotopy type as their cohomology algebras (which are considered as differential graded algebras with differential 0).

1.2 Motivating Problem

The problem in rational homotopy theory that motivated our current study is the problem of computing $\mathcal{M}_H$, the moduli space of rational homotopy types whose cohomology is isomorphic to a fixed graded commutative algebra H. This is studied by [SS12] using the bigraded and filtered models of differential graded algebras of [HS79]. $\mathcal{M}_H$ always has a point corresponding to the formal rational homotopy type, which can be represented by H itself, with differential 0, or by the bigraded minimal model of H that [HS79] provides. In this context, it is useful to study Massey products, a type of higher order operation in cohomology, as invariants of the rational homotopy type, including as obstructions to formality (in the sense that the formal rational homotopy type, all higher order Massey products are 0).

For the case when $H^i = 0$ for $i \geq n$, we were led to wonder how much information is needed to distinguish rational homotopy types in $\mathcal{M}_H$. Specifically, as is remarked in [GM13, Ch. 8], for simply connected CW complexes of dimension n, the process of building the Postnikov tower for the CW complex becomes "formal" after the nth stage is constructed. Therefore, the rational homotopy type should be determined by the n-type in this context, and the determination of $\mathcal{M}_H$ when $H^i = 0$ for $i > n$ should be determined by the distinct n-types with cohomology isomorphic to H through degree n. This is what we confirm with the notion of cohomological n-equivalences of differential graded algebras. These are maps that induce an isomorphism on cohomology in degrees $\leq n$ and an injection on degree $n+1$ cohomology. This definition, in particular the condition on $n+1$, was chosen to match the properties of the stages of the minimal model (see Section 2.2). We point out in Chapter 2 a few references where similar concepts have appeared. This formulation of n-equivalence ensures that our n-types are uniquely represented up to isomorphism by a n-minimal algebra (Proposition 3).

Finally, with the definition of n-equivalence in hand along with the corresponding localized category, we can anticipate some future work. Namely, we hope to modify the theory of perturbations in [SS12, HS79] for calculating the moduli space of n-types with cohomology isomorphic to a fixed graded algebra concentrated in finitely many degrees in a way that avoids infinitely generated models, at least for cohomology algebras of finite type. Moreover, it seems reasonable for when H is not concentrated in finitely many degrees, that we could approximate $\mathcal{M}_H$ by considering the sequence of truncated algebras $H^{\leq n}$ and the corresponding moduli spaces $\mathcal{M}_{H^{\leq n}}$ for $n = 0, 1, 2, 3, \ldots$. Furthermore, as is often the case in rational homotopy theory, one can imagine a "dual" theory of n-equivalences and n-types in terms of differential graded Lie algebras (see [FT81] for a summary of the dual concepts).

1.3 Outline of Topics covered

In Chapter 2 we begin by reviewing and fixing notation for graded algebra. We introduce cohomological n-equivalences as well as the main examples when over a field of characteristic 0, namely n-minimal models and n-Sullivan models. Finally, we briefly recall the definition of homotopy of maps between differential graded algebras and some relevant properties.

In Chapter 3 we study the category of differential graded commutative algebras localized with respect to cohomological n-equivalences. We briefly review localization of categories and then describe the categorical framework of Cartan-Eilenberg categories [GNPR10]. This provides a convenient conceptual backdrop for comparing categories localized with respect to a class of weak equivalences and with a notion of homotopy equivalence. One of our goals is to show that just as for rational homotopy theory of dgc algebras (that is when we localize by the class of quasi-isomorphisms), the category of dgc algebras localized by cohomological n-equivalences is equivalent to the homotopy category of cofibrant objects, which are n-Sullivan algebras in our context (Theorem 5). Then [GNPR10] provides a theorem that reduces this goal to checking that n-Sullivan algebras satisfy the the same lifting properties with respect to n-equivalences that Sullivan algebras do with respect to quasi-isomorphisms. The verification of these lifting lemmas is written out in detail.

In the last part of Chapter 3, we make precise the “finite determination principle” that originally motivated us. Specifically, when $H^i = 0$ for $i > n$, then rational homotopy types with cohomology isomorphic to H are determined by n-types whose cohomology agrees with H through degree n. We actually prove a more precise version of this statement that allows some nonzero cohomology in degree $n+1$ by noticing some degree $n+1$ cohomology is invariant under n-equivalences. Finally, we use this finite determination principle in examples where we can determine $\mathcal{M}_H$ using n-equivalences for the appropriate value of n.

Finally, in Chapter 4 we prove that Massey products in cohomology through degree $n+1$ are preserved by n-equivalences (Theorem 8) . In this sense n-equivalences “detect” Massey products that n-quasi-isomorphisms would not be able to. In other words, no nonzero subset of H^{n+1} is invariant under any n-quasi-isomorphism in general, but in this chapter we show that Massey products in degree $n+1$ are an invariant of an n-type. To this end, we review the definition of higher Massey products and their defining systems from [Kra66]. To prove that Massey products are preserved by n-equivalences, we establish some lemmas about modifying defining systems and pulling them back along n-equivalences.

Chapter 2

Cohomological n-equivalence

2.1 Definitions and Notation

In this section, we establish some basic terminology and notation concerning graded algebra. In this section, k will denote a commutative ring with unit 1. For the most part, we are following the notation of [FHT01, Chapter 3].

- A *graded k-module* V will be a collection of k-modules indexed by the non-negative integers. We will write $V = (V^q)_{q\geq 0}$. Elements of V^q are said to have *degree* q, and we write $|x| = q$ or $\deg x = q$ for $x \in V^q$.

- We say a graded module V has *finite type* if it is finitely generated in each degree.

- If k is a field, then V is called a *graded vector space.*

- Submodules, quotients, direct sums, and products are defined "degree-wise." The tensor product $V \otimes W$ of two graded vector spaces is given by $(V \otimes W)^q = \oplus_{i+j=q} V^i \otimes W^j$.

- A *degree ℓ linear morphism* $f : V \to W$ of graded modules is a collection of k-linear maps $f_q : V^q \to W^{q+\ell}$ for $q \geq 0$. If we refer to a linear morphism without specifying a degree, then we mean a degree 0 linear morphism.

- A *graded k-algebra* is a graded k-module $A = (A^q)_{q\geq 0}$ with an associative multiplication $A \otimes A \to A$ (usually written $x \otimes y \mapsto xy$) with an identity element $1 \in A^0$. An morphism of graded algebras $f : A \to B$ is a degree 0 linear map satisfying $f(1) = 1$ and $f(xy) = f(x)f(y)$ for all $x, y \in A$ and $f(1) = 1$.

- If k contains $\frac{1}{2}$, then A graded module V generates a *free commutative graded algebra* ΛV by taking the tensor algebra on V and forming the quotient by the ideal generated by the elements $v \otimes w - (-1)^{|v|\cdot|w|} w \otimes v$. If V has a basis $\{v_1, v_2, \ldots\}$, we write $\Lambda(v_1, v_2, \ldots)$ for ΛV.

Definition 2. A morphism of graded k-modules $f : V \to W$ is an *n-surjection* if f_q is surjective for $q \leq n$.

Our main objects of study will be differential graded algebras, or dg algebras for short.

- A graded algebra A is *commutative* if $xy = (-1)^{|x||y|} yx$ for all $x, y \in A$.

- A *degree ℓ derivation* $\delta : A \to A$ is a degree ℓ linear map such that $\delta(xy) = \delta(x)y + (-1)^{\ell|x|} x\delta(y)$.

- A *differential graded algebra* (abbrv. dg algebra) is a pair (A, d) with A a graded algebra and differential d: a degree 1 derivation such that $d^2 = d \circ d = 0$. Usually the algebra corresponding to the differential is clear from the context, but if specificity is needed, then we shall write (A, d_A). Since we will almost always be referring to differential graded algebras, we will often refer to (A, d) simply as A.

- We will write dgc algebra for 'differential graded commutative algebra.' We may write $d = d_A$ when we need to distinguish between differentials.

- The *cohomology algebra* of a dg algebra is $H(A) = H^*(A) = \ker(d)/\operatorname{im}(d)$, which is indeed a graded algebra. If (A, d) is a dgc algebra, then $H(A)$ is commutative graded algebra.

- A *morphism of dg algebras* $f : A \to B$ is an algebra morphism that commutes with the differentials: $fd_A = d_B f$, or $fd = df$ for short.

- A dg algebra morphism as above induces a morphism of graded algebras $H(A) \to H(B)$, which we will denote by f^* or $H(f)$. If $H^i(f)$ is an isomorphism for all i , then we say that f is a *quasi-isomorphism*, and if it is an isomorphism for $0 \leq i \leq n$, we say f is a n-quasi-isomorphism.

- A dg algebra A is *n-connected* if $H^0(A) = k$ and $H^i(A) = 0$ for $0 < i \leq n$ (sometimes this is referred to as *cohomologically n-connected*). We refer to 0-connected as *connected*.

Since k is typically fixed in the contexts we consider below, we will write **DGCA** for the category of dgc algebras over k and $\mathbf{DGCA}(A, B)$ for the set of morphisms of dg algebras $A \to B$.

Our purpose in this paper is to see some of the consequences of the following definition.

Definition 3. A dg algebra morphism $f : A \to B$ is called a *cohomological n-equivalence* if $H^i(f) : H^i(A) \to H^i(B)$ is an isomorphism for $i \leq n$ and $H^{n+1}(f)$ is a monomorphism. More generally, we say dg algebras A and B belong to the same *cohomological n-type* if they can be connected by a path of cohomological n-equivalences (not necessarily composable).

