Cohomology of Local Systems on Siegel Threefolds with Square-free Parahoric Level

by

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(Under the Direction of Daniel Litt)

Abstract

The present thesis provides a survey of results and techniques from the Langlands program which allow for the computation of the ℓ-adic cohomology of local systems on Shimura varieties in terms of automorphic representations. We implement this strategy for the group $G = \mathrm{GSp}_4$ and obtain results on the ℓ-adic cohomology of local systems on Siegel modular threefolds. We then specialize these results to the case of square-free parahoric level in order to obtain explicit computations using results of Rösner on depth zero parahoric restriction. We further specialize these results to obtain explicit computations of the ℓ-adic cohomology of local systems on the moduli of principally polarized Abelian surfaces with full level 2 structure, and resolve conjectures of Bergström-Faber-van der Geer in this case.

Index words: Automorphic forms, Cohomology of Shimura varieties, Eisenstein cohomology, Siegel modular threefolds

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Introduction

The main goal of this thesis is to give an essentially self-contained treatment of the explicit computation of the cohomology of local systems on modular curves and on Siegel modular threefolds, particularly in the case of square-free parahoric level. Several applications are given: conjectural congruences for (Siegel) modular forms, conjectural extensions of mixed motives appearing in the cohomology of Shimura varieties, and finally explicit computations around arithmetic statistics for point counts on Abelian varieties over finite fields, the last of which is the main focus of the first part of the thesis. The second part of the thesis, which comprises a majority of its length, is devoted to these explicit computations of cohomology.

The number of new results in this thesis is somewhat modest relative to its length. To wit, a large portion of the thesis is devoted to exposition around the methods used in the explicit computation of cohomology of local systems on Shimura varieties, which are surely familiar to experts in this topic and which are generously scattered throughout the present literature. That being said, there are a few reasons why we have gone to such lengths to exposit this material:

- Although there is an extensive literature on the cohomology of Shimura varieties and the methods used to compute this, most sources focus only on individual parts of these computations rather than giving a comprehensive treatment of the topic: many sources focus on the intersection cohomology or the Eisenstein cohomology alone, and many sources only consider the case of the trivial local system rather than more general local systems. The main issue is that there are enough differences in notation and conventions between various treatments that it is difficult to combine individual parts of the literature without substantial refactoring. This seems to be a common theme: for example as Langlands-Ramakrishnan remark in their volume on the zeta functions of Picard modular surfaces [75], to which many authors contributed:

> "We have, in our editorial capacity, forgone any attempt to impose absolute uniformity of notation or definitions on the authors; it would have been beyond our powers of persuasion. The authors, in additional to their particular goals also had this common topic in mind, and were by and large consistent, but the reader is appraised that the effort of reconciling all signs and all possible variants of the defining data is left to him, if his own purposes require it. The editors try occasionally in this account to mediate between conflicting notations, but in no systematic way, nor do they guarantee their interpretations are always correct, or that they have been completely successful in keeping abreast of changes of notation in successive versions of texts"

In many ways, the same appologia apply to our own work, and there are surely some inconsistencies in notation which remain at the time of writing. That being said, our purposes require exactly the reconciliation which Langlands-Ramakrishnan are alluding to, and much of the expository work in this thesis is concerned with these efforts.

- Despite the extensive literature on the cohomology of Shimura varieties, there are surprisingly few sources which are concerned with complete and explicit computations. One reason for this is that there are some disconnects between the utility of the cohomology of Shimura varieties to the broader goals of the Langlands program, and the utility of such explicit computations to various applications in algebraic geometry. To the extent that a large part of the Langlands program is concerned with the construction of Galois representations attached to automorphic representations, computations involving the cohomology of non-compact Shimura varieties are in many ways a needlessly complicated way to realize this correspondence: indeed, any Galois representation which appears in the cohomology of non-compact Shimura varieties already appears in the cohomology of compact Shimura varieties, where many technical difficulties involving Eisenstein cohomology can be ignored. On the other hand, many applications to algebraic geometry require the consideration of various moduli spaces which happen to be non-compact Shimura varieties, and many experts in algerbaic geometry who would benefit from such explicit computations are not experts in the Langlands program, and such computations are prohibitively difficult without a great deal of familiarity with certain aspects of the Langlands program.
- Some of the existing literature pertaining to the cohomology of local systems on the moduli of principally polarized Abelian surfaces $\mathcal{A}_2$ are in some ways slightly unsatisfactory, although the present writing could hardly exist without them, and a great intellectual debt is owed to those who contributed to this literature.

The main sources are Harder's paper [55] treating the Eisenstein cohomology of local systems on $\mathcal{A}_2$ and Petersen's paper [96] on the cuspidal cohomology of local systems on $\mathcal{A}_2$ which relies heavily on Flicker's book [33] on automorphic representations for PGSp_4. Harder's paper does not adequately treat the case of Borel Eisenstein cohomology (the hardest case in this computation!) and merely states the answer; we end up taking a quite different approach compared to Harder's paper, relying on the cohomology of the Franke-Schwermer filtration rather than the cohomology of the boundary of the Borel-Serre compactification. Petersen's paper is quite satisfactory, but spends some amount of time remarking on various (arguably minor) typos in Flicker's book which would lead to false results if taken without modification. Moreover, the results of Flicker are inadequate for the computations involving level structure considered in this thesis, and we instead rely on more recent results of Gee-Taïbi [43] which provide the required endoscopic classification for automorphic representations of GSp_4 possibly with nontrivial central character as conjectured by Arthur [6], as the results of Flicker for automorphic representations of PGSp_4 can only treat the case of trivial central character.

We should also mention the enormous intellectual debt that is owed to Rösner's thesis [100] which more or less completely treats the computations of intersection and cuspidal cohomology treated in this thesis. In particular, many of the tables of representations appearing at the end of this thesis, particularly those related to representations of $\mathrm{GSp}_4(\mathbb{F}_q)$ and parahoric restrictions of depth 0 representations of $\mathrm{GSp}_4(F)$ are maybe a bit shamelessly borrowed from Rösner's thesis, if only for the sake of completeness of exposition. There are a few key differences:

- Rösner's thesis is largely concerned with the example of the cohomology of local systems on the moduli of principally polarized Abelian surfaces with full level 2 structure $\mathcal{A}_2[2]$, and does not analyze the contributions to cohomology coming from Soudry lifts, as these do not appear in this example. Since we are concerned with the cohomology of local systems on the moduli of principally polarized Abelian surfaces with full level N structure $\mathcal{A}_2[N]$ for N square-free more generally, we are forced to analyze this remaining case in some amount of detail.
- Rösner's thesis makes no attempt to justify the Galois action on intersection cohomology, and makes no mention of how the Langlands-Kottwitz method is used. Some of these details are treated in Weissauer's book on Siegel modular threefolds with particular focus on the case of Yoshida lifts, but much of the trace formula computations there make use of CAP localization which discards the terms which intervene in the remaining cases of Saito-Kurokawa lifts and Soudry lifts. Although we regrettably fall short of giving a comprehensive

treatment of the necessary background required to apply the Langlands-Kottwitz method, a portion of the relevant computations are included in this thesis.

- Rösner's thesis makes the unfortunate choice of using different notations for representations of $\mathrm{GSp}_4(\mathbb{F}_q)$ for q even and q odd, consistent with the differing notations between work of Enomoto [30] treating the former and work of Shinoda [109] treating the latter. At the risk of introducing more differing notations into the literature, we choose to use consistent notation for both of these situations, which we hope to clarify in future writing expositing the Deligne-Lusztig theory used to establish these results in the first place.

Finally we should mention the work of Grbac-Gröbner [48] which informed many of the computations of Eisenstein cohomology appearing in this thesis. Again there are a few key differences:

- The work of Grbac-Gröbner provides partial computations of the Eisenstein cohomology with coefficients in highest weight representations for Sp_4, over totally real number fields in general. By contrast, this thesis is concerned with the same computation for GSp_4, but only over $\mathbb{Q}$. We still run into the same issues of understanding certain connecting morphisms which obstruct complete computations as in work of Grbac-Gröbner; nevertheless, these uncertainties are isolated to a much smaller range when working over $\mathbb{Q}$, and in many cases these issues are resolved by considerations of cohomological dimension for congruence subgroups of $\mathrm{Sp}_4(\mathbb{Q})$. As we will explain later, the computations of Euler characteristics are unaffected by these issues.
- The work of Grbac-Gröbner makes no mention of the Galois action on Eisenstein cohomology; after all, their computation is for Sp_4 and topological in nature, and these considerations are only relevant for GSp_4 in the setting of Shimura varieties. Again, we fall a bit short of giving a satisfactory treatment of the Galois action on Eisenstein cohomology; one would have to expand the parabolic terms in the Arthur-Selberg trace formula as work of Laumon [76] does in the case of the constant local system, and then explain the application of the Langlands-Kottwitz method relating such parabolic terms for suitable test functions to the weighted intersection cohomology of local systems as in work of Morel [87], which is beyond the scope of the present thesis.

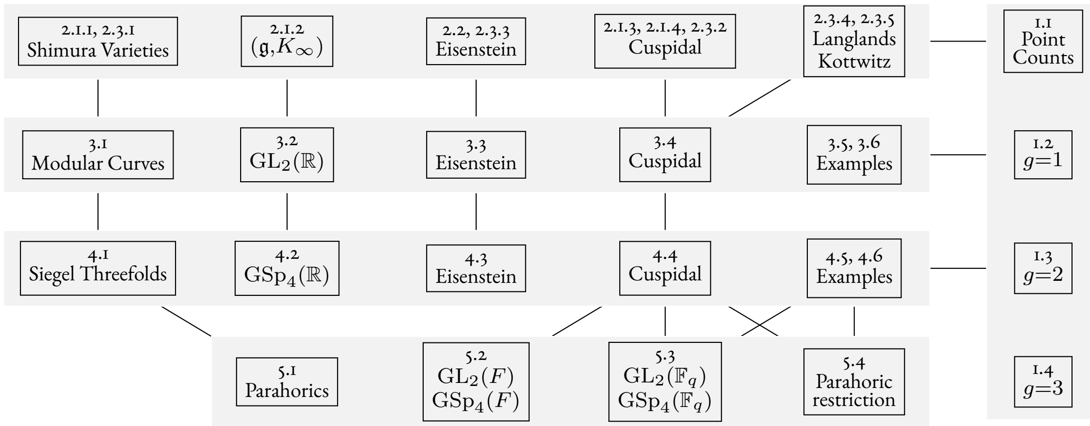
Having stated some of these caveats and comparisons to previous work, it is worth mentioning what is actually new:

- As mentioned above, we compute the Eisenstein cohomology of local systems on Siegel modular threefolds under certain assumptions on the behavior of connecting morphisms. The main results are 4.3.12 (following from 4.3.10), 4.3.21 (following from 4.3.19), and more notably 4.3.34 (following from 4.3.28).

- We verify the conjectures of Bergström-Faber-van der Geer [14] on the Euler characteristics of local systems on $\mathcal{A}_2[2]$, and strengthen these conjectures to a complete determination of the cohomology of local systems on $\mathcal{A}_2[2]$, up to the caveats mentioned above. There is strong computational evidence that the expression for the Euler characteristics of local systems on $\mathcal{A}_2[2]$ appearing in this thesis are correct; surprisingly, almost half the entries in the table of Euler characteristics appearing in work of Bergström-Faber-van der Geer differ substantially from our results, but our results agree completely with the outputs of the computer program these authors used to formulate these conjectures in the first place. Some of the discrepancies in their tables contradict known results of Harder and Petersen on the cohomology of local systems on $\mathcal{A}_2$, and some contradict results of Rösner on the cuspidal cohomology of local systems on $\mathcal{A}_2[2]$; by contrast the computer program (kindly shared by Bergström) has a quite robust implementation, relying on direct computations of point counts of genus 2 curves over finite fields. The main results are 4.6.5 (following from 4.6.1 and 4.6.4 which reproves results of Rösner) and 4.6.6, 4.6.7, 4.6.8 (under the same assumptions on the behavior of connecting morphisms).
- We reprove a result of Hoffman-Weintraub [57] on the cohomology of $\mathcal{A}_2[3]$: whereas the results of Hoffman-Weintraub explore this computation using rather explicit algebraic geometry, their computation takes the better part of a fourty page paper without much hope of generalizing to other local systems; by contrast, we treat the same computation in only a few pages. The main results are 4.7.4, 4.7.5.

Much remains to be done, especially around an adequate treatment of the Langlands-Kottwitz method for intersection cohomology and its generalization to weighted intersection cohomology, and explicit computations around the Arthur-Selberg trace formula and its stabilization required to execute this method.

Organization A general map of the sections of the thesis, and relations between them, can be displayed as follows:



The content of these sections is summarized as follows:

(i) Chapter 1 gives a gentler introduction to the problem of explicit computations of the cohomology of local systems on Shimura varieties by focusing on the example of the moduli stacks $\mathcal{A}_g$ of principally polarized Abelian varieties of dimension g, which are Siegel modular varieties of the simplest possible level. Results on the cohomology of local systems on $\mathcal{A}_1$ and $\mathcal{A}_2$, as well as conjectures of Bergström-Faber-van der Geer on the cohomology of local systems on $\mathcal{A}_3$ are reviewed in detail, and in each case we apply these results and conjectures to explicit computations around arithmetic statistics for point counts on Abelian varieties over finite fields.

(ii) Chapter 2 reviews and develops the main setup used in the discussion of the cohomology of local systems on Shimura varieties. Roughly speaking the computation of the cohomology of local systems on Shimura varieties breaks up into two main steps: the intersection and cuspidal cohomology, and the Eisenstein cohomology. In each of these cases one can consider the computation in the setting of singular or de Rham cohomology, or in the setting of ℓ-adic cohomology.

- Section 2.1 discusses the basic setup around Shimura varieties and the intersection cohomology of their minimal compactifications. In 2.1.1 we recall some basic definitions around Shimura varieties, their Hecke correspondences, and the local systems they carry. In 2.1.2 we recall the definition of $(\mathfrak{g}, K_\infty)$-cohomology and clarify some subtleties around central characters and component groups, and then explain how the de Rham cohomology of Shimura varieties can be computed as $(\mathfrak{g}, K_\infty)$-cohomology (albeit intractably). Moving towards a more tractable strategy, in 2.1.3 we recall the definition of L^2-

cohomology and then explain how the L^2-cohomology (respectively the cuspidal cohomology) of Shimura varieties is related to the $(\mathfrak{g}, K_\infty)$-cohomology of the automorphic discrete spectrum L^2_{disc} (respectively the automorphic cuspidal spectrum L^2_{cusp}) In 2.1.4 we recall the definition of intersection cohomology, and then recall Zucker's conjecture which relates L^2-cohomology to intersection cohomology.

- Section 2.2 discusses the basic setup around the Eisenstein cohomology of Shimura varieties as the $(\mathfrak{g}, K_\infty)$-cohomology of a slightly larger space of automorphic forms. In 2.2.1 we recall further results around automorphic forms and their decomposition according to cuspidal support, as well as reviewing necessary results on automorphic Eisenstein series. In 2.2.2 we recall the definition of the Franke-Schwermer filtration on spaces of automorphic forms defined in terms of residues and derivatives of automorphic Eisenstein series, and explain how the associated spectral sequence in $(\mathfrak{g}, K_\infty)$-cohomology computes the cohomology of local systems on Shimura varieties.
- Section 2.3 reinterprets the above constructions in the ℓ-adic setting, and sketches how the Galois action on the ℓ-adic cohomology of Shimura varieties is computed by the Langlands-Kottwitz method. In 2.3.1 we fix some assumptions on the existence of integral models of Shimura varieties and their minimal compactifications compatible with the previously constructed Hecke correspondences and local systems. In 2.3.2 we quickly recall the definition of ℓ-adic intersection cohomology and explain the compatibility with the previous construction of L^2-cohomology. In 2.3.3 we discuss the more subtle definition of ℓ-adic perverse sheaves carrying weight filtrations and the associated weighted t-structure, explain how the associated spectral sequence in weighted ℓ-adic cohomology computes the ℓ-adic cohomology of local systems on Shimura varieties, and explain the compatibility with the previous construction of Eisenstein cohomology. In 2.3.4 we explain some of the details around the Langlands-Kottwitz method the stabilization of Arthur's trace formula. In 2.3.5 we review Arthur's conjectures on the decomposition of the automorphic discrete spectrum L^2_{disc} in terms of global A-parameters and explicit multiplicity formulas.

(iii) Chapter 3 applies the setup from the second chapter to the case of modular curves. The main purpose of including this chapter is to provide a much more example of the general theory developed in the second chapter as a warmup before considering a much more complicated example in the next chapter.

- Section 3.1 discusses the basic setup around modular curves as Shimura varieties and their corresponding moduli problems, and addresses the construction of the local systems whose cohomology we are interested in.
- Section 3.2 collects results on discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$, including the computation of $(\mathfrak{g}, K_\infty)$-cohomology.
- Section 3.3 considers the Eisenstein cohomology of local systems on modular curves. In 3.3.1 we review and reprove results on the locations of poles of automorphic Eisenstein series for GL_2. In 3.3.2 we execute the relevant computations of $(\mathfrak{g}, K_\infty)$-cohomology, and explain what assumptions are made on the behavior of connecting morphisms which are not addressed in the present thesis.
- Section 3.4 considers the cuspidal cohomology of local systems on modular curves. We sketch how the Langlands-Kottwitz method is used to compute the relevant Galois representations.
- Section 3.5 quickly reproves the example of the cohomology of local systems on $\mathcal{A}_1$ in terms of the above results. We then remark on mod ℓ congruences for $\mathrm{SL}_2(\mathbb{Z})$ cusp forms and explain how this relates to the previously constructed ℓ-adic Galois representations.
- Section 3.6 explains a further example of the cohomology of local systems on $\mathcal{A}_1[2]$ in terms of the above results, this time making use of parahoric restriction for $\mathrm{GL}_2(\mathbb{Q}_2)$.

(iv) The fourth chapter applies the setup from the second chapter to the case of Siegel modular threefolds. This is the main computational content of the thesis.

- Section 4.1 discusses the basic setup around Siegel modular threefolds as Shimura varieties and their corresponding moduli problems, and addresses the construction of the local systems whose cohomology we are interested in.
- Section 4.2 collects results on representations of $\mathrm{Sp}_4(\mathbb{R})$ and $\mathrm{GSp}_4(\mathbb{R})$, and summarizes results on their $(\mathfrak{g}, K_\infty)$-cohomology, which are used heavily in later sections.
- Section 4.3 considers the Eisenstein cohomology of local systems on modular curves. In 4.3.1 we review and reprove results on the locations of poles of automorphic Eisenstein series for GSp_4. We then execute the relevant computations of $(\mathfrak{g}, K_\infty)$-cohomology, where the main computation involves three main cases, one for each standard parabolic $\mathbb{Q}$-subgroup of GSp_4. In 4.3.2 we treat the case of the Siegel parabolic subgroup, in 4.3.3 we treat the case of the Klingen parabolic subgroup, and in 4.3.5

we treat the case of the Borel parabolic subgroup. In all of these cases we explain what assumptions are made on the behavior of connecting morphisms which are not addressed in the present thesis.

- Section 4.4 considers the cuspidal cohomology of local systems on modular curves. In 4.4.1 we review the structure of the relevant Arthur parameters for GSp_4. In 4.4.2 we sketch some of the background needed to apply the Langlands-Kottwitz method for GSp_4, namely the choice of test functions and the basic shape of the trace formula and endoscopic character identities which appear. In 4.4.3 we review some definitions and results around local and global theta lifts which are used to construct the relevant packets of representations for GSp_4. We then execute the relevant computations of $(\mathfrak{g}, K_\infty)$-cohomology and applications of the Langlands-Kottwitz method, where the main computation involves six main cases, one for each type of A-parameter for GSp_4. In 4.4.4 we treat the case of general type A-parameters, in 4.4.5 we treat the case of endoscopic A-parameters and their parahoric restrictions, in 4.4.6 we treat the case of Siegel-CAP A-parameters and their parahoric restrictions. In 4.4.7 we treat the case of Klingen-CAP A-parameters, although we do not analyze their parahoric restriction in as much detail. In 4.4.8 we analyze the case of Borel-CAP A-parameters and explain why they cannot contribute to cohomology. Finally in 4.4.9 we treat the simplest case of 1-dimensional type A-parameters.
- Setion 4.5 quickly reproves the example of the cohomology of local systems on $\mathcal{A}_2$ in terms of the above results. We then recall Harder's conjectures on mod ℓ congruences for $\mathrm{Sp}_4(\mathbb{Z})$ Siegel cusp forms and explain how this relates to the previously constructed ℓ-adic Galois representations.
- Section 4.6 explains new results on the cohomology of local systems on $\mathcal{A}_2[2]$.
- Section 4.7 reproves some known results on the cohomology of $\mathcal{A}_2[3]$.

(v) The fifth chapter is an appendix which collects results from representation theory which are used in the third and especially the fourth chapters.

- Section 5.1 recalls definitions around Bruhat-Tits theory and parahoric subgroups. In 5.1.1 we quickly summarize the standard parahoric subgroups for GL_2 which are assumed in section 3.1, and in 5.1.2 we quickly summarize the standard parahoric subgroups for $\mathrm{GSp}_4(F)$ which are assumed in section 4.1.
- Section 5.2 quickly summarizes some results about representations of relevant p-adic groups. In 5.2.1 we summarize the classification of non-supercuspidal representations of $\mathrm{GL}_2(F)$ along with their L-

factors and ϵ factors. In 5.2.2 we summarize the classification of non-supercuspidal representations of $\mathrm{GSp}_4(F)$ along with their L-factors and ϵ factors.

- Section 5.3 summarizes some results about representations of relevant finite groups. In 5.3.1 we summarize the classification of representations of $\mathrm{GL}_2(\mathbb{F}_q)$ and related groups. In 5.3.2 we summarize the classification of representations of $\mathrm{GSp}_4(\mathbb{F}_q)$ (treating the cases of characteristic $p > 2$ and characteristic $p = 2$ separately but with uniform notation), in particular listing the various parabolic restrictions.
- Section 5.4 summarizes some results about depth 0 parahoric restrictions of relevant groups. In 5.4.1 we summarize the depth 0 parahoric restriction of representations of $\mathrm{GL}_2(F)$. In 5.4.2 we summarize the depth 0 parahoric restriction of representations of $\mathrm{GSp}_4(F)$.

Chapter 1

Arithmetic Statistics

Introduction

Let $[\mathcal{A}_g(\mathbb{F}_q)]$ be the set of isomorphism classes of principally polarized Abelian varieties of dimension g over $\mathbb{F}_q$. The cardinality $\#[\mathcal{A}_g(\mathbb{F}_q)]$ is finite; of course, for each $[A,\lambda] \in [\mathcal{A}_g(\mathbb{F}_q)]$ the cardinality $\#A(\mathbb{F}_q)$ is finite, and is constant in its isogeny class. One would like to understand how the point counts of principally polarized Abelian varieties over $\mathbb{F}_q$ distribute.

Experience informs us that such point counting problems are better behaved when weighted by the number of automorphisms. To that end let $\mathcal{A}_g(\mathbb{F}_q)$ be the groupoid of principally polarized Abelian varieties of dimension g over $\mathbb{F}_q$. Consider the groupoid cardinality

$$\#\mathcal{A}_g(\mathbb{F}_q) = \sum_{[A,\lambda]\in[\mathcal{A}_g(\mathbb{F}_q)]} \frac{1}{\#\mathrm{Aut}_{\mathbb{F}_q}(A,\lambda)}$$

For example, one has (classically for $g = 1$, by Lee-Weintraub [78, Corollary 5.2.3] for $g = 2$ and by Hain [53, Theorem 1] for $g = 3$):

$$\begin{aligned}
\#\mathcal{A}_1(\mathbb{F}_q) &= q \\
\#\mathcal{A}_2(\mathbb{F}_q) &= q^3 + q^2 \\
\#\mathcal{A}_3(\mathbb{F}_q) &= q^6 + q^5 + q^4 + q^3 + 1
\end{aligned}$$

Consider the natural probability measure $\mu_{\mathcal{A}_g(\mathbb{F}_q)}$ on $[\mathcal{A}_g(\mathbb{F}_q)]$ such that $[A,\lambda] \in [\mathcal{A}_g(\mathbb{F}_q)]$ has mass weighted by the number of automorphisms:

$$\mu_{\mathcal{A}_g(\mathbb{F}_q)}([A,\lambda]) = \frac{1}{\#\mathcal{A}_g(\mathbb{F}_q)\#\mathrm{Aut}_{\mathbb{F}_q}(A,\lambda)}$$

On the discrete probability space $([\mathcal{A}_g(\mathbb{F}_q)], 2^{[\mathcal{A}_g(\mathbb{F}_q)]}, \mu_{\mathcal{A}_g(\mathbb{F}_q)})$ consider the random variable $\#A_g(\mathbb{F}_q) : [\mathcal{A}_g(\mathbb{F}_q)] \to \mathbb{Z}$ assigning to $[A,\lambda] \in [\mathcal{A}_g(\mathbb{F}_q)]$ the point count $\#A(\mathbb{F}_q)$. Our goal is to understand, among other things, the expected values $\mathbb{E}(\#A_g(\mathbb{F}_q))$, and more generally the higher moments $\mathbb{E}(\#A_g(\mathbb{F}_q)^n)$ with respect to the natural probability measure $\mu_{\mathcal{A}_g(\mathbb{F}_q)}$.

