HENRY THOMAS LINGEFJÄRD
Mathematical Modeling by Prospective Teachers Using Technology
(Under the direction of JEREMY KILPATRICK)

Three studies were conducted to investigate prospective mathematics teachers' understanding of mathematical modeling when using technology solve a variety of problems. The purpose was neither to verify an existing theory nor to test a priori hypotheses. Rather, the intent was to develop a framework for exploring the students' difficulties with mathematical modeling by observing and interviewing them in the context of a regular, if unique, course on mathematical modeling. The framework illustrates how different sources of authority as well as conceptions and misconceptions of mathematics and mathematics modeling play different roles in the mathematical modeling process. Technology acted both as a tool and as a source of authority in this process.

The studies were conducted at the University of Gothenburg during the fall semester of 1997, the spring semester of 1998, and the fall semester of 1998. A qualitative approach was used in which special attention was focused on a small group of students working together in the laboratory. Data were collected from questionnaires, videotaped interviews, observations, and written documents such as course assignments and examinations.

The first study revealed that the students in general favored the use of technology, especially when solving complex mathematical modeling problems. On the other hand, they easily "got lost" and trusted the technology far too much when working on mathematical modeling problems, thereby neglecting a necessary validity check. This trust, in turn, seemed to profoundly disturb their ability to relate mathematical models to reality. The second study, in addition to verifying the findings from the first, indicated that the students had misconceptions associated with their knowledge of mathematics, of technology, and of problem contexts. A major finding of the third study concerned a transformation of authority that occurred after the first few weeks of the course. The students became rather uncritical of the results they got from the computer or graphing calculator despite the fact that in lectures and laboratory sessions they had been urged to be very cautious when employing software to select models. All three studies confirmed the essential role played by the validation part of mathematical modeling when technology is present.

INDEX WORDS: Mathematical Modeling, Assessment, Authority, Responsibility, Open-Ended Questions, Technology, Teacher Education.

MATHEMATICAL MODELING BY PROSPECTIVE TEACHERS USING TECHNOLOGY

by

HENRY THOMAS LINGEFJÄRD

M.S., University of Gothenburg, Sweden, 1985

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Approved:

Major Professor: Jeremy Kilpatrick

Committee: Edward J. Davis
C. Henry Edwards
George M. A. Stanic
James W. Wilson

Electronic Version Approved:

Gordhan L. Patel
Dean of the Graduate School
The University of Georgia
May 2000

DEDICATION

To My Family

ACKNOWLEDGMENTS

Excuse me, gentle reader if oughte be amisse,
straung paths ar not troden al truly at the first:
the way muste needes be comberous,
wher none hathe gone before...
(Robert Recorde, 1551: *The Pathway to Knowledge*)

The completion of this dissertation has been far from a solitary achievement. The support and encouragement of family, friends, and colleagues over the past six years have helped me to fulfill this dream. I will try to acknowledge a few of them here

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CHAPTER 1

THE PROBLEM AND ITS BACKGROUND

The heart of applied mathematics is the injunction "Here is a situation; think about it." The heart of our usual mathematics teaching, on the other hand, is: "Here is a problem; solve it" or "Here is a theorem; prove it." We have very rarely, in mathematics, allowed the student to explore a situation for himself and find out what the right theorem to prove or the right problem to solve might be.
Henry Pollak (1970)

Model and *modeling* are common expressions with many seemingly different meanings. We are introduced to new car models that we are supposed to feel attracted to, to picture ourselves in possession of the new car. Architects use models of a landscape or a house to illustrate a product they want to sell. In the fashion industry, a model is a person who wears clothes that other people watching can imagine themselves wearing. Fashion models are selected because they possess certain idealized human characteristics, which change from time to time but always refer to ideals such as thinness, height, skin color, and attitude. Children use many models of reality in their toy cars, dolls, trains, and so forth. All modeling activities have at least two aspects in common: They use a model in order to think about or introduce the related reality, and the model is something more or less idealized or simplified.

The process of *mathematical modeling* also has a variety of definitions. As used in secondary mathematics, it ordinarily entails taking a situation, usually one from the real world, and using variables and one or more elementary functions that fit the phenomena under consideration to arrive at a conclusion that can then be interpreted in light of the original situation. Pollak (1970) argued that we seldom challenge students to study a situation and try to make a model of it for analyzing the situation.

> A carefully organized course in mathematics is sometimes too much like a hiking trip in the mountains that never leaves the well-worn trails. The tour manages to visit a steady sequence of the "high spots" of the natural scenery. It carefully avoids all false starts, dead-ends, and impossible barriers, and arrives by five o'clock every afternoon at a well-stocked cabin. The order of difficulty is carefully controlled, and it is obviously a most pleasant way to proceed. However, the hiker misses the excitement of risking an enforced camping out, of helping locate a trail, and of making his way cross-country with only intuition and a compass as a guide. "Cross-country" mathematics is a necessary ingredient of a good education. (p. 329)

Prospective teachers need to understand a great variety of topics and approaches in mathematics. Today these topics include concepts, principles, methods, and procedures that were not traditionally part of school or college mathematics but that many secondary school students may now address very well through the use of computers and graphing calculators. Applied mathematics as a field and the process of mathematical modeling in particular are one part of the mathematical curriculum that may be broadened and enhanced through the use of technology. The presence of technology in today's classrooms may assist teachers in implementing Pollak's vision of cross-country mathematics, since tedious, routine calculations can be done by the technology and a much greater number of realistic, open-ended situations can be modeled. In Sweden, the United States, and many other countries, the availability of the graphing calculator, with its built-in regression analysis capability for comparing a number of mathematical models, has changed the school mathematics curriculum. Today, secondary school students can handle problems that were not even possible in college mathematics only a decade ago

For at least 3 decades, many authors with different perspectives have discussed the role of applications and modeling in the curriculum. The 1979 yearbook of the National Council of Teachers of Mathematics (NCTM), *Applications in School Mathematics*, contains articles illustrating the variety of those perspectives. In recent years, interest in mathematical modeling has increased among mathematics educators. The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), for example, stressed its importance. Given the potential value of technology for enhancing learning, students can undertake some realistic modeling problems and thereby develop their ideas about and their understanding of mathematics. The intent of the NCTM's recommendations regarding the curriculum is that through a consideration of real-world problems—problems that capture students' interest and that might readily arise in daily life—students will gain both an appreciation of the power of mathematics and some essential mathematical skills. In the report *Heeding the Call for Change* (Steen, 1992), published by the Mathematical Association of America, a group of collegiate mathematics educators suggested that "the key [in selecting such problems] is to have the contexts relate to students' interest, daily life, and likely work settings" (p. 100).

The three studies reported herein arose from my experience in teaching mathematics to prospective mathematics teachers at the University of Gothenburg and were stimulated in part by the ongoing evolution of technology. During the past 3 decades, personal computational technology has evolved from four-function calculators in the 1970s through scientific calculators in the 1980s to graphing and symbolic calculators in the 1990s. Today, most students who study high school or college mathematics also have easy access to computers equipped with a variety of mathematical tool systems. The evolution in technology has affected the content of some courses in mathematics for teachers and many times also the way those courses are taught.

During the last 5 years or so, there has been a distinct change in some of the courses in the program for prospective mathematics teachers at Gothenburg. In the mid-1990s, technology was introduced as an isolated part of the program, often through a visit to the computer laboratory. Today, the program includes courses in which the technology is an integral part of the syllabus, including the assessment. In my case, the course in mathematical modeling I teach every semester was changing dramatically during that time. The new software tools that became available in addition to the perspectives I brought with me from the University of Georgia encouraged me to restructure the course together with a colleague and focus much more on mathematical modeling. As a consequence, I started to look more closely at the students' conceptions of mathematical modeling. It should be explained at this point that the mathematical modeling that I studied is the kind in which students work with data drawn from real life. Although the data are usually somewhat simplified, the problems are more open and less constrained than, for example, standard mathematical word problems.

Thanks to the technology, students today can find or construct models for more complicated situations than before, but as a consequence they seem to encounter more and more problems with understanding and interpreting the results provided by that technology. Because I had observed students mistaking model for reality, I undertook

Study 1 to find out why they seem to forget reality when using sophisticated software to model problems. I noticed that when students wrote up their solutions to modeling problems, they revealed previously unexposed mathematical misconceptions in trying to explain and argue for a model they had produced. Consequently, I undertook Study 2 to investigate their conceptions and misconceptions in the modeling process. I then saw that students were losing faith in their own mathematical knowledge, thereby trusting in obvious distortions of the relation between mathematical model and experiential reality. I wanted to investigate why students sometimes shifted their sources of authority during the modeling process from mathematics to the computer, and at the same time they seemed unwilling to take full responsibility for their own learning and performance. That led me to undertake Study 3.

Mathematical Modeling

As often portrayed, the first step in the mathematical modeling process is the formulation of a real-world problem in mathematical terms—that is, the construction of a *mathematical model* consisting of variables that describe the situation and equations that relate these variables. The real-world problem is then translated into a mathematical problem that is analyzed and perhaps solved. Finally, the mathematical results obtained are interpreted in the context of the original real-world situation in an attempt to answer the question originally posed (Pollak, 1970; Mason, 1988).

In Figure 1, the left-hand column represents the real world, the right-hand column represents the mathematical world, and the middle column represents the connection between the two. In the middle column, the problem is simplified and formalized, and

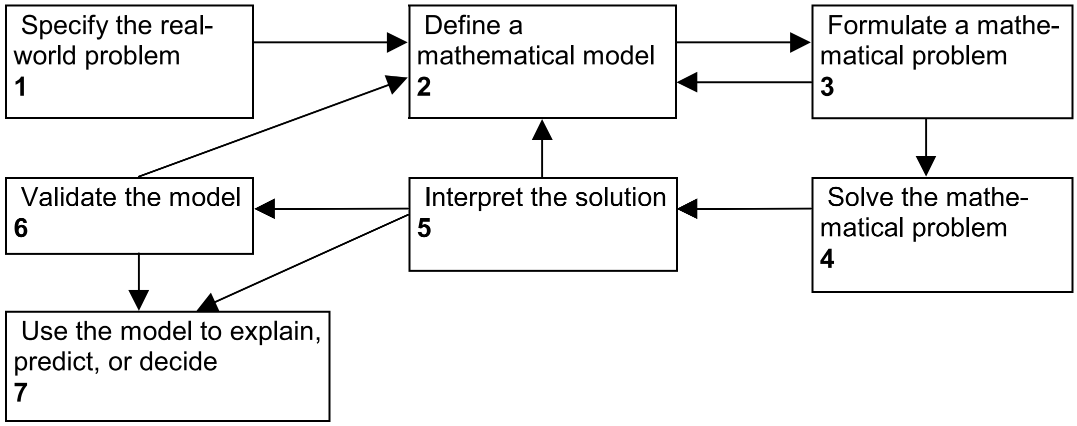


Figure 1. Main stages in modeling (adapted from Mason, 1988, p. 209).

then the mathematical results obtained are translated back into terms meaningful in the original real-world situation. In a straightforward modeling process, one might be able to go through Stages 1 through 7 in sequence. But mathematical modeling is not always straightforward, especially when realistic results are expected. There often is a tradeoff between a model sufficiently simple that a mathematical solution is feasible and one sufficiently complex that it faithfully mirrors the real-world situation. If the model originally defined is too simple to be realistic, the mathematical results may not translate into valid real-world results. In that case, one might have to return from Stage 6 to Stage 2 and repeat the process using a more sophisticated model. In many cases, particularly in the social sciences, it is difficult to carry out the Stage 6 validation step at all, and one might simply proceed directly from Stage 5 to Stage 7. In other cases, when the mathematical model is so sophisticated that the mathematics is intractable, one might have to return to Stage 2 and simplify the model in order to make a mathematical solution feasible. But then the validation step of Stage 6 might indicate that the model is now too simple to yield correct real-world results. There is an inevitable tradeoff, therefore, between what is physically realistic and what is mathematically possible. The construction of a model that adequately bridges this gap between realism and feasibility is the most crucial and delicate step in the process.

Skovsmose (1994) distinguishes between two types of mathematical modeling; namely, *pointed modeling* and *extended modeling*. When we perform pointed modeling, the problem we are dealing with is transformed into a formal language, in terms of which we try to solve the original problem. Pointed modeling is the type whose stages were just discussed. But extended modeling is different. In this case, mathematical modeling is used not to describe a specific problem situation but to provide a general foundation for a technological process. Mathematics becomes part of the conceptual framework we use to interpret and interpret the reality of our modern world. Through that framework our daily lives are structured mathematically—how we measure distance, space, time, and so forth. A pointed mathematical model must be based on some sort of specific interpretation of reality.

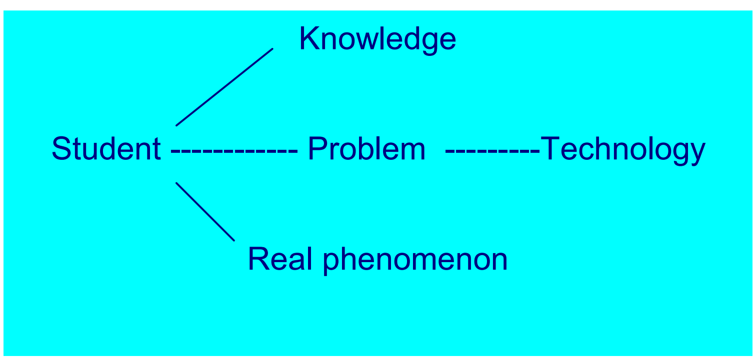


Figure 2. Main components of the modeling process.

At the beginning of my research, I considered the elements in Figure 2 as basic components of the modeling process. The student is interacting with the problem, using her or his knowledge together with the technology to model a real phenomenon. The framework guided my research as I attempted to look more closely at the interaction and to identify some of the sources of the difficulties students have with mathematical modeling. I was also looking for additional components of the mathematical modeling process, thereby exploring the possibility of developing the framework further. In the next section I will describe some of the arguments that are used to motivate the presence and emphasis of mathematical modeling in different curricula.

Arguments for Teaching Modeling

Niss (1989) presented various arguments as to why applications and modeling belong in the curriculum. Blum and Niss (1989, p. 5) defined five arguments that I have termed as follows: formative, critical, practical, cultural, and instrumental. Niss explained these arguments, which are not totally separate and distinct:

> Applications and modelling should be part of the mathematics curriculum in order to
>
> 1. foster among students general *creative* and *problem solving* attitudes, activities and competences.
>
> 2. generate, develop and qualify a *critical potential* in students towards the use (and misuse) of mathematics in extra-mathematical contexts.
>
> 3. prepare students to being able to *practice applications and modelling*—in other teaching subjects; as private individuals or as citizens, at present or in the future; or in their professions.
>
> 4. establish a representative and balanced *picture of mathematics*, its character and role in the world. Such a picture must encompass all essential aspects of mathematics, and the application of mathematics and mathematical modelling in other areas *do* form one such aspect.
>
> 5. assist students' *acquisition* and *understanding of mathematical concepts*, notions, methods, results and topics, either to give a fuller body to them, or to provide motivation for the study of certain mathematical disciplines. (pp. 23-24)

The Formative Argument

The formative argument views modeling as “oriented towards fostering overall, explorative, creative and problem solving capacities, as well as open-mindedness and self-reliance” (Blum & Niss, 1989, p. 5). The argument seems to relate to the belief that abilities in a subject—in this case, mathematics—may very well be transferred and used in other areas and other contexts. It assumes that if students work with realistic problems with a context they can understand, they can become self-confident investigators and at the same time learn problem solving. D’Ambrosio (1989) even considers the process of mathematical modeling to be the essence of creativity:

> This is the very essence of the intelligent inquiry which distinguishes Homo sapiens from other species. Hence, modelling is the essential feature of human intellectual behaviour. (p. 23)

He also claims that the use of mathematical modeling in the mathematics classroom opens up the subject and prevents it from being presented and understood as closed and complete. Instead, students who are involved in the process of mathematical modeling will understand that mathematics is a science that is growing and developing all the time. Consequently, these students will become more creative, and mathematical modeling will contribute to their personal development.

The Critical Argument

The critical argument is related to the goal of creating a general critical perspective among students. They should learn to be critical of the mathematical models used in all parts of society. If we wish to help students develop a critical perspective on the uses of mathematics, then we should teach them about mathematical modeling. A critical perspective can be taken toward a specific model or toward the ways in which society uses such models. This dissertation deals mainly with the former. A critique of a mathematical model concerns its mathematical content and the potential and actual uses of that content. It acts mainly on questions in the universe of mathematical content but with natural links to the society from which it chooses phenomena to model.

Arguments for the use of mathematical modeling as a way to foster critical perspectives among students often relate to the building of a democracy. Mathematics is seen as a force shaping society and as an aid in planning, acting, and making decisions in many different arenas of the social and natural sciences. By adapting a critical perspective, students examine the hidden assumptions in the ways mathematics is used or misused.

The Practical Argument

The practical argument is based on the assumption that mathematical modeling must be taught if schooling is to produce people who can create and use models in their professional and personal lives. This argument is especially relevant for prospective teachers, who will need to do that teaching. The practical argument can be used as a tool to solve many different problems outside mathematics. When one talks about the usefulness of mathematics, it is important to ask for whom the mathematics is important. All students will in fact use mathematical models in their daily life, whether or not they construct those models. Modeling is important for citizens of many different countries, as Banu (1991) notes:

> The main objectives of teaching mathematics in developing countries like Bangladesh are as follows:
>
> - To increase the students' ability and skills in mental calculations and estimations and applying the rules of calculations to the practical problems faced in daily life.

- To encourage the students to develop mathematical models depending on the requirements of the country, by making complete use of the local resources. (p. 118)

The Cultural Argument

The cultural argument springs from a desire that all students should see the richness of mathematics. They should see the whole picture, a picture that must encompass all essential aspects of mathematics. Mathematics is a science, a visible part of the culture, and as such it interacts with the physical world. But mathematics also plays a role in social phenomena, and as such it is an invisible part of the culture. One way to reveal the multi-dimensional picture of mathematics is to engage students in the modeling process and thereby illuminate the experimental side of mathematics. By *experimental,* I mean that students can be aware that mathematics is not only used but also constructed or at least modified. It is reorganized, specialized, or otherwise adjusted to the specific modeling situation. The solution is not just found but constructed within a cultural matrix. It is important that students at least occasionally adopt a "meta" view of mathematics by trying to step outside it and look at it while it is serving as a tool in other subjects.

> Sociology, political sciences, psychology and even literary fields are joining economics and other subjects traditionally placed among the humanities in claiming creditability through the use of mathematical models. In some cases it is no more than the use of mathematical jargon. But it is the first step of adopting a mathematical way of thought in dealing with their subjects. … Clearly this is an indicator of how influential mathematics is in modern society. Mathematical thinking has acquired unprecedented prestige. Maybe this is the main reason why mathematics is kept with such intensity as a major school subject. (D'Ambrosio, 1989, p. 26)

The Instrumental Argument

The instrumental argument claims that the use of mathematical modeling in the teaching of mathematics will "assist students' acquisition and understanding of mathematical concepts" (Blum & Niss, 1989, p. 24). The aim of modeling is both to motivate students to study mathematics and to contribute to the meaning of mathematical concepts. The argument is related to students' learning, and the model is seen as an educational tool that may provide a concrete example on which students may construct knowledge. To concretize the mathematics involved may make it possible for students to connect their studies to their previous experience and thereby construct new knowledge more easily. There is a complementarity in which, on the one hand, models support the understanding of the concepts used and, on the other hand, conceptual understanding is necessary when modeling real-world phenomena.

Points of Departure

These five arguments as to why we should teach mathematical modeling are of different character, but some of them overlap others. None conflicts with any other. Different weightings of the arguments produce different views on the teaching and learning of mathematics. The *formative argument* and the *instrumental argument* both take as their point of departure the student and her or his process of personal development or learning. The *practical argument* begins with the student or the society, stressing usefulness for one or the other. The *critical argument* is focused only on the society, whereas the *cultural argument* is the only one that connects its point of departure to mathematics. It is the only argument that is centered on the aim of illuminating mathematics as a science.

Blum (1991) claims that the critical argument is contained within the others, especially the cultural argument. The practical argument and the cultural argument may both be seen as two arguments that take a critical perspective. The aim of the practical argument is to equip individuals to deal with situations outside mathematics by providing them with experiences of mathematical modeling, and those may very well include a critical dimension. Further, it is often relevant to make a critical evaluation of internal problems in a mathematical model and of its relation to the real world if the model proves to be useful.

Goals for Teaching Modeling

Regardless of how we distinguish among the different arguments or try to find additional ones, if we accept them, we somehow arrive at the teaching situation. What should be learned by the students, and how should we assess it? Clayton (1999) discussed the goals for mathematical modeling and the importance of the possible outcome for society. His discussion links back to the arguments for including mathematical modeling in the school curriculum:

> My conclusions lead me to suggest that an important aim of mathematics education should be to make students properly aware of the value of mathematical modelling in a wide range of situations, and to train them how to apply IT [information technology] tools most effectively. The benefits that will accrue are essential for the survival and future growth of commerce, industry and science, and there are opportunities for them to be realized at every level of employment.
>
> To help our young people acquire the necessary skills and use them profitably in their later employment, I hope that schools and colleges will be enabled and encouraged to maintain a balanced mathematics syllabus that includes:
>
> - mathematical techniques and analysis methods taught in contexts that show how they can be used
> - the principles and application of mathematical modelling
> - numerical methods including direct simulation, and the use of appropriate technology

> • the effects of uncertainty—how they can be measured and analysed.
>
> In this type of learning, IT, with its power to produce graphical images and manipulate symbols, objects, or numbers, has an important role as a tutorial assistant: illustrating mathematical concepts, encouraging directed investigations, and aiding visualization in the exploration and transformation of data.
>
> The scope of mathematics can usefully be broadened to provide the basis for a disciplined approach to problem solving and IT tools are used to enhance understanding and derive quantitative results in a wide variety of subject areas. If such activities are carefully planned they can be used to ensure that the principles of verification, validation, and accuracy estimation are understood and properly applied in the construction and application of mathematical models. (pp. 27-27)

In a teaching situation that results from the arguments and goals listed above, one needs to consider that for a student who takes part in a modeling process the activity should work in two positive ways. The modeling activity should be a way to express the student's mathematical competence and simultaneously develop that competence further. This complementarity is characteristic of the kinds of modeling activities discussed throughout this dissertation and corresponds to the goals of being able to perform a modeling process and at the same time to know about the process. In my view there are a number of competencies that students constantly should have developed and keep developing when engaging in modeling. These are competencies in doing mathematics; using everyday knowledge of phenomena being modeled; performing the modeling process itself; validating mathematical models; reflecting on and critiquing mathematical models; and explaining, describing, and otherwise communicating mathematical models.

Modeling in the Mathematics Curriculum

The Swedish Curriculum

Documents setting forth the Swedish national curriculum emphasize the modeling process in both the curriculum for the compulsory school and the curriculum for the upper secondary school, or *gymnasium* (Grades 10 to 12). The principal argument given by the government for the teaching of mathematical modeling is that situations of all sorts in which mathematical models can be used surround us in our daily life:

> Mathematical applications in everyday life, social life and scientific activity provide formulations of problems in terms of mathematical models, which are studied using mathematical methods. The value of the results achieved depends on how well the model describes the problem. In recent years the development of powerful computers has made it possible to apply more accurate mathematical models and methods in activities than were feasible earlier. This technology has also led to the development of new research areas in mathematics, which in their turn have led to new applications. (Skolverket, 1997, p. 9, my translation)

Not only is it important that society uses many mathematical models, but students should also be part of the modeling process.

> The importance of mathematical models has increased in today's society. Everything that takes place in a computer, for instance, is a result of some sort of model. It is very important that this area is part of the mathematics we teach. (Skolverket, 1997, p. 18, my translation)

Since Sweden has a goal-oriented curriculum that sets forth ends but not means, there are no detailed guidelines about how the modeling described above is to be accomplished. Textbook authors bear some responsibility for suggesting modeling activities, as do the developers of the local school curriculum plan. The curriculum goal is described as follows:

> Students should develop their ability to construct, refine, and use mathematical models together with critical judgments of the model's qualifications, possibilities, and limitations. (Swedish Ministry of Education, 1994, p. 22, my translation)

The NCTM Standards

Its emphasis on mathematics as an essential subject for all other subjects was one reason for the 1989 NCTM *Standards* to promote mathematical modeling. In the view of the authors of the *Standards* document, the natural place for doing mathematical modeling was the high school curriculum:

> Because mathematics is a foundation discipline for other disciplines and grows in direct proportion to its utility, we believe that the curriculum for all students must provide opportunities to develop an understanding of mathematical models, structures, and simulations applicable to many disciplines. . . .
>
> Another premise of the standards is that problem situations must keep pace with the maturity—both mathematical and cultural—and experience of the students. For example, the primary grades should emphasize the empirical language of the mathematics of whole numbers, common fractions, and descriptive geometry. In the middle grades, empirical mathematics should be extended to other numbers, and the emphasis should shift to building the abstract language of mathematics needed for algebra and other aspects of mathematics. High school mathematics should emphasize functions, their representations and uses, modeling, and deductive proofs. (pp. 7, 10-11)

A stronger emphasis on mathematical modeling can be found in the "Standards 2000" discussion draft (NCTM, 1998). Mathematical modeling is mentioned in Standard 2, Patterns, Functions and Algebra, and in Standard 10, Representation. The draft makes it explicit that mathematical modeling is an important way to teach mathematics to relatively young students:

> One of the most powerful uses of mathematics is the mathematical modeling of phenomena. Using of symbolic notation is central to modeling. For example, distribution and communication networks, laws of physics, population models, and statistics for a data set can all be expressed in symbolic language. Algebra is implicit in any moderately sophisticated use of spreadsheets. If the relations among a set of numerical categories are well understood, this understanding will be expressed in the language of variables, functions, and variables.
>
> Connections between mathematics and the sciences often become apparent when students engage in the modeling of physical phenomena, such as finding the speed of light in water, determining proper doses of medicine, or optimizing locations of fire stations in forests. (pp. 60, 328)

The Core-Plus Curriculum

One example of a secondary curriculum in line with the NCTM Standards that attempts to deal explicitly with modeling is the Core-Plus Curriculum. On 6 October 1999, U.S. Assistant Secretary of Education Kent McGuire announced the selection of 10 mathematics education programs as "exemplary and promising." The Core-Plus curriculum program was one of 5 chosen as exemplary. The Core-Plus Mathematics Project (CPMP) was funded by the National Science Foundation to develop student and teacher materials for a complete 3-year high school mathematics curriculum for all students, plus a 4-year course continuing the preparation of students for college mathematics. The curriculum is said to build upon the theme of *mathematics as sense making.* Through investigation of real-life situations, like the Medicine and Mathematics section on page 445-447, students are supposed to develop a deep understanding of important mathematics that makes sense to them and that, in turn, can enable them to make sense out of new situations and problems.

The preface to the Core-Plus textbook for Course 1, Part 2, says the following:

> *Mathematical Modeling.* The curriculum emphasizes mathematical modeling and modeling concepts including data collection, representation, prediction, and simulation. The modeling perspective permits student to experience mathematics as a means of making sense of data and problems that arise in diverse contexts within and across cultures. (Coxford, Fey, Hirsch, Schoen, Burrill, Hart, & Watkins, 1997, p. x)

In fact, the Core-Plus curriculum is built around the modeling process. The emphasis on mathematical modeling can be seen in the titles of the seven chapters in Course 1: Patterns in Data, Patterns of Change, Linear Models, Graph Models, Patterns in Space and Visualization, Exponential Models, and Simulation Models.

The Impact of Technology on Modeling

The observation is no longer new that teachers need to do other things than just teach paper-and-pencil arithmetic (in a broad sense) when computing technology is present.

> The ready availability of versatile calculators and computers establishes new ground rules for mathematics education. Template exercises and mimicry mathematics—the staple diet of today's tests—will diminish under the assault of machines that specialize in mimicry. Instructors will be forced to change their approach and their assignments. It will no longer do for teachers to teach as they were taught in the paper-and-pencil era. (National Research Council, 1989, p. 63)

As a consequence, mathematics teachers as well as prospective mathematics teachers today need an understanding of mathematics that allows them to produce and interpret technology-generated results, to develop and evaluate alternative solution paths, and to recognize and understand the mathematical limitations of particular technological tools. To exploit new technology in their daily practice, teachers must be well informed about its place and role in the didactical process (Balacheff & Kaput, 1996). The mathematics education community must address many questions, including how to prepare prospective secondary school mathematics teachers to function in a technology-enhanced environment.

The potential impact of technology on the school mathematics curriculum is also evident in many recommendations for curricula around the world. From the calls for reform in school mathematics of the 1989 NCTM Standards documents through the Swedish curriculum of 1994 to NCTM's 1998 Standards 2000 discussion draft, one can see a change in guidelines in which technology seems to take an increasingly natural role in the teaching of mathematics:

> *The Information Society.* This social and economic shift can be attributed, at least in part, to the availability of low-cost calculators, computers, and other

> technology. The use of this technology has dramatically changed the nature of the physical, life, and social sciences; business; industry; and government. The relatively slow mechanical means of communication—the voice and the printed page—have been supplemented by electronic communication, enabling information to be shared almost instantly with persons—or machines—anywhere. Information is the new capital and the new material, and communication is the new means of production. The impact of this technological shift is no longer an intellectual abstraction. It has become an economic reality. Today, the pace of economic change is being accelerated by continued innovation in communications and computer technology. (NCTM, 1989, p. 3)

> Students should develop an ability to use the visual and computational capabilities of calculators and computers. (Swedish Ministry of Education 1994, p. 33, my translation)

> Given continuing rapid change in technology, both within and outside the school environment, this draft of *Principles and Standards* goes further that the original *Standards* documents in describing the role of technology. A principle about technology is now included. The discussion considers not only how technology might best support mathematics learning but also how the presence of technology implies shifts in mathematical content emphasis and the way in which students' thinking might be qualitatively different. Also in response to changing technologies, this document is provided in electronic form. (NCTM, 1998, p. 17)

Its is hard to understand exactly how the authors of this part of the NCTM's 1998 Standards 2000 discussion draft picture how the presence of technology would imply a shift in the way in which students' thinking might be qualitatively different. Presumably the shift is in a positive direction, although is unclear to me how the same authors consider that this change should be measured.

Conceptions and Misconceptions Related to Mathematical Modeling

When a teacher encourages students to participate in the process of modeling a real world phenomenon, the students must use the mathematical tools at their disposal. Two questions are relevant: namely, what information relevant to the mathematical situation or problem at hand do the students possess, and how is that information accessed and used? Schoenfeld (1992) described these as analogous to questions about the contents of a library: What is in the library, and how do patrons gain access to the contents?

> The answer to the first question is contained in the catalogue: a list of books, records, tapes, and other materials the library possesses. The contents are what interest you if you have a particular problem or need particular resources. How the books are catalogued or how you gain access to them is somewhat irrelevant (especially if the ones you want aren't in the catalogue). On the other hand, once you are interested in finding and using something listed in the catalogue, the situation changes. How the library actually works becomes critically important: Procedures for locating a book on the shelves, taking it to the desk, and checking it out must be understood. Note, incidentally, that these procedures are largely independent of the contents of the library. One would follow the same set of procedures for accessing any two books in the general collection. (p. 349)

The same could metaphorically be seen as true for assessing the mathematical knowledge a student brings to a problematic situation. When students are given mathematical modeling problems, they are expected to know and have access to a lot of mathematical tools. This set of tools includes some basic mathematical concepts such as functions, limits, infinity, derivatives, and integrals. It is important that their basic conceptions remain true, stable, and unchanged by the impact of computers and calculators. Even though we all would like our students' thinking to become qualitatively different and better, it is equally important that technology does not destroy what we want to remain as it is in students' thinking.

Vinner (1991) gave examples of how high school students had difficulty defining and identifying what a function is. In his research, only about one-third of a group of 147 students could both give a correct definition and answer correctly three questions about functions (p. 75). Many of the students also had misconceptions about the concept of mathematical limit. My experience is that students often mistake a very large number for infinity, and the concept of limit is then adjusted to reach that very large number. This misconception is exemplified in the use of Excel software when students try to evaluate a function at infinity by using a large number of cells.

Misconceptions about functions when using technology appear in at least two contexts. First, students may believe that a real-world phenomenon behaves in a way that is impossible or unlikely. They may contend that growth is exponential, for example, when other evidence shows that an exponential function makes no sense for this phenomenon. Second, they may uncritically accept trendlines or regression curves provided by the software that then lead to contradictions. For example, students may conclude from a linear model that record times for an athletic event will eventually drop to zero or below. The uncritical use of technology is related to the authority that students see it as possessing.