One simple example of a n-equivalence from a dg algebra A is to let $I = A^{>n+1}$ and let (B, d) be the quotient algebra $B = A/I$ with the induced differential. Then the quotient map $A \to B$ provides a cohomological n-equivalence. Over a field of characteristic 0, we give more useful examples in the following section.

Remark 1. If $\eta : A \to B$ is a dgc algebra map, then η is also a morphism of the underlying cochain complexes, and so the mapping cone $C(\eta)$ can be considered. Then η is a cohomological n-equivalence if and only if $H^i(C(\eta)) = 0$ for $0 \leq i \leq n + 1$ (see [GM13, Section 10.3], where $C(\eta)$ is denoted M_η).

Since we typically do not refer to any other types of n-equivalence, we will often drop 'cohomological.' The concept of cohomological n-equivalence has appeared in [Ark11] for topological spaces, and it is dual to the concept of homotopical n-equivalence of spaces, namely maps $f : X \to Y$ such that $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leq n$ and $\pi_{n+1}(f)$

is a surjection. The condition has also appeared in [SY03] and in [CC17] (under the name n-quasi-isomorphism). Moreover, in some sources the conditions on H^i or π_i are shifted down one degree (that is H^n is injective and π_n is surjective with isomorphisms in degrees $\leq n-1$).

What we hope to show in the following chapters is that cohomological n-equivalence is sufficient for studying the "homotopical information" in degrees up to n. More specifically, while the category **DGCA** has a model structure with the class of weak equivalences being quasi-isomorphisms, it does not have a model structure whose class of weak equivalences is cohomological n-equivalences. However, we show in Chapter 3 that the localization by the class of weak-equivalences still shares many of the results that the localization with respect to quasi-isomorphisms has. For example, we can prove that morphisms from n-Sullivan algebras can be uniquely lifted along n-equivalences up to homotopy, and consequently that n-equivalences induce bijections on the set of homotopy classes of maps having n-Sullivan algebras as a source (see Definition 9 for n-Sullivan algebras). Similarly, n-surjective dg algebra morphisms can play the role of fibrations in **DGCA**, which are degree-wise surjections, and thus n-surjective n-equivalences have properties that trivial fibrations have in a model category. The following property of n-surjective n-equivalences will be used several times in Chapter 3, in particular for establishing the lifting properties of n-equivalences.

Lemma 1. *Suppose that $\eta : A \to B$ is a n-surjective n-equivalence of dg algebras. If $a \in A$ and $d\eta a = 0$ (equivalently $\eta da = 0$) with $\deg a \leq n$, then there exists a cocycle $z \in A$ and an element $a' \in A$ such that $\eta a = \eta z + d\eta a' = \eta(z + da')$ (this says that cocycles in the image of η have preimages which are cocycles).*

Proof. Suppose $a \in A$ satisfies $d\eta a = 0$, that is ηa is a cocycle in B. Since η is a n-equivalence, surjectivity of η^* implies the existence of a cocycle $z \in A$ such that $\eta^*[z] = [\eta a]$. Thus, ηz and ηa differ by a coboundary; $\eta a = \eta z + db$ for some $b \in B$. Finally,

n-surjectivity implies that there is $a' \in A$ with $\eta a' = b$, and so $\eta a = \eta z + d\eta a' = \eta(z + da')$, as claimed. □

2.2 Examples of n-Models

In this section, k will denote a field of characteristic 0.

(a) We can construct n-Sullivan models of connected dgc algebras. The construction of Sullivan models in [FHT01, Ch. 12] can be adapted to make models $f : \Lambda V \to A$ where ΛV is a Sullivan algebra generated in degrees $\leq n$, which we refer to as a n-Sullivan algebra. Since ΛV is generated in degrees $\leq n$, we can expect ΛV to be finitely generated when $H(A)$ is 1-connected and of finite type.

(b) See [FHT01, FHT15, Sul77] for the theory of minimal models. We will recall some important facts. A minimal algebra is in particular a Sullivan algebra. If $\rho : M \to A$ is the minimal model of (A, d), write $M(n)$ for the subalgebra of M generated by elements of degree $\leq n$ and $\rho_n = \rho|_{M(n)}$. Then M is determined by the properties: M is the increasing union of subalgebras $\mathbb{Q} = M(0) \subseteq M(1) \subseteq M(2) \subseteq M(3) \subseteq \cdots$ with maps $\rho_n : M(n) \to A$ such that

  (i) $M(n)$ is a minimal dgc algebra generated by elements in degrees $\leq n$,

  (ii) ρ_n induces an isomorphism on cohomology in degrees $\leq n$,

  (iii) ρ_n induces an injection on cohomology in degree $n + 1$.

So $\rho_n : M(n) \to A$ is a cohomological n-equivalence (in fact, this motivated our definition of cohomological n-equivalence), and we call it the *n-minimal model* of A. In [FHT15, page 109], it is shown that n-minimal models are unique up to isomorphism. Sometimes we may wish to speak of a *n-minimal algebra* on its own, not paired with a n-equivalence to any other dgc algebra, in which case we mean a minimal dgc algebra ΛV such that V is concentrated in degrees $\leq n$.

We will see some concrete examples of n-minimal models in the examples at the end of Chapter 3 (e.g. Example 2).

Chapter 3

Localizing **DGCA** with respect to n-equivalences

We begin by briefly recalling the notion of localization of a category. Then we review the formalism of categories with strong and weak equivalences, as described by [GNPR10]. This framework helps us compare the category of dgc algebras localized by n-equivalences with the "homotopy category" of n-Sullivan algebras with homotopy classes of maps as morphisms.

3.1 Categorical Formalism

3.1.1 Localization of Categories

First, we briefly review localization of categories. Let $\mathcal{C}$ be a category and S a class of morphisms in C. The localization is a category $\mathcal{C}[S^{-1}]$ and a functor $\gamma : \mathcal{C} \to \mathcal{C}[S^{-1}]$ with the following universal property: If $F : \mathcal{C} \to \mathcal{D}$ is any functor such that $F(s)$ is an isomorphism for all $s \in S$, then F factors uniquely through $\mathcal{C}[S^{-1}]$, i.e. there exists $F' : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = F' \circ \gamma$. The category $\mathcal{C}[S^{-1}]$ can be constructed by formally inverting the morphisms in $\mathcal{S}$ so that they become isomorphisms in $\mathcal{C}[S^{-1}]$. See [GNPR10, GZ12, GM03] for details on the construction of $S^{-1}\mathcal{C}$. In particular, for objects $X, Y \in \mathcal{C}$, the morphisms $X \to Y$ in $\mathcal{C}[S^{-1}]$ are equivalence classes of paths of the form $X \overset{f_0}{\text{——}} C_1 \overset{f_1}{\text{——}} \cdots C_k \overset{f_k}{\text{——}} Y$ where f_i represents a left or right pointing arrow in $\mathcal{C}$ such that the "backwards" arrows (the ones pointing left) are members of S. The equivalence relation on these paths is generated by the following operations:

(i) Omitting the identity map,

(ii) replacing adjacent maps pointing in the same direction by their composition,

(iii) omitting two adjacent maps when they are the same, but point in opposite directions.

In particular, item (iii) above is what formally inverts the members of S in $\mathcal{C}[S^{-1}]$.

Definition 4. The class S is called *saturated* if for $f \in \mathrm{Mor}_0 \mathcal{C}$, γf is an isomorphism in $\mathcal{C}[S^{-1}]$ if and only if $f \in S$. The *saturation of* S, denoted $\overline{S}$, is the smallest saturated class of morphisms containing S.

Remark 2. $\overline{S}$ can also be characterized as the preimage of S under γ.

Finally, we will require the notion of *relative localization* of a subcategory for the results from [GNPR10] that appear in the following section (specifically, Theorem 3 and its consequence Theorem 5).

Definition 5. Let $\mathcal{C}$ be a category, S a class of morphisms in $\mathcal{C}$, and $\mathcal{M}$ a full subcategory of $\mathcal{C}$. The *relative localization* of the subcategory $\mathcal{M}$ of $\mathcal{C}$ with respect to S, denoted by $\mathcal{M}[S^{-1}, \mathcal{C}]$, is the full subcategory of $\mathcal{C}[S^{-1}]$ whose objects are those of $\mathcal{M}$.

3.1.2 Categories with Strong and Weak equivalences

Here we review some concepts from [GNPR10]. Let $\mathcal{C}$ be a category and S and W classes of morphisms in $\mathcal{C}$ that are closed under composition and such that $\mathcal{S} \subseteq \overline{\mathcal{W}}$. Members of S are called strong equivalences and members of W are called weak equivalences.

Because $S \subseteq \overline{W}$, the localization $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ factors through the localization $\delta : \mathcal{C} \to \mathcal{C}[S^{-1}]$:

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\delta"'] \arrow[dr, "\gamma"] & \\
\mathcal{C}[S^{-1}] \arrow[r, "\gamma'"'] & \mathcal{C}[W^{-1}]
\end{tikzcd}$$

where $\gamma' : \mathcal{C}[S^{-1}] \to \mathcal{C}[W^{-1}]$ is identified with the localization $\mathcal{C}[W^{-1}] \cong \mathcal{C}[S^{-1}][\delta(W)^{-1}]$.

Definition 6. Call an object M of $\mathcal{C}$ *cofibrant* if for every weak equivalence $w : Y \to X$, the map

$$\mathcal{C}[S^{-1}](M,Y) \to \mathcal{C}[S^{-1}](M,X) \qquad g \mapsto w \circ g$$

is a bijection. In other words, every map $f : M \to X$ in $\mathcal{C}[S^{-1}]$ lifts uniquely in $\mathcal{C}[S^{-1}]$ to $\widetilde{f}$ such that

$$\begin{tikzcd} & Y \arrow[d, "w"] \\ M \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes in $\mathcal{C}[S^{-1}]$.