For example, one has the expected values (classically for $g=1$, by Lee [79, Corollary 1.4] for $g=2$, and by 1.4.6 for $g=3$):

$$\begin{aligned}
\mathbb{E}(\#A_1(\mathbb{F}_q)) &= q+1 \\
\mathbb{E}(\#A_2(\mathbb{F}_q)) &= q^2+q+1-\frac{1}{q^3+q^2} \\
\mathbb{E}(\#A_3(\mathbb{F}_q)) &= q^3+q^2+q+1-\frac{q^2+q}{q^6+q^5+q^4+q^3+1}
\end{aligned}$$

and one has the expected values (classically for $g=1$, by Lee [79, Corollary 1.5] for $g=2$, and by 1.4.6 for $g=3$):

$$\begin{aligned}
\mathbb{E}(\#A_1(\mathbb{F}_q)^2) &= q^2+3q+1-\frac{1}{q} \\
\mathbb{E}(\#A_2(\mathbb{F}_q)^2) &= q^4+3q^3+6q^2+3q-\frac{5q^2+5q+3}{q^3+q^2} \\
\mathbb{E}(\#A_3(\mathbb{F}_q)^2) &= q^6+3q^5+6q^4+10q^3+6q^2+2q-2-\frac{8q^5+14q^4+12q^3+7q^2-2q-7}{q^6+q^5+q^4+q^3+1}
\end{aligned}$$

Many more expected values are computed and displayed in 1.2.4, 1.3.4, and 1.4.7 later in the paper.

The above expected values are obtained by applying the Grothendieck-Lefschetz trace formula to the ℓ-adic cohomology of the universal family of principally polarized Abelian varieties in order to produce the required point counts over finite fields. Let $\mathcal{A}_g$ be the moduli of principally polarized Abelian varieties of dimension g and let

$\pi : \mathcal{X}_g \to \mathcal{A}_g$ be the universal family of Abelian varieties over $\mathcal{A}_g$. Consider the n-fold fiber product:

$$\pi^n : \mathcal{X}_g^{\times n} := \underbrace{\mathcal{X}_g \times_{\mathcal{A}_g} \ldots \times_{\mathcal{A}_g} \mathcal{X}_g}_{n} \to \mathcal{A}_g$$

Then the expected value $\mathbb{E}(\#A_g(\mathbb{F}_q)^n)$ is related to the groupoid cardinality $\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)$:

$$\mathbb{E}(\#A_g(\mathbb{F}_q)^n) = \frac{\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)}{\#\mathcal{A}_g(\mathbb{F}_q)}$$

In order to compute the groupoid cardinalities $\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)$ it is enough to compute the compactly supported Euler characteristic $e_{\mathrm{c}}(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell) := \sum_{i \geq 0}(-1)^i H_{\mathrm{c}}^i(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$ as an element of the Grothendieck group of ℓ-adic Galois representations, in which case by applying the Grothendieck-Lefschetz trace formula we have:

$$\#\mathcal{X}_g^{\times n}(\mathbb{F}_q) = \mathrm{tr}(\mathrm{Frob}_q | e_{\mathrm{c}}(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)) := \sum_{i \geq 0}(-1)^i \mathrm{tr}(\mathrm{Frob}_q | H_{\mathrm{c}}^i(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell))$$

Note that since $\mathcal{X}_g^{\times n}$ is the complement of a normal crossings divisor of a smooth proper Deligne-Mumford stack over $\mathbb{Z}$ (see [31, Chapter VI, Theorem 1.1]), the ℓ-adic étale cohomology $H^i(\mathcal{X}_{g,\overline{\mathbb{Q}}}^{\times n}, \mathbb{Q}_\ell)$ is unramified for all primes $p \neq \ell$ (so that the action of Frob_p is well-defined) and is isomorphic to the ℓ-adic étale cohomology $H^i(\mathcal{X}_{g,\overline{\mathbb{F}}_p}^{\times n}, \mathbb{Q}_\ell)$ as a representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ factoring through the surjection $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Consequently we will use the cohomology over $\overline{\mathbb{Q}}$ and the cohomology over $\overline{\mathbb{F}}_p$ somewhat interchangeably, dropping either of these fields from the subscript whenever stating results which are true for both of these situations, as we have done above.

The computation requires three results: the first result 1.1.7, due to Deligne, involves the degeneration of the Leray spectal sequence computing $H^*(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$ in terms of the cohomology of the ℓ-adic local systems $\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell$ on $\mathcal{A}_g$, the second result 1.1.9 expresses the local systems $\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell$ in terms of the local systems $\mathbb{V}_\lambda$ on $\mathcal{A}_g$ corresponding to the irreducible representation of Sp_{2g} of highest weight λ, and the third result (1.2.1 for $g = 1$ due to Eichler-Shimura, 1.3.1 for $g = 2$ due to Lee-Weintraub and Petersen, and 1.4.1 for $g = 3$ due to Hain and Bergström-Faber-van der Geer) computes the cohomology of ℓ-adic cohomology of the local systems $\mathbb{V}_\lambda$ on $\mathcal{A}_g$. These results about the cohomology of local systems relies on the work of many people and results of the Langlands program as input.

Indeed, the expected values displayed so far might give the impression that the compactly supported Euler characteristics $e_c(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$ are Tate type, so that the point counts $\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)$ are polynomial in q. This is not true in general: the compactly supported Euler characteristics $e_c(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$ in general involve ℓ-adic Galois representations attached to vector-valued Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$, so that the point counts $\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)$ in general involve traces of Hecke operators on spaces of vector-valued Siegel modular forms. The relation between traces of Frobenius and traces of Hecke operators is ultimately obtained by the Langlands-Kottwitz method by comparing the Grothendieck-Lefschetz trace formula to the stabilization of the Arthur-Selberg trace formula [70]; while this strategy is overly sophisticated in the case $g = 1$, it is the strategy used in the work of Petersen [96] in the case $g = 2$ and by unpublished work of Taïbi [112] in the case $g \geq 3$.

Summary of Results For $g = 1, 2$ we know enough about the cohomology of local systems on $\mathcal{A}_g$ to compute $H^i(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$ as an ℓ-adic Galois representation (up to semisimplification). In the case $g = 1$ a classical result of Eichler-Shimura (see for example [15, Theorem 2.3]) implies the following result:

Theorem. 1.2.3 The cohomology $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ is Tate type for all i and all $1 \leq n \leq 9$. The cohomology $H^i(\mathcal{X}_1^{\times 10}, \mathbb{Q}_\ell)$ is Tate type for all $i \neq 11$, whereas for $i = 11$ we have

$$H^{11}(\mathcal{X}_1^{\times 10}, \mathbb{Q}_\ell) = \mathbb{S}_{\Gamma(1)}[12] + \mathbb{L}^{11} + 99\mathbb{L}^{10} + 1925\mathbb{L}^9 + 12375\mathbb{L}^8 + 29700\mathbb{L}^7$$

where $\mathbb{S}_{\Gamma(1)}[12]$ is the 2-dimensional ℓ-adic Galois representation attached to the weight 12 cusp form $\Delta \in S_{12}(\Gamma(1))$. In particular the compactly supported Euler characteristic $e_c(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 10$.

In the case $g = 2$ results of Lee-Weintraub [78, Corollary 5.2.3] and Petersen [96, Theorem 2.1] imply following result:

Theorem. 1.3.3 The cohomology $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ is Tate type for all i and all $1 \leq n \leq 6$. The cohomology $H^i(\mathcal{X}_2^{\times 7}, \mathbb{Q}_\ell)$ is Tate type for all $i \neq 17$, whereas for $i = 17$ we have

$$H^{17}(\mathcal{X}_2^{\times 7}, \mathbb{Q}_\ell) = \mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^{17} + 1176\mathbb{L}^{15} + 63700\mathbb{L}^{13} + 6860\mathbb{L}^{12} + 321048\mathbb{L}^{11} + 294440\mathbb{L}^{10} + \mathbb{L}^9$$

where $\mathbb{S}_{\Gamma(1)}[18]$ is the 2-dimensional ℓ-adic Galois representation attached to the weight 18 cusp form $f_{18} = \Delta E_6 \in S_{18}(\Gamma(1))$. In particular the compactly supported Euler characteristic $e_c(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 7$.

The cohomology groups $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for $1 \leq n \leq 10$ and $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for $1 \leq n \leq 7$ are displayed in the tables 1 and 2 at the end of the paper. The Euler characteristics $e_c(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for $1 \leq n \leq 10$ and $e_c(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for $1 \leq n \leq 7$ are displayed along with these theorems later in the paper.

In the case $g = 3$ there are precise conjectures of Bergström-Faber-van der Geer [15, Conjecture 7.1] about the compactly supported Euler characteristics of local systems on $\mathcal{A}_3$ as an element of the Grothendieck group of ℓ-adic Galois representations. These conjectures are now known at least for small highest weight λ using dimension formulas for spaces of vector-valued Siegel modular forms for $\mathrm{Sp}_6(\mathbb{Z})$ obtained by Taïbi [113]. These conjectures, along with a result of Hain [53, Theorem 1] implies the following result:

Theorem. 1.4.6 Assume conjectures 1.4.1 and 1.4.2. Then the Euler characteristic $e_c(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ is Tate type for all $1 \leq n \leq 5$. The compactly supported Euler characteristic $e_c(\mathcal{X}_3^{\times 6}, \mathbb{Q}_\ell)$ is given by:

$$\begin{aligned}
e_c(\mathcal{X}_3^{\times 6}, \mathbb{Q}_\ell) = {} & (\mathbb{L}^6 + 21\mathbb{L}^5 + 120\mathbb{L}^4 + 280\mathbb{L}^3 + 309\mathbb{L}^2 + 161\mathbb{L} + 32)\mathbb{S}_{\Gamma(1)}[0,10] \\
& + \mathbb{L}^{24} + 22\mathbb{L}^{23} + 253\mathbb{L}^{22} + 2024\mathbb{L}^{21} + 11362\mathbb{L}^{20} + 46613\mathbb{L}^{19} \\
& + 146665\mathbb{L}^{18} + 364262\mathbb{L}^{17} + 720246\mathbb{L}^{16} + 1084698\mathbb{L}^{15} + 1036149\mathbb{L}^{14} + 38201\mathbb{L}^{13} \\
& - 1876517\mathbb{L}^{12} - 3672164\mathbb{L}^{11} - 4024657\mathbb{L}^{10} - 2554079\mathbb{L}^{9} + 101830\mathbb{L}^{8} + 2028655\mathbb{L}^{7} \\
& + 2921857\mathbb{L}^{6} + 2536864\mathbb{L}^{5} + 1553198\mathbb{L}^{4} + 687157\mathbb{L}^{3} + 215631\mathbb{L}^{2} + 45035\mathbb{L} + 4930
\end{aligned}$$

where $\mathbb{S}_{\Gamma(1)}[0,10] = \mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^9 + \mathbb{L}^8$ is the 4-dimensional ℓ-adic Galois representation attached to the Saito-Kurokawa lift $\chi_{10} \in S_{0,10}(\Gamma(1))$ of the weight 18 cusp form $f_{18} = \Delta E_6 \in S_{18}(\Gamma(1))$. In particular the compactly supported Euler characteristic $e_c(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 6$.

The Euler characteristics $e_c(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ for $1 \leq n \leq 6$ are displayed along with these theorems later in the paper. In view of arguments by Bergström-Faber [12], using the classification results of Chevevier-Taïbi [26], these computations are unconditional for $1 \leq n \leq 5$ on the basis of point counts, and are only conditional on the Euler characteristic $e_c(\mathcal{A}_3, \mathbb{V}_{6,6,6}) = \mathbb{S}_{\Gamma(1)}[0,10] - \mathbb{L}^9 - \mathbb{L}^8 + 1$ in the case $n = 6$.

We have continued these computations until reaching the first modular contributions: in the case $g = 1$ the contribution is through the discriminant cusp form $\Delta \in S_{12}(\Gamma(1))$ which contributes the irreducible 2-dimensional ℓ-adic Galois representation $\mathbb{S}_{\Gamma(1)}[12]$, and in the case $g = 2$ and $g = 3$ the contributions are through the Saito-Kurokawa lift $\chi_{10} \in S_{0,10}(\Gamma(1))$ which contributes the irreducible 2-dimensonal ℓ-adic Galois representation $\mathbb{S}_{\Gamma(1)}[18]$. One can continue further, where for $g = 2$, in the case $n = 11$ we have contributions from the vector-valued Siegel modular forms $\chi_{6,8} \in S_{6,8}(\Gamma(1))$ and $\chi_{4,10} \in S_{4,10}(\Gamma(1))$ of general type (see [116, Section 25] for the relevant dimensions), which contribute the irreducible 4-dimensional ℓ-adic Galois representations $\mathbb{S}_{\Gamma(1)}[6, 8]$ and $\mathbb{S}_{\Gamma(1)}[4, 10]$ (see [120, Theorem I, Theorem II]). For $g = 3$, in the case $n = 9$ we have a contribution from an 8-dimensional ℓ-adic Galois representation $\mathbb{S}_{\Gamma(1)}[3, 3, 7]$ which decomposes into a 1-dimensional ℓ-adic Galois representation of Tate type and an irreducible 7-dimensional ℓ-adic Galois representation (see [15, Example 9.1]), which is explained by a functorial lift from the exceptional group G_2 predicted by [49]. This is to say that if one continues a bit further, one encounters more complicated ℓ-adic Galois representations in cohomology governing these arithmetic statistics. We end up using each of these contributions to deduce that $e_{\mathrm{c}}(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$ is not Tate type above a certain range.

Relation to Other Work Much work has been done regarding the cohomology of local systems on $\mathcal{M}_{g,n}$ and its compactification (see [95] for a survey, and for example [11], [12], [13], [16], [21], [23], [22], [45], [81]), and likewise for $\mathcal{A}_g$ and its compactifications (see [61] for a survey, and for example [15] [25], [50], [51], [53], [59], [60], [78], [96]).

The method we have used to investigate arithmetic statistics for varieties over finite fields is hardly new: it is explained very clearly by Lee [79] in the case $g = 2$, where the computations of $H^i(\mathcal{X}_2, \mathbb{Q}_\ell)$ and $H^i(\mathcal{X}_2^{\times 2}, \mathbb{Q}_\ell)$ appear. The computations in the case $g = 3$ are new, but use the same method. The theme of identifying in which range modular contributions appear in the cohomology of fiber powers of the universal Abelian variety represents a departure from this previous work.

The work of Achter-Altuğ-Garcia-Gordon [1] takes a rather different approach to the study arithmetic statistics for principally polarized Abelian varieties over $\mathbb{F}_q$, starting from a theorem of Kottwitz relating masses of isogeny classes to volumes of tori and twisted orbital integrals, and then relating these to a product of local factors $\nu_v([A, \lambda], \mathbb{F}_q)$ over all places v of $\mathbb{Q}$. By contrast, almost every result we have used about the Galois action on the ℓ-adic cohomology of local systems on $\mathcal{A}_g$ relies on the Langlands-Kottwitz method relating traces of Frobenius to traces of Hecke operators, starting from the same theorem of Kottwitz and ultimately relating this to the stabilization of the Arthur-Selberg trace formula. It may be interesting to relate these two approaches, for instance by reexamining the computations in this paper in terms of explicit computations of twisted orbital integrals.

1.1 Arithmetic Statistics and Cohomology of Moduli Stacks

We now explain the method we use to study point counts of Abelian varieties over finite fields in terms of the ℓ-adic cohomology of their moduli stacks, following Lee [79].

1.1.1 Groupoid Cardinality

Recall that a groupoid is a category such that every morphism is an isomorphism. A thin groupoid is a groupoid such that every morphism is an automorphism. Given a groupoid $\mathcal{G}$ we can produce a thin groupoid $[\mathcal{G}]$ by choosing representatives for each isomorphism class in $\mathcal{G}$. We can think of a thin groupoid as a set $[\mathcal{G}]$ of isomorphism classes of objects along with the automorphism group $\mathrm{Aut}_{\mathcal{G}}(X)$ for each object $X \in [\mathcal{G}]$.

Let $\mathcal{G}$ be a groupoid and let $[\mathcal{G}]$ be a corresponding thin groupoid. Consider the groupoid cardinality

$$\#\mathcal{G} = \sum_{X \in [\mathcal{G}]} \frac{1}{\#\mathrm{Aut}_{\mathcal{G}}(X)}$$

We say that $\mathcal{G}$ is tame if $\#\mathcal{G} < \infty$. For $\mathcal{G}$ a discrete groupoid on a set X we have $\#\mathcal{G} = \#X$. In general groupoid cardinality behaves like an Euler characteristic:

Proposition 1.1.1. Groupoid cardinality satisfies, and is determined by, the following properties:

(i) For the trivial groupoid $*$ with 1 object we have $\#* = 1$.

(ii) For groupoids $\mathcal{G}$ and $\mathcal{G}'$ with a homotopy equivalence $\mathcal{G} \simeq \mathcal{G}'$ we have $\#\mathcal{G} = \#\mathcal{G}'$.

(iii) For groupoids $\mathcal{G}$ and $\mathcal{G}'$ we have $\#(\mathcal{G} \amalg \mathcal{G}') = \#\mathcal{G} + \#\mathcal{G}'$.

(iv) For $\mathcal{G}' \to \mathcal{G}$ a degree n cover of groupoids we have $\#\mathcal{G}' = n\#\mathcal{G}$.

Example 1.1.2. Let G be a finite group and let BG be the groupoid with 1 object $*$ with $\mathrm{Aut}_{BG}(*) = G$. Then we have

$$\#BG = \frac{1}{\#G}$$

In general for a groupoid $\mathcal{G}$ we have a homotopy equivalence $\mathcal{G} \simeq \coprod_{X \in [\mathcal{G}]} B\mathrm{Aut}_{\mathcal{G}}(X)$, so we recover the definition of groupoid cardinality

$$\#\mathcal{G} = \# \coprod_{X \in [\mathcal{G}]} B\mathrm{Aut}_{\mathcal{G}}(X) = \sum_{X \in [\mathcal{G}]} \#B\mathrm{Aut}_{\mathcal{G}}(X) = \sum_{X \in [\mathcal{G}]} \frac{1}{\#\mathrm{Aut}_{\mathcal{G}}(X)}$$

Example 1.1.3. (Class formula) Let X be a set and let G be a finite group acting on X. Let $X//G$ be the action groupoid with objects X and with morphisms $\mathrm{Hom}_{X//G}(x, y) = \{g \in G | g(x) = y\}$. The projection $X \to X//G$ is not a cover unless G acts freely on X. Consider the cover $X \times EG \to (X \times EG)//G$ where $EG = G//G$ with G acting on itself by conjugation, which is a degree $\#G$ cover of groupoids since G acts freely on $X \times EG$. We have homotopy equivalences $X \times EG \simeq X$ and $(X \times EG)//G \simeq X//G$, so it follows that

$$\#X//G = \#(X \times EG)//G = \frac{\#(X \times EG)}{\#G} = \frac{\#X}{\#G}$$

Example 1.1.4. Let Fin be the groupoid of finite sets and bijections. Then we have

$$\#\mathrm{Fin} = \sum_{n \geq 0} \frac{1}{n!} = e$$

More generally let S be a finite set of cardinality $\#S = \lambda$ and let Fin^S be the groupoid of S-colored finite sets and S-colored bijections. Then we have

$$\#\mathrm{Fin}^S = \sum_{n \geq 0} \frac{\lambda^n}{n!} = e^\lambda$$

1.1.2 Moduli of Abelian Varieties

Let $\mathcal{A}_g$ be the moduli stack of principally polarized Abelian varieties of dimension g which is a smooth Deligne-Mumford stack of dimension $\dim(\mathcal{A}_g) = \frac{g(g+1)}{2}$ over $\mathbb{Z}$ (and hence over any $\mathbb{F}_q$ by base change) and let $\mathcal{A}_g(\mathbb{F}_q)$ be the groupoid of principally polarized Abelian varieties of dimension g over $\mathbb{F}_q$. Let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ be the universal family of Abelian varieties over $\mathcal{A}_g$. For $n \geq 1$ consider the n-th fiber power of the universal family

$$\pi^n : \mathcal{X}_g^{\times n} := \underbrace{\mathcal{X}_g \times_{\mathcal{A}_g} \ldots \times_{\mathcal{A}_g} \mathcal{X}_g}_{n} \to \mathcal{A}_g$$

which is a smooth Deligne-Mumford stack of dimension $\dim(\mathcal{X}_g^{\times n}) = \frac{g(g+1)}{2} + ng$ over $\mathbb{Z}$ (and hence over any $\mathbb{F}_q$ by base change). The fiber of $\pi^n : \mathcal{X}_g^{\times n} \to \mathcal{A}_g$ over a point $[A, \lambda] \in \mathcal{A}_g$ is the product A^n, so the point counts $\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)$ encode the point counts $\#A(\mathbb{F}_q)^n$ averaged over their moduli and weighted by the number of automorphisms.

By definition the expected value $\mathbb{E}(\#A_g(\mathbb{F}_q))$ of the random variable $\#A_g(\mathbb{F}_q)$ with respect the probability measure $\mu_{\mathcal{A}_g(\mathbb{F}_q)}$ defined in the introduction is given

$$\mathbb{E}(\#A_g(\mathbb{F}_q)^n) = \sum_{[A,\lambda] \in [\mathcal{A}_g(\mathbb{F}_q)]} \frac{\#A(\mathbb{F}_q)^n}{\#\mathcal{A}_g(\mathbb{F}_q)\#\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)}$$

which are related to the groupoid cardinality $\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)$ as follows:

Proposition 1.1.5. (Compare to [79, Lemma 6.8]) The expected value $\mathbb{E}(\#A_g(\mathbb{F}_q)^n)$ is given by

$$\mathbb{E}(\#A_g(\mathbb{F}_q)^n) = \frac{\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)}{\#\mathcal{A}_g(\mathbb{F}_q)}$$

Proof. Let $[A, \lambda] \in [\mathcal{A}_g(\mathbb{F}_q)]$ and consider the action of $\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)$ on A^n. Let $[A(\mathbb{F}_q)^n] = A(\mathbb{F}_q)^n // \mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)$ be the action groupoid. For $\underline{x} \in A(\mathbb{F}_q)^n$ let $\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda; \underline{x}) \subseteq \mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)$ be the subgroup stabilizing $\underline{x}$, and let $\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda) \cdot \underline{x}$ be the $\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)$-orbit of $\underline{x}$. By the orbit-stabilizer theorem we have

$$\sum_{[\underline{x}] \in [A(\mathbb{F}_q)^n]} \frac{1}{\#\mathrm{Aut}(A, \lambda; \underline{x})} = \sum_{[\underline{x}] \in [A(\mathbb{F}_q)^n]} \frac{\#(\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda) \cdot \underline{x})}{\#\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)} = \frac{\#A(\mathbb{F}_q)^n}{\#\mathrm{Aut}_{\mathbb{F}_q}(A, \lambda)}$$

It follows that

$$
\begin{aligned}
\mathbb{E}(\#A_g(\mathbb{F}_q)^n) &= \sum_{[A,\lambda]\in[\mathcal{A}_g(\mathbb{F}_q)]} \frac{\#A(\mathbb{F}_q)^n}{\#\mathcal{A}_g(\mathbb{F}_q)\#\mathrm{Aut}_{\mathbb{F}_q}(A,\lambda)} \\
&= \frac{1}{\#\mathcal{A}_g(\mathbb{F}_q)} \sum_{[A,\lambda]\in[\mathcal{A}_g(\mathbb{F}_q)]} \sum_{[\underline{x}]\in[A(\mathbb{F}_q)^n]} \frac{1}{\#\mathrm{Aut}_{\mathbb{F}_q}(A,\lambda;\underline{x})} \\
&= \frac{1}{\#\mathcal{A}_g(\mathbb{F}_q)} \sum_{[A,\lambda;\underline{x}]\in[\mathcal{X}_g^{\times n}(\mathbb{F}_q)]} \frac{1}{\#\mathrm{Aut}_{\mathbb{F}_q}(A,\lambda;\underline{x})} = \frac{\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)}{\#\mathcal{A}_g(\mathbb{F}_q)} \qquad \square
\end{aligned}
$$

Finally, we will consider the moment generating function

$$
M_{\#A_g(\mathbb{F}_q)}(t) = \sum_{n\geq 0} \mathbb{E}(\#A_g(\mathbb{F}_q)^n)\frac{t^n}{n!} = \sum_{n\geq 0} \frac{\#\mathcal{X}_g^{\times n}(\mathbb{F}_q)}{\#\mathcal{A}_g(\mathbb{F}_q)}\frac{t^n}{n!}
$$

1.1.3 Grothendieck-Lefschetz Trace Formula

Now let $\mathcal{X}$ be a Deligne-Mumford stack of finite type over $\mathbb{F}_q$, and fix a prime ℓ not dividing q. For $\mathbb{V}$ an étale $\mathbb{Q}_\ell$-sheaf on $\mathcal{X}$ along with a choice of $\mathbb{Z}_\ell$-lattice $\mathbb{V}_0$ write $H^i(\mathcal{X},\mathbb{V})$ for the ℓ-adic étale cohomology $H^i_{\mathrm{et}}(\mathcal{X}_{\overline{\mathbb{F}}_q},\mathbb{V}) = \varprojlim_n H^i_{\mathrm{et}}(\mathcal{X}_{\overline{\mathbb{F}}_q},\mathbb{V}_0/\ell^n)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ and write $\phi_q : H^i(\mathcal{X},\mathbb{V}) \to H^i(\mathcal{X},\mathbb{V})$ for the arithmetic Frobenius. Similarly, write $H^i_{\mathrm{c}}(\mathcal{X},\mathbb{V})$ for the compactly supported ℓ-adic étale cohomology $H^i_{\mathrm{c,et}}(\mathcal{X}_{\overline{\mathbb{F}}_q},\mathbb{V}) = \varprojlim_n H^i_{\mathrm{c,et}}(\mathcal{X}_{\overline{\mathbb{F}}_q},\mathbb{V}_0/\ell^n)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ and write $\mathrm{Frob}_q : H^i(\mathcal{X},\mathbb{V}) \to H^i(\mathcal{X},\mathbb{V})$ for the geometric Frobenius.