Authority and Responsibility in Modeling

The concepts of authority and responsibility are relevant to the behavior of at least some students in the modeling course in Gothenburg. Perry (1968) found that most Harvard students typically developed personal responsibility during their years at the university. Those same students had come to the university with a notion that true authority or truth resided in the heads of their professors.

> When I went to my first lecture, what the man said was just like God's word, you know. I believed everything he said, because he was a professor, and he's a Harvard professor, and this was a respected position. (Perry, 1969, p. 61)

Students who study mathematics normally have at least two authorities they can rely on: the textbook and the teacher. The relation between those authorities may vary over time. Sometimes the textbook even becomes equal to the content of the mathematics course, especially at the university level, where students and teachers meet less often than in pre-university studies. Everything is in the textbook: the theorems, the problems, the exercises, and the answers in the back. In my experience, most freshmen who take a university or college course in mathematics like, for instance, linear algebra believe that the textbook has absolute authority. That is, they believe that the form in which the content is presented through definition, example, theorem, and proof must be the correct way to present mathematics, and they also believe that the mathematical content in such a book must be completely correct and therefore worth taking as authoritative.

The development of a student's trust in his or her understanding and in the value of the solution to a problem leads to the important question of who owns the mathematics. Students with a strong tendency to trust "the authority" often give up their ownership of ideas and of problem-solving strategies when questioned by a teacher. A textbook or a teacher might even present and explain "the right way to solve problems" so strongly that it can create feelings of *algebraic guilt* (Dunham 1990, 1993; Dahland & Lingefjärd, 1996) in students when they use nonstandard methods. That is, students think there is a proper, algebraic way to solve a problem that they ought to have followed.

What happens to the students' authority in a course in which there is no textbook and in which the content is based on all the mathematics they previously studied, together with information from a variety of sources such as the Internet, friends, or resources in library? The instructors in the modeling course at the University of Gothenburg have tried to act more as coaches and supervisors than as lecturers and carriers of mathematical truth. They have hoped that the students would take full responsibility for their own learning and base authority for that learning mainly on their own knowledge.

After conducting Studies 1 and 2, therefore, I was surprised at the direction the shift of authority took in some of the students in the course. The result provided by technology seemed to override their own mathematical background. I was also surprised by the strong efforts they sometimes made to convince me that their model would work. Some shift in authority might have been expected given that the students were using computers to solve open problems without a specific mathematics textbook. But I did not expect that the students would take the calculator or computer as the sole mathematical authority.

Both the background document for the Swedish curriculum and the Standards 2000 discussion draft (NCTM, 1998) emphasize the responsibility schoolchildren should take for their own learning:

> The students should, with increasing maturity, be encouraged to take a greater personal responsibility for their learning. The curriculum committee will consider if this should give expression for the gymnasium to some sort of study contract between the students and the school. (Swedish Ministry of Education, 1992, pp. 337-338, my translation)

> Teachers and students together share responsibility for mathematics learning. Each student is responsible for making sense of mathematics. The development of deeper understanding and meaning succeeds through a process of struggling with new concepts and incorporating that information into existing knowledge. With that expectation in mind, students will view the challenge of mathematics as a normal part of learning, rather than a signal to give up and consider oneself a failure. Learning mathematics may not always be fun, but it can be engaging and rewarding. When students successfully solve a difficult problem or finally understand a complex idea, they experience a very special feeling of accomplishment. (NCTM, 1998, p. 36)

The same responsibility should also hold true for university students. The problems of exhorting prospective teachers to become responsible for their own learning have been discussed by Ekholm (1997) and Povey (1995), among others.

Research Questions

It is natural to conclude that mathematics educators need to develop ways in which to empower prospective mathematics teachers to do and teach mathematical modeling in technology-enhanced environments. At the same time, it is equally important that researchers learn more about the effects that the use of calculators and computers will have when these fledgling teachers construct mathematical models of real-world phenomena. My experience from the modeling course was that the students increasingly seemed to trust the result from the computer or graphing calculator even if it contradicted other mathematical ideas they had. Many students had approached me arguing for their right to blindly accept and copy a solution from a graphing calculator or computer, thereby accepting the machine's authority. How was that possible? What started as my vague notion about students' technology-driven errors grew into a strong curiosity. Why did so many students appear to abandon their own common sense and mathematical knowledge when working with sophisticated mathematical tools?

The studies reported in this dissertation dealt with the thinking and actions of prospective Swedish mathematics teachers preparing to teach in Grades 4 to 9 or in the gymnasium (Grades 10 to 12) when taking a course in mathematical modeling. My initial focus was on their understanding of modeling and how they related mathematical models to the real world. It began with several questions that stimulated these attempts to understand the complexity of the phenomena at hand: What do we know about prospective teachers' understanding of mathematical modeling in general? What do we know about prospective mathematics teachers' ability to access and use their background in mathematics when solving modeling problems? What do we know about the impact of technology on prospective mathematics teachers' understanding of mathematics? Finally, what methodology can be used to study prospective mathematics teachers' understanding of mathematical modeling in a technology-enhanced environment?

Those questions were eventually distilled into the following three broad research questions:

- What beliefs do prospective teachers have about technology, mathematical models, and reality?
- What conceptions and misconceptions do prospective teachers exhibit when solving mathematical modeling problems using technology?
- What do prospective teachers take as the authority when solving mathematical modeling problems using technology?

The next chapter contains a review of the literature relevant to the research questions. Chapter 3 introduces the reader to the Swedish educational system and describes the course in mathematical modeling. Chapter 4 explains the first study, in fall 1997, which explored the views of students in the course. Chapter 5 describes the spring 1998 study, which investigated the question of students' conceptions and misconceptions when using models. Chapter 6 discusses the fall 1998 study, which dealt with the sources of authority students used and the responsibility they accepted. The final chapter, chapter 7, contains the summary, conclusions, and implications of the three studies.

CHAPTER 2

RELATED LITERATURE

No clear line of research has directly addressed the ways in which prospective mathematics teachers learn, do, and think about mathematical modeling in the presence of technology. Nevertheless, research and development work in several related areas may suggest where potential promise or difficulty resides. Several topics are particularly relevant:

- Conceptions of mathematical modeling,
- Effects of technology use on mathematics learning,
- Mathematical misconceptions when using technology,
- Assessment of mathematical modeling, and
- Authority and responsibility when learning mathematics.

In this chapter, I analyze and review literature that was important for the three studies. The chapter has five sections, each dealing with the research relevant to one of the topics above.

Conceptions of Modeling

Studies of the conceptions of mathematical modeling held by prospective teachers are rare. Nevertheless, there is evidence about how other groups of college students understand various aspects of the modeling process (Clement, 1982; Clement, Lockhead, & Monk, 1981). The work of Clement and his associates suggests that college students mathematically at or above the level of college calculus have difficulty generating simple equations to represent real-world relationships among quantities, even when those quantities are exact and not approximate. Furthermore, the researchers' study of interview protocols indicated that the students' errors were the products of their misconceptions and not just temporary or accidental. Wollman (1983) indicated that an unspecified number of elementary education majors were unable to translate a verbal description into an equation although they correctly answered comparable questions about real-world situations and about equations of the form $y = kx$. Hence, successful performance of related mathematical skills does not necessarily imply that students can make correct translations.

The studies of errors and misconceptions in elementary algebra by Clement and others reveals that college students, including students with majors in mathematics-intensive areas, have difficulty developing very simple mathematical models for verbal descriptions with which these students are probably familiar from their high school algebra experiences. The extensive mathematical background of prospective mathematics teachers may or may not enhance their ability to develop such models. The effects of introducing technology into the modeling setting, even when the setting is as oversimplified as it is in standard word problems from algebra, have not been studied.

In a study that used open-ended mathematical modeling activities with preservice mathematics teachers, Trelinski (1983) tried to assess the degree to which graduate mathematics students training to become teachers were ready to introduce mathematical modeling to their own students. Presumably from her analysis of their written work on a modeling problem from chemistry, Trelinski claimed that each student typically tried only one major solution path and that many students seemed to follow no mathematically consistent scheme.

In Trelinski's study, there were 223 students (from the mathematics departments of three colleges of education and three universities in Poland) who provided adequate information about their progress. She identified three general mathematical approaches to chemical absorption: as a discrete process, as a continuous process, and as an ongoing process by which a finite, decreasing amount of the chemical was absorbed periodically. No student presented a complete solution, but 4% constructed formal (symbolic) models, 26% gave an enriched description of the process, 22% gave a visual model, 39% offered a tentative model, and only 8% produced models that could be verified. According to Trelinski, the students often missed variables, yet it appeared that at least some knew their models were flawed. The students also seemed to make assumptions, perhaps unconsciously, about the whole process and then failed to use these assumptions in a consistent way. Trelinski concluded that the prospective teachers did not demonstrate a natural transfer of their (supposedly) abstract mathematical knowledge to the modeling situation. Hence, she suggested that mathematical modeling should be included in the teacher-training curriculum as well as in the school curriculum.

The research reveals that both on short-answer paper-and-pencil tests and in open-ended interviews, college students, including prospective teachers, in several countries respond inadequately in mathematical modeling situations. This observation indicates that prospective teachers are not calling upon the relevant mathematical ideas that were supposedly part of their coursework. This evidence, however, is insufficient in an important way. Some of these students may actually have some understanding of the necessary ideas. There has been little attempt in most studies to determine through questionnaires, observations, or interviews, whether the participants could connect any of their existing understanding of the relevant ideas to the modeling question posed. In the case of prospective teachers, the possibility that they made no connection is particularly distressing. Even if preservice teachers possess an understanding of fundamental mathematical concepts and principles, this knowledge is useless unless it is accessible.

Effects of Technology Use

Research focused upon the use of technology and its effects on achievement has increased greatly during the second half of the 1980s and the 1990s. Two streams are visible: the use of graphics (calculators and computers) and the use of computer algebra software packages.

Graphical Representation

The method of visualization in the teaching of mathematics is probably very old. With the present technology, we can visualize mathematics in a way that, for instance, Hilbert and Cohn-Vossen (1932/1956) described in the preface of *Geometry and the Imagination:*

> In mathematics, … we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations…. With the aid of visual imagination [*Anschauung*] we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof…. In this manner, geometry being as many faceted as it is and being related to the most diverse branches of mathematics, we may even obtain a summarizing survey of mathematics as a whole, and a valid idea of the variety of its problems and the wealth of ideas it contains. (p. iii)

Some years after hand-held technology in the form of the graphing calculator, together with attractive (relatively cheap and having good graphics) software, spread throughout upper secondary and university education in many countries, evidence of the impact of visual representation in mathematics instruction started to appear. Beckmann (1990), comparing four treatments in a first-semester college calculus course, investigated students' understanding of selected calculus concepts through graphical representation. She concluded that "developing calculus concepts through the use of a graphical representation system, especially as presented through computer graphics, can positively affect student understanding and interest without negatively influencing skill acquisition" (p. 107).

At the same time, studies started to be published that not only showed positive results of technology use but also revealed unexpected results. Dunham (1990) investigated relationships between confidence and performance in a college precalculus course that fully integrated graphing calculators. She measured the confidence and performance of 213 students in the first and last weeks of a 10-week course. She also interviewed 8 students who showed high confidence and 8 who showed low confidence after each of the four examinations to obtain information about shifts in attitude and patterns of technology use. One of her results was that some students felt algebraic guilt (see page 22) about what they called "cheating," that is, choosing the "easy way" of solving problems by using a graphing calculator.

Ruthven (1990) compared the achievement of high school students in the Graphic Calculators in Mathematics Project in England who regularly used graphing calculators in class with that of students in a control group. The project group showed superior performance (with the use of a graphing calculator) on symbolization items that called for an algebraic description of a graph, but performance did not differ on interpretation items that called for the extraction of information from a verbally contextualized graph. Ruthven and Dunham found no clear evidence that graphing calculators and visualization actually provided the major step in the progress of the mathematical learner that many had hoped for when visualization became everybody's property. But many of the published studies did obtain positive results.

As an example of research in which technology yielded positive effects, Quesada and Maxwell (1994) showed that when graphing calculators were allowed on a comprehensive final examination in a precalculus course, the students' performance was substantially higher than when calculators were not allowed. The study extended for three semesters, and all three experimental groups had significantly higher scores on the examination than the control groups did.

In 1996, Dahland and Lingefjärd investigated how upper secondary students in the natural science program of four Swedish gymnasiums were able to use graphing calculators to solve mathematical problems and how they interpreted the solutions the graphing calculator presented on its graphing screen. A clear tendency was found toward uncritical acceptance of the visible graph. In that sense, the study was a wake-up call for the mathematics teaching community to start looking more closely into the results the students derived from their calculators and how they were presented. This study also found the presence of algebraic guilt (see page 22) in students. Some students in the study chose to calculate mentally or with paper and pencil, thereby making more errors, rather than to use the graphing calculator.

Computer Algebra Software

The other part of the progress in technology that took place over the past several decades in the teaching of mathematics was the possibility of programming computers and calculators, followed by the arrival of symbolic manipulation capabilities. The possibility of finding derivatives and integrals, among other manipulations, on a machine instead of with paper and pencil called for research into how this advance would affect the teaching of mathematics. In 1988, Heid used a computer algebra system (muMath), a function grapher program (Graph Functions, Fit Functions to Data, Table of Values), and other demonstration programs as tools in a concept-oriented introductory calculus course. Two experimental classes of an applied college calculus course studied calculus concepts using graphical and symbol-manipulation computer programs to perform routine manipulations during the first 12 weeks of the semester. The last 3 weeks were spent on skill development. In the control class, the emphasis was on skills in demonstration assignments, quizzes, and examinations in all 15 weeks of the semester. Heid found that the concepts of calculus could be learned without concurrent or previous mastery of the usual algorithmic skills of computing derivatives and integrals and of sketching curves.

Palmiter (1986, 1991) investigated the use of a computer algebra system in an introductory college calculus course. She concluded that with the use of a computer algebra system, integral calculus could be taught in substantially less time, eliminating the teaching of integration techniques and yet producing equivalent or better conceptual understanding.

Misconceptions When Using Technology

The fact that mathematical modeling in general and the choice of a "correct" model in particular can indeed confuse students was reported by Searcy (1997), who studied in depth one student's misconceptions of the relation between model and reality when predicting the population growth of Sacramento, California. The student mistook the computer-generated result for reality and indicated that by the use of a growth model, she (or the model) could "determine" the population of Sacramento. Much like what I have observed with my students, Searcy found that the mathematics that is hidden inside a computer software program or a calculator may very well, for many students, be a black box:

> Probably one of the major implications from the study is directed at the use of technology in this course. We have seen an example of a student whose procedural disposition allowed her to treat a spreadsheet template as a black box. She needed hands-on experience with the model's algebraic representation. It took her a long time to be able to connect that representation with its graph. Her ultimate objective with the template was to get the lowest average error. This she could do very well, but she seemed to miss out on understanding much of the mathematics associated with this assignment. Granted, many of the concepts she had difficulties with were addressed by the instructor in class. However, to reach this student, she needed to actively participate in the development of concepts, such as average error and model. She needed to be held accountable for these notions. (p. 159)

It is interesting that so many similarities can be found between a study of a student in a mathematics course at the University of Georgia and my observations of prospective teachers in Gothenburg. Another study that arrived at conclusions about students studying functions, proofs, and modeling was reported by Zbiek in 1993.

Zbiek studied 13 prospective mathematics teachers who were doing mathematical modeling with the help of technology. Her students recognized complex relationships among variables in the real world, but oversimplified the relationships they attempted to mathematize. The models her students constructed were usually developed using either simple mathematical operations together with personal experience or numerically correct but situationally irrelevant functions generated by the computer to fit the data.

Zbiek (1993, 1998) also found that the reasoning of prospective mathematics teachers in a technology-enriched environment revolved around loose associations and contradictions among various realms of reasoning, such as results from tools, real world data, and personal experience. She discussed the complexity of that reasoning: Are the students' errors situated in their mathematical background, the use of computers, or both?

> The data gathered during the current study suggest that the prospective secondary mathematics teachers have conceptions of functions and mathematical models that are dominated by strong beliefs as well as by fundamental inconsistencies. Their reasoning processes frequently involve connections among several ideas yet fail to provide conclusive mathematical ideas. Why do they exhibit substantial misunderstandings with respect to these fundamental mathematical ideas? Part of the problem may be their uses of computing tools and the role of their formal mathematical background in the presence of these tools. (1993, pp. 205-206)

Among other things, Zbiek observed that several of her students failed to use realistic endpoints for the domain when using graphing tools to construct graphs for real situations. She concluded, "The interviews with the prospective teachers, however, do suggest that the naive belief that a model could be chosen on the basis of goodness-of-fit value alone still guided most of the subjects' work" (1993, p. 170).

Lanier (1999) followed three college students in a mathematical modeling course, investigating their understanding of linear modeling when using a spreadsheet template to model data. The way in which spreadsheets and graphing calculators represent a geometric point by means of a visible entity seemed to have caused some confusion for at least one of her students. The student described the fit of one model as a line that was "barely touching the bottom" of a point and crossing "through the middle" of another point (pp. 55-56). The student seemed to have the impression that data points were physical entities from the spreadsheet used in the course and did not appear to know or consider the geometric idea that a point has neither size nor shape.

Zbiek's and Lanier's studies, like Searcy's, have strong connections to my observations in Gothenburg. Although the participants in those studies tended to be younger than my students, there appear to be striking similarities in the behavior of college students in Pennsylvania, Georgia, and Sweden when they use technology to do mathematical modeling.

Assessment of Modeling

One way to attempt the assessment of almost anything students do, such as playing sports, taking school subjects, or reading literature, is to make observations and assess each part of the students' learning and performance. But how should this be done in a complex situation such as mathematical modeling?

> We are prepared to risk our skin by claiming that assessment of applications and modelling is easy. As mentioned earlier, assessment is not easy if we (have to) stick to conventional modes and practices. In that case sound assessment is rather very difficult if not impossible. (Niss, 1993, p. 48)

If one adds the component of existing technology, sound assessment becomes even more complicated unless "conventional modes and practices" are changed. What support should be provided by technology when students are being assessed is a difficult issue and is the subject of ongoing discussion in several places around the world. A phrase often mentioned together with the use of technology is *authentic assessment* or *authentic performance assessment,* which, according to Clarke (1996), refers to mathematical tasks that are meaningful for the student, represent applications of mathematics, and include activities that are, in some sense, also carried out by mathematicians. Mathematical activities like these are natural to combine simultaneously with technological aids, as the technologies naturally affect the selection of tasks. Basic routine problems, designed for a traditional paper-and-pencil test, may appear trivial and irrelevant in the light of technology. For example, the task of manually drawing or sketching complicated curves may be difficult to defend as a relevant activity in the presence of graphing calculators. The use of technology such as computer programs and graphing calculators naturally affects the evaluation situation and also what we mean by *assessment* (Webb 1992).

An essential consideration is whether students using, say, a computer program when they are learning should therefore be allowed to interact with that program when being assessed in mathematics. The motivation for such a consideration is embedded in the view that assessment should mirror teaching. Or as Clarke (1996) argued, "If a technological tool is employed with legitimacy for the completion of a mathematics task in an instructional setting, then the same tool should be available in an assessment setting" (p. 342). In addition to connecting teaching and assessment, the technology can also offer a better possibility of documentation, visualization, and reporting.

The involvement of the students in assessment is likely to shape the educational process. Undoubtedly any advice or instruction to a student on how to express the intended outcome will affect the way in which that student and his or her peers present the solution. It can be seen as essential to students' learning that they are well informed about the critical points that will be assessed and about the grading system to be used by the instructors.

> Self assessment is also important, as it help the students to understand and evaluate the task which have been undertaken. Clearly, students who cannot recognise high-quality work produced by their peers (or by themselves) have little claim to soundly-based knowledge. The inclusion of any student self-assessed

> mark in the overall rating given to a project is a matter for debate, as is the way in which that assessment is carried out. (Izard, 1997, p. 121)

Students become more involved in the process of evaluation as the characteristics of good mathematical performance are clarified. Evaluation may be seen as a substantial part of the didactical contract being negotiated between student and teacher (Brousseau, 1997). Through this interplay, the students can learn to identify the criteria for qualitatively good performance. Further, they can also learn what is regarded as unsatisfactory, fair, good, or very good performance.

Students should also be informed of the dilemma faced by the instructors in a course when modeling problems and technology are used. It is very difficult to create new modeling problems for a new group of students who are supposed to use new computer tools to perform the modeling (which itself is new to them) and simultaneously to have exact and well-tested criteria for assessing different outcomes of the modeling process.

> We should accept that assessment of applications and modelling has to be exercised as an intricately balanced judging of a vast variety of components in a complex and often fuzzy structure. This implies elements of subjectivity and disagreement, and it implies that assessment takes time and cannot be standardised. It does **not** imply that assessment cannot be exercised on a sound foundation of reflection and reasoning and articulate criteria and be subject to clear communication. It also does not imply that assessment cannot be summative and, if necessary, result in marks that may be given in ways which are fairly robust to changes of assessors. (Niss, 1993, p. 48)

One major part of the assessment is what type of problems the students should be assessed on. Silver and Kilpatrick (1989) argued for the use of open-ended problems in the assessment of mathematical problem solving, thereby moving from facts and procedures to concepts and structures.

> Mathematics and mathematics education instruction should enable all learners to experience mathematics as a dynamic engagement in solving problems. These experiences should be designed deliberately to help teachers rethink the conceptions of what mathematics is, what a mathematics class is like, and how mathematics is learned. Instruction should be organized around searching for solutions of problems and should include continuing opportunities to talk about mathematics. Working in groups is an excellent way for learners to explore, develop mathematical arguments, conjecture, validate possible solutions, and identify connections among mathematical ideas. In such experiences, teachers should be encouraged to generalize solutions and communicate results from their explorations of mathematical ideas visually, in writing, or through dialogue and discussion. (NCTM, 1990, p. 128)

If mathematics teachers allow group work, discussion, and information gathering in libraries and over the Internet, and also want students to learn more mathematics in collaborative work, they then face great demands on the types of problems they should pose. George Pólya (1992) once defined a *problem* in the following way:

> In general, a desire may or may not lead to a problem. If the desire brings to my mind immediately, without any difficulty, some obvious action that is likely to attain the desired object, there is no problem. If, however, no such action occurs to me, there is a problem. Thus, to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable, aim. To solve a problem, means to find such action. (p. 117)

Schoenfeld (1983, 1992) argued that a problem needs to have a great deal of uncertainty to be called a problem.

> A problem is only a *Problem* (as mathematicians use the term) if you don't know how to go about solving it. A problem that holds no "surprises" in store, and that can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an *exercise*. (1983, p. 41)

The problems teachers choose also need to provide the students with opportunities to express what they have learned in the course and in previous courses. On an examination, such problems should as much as possible focus on qualitative reasoning and not on the reproduction of facts and basic routines. Allowing students to use technology such as graphing calculators and mathematical software on their examinations makes it even more important to select problems that are relevant to the use of that technology. At the same time that the problem should remain non-trivial in the presence of technological tools, their use should not be the only performance component that is essential and leads to success. A relevant problem should encourage students to make various assumptions and use various strategies in which technology can serve as an aid but never as a goal.

There is no doubt that the content being taught and assessed is changing over time—just look at any school or university mathematics examination from 100 years ago. But to what extent should teachers change the content when teachers and students have access to tools that can do the graphing and the symbolic manipulation? Clearly teachers need to assess more than just the ability to perform such operations. The graphing and symbolic manipulation can be part of the task, but they are no longer sufficient. The question of whether to require technology for solving problems is harder. In the kind of assessment done in the course described in this dissertation, technology is needed for doing part of the problems, but in many problems values could be estimated by using paper and pencil coupled with good thinking. Computers with a spreadsheet are common today, however, and most students rapidly discover that a spreadsheet can be used instead of paper and pencil, but in almost the same way.

Blomhøj (1993) engaged a group of 14- to 16-year-old students in modeling activities with a specially designed spreadsheet. He found that the students did not find the spreadsheet was a barrier when they were setting up a model. Instead, they often expressed a given relation between variables in the model more easily in spreadsheet notation than in words. Blomhøj's study also underlines the importance of having an ongoing dialogue with students when they are working on open-ended tasks. Students who work on open-ended modeling problems need feedback from other students or from a teacher if they are to progress and to stretch the limits of the activity and their own mathematical knowledge.

> As for the dialog with the pupils, the project demonstrates a huge difference in the course of activities of pupils when they are left to themselves (in groups of two), only now and then getting encouragement and minor support through dialogue with a teacher. If the pupils are not challenged by such teacher intervention from time to time, many of them will adopt a humdrum way of working, in which they are very much fettered by the initial presentation of the problem, and by the exercises in the text. If, on the other hand, the teacher succeeds in challenging the pupils during the activity, most of the pupils will develop a reflective activity, characterised by the creation and examination of new problems. Furthermore, the pupils will tend to lean quite heavily on their knowledge about an experience with the problems in question, when analysing and evaluating the model. (p. 267)

Blomhøj's conclusion points to the responsibility to learn that all students should have and to the sources of authority the students are likely to identify when working more with computers than with textbooks and when the teacher acts more like a coach than a traditional lecturer.

Authority and Responsibility

During their schooling, students inevitably try to identify, interpret, and follow authority. One interpretation of this social behavior is that the search for trustworthy authority is part of the human survival instinct. That instinct does not disappear when students begin their university studies, although the search for authorities or survival structures may be more hidden the older and more sophisticated they get. Perry (1968), after interviewing a Harvard student, characterized the student's beliefs as follows:

> He appeared to assume that the ultimate reference in all matters of knowledge and conduct lay in the set of right answers. He saw these answers as the possession of

> authorities, to be dispensed by them through explicit exposition or detailed advice. A student's responsibility, in turn, was to acquire the answers (and to compete for authority's approval) through honest hard work. (p. 23)

As suggested in chapter 1, students' faith in their own understanding and ability seems closely connected to the educational and social question of ownership of the mathematics. Since students easily trust authority, they may also easily surrender their right to their own ideas and problem-solving strategies. The textbook, the teacher, or both may present mathematics and reason about mathematics in such a way that students develop algebraic guilt. Dunham (1990, 1993) uses the expression *algebraic guilt* in several papers to underline the fact that some students see the use of calculators as "too easy." The students seem to feel that such an approach is not "mathematical" enough, that it would be more valuable for them to use algebraic techniques. It is like getting away with something to use the graphing calculator—"almost like cheating" (Dunham, 1993). If a student has this general attitude towards the use of a graphing calculator, he or she may also hesitate to describe the role of the calculator in the current solution, and the presentation will consequently be incomplete or misleading.

With a strong and skillful teacher, it is easy for students to believe that the teacher is the authority on mathematics:

> Unless the child intuitively realizes that standard formalisms are an agreed upon means of expressing and communicating mathematical thought, they can only be construed as arbitrary dictates of an authority. Academic mathematics is then totalitarian mathematics. The child's overall goal might then become to satisfy the demands of the authority rather than to learn academic mathematics per se. This goal can be achieved, at least short-term, by either covertly constructing and using self-generated methods or by attempting to memorize superficial aspects of formal, codified procedures. If the latter approach is adopted, mathematics becomes an activity in which one applies superficial, instrumental rules. (Cobb, 1986, p. 7)

One important aspect of any culture is what constitutes authority in that culture and how that authority deals with people when they think in a critical way or a way that deviates from the "correct," "normal," or "expected" path. In Sweden and probably elsewhere, many people in educational institutions that through policy statements claim they encourage independent, free, and critical thinking do not really know how to deal with students who follow that advice. Students who constantly ask possibly relevant questions about important issues and events in almost any subject are likely to be considered difficult or troublesome. To teach people to question, doubt, argue, experiment, and be critical about mathematics may very well be seen as a threat to established institutions and to the authority system in the culture of school mathematics. It would certainly be very hard to figure out how to lecture in an economically sound fashion about some part of mathematics, like abstract algebra, to a large group of students who regularly argued and expressed doubts and critical views.

If we were to ask almost anyone about the nature of mathematics who is outside the discourse of mathematics and mathematics education, we would most likely get the answer that mathematics is a science from which errors have been eliminated and in which the truth is considered timeless and absolute. From our first day in school, we are taught that there is at least one subject that never changes. Mathematics keeps that image throughout the school years, and in many societies, it is used as an ability selector. Mathematics is often a dominant part of so-called aptitude tests around the world and closely connected to the common definition of intelligence.

The student in Searcy's (1997) study was very much in favor of using a textbook. She expressed the source of authority that lies in a textbook as a portable teacher, always with you:

> During our first meeting, she told me that one of the reasons she was hesitant about using the computer was because it would mean that she would not have a textbook in her possession. In high school, she used her mathematics textbooks often to supplement what she wrote down in class notes. Whenever she did not understand a topic in class, she would read about it from her textbook and "teach" herself. Access to the applied college algebra on-line text from a computer laboratory was not the same as having her own textbook readily available when she needed it. (p. 48)

It is natural that a student who is used to having a textbook to support her studies would feel uncomfortable when the course does not have a textbook. Where does the trust in a textbook go when it is replaced by a computer? Zbiek (1993) reported that some of the students in her study abandoned their own version of a graph for a specific given problem in favor of the computer-generated graph because "obviously the CALC would know more than I would." Another student personalized the computer software, giving the system powerful mathematical ability: "I was hoping the computer kind of would graph something that I could see…that would show that my graph is somewhat right" (p. 226).

Lanier (1999) did not explicitly discuss any transformation of authority, but she observed that her students became very focused, almost obsessed on one particular method to evaluate models:

> Throughout the semester the students were asked to use a spreadsheet template and to consider the average error in determining optimal models. The use of this average error became a dominant procedure for them in the course. They accepted this procedure as the authority for determining the best model within a given modeling situation and even across modeling situations. This use of average error seemed to become an obsession with the students. Every

> conversation, observation, and written document contained numerous referrals to average error and its role in finding an optimal linear model. (p. 85)

Every student likely searches for stable ground in his or her choice of a major or a career, whether it be mathematics or not. A pattern emerging from the studies of mathematical modeling by Searcy, Zbiek and Lanier, as well as from my own research, is that the computer software moves from being one tool among many in the problem-solving or mathematical-modeling process to being the most important or maybe the only tool that students use. It seems important, therefore, to help students see that when they start to solve a problem, they need to decide whether and how technology can help in the solution. In those parts of the problem in which technology is not necessary, it ought to be avoided because it may lead the student astray.

CHAPTER 3

THE CONTEXT FOR THE STUDIES

This chapter contains a brief introduction to the Swedish system of education followed by a discussion of the teacher education program at the University of Gothenburg of which the course in mathematical modeling is a part. The course itself is then described in more detail.

The Swedish Education System

The Swedish School Law stipulates that all children and young people must have access to education of equal value. All pupils enjoy this right, irrespective of their gender, place of residence, or social and economic status. Consideration must be afforded to pupils with special needs. The School Law specifies that education should "provide pupils with knowledge and skills, and, working together with their homes, promote their harmonious development towards becoming responsible human beings and members of society" (Swedish Ministry of Education 1997, p. 15, my translation). The School Law also provides adults with the right to education. This can be provided in municipal adult education, which is called *komvux*. The word *kom* is short for *kommunal* (municipal), and *vux* is short for *vuxen* (adult). Adult education started as a municipal movement in the 1960s. Adult education for those who are physically or mentally challenged is in the same fashion called *särvux*, where *sär* is short for *särskild* (special).

Compulsory education includes basic schools, schools for the Saami peoples of northern Sweden, special schools (for children with impaired sight, hearing, or speech), and schools for the mentally challenged. The nine years of compulsory schooling are for all children between the ages of seven and sixteen years.

After completing their compulsory education, all students have the right to study in a voluntary upper secondary school, the *gymnasium*. Although the gymnasium is voluntary, there are almost no students from compulsory school who do not apply. A gradual change in gymnasium programs started in 1992, and by 1995 the changes had been made. Various options and special courses were replaced by three-year programs, and all students who left the gymnasium became eligible for studies at a university or college. About a quarter of the specialized courses were moved to adult education. In addition, the grading system changed from norm-referenced marks to criterion-referenced marks. In the 1995-1996 school year, these changes were fully implemented.

The school year in Sweden normally begins during the second half of August and ends the second week of June, for a total of some 40 weeks or about 170 instructional days. A school week is five days long, from Monday to Friday. The longest holiday during the school year is from around 20 December to the second week of January.