In [GNPR10], the following equivalent condition for an object to be cofibrant is proven.

Theorem 2 (GNPR10)**.** *An object M of $\mathcal{C}$ is cofibrant if and only if*

$$\gamma' : \mathcal{C}[S^{-1}](M,X) \to \mathcal{C}[W^{-1}](M,X) \qquad g \mapsto w \circ g$$

is a bijection for every $X \in \operatorname{Ob}\mathcal{C}$.

Definition 7. Given $X \in \operatorname{Ob}\mathcal{C}$, a *cofibrant left model* for A is a weak equivalence $M \to X$, where M is cofibrant. A category with strong and weak equivalences is called a *left Cartan-Eilenberg* category if every object in $\mathcal{C}$ has a cofibrant left model.

Every connected dgc algebra has a Sullivan model (Example 1), which are the cofibrant objects in $\mathbf{DGCA}_0$. So $\mathbf{DGCA}_0$ is a Cartan-Eilenberg category with W given by quasi-isomorphisms and S given by homotopy equivalences. We will modify this to hold for n-equivalences below.

Now we summarize some results for the special case when the class of strong equivalences comes from a *congruence* [ML13, pg. 52]. If $\sim$ denotes the congruence, then we can take S to be the maps $f \in \operatorname{Mor}\mathcal{C}$ such that if $f : X \to Y$, then there exists $g : Y \to X$ with $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. In this context, S and $\sim$ are *compatible* if $f \sim g$ implies $\delta f = \delta g$ in $\mathcal{C}[S^{-1}]$. We can form the category $\mathcal{C}/\!\sim$ whose objects are the same as $\mathcal{C}$ and whose morphisms are given by $\mathcal{C}(X,Y)/\sim$ for $X,Y \in \operatorname{Ob}\mathcal{C}$. In general, $\mathcal{C}/\sim$ and $\mathcal{C}[S^{-1}]$ are not equivalent, but they are equivalent if S and $\sim$ are compatible.

Proposition 1 (GNPR10). *If S and $\sim$ are compatible, then $\mathcal{C}/\sim$ and $\mathcal{C}[S^{-1}]$ are canonically isomorphic.*

Proof. The compatibility condition implies that the localizing functor δ factors through quotient functor $\pi : \mathcal{C} \to \mathcal{C}/\sim$, that is there exists $\phi : \mathcal{C}/\sim \longrightarrow \mathcal{C}[S^{-1}]$ such that $\phi \circ \pi = \delta$. One can check that π has the universal property of localization with respect to S, and hence that there is a canonical isomorphism $\mathcal{C}[S^{-1}] \cong \mathcal{C}/\sim$. □

The congruence $\sim$ and a compatible class of morphisms for it above is exemplified by the homotopy relation on maps in various categories with S the class of homotopy equivalences, including homotopy of dgc algebras described in the next section (Definition 8).

Finally, [GNPR10] provides the following "recognition theorem."

Theorem 3 (GNPR10). *Let $(\mathcal{C}, S, W)$ be a category with strong and weak equivalences and $\mathcal{M}$ a full subcategory of $\mathcal{C}$. Suppose that*

(i) For any $w : Y \to X$ in W and any $f \in \mathcal{C}(M,X)$ with $M \in \operatorname{Ob}\mathcal{M}$, there exists a morphism $g \in \mathcal{C}[S^{-1}](M,Y)$ such that $w \circ g = f$ in $\mathcal{C}[S^{-1}]$.

(ii) For any $w : Y \to X$ in W and $M \in \operatorname{Ob}\mathcal{M}$,

$$\mathcal{C}[S^{-1}](M,Y) \to \mathcal{C}[S^{-1}](M,X)$$
$$g \mapsto w \circ g$$

is injective,

(iii) for each $X \in \operatorname{Ob}\mathcal{C}$, there exists a weak equivalence $M \to X$ with $M \in \operatorname{Ob}\mathcal{M}$.

Then the following are true:

(1) every object of $\mathcal{M}$ is cofibrant,

(2) $(\mathcal{C}, S, W)$ is a left Cartan-Eilenberg category, and

(3) the functor $\mathcal{M}[S^{-1}, \mathcal{C}] \to \mathcal{C}[W^{-1}]$ is an equivalence of categories.

The key distinction is in assumption (i), which says that cofibrant objects in $\mathcal{C}$ can be recognized by checking the lifting property on morphisms $f \in \operatorname{Mor} \mathcal{C}$ (by definition, cofibrant objects are required to have the lifting property on all $f \in \operatorname{Mor} \mathcal{C}[S^{-1}]$, which in general would mean working with paths of morphisms in $\mathcal{C}$). Thus, this hypothesis of the theorem make it easier to identify a subcategory of cofibrant models. Most of the work in the following section is verifying the hypotheses (i) and (ii), so that we can apply the conclusion to **DGCA** and the class of n-equivalences.

3.2 Application to n-Equivalences in **DGCA**

In this section we work over a field k of characteristic 0. We shall apply the above formalism to the category **DGCA**. Let $\mathbf{DGCA}_0$ denote the full subcategory of **DGCA** consisting of cohomologically connected dgc algebras (that is, dgc algebras A such that $H^0(A) = 0$).

Let W_n be the class of cohomological n-equivalences and construct $\mathbf{DGCA}[W_n^{-1}]$, as described in section 3.1.1 above. Let S be generated by the congruence $\sim$, which is the equivalence relation on $\mathbf{DGCA}(A, B)$ transitively generated by homotopy equivalence $\simeq$ of dg algebra maps.

Unlike the axioms for weak equivalences in model categories or similar axiomatic approaches to homotopy theory, W_n does not have the two-out-of-three property. In general, a class of morphisms W in a category is said to have the *two-out-of-three property* if for any composable morphisms f and g of the category, any two of f, g, or $g \circ f$ belonging to W implies the third morphism belongs to W as well. However, W_n does satisfy two out of the three conditions of the two-out-of-three property.

Lemma 2. *Let f and g be composable morphisms in* **DGCA**.

(i) The class W_n is closed under composition, that is $f, g \in W_n$ implies $g \circ f \in W_n$.

(ii) If $g \circ f$ and g are n-equivalences, then f is also a n-equivalence.

Proof. (i) This follows from the definition of n-equivalence since isomorphisms and monomorphisms are closed under composition.

(ii) In general, if g and $g \circ f$ are isomorphisms in a category, then so is f. Similarly if g and $g \circ f$ are monomorphisms in a category. The lemma follows from applying these facts to $H^i(f)$ for $0 \leq i \leq n$ and $H^{n+1}(f)$, respectively.

□

We recall the homotopy relation for differential graded algebras. Recall that $\Lambda(t, dt)$ denotes the free dgc algebra generated by t in degree 0 and dt in degree 1 with differential $d(t) = dt$. In particular, $H^0(\Lambda(t, dt)) = k$ and $H^i(\Lambda(t, dt)) = 0$ for $i > 0$. Moreover, $\Lambda(t, dt)$ has two augmentations $\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \to k$ where $\varepsilon_i(t) = i$ and $\varepsilon_i(dt) = 0$ (in short, the evaluations $t \mapsto 0, 1$).

Definition 8. Let $f_0, f_1 : A \to B$ be two dg algebra morphisms. A *homotopy* between f_0 and f_1 is a morphism $H : A \to B \otimes \Lambda(t, dt)$ such that $\varepsilon_0 \circ H = f_0$ and $\varepsilon_1 \circ H = f_1$. In this case, we say f_0 and f_1 are *homotopic* and write $f_0 \sim f_1$.

We will be mainly interested in homotopic maps having a Sullivan algebra as the domain because of the following results.

Proposition 2. *Let $(\Lambda V, d)$ be a Sullivan algebra and A any dgc algebra.*

(a) If $f_0, f_1 : \Lambda V \to A$ are homotopic maps of dg algebras, then $f_0^* = f_1^*$, that is f_0 and f_1 induce the same map on cohomology.

(b) On the set $\mathbf{DGCA}(\Lambda V, A)$, *$\sim$ is an equivalence relation. We denote the set of homotopy classes of maps by $[\Lambda V, A]$.*

Proof. These are Propositions 12.7 and 12.8 in [FHT01]. □

3.2.1 Lifting Lemma

Recall the following definition from Chapter 2.

Definition 9. A n-Sullivan algebra is a Sullivan algebra $(\Lambda V, d)$ generated in degrees $\leq n$, i.e. $V^q = 0$ for $q > n$.

Let **nSull** denote the full subcategory of **DGCA** whose objects are n-Sullivan algebras. The following lifting lemma proof is adapted from [FHT01] to work for n-equivalences.

Lemma 3 (Lifting Lemma)**.** *Let $\eta : A \to B$ be a n-surjective cohomological n-equivalence and let $f : \Lambda V \to B$ be a dg algebra map, where $(\Lambda V, d)$ is a n-Sullivan algebra. Then there exists a lift $\widetilde{f} : \Lambda V \to A$ such that $f = \eta \circ \widetilde{f}$:*

$$\begin{tikzcd} & A \arrow[d, "\eta"] \\ \Lambda V \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Because $(\Lambda V, d)$ is a Sullivan algebra, we can write it as an increasing union of graded vector spaces $V(0) \subseteq V(1) \subseteq \cdots$ such that $V(k) = V(k-1) \oplus V_k$, where $d(V_k) \subseteq \Lambda V(k-1)$. We construct $\widetilde{f}$ inductively.

