Schwarz-Christoffel Transformations

by

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(Under the direction of Edward Azoff)

Abstract

The Riemann Mapping Theorem guarantees that the upper half plane is conformally equivalent to the interior domain determined by any polygon. Schwarz-Christoffel transformations provide explicit formulas for the maps that work. Popular textbook treatments of the topic range from motivational and contructive to proof-oriented. The aim of this paper is to combine the strengths of these expositions, filling in details and adding more information when necessary. In particular, careful attention is paid to the crucial fact, taken for granted in most elementary texts, that all conformal equivalences between the domains in question extend continuously to their closures.

Index words: Complex Analysis, Schwarz-Christoffel Transformations, Polygons in the Complex Plane

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Table of Contents


Page

Acknowledgments . . . . . . . . . . . . . . . . . . . . . . . . . . . . . iv

Chapter

1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

2 Background Information . . . . . . . . . . . . . . . . . . . . . . . . 5

2.1 Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . 5

2.2 Linear Curves and Polygons . . . . . . . . . . . . . . . . . . 8

3 Two Examples and Motivation for the Formula . . . . . . . . . . . 13

3.1 Prototypical Examples . . . . . . . . . . . . . . . . . . . . . 13

3.2 Motivation for the Formula . . . . . . . . . . . . . . . . . . 16

4 Properties of Schwarz-Christoffel Candidates . . . . . . . . . . . 19

4.1 Well-Definedness of $f$ . . . . . . . . . . . . . . . . . . . . 20

4.2 The limit of $f(z)$ as $|z| \to \infty$ . . . . . . . . . . . . . . . . . 21

4.3 Continuity of $f$ . . . . . . . . . . . . . . . . . . . . . . . 24

4.4 Injectivity of $f(z)$ . . . . . . . . . . . . . . . . . . . . . . 25

5 Proof of the Theorem . . . . . . . . . . . . . . . . . . . . . . . . 33

5.1 Continuity at the Boundary of $\mathbb{H}$ . . . . . . . . . . . . . . 33

5.2 Completion of the Proof . . . . . . . . . . . . . . . . . . . . 35

5.3 Infinite Prevertex . . . . . . . . . . . . . . . . . . . . . . . 37

5.4 Vertices With $\alpha_k = 1$ . . . . . . . . . . . . . . . . . . . . . 38

5.5 Degrees of Freedom in the Formula . . . . . . . . . . . . . . 39

6 Examples and Variants . . . . . . . . . . . . . . . . . . . . . . . . 42
6.1 Polygons With One and Two Vertices . . . . . . . . . . . . 42
6.2 Triangles . . . . . . . . . . . . . . . . . . . . . . . . . . . 45
6.3 Variants of the Schwarz-Christoffel Transformation . . 51

Bibliography . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 56

Appendix

A Tri.mws . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 57

B Quad.mws . . . . . . . . . . . . . . . . . . . . . . . . . . . . 59

# Chapter 1

## Introduction

Given a polygonal curve $\Gamma$, its interior $P$ is a simply connected domain. Thus, by the Riemann Mapping Theorem, there exists a function $S$ that conformally maps the upper half plane onto $P$. The Schwarz-Christoffel theorem provides a concrete description of such maps.

Here is a typical textbook statement of the theorem:

> **Theorem**: *Let $P$ be the interior of a polygon $\Gamma$ having vertices $w_1, \dots w_n$ and interior angles $\alpha_1\pi \dots \alpha_n\pi$ in counterclockwise order. Let $S$ be any conformal, one-to-one map from the upper half plane $\mathbb{H}$ onto $P$ satisfying $S(\infty) = w_n$. Then $S$ can be written in the form:*
>
> $$S(z) = A + C \int_{z_0}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta \qquad (1)$$
>
> *where $A$ and $C$ are complex constants, and $z_0 < z_1 < \cdots < z_{n-1}$ are real numbers satisfying $S(z_k) = w_k$ for $k = 1, \dots n-1$.*

Functions of the form in Equation (1) are called **Schwarz-Christoffel candidates**[1]. Furthermore, a Schwarz-Christoffel candidate is a **Schwarz-Christoffel Transformation** if it does indeed conformally map the upper half plane $\mathbb{H}$ onto the interior of a polygon.

To make total sense of this theorem, several issues have to be addressed. First, and most fundamentally, the map $S$ from Equation (1) refers to values of $S$ on the extended real axis, but this set is not part of the upper half plane. Therefore, to be able to discuss $S(z_1), \dots, S(\infty)$, it is important to extend the definition of $S$ to the closure of $\mathbb{H}$.

[1] This terminology is nonstandard.

Secondly, notice that Equation (1) involves improper contour integrals. We need to specify which contours joining $z_0$ to $z$, are admissible, show that the resulting integrals converge, and are in fact independent of the particular contour is chosen.

Also, the theorem mandates that $S(\infty) = w_n$. We shall discuss the seriousness of this stipulation, as well as how much freedom we are allowed with the parameters $A$, $C$, $z_0$, $\ldots$ $z_{n-1}$ in the map $S$.

After these issues have been addressed, we can then formally prove the theorem.

We start the paper with a careful setup of notations and terms in Chapter 2. We begin Chapter 3 by proving the theorem for prototypical cases when $P$ is a half or quarter plane. This will then motivate us to construct a Schwarz-Christoffel candidate $f$ for the general case. In Chapter 4, we show that $f$ is indeed a Schwarz-Christoffel Transformation if and only if its image curve does not cross itself.

In Chapter 5, we prove the theorem. That is, if a function $S$ takes the upper half plane conformally onto the interior of a polygon, then it is of the form in Equation (1). Here, we carefully address the initial issue at hand: that any conformal map $S$ from the upper half plane $\mathbb{H}$ to the interior of a polygon extends to be continuous on the closure of $H$ in the Riemann Sphere.

Finally, in Chapter 6, several examples are given, as well as variants of the Schwarz-Christoffel transformation for different domains.

The two main references for this paper are *Complex Variables and Applications* by James Brown and Ruel Churchill and *Schwarz-Christoffel Mapping* by Tobin Driscoll and Lloyd Trefethen. Brown's book has excellent insights in the motivation for the construction of the candidate functions as well as showing that many of the desirable properties are met. The expository sections as well as the homework exercises provide a clear outline for our work in Chapters 3 and 4. However, no proof of the theorem is given in this text.

Driscoll's book provides a proof of the theorem as well as numerical approaches on how to calculate the prevertices $z_k$. Neither of the two books, however, addresses the crucial issue of continuity at the boundary.

We refer to two treatments of this last issue. Ch. Pommerenke's *Boundary Behaviour of Conformal Maps* addresses the general question of conformal map extensions, but his proofs are rather sophisticated. For the purposes of our paper, it suffices to instead follow the less general but more elementary approach taken in *Complex Analysis* by Lars Ahlfors.

We conclude this introduction with a brief historical background on the two men playing title roles in the realm of Schwarz-Christoffel Transformations.

The transformation is named after German mathematicians Elwin Bruno Christoffel and Hermann Amandus Schwarz, both of whom discovered it independently.

Elwin Bruno Christoffel was born in Montjoie Aachen (now Monschau), Germany on 10 November 1829. He attended the University of Berlin, where he was taught by some renowned mathematicians, one of which were Peter Dirichlet. Christoffel achieved his doctorate in 1856 with his dissertation on the motion of electricity in homogeneous bodies.

Christoffel's work on the Schwarz-Christoffel transformations were published in four papers between 1868 and 1870. The first paper was published while he was in Zurich, where he was chair at the Polytechnicum from 1862 to 1869. The last three papers were published while he was in Berlin, where he was chair at the University of Technology of Berlin.[2]

Hermann Amandus Schwarz was born in Hermsdorf, Silesia (now Poland) on 25 January 1843. He attended the Technical University of Berlin, where he initially intended to take a degree in chemistry. It was there that two of his teachers, Ernst Kummer and Karl Weierstrass influenced him to switch to Mathematics. Schwarz received his doctorate in 1864 under these two men.

[2] More information on Christoffel can be accessed at this website: http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Christoffel.html

Schwarz's discovery of the Schwarz-Christoffel formula occurred independently of Christoffel's work in the late 1860s while working as a privatdozent at the University of Halle in Germany. He published his work on this subject in two papers in 1869.[3]

[3] More information on Schwarz can be accessed at this website:
http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Schwarz.html

# Chapter 2

# Background Information

Before we embark on the construction and analysis of the Schwarz-Christoffel theorem, let us consider a few facts that we will need. Throughout the paper, we shall use the following notation for two open subsets of $\mathbb{C}$:

$$\mathbb{H} : \{z \in \mathbb{C} : Im(z) > 0\}. \qquad \mathbb{D} : \{z \in \mathbb{C} : |z| < 1\}.$$

## 2.1 Preliminaries

We now also remind ourselves of some necessary background information on complex analysis.

### 2.1.1 Branch Points and Branch Cuts

Whenever a function $f$ is multi-valued, we have to introduce branch cuts in the complex plane in order to allow for analyticity. For example, the function $f(z) = \log(z)$ is a multi-valued function, but taking the branch point at the origin, and taking the branch cut $\{x : x < 0\}$ allows the logarithm function to be single valued.

Throughout this paper, we will be working with functions of the form $f(z) = (z - z_k)^r$, where $z_k$ is a complex number and $r$ is a real number.

Thus, $f(z) = (z - z_k)^r = \exp\left(r \log(z - z_k)\right)$.

The standard branch cut used in complex analysis is the negative real axis, which restricts all complex arguments between $-\pi$ and $\pi$. However, the statement of the Theorem involves powers of negative real numbers, so we shall instead use the branch cut $\{y\boldsymbol{i} : y < 0\}$. This then restricts all complex arguments between $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

### 2.1.2 Möbius Transformations

Möbius Transformations from the complex plane to itself are any function $w = f(z)$ of the form:

$$f(z) = \frac{A + Cz}{B + Dz}$$

where $AD - BC \neq 0$. Another way of stating this is that it is a composition of a finite collection of the following:

i. $w = a + cz$, $c \neq 0$;

ii. $w = \frac{c}{z}$, $c \neq 0$

In this paper, we are especially concerned with Möbius Transformations which map $\mathbb{H}$ onto $\mathbb{H}$. Such a transformation maps the extended real axis to itself, and thus the above parameters $A, B, C, D$ can be chosen to be real. If $D = 0$, then we have real numbers $a, c$ so that $w = c(z - a)$ and this will only preserve $\mathbb{H}$ when $c > 0$. The remaining maps take the form $w = b + \frac{c}{z-a}$ with $a, b, c \in \mathbb{R}$, and these will only preserve $\mathbb{H}$ when $c < 0$. Therefore the Möbius functions which map $\mathbb{H}$ onto $\mathbb{H}$ are:

i. $w = c(z - a)$, $a$ in $\mathbb{R}$, $c > 0$;

ii. $w = b + \frac{c}{z-a}$, $a, b \in \mathbb{R}, c < 0$

### 2.1.3 Schwarz' Lemma and Reflection Principle

Here, we state two important results by Schwarz. They will prove to be useful later in the paper.

> **Schwarz's Lemma**: *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. If $f(0) = 0$ then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Moreover, if $|f(z)| = |z|$ for some $z \in \mathbb{D}\backslash\{0\}$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some $|c| = 1$.*

Schwarz's Lemma implies that every conformal equivalence between $\mathbb{D}$ and itself is implemented by a Möbius transformation. Indeed, suppose $f$ maps $\mathbb{D}$ conformally onto itself. Fix a Möbius transformation $T$ which sends $f(0)$ to 0 and maps $\mathbb{D}$ into itself. Schwarz's Lemma then tells us that there is a $c$ so that $(T \circ f)(z) = cz$ for all $z \in \mathbb{D}$. In particular, $T \circ f$ is the restriction of a Mobius transformation to $\mathbb{D}$. Thus the same is true for $f$, as desired.

> **Schwarz's Reflection Principle**: *Let $\Omega$ be a symmetric region, and set $\Omega^+ :$ $\Omega \cap \mathbb{H}$ and $\sigma := \Omega \cap \mathbb{R}$. Suppose that $v$ is continuous on $\Omega^+ \cup \sigma$, harmonic in $\Omega^+$, and zero on $\sigma$. Then $v$ has a harmonic extension to $\Omega$ which satisfies the symmetry relation $v(\overline{z}) = -v(z)$. In the same situation, if $v$ is the imaginary part of an analytic function $f$ in $\Omega^+$, then $f$ can be extended to an analytic function on all of $\Omega$ by the formula $f(\overline{z}) = \overline{f(z)}$.*

### 2.1.4 Riemann Mapping Theorem

Given two domains $D$ and $D'$, we say that $f$ maps $D$ **conformally** onto $D'$ provided that:

(i) $f$ is analytic throughout $D$

(ii) $f$ is injective;

(iii) $f(D) = D'$.

Now recall the Riemann Mapping Theorem:

> *Let $U$ be a simply connected open proper subset of the complex plane $\mathbb{C}$, and let $z_0 \in U$. Then there exists a unique function $f$ that maps $U$ conformally onto $\mathbb{D}$ so that $f(z_0) = 0$ and $f'(z_0) > 0$. A corollary to this then guarantees that any two simply connected proper subsets of $\mathbb{C}$ can be mapped conformally onto each other.*

## 2.2 Linear Curves and Polygons

We define a closed curve to be a **linear curve** if it is a concatenation of finitely many line segments $\gamma_k$. We define the point at which two segments meet to be a **vertex**. Since we will allow $w = \infty$ to be a vertex, we shall consider a ray as a line segment with the point $\infty$ as one of its terminal points.

A linear curve that does not intersect itself, i.e., is simple, is called a **polygon**. Polygons have desirable properties that not all linear curves have. One such property is that polygons satisfy the Jordan Curve Theorem.[1]

Given a polygon $\Gamma$ , let $P$ denote its interior with vertices $w_1$, $w_2$, $\ldots$, $w_n$ and interior angles $\alpha_1\pi$, $\alpha_2\pi$, $\ldots$, $\alpha_n\pi$. The numbering of these vertices and angles are taken with respect to the interior of the polygon. As one traverses the edges of the polygon counterclockwise from vertex $w_k$ to vertex $w_{k+1}$, the interior $P$ should lie to the left.

The **interior angle** at a vertex is defined to be the angle created by sweeping counterclockwise from the outgoing side to the incoming side. Similarly, we define the **turning angle** at that vertex as the angle created by extending the incoming side and sweeping from the extended incoming side towards the outgoing side so that the magnitude of this sweeping motion, either clockwise or counterclockwise is less than $\pi$. [See Fig. 2.1] Thus the interior and turning angles add to $\pi$ radians.

We shall specify that for a finite vertex $w_k$, its interior angle $\alpha_k\pi$ where $\alpha_k \in (0,\, 2)$.[2] For an infinite vertex $w_k$, we specify $\alpha_k \in [-2,\, 0]$.