Levels of Education

Preschool Education

New regulations regarding the Swedish child-care system came into force in 1995. Among other responsibilities, the local authorities must see to it that children between the ages of 1 and 12 years with parents who are working or studying (or who have children with special needs) are supplied with child care. A place should be found within 3 to 4 months after an application. Every child's needs should be central when organizing child care, and those children who need special support should be given it. In January 1998, a new school form was initiated: preschool classes. The local authorities provide every six-year-old the possibility of 525 class hours in preschool (free of charge). By October 1998, 91 percent of all six-year-olds were participating in this program. For 1998 as a whole, 73 percent of all children ages one to five were in preschool compared with 66 percent in 1995. Arrangements vary between the municipalities; for example, preschool classes may be located in or coordinated by a compulsory school, or they may be linked to some other function of municipal child care.

Compulsory Education

The basic school may be either municipally run or independent. Most children attend a municipal school near their homes, but pupils and their parents have the right to select another municipal school or a school independent of the local authority. Slightly more than 2 percent of all students attended an approved independent school in 1995, and the percentage increased to three percent in 1998.

In the fall of 1994, a new national curriculum for the compulsory school came into effect. This curriculum is common to the basic schools whether municipal or independent, the Saami schools, and the schools for those who are physically or mentally challenged. In 1995, all students except eighth and ninth graders were governed by the new curriculum; the 1997-1998 school year was the first in which no grade studied according to the old curriculum. The compulsory schools now have more freedom to select for themselves the grade in which students will be able to choose a foreign language to study. Mathematics and science are compulsory subjects from Grades 1 to 9. At this level, each subject is taught as a separate subject with its own syllabus. The syllabuses indicate the purpose, content, and objectives for teaching each subject. From Grades 1 to 9, students spend a minimum of 6 665 hours of study time on the school subjects. Of these, 900 hours are spent on mathematics and 800 on science, including biology, chemistry, physics, and technology. The objectives are of two kinds: (a) opportunities the school must provide all pupils, and (b) targets the school must pursue. For example, among the objectives for mathematics is the following:

> The school should aim to ensure in its teaching of mathematics that pupils
>
> - set up and use simple mathematical models as well as critically examine the preconditions, limitations, and uses of such models. (Swedish Ministry of Education, 1994, p. 33, my translation)

The paragraph on what the school must pursue in this topic says the following:

> Targets that pupils should have attained by the end of the ninth year in school—
>
> Pupils should
>
> - be able to interpret, compile, analyze, and evaluate data in tables and diagrams. (Swedish Ministry of Education, 1994, p. 35, my translation)

To coincide with the introduction of the new curriculum and syllabuses, a new criterion-referenced system of marks is coming into force. Under this new system, marks are awarded on a three-grade scale from the eighth year of schooling onward. (No marks are given before Grade 8.) The marks are Pass, Pass with Distinction, and Pass with Special Distinction. A pupil who does not achieve the goals set out in the syllabus for the ninth year does not receive a mark in that subject, but has the right to a written assessment from the teacher. Each mark is to relate the pupil's achievement to the national objective stated in the syllabus for the subject.

Upper Secondary Education

Of all the students in Sweden who completed Grade 9 in 1995, 98 percent went on to the gymnasium. Nationwide, gymnasiums offer seventeen 3-year programs, all of which are intended to provide a general secondary education and eligibility for higher education. A large gymnasium might offer 12 or 13 programs, whereas a small gymnasium would offer only 6 or 7. In addition to the national programs, there are also specially designed and individual programs. The national programs are art, business administration, children's recreation, construction, electrical engineering, energy, food, handicrafts, health care, hotel and restaurant management, industrial engineering, media studies, natural resource management, natural sciences, social sciences, technology, and vehicular engineering. These programs cover 2 150 hours for the natural and social science programs and 2 370 hours for the remainder. Each program provides a set number of hours for the various subjects. Time may be divided between subjects. All national programs contain eight core subjects: English, art, physical and health education, mathematics, natural science, civics, Swedish (or Swedish as a second language), and religious education.

Mathematics is organized into five courses that are known as A to E. Course A in mathematics (110 hours) is compulsory for all students in all programs. Sequence A+B (150 hours) is compulsory for the liberal arts branch of the social science program. Sequence A+B+C (180 hours) is compulsory for the economics and social science branches. Students in the natural science program must take courses A to D (240 hours) in mathematics and may choose to take the optional course E (60 hours) as well. Only the natural science program gives students a direct opportunity to apply to any of the three university programs to become a mathematics teacher. Those programs are the programs for mathematics and natural science teachers for Grades 1-7, mathematics and natural science teachers for Grades 4-9, and mathematics and another subject teachers for the gymnasium. Students who lack the necessary courses from their gymnasium program and who are more than 20 years old can complete their education in an adult education (*komvux*) institution.

Higher Education

In 1997-1998, higher education was available at approximately 70 Swedish universities or colleges. Higher education was reformed in 1993, and the central establishment of programs, which had been in effect since 1977, was replaced by local decisions concerning programs to be offered. The Swedish Parliament now creates only a framework for the activity of universities and colleges, and a degree ordinance states which degrees are to be awarded. The number of students in higher education increased by 60 percent from 1988-1989 (when there were slightly less than 200 000 students) to 1997-1998 (when there were just over 300 000). Admission capacity has increased, educational programs have been extended, and the students stay in the system longer. The number of degrees awarded, however, has not increased to a comparable extent. The Swedish population increased from 8 458 888 to 8 854 322 during the same time.

Programs of study. Higher education in Sweden has two kinds of eligibility requirements: general and specific. The general requirements are common to all higher education institutions and are as follows:

1. The completion of gymnasium, adult gymnasium, folk high school, or a foreign secondary school (for a minimum of 12 years of schooling), or the attainment of 25 years of age plus 4 years of work experience.

2. A knowledge of Swedish and English equivalent to the final year of the gymnasium. Although instruction in higher education is in the Swedish language, a great deal of the literature is in English, which is the reason that English is required. For visiting students, a one-year intensive course in Swedish is offered at most of universities.

The specific requirements vary according to the field of study. Competition is usually keen because there is a *numerus clausus* (quota) for each higher education faculty. Students are admitted to a faculty according to their gymnasium marks (or the equivalent) in relevant courses or their scores on a special test: the national University Aptitude Test. Some faculties are also introducing other tests for specific programs.

The academic year, from the end of August to the beginning of June, is 40 weeks, which include periods for examination preparation and thesis writing. Full-time students average 40 hours of study a week, including lectures, laboratories, and independent studies. One such week is measured and labeled as 1 university point. In most programs, grades are given on a three-level scale: Well Pass, Pass, and Not Pass. Some programs, however, use only a two-level scale: Pass and Not Pass. And others, like law and engineering, use scales with more than two levels expressed in letters or numbers. In general, degrees based on 120-140 points are translated into English as *bachelor's* *degrees,* and degrees based on 160 points or more as *master's degrees.* Degrees from programs with fewer than 120 points used to be translated as *university certificate* but are now termed *university diploma.*

Graduate education. Graduate education is offered at the Universities of Stockholm, Uppsala, Linköping, Lund, Gothenburg, and Umeå; the University of Agricultural Sciences in Uppsala; the Royal Institute of Technology and Karolinska Institute in Stockholm; the Stockholm School of Economics; Chalmers University of Technology; Luleå University of Technology; and the University College of Jönköping. In the Swedish system, doctoral studies are systematically planned with courses and a doctoral dissertation. It is in principle possible to complete the doctoral program after 4 years of full-time study, but the average time is around 6 years. Each student is given individual supervision, and the dissertation is defended in public against an opponent (external examiner), often from abroad. The dissertation may be written either as a monograph or as a so-called composite dissertation, consisting of a number of published research papers and a summary. It is published, and a copy is distributed to all universities.

Apart from the doctoral degree, there is a licentiate degree, which is a research degree having a shorter qualifying period (a minimum of 2 years of courses) and a less extensive dissertation than the doctoral dissertation. The licentiate dissertation is defended in a seminar against an opponent. The licentiate was reintroduced in the 1980s by the technical faculty, which is the faculty having the most licentiates because of the demand from industry. Before 1972, there was an older kind of licentiate degree with different criteria. Completion of a postdoctoral research study may lead to the title of *docent,* but special positions are no longer available for docents in the higher education system.

Teacher Education

Swedish teacher education has changed in the past few years. Students used to be trained to teach all subjects to students in Grades 1 to 6, or to teach a set combination of subjects to students in Grade 7 to 9. In 1988, a new program was introduced, a "program for teaching in compulsory school," parts of which are common to all teachers in the compulsory school (Grades 1 to 9). Teachers are trained to teach in Grades 1 to 7 (Swedish and social sciences, or mathematics and natural sciences) or in Grades 4 to 9 with specialization in one of five fields:

- Swedish and foreign languages
- Social sciences
- Natural sciences
- Mathematics and natural sciences
- A practical or artistic subject plus another subject

The length of the program is 140 to 180 points, or 3.5 to 4.5 years of full-time study.

Teacher education for the gymnasium is most often based on a combination of subjects within a field such as mathematics and the natural sciences, languages, or the humanities, for a total of 180 to 220 points. Since 1992, it has been possible in principle to combine any two subjects within a teacher-education program. From 60 to 80 points are taken in each subject, which, along with additional subjects makes the student eligible for doctoral studies. It is also possible to study the subjects in a university faculty first and then to take a 40-point education program at the teacher-education institute or faculty that includes pedagogy, teaching methods, and teaching practice.

Government regulations set out the requirements for obtaining a diploma for teaching Grades 1 to 7, Grades 4 to 9, or the gymnasium. Apart from these general regulations, universities and colleges are free to decide on their own goals and on how the programs are to be organized.

Program in Preschool Education

Before 1977, preschool teachers were educated in special preschool-training seminaries. In that year, the education of preschool teachers and teachers of after-school recreation was brought together into the higher education system. As part of the 1993 higher education reform, programs and qualifications for preschool teaching and recreation were merged into a single program, providing a diploma in child and youth education with different options for specialization. It is not required that these students study mathematics, although they must study Swedish, English, and natural science.

Program in Compulsory Education

Elementary school teachers used to be educated at special teacher-training colleges or in seminaries. In 1977, their education was brought into the higher education system. In the fall of 1988, following a decision by the Parliament, a new teacher-education program was introduced for the compulsory school. It replaced the previous generalist programs for teachers at the elementary and middle school levels of the compulsory school (Grades 1 to 6) as well as the subject-matter programs for the Grades 7 to 9. An important principle in changing teacher education for the compulsory school was that the school should be regarded as a coherent unit. The education was aimed at creating a more integrated culture for teachers in the compulsory school, with no direct connection to the gymnasium.

A new teacher-education program was established in 1992-1993 for the higher age groups in the compulsory school. It aimed at providing a greater focus on subject knowledge and at stimulating variety and diversity in education. As part of the 1993 higher education reform, there is now a single compulsory school teaching diploma with specialization at different grades: Grades 1 to 7 or Grades 4 to 9.

Program in Upper Secondary Education

Prior to the 1977 higher education reform, students were able to supplement their theoretical studies at a university or college with practical pedagogical education from a teacher-training college in order to become a gymnasium teacher. After 1977, the theoretical studies for gymnasium teachers were organized into programs that were restricted along subject-matter lines and that lasted from 4 to 5.5 years. The practical pedagogical program of 40 points (one year), originally intended for those who had completed at least an undergraduate program in the same subject area as that taught in the gymnasium, was still offered. In 1992-1993, the program was expanded to new categories of prospective teachers, particularly those who at a relatively late date wished to enter the teaching profession.

The education of gymnasium teachers aims as far as possible at combining education to teach theoretical subjects with education to teach vocational subjects so that prospective teachers are better prepared for the gymnasium, in which the need for cooperation between teachers from different fields will be greater. The flexibility of the teacher-education program should increase opportunities to match the supply of trained teachers to the gymnasium's need for different categories of teachers. As a result of the 1993 reform of the university and college system, there is a single upper-secondary teaching diploma focusing on all subjects (either theoretical, vocational, or a combination).

Mathematical Modeling at the University of Gothenburg

At the University of Gothenburg, students preparing to become teachers of mathematics and natural science for Grades 4 to 9 or for the gymnasium (Grades 10 to 12) take courses in mathematics that are offered by either the department of mathematics or by the department of mathematics education. Their first semester of studies in mathematics is in a one-semester sequence called MAL200, which includes courses in number theory, discrete mathematics, geometry, and algebraic structures. Depending on which program they are in, in their third, fifth, or seventh semester, they take the next semester sequence in mathematics, called Mathematics, Compulsory School Teacher Education, Intermediate Level (MAL400). MAL400 is divided into four courses: Geometry and Linear Algebra (MAL400a), Real Analysis (MAL400b), Statistics (MAL400c), and Mathematical Modeling (MAL400d). Normally, prospective teachers in Swedish university programs for Grades 4 to 9 stop at the 400 level, and prospective teachers in the programs for the gymnasium (Grades 10 to 12) take additional sequences at the 600 level and the 800 level. Two of the courses in the MAL200 sequence for teachers and one of the courses in the MAL400 sequence for teachers (MAL400d) are taught by the department of mathematics education. The structure of MAL400 has so far been that the courses MAL400a and MAL400b are given parallel with each other over the first half of the semester and the courses MAL400c and MAL400d are given parallel with each other over the second half of the semester. Each of the courses is given over 10 weeks (twice the usual length of a 5-point course).

The students come into MAL400d with either one or two semesters of courses in general pedagogy. The first course block, taken by all students in the teacher education program during the first semester, is called GEM11. It includes topics such as science in society, teaching and learning in the school, how children and adolescents develop language, and perspectives on scientific subjects in teacher education. It also includes 3 weeks of student teaching. If the students take the mathematics sequence MAL400a-d in their fifth semester, they will have taken the second general pedagogy block called GEM41. This block includes lifelong human growth and development, the study of teaching, and 4 weeks of student teaching.

Origins and Objectives of the Course

In response to requests from students and faculty members in the department of mathematics and in the department of mathematics education at the University of Gothenburg, an elective course on the use of technology in teaching mathematics and science was offered as part of the teacher education program in mathematics and natural science from 1989 through 1995. In spring 1996, the university mandated the present course structure for the program, and the course MAL400d, Mathematical Modeling, was put into the 400 sequence as a regular mathematics course for all students in the program.

The mathematical modeling course was first offered in the fall of 1996. From the beginning, it had a clear focus on modeling. Because of its subject matter and its strong emphasis on the use of graphing calculators and a variety of software, not everyone in the department was prepared to teach it. Only three teachers have taught it to date. Normally there are two or three teachers each semester, depending on the size of the class.

The course has several objectives. One is that students should learn how to handle technology and become experienced in using it. Another objective is that students should learn how to describe and handle the mathematics used in daily life and in modeling problems involving real phenomena. The course also prepares students for further studies in mathematics. When students study geometry in the 600-level sequence, many of them return to the MAL400d computer laboratory to use The Geometer's Sketchpad software since the department of mathematics does not provide that kind of software in their own computer laboratory. The course is the only such mathematical modeling course in the Swedish program for teacher education.

Organization

The course is designed to give the students insight into how they could solve extended problems using mathematical modeling by drawing on technology and their background in mathematics. The students usually have very limited experience collaborating in solving extended problems requiring approaches unlike those practiced in class or in using mathematical literature other than the normally required textbook to support their arguments. Theoretical views of mathematical didactics and of mathematical modeling in education are discussed as part of the course in seminars on the literature, in written assignments, and in the students' work in the computer laboratory. The software used in the course is mainly The Geometer's Sketchpad (Jackiw, 1995), PC Logo (Daumling, 1997), Excel (Microsoft, 1997), and CurveExpert (Hyams, 1996). Graphing calculators are available on loan to those students who do not have one of their own. The students also can access a page devoted to the course on the World-Wide Web. This Web page contains instructional material (in the so-called dynamic HTML code) regarding the use of graphing calculators, GSP, and PC Logo. The schedule for the course is also there, together with guidelines on how to write up assignments and final-examination responses. For those students who miss one of the literature seminars that are part of the course, an alternative assignment is presented on the Web page.

Depending on the number of students, the organization of the course may change somewhat. When the course is offered in the spring, there are seldom more than 30 students, whereas in the fall, there are typically around 75 students. This difference results from the fact that the University of Gothenburg accepts almost three times as many students into the teacher-education program each fall as in the spring. The only change in the organization of the course is that the larger fall course is divided into three groups, each treated like one spring course in terms of laboratory hours, seminars on the literature, and assignments.

The MAL400d class sessions for the 10 weeks are held on Mondays, Tuesdays, and Wednesdays, with lectures once a week on Monday or Tuesday morning and laboratories or an occasional literature seminar at the other times. Two computer laboratories are used for the course: one large laboratory with 15 computers and a smaller one with 8 computers.

Lectures

The software used in the course and the graphing calculator are introduced through an example or by solving a specific problem. As the instructors introduce the graphing calculator, they also discuss mathematical modeling in terms of data analysis, regression analysis, and curve fitting. Usually during the second lecture, the process of mathematical modeling is introduced and discussed explicitly. For example, in the fall 1997, the lecturer discussed the stages in the modeling process that are presented in chapter 1. After the stepwise introduction, the lecturer showed Figure 1 (see p. 6) as a summary of what had been said and as a support for further discussion. A PowerPoint version of the figure was available for downloading from the course Web page.

The lecturer also told the students how important it was to identify the difference between curve fitting and modeling and to validate every model based upon realistic reasoning about the mathematical model and the phenomena it was supposed to model. This point was made not only in the second lecture but also when the lecturer discussed the grading of Assignments 1 and 2. One of the lectures also focused on the difference between drawing or sketching by hand and the construction work done with a tool like The Geometer's Sketchpad.

Two main ideas dominate the use of technology in the course. First, and most important, the students are given the opportunity to see how technology may affect the teaching, learning, and assessment of school mathematics. Second, it is assumed that the students can deepen their own understanding of geometry, mathematical modeling, and proof if they use technology when solving problems.

Literature Seminar and Laboratory Work

There are two literature seminar meetings during the course. The aim is that students should be given the opportunity to read articles closely connected to the content of the course. Some of the articles are in Swedish, and some are in English. The ideas and perspectives presented in them are discussed during the seminar. Every article is assigned to a pair of students who are responsible for presenting the gist of the article and starting a discussion of its content. Every student reads two articles, one that she or he is responsible for in the seminar and one that she or he will be informed enough about to support the student who is responsible. Articles for the first seminar meeting are handed out the first week of the course, and the first meeting is held during the fourth or fifth week. At the end of the first meeting, the articles for the second meeting are handed out. That meeting is held during the eighth week.

The two computer laboratories of the department of mathematics education in Gothenburg have fairly new or "good" computing equipment; no machine is older than 3 years. All currently run Windows 95 or 98, and MS Office 97 has been installed on them since 1997. Every computer is connected to the Internet and to a printer. As part of the modeling course, the students work in a computer laboratory with all the software described above and with access to the Internet. Their reports consist partly of paper documents and partly of computer files on a disk. They can communicate with the instructors from home by electronic mail and fax. The computer laboratory is reserved for the exclusive use of students in the course Monday to Wednesday between 8.00 and 17.00 for 10 weeks.

Assignments and Assessment

There are two assignments in the course. The first consists of small, relatively closed problems that the students are supposed to solve with the help of PC Logo, The Geometer's Sketchpad, graphing calculators, or Excel. A typical problem in the first assignment might ask students to perform a geometrical construction with both The Geometer's Sketchpad and PC Logo and then discuss how their strategies were different depending on the software. The second assignment is generally broader, more open, and more focused on the modeling process. The students can ordinarily choose the kinds of computer aids they would like to employ in the modeling process and whether or not to use a graphing calculator. The first assignment is typically handed out during the first week of the course and is due the third week. The second assignment is handed out during the third week and is due the seventh week. The assignments serve partly as instructional material and partly as preparation for the final examination. The final examination is a take-home examination that is handed out the last week of the course and is due the following week.

A fundamental idea in the assessment of the students is that assessment and teaching are integrated. This integration is visible in the selection of course material, in how the teaching is conducted, and in how the students' performance is evaluated. Since the students are all prospective mathematics teachers, it is natural to discuss the developmental work being done on assessment at the national level. In Sweden, as in countries like the United States, England, and Australia, alternative assessment strategies are discussed and tested with the purpose of adopting a more qualitative perspective when assessing students' mathematical performance.

Sweden conducts a national test in mathematics for all students in Grade 9. The criteria recommended for the evaluation of this test are basically the same as those used in the modeling course:

- The mathematical content—Is it correct in terms of notation, figures, diagrams, and conclusions?
- The report—Is it written in a language and style that is structured, clear, and distinct?
- The problem—Is it solved, generalized, explored, and investigated to the limit of all available resources?

In a course of this type, assessment may be difficult because of questions regarding how and in what situations the student builds her or his mathematical competence. The course also attempts to challenge the students' attitude toward and knowledge of mathematics, and the effects of that challenge need to be assessed.

After the course, the students are asked to evaluate it by responding to questions regarding the content and organization. The student are sometimes very frustrated by the difficulties they have in attempting to solve the modeling problems and by the fact that their work in the course sometimes reveals how much mathematics they have misunderstood. They almost always conclude, however, that more courses should be like this one because the enjoy the way the assignments and the final examination are set up, the collaboration between students, and the friendly atmosphere.

The mathematicians at the University of Gothenburg typically criticize the mathematics courses given by the department of mathematics education, because the courses are not mathematical enough and far too methodologically oriented. They have made an exception for this course, however, which they seem to think of as one that is both mathematically correct and well thought out. Several papers have been published about the course (Holmquist & Lingefjärd, 1997a, 1997b; Lingefjärd & Holmquist, 1997, 1999, in press; Lingefjärd & Kilpatrick, 1998), and it has been discussed in presentations at national and international conferences.

CHAPTER 4

MODELS AND REALITY (STUDY 1)

But the classroom is a social world that is strange and mysterious, how can what is going on be understood, how can it be recorded, and how can it be interpreted.
Stephen Ball (1982)

A growing awareness of students' misinterpretations of computer- and calculator-generated results prompted me to conduct an investigation into how students think about mathematical modeling using technology. The students' responses when I taught modeling, encouraging them to use both sophisticated graphing calculators and no-less-sophisticated computer software when solving mathematical problems, made me realize how hard it may be for students to validate the results they get. Do they "abandon reality" and their own mathematical knowledge?

I undertook a study to address the following questions:

- What views do students in a course on mathematical modeling have about technology?
- To what extent do students in a course on mathematical modeling believe in results from calculators and computers?
- How do students relate mathematical models to reality when using calculators and computers?

Method

The study was conducted in the MAL400d course at the University of Gothenburg in fall 1997. Until the middle of the term, I co-taught the course with one of my colleagues, giving roughly half of the lectures. At that point, as explained below, I began interviews and observations of lectures and laboratory sessions. The participants included the entire class as well as a study group of five students whose work was studied more intensely.

Participants

The Class

The 71 students in the class ranged in age from 20 to 49, with a median age of 29 and a mode of 21. Forty-two were women, and 29 were men. The distribution of their ages, which is shown in Figure 3, was typical of students in the program for prospective mathematics teachers at the University of Gothenburg.

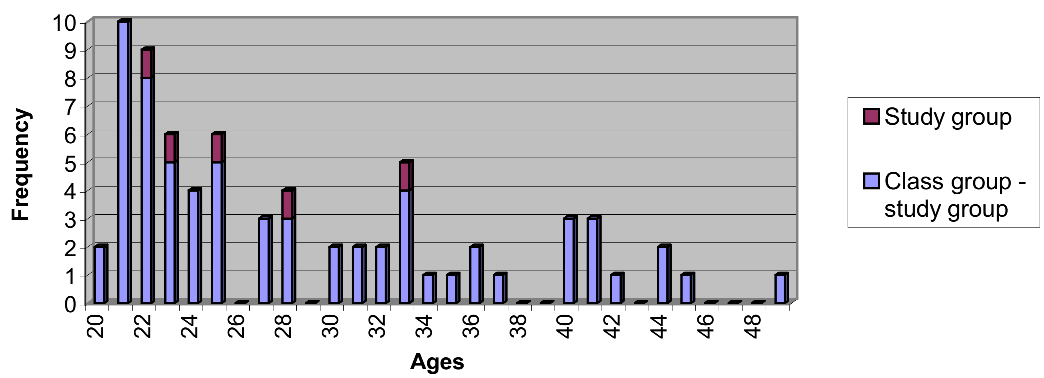


Figure 3. Age distribution for the fall 1997 class (with study group members shaded).

Some students had gone directly to the university from the gymnasium; others had worked a few years, been in other university programs, or served in the military. Two students had been teaching mathematics in the compulsory school in Iraq, three students had received their gymnasium education in the former Yugoslavia, two were from Finland, and one was from Romania. The remaining students had all received their primary and secondary education in Sweden.

As is typical for the course, the students had all taken essentially the same mathematics courses in the mathematics department at the university, but sometimes in a different order. The two mathematics courses that were closest to the modeling course were the real analysis course and the linear algebra course. Students who did not earn a grade of Pass or Well Pass on their final examination for a course were required to repeat the examination. They were allowed to repeat the examination as often as necessary until they received a passing grade. Although some of the participating students had not attempted the examinations for real analysis or linear algebra, most had passed both examinations on the first attempt. One student, however, had not passed the real analysis examination until the third attempt, just like another, who had taken the linear algebra examination three times before passing it. The students' performance on the examinations is shown in the second column of Table 1.

The Study Group

Selection. Just before the midpoint of the course, using students' responses to an entry questionnaire and two assignments, I chose five students to follow more closely. These five students appeared very confident of their mathematical ability, but their written work showed quite a few misconceptions. I told them that I was interested in them and their work. I said that to learn more about how prospective teachers learn and do mathematics in the presence of technology, I wanted to observe their modeling work and conduct some interviews with them. I indicated that they would not need to spend any extra time or do any more work beyond what was otherwise expected in the course. Instead, they would most likely benefit from my presence as an observer since I could help them with technical and mathematical problems.

Table 1
Number of Students in Each Group Earning a Pass (P) or Well Pass (WP) on the Analysis and Linear Algebra Examinations After Different Numbers of Attempts

Course	**Group**	
	Class	**Study**
Real Analysis		
No attempt	2	1
First attempt (WP or P)	61	3
Second attempt (WP or P)	5	1
Third attempt (P)	1	0
Linear Algebra		
No attempt	2	0
First attempt (WP or P)	58	4
Second attempt (WP or P)	5	1
Third attempt (P)	1	0

The five students agreed and continued to work together the rest of the term in one of the two computer laboratories that were used for the course. We formed a group that ordinarily met in the afternoons when the laboratories were less crowded. At the first meeting, I informed them as to how I would observe and take notes in class and how I would observe their work in the laboratory. I also informed them that I would not be grading their assignments or the final take-home examination. All five welcomed this opportunity to have a “teacher of their own.”

Description. Of the five students, three were women and two men, which was about the same gender distribution as in the whole class. Their pseudonyms for this report are Adam, Beatrice, Carl, Doris, and Eva. They ranged in age from 22 to 33, with a median age of 26. As Figure 3 shows, the distribution of their ages roughly approximated that of the younger members of the class as a whole. Data on how the study group performed on the linear algebra and real analysis examinations can be found in the last column of Table 1. The table shows that four of the students in the study group had passed their real analysis and linear algebra examinations on the first or second attempt. Since the second examination had to have been taken by the second week of the modeling course, the data in the table show that these four had completed real analysis and linear algebra early in the course. One of the students, Adam, did not even try to take the real analysis examination for reasons he chose not to reveal. Nevertheless, he showed great confidence in his mathematical knowledge and ability. He was sure that he would go on to study much more mathematics and perhaps become a mathematician instead of a mathematics teacher in the gymnasium. All five students indicated that they had taken a “full course load” in the gymnasium, which meant that in the gymnasium they had studied differential equations or complex numbers or both. This information suggests that the students in the study group had a somewhat stronger mathematical background than the class as a whole, since 25 students in the class declared that they had never studied differential equations or complex numbers.

All five of the students had graphing calculators. Three had a Texas Instruments TI-82 calculator, and two had a TI-81. They also had access to computers outside the computer laboratory, either at home, at their parents' house, or at a friend's house. All five were running Windows on these machines. They had all MS-Office with Word and Excel and had successfully downloaded and installed The Geometer's Sketchpad Demo and CurveExpert.

Instruments

The data I collected during the course came from multiple sources: a questionnaire, observations of some lecture sessions and laboratories, interviews, students' written assignments, and the final examination. I developed several instruments for data collection that are discussed below.

Entry Questionnaire

The entry questionnaire for this study was a short three-page questionnaire. Its purpose was to provide background information about the students' views as to when calculators, graphing calculators, and computers should be allowed in the mathematics classroom. It also provided information about the students' confidence and trust in their own computational skills and in results given by their calculators and computer software. Two problems asked how the students viewed answers given by themselves, their calculator, and various software programs to a set of three calculations. A copy of the entry questionnaire is in Appendix A.

First Interview Protocol

The major intent of the first interview was to clarify the written responses of the students in the study group to the entry questionnaire and the first and second assignments, as well as to obtain further baseline information about their understanding of mathematical modeling and their beliefs about computer- or calculator-generated results. I asked the students to perform two calculations on their calculator that, because of rounding by the calculator, did not yield corrects answers and thereby challenged the students' beliefs. The first assignment included geometrical tasks for GSP and PC LOGO and two modeling problems. The students were expected to use Excel primarily for the first modeling problem but could also employ other software or a graphing calculator. The second assignment included a geometrical modeling task and a more-open modeling task for which the students had to choose a strategy of their own. A copy of the modeling tasks in the two assignments can be found in Appendix B, and a copy of the protocol schedule for the first interview can be found in Appendix C.

Second Interview Protocol

The purpose of the second interview was to explore each study group member's understanding and views of mathematical modeling in light of technology. The interview also made use of the final examination that the students had submitted, which had been graded by the other instructor. The solutions the students had turned in and the grades they had received were discussed. In particular, the solutions to the two modeling problems were analyzed, and the students were asked about possible errors they had made. A copy of the protocol schedule for the second interview can be found in Appendix D.

During the second interview, the student and I considered two main questions. The first was intended to explore how the student related the discussion we had in the first interview regarding miscalculations by machines to their responses to the examination questions. The second question dealt primarily with the student's conceptions and strategies when validating the mathematical model she or he had constructed and that supposedly modeled a real phenomenon.

Procedure

All 71 students were given the entry questionnaire during the first meeting of the class. Five students did not respond to all the questions. I recruited the five students for the study group during the fifth week, after reviewing the class's responses to the questionnaire and to the two assignments.

I was present in one of the computer laboratories on at least two of the three days every week except when the students were gathered for a lecture or for a literature seminar or when I was interviewing one of the students. I did not give any lectures in the course after the point in the middle of the term when I started to follow and observe the study group.

Assignments

The students in MAL400d in fall 1997 completed two assignments (see Appendix B for the parts of the assignments used for this study) during the first half of the course, submitting a written report for each one. Each report included several pages of answers to the problems in the assignment. The reports also contained the students' thoughts on how they had used computing tools, as well as their observations, notes, and conclusions about the problems they had solved and the explorations they had conducted. I made a photocopy of every student's laboratory reports.

Laboratory Notes

During some of the laboratory sessions, I took notes about tool use and monitor displays as the students did the assignments, observing both individual students and groups working together in the laboratory. During the second half of the course, special attention was directed towards the study group.

Final Examination

The students in MAL400d in fall 1997 were given a take-home final examination consisting of three problems. The examination was given out on Monday, 12 January 1998, and the students were informed that it should be handed in by noon Friday, 16 January 1998. After all of the students had handed in the papers, I made a photocopy of each one. The two final examination problems used in the study are given in Appendix E.

Interviews

I twice interviewed each of the five study group students individually. The first interview was conducted in the computer laboratory and the second in a small room near the computer laboratory. The first interview was conducted during the fifth week and lasted approximately one hour. The second interview was conducted the first week after the course ended and was also about an hour. During the first interview, the student sat in the computer laboratory with a nearby computer running Windows 95 with Excel, The Geometer's Sketchpad, and CurveExpert software. The student also had her or his own graphing calculator to use. During the second interview, the student sat in a small room with a nearby whiteboard. I took detailed notes during each interview. These notes included descriptions of the student's work with the computing tools, nonverbal components of the conversation, and references to previous written material. I also kept copies of each student's written work.

Results

In the first section below, I discuss the students' views of technology. The second section examines their beliefs about the results technology gave them. The third considers how they related models to reality.