Assume $\widetilde{f}$ has been constructed on $(\Lambda V(k), d)$ and that $f = \eta \circ \widetilde{f}$. Let $v \in V_{k+1}$, and note that $\deg v \leq n$. Since η is n-surjective and $\deg v \leq n$, there exists an $a' \in A$ such that $\eta a' = fv$. Moreover, $dv \in \Lambda V(k)$ by the Sullivan condition, and so $\widetilde{f}$ is defined on dv. In fact, $\widetilde{f}dv$ is a cocycle in (A, d) since

$$d\widetilde{f}dv = \widetilde{f}d^2v = 0.$$

Hence, we have that $\eta^*[\widetilde{f}dv] = f^*[dv] = f^*0 = 0$, which implies that $[\widetilde{f}dv] = 0$ by the assumption that η is an n-equivalence and $\deg(dv) \leq n+1$. So there is some $a'' \in A$ such that $da'' = \widetilde{f}dv$.

Our goal is to show that we can choose a single $a \in A$ that satisfies both the equations $\eta a' = fv$ and $da'' = \widetilde{f}dv$ from above. Start by choosing a' as before, but choose $a'' \in A$ such that $da'' = da' - \widetilde{f}dv$. We can do this since the class $[da' - \widetilde{f}dv]$ is in $\ker \eta_* = 0$:

$$\eta da' = d\eta a' = dfv = fdv = \eta\widetilde{f}dv \qquad \Rightarrow \qquad \eta(da' - \widetilde{f}dv) = 0.$$

It follows that $d\eta a'' = 0$ (by the equation above). Thus, Lemma 1 says there exists a cocycle $z \in A$ and element $a''' \in A$ such that $\eta(a'') = \eta z + d\eta a''' = \eta(z + da''')$.

Set $z' = z + da'''$, which is a cocycle in (A, d), and set $a = a' - a'' + z'$. Now a should satisfy the two necessary equations for $\widetilde{f}v$ to be well-defined if we set $f\tilde{v} = a$:

$$\eta a = \eta(a' - a'' + z') = \eta a' - \eta(a'' - z') = fv - 0 = fv$$

and

$$da = d(a' - a'' + z') = d(a' - a'') = \widetilde{f}dv.$$

Therefore, we can extend $\widetilde{f}$ to $V(k+1)$, completing the induction. $\square$

Now we work towards establishing the more general lifting property removing the n-surjectivity condition.

Remark 3. In the following lemma, we will need the fact that cocycles of the dgc algebra $B \otimes \Lambda(t, dt)$ for any dgc algebra B can be put in the form $b \otimes 1$ or $b \otimes fdt$, where b is a cocycle in B and $f = f(t)$ is a polynomial in t with coefficients in k (so fdt represents an arbitrary degree 1 element of $\Lambda(t, dt)$). Furthermore, a cocycle $b \otimes f(t)dt$ is actually a coboundary since if F is any antiderivative of f, then

$$d((-1)^{|b|}b \otimes F)) = (-1)^{|b|}db \otimes F + b \otimes dF = b \otimes fdt.$$

Thus, any nonzero cohomology class is represented by a cocycle of the form $b \otimes 1$. Indeed, the inclusion $B \to B \otimes \Lambda(t, dt)$, $b \mapsto b \otimes 1$ is a quasi-isomorphism.

Lemma 4. *Let $\eta : A \to B$ be a n-surjective cohomological n-equivalence. Let $\mathcal{P}$ be the pull back in the diagram below:*

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \times A \arrow[d, "\eta\times\eta"] \\
B \otimes \Lambda(t, dt) \arrow[r, "(1_B\varepsilon_0, 1_B\varepsilon_1)"'] & B \times B
\end{tikzcd}$$

Finally, let $\rho : A \otimes \Lambda(t, dt) \to \mathcal{P}$ *be the map induced by:*

$$(1_A\varepsilon_0, 1_A\varepsilon_1) : A \otimes \Lambda(t, dt) \to A \times A \quad \textit{and} \quad \eta \otimes \mathrm{id} : A \otimes \Lambda(t, dt) \to B \otimes \Lambda(t, dt)$$

Then $\rho = (\eta \otimes \mathrm{id}, 1_A\varepsilon_0, 1_A\varepsilon_1)$ *is also a* n*-surjective* n*-equivalence.*

Remark 4. The above lemma provides the same result as Lemma 2.18 in [Cir15], where the fiber product we call $\mathcal{P}$ is called the *double mapping path* $\mathcal{P}(\eta, \eta)$ in [Cir15]. In this context, Cirici uses a formalism of P-categories, which are categories with a functorial path P and classes of weak equivalences and fibrations satisfying axioms sufficient for doing homotopy theory. Lemma 2.18 is then needed for the same reason as we need below in the proof of lifting property, namely to lift homotopies through a trivial fibration. The analysis in this work can be seen through this lens as well, indeed Cirici uses **DGCA** with the path $P : A \mapsto A \otimes \Lambda(t, dt)$ as an example of a P-category with the usual weak equivalences and fibrations, that is quasi-isomorphisms and surjections, respectively. However, weak equivalences in [Cir15] are assumed to have the two-out-of-three property, and Lemma 2.18 uses the two-out-of-three property in a way that the class of cohomological n-equivalences does not satisfy. However, as we will see below in the proof of 4, the result of Lemma 2.18 in our context can be checked directly without reference to the two-out-of-three property.

Before proving the above lemma, we will characterize the coboundaries in $\mathcal{P}$.

Lemma 5. *The coboundaries in* $\mathcal{P}$ *are of the form* $(b \otimes f dt, z, z')$ *where* b *is a cocycle in* B*,* f *is a polynomial in* t*, and* z, z' *are cocycles in* A*.*

Proof. To see this, we first show that if $(0, 0, da) \in \mathcal{P}$ for some $a \in A$, then $(0, 0, da)$ is indeed a coboundary. The compatibility condition on $(0, 0, da) \in \mathcal{P}$ means that $\eta da = 0$, and so by Lemma 1 there is a cocycle $z \in A$ and element $a' \in A$ with $\eta a = \eta(z + da')$. Then $(\eta(a) \otimes 1, z + da', a)$ is easily checked to satisfy the compatibility condition for $\mathcal{P}$ and $d(\eta(a) \otimes 1, z + da', a) = (0, 0, da)$. The same argument shows that if $(0, da, 0) \in \mathcal{P}$, then $(0, da, 0)$ is also a coboundary.

Now we consider an element of the form $(b\otimes f dt, z, z')$ as in the claim. The compatibility condition for $(b\otimes f dt, z, z')$ being in $\mathcal{P}$ implies that $\eta(z) = 0 = \eta(z')$, and so $(b\otimes f dt, 0, 0)$, $(0, z, 0)$, $(0, 0, z')$ are all elements of $\mathcal{P}$. So it suffices to show that $(b\otimes f dt, 0, 0)$, $(0, z, 0)$ and $(0, 0, z')$ are both coboundaries. For $(b\otimes f dt, 0, 0)$, if F is an antiderivative of f, then $(b\otimes F, F(0), F(1))$ is in $\mathcal{P}$ (η is the identity on the underlying field k) and $d(b\otimes F, F(0), F(1)) = (b\otimes f dt, 0, 0)$. To see that $(0, z, z')$ is a coboundary, note that $\eta(z) = \eta(z') = 0$ implies that $z = da$ and $z' = da'$ for some $a, a' \in A$ since η is a n-equivalence. Thus, $(0, z, 0) = (0, da, 0)$ and $(0, 0, z') = (0, 0, da')$ are coboundaries by the special case considered above. $\square$

Proof of Lemma 4. First we check that ρ is n-surjective. It suffices to find preimages for elements in $\mathcal{P}$ of the form $(b\otimes 1, a, a')$, $(b\otimes t^k, a, a')$, and $(b\otimes t^k dt, a, a')$ for $a, a' \in A^{\leq n}$, and $b \in B^{\leq n}$ (except in the third case, where $\deg b < n$) since these elements linearly span $\mathcal{P}$.

For $(b\otimes 1, a, a') \in \mathcal{P}$, we have by definition of $\mathcal{P}$ that $b = \eta(a) = \eta(a')$. Then the element $a\otimes(1-t) + a'\otimes t \in A\otimes\Lambda(t, dt)$ is a preimage of $(b\otimes 1, a, a')$:

$$
\begin{aligned}
\rho(a\otimes(1-t) + a'\otimes t) &= (\eta(a)\otimes(1-t) + \eta(a')\otimes t, a\varepsilon_0(1-t) + a'\varepsilon_0(t), a\varepsilon_1(1-t) + a'\varepsilon_1(t)) \\
&= (b\otimes(1-t) + b\otimes t, a + 0, 0 + a') \\
&= (b\otimes 1, a, a')
\end{aligned}
$$

Similarly for an element of the form $(b\otimes t^k, a, a')$ with $k > 0$. This satisfies $\eta(a) = 0$ and $\eta(a') = c$ by the construction of $\mathcal{P}$, and so $a\otimes(1-t^k) + a'\otimes t^k$ is easily seen to be a preimage again. Finally, suppose we have $(b\otimes t^k dt, a, a') \in \mathcal{P}$ with $b \in B^{<n}$ and $k \geq 0$. Again using the construction of $\mathcal{P}$, we have that $\eta(a) = \eta(a') = 0$ (since ε_0 and ε_1 both map dt to 0). Since η is n-surjective, we can choose $a'' \in A$ such that $\eta(a'') = b$. Now $a''\otimes t^k dt + a\otimes(1-t) + a'\otimes t$ can be seen to be a preimage of $(b\otimes t^k dt, a, a')$ under ρ. This establishes the n-surjectivity of ρ.