Furthermore, we want our polygons, even with infinte vertices, to make a complete turn of $2\pi$. That is, we specify that

$$\sum_{k=1}^{n}(1-\alpha_k) = 2. \tag{2a}$$

[1]**Jordan Curve Theorem**: Any continuous simple closed curve in the plane, separates the plane into two disjoint regions, the inside and the outside.

[2]The case when $\alpha_k = 2$ forms an interesting scenario, which will be covered later.

or the turning angles sum up to $2\pi$. Alternatively, we also have,

$$\sum_{k=1}^{n}(\alpha_k) = n - 2. \tag{2b}$$

Note that linear curves that are not polygons need not abide by this rule.

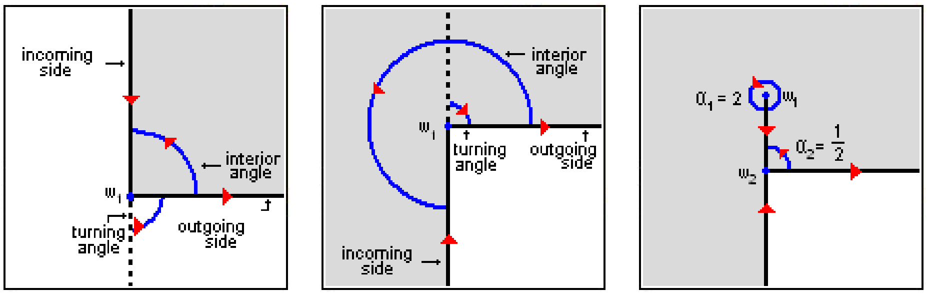



Figure 2.1: Interior and Turning Angles

The turning angle on the left is positive, while the turning angle in the middle is negative.
The figure on the right illustrates a slit.

With these stipulations, the interior angles of infinite vertices could be deduced.

### 2.2.1 Boundary Properties of Polygons

Notice that the interior of any polygon $P$ is an open set. Thus, we are guaranteed a conformal $S$ from the upper half plane $\mathbb{H}$ onto $P$.

Furthermore, as we will discuss later, this function $S$ has a continuous extension that maps the real axis to the boundary of the polygon. Thus, for each of the vertices $w_k$, there exists a unique prevertex $z_k$ so that $f(z_k) = w_k$.

### 2.2.2 Slits

Note that if $\alpha_k$ is 2, the incoming and outgoing sides are collinear and thus forms a slit, with the vertex $w_k$ as the tip. [See Fig. 2.2] This, of course, violates the stipulation that the polygon be simple, since there are infinitely many points of intersection.

Let us, for a moment, consider what happens to these slits. Take $z = a + b\boldsymbol{i}$. Let $c, t$ be real numbers. Consider the diagram on Fig 2.2.

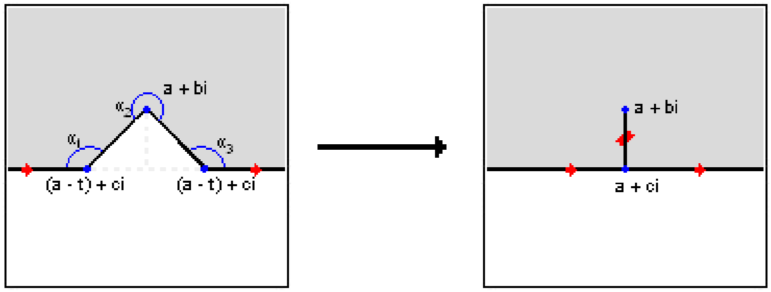



Figure 2.2: Slits

Notice that as $t \to 0$, $\alpha_1$ and $\alpha_3 \to \frac{1}{2}$ and $\alpha_2 \to 2$. Moreover, note that the interior of the polygon grows as $t \to 0$. Furthermore, even though we have this weird phenomenon, none of the interior points are accounted for more than once. Thus the function from $H$ to $P$, which the Riemann Mapping theorem claims to exist, is still one-to-one. Thus, we allow for the interior angle $\alpha\pi$ to be $2\pi$.

Note, however, this function, when extended to the boundary, will not possess that same one-to-one characteristic.

### 2.2.3 Infinite Vertices

Polygons with infinite vertices often arise in the applications of the Schwarz-Christoffel formula. Note that by knowing the interior angles of finite vertices, we can deduce what the interior angle corresponds to the infinite vertex, if only one exists. If more than one exists, we would need more information to deduce the turning angle that corresponds to each infinite vertex.

Fig. 2.3 illustrates some examples of polygons with infinite vertices.

Staring with Diagram A, we have that $\alpha_1 = \alpha_2 = \frac{1}{2}$. And since we have three vertices, Equation (2b) gives us that $\alpha_1 + \alpha_2 + \alpha_3 = 3 - 2 = 1$. Hence $\alpha_3 = 0$. Moreover, Diagrams C and E also have three vertices and will yield that $\alpha_3 = -1$ and $-2$ respectively.

Diagram B has two vertices, and thus $\alpha_1 + \alpha_2 = 0$. Since $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$. Similarly, Diagram D also has two vertices and will yield $\alpha_2 = -\frac{3}{2}$.

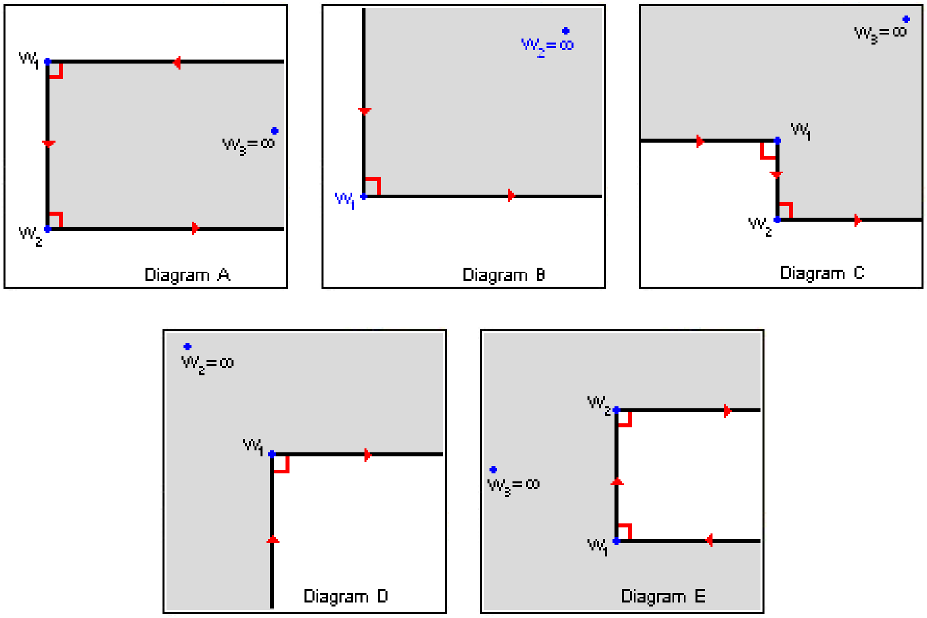



Figure 2.3: Polygons With One Infinite Vertex

Illustrated here are polygons with one infinite vertex with various interior angles. One may deduce this angle measure from the other interior angles given.

Another way to consider infinite vertices is taking a standard polygon and allowing one or more of its sides to limit to infinity. Some examples are illustrated in Fig. 2.4. The diagram on the left, as $w_3 \to \infty$, becomes a 90° rotation of Diagram A from Fig. 2.3.

The diagram on the right illustrates how polygons with two infinite vertices behave. As $w_2$ and $w_4$ move infinitely far to the left and to the right, respectively, an infinite strip is created. As a result, the interior angles $\alpha_k\pi$ for each $w_k$, k = 1, … 4, we have $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = \alpha_4 = 1$.


Figure 2.4: More Infinite Vertices

Illustrated here are polygons with finite vertices where some vertices are taken to infinity.

Here are two more examples:


Figure 2.5: More Infinite Vertices

Illustrated here are polygons with finite vertices where some vertices are taken to infinity.

With the diagram on the left, as $w_1$ and $w_4$ go infinitely far to the left and right, respectively, an infinite strip with a dent is created. The resulting interior angles yield that $\alpha_1 = \alpha_4 = -1$ and $\alpha_2 = \alpha_3 = \frac{3}{2}$.

The diagram on the right is again a polygon with four vertices, one of which $w_4$ is already at infinity, while another, $w_2$, is shifted infinitely far to the left. The result is the complex plane that has two slits. The interior angles that arise have $\alpha_2 = \alpha_4 = 0$ and $\alpha_1 = \alpha_3 = 2$.

Now that we have a good grasp of the polygonal domains that we will be mapping to, let us consider what function will be used to create our desired output.

# Chapter 3

## Two Examples and Motivation for the Formula

### 3.1 Prototypical Examples

We begin with a full discussion of Theorem 1 for two special polygons.

Let us first consider the polygon $\Gamma_1$ consisting of the extended real axis. It has a single vertex, which which we take to be $w_1 = \infty$. By Equation (2a), $\alpha_1 = -1$. The domain staying to the left of $\Gamma_1$ as we proceed from left to right is the upper half plane $\mathbb{H}$. In this setting, the main Theorem becomes:

> **Example**: *Let $S$ be any conformal, one-to-one map from the upper half plane $\mathbb{H}$ onto $\mathbb{H}$ satisfying $S(\infty) = \infty$. Then $S$ can be written in the form:*
>
> $$S(z) = A + C \int_{z_0}^{z} 1 d\zeta$$
>
> *where $A$ and $C$ are complex constants*

**Proof:** Fix a Möbius transformation $T$ which maps $\mathbb{D}$ to $\mathbb{H}$. Then $g := T^{-1} \circ f \circ T$ maps $\mathbb{D}$ conformally onto itself, so Schwarz's Lemma tells us that $g$ is implemented by a Möbius transformation. Thus, $f = T \circ g \circ T^{-1}$ is also implemented by a Möbius transformation. This then allows for the continuous extension of $f$ onto the boundary of $\mathbb{H}$.

Furthermore, as stated in chapter 2, all Möbius transformations sending $\mathbb{H}$ onto itself take one of the forms:

(i.) $f(z) = c(z - a)$ where $a \in \mathbb{R}$, $c > 0$.

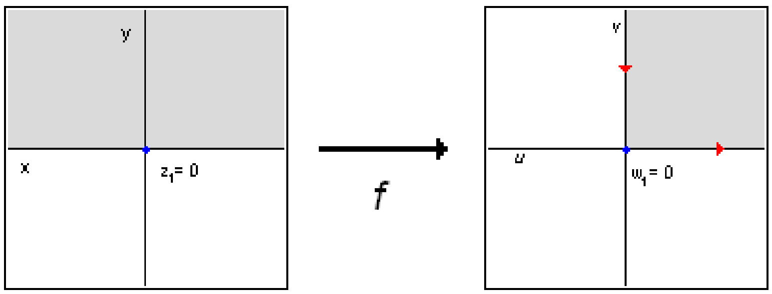
(ii.) $f(z) = b + \frac{c}{z-a}$ where $a, b \in \mathbb{R}, c < 0$.

And since in our case, we want $f(\infty) = w_1 = \infty$, the second case is not attainable. Therefore,

$$f(z) = c(z-a) = c\int_a^z 1 d\zeta$$

and $f$ is indeed in the Schwarz-Christoffel form.

For our second polygon, we take $\Gamma_2$, consisting of the union of the non-negative real and imaginary axes. The two verices of this polygon are $0, \infty$. The interior domain is the first quadrant, denoted $P_2$. By convention, the infinite vertex will have a corresponding interior angle of $-\frac{1}{2}$ while the finite vertex will have an interior angle of $\frac{1}{2}$.



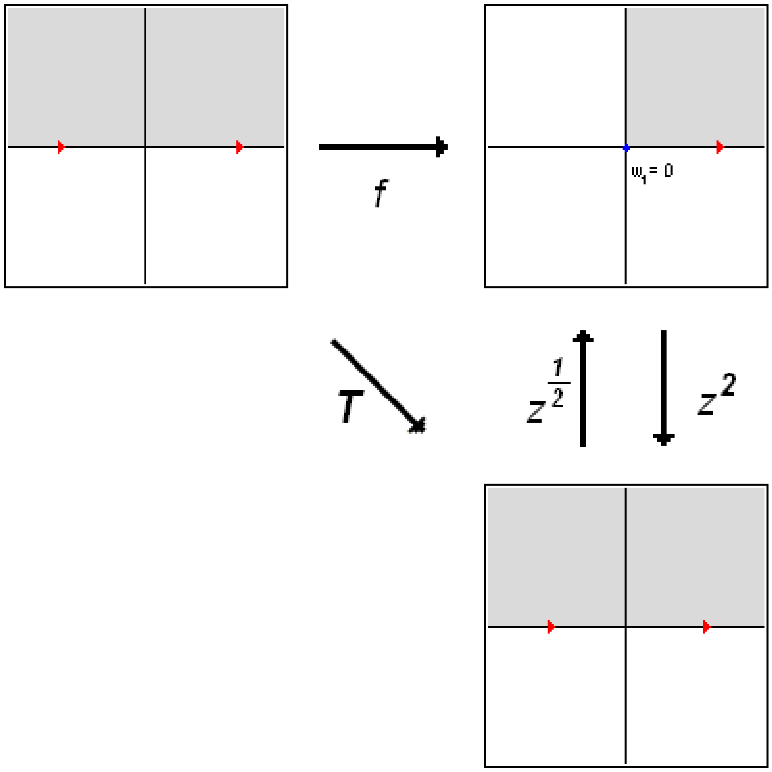

Now we consider which functions $f$ would map $\mathbb{H}$ onto $P_2$. The main theorem becomes:

**Example**: Let $S$ be any conformal, one-to-one map from the upper half plane $\mathbb{H}$ onto the first quadrant $P_2$. Then $S$ extends to a homeomorphism from the closure of $\mathbb{H}$ to the closure of $P_2$. If $S(z_1) = 0$ and $S(\infty) = \infty$, then $S$ can be written in the form:

$$S(z) = A + C\int_{z_1}^z (\zeta - z_1)^{-\frac{1}{2}} d\zeta$$

On the other hand, if $S(z_1) = \infty$ and $S(\infty) = 0$, then $S$ can be written in the form

$$S(z) = A + C\int_0^z (\zeta - z_1)^{-\frac{3}{2}} d\zeta$$




**Proof:** Notice that if we compose $S$ with $\phi(z) = z^2$, the result $T : z \mapsto (\phi \circ S)(z) = (f(z))^2$ is a map that sends $\mathbb{H}$ onto itself. Thus $T$ is a Mobius transformation taking one of the two forms described earlier. Hence, $S(z) = (\phi^{-1} \circ T)(z) = (T(z))^{\frac{1}{2}}$ and this extends to be continuous on the real axis.

The function $S$ bijectively maps the negative real axis to the positive imaginary axis and the positive real axis is bijectively mapped onto itself.