Technology Use

When Learning Mathematics

Asked if they thought one should be allowed to use a calculator or graphing calculator when learning mathematics, most students in the class said yes, and none said no (see Table 2). One wrote, "Even if I type a formula into a calculator, I learn something. Why is it better to memorize?" One student who favored use of calculators only on special occasions wrote, "Mathematics is what you do in your head. The calculator only does approximations." The students were less likely to favor computer use when learning. Ten would never allow computers, and the majority would allow them to be used only sometimes. They wrote, for example, "The computer may be a good tool when writing about mathematics." "There are far more disadvantages than advantages in using a computer when learning mathematics, since you can't take it with you to the examination." Adam wrote "If I can't do the problems with paper, pencil, and my mind, then I'm not good enough in mathematics to teach in the gymnasium," and Eva claimed "I always have a calculator at my desk when I study, but I don't use it very much."

Table 2
Distribution of Class Responses to Questions About Allowing Various Technological Aids When Learning Mathematics

Aid	Yes	Sometimes	No
Calculator	46	20	0
Graphing calculator	41	25	0
Computer	21	35	10

When Being Examined in Mathematics

The students' attitudes about the use of technological aids on examinations were less favorable overall (see Table 3), with no one giving an unqualified yes. As with learning mathematics, the students were more favorable toward the use of calculators on examinations at times, especially if they were not graphing calculators. Many thought that allowing graphing calculators depended on what was being examined. Students wrote, "There should be examinations where you are required to have a graphing calculator. There should also be examinations where you have no use for one or do not need it." "You must understand what you are doing." When it came to computers, just under a quarter did not know where they stood, and another quarter thought computers should never be allowed on mathematics examinations. One attitude concerned "fairness" with respect to the examination situation: "How could someone [the teacher] ever check what's in there? I mean, someone could have tons of software and information in the computer." Several students mentioned the question of equity among students: "Some students can afford a fancy computer at home, others cannot, and that will create a huge difference in the examination situation." One student who did not favor computer use wrote: "Mathematics is really a thinking game and should be performed mentally, possibly with the use of notes on paper but never with technology. When we use technology, it just becomes applications."

Table 3
Distribution of Class Responses to Questions About Allowing Various Technological Aids When Being Examined in Mathematics

Aid	**Sometimes**	**No**	**Depends on what is examined**	**Don't know**
Calculator	50	0	16	0
Graphing calculator	35	0	31	0
Computer	17	17	17	15

Among the study group members, Carl considered it impossible to distinguish between calculator and computers, and Doris indicated that "calculators and computers should only be allowed if they are beneficial to one's understanding." Adam claimed that the calculator probably hurt more of the students' thinking than it helped, whereas Beatrice and Eva took the opposite attitude. Beatrice said, "If calculating devices help with tedious calculations, then students should be allowed to use them." Eva proposed that tests be divided into different parts: one part that would require calculators, and another that would not allow calculators. It was clear from their responses that these students, like the rest of the class, had quite different opinions about how to assess at a time when the available technology was so powerful.

Confidence in Technology

Questionnaire Responses

A majority of the students viewed their calculator as a reliable instrument, trusting its hidden mathematical skills more than their own mathematical knowledge and sometimes even more than they trusted a teacher. As part of the entry questionnaire, the students were given a problem that consisted of a sequence of questions about powers of numbers (Figure 4). When asked to calculate $(8)^{1/3}$ mentally or with paper and pencil, all 66 of the students wrote 2 as the root, and all said they believed and trusted their result. Asked to calculate $(-8)^{1/3}$ the same way, 62 students answered –2, and all of them believed and trusted their result. The remaining 4 students gave a complex number such as $1 + \sqrt{3}i$ as their answer and said they were not sure of their result. Asked to calculate $(-8)^{2/3}$, 55 students gave 4 as their answer, 4 gave a complex number, and 10 did not know the answer. Asked if they believed and trusted their result, 50 said yes, and 16 said no.

a. Calculate by mental arithmetic or by paper-and-pencil arithmetic the result of $8^{1/3}$. Do you believe and trust your result?
b. Calculate by mental arithmetic or by paper-and-pencil arithmetic the result of $(-8)^{1/3}$. Do you believe and trust your result?
c. Calculate by mental arithmetic or by paper-and-pencil arithmetic the result of $(-8)^{2/3}$. Do you believe and trust your result?
d. Compare the results you got on (a), (b), and (c) with the result that your calculator gives for the same problems.

Figure 4. Problem 1 of the entry questionnaire.

The students were then asked to do these same problems with their calculator and compare the results they got. Ten students said that their results were okay, 22 said that the results did not match, and 34 said they were confused and did not understand. Of the

66 students, 15 indicated that they got an error message on their calculator (probably because they had mistyped the expression).

a. We have calculated the result of $(–8)^{1/3}$ with the help of Derive, Maple and MATLAB. The results were:

Derive:	$1 + \sqrt{3}\, i$
Maple:	1.000000000 + 1.732050808*I
MATLAB:	1.000000000 + 1.732050808*I

What do you now think of your own answer and the answer from your calculator for this question? Do you still trust your calculator? Do you trust your own calculation?

b. We have calculated the result of $(–8)^{2/3}$ with the help of Derive, Maple and MATLAB. The results were:

Derive:	$-2 + 2\cdot\sqrt{3}\, i$
Maple:	-2.000000000 + 3.464101615*I
MATLAB:	-2.000000000 + 3.464101615*I

What do you now think of your own answer and the answer from your calculator for this question? Do you still trust your calculator? Do you trust your own calculation?

c. Let us try to use some calculation rules on the expression $(–8)^{1/3}$:

$$(-8)^{1/3} = (-8)^{2/6} = ((-8)^2)^{1/6} = (64)^{1/6} = 2$$

Figure 5. Problem 2 of the entry questionnaire.

The great majority of the students appeared to believe that the solution to $(–8)^{1/3}$ was –2 and nothing else, although their response was equivalent to solving the equation $x^3 = –8$ for the real root only. Although they had studied abstract algebra, they did not apply the fundamental theorem of algebra.

In a second problem (Figure 5), the students were confronted with answers to $(–8)^{1/3}$ and $(–8)^{2/3}$ from Derive, Maple, and MATLAB. Fifty-two of the students responded that they thought that the answer $(–8)^{1/3} = 1 + \sqrt{3}\, i$ was a software error. When the result of $(–8)^{2/3}$ was presented as $-2 + 2\sqrt{3}\, i$, then 35 students decided that they had not understood the question, 11 said that the software programs were probably correct, and the rest left the response blank. One student wrote, "This is weird. I don't understand why the computer gets a different value than I get and my calculator gets. Is this a complex number? I don't see why."

At the end of the second problem (Figure 5c), the students were shown a miscalculation of $(-8)^{1/3}$ that led to the answer 2. Eight students wrote that they now were suspicious of their previous answers. Only 5 students now trusted their own calculation, 20 did not, and 41 did not know whether they trusted it or not. One student said the following: "This is amazing; I'm totally astonished. With my way to calculate $(-8)^{1/3}$, I got –2, which is obviously wrong. But I can't see why."

Interview Responses

Only one of the five study group students, Eva, was successful in getting a correct answer to one of the two calculator problems from the first interview (see Appendix C). She saw that $28923761^2 - 28923760^2$ could be expressed as $(x + 1)^2 - x^2 = 2x + 1$, which gives $2 \cdot (28923760) + 1$ or 57847521. The other four all got incorrect answers of one sort or another and had difficulty understanding why. When I talked with them one by one about the fact that all machines have limitations in the way numbers are stored and represented, they all had the same puzzled expression. When I explained that for a machine with n digits of accuracy, the result of $1 + 0.00\ldots01$ with n zeros must be 1, Adam, Beatrice, and Doris indicated that it was the responsibility of the manufacturers to see that such errors did not occur. Carl claimed that if you cannot trust calculators and computer software, then you are in an impossible situation: "You simply must trust the calculator but should use your head and make estimates when you can."

In the second interview, when the students were given Worksheet 1 on critical points of a function (Appendix D), four of the five students said that they would use another software package to check the result. If the answer was the same, they would consider it correct. This response indicated that they had learned to trust computer-generated results increasingly during the course. In contrast, the remaining student Eva declared that she had become more critical than before of using technological support to do calculations, since it was so difficult to see through them. She argued that she would try to differentiate by hand, graph the function $f(x)$ and its derivative $f'(x)$ with a computer or graphing calculator, and from that visual information try to validate the given result.

Laboratory Observations

My first laboratory observation occurred when all the students got back their graded second assignment. The study group met to discuss the assignment and the question marks they found on it, together with a request from the lecturer to make some comments. Three members of the group (Adam, Doris, and Eva) also had to do further work on Problem 1B from their first assignment, since they had been graded Not Pass. The five of them were not working together but seemed to be divided into three sets: Adam and Beatrice were telling Carl and Eva, respectively, what to key in, while Doris was passive, barely listening. When discussing the first assignment, they seemed to talk more about the limits of Excel than about the substantive features of a solution. Adam suggested that they "drag down and copy to more cells. Why don't we use 4000 cells instead?" Eva expressed an opposite opinion: "If that is what's asked for, then why don't we use 10 000 cells instead? Or 12 000?" showing some insight into the difference between thousands of cells and infinity.

The second time I met them in the laboratory, a more definite group spirit seemed to be present. The group was struggling with their calculators and asked me for some help. Although they had been using their calculators to do regression analysis and curve fitting, no one made any connections between this technique and the recursive problem in Assignment 1B where the scatter plot of $R(k)$ of course could easily be fitted to a Trendline in Excel. Eva showed me an analytical solution of the divergence of $R(k)$, which she was very proud of:

> You just set $R(k) = R(k+1)$ when k goes to infinity. Let $R(k) = A$, and you get $A = 1 + k/A$. That gives $A^2 = A + k$, which can be written $A^2 - A - k = 0$, with the solution $\frac{1}{2}[1 \pm \sqrt{(4k + 1)}]$. It is obvious that the positive solution is divergent when $k \to \infty$.

I got the impression that Eva was the only one in the study group who relied on her mathematical competencies and resisted putting complete trust in the technology to solve the problem for her.

The third time I met the group, the final examination had been given out, and the students were occupied with the problems. The e-mail problem (see Figure 6 below, p. 84) created a lot of discussion. Carl decided to contact someone at the Swedish National Statistical Bureau and try to find out whether it is possible to answer such questions. Adam had downloaded and installed CurveExpert on his home computer and was very enthusiastic and confident about his newfound ability to solve any kind of mathematical modeling problem. He demonstrated CurveFinder to the rest of the group:

> Look here, see this set of data points. Now I just run CurveFinder, and it gives me the best fit out of hundreds of models. It's fantastic!

The course at that point had apparently brought the students to a level at which they could use the technology in a relatively advanced but uncritical way. The only skepticism and critical view I could sense was that of Eva, although her view could have been grounded in a stubborn desire to search for the "true mathematical solution" before using technological aids. The laboratory observations made me suspect that the available technology—graphing calculators, Excel, and CurveExpert together—created an environment that was hard to analyze and stay critical of while simultaneously using it to solve mathematical modeling problems.

Models and Reality

The fact that a modeling process should contain a validation phase was definitely new to most students. Through the lectures and laboratories, the lecturers tried to make the students aware of the need to validate their mathematical models by appealing to other sources of information or by common sense. Assignment 1B (given a recursive function, sketch its graph, investigate its behavior for an infinite argument, and argue for your choice of method; see Appendix B) is an example of the type of task used to encourage students to validate models. Nevertheless, this phase seemed to be the easiest for students to omit, most likely because many years of looking for the right answer fosters a behavior that is not that easy to change.

The capacity of Excel to calculate with a precision of 30 decimal places convinced the five study group students that they had solved Assignment 1B correctly when, in fact, Beatrice and Carl had been graded Not Pass on it. Both of them had assumed that the number of cells in Excel was infinite. Beatrice had written such comments as, “You see that it will continue like this forever.” Carl wrote, “If only Excel had had twice as many cells, I could have proved what the limit would have been.”

For Assignment 1C about the population of Sweden in the year 2000 (Appendix B), all five had used exponential regression on their calculators, obtaining answers like $y = 1.180655999 \cdot 10^{-5} \cdot 1.006829125^{x}$, which yields 9.63 million for $x = 2000$, with a very slight difference between the TI-81 and the TI-82 responses. I asked the students how sure they were of this result. For instance, were they sure to the nearest hundred thousand or to the nearest ten thousand? Could they be off by half a million, which is well over the number of people living in Gothenburg? All of them tried to convince me that this extremely large result was not evidence against their model and that it was likely that Sweden would grow faster in the future with immigration and other factors. Even Eva, who had expressed the most doubt about technologically generated results, held on to her model and prediction.

When I went over the results of the final examination with the study group, all of them wanted to discuss Problem 3 with me. It was about e-mail traffic (Figure 6). Just by examining the data given in the problem year by year, one could see that the number of messages during the whole of 1994 exceeded that of 1993 by about 1200%. The increase in 1995 was about 400% more messages than 1994, in 1996 about 200% more than in 1995, and in 1997 about 180% more than in 1996. The month of June accounted in general for about 70% of the yearly average for every year in the period.

Most of the students in the course had employed CurveExpert to conduct a curve-fitting process and then chose the “best” model by using the regression coefficient. The tool CurveFinder in CurveExpert rates the available models according to the value of that

The number of people who communicate over the Internet by e-mail has increased greatly over the last 3 to 4 years. The graph describes the increasing use of e-mail in the teachers college at the University of Gothenburg since April 1994. See Figure 1.

The numbers that were used to draw this graph can be downloaded from the course Web page.

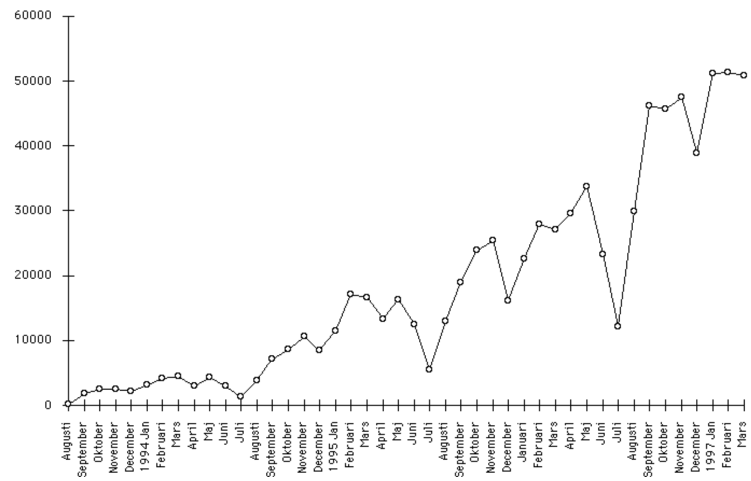


Figure 1

A reasonable assumption would be to expect the frequency of e-mail to continue to increase. Your task now is to help the technical advisors in the teachers college calculate the amount of e-mail expected in June 2000.

To do that, you need to construct or determine a mathematical model according to suitable principles discussed in class and then use this model to provide the technical advisors with the numbers they need.

Write a report that describes your strategies, your analysis, and your conclusions in a satisfactory and adequate way. Give careful, detailed explanations and justifications of all assumptions and calculations in your solution, put numbers on figures and tables, do not mix assumptions and conclusions, and use correct mathematical notation.

Figure 6. Problem 3 from the final examination.

coefficient. All five students in the study group had taken this approach. During class, the lecturers had frequently discussed the difference between fitting a curve and obtaining a “suitable” model. What was obvious with the e-mail phenomenon was that the data showed a tendency toward rapid growth. CurveExpert gave an exponential fit as the first choice and a power fit as the second.

The graded papers for the study group showed the following results:

STUDENT	MODEL	RESULT	GRADE
Adam	*Power*	*140 000*	*Not Pass*
Beatrice	*Power*	*164 000*	*Not Pass*
Carl	*Power*	*135 000*	*Not Pass*
Doris	*Logistic*	*91 000*	*Pass*
Eva	*Power*	*250 000*	*Not Pass*

Most students in the class, including Beatrice and Eva, accepted that the instructors did not consider a power or an exponential fit passing, since both models assumed a growing rate of increase in e-mail that was contradicted by the data. Adam and Carl, however, decided to try to convince me that their calculations were correct. During the discussion, it turned out that they truly believed that the values showed exponential growth. They were convinced that the growth of e-mail messages was exponential and that people's use of e-mail would follow that trend well beyond the turn of the century. They even invented scenarios to try to convince me that each faculty member in the teachers college would be sending and receiving over 100 e-mail messages a day in June 2000. Or possibly that the staff would grow. Or that the summer school would grow (most faculty leave for summer vacation in mid-June). In sum, both Adam and Carl deeply believed that the model was true and that the reality was false.

When I asked them what the characteristics of this exponential growth were, they explained that an exponential function is "strongly" growing in R^2 and that they saw no reason to believe that the tendency of e-mail would change from strong growth to decreasing growth in the next two years. When I asked them what "strong growth" meant, they agreed after some discussion that it could be related to the derivative. When I showed them the contradictory data that the slope between pairs of consecutive years after 1994 was diminishing, whereas it should be growing if the function were exponential, it was as if their world had fallen apart. They had forgotten that such an "easy" check could be used to determine the validity of a model.

It was obvious, even after this very limited study of how students handle modeling situations in the presence of technology, how easy it can be for students to "get lost" and trust the technology far too much, thereby avoiding a necessary validity check. This trust, in turn, seems to profoundly disturb students' ability to relate mathematical models to reality. Whether this trust in technology is related to a limited knowledge or understanding was hard to say. I decided to undertake a second study to learn more about this phenomenon.

CHAPTER 5

CONCEPTIONS AND MISCONCEPTIONS (STUDY 2)

The findings of Study 1, especially those from the discussion concerning the e-mail problem, made me realize how complex it might be for preservice teachers to relate mathematical models to reality when using software tools to generate them. When the students in Study 1 tried to convince me about the change in reality that would justify their model, I became curious. What conceptions and misconceptions about mathematical modeling might they have? I decided to follow a small group of students more closely in Study 2, and to conduct interviews and make video recordings to find out more about how the students interacted with the technology when solving modeling problems and how they thought about what they were doing.

Research Questions

The purpose of Study 2 was to address the following specific questions:

- How do students in a course on mathematical modeling view mathematics, technology, and their choice of a career?
- How do students in a course on mathematical modeling use calculators and computers to generate models?

- What conceptions and misconceptions about mathematical modeling lie behind students' decisions to believe more in a mathematical model than in real-world phenomena?

Method

Participants

The Class

The MAL400d course in spring 1998 enrolled 30 students. Fifteen were women, and 15 were men. The students ranged in age from 21 to 39, with a median age of 26 and a mode of 27. The distribution of their ages, which is shown in Figure 7, was like that of Study 1. One student had been a teacher of high school chemistry in Iraq, and another had completed the gymnasium in France before coming to Sweden. The remaining students had all received their primary and secondary schooling in Sweden. All the students had taken the basic mathematics courses offered in the Swedish gymnasium. One student, however, had not taken a course in trigonometric functions, and between 8 and 13 either had not taken courses in differential equations, complex analysis, statistics, and probability, or did not remember whether they had taken them. The latter two courses are not always offered in the Swedish gymnasium and also would not have been taken by the students from Iraq and France.

As expected, the students had all taken essentially the same mathematics courses in the mathematics department at the university. Two of the students had transferred into the teacher education program from the engineering program; they had substituted a course in multivariate real analysis for a course in geometry that the others had taken.

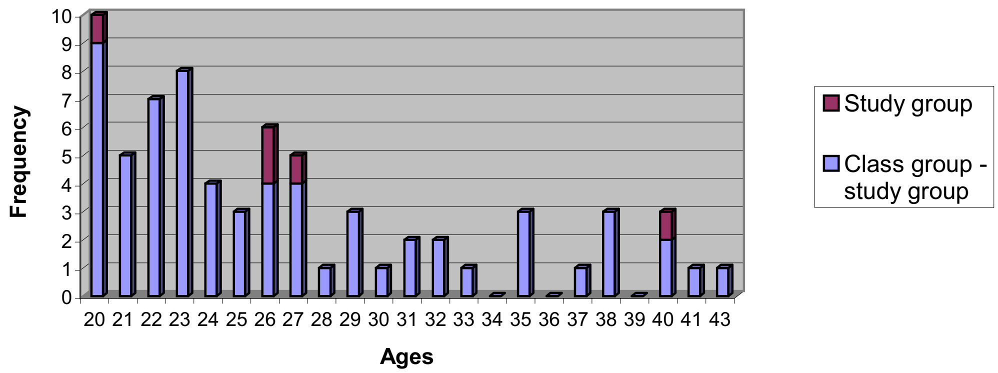


Figure 7. Age distribution for the spring 1998 class (with study group members shaded).

Although some of the participating students had not attempted the examinations for real analysis or linear algebra, most had passed both examinations on the first attempt. One student, however, did not pass the real analysis examination until the third attempt, and another took the linear algebra examination six times before passing it. The students' performance on the examinations is shown in the second column of Table 4.

The Study Group

Selection. At the beginning of the course, I told the class that I wanted 10 students to join me in a special group that would be working together in a separate small computer laboratory, with me observing them. I mentioned my desire to know more about how prospective teachers learn and do mathematics in the presence of technology. I indicated that the volunteers would not be required to spend extra time or do any more work beyond what was otherwise expected in the course. Instead, they would most likely benefit from my presence since I could help them with various problems.

Table 4
Number of Students Earning a Pass (P) or Well Pass (WP) on the Analysis and Linear Algebra Examinations After Different Numbers of Attempts

Course	**Group**	
	Class	**Study**
Real Analysis		
No attempt	4	0
First attempt (WP or P)	15	6
Second attempt (WP or P)	2	2
Third attempt (P)	1	0
Linear Algebra		
No attempt	4	0
First attempt (WP or P)	17	6
Second attempt (WP or P)	2	1
Sixth attempt (P)	1	0

The 15 students who volunteered came to a first meeting that afternoon after class. I gave them further information about how the interviews would be documented with videotape, how I would be observing them and taking notes in class, and how I would be observing their work in the computer laboratory. At the end of the meeting, 10 students signed up for the study group. After the first week, two of the students decided that they preferred to work at home instead of in the computer laboratory and left the study group. Consequently, the final study group consisted of eight students.

Description. Of the eight students, six were women and two men, which differed from the equal gender distribution in the class as a whole. Their pseudonyms for this report are Felicia, George, Hannah, Irene, Jacob, Kristine, Linda, and Monica. They ranged in age from 23 to 33, with a median age of 26.5. As Figure 7 shows, their ages roughly approximated those of the class as a whole, although the range was not as great.

Data on how the study group performed on the linear algebra and real analysis examinations can be found in the last column of Table 4. The table shows that all the students in the study group had passed their real analysis and linear algebra examinations on the first or second attempt. All eight indicated that they had taken statistics and probability in the gymnasium, and five of them also had taken gymnasium courses in differential equations and complex numbers. This information suggests that the students in the study group had a somewhat stronger mathematical background than the class as a whole.

Six of the students had graphing calculators, while two of them had to borrow TI-82 calculators from the department. Three had a Texas Instruments TI-82 calculator, and the other three had CASIO *fx*-9700GE calculators. They all had access to computers outside the computer laboratory, either at home, at their parents' house, or at a friend's house. Seven of them were using Windows machines, and one student was using a Macintosh. They all had MS-Office with Word and Excel and had successfully downloaded and installed The Geometer's Sketchpad Demo and CurveExpert (only on the Windows machines).

Instruments

The data for the study came from multiple sources: a questionnaire, videotaped interviews, observations of the lecture sessions and the laboratories, students' laboratory reports, their written assignments, and the final examination. I did not give any lectures in the course, nor did I grade assignments or the final examination. The instruments I developed are discussed below.

Entry Questionnaire

The entry questionnaire for this study was different from the one used in Study 1. Part of its purpose was to obtain background information about the students' mathematical and educational experiences and their goals. The questionnaire also provided information about their understanding of central mathematical ideas like function, proof, and mathematical modeling and about their access to and knowledge about calculators and computers. One item asked how the students viewed answers given by their calculator to a set of four calculation problems, which were slightly different from these in Study 1. There were also three mathematical modeling problems and two geometrical problems for them to solve that were intended to measure their ability to sketch graphical models to illustrate real world phenomena and to construct geometrical objects with paper and pencil. A copy of the entry questionnaire with selected problems is in Appendix F.

First Interview Protocol

The major intent of the first interview was to clarify the written responses of the students in the study group to the entry questionnaire and to obtain further baseline information about their understanding of mathematical modeling and their beliefs about computer- or calculator-generated results. To follow up on the students' responses to the entry questionnaire, I also asked them to suggest a model for another real-world phenomenon (the change in radius of a videotape reel) and to do the same calculations on the calculator as in Study 1 (p. 77). A copy of the protocol for the first interview can be found in Appendix G.

Second Interview Protocol

The purpose of the second interview was to further explore the study group members' understanding and views of mathematical modeling in light of technology. The interview also made use of the written assignments that the students had submitted, which had been graded by the instructor. The first assignment included modeling tasks for GSP, PC-LOGO, Excel, and graphing calculators, and the second assignment had one large modeling problem in which the students were supposed to use Excel primarily but also could employ other suitable software or a graphing calculator. The solutions the students had turned in and the grades they had been given by the instructors were discussed. In particular, the solutions to the two modeling problems were analyzed, and the students were asked about the possible errors they had made.

During the interview, the student and I considered two main questions. The first question was intended to explore how the student related mathematical models to reality when using software tools to generate the models. The second question, as in Study 1, dealt primarily with the student's conceptions and strategies when validating the mathematical model she or he had constructed and that supposedly modeled a real phenomenon. A copy of the protocol for the second interview can be found in Appendix H.

Procedure

All 30 students were given the entry questionnaire during the first meeting of the class. Three students did not respond to all the questions. I recruited the volunteers for the study group after administering the questionnaire.

The study group met in the smaller laboratory that had eight computers. I was present in the laboratory on the 3 days a week that class was held except when the students were gathered for a lecture or for a literature seminar or when I was interviewing one of the students.

Class Notes

I sat in on all lectures except one during the course, taking notes on important issues raised by the instructor or the students. I also noted the computer, calculator, overhead projector, and overhead computer projection displays that were used and viewed by the instructor and the students during the class sessions. I described or collected any papers or other written and visual materials to which the students or the instructor referred, including PowerPoint presentations and material on the course Web page. The descriptions of these materials were drawn from brief notes I took during the class, from my recollection of the class events, or from conversations captured on the videotapes in which the students or I made references to the materials.

Assignments

The students in MAL400d in spring 1998 completed two computer laboratory assignments (see Appendix I for the parts of the assignments used for this study), submitting a written report for each one. I made a photocopy of every student's reports.

Laboratory Notes

During the laboratory sessions, I took notes about tool use and monitor displays as the students did the assignments, observing both individual students and groups working together in the laboratory.

Final Examination

The students in MAL400d in spring 1998 were given a take-home final examination consisting of three problems. The examination was given out on Friday, 29 May 1998, and was to be handed in by noon Friday, 5 June 1998. I made a photocopy of each paper handed in. The two final examination problems used in the study are given in Appendix J.

Interviews

I twice interviewed each of the study group students individually. Both interviews were videotaped and were conducted in a small room near the computer laboratories. The first one took place during the second week and lasted approximately 45 minutes. The second interview took place during the eighth week and lasted approximately 90 minutes. During each interview, the student sat at a small table on which were a notebook computer running Windows 95 with Excel, The Geometer's Sketchpad, and CurveExpert software; a Casio *fx*-9700GE graphing calculator; a Texas Instruments TI-82 graphing calculator; paper and pencil; and a small microphone attached to the video camera. Many students also had their own graphing calculator to use.

The video camera was mounted approximately 2 meters away from the table. It was focused on the student's hands and the working area, including the computer monitor. I took detailed notes during the interview conversation to supplement the video recording. These notes included descriptions of the students' work with the computing tools, nonverbal components of the conversation, and references to previous written material. I also kept copies of each student's written work

Weekly Summary

Throughout the course I kept a journal with weekly summaries. I was functioning as a colleague to the two course instructors, as an observer during lectures and seminars, and as an interviewer, teacher, and researcher in the computer laboratory. Each weekly summary contained my intentions, observations, and reflections regarding what I had observed during the week and how I would orchestrate the following week in order to serve as both a support and a noncommittal observer. I included summaries of the class activities, a description of nonverbal interactions, descriptions of the students' computer work, notes from discussion around the coffee table, and any other events that that were not recorded as part of the normal class and laboratory interactions. I also recorded my subjective reactions to the progress of individual students and to the progress and flow of the course.

Results

The first section below deals with results drawn from the entry questionnaire and the initial interview of the study group students concerning their views of mathematics, technology, and their choice of a career. In the second section, I look at how the study group students used calculators and computers in solving problems. And in the third section, I consider the conceptions and misconceptions they exhibited.

Students' Views

Asked when and why they had decided to become a mathematics teacher, most students in the class wrote they had decided either in the gymnasium or closely thereafter. Two of the students had applied for other programs and then been rejected, so for them mathematics teaching was a second choice. Most students said something to the effect that mathematics was an important or interesting subject and that therefore they had decided to teach it. For example, one student wrote, “I have always liked mathematics, and I consider it be an important subject. Many times mathematics is given an undeservedly bad reputation because of bad teaching—I would like to be part of changing that.” Two students in the study group said in the interview that they had chosen to become mathematics teachers because the subject is so easy to teach: “You just need a book and a piece of chalk, that’s all. ”

When asked about the best way to prepare students for employment as mathematics teachers in the next century, roughly half the students thought that a strong background in mathematics was important, but almost as many thought courses in methodology together with extended practice were equally important. A few students mentioned the importance of methods for teaching children with learning disabilities. Only two mentioned that courses on using the computer as a learning tool would be valuable. After a couple of weeks, during interviews, all students in the study group expressed opinions about how important it was to know how to use and master calculators and computers when teaching mathematics, so the course appeared to have changed their views in that sense.

In reference to the evolution of technology as it affects the curriculum, assessment, and instruction, most students seemed to have a moderate view as to what could or should be done. At least half of the students expressed some concern over the danger of totally accepting the use of technology, wanting to make sure that useful things were done with it and that, for example, the curriculum determined the technology and not the other way around. Almost half thought it important to continue to have some assessment without technology through traditional paper-and-pencil tests. A few were very positive, wanting practice to keep up with advances in technology and expressing opinions such as the following:

> A lot of mathematics today (all applied mathematics?) is done with technology. If you are not learning to do mathematics with technology, then you are not getting a complete mathematics education.

In the last interview and when the course was almost over, all but one in the study group declared that assessment must change because computers and calculators can do so much more mathematics and because "technology skill" is growing every year.

As in Study 1, a majority of the students viewed their calculator as a reliable instrument, trusting its hidden mathematical skills more than their own mathematical knowledge and sometimes even more than their teacher. When asked to calculate $(-8)^{1/3}$, most of the students answered "negative two" if the calculator said so, but answered "not sure" if their calculator gave an error message (probably because the student had mistyped the expression).

A majority of the students considered mathematics to be a language or a way to structure, describe, and understand the world. A few considered mathematics to be problem solving. Consequently, many viewed *doing mathematics* as an activity in which we practice that language or use those structures to describe the world around us. Some students also considered *doing mathematics* as a practice of logical thinking. As a likely consequence of the course, all students but one in the study group considered it important to write and communicate mathematics by the time the course was nearly over.

When the students were asked what a function is, the responses they gave suggested that the course in real analysis might have influenced their thinking. More than half said that it is a one-to-one relation between two sets, and almost a quarter considered it to be a relation between two or more variables. A proof, a few of them said, is either to convince someone or what you do while you are proving something like a theorem, a circular statement in itself!

Mathematical modeling was harder to describe and define; more than a third of the students did not know how to define the term. The rest were divided among those who believed that mathematical models are what you use when you solve a mathematical problem: those who viewed them as descriptions of mathematical situations, such as graphs or formulas; and three students who remarked that they considered a mathematical model to be a teaching method, like the chocolate bar model for teaching fractions. When given the same question in the interviews, the study group students described a mathematical model in words similar to those of Monica: "a mathematical symbolic metaphor for the relations within and outside a real world phenomenon," probably remembered from one of the given lectures. Nevertheless, all students in the study group seemed to be confident in knowing many different way to describe both extended and pointed models (see p. 7).

Although all of the students in the course had used calculators and had used them in mathematics courses, only 14 had used a graphing calculator to draw a curve. All of the students had used computers previously, with 25 students having experience with computers running Windows and 5 having experience with Macintosh computers. Most of them had done word processing; quite a few had been out on the Web; and some had been using spreadsheets, playing games, and so on. Sixteen students said that they had used computers in a mathematics course at the university. It was surprising that more did not say so, since all students normally do a small assignment on function graphs in the MAL200 sequence. Many students may have forgotten that computer laboratory.