When $\mathcal{X}$ is smooth and has constant dimension the groupoid cardinality $\#\mathcal{X}(\mathbb{F}_q)$ can be computed by a Grothendieck-Lefschetz trace formula as the alternating sum of traces of arithmetic (geometric) Frobenius on the (compactly supported) ℓ-adic cohomology of $\mathcal{X}$:

Proposition 1.1.6. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack of finite type and constant dimension d over $\mathbb{F}_q$. Then we have

$$
\#\mathcal{X}(\mathbb{F}_q) = q^d \sum_{i\geq 0}(-1)^i \mathrm{tr}(\phi_q|H^i(\mathcal{X},\mathbb{Q}_\ell)) = \sum_{i\geq 0}(-1)^i\mathrm{tr}(\mathrm{Frob}_q|H^i_{\mathrm{c}}(\mathcal{X},\mathbb{Q}_\ell))
$$

Proof. The first equality follows by [8, Theorem 2.4.5], noting that the étale cohomology of Deligne-Mumford stacks agrees with the smooth cohomology used in this theorem. The second equality follows by Poincare duality (see [122, Proposition 2.30] for the case of Deligne-Mumford stacks), noting that $q^d\mathrm{tr}(\phi_q|H^i(\mathcal{X},\mathbb{Q}_\ell)) = \mathrm{tr}(\mathrm{Frob}_q|H_c^{2d-i}(\mathcal{X},\mathbb{Q}_\ell))$. $\square$

It follows that we have

$$\mathbb{E}(\#A_g(\mathbb{F}_q)^n) = \frac{\mathrm{tr}(\mathrm{Frob}_q|e_c(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell))}{\mathrm{tr}(\mathrm{Frob}_q|e_c(\mathcal{A}_g,\mathbb{Q}_\ell))} := \frac{\sum_{i\geq 0}(-1)^i\mathrm{tr}(\mathrm{Frob}_q|H_c^i(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell))}{\sum_{i\geq 0}(-1)^i\mathrm{tr}(\mathrm{Frob}_q|H_c^i(\mathcal{A}_g,\mathbb{Q}_\ell))}$$

It remains to compute the Euler characteristics $e(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell) := \sum_{i\geq 0}(-1)^i H^i(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell)$, or Poincare dually the compactly supported Euler characteristics $e_c(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell) := \sum_{i\geq 0}(-1)^i H_c^i(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell)$, as elements of the Grothendieck group of ℓ-adic Galois representations.

1.1.4 Leray Spectral Sequence

Now that we have related the moments $E(\#A_g(\mathbb{F}_q)^n)$ to the cohomology of $\mathcal{X}_g^{\times n}$, we would like to compute the cohomology of $\mathcal{X}_g^{\times n}$ in terms of the cohomology of local systems on $\mathcal{A}_g$. We observe that the Leray spectral sequence for the morphism $\pi^n : \mathcal{X}_g^{\times n} \to \mathcal{A}_g$ degenerates at the E_2-page, as it does for smooth projective morphisms of schemes:

Proposition 1.1.7. (Compare to [79, Proposition 2.8]) We have a spectral sequence

$$E_2^{i,j} = H^i(\mathcal{A}_g,\mathbb{R}^j\pi_*\mathbb{Q}_\ell) \Rightarrow H^{i+j}(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell)$$

which degenerates at the E_2-page, and we have a spectral sequence

$$E_2^{i,j} = H_c^i(\mathcal{A}_g,\mathbb{R}^j\pi_*\mathbb{Q}_\ell) \Rightarrow H_c^{i+j}(\mathcal{X}_g^{\times n},\mathbb{Q}_\ell)$$

which degenerates at the E_2-page.

Proof. Let $N \geq 3$ and let $\mathcal{A}_g[N]$ be the moduli stack of principally polarized Abelian varieties of dimension g with full level N structure, which is a smooth quasi-projective scheme over $\mathbb{Z}[\frac{1}{N}]$ (and hence over any $\mathbb{F}_q$ for $q = p^k$ with $p \nmid N$ by base change). Let $\pi : \mathcal{X}_g[N] \to \mathcal{A}_g[N]$ be the universal family of Abelian varieties over $\mathcal{A}_g[N]$. For

$n \geq 1$ consider the n-th fiber power of the universal family

$$\pi^n : \mathcal{X}_g[N]^{\times n} = \underbrace{\mathcal{X}_g[N] \times_{\mathcal{A}_g} \ldots \times_{\mathcal{A}_g} \mathcal{X}_g[N]}_{n} \to \mathcal{A}_g[N]$$

which is a smooth quasi-projective scheme over $\mathbb{Z}[\frac{1}{N}]$ (and hence over any $\mathbb{F}_q$ for $q = p^k$ with $p \nmid N$ by base change). Since $\pi^n : \mathcal{X}_g[N]^{\times n} \to \mathcal{A}_g[N]$ is a smooth projective morphism, the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathcal{A}_g[N], \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell) \Rightarrow H^{i+j}(\mathcal{X}_g[N]^{\times n}, \mathbb{Q}_\ell)$$

degenerates at the E_2-page (see for example [**?** , Proposition 2.4] and [29, Theorem 4.1.1]), so we have an isomorphism

$$\bigoplus_{i+j=k} H^i(\mathcal{A}_g[N], \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell) \simeq H^k(\mathcal{X}_g[N]^{\times n}, \mathbb{Q}_\ell)$$

of ℓ-adic Galois representations up to semisimplification. Now by the Hochschild-Serre spectral sequence [82, Theorem 2.20] for the $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$-quotient $\mathcal{A}_g[N] \to \mathcal{A}_g$ we have

$$H^i(\mathcal{A}_g[N], \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell)^{\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})} \simeq H^i(\mathcal{A}_g, \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell)$$

and by the Hochschild-Serre spectral sequence for the $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$-quotient $\mathcal{X}_g[N]^{\times n} \to \mathcal{X}_g^{\times n}$ (with $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ acting diagonally) we have

$$\bigoplus_{i+j=k} H^i(\mathcal{A}_g[N], \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell)^{\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})} \simeq H^k(\mathcal{X}_g[N]^{\times n}, \mathbb{Q}_\ell)^{\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})} \simeq H^k(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$$

so by naturality of the Leray spectral sequence we can take $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$-invariants and it follows that the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathcal{A}_g, \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell) \Rightarrow H^{i+j}(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell)$$

degenerates at the E_2-page. The proof for the Leray spectral sequence for compactly supported cohomology is similar, and follows by Poincare duality, noting that $\mathbb{R}^j\pi^n_!\mathbb{Q}_\ell \simeq \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell$ since π^n is proper. $\square$

Corollary 1.1.8. We have

$$e(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell) = \sum_{j \geq 0} (-1)^j e(\mathcal{A}_g, \mathbb{R}^j \pi_*^n \mathbb{Q}_\ell)$$

and we have

$$e_c(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell) = \sum_{j \geq 0} (-1)^j e_c(\mathcal{A}_g, \mathbb{R}^j \pi_*^n \mathbb{Q}_\ell)$$

as an element of the Grothendieck group of ℓ-adic Galois representations.

1.1.5 Künneth Formula

We can make one further simplification by using the Künneth formula to express the ℓ-adic sheaves $\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell$ in terms of the ℓ-adic sheaves $\mathbb{R}^j \pi_* \mathbb{Q}_\ell$:

Proposition 1.1.9. We have an isomorphism

$$\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell \simeq \bigoplus_{\substack{\lambda \vdash j \\ \lambda = (1^{j_1} \ldots n^{j_n})}} \bigotimes_{1 \leq i \leq n} \wedge^{j_i} \mathbb{V}$$

where $\mathbb{V} = \mathbb{R}^1 \pi_* \mathbb{Q}_\ell$ is the ℓ-adic local system on $\mathcal{A}_g$ whose fiber over $[A, \lambda] \in \mathcal{A}_g$ is $H^1(A, \mathbb{Q}_\ell)$ corresponding to the standard representation of Sp_{2g}.

Proof. By the Künneth formula (see [122, Corollary 2.20] for the case of Deligne-Mumford stacks) we have have an isomorphism $\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell \simeq \bigoplus_{j_1 + j_2 = j} (\mathbb{R}^{j_1} \pi_*^{n-1} \mathbb{Q}_\ell) \otimes (\mathbb{R}^{j_2} \pi_* \mathbb{Q}_\ell)$, so by induction on n it follows that

$$\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell \simeq \bigoplus_{\substack{\lambda \vdash j \\ \lambda = (1^{j_1} \ldots n^{j_n})}} \bigotimes_{1 \leq i \leq n} \mathbb{R}^{j_i} \pi_* \mathbb{Q}_\ell$$

Now the result follows since $\mathbb{R}^j \pi_* \mathbb{Q}_\ell \simeq \wedge^j \mathbb{V}$ is the ℓ-adic local sytem on $\mathcal{A}_g$ whose fiber over $[A, \lambda] \in \mathcal{A}_g$ is $H^j(A, \mathbb{Q}_\ell) \simeq \wedge^j H^1(A, \mathbb{Q}_\ell)$. □

For $\lambda = (\lambda_1 \geq \ldots \geq \lambda_g \geq 0)$ a highest weight for Sp_{2g} let $\mathbb{V}_\lambda$ be the ℓ-adic local system on $\mathcal{A}_g$ occurring in $\mathrm{Sym}^{\lambda_1 - \lambda_2}(\mathbb{V}) \otimes \ldots \otimes \mathrm{Sym}^{\lambda_{g-1} - \lambda_g}(\wedge^{g-1} \mathbb{V}) \otimes \mathrm{Sym}^{\lambda_g}(\wedge^g \mathbb{V})$ corresponding to the irreducible highest weight representation V_λ of Sp_{2g}. The tensor product of highest weight representations decomposes as a direct sum of highest weight representations with multiplicities

$$\mathbb{V}_\lambda \otimes \mathbb{V}_{\lambda'} = \bigoplus_{\lambda''} m_{\lambda,\lambda',\lambda''} \mathbb{V}_{\lambda''}$$

where the multiplicities $m_{\lambda,\lambda',\lambda''}$ can be computed in terms of Littlewood-Richardson coefficients and the image of the specialization morphism from the universal character ring (see [67, Theorem 3.1] and [68, Section 2.2], though we will not use this description in later computations).

It follows that we have a decomposition

$$\mathbb{R}^j \pi_*^n \mathbb{Q}_\ell = \bigoplus_{\lambda} \mathbb{V}_\lambda(\tfrac{|\lambda|-j}{2})^{\oplus m_\lambda^{j,n}}$$

where the $\mathbb{V}_\lambda$ are irreducible ℓ-adic local systems on $\mathcal{A}_g$ with multiplicity $m_\lambda^{j,n} \geq 0$, and where $|\lambda| = \lambda_1 + \ldots + \lambda_g$. Then we have

$$e_{\mathrm{c}}(\mathcal{X}_g^{\times n}, \mathbb{Q}_\ell) = \sum_{j \geq 0} (-1)^j \sum_{\lambda} m_\lambda^{j,n} e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{V}_\lambda)(\tfrac{|\lambda|-j}{2}) = \sum_{\lambda} f_\lambda^n(\mathbb{L}) e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{V}_\lambda)$$

as elements of the Grothendieck group of ℓ-adic Galois representations, where $f_\lambda^n(\mathbb{L}) = \sum_{j \geq 0} (-1)^j m_\lambda^{j,n} \mathbb{L}^{\frac{j-|\lambda|}{2}}$ is a polynomial in the Lefschetz motive $\mathbb{L} = \mathbb{Q}_\ell(-1)$, in which case by applying the Grothendieck-Lefschetz trace formula we obtain

$$\mathbb{E}(\#A(\mathbb{F}_q)^n) = \frac{\sum_\lambda \mathrm{tr}(\mathrm{Frob}_q | f_\lambda^n(\mathbb{L}) e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{V}_\lambda))}{\mathrm{tr}(\mathrm{Frob}_q | e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{Q}_\ell))} = \frac{\sum_\lambda f_\lambda^n(q) \mathrm{tr}(\mathrm{Frob}_q | e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{V}_\lambda))}{\mathrm{tr}(\mathrm{Frob}_q | e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{Q}_\ell))}$$

We have reduced the problem of computing the moments $E(\#A(\mathbb{F}_q)^n)$ to the problem of computing the multiplicities $m_\lambda^{j,n}$, and to the problem of computing the Euler characteristics $e_{\mathrm{c}}(\mathcal{A}_g, \mathbb{V}_\lambda)$ as elements of the Grothendieck group of ℓ-adic Galois representations. The first problem is straightforward, although it is perhaps not so easy to produce clean expressions for these coefficients except for small g. The second problem is more difficult: explicit computations are only known for $g = 1$ by results of Eichler-Shimura, for $g = 2$ by results of Lee-Weintraub [78] and Petersen [96], and for $g = 3$ by results of Hain [53] and conjectures of Bergström-Faber-van der Geer [15]. We will summarize these computations at the end of the paper.

1.2 Arithmetic Statistics for Elliptic Curves

In this section we will summarize what is known about the cohomology of local systems on $\mathcal{A}_1$, and then use this to deduce some results about arithmetic statistics for elliptic curves over finite fields.

1.2.1 Cohomology of Local Systems on $\mathcal{A}_1$

Let $\mathcal{A}_1$ be the moduli stack of elliptic curves, which is a smooth Deligne-Mumford stack of dimension 1 over $\mathbb{Z}$. Let $\pi : \mathcal{X}_1 \to \mathcal{A}_1$ be the universal elliptic curve over $\mathcal{A}_1$ and let $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}_\ell$ be the ℓ-adic local system on $\mathcal{A}_1$ corresponding to the standard representation of SL_2. For $\lambda \geq 0$ an integer let $\mathbb{V}_\lambda = \mathrm{Sym}^\lambda(\mathbb{V})$ be the ℓ-adic local system on $\mathcal{A}_1$ corresponding to the irreducible $\lambda + 1$-dimensional representation of SL_2. For λ odd we have $H^*(\mathcal{A}_1, \mathbb{V}_\lambda) = 0$ since $-\mathrm{id} \in \mathrm{SL}_2(\mathbb{Z})$ acts by multiplication by $(-1)^\lambda$ on the stalks of $\mathbb{V}_\lambda$.

Let $\mathbb{S}_{\Gamma(1)}[\lambda + 2] = \bigoplus_f \rho_f$ be the ℓ-adic Galois representation corresponding to cusp forms of weight $\lambda + 2$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$: for each eigenform $f \in S_{\lambda+2}(\Gamma(1))$ we have a 2-dimensional ℓ-adic Galois representation ρ_f, and we have

$$\mathrm{tr}(\mathrm{Frob}_p|\mathbb{S}_{\Gamma(1)}[\lambda + 2]) = \mathrm{tr}(T_p|S_{\lambda+2}(\Gamma(1)))$$

for every prime p, which determines $\mathbb{S}_{\Gamma(1)}[\lambda+1]$ as an element of the Grothendieck group of ℓ-adic Galois representations. The ℓ-adic Galois representation ρ_f is irreducible as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

By Deligne every eigenform $f \in S_{\lambda+2}(\Gamma(1))$ satisfies the Ramanujan conjecture: the roots of the characteristic polynomial

$$1 - \lambda_p(f)x + p^{2(\lambda+1)}x^2$$

of the Frobenius ϕ_p acting on the ℓ-adic Galois representation ρ_f for $\ell \neq p$, have absolute value $p^{-\frac{\lambda+1}{2}}$.

By work of Eichler-Shimura and Deligne we have the following:

Proposition 1.2.1. [15, Theorem 2.3] For $\lambda > 0$ even the compactly supported cohomology $H^*_c(\mathcal{A}_1, \mathbb{V}_\lambda)$ is concentrated in degree 1 where

$$H^1_c(\mathcal{A}_1, \mathbb{V}_\lambda) = \mathbb{S}_{\Gamma(1)}[\lambda+2] \oplus 1$$

Poincare dually, for $\lambda > 0$ even the cohomology $H^*(\mathcal{A}_1, \mathbb{V}_\lambda)$ is concentrated in degree 1 where

$$H^1(\mathcal{A}_1, \mathbb{V}_\lambda) = \mathbb{S}_{\Gamma(1)}[\lambda+2] \oplus \mathbb{L}^{\lambda+1}$$

In particular, for $\lambda > 0$ even we have

$$e_c(\mathcal{A}_1, \mathbb{V}_\lambda) = -\mathbb{S}_{\Gamma(1)}[\lambda+2] - 1$$

as an element of the Grothendieck group of ℓ-adic Galois representations. Poincare dually, for $\lambda > 0$ even we have

$$e(\mathcal{A}_1, \mathbb{V}_\lambda) = -\mathbb{S}_{\Gamma(1)}[\lambda+2] - 1$$

as an element of the Grothendieck group of ℓ-adic Galois representations.

This remains true for $\lambda = 0$ if we set $\mathbb{S}_{\Gamma(1)}[2] := -\mathbb{L} - 1$: the compactly supported cohomology $H^*_c(\mathcal{A}_1, \mathbb{Q}_\ell)$ is concentrated in degree 2 where $H^2_c(\mathcal{A}_1, \mathbb{Q}_\ell) = \mathbb{L}$. Poincare dually, the cohomology $H^*(\mathcal{A}_1, \mathbb{Q}_\ell)$ is concentrated in degree 0 where $H^0(\mathcal{A}_1, \mathbb{Q}_\ell) = 1$.

The identification of the action of Frobenius on ℓ-adic cohomology with the action of Hecke operators on spaces of modular forms can be understood geometrically in the following way. Let $\mathcal{T}_1(p)$ be the moduli stack of degree p isogenies of elliptic curves over $\mathbb{Z}$. We have the Hecke correspondence

$$\begin{tikzcd} & \mathcal{T}_1(p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{A}_1 & & \mathcal{A}_1 \end{tikzcd}$$

The Hecke correspondence induces an endomorphism T_p of $H^1(\mathcal{A}_1, \mathbb{V}_\lambda)$ which decomposes as $T_p = \mathrm{Frob}_p + \mathrm{Ver}_p$ where Frob_p is the Frobenius and Ver_p is the Verchiebung, and satisfies $\mathrm{Frob}_p\mathrm{Ver}_p = \mathrm{Ver}_p\mathrm{Frob}_p = p^{\lambda+1}$ so that $\mathrm{Ver}_p = p^{\lambda+1}\phi_p$ and $T_p = \phi_p^{-1} + p^{\lambda+1}\phi_p$, where ϕ_p is the absolute Frobenius.

On one hand the action of ϕ_p on the inner cohomology

$$H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda) = \mathrm{im}(H^1_c(\mathcal{A}_1, \mathbb{V}_\lambda) \to H^1(\mathcal{A}_1, \mathbb{V}_\lambda)) = \mathbb{S}_{\Gamma(1)}[\lambda+2]$$

is given as follows: the inner cohomology $H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda)$ admits an inner product such that F_p and V_p are adjoint, so we have

$$\begin{aligned}
\mathrm{tr}(\phi_p | H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda)) &= p^{-\lambda-1}\mathrm{tr}(\mathrm{Ver}_p | H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda)) \\
&= p^{-\lambda-1}\tfrac{1}{2}(\mathrm{tr}(\mathrm{Frob}_p | H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda)) + \mathrm{tr}(\mathrm{Ver}_p | H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda))) \\
&= p^{-\lambda-1}\tfrac{1}{2}\mathrm{tr}(T_p | H^1_!(\mathcal{A}_1, \mathbb{V}_\lambda)) \\
&= p^{-\lambda-1}\tfrac{1}{2}(\mathrm{tr}(T_p | S_{\Gamma(1)}[\lambda+2]) + \mathrm{tr}(\overline{T}_p | \overline{S}_{\Gamma(1)}[\lambda+2])) \\
&= p^{-\lambda-1}\mathrm{tr}(T_p | S_{\Gamma(1)}[\lambda+2])
\end{aligned}$$

On the other hand the action of ϕ_p on the Eisenstein cohomology

$$H^1_{\mathrm{Eis}}(\mathcal{A}_1, \mathbb{V}_\lambda) = \mathrm{coker}(H^1_c(\mathcal{A}_1, \mathbb{V}_\lambda) \to H^1(\mathcal{A}_1, \mathbb{V}_\lambda)) = \mathbb{E}_{\Gamma(1)}[\lambda+2]$$

is given as follows: since $E_{\Gamma(1)}[\lambda+2]$ is 1-dimensional T_p and ϕ_p act by multiplication by scalars. We have $T_p = \sigma_{\lambda+1}(p) = \sum_{d|p} d^{\lambda+1} = 1 + p^{\lambda+1}$ and $T_p = \phi_p^{-1} + p^{\lambda+1}\phi_p$ so that $\phi_p^{-1} + p^{\lambda+1}\phi_p = 1 + p^{\lambda+1}$ hence $\phi_p = 1$ or $\phi_p = p^{-\lambda-1}$. But $\phi_p = 1$ is excluded by considerations of weight, so we have

$$\mathrm{tr}(\phi_p | H^1_{\mathrm{Eis}}(\mathcal{A}_1, \mathbb{V}_\lambda)) = p^{-\lambda-1}$$

so that $\mathbb{E}_{\Gamma(1)}[\lambda+2] = \mathbb{L}^{\lambda+1}$.

We will use the following values for the Euler characteristics $e_c(\mathcal{A}_1, \mathbb{V}_\lambda)$, which are obtained by combining 1.2.1 with the vanishing of the spaces $S_{\lambda+2}(\Gamma(1))$ for all $\lambda \geq 0$ with $\lambda \leq 9$:

λ	$e_c(\mathcal{A}_1, \mathbb{V}_\lambda)$
0	$\mathbb{L}$
2	-1
4	-1

λ	$e_c(\mathcal{A}_1, \mathbb{V}_\lambda)$
6	-1
8	-1
10	$-\mathbb{S}_{\Gamma(1)}[12] - 1$

The space $S_{12}(\Gamma(1))$ is spanned by the discriminant cusp form

$$\Delta = \sum_{n \geq 1} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

which contributes an irreducible 2-dimensional ℓ-adic Galois representation $\mathbb{S}_{\Gamma(1)}[12]$ to $H^1(\mathcal{A}_1, \mathbb{V}_{10})$, with the property that $\mathrm{tr}(\mathrm{Frob}_p | \mathbb{S}_{\Gamma(1)}[12]) = \tau(p)$, which is not polynomial in p.

The Langlands correspondence predicts in this case that an irreducible 2-dimensional Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ contributing to the cohomology $H^*(\mathcal{A}_1, \mathbb{V}_\lambda)$ must come from a cuspidal automorphic representation π of $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ with π_∞ a holomorphic discrete series representation. As the $(\mathfrak{sl}_2, \mathrm{U}(1))$-cohomology of such discrete series representations is concentrated in middle degree 1 by [119], such a contribution can only occur in $H^1(\mathcal{A}_1, \mathbb{V}_\lambda)$.

1.2.2 Examples: Cohomology of $\mathcal{X}_1^{\times n}$ through $n = 10$

In this section we compute $H^*(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for all $n \geq 0$. The case $n = 10$ is the first case where $H^*(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ is not of Tate type, owing to a contribution from the discriminant cusp form $\Delta \in S_{12}(\Gamma(1))$.

We start by computing the local systems $\mathbb{R}^* \pi^n_* \mathbb{Q}_\ell = \bigoplus_{0 \leq j \leq 2n} \mathbb{R}^j \pi^n_* \mathbb{Q}_\ell$ up to $n = 10$. For this it suffices to consider the local systems $\mathbb{R}^j \pi_* \mathbb{Q}_\ell$ for $0 \leq j \leq n$ even with Tate twists omitted since $\mathbb{R}^j \pi^n_* \mathbb{Q}_\ell$ is pure of weight j (so the missing Tate twists can be inferred from the weights of the local systems $\mathbb{V}_\lambda$) and $\mathbb{R}^{2n-j} \pi^n_* \mathbb{Q}_\ell \simeq \mathbb{R}^j \pi^n_* \mathbb{Q}_\ell(-n-j)$.