Now we establish that ρ is a n-equivalence. For injectivity of ρ^*, suppose $\rho^*[z\otimes 1] = 0$ with $z \in A$ a cocycle. Since $\rho(z\otimes 1) = (\eta(z)\otimes 1, z, z)$, $\rho^*[z\otimes 1] = 0$ means in particular that z is a coboundary in A. So there is $a \in A$ and such that $da = z$, and thus $d(a \otimes 1) = z \otimes 1$. Note that ρ^* is in fact injective in every degree.

It follows from Remark 3 that cocycles in $\mathcal{P}$ are of the form $(b \otimes 1, z, z')$ or $(b \otimes f dt, z, z')$ where $b \in B$ and $z, z' \in A$ are all cocycles and f is a polynomial in t. The lemma directly above shows that $(b \otimes f dt, z, z')$ is a coboundary. So to check surjectivity of ρ^* it suffices to see that cocycles of the form $(b \otimes 1, z, z')$ have a cocycle preimage where $|b|, |z|, |z'| \leq n$. As an element of $\mathcal{P}$, $(b \otimes 1, z, z') \in \mathcal{P}$ implies $\eta(z) = \eta(z') = b$. In particular, $\eta^*[z] = \eta^*[z']$, which implies $z' = z + da$ for some $a \in A$ by the injectivity of η^*. Then $z \otimes 1$ is a cocycle of $A \otimes \Lambda(t, dt)$ and

$$\rho^*[z \otimes 1] = [(\eta z \otimes 1, z, z)] = [(b \otimes 1, z, z)] = [(b \otimes 1, z, z')]$$

since $z' = z + da$ implies that $(b \otimes 1, z, z') - (b \otimes 1, z, z) = (0, 0, da)$ (so $\rho(z \otimes 1)$ and $(b \otimes 1, z, z')$ differ by a coboundary). This shows that ρ^* is surjective in degrees $\leq n$. $\square$

With the previous two results, Lemmas 3 and 4, the following more general lifting property (removing the n-surjectivity condition) can be proven by formally following the proof in [FHT01].

Theorem 4 (Lifting Property)**.** *Let $\eta : (A, d) \to (B, d)$ be any cohomological n-equivalence and let $f : (\Lambda V, d) \to (B, d)$ be a morphism from a n-Sullivan algebra $(\Lambda V, d)$. Then there exists $\widetilde{f} : (\Lambda V, d) \to (A, d)$ such that $\eta \circ \widetilde{f} \sim f$. Consequently, composing with η induces a bijection*

$$\eta_{\#} : [\Lambda V, A] \to [\Lambda V, B], \qquad \eta_{\#}(g) = \eta \circ g.$$

Proof. Because we have the previous lifting lemma 3, we can follow the same argument as in Proposition 12.9 of [FHT01]. We begin by assuming η is n-surjective, in which case the previous lifting lemma implies that $\eta_{\#}$ is surjective. To prove injectivity, suppose we have $f_0, f_1 : \Lambda V \to A$ such that $\eta \circ f_0 \simeq \eta \circ f_1$. Let $H : \Lambda V \to B \otimes \Lambda(t, dt)$ be a homotopy from $\eta \circ f_0$ to $\eta \circ f_1$. Let $\mathcal{P}$ and $\rho : A \otimes \Lambda(t, dt) \to \mathcal{P}$ be as in the above lemma 4. The diagram below commutes:

$$\begin{tikzcd} \Lambda V \arrow[r, "(f_0,f_1)"] \arrow[d, "H"'] & A \times A \arrow[d, "(\eta,\eta)"] \\ B \otimes \Lambda(t,dt) \arrow[r, "(1_B\varepsilon_0, 1_B\varepsilon_1)"'] & B \times B \end{tikzcd}$$

So there is an induced map $\overline{H} = (H, f_0, f_1) : \Lambda V \to \mathcal{P}$. Since ρ is a n-surjective n-equivalence by Lemma 4, we can lift the map $\overline{H}$ to a map $\widetilde{H} : \Lambda V \to A \otimes \Lambda(t, dt)$ by lemma 3 such that $H = \rho \circ \widetilde{H}$ in the following commutative diagram:

$$\begin{tikzcd} & A \otimes \Lambda(t,dt) \arrow[d, "\rho"] \arrow[dr, "(\varepsilon_0,\varepsilon_1)"] & \\ \Lambda V \arrow[ur, "\widetilde{H}"] \arrow[r, "\overline{H}"'] & \mathcal{P} \arrow[r, "\pi_1"'] & A \times A \end{tikzcd}$$

It follows that $(\varepsilon_0, \varepsilon_1) \circ \widetilde{H} = \pi_1 \circ \overline{H} = (f_0, f_1)$ (since $\overline{H} = (H, f_0, f_1)$), i.e. that $\varepsilon_0 \circ \widetilde{H} = f_0$ and $\varepsilon_1 \circ \widetilde{H} = f_1$. Therefore, we have shown that $\eta \circ f_0 \simeq \eta \circ f_1$ implies $f_0 \simeq f_1$.

The general case where the n-surjectivity hypothesis on η is dropped follows from the above special case using the same argument as in [FHT01] using their "surjective trick" which factors η into $A \xrightarrow{\lambda} A \otimes E \xrightarrow{\sigma} B$, where λ and σ are quasi-isomorphisms and σ is surjective. By reviewing the construction of E and σ, one can easily see that σ is a surjective n-equivalence when η is a n-equivalence. $\square$

Now, we show that morphisms between n-Sullivan algebras in the localized category $\mathbf{DGCA}_0[W_n^{-1}]$ can be regarded as homotopy classes of maps as opposed to equivalence classes of zig-zags in $\mathbf{DGCA}$.

Lemma 6. *On the categories* $\mathbf{nSull}$ *and* $\mathbf{DGCA}$*, the class S of equivalences generated by the congruence $\sim$ is compatible with $\sim$.*

Proof. See [Cir15]. $\square$

Thus, it follows from Proposition 1 that $\mathbf{DGCA}[S^{-1}] \cong \mathbf{DGCA}/\sim$ and $\mathbf{nSull}[S^{-1}, \mathbf{DGCA}] \cong \mathbf{nSull}/\sim$.

The lifting property, Theorem 4, verifies that hypotheses (i) and (ii) of Theorem 3 hold, where $\mathcal{M} = \mathbf{nSull}$ and $\mathcal{C} = \mathbf{DGCA}_0$. Hypothesis (iii) of Theorem 3 follows from the existence of n-Sullivan models for every connected dgc algebra (Section 2.2). Therefore, using the above compatibility lemma and Theorem 3, we have the following equivalence of categories.

Theorem 5. *The following categories are equivalent:*

$$\mathbf{DGCA}_0[W_n^{-1}] \simeq \mathbf{nSull}[S^{-1}, \mathbf{DGCA}_0] \cong \mathbf{nSull}/\sim$$

3.2.2 The Finite Determination Principle

For a connected dgc algebra A, let $[A]_n$ denote the *cohomological* n-type of A, that is the collection of dgc algebras that are n-equivalent to A. As mentioned in Section 2.2, every dgc algebra A has n-minimal and minimal models which are unique up to isomorphism. We will denote the minimal model of A by $\rho : M_A \to A$ and the corresponding n-minimal model $M_A(n) \subseteq M$ by $\rho_n : M_A(n) \to A$.

Remark 5. Since every connected dgc algebra has a n-minimal model, if **nMin** denotes the full subcategory of n-minimal dgc algebras, then $(\mathbf{DGCA}_0, S, W_n)$ is also a Sullivan category in the sense of [GNPR10], and we can replace **nSull** with **nMin** in the above Theorem 5.

Proposition 3. *Suppose A and B are isomorphic in $\mathbf{DGCA}_0[W_n^{-1}]$, i.e. there is a path of cohomological n-equivalences between A and B. Then there is an isomorphism $M_A(n) \to M_B(n)$. Consequently, any n-type of a dgc algebra can be represented uniquely up to isomorphism by a n-minimal dgc algebra.*

Proof. It suffices to check that if $\eta : A \to B$ is an n-equivalence, then $M_A(n) \cong M_B(n)$. Since $M_A(n) \to A \to B$ is a composition of n-equivalences, we see that $M_A(n) \to B$ is a n-minimal model for B. Since n-minimal models are unique up to isomorphism, it follows that $M_A(n) \cong M_B(n)$. $\square$

By the above proposition, we can fix a n-minimal algebra $M(n)$ for a particular n-type such that every dgc algebra A in that n-type has a n-equivalence $M(n) \to A$. This leads us to make the following definition.

Definition 10. For a dgc algebra A and n-minimal model $\rho_n : M(n) \to A$, let $H^i_{n\pi}(A)$ denote the image of $H^i(M(n))$ under ρ_n for $i \leq n+1$ and let $H^i_{n\pi}(A) = 0$ for $i \geq n+2$. We refer to $H_{n\pi}$ as the *cohomology of the n-type* $[A]_n$.