The mathematical modeling achievement test at the end of the questionnaire (Appendix F) revealed different conceptions of the applicable model for each of the problems. The fact that only one student in the class modeled the growth of the population of Sweden (as well as most west European countries) as a logistic curve was disappointing. It is important to note that the students did not have to name the model, and most of them just sketched the graph on the paper.

Even more disappointing was the fact that 13 students modeled as a straight line the graph over time of the height of a chocolate bar falling from 330 meters. In context, this means the chocolate bar had no acceleration, although the students should have known otherwise since they were studying to become mathematics and natural science teachers.

It is not so surprising that many students considered the growth of bacteria to be exponential; the fact that bacterial growth is defined by doubling is often used to introduce exponential functions in the Swedish gymnasium. Only five students noted that the amount of oxygen and energy was limited in this case.

Uses of Calculators and Computers in Modeling

Six of the study group students—Felicia, George, Hannah, Irene, Jacob, and Kristin—had graphing calculators; Linda and Monica borrowed TI-82s from the department of mathematics education. None had experience in using calculator facilities such as those for regression analysis or for evaluating the derivative of a function at a specific point. Instead, they had been using the calculator for basic computations and drawing standard graphs. All eight had used computers in different environments: at home, at work, at friends' places, and so forth. They were all more comfortable with machines running Windows than with other platforms, although Linda had worked on a Macintosh a couple of years before. The software they had been using was more or less as expected: for word processing, playing music, browsing the Internet, or e-mail communication. George and Hannah had used spreadsheets but only a few times. Linda had worked as a graphic designer for a couple of years. None seemed uncertain or expressed any feeling of being a beginner when we talked about their knowledge of and experience with computers.

All the eight students in the study group believed that technology would be part of their future lives as teachers and in their personal lives. George and Jacob were convinced that a mathematics teacher of today must know a lot about technology to be hired and that the students might know more anyway. Monica thought that it was important to know the potential and limitations of technology so that you could choose as a teacher to use it or not. All the rest thought that it was important to know how to use technology, and how to do it in a methodologically correct way. When asked if they believed that assessment should or would change depending on the technology, many said they had not thought that much about assessment, although they had been informed about the take-home final examination the first day of class. I asked them to think of the assessment as a major part of the eventual change occurring in the teaching of mathematics in a technology-enriched environment.

None of the eight study group students succeeded in solving either of the two calculator questions in the first interview (see Appendix G), and no one could explain why the calculator gave a wrong result. Linda and I had the following discussion about the first question, which involved calculating $\dfrac{123456 \cdot 10^4 - 1}{10^9 - 1}$:

I: So you think that the answer to Problem 1 is 1.23456?

L: Yes, well, that's what my calculator says.

I: Do you think it is right?

L: Well, I guess so. Why shouldn't it be?

I: Well, look at the expression again. Think of 123456 as *X* and write it again, please.

L: You mean like this? She writes $\dfrac{X \cdot 10^4 - 1}{10^9 - 1}$

I: Yes. If you look at it, and reflect on the answer of 1.23456, where do you think the –1 went?

L: I don't know, I guess … [She hesitates.] … I don't know. Maybe they cancel out?

I: Linda, come on. You've been studying mathematics for 12 years in school and for almost a year at the university, and you think they "cancel out"? What kind of rule are you referring to? Why do you think the –1 "cancels out"?

L: Well, how could it be 1.23456 if it's not canceled out? I used parentheses [when entering the numbers into the calculator].

Linda did not like the idea of a calculator that miscalculates, and she seemed to prefer to invent new algebraic rules to support her calculations. Perhaps this misconception of how you divide by common factors in a rational expression was related to the context of the situation. Linda did have good grades from the gymnasium and the previous mathematics courses. Monica, on the other hand, became very interested in the error the machine produced.

M: That's neat. I didn't know it was that easy to show how dangerous it could be to rely on machine calculations. Do you have more problems like that?

I: Well, if you think about the two problems, you can very easily construct as many such problems as you want. You see?

M: Do you mean that any number in that algebraic expression would force the calculator to miscalculate? Does it have to be 10^4 and 10^9 in the denominator?

I: Well, that's an interesting question. Why don't you check to see what you get if you put in 10^3 and 10^8 instead?

M: Well, let's see. I get 1.234560002. So we couldn't see the 2 before, I guess. Neat. I like this. I can use this in my student teaching. Is this true for all calculators?

I: In fact, for all machines, but the precision depends on the machine.

Linda and Monica represented the two extremes in the study group: those who thought it was embarrassing to answer problems they should have understood but did not and those who had enough self-confidence to see the problem as a learning opportunity.

The second interview started with a general discussion of how the students felt about the problems they were working on and about all the technology they were using to model with. Hannah and Kristine were somewhat reluctant to accept that many of the assignments could not be solved with paper and pencil only. They argued about the fact that it took them such a long time to learn the software. Kristine said:

K: There should be an Excel course in the beginning, instead of us sitting here and learning Excel at the same time as we do the mathematics. There are just so many different commands you need to know and learn, and I don't have the time.

I: So how do you see your own responsibility for your learning?

K: Well, I don't think it is fair that I should be forced to discuss with my classmates and even friends at home and ask them for advice. It shouldn't be like that.

Jacob was upset with all the writing that was required for the assignments: “I don’t want to be assessed on my writing skills in mathematics, only on how I do the mathematics.”

My first laboratory observation occurred the first week of the course during the first scheduled laboratory. The eight students had a computer laboratory with a computer for each person. Most, however, preferred to work together: George and Jacob, Hannah and Irene, and Felicia and Monica. Only Kristine and Linda preferred to sit alone in front of the computer. Otherwise, the students seemed to know each other quite well and did not hesitate to make contact with one another. No one seemed to dominate in the group, although both Monica and Linda asked good questions and seemed well respected by the others. Although George and Hannah were the only ones who had previously worked with spreadsheets, the rest of the group only needed a couple of hours to eliminate that difference in practical skill. During that first day we went over some practical issues like how to save files in Excel so that the students could work with them on their home computer even though they ran Excel 97 in school and Excel 95 at home. We also looked at basic commands in Excel and even how to make graphs in Excel and paste them into Word documents. All except Hannah seemed confident and relaxed with the computer.

The second observation was the following week, and the students had started to work on their first assignment (Appendix I). We discussed Assignment 1B with the recursive function $R(k)$ and Assignment 1C about the future number of employees at the computer firm. I asked them about limit: “Since you are all coming directly from a course in real analysis, how do you view the concept of limit?”

I knew that they had seen among others the following formal definition:

Definition. Limit when $x \to \infty$.
Assume that f is a function defined on an interval $x \geq a$.
We say that $f(x) \to A$ when $x \to \infty$ if for every $\varepsilon > 0$ there exists a ω such that

$$|f(x) - A| < \varepsilon \text{ for all } x > \omega.$$

We also write $\lim_{x \to \infty} f(x) = A$ and say that $f(x)$ approaches or converges to A when $x \to \infty$.

(Hellström, Morander, & Tengstrand, 1996, p. 132)

Despite the recent course in real analysis and all the calculus they had taken in the gymnasium, the students expressed concern about not really knowing what a limit was. I asked the general question: Did they see limit as a value or a computation? We then had a discussion about what a limit really is. When they started to work on Assignment 1B with Excel, all of them seemed to adopt a view of limit as a big value waiting somewhere, somewhere beyond the 65 536 cells in Excel. I asked them if they knew how many cells Excel 97 had.

"Infinitely many?" Jacob asked, followed by laughter in the group.

"No, not really. Please give me a serious guess."

"Well, maybe 100 000?"

I told the students that the number of cells is based on the binary system natural to computers and indicates the amount of memory the developers allocate to each sheet in Excel. The number 2^{16}, or 65 536, is the possible number of cells in a column. But how does that connect to the limit of $R(k)$ when $k \to \infty$?

Even though the students seemed clear about the difference between the amount of cells in Excel and the limit of $R(k)$, they did not express any insight into, for instance, a way to bound $R(k)$ between two known functions or to give an algebraic solution to the problem. An analytical approach to this problem with the help of the real analysis they had just studied seemed to be out of their reach at this point. They preferred to search for a function or formula in Excel that would solve the problem.

The third time I met them, they were all working on Assignment 1, which was due in several days and on some practice problems for different software programs. The general idea among the students was that there must exist ways to make Excel calculate limits and solve equations, and they wanted me to tell them how. Monica was working on a problem from the problem collection.

M: Look, here is a problem I'd like to ask you about. It says,
Find the positive solution to 4 decimals of the equation

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots+\frac{x^{100}}{100}=5000$$

I: Yes?

M: Well, I haven't found how to write an equation in Excel. Can you show me? How do I write *X*?

I: You mean that you would like to find an equation solver in Excel? One that solves equations algebraically?

M: Yes, that's exactly what I'm looking for.

I: What do you know about the solving of equations of degree *n*? How large may *n* be?

M: I'm not sure I understand your question. Please just show me where to find the equation solver.

Many of the students showed a surprisingly weak knowledge of basic mathematical ideas or illustrated misconceptions of what actually can be done algebraically in mathematics.. I thought that at least some of them would know that already in 1824 Nils Abel proved the impossibility of solving algebraically a general equation of the fifth degree. When I told Monica how to solve the equation by generating numbers for a certain value of *X* and then either trying to find it yourself or using the problem solver in Excel, she became very disappointed. Jacob and George offered to demonstrate:

J: You see, you first create a column with the natural numbers, 1 to 100, which are both in the denominator and in the exponent. Let us take the A column.

M: Okay, and then what?

J: Then you make a rough estimate of *X*.

M: How would I do that?

G: Well, it's not 1, is it? And it is not 2, since that would become too large.

M: Well, I guess it is somewhere in between then.

G: Therefore we start with the value 1 for *X* and then increase slowly. Let us put the *X* value in cell D1.

M: I'm not sure I understand this.

G: Now we create the first term by entering the following in B1: =(D1^A1)/A1. You see how we combine the exponent and the denominator with the X value? Then copy it down and add the terms together by using SUM.

M: This is not solving an equation. It's like cheating!

It seems as if Monica was close to expressing the feeling of algebraic guilt discussed in chapter 1 (p. 22). She expressed concern that solving equations in the way demonstrated by Jacob and George was far too simple and would give almost anyone the power to solve complicated equations without any idea of what was going on.

Conceptions and Misconceptions About Modeling

Since most of the students in the class, including the study group, had difficulty explaining in the questionnaire what a mathematical model was, I started each first interview by asking what the student knew about mathematical modeling. Only Felicia had expressed in the questionnaire a mathematical view that referred to the unit circle. She had considered the unit circle to be a mathematical model of sine and cosine (not expressing sine and cosine as models of mathematical relations in triangles). Hannah based her response on teaching methods and referred to the chocolate bar model for teaching fractions. The rest of the group wrote such comments as, “I don’t know” or “I think I have heard about it, but right now I can’t remember.”

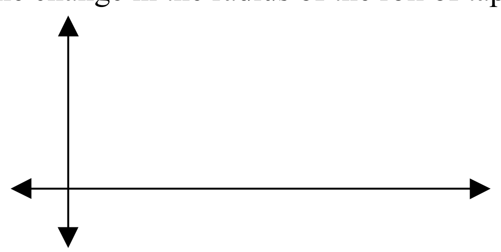
Most of them could see, after a short conversation during the interview, many models that they used in their daily lives. Most had never thought of what Skosmose (1994) calls *extended models* and how they seem to be everywhere in our lives. They all considered the questionnaire difficult, and none had managed to describe correctly all three models in the achievement test. Jacob had done the best since he managed to describe both the growth of the Swedish population and the bacteria growth correctly. He presented, however, a strange graph of how the chocolate bar would fall from the Eiffel Tower: an S-shaped curve.

In the interview, Felicia said that she was embarrassed that she could not describe the “free fall” of the chocolate bar either on the test or then; she should have remembered it but could not. On the achievement test, Linda had sketched half of an inverted parabola but had written a linear expression. In the interview, she said, “I think it’s a parabola, but I can’t explain why.”

All together, the questionnaire responses from the study group were as confused as for the whole class. The general opinion among the eight students in the study group was that the modeling problems were not too hard. Instead, they argued that they were far too inexperienced in describing real-world phenomena. As far as they could recall, none had ever been asked to describe a real-world phenomenon with a mathematical model. All but Hannah expressed some embarrassment that they could not graph or otherwise describe mathematical models of such phenomena; after all, they were studying to become mathematics and natural science teachers.

I gave them the following modeling question in the first interview (see Appendix G):

> Consider an ordinary videotape placed in a videotape recorder. When the tape is played, it is transferred from one reel to the other with *constant* speed. Illustrate with a graph the change in the radius of the roll of tape on the first reel.



Only Monica and Jacob claimed that the radius of the roll on the first reel must decrease faster and faster, assuming constant tape speed. Felicia, George, Irene, Kristine, Linda, and Monica agreed gradually that the graph could not be linear but had difficulty organizing the graph, labeling the axes, and so on. Hannah was the only one who could neither understand nor even agree that the radius must decrease in a nonlinear way. Neither arguments nor metaphors helped—she had decided to stay with her opinion, no matter what. So I asked her to sketch a mathematical model of how the volume of air in the lungs depends on time.

H: What do you mean? I don't know the volume of the air in my lungs.

I: I understand that. But if you start with the volume when the lung is full of air, how would that change when you are breathing?

H: So you mean that I just graph. . . . [She draws what looks like a sine curve.] Is this what you want?

I: Is that how you think the volume of air in your lungs changes over time?

H: Yes.

I: Very well. Why don't you breathe, and try to picture the movement of your lungs, the amount of air in them, and so forth? Compare that to the graph you have drawn, and see if they match.

H: [She sits quietly and breathes, looking at the sine curve she has sketched.] Yes, I think this is accurate.

I: So what about the smooth and gradual changes on the top and the bottom of the curve. Can you breathe like that and show me?

H: [Hannah breathes deeply and constantly, becoming red in the face, apparently because of her embarrassment at finding how difficult it is to breathe in such a manner.] I guess it is hard to breathe like that, but I can't think of any other curve. [She seems uncomfortable and impatient, so I leave the problem there.]

In the second interview, after a general discussion of about 10 minutes, I gave each student the modeling problem concerning gold medallists in the woman's 200-meter event from selected summer Olympics (see Appendix H). I asked the student to think about it for a few minutes and then explain to me how he or she would proceed with the data. All of them, including Hannah and Kristine, went directly to Excel and started to input the data points. I asked each student to write down what he or she was doing so that we could then discuss it together. The following dialogue occurred between George and me:

G: I'm entering the data so that I can get a graph of the results (Figure 8). … Look here, it seems that the winning time is steadily going down.

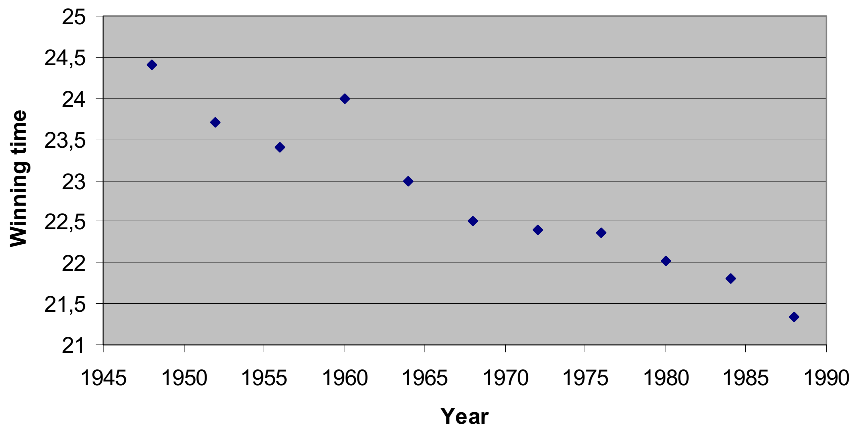


Figure 8. Winning times for the Olympic 200-meter race for women.

I: You would say that it is going down, okay. But how? Please express it with a mathematical model, a function.

G: Well, if I draw a line here, I would say that it will be down under 21 seconds rather soon. Well, it already is, in fact. Is that true? I'm not so good at remembering results.

I: What if you look ahead? Like the Olympic games in 2000. What will the winning time be then?

G: Well, under 20 seconds it seems. Is that possible?

I: No, I don't think so. So what do you think is wrong?

G: You mean with the model or with the data points?

I: Well, I mean with your interpretation of the scatter plot you have. Why do you assume that it is linear? Even if the data look linear, do you think that can be a trend?

G: I guess not. I guess it can't go on forever. It must just stop at some point.

I: You mean just stop? Just like that? First a linear trend and then a stop?

G: Well, I guess not. Maybe more approaching it like this [George sketches something that looks like a negative exponential function].

I: What would you do to improve your model?

G: Well, I guess I would try to get more results, from the events that are missing.

At this point, I gave George a copy of Worksheet 2 (Appendix H). He immediately started to enter the data points into Excel and arrived at the scatter plot in Figure 9.

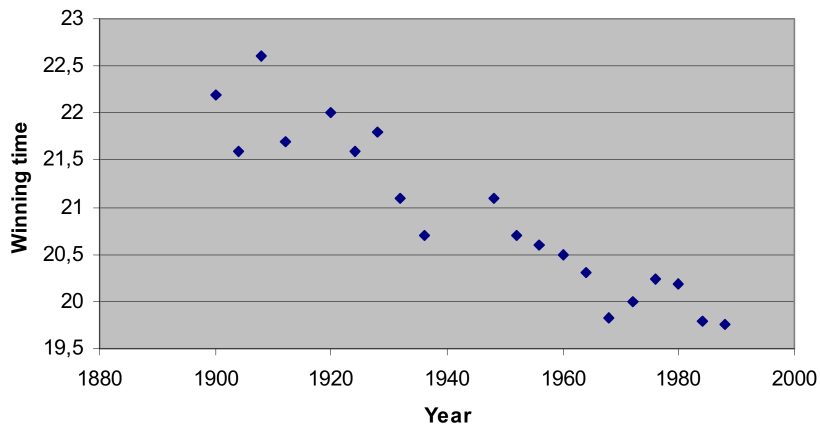


Figure 9. Winning times for the Olympic 200-meter race for men.

I: So what do you think about the trend now?

G: Obviously it is going down, but it's hard to tell when and how. I could put a line in there between the data points, but it would not help me. This is so complicated. I'm not at all sure what to do with all this.

I then gave George the question in Worksheet 3. He did not seem to have any problem with combining the data from Worksheet 1 and Worksheet 2 and arrived at the figure in Figure 10.

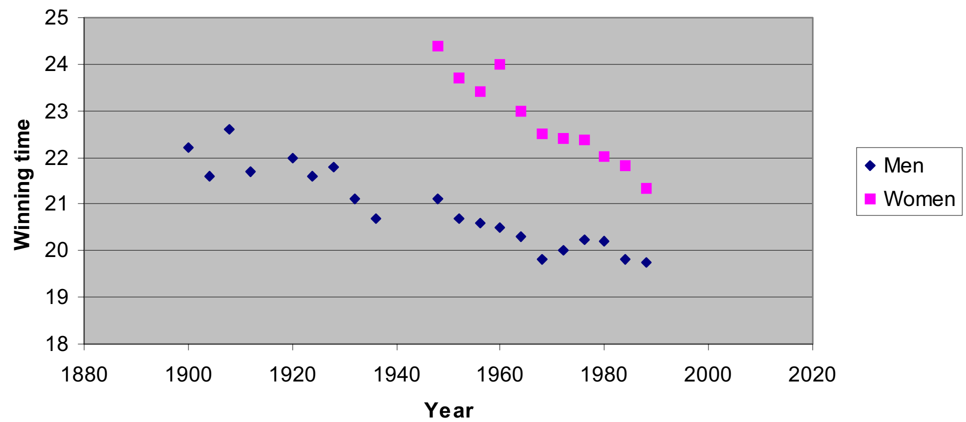


Figure 10. Winning times for the Olympic 200-meter races for women and men.

G: Let's see what to do. I think I'll do a linear regression analysis in Excel to start with.

I: I thought you said you did not believe it was linear.

G: No, but I'd like to see where they cross.

I: You would, would you? Well, let me see how you do this.

George was apparently almost forced by the scatter plot to believe that the trends would cross, thereby assuming linearity without realizing it. He had also locked himself into a model that leads to the conclusion that women will eventually outrun men. By using the trendline tool in Excel, George arrived at the result in Excel shown in Figure 11.

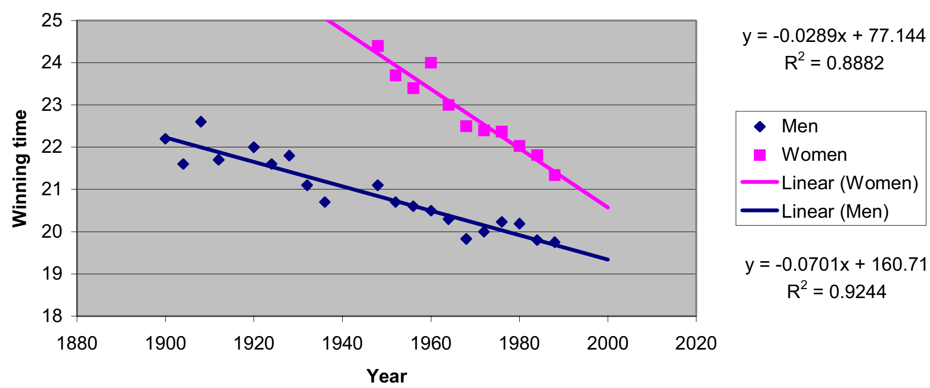


Figure 11. Winning times for the Olympic 200-meter races for women and men with linear regression lines.

I: So how will you proceed now?

G: I just set up the equation $-0.0289x + 77.144 = -0.0701x + 160.71$ and solve it. It gives $x = 2028.30$.

I: Very good. What does it mean?

G: I guess it means that, that women will outrun men in some 30 years or so.

I: Do you believe that?

G: No, not really ... but it does seem as if the women are getting closer. Maybe they will run almost as fast.

I: What do you think the lack of results from early in the century for women might mean for your modeling?

G: I guess I would get another value for x if I had some more results to enter.

I: How would you validate your model? Is it reliable?

G: I guess I could look up data from the Olympic games of 1992 and 1996 and check to see if my model predicts the actual winning times then.

By this point in our discussion, George had seemingly accepted as true the linear model that he did not believe in at the beginning. The obvious result over the long run—the prediction of an eventual record time of 0 seconds in the 200-meter race for both women and men—did not occur to George at all.

Finally, I asked George if he had any alternative way to approach this problem. What about looking at the differences between the slowest and fastest winning times for a 10-year period? Or constructing a model with a horizontal asymptote?

George did not seem to notice that the equation he solved resulted in a winning time of about 18.5 seconds by 2028. To achieve that would mean that a 200-meter runner would need to maintain a pace of 9.25 seconds per 100 meters, well under the present world record for that distance. Was that likely? George admitted that it was not. What about the predicted time for the year 1600? What did the model give? A result of over 30 seconds for women running 200 meters then? Was it likely that they would run that slowly?

The interview with George and the other students in the study group revealed that the computing tool seemed to take over and that the students lacked the necessary critical thinking they so badly needed in order to perform the modeling process with caution. Even though they were all inexperienced in terms of modeling, they seemed to make many mathematical mistakes I did not anticipate. The obvious misconception exemplified by George was that a trendline or a regression line actually means something in real life, although George was the one who put it there. Another misconception that became visible in the problem about population growth in Middletown (Assignment 2B, Appendix I), was that student preferred a simple model that could then be constructed in

Excel rather than considering the way the model fit the real phenomena. The fact that the linear model was so easy to construct and find an equation for persuaded Felicia, George, Hannah, Irene, Kristine, and Linda to argue that it was the best model to describe the population growth. When I asked them whether it was not more likely that the growth should have an upper bound, they claimed that "it could be one of those towns that just keeps growing." Jacob and Monica both claimed that if you "invent" an upper bound in the third growth model, then you are acting as if you know the result and the only task the model has is to give you the rate of growth.

> *M:* For all we know, the population growth could behave like a sine wave. Maybe they'll find oil in another town 1000 kilometers away, and everybody will move there. Then, in another 100 years or so, they'll find gold in Middletown, and a lot of people will move back there again. We can't know that!

In fact, Monica was the only student who took a broader perspective on the modeling activities. They were set up to help the student learn to use different models, but most of the students did not reflect on an activity the way Monica did.

The hourglass problem in the final examination (Appendix J) showed another inconsistency in the students' mathematical ideas. Many of them could not begin by picturing how the amount of fluid in an hourglass would run out of one cone into the other. They did not see the relation between the height and volume of the liquid. Quite a few took it for granted that a scale on the hourglass would have equal intervals along the glass, a misconception related to a strong belief in a linear relation between height and volume in a cone. As a consequence, all the other models used in the problem became distorted.

Problem 3 (from Edwards & Hamson, 1996, pp. 153-154)

a. In an experiment to investigate how the weight of a tomato crop can be increased by applying fertilizer, the yield $\boldsymbol{y}$ (kg/m^2) for various amounts $\boldsymbol{x}$ (kg/m^2) of fertilizer was found to be as follows:

Fertilizer, *x* (kg/m^2)	0	0.2	0.5	0.8	1.0	1.5
Profit, *y* (kg/m^2)	1.00	1.20	1.40	1.50	1.55	1.65

Fit a model of the form $\boldsymbol{y = L + (y_0 - L)e^{-kx}}$ where $\boldsymbol{L}$ is the value that $\boldsymbol{y}$ approaches as $\boldsymbol{x}$ approaches infinity.

b. To investigate the effect of temperature on the yield, a number of plots were maintained at different temperatures and gave the following results:

Temperature, *T* (°C)	10	15	20	25	30
Profit, *y* (kg/m^2)	1.0	2.5	4.0	4.5	4.8

Fit a similar model to that of Part **(a)**, relating $\boldsymbol{y}$ and $\boldsymbol{T}$**.**

c. The total cost for heating, $\boldsymbol{C}$ (£/m^2) at constant temperature $\boldsymbol{T}$ was found to be as follows:

Temperature, *T* (°C)	10	15	20	25	30
Cost, *C* (£/m^2)	10	25	45	70	100

Fit the simplest model for $\boldsymbol{C}$ as a function of $\boldsymbol{T}$.

d. If the cost of fertilizer is £10 per kg and each kg of tomatoes is worth £5, write down a model for the net value of the crop obtained from one m^2 of compost in terms of $\boldsymbol{x}$ and $\boldsymbol{T}$.

Figure 12: The tomato problem from the final examination in Study 2.

The tomato problem in the final examination (Figure 12) caused an unexpected commotion, since the students discovered that under the present exchange rate between the Swedish crown and the British pound, applying fertilizer was just not profitable. Some of the students also complained about the expected model for the profit based on the amount of fertilizer. Jacob wrote the following:

> Already here we see a weakness with this kind of formula. An unlimited amount of fertilizer would, as every amateur farmer knows, lead to nothing other than zero in terms of yield, since the fertilizer would burn the crop to death.

Despite their view that it was unrealistic, most of the students performed well on the tomato problem. Two exceptions in the study group were Hannah and Linda. They both ran into problems from the beginning when trying to find *L* and *k* plus y_0 in the first part of the problem. During one of the laboratory sessions when they were working on the tomato problem, I observed the following discussion:

H: We must determine *L* and *k*, but how should we do that?

L: I guess we can enter the data set into Excel or CurveExpert.

They decided to work in both Excel and CurveExpert as a team. For several minutes while I waited, Hannah entered the data points into Excel, and Linda did the same with CurveExpert.

H: Now what? What do we do to find all those variables? [She did not notice that most of the symbols were constants.]

L: We'd better use CurveExpert for this. It has many more models in it, compared to Excel.

I left them for half an hour, and when I returned they were still struggling with finding the values of *L*, *k* and y_0.

H: Thomas, could you please tell us how to do this in CurveExpert? We've created the formula, but now we can't finish since CurveExpert is asking for good guesses for *L*, *k* and y_0 before it starts the regression.

I: Maybe you already know the values of some of those constants, don't you? What do you think y_0 stands for?

L: Maybe a starting value.

I: For what?

H: Maybe for the fertilizer. No, wait. It says the yield here in the problem. I guess it is the yield. Now what could y_0 be?

I: Why don't you take a look at the table and test the first value of *y*? It could not be smaller, could it?

Because of their difficulties in understanding the tomato problem, Hannah and Linda had severe problems with it. It was much too open-ended and had too many possibilities. It took them a long time to see that L and y_0 could be determined directly from the table. That fact, in turn, made them trust the computer software too much at every step of the modeling process and thereby to shift their source of authority entirely to CurveExpert and Excel.

> *H:* This was a terrible, terrible task. I never understood what to do. And Linda and I were at least two hours behind everybody else in the study group from the start. We never understood what CurveExpert and Excel did with those models. And how could you, how could you give us a problem that was so terribly loose and open? It doesn't even end with a finite answer. If I knew where to complain, I would do that. I think this was a very bad problem.

Most of the students in the study group, as well as most of the students in the class as a whole, expressed the opinion that it was important to have a solid knowledge of both technology and mathematics when becoming a mathematics teacher. Nevertheless, the students constantly complained about the rigor they considered the instructors were using when grading the papers. For many students, it seemed that the emphasis on validation of models and on a critical attitude toward computer-generated results was just so much smoke or fuzz. They were more eager to learn the technical aspects of technology use for their future careers than to develop a critical attitude.

The view that a mental image is needed of the phenomenon one is supposed to model was strengthened by the results of this study. Hannah, who had severe problems picturing for herself both her breathing and the behavior of the videotape recorder, also had major difficulties with the modeling process in the assignments and the final examination problems. The construction of a reliable and valid mathematical model does not come without a reasonably good mental image of what one is expected to represent mathematically. The study reaffirmed the findings of the first study regarding misconceptions in modeling, detailing what some of those might be. The students, for instance, had severe problems in deciding when a model can be used to predict something that happens outside the domain of the data points, and when it can not. They also had major difficulties in allowing their common sense to guide their way through computer results in the modeling process. Many students also showed uncertainty in linking their expertise in mathematics and natural science together in the modeling process. It also raised some questions about the sources of authority students were turning to for their modeling work that I addressed in a third study.

CHAPTER 6

AUTHORITY AND RESPONSIBILITY (STUDY 3)

The findings from Study 1 concerning the students' tendency to select a model in favor of reality and from Study 2, where their conceptions and misconceptions related to mathematical modeling were revealed, led to a third study. I wanted to explore further how students were responding to open-ended tasks in assessment and to efforts to have them accept responsibility for learning. I did not understand why even successful students sometimes abandoned their faith in their mathematical skill and began to trust results from the computer instead. I wanted to know how technology fit in among various sources of authority as they solved mathematical modeling problems. I decided to undertake a third study to address the following questions:

Research Questions

- How do students in a course on mathematical modeling view open-ended tasks?
- How well do students in a course on mathematical modeling accept responsibility for their own learning?
- What sources of authority can be observed among students who use calculators and computers to solve mathematical modeling problems?

Method

Participants

The Class

The MAL400d course in fall 1998 enrolled 70 students. Thirty-nine were women, and 31 were men. The students ranged in age from 20 to 43, with a median age of 27 and a mode of 20. The distribution of their ages, shown in Figure 12, was typical of students in the program. Some students had gone directly to the university from the gymnasium; others had worked a few years, been in other university programs, or served in the military. Two students had received their gymnasium education in the former Yugoslavia, one student was from Iran, and one was from Iraq. The remaining students had all received their primary and secondary education in Sweden.

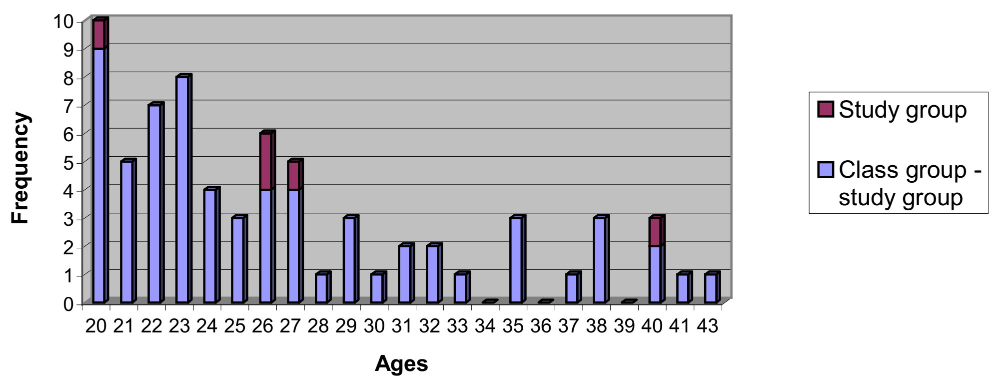


Figure 13. Age distribution for the fall 1998 class (with study group members shaded).