Proposition 1.2.2. (i) The local system $\mathbb{R}^*\pi_*\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,2$ and given by

j	$\mathbb{R}^j\pi_*\mathbb{Q}_\ell$
0	$\mathbb{V}_0$

(ii) The local system $\mathbb{R}^*\pi_*^2\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,4$ and given by

j	$\mathbb{R}^j\pi_*^2\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$\mathbb{V}_2 + 3\mathbb{V}_0$

(iii) The local system $\mathbb{R}^*\pi_*^3\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,6$ and given by

j	$\mathbb{R}^j\pi_*^3\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$3\mathbb{V}_2 + 6\mathbb{V}_0$

(iv) The local system $\mathbb{R}^*\pi_*^4\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,8$ and given by

j	$\mathbb{R}^j\pi_*^4\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$6\mathbb{V}_2 + 10\mathbb{V}_0$
4	$\mathbb{V}_4 + 15\mathbb{V}_2 + 20\mathbb{V}_0$

(v) The local system $\mathbb{R}^*\pi_*^5\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,10$ and given by

j	$\mathbb{R}^j\pi_*^5\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$10\mathbb{V}_2 + 15\mathbb{V}_0$
4	$5\mathbb{V}_4 + 45\mathbb{V}_2 + 50\mathbb{V}_0$

(vi) The local system $\mathbb{R}^*\pi_*^6\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,12$ and given by

j	$\mathbb{R}^j\pi_*^6\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$15\mathbb{V}_2 + 21\mathbb{V}_0$
4	$15\mathbb{V}_4 + 105\mathbb{V}_2 + 105\mathbb{V}_0$
6	$\mathbb{V}_6 + 35\mathbb{V}_4 + 189\mathbb{V}_2 + 175\mathbb{V}_0$

(vii) The local system $\mathbb{R}^*\pi_*^7\mathbb{Q}_\ell$ is concentrated in degrees $0,\ldots,14$ and given by

j	$\mathbb{R}^j\pi_*^7\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$21\mathbb{V}_2 + 28\mathbb{V}_0$
4	$35\mathbb{V}_4 + 210\mathbb{V}_2 + 196\mathbb{V}_0$
6	$7\mathbb{V}_6 + 140\mathbb{V}_4 + 588\mathbb{V}_2 + 490\mathbb{V}_0$

(viii) The local system $\mathbb{R}^*\pi^8_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 16$ and given by

j	$\mathbb{R}^j\pi^8_*\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$28\mathbb{V}_2 + 36\mathbb{V}_0$
4	$70\mathbb{V}_4 + 378\mathbb{V}_2 + 336\mathbb{V}_0$
6	$28\mathbb{V}_6 + 420\mathbb{V}_4 + 1512\mathbb{V}_2 + 1176\mathbb{V}_0$
8	$\mathbb{V}_8 + 63\mathbb{V}_6 + 720\mathbb{V}_4 + 2352\mathbb{V}_2 + 1764\mathbb{V}_0$

(ix) The local system $\mathbb{R}^*\pi^9_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 18$ and given by

j	$\mathbb{R}^j\pi^9_*\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$36\mathbb{V}_2 + 45\mathbb{V}_0$
4	$126\mathbb{V}_4 + 630\mathbb{V}_2 + 540\mathbb{V}_0$
6	$84\mathbb{V}_6 + 1050\mathbb{V}_4 + 3402\mathbb{V}_2 + 2520\mathbb{V}_0$
8	$9\mathbb{V}_8 + 315\mathbb{V}_6 + 2700\mathbb{V}_4 + 7560\mathbb{V}_2 + 5292\mathbb{V}_0$

(x) The local system $\mathbb{R}^*\pi^{10}_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 20$ and given by

j	$\mathbb{R}^j\pi^{10}_*\mathbb{Q}_\ell$
0	$\mathbb{V}_0$
2	$45\mathbb{V}_2 + 55\mathbb{V}_0$
4	$210\mathbb{V}_4 + 990\mathbb{V}_2 + 825\mathbb{V}_0$
6	$210\mathbb{V}_6 + 2310\mathbb{V}_4 + 6930\mathbb{V}_2 + 4950\mathbb{V}_0$
8	$45\mathbb{V}_8 + 1155\mathbb{V}_6 + 8250\mathbb{V}_4 + 20790\mathbb{V}_2 + 13860\mathbb{V}_0$
10	$\mathbb{V}_{10} + 99\mathbb{V}_8 + 1925\mathbb{V}_6 + 12375\mathbb{V}_4 + 29700\mathbb{V}_2 + 19404\mathbb{V}_0$

We now explain the entries of the table of $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for $1 \leq n \leq 10$. We consider the spectral sequence $E_2^{i,j} = H^i(\mathcal{A}_1, \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell) \Rightarrow H^{i+j}(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$.

(i) We have a contribution to H^0 from $H^0(\mathcal{A}_1, \mathbb{V}_0) = 1$. It follows that the E_2-page is given

$$\begin{array}{c|cc} 2 & \mathbb{L} & 0 \\ 0 & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

(ii) We have a contribution to H^1 from $H^1(\mathcal{A}_1, \mathbb{V}_2) = \mathbb{L}^3$. It follows that the E_2-page is given

$$\begin{array}{c|cc} 4 & \mathbb{L}^2 & 0 \\ 2 & 3\mathbb{L} & \mathbb{L}^3 \\ 0 & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

(iii) We the same contributions as before. It follows that the E_2-page is given

$$\begin{array}{c|cc} 6 & \mathbb{L}^3 & 0 \\ 4 & 6\mathbb{L}^2 & 3\mathbb{L}^4 \\ 2 & 6\mathbb{L} & 3\mathbb{L}^3 \\ 0 & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

(iv) We have a contribution to H^1 from $H^1(\mathcal{A}_1, \mathbb{V}_4) = \mathbb{L}^5$. It follows that the E_2-page is given

$$\begin{array}{c|cc} 8 & \mathbb{L}^4 & 0 \\ 6 & 10\mathbb{L}^3 & 6\mathbb{L}^5 \\ 4 & 20\mathbb{L}^2 & \mathbb{L}^5 + 15\mathbb{L}^4 \\ 2 & 10\mathbb{L} & 6\mathbb{L}^3 \\ 0 & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

(v) We the same contributions as before. It follows that the E_2-page is given

$$\begin{array}{c|cc} 10 & \mathbb{L}^5 & 0 \\ 8 & 15\mathbb{L}^4 & 10\mathbb{L}^6 \\ 6 & 50\mathbb{L}^3 & 5\mathbb{L}^6 + 45\mathbb{L}^5 \\ 4 & 50\mathbb{L}^2 & 5\mathbb{L}^5 + 45\mathbb{L}^4 \\ 2 & 15\mathbb{L} & 10\mathbb{L}^3 \\ 0 & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

(vi) We have a contribution to H^1 from $H^1(\mathcal{A}_1, \mathbb{V}_6) = \mathbb{L}^7$. It follows that the E_2-page is given

$$\begin{array}{c|cc} 12 & \mathbb{L}^6 & 0 \\ 10 & 21\mathbb{L}^5 & 15\mathbb{L}^7 \\ 8 & 105\mathbb{L}^4 & 15\mathbb{L}^7 + 105\mathbb{L}^6 \\ 6 & 175\mathbb{L}^3 & \mathbb{L}^7 + 35\mathbb{L}^6 + 189\mathbb{L}^5 \\ 4 & 105\mathbb{L}^2 & 15\mathbb{L}^5 + 105\mathbb{L}^4 \\ 2 & 21\mathbb{L} & 15\mathbb{L}^3 \\ 0 & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

(vii) We the same contributions as before. It follows that the E_2-page is given

14	$\mathbb{L}^7$	0
12	$28\mathbb{L}^6$	$21\mathbb{L}^8$
10	$196\mathbb{L}^5$	$35\mathbb{L}^8 + 210\mathbb{L}^7$
8	$490\mathbb{L}^4$	$7\mathbb{L}^8 + 140\mathbb{L}^7 + 588\mathbb{L}^6$
6	$490\mathbb{L}^3$	$7\mathbb{L}^7 + 140\mathbb{L}^6 + 588\mathbb{L}^5$
4	$196\mathbb{L}^2$	$35\mathbb{L}^5 + 210\mathbb{L}^4$
2	$28\mathbb{L}$	$21\mathbb{L}^3$
0	1	0
	0	1

(viii) We have a contribution to H^1 from $H^1(\mathcal{A}_1, \mathbb{V}_8) = \mathbb{L}^9$. It follows that the E_2-page is given

16	$\mathbb{L}^8$	0
14	$36\mathbb{L}^7$	$28\mathbb{L}^9$
12	$336\mathbb{L}^6$	$70\mathbb{L}^9 + 378\mathbb{L}^8$
10	$1176\mathbb{L}^5$	$28\mathbb{L}^9 + 420\mathbb{L}^8 + 1512\mathbb{L}^7$
8	$1764\mathbb{L}^4$	$\mathbb{L}^9 + 63\mathbb{L}^8 + 720\mathbb{L}^7 + 2352\mathbb{L}^6$
6	$1176\mathbb{L}^3$	$28\mathbb{L}^7 + 420\mathbb{L}^6 + 1512\mathbb{L}^5$
4	$336\mathbb{L}^2$	$70\mathbb{L}^5 + 378\mathbb{L}^4$
2	$36\mathbb{L}$	$28\mathbb{L}^3$
0	1	0
	0	1

(ix) We the same contributions as before. It follows that the E_2-page is given

18	$\mathbb{L}^9$	0
16	$45\mathbb{L}^8$	$36\mathbb{L}^{10}$
14	$540\mathbb{L}^7$	$126\mathbb{L}^{10} + 630\mathbb{L}^9$
12	$2520\mathbb{L}^6$	$84\mathbb{L}^{10} + 1050\mathbb{L}^9 + 3420\mathbb{L}^8$
10	$5292\mathbb{L}^5$	$9\mathbb{L}^{10} + 315\mathbb{L}^9 + 2700\mathbb{L}^8 + 7560\mathbb{L}^7$
8	$5292\mathbb{L}^4$	$9\mathbb{L}^9 + 315\mathbb{L}^8 + 2700\mathbb{L}^7 + 7560\mathbb{L}^6$
6	$2520\mathbb{L}^3$	$84\mathbb{L}^7 + 1050\mathbb{L}^6 + 3420\mathbb{L}^5$
4	$540\mathbb{L}^2$	$126\mathbb{L}^5 + 630\mathbb{L}^4$
2	$45\mathbb{L}$	$36\mathbb{L}^3$
0	1	0
	0	1

(x) We have a contribution to H^1 from $H^1(\mathcal{A}_1, \mathbb{V}_{10}) = \mathbb{S}_{\Gamma(1)}[12] + \mathbb{L}^{11}$. It follows that the E_2-page is given

20	$\mathbb{L}^{10}$	0
18	$55\mathbb{L}^{9}$	$45\mathbb{L}^{11}$
16	$825\mathbb{L}^{8}$	$210\mathbb{L}^{11} + 990\mathbb{L}^{10}$
14	$4950\mathbb{L}^{7}$	$210\mathbb{L}^{11} + 2310\mathbb{L}^{10} + 6930\mathbb{L}^{9}$
12	$13860\mathbb{L}^{6}$	$45\mathbb{L}^{11} + 1155\mathbb{L}^{10} + 8250\mathbb{L}^{9} + 20790\mathbb{L}^{8}$
10	$19404\mathbb{L}^{5}$	$\mathbb{S}_{\Gamma(1)}[12] + \mathbb{L}^{11} + 99\mathbb{L}^{10}$ $+1925\mathbb{L}^{9} + 12375\mathbb{L}^{8} + 29700\mathbb{L}^{7}$
8	$13860\mathbb{L}^{4}$	$45\mathbb{L}^{9} + 1155\mathbb{L}^{8} + 8250\mathbb{L}^{7} + 20790\mathbb{L}^{6}$
6	$4950\mathbb{L}^{3}$	$210\mathbb{L}^{7} + 2310\mathbb{L}^{6} + 6930\mathbb{L}^{5}$
4	$825\mathbb{L}^{2}$	$210\mathbb{L}^{5} + 990\mathbb{L}^{4}$
2	$55\mathbb{L}$	$45\mathbb{L}^{3}$
0	1	0
	0	1

Taking $H^k(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell) = \bigoplus_{i+j=k} H^i(\mathcal{A}_1, \mathbb{R}^j\pi_*^n\mathbb{Q}_\ell)$ yields the result.

Theorem 1.2.3. The cohomology $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ is Tate type for all i and all $1 \le n \le 9$. In this range the compactly supported Euler characteristics are given by:

$$
\begin{aligned}
e_c(\mathcal{X}_1, \mathbb{Q}_\ell) &= \mathbb{L}^2 + \mathbb{L} \\
e_c(\mathcal{X}_1^{\times 2}, \mathbb{Q}_\ell) &= \mathbb{L}^3 + 3\mathbb{L}^2 + \mathbb{L} - 1 \\
e_c(\mathcal{X}_1^{\times 3}, \mathbb{Q}_\ell) &= \mathbb{L}^4 + 6\mathbb{L}^3 + 6\mathbb{L}^2 - 2\mathbb{L} - 3 \\
e_c(\mathcal{X}_1^{\times 4}, \mathbb{Q}_\ell) &= \mathbb{L}^5 + 10\mathbb{L}^4 + 20\mathbb{L}^3 + 4\mathbb{L}^2 - 14\mathbb{L} - 7 \\
e_c(\mathcal{X}_1^{\times 5}, \mathbb{Q}_\ell) &= \mathbb{L}^6 + 15\mathbb{L}^5 + 50\mathbb{L}^4 + 40\mathbb{L}^3 - 30\mathbb{L}^2 - 49\mathbb{L} - 15 \\
e_c(\mathcal{X}_1^{\times 6}, \mathbb{Q}_\ell) &= \mathbb{L}^7 + 21\mathbb{L}^6 + 105\mathbb{L}^5 + 160\mathbb{L}^4 - 183\mathbb{L}^2 - 139\mathbb{L} - 31 \\
e_c(\mathcal{X}_1^{\times 7}, \mathbb{Q}_\ell) &= \mathbb{L}^8 + 28\mathbb{L}^7 + 196\mathbb{L}^6 + 469\mathbb{L}^5 + 280\mathbb{L}^4 - 427\mathbb{L}^3 - 700\mathbb{L}^2 - 356\mathbb{L} - 63 \\
e_c(\mathcal{X}_1^{\times 8}, \mathbb{Q}_\ell) &= \mathbb{L}^9 + 36\mathbb{L}^8 + 336\mathbb{L}^7 + 1148\mathbb{L}^6 + 1386\mathbb{L}^5 - 406\mathbb{L}^4 - 2436\mathbb{L}^3 - 2224\mathbb{L}^2 - 860\mathbb{L} - 127 \\
e_c(\mathcal{X}_1^{\times 9}, \mathbb{Q}_\ell) &= \mathbb{L}^{10} + 45\mathbb{L}^9 + 540\mathbb{L}^8 + 2484\mathbb{L}^7 + 4662\mathbb{L}^6 + 1764\mathbb{L}^5 - 6090\mathbb{L}^4 - 9804\mathbb{L}^3 - 6372\mathbb{L}^2 - 2003\mathbb{L} - 255
\end{aligned}
$$

The cohomology $H^i(\mathcal{X}_1^{\times 10}, \mathbb{Q}_\ell)$ is Tate type for all $i \neq 11$, whereas for $i = 11$ we have

$$H^{11}(\mathcal{X}_1^{\times 10}, \mathbb{Q}_\ell) = \mathbb{S}_{\Gamma(1)}[12] + \mathbb{L}^{11} + 99\mathbb{L}^{10} + 1925\mathbb{L}^9 + 12375\mathbb{L}^8 + 29700\mathbb{L}^7$$

where $\mathbb{S}_{\Gamma(1)}[12]$ is the 2-dimensional Galois representation attached to the weight 12 cusp form $\Delta \in S_{12}(\Gamma(1))$. In this case the compactly supported Euler characteristic is given by:

$$\begin{aligned} e_{\mathrm{c}}(\mathcal{X}_1^{\times 10}, \mathbb{Q}_\ell) = -&\mathbb{S}_{\Gamma(1)}[12] \\ &+ \mathbb{L}^{11} + 55\mathbb{L}^{10} + 825\mathbb{L}^9 + 4905\mathbb{L}^8 + 12870\mathbb{L}^7 + 12264\mathbb{L}^6 \\ &- 9240\mathbb{L}^5 - 33210\mathbb{L}^4 - 33495\mathbb{L}^3 - 17095\mathbb{L}^2 - 4553\mathbb{L} - 511 \end{aligned}$$

In particular the compactly supported Euler characteristic $e_{\mathrm{c}}(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 10$.

Proof. Follows by combining 1.1.7 and 1.1.9 with 1.2.1. In this case the multiplicities $m_\lambda^{j,n}$ are easily computed using the fact that

$$\mathbb{V}_{\lambda_1} \otimes \mathbb{V}_{\lambda_2} = \mathbb{V}_{\lambda_1+\lambda_2} \oplus \mathbb{V}_{\lambda_1+\lambda_2-2} \oplus \ldots \oplus \mathbb{V}_{|\lambda_1-\lambda_2|}$$

To argue that $e_{\mathrm{c}}(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 10$ note that $H^{11}(\mathcal{X}_1^{\times 10}, \mathbb{Q}_\ell)$ (which is not Tate type, owing to the irreducible 2-dimensional contribution $\mathbb{S}_{\Gamma(1)}[12]$ to $H^1(\mathcal{A}_1, \mathbb{V}_{10})$) appears as a summand in $H^{11}(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for all $n \geq 10$ by the Künneth formula. This contribution cannot be cancelled in the Euler characteristic: since the contribution occurs in $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for i odd, any contribution leading to cancellation would have to occur in $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for i even. Since $H^*(\mathcal{A}_1, \mathbb{V}_\lambda) = 0$ for $\lambda > 0$ odd, any contribution to $H^i(\mathcal{X}_1^{\times n}, \mathbb{Q}_\ell)$ for i even would have to come from a contribution to $H^0(\mathcal{A}_1, \mathbb{V}_\lambda)$ (since $H^2(\mathcal{A}_1, \mathbb{V}_\lambda) = 0$ for all $\lambda \geq 0$), but there are no irreducible 2-dimensional contributions in this case: the only irreducible 2-dimensional contributions come from the contribution $\mathbb{S}_{\Gamma(1)}[\lambda + 2]$ to $H^1(\mathcal{A}_1, \mathbb{V}_\lambda)$. ◻

1.2.3 Point Counts for Elliptic Curves

We now consider the point counts and consequences for arithmetic statistics which come from the above computations.

Since $e_{\mathrm{c}}(\mathcal{A}_1, \mathbb{Q}_\ell) = \mathbb{L}$ we have

$$\#\mathcal{A}_1(\mathbb{F}_q) = q$$

Since $e_c(\mathcal{X}_1, \mathbb{Q}_\ell) = \mathbb{L}^2 + \mathbb{L}$ we have

$$\#\mathcal{X}_1(\mathbb{F}_q) = q^2 + q$$

It follows that we have the expected value

$$\mathbb{E}(\#A_1(\mathbb{F}_q)) = \frac{q^2 + q}{q} = q + 1$$

For E an elliptic curve the Weil conjectures yield

$$\#E(\mathbb{F}_q) = q + aq^{\frac{1}{2}} + 1$$

where a is a sum of 2 roots of unity. In particular we have

$$|\#E(\mathbb{F}_q) - (q+1)| \leq 2q^{\frac{1}{2}}$$

On the other hand by the Honda-Tate correspondence for elliptic curves there exists an elliptic curve E over $\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = q + 1$ corresponding to the case $a = 0$. In particular we have

$$\min_{[E]\in[\mathcal{A}_1(\mathbb{F}_q)]} \#E(\mathbb{F}_q) = q + 1$$

Comparing this to the computation of the expected value $\mathbb{E}(\#A(\mathbb{F}_q))$ yields

$$\lim_{q\to\infty} |\mathbb{E}(\#A_1(\mathbb{F}_q)) - \min_{[E]\in[\mathcal{A}_1(\mathbb{F}_q)]} \#E(\mathbb{F}_q)| = 0$$

Since $e_c(\mathcal{X}_1^{\times 2}, \mathbb{Q}_\ell) = \mathbb{L}^3 + 3\mathbb{L}^2 + \mathbb{L} - 1$ we have

$$\#\mathcal{X}_1^{\times 2}(\mathbb{F}_q) = q^3 + 3q^2 + q - 1$$

It follows that we have the expected value

$$\mathbb{E}(\#A_1(\mathbb{F}_q)^2) = \frac{q^3+3q^2+q-1}{q} = q^2+3q+1-\frac{1}{q}$$

and we have the variance

$$\mathrm{Var}(\#A_1(\mathbb{F}_q)) = \mathbb{E}(\#A_1(\mathbb{F}_q)^2) - \mathbb{E}(\#A_1(\mathbb{F}_q))^2 = q - \frac{1}{q}$$

We can continue in this way to obtain the first 9 terms of the moment generating function:

Corollary 1.2.4. The first 9 terms of the moment generating function $M_{\#A_1(\mathbb{F}_q)}(t)$ are rational functions in q:

$$\begin{aligned}
&1+(q+1)t\\
&\quad+(q^2+3q+1-\tfrac{1}{q})\tfrac{t^2}{2!}\\
&\quad+(q^3+6q^2+6q-2-\tfrac{3}{q})\tfrac{t^3}{3!}\\
&\quad+(q^4+10q^3+20q^2+4q-14-\tfrac{7}{q})\tfrac{t^4}{4!}\\
&\quad+(q^5+15q^4+50q^3+40q^2-30q-49-\tfrac{15}{q})\tfrac{t^5}{5!}\\
&\quad+(q^6+21q^5+105q^4+160q^3-183q-139-\tfrac{31}{q})\tfrac{t^6}{6!}\\
&\quad+(q^7+28q^6+196q^5+469q^4+280q^3-427q^2-700q-356-\tfrac{63}{q})\tfrac{t^7}{7!}\\
&\quad+(q^8+36q^7+336q^6+1148q^5+1386q^4-406q^3-2436q^2-2224q-860-\tfrac{127}{q})\tfrac{t^8}{8!}\\
&\quad+(q^9+45q^8+540q^7+2484q^6+4662q^5+1764q^4-6090q^3-9804q^2-6372q-2003-\tfrac{255}{q})\tfrac{t^9}{9!}
\end{aligned}$$

1.3 Arithmetic Statistics for Abelian Surfaces

In this section we will summarize what is known about the cohomology of local systems on $\mathcal{A}_2$, and then use this to deduce some results about arithmetic statistics for principally polarized Abelian surfaces over finite fields.

The moduli of curves of genus 2 curves and the moduli of principally polarized Abelian surfaces are quite similar. The Torelli morphism $\tau : \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ is an open immersion, and we have a stratification

$$\mathcal{A}_2 = \tau(\mathcal{M}_2) \amalg \mathrm{Sym}^2(\mathcal{A}_1)$$

In other words, every principally polarized Abelian surface is either the Jacobian of a genus 2 (hyperelliptic) curve, or a product of elliptic curves. Consequently, the point counts in this situation can be reduced to point counts on curves of genus ≤ 2, and one can use results about the cohomology of local systems on $\mathcal{M}_2$ to study arithmetic statistics for genus 2 curves over finite fields in the same way that we do for principally polarized Abelian surfaces over finite fields. We have chosen not to investigate this direction.

1.3.1 Cohomology of Local Systems on $\mathcal{A}_2$

Let $\mathcal{A}_2$ be the moduli stack of principally polarized Abelian surfaces, which is a smooth Deligne-Mumford stack of dimension 3 over $\mathbb{Z}$. Let $\pi : \mathcal{X}_2 \to \mathcal{A}_2$ be the universal Abelian surface over $\mathcal{A}_2$ and let $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}_\ell$ be the ℓ-adic local system on $\mathcal{A}_2$ corresponding to the standard representation of Sp_4. For $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$ a dominant integral highest weight for Sp_4 let $\mathbb{V}_\lambda$ be the ℓ-adic local system on $\mathcal{A}_2$ corresponding to the irreducible representation of Sp_4 of highest weight λ, occurring in $\mathrm{Sym}^{\lambda_1-\lambda_2}(\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_2}(\wedge^2\mathbb{V})$. For $\lambda_1 + \lambda_2$ odd we have $H^*(\mathcal{A}_2, \mathbb{V}_\lambda) = 0$ since $-\mathrm{id} \in \mathrm{Sp}_4(\mathbb{Z})$ acts by multiplication by $(-1)^{\lambda_1+\lambda_2}$ on the stalks of $\mathbb{V}_{\lambda_1,\lambda_2}$.

Let $\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] = \bigoplus_f \rho_f$ be the ℓ-adic Galois representation corresponding to vector-valued Siegel cusp forms of weight $(\lambda_1 - \lambda_2, \lambda_2 + 3)$ for $\Gamma(1) = \mathrm{Sp}_4(\mathbb{Z})$: for each eigenform $f \in S_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1))$ we have a 4-dimensional ℓ-adic Galois representation ρ_f, and we have

$$\mathrm{tr}(\mathrm{Frob}_p | \mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]) = \mathrm{tr}(T_p | S_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1)))$$

for every prime p, which determines $\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ as an element of the Grothendieck group of ℓ-adic Galois representations.

As a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the ℓ-adic Galois representation ρ_F need not be irreducible: it is reducible for instance when $F \in S_{0,k}(\Gamma(1))$ is the Saito-Kurokawa lift of a cusp form $f \in S_{2k-2}(\Gamma(1))$ (see [116, Theorem 21.1] for a description of the Saito-Kurokawa lift), in which case $\rho_F \simeq \rho_f + \mathbb{L}^{k-1} + \mathbb{L}^{k-2}$ up to semisimplification. On the other hand if $F \in S_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1))$ is a vector-valued Siegel modular form of general type, the ℓ-adic Galois representation ρ_F is irreducible as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ (see [120, Theorem I, Theorem II]). Write $\mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ for the ℓ-adic Galois representation corresponding to vector-valued Siegel cusp forms of general type.