Clearly, $H^i_{n\pi}(A) = H^i(A)$ for $i \leq n$ since n-equivalences induce isomorphisms in these degrees. However, $H^{n+1}_{n\pi}(A)$ and $H^{n+1}(A)$ do not agree in general, and so we are left to wonder what part of degree $n+1$ cohomology is invariant under any n-equivalence. The subset $H^{n+1}_{n\pi}(A)$ can be regarded as the largest invariant subset. Indeed, if $\eta : A \to B$ is a n-equivalence and $\rho : M_A(n) \to A$ and $\sigma : M_B(n) \to B$ are n-minimal models, then η lifts to an isomorphism $\bar{\eta} : M_A(n) \to M_B(n)$ such that

$$\begin{tikzcd}
M_A(n) \arrow[r, "\bar{\eta}"] \arrow[d, "\rho"'] & M_B(n) \arrow[d, "\sigma"] \\
A \arrow[r, "\eta"'] & B
\end{tikzcd}$$

commutes up to homotopy. However, homotopic maps induce the same map on cohomology (Proposition 2), and so the digram commutes after applying H^{n+1}. In particular, this means that

$$\eta^*(H^{n+1}_{n\pi}(A)) = \eta^*\rho^*(H^{n+1}(M_A(n))) = \sigma^*\bar{\eta}^*(H^{n+1}(M_A(n))) = \sigma^*(H^{n+1}(M_B(n))) = H^{n+1}_{n\pi}(B).$$

This shows that $H_{n\pi}([A]_n)$ is independent (up to isomorphism) of the choice of representative A of an n-type. If $M(n)$ is the n-minimal representative of a n-type, then the cohomology of the n-type is by definition $H^{\leq n+1}(M(n))$, and so is the *largest* invariant subset because $\mathrm{id} : M(n) \to M(n)$ is a n-equivalence that induces the identity on $H^{\leq n+1}(M(n))$. A slogan for this result is that n-minimal algebras have "minimal" cohomology in degrees $\leq n+1$ among representatives of their n-types.

Remark 6. As mentioned above, $H_{n\pi}^{\leq n}(A) = H^{\leq n}(A)$, but $H_{n\pi}^{n+1}(A)$ is only a subset of $H^{n+1}(A)$ in general. If $\alpha \in H^{n+1}(A)$ is decomposable, then α is a linear combination of products of lower degree cohomology classes, and since these lower degree classes are all preserved by n-equivalences, α will be preserved as well by the injectivity condition on H^{n+1} for n-equivalences. Therefore, $H_{n\pi}^{n+1}(A)$ always contains the subset of decomposable cohomology. However, for some n-types, there may also be indecomposable classes in $H_{n\pi}^{n+1}(A)$. We will prove in Chapter 4 that if $\alpha \in H^{n+1}(A)$ is represented by a Massey product cocycle (which represents an indecomposable class in cohomology, in general), then α is preserved by any n-equivalence, i.e. $\alpha \in H_{n\pi}^{n+1}(A)$.

In fact, we expect $H_{n\pi}^{n+1}(A)$ to coincide with $\ker \zeta^* = (H^+ \cdot H^+) \oplus K$ in degree $n+1$ in the notation of [HS79], where ζ^* is the dual of the Hurewicz map. This indicates that $H_{n\pi}^{n+1}(A)$ is spanned by decomposable classes and those represented by matric Massey products based on Remark 8.4(2) in [HS79] (see [May69] for details on matric Massey products).

We give the following result as an analogy to the "finite determination" statement in [GM13, page 71] that for simply connected CW complexes of dimension n, the construction of the Postnikov tower for X is *formal* after the nth Postnikov section (though the statement below does not require simple connectivity). It is a consequence of the uniqueness of minimal models.

Proposition 4. *Suppose that $M(n)$ is a n-minimal dgc algebra, and let M be a minimal algebra such that $H^{n+1}(M) = H^{n+1}(M(n))$ and $H^i(M) = 0$ for $i \geq n+2$, then M is unique up to isomorphism with these properties.*

Combined with the above ideas, we can write a more precise "finite determination principle."

Theorem 6. *If A and B are dgc algebras with of the same n-type with n-minimal model $M(n)$. Moreover, assume that $H^{n+1}(A) \cong H^{n+1}(M(n)) \cong H^{n+1}(B)$ and that*

$H^i(A) = H^i(B) = 0$ for $i \geq n+2$. Then A and B have isomorphic minimal models, i.e. are of the same rational homotopy type.

Proof. By assumption we have that A and B have the same cohomology in all degrees. Moreover, we can regard the n-minimal models $M(n) \to A$ and $M(n) \to B$ as the nth stage in the construction of M_A and M_B. So M_A and M_B are minimal algebras constructed from $M(n)$ satisfying $H^{\leq n+1} = H^{\leq n+1}(M(n))$ and $H^i(M) = 0$ for $i \geq n+2$, and so $M_A \cong M_B$ by the above proposition. □

Remark 7. (1) Even though we are assuming A and B have isomorphic cohomology in the above theorem, a priori the isomorphism on cohomology is realized though a path of n-equivalences, where the intermediate dgc algebras in the path may have cohomology in higher degrees (as the n-minimal model does typically). However, the conclusion that A and B have the same rational homotopy type means they can be connected by a path of quasi-isomorphisms (where the intermediate dgc algebras must therefore have the same cohomology as A and B).

(2) A weaker version of the above theorem would be that $H^i(A) = H^i(B) = 0$ for $i \geq n+1$. In this case, it follows that $H^{n+1}(M(n)) = 0$ and the same conclusion holds, and so a theory based on n-quasi-isomorphisms would give the same result. So the above theorem shows that we can detect some "homotopical information" in degree $n+1$ with n-equivalences that n-quasi-isomorphisms would not be able to.

The above results show that just as in rational homotopy theory (i.e. localizing the class of quasi-isomorphisms), the n-minimal models of a n-type contain all the "homotopy theoretic" information of the n-type. Moreover, as the first remark above indicates, n-equivalences allow for a wider choice of models of a dgc algebra with finite dimensional cohomology. While the full minimal model $M \to A$ of a dgc algebra A with finite cohomology is a quasi-isomorphism, M will often require an infinite number of generators, whereas a n-minimal model of A will be finitely generated, at least if $H(A)$ is simply connected and of finite type. For example, for a fixed graded commutative algebra H, we can consider the space $\mathcal{M}_H$ of all rational homotopy types of dgc algebras that have cohomology isomorphic to H. When $H^i = 0$ for $i \gg 0$ and is of finite type, then the above analysis indicates that we can classify these rational homotopy types by their n-minimal models as opposed to their full minimal models, and since the n-minimal models are finitely generated in general, the determination of $\mathcal{M}_H$ should be simpler.

To illustrate the above comments concerning $\mathcal{M}_H$, consider the following examples from [HS79]. In both cases we consider a commutative graded algebra H such that $H^i = 0$ for $i > n$, for some n. We then show that the n-types that have cohomology H determine the rational homotopy types with cohomology H.

Example 1. Let $H = H(S^2 \vee S^2; k)$ (recall that k is a field of characteristic 0 in this section). We can see from the finite determination principle that $\mathcal{M}_H$ consists of a single rational homotopy type, i.e. that H is intrinsically formal. Then $H^0 = k$ and and $H^2 = k^2$, with $H^i = 0$ otherwise. Thus, if $\overline{x}$ and $\overline{y}$ denote generators of degree 2 for H, then $\rho_2 : \Lambda(x, y) \to H$ is a 2-minimal model where $|x| = |y| = 2$, $dx = dy = 0$ and $\rho_2 x = \overline{x}$, $\rho_2 y = \overline{y}$ (note that we think of H as a dgc algebra with differential $d = 0$ when speaking of any n-minimal model for it). Because of the small size of this example, this is clearly the only 2-minimal algebra M such that cohomology $H^{\leq 2}(M) \cong H(S^2 \vee S^2)$. In other words, there is a unique 2-type with cohomology isomorphic to $H(S^2 \vee S^2)$ through degrees ≤ 2, represented by the 2-minimal algebra $\Lambda(x, y)$. By the finite determination principle, Theorem 6, we can conclude that there is a single rational homotopy type with cohomology isomorphic to H, i.e. we recover the fact that $S^2 \vee S^2$ is intrinsically formal (which is verified by [HS79] via their obstruction theory).

Note that the minimal model $\rho : M \to H$ would be infinitely generated. This can be anticipated from Hilton's Theorem [Hil55], which implies that $S^2 \vee S^2$ will have infinitely many nonzero rational homotopy groups, along with the fact that the generators of a minimal model of a space (say, simply-connected) are in one-to-one correspondence with rational homotopy classes of the space. But we can get a feeling concretely for why M will be infinitely generated. Note that $\Lambda(x, y)$ is not quasi-isomorphic to H. Indeed, $H(\Lambda(x, y))$ has much higher cohomology that needs to be killed as we build the stages of the full minimal model for H. For example, $[x^2], [xy], [y^2] \in H^4(\Lambda(x, y))$ are independent nonzero cohomology classes, and so for the 3-minimal model $\rho_3 : M(3) \to H$, we would add degree 3 generators A, B, C such that $dA = x^2$, $dB = xy$, and $dC = y^2$ and $\rho_3 A = \rho_3 B = \rho_3 C = 0$, giving the 3-minimal algebra $M(3) = \Lambda(x, y, A, B, C)$. However, $H^5(M(3))$ is now two-dimensional, spanned by the classes $[-Ay + xB]$ and $[-By + xC]$ (which are representatives of the Massey triple products $\langle x, x, y\rangle$ and $\langle x, y, y\rangle$, respectively). So for the 4-minimal model, two more degree 4 generators, E and F are added to kill this cohomology in degree 5: $dE = -Ay + xB$ and $dF = -By + xC$ and mapped to 0 under ρ_4. This process of adding generators to kill higher cohomology continues to build the full minimal model of H, but we avoid it by considering the 2-type of H and invoking the finite determination principle.