As is typical for the course and was true for the other two studies, the students had all taken essentially the same mathematics courses in the mathematics department at the university, but sometimes in a different order. Although some of the participating students had not attempted the examinations for real analysis or linear algebra, most had passed both examinations on the first attempt. It is notable that as many as 20 students had not yet passed the linear algebra examination after the first three attempts, which may indicate that the class as a whole was weaker in mathematics than previous classes. The students' performance on the examinations is shown in the second column of Table 5.

Table 5
Number of Students in Each Group Earning a Pass (P) or Well Pass (WP) on the Analysis and Linear Algebra Examinations After Different Numbers of Attempts

Course	**Group**	
	Class	**Laboratory**
Real Analysis		
No attempt	3	
First attempt (WP or P)	43	5
Second attempt (WP or P)	11	0
Third attempt (P)	1	0
Linear Algebra		
No attempt	5	0
First attempt (WP or P)	16	3
Second attempt (WP or P)	12	2
Third attempt (P)	12	0

The Study Group

Selection. In very much the same way as in Study 2, I told the students that I wanted a small group to work more closely with. I explained that I was doing a research study of how prospective teachers learn and do mathematics in the presence of technology and that we sometimes would be working in the smaller of the two computer laboratories. The students were informed that I would conduct some interviews and observations of their work in the computer laboratory but also in a special interview room. I told them that the volunteers would benefit from my presence, at least that was my experience from the two previous studies. I also made it clear that the volunteers would not be required to spend extra time or do any more work beyond what was otherwise expected in the course. Eight students volunteered.

I met the eight students the next morning and gave them further information about the setup of the study and the videotaped interview that would cover work with and without a computer. I said that I would be observing them and taking notes in class. I also informed them that I would not be grading the assignments or the take-home final examination for the students who signed up for the study group. Three of the students decided that they would prefer to work more on their own and thus did not want to sign up for the study group. As a consequence, the study group in Study 3 consisted of five students.

Description. Of the five students, four were female and one male, which differed from the nearly equal distribution in the whole class. Their pseudonyms for this report are Nina, Olga, Patricia, Robert, and Sarah. They ranged in age from 20 to 40, with a median age of 28. As Figure 12 shows, the distribution of their ages roughly approximated those of the class as a whole. Data on how the focus group performed on the linear algebra and real analysis examinations can be found in the last column of Table 5. The table shows that all of the students in the study group had passed their real analysis and linear algebra examinations on the first or second attempt. Since the second examination had to have been taken by the second week of the modeling course, all these students had completed real analysis and linear algebra early in the course. All five indicated that they had taken a “full course load” in the gymnasium, which meant that in the gymnasium they had studied differential equations or complex numbers or both. Similar to the first two studies, this information suggests that the students in the focus group had a somewhat stronger mathematical background than the class as a whole. When the lecturer on the first day asked how many that who had studied differential equations or complex numbers, just about 50% answered with a yes.

All five of the students had graphing calculators. Two of them had a Texas Instruments TI-82 calculator, one had a Texas Instruments TI-83 calculator, and two had a CASIO *fx*-9800G calculator. They also had access to computers outside the computer laboratory, either at home, at their parents’ house, or at a friend’s house. All five were running Windows on these machines. They had all MS-Office with Word and Excel and had successfully downloaded and installed The Geometer’s Sketchpad Demo and CurveExpert.

Instruments

The data came from multiple sources: an entry questionnaire, observations of lecture sessions and laboratories, interviews, students’ written assignments, and the final examination. I developed several instruments for data collection that are discussed below.

Entry Questionnaire

The entry questionnaire for this study was a short five-page questionnaire, basically a shorter version of the questionnaire in Study 2. Its purpose was to provide information about the students’ understanding of central mathematical ideas like function, proof, and mathematical modeling and about their access to and knowledge of calculators and computers. The mathematical modeling achievement test was extended to five mathematical modeling problems for them to solve that were intended to measure their ability to sketch graphical models or otherwise illustrate models for real world phenomena. A copy of the entry questionnaire is in Appendix K.

First Interview Protocol

The first interview was intended primarily to clarify the written responses of the students in the study group to the entry questionnaire and to obtain further baseline information about their understanding of mathematical modeling and their beliefs about computer- or calculator-generated results. I had a rather long discussion with each student about the modeling problems in the entry questionnaire. I also asked questions about the student's previous knowledge of and experience with computers and calculators, and I gave the student four calculator problems to explore and comment on. I wanted to see how their mathematical knowledge would stand up against results from a graphing calculator or computer software. A copy of the protocol schedule for the first interview can be found in Appendix L.

Second Interview Protocol

The purpose of the second interview, as before, was to further explore the study group members' understanding and views of mathematical modeling in light of technology. The interview also made use of the written assignments on mathematical modeling that the students had submitted, which had been graded by the instructor. As in Study 2, the first assignment included modeling tasks for GSP, PC-LOGO, Excel, and graphing calculators, and the second assignment had one large modeling problem in which the students were supposed to use Excel primarily but also could employ other suitable software or a graphing calculator. The solutions the students had turned in and the grades they had been given by the instructors were discussed. In particular, the solutions to the two modeling problems were analyzed, and the student was asked about the possible errors she or he had made.

During the interview, the student and I considered two main questions. As in Study 2, the first question was intended to explore how the student related mathematical models to reality when using software tools to generate the models. The second question dealt primarily with the student's beliefs of where the authority lies in a modeling process. A copy of the protocol schedule for the second interview can be found in Appendix M.

Third Interview Protocol

The purpose of the third interview was to try to identify whether there were any changes in the students' view of mathematical modeling after 10 weeks. The interview also made use of the take-home final examination, which had been graded by the instructors. The solutions the students had turned in and the grades they had been given by the instructors were discussed. In particular, the two solutions to the two modeling problems were analyzed, and the student was asked about the possible errors she or he had made.

During the third interview, the student and I considered two main questions. The first question was intended to explore how the student thought that she or he had changed her or his view of mathematical modeling during the course. The second question dealt primarily with the student's beliefs as to where the authority lies in a modeling process like the ones she or he was involved in. A copy of the protocol schedule for the third interview can be found in Appendix N.

Procedure

All 70 students were given the entry questionnaire during the first meeting of the class. Two students did not respond to all the questions. As in Study 2, I recruited the volunteers for the study group after administering the questionnaire. The study group stayed in the smaller computer laboratory for most of the computer laboratory time. I was present in the laboratory on the days when class was held except when the students were gathered for a literature seminar or when I was interviewing students.

Class Notes

I sat in on most lectures during the course, taking notes on important issues raised by the instructor or the students. As in Study 2, I also noted the computer, calculator, overhead projector, and overhead computer projection displays that were used and viewed by the instructor and the students during the class sessions. I described or collected any papers or other written and visual materials to which the students or the instructor referred, including PowerPoint presentations and material on the course's Web page.

Assignments

The students in MAL400d in fall 1998 completed two assignments (see Appendix O) during the first half of the course, submitting a written report for each one. Each report included several pages of answers to the problems in the assignment. The reports also contained the students' thoughts on how they had used computing tools, as well as their observations, notes, and conclusions about the problems they had solved and the explorations they had conducted. I made a photocopy of every student's reports.

Laboratory Notes

During the laboratory sessions, I took notes about tool use and monitor displays as the students did the assignments, observing both individual students and groups working together in the laboratory. The notes included enough detail about the students' work to enable me to reconstruct what the students did and saw while they used each tool.

Final Examination

The students in MAL400d in fall 1998 were given a take-home final examination consisting of three problems. The examination was given out on Monday, 11 January 1999 and was to be handed in by 4 p.m. Monday, 18 January 1999. After all of the students had handed in the papers, I made a photocopy of each one. Two of the final examination problems are given in Appendix P.

Interviews

I interviewed each of the study group students individually three times. All three interviews were videotaped and were conducted in a small room near the computer laboratories. The first interview took place during the second week and lasted approximately one hour. The second interview took place during the seventh week and lasted approximately 90 minutes. The third interview was conducted the first week after the course ended and was again about an hour. During the first interview, the student sat in the computer laboratory with a nearby computer running Windows 95 with Excel, The Geometer's Sketchpad, and CurveExpert software; a Casio CFX-9800G graphing calculator; a Texas Instruments TI-83 graphing calculator; paper and pencil; and a small microphone attached to the video camera. Many students also had their own graphing calculator to use. During the second and third interviews, the student sat in a small room with a nearby whiteboard. The interviews were recorded using the same setup and procedure as in Study 2.

Results

Views of Open-Ended Tasks

When the teaching of mathematics is changed from traditional routine tasks to open-ended tasks and when students are encouraged to seek and develop knowledge in a group or on their own, then all participants in a course—students and teacher—are faced with new challenges. To identify and discuss these challenges early in the modeling course and to let the students know what was expected of them, the instructors informed them on several occasions at the beginning of the course about the nature of the problems and how they would be assessed.

What sort of response might be expected from a student who is given a problem like the Olympic swimming problem in Assignment 2B (Figure 14)? With an open-ended task, the student faces several difficult questions at once. What kind of mathematical model is suitable here? How should I construct or generate this model? With access to modern technology in the form of graphing calculators and efficient computer software, the student can, already at the beginning of the modeling process, produce a visual image. The problem places serious demands on the student's ability to make assumptions, make and interpret decisions, and describe a situation that will lead to a variety of outcomes and generalizations. The problem is intended to lead the student initially to experiment with the data given and thereby produce results that can be tested and interpreted in relation to the real-world situation. The possibility of continuously visualizing relations and a sequence of events is especially important.

Assignment 2B

Table 1 illustrates the winning time for the gold medal in the women's 100-meter freestyle in the Olympic games during the last century.

Table 1

Year	Time (seconds)
1912	82.2
1920	73.6
1924	72.4
1928	71.0
1932	66.8
1936	65.9
1948	66.3
1952	66.8
1956	62.0
1960	61.2
1964	59.5
1968	60.0
1972	58.59
1976	55.65
1980	54.79
1984	55.92
1988	54.93
1992	54.64
1996	54.50

Use the given data to create a mathematical model that can predict future results in general and also answer the question: What will the winning time be in the women's 100-meter freestyle in the 2000 Olympics?

Figure 14. Winning time for the gold medal in the women's 100-meter freestyle in the Olympic games

When they were given the first assignment, there was a sprawling, animated discussion among the students in the class, including the students in the study group, about the open-ended task they were expected to solve and the way they were supposed to write up their solutions. This reaction was expected, since the students came from studies in mathematics that had not trained them in solving mathematical problems without definite answers, in writing about mathematics, or in communicating mathematics. The students' reactions to the assignments often covered both the mathematical content and how it was expected to be presented.

The second interview began with a discussion of the student's first and second assignments, which had been graded and returned to them. What problems did the student see with either the assignment or with the grading? None said at first that they saw any problems at all with Assignment 1; instead, they seemed to think that they had learned a lot from the problems in the assignment about how to construct and validate mathematical models. Nina, Olga, and Patricia eventually said that they all had been graded Not Passed on their first attempt to solve the Olympic-swimming problem. When discussing with each one the errors she had made, I pointed out that the instructors had criticized her trust in the computer-generated model rather than in her reasoning, presumably based in reality. Nina said:

> What do you mean? I can't know anything about the swimming results for the Olympics in the year 2000, can I? It's not fair to ask me for assumptions or knowledge that lies outside the task.

This reaction was typical at the beginning and during the first half of the course. It reflected the student view that "this should be enough, shouldn't it?" It was as if the students expected that answers to all mathematical questions should be limited and exact, and that you should always be certain of when the answer was reached. Another problem with open-ended tasks was that the students were evaluated not only on their mathematics but also on their Swedish, and specifically on argumentation, assumptions, validation, and reflections. "I don't want to be graded on how I write," Linda said, "just on how I calculate and perform the mathematics." Virtually all of the study group students felt the same way at the beginning of the course, although many gradually began to appreciate the benefits of being forced to write about their solution strategies.

In the final examination, one question dealt with the effects of insulating a house (Figure 15). The gas problem opened as an easy problem, but created unexpected difficulties for all the students in the study group, as well as most students in the rest of the class.

Problem 3 (from Edwards & Hamson, 1996, pp. 155-156)

Recently it has become more and more interesting to use natural gas to heat homes.

a. Table 1 gives the weekly gas consumption (m^3) and average outside temperature (°C) for a particular house before the installation of cavity wall insulation.

Table 1

Temperature (°C)	-1	0	2	4	5	7	10
Gas (m^3)	206.6	195.6	173.2	149.4	115.7	116.0	82.4

Construct the simplest possible model to describe the correlation between weekly gas consumption and outside temperature.

b. Table 2 gives similar data for the same house after insulation.

Table 2

Temperature (°C)	-1	0	1	3	6	8	10
Gas (m^3)	134.4	127.6	120.6	110.1	89.4	72.7	59.4

Construct the simplest possible model to describe the correlation between weekly gas consumption and outside temperature after insulation.

c. Table 3 gives monthly averages of the outside temperature at the location of this house from October to May.

Table 3

Month	O	N	D	J	F	M	A	M
°C	10.3	6.7	4.4	3.4	3.8	5.7	8.7	11.5

Find an appropriate model to describe the annual variation of the average temperature over the year.

d. Write an expression for the amount of gas saved in one year by having insulation, and calculate a numerical answer for the amount of gas saved.

Figure 15. Problem 3 from the final examination.

After finding the first two linear models in (a) and (b), the students ran into difficulty because to finish the problem they needed a periodic model from October to October to illustrate the temperature changes. As a consequence, a majority of the class and all the students in the study group except Sarah expressed the opinion that a model in every part of the problem is exclusively chosen by the technology. The selection principle is dominated by the ranking of the values of the correlation coefficient. Nina wrote,

> In order to find a model, I used the software CurveExpert. I picked a continuous and periodic function since that would also provide me with temperature values for the summer months. I decided that the function $y = a + b \cdot \sin(ct + d)$ was the most suitable. After further investigation, I found that this function is not quite periodic.

Sarah, on the other hand, decided rather early in the modeling process to choose or construct her own model, not to be offered one. She wrote in her examination:

> When I enter those values into CurveExpert and apply curve fitting of a model like $y = a + b \cdot \cos(cx + d)$, in which I define $c = 2\pi/12$ since this will force a periodicity equal to 12 months, then I get the following model:
> $y = 8 + 4.9 \cdot \cos(0.62 + \pi/6)$. (Figure 16.)

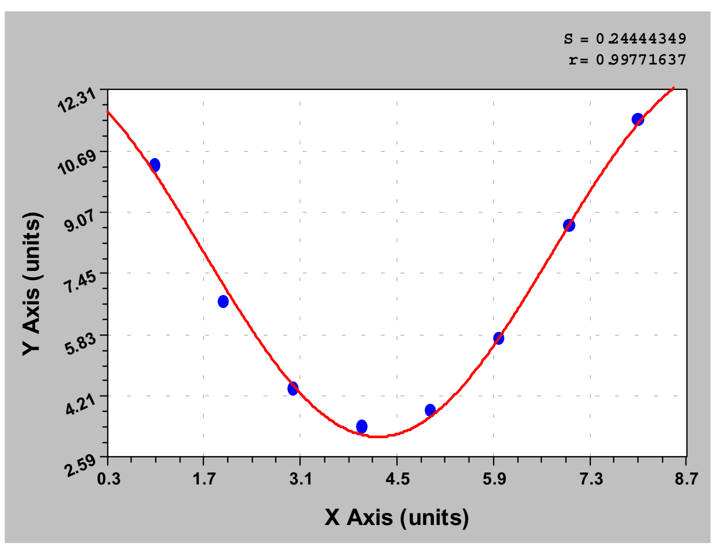


Figure 16. Regression curve generated by CurveExpert

Three of the other students in the study group, Nina, Olga and Patricia, selected a model based on the calendar year, which meant that they arranged the monthly averages of the outside temperature from January to December, thereby yielding a different figure (Figure 17).

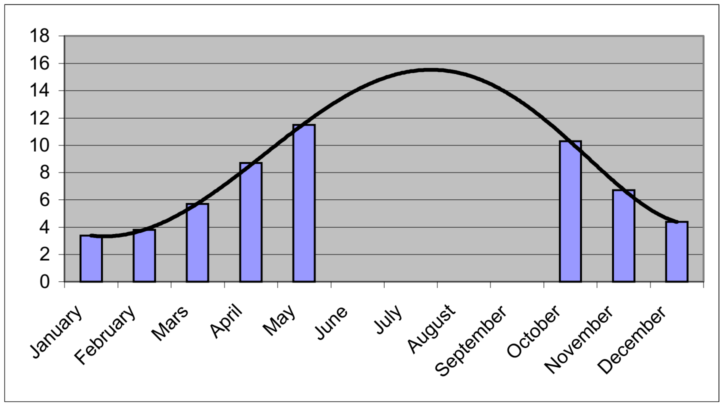


Figure 17. Illustration of the modeling of outside temperature from given data.

Both Nina and Olga became so confused by this modeling process that they used the computer again, seemingly without really needing it. Having obtained a pretty good model, they took all the points Excel used to draw the sine curve and generated the same model once more in CurveExpert. Then they integrated over the year, and presumably since they did not have the Excel figure in front of them, they forgot to exclude the "summer gap" and got an amount of gas saved that was at least twice as much as the data suggested. Patricia, on the other hand, tried to interpret the graph and claimed:

> When I see the graph and that the curve is "empty" during the warmest months of the summer, then I realize that it would be very stupid to use gas for heating when it is warmer outside than inside. And I just exclude this interval from my calculation.

By keeping careful track of every step she took in the modeling process, and by checking on the Internet for companies that manufacture insulation, Patricia was the only other person in the study group besides Sarah to get a passing score on the gas problem. Both

Sarah and Patricia were graded Well Passed. Patricia managed both to solve the problem in a very elegant way and also to connect her solution to claims made by insulation companies in the south of Sweden, where the temperatures were close to those given in the problem.

The mean temperature in Table 3 of Figure 15 over the 8 months can be roughly estimated to 7°C, and Tables 1 and 2 indicate that the amount of gas saved for a temperature of 7°C is about 35 m^3 a week. A very rough estimate is that the amount of gas saved over a year probably does not exceed 1200 m^3. On the other hand, Table 3 shows that 5 months have average temperature below 7°C and therefore that the amount of saved gas is not below 700 m^3.

The only student in the study group who did anything in the direction of a simple arithmetic calculation and stayed with it was Sarah. Her approach helped her during the modeling process to make important and influential decisions:

> Since I know that the amount of gas saved should be somewhere around 1200 m^3, I can check my models in order to "fit" the model to the real-world solution, instead of the other way around.

Robert got very upset after getting back his final examination paper. He had neglected to take into account that the gas consumption was expressed in weeks whereas the average outside temperature for the geographical location was in months. His model thus became a mixture of two different units of time, and his result about a tenth of what it should have been.

> *R:* I know this, I know this stuff. I know that I know this. I'm good with integrals. I can't believe that I made this error and that I didn't do a rough check before. I've spent hours and hours making nice graphs and formatting the mathematical text. And then everything is wrong!

I: What happened to your strategy of always doing a mental and paper-and-pencil check first?

R: I don't know. I guess I got carried away, and for some reason I thought that I didn't need the common-sense check. I just became obsessed with the problem!

I: Do you think the problem was too complicated?

R: No, the problem was great. It's just that I'm upset with myself. The problem sort of rips your clothing off and shows how much of the mathematics you have understood. I think it is healthy to face problems like this and to be forced to write about them, but at the same time it is almost too revealing!

Another way of looking at the task was expressed by Olga (who had forgotten to take away the empty summer months in the middle of Figure 17 when she integrated):

> It's so typical of you guys in this course. It's always complicated and hard, never easy. It is just so typical.

Although Olga was complaining about a specific open-ended task, she was also expressing an opinion that could reflect her view of the responsibility she ought to take for her own learning.

Students' Responsibility for Their Learning

Closely intertwined with students' views of open-ended tasks seemed to be their views of the responsibility they had for their learning. When Linda, Nina, or Olga complained about the nature of the assignments, it might very well have been that they did not want to take full responsibility for their own learning. Responsibility in all different shapes that we as humans meet is something we must learn to accept, and for some students it may take a long time for that to happen. Let us return to the dialogue with Nina that started in the previous section:

N: What do you mean? I can't know anything about the swimming results for the Olympics in the year 2000, can I? It's not fair to ask me for assumptions or knowledge that lies outside the task.

I: Well, you could make an assumption based upon results we have now and make a likely estimate, couldn't you? I mean, you have chosen a model in Excel that gives a winning time around 50 seconds. Don't you think that is far from reasonable?

N: But the model did fit the data points very well.

I: The instructor has written that you must have a model with a reasonable limit. Your model should not approach zero or infinity when x increases. Do you understand that concern?

N: Not really. There is nothing that says that the winning time couldn't be around 50 seconds. Besides, it seemed difficult to add a limit in Excel, and the logarithmic model that Excel provides us with does not allow any limit to be added.

I: But you can always increase a function by adding a constant, can't you? Compare $f(x) = 1/x$ and $g(x) = 1/x + 10$. You see?

N: I didn't know how to do that in Excel, so I let Excel give me a model. I get tired when the problems don't get solved easily, and then I prefer to let Excel do the job. I know that I should know how to deal with all this. My brother who's in the gymnasium does modeling problems in Math D. But I've always gotten high grades in mathematics without explaining what I actually do. And when there are too many balls in the air, I get stressed.

Nina, like many other students, tried to combine the traditional way of doing mathematical problems on a time schedule, where every problem leads to a definite answer in an expected length of time, with the exploratory way that the instructors hoped for with the Olympic swimming problem. It is notable that many students in the class complained about the lack of time they felt they had. In the study group, Nina, Olga, and Patricia, in particular, continuously commented on the lack of time and on the injustice they thought was hidden in the fact that they had to learn more than just modeling in the course. Patricia said, *P:* Why don't you [the instructors] design a special one-week course for us to learn The Geometer's Sketchpad, Excel, CurveExpert, and other software? Then we wouldn't need to sit here in the afternoon, struggling with the computers. I spent all last weekend learning how to handle trendlines in Excel and regression lines in CurveExpert. It shouldn't be like that! It's not fair!

I: So what would be fair? Do you want to have more free time?

P: Yes. I don't want to work late in the evening or on the weekend!

Olga had a view of the time to be spent on academic study that paralleled the time ordinarily spent working full time at a regular job:

> Well, all study should be done within a 40-hour week, I think. Otherwise it is more than full-time study.

It is interesting that a prospective mathematics teacher would have this view of mathematics as a finite volume of knowledge that should be covered, inch by inch, at the rate of a "regular job." That view is especially surprising since the national curriculum in Sweden urges, "The students should, with increasing maturity, be encouraged to take a greater personal responsibility for their learning" (Swedish Ministry of Education, 1992, p. 337, my translation). What applies to the schoolchild should apply to the prospective teacher as well.

Sources of Authority

Just as the students' views of the open-ended tasks were intertwined with their views of the responsibility they had for their learning, similar connections might well have existed between these views and their views about sources of authority. The main difference is that opinions about problems and assessment together with opinions about responsibility were much easier to get from the students. In general, most students had not thought very much about the concept of authority or where its sources might be. A third look at the conversation with Nina reveals how uncertain students sometimes were about sources of authority they were using.

I: But you had about two weeks to solve this, didn't you?

N: Yes. I guess I get stressed by the fact that I know that I should know this. I mean, I've been a good student in mathematics all my life. Now when you ask me to explain why I chose a model, I realize that there's a lot I never understood. I always was good at looking at the examples in the textbook and then repeating about the same strategies on the problems. And then I could check in the answer section of the textbook. But in this course I can't do that, since every problem is different, and since I should develop and write about my own strategy.

I: Would you say that your source of authority used to be the textbook examples and the answers in the back of the textbook? And now, when you don't have a textbook, where is the authority then?

N: As I said, I trust Excel and CurveExpert to do the modeling work and give a function I can use. I guess I rely on them and the calculator as much as I did on the textbook examples in some sense.

I: Don't you see the instructors here as a source of authority?

N: I guess I do, but then they never really answer a question on the assignment with a definite answer. They always ask a question back. At least Excel answers.

Nina seemed to see the source of authority as a place where one can get immediate answers. She said that she used to rely heavily on the textbook, and she expressed severe discomfort over the fact that the course did not have a textbook with examples, a clear description of what the examination would look like, a set of representative problems, or something similar.

At the time of the first interview, the course had brought the students in the study group fairly close together regarding their views of technology. No one hesitated when shifting between different software programs to draw graphs or identify models. Even Robert did most of his work directly on the computer, although he was careful with his interpretations. The only skepticism about the models they generated that I could sense came from Sarah, who argued that the more complicated the models became, the more dangerous it was to use computers:

> It's as though we become seduced by the fancy graphs and the quickly generated results with all the decimal places. And if the model is complicated, you really don't have any chance to follow the calculations.

Sarah supported my previous observations that the available technology—graphing calculators, Excel, and CurveExpert together—created an environment that was hard for the students to analyze and remain critical of while simultaneously using it to solve mathematical modeling problems.

Sarah's observation that the more complicated the situation is, the harder it is to see through the modeling process provided by computer software was underlined in the results of the final examination question on insulation discussed above in the section on open-ended tasks. After Robert expressed dismay at his performance, recognizing that it was healthy to work on such problems but finding it "almost too revealing," I asked a follow-up question.

> *I:* So what is your opinion about the trust you now put in computer-generated results?
>
> *R:* I don't know. I think that I'm as skeptical as before; at least I still know that I need to be in control. At the same time, when you learn to use computers, it is hard not to use them all the time. I've heard about people who have problems writing letters by hand after using computers for a long time, and now I can believe that. It's almost the same with me. Now that I've learned to make nice graphs in Excel and CurveExpert, to cut and paste into Word documents, I hesitate to do mental or paper-and-pencil estimates first. It is like I'm drawn to the computer first instead, and then it's hard to stop or look back.

This opinion was supported by the dialogue Robert and I had regarding the other modeling problem in the final examination, which dealt with medicine injected directly into the blood (see Appendix P). The model was more or less given, and the first task was simply to determine "good values" for three parameters so the model would fit measured values. The second part of the problem, however, was more open. The students were asked to decide if it was possible to determine the amount of medicine that was injected and to give good arguments for the choice they made.

Robert enjoyed the problem and thought it was easy: "Six given data points and only three parameters, *a, b,* and *c*—this is piece of a cake."

I: Really?

R: Yes, I just run CurveFinder with the given function, and it will give me the best fit. Then I can use CurveExpert to integrate under the curve.

I: Integrate? Is that for the maximal concentration?

R: No, it's for the amount of medicine given. I just find the derivative of the function for the maximal concentration. But CurveExpert can integrate just with a mouse click; it's so cool.

I: And you'll get the total amount of injected medicine by integration?

R: Sure, what else could it be?

I did not interfere with Robert's problem solving by asking him for the lower and upper limits for his integration.

Given a chance to decide by themselves if it was possible to determine the total amount of injected medicine, all five students in the study group, together with 90% of the whole class, decided that it definitely was possible, but their methods and results varied. Nina, Olga, and Patricia could not determine if the value (0, 0), meaning no medicine before the injection, should be added or not. Sarah wrote, "I assume the medicine is not present in the patient's body before the injection, and therefore I added the value (0, 0) to my data set in order to have a natural starting point for my integration."

Since Sarah declared this strategy rather openly, the other four students followed her example. Unfortunately, they chose the wrong problem on which to trust Sarah's authority. She integrated between 0 and a large number, thereby forgetting about the recirculation of the blood and the obviously fast-working medicine. Patricia had written, "One can see that it is a medicine that reaches a peak already after about 5 seconds and then descends. It could be some sort of adrenaline that leaves the blood quickly." Even Patricia, however, became trapped by Sarah's authority and possibly also the authority of CurveExpert. She allowed CurveExpert to integrate far out of the range of the data set and beyond what her mathematical model could provide.

P: I just integrated.

I: Yes, you did. Did you get a good grade on it?

P: No. I was so happy when I found the second model with the new data point (0, 0) added in CurveExpert that I just picked the largest *x* that CurveExpert would converge for—I think it was 180—and then I used it.

I: But there's a contradiction between what you wrote about the medicine being a fast medicine like adrenaline and integrating between 0 and 180, isn't there?

P: Sure. I just wanted it over with, and I forgot about my idea about the medicine. My mother's a doctor, you know, and… Well, it's so embarrassing. I answered with an amount that's ten times as high as possible.

Only Robert used the validity check that the instructors and I had talked so much about. He called a drugstore and spoke to a pharmacist, who told him that a medicine with a maximum concentration after about 5 seconds would most likely leave the body in about 30 seconds. Robert sought an authority outside the course.

Olga was as open with her feelings as Robert and Patricia were. On the gas problem, she and Nina had arrived at 4000 m^3 for the amount of gas saved, and Olga had major difficulties understanding the link between the models in (a) and (b) and the model she should construct in order to solve (d) (see Figure 13).

O: It's so typical of you guys in this course. It's always complicated and hard, never easy. It is just so typical.

I: Do you mean the integral?

O: I mean the whole thing. How do you think we are supposed to use computers and the results we get from them when we are criticized this way by you teachers? I worked a lot on this problem, and I used both CurveExpert and Excel to generate models. So how could it be wrong?

I: I think that the grader's comments on your paper criticize you for trusting the computer a lot and not using your common sense. It is not likely from the data you have that an amount of 4000 m^3 gas could be saved, is it?

O: Well, that's what I got when I calculated the area under the curve with CurveExpert.

I: And you trust the answer from the computer?

O: Of course I do. That's what this course is all about, isn't it? We should use computers, shouldn't we?

I: But if you put something wrong in, then you will get something wrong out, right?

The modeling process in the gas problem eventually led to an integral expression. The analytical reasoning connected to the evaluation of this integral is something that several students considered a task for the software, not for themselves. As a result, they bypassed the important question about the model's validity for the whole year. They sometimes ended up with model describing (and calculating) negative gas consumption for a couple of the summer months.

In the last interview, the students were asked to think about how they had studied mathematics during the course and about the lack of textbook among several other things.

Interestingly, all but Olga invented a new category in their answers that might be called "hard but not to be missed." Olga simply considered the course to have been very hard.

All of them had missed a textbook, two saying very much and the others saying much. Their answers to how much they usually trusted a mathematics textbook ranged from not at all to very much. All five usually trusted their mathematics teacher very much. Their answers to the question concerning their trust in results from computer software and calculators were ambiguous. Robert wrote: "Too much, at the same time that I don't trust them at all." None believed that this course had changed their trust toward different sources, but both Sarah and Robert said that their first thought was CurveExpert when I asked them about sources of authority.

None of the students in the laboratory group referred to their peers as sources of authority, yet the collaboration and activity that took place during all the hours they spent on the mathematical modeling problems in the computer laboratory were mostly built upon exchanges of ideas between students. Undoubtedly, many of the students were affected by their peers' ideas during those meetings.

In all of the interviews there was of course a very special source of authority present, namely the researcher. None of the students identified or named me as a source of authority, although the interviews and conversations in the computer laboratory most likely made the students realize that I was well oriented in the field of mathematical modeling and technology.

CHAPTER 7

SUMMARY AND CONCLUSIONS

We recognize, of course, a methodical imperative that cognitive processes must be inferred from behavioral evidence. But cognitive studies are concerned with what people know, and there is no simple relation between what they know and what they do. The real problem is to see beyond their behavior to the underlying rules and concepts that characterize this knowledge.
—Center for Cognitive Studies, Harvard University, Annual Report, 1961

Calculators and computer software have advanced to the point where they now have more than enough power and capability to perform mathematical computations and to display representations of mathematical ideas that people could only dream about a few decades ago. New areas of mathematics have entered rapidly into the undergraduate mathematics curriculum, and as mentioned in the first chapter, mathematical modeling is one such area. It may seem ironic, but compared with students from only a generation ago, students today need to know even more mathematics if they are to understand the results they get from the powerful calculators and computer software they use when doing mathematical modeling. The practical experience I had from my own teaching at the University of Gothenburg led me to conduct the studies reported in this dissertation.

Summary

I conducted three studies at the University of Gothenburg during the fall 1997, spring 1998, and fall 1998 semesters. The purpose was neither to verify an existing theory nor to test a priori hypotheses. Rather, my intent was to develop a framework for exploring the students' difficulties with mathematical modeling by observing and interviewing them in the context of a regular, if unique, course on mathematical modeling.