Let $\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] = \mathrm{Gr}^W_{\lambda_1+\lambda_2+3}\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ be the ℓ-adic Galois representation corresponding to Siegel cusp forms satisfying the generalized Ramanujan conjecture, that is those cusp forms $f \in S_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1))$ such that the roots of the characteristic polynomial

$$1 - \lambda_p(f)x + (p\lambda_{1,p^2}(f) + (p^3+p)\lambda_{2,p^2}(f))x^2 - \lambda_p(f)p^{\lambda_1+\lambda_2+3}x^3 + p^{2(\lambda_1+\lambda_2+3)}x^4$$

of the Frobenius ϕ_p acting on the ℓ-adic Galois representation ρ_f for $\ell \neq p$, have absolute value $p^{-\frac{\lambda_1+\lambda_2+3}{2}}$. By Weissauer, [120, Theorem II], the Siegel cusp forms not satisfying the generalized Ramanujan conjecture are exactly the Saito-Kurokawa lifts.

Now we have a decomposition

$$\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] = \mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] \oplus \mathbb{S}^{\mathrm{lift}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$$

where $\mathbb{S}^{\mathrm{lift}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ is given

$$\mathbb{S}^{\mathrm{lift}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] = \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \\ 0 & \text{otherwise} \end{cases}$$

In particular we have $\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] = \mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ except in the case $\lambda_1 = \lambda_2$ even where $\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ is obtained by removing two summands of Tate type from $\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3]$ for each eigenform $f \in S_{\lambda_1+\lambda_2+4}(\Gamma(1))$.

We also have contributions from non-holomorphic Yoshida lifts: we have a decomposition

$$H^3_!(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = \mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] \oplus H^3_{\mathrm{endo}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$$

where $H^3_{\mathrm{endo}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ is given

$$H^3_{\mathrm{endo}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1}$$

where $s_{\Gamma(1)}[k]$ is the dimension of the space of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

For an integer $k \geq 2$ let $s_{\Gamma(1)}[k]$ be the dimension of the space of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Let $s_{\Gamma(1)}[k]_{L(\frac{1}{2})=0}$ be the dimension of the subspace of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ with vanishing central L-value $L(f, \frac{k}{2}) = 0$. Let $s_{\Gamma(1)}[k]_{L(\frac{1}{2})\neq 0}$ be the dimension of the subspace of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ with nonvanishing central L-value $L(f, \frac{k}{2}) \neq 0$. Note that by the functional equation, $\mathrm{ord}_{s=\frac{k}{2}} L(f, s)$ is odd for $k \equiv 2 \mod 4$ (in which case we have $s_{\Gamma(1)}[k]_{L(\frac{1}{2})=0} = s_{\Gamma(1)}[k]$) and is even for $k \equiv 0 \mod 4$ (in which case we have $0 \leq s_{\Gamma(1)}[k]_{L(\frac{1}{2})\neq 0} \leq s_{\Gamma(1)}[k]$). By Maeda's conjecture (which is special to the case of level 1), for $k \equiv 0 \mod 4$ we should have $s_{\Gamma(1)}[k]_{L(\frac{1}{2})=0} = 0$.

By work of Petersen, using work of Harder [55] and Flicker [33] as input, we have the following:

Proposition 1.3.1. [96, Theorem 2.1] For $\lambda_1 \geq \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 > 0$ even the compactly supported cohomology $H^*_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ is concentrated in degrees 2, 3, 4, and given up to semisimplification by

$$
\begin{aligned}
H^2_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) &= \mathbb{S}_{\Gamma(1)}[\lambda_2+2] + s_{\Gamma(1)}[\lambda_1-\lambda_2+2] \\
&\quad + \begin{cases} s'_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} \\
H^3_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) &= \mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1-\lambda_2, \lambda_2+3] + \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1+\lambda_2+4] & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \\
&\quad + s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} + \mathbb{S}_{\Gamma(1)}[\lambda_1+3] \\
&\quad + \begin{cases} s'_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases} + \begin{cases} 1 & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L} & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases} \\
H^4_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) &= \begin{cases} s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Poincare dually, for $\lambda_1 \geq \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 > 0$ even the cohomology $H^*(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ is concentrated in degrees 2, 3, 4, and given up to semisimplification by

$$H^2(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$H^3(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = \mathbb{S}^{\text{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] + \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$+ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1} + \mathbb{S}_{\Gamma(1)}[\lambda_1 + 3]\mathbb{L}^{\lambda_2+1}$$

$$+ \begin{cases} s'_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+2} & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+2} & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^4(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = \mathbb{S}_{\Gamma(1)}[\lambda_2 + 2]\mathbb{L}^{\lambda_1+2} + s_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_1+\lambda_2+3}$$

$$+ \begin{cases} s'_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Note that the above formulas simplify greatly assuming Maeda's conjecture.

We will use the following values for the cohomology groups $H^i(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ which are obtained by combining 1.3.1 with the vanishing of the spaces $S_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1))$ for all $\lambda_1 \geq \lambda_2 \geq 0$ with $\lambda_1, \lambda_2 \leq 7$ except for $\lambda_1 = \lambda_2 = 7$:

(λ_1,λ_2)	$H^2(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2})$	$H^3(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2})$	$H^4(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2})$
$(2,0)$		$\mathbb{L}^4$	
$(1,1)$		$\mathbb{L}^5$	
$(4,0)$		$\mathbb{L}^6$	
$(3,1)$			
$(2,2)$			
$(6,0)$		$\mathbb{L}^8$	
$(5,1)$			
$(4,2)$			$\mathbb{L}^9$
$(3,3)$		$\mathbb{L}^9$	
$(7,1)$		$\mathbb{L}^9$	
$(6,2)$		$\mathbb{L}^8$	$\mathbb{L}^{11}$
$(5,3)$		$\mathbb{L}^7$	
$(4,4)$	$\mathbb{L}^5$		
$(7,3)$			
$(6,4)$			$\mathbb{L}^{13}$
$(5,5)$		$\mathbb{L}^{13}$	
$(7,5)$		$\mathbb{L}^9$	
$(6,6)$	$\mathbb{L}^7$		
$(7,7)$		$\mathbb{S}_{\Gamma(1)}[18]+\mathbb{L}^{17}+\mathbb{L}^9$	

In particular, setting $\mathbb{S}_{\Gamma(1)}[2] := -\mathbb{L}-1$ and $s_{\Gamma(1)}[2] := -1$, for $\lambda_1 \geq \lambda_2 \geq 0$ with $\lambda_1+\lambda_2>0$ even we have

$$e_{\mathrm{c}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2}) = -\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2+3] + e_{\mathrm{c,extr}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2})$$

as an element of the Grothendieck group of ℓ-adic Galois representations, where $e_{\mathrm{c,extr}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2})$ is given

$$\begin{aligned} e_{\mathrm{c,extr}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1,\lambda_2}) &= -s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} \\ &\quad + s_{\Gamma(1)}[\lambda_1-\lambda_2+2] - s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} \\ &\quad + \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_2+2]+1 & \lambda_1 \text{ even} \\ -\mathbb{S}_{\Gamma(1)}[\lambda_1+3] & \lambda_1 \text{ odd} \end{cases} \end{aligned}$$

Poincare dually, for $\lambda_1 \geq \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 > 0$ even we have

$$e(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = -\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] + e_{\mathrm{c,extr}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$$

as an element of the Grothendieck group of ℓ-adic Galois representations, where $e_{\mathrm{extr}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ is given

$$\begin{aligned} e_{\mathrm{extr}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = &-s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1} \\ &+ s_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_1+\lambda_2+3} - s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+2} \\ &+ \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_2 + 2]\mathbb{L}^{\lambda_1+2} + \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 \text{ even} \\ -\mathbb{S}_{\Gamma(1)}[\lambda_1 + 3]\mathbb{L}^{\lambda_2+1} & \lambda_1 \text{ odd} \end{cases} \end{aligned}$$

This remains true for $(\lambda_1, \lambda_2) = (0, 0)$ if we set $\mathbb{S}_{\Gamma(1)}[0, 3] := -\mathbb{L}^3 - \mathbb{L}^2 - \mathbb{L} - 1$: by [78, Corollary 5.2.3] the compactly supported cohomology $H^*_c(\mathcal{A}_2, \mathbb{Q}_\ell)$ is concentrated in degrees 4 and 6 where $H^4_c(\mathcal{A}_2, \mathbb{Q}_\ell) = \mathbb{L}^2$ and $H^6_c(\mathcal{A}_2, \mathbb{Q}_\ell) = \mathbb{L}^3$, in particular $e_c(\mathcal{A}_2, \mathbb{Q}_\ell) = \mathbb{L}^3 + \mathbb{L}^2$. Poincare dually, the cohomology $H^*(\mathcal{A}_2, \mathbb{Q}_\ell)$ is concentrated in degrees 0 and 2 where $H^0(\mathcal{A}_2, \mathbb{Q}_\ell) = 1$ and $H^2(\mathcal{A}_2, \mathbb{Q}_\ell) = \mathbb{L}$, in particular $e(\mathcal{A}_2, \mathbb{Q}_\ell) = \mathbb{L} + 1$.

We will use the following values for the Euler characteristics $e_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$, which are obtained by combining 1.3.1 with the vanishing of the spaces $S_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1))$ for all $\lambda_1 \geq \lambda_2 \geq 0$ with $\lambda_1, \lambda_2 \leq 7$ except for $\lambda_1 = \lambda_2 = 7$:

(λ_1, λ_2)	$e_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$
$(0,0)$	$\mathbb{L}^3 + \mathbb{L}^2$
$(2,0)$	$-\mathbb{L}$
$(1,1)$	-1
$(4,0)$	$-\mathbb{L}$
$(3,1)$	0
$(2,2)$	0
$(6,0)$	$-\mathbb{L}$
$(5,1)$	0
$(4,2)$	1
$(3,3)$	-1

(λ_1, λ_2)	$e_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$
$(7,1)$	$-\mathbb{L}^2$
$(6,2)$	$-\mathbb{L}^3 + 1$
$(5,3)$	$-\mathbb{L}^4$
$(4,4)$	$\mathbb{L}^6$
$(7,3)$	0
$(6,4)$	1
$(5,5)$	-1
$(7,5)$	$-\mathbb{L}^6$
$(6,6)$	$\mathbb{L}^8$
$(7,7)$	$-\mathbb{S}_{\Gamma(1)}[18] - \mathbb{L}^8 - 1$

The space $S_{0,10}(\Gamma(1))$ is spanned by the Igusa cusp form (see [94]):

$$\begin{aligned}\chi_{10} &= (q^{-1}-2+q)q_1q_2 - (2q^{-2}+16q^{-1}-36+16q+2q^2)(q_1^2q_2+q_1q_2^2)\\ &\quad + (q^{-3}+36q^{-2}+99q^{-1}-272+99q+36q^2+q^3)(q_1^3q_2+q_1q_2^3)\\ &\quad + (4q^{-3}+72q^{-2}+252q^{-1}-656+252q+72q^2+4q^3)q_1^2q_2^2+\dots\end{aligned}$$

which is a Saito-Kurokawa lift of the weight 18 cusp form $f_{18} = \Delta E_6 \in S_{18}(\Gamma(1))$ and contributes an irreducible 2-dimensional ℓ-adic Galois representation $\mathbb{S}_{\Gamma(1)}[18]$ to $H^3_c(\mathcal{A}_2, \mathbb{V}_{7,7})$ (see for example [96, 4.3.5]) with the property that $\mathrm{tr}(\mathrm{Frob}_p|\mathbb{S}_{\Gamma(1)}[18]) = \lambda_p(f_{18})$ (the eigenvalue of the Hecke operator T_p on f_{18}), which is not polynomial in p; the remaining summands $\mathbb{L}^9$ and $\mathbb{L}^8$ of the 4-dimensional ℓ-adic Galois representation $\mathbb{S}_{\Gamma(1)}[0,10] = \mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^9 + \mathbb{L}^8$ do not contribute to $H^3(\mathcal{A}_2, \mathbb{V}_{7,7})$.

We will use another contribution which does not appear in the above table but which was mentioned in the introduction. The space $S_{6,8}(\Gamma(1))$ is spanned by the vector-valued cusp form (see [27, Section 8])

$$\begin{aligned}\chi_{6,8} &= \begin{pmatrix} 0\\ 0\\ q^{-1}-2+q\\ 2(q-q^{-1})\\ q^{-1}-2+q\\ 0\\ 0\end{pmatrix} q_1q_2 + \begin{pmatrix} 0\\ 0\\ -2(q^{-2}+8q^{-1}-18+8q+q^2)\\ 8(q^{-2}+4q^{-1}-4q-q^2)\\ -2(7q^{-2}-4q^{-1}-6-4q+7q^2)\\ 12(q^{-2}-2q^{-1}+2q-q^2)\\ -4(q^{-2}-4q^{-1}+6-4q+q^2)\end{pmatrix} q_1q_2^2\\ &\quad + \begin{pmatrix} -4(q^{-2}-4q^{-1}+6-4q+q^2)\\ 12(q^{-2}-2q^{-1}+2q-q^2)\\ -2(7q^{-2}-4q^{-1}-6-4q+7q^2)\\ 8(q^{-2}+4q^{-1}-4q-q^2)\\ -2(q^{-2}+8q^{-1}-18+8q+q^2)\\ 0\\ 0\end{pmatrix} q_1^2q_2 + \begin{pmatrix} 16(q^{-3}-9q^{-1}+16-9q+q^3)\\ -72(q^{-3}-3q^{-1}+3q-q^3)\\ 128(q^{-3}-2+q^3)\\ -144(q^{-3}+5q^{-1}-5q-q^3)\\ 128(q^{-3}-2+q^3)\\ -72(q^{-3}-3q^{-1}+3q-q^3)\\ 16(q^{-3}-9q^{-1}+16-9q+q^3)\end{pmatrix} q_1^2q_2^2+\dots\end{aligned}$$

which is of general type and contributes an irreducible 4-dimensional Galois representation $\mathbb{S}_{\Gamma(1)}[6,8]$ to $H^3_c(\mathcal{A}_2, \mathbb{V}_{11,5})$ (see for example [96, 4.3.1]) with the property that $\mathrm{tr}(\mathrm{Frob}_p|\mathbb{S}_{\Gamma(1)}[6,8]) = \lambda_p(\chi_{6,8})$ (the eigenvalue of the Hecke operator T_p acting on $\chi_{6,8}$) which is not polynomial in p.

The Langlands correspondence predicts in this case that an irreducible 4-dimensional Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ (which is the composition of a Spin_5 Galois representation $\rho' : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Spin}_5(\overline{\mathbb{Q}}_\ell) = \widehat{\mathrm{PGSp}_4}$ with the 4-dimensional spin representation $\mathrm{spin} : \mathrm{Spin}_5(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$) contributing to the cohomology $H^*(\mathcal{A}_2, \mathbb{V}_\lambda)$ must come from a packet of cuspidal automorphic representations π of $\mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})$ with π_∞ varying over all members of a discrete series L-packet. As the $(\mathfrak{sp}_4, \mathrm{U}(2))$-cohomology of such discrete series representations is concentrated in middle degree 3 by [119], such a contribution can only occur in $H^3(\mathcal{A}_2, \mathbb{V}_\lambda)$.

1.3.2 Examples: Cohomology of $\mathcal{X}_2^{\times n}$ through $n = 7$

In this section we compute $H^*(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ up to $n = 7$. The case $n = 7$ is the first case where $H^*(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ is not of Tate type, owing to a contribution from the Saito-Kurokawa lift $\chi_{10} \in S_{0,10}(\Gamma(1))$. In the case $n = 11$ we have contributions from the vector-valued Siegel modular forms $\chi_{6,8} \in S_{6,8}(\Gamma(1))$ and $\chi_{4,10} \in S_{4,10}(\Gamma(1))$ of general type, though we have no reason do these computations explicitly in this range.

We start by computing the local systems $\mathbb{R}^*\pi_*^n\mathbb{Q}_\ell = \bigoplus_{0\leq j\leq 4n} \mathbb{R}^j\pi_*^n\mathbb{Q}_\ell$ up to $n = 7$. For this it suffices to consider the local systems $\mathbb{R}^j\pi_*\mathbb{Q}_\ell$ for $0 \leq j \leq 2n$ even with Tate twists omitted since $\mathbb{R}^j\pi_*^n\mathbb{Q}_\ell$ is pure of weight j (so the missing Tate twists can be inferred from the weights of the local systems $\mathbb{V}_\lambda$) and $\mathbb{R}^{4n-j}\pi_*^n\mathbb{Q}_\ell \simeq \mathbb{R}^j\pi_*^n\mathbb{Q}_\ell(-2n-j)$.

We used SAGE (the source code for this is commented out above this line in the source code for this document) to compute the following:

Proposition 1.3.2. (i) The local system $\mathbb{R}^*\pi_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 4$ and given by

j	$\mathbb{R}^j\pi_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$\mathbb{V}_{0,0} + \mathbb{V}_{1,1}$

(ii) The local system $\mathbb{R}^*\pi_*^2\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 8$ and given by

j	$\mathbb{R}^j\pi_*^2\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$3\mathbb{V}_{0,0} + 3\mathbb{V}_{1,1} + \mathbb{V}_{2,0}$
4	$6\mathbb{V}_{0,0} + 4\mathbb{V}_{1,1} + 3\mathbb{V}_{2,0} + \mathbb{V}_{2,2}$

(iii) The local system $\mathbb{R}^*\pi_*^3\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 12$ and given by

j	$\mathbb{R}^j\pi_*^3\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$6\mathbb{V}_{0,0} + 6\mathbb{V}_{1,1} + 3\mathbb{V}_{2,0}$
4	$21\mathbb{V}_{0,0} + 21\mathbb{V}_{1,1} + 18\mathbb{V}_{2,0} + 6\mathbb{V}_{2,2} + 3\mathbb{V}_{3,1}$
6	$28\mathbb{V}_{0,0} + 36\mathbb{V}_{1,1} + 28\mathbb{V}_{2,0} + 9\mathbb{V}_{2,2} + 8\mathbb{V}_{3,1} + \mathbb{V}_{3,3}$

(iv) The local system $\mathbb{R}^*\pi^4_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 16$ and given by

j	$\mathbb{R}^j\pi^4_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$10\mathbb{V}_{0,0} + 10\mathbb{V}_{1,1} + 6\mathbb{V}_{2,0}$
4	$55\mathbb{V}_{0,0} + 65\mathbb{V}_{1,1} + 60\mathbb{V}_{2,0} + 20\mathbb{V}_{2,2} + 15\mathbb{V}_{3,1} + \mathbb{V}_{4,0}$
6	$136\mathbb{V}_{0,0} + 200\mathbb{V}_{1,1} + 190\mathbb{V}_{2,0} + 74\mathbb{V}_{2,2} + 80\mathbb{V}_{3,1} + 10\mathbb{V}_{3,3}$
	$+10\mathbb{V}_{4,0} + 6\mathbb{V}_{4,2}$
8	$190\mathbb{V}_{0,0} + 275\mathbb{V}_{1,1} + 280\mathbb{V}_{2,0} + 120\mathbb{V}_{2,2} + 125\mathbb{V}_{3,1} + 16\mathbb{V}_{3,3}$
	$+20\mathbb{V}_{4,0} + 15\mathbb{V}_{4,2} + \mathbb{V}_{4,4}$

(v) The local system $\mathbb{R}^*\pi^5_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 20$ and given by

j	$\mathbb{R}^j\pi^5_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$15\mathbb{V}_{0,0} + 15\mathbb{V}_{1,1} + 10\mathbb{V}_{2,0}$
4	$120\mathbb{V}_{0,0} + 155\mathbb{V}_{1,1} + 150\mathbb{V}_{2,0} + 50\mathbb{V}_{2,2} + 45\mathbb{V}_{3,1} + 5\mathbb{V}_{4,0}$
6	$470\mathbb{V}_{0,0} + 750\mathbb{V}_{1,1} + 780\mathbb{V}_{2,0} + 330\mathbb{V}_{2,2} + 395\mathbb{V}_{3,1} + 50\mathbb{V}_{3,3}$
	$+75\mathbb{V}_{4,0} + 45\mathbb{V}_{4,2} + 5\mathbb{V}_{5,1}$
8	$1065\mathbb{V}_{0,0} + 1800\mathbb{V}_{1,1} + 2000\mathbb{V}_{2,0} + 960\mathbb{V}_{2,2} + 1200\mathbb{V}_{3,1} + 190\mathbb{V}_{3,3}$
	$+285\mathbb{V}_{4,0} + 225\mathbb{V}_{4,2} + 15\mathbb{V}_{4,4} + 40\mathbb{V}_{5,1} + 10\mathbb{V}_{5,3}$
10	$1377\mathbb{V}_{0,0} + 2425\mathbb{V}_{1,1} + 2700\mathbb{V}_{2,0} + 1324\mathbb{V}_{2,2} + 1725\mathbb{V}_{3,1} + 300\mathbb{V}_{3,3}$
	$+425\mathbb{V}_{4,0} + 351\mathbb{V}_{4,2} + 25\mathbb{V}_{4,4} + 75\mathbb{V}_{5,1} + 24\mathbb{V}_{5,3} + \mathbb{V}_{5,5}$

(vi) The local system $\mathbb{R}^*\pi^6_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 24$ and given by

j	$\mathbb{R}^j\pi^6_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$21\mathbb{V}_{0,0} + 21\mathbb{V}_{1,1} + 15\mathbb{V}_{2,0}$
4	$231\mathbb{V}_{0,0} + 315\mathbb{V}_{1,1} + 315\mathbb{V}_{2,0} + 105\mathbb{V}_{2,2} + 105\mathbb{V}_{3,1} + 15\mathbb{V}_{4,0}$
6	$1309\mathbb{V}_{0,0} + 2205\mathbb{V}_{1,1} + 2415\mathbb{V}_{2,0} + 1071\mathbb{V}_{2,2} + 1365\mathbb{V}_{3,1} + 175\mathbb{V}_{3,3}$
	$+315\mathbb{V}_{4,0} + 189\mathbb{V}_{4,2} + 35\mathbb{V}_{5,1} + \mathbb{V}_{6,0}$
8	$4389\mathbb{V}_{0,0} + 8085\mathbb{V}_{1,1} + 9465\mathbb{V}_{2,0} + 4851\mathbb{V}_{2,2} + 6615\mathbb{V}_{3,1} + 1155\mathbb{V}_{3,3}$
	$+1890\mathbb{V}_{4,0} + 1539\mathbb{V}_{4,2} + 105\mathbb{V}_{4,4} + 420\mathbb{V}_{5,1} + 105\mathbb{V}_{5,3} + 21\mathbb{V}_{6,0} + 15\mathbb{V}_{6,2}$
10	$8877\mathbb{V}_{0,0} + 17325\mathbb{V}_{1,1} + 20790\mathbb{V}_{2,0} + 11319\mathbb{V}_{2,2} + 16170\mathbb{V}_{3,1} + 3255\mathbb{V}_{3,3}$
	$+5040\mathbb{V}_{4,0} + 4536\mathbb{V}_{4,2} + 405\mathbb{V}_{4,4} + 1470\mathbb{V}_{5,1} + 504\mathbb{V}_{5,3} + 21\mathbb{V}_{5,5}$
	$+105\mathbb{V}_{6,0} + 105\mathbb{V}_{6,2} + 15\mathbb{V}_{6,4}$
12	$11242\mathbb{V}_{0,0} + 22176\mathbb{V}_{1,1} + 26950\mathbb{V}_{2,0} + 14994\mathbb{V}_{2,2} + 21560\mathbb{V}_{3,1} + 4480\mathbb{V}_{3,3}$
	$+6930\mathbb{V}_{4,0} + 6426\mathbb{V}_{4,2} + 630\mathbb{V}_{4,4} + 2156\mathbb{V}_{5,1} + 784\mathbb{V}_{5,3} + 36\mathbb{V}_{5,5}$
	$+175\mathbb{V}_{6,0} + 189\mathbb{V}_{6,2} + 35\mathbb{V}_{6,4} + \mathbb{V}_{6,6}$

(vii) The local system $\mathbb{R}^*\pi^7_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 28$ and given by

j	$\mathbb{R}^j\pi^7_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0}$
2	$28\mathbb{V}_{0,0} + 28\mathbb{V}_{1,1} + 21\mathbb{V}_{2,0}$
4	$406\mathbb{V}_{0,0} + 574\mathbb{V}_{1,1} + 588\mathbb{V}_{2,0} + 196\mathbb{V}_{2,2} + 210\mathbb{V}_{3,1} + 35\mathbb{V}_{4,0}$
6	$3136\mathbb{V}_{0,0} + 5488\mathbb{V}_{1,1} + 6216\mathbb{V}_{2,0} + 2842\mathbb{V}_{2,2} + 3780\mathbb{V}_{3,1} + 490\mathbb{V}_{3,3}$
	$+980\mathbb{V}_{4,0} + 588\mathbb{V}_{4,2} + 140\mathbb{V}_{5,1} + 7\mathbb{V}_{6,0}$
8	$14602\mathbb{V}_{0,0} + 28420\mathbb{V}_{1,1} + 34440\mathbb{V}_{2,0} + 18424\mathbb{V}_{2,2} + 26460\mathbb{V}_{3,1} + 4900\mathbb{V}_{3,3}$
	$+8435\mathbb{V}_{4,0} + 7014\mathbb{V}_{4,2} + 490\mathbb{V}_{4,4} + 2345\mathbb{V}_{5,1} + 588\mathbb{V}_{5,3}$
	$+196\mathbb{V}_{6,0} + 140\mathbb{V}_{6,2} + 7\mathbb{V}_{7,1}$
10	$42448\mathbb{V}_{0,0} + 88480\mathbb{V}_{1,1} + 110985\mathbb{V}_{2,0} + 64288\mathbb{V}_{2,2} + 97020\mathbb{V}_{3,1} + 21280\mathbb{V}_{3,3}$
	$+34300\mathbb{V}_{4,0} + 32340\mathbb{V}_{4,2} + 3220\mathbb{V}_{4,4} + 12740\mathbb{V}_{5,1} + 4557\mathbb{V}_{5,3} + 196\mathbb{V}_{5,5}$
	$+1456\mathbb{V}_{6,0} + 1470\mathbb{V}_{6,2} + 210\mathbb{V}_{6,4} + 112\mathbb{V}_{7,1} + 35\mathbb{V}_{7,3}$
12	$79849\mathbb{V}_{0,0} + 172018\mathbb{V}_{1,1} + 220276\mathbb{V}_{2,0} + 133084\mathbb{V}_{2,2} + 205310\mathbb{V}_{3,1} + 48580\mathbb{V}_{3,3}$
	$+76636\mathbb{V}_{4,0} + 76566\mathbb{V}_{4,2} + 8890\mathbb{V}_{4,4} + 32438\mathbb{V}_{5,1} + 13132\mathbb{V}_{5,3} + 763\mathbb{V}_{5,5}$
	$+4312\mathbb{V}_{6,0} + 4872\mathbb{V}_{6,2} + 980\mathbb{V}_{6,4} + 28\mathbb{V}_{6,6} + 490\mathbb{V}_{7,1} + 224\mathbb{V}_{7,3} + 21\mathbb{V}_{7,5}$
14	$98296\mathbb{V}_{0,0} + 214424\mathbb{V}_{1,1} + 275968\mathbb{V}_{2,0} + 168756\mathbb{V}_{2,2} + 262640\mathbb{V}_{3,1} + 63700\mathbb{V}_{3,3}$
	$+99568\mathbb{V}_{4,0} + 101136\mathbb{V}_{4,2} + 12200\mathbb{V}_{4,4} + 43904\mathbb{V}_{5,1} + 18424\mathbb{V}_{5,3} + 1176\mathbb{V}_{5,5}$
	$+6076\mathbb{V}_{6,0} + 7056\mathbb{V}_{6,2} + 1520\mathbb{V}_{6,4} + 49\mathbb{V}_{6,6} + 784\mathbb{V}_{7,1} + 392\mathbb{V}_{7,3} + 48\mathbb{V}_{7,5} + \mathbb{V}_{7,7}$

We now explain the entries of the table of $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for $1 \le n \le 7$. We consider the spectral sequence $E_2^{i,j} = H^i(\mathcal{A}_2, \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell) \Rightarrow H^{i+j}(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$.