The following example builds on the one above.

Example 2. Let $H = H(S^2 \vee S^2 \vee S^5)$. In this case, $H^2 = k^2$ and $H^5 = k$, and all products are 0. Let $\overline{z} \in H^5$ denote a generator. Because H is concentrated in degrees ≤ 5, we shall consider all 5-types that have cohomology isomorphic to H in degrees ≤ 5. Once again, any distinct 5-type will give a distinct rational homotopy type by the finite determination principle. The 5-minimal model $\rho_5 : M(5) \to H$ is built from the 4-minimal algebra $\Lambda(x, y, A, B, C, E, F)$ of the previous example by adding a generator z of degree 5 with $dz = 0$ with $\rho_5 z = \overline{z}$. So the 5-minimal algebra $(\Lambda(x, y, A, B, C, E, F, z), d)$ represents the 5-type of the algebra H, and corresponds to the the formal rational homotopy type of H. In [HS79], it is shown using Halperin-Stasheff's filtered models that there is one other rational homotopy type having cohomology isomorphic to H, i.e. that $\mathcal{M}_H$ is a two-point space. Specifically, consider the dgc algebra $(\Lambda(x, y, A, B, C, E, F, z), D)$ where $D = d$ on each generator except E and F, where $DE = dE + c_1 z$ and $DF = dF + c_2 z$ where

$c_1, c_2 \in k$ are not both 0. With the new differential D, $(\Lambda(x, y, A, B, C, E, F, z), D)$ is no longer a minimal algebra, but it is still a 5-Sullivan model, and it is shown in [HS79] that for any choice of c_1, c_2 with $(c_1, c_2) \neq (0, 0)$, the resulting algebras are isomorphic, and so they represent the same 5-type. Incidentally, a 5-minimal model for the non-formal space is $(\Lambda(x, y, A, B, C, E), d)$, where the generator F in the 4-minimal model of H is left out to allow one of the degree 5 Massey product cocycles mentioned in the preceding example to "survive." So once again, by the finite determination principle, the fact that there are exactly two 5-types whose cohomology through degree 5 agrees with H implies that there are two rational homotopy types with cohomology types having cohomology H.

To compare with the approach in [HS79], once they establish that the filtered models mentioned above represent the same rational homotopy type, they must also argue that perturbations of the full minimal model of H do not represent any other rational homotopy types since the full minimal model will have generators of arbitrarily high degree, and thus the possibility for more perturbations than the ones that can be seen on the 5-model. They are able to prove their claim using their obstruction theory. From our point of view, however, the 5-Sullivan models are sufficient to distinguish the distinct rational homotopy types.

Chapter 4

Massey Decomposable Cohomology

Under our concept of cohomological n-equivalence, it is clear that the cohomology through degree n is an invariant among representatives of a cohomological n-type since n-equivalences are defined to induce isomorphisms on cohomology through degree n. However, the injectivity condition in degree $n+1$ makes it less clear what subset of H^{n+1} is invariant under n-equivalence. In general, if $\alpha \in H^{n+1}(A)$ for some connected dg algebra A is decomposable, then α is a linear combination of products of lower degree cohomology classes, and since these lower degree classes are all preserved by n-equivalences, α will be as well. However, there are some cases when α will be preserved under any n-equivalence even if it is indecomposable as an element of the algebra $H(A)$, e.g. when α is a Massey product.

4.1 Review of Massey Products

In this section, we only assume k is a commutative ring with unity. We also do not assume the dg algebras are commutative. First we recall the definition of p-fold Massey product (following [Kra66], with different indices). For $a \in A$, $\overline{a} = (-1)^{|a|}a$.

Definition 11. Let (A, d) be a dg algebra and let $a_{ij} \in A$ for $1 \leq i \leq j \leq p$ excluding $(i, j) = (1, p)$ be elements such that a_{ii} is a cocycle representative of a cohomology class a_i and

$$da_{ij} = \sum_{k=i}^{j-1} \overline{a}_{ik} a_{k+1,j}.$$

The p-fold Massey product is the set of cohomology classes denoted by $\langle a_1, \ldots, a_p \rangle$ that are represented by the cocycles

$$\sum_{k=1}^{p-1} \overline{a}_{ik} a_{k+1,p}$$

for some choice of elements a_{ij} as above. We say that $\langle a_1, \ldots, a_p \rangle$ is *trivial* or *vanishes* if it can be represented by the 0 cohomology class.

The degree of a p-fold Massey product as above is $2 + \sum_{k=1}^{p}(|a_i| - 1) = 2 - p + \sum_{i=1}^{p} |a_i|$. When convenient, we will write

$$\mu_{ij} = \sum_{k=i}^{j-1} \overline{a}_{ik} a_{k+1,j}$$

so that $da_{ij} = \mu_{ij}$. Notice that μ_{ij} is a representative of $\langle a_i, \ldots, a_j \rangle$.

We also introduce the following refinement of the notion of a defining system.

Definition 12. Let $a_i \in H(A, d)$ for $1 \leq i \leq p$. For $1 \leq \ell \leq p-1$, an *ℓ-stage system* for $(a_1, \ldots, a_p)$ is a collection $a_{ij} \in A$ for $1 \leq i \leq j \leq p$ such that $j - i < \ell$,

(1) a_{ii} is a cocycle representative of the cohomology class a_i,

(2)
$$da_{ij} = \sum_{k=i}^{j-1} \overline{a}_{ik} a_{k+1,j}.$$

We write $\langle a_1, \ldots, a_p \rangle_\ell$ for the set of ℓ-stage systems for $(a_1, \ldots, a_p)$ (we show below the sense in which this is independent of the choice of cocycle representatives).

Notice that whereas $\langle a_1, \ldots, a_p \rangle$ is a set of cohomology classes, $\langle a_1, \ldots, a_p \rangle_\ell$ is a set of collections of cochains.

Example 3. (i) A 1-stage system for $(a_1, \ldots, a_p)$ is nothing more than a choice of cocycle representatives $(a_{11}, \ldots, a_{pp})$ for these cohomology classes.

(ii) A $(p-1)$-stage system is a defining system in the above sense.

It is illustrative to picture ℓ-stage systems as partial upper triangular matrices. For example, a 2-stage system for (a_1, a_2, a_3, a_4) would be

$$\begin{pmatrix} a_{11} & a_{12} & * & * \\ & a_{22} & a_{23} & * \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix}$$

Given an ℓ-stage system, we can form the $(\ell+1)$-fold Massey product representatives μ_{ij} for $j - i = \ell$. We can then extend the ℓ-stage system to a $(\ell+1)$-stage system *iff* there exist $a_{ij} \in A$ such that $da_{ij} = \mu_{ij}$, i.e. *iff* we can fill in the ℓth super-diagonal in the above matrix (where the 0th super-diagonal means the diagonal a_{ii}).

4.2 Lemmas on Defining Systems

The following lemma generalizes Kraines's theorem that changing the cocycle representatives of a Massey product by coboundaries will not change the Massey product, i.e. the Massey product is independent of the cocycle representatives of the cohomology factors $a_1, \ldots, a_p$ (compare [Kra66]). Given an ℓ-stage system, we use similar formulas to show that we can make a new ℓ-stage system if we start by changing one entry by a coboundary.

Lemma 7. *Let (a_{ij}) be a ℓ-stage system for $(a_1, \ldots, a_p)$ in a dg algebra A, and let $a'_{mn} = a_{mn} + db$ for some $b \in A$. Then we can construct a ℓ-stage system (a'_{ij}) by the following formulas:*

(i) $a'_{ij} = a_{ij}$ *if* $i \neq m$ *and* $j \neq n$,

(ii) $a'_{in} = a_{in} + a_{i,m-1}b$ *for* $i < m$,

(iii) $a'_{mj} = a_{mj} - \bar{b}a_{n+1,j}$ *for* $j > n$.

Moreover, the Massey Product cocycles μ'_{ij} for the modified system (a'_{ij}) are either identical or cohomologous to μ_{ij} (in fact, $\mu'_{ij} = \mu_{ij}$ if and only if $i \neq m$ and $j \neq n$). Notice that only entries in the column and row above and to the right of a_{mn} are modified.

Proof. We will write $\mu'_{ij} = \sum_{k=i}^{j-1} \overline{a'}_{ik} a'_{k+1,j}$. It is clear that condition (1) of Definition 12 is satisfied by (a'_{ij}) since either $a'_{ii} = a_{ii}$ or $a'_{mm} = a_{mm} + db$ (in the case $n = m$).

Keeping in mind that only entries in the column and row above and to the right of a_{mn} are modified, $\mu'_{ij} = \mu_{ij}$, so $da'_{ij} = da_{ij} = \mu_{ij} = \mu'_{ij}$ if $i > m$ or $j < n$. We will check that $da'_{ij} = \mu'_{ij}$ when $i \leq m$ and $j \geq n$ in three cases: (i) $i < m$ and $j > n$, (ii) $j = n$, and (iii) $i = m$. Note that the difficulty here is that in the formula for μ'_{ij} now contains modified and unmodified entries of the original ℓ-stage system (a_{ij})

(i) In this case, we show that $\mu'_{ij} = \mu_{ij}$ and that $da'_{ij} = \mu_{ij}$. In this case, the sum for μ'_{ij} contains only two modified terms, $\overline{a}'_{i,m-1} a'_{mj}$ and $\overline{a}'_{in} a'_{n+1,j}$. These modified terms simplify as

$$\begin{aligned}
\overline{a}'_{i,m-1} a'_{mj} + \overline{a}'_{in} a'_{n+1,j} &= \overline{a}_{i,m-1}(a_{mj} - \overline{b} a_{n+1,j}) + (\overline{a}_{in} + \overline{a}_{i,m-1}\overline{b}) a_{n+1,j} \\
&= \overline{a}_{i,m-1} a_{mj} - \overline{a}_{i,m-1}\overline{b} a_{n+1,j} + \overline{a}_{in} a_{n+1,j} + \overline{a}_{i,m-1}\overline{b} a_{n+1,j} \\
&= \overline{a}_{i,m-1} a_{mj} + \overline{a}_{in} a_{n+1,j},
\end{aligned}$$

i.e. the modified terms simplify to match the unmodified terms in μ_{ij}. Therefore, we have that $\mu'_{ij} = \mu_{ij}$ Since $a'_{ij} = a_{ij}$, this verifies that $da'_{ij} = da_{ij} = \mu_{ij} = \mu'_{ij}$.