The course is a 10-week modeling course for prospective mathematics teachers for Grades 4 to 9 or for the gymnasium. It is designed to give students insight into how they can solve extended problems using mathematical modeling by drawing on technology and their background in mathematics.

Models and Reality

The first study was conducted during the fall of 1997 with a class of 71 students and a study group consisting of 5 students from the class. I started to follow, observe, and interview the study group students after almost 5 weeks of the course had elapsed. The five students continued to work together during the rest of the course in one of the two computer laboratories.

The research questions for the study were:

- What views do students in a course on mathematical modeling have about technology?
- To what extent do students in a course on mathematical modeling believe in results from calculators and computers?
- How do students relate mathematical models to reality when using calculators and computers?

The findings were that the students in general favored the use of technology, especially when solving complex mathematical modeling problems. On the other hand, they easily "got lost" and trusted the technology far too much when working on mathematical modeling problems, thereby neglecting a necessary validity check. This trust, in turn, seemed to profoundly disturb their ability to relate mathematical models to reality.

Conceptions and Misconceptions

The second study was conducted during the spring of 1998 with a class of 30 students and a study group of 8 students from the class. I followed, observed, and interviewed the study group students throughout the whole course and in much more detail than in Study 1. The eight students worked closely together during all 10 weeks in one of the computer laboratories.

The research questions for the second study were:

- How do students in a course on mathematical modeling view mathematics, technology, and their choice of a career?
- How do students in a course on mathematical modeling use calculators and computers to generate models?
- What conceptions and misconceptions about mathematical modeling lie behind students' decisions to believe more in a mathematical model than in real-world phenomena?

The results from Study 2 were that most students in the class considered mathematics to be an important or interesting subject and that therefore they had decided to become teachers. At the same time, only a few of the students saw time spent using technology as important preparation for their profession, compared with learning teaching methods or learning more mathematics. Many saw their choice of career as providing a way to correct something that was wrong, namely poor teaching of mathematics. A majority considered mathematics to be a language or a means to structure, describe, and understand the world.

The students in the study group were confident and assured when using technology, and all appeared ready to attack almost any problem with a calculator or a computer. They typically had difficulty, however, distinguishing between the technical skill they needed when using technology in teaching mathematics and the critical view one must have in interpreting the results technology gives. Their misconceptions seemed to be of two kinds: mathematically grounded misconceptions and misconceptions based on a strong belief in the technical aspect of the modeling process. An example of the former was a belief that a straight-line model adequately describes the height of an object in free fall at a given time. A misconception of the latter kind was that the calculator or computer always calculates correctly, even when it contradicts one's common sense.

Authority and Responsibility

The third study was conducted during the fall of 1998 with class of 70 students and a study group of 5 students from the class. I followed, observed, and interviewed the study group students throughout the whole course in a similar way as in Study 2. As before, the five students worked together during the 10 weeks.

The research questions for the third study were:

- How do students in a course on mathematical modeling view open-ended tasks?
- How well do students in a course on mathematical modeling accept responsibility for their own learning?
- What sources of authority can be observed among students who use calculators and computers to solve mathematical modeling problems?

Most of the 70 students in the class and 4 of the 5 in the study group were favorable toward open-ended tasks as long as they were not required to use their mathematical knowledge extensively. Being required to explain, argue for, and reason about a solution or a modeling process that might yield a variety of possible solutions imposed higher standards for the students' mathematical modeling performance and consequently for their responsibility for their own learning. Few students in the class or in the study group were ready to take on that responsibility, which is closely connected to those authorities the student identified with or trusted during the learning process. Many looked for authority from their peers in the classroom, from information sources on the Web, and most of all from their computer-generated results. Other sources of authority were textbooks from other courses and the lecturers in the course.

Conclusions

The major conclusion of this research is that prospective teachers with a good background in mathematics from the gymnasium and from other courses at the university, together with a normal skepticism towards technology and technology-generated results, can during a 10-week course change their source of authority and become seduced by the technology.

For example, one prospective teacher, very academically motivated and ambitious, with high grades in mathematics from the gymnasium and in the mathematics courses at the university, started the course with a clear and distinct ambition to learn the technology but at the same time to stay in control of the mathematics. Almost without knowing it, he changed his view about control and allowed the computer software to become more and more dominant. By the end of the 10 weeks, his problem-solving strategies had changed from him using mathematics first to trying to employ the computer software as soon as possible, jumping directly into the mathematical-modeling process.

But prospective teachers tend to approach modeling in different ways. For example, one prospective teacher in the second study was eager to learn the tools in every different software program she came across to see what they could do. She was in some sense naive in her belief as to what the software could do and by the end of the course preferred to work as much as possible with paper and pencil, expressing the opinion that solving problems with the help of Excel was "like cheating." She was also one of the few who took a broad perspective on what she learned about the errors that technology can produce. She viewed that information as an important part of the process of becoming a teacher, while other students saw it just as an annoying obstacle. She never really changed her source of authority, remaining confident in her own mathematical knowledge. Interestingly her grades from the gymnasium and from the university courses were not nearly as impressive as those of the previous prospective teacher.

The results of my studies also apply directly to the mathematical preparation of prospective mathematics teachers for Grades 4 to 9 and for the gymnasium. All the students interviewed and observed were successful and had satisfactorily completed at least six mathematics courses at the university. Nevertheless, my data contain multiple examples of the fragile and fragmented nature of these students' knowledge of mathematics. That so many of the students seemed to have forgotten many topics, even those studied at the gymnasium level, is not so difficult to deal with as the fact that many of the students introduced and defended contradictory ideas despite their records of satisfactory mathematical achievement. These results challenge many of the foundations of how prospective mathematics teachers are taught and assessed. The results I have reported here can be seen as supporting the position that prospective mathematics teachers for the secondary grades may not need more mathematics courses as much as they need different learning experiences. Such experiences should engage them in reasoning and in constructing mathematical models, in assessing the extent to which a mathematical argument is valid, and in developing, comparing, and evaluating alternative solution processes.

I also found that prospective mathematics teachers very well can learn quite a lot about mathematical modeling, both as a process and as a performance. They can learn how to construct, discuss, and argue for the validity of results and conclusions. Many prospective teachers do not have a good grasp of the different mathematical ideas and topics they have studied in different courses. A course such as the modeling course I studied makes a good conclusion to their studies. It can help them consolidate their previous knowledge and deepen their understanding of mathematics. They can develop greater insight into their own thinking and possible misconceptions that they have succeeded in hiding or overlooking.

Change is clearly very hard for many prospective teachers in confronting a course in mathematical modeling such as the one studied. They are in a baffling, difficult situation. They come from many years of mathematical studies in which the primary teaching methodology has called for the reproduction of knowledge, and in which there is a clear structure consisting of textbook, instructions for studying, and regular lectures that illustrate the content of the textbook and of the course. And at the end of the course, there are often old examinations to use for practice. Then they come into a course with no textbook, with all previous studied mathematics as the content base, and in which they must solve open-ended problems. Their view of mathematics and of studies in mathematics as having a fixed character has been well established and is difficult to shake.

In such a shift among many of the major components that build prospective teachers' views of mathematics, many seem to abandon their trusted sources of authority. In contradiction to what might be expected, not all of the prospective mathematics teachers turn to their more mathematically skilled peers. Instead, some of them try alone or in pairs to find the ultimate computer-based tool or through a tool the ultimate and correct model, presented by the software and judged by the strength of a statistical correlation and not by how well it fits real phenomena.

Although the course on mathematical modeling is organized in a way that allows and encourage the students to take the initiative to construct mathematical models of their own free choice and seek relevant information wherever they think they can find it, many students seem uncomfortable in this situation. Povey (1995) described a similar teaching experience in a course for prospective teachers in which the Logo software was used in the following way:

> As soon as possible, students are encouraged to understand that whilst the tutor is more experienced in Logo than the students and has better access to information about Logo than they do initially, since the students are setting the problems, it will be an everyday occurrence that students will know more than their teachers about aspects of their work. An atmosphere is sought in which some pairs quickly become experts about particular things in which others are likely to be interested as tools in their, different, inquiries. Knowledge and authority are shared. (p. 141)

What Povey means by "setting the problem" is the well-known situation in which students have been working for some time on a task using a computer. When they call the teacher for help, they expect the teacher to immediately disclose the error that has been made and that is causing their problem. It was never evident in the course I studied, however, that the students ever experienced themselves as knowing more than the teachers, but different students naturally became sources of authority for other students, for better or worse. Sometimes knowledge and authority go hand in hand, sometimes they do not. It can be very hard for a student to judge whether or not she or he should have confidence in a peer. The aspect of negotiation with peers was something many students found difficult.

A Framework for Mathematical Modeling

The framework presented in chapter 1 that guided me when I began these studies had become substantially more complicated by the time I had analyzed all the results. Along the way, I identified several additional components of the mathematical modeling process, among them relations with sources of authority that I did not have in mind at the outset. The resulting framework (Figure 18) is constructed to suggest the sources of difficulty that students encounter in doing mathematical modeling, and it also reveals the general structure of the process as performed by a student.

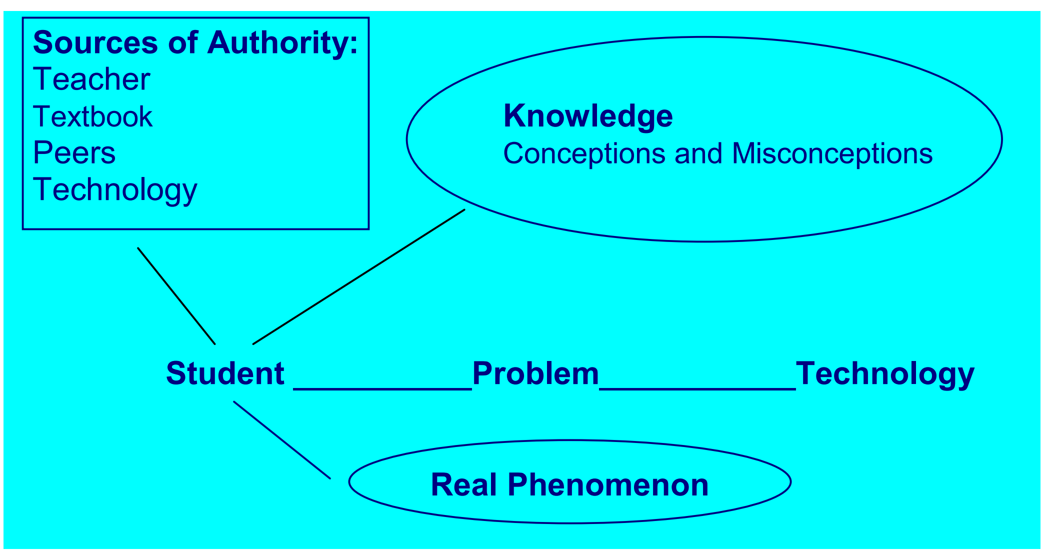


Figure 18. Main components of the mathematical modeling process.

In the complexity of the modeling process in a classroom or computer laboratory, with students working sometimes in a group and sometimes alone at the computer, it is hard to describe all the relations that occur. The discussions in which the students take part nearly always have a third, silent partner: the computer software and its result. This third partner changes the relationship between the students and the instructor. Much of the research reported in this dissertation dealt implicitly with how this relationship was affected by the modeling process as well as the presence of technology.

As the student solves the problem using technology and attempting to understand the connection between the problem and the real phenomenon it refers to, various factors shown in the figure come into play. The student's knowledge of mathematics and about mathematical modeling includes conceptions and misconceptions he or she brings to the modeling situation. In trying to understand and solve the problem, the student also draws on various sources of authority. Observe that the technology functions both as a tool to help the student solve the problem and as a source of authority. The connections shown are just figurative and are not meant to be hierarchical in any sense.

As an illustration of the complexity in the mathematical modeling process, consider the discussion related to the students' performance on the gas problem (see page 133) in the final examination in Study 3. All of the students in the study group successfully used the technology mainly as a tool for the first two parts of the problem. In the next step of the mathematical modeling process, most of the students in the study group were influenced by the way the technology illustrated the third part of the problem, and thus the technology acted as a source of authority. Some of the students also found sources of authority among their peers or on the Internet, thereby referring to the real phenomenon of insulation. The necessity to set up an integral expression when solving the fourth part of the problem revealed students' conceptions and misconceptions. Finally, the interpretation of the students' solution revealed conceptions and misconceptions of mathematical modeling and of how to validate a mathematical model.

That students make mistakes for a variety of reasons and misinterpret the results of mathematical models is not a new phenomenon. Usiskin (1979) gave several examples of problems that might very well mislead students into false assumptions and conclusions. One such problem was the Evolution of the Mile Record (pp. 434-437). With this problem, Usiskin showed how a supposedly correct straight line fitted to the data points representing the world records for the British Mile from 1875 to 1975 would actually yield a record of 0 seconds in the year 2550.

The fact that various sources of mistakes too often and easily can be summarized as misconceptions is described in depth by Shaughnessy (1985). In his article "Problem-Solving Derailers: The Influence of Misconceptions on Problem-Solving Performance, " Shaughnessy identified several sources of errors: lack of appropriate knowledge structure or knowledge organization, algorithmic bugs, lack of problem-solving strategies, relinquished executive control, belief system, folklore paradigms, and inadequate problem representation, and ill-chosen schema. In the case of mathematical modeling, it is quite obvious that unfamiliarity with or uncertainty about the phenomenon to be modeled is a great hindrance to the mathematical-modeling process.

Teacher As Researcher

When observing and conducting research in a program of mathematics teacher education in which one is known to be a member of the faculty, the researcher should be aware that the students are likely to see him or her as both a researcher and a teacher. Even though I did not participate in the grading process during the second two studies, I was still a member of the teaching faculty and could observe that the students reacted differently to that. Many assessment tasks contained phrases that I could identify as coming from me. Several students expected me to give them a hint or sometimes even the right answer to a mathematical modeling problem in exchange for their participation in interviews and laboratory observations. A few also accused me of an unfair and rigorous grading process and did not believe me when I claimed that I was only observing. And in fact, I *was* doing more than just observing, since I took part in many discussions in interviews or in computer laboratory sessions. To be a participant observer naturally means that the researcher is part of the activities she or he observes.

It is hard to avoid the "teacher personality" and just be a researcher when one already has some knowledge of the participants' qualities and abilities. Most of the interviews in the studies provided opportunities for learning, which can be seen in the dialogues presented in the earlier chapters. These dialogues need to be seen as showing progress both for the students with respect to their mathematical modeling and for me with respect to learning more about the students' conceptions, views, and perspectives.

Regardless of whether students are positive or negative about being interviewed in the presence of a video camera, their attitudes toward what is happening affect a study's results. A strong positive attitude toward being "a research object" may distort a participant's response in the form of exaggerated engagement with the task and an obvious desire to make a good impression. I experienced that phenomenon with only one student, and I told her that she did not need to please me with her answers. Had another researcher conducted the research who was not as active as I was in the program and the course, however, the results might well have been different.

On the other hand, it is difficult to make close observations of how students handle a mathematical-modeling process without being an active observer. A researcher who is not familiar with or involved in a course would undoubtedly experience severe problems in gaining access to teaching materials, assessments, grading strategies, and various unspoken course policies. A researcher who is a novice in the content that the students are expected to learn would probably be seen as less threatening when it comes to grading assignments and tests, but at the same time such a researcher would be bound to miss a lot of the information provided.

Although I attempted to introduce as little distortion as possible into the natural flow of the students' thought processes and problem-solving strategies, I must acknowledge that by carefully choosing follow-up questions and suggesting to the students that they explore questions with calculators or computer software, I probably crossed the line between researcher and teacher many times. The interview transcripts document that the students often came up with important ideas in the mathematical modeling process after my interruptions during the interviews. It should be pointed out, however, that I rarely provided direct answers. Instead, I led the students to discover new ideas for themselves. Of course, these interviews were not typical classroom learning environments available to the rest of the class but instead were isolated moments for the students in the study group. A skillful student probably could have encouraged me to go on questioning and to provide leads to solutions, thereby creating a more or less private lesson in mathematical modeling during the interview.

An interesting direction for further research might be to study whether the results I obtained would be even more dramatic if the researcher were not also part of the faculty giving the course and thus a source of potential authority. The position between faculty members and students in terms of authority is part of the skewed "balance of power" in any interview between a faculty member and a student. Even though we in Sweden consider the relation between students and teachers to be open and equal in status—not just through the way we address one another but through other social customs and legislation dealing with equity—it is understandable if the students feel they are in a position of inferiority. The question of how this complex web of authority and power affects the validity and generality of research such as this remains to be resolved.

Implications for Research and Teaching Practice

Perhaps the most obvious effect of the three studies was how they changed the course, which in turn changed the research. As a result of my findings in the first study, the next time the course was given there was more emphasis on the validation part of the mathematical modeling process, and all students were briefly informed of the results of Study 1. After the second study, the course lectures emphasized both validation and common misconceptions. As a result of my third study, responsibility, authority, and peer tutoring became additional topics in the course. The course now also engages the students in evaluating their peers' work in order to encourage them to be more careful about how they write about and argue for their mathematical models. There is no guarantee that these changes in the course will manage to keep future students from mistaking a computer-generated model for reality, but the ways in which the results have been used do illustrate the possibly fruitful interaction between research and teaching practice.

It is clear that the progress of computing technology is far from ended. We can expect the calculator of tomorrow to do at least as much as and maybe more than what the computer software of today does. And courses in mathematical modeling are important for prospective mathematics teachers as well as for other students who study mathematics. To reveal, and even better to avoid, the phenomena I found, teachers of courses on mathematical modeling must pay great attention to they way they set up, conduct, and grade their assessments. With technology, it is sometimes very easy, much too easy, for students to provide the correct answer without really understanding what the problem is about. Without assessment situations that make use of the technology and involve the students in critical thinking about what the technology offers in terms of possibilities and solutions, we may very well create students who are dependent on technology and not critical and insightful users of it.

From my observations in the studies, I conclude that there is a great risk that technology will create a new sort of authority. Calculators have been known for some years to make students "slaves" in the sense that even a simple multiplication like 5 times 8 will be carried out on the machine if available. Anyone who has been to a shop in the Western world has probably seen clerks using calculators in this unthinking fashion.

As computers become increasingly common in every classroom and are used to help students and teachers with many tedious tasks, teachers need to pay careful attention to the kind of problems they give students. I am convinced that assignments like those used in the modeling course can function as a tool to promote the shift away from facts and standard procedures to conceptual reasoning. They can also reveal qualities in the students' beliefs about concepts and mathematical structures.

When students are forced to explain and argue for their models, they disclose inaccuracies and misunderstandings in a way that would be hidden otherwise. If teachers, for instance, ask students to calculate an integral expression, how do they know whether subsequent errors arise from the routine or the conceptual part of the solution process? With today's technology, the routine part of solving integrals is only a question of pressing the right button or giving the correct command.

Without the kind of assignments and assessments I have described, many students may pass through the education system without encountering any real challenge to what they actually know in mathematics. These studies of how students handle modeling situations in the presence of technology suggest that teachers at all levels need to be cautious about what students understand and the interpretations they make during the modeling process. I have demonstrated how easy it seems to be for students to "get lost" and trust the technology far too much, thus avoiding a necessary validity check. A clear focus on the validation part of mathematical modeling is obviously more essential in the presence of technology than ever before.

Equally or perhaps even more important is to engage students in discussions about possible models even before they start the modeling process, which would slow the rate at which technology enters, allowing mental processes to keep up. It is important to recognize that the mathematical modeling course in which the studies were done is a course in which students work with open-ended problems but do not generate the data themselves by going out and measuring a phenomenon with some instruments. The time available in the course does not allow such activities. An interesting research study would be to repeat all or part of these studies in such a course. An implication for practice might be that instructors of such a course might look for ways in which students might gather their own data for at least some tasks. One might then hope that would lead to a more open and free discussion among the students of what mathematical model to construct and how to construct it together with a discussion of what validity that model would possibly have. In a recent version of the course, the students were asked to study a specific phenomenon (the center of mass of syrup as it flows from a bottle) at home, thereby generating several data points for further mathematical modeling. I have no evidence, however, that this approach led to a better understanding of the mathematical model involved in the problem.

A possible next step, which would call for another major change in the structure of the modeling course, would be to require or encourage students to discover or generate their own data sets before they start thinking about what mathematical model they are looking for. Then evaluations of the modeling process might reveal more of the students' interesting mathematical ideas and fewer of their technology-related misconceptions. Possible implications for research could be to investigate if the process of mathematical modeling with use of many different mathematical ideas really do deepen the students' understanding of mathematics. Anther possible research idea is to examine the peer-to-peer negotiation that takes place in a setup like the one used in the modeling course. In what way can the active discussion be used further to strengthen the students' modeling and other mathematical capabilities?

The reader of this dissertation must keep in mind that the three studies reported here and the tentative conclusions I have drawn from my observations concerned the effects on students' conceptions and views of specific mathematical modeling problems in a specific technology-enriched environment in a specific instructional setting. The number of students studied in depth was small: five from the first study, eight from the second study, and five from the third study. It should also be emphasized that the personal construction of mathematical knowledge is varied and complex, depending on a great number of variables, many of which were not accounted for in these studies. Any observations and generalizations made in this work must be considered within the above context.

The three studies explore an area of growing concern in the learning and doing of mathematics, namely, the prospective mathematics teacher's developing understanding of mathematics and of mathematical modeling in a technology-enriched environment. As prospective mathematics teachers who are themselves products of technology-intensive classrooms enter programs to prepare mathematics teachers, some of the questions concerning naive uses of technology will possibly fade. The future will present different demands and challenges, but the results from this research can form a base for discussing issues as technology-based and technology-free mathematics learning. The ultimate challenge will continue to be how to empower prospective mathematics teachers to become and remain active learners and doers of mathematics.

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APPENDIX A

ENTRY QUESTIONNAIRE FOR STUDY 1

To make this course as worthwhile and as useful as possible, we would like to explore your views on calculators and computers, and we hope that you will participate by answering the questions below.

Name:

Program:

Do you think that one should be allowed to use a calculator:
  When learning mathematics?
  When being examined in mathematics?

Do you think that one should be allowed to use a graphing calculator:
  When learning mathematics?
  When being examined in mathematics?

Do you think that one should be allowed to use a computer:
  When learning mathematics?
  When being examined in mathematics?

……………………………………………………………………………………………

On the following pages we have posed some problems we would like you to solve. Your solutions will not be counted in your grade for this course; instead, they will help us conduct some good discussions on issues that are important for the course. Please help us by answering as accurately as possible.

Thank you for your cooperation!

……………………………………………………………………………………………

Problem 1

a. Calculate by mental arithmetic or by paper-and-pencil arithmetic the result of $8^{1/3}$.
Do you believe and trust your result?

b. Calculate by mental arithmetic or by paper-and-pencil arithmetic the result of $(-8)^{1/3}$.
Do you believe and trust your result?

c. Calculate by mental arithmetic or by paper-and-pencil arithmetic the result of $(-8)^{2/3}$.
Do you believe and trust your result?

d. Compare the results you got on (a), (b), and (c) with the result that your calculator gives for the same problems.

Problem 2

a. We have calculated the result of $(-8)^{1/3}$ with the help of Derive, Maple and MATLAB. The results were:

Derive:	$1 + \sqrt{3}\, i$
Maple:	1.000000000 + 1.732050808*I
MATLAB:	1.000000000 + 1.732050808*I

What do you now think of your own answer and the answer from your calculator for this question? Do you still trust your calculator? Do you trust your own calculation?

b. We have calculated the result of $(-8)^{2/3}$ with the help of Derive, Maple and MATLAB. The results were:

Derive:	$-2 + 2\cdot\sqrt{3}\, i$
Maple:	–2.000000000 + 3.464101615*I
MATLAB:	–2.000000000 + 3.464101615*I

What do you now think of your own answer and the answer from your calculator for this question? Do you still trust your calculator? Do you trust your own calculation?

c. Let us try to use some calculation rules on the expression $(-8)^{1/3}$:

$$(-8)^{1/3} = (-8)^{2/6} = ((-8)^2)^{1/6} = (64)^{1/6} = 2$$

What do you now think of your own answer and the answer from your calculator for this question? Do you still trust your calculator? Do you trust your own calculation?

APPENDIX B

SELECTED ASSIGNMENTS FOR STUDY 1

Assignment 1B

Let $R(1) = 1$, and for $k \geq 1$, $k \in N$,

set $R(k+1) = 1 + \dfrac{k}{R(k)}$.

Use Excel to sketch $R(k)$.

Investigate what happens with $R(k)$ when $k \to \infty$.

Report your analysis and provide an argument for your choice of method.

Assignment 1C

The population of Sweden has varied according to the figures in the table below:

Year	1750	1800	1850	1900	1950	1990
Population (in millions)	1.78	2.35	3.48	5.14	7.04	8.59

Use a graphing calculator or Excel and fit an adequate function to the data set, and predict the population of Sweden in the year 2000.

Assignment 2B

The school in which you get your first position has decided to develop a strong program of sports in different areas. As part of the financing for the different sport activities, the school is planning for parents, relatives, and others to subscribe to a one-year entrance pass. All 811 families in the neighborhood of the school were interviewed and among other questions were asked the following:

> "What is the highest price you would be willing to pay for a one-year pass that would cover all sport activities?"

The results of the questionnaire are presented in Table 1.

Table 1

Price for a one-year pass	Number of families who would pay this price
250	145
375	80
450	45
475	85
575	120
675	80
750	60
875	150

Use this information to help the school board decide the optimal price for the one-year pass.

APPENDIX C

FIRST INTERVIEW PROTOCOL FOR STUDY 1

Introduction

I'm glad that we have this opportunity to talk. I hope that you can forget the tape recorder I have running on the table, but please respond in a clear, distinct voice so that later I can analyze the interview on the audiotape.

Background

There are several reasons that I have scheduled interviews with all of you the next two weeks:

1. You responded to some different questions in the questionnaire. I'm eager to make sure that I have understood your written responses to the questionnaire.

2. Second, I would like to learn more about your trust or belief in computer- and calculator-generated results.

Technology

What kind of calculator do you use? How much do you know about it?

Do you have a computer at home?
 If yes, what kind of computer? What kind of software do you run on it?
 If no, do you have access to computers elsewhere?

Calculator Questions

[The student was given a sheet of paper with the following two questions. Below each question are the follow-up questions that were asked.]

Calculate $\dfrac{\mathbf{123456 \cdot 10^4 - 1}}{\mathbf{10^9 - 1}}$ **with your calculator.**

What did you get?
Is it true?
How do you know?
How sure are you?

Calculate $\mathbf{28{,}923{,}761^2 - 28{,}923{,}760^2}$ **with your calculator.**

What did you get?
Is it true?
How do you know?
How sure are you?

What possible limitations in the way machines perform mathematics do you think could affect the answers to the two problems above?

How do you view the modeling problems you have worked on? Do you have any questions or any comments about the work you have done in Excel and with your graphing calculator?

APPENDIX D

SECOND INTERVIEW PROTOCOL FOR STUDY 1

Introduction

I'm glad that we have this second opportunity to talk. I hope that you can forget the tape recorder I have running on the table, but please respond in a clear, distinct voice so that later I can analyze the interview on the audiotape.

Background

There are several reasons that I have scheduled this second interview with all of you this week:

1. You have been solving different modeling problems during this course, using different computer tools and graphing calculators. I now want to know more about how you decide to work on mathematical problems with the help and assistance of advanced calculators and computers.

2. I want to know more about how you think about and react to real-world problems that you are given to model.

Questions

What do you think of the results from calculators and computers when solving mathematical problems now that you have solved quite a few problems during the course and all that time have been using computer and graphing-calculator tools?

[Here I referred to all the written work done by the student in the course. Together, we went over the assignments the student had done and the take-home final exam.]

Do you have any specific questions concerning your final exam?
Are you now comfortable with all the computing tools?

Problem

I will give you a mathematical problem I want you to start thinking about, describing your solution strategies to me. Please tell me what you know that you can relate to the problem at hand. Tell me what tools you intend to use and how. I will not tell you how to solve the problem, but I will help you to deal with commands and error messages. It is important that you understand that I expect you to do as much as possible on your own. Since I'm more interested in what you do than that you reach a possible "right" answer, I will not always give you direct answers. [At this point, I gave the student Worksheet 1.]

Worksheet 1

An advanced mathematical computer software program was employed to find critical points of the function $f(x) = 3x \cdot 2^{(-x)}$. The result was

Type	*x*	*y*
Zero	0	0
Max	1.445	1.595

How would you approach this problem and validate the result?

[Worksheet 2 was given out after the student had been working approximately 15 minutes.]

Worksheet 2

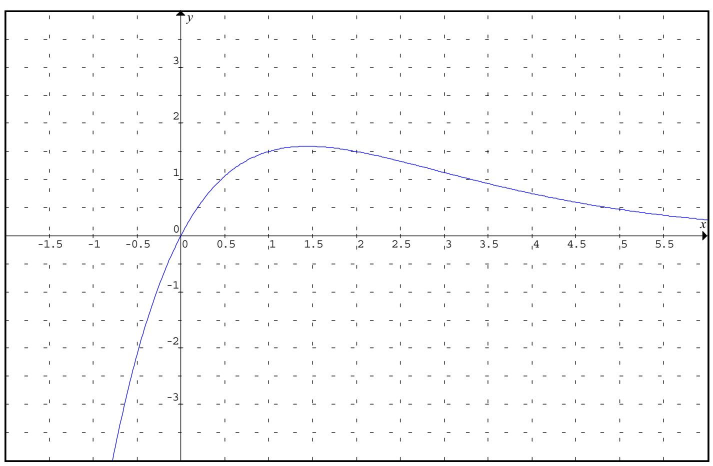


Figure 1: Sketch of the function $f(x) = 3x \cdot 2^{(-x)}$.

In what way does this figure support your opinion about the result in the problem?

[Worksheet 3 was given out after the student had been working approximately 10 minutes more.]

Worksheet 3

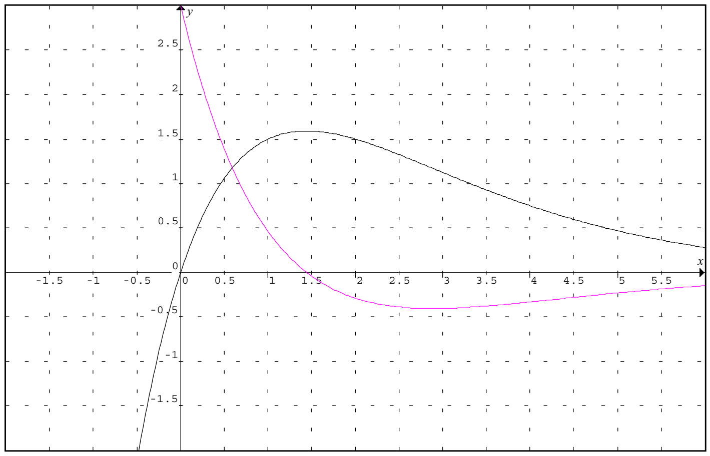


Figure 2: Sketch of the function $f(x) = 3x \cdot 2^{(-x)}$ and its derivative.

In what way does this figure support your opinion about the result in the problem?

[After the student had had a chance to work on these problems, giving answers in writing and aloud, I asked some follow-up questions.]

When I asked you about the strategy you would use with our without technology in order to solve, this problem you told me the following —. [At this point, I would show the students my handwritten notes and the computer or calculator display of their work and would ask them to discuss their response. My purpose was to verify whether I had understood the response correctly and to see if they had additional comments to make about it.]

APPENDIX E

FINAL EXAM PROBLEMS FOR STUDY 1

Problem 2

At the right, we see a well-known profile in architecture and construction art, namely, *the Gateway Arch* in St Louis. See Figure 1.

The arch consists of a hollow steel construction, where the inner centerline closely follows a certain mathematical relation:

$$y = A - B\cosh\frac{3x}{C}$$

Figure 1

Skyline of Saint Louis, Missouri

The skyline of the city of Saint Louis, Missouri, is dominated by the stainless steel Gateway Arch, rising 192 m (630 ft) high. It was designed by Eero Saarinen and completed in 1965, after the famed architect's death. Saint Louis lies on the banks of the Mississippi River.

Source: Encarta 97

Here we see a two-dimensional projection of the arch, with y as the height above the ground (in meters) and x as the horizontal distance (in meters) from the vertical axis of symmetry. See Figure 2.

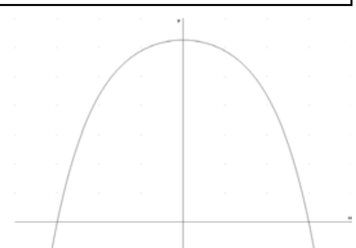


Figure 2

With the help of optical measurement equipment, it is possible to measure approximate values for x and y. Fourteen of these values are shown in Table 1.