The integral expression for $S$ depends on how the vertices $0, \infty$ are ordered. In either case, the statement of the Theorem requires our prevertices to be $z_1 \in \mathbb{R}$ and $z_2 = \infty$, so if $w_2 = \infty$, then $T(z) = c(z - z_1)$ for some $c > 0$, and we have:

$$\begin{aligned} S(z) &= c(z - z_1)^{\frac{1}{2}} \\ &= \frac{1}{2} c \int_{z_1}^{z} (\zeta - z_1)^{-\frac{1}{2}} d\zeta \\ &= A + C \int_{z_1}^{z} (\zeta - z_1)^{-\frac{1}{2}} d\zeta \end{aligned}$$

Here, the vertex is $w_1 = 0$, with $\alpha_1 = \frac{1}{2}$ and its prevertex is at $z = z_1$. Thus, $S$ is in the Schwarz-Christoffel form.

On the other hand, if $w_1 = \infty$, then $S(z_1) = \infty$ and $S(\infty) = 0$, so $T(z) = \dfrac{c}{z - z_1}$. We then have:

$$\begin{aligned} S(z) &= \left(\frac{c}{z - z_1}\right)^{\frac{1}{2}} \\ &= -\frac{1}{2}|c|^{\frac{1}{2}} \int_{-\infty}^{z} (\zeta - z_1)^{-\frac{3}{2}} d\zeta \\ &= A + C \int_{-\infty}^{z} (\zeta - z_1)^{-\frac{3}{2}} d\zeta \end{aligned}$$

Here, the vertex used is $w_2 = \infty$, with $\alpha_2 = -\frac{1}{2}$ and its prevertex is at $z = z_1$. Thus again, $f$ is in the Schwarz-Christoffel form.

Note, however, that the singularity at $\zeta = z_1$ in the integrand is too strong for the contour integral two make sense, so care must be taken so that $z_1$ is not on the contour of integration. One way to remedy this is to use the point at $\infty$ as the lower limit of integration. If $z < z_1$, take the contour from $-\infty$ to $z$ along the real axis. Similarly, if $z > z_1$, then take the contour $\infty$ to $z$ again on the real axis.

Therefore, to achieve the turning angle that we want, we must use a power function, with the exponent being one less than the desired interior angle.

### 3.2 Motivation for the Formula

Notice here that for the first case, as one traverses the path from $-\infty$ towards 0 on the negative real axis, the image traverses from $\infty$ towards 0 on the positive imaginary axis. Similarly, as one traverses from 0 towards $\infty$ on the positive real axis, the image traverses from 0 towards $\infty$ on the positive real axis as well. Thus, we have a turning angle of $\frac{\pi}{2}$ at the point $z = z_1$.

For the second case, the negative real axis is mapped onto the positive real axis (since $c > 0$) and the real axis is mapped onto the positive imaginary real axis.

In either case, the turning angle is the exponent of the integrand.

As far as generalizing the 2-gon case, the argument would be the same for any $\alpha_1 \in (0, 2)$ and for any $z_0 \in \mathbb{R}$.

Now consider a polygon with two finite vertices $w_1$ and $w_2$ and interior angles $\alpha_1\pi$ and $\alpha_2\pi$ respectively. It will be proven later that there exist prevertices $z_1$ and $z_2$ so that $f(z_k)$ $= w_k$. Without loss of generality, let $z_1 < z_2$.

Consider $f'(z) = (z - z_1)^{\alpha_1 - 1}(z - z_2)^{\alpha_2 - 1}$. This function is conformal, excluding the prevertices, and we have that

$$\begin{aligned}\arg f'(z) &= \arg(z - z_1)^{\alpha_1 - 1}(z - z_2)^{\alpha_2 - 1}\\ &= (\alpha_1 - 1)\arg(z - z_1) + (\alpha_2 - 1)\arg(z - z_2)\end{aligned}$$

Again, we consider what our function does to the real axis using the same branch cut.

Note that we have, for $z \in \mathbb{R}$,

$$\arg(z - z_k) = \begin{cases} 0 & \text{if } z > z_k, \\ \pi & \text{if } z < z_k \end{cases}$$

So with this information, we have that

$$\arg f'(z) = \begin{cases} 0 & \text{if } z_1 < z_2 < z, \\ (\alpha_2 - 1)\pi & \text{if } z_1 < z < z_k \\ (\alpha_2 - 1)\pi + (\alpha_1 - 1)\pi & \text{if } z < z_1 < z_2. \end{cases}$$

This then gives us the desired turning angles that we want.

If we repeat this process for vertices $w_1, \ldots, w_n$ with real prevertices $z_1, \ldots, z_n$ and interior angles $\alpha_1\pi, \ldots\alpha_n\pi$, we would have the following function that gives the desired turning angles

$$f'(z) = \prod_{k=1}^{n}(z - z_k)^{(\alpha_k - 1)}$$

For more flexibility , we add one more complex parameter into the formula. By multiplying $f'(z)$ by a fixed complex number $C$, we allow for a rotation and dilation of our polygon, since

$\arg Cf'(z) = \arg C + \arg f'(z)$.

$$f'(z) = C \prod_{k=1}^{n} (z - z_k)^{(\alpha_k - 1)}$$

Moreover, adding a complex number $A$ to the function $f$ allows us to shift the polygon around the complex plane. Thus by construction, we want our function to be

$$f(z) = A + C \int_{z_1}^{z} \prod_{k=1}^{n} (\zeta - z_k)^{(\alpha_k - 1)} d\zeta$$

The lower integration limit is left unspecified in Driscoll's text because the evaluation of the antiderivative for this value can merely be absorbed into our complex variable $A$. However, we shall specify it in this paper to be $z_1$.

Furthermore, notice that the equation above and Equation (1) have one slight difference: the index on the product. We shall explain in Chapter 5 why we are allowed to neglect this last product term. For now, we shall just take it as fact.

## Chapter 4

## Properties of Schwarz-Christoffel Candidates

We now continue by analyzing the function we created in the previous chapter. We propose the following:

> **Proposition**: *Let* $z_1 < z_2 < \ldots < z_{n-1} \in \mathbb{R}$ *and* $\displaystyle\sum_{k=1}^{n} \alpha_k - 1 = 2$*. Let* $z_n = \infty$ *and* $g(z) = \displaystyle\prod_{k=1}^{n-1} (z - z_k)^{\alpha_k - 1}$*. If the contour from* $z_1$ *to* $z$ *stays in the region of analyticity of* $g$*, then the map*
>
> $$f(z) = \int_{z_1}^{z} g(\zeta) d\zeta$$
>
> *maps* $\mathbb{R} \cup \infty$ *to a linear curve* $\Gamma$ *with vertices at* $f(z_k)$ *and interior angles* $\alpha_k \pi$*. Moreover, if* $\Gamma$ *is simple, then* $F$ *maps* $\mathbb{H}$ *conformally to the interior of a polygon* $P$*.*

In Chapter 3, we showed by construction that this function makes a turn of $\alpha_k - 1$ at the point $f(z_k)$. Thus $f$ gives us the interior angles of a polygon that we desire and therefore the image of the extended real axis is a linear curve. So now we are left with the following to prove:

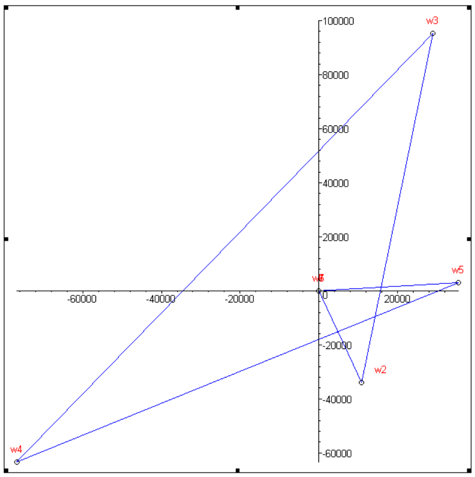
i. $f$ is well-defined.

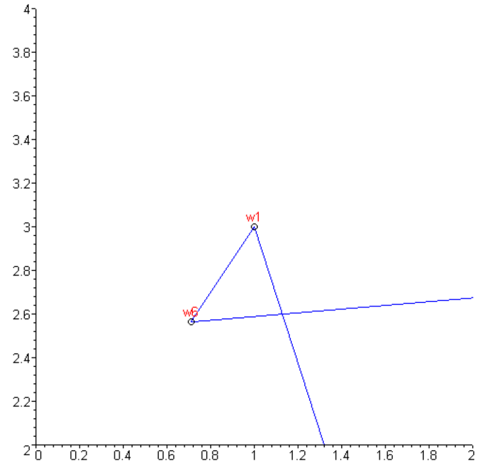
ii. $\displaystyle\lim_{z \to \infty} f(z)$ exists.

iii. $f$ extends to be continuous on the closure of $\mathbb{H}$.

iv. If $\Gamma$ is simple, $f$ is one to one and maps $\mathbb{H}$ onto the interior of $P$.

Notice that if $\Gamma$ is not simple, then injectivity, and thus conformality of our map will be impossible to attain. In Johnston's paper[4], he creates a function in the form from Equation (1) with interior angles satisfying Equation (2a) and gets the following as the image of $\mathbb{H}$:




The image on the right is just a magnification of the linear curve around the points $w_1$ and $w_6$. This is a good, concrete example that functions of the form from Equation (1) need not yield a polygon.

Now let us consider at what points $g(z)$ is analytic. Notice that by the product and chain rules of derivatives, $g'(z)$ will exist everywhere in the closure of $\mathbb{H}$, with the exception of the prevertices $z_k$'s. Thus the region of analyticity of $g$ is $\mathbb{H} \cup \mathbb{R} \setminus \{z_k\}_{k=1}^{n}$. We shall denote this domain as $\mathfrak{R}$.

For this entire chapter, we shall define $f(z)$ and $g(z)$ as such:

$$g(z) = \prod_{k=1}^{n-1} (z - z_k)^{\alpha_k - 1}$$

$$f(z) = \int_{z_1}^{z} g(\zeta) d\zeta$$

The explorations in this chapter come from section 94 of Brown [1].

## 4.1 Well-Definedness of $f$

Whenever a function is created, this should be the initial question: whether or not the output values are independent of any arbitrary choices made in computing it. In our case, our function

$$f(z) = \int_{z_1}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta$$

is a contour integral where the contour stays in the region of analyticity, namely $\mathfrak{R}$.

What we want here is path independence. So, let two contours, $C_1$ and $C_2$, both have endpoints starting at $z_1$ and ending at $z$. Moreover, let $C_1$ and $C_2$ lie entirely in $\mathfrak{R}$. [See Figure 4.1]

If we create a new contour $C = C_1 + -C_2$, notice that $C$ then is a closed contour, and $g(z)$ is analytic in the convex set $\mathfrak{R}$ containing it. [See Figure 4.1] Hence, the integral will yield a value of zero. Therefore,

$$\begin{aligned} 0 \quad = \quad \int_C g(\zeta) d\zeta &= \int_{C_1} g(\zeta) d\zeta + \int_{-C_2} g(\zeta) d\zeta \\ -\int_{-C_2} g(\zeta) d\zeta &= \int_{C_1} g(\zeta) d\zeta \\ \int_{C_2} g(\zeta) d\zeta &= \int_{C_1} g(\zeta) d\zeta \end{aligned}$$

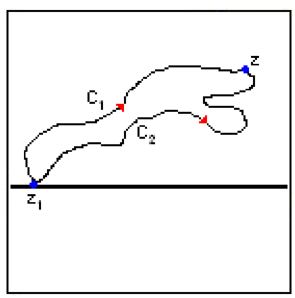



Figure 4.1: Contour Integral

Thus, we have path independence.

## 4.2 The limit of $f(z)$ as $|z| \to \infty$

We now consider how $f(z)$ behaves as $|z| \to \infty$. Let $R >$ max $|z_k|$ for k = 1, … $n-1$. For sufficiently large $R$, we have that when $|z| > R$,

$$|z - z_k| < |z| + |z_k| < |z| + |z| = 2|z|$$

Furthermore,

$$\frac{|z|}{2|z-z_k|} < 1$$

This yields the inequality

$$\frac{|z|}{2} < |z-z_k| < 2|z| \tag{4}$$

We then bound $g(z)$.

$$\begin{aligned}
|g(z)| &= \left|\prod_{k=1}^{n-1}(z-z_k)^{(\alpha_k-1)}\right| \\
&= \left|\prod_{k=1}^{n-1}(z-z_k)^{(\alpha_k-1)}\right| \\
&< \left|\prod_{k=1}^{n-1}(2|z|)^{(\alpha_k-1)}\right| && (5)\\
&< \frac{2^{-1-\alpha_n}}{|z|^{\alpha_n+1}} && (6)\\
&< \frac{M}{|z|^{\alpha_n+1}} && (7)
\end{aligned}$$

Note that (5) follows from Inequality (4) and (6) follows from Equation (2a), where the sum of the exterior angles need to add to 2. This bounds $f(z)$.

Our goal here now is to show that as $|z| \to \infty$, $f(z)$ approaches a fixed number. We do this by showing that by taking an arbitrary sequence $\{z_m\}$ that goes to $\infty$ and showing that $f(\{z_m\})$ is Cauchy. Take two complex numbers $p$ and $q$ and consider $|F(p) - F(q)|$, where the contour used in $f(z)$ stays in the region of analyticity. Thus, we have

$$\begin{aligned}
|f(p) - f(q)| &= \left|\int_{z_0}^{p} g(\zeta)d\zeta - \int_{z_0}^{q} g(\zeta)d\zeta\right| \\
&= \left|\int_{\mathrm{C}} g(\zeta)d\zeta\right|
\end{aligned}$$

where C again is a contour that stays within the region of analyticity. Notice that this contour integral is path independent, so we can use horizontal and vertical lines to do this analysis.

Let $p = a_1 + b_1\boldsymbol{i}$ and $q = a_2 + b_2\boldsymbol{i}$.

There are two cases here. The first case is if both $p$ and $q$ are real. The second case is if at least one has a positive imaginary part.


Figure 4.3: Contour Integral

Illustrated here are the two possible locations for $p$ and $q$.

Let us work with the first case. The second case is analogous. Let $b > 0$.

$$\begin{aligned}
|f(p) - f(q)| &= \left| \int_{\mathrm{C}} g(\zeta) d\zeta \right| \\
&= \left| \int_{\mathrm{C}_1} g(\zeta) d\zeta \right| + \left| \int_{\mathrm{C}_2} g(\zeta) d\zeta \right| + \left| \int_{\mathrm{C}_3} g(\zeta) d\zeta \right| \\
&= \left| \int_{p}^{p+bi} f'(\zeta) d\zeta \right| + \left| \int_{p+bi}^{q+bi} f'(\zeta) d\zeta \right| + \left| \int_{q+bi}^{q} f'(\zeta) d\zeta \right| \\
&= \left| \boldsymbol{i} \int_0^b g(p + t\boldsymbol{i}) dt \right| + \left| \int_p^q g(t + b\boldsymbol{i}) dt \right| - \boldsymbol{i} \left| \int_0^b g(q + t\boldsymbol{i}) dt \right|
\end{aligned}$$

We consider the contour integrals separately. Note now that our limits of integration are now real valued.

$$\begin{aligned}
\left| \int_0^b g(p + t\boldsymbol{i}) dt \right| &\leq \int_0^b |g(p + t\boldsymbol{i})| \, dt \\
&\leq \int_0^b \frac{M dt}{|p + t\boldsymbol{i}|^{1+\alpha_n}} \qquad (8)
\end{aligned}$$

Inequality (8) follows from Inequality (7). And since $\alpha_n > 0$, (8) $\to 0$ as $b \to \infty$.