(ii) In this case, the only factors in the expansion of μ'_{in} are a'_{kn} for $i+1 \leq k \leq m$ and $a'_{mn} = a_{mn} + db$. Thus, we have

$$\begin{aligned}
\mu'_{in} &= \sum_{k=i}^{n-1} \overline{a'}_{ik} a'_{k+1,n} \\
&= \sum_{k=i}^{m-2} \overline{a}_{ik} a'_{k+1,n} + \overline{a'}_{i,m-1} a'_{m,n} + \sum_{k=m}^{n-1} \overline{a}_{ik} a_{k+1,n} \\
&= \sum_{k=i}^{m-2} \overline{a}_{ik}(a_{k+1,n} + a_{k+1,m-1}b) + \overline{a}_{i,m-1}(a_{mn} + db) + \sum_{k=m}^{n-1} \overline{a}_{ik} a_{k+1,n} \\
&= \mu_{in} + \overline{a}_{i,m-1} db + \sum_{k=i}^{m-2} \overline{a}_{ik} a_{k+1,m-1} b \\
&= \mu_{in} + \overline{a}_{i,m-1} db + \mu_{i,m-1} b \\
&= \mu_{in} + \overline{a}_{i,m-1} db + da_{i,m-1} b.
\end{aligned}$$

Now it is clear that $da'_{in} = \mu'_{in}$:

$$da'_{in} = d(a_{in} + a_{i,m-1}b) = da_{in} + da_{i,m-1}b + \overline{a}_{i,m-1}db = \mu'_{in}.$$

Note that this also shows that $\mu'_{in} = \mu_{in} + d(a_{i,m-1}b)$.

(iii) This is similar to the above case.

□

Now we see one way we can control ℓ-stage systems under cohomological n-equivalence.

Theorem 7. *Let $f : A \to B$ be a cohomological n-equivalence of connected dg algebras, and let $E = (b_{ij})$ be a ℓ-stage system in B for $b_1, \ldots, b_p \in H^*(B)$. Then there are a ℓ-stage systems $D = (a_{ij})$ and $E' = (b'_{ij})$ in (A, d) and (B, d), respectively, such that $f(D) = E'$ and the Massey product cocycles for E and E' are cohomologous.*

Proof. We proceed by induction on $k = j - i$. Specifically, we show that for $k = 1, \ldots, \ell - 1$ there exists a k-stage system $D_k = (a_{ij})$ in A and an ℓ-stage system $E' = (b_{ij})$ such that $f(D_k) = E'$ and the $(k+1)$-fold and lower Massey product cocycles for E' are cohomologous to the corresponding ones for E. For $k = 1$, this amounts to choosing cocycles $a_{ii} \in A$ for $1 \le i \le p$ such that $f(a_{ii}) = b_{ii} + dc_{ii}$ for some cochains $c_{ii} \in B$. This is possible because f is an n-equivalence. We take E' to be the ℓ-stage system obtained by repeatedly applying the previous lemma to modify E. As that lemma shows, the Massey product cocycles for E' are cohomologous to those of E.

Now assume that the claim holds for some value k where $1 \le k \le \ell - 1$. First we show that D_k can be extended to a $(k+1)$-stage system. Assume that $f(D_k) = E$, that is $f(a_{ij}) = b_{ij}$ for $j - i \le k$. In particular, because f is a dg algebra homomorphism $f(\nu_{ij}) = \mu_{ij}$, where ν_{ij} is the Massey cocycle for D_k. Since E is an ℓ-stage system and $k \le \ell - 1$, we have $b_{ij} \in E$ such that $db_{ij} = \mu_{ij}$ for $j - i = k + 2$. Therefore, $f(\nu_{ij}) = db_{ij}$, which means that $[\nu_{ij}]$ is in the kernel of f_* for $j - i = k + 2$. Since f is an n-equivalence, this kernel is trivial, and so $\nu_{ij} = da_{ij}$ for some $a_{ij} \in A$ for $j - i = k + 2$, which extends D_k to a $(k+1)$-stage system. Note that $df(a_{ij}) = f(da_{ij}) = \mu_{ij} = db_{ij}$, and so $f(a_{ij}) = b_{ij} + dc_{ij}$ for some $c_{ij} \in B$. Once again, by repeatedly applying the previous lemma about modifying defining systems by coboundaries, we can construct an ℓ-stage system E' whose $(k+1)$-diagonal is $b_{ij} + dc_{ij}$, and hence satisfies $f(a_{ij}) = b'_{ij}$ for $j - i \le k + 1$. This completes the induction. □

4.3 Massey Decomposable Elements of Cohomology

In light of the above propositions, we introduce the following definition.

Definition 13. Let $n \ge 0$ and (A, d) be a dg algebra. The *Massey decomposable elements* of $H(A)$ are cohomology classes in the span of classes that are decomposable or are representatives of a Massey product. This defines a subalgebra of $H(A)$ that we will denote by $H_{\mathrm{Mas}}(A)$.

As a corollary to Theorem 7 on defining systems, we have the following.

Theorem 8. *Let $f : (A, d) \to (B, d)$ be a cohomological n-equivalence between connected dg algebras. Then f^* induces a bijection on the Massey decomposable elements in degrees $\leq n + 1$.*

Proof. If $\beta \in H^{n+1}(B)$ is decomposable as an element of the algebra $H(B)$, then β is a linear combination of the cohomology factors from degrees $1, 2, \ldots, n$. Each of these lower degree factors has a unique preimage under f^* since f^* is an isomorphism in these degrees, and so the corresponding linear combination of preimage factors in H^{n+1} will be a preimage of β. This preimage is unique because f^* is injective in degree $n + 1$.

Now suppose β can be represented by a Massey product cocycle. For any choice of defining system for β, Theorem 7 provides a defining system for a Massey product in A whose image is cohomologous to β. Conversely, any Massey product $H^{n+1}(A)$ is mapped to a Massey product in $H^{n+1}(B)$ since as an algebra morphism, f preserves the equations that define Massey products. □

Remark 8. (1) The reason for the term *Massey decomposable* is that $\alpha \in H(A)$ can be indecomposable with respect to the algebra multiplication on $H(A)$, but should still be considered "decomposable" if it represents a Massey product. Specifically, since a Massey product in degree $n + 1$ consists of information (i.e. defining systems) from degrees $\leq n$, it should be considered "decomposable," even though the cohomology class may actually be indecomposable in terms of the algebra structure on $H(A)$. Calling these classes Massey decomposable is meant to emphasize that they are determined in lower degree.

(2) Theorem 8 above shows that H^{n+1}_{Mas} is preserved by n-equivalences. When working over a field of a characteristic 0, as in Section 3.2.2, we defined $H_{n\pi}$ to represent the largest invariant subset of cohomology of a n-type. So Theorem 8 says that $H_{\mathrm{Mas}} \subseteq H_{n\pi}$. In light of remark 6 in Section 3.2.2, it seems plausible that H_{Mas} could be generalized to include cohomology in the span of classes represented by matric

Massey products, in which case $H_{\mathrm{Mas}}^{\leq n+1}$ would presumably be equal to $H_{n\pi}$, or in other words the cohomology preserved by under any n-equivalence would be exactly H_{Mas} if matric Massey products are considered. We leave this possible modification for future consideration.

Consider the following examples.

Example 4. (1) Let $(A, d) = (\Lambda(x, y, z, A, B), d)$ where $dx = dy = dz = 0$, $dA = xy$, and $dB = yz$, and $|x| = 2$, $|y| = 2$, and $|z| = 2$ (the degrees can be set arbitrarily, but we choose to be concrete). In this case, the Massey triple product $\langle [x], [y], [z] \rangle$ is defined and represented by $\mu = \overline{A}z + \overline{x}B$ in degree 5. The class $[\mu]$ is indecomposable with respect to the algebra structure of $H(A)$ (which is easily seen for degree reasons in this case since the only other indecomposable cohomology in lower degrees are the classes $[x]$, $[y]$, and $[z]$). Thus, $[\mu]$ is an example of a Massey decomposable element that is indecomposable with respect to the algebra structure on $H(A)$.

Note over a field of characteristic 0, A is in fact a 4-minimal algebra and so the identity $A \to A$ is can be regarded as a cohomological 4-equivalence. Thus, $[\mu] \in H_{\mathrm{Mas}}^5(A) \subseteq H_{4\pi}^5(A)$ is an indecomposable degree 5 invariant of the 4-type of A.

(2) A similar statement can be made about the 4-minimal model of $H = H(S^2 \vee S^2 \vee S^5)$ from Example 2 in Chapter 3. There are two indecomposable classes in degree 5 represented by Massey triple products, and so the 4-type of that minimal model has two indecomposable invariant degree 5 cohomology classes.

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