(i) We have contributions to H^0 and H^2 from $H^0(\mathcal{A}_2, \mathbb{V}_{0,0}) = 1$ and $H^2(\mathcal{A}_2, \mathbb{V}_{0,0}) = \mathbb{L}$ and to H^3 from $H^3(\mathcal{A}_2, \mathbb{V}_{1,1}) = \mathbb{L}^5$. It follows that the E_2-page is given

$$\begin{array}{c|cccc}
4 & \mathbb{L}^2 & 0 & \mathbb{L}^3 & 0 \\
2 & \mathbb{L} & 0 & \mathbb{L}^2 & \mathbb{L}^5 \\
0 & 1 & 0 & \mathbb{L} & 0 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

(ii) We have contributions to H^3 from $H^3(\mathcal{A}_2, \mathbb{V}_{1,1}) = \mathbb{L}^5$ and $H^3(\mathcal{A}_2, \mathbb{V}_{2,0}) = \mathbb{L}^4$. It follows that the E_2-page is given

$$\begin{array}{c|cccc}
8 & \mathbb{L}^4 & 0 & \mathbb{L}^5 & 0 \\
6 & 3\mathbb{L}^3 & 0 & 3\mathbb{L}^4 & 3\mathbb{L}^7 + \mathbb{L}^6 \\
4 & 6\mathbb{L}^2 & 0 & 6\mathbb{L}^3 & 4\mathbb{L}^6 + 3\mathbb{L}^5 \\
2 & 3\mathbb{L} & 0 & 3\mathbb{L}^2 & 3\mathbb{L}^5 + \mathbb{L}^4 \\
0 & 1 & 0 & \mathbb{L} & 0 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

(iii) We have contributions to H^3 from $H^3(\mathcal{A}_2, \mathbb{V}_{3,3}) = \mathbb{L}^9$. It follows that the E_2-page is given

12	$\mathbb{L}^6$	0	$\mathbb{L}^7$	0
10	$6\mathbb{L}^5$	0	$6\mathbb{L}^6$	$6\mathbb{L}^9 + 3\mathbb{L}^8$
8	$21\mathbb{L}^4$	0	$21\mathbb{L}^5$	$21\mathbb{L}^8 + 18\mathbb{L}^7$
6	$28\mathbb{L}^3$	0	$28\mathbb{L}^4$	$\mathbb{L}^9 + 36\mathbb{L}^7 + 28\mathbb{L}^6$
4	$21\mathbb{L}^2$	0	$21\mathbb{L}^3$	$21\mathbb{L}^6 + 18\mathbb{L}^5$
2	$6\mathbb{L}$	0	$6\mathbb{L}^2$	$6\mathbb{L}^5 + 3\mathbb{L}^4$
0	1	0	$\mathbb{L}$	0
	0	1	2	3

(iv) We have contributions to H^2 from $H^2(\mathcal{A}_2, \mathbb{V}_{4,4}) = \mathbb{L}^5$, to H^3 from $H^3(\mathcal{A}_2, \mathbb{V}_{4,0}) = \mathbb{L}^6$, and to H^4 from $H^4(\mathcal{A}_2, \mathbb{V}_{4,2}) = \mathbb{L}^9$. It follows that the E_2-page is given

16	$\mathbb{L}^8$	0	$\mathbb{L}^9$	0	0
14	$10\mathbb{L}^7$	0	$10\mathbb{L}^8$	$10\mathbb{L}^{11} + 6\mathbb{L}^{10}$	0
12	$55\mathbb{L}^6$	0	$55\mathbb{L}^7$	$66\mathbb{L}^{10} + 60\mathbb{L}^9$	0
10	$136\mathbb{L}^5$	0	$136\mathbb{L}^6$	$10\mathbb{L}^{11} + 210\mathbb{L}^9 + 190\mathbb{L}^8$	$6\mathbb{L}^{11}$
8	$190\mathbb{L}^4$	0	$191\mathbb{L}^5$	$16\mathbb{L}^{10} + 295\mathbb{L}^8 + 280\mathbb{L}^7$	$15\mathbb{L}^{10}$
6	$136\mathbb{L}^3$	0	$136\mathbb{L}^4$	$10\mathbb{L}^9 + 210\mathbb{L}^7 + 190\mathbb{L}^6$	$6\mathbb{L}^9$
4	$55\mathbb{L}^2$	0	$55\mathbb{L}^3$	$66\mathbb{L}^6 + 60\mathbb{L}^5$	0
2	$10\mathbb{L}$	0	$10\mathbb{L}^2$	$10\mathbb{L}^5 + 6\mathbb{L}^4$	0
0	1	0	$\mathbb{L}$	0	0
	0	1	2	3	4

(v) We have contributions to H^3 from $H^3_{\mathrm{c}}(\mathcal{A}_2, \mathbb{V}_{5,3}) = \mathbb{L}^7$ and $H^3_{\mathrm{c}}(\mathcal{A}_2, \mathbb{V}_{5,5}) = \mathbb{L}^{13}$. It follows that the E_2-page is given

20	$\mathbb{L}^{10}$	0	$\mathbb{L}^{11}$	0	0
18	$15\mathbb{L}^9$	0	$15\mathbb{L}^{10}$	$15\mathbb{L}^{13} + 10\mathbb{L}^{12}$	0
16	$120\mathbb{L}^8$	0	$120\mathbb{L}^9$	$160\mathbb{L}^{12} + 150\mathbb{L}^{11}$	0
14	$470\mathbb{L}^7$	0	$470\mathbb{L}^8$	$50\mathbb{L}^{13} + 825\mathbb{L}^{11} + 780\mathbb{L}^{10}$	$45\mathbb{L}^{13}$
12	$1065\mathbb{L}^6$	0	$1080\mathbb{L}^7$	$190\mathbb{L}^{12} + 2085\mathbb{L}^{10} + 2010\mathbb{L}^9$	$225\mathbb{L}^{12}$
10	$1377\mathbb{L}^5$	0	$1402\mathbb{L}^6$	$\mathbb{L}^{13} + 300\mathbb{L}^{11} + 2850\mathbb{L}^9 + 2724\mathbb{L}^8$	$351\mathbb{L}^{11}$
8	$1065\mathbb{L}^4$	0	$1080\mathbb{L}^5$	$190\mathbb{L}^{10} + 2085\mathbb{L}^8 + 2010\mathbb{L}^7$	$225\mathbb{L}^{10}$
6	$470\mathbb{L}^3$	0	$470\mathbb{L}^4$	$50\mathbb{L}^9 + 825\mathbb{L}^7 + 780\mathbb{L}^6$	$45\mathbb{L}^9$
4	$120\mathbb{L}^2$	0	$120\mathbb{L}^3$	$160\mathbb{L}^6 + 150\mathbb{L}^5$	0
2	$15\mathbb{L}$	0	$15\mathbb{L}^2$	$15\mathbb{L}^5 + 10\mathbb{L}^4$	0
0	1	0	$\mathbb{L}$	0	0
	0	1	2	3	4

(vi) We have contributions to H^2 from $H^2(\mathcal{A}_2, \mathbb{V}_{6,6}) = \mathbb{L}^7$, to H^3 from $H^3(\mathcal{A}_2, \mathbb{V}_{6,0}) = \mathbb{L}^8$ and $H^3(\mathcal{A}_2, \mathbb{V}_{6,2}) = \mathbb{L}^8$, and to H^4 from $H^4(\mathcal{A}_2, \mathbb{V}_{6,2}) = \mathbb{L}^{11}$ and $H^4(\mathcal{A}_2, \mathbb{V}_{6,4}) = \mathbb{L}^{13}$. It follows that the E_2-page is given

24	$\mathbb{L}^{12}$	0	$\mathbb{L}^{13}$	0	0
22	$21\mathbb{L}^{11}$	0	$21\mathbb{L}^{12}$	$21\mathbb{L}^{15} + 15\mathbb{L}^{14}$	0
20	$231\mathbb{L}^{10}$	0	$231\mathbb{L}^{11}$	$330\mathbb{L}^{14} + 315\mathbb{L}^{13}$	0
18	$1309\mathbb{L}^{9}$	0	$1309\mathbb{L}^{10}$	$175\mathbb{L}^{15} + \mathbb{L}^{14} + 2520\mathbb{L}^{13} + 2415\mathbb{L}^{12}$	$189\mathbb{L}^{15}$
16	$4389\mathbb{L}^{8}$	0	$4494\mathbb{L}^{9}$	$1155\mathbb{L}^{14} + 21\mathbb{L}^{13} + 9990\mathbb{L}^{12} + 9570\mathbb{L}^{11}$	$15\mathbb{L}^{15} + 1539\mathbb{L}^{14}$
14	$8877\mathbb{L}^{7}$	0	$9282\mathbb{L}^{8}$	$21\mathbb{L}^{15} + 3255\mathbb{L}^{13} + 105\mathbb{L}^{12} + 22470\mathbb{L}^{11} + 21294\mathbb{L}^{10}$	$15\mathbb{L}^{15} + 105\mathbb{L}^{14} + 4536\mathbb{L}^{13}$
12	$11242\mathbb{L}^{6}$	0	$11873\mathbb{L}^{7}$	$36\mathbb{L}^{14} + 4480\mathbb{L}^{12} + 175\mathbb{L}^{11} + 29295\mathbb{L}^{10} + 27734\mathbb{L}^{9}$	$35\mathbb{L}^{14} + 189\mathbb{L}^{13} + 6426\mathbb{L}^{12}$
10	$8877\mathbb{L}^{5}$	0	$9282\mathbb{L}^{6}$	$21\mathbb{L}^{13} + 3255\mathbb{L}^{11} + 105\mathbb{L}^{10} + 22470\mathbb{L}^{9} + 21294\mathbb{L}^{8}$	$15\mathbb{L}^{13} + 105\mathbb{L}^{12} + 4536\mathbb{L}^{11}$
8	$4389\mathbb{L}^{4}$	0	$4494\mathbb{L}^{5}$	$1155\mathbb{L}^{10} + 21\mathbb{L}^{9} + 9990\mathbb{L}^{8} + 9570\mathbb{L}^{7}$	$15\mathbb{L}^{11} + 1539\mathbb{L}^{10}$
6	$1309\mathbb{L}^{3}$	0	$1309\mathbb{L}^{4}$	$175\mathbb{L}^{9} + \mathbb{L}^{8} + 2520\mathbb{L}^{7} + 2415\mathbb{L}^{6}$	$189\mathbb{L}^{9}$
4	$231\mathbb{L}^{2}$	0	$231\mathbb{L}^{3}$	$330\mathbb{L}^{6} + 315\mathbb{L}^{5}$	0
2	$21\mathbb{L}$	0	$21\mathbb{L}^{2}$	$21\mathbb{L}^{5} + 15\mathbb{L}^{4}$	0
0	1	0	$\mathbb{L}$	0	0
	0	1	2	3	4

(vii) We have contributions to H^3 from $H^3(\mathcal{A}_2, \mathbb{V}_{7,1}) = \mathbb{L}^9$, $H^3(\mathcal{A}_2, \mathbb{V}_{7,5}) = \mathbb{L}^9$, and $H^3(\mathcal{A}_2, \mathbb{V}_{7,7}) = \mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^{17} + \mathbb{L}^9$. It follows that the E_2-page is given

28	$\mathbb{L}^{14}$	0	$\mathbb{L}^{15}$	0	0
26	$28\mathbb{L}^{13}$	0	$28\mathbb{L}^{14}$	$28\mathbb{L}^{17} + 21\mathbb{L}^{16}$	0
24	$406\mathbb{L}^{12}$	0	$406\mathbb{L}^{13}$	$609\mathbb{L}^{16} + 588\mathbb{L}^{15}$	0
22	$3136\mathbb{L}^{11}$	0	$3136\mathbb{L}^{12}$	$490\mathbb{L}^{17} + 7\mathbb{L}^{16} + 6468\mathbb{L}^{15} + 6216\mathbb{L}^{14}$	$588\mathbb{L}^{17}$
20	$14602\mathbb{L}^{10}$	0	$15092\mathbb{L}^{11}$	$4900\mathbb{L}^{16} + 203\mathbb{L}^{15} + 36995\mathbb{L}^{14} + 35028\mathbb{L}^{13}$	$140\mathbb{L}^{17} + 7014\mathbb{L}^{16}$
18	$42448\mathbb{L}^{9}$	0	$45668\mathbb{L}^{10}$	$196\mathbb{L}^{17} + 21280\mathbb{L}^{15} + 1568\mathbb{L}^{14} + 124250\mathbb{L}^{13} + 115542\mathbb{L}^{12}$	$210\mathbb{L}^{17} + 1470\mathbb{L}^{16} + 32340\mathbb{L}^{15}$
16	$79849\mathbb{L}^{8}$	0	$88767\mathbb{L}^{9}$	$763\mathbb{L}^{16} + 48580\mathbb{L}^{14} + 4802\mathbb{L}^{13} + 253526\mathbb{L}^{12} + 233429\mathbb{L}^{11}$	$980\mathbb{L}^{16} + 4872\mathbb{L}^{15} + 76566\mathbb{L}^{14}$
14	$98296\mathbb{L}^{7}$	0	$110545\mathbb{L}^{8}$	$\mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^{17} + 1176\mathbb{L}^{15} + 63700\mathbb{L}^{13}$ $+6860\mathbb{L}^{12} + 321048\mathbb{L}^{11} + 294440\mathbb{L}^{10} + \mathbb{L}^{9}$	$1520\mathbb{L}^{15} + 7056\mathbb{L}^{14} + 101136\mathbb{L}^{13}$
12	$79849\mathbb{L}^{6}$	0	$88767\mathbb{L}^{7}$	$763\mathbb{L}^{14} + 48580\mathbb{L}^{12} + 4802\mathbb{L}^{11} + 253526\mathbb{L}^{10} + 233429\mathbb{L}^{9}$	$980\mathbb{L}^{14} + 4872\mathbb{L}^{13} + 76566\mathbb{L}^{12}$
10	$42448\mathbb{L}^{5}$	0	$45668\mathbb{L}^{6}$	$196\mathbb{L}^{13} + 21280\mathbb{L}^{11} + 1568\mathbb{L}^{10} + 124250\mathbb{L}^{9} + 115542\mathbb{L}^{8}$	$210\mathbb{L}^{13} + 1470\mathbb{L}^{12} + 32340\mathbb{L}^{11}$
8	$14602\mathbb{L}^{4}$	0	$15092\mathbb{L}^{5}$	$4900\mathbb{L}^{10} + 203\mathbb{L}^{9} + 36995\mathbb{L}^{8} + 35028\mathbb{L}^{7}$	$140\mathbb{L}^{11} + 7014\mathbb{L}^{10}$
6	$3136\mathbb{L}^{3}$	0	$3136\mathbb{L}^{4}$	$490\mathbb{L}^{9} + 7\mathbb{L}^{8} + 6468\mathbb{L}^{7} + 6216\mathbb{L}^{6}$	$588\mathbb{L}^{9}$
4	$406\mathbb{L}^{2}$	0	$406\mathbb{L}^{3}$	$609\mathbb{L}^{6} + 588\mathbb{L}^{5}$	0
2	$28\mathbb{L}$	0	$28\mathbb{L}^{2}$	$28\mathbb{L}^{5} + 21\mathbb{L}^{4}$	0
0	1	0	$\mathbb{L}$	0	0
	0	1	2	3	4

Taking $H^k(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell) = \bigoplus_{i+j=k} H^i(\mathcal{A}_2, \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell)$ yields the result.

Theorem 1.3.3. The cohomology $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ is Tate type for all i and all $1 \leq n \leq 6$. In this range the compactly supported Euler characteristics are given:

$$\begin{aligned}
e_c(\mathcal{X}_2, \mathbb{Q}_\ell) &= \mathbb{L}^5 + 2\mathbb{L}^4 + 2\mathbb{L}^3 + \mathbb{L}^2 - 1 \\
e_c(\mathcal{X}_2^{\times 2}, \mathbb{Q}_\ell) &= \mathbb{L}^7 + 4\mathbb{L}^6 + 9\mathbb{L}^5 + 9\mathbb{L}^4 + 3\mathbb{L}^3 - 5\mathbb{L}^2 - 5\mathbb{L} - 3 \\
e_c(\mathcal{X}_2^{\times 3}, \mathbb{Q}_\ell) &= \mathbb{L}^9 + 7\mathbb{L}^8 + 27\mathbb{L}^7 + 49\mathbb{L}^6 + 46\mathbb{L}^5 + 3\mathbb{L}^4 - 42\mathbb{L}^3 - 53\mathbb{L}^2 - 24\mathbb{L} - 7 \\
e_c(\mathcal{X}_2^{\times 4}, \mathbb{Q}_\ell) &= \mathbb{L}^{11} + 11\mathbb{L}^{10} + 65\mathbb{L}^9 + 191\mathbb{L}^8 + 320\mathbb{L}^7 + 257\mathbb{L}^6 \\
&\quad - 65\mathbb{L}^5 - 425\mathbb{L}^4 - 474\mathbb{L}^3 - 273\mathbb{L}^2 - 73\mathbb{L} - 14 \\
e_c(\mathcal{X}_2^{\times 5}, \mathbb{Q}_\ell) &= \mathbb{L}^{13} + 16\mathbb{L}^{12} + 135\mathbb{L}^{11} + 590\mathbb{L}^{10} + 1525\mathbb{L}^9 + 2292\mathbb{L}^8 + 1527\mathbb{L}^7 \\
&\quad - 1285\mathbb{L}^6 - 4219\mathbb{L}^5 - 4730\mathbb{L}^4 - 2814\mathbb{L}^3 - 923\mathbb{L}^2 - 135\mathbb{L} - 21 \\
e_c(\mathcal{X}_2^{\times 6}, \mathbb{Q}_\ell) &= \mathbb{L}^{15} + 22\mathbb{L}^{14} + 252\mathbb{L}^{13} + 1540\mathbb{L}^{12} + 5683\mathbb{L}^{11} + 13035\mathbb{L}^{10} + 17779\mathbb{L}^9 + 8660\mathbb{L}^8 \\
&\quad - 17614\mathbb{L}^7 - 44408\mathbb{L}^6 - 48770\mathbb{L}^5 - 30667\mathbb{L}^4 - 10437\mathbb{L}^3 - 1391\mathbb{L}^2 + 142\mathbb{L} + 2
\end{aligned}$$

The cohomology $H^i(\mathcal{X}_2^{\times 7}, \mathbb{Q}_\ell)$ is Tate type for all $i \neq 17$, whereas for $i = 17$ we have

$$H^{17}(\mathcal{X}_2^{\times 7}, \mathbb{Q}_\ell) = \mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^{17} + 1176\mathbb{L}^{15} + 63700\mathbb{L}^{13} + 6860\mathbb{L}^{12} + 321048\mathbb{L}^{11} + 294440\mathbb{L}^{10} + \mathbb{L}^9$$

where $\mathbb{S}_{\Gamma(1)}[18]$ is the 2-dimensional ℓ-adic Galois representation attached to the weight 18 cusp form $f_{18} = \Delta E_6 \in S_{18}(\Gamma(1))$. In this case the compactly supported Euler characteristic is given:

$$\begin{aligned}
e_c(\mathcal{X}_2^{\times 7}, \mathbb{Q}_\ell) = &-\mathbb{S}_{\Gamma(1)}[18] \\
&+ \mathbb{L}^{17} + 29\mathbb{L}^{16} + 434\mathbb{L}^{15} + 3542\mathbb{L}^{14} + 17717\mathbb{L}^{13} + 56924\mathbb{L}^{12} + 118692\mathbb{L}^{11} + 145567\mathbb{L}^{10} + 37850\mathbb{L}^9 \\
&- 226570\mathbb{L}^8 - 487150\mathbb{L}^7 - 529851\mathbb{L}^6 - 342930\mathbb{L}^5 - 121324\mathbb{L}^4 - 9491\mathbb{L}^3 + 9018\mathbb{L}^2 + 3164\mathbb{L} + 223
\end{aligned}$$

In particular the compactly supported Euler characteristic $e_c(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 7$.

Proof. Follows by combining 1.1.7 and 1.1.9 with 1.3.1. In this case we computed the multiplicities $m_\lambda^{j,n}$ with a SAGE program (available on request).

To argue that $e_c(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 7$ note that $H^{17}(\mathcal{X}_2^{\times 7}, \mathbb{Q}_\ell)$ (which is not Tate type, owing to the irreducible 2-dimensional contribution $\mathbb{S}_{\Gamma(1)}[18]$ to $H^3(\mathcal{A}_2, \mathbb{V}_{7,7})$) appears as a summand in $H^{17}(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for all $n \geq 7$ by the Künneth formula. This contribution cannot be cancelled in the Euler characteristic, at least for $7 \leq n \leq 15$: since the contribution occurs in $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for i odd, any contribution leading to cancellation would have to occur in $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for i even. Since $H^*(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = 0$ for $\lambda_1+\lambda_2 > 0$ odd, any contribution to $H^i(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for i even would have to come from a contribution to $H^j(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ for $j = 0, 2, 4$ (since $H^6(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2}) = 0$ for all $\lambda_1 \geq \lambda_2 \geq 0$). The only irreducible 2-dimensional contributions that occur in this way come from the contribution $\mathbb{S}_{\Gamma(1)}[\lambda_2+2]\mathbb{L}^{\lambda_1+2}$ to $H^4(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ (Poincare dual to the contribution $\mathbb{S}_{\Gamma(1)}[\lambda_2+2]$ to $H^2_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ in [96, Theorem 2.1]), which would require $\lambda_2 = 16$ for cancellation.

Now note that $H^{19}(\mathcal{X}_2^{\times 11}, \mathbb{Q}_\ell)$ (which is not Tate type, owing to the irreducible 4-dimensional contribution $\mathbb{S}_{\Gamma(1)}[6,8]$ to $H^3(\mathcal{A}_2, \mathbb{V}_{11,5})$) appears as a summand in $H^{19}(\mathcal{X}_2^{\times n}, \mathbb{Q}_\ell)$ for all $n \geq 11$ by the Künneth formula. This contribution cannot be cancelled in the Euler characteristic: by the same reasoning as above any contribution leading to cancellation would have to come from a contribution to $H^j(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ for $j = 0, 2, 4$, but there are no irreducible 4-dimensional contributions in this case: the only irreducible 4-dimensional contributions come from the contribution $\mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1-\lambda_2, \lambda_2+3]$ to $H^3(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ (Poincare dual to the contribution $\mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1-\lambda_2, \lambda_2+3]$ to $H^3_c(\mathcal{A}_2, \mathbb{V}_{\lambda_1,\lambda_2})$ in [96, Theorem 2.1]). $\square$

Note that the contribution $\mathbb{S}_{\Gamma(1)}[0,10]$ should always persist, but we cannot argue this without estimates on the multiplicities $m_\lambda^{j,n}$.

1.3.3 Point Counts for Abelian Surfaces

We now consider the point counts and consequences for arithmetic statistics which come from the above computations.