A similar argument would give us that the other two contour integrals limit to zero as $p, q \to \infty$ and as $b \to \infty$, respectively.

Hence, $\lim_{z \to \infty} f(z) = W_n$ for some complex value $W_n$.

Thus, we specify $f(\infty) = A + CW_n = w_n$.

The argument for case two is analogous.

Therefore, as $|z| \to \infty$, $|f(z)|$ is bounded.

### 4.3 Continuity of $f$

We now consider the continuity of $f(z)$. As is, $f$ is not continuous at $z = z_j$, since $f(z_j)$ is undefined if $\alpha_j < 1$. Note that

$$g(z) = \prod_{k=1}^{n-1} (z - z_k)^{\alpha_k - 1}.$$

The only factor of $g(z)$ that is not analytic at $z = z_1$ is $(z - z_k)^{\alpha_1 - 1}$. So let

$$\phi(z) = \prod_{k=2}^{n-1} (z - z_k)^{\alpha_k - 1}.$$

Note that $\phi(z)$ is analytic at $z = z_1$ and thus has a Taylor expansion series in an open disk $|z - z_1| < R_1$. This gives us that

$$\begin{aligned}
g(z) &= (z - z_1)^{\alpha_1 - 1}\phi(z) \\
&= (z - z_1)^{\alpha_1 - 1}\left(\phi(z_1) + \frac{\phi'(z_1)}{1!}(z - z_1) + \frac{\phi''(z_1)}{2!}(z - z_1)^2 + \dots\right) \\
&= (z - z_1)^{\alpha_1 - 1}\phi(z_1) + (z - z_1)^{\alpha_1}\left(\frac{\phi'(z_1)}{1!} + \frac{\phi''(z_1)}{2!}(z - z_1) + \dots\right) \\
&= (z - z_1)^{\alpha_1 - 1}\phi(z_1) + (z - z_1)^{\alpha_1}\psi(z)
\end{aligned}$$

where:

$$\psi(z) = \frac{\phi'(z_1)}{1!} + \frac{\phi''(z_1)}{2!}(z - z_1) + \dots$$

Notice that $\alpha_1 > 0$, so the second term above is a continuous function on $z$ throughout the upper half disk if it is assigned the value 0 at $z = z_1$. Moreover, $\psi$ is analytic and thus continuous throughout the entire open disk.

Thus, for a countour and a point $z_0$ that lie in the half disk, the integral

$$\int_{z_0}^{z} (\zeta - z_1)^{\alpha_1}\psi(\zeta)d\zeta$$

is a continuous function of $z$ at $z = z_1$. Furthermore, the integral

$$\int_{z_0}^{z} (\zeta - z_1)^{\alpha_1 - 1}d\zeta \;= \frac{1}{\alpha_1}\left((z - z_1)^{\alpha_1} - (z_0 - z_1)^{\alpha_1}\right)$$

also represents a continuous function of $z$ at $z = z_1$ if it is given the value of its limit as $z$ approaches $z_1$ in the half disk. Thus $\int_{z_0}^{z} g(\zeta)d\zeta$ is continuous at $z = z_1$. Similarly, $f(z)$ is continuous at $z = z_1$ since it can be written as a contour in $\mathfrak{R}$ from $a$ to $z_0$ and then from $z_0$ to $z$.

The argument works for all $z_k$'s. Therefore $f(z)$ is continuous at $z = z_0$. Thus $f$ is continuous.

## 4.4 Injectivity of $f(z)$

Before we tackle this obstacle, let us recall a very useful theorem.

**The Argument Principle**: *If $h$ is analytic and nonzero at each point of a simple closed positively oriented contour $C$ and is meromorphic inside $C$, then*

$$\frac{1}{2\pi\mathbf{i}}\int_C \frac{h'(z)}{h(z)}dz \;= N_0(h) - N_p(h)$$

*where $N_0(h)$ and $N_p(h)$ are, respectively, the number of zeros and poles of $h$ inside $C$ with multiplicity included.*

So in this expression, let $h(z) = f(z) - w_0$, where $w_0$ is either inside or outside the polygon $P$, which is a Jordan curve. Since $w_0$ is not on the polygonal curve, then $f(z) \neq w_0$ for all $z \in \mathbb{R}$.

Consider the following contour $C$, which is illustrated below. Let $C_1$ be the contour consisting of the upper half of the circle $|z| = R$ with counterclockwise orientation. Now, each $z_k$ will define two contours, which will be dependent on a positive real value $\rho_k$, which is arbitrarily small. For $z_1$, we define the contour $\Gamma_1$ to be the upper half circle $|z - z_1|$ $= \rho_1$ with clockwise orientation. We also define $R_1$ to be the line segment $[-R, z_1 - \rho_1]$. For $z_2$, we define the contour $\Gamma_2$ similar to $\Gamma_1$ and we define $R_2$ to be the line segment $[z_1 + \rho_1, z_2 - \rho_2]$. We do this for all $z_k$'s for $k = 1, 2, ...n - 1$ with $R_n$ being the line segment connecting $[z_{n-1} - \rho_{n-1}, R]$. The contour $C$ is then the concatenation of all these contours, namely:

$$C = C_1 + R_1 + \Gamma_1 + \ldots R_{n-1} + \Gamma_{n-1} + R_n$$

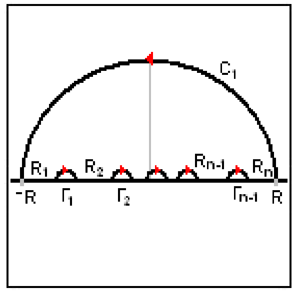



Figure 4.2: Contour Integral

Since $C$ lies entirely in the region of analyticity of $f$, then $N_p(f) = 0$. Then the number of points $z$ interior to $C$ such that $f(z) = w_0$ is

$$N_0 = \frac{1}{2\pi \boldsymbol{i}} \int_C \frac{f'(z)}{f(z) - w_0} dz.$$

We then calculate this integral.

$$\frac{1}{2\pi \boldsymbol{i}} \int_C = \frac{1}{2\pi \boldsymbol{i}} \left( \int_{C_1} + \sum_{k=1}^{n-1} \int_{\Gamma_k} + \sum_{k=1}^{n-1} \int_{R_k} \right)$$

The first summand is the contour integral of the top half of the circle $|z| = R$. The value

$$\left| \frac{1}{2\pi \boldsymbol{i}} \int_{C_1} \frac{f'(z)}{f(z) - w_0} dz \right|$$

can be estimated as follows. From Inequality (6), the integrand of the numerator is bounded above by a constant value $M$ as $r \to \infty$. Similarly, by Equation (4) of that same section, the denominator limits to $r^{\alpha_n + 1}$ as $r \to \infty$. And by using a change of variables, this modulus becomes

$$\left| \frac{1}{2\pi} \int_0^{\pi} \frac{f'(r\boldsymbol{e}^{\boldsymbol{i}t})}{f(r\boldsymbol{e}^{\boldsymbol{i}t}) - w_0} (r\boldsymbol{e}^{\boldsymbol{i}t}) dt \right| \leq \frac{M}{2r^{\alpha_n}} \tag{9}$$

Since $\alpha_n > 0$, (9) then goes to zero as $r \to \infty$.

Our second summand is the contour integral of the top half of the circle $|z - z_k| = \rho_k$. Here, we will consider the $k = 1$ case and the rest are analogous. Moreover, we shall let the circle have radius $\rho$, that is $\rho_1 = \rho$.

The value

$$\left|\frac{1}{2\pi \boldsymbol{i}}\int_{\Gamma_1}\frac{f'(z)}{f(z)-w_0}dz\right|$$

can be estimated as follows. Because $f$ is continuous on the upper half plane and on the real axis, denominator of the integrand is bounded below by a real value $M$.

Now we consider $|f'(z)|$ as $\rho \to 0$. This conceptually is the same as limiting $z \to z_1$. Thus, for some real value $B$, we have:

$$|f'(z)| = \prod_{k=1}^{n-1}|z-z_k|^{\alpha_k-1} = (z-z_1)^{\alpha_1-1}\prod_{k=2}^{n-1}|z-z_k|^{\alpha_k-1} \leq B\rho^{\alpha_k-1}$$

Moreover, the length of the contour is $\pi\rho$. Thus we have the following inequality[1]:

$$\left|\frac{1}{2\pi \boldsymbol{i}}\int_{\Gamma_1}\frac{f'(z)}{f(z)-w_0}dz\right| \leq \frac{B\rho^{\alpha_k-1}}{2\pi M}(\pi\rho) = \frac{B\rho^{\alpha_k}}{2M}$$

which goes to zero as $\rho \to 0$, since $\alpha_k > 0$.

Thus, for each of the $\Gamma_k$, the contour integral approaches 0 as $\rho_k \to 0$.

Thus, as $R \to \infty$ and $\rho_k \to 0$, $N_0 = \dfrac{1}{2\pi \boldsymbol{i}}\displaystyle\int_{\mathbb{R}}\frac{f'(z)}{f(z)-w_0}dz$.

And by the variable change $w = f(z)$, we have that:

$$N_0 = \frac{1}{2\pi \boldsymbol{i}}\int_{\Gamma}\frac{dw}{w-w_0} = \frac{1}{2\pi \boldsymbol{i}}\int_{\mathbb{R}}\frac{f'(z)}{f(z)-w_0}dz$$

The left hand side of the equation is 1 if $w_0$ is inside the contour and is 0 if $w_0$ is outside the contour. Therefore, $f(z)$ is injective. Moreover, this also shows that $f(\mathbb{H}) = P$.

This then proves that our function $f$ maps the upper half plane to the interior of a a specific polygon. Now let us consider how far we can go with this constructed function. Given a linear curve $\Gamma$ we recall that the side lengths starting at the vertex $z_k$ to be:

$$\begin{array}{lll} \displaystyle\int_{z_k}^{z_{k-1}}|g\zeta|d\zeta & \quad\text{for}\quad & k = 1\ldots n-1 \\ \displaystyle\int_{-\infty}^{z_1}|g\zeta|d\zeta & \quad\text{for}\quad & k = n \end{array}$$

[1]ML Theorem: Given a contour $C$, if —f(z)— ¡ $M$ for all values of f on $C$, and if $L$ is the length of $C$, then $\left|\int_C f(z)dz\right| \leq ML$

### 4.4.1 Triangles

From geometry, certain known information about a triangle uniquely determines it. For example, knowing three side lengths uniquely determine a triangle up to position. However, if we knew just three angles, then the triangle is uniquely determined up to both position and similarity. For example, if you knew that a triangle has three interior angles of $\frac{\pi}{3}$, then it is an equilateral triangle, but could have any real equal side lengths and could be positioned anywhere on the complex plane.

Note that $f$ uniquely determines the turning angles of the polygonal image of the real axis by the function $f$. So now take an arbitrary triangle $\triangle$ with vertices $w_1$, $w_2$, and $w_3$ and turning angles $\alpha_1$, $\alpha_2$, and $\alpha_3$, respectively. Now choose $z_1 < z_2 < z_3 = \infty$ and define:

$$f(z) = \int_{z_1}^{z} (\zeta - z_1)^{\alpha_1 - 1}(\zeta - z_2)^{\alpha_2 - 1} d\zeta$$

The resulting triangle is similar to $\triangle$ up to position and side length ratio. Thus, to fix this problem, we introduce two complex numbers $A$ and $C$. The Schwarz-Christoffel transformation for a triangle is then:

$$S(z) = A + C\int_{z_1}^{z} (\zeta - z_1)^{\alpha_1 - 1}(\zeta - z_2)^{\alpha_2 - 1} d\zeta$$

Here, the complex number $C$ fixes any rotation or scaling that may be needed. The complex number $A$ then shifts the triangle to the desired position. By construction of $f$, $S(z_1) = A = w_1$. Appended is a MAPLE program TRI.mws that illustrates this point.

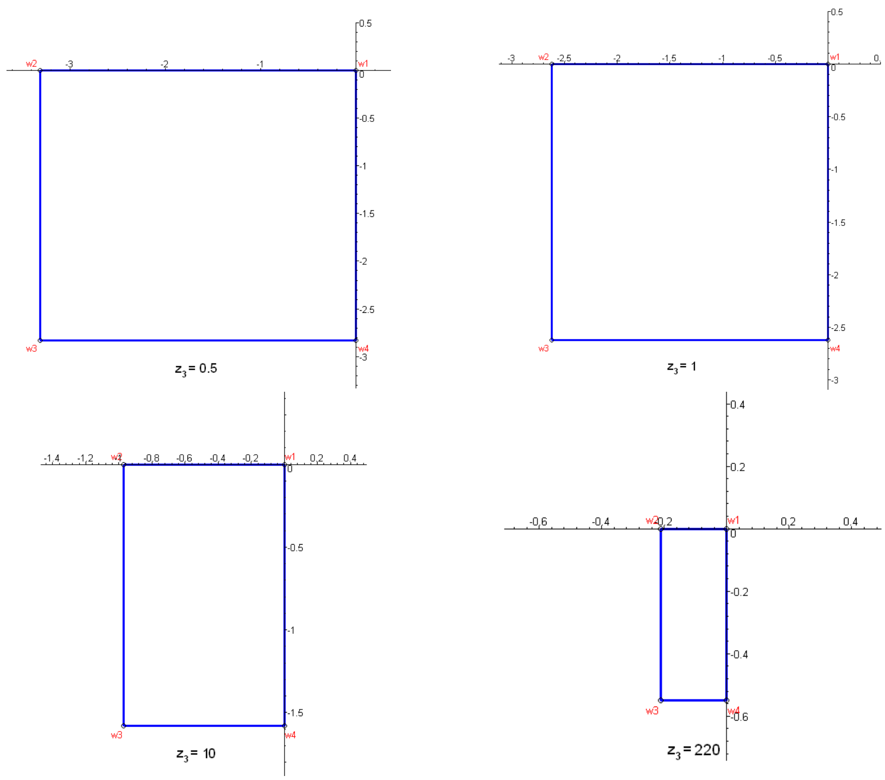
Hence, for any triangle $\triangle$, the Schwarz-Christoffel transformation $S(z)$ depends on only two variables: $A$, and $C$.

Furthermore, with closed triangles, self intersection is impossible. Thus all three sided linear curves are triangles. This cannot be said about quadrilaterals, however. This will be illustrated in the next section.

### 4.4.2 Quadrilaterals

Now let us consider quadrilaterals. Suppose we have a square of unit length, with vertices at $w_1 = 1$, $w_2 = 2$, $w_2 = 2 + \boldsymbol{i}$ and $w_3 = 1 + \boldsymbol{i}$. By the Riemann mapping theorem, there exists a function $h$ that maps $\mathbb{H}$ onto this square. Ideally, we want real pre-vertices $z_1, z_2, z_3, z_4$ so that $h(z_k) = w_k$.