Estimate the length of the Gateway Arch.

Table 1

x	y
-92.5	0.0
-77.5	76.3
-62.5	128.1
-47.5	159.2
-32.5	177.3
-17.5	186.9
-2.5	190.4
2.5	190.4
17.5	186.9
32.5	177.3
47.5	159.2
62.5	128.1
77.5	76.3
92.5	0.0

Problem 3

The number of people who communicate over the Internet by e-mail has increased greatly over the last 3 to 4 years. The graph describes the increasing use of e-mail in the teachers college at the University of Gothenburg since April 1994. See Figure 1.

The numbers that were used to draw this graph can be downloaded from the course Web page.

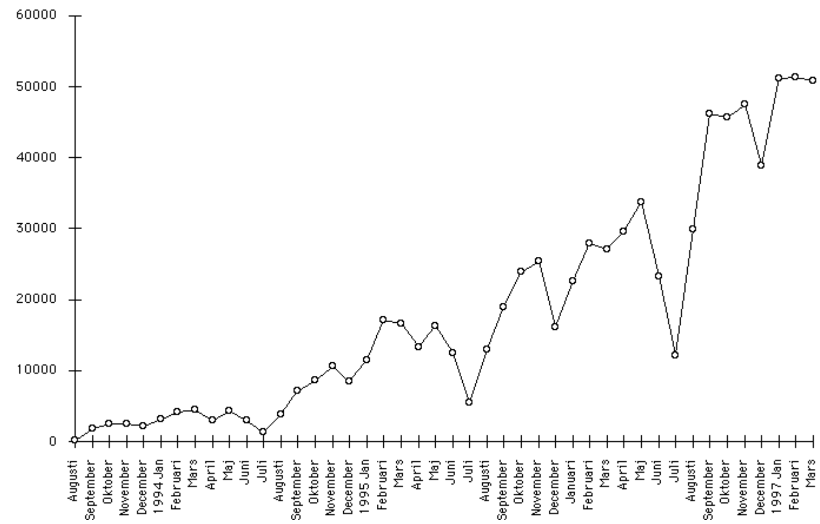


Figure 1

A reasonable assumption would be to expect the frequency of e-mail to continue to increase. Your task now is to help the technical advisors in the teachers college calculate the amount of e-mail expected in June 2000.

To do that, you need to construct or determine a mathematical model according to suitable principles discussed in class and then use this model to provide the technical advisors with the numbers they need.

Write a report that describes your strategies, your analysis, and your conclusions in a satisfactory and adequate way. Give careful, detailed explanations and justifications of all assumptions and calculations in your solution, put numbers on figures and tables, do not mix assumptions and conclusions, and use correct mathematical notation.

APPENDIX F

ENTRY QUESTIONNAIRE FOR STUDY 2

General Information

Name:

Date of birth (y/m/d):

Address:

Phone:

E-mail:

All data will be treated confidentially.

Academic Information

Gymnasium program:

Courses:

Algebra and Functions:	Yes	No
Exponential Functions:	Yes	No
Trigonometry:	Yes	No
Trigonometric Functions:	Yes	No
Derivatives and Integrals:	Yes	No
Differential Equations:	Yes	No
Complex Analysis:	Yes	No
Statistics:	Yes	No
Probability:	Yes	No

University mathematics program:

Algebra 1
Algebraic Structures
Algebra and Combinatorics
Discrete Mathematics
Real Analysis
Real Analysis in Several Variables
Linear Algebra
Geometry
Statistics and Probability

Professional Information (selected questions)

Teaching experience (include student teaching and other field experiences):
When did you decide to become a mathematics teacher?
Why did you decide to become a mathematics teacher?
What experiences, courses, activities, etc., would be the most helpful to people preparing today to teach mathematics in the next century?

In what way do you think the ongoing evolution of technology will affect the following aspects of your professional life as a teacher?

The curriculum
Assessment
Instruction

How do you view the response you get from your calculator when you evaluate the following? Do you trust the answers you get:

a. More than yourself?
b. More than your teacher?

2^3 $(-2)^3$ $8^{1/3}$ $(-8)^{1/3}$

Mathematics Information

What is mathematics?

What does it mean to "do mathematics"?

Explain what each of the following words and phrases means as it relates to mathematics. If a word or phrase is unfamiliar to you write *unfamiliar,* and then write your best guess as to what the word or phrase might mean.

function
proof
mathematical modeling

Technology Information

What brand(s) of calculator(s) do you have or have you used?

What features do or did the calculator(s) have?

For what do or did you use the calculator(s)?

Did you ever use a calculator in a mathematics course? Yes No
If yes, when?

Have you ever used a computer? Yes No

If you never used a computer, skip the next four questions:

What brand(s) of computer(s) have you used?

What features did the computer(s) have?

For what purpose did you use the computer(s)?

Did you ever use a computer in a mathematics course? Yes No
If yes, when?

Mathematical Modeling Achievement Test

Try to solve each of these problems. Show all of your work in the space provided. Please use the back of this paper if you need more space.

For the following problems, label the axes and sketch a reasonable graph showing how the dependent variable varies with the independent one. Actual values are not important here. If possible, give a function expression that describes your model.

1. The population of Sweden has varied according to the figures in the table below.

Year	1750	1800	1850	1900	1950	1990
Population (in millions)	1.78	2.35	3.48	5.14	7.04	8.59

2. You drop a chocolate bar from the top of the 330-meter-high Eiffel Tower in Paris. The distance of the bar above the ground depends on the number of seconds that have elapsed since you dropped it.

3. A culture of yeast is placed into a restricted enclosure to grow. The amount of yeast depends on the time and the living conditions.

APPENDIX G

FIRST INTERVIEW PROTOCOL FOR STUDY 2

Introduction

I'm glad that we have this opportunity to talk. I hope that you can forget the video camera I have running in the background, but please respond in a clear, distinct voice so that later I can analyze the interview on the videotape.

Background

There are several reasons that I have scheduled interviews with all of you the next two weeks:

1. You responded to a lot of questions in the questionnaire. Because you all have different backgrounds, it was hard to construct a questionnaire that would be equally appropriate for everyone. Therefore, I'm eager to make sure that I have understood your written responses to the questionnaire.

2. Second, I want to know more about your understanding of mathematical modeling in general and in particular when we work with computing tools as we do in this course.

3. Third, I would like to learn more about your trust or belief in computer- and calculator-generated results.

Mathematical Modeling

Have you ever heard of mathematical modeling?
- If yes, in what context?
  - In class: Could you tell me about that experience?
  - Outside class: Could you tell me about that experience?
- If no, what do you think mathematical modeling might be?

What did you think of the three modeling problems that you were asked to "solve" in the questionnaire?
- If they were hard, in what way?
  - What were you missing, or what did you need to make them less hard?
- If they were impossible, how is that?
- If they were easy, have you been modeling this kind of problem before?

Technology

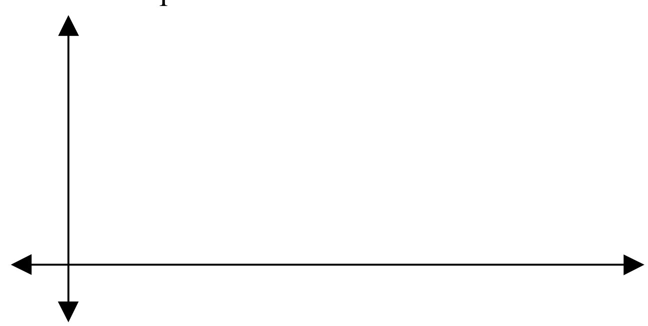
What kind of calculator do you use? How much do you know about it?

Do you have a computer at home?
  If yes, what kind of computer? What kind of software do you run on it?
  If no, do you have access to computers elsewhere?

Modeling Question

[The student was given a sheet of paper with the following question and asked to think aloud while solving it.]

Consider an ordinary videotape placed in a videotape recorder. When the tape is played, it is transferred from one reel to the other with *constant* speed. Illustrate with a graph the change in the radius of the roll of tape on the first reel.



Calculator Questions

[The student was given a sheet of paper with the following two questions. Below each question are the follow-up questions that were asked.]

Calculate $\dfrac{123456 \cdot 10^4 - 1}{10^9 - 1}$ with your calculator.

What did you get?
Is it true?
How do you know?
How sure are you?

Calculate $28{,}923{,}761^2 - 28{,}923{,}760^2$ with your calculator.

What did you get?
Is it true?
How do you know?
How sure are you?

APPENDIX H

SECOND INTERVIEW PROTOCOL FOR STUDY 2

Introduction

I'm glad that we have this second opportunity to talk. I hope you can forget the video camera I have running in the background, but please respond in a clear, distinct voice so that later I can analyze the interview on the videotape.

Background

There are several reasons that I have scheduled this second interview with all of you this week.

1. You have been solving different modeling problems during this course, using different computer tools and graphing calculators. I now want to know more about how you relate mathematical models to reality when you are using all this technological aid that is available.

2. I want to know more about how you consider and react to real-world problems that you are given to model.

Questions

What do you think of mathematical modeling now that you have solved quite a few problems during the course and all the time have been using computer and graphing-calculator tools?

[Here I referred to all the written work done by the student up to that point in the course. Together, we went over the assignments the student had done.]

Are you comfortable with the computing tools?

Problem

I will give you a real-world problem I want you to start thinking about, describing your solution strategies to me. Please tell me what you know that you can relate to the problem at hand. Tell me what tools you intend to use and how. I will not tell you how to solve the problem, but I will help you to deal with commands and error messages.

It is important that you understand that I expect you to do as much as possible on your own. Since I'm more interested in what you do than that you reach a possible "right" answer, I will not always give you direct answers. [At this point, I gave the student Worksheet 1.]

Worksheet 1

Please consider the following known facts. How do you want to go ahead and work with these statistics?

Gold Medallists in the Women's 200-Meter Event

Year	***Name, Country***	***Time in seconds***
1988	**F. Griifith-Joyner, USA**	**21.34**
1984	**V. Brisco-Hooks, USA**	**21.81**
1980	**B. Wockel, East Germany**	**22.03**
1976	**B. Eckert, East Germany**	**22.37**
1972	**R. Stecher, East Germany**	**22.40**
1968	**I. Szewinska, Poland**	**22.5**
1964	**E. McGuire, USA**	**23.0**
1960	**W. Rudolph, USA**	**24.0**
1956	**B. Cuthbert, Australia**	**23.4**
1952	**M. Jackson, Australia**	**23.7**
1948	**F. Blankers-Koen, Netherlands**	**24.4**

[Worksheet 2 was given out after the student had been working approximately 15 minutes.]

Worksheet 2

Consider the following additional known facts. How do you want to go ahead and work with these statistics? How does it affect your strategies from the first data set?

Gold Medallists in the Men's 200-Meter Event

Year	***Name, Country***	***Time in seconds***
1988	**J. DeLoach, USA**	**19.75**
1984	**C. Lewis, USA**	**19.80**
1980	**P. Minnea, Italy**	**20.19**
1976	**D. Quarrie, Jamaica**	**20.23**
1972	**V. Borzov, USSR**	**20.00**
1968	**T. Smith, USA**	**19.83**
1964	**H. Carr, USA**	**20.3**
1960	**L. Berruti, Italy**	**20.5**
1956	**B. Marrow, USA**	**20.6**
1952	**A. Stanfield, USA**	**20.7**
1948	**M. Patton, USA**	**21.1**
1936	**J. Owens, USA**	**20.7**
1932	**E. Tolan, USA**	**21.1**
1928	**P. Williams, USA**	**21.8**

1924	**J. Scholtz, USA**	**21.6**
1920	**A. Woodring, USA**	**22.0**
1912	**R. Craig, USA**	**21.7**
1908	**R. Kerr, Canada**	**22.6**
1904	**A. Hahn, USA**	**21.6**
1900	**W. Tewksbury**	**22.2**

[When the student had answered the questions on Worksheet 2, I gave him or her a copy of Worksheet 3, which contained questions requiring a comparison of data from the two previous worksheets.]

Worksheet 3

Do you think that women may soon outrun men? How fast do you think women and men will run in 100 years? In 200 years?

Can you construct a mathematical model that will allow you to make predictions and comparisons of the speeds for women and men in the 200-meter run of future Olympic Games?

[After the student had had a chance to work on these problems giving answers in writing and aloud, I asked some follow-up questions.]

When I asked you about the relationship that you thought might exist among the statistics you got in Worksheets 1 and 2, and what model you were thinking of, you told me the following—. [At this point, I would show the students my handwritten notes and the computer or calculator display of their work and would ask them to discuss their response. My purpose was to verify whether I had understood the response correctly and to see if they had additional comments to make about it.]

Please use a computer, a graphing calculator, or just your paper and pencil and tell me more about what a mathematical model of that relationship would look like?

Would you go ahead and construct a model of the kind you just described? Please tell me what you are doing while you work.

How good do you think this model is? Are there any changes that you would like to make in the model? Why is that?

In what way does the model support your assumptions from before? Does the model help you verify the possible outcome that you suggested earlier? How might you test the validity of the model?

APPENDIX I

SELECTED ASSIGNMENTS FOR STUDY 2

Assignment 1B

Let $R(1) = 1$, and for $k \geq 1$, $k \in \mathrm{N}$,

set $R(k+1) = 1 + \dfrac{k}{R(k)}$.

Sketch $R(k)$.

Investigate what happens with $R(k)$ when $k \to \infty$.

Report your analysis and provide an argument for your choice of method.

Assignment 1C

A successful computer firm has had the following average number of employees per year:

Year	1986	1987	1988	1989	1990	1991	1992	1993
Employees	10	2	14	17	20	24	29	34

Use your graphing calculator with the tools for curve fitting that are in there and try to predict the number of employees the year 1994 and the year 1998. How strong do you consider your prediction to be?

Assignment 2B

You have been hired as a consultant by the city council of Middletown, and you are now asked to write a report that predicts the population growth in Middletown from now until 2050. Your report should be written in a visually attractive and professional manner, with separate title page, introduction, and clear and distinct graphics, and with mathematical formulas preferably written with an equation editor and included in the body of the text. The report should be well written, easy to read, and at the same time scientifically correct. Papers should be organized in a folder, unstapled, and with every sheet marked with a page number, your name, and your federal ID number.

Perform your calculations mainly in Excel or another powerful spreadsheet, and collect your graphical illustrations from the same software. The report should contain a complete report of how you solved the problem with the help of Excel or the spreadsheet of your choice. Other technical aids like calculators or other mathematical software may be used but should be accounted for.

Part 1: Natural growth: $N(t) = N_0(1 + r)^t$

Table 1: Population Growth in Middletown

Year	t	Population
1960	0	6000
1970	10	8000
1980	20	9500
1990	30	10500

Start by calculating the annual percentage growth in Middletown during the 30-year period from 1960 to 1990. This growth will serve as a first estimate of the yearly rate of growth r during the long period from 1960 to 2050. The measured population in 1960 in Middletown is similarly a possible first estimate of the initial population, N_0.

Then use those estimates for r and N to construct a natural growth model that, as you see it, "best" describes the population growth in Middletown—that is, where the "error" is as small as possible. Below you will find a description of a possible error analysis. Then employ your model to calculate the number of inhabitants in Middletown in 2050. Your report should, in addition to the number of inhabitants in Middletown in 2050, also include a table for the population in 2000, 2010, 2020, 2030, and 2040.

Part 2: Linear growth: $N(t) = N_0 + p\cdot t$

In this case, N_0 is the initial population, and p the annual change in growth. Perform the same modeling process as in Part 1 but for a linear growth model.

Part 3: Bounded growth: $N(t) = \dfrac{M \cdot N_0}{N_0 + (M - N_0)\cdot 2^{-k\cdot M\cdot t}}$

Here N_0 is the initial population, k is a pure growth parameter, and M is an upper bound for the population of Middletown. Perform the same modeling process as in Parts 1 and 2 but for a bounded growth model.

Error Analysis

For every model you construct, calculate a measure of the error between your model and the given data according to the following scheme:

1. Square the difference between given and estimated value.
2. Sum the squares.
3. Divide the sum by the number of data points.
4. Calculate the square root of the value obtained in 3.

The value yielded by this analysis, based on the values from Table 1 and from your tested functions, may be used to measure the error between the two data sets. Naturally it is important to minimize this error.

APPENDIX J

FINAL EXAM PROBLEMS FOR STUDY 2

Problem 2

The hourglass to the right (see Figure 1) consists of two identical symmetric cones, where the cone's radius is equal to its height. A narrow opening through which a colored fluid flows from the upper cone to the lower cone connects the two cones. When one turns the hourglass, it takes 7½ minutes for the fluid to complete the transfer from one cone to the other. The draining velocity is 0.5 cm^3/sec. The flow is considered to be constant, and the height of the opening is insignificant.

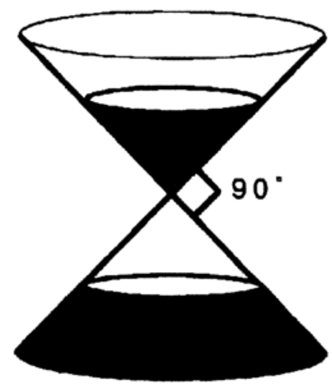


Figure 1

In order to mark a scale on the hourglass, the following measurements were registered for the lower cone:

Volume (cm^3)	Height of fluid (cm)
25	0.23
50	0.48
75	0.76
100	1.06
150	1.82
200	3.08

Give a complete description of how to mark the two cones of the hourglass so that the time can be read from it from either the upper or lower cone.

Problem 3 (from Edwards & Hamson, 1996, pp. 153-154)

a. In an experiment to investigate how the weight of a tomato crop can be increased by applying fertilizer, the yield ***y*** (kg/m^2) for various amounts ***x*** (kg/m^2) of fertilizer was found to be as follows:

Fertilizer, *x* (kg/m^2)	0	0.2	0.5	0.8	1.0	1.5
Profit, *y* (kg/m^2)	1.00	1.20	1.40	1.50	1.55	1.65

Fit a model of the form $y = L + (y_0 - L)e^{-kx}$ where ***L*** is the value that ***y*** approaches as ***x*** approaches infinity.

b. To investigate the effect of temperature on the yield, a number of plots were maintained at different temperatures and gave the following results:

Temperature, *T* (°C)	10	15	20	25	30
Profit, *y* (kg/m^2)	1.0	2.5	4.0	4.5	4.8

Fit a similar model to that of Part **(a)**, relating ***y*** and ***T.***

c. The total cost for heating, ***C*** (£/m^2) at constant temperature ***T*** was found to be as follows:

Temperature, *T* (°C)	10	15	20	25	30
Cost, *C* (£/m^2)	10	25	45	70	100

Fit the simplest model for ***C*** as a function of ***T***.

d. If the cost of fertilizer is £10 per kg and each kg of tomatoes is worth £5, write down a model for the net value of the crop obtained from one m^2 of compost in terms of ***x*** and ***T***.

APPENDIX K

ENTRY QUESTIONNAIRE FOR STUDY 3

General Information

Name:

Date of birth (y/m/d):

Address:

Phone:

E-mail:

All data you report will be treated confidentially.

Mathematics Information

Explain what each of the following words and phrases means as it relates to mathematics. If a word or phrase is unfamiliar to you write *unfamiliar,* and then write your best guess as to what the word or phrase might mean.

function
proof
mathematical modeling

Technology Information

What brand(s) of calculator(s) do you have or have you used?

What features do or did the calculator(s) have?

For what do or did you use the calculator(s)?

Did you ever use a calculator in a mathematics course? Yes No
If yes, when?

Have you ever used a computer? Yes No

If you never used a computer, skip the next three questions:

What brand(s) of computer(s) have you used?

What features did the computer(s) have?

Did you ever use a computer in a mathematics course? Yes No
If yes, when?

Mathematical Modeling Achievement Test

Try to solve each of these problems. Show all of your work in the space provided. Please use the back of this paper if you need more space.

For the following problems, label the axes and sketch a reasonable graph showing how the dependent variable varies with the independent one. Actual values are not important here. If possible, give a function expression that describes your model.

1. The population of Sweden has varied according to the figures in the table below.

Year	1750	1800	1850	1900	1950	1990
Population (in millions)	1.78	2.35	3.48	5.14	7.04	8.59

2. You drop a chocolate bar from the top of the 330-meter-high Eiffel Tower in Paris. The distance of the bar above the ground depends on the number of seconds that have elapsed since you dropped it.

3. A culture of yeast is placed into a restricted enclosure to grow. The amount of yeast depends on the time and the living conditions.

4. Consider an ordinary videotape placed in a videotape recorder. When the tape is played, it is transferred from one reel to the other with *constant* speed. Illustrate with a graph the change in the radius of the roll of tape on the first reel.

5. Speaking on the radio at the end of 1980, the former German finance minister, Martin Bangemann, said, among other things, the following:

   The top value of the US dollar is now 3.47 DM. Two years ago, before the decline in the stock market, we had an exchange rate of $1 for 1.80 DM. That means that the U.S. dollar has lost almost 100% of its value.

   Critique Bangemann's statement and illustrate it with a mathematical representation.

APPENDIX L

FIRST INTERVIEW PROTOCOL FOR STUDY 3

Introduction

I'm glad that we have this opportunity to talk. I hope that you can forget the video camera I have running in the background, but please respond in a clear, distinct voice so that later I can analyze the interview on the videotape.

Background

There are several reasons that I have scheduled interviews with all of you the next two weeks:

1. You responded to a lot of questions in the questionnaire. Because you all have different backgrounds, it was hard to construct a questionnaire that would be equally appropriate for everyone. Therefore, I'm eager to make sure that I have understood your written responses to the questionnaire.

2. Second, I want to know more about your understanding of mathematical modeling in general and in particular how you think and reflect on mathematical modeling when we work with computing tools as we do in this course.

3. Third, I would like to learn more about your view of what authority you trust or believe in when you validate computer- and calculator-generated results.

Technology

What kind of calculator do you use? How much do you know about it?
  How much did you use calculators in your previous studies?
  Have you ever reflected on the accuracy of results we get from calculators?
  Has anyone ever mentioned anything about how a machine calculates?
  Do you have an opinion yourself about any major differences between how you calculate and a machine calculates?

Do you have a computer at home?
If yes, what kind of computer? What kind of software do you run on it?
Have you done any mathematical calculations on your computer?
Do you trust those results? How much?
If no, do you have access to computers elsewhere?

Calculator Questions

[The student was given a sheet of paper with the following three questions. Below each question are the follow-up questions that were asked.]

Calculate $(-8)^{2/3}$ with your calculator.

What did you get?
Is it true?
How do you know?
How sure are you?

Calculate $\dfrac{123456 \cdot 10^4 - 1}{10^9 - 1}$ with your calculator.

What did you get?
Is it true?
How do you know?
How sure are you?

Calculate $28{,}923{,}761^2 - 28{,}923{,}760^2$ with your calculator.

What did you get?
Is it true?
How do you know?
How sure are you?

What possible limitations in the way machines perform mathematics do you think could affect the answer to the three problems above?

[After a discussion I gave them the following problem:]

Use your graphing calculator and sketch the graph of

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

for x in a range first below and then above x = 1013.

APPENDIX M

SECOND INTERVIEW PROTOCOL FOR STUDY 3

Introduction

I'm glad that we have this second opportunity to talk. I hope you can forget the video camera I have running in the background, but please respond in a clear, distinct voice so that later I can analyze the interview on the videotape.

Background

There are several reasons that I have scheduled this second interview with all of you this week.

1. You have been solving different modeling problems during this first half of the course, using different computer tools and graphing calculators. I now want to know more about how you reflect on mathematical modeling when you use the assistance of advanced calculators and computers.

2. I want to know more about how you think about the results you get from the machine you are using and how you relate those results to our discussion in the first interview.

Questions

What do you think of mathematical modeling now that you have solved quite a few problems during the course and all the time have been using computer and graphing-calculator tools?

[Here I referred to all the written work done by the student up to that point in the course. Together, we went over the assignments the student had done.]

Are you comfortable with the computing tools?

Problem

I will give you a real-world problem I want you to start thinking about, describing your solution strategies to me. Please tell me what you know that you can relate to the problem at hand. Tell me what tools you intend to use and how. I will not tell you how to solve the problem, but I will help you to deal with commands and error messages. It is important that you understand that I expect you to do as much as possible on your own. Since I'm more interested in what you do than that you reach a possible "right" answer, I will not always give you direct answers. [At this point, I gave the student Worksheet 1. The table was adapted from *Collaborative Explorations for Algebra* by Pepe, Ray, and Langkamp, 1993.]

Worksheet 1

Please consider the following known facts. How do you want to go ahead and work with these statistics?

The table below shows some winning distances for the Olympic long jump from 1948 to 1988.

Year	Men (length in feet)
1948	25.7
1956	25.7
1964	26.5
1972	27.5
1980	28.3
1988	28.6

[Worksheet 2 was given out after the student had been working approximately 15 minutes while I asked some questions in addition to what the student were doing.]

Worksheet 2

Consider the following additional known facts. How do you want to go ahead and work with these statistics? How does it affect your strategies from the first data set?

Year	Men (length in feet)	Women (length in feet)
1948	25.7	18.7
1956	25.7	20.8
1964	26.5	22.2
1972	27.5	22.3
1980	28.3	23.2
1988	28.6	24.3

[When the student had answered the questions on Worksheet 2, I gave him or her a copy of Worksheet 3, which contained questions requiring a comparison of data from the two previous worksheets.]

Worksheet 3

Do you think that women may soon jump longer than men? How far do you think women and men will jump in 100 years? In 200 years?

[After the student had had a chance to work on these problems, giving answers in writing and aloud, I asked some follow-up questions.]

When I asked you about the relationship that you thought might exist among the statistics you got in Worksheets 1 and 2, and what model you were thinking of, you told me the following—. [At this point, I would show the students my handwritten notes and the computer or calculator display of their work and would ask them to discuss their response. My purpose was to verify whether I had understood the response correctly and to see if they had additional comments to make about it.]

Please use a computer, a graphing calculator, or just your paper and pencil and tell me more about what a mathematical model of that relationship would look like?

Would you go ahead and construct a model of the kind you just described? Please tell me what you are doing while you work.

How good do you think this model is? Are there any changes that you would like to make in the model? Why is that?

In what way does the model support your assumptions from before? Does the model help you verify the possible outcome that you suggested earlier? How might you test the validity of the model? Please give some opinions.

Let me add some questions to the ones we have been discussing. If the women's event had been held in the first recorded Olympics in 776 BC, what does your model predict for the women's long jump distance? For the men's long jump distance? Would you like to comment on your earlier opinions about the validity and consequences of your model?

When you try to answer all these questions, where would you say that you seek arguments? From within yourself and your total mathematical knowledge, or is it based somewhat on the limitations and possibilities of calculators and computers?

APPENDIX N

THIRD INTERVIEW PROTOCOL FOR STUDY 3

Introduction

I'm glad that we have this third opportunity to talk. I hope you can forget the video camera I have running in the background, but please respond in a clear, distinct voice so that later I can analyze the interview on the videotape.

Background

There are several reasons that I have scheduled this last interview with all of you this week.

1. You have been solving different modeling problems during this course, using different computer tools and graphing calculators. I now want to have a last opportunity to know more about how you reflect on mathematical modeling when you use the assistance of advanced calculators and computers.

2. I also want to know more about how you think about and look at the results you got during all the exercises, the assignments, and the take-home final exam in this course.

3. Finally, I'd like to ask you some questions about authority.

Questions

What is your opinion about mathematical modeling now that you have solved quite a few problems during the course and all that time have been using computer and graphing-calculator tools?

[Here I referred to all the written work done by the student including the take-home final exam. Together, we went over the assignments and the final exam the student had done.]

Are you even more comfortable with the computing tools now?

What do you think of the modeling problems in the take-home final exam? What did you do in order to solve them?

What did you think of the Gas Problem? What was difficult about it? How did you validate your model?
Would you like to discuss your own mathematical knowledge now that you have taken this course and are reflecting on your performance?

I like to you respond to some questions below. Please take your time. I want us to discuss your responses afterwards. [I gave the student a sheet with the following questions.]

What is your opinion about this way to study mathematics? Do you consider it to be:

Very hard?
Hard?
Rather easy?
Easy and attractive?

To what degree have you missed a conventional textbook in this course?

Very much?
Much?
Almost not at all?
Not at all?

To what degree do you usually trust your mathematics textbook?

Very much?
Much?
Almost not at all?
Not at all?

To what degree do you usually trust your mathematics teacher?

Very much?
Much?
Almost not at all?
Not at all?

To what level do you usually trust results from computer software and calculators?

Very much?
Much?
Almost not at all?
Not at all?

To what level has this course changed the your trust to different sources?

Very much?
Much?
Almost not at all?

Not at all?

When I ask you about sources of authority, what do you think of first? Second?

APPENDIX O

SELECTED ASSIGNMENTS FOR STUDY 3

Assignment 1B

Let $R(1) = 1$, and for $k \geq 1$, $k \in$ N,

set $R(k+1) = 1 + \dfrac{k}{R(k)}$.

Sketch $R(k)$.

Investigate what happens with $R(k)$ when $k \to \infty$.

Report your analysis and provide an argument for your choice of method.

Assignment 1C

The population of Sweden has varied according to the figures in the table below:

Year	1750	1800	1850	1900	1950	1990
Population (in millions)	1.78	2.35	3.48	5.14	7.04	8.59

Fit an adequate function to the data set, and predict the population of Sweden in the year 2000.

Assignment 2B

Table 1 illustrates the winning time for the gold medal in the women's 100-meter freestyle in the Olympic games during the last century.

Table 1

Year	Time
1912	82.2
1920	73.6
1924	72.4
1928	71.0
1932	66.8
1936	65.9
1948	66.3
1952	66.8
1956	62.0
1960	61.2
1964	59.5
1968	60.0
1972	58.59
1976	55.65
1980	54.79
1984	55.92
1988	54.93
1992	54.64
1996	54.50

Use the given data to create a mathematical model that can predict future results in general and also answer the question: What will the winning time be in the women's 100-meter freestyle in the 2000 Olympics?

APPENDIX P

FINAL EXAM PROBLEMS FOR STUDY 3

Problem 2

If a medicine is injected directly into the circulation of the blood, it will be assimilated relatively fast. With the help of advanced measuring devices, it is possible to measure values of the concentration of a medicine in the blood a short time after an injection. The following values of the concentration ***K(t)*** of a certain medicine were measured at the corresponding time of ***t*** (seconds):

$$t = [3,\ 9,\ 12,\ 18,\ 24,\ 30];$$
$$K(t) = [4.1,\ 4.3,\ 3.9,\ 3{,}4,\ 3.1,\ 2.7].$$

In order to be able to optimize different methods of treatment where this medicine is involved, a researcher would like to know at what time ***t*** the concentration is maximal. From earlier experiments she knows that the following relation usually describes a model for this situation quite well:

$$K(t) = c + a \cdot e^{-0.47 \cdot t} + b \cdot e^{-0.06 \cdot t}.$$

Your task is to:

a. Decide the time of maximum concentration.

b. Decide if it is possible to measure the total amount of injected medicine, and if so, use the model to give a good estimate. If you do not think it is possible to determine that amount, explain why and what is missing from the information in order to do so. Also, explain how you would measure the amount of injected medicine if that piece of missing information were provided.

In order to solve these two problems, you must find the ”best” mathematical model for the data you have. Your reasoning and solution process must include a relevant error analysis that supports your opinion about why your model and your solution are correct.

Problem 3 (from Edwards & Hamson, 1996, pp. 155-156)

Recently it has become more and more interesting to use natural gas to heat homes.

a. Table 1 gives the weekly gas consumption (m^3) and average outside temperature (°C) for a particular house before the installation of cavity wall insulation.

Table 1

Temperature (°C)	-1	0	2	4	5	7	10
Gas (m^3)	206.6	195.6	173.2	149.4	115.7	116.0	82.4

Construct the simplest possible model to describe the correlation between weekly gas consumption and outside temperature.

b. Table 2 gives similar data for the same house after insulation.

Table 2

Temperature (°C)	-1	0	1	3	6	8	10
Gas (m^3)	134.4	127.6	120.6	110.1	89.4	72.7	59.4

Construct the simplest possible model to describe the correlation between weekly gas consumption and outside temperature after insulation.

c. Table 3 gives monthly averages of the outside temperature at the location of this house from October to May.

Table 3

Month	O	N	D	J	F	M	A	M
°C	10.3	6.7	4.4	3.4	3.8	5.7	8.7	11.5

Find an appropriate model to describe the annual variation of the average temperature over the year.

d. Write an expression for the amount of gas saved in one year by having insulation, and calculate a numerical answer for the amount of gas saved.