Since $e_c(\mathcal{A}_2, \mathbb{Q}_\ell) = \mathbb{L}^3 + \mathbb{L}^2$ we have

$$\#\mathcal{A}_2(\mathbb{F}_q) = q^3 + q^2$$

Since $e_c(\mathcal{X}_2, \mathbb{Q}_\ell) = \mathbb{L}^5 + 2\mathbb{L}^4 + 2\mathbb{L}^3 + \mathbb{L}^2 - 1$ we have

$$\#\mathcal{X}_2(\mathbb{F}_q) = q^5 + 2q^4 + 2q^3 + q^2 - 1$$

It follows that we have the expected value

$$\begin{aligned}\mathbb{E}(\#A_2(\mathbb{F}_q)) &= \frac{q^5+2q^4+2q^3+q^2-1}{q^3+q^2} \\ &= q^2+q+1-\frac{1}{q^3+q^2}\end{aligned}$$

For A a principally polarized Abelian surface the Weil conjectures yield

$$\#A(\mathbb{F}_q) = q^2 + a_3 q^{\frac{3}{2}} + a_2 q + a_1 q^{\frac{1}{2}} + 1$$

where a_i is a sum of n_i roots of unity for $n_i = 4, 6, 4$ for $i = 1, 2, 3$ respectively. In particular we have

$$|\#A(\mathbb{F}_q) - (q^2+1)| \le 4q^{\frac{3}{2}} + 6q^2 + 4q^{\frac{1}{2}}$$

On the other hand by the Honda-Tate correspondence for Abelian surfaces there exists a simple Abelian surface A over $\mathbb{F}_q$ with $\#A(\mathbb{F}_q) = q^2+q+1$, corresponding to the case $a_3 = a_1 = 0$. In particular we have

$$\min_{[A,\lambda]\in[\mathcal{A}_2(\mathbb{F}_q)]} \#A(\mathbb{F}_q) = q^2+q+1$$

Comparing this to the computation of the expected value $\mathbb{E}(\#A_2(\mathbb{F}_q))$ yields

$$\lim_{q\to\infty} |\mathbb{E}(\#A_2(\mathbb{F}_q)) - \min_{[A,\lambda]\in[\mathcal{A}_2(\mathbb{F}_q)]} \#A(\mathbb{F}_q)| = 0$$

Since $e_c(\mathcal{X}_2^{\times 2}, \mathbb{Q}_\ell) = \mathbb{L}^7 + 4\mathbb{L}^6 + 9\mathbb{L}^5 + 9\mathbb{L}^4 + 3\mathbb{L}^3 - 5\mathbb{L}^2 - 5\mathbb{L} - 3$ we have

$$\#\mathcal{X}_2^{\times 2}(\mathbb{F}_q) = q^7 + 4q^6 + 9q^5 + 9q^4 + 3q^3 - 5q^2 - 5q - 3$$

It follows that we have the expected value

$$\begin{aligned}\mathbb{E}(\#A_2(\mathbb{F}_q)^2) &= \frac{q^7+4q^6+9q^5+9q^4+3q^3-5q^2-5q-3}{q^3+q^2} \\ &= q^4+3q^3+6q^2+3q-\frac{5q^2+5q+3}{q^3+q^2}\end{aligned}$$

and we have the variance

$$\begin{aligned}\mathrm{Var}(\#A_2(\mathbb{F}_q)) &= \mathbb{E}(\#A_2(\mathbb{F}_q)^2) - \mathbb{E}(\#A_2(\mathbb{F}_q))^2 \\ &= q^3 + 3q^2 + q - 1 - \frac{3q^5 + 6q^4 + 4q^3 + q^2 + 1}{(q^3 + q^2)^2}\end{aligned}$$

We can continue in this way to obtain the first 6 terms of the moment generating function:

Corollary 1.3.4. The first 6 terms of the moment generating function $M_{\#A_2(\mathbb{F}_q)}(t)$ are rational functions in q:

$$\begin{aligned}
&1 + (q^2 + q + 1 - \frac{1}{q^3 + q^2})t \\
&+ (q^4 + 3q^3 + 6q^2 + 3q - \frac{5q^2 + 5q + 3}{q^3 + q^2})\frac{t^2}{2!} \\
&+ (\begin{matrix} q^6 + 6q^5 + 21q^4 + 28q^3 \\ +18q^2 - 15q - 27 \end{matrix} - \frac{26q^2 + 24q + 7}{q^3 + q^2})\frac{t^3}{3!} \\
&+ (\begin{matrix} q^8 + 10q^7 + 55q^6 + 136q^5 + 184q^4 \\ +73q^3 - 138q^2 - 287q - 187 \end{matrix} - \frac{86q^2 + 73q + 14}{q^3 + q^2})\frac{t^4}{4!} \\
&+ (\begin{matrix} q^{10} + 15q^9 + 120q^8 + 470q^7 + 1055q^6 + 1237q^5 \\ +290q^4 - 1575q^3 - 2644q^2 - 2086q - 728 \end{matrix} - \frac{195q^2 + 135q + 21}{q^3 + q^2})\frac{t^5}{5!} \\
&+ (\begin{matrix} q^{12} + 21q^{11} + 231q^{10} + 1309q^9 + 4374q^8 + 8661q^7 + 9118q^6 \\ -458q^5 - 17156q^4 - 27252q^3 - 21518q^2 - 9149q - 1288 \end{matrix} - \frac{103q^2 - 142q - 2}{q^3 + q^2})\frac{t^6}{6!}
\end{aligned}$$

1.4 Arithmetic Statistics for Abelian Threefolds

In this section we will summarize what is known and conjectured about the cohomology of local systems on $\mathcal{A}_3$, and then use this to deduce some results about arithmetic statistics for principally polarized Abelian threefolds over finite fields.

The Torelli morphism $\tau : \mathcal{M}_3 \to \mathcal{A}_3$ has degree 2 and is ramified along the hyperelliptic locus $\mathcal{H}_3 \subseteq \mathcal{M}_3$, and we have a stratification

$$\mathcal{A}_3 = \tau(\mathcal{M}_3 - \mathcal{H}_3) \amalg \tau(\mathcal{H}_3) \amalg (\tau(\mathcal{M}_2) \times \mathcal{A}_1) \amalg \mathrm{Sym}^3(\mathcal{A}_1)$$

In other words, every principally polarized Abelian threefold is either the Jacobian of a genus 3 (either plane quartic or hyperelliptic) curve, or a product of the Jacobian of a genus 2 curve and an elliptic curve, or a product of elliptic curves. Consequently, the point counts in this situation can be reduced to point counts on curves of genus ≤ 3, and one can use results about the cohomology of local systems on $\mathcal{M}_3$ to study arithmetic statistics for genus 2 curves over finite fields in the same way that we do for principally polarized Abelian surfaces over finite fields. We have chosen not to investigate this direction, but we mention it especially because this strategy of point counts on curves of genus ≤ 3 was what originally led to the conjectures of Bergström-Faber-van der Geer which we use in this section.

1.4.1 Cohomology of Local Systems on $\mathcal{A}_3$

Let $\mathcal{A}_3$ be the moduli stack of principally polarized Abelian threefolds, which is a smooth Deligne-Mumford stack of dimension 6 over $\mathbb{Z}$. Let $\pi : \mathcal{X}_3 \to \mathcal{A}_3$ be the universal Abelian threefold over $\mathcal{A}_3$ and let $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}_\ell$ be the ℓ-adic local system on $\mathcal{A}_3$ corresponding to the standard representation of Sp_6. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0)$ a dominant integral highest weight for Sp_6 let $\mathbb{V}_\lambda$ be the ℓ-adic local system on $\mathcal{A}_3$ corresponding to the irreducible representation of Sp_6 of highest weight λ, occurring in $\mathrm{Sym}^{\lambda_1-\lambda_2}(\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_3}(\wedge^3\mathbb{V})$. For $\lambda_1+\lambda_2+\lambda_3$ odd we have $H^*(\mathcal{A}_3, \mathbb{V}_\lambda) = 0$ since $-\mathrm{id} \in \mathrm{Sp}_6(\mathbb{Z})$ acts by multiplication by $(-1)^{\lambda_1+\lambda_2+\lambda_3}$ on the stalks of $\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3}$.

Let $\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_3+4] = \bigoplus_f \rho_f$ be the ℓ-adic Galois representation corresponding to vector-valued Siegel cusp forms of weight $(\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_3+4)$ for $\Gamma(1) = \mathrm{Sp}_6(\mathbb{Z})$: for each eigenform $f \in S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ we have an 8-dimensional ℓ-adic Galois representation ρ_f, and we have

$$\mathrm{tr}(\mathrm{Frob}_p|\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_3+4]) = \mathrm{tr}(T_p|S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1)))$$

for every prime p, which determines $\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_3+4]$ as an element of the Grothendieck group of ℓ-adic Galois representations.

As a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the ℓ-adic Galois representation ρ_F need not be irreducible, for example if F is one of the lifts predicted by [15, Conjecture 7.7]. On the other hand if $F \in S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ is a vector-valued Siegel modular form of general type, the ℓ-adic Galois representation ρ_F is predicted to be irreducible as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Write $\mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]$ for the ℓ-adic Galois representation corresponding to vector-valued Siegel cusp forms of general type.

Let $\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]=\mathrm{Gr}^W_{\lambda_1+\lambda_2+\lambda_3+4}\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]$ be the ℓ-adic Galois representation corresponding to Siegel cusp forms satisfying the generalized Ramanujan conjecture, that is those cusp forms $f\in S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ such that the roots of the characteristic polynomial of the Frobenius ϕ_p acting on the ℓ-adic Galois representation ρ_f for $\ell\neq p$, have absolute value $p^{-\frac{\lambda_1+\lambda_2+\lambda_3+4}{2}}$.

By [15, Conjecture 7.7] one predicts the following lifts. For eigenforms $f\in S_{\lambda_2+3}(\Gamma(1)), g\in S_{\lambda_1+\lambda_3+5}(\Gamma(1))$, and $h\in S_{\lambda_1-\lambda_3+3}(\Gamma(1))$ there should exist an eigenform $F\in S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ with spinor L-function

$$L(F,s)=L(f\otimes g,s)L(f\otimes h,s-\lambda_3-1)$$

For $\lambda_1=\lambda_2$ and for eigenforms $f\in S_{\lambda_3+2}(\Gamma(1))$ and $g\in S_{2\lambda_1+6}(\Gamma(1))$ there should exist an eigenform $F\in S_{0,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ with spinor L-function

$$L(F,s)=L(f,s-\lambda_1-2)L(f,s-\lambda_1-3)L(f\otimes g,s)$$

For $\lambda_2=\lambda_3$ and for eigenforms $f\in S_{\lambda_1+4}(\Gamma(1))$ and $g\in S_{2\lambda_2+4}(\Gamma(1))$ there should exist an eigenform $F\in S_{\lambda_1-\lambda_2,0,\lambda_3+4}(\Gamma(1))$ with spinor L-function

$$L(F,s)=L(f,s-\lambda_2-1)L(f,s-\lambda_2-2)L(f\otimes g,s)$$

For example for $\Delta\in S_{12}(\Gamma(1))$ and $f_{20}=\Delta E_4^2\in S_{20}(\Gamma(1))$ we have the Miyawaki lift $\chi_{12}\in S_{0,0,12}(\Gamma(1))$ with spinor L-function $L(\chi_{12},s)=L(\Delta,s-9)L(\Delta,s-10)L(\Delta\otimes f_{20},s)$.

Now by [15, Conjecture 7.11] we should have a decomposition

$$\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]\simeq\mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]\oplus\mathbb{S}^{\mathrm{lift}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]$$

where $\mathbb{S}^{\mathrm{lift}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]$ is given

$$\begin{aligned}\mathbb{S}^{\mathrm{lift}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4] &\simeq s_{\Gamma(1)}[\lambda_1-\lambda_3+3]\mathbb{S}_{\Gamma(1)}[\lambda_2+3]\otimes\mathbb{S}_{\Gamma(1)}[\lambda_1+\lambda_3+5]\\ &\oplus\begin{cases}\mathbb{S}_{\Gamma(1)}[\lambda_3+2]\otimes\mathbb{S}_{\Gamma(1)}[2\lambda_1+6] & \lambda_1=\lambda_2 \text{ and } \lambda_3>0\\ 0 & \text{otherwise}\end{cases}\\ &\oplus\begin{cases}\mathbb{S}_{\Gamma(1)}[\lambda_1+4]\otimes\mathbb{S}_{\Gamma(1)}[2\lambda_2+4] & \lambda_2=\lambda_3\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

We also have contributions from non-holomorphic endoscopic lifts: we should have a decomposition

$$H^6_!(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})=\mathbb{S}^{\mathrm{Ram}}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]\oplus H^6_{\mathrm{endo}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$$

where $H^6_{\mathrm{endo}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$ is given

$$\begin{aligned}H^3_{\mathrm{endo}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3}) &= s_{\Gamma(1)}[\lambda_2+\lambda_3+4]\mathbb{S}_{\Gamma(1)}[\lambda_1+4]\otimes\mathbb{S}_{\Gamma(1)}[\lambda_2-\lambda_3+2]\mathbb{L}^{\lambda_3+1}\\ &\oplus s_{\Gamma(1)}[\lambda_1-\lambda_3+3]\mathbb{S}_{\Gamma(1)}[\lambda_2+3]\otimes\mathbb{S}_{\Gamma(1)}[\lambda_1+\lambda_3+5]\\ &\oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+6]\mathbb{S}_{\Gamma(1)}[\lambda_3+2]\otimes\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+2}\end{aligned}$$

By work of Bergström-Faber-van der Geer, one conjectures the following:

Conjecture 1.4.1. [15, Conjecture 7.1] For $\lambda_1\geq\lambda_2\geq\lambda_3\geq 0$ with $\lambda_1+\lambda_2+\lambda_3>0$ even we have

$$e_{\mathrm{c}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})=\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4]+e_{\mathrm{c,extr}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$$

as an element of the Grothendieck group of ℓ-adic Galois representations, where $e_{\mathrm{c,extr}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$ is given by

$$\begin{aligned}e_{\mathrm{c,extr}}(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3}) &= -e_{\mathrm{c}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1+1,\lambda_2+1})-e_{\mathrm{c,extr}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1+1,\lambda_2+1})\otimes\mathbb{S}_{\Gamma(1)}[\lambda_3+2]\\ &+e_{\mathrm{c}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1+1,\lambda_3})+e_{\mathrm{c,extr}}(\mathcal{A}_2,\mathbb{V}_{\lambda_1+1,\lambda_3})\otimes\mathbb{S}_{\Gamma(1)}[\lambda_2+3]\\ &-e_{\mathrm{c}}(\mathcal{A}_2,\mathbb{V}_{\lambda_2,\lambda_3})-e_{\mathrm{c,extr}}(\mathcal{A}_2,\mathbb{V}_{\lambda_2,\lambda_3})\otimes\mathbb{S}_{\Gamma(1)}[\lambda_1+4]\end{aligned}$$

Poincare dually, for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 > 0$ even we have

$$e(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3}) = \mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 + 4] + e_{\mathrm{extr}}(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$$

as an element of the Grothendieck group of ℓ-adic Galois representations, where $e_{\mathrm{extr}}(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$ is given by

$$\begin{aligned}
e_{\mathrm{extr}}(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3}) = &-e(\mathcal{A}_2, \mathbb{V}_{\lambda_1+1,\lambda_2+1})\mathbb{L}^{\lambda_3+1} - e_{\mathrm{extr}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1+1,\lambda_2+1}) \otimes \mathbb{S}_{\Gamma(1)}[\lambda_3 + 2] \\
&+ e(\mathcal{A}_2, \mathbb{V}_{\lambda_1+1,\lambda_3})\mathbb{L}^{\lambda_2+2} + e_{\mathrm{extr}}(\mathcal{A}_2, \mathbb{V}_{\lambda_1+1,\lambda_3}) \otimes \mathbb{S}_{\Gamma(1)}[\lambda_2 + 3] \\
&- e(\mathcal{A}_2, \mathbb{V}_{\lambda_2,\lambda_3})\mathbb{L}^{\lambda_1+3} - e_{\mathrm{extr}}(\mathcal{A}_2, \mathbb{V}_{\lambda_2,\lambda_3}) \otimes \mathbb{S}_{\Gamma(1)}[\lambda_1 + 4]
\end{aligned}$$

This remains true for $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$ if we set $\mathbb{S}_{\Gamma(1)}[0, 0, 4] := \mathbb{L}^6 + \mathbb{L}^5 + \mathbb{L}^4 + 2\mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1$: by [53, Theorem 1] the compactly supported cohomology $H^*_c(\mathcal{A}_3, \mathbb{Q}_\ell)$ is concentrated in degrees 6, 8, 10, and 12 where $H^{12}_c(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^6$, $H^{10}_c(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^5$, $H^8_c(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^4$, and $H^6_c(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^3 + 1$, in particular $e_c(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^6 + \mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^3 + 1$. Poincare dually, the cohomology $H^*(\mathcal{A}_3, \mathbb{Q}_\ell)$ is concentrated in degrees 0, 2, 4, 6 and given by $H^0(\mathcal{A}_3, \mathbb{Q}_\ell) = 1$, $H^2(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}$, $H^4(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^2$, and $H^6(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^6 + \mathbb{L}^3$, in particular $e(\mathcal{A}_3, \mathbb{Q}) = \mathbb{L}^6 + \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1$.

As explained in [15, Section 8] this conjecture was made after extensive point counts for curves up to genus 3 over finite fields. In particular by [15, Remark 8.2] the conjecture is true for all $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1+\lambda_2+\lambda_3 \leq 6$ on the basis of these point counts since $S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ has dimension 0 in these cases by [113]. By the arguments in [12], using the classification results of [26], the conjecture is true for all $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 + \lambda_2 + \lambda_3 \leq 16$ on the basis of these point counts. The conjecture is claimed to be proven unconditionally by unpublished work of Taïbi [112].

We will use the following values for the Euler characteristics $e_c(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$, which are obtained by combining 1.4.1 with the vanishing $S_{\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_3+4}(\Gamma(1))$ for all $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ with $\lambda_1, \lambda_2, \lambda_3 \leq 6$ obtained by [113] (compare to the tables at the end of [15]):

$(\lambda_1,\lambda_2,\lambda_3)$	$e_c(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$
$(0,0,0)$	$\mathbb{L}^6+\mathbb{L}^5+\mathbb{L}^4+\mathbb{L}^3+1$
$(2,0,0)$	$-\mathbb{L}^3-\mathbb{L}^2$
$(1,1,0)$	$-\mathbb{L}$
$(4,0,0)$	$-\mathbb{L}^3-\mathbb{L}^2$
$(3,1,0)$	0
$(2,2,0)$	0
$(2,1,1)$	1
$(6,0,0)$	$-2\mathbb{L}^3-\mathbb{L}^2$
$(5,1,0)$	$-\mathbb{L}^4$
$(4,2,0)$	$-\mathbb{L}^5+\mathbb{L}$
$(4,1,1)$	1
$(3,3,0)$	$\mathbb{L}^7-\mathbb{L}$
$(3,2,1)$	0
$(2,2,2)$	1
$(6,2,0)$	$\mathbb{L}$
$(6,1,1)$	$-\mathbb{L}^2+1$
$(5,3,0)$	0
$(5,2,1)$	0
$(4,4,0)$	0
$(4,3,1)$	0
$(4,2,2)$	$\mathbb{L}^4$
$(3,3,2)$	$-\mathbb{L}^6+1$

$(\lambda_1,\lambda_2,\lambda_3)$	$e_c(\mathcal{A}_3,\mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$
$(6,4,0)$	$-\mathbb{L}^7+\mathbb{L}$
$(6,3,1)$	$-\mathbb{L}^2$
$(6,2,2)$	0
$(5,5,0)$	$\mathbb{L}^9-\mathbb{L}$
$(5,4,1)$	0
$(5,3,2)$	$-\mathbb{L}^3$
$(4,4,2)$	0
$(4,3,3)$	$-\mathbb{L}^4+1$
$(6,6,0)$	$\mathbb{S}_{\Gamma(1)}[0,10]+\mathbb{L}^{10}$
$(6,5,1)$	$-\mathbb{L}^2$
$(6,4,2)$	$\mathbb{L}^6-1$
$(6,3,3)$	1
$(5,5,2)$	$-\mathbb{L}^8-\mathbb{L}^3+1$
$(5,4,3)$	0
$(4,4,4)$	$-\mathbb{L}^6+1$
$(6,6,2)$	$\mathbb{S}_{\Gamma(1)}[0,10]-\mathbb{L}^9+\mathbb{L}^3$
$(6,5,3)$	$\mathbb{L}^4$
$(6,4,4)$	0
$(5,5,4)$	$-\mathbb{L}^8+1$
$(6,6,4)$	$\mathbb{S}_{\Gamma(1)}[0,10]-\mathbb{L}^9$
$(6,5,5)$	$-\mathbb{L}^6+1$
$(6,6,6)$	$\mathbb{S}_{\Gamma(1)}[0,10]-\mathbb{L}^9-\mathbb{L}^8+1$

We will use another contribution which does not appear n the above table. For $\lambda=(9,6,3)$ we have a contribution from an 8-dimensional Galois representation $\mathbb{S}_{\Gamma(1)}[3,3,7]$ which decomposes into a 1-dimensional Galois representation and an irreducible 7-dimensional Galois representation (see [15, Example 9.1]), which is explained by a functorial lift from the exceptional group G_2 predicted by [49].

The Langlands correspondence predicts in this case that an irreducible 8-dimensional Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_8(\overline{\mathbb{Q}}_\ell)$ (which is the composition of a Spin_7 Galois representation $\rho' : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell) = \widehat{\mathrm{PGSp}_6}$ with the 8-dimensional spin representation $\mathrm{spin} : \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}_8(\overline{\mathbb{Q}}_\ell)$) contributing to the cohomology $H^*(\mathcal{A}_3, \mathbb{V}_\lambda)$ must come from a packet of cuspidal automorphic representations π of $\mathrm{PGSp}_6(\mathbb{A}_\mathbb{Q})$ with π_∞ varying over all members of a discrete series L-packet. As the $(\mathfrak{sp}_6, \mathrm{U}(3))$-cohomology of such discrete series representations is concentrated in degree 3, such a contribution can only occur in $H^6(\mathcal{A}_3, \mathbb{V}_\lambda)$.

As explained in [49], any $\rho' : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$ factoring through the inclusion $\widehat{\mathrm{G}}_2 = \mathrm{G}_2(\overline{\mathbb{Q}}_\ell) \hookrightarrow \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell) = \widehat{\mathrm{PGSp}_6}$ of the stabilizer of a non-isotropic vector in the 8-dimensional spin representation must come from a packet of cuspidal automorphic representations π of $\mathrm{G}_2(\mathbb{A}_\mathbb{Q})$ which lifts to a packet of cuspidal automorphic representations π' of $\mathrm{PGSp}_6(\mathbb{A}_\mathbb{Q})$ with π'_∞ varying over all but one member of a discrete series L-packet, and again such a contribution can only occur in $H^6(\mathcal{A}_3, \mathbb{V}_\lambda)$; the remaining 1-dimensional Tate-type contribution comes from the cycle class of a Hilbert modular threefold in this Siegel modular 6-fold.

We record these predictions as the following conjecture:

Conjecture 1.4.2. Any irreducible ℓ-adic Galois representation of dimension 7 or 8 occurring in $H^*(\mathcal{A}_3, \mathbb{V}_\lambda)$ can only occur in $H^6(\mathcal{A}_3, \mathbb{V}_\lambda)$.

1.4.2 Examples: Euler Characteristics of $\mathcal{X}_3^{\times n}$ through $n = 6$

In this section we compute $e_\mathrm{c}(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ up to $n = 6$. The case $n = 6$ is the first case where $e_\mathrm{c}(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ is not of Tate type, again owing to a contribution from the Saito-Kurokawa lift $\chi_{10} \in S_{0,10}(\Gamma(1))$, but now in a much more complicated way.

We start by computing the local systems $\mathbb{R}^*\pi^n_*\mathbb{Q}_\ell = \bigoplus_{0 \le j \le 6n} \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell$ up to $n = 6$. For this it suffices to consider the local systems $\mathbb{R}^j\pi_*\mathbb{Q}_\ell$ for $0 \le j \le 3n$ even with Tate twists omitted since $\mathbb{R}^j\pi^n_*\mathbb{Q}_\ell$ is pure of weight j (so the missing Tate twists can be inferred from the weights of the local systems $\mathbb{V}_\lambda$) and $\mathbb{R}^{6n-j}\pi^n_*\mathbb{Q}_\ell \simeq \mathbb{R}^j\pi^n_*\mathbb{Q}_\ell(-3n-j)$.