Let us consider the function $f$ from the previous chapter, with $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$. We know that it takes the upper half plane and maps it conformally onto the interior of some quadrilateral with four right angles, thus a rectangle. The lengths, of course, need not be equal for all four sides. Here are some examples, using the MAPLE program QUAD.MWS appended at the end. Let us choose prevertices $z_1 = -1$, $z_2 = 0$, and $z_4 = \infty$.




Rectangles for various $z_3$, with $A = 0$ and $C = 1$.

Notice that any choice of $z_3$ will yield a rectangular image, but it will not necessarily give us the desired rectangle.

Let us consider what happens when $z_3 = 1$. Define:

$$F(a,b) = \int_a^b |z+1|^{-\frac{1}{2}}|z|^{-\frac{1}{2}}|z-1|^{-\frac{1}{2}}dz$$

Notice that $F(-1,0,z)$ is then the side length of the rectangle $R_1$. We then shall prove that the other three lengths are the same. Consider $F(0,1)$ and let $-t = z$, which gives us that $-dt = dz$.

$$\begin{aligned} F(0,1) &= \int_0^1 |z+1|^{-\frac{1}{2}}|z|^{-\frac{1}{2}}|z-1|^{-\frac{1}{2}}dz \\ &= -\int_{-1}^0 |-t+1|^{-\frac{1}{2}}|-t|^{-\frac{1}{2}}|-t-1|^{-\frac{1}{2}}dz \\ &= \int_0^1 |t-1|^{-\frac{1}{2}}|t|^{-\frac{1}{2}}|t+1|^{-\frac{1}{2}}dt \\ &= F(-1,0) \end{aligned}$$

Now we consider $F(1,\infty)$ and let $-\dfrac{1}{t} = z$, which gives us that $\dfrac{dt}{t^2} = dz$.

$$\begin{aligned} F(1,\infty) &= \int_1^\infty |z+1|^{-\frac{1}{2}}|z|^{-\frac{1}{2}}|z-1|^{-\frac{1}{2}}dz \\ &= \int_{-1}^0 \left|-\frac{1}{t}+1\right|^{-\frac{1}{2}}\left|-\frac{1}{t}\right|^{-\frac{1}{2}}\left|-\frac{1}{t}-1\right|^{-\frac{1}{2}}\frac{dt}{t^2} \\ &= \int_{-1}^0 \left(\left|-\frac{1}{t}+1\right|^{-\frac{1}{2}}|t|^{-\frac{1}{2}}\right)\left(\left|-\frac{1}{t}\right|^{-\frac{1}{2}}|t|\right)\left(\left|-\frac{1}{t}-1\right|^{-\frac{1}{2}}|t|^{-\frac{1}{2}}\right)dt \\ &= \int_1^\infty |-1+t|^{-\frac{1}{2}}|-t|^{-\frac{1}{2}}|-1-t|^{-\frac{1}{2}}dt \\ &= \int_1^\infty |t-1|^{-\frac{1}{2}}|t|^{-\frac{1}{2}}|t+1|^{-\frac{1}{2}}dt \\ &= F(-1,0) \end{aligned}$$

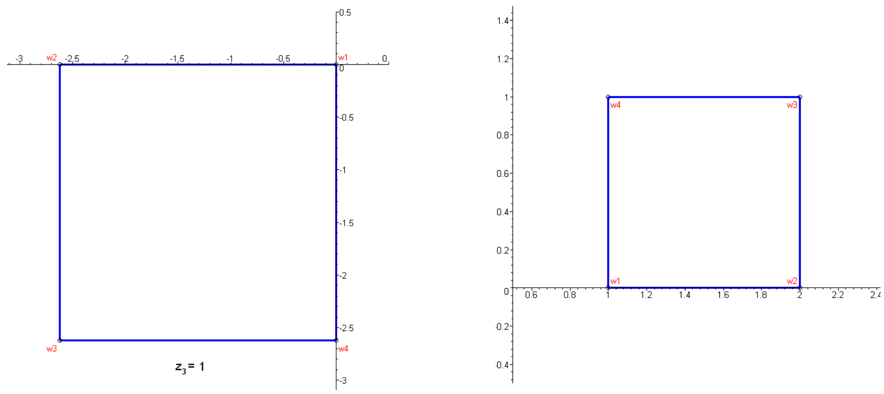
Finally, we have $F(\infty,-1)$. The argument is similar to the above. Moreover, since we have three congruent sides for a rectangle, we know that it has to be a square.

So to create the desired square from the beginning of the section, let $A = 1$ and

$$C = \frac{\exp(\pi \boldsymbol{i})}{L}$$

where:

$$L = \int_{z_1}^{z_2} |g\zeta| d\zeta$$




For both rectangles, $z_3 = 1$, thus it is a square. The square on the right is the rotated, resized, and translated version that we want.

As we will see later, the Schwarz-Christoffel map $S(z) = A + Cf(z)$ depend on three variables: $A$, $C$, and $z_3$.

The following argument shows us that any rectangle is attainable using this form.

Consider the rectangle $R_n$ with prevertices $-1, 0, n, \infty$. The length of one side of the rectangle is given by:

$$s_1(n) := \int_{-1}^{0} \frac{1}{\sqrt{x(x+1)(x-n)}}\, dx$$

Since $\sqrt{n-x} \geq \sqrt{n}$ for all $x \in [-1, 0]$, we have:

$$s_1(n) \leq \frac{1}{\sqrt{n}} \int_{-1}^{0} \frac{1}{\sqrt{-x(x+1)}}\, dx$$

Substituting $y = -x$ and completing the square inside the radical shows the latter integral is $\pi$. Thus $s_1(n) \leq \dfrac{\pi}{\sqrt{n}}$.

Next, consider the length of the adjacent side. It is given by:

$$s_2(n) := \int_{0}^{n} \frac{1}{\sqrt{x(x+1)(n-x)}}\, dx$$

Since $n - x \leq n$ throughout this interval, we can bound $s_2(n)$ from below:

$$s_2(n) \geq \frac{1}{\sqrt{n}} \int_{0}^{n} \frac{1}{x+1}\, dx = \frac{\ln(n+1)}{\sqrt{n}}$$

Dividing, we conclude $\lim_{n\to\infty} \frac{s_2(n)}{s_1(n)} = \infty$. Since we already know $\frac{s_2(1)}{s_1(1)} = 1$, the Intermediate Value Theorem[2] tells us we can make the ratio of adjacent sides of our rectangle be any number in $[1, \infty)$, and hence prevertices can be chosen to achieve any desired rectangle.

[2]**Intermediate Value Theorem**: If a function $f$ is continuous on $[a, b]$ and if $c$ is between $f(a)$ and $f(b)$, then there exists $d \in [a, b]$ so that $f(d) = c$.

Chapter 5

# Proof of the Theorem

In the preceding Chapter, we examined properties of Schwartz-Christoffel candidates, i.e., functions defined by Formula (1) of the Introduction. Notice that for $n > 3$, we haven't settled the following: when the image of $S$ is actually an n-gon and not just an n-sided linear curve, whether all n-gons are attainable from this form, and whether all conformal mappings from $\mathbb{H}$ to the interior of an n-gon take the form from Equation(1).

The first question does not seem to have a theoretically satisfying answer, though one can use techniques from numerical analysis to examine the matter for specific candidate functions. In this chapter, we will settle the second and third questions by proving the theorem stated in the introduction. But before we do so, we shall address an important preliminary: continuity on the boundary of $\mathbb{H}$.

## 5.1 Continuity at the Boundary of $\mathbb{H}$

There are definitive results in the literature concerning the possibility of extending a conformal map to a homeomorphism between the closures of the domains in question.

**Proposition 1.** *Suppose $f$ maps $\mathbb{D}$ conformally onto a domain $G$ in the extended complex plane. Then the following are equivalent:*

*(i)* *$f$ has a continuous extension to $\overline{\mathbb{D}}$ iff $\partial G$ is locally connected.*

*(ii)* *$f$ has an injective continuous extension to $\overline{\mathbb{D}}$ iff $\partial G$ is a Jordan curve.*

*(iii)* *$f$ has a bijective continuous extension to $\overline{\mathbb{D}}$ iff $\partial G$ is a Jordan curve.*

(ii) is due to Carathodory. Modern treatments of (i) and (ii) can be found in Ch. Pommerenke's text [3]. (iii) is an easy consequence of (ii).

Fix a Mobius transformation $T$ sending $\mathbb{H}$ onto $\mathbb{D}$. Then $T$ also provides a homeomorphism between the closures of these domains and thus $\mathbb{D}$ can be replaced by $\mathbb{H}$ in Proposition 1. Since polygons are Jordan curves, the following is a very special case of Part (iii) of that Proposition.

**Proposition 2.** *Suppose that $\Gamma$ is a polygon, with $P$ as its interior and let $f$ map the upper half plane $\mathbb{H}$ conformally onto $P$. Then $f$ extends continuously to the closure of $\mathbb{H}$ (to be denoted $\mathbb{H}^{cl}$). More precisely, there is a homeomorphism $F$ from $\mathbb{H}^{cl}$ onto $P \cup \Gamma$ satisfying $F\mid_{\mathbb{H}} = f$.*

Our goal in this section is a self-contained treatment of Proposition 2. We follow L. Ahlfors' approach in [5].

**Lemma**: *Fix $f, \Gamma$, and $P$ as in Proposition 2, and suppose that $a \in \Gamma$ is not a vertex. Then $h := f^{-1}$ extends to an analytic function $H$ in a disc centered at $a$, and $H'(a) \neq 0$.*

**P**roof. Fix $b \in P$ and choose $r > 0$ so that the closure of the disc $B_r(a)$ of radius $r$ centered at $a$ (1) excludes $b$ and (2) only intersects the edge of $\Gamma$ on which $a$ lies. Write $E := P \cap B_r(a)$; this is a half disc centered at $a$ and lying completely in $P$. Fix a Mobius transformation $T$ which maps $\mathbb{H}$ onto $\mathbb{D}$ and sends $b$ to 0, and write $g := T \circ f$. Since $0 \notin g(E)$, we can define an analytic function $L$ on $E$ by $L(z) := \int_a^z \frac{g'(w)}{g(w)}\, dw$. This is a branch of $\log g(z)$. Write $u$ for the real part of $L$ and $\sigma$ for the open line segment which is the intersection of $B_r(a)$ with $\Gamma$. Now suppose $(z_n)$ is a sequence in $E$ which approaches a point $p$ on $\sigma$ The conformality of $L$ tells us that the sequence $(L(z_n))$ is eventually disjoint from every compact subset of $g(E)$. In particular, this means $\lim_{n\to\infty} u(z_n) = 0$ and so defining $u$ to be zero on $\sigma$ makes $u$ continuous on $P \cup \sigma$. We can now apply the Schwarz Lemma to get a harmonic extension $U$ of $u$ to $B_r(a)$. Now $U$ is the real part of an analytic function $M$ on $\mathbb{D}$, and so $H := \exp \circ T^{-1} \circ M$ provides an analytic extention of $h$.

Now if $H'(a)$ were zero, then applying $H$ to opposite rays emanating from $a$ would result in curves such that the angle between them would exceed $\pi$ radians. This, however, contradicts the fact that both image curves must lie in the closed upper half plane.

**Proof of Proposition 2.** Let $h = f^{-1}$ as in the preceding Lemma. Consider a vertex $a = w_k$ of $\Gamma$. Then the function $z \mapsto z^{\frac{1}{\alpha_k}}$ straightens out the angle between the two edges on which $a$ lies, and thus the Lemma shows that $h$ extends to a continuous function $H$ on all of $P \cup \Gamma$. Since we already know that $H'$ does not vanish on any open edge of $\Gamma$, the mean value theorem tells us that $H$ is actually injective on each closed edge. In particular, it can't "reverse direction" on adjacent edges and so $H|_\Gamma$ is injective. Since we also know that $h'$ does not vanish on $P$, the open mapping theorem tells us that $H(\Gamma) \cap H(P) = \emptyset$, and we see that $H$ is injective on all of $P \cup \Gamma$. The fact that $\Gamma$ is a closed curve tells us that $H(\Gamma)$ exhausts the extended real axis. The proof is concluded by taking $F := H^{-1}$.

## 5.2 Completion of the Proof

We start by restating our goal. For simplification purposes, we consider the case where all prevertices are finite. This is valid because we can send, by a Möbius Transformation, the extended real axis to itself, with infinity being mapped to a finite value. This then makes the index in product of the integrand range from 1 to $n$.

> **Theorem**: *Let $P$ be the interior of a polygon $\Gamma$ having vertices $w_1, \ldots w_n$ and interior angles $\alpha_1\pi \ldots \alpha_n\pi$ in counterclockwise order. Let $S$ be any conformal, one-to-one map from the upper half plane $\mathbb{H}$ onto $P$ such that $S(\infty)$ is on the edge of $\Gamma$ joining $w_n$ to $w_1$. Then $S$ can be written in the form:*
>
> $$S(z) = A + C\int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - z_k)^{\alpha_k - 1} d\zeta \tag{1}$$
>
> *where $A$ and $C$ are complex constants, and $z_0 < z_1 < \cdots < z_n$ are real numbers satisfying $S(z_k) = w_k$ for $k$ = 1, $\ldots$ $n-1$.*

The following argument is mostly from Driscoll [2].

Proposition 2 of the preceding section tells us that $S$ can be continuously extended to the closed upper half plane. For $k = 1, \ldots, n$, set $z_k := S^{-1}(w_k)$; these are called prevertices. The reflection principle allows us to extend $S$ and $S'$ analytically across the real axis everywhere except at these prevertices. The technique of straightening angles at vertices of $\Gamma$ used in the proof of Proposition 2 shows that for each $k$, the function $z \mapsto (z - z_k)^{1-\alpha_k} S'(z)$ has an analytic extension in a neighborhood of $z_k$. Thus we can write

$$S'(z) = (z - z_k)^{\alpha_k - 1}\psi(z)$$

for some function $\psi(z)$ analytic in a neighborhood of $z_k$. Thus, we have the following:

$$\begin{aligned}
S'(z) &= (z - z_k)^{\alpha_k - 1}\psi(z) \\
S''(z) &= (\alpha_k - 1)(z - z_k)^{\alpha_k - 2}\psi(z) + \psi'(z)(z - z_k)^{\alpha_k - 1} \\
\frac{S''(z)}{S'(z)} &= \frac{(\alpha_k - 1)(z - z_k)^{\alpha_k - 2}\psi(z)}{(z - z_k)^{\alpha_k - 1}\psi(z)} + \frac{\psi'(z)(z - z_k)^{\alpha_k - 1}}{(z - z_k)^{\alpha_k - 1}\psi(z)} \\
&= \frac{\alpha_k - 1}{z - z_k} + \frac{\psi'(z)}{\psi(z)}
\end{aligned}$$

This, therefore, implies that $\dfrac{S''(z)}{S'(z)}$ has a simple pole with residue $\alpha_k - 1$ at $z = z_k$. Therefore, the following function is entire:

$$\frac{S''(z)}{S(z)} - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - z_k} \tag{10}$$

Moreover, since all of the prevertices are finite, then $S$ is analytic at $z = \infty$. A Laurent expansion there implies that $\dfrac{S''(z)}{S'(z)} \to 0$ as $z \to \infty$. Also, each term of the summand goes to 0 as $z \to \infty$. Thus we have an entire bounded function. So, by Liouville's Theorem[1], we have that Expression (21) is constant and hence is identically 0.

[1]**Liouville's Theorem**: If $f$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Now we integrate. Notice that $(\log(S'))' = \dfrac{S''(z)}{S'(z)}$ so we use this substitution.

$$
\begin{aligned}
(\log(S'))' &= \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - z_k} \\
\log(S') &= \int \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - z_k} dz \\
\log(S') &= \sum_{k=1}^{n} \int \frac{\alpha_k - 1}{z - z_k} dz \\
\log(S') &= \sum_{k=1}^{n} (\alpha_k - 1)\ln(z - z_k) + C_1 && C_1 \in \mathbb{C} \\
\log(S') &= \sum_{k=1}^{n} \ln\left((z - z_k)^{(\alpha_k - 1)}\right) + C_1 \\
S' &= \exp\left(\sum_{k=1}^{n} \ln\left((z - z_k)^{(\alpha_k - 1)}\right) + C_1\right) \\
S' &= \exp\left(\sum_{k=1}^{n} \ln\left((z - z_k)^{(\alpha_k - 1)}\right)\right)\exp(C_1) \\
S' &= C \prod_{k=1}^{n} \exp(\ln\left((z - z_k)^{(\alpha_k - 1)}\right) && C = \exp(C_1) \\
S' &= C \prod_{k=1}^{n} (z - z_k)^{(\alpha_k - 1)} \\
S &= A + C \int \prod_{k=1}^{n} (z - z_k)^{(\alpha_k - 1)} && A \in \mathbb{C}
\end{aligned}
$$

We then have the desired Schwarz-Christoffel form.

### 5.3 Infinite Prevertex

It is often convenient to use $z = \infty$ as one of the prevertices. By convention, we shall take $z_n$ to be the infinite prevertex. It will be proved later that our function gets simplified to the following:

**Theorem**: *Let $P$ be the interior of a polygon $\Gamma$ having vertices $w_1, \ldots w_n$ and interior angles $\alpha_1\pi \ldots \alpha_n\pi$ in counterclockwise order. Let $S$ be any conformal, one-to-one map from*

*$\mathbb{H}$ to $P$ with $S(\infty) = w_n$. Then $S$ can be written in the form:*

$$S(z) = A + C \int_{z_0}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta \tag{3}$$

*for some complex constants $A$ and $C$ and extended reals $z_1 < z_2 < \ldots < z_{n-1} < z_n = \infty$ where $S(z_k) = w_k$ for $k$ = 1, $\ldots$ $n-1$.*

## 5.4 Vertices With $\alpha_k = 1$

It was earlier specified that for the interior angle $\alpha_k\pi$ at a finite vertex $w_k$, $0 < \alpha_k \leq 2$. Certainly, then, it is admissible for $\alpha_k = 1$, and thus its turning angle is 0. Graphically, this would mean that a vertex is located on an edge of the polygon.

Moreover, for any n-gon, we can create an (n+1)-gon by adding another vertex with turning angle 0 anywhere along the edges. Equation (2a) is still preserved, for adding such a vertex does not change our summation.

How is Equation (3) affected by adding a vertex with turning angle 0? Suppose we insert such a vertex $w_0$ anywhere on our polygon, say between $w_j$ and $w_{j+1}$ where $1 \leq j < n-1$. Equation (3) then becomes:

$$f(z) = \int_{z_0}^{z} (\zeta - z_0)^{(\alpha_0 - 1)} \prod_{k=1}^{j} (\zeta - z_k)^{(\alpha_k - 1)} \prod_{k=j+1}^{n-1} (\zeta - z_k)^{(\alpha_k - 1)} d\zeta$$

And since $\alpha_0 = 1$, the additional product term in the integral becomes $(\zeta - z_0)^0 = 1$, which does not affect the rest of the integrand.

With this in mind, we can then ignore such "vertices" (or add them for convenience), for they add no pertinent information into our calculations. For example, although we have a quadrilateral on the right diagram in Figure 2.3, $w_1$ and $w_3$ both have turning angles of 0 as $w_2$ and $w_4$ are moved infinitely far away from the right respectively. Thus we can neglect $w_1$ and $w_3$ in our diagram and we are down to a 2-gon, with both vertices infinite.

## 5.5 Degrees of Freedom in the Formula

If you look back at the theorem, notice that we have that $S(\infty) = w_n$ as part of the quantifiers in our hypothesis. We will discuss in this section this interesting stipulation.

Let $S$ be a conformal map that sends $\mathbb{H}$ onto the interior of a polygon $P$. From Section 5.1, we have that $S$ extends continuously to $\mathbb{H}^{cl}$ onto the polygon $\Gamma$. Thus, the vertices $w_k$ of $P$ have a real valued preimage $x_k$ so that $S(x_k) = w_k$.

Section 4.4 gave us that $\lim\limits_{z\to\infty} |S(z)|$ is bounded. Moreover, section 5.1 gave us that $S(\infty)$ is on the polygon $\Gamma$ since $\infty$ is on the boundary of the upper half plane $\mathbb{H}$. Thus, if $\infty$ is not already a prevertex, we can introduce $S(\infty)$ as a vertex on $\Gamma$ with a turning angle of $\pi$.

Thus, by renumbering and inserting new vertices, we can guarantee that $S(\infty)$ is indeed $w_n$.

From construction in Chapter 3, one would think that we should let the product in the integrand index up to $n$. This would be the case if $\infty$ was a prevertex of a vertex with turning angle $\pi$.

Now if we were given $S(z)$ with no $z_k = \infty$, then we use a Möbius transformation so that one of the $z_k$'s $= \infty$. Let:

$$S(z) = A + C \int_{z_1}^{z} \prod_{k=1}^{n} (\zeta - z_k)^{\alpha_k - 1} d\zeta$$

If we followed the two types of Möbius Transformations that sends $\mathbb{H}$ onto $\mathbb{H}$, notice that the first form would not make any of the current $z_k$'s go to $\infty$. With the second case, however, if one of the $z_k$'s $= 0$, then $T(0) = \infty$. So we consider this case. Take $z_j = 0$, $1 \le j \le n$.

Using the following Möbius transformation, let:

$$\zeta = -t^{-1}. \qquad \text{Then} \qquad d\zeta = t^{-2}.$$

$$
\begin{aligned}
S(z) &= A + C\int_{z_1}^{z}\prod_{k=1}^{n}(\zeta - z_k)^{\alpha_k - 1}d\zeta \\
&= A + C\int_{-\frac{1}{z_1}}^{n}\prod_{k\neq j}\left(\left(-\frac{1}{t} - \frac{1}{z_k}\right)^{\alpha_k-1}\right)\left(-\frac{1}{t}\right)^{\alpha_j-1}t^{-2}dt \\
&= A + C\int_{-\frac{1}{z_1}}^{n}\prod_{k\neq j}\left((t+z_k)^{\alpha_k-1}\left(\frac{1}{t}\right)^{\alpha_k-1}(-1)^{\alpha_k-1}\right)\left(\frac{1}{t}\right)^{\alpha_j-1}(-1)^{\alpha_j-1}t^{-2}dt \\
&= A + C\int_{-\frac{1}{z_1}}^{n}\prod_{k\neq j}(t+z_k)^{\alpha_k-1}\prod_{k=1}^{n}\left(\frac{1}{t}\right)^{\alpha_k-1}t^{-2}dt \\
&= A + C\int_{-\frac{1}{z_1}}^{n}\prod_{k\neq j}(t+z_k)^{\alpha_k-1}dt
\end{aligned}
$$

Thus we now have one less factor. The final step at the second product uses Equation (2a) to cancel the factors involving $t$.

Now as for choosing prevertices for our function $S(z)$, we propose the following:

**Proposition**: *Let $S$ be a Schwarz-Christoffel transformation from $\mathbb{H}$ to the interior $P$ of a polgyon $\Gamma$. Then we can choose to prevertices $z_1$ and $z_2$ so that $f(z_1)$ $= w_1$ and $f(z_2) = w_2$ for some Schwarz-Christoffel transformation $f$ from $\mathbb{H}$ to $P$.*

Note again that since $S$ extends continuosuly to its boundary, we are guaranteed real prevertices for each vertex. Now, note that there exists a Möbius transformation $g$: $\mathbb{H} \to \mathbb{H}$ so that $g(z_1) = x_1$, $g(z_2) = x_2$ and $g(\infty) = x_n$, where $z_1, z_2 \in \mathbb{R}$

Now consider another function $h$ that conformally sends $\mathbb{H}$ onto $P$, such that $h(z_1) = w_1$, $h(z_2) = w_2$ and $h(\infty) = w_n$. Note that $h^{-1} \circ S$ sends $\mathbb{H}$ to itself. Moreover, we have that:

$$(h^{-1} \circ S)(z_1) = h^{-1}(S(z_1)) = h^{-1}(w_1) = z_1$$

Similiarly, $(h^{-1} \circ S)(z_2) = z_2$. And since $z_1$ and $z_2$ are distinct, they both cannot be zero. So applying Schwarz' Lemma, $h^{-1} \circ S$ is then a rotation. And, since $h^{-1} \circ S : \mathbb{H} \to \mathbb{H}$, we have that $h^{-1} \circ S = z$, thus it is a conformal map and can be identified by three points.

Finally we analyze the complex constants $A$ and $C$. Suppose that we have:

$$S(z) = A_1 + C_1 \int_{z_1}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta$$
$$= A_2 + C_2 \int_{z_2}^{z} \prod_{k=1}^{n-1} (\zeta - \tilde{z_k})^{\alpha_k - 1} d\zeta$$

Note that without loss of generality, we can make the lower limit of the integral the same, and it would only alter the complex constant term $A_k$.

$$S(z) = A_1 + C_1 \int_{z_1}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta = \tilde{A}_2 + C_2 \int_{z_1}^{z} \prod_{k=1}^{n-1} (\zeta - \tilde{z_k})^{\alpha_k - 1} d\zeta$$

Note that $S(z_1)$ would make the integral equal zero. Therefore, $A_1 = \tilde{A}_2$. Now consider the antiderivative of both versions of $S(z)$.

$$S'(z) = C_1 \int_{z_1}^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta = C_2 \int_{z_1}^{z} \prod_{k=1}^{n-1} (\zeta - \tilde{z_k})^{\alpha_k - 1} d\zeta$$

Note that this shows where the poles of $S(z)$ occur, meaning that the set $\{z_k\}$ and the set $\{\tilde{z_k}\}$ is just a permutation of the same elements. Thus the product terms are the same and hence $C_1 = C_2$.

So overall, $S(z)$ depend on $n - 3$ prevertices and the complex constants $A$ and $C$.

## Chapter 6

## Examples and Variants

Now that we have all this machinery, let us consider regions with n vertices.

### 6.1 Polygons With One and Two Vertices

We have discussed all polygons with one vertex in Chapter 3. (See Fig. 6.1) It turns out that the Schwarz-Christoffel Transformation for an arbitrary one-gon is:

$$S(z) = A + C \int_0^z d\zeta$$

for some complex constants $A$ and $C$.

We also discussed one polygon with two vertices in the same chapter. There is, however, another polygon with two vertices. Let our 2-gon have vertices $w_1$ and $w_2$ with prevertices $z_1$, $z_2$. As for the interior angles $\alpha_1\pi$ and $\alpha_2\pi$, note that we need $(1-\alpha_1) + (1 - \alpha_2) = 2$. Thus, $\alpha_1 + \alpha_2 = 0$. This gives rise to two cases.


Figure 6.1: Polygons of One or Two Vertices

The diagram on the left shows a polygon with one vertex, while the diagram on the right shows a polygon with two infinite vertices.

The first case for a polygon with two vertices gives us that $\alpha_1 = -\alpha_2$, neither of which is zero. (See Fig. 6.2) Thus if one is positive, the other is negative. Hence, one vertex is infinite. This is exactly the case described in Chapter 3, and the Schwarz-Christoffel transformation following:

$$S(z) = A + C\int_{z1}^{z} (\zeta - z_1)^{\alpha_1 - 1} d\zeta$$

where again, $A$ and $C$ are complex constants.

A way to visualize polygons of this form is by taking a quadrilateral (as shown in Fig 6.2) and limiting $w_4 \to \infty$. This forces $\alpha_1$ and $\alpha_3 = 1$, while keeping $\alpha_2$ intact. Thus $w_1$ and $w_3$ can be ignored and we are left with two vertices, one finite ($w_2$) and the other infinite $w_4$.

Our example from section 3.3 is a polygon with two vertices, one being infinite and the other is finite. Thus it is of this form with $A = 0$, $C = 1$ and $\alpha_1 = \frac{1}{2}$.

The second case is when $\alpha_1 = \alpha_2 = 0$. If this is the case, note that this then implies that both vertices are infinite. (See Fig. 6.1) Here, the Schwarz-Christoffel transformation is

$$\begin{aligned} S(z) &= A + C\int^{z} (\zeta - z_1)^{(0-1)} d\zeta \\ &= A + C\log(z - z_1) + C\log(z_1 - z_1) \\ &= C\log(z - z_1) + A_1 \end{aligned}$$

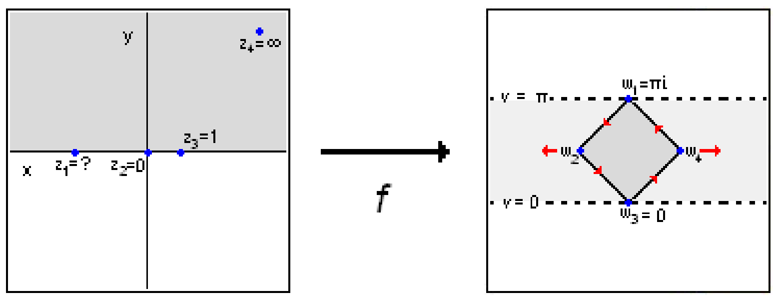

Figure 6.2: Polygons of Two Vertices

The diagram on the left is a quadrilateral with one vertex going to infinity, while another vertex has a stationary interior angle. The diagram on the right is a specific example, where $\alpha_1 = \frac{7}{4}$.

Here, it is obvious that $A_1$ has to be $\infty$. And since we specified $A = f(z_0)$, this further stresses the necessity of both vertices to be at $\infty$.

Polygons of this form can be interpreted geometrically as in Figure 2.3 and then ignoring the two vertices with turning angle 0.

Here's a concrete example of mapping $\mathbb{H}$ onto a 2-gon: Let us consider mapping the upper half plane $H$ to an infinite strip bounded above by the line $\Im(z) = \pi$ and bounded below by the real axis. This is implemented by the function $\log(z)$. Let us retrieve this function using Equation (3).




This infinite strip can be visualized like the diagram on the right in Fig. 2.3. We have that $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = \alpha_4 = 0$. We are allowed to specify three prevertices. We first let $z_4 = \infty$. Thus Equation (3) becomes:

$$\begin{aligned} f(z) &= A + C\int^{z} (\zeta - z_1)^{(1-1)}(\zeta - z_2)^{(0-1)}(\zeta - z_3)^{(1-1)} d\zeta \\ &= A + C\int^{z} (\zeta - z_2)^{(-1)} d\zeta \\ &= A + C\log(z - z_2) \end{aligned}$$

Now, by choosing two more prevertices, $z_2 = 0$, $z_3 = 1$ our function becomes:

$$f(z) = A + C\log(z)$$

Moreover, we have that $f(3) = 0 = A +$ C $\log(1)$, thus $A = 0$. Now we consider our unspecified prevertex, $z_1$. We are given that $f(z_1) = \pi \boldsymbol{i} = C\log(z_1) = C\ln|z_1| + C\pi\boldsymbol{i}$. Equating imaginary parts gives us that $C\pi\boldsymbol{i} = \pi\boldsymbol{i}$, thus $C = 1$. Equating real parts gives us that $C\ln|z_1| = 0$, thus $|z_1| = 1$, or $z_1 = -1$ or 1. Since $z_3 = 1$, then $z_1 = -1$.

Thus we have the equation $f(z) = \log(z)$. We then check to see if $f(z_4) = w_4$. Note that as $z \to \infty$, $f(z) \to \infty = w_4$. Now we are sure that we have the right equation.

This takes care of n-gons for n = 1 and 2.

## 6.2 Triangles

Note that since we are allowing $w = \infty$ to be a polygonal vertex, triangles that we will consider need not be the conventional bounded triangles with finite area. Throughout this section, let our triangle $\Gamma$ have vertices $w_1$, $w_2$, $w_3$, have interior angles $\alpha_1\pi$, $\alpha_2\pi$, $\alpha_3\pi$ and have prevertices $z_1$, $z_2$, $z_3$ respectively. Equation (2b) gives us that $\alpha_1 + \alpha_2 + \alpha_3 = 3 - 2$. Therefore, $1 = \alpha_1 + \alpha_2 + \alpha_3$. Moreover, if a vertex $w$ is infinite, then the corresponding interior angle is $\alpha\pi$, where $\alpha \in [-2, 0]$. Thus we can have at most two infinite vertices.


Figure 6.3: Triangles With Two Infinite Vertices

The diagram on the left shows a triangle with two infinite vertices. For the diagram on the right, as $w_1$ moves infinitely far to the left $\alpha_2 \to 2$, thus creating a slit.

### 6.2.1 Triangles with two infinite vertices

Triangles with two infinite vertices arise when two interior angles are not positive. Let $w_1$ and $w_3$ be the triangle's two infinite vertices. Then $\alpha_2 = 1 - \alpha_1 - \alpha_3$. Thus, $\alpha_2 \geq 1$. Geometrically, this can be visualized by taking a pentagon and allowing two vertices to go to infinity. (See diagram on the left of Fig. 6.3) As $w_2$ and $w_5$ are moved to infinity (with their arguments preserved), notice that $\alpha_1$ remains unaltered, while $\alpha_3$ and $\alpha_4$ limit to 1. Thus they can be ignored and we result in the desired triangle: an infinite strip with a kinked side.

A special case of this type of triangle is when $\alpha_2 = 2$, which results in the kinked side becoming a slit. (See diagram on the left of Fig. 6.3)

As for the Schwarz-Christoffel transformation of this form, we cannot write it explicitly. This is because we need $\alpha_2$ in our integrand, which could be any real number between 0 and $2\pi$.

### 6.2.2 Triangles with one infinite vertex

Now we consider triangles with just one infinite vertex. Because there is only one, we use Equation (2a) to calculate its corresponding interior angle. Without loss of generality, let $w_3$ be our infinite vertex. Recall then that $\alpha_3 \in [-2, 0]$ and that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. This gives us some restrictions, but we still have a broad range of triangles with one infinite vertex.

In general, the Schwarz-Christoffel transformation for a triangle is:

$$S(z) = A + C \int_{z_1}^{z} (\zeta - z_1)^{(\alpha_1 - 1)} (\zeta - z_2)^{(\alpha_2 - 1)} d\zeta \tag{11}$$

We could then manipulate this function as follows. By choosing $z_1 = 0$ and $z_2 = 1$, we have:

$$\begin{aligned} S(z) &= A + C \int_{z_1}^{z} (\zeta - z_1)^{(\alpha_1 - 1)} (\zeta - z_2)^{(\alpha_2 - 1)} d\zeta \\ &= A + C \int_{0}^{z} (\zeta)^{(\alpha_1 - 1)} (\zeta - 1)^{(\alpha_2 - 1)} d\zeta \\ &= A + C \int_{0}^{z} (\zeta)^{(\alpha_1 - 1)} (1 - \zeta)^{(\alpha_2 - 1)} (-1)^{(\alpha_2 - 1)} d\zeta \qquad (12) \\ &= A + C \int_{0}^{z} (\zeta)^{(\alpha_1 - 1)} (1 - \zeta)^{(\alpha_2 - 1)} d\zeta \qquad (13) \end{aligned}$$

The power of $-1$ in Equation (12) is then absorbed into our complex constant $C$.

Equation (13) then gives us that for general triangles with one vertex at infinity, we encounter the incomplete beta function

$$B_z(p, q) = \int_{0}^{z} (\zeta)^{p} (1 - \zeta)^{q-1} d\zeta$$

Thus:

$$S(z) = A + C B_z(\alpha_1, \alpha_2)$$

The diagram on the right in Fig. 2.1 as well as Diagrams A, C, and E in Fig 2.2 are examples of triangles with one infinite vertex. Let us then try to give the exact Schwarz-Christoffel equation for these polygons. For convenience, here are those diagrams again:

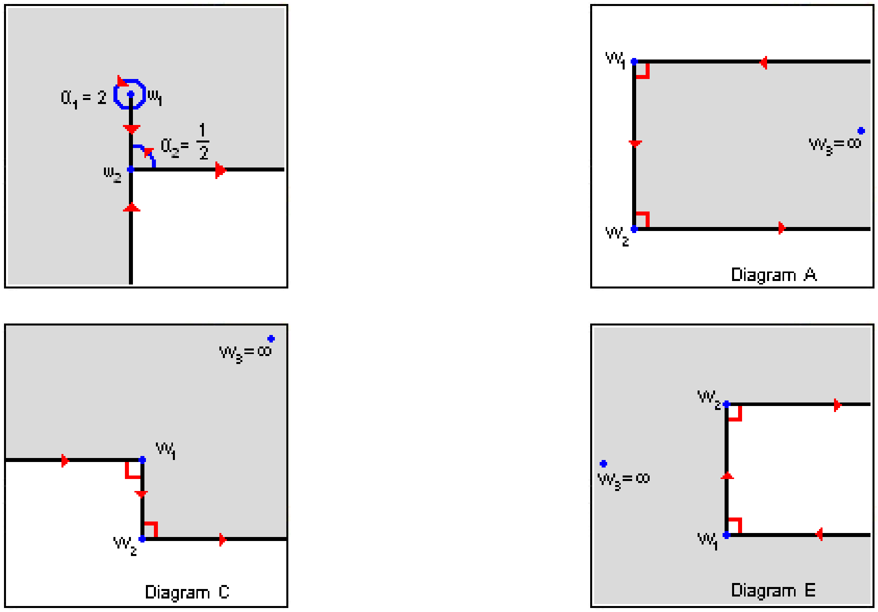



Figure 6.4: Triangles With One Infinite Vertex

### 6.2.3 Specific Examples

With the aid of a table of integrals, Equation (11) can actually be computed for nice values of $\alpha_1$ and $\alpha_2$. Here, we explore such instances.

#### Slit Triangle

Using the diagram with the slit, we have: $\alpha_1 = 2$, $\alpha_2 = \frac{1}{2}$. Thus, $\alpha_3 = -\frac{3}{2}$. The Schwarz-Christoffel Transformation in this case is

$$\begin{aligned} S(z) &= A + C\int_{z_1}^{z} (\zeta - z_1)^{(2-1)}(\zeta - z_2)^{(\frac{1}{2}-1)}d\zeta \\ &= A + C\int_{z_1}^{z} (\zeta - z_1)(\zeta - z_2)^{-\frac{1}{2}}d\zeta \end{aligned}$$

Choose $z_1 = 0$ and $z_2 = 1$. Using a table of integrals, we have

$$\begin{aligned} S(z) &= A + C\int_{0}^{z} \zeta(\zeta - 1)^{-\frac{1}{2}}d\zeta \\ &= A + \frac{2}{3}C(z+2)\sqrt{z-1} \\ &= A + C(z+2)\sqrt{z-1} \end{aligned}$$

If, further, we specify that $w_1 = 2\boldsymbol{i}$ and $w_2 = 0$, we get that $2\boldsymbol{i} = S(0) = A + 2C\boldsymbol{i}$, thus $A = 0$ and $C = 1$.

So for this specific slit triangle, our transformation is

$$S(z) = (z+2)\sqrt{z-1}.$$

And as $z \to \infty$, $f(z) \to \infty$. Thus we have the desired function.

Diagram A

Using Diagram A, we have: $\alpha_1 = \alpha_2 = \frac{1}{2}$. Thus, $\alpha_3 = 0$. The Schwarz-Christoffel Transformation in this case is

$$\begin{aligned} S(z) &= A + C\int^z (\zeta - z_1)^{(\frac{1}{2}-1)}(\zeta - z_2)^{(\frac{1}{2}-1)}d\zeta \\ &= A + C\int^z (\zeta - z_1) - \frac{1}{2}(\zeta - z_2)^{-\frac{1}{2}}d\zeta \\ &= A + C\int^z \frac{d\zeta}{\sqrt{(\zeta - z_1)(\zeta - z_2)}} \end{aligned}$$

By specifying $z_1 = -1$ and $z_2 = 1$, and with the aide of a tale of integrals, we have

$$\begin{aligned} S(z) &= A + C\int^z \frac{d\zeta}{\sqrt{(\zeta + 1)(\zeta - 1)}} \\ &= A + C\cosh^{-1}(z) \end{aligned}$$

Again, by further specifying our polygon to have vertices $w_1 = \pi\boldsymbol{i}$ and $w_2 = 0$, we have $\pi\boldsymbol{i} = S(-1) = A + C\cosh^{-1}(-1) = \pi\boldsymbol{i}$. Thus, $A = 0$ and $C = 1$.

So for this specific version of Diagram A, our transformation is

$$S(z) = \cosh^{-1}(z).$$

Diagram C

Using Diagram A, we have: $\alpha_1 = \frac{3}{2}$ and $\alpha_2 = \frac{1}{2}$. Thus, $\alpha_3$ = -1. The Schwarz-Christoffel Transformation in this case is

$$\begin{aligned} S(z) &= A + C\int^z (\zeta - z_1)^{(\frac{3}{2}-1)}(\zeta - z_2)^{(\frac{1}{2}-1)}d\zeta \\ &= A + C\int^z (\zeta - z_1)\frac{1}{2}(\zeta - z_2)^{-\frac{1}{2}}d\zeta \\ &= A + C\int^z \frac{\sqrt{\zeta - z_1}}{\sqrt{\zeta - z_2}}d\zeta \end{aligned}$$

Let $z_1 = -1$ and $z_2 = 1$. Our equation then becomes

$$\begin{aligned} S(z) &= A + C\int^z \frac{\sqrt{\zeta + 1}}{\sqrt{\zeta - 1}}d\zeta \\ &= A + C\int^z \frac{\zeta + 1}{\sqrt{\zeta^2 - 1}}d\zeta \end{aligned}$$

Integration by parts is then applied here. Let

$$u = \zeta + 1 \qquad \text{and} \qquad dv = \frac{d\zeta}{\sqrt{\zeta^2 - 1}}.$$

Then

$$du = d\zeta \qquad \text{and} \qquad v = \cosh^{-1}(\zeta).$$

Our equation then becomes

$$\begin{aligned} S(z) &= A + C\left((z+1)\cosh^{-1}(z) - \int^z \cosh^{-1}(\zeta)d\zeta\right) \\ &= A + C\left((z+1)\cosh^{-1}(z) - z\cosh^{-1}(z) - (1+z)\sqrt{\frac{z+1}{z-1}}\right) \\ &= A + C\left(\cosh^{-1}(z) + (z+1)\frac{\sqrt{z^2-1}}{z+1}\right) \\ &= A + C(\cosh^{-1}(z) + \sqrt{z^2-1}) \end{aligned}$$

Now let us consider the polygon of this form where $w_1 = \pi\boldsymbol{i}$ and $w_2 = 0$. We have that

$$0 = S(1) = A + C(\cosh^{-1}(1) + \sqrt{1-1}) = A + C(0).$$

Thus $A = 0$. Moreover, we have that

$$\pi \boldsymbol{i} = S(-1) = C(\cosh^{-1}(-1) + \sqrt{-1-1}) = C(\pi \boldsymbol{i} + \sqrt{2}\pi \boldsymbol{i}).$$

Thus,

$$C = \frac{\pi}{\pi + \sqrt{2}}.$$

So for this specific version of Diagram C, our equation is

$$S(z) = \frac{\pi}{\pi + \sqrt{2}} \left( \cosh^{-1}(z) + \sqrt{z^2 - 1} \right)$$

### 6.2.4 Diagram E

Finally, using Diagram E, we have: $\alpha_1 = \alpha_2 = \frac{3}{2}$. Thus, $\alpha_3 =$ -2. The Schwarz-Christoffel Transformation in this case is

$$\begin{aligned} S(z) &= A + C \int^z (\zeta - z_1)^{(\frac{3}{2}-1)} (\zeta - z_2)^{(\frac{3}{2}-1)} d\zeta \\ &= A + C \int^z (\zeta - z_1)\frac{1}{2} (\zeta - z_2)^{\frac{1}{2}} d\zeta \end{aligned}$$

Let $z_1 = -1$, $z_2 = 1$. Our function then becomes

$$\begin{aligned} S(z) &= A + C \int^z \sqrt{\zeta + 1}\sqrt{\zeta - 1} d\zeta \\ &= A + C \int^z \sqrt{\zeta^2 - 1} d\zeta \\ &= A + C \left( \frac{1}{2} \left( z\sqrt{z^2 - 1} - \cosh^{-1}(z) \right) \right) \qquad (14) \\ &= A + C(z\sqrt{z^2 - 1} - \cosh^{-1}(z)) \end{aligned}$$

Here, Equation (14) is derived from a table of integrals.

Now let us again consider a polygon of this form with specified vertices. Let $w_1 = 0$ and $w_2 = \pi \boldsymbol{i}$. Thus, $0 = S(-1) = A + C(-\sqrt{(-1)^2 - 1} - \cosh^{-1}(-1)) = A + -C(\pi \boldsymbol{i})$. Thus, we have $A = C\pi \boldsymbol{i}$. Also, we have $\pi \boldsymbol{i} = S(1) = \pi \boldsymbol{i} + C(\sqrt{(1)^2 - 1} - \cosh^{-1}(1)$. This gives us that $0 = C(0)$, which yields no further information. Hence, here we have infinitely many values for $A$ and $C$ that will give us the desired result. We make life easy and let $C = 1$, which makes $A = \pi \boldsymbol{i}$.

This takes care of four polygons of this form. Notice that they aren't too hard, but one would need a good table of integrals or a computer algorithm system like MAPLE for other obscure interior angles. We then venture on to one more type of triangle, the type with finite area that everyone is accustomed to.

### 6.2.5 Triangles With No Infinite Vertices

While easier to visualize, triangles with no infinite vertices are much harder to calculate. Again, we have that the Schwarz-Christoffel transformation is $S(z) = A + CB_z(\alpha_1, \alpha_2)$

## 6.3 Variants of the Schwarz-Christoffel Transformation

In this section, we will apply Riemann's mapping theorem to polygons from other domains. We will consider another domain here, the unit disc $D$. Of course, other domains can also be used. The explorations in this chapter are from Driscoll's book in Chapter 4[2]. There, he considers other domains including strips, rectangles, and fractals. He also gives a variation of Schwarz-Christoffel transformation in which the target space is the exterior of the polygon.

### 6.3.1 Schwarz-Christoffel Mapping on $\mathbb{D}$

The Riemann Mapping theorem states that we have a one-to-one analytic function between any two simply connected open domains. So far, we have worked only with the upper half plane. In this chapter, we will venture into the open unit disc $D$ and work with the analogous information.

Notice that one way to extract the Schwarz-Christoffel formula for mapping the unit disc to a desired polygon is by using composiion of functions. First map $\mathbb{D}$ to $\mathbb{H}$, then use the machinery we previously derived to then map $\mathbb{H}$ to the polygon.

We then need a function $g(z)$ that maps the boundary of the unit disc to the real axis of the complex plane.

So the plan is as follows. Given a polygon $P$, we want

$$h := \mathbb{D} \to P \quad h(z) = S(g(z))$$

where $S$ is the Schwarz-Christoffel transformation that maps $\mathbb{H}$ conformally onto $P$.

### 6.3.2 Construction

We analyze $h'(z)$. Note that $g(z) = \dfrac{1-z}{1+z}\boldsymbol{i}$ is the Möbius transformation that sends the unit disc to the upper half plane, thus we have

$$\begin{aligned} g'(z) &= -\boldsymbol{i}(1+z)^{-1} - \boldsymbol{i}(1-z)(1+z)^{-2} \\ &= \frac{-2\boldsymbol{i}}{(1+z)^2} \end{aligned}$$

$$\begin{aligned} h'(z) &= S'(g(z))g'(z) \\ &= Cg'(z)\prod_{k=1}^{n}(g(z)-g(z_k))^{\alpha_k-1} \\ &= C\frac{-2\boldsymbol{i}}{(1+z)^2}\prod_{k=1}^{n}\left(\frac{1-z}{1+z}\boldsymbol{i} - \frac{1-z_k}{1+z_k}\boldsymbol{i}\right)^{\alpha_k-1} \\ &= C\frac{-2\boldsymbol{i}}{(1+z)^2}\prod_{k=1}^{n}\left(\frac{2(z-z_k)}{(1+z)(1+z_k)}\right)^{\alpha_k-1} \end{aligned} \tag{15}$$

Here we can take the $-2\boldsymbol{i}$ as well as the 2 inside the product and absorb it into our arbitrary complex constant $C$. Let

$$L = \prod_{k=1}^{n}\frac{1}{1-z}^{\alpha_k-1}$$

and

$$M = \prod_{k=1}^{n}\frac{1}{1-z_k}^{\alpha_k-1}.$$

Consider $L$. By Equation (2a), the exponent adds to $-2$, thus we have that the product is $(1-z)^2$.

Now consider $M$. Note that this product will be some complex number, which could then be absorbed into our arbitrary constant $C$.

So Equation (4) becomes

$$h'(z) = C \prod_{k=1}^{n} (z - z_k)^{(\alpha_k - 1)}$$

And thus,

$$h(z) = A + C \int^{z} \prod_{k=1}^{n} (\zeta - z_k)^{(\alpha_k - 1)} d\zeta \tag{16}$$

This formula should look familiar. This is the same Schwarz-Christoffel formula for mapping $\mathbb{H}$ to a polygon, with the exception that we now cannot use $z_n = \infty$ as one of the vertices since $\infty$ is not on $\mathbb{D}$'s boundary.

We then proceed to find the analogous transformations for polygons with n vertices.

### 6.3.3 Polygons

Using the information from section 5, the angular analysis will be the same. For polygons with one vertex, $w_0 = \infty$ and $\alpha_1 = -1$. Thus our formula is

$$\begin{aligned} h(z) &= A + C \int^{z} (\zeta - z_1)^{(-1-1)} d\zeta \\ &= A + C \int^{z} (\zeta - z_1)^{-2} d\zeta \\ &= A + C(z - z_k)^{-1} \end{aligned}$$

For polygons with two vertices, remember that we had two cases. If $\alpha_1 = \alpha_2 = 0$, we have

$$\begin{aligned} h(z) &= A + C \int^{z} (\zeta - z_1)^{(-1)} (\zeta - z_2)^{(-1)} d\zeta \\ &= A + C \int^{z} (\zeta - z_1)^{(-1)} - (\zeta - z_2)^{(-1)} d\zeta \qquad (17) \\ &= A + C \log \left( \frac{z - z_1}{z - z_2} \right) \end{aligned}$$

Equation (17) uses partial fractions. As for the other case, we have

$$h(z) = A + C \int^{z} (\zeta - z_1)^{(\alpha_1 - 1)} (\zeta - z_2)^{(\alpha_2 - 1)} d\zeta$$

This again evokes the incomplete beta function, which we earlier encountered for triangles with one infinite vertex.

As in Chapter 6, for specific angles, we can use a table of integrals to pinpoint the exact transformation. We do that in the next section.

### 6.3.4 Examples

#### Example 1

Let us now take $\mathbb{D}$ and map it to the first and fourth quadrants of the complex plane. Note that the target space has one vertex $w_1 = \infty$. Thus, the Schwarz-Christoffel mapping is

$$h(z) = A + C(z - z_1)^{-1}.$$

Remember that we are allowed to specify three prevertices. So let $z_1 = 1$. It is then clear that

$h(1) = \infty = w_1$. To pinpoint $A$ and $C$, we consider two more points on the unit circle. Let

$$h(-1) = 0 \qquad \text{and} \qquad h(-\boldsymbol{i}) = -\boldsymbol{i}.$$

Then, $0 = h(-1) = A - \frac{1}{2}C$ which implies that $A = \frac{1}{2}C$. Also,

$$-\boldsymbol{i} = h(-\boldsymbol{i}) = A + \frac{C}{-1-\boldsymbol{i}} = A - \frac{1}{2}C + \frac{1}{2}C\boldsymbol{i}.$$

Hence, $C = -2$ and $A = -1$.

So the function that maps the unit disk to the first and fourth quadrant is

$$h(z) = -1 - \frac{2}{z-1}$$

which is a Möbius transformation.

#### Example 2

Now we shall try to map $\mathbb{D}$ onto an infinite strip between the lines $\Im(z) = 0$ and $\Im(z) = \pi$. Note that this has two vertices which are infinite. Thus, the Schwarz-Christoffel mapping is

$$h(z) = A + C\log\left(\frac{z-z_1}{z-z_2}\right).$$

Choosing two points, we select $z_1 = -1$, $z_2 = 1$. Thus, we get that $h(-1) = h(1) = \infty$. To pinpoint the complex constants $A$ an $C$, we map $-\boldsymbol{i}$ to $\pi\boldsymbol{i}$.

$$\begin{aligned} \pi\mathbf{i} = h(-\boldsymbol{i}) &= A + C\log\left(\frac{i-1}{i+1}\right) \\ &= A + C\log(\boldsymbol{i}) \\ &= A + C\ln(1) + C\frac{\pi}{2}\boldsymbol{i} \end{aligned}$$

Equating real and complex parts then gives us that $C = 2$ and $A = 0$.

So the function that maps the unit disk to this infinite strip is

$$h(z) = 2\log\left(\frac{z-1}{z+1}\right)$$

# Bibliography


[1] Brown, James Ward; and Churchill, Ruel V. (1996) *Complex Variables and Applications.* 6th ed. New York, NY: McGraw-Hill, Inc.

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[3] Pommerenke, Ch. (1991) *Boundary Behaviour of Conformal Maps.* New York, NY: Springer-Verlag.

[4] Johnston, Elgin *A "Counterexample for the Schwarz-Christoffel Transform"* The American Mathematical Monthly Dec. 1983.

[5] Ahlfors, Lars. (1979) *Complex Analysis* 3rd ed. New York, NY: McGraw-Hill Publishing Company.

# Appendix A

## Tri.mws

Here's the MAPLE program tri.mws:

```
> restart;
> Digits:=5:
> argv:=z->argument(z*exp(-I*Pi/2))+Pi/2:
> pow:=(z,a)->exp(a*(ln(abs(z))+I*argv(z))):
> h:=(s,t)->int(((abs(z-z0))^(alpha0-1))*((abs(z-z1))^(alpha1-1)),z=s..t);
> alpha0:=1/3: alpha1:=1/3:
> z0:=-1.0: z1:=0.0: z2:=infinity:
> h(z0,z1);
> h(z1,z2);
> evalf(h(-infinity,z0));
> A:=1.0+6.0*I:C:=5/h(z0,z1)*exp(-Pi*I/3):
A is your starting point w0 and C is the value that corrects the
rotation of the polygon and the length of the sides.
> SC:=(s,t)->C*h(s,t):
> w0:=A;
> w1:=w0+pow(-1,alpha1-1)*SC(z1,z2);
> w2:=w1+SC(z1,z2);
> x0 := Re(w0): y0 := Im(w0): x2 := Re(w2): y2 := Im(w2): x1 := Re(w1): y1 := Im(w1):
> z01 := t -> x0 + (x1-x0)*t + I*(y0 + (y1-y0)*t):
z12 := t -> x1 + (x2-x1)*t + I*(y1 + (y2-y1)*t): z20 := t -> x2 +
(x0-x2)*t + I*(y2 + (y0-y2)*t):
> with(plots,textplot,display):
> points:=[[x0,y0],[x1,y1],[x2,y2],[x3,y3]]:
> plot1:=plot([points],style=point,symbol=circle,
symbolsize=15,color=black):
> plot2:=plot([[evalf(Re(z01(t))),evalf(Im(z01(t))), t=0..1],
[evalf(Re(z12(t))),evalf(Im(z12(t))), t=0..1],
> [evalf(Re(z20(t))),evalf(Im(z20(t))), t=0..1]], labels=[‘ u‘,‘ v‘],
color=[blue,blue,blue,blue],thickness=3):
> d:=min(y0-1/2,y1-1/2,y2-1/2): u:=max(y0+1/2,y1+1/2,y2+1/2):
> l:=min(x0-1/2,x1-1/2,x2-1/2): r:=max(x0+1/2,x1+1/2,x2+1/2):
> plot3:=textplot([x0+(r-l)/50,y0+(u-d)/50,’w0’],color=red):
```

```
> plot4:=textplot([x1-(r-l)/50,y1+(u-d)/50,'w1'],color=red):
> plot5:=textplot([x2-(r-l)/50,y2-(u-d)/50,'w2'],color=red):
> display(plot1,plot2,plot3,plot4,plot5,view=[l..r,d..u]);
```

# Appendix B

## Quad.mws

Here's the MAPLE program quad.mws:

```
> restart;
> Digits:=5:
> argv:=z->argument(z*exp(-I*Pi/2))+Pi/2:
> pow:=(z,a)->exp(a*(ln(abs(z))+I*argv(z))):
> h:=(s,t)->int(((abs(z-z[1]))^(alpha1-1))*
  ((abs(z-z[2]))^(alpha2-1))
  *((abs(z-z[3]))^(alpha3-1)),z=s..t):
> A:=0:C:=1:alpha1:=1/2: alpha2:=1/2: alpha3:=1/2:
> z[1]:=0.0: z[2]:=1.0: z[3]:=2.0: z[4]:=infinity:
> w[1]:=A;
> w[2]:=w[1]+C*pow(-1,alpha2-1)*pow(-1,alpha3-1)*h(z[1],z[2]);
> w[3]:=w[2]+C*pow(-1,alpha3-1)*h(z[2],z[3]);
> w[4]:=w[3]+C*h(z[3],z[4]);
> h(z[1],z[2]);
> h(z[2],z[3]);
> h(z[3],infinity);
> evalf(h(-infinity,z[1]));
> for k from 1 to 4 do
> x[k]:=Re(w[k]):
> y[k]:=Im(w[k]):
> end do:
> z01 := t -> x[1] + (x[2]-x[1])*t + I*(y[1] + (y[2]-y[1])*t):
  z12 := t -> x[2] + (x[3]-x[2])*t + I*(y[2] + (y[3]-y[2])*t):
  z23 := t -> x[3] + (x[4]-x[3])*t + I*(y[3] + (y[4]-y[3])*t):
  z30 := t -> x[4] + (x[1]-x[4])*t + I*(y[4] + (y[1]-y[4])*t):
> with(plots,textplot,display):
> points:=[[x[1],y[1]],[x[2],y[2]],[x[3],y[3]],[x[4],y[4]]]:
> plot1:=plot([points],style=point,symbol=circle, symbolsize=15,color=black):
> plot2:=plot([[evalf(Re(z01(t))),evalf(Im(z01(t))), t=0..1],
  [evalf(Re(z12(t))),evalf(Im(z12(t))), t=0..1],
> [evalf(Re(z23(t))),evalf(Im(z23(t))), t=0..1], [evalf(Re(z30(t))),
  evalf(Im(z30(t))), t=0..1]], labels=[‘u‘,‘v‘], color=[blue,blue,
```

```
  blue,blue],thickness=3,scaling=constrained):
> d:=min(y[1]-0.5,y[2]-0.5,y[3]-0.5,y[4]-0.5):
  u:=max(y[1]+0.5,y[2]+0.5,y[3]+0.5,y[4]+0.5):
> l:=min(x[1]-0.5,x[2]-0.5,x[3]-0.5,x[4]-0.5):
  r:=max(x[1]+0.5,x[2]+0.5,x[3]+0.5,x[4]+0.5):
> plot3:=textplot([x[1]+(r-l)/50,y[1]+(u-d)/50,'w1'],color=red):
> plot4:=textplot([x[2]-(r-l)/50,y[2]+(u-d)/50,'w2'],color=red):
> plot5:=textplot([x[3]-(r-l)/50,y[3]-(u-d)/50,'w3'],color=red):
> plot6:=textplot([x[4]+(r-l)/50,y[4]-(u-d)/50,'w4'],color=red):
> display(plot1,plot2,plot3,plot4,plot5,plot6,view=[l..r,d..u]);
```