We used SAGE (the source code for this is commented out above this line in the source code for this document) to compute the following:

Proposition 1.4.3. (i) The local system $\mathbb{R}^*\pi_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 6$ and given by

j	$\mathbb{R}^j\pi_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0,0}$
2	$\mathbb{V}_{0,0,0} + \mathbb{V}_{1,1,0}$

(ii) The local system $\mathbb{R}^*\pi^2_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 12$ and given by

j	$\mathbb{R}^j\pi^2_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0,0}$
2	$3\mathbb{V}_{0,0,0} + 3\mathbb{V}_{1,1,0} + \mathbb{V}_{2,0,0}$
4	$6\mathbb{V}_{0,0,0} + 9\mathbb{V}_{1,1,0} + 3\mathbb{V}_{2,0,0} + 3\mathbb{V}_{2,1,1} + \mathbb{V}_{2,2,0}$
6	$10\mathbb{V}_{0,0,0} + 11\mathbb{V}_{1,1,0} + 6\mathbb{V}_{2,0,0} + 4\mathbb{V}_{2,1,1} + 3\mathbb{V}_{2,2,0} + \mathbb{V}_{2,2,2}$

(iii) The local system $\mathbb{R}^*\pi^3_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 18$ and given by

j	$\mathbb{R}^j\pi^3_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0,0}$
2	$6\mathbb{V}_{0,0,0} + 6\mathbb{V}_{1,1,0} + 3\mathbb{V}_{2,0,0}$
4	$21\mathbb{V}_{0,0,0} + 36\mathbb{V}_{1,1,0} + 18\mathbb{V}_{2,0,0} + 15\mathbb{V}_{2,1,1} + 6\mathbb{V}_{2,2,0} + 3\mathbb{V}_{3,1,0}$
6	$56\mathbb{V}_{0,0,0} + 98\mathbb{V}_{1,1,0} + 63\mathbb{V}_{2,0,0} + 55\mathbb{V}_{2,1,1} + 36\mathbb{V}_{2,2,0}$
	$+10\mathbb{V}_{2,2,2} + 18\mathbb{V}_{3,1,0} + 8\mathbb{V}_{3,2,1} + \mathbb{V}_{3,3,0}$
8	$81\mathbb{V}_{0,0,0} + 168\mathbb{V}_{1,1,0} + 105\mathbb{V}_{2,0,0} + 105\mathbb{V}_{2,1,1} + 66\mathbb{V}_{2,2,0}$
	$+18\mathbb{V}_{2,2,2} + 42\mathbb{V}_{3,1,0} + 24\mathbb{V}_{3,2,1} + 6\mathbb{V}_{3,3,0} + 3\mathbb{V}_{3,3,2}$

(iv) The local system $\mathbb{R}^*\pi^4_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 24$ and given by

j	$\mathbb{R}^j\pi^4_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0,0}$
2	$10\mathbb{V}_{0,0,0} + 10\mathbb{V}_{1,1,0} + 6\mathbb{V}_{2,0,0}$
4	$55\mathbb{V}_{0,0,0} + 100\mathbb{V}_{1,1,0} + 60\mathbb{V}_{2,0,0} + 45\mathbb{V}_{2,1,1} + 20\mathbb{V}_{2,2,0} + 15\mathbb{V}_{3,1,0} + \mathbb{V}_{4,0,0}$
6	$220\mathbb{V}_{0,0,0} + 466\mathbb{V}_{1,1,0} + 330\mathbb{V}_{2,0,0} + 310\mathbb{V}_{2,1,1} + 200\mathbb{V}_{2,2,0}$
	$+50\mathbb{V}_{2,2,2} + 150\mathbb{V}_{3,1,0} + 64\mathbb{V}_{3,2,1} + 10\mathbb{V}_{3,3,0} + 10\mathbb{V}_{4,0,0} + 10\mathbb{V}_{4,1,1} + 6\mathbb{V}_{4,2,0}$
8	$550\mathbb{V}_{0,0,0} + 1360\mathbb{V}_{1,1,0} + 1005\mathbb{V}_{2,0,0} + 1075\mathbb{V}_{2,1,1} + 740\mathbb{V}_{2,2,0}$
	$+220\mathbb{V}_{2,2,2} + 636\mathbb{V}_{3,1,0} + 384\mathbb{V}_{3,2,1} + 100\mathbb{V}_{3,3,0} + 45\mathbb{V}_{3,3,2}$
	$+55\mathbb{V}_{4,0,0} + 65\mathbb{V}_{4,1,1} + 60\mathbb{V}_{4,2,0} + 20\mathbb{V}_{4,2,2} + 15\mathbb{V}_{4,3,1} + \mathbb{V}_{4,4,0}$
10	$946\mathbb{V}_{0,0,0} + 2530\mathbb{V}_{1,1,0} + 1910\mathbb{V}_{2,0,0} + 2200\mathbb{V}_{2,1,1} + 1550\mathbb{V}_{2,2,0}$
	$+490\mathbb{V}_{2,2,2} + 1410\mathbb{V}_{3,1,0} + 960\mathbb{V}_{3,2,1} + 280\mathbb{V}_{3,3,0} + 150\mathbb{V}_{3,3,2}$
	$+136\mathbb{V}_{4,0,0} + 200\mathbb{V}_{4,1,1} + 190\mathbb{V}_{4,2,0} + 74\mathbb{V}_{4,2,2} + 80\mathbb{V}_{4,3,1} + 10\mathbb{V}_{4,3,3} + 10\mathbb{V}_{4,4,0} + 6\mathbb{V}_{4,4,2}$
12	$1155\mathbb{V}_{0,0,0} + 3080\mathbb{V}_{1,1,0} + 2387\mathbb{V}_{2,0,0} + 2750\mathbb{V}_{2,1,1} + 2000\mathbb{V}_{2,2,0}$
	$+664\mathbb{V}_{2,2,2} + 1815\mathbb{V}_{3,1,0} + 1280\mathbb{V}_{3,2,1} + 375\mathbb{V}_{3,3,0} + 204\mathbb{V}_{3,3,2}$
	$+190\mathbb{V}_{4,0,0} + 275\mathbb{V}_{4,1,1} + 280\mathbb{V}_{4,2,0} + 120\mathbb{V}_{4,2,2} + 125\mathbb{V}_{4,3,1} + 16\mathbb{V}_{4,3,3} + 20\mathbb{V}_{4,4,0} + 15\mathbb{V}_{4,4,2} + \mathbb{V}_{4,4,4}$

(v) The local system $\mathbb{R}^*\pi^5_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 30$ and given by

j	$\mathbb{R}^j\pi^5_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0,0}$
2	$15\mathbb{V}_{0,0,0} + 15\mathbb{V}_{1,1,0} + 10\mathbb{V}_{2,0,0}$
4	$120\mathbb{V}_{0,0,0} + 225\mathbb{V}_{1,1,0} + 150\mathbb{V}_{2,0,0} + 105\mathbb{V}_{2,1,1} + 50\mathbb{V}_{2,2,0} + 45\mathbb{V}_{3,1,0} + 5\mathbb{V}_{4,0,0}$
6	$680\mathbb{V}_{0,0,0} + 1590\mathbb{V}_{1,1,0} + 1200\mathbb{V}_{2,0,0} + 1155\mathbb{V}_{2,1,1} + 750\mathbb{V}_{2,2,0} + 175\mathbb{V}_{2,2,2}$
	$+675\mathbb{V}_{3,1,0} + 280\mathbb{V}_{3,2,1} + 50\mathbb{V}_{3,3,0} + 75\mathbb{V}_{4,0,0} + 70\mathbb{V}_{4,1,1} + 45\mathbb{V}_{4,2,0} + 5\mathbb{V}_{5,1,0}$
8	$2565\mathbb{V}_{0,0,0} + 7050\mathbb{V}_{1,1,0} + 5645\mathbb{V}_{2,0,0} + 6300\mathbb{V}_{2,1,1} + 4500\mathbb{V}_{2,2,0} + 1365\mathbb{V}_{2,2,2}$
	$+4455\mathbb{V}_{3,1,0} + 2760\mathbb{V}_{3,2,1} + 750\mathbb{V}_{3,3,0} + 315\mathbb{V}_{3,3,2}$
	$+600\mathbb{V}_{4,0,0} + 735\mathbb{V}_{4,1,1} + 675\mathbb{V}_{4,2,0} + 210\mathbb{V}_{4,2,2} + 175\mathbb{V}_{4,3,1} + 15\mathbb{V}_{4,4,0}$
	$+75\mathbb{V}_{5,1,0} + 40\mathbb{V}_{5,2,1} + 10\mathbb{V}_{5,3,0}$
10	$6777\mathbb{V}_{0,0,0} + 20700\mathbb{V}_{1,1,0} + 17125\mathbb{V}_{2,0,0} + 21000\mathbb{V}_{2,1,1} + 15625\mathbb{V}_{2,2,0} + 5250\mathbb{V}_{2,2,2}$
	$+16425\mathbb{V}_{3,1,0} + 12000\mathbb{V}_{3,2,1} + 3750\mathbb{V}_{3,3,0} + 2025\mathbb{V}_{3,3,2}$
	$+2476\mathbb{V}_{4,0,0} + 3675\mathbb{V}_{4,1,1} + 3650\mathbb{V}_{4,2,0} + 1449\mathbb{V}_{4,2,2} + 1575\mathbb{V}_{4,3,1} + 175\mathbb{V}_{4,3,3} + 225\mathbb{V}_{4,4,0} + 126\mathbb{V}_{4,4,2}$
	$+474\mathbb{V}_{5,1,0} + 376\mathbb{V}_{5,2,1} + 150\mathbb{V}_{5,3,0} + 75\mathbb{V}_{5,3,2} + 24\mathbb{V}_{5,4,1} + \mathbb{V}_{5,5,0}$
12	$12965\mathbb{V}_{0,0,0} + 41630\mathbb{V}_{1,1,0} + 35430\mathbb{V}_{2,0,0} + 45325\mathbb{V}_{2,1,1} + 34875\mathbb{V}_{2,2,0}$
	$+12530\mathbb{V}_{2,2,2} + 37665\mathbb{V}_{3,1,0} + 29920\mathbb{V}_{3,2,1} + 9875\mathbb{V}_{3,3,0} + 5805\mathbb{V}_{3,3,2}$
	$+6165\mathbb{V}_{4,0,0} + 9800\mathbb{V}_{4,1,1} + 10290\mathbb{V}_{4,2,0} + 4585\mathbb{V}_{4,2,2} + 5250\mathbb{V}_{4,3,1} + 735\mathbb{V}_{4,3,3} + 925\mathbb{V}_{4,4,0} + 675\mathbb{V}_{4,4,2} + 35\mathbb{V}_{4,4,4}$
	$+1510\mathbb{V}_{5,1,0} + 1440\mathbb{V}_{5,2,1} + 660\mathbb{V}_{5,3,0} + 425\mathbb{V}_{5,3,2} + 200\mathbb{V}_{5,4,1} + 40\mathbb{V}_{5,4,3} + 15\mathbb{V}_{5,5,0} + 10\mathbb{V}_{5,5,2}$
14	$17775\mathbb{V}_{0,0,0} + 58920\mathbb{V}_{1,1,0} + 50550\mathbb{V}_{2,0,0} + 66255\mathbb{V}_{2,1,1} + 51450\mathbb{V}_{2,2,0} + 18900\mathbb{V}_{2,2,2}$
	$+56565\mathbb{V}_{3,1,0} + 46560\mathbb{V}_{3,2,1} + 15825\mathbb{V}_{3,3,0} + 9675\mathbb{V}_{3,3,2}$
	$+9555\mathbb{V}_{4,0,0} + 15750\mathbb{V}_{4,1,1} + 16800\mathbb{V}_{4,2,0} + 7770\mathbb{V}_{4,2,2} + 9170\mathbb{V}_{4,3,1} + 1400\mathbb{V}_{4,3,3} + 1695\mathbb{V}_{4,4,0} + 1305\mathbb{V}_{4,4,2} + 75\mathbb{V}_{4,4,4}$
	$+2655\mathbb{V}_{5,1,0} + 2720\mathbb{V}_{5,2,1} + 1325\mathbb{V}_{5,3,0} + 930\mathbb{V}_{5,3,2} + 480\mathbb{V}_{5,4,1} + 120\mathbb{V}_{5,4,3} + 50\mathbb{V}_{5,5,0} + 45\mathbb{V}_{5,5,2} + 5\mathbb{V}_{5,5,4}$

(vi) The local system $\mathbb{R}^*\pi^6_*\mathbb{Q}_\ell$ is concentrated in degrees $0, \ldots, 36$ and given by

q	$\mathbb{R}^q\pi^6_*\mathbb{Q}_\ell$
0	$\mathbb{V}_{0,0,0}$
2	$21\mathbb{V}_{0,0,0} + 21\mathbb{V}_{1,1,0} + 15\mathbb{V}_{2,0,0}$
4	$231\mathbb{V}_{0,0,0} + 441\mathbb{V}_{1,1,0} + 315\mathbb{V}_{2,0,0} + 210\mathbb{V}_{2,1,1} + 105\mathbb{V}_{2,2,0} + 105\mathbb{V}_{3,1,0} + 15\mathbb{V}_{4,0,0}$
6	$1771\mathbb{V}_{0,0,0} + 4389\mathbb{V}_{1,1,0} + 3465\mathbb{V}_{2,0,0} + 3360\mathbb{V}_{2,1,1} + 2205\mathbb{V}_{2,2,0} + 490\mathbb{V}_{2,2,2}$
	$+2205\mathbb{V}_{3,1,0} + 896\mathbb{V}_{3,2,1} + 175\mathbb{V}_{3,3,0} + 315\mathbb{V}_{4,0,0} + 280\mathbb{V}_{4,1,1} + 189\mathbb{V}_{4,2,0} + 35\mathbb{V}_{5,1,0} + \mathbb{V}_{6,0,0}$
8	$9339\mathbb{V}_{0,0,0} + 27489\mathbb{V}_{1,1,0} + 23100\mathbb{V}_{2,0,0} + 26460\mathbb{V}_{2,1,1} + 19305\mathbb{V}_{2,2,0} + 5880\mathbb{V}_{2,2,2}$
	$+20790\mathbb{V}_{3,1,0} + 13056\mathbb{V}_{3,2,1} + 3675\mathbb{V}_{3,3,0} + 1470\mathbb{V}_{3,3,2}$
	$+3465\mathbb{V}_{4,0,0} + 4305\mathbb{V}_{4,1,1} + 3969\mathbb{V}_{4,2,0} + 1176\mathbb{V}_{4,2,2} + 1050\mathbb{V}_{4,3,1} + 105\mathbb{V}_{4,4,0}$
	$+735\mathbb{V}_{5,1,0} + 384\mathbb{V}_{5,2,1} + 105\mathbb{V}_{5,3,0} + 21\mathbb{V}_{6,0,0} + 21\mathbb{V}_{6,1,1} + 15\mathbb{V}_{6,2,0}$
10	$35112\mathbb{V}_{0,0,0} + 116424\mathbb{V}_{1,1,0} + 101640\mathbb{V}_{2,0,0} + 129360\mathbb{V}_{2,1,1} + 99330\mathbb{V}_{2,2,0} + 34440\mathbb{V}_{2,2,2}$
	$+113190\mathbb{V}_{3,1,0} + 86016\mathbb{V}_{3,2,1} + 28050\mathbb{V}_{3,3,0} + 15120\mathbb{V}_{3,3,2}$
	$+21021\mathbb{V}_{4,0,0} + 31605\mathbb{V}_{4,1,1} + 32109\mathbb{V}_{4,2,0} + 12789\mathbb{V}_{4,2,2} + 14175\mathbb{V}_{4,3,1} + 1470\mathbb{V}_{4,3,3} + 2205\mathbb{V}_{4,4,0} + 1176\mathbb{V}_{4,4,2}$
	$+6699\mathbb{V}_{5,1,0} + 5376\mathbb{V}_{5,2,1} + 2205\mathbb{V}_{5,3,0} + 1050\mathbb{V}_{5,3,2} + 384\mathbb{V}_{5,4,1} + 21\mathbb{V}_{5,5,0}$
	$+231\mathbb{V}_{6,0,0} + 315\mathbb{V}_{6,1,1} + 315\mathbb{V}_{6,2,0} + 105\mathbb{V}_{6,2,2} + 105\mathbb{V}_{6,3,1} + 15\mathbb{V}_{6,4,0}$
12	$97097\mathbb{V}_{0,0,0} + 346577\mathbb{V}_{1,1,0} + 311850\mathbb{V}_{2,0,0} + 421960\mathbb{V}_{2,1,1} + 336105\mathbb{V}_{2,2,0} + 126280\mathbb{V}_{2,2,2}$
	$+396396\mathbb{V}_{3,1,0} + 335104\mathbb{V}_{3,2,1} + 117425\mathbb{V}_{3,3,0} + 71190\mathbb{V}_{3,3,2}$
	$+79821\mathbb{V}_{4,0,0} + 132055\mathbb{V}_{4,1,1} + 142065\mathbb{V}_{4,2,0} + 64729\mathbb{V}_{4,2,2} + 77175\mathbb{V}_{4,3,1} + 11025\mathbb{V}_{4,3,3} + 14630\mathbb{V}_{4,4,0} + 10521\mathbb{V}_{4,4,2} + 490\mathbb{V}_{4,4,4}$
	$+32879\mathbb{V}_{5,1,0} + 32256\mathbb{V}_{5,2,1} + 15345\mathbb{V}_{5,3,0} + 9800\mathbb{V}_{5,3,2} + 4864\mathbb{V}_{5,4,1} + 896\mathbb{V}_{5,4,3} + 441\mathbb{V}_{5,5,0} + 280\mathbb{V}_{5,5,2}$
	$+1309\mathbb{V}_{6,0,0} + 2205\mathbb{V}_{6,1,1} + 2415\mathbb{V}_{6,2,0} + 1071\mathbb{V}_{6,2,2} + 1365\mathbb{V}_{6,3,1} + 175\mathbb{V}_{6,3,3} + 315\mathbb{V}_{6,4,0} + 189\mathbb{V}_{6,4,2} + 35\mathbb{V}_{6,5,1} + \mathbb{V}_{6,6,0}$
14	$198627\mathbb{V}_{0,0,0} + 745437\mathbb{V}_{1,1,0} + 683025\mathbb{V}_{2,0,0} + 960960\mathbb{V}_{2,1,1} + 781605\mathbb{V}_{2,2,0} + 307230\mathbb{V}_{2,2,2}$
	$+944181\mathbb{V}_{3,1,0} + 849024\mathbb{V}_{3,2,1} + 310695\mathbb{V}_{3,3,0} + 200640\mathbb{V}_{3,3,2}$
	$+200151\mathbb{V}_{4,0,0} + 349755\mathbb{V}_{4,1,1} + 388080\mathbb{V}_{4,2,0} + 189189\mathbb{V}_{4,2,2} + 234465\mathbb{V}_{4,3,1} + 38325\mathbb{V}_{4,3,3} + 48510\mathbb{V}_{4,4,0} + 38661\mathbb{V}_{4,4,2} + 2415\mathbb{V}_{4,4,4}$
	$+96789\mathbb{V}_{5,1,0} + 105216\mathbb{V}_{5,2,1} + 53970\mathbb{V}_{5,3,0} + 38760\mathbb{V}_{5,3,2} + 21504\mathbb{V}_{5,4,1} + 5376\mathbb{V}_{5,4,3} + 2541\mathbb{V}_{5,5,0} + 2205\mathbb{V}_{5,5,2} + 210\mathbb{V}_{5,5,4}$
	$+4389\mathbb{V}_{6,0,0} + 8085\mathbb{V}_{6,1,1} + 9465\mathbb{V}_{6,2,0} + 4851\mathbb{V}_{6,2,2} + 6615\mathbb{V}_{6,3,1} + 1155\mathbb{V}_{6,3,3}$
	$+1890\mathbb{V}_{6,4,0} + 1539\mathbb{V}_{6,4,2} + 105\mathbb{V}_{6,4,4} + 420\mathbb{V}_{6,5,1} + 105\mathbb{V}_{6,5,3} + 21\mathbb{V}_{6,6,0} + 15\mathbb{V}_{6,6,2}$
16	$304122\mathbb{V}_{0,0,0} + 1174089\mathbb{V}_{1,1,0} + 1086765\mathbb{V}_{2,0,0} + 1562610\mathbb{V}_{2,1,1} + 1285305\mathbb{V}_{2,2,0} + 517440\mathbb{V}_{2,2,2}$
	$+1573719\mathbb{V}_{3,1,0} + 1462272\mathbb{V}_{3,2,1} + 547470\mathbb{V}_{3,3,0} + 364980\mathbb{V}_{3,3,2}$
	$+343266\mathbb{V}_{4,0,0} + 617925\mathbb{V}_{4,1,1} + 696234\mathbb{V}_{4,2,0} + 350889\mathbb{V}_{4,2,2} + 443940\mathbb{V}_{4,3,1} + 77385\mathbb{V}_{4,3,3} + 95844\mathbb{V}_{4,4,0} + 79926\mathbb{V}_{4,4,2} + 5565\mathbb{V}_{4,4,4}$
	$+181104\mathbb{V}_{5,1,0} + 206976\mathbb{V}_{5,2,1} + 110040\mathbb{V}_{5,3,0} + 83160\mathbb{V}_{5,3,2} + 48384\mathbb{V}_{5,4,1} + 13440\mathbb{V}_{5,4,3} + 6321\mathbb{V}_{5,5,0} + 5985\mathbb{V}_{5,5,2} + 735\mathbb{V}_{5,5,4}$
	$+8877\mathbb{V}_{6,0,0} + 17325\mathbb{V}_{6,1,1} + 20790\mathbb{V}_{6,2,0} + 11319\mathbb{V}_{6,2,2} + 16170\mathbb{V}_{6,3,1} + 3255\mathbb{V}_{6,3,3}$
	$+5040\mathbb{V}_{6,4,0} + 4536\mathbb{V}_{6,4,2} + 405\mathbb{V}_{6,4,4} + 1470\mathbb{V}_{6,5,1} + 504\mathbb{V}_{6,5,3} + 21\mathbb{V}_{6,5,5} + 105\mathbb{V}_{6,6,0} + 105\mathbb{V}_{6,6,2} + 15\mathbb{V}_{6,6,4}$
18	$350714\mathbb{V}_{0,0,0} + 1364454\mathbb{V}_{1,1,0} + 1268190\mathbb{V}_{2,0,0} + 1834560\mathbb{V}_{2,1,1} + 1515780\mathbb{V}_{2,2,0} + 615440\mathbb{V}_{2,2,2}$
	$+1862784\mathbb{V}_{3,1,0} + 1748992\mathbb{V}_{3,2,1} + 659120\mathbb{V}_{3,3,0} + 443520\mathbb{V}_{3,3,2}$
	$+410454\mathbb{V}_{4,0,0} + 744800\mathbb{V}_{4,1,1} + 844074\mathbb{V}_{4,2,0} + 430122\mathbb{V}_{4,2,2} + 546840\mathbb{V}_{4,3,1} + 97020\mathbb{V}_{4,3,3} + 119854\mathbb{V}_{4,4,0} + 101430\mathbb{V}_{4,4,2} + 7370\mathbb{V}_{4,4,4}$
	$+222376\mathbb{V}_{5,1,0} + 258048\mathbb{V}_{5,2,1} + 138600\mathbb{V}_{5,3,0} + 106260\mathbb{V}_{5,3,2} + 62720\mathbb{V}_{5,4,1} + 17920\mathbb{V}_{5,4,3} + 8379\mathbb{V}_{5,5,0} + 8085\mathbb{V}_{5,5,2} + 1035\mathbb{V}_{5,5,4}$
	$+11242\mathbb{V}_{6,0,0} + 22176\mathbb{V}_{6,1,1} + 26950\mathbb{V}_{6,2,0} + 14994\mathbb{V}_{6,2,2} + 21560\mathbb{V}_{6,3,1} + 4480\mathbb{V}_{6,3,3}$
	$+6930\mathbb{V}_{6,4,0} + 6426\mathbb{V}_{6,4,2} + 630\mathbb{V}_{6,4,4} + 2156\mathbb{V}_{6,5,1} + 784\mathbb{V}_{6,5,3} + 36\mathbb{V}_{6,5,5} + 175\mathbb{V}_{6,6,0} + 189\mathbb{V}_{6,6,2} + 35\mathbb{V}_{6,6,4} + \mathbb{V}_{6,6,6}$

Not enough is presently known about the individual cohomology groups of local systems on $\mathcal{A}_3$ to compute the individual cohomology groups of $\mathcal{X}_3^{\times n}$. To compute the individual cohomology groups of $\mathcal{X}_3$ we only need to know the individual cohomology groups of $\mathbb{V}_{0,0,0}$ and of $\mathbb{V}_{1,1,0}$. We know this in the first case by the earlier result of Hain, and in the second case we have the following result of Hulek-Tommasi:

Proposition 1.4.4. [61, Lemma 35] The cohomology $H_c^*(\mathcal{A}_3, \mathbb{V}_{1,1,0})$ is concentrated in degrees 5 (and possibly in degrees 8 and 9) and given by $H_c^5(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = \mathbb{L}$ (and by $H_c^8(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = H_c^9(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = \varepsilon\mathbb{L}^4$ for some $\varepsilon \in \{0, 1\}$, presumably $\varepsilon = 0$).

The indeterminacy in the above lemma involves the only possibly nontrivial differenial in the Gysin long exact sequence associated to the closed embedding $\mathcal{A}_3^{\mathrm{red}} \hookrightarrow \mathcal{A}_3$:

$$\mathbb{L}^5 = H_c^8(\mathcal{A}_3^{\mathrm{red}}, \mathbb{V}_{1,1,0}) \to H_c^9(\mathcal{M}_3, \mathbb{V}_{1,1,0}) = \mathbb{L}^5$$

but it seems difficult to analyze this differential directly. It may be possible to identify the contributions $H_c^8(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = H_c^9(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = \varepsilon\mathbb{L}^4$ as contributions to compactly supported Eisenstein cohomology for one of the standard parabolic subgroups of Sp_6, and then compute the relevant Eisenstein cohomology groups, but this is far beyond the scope of this paper. We will have to settle with the following theorem:

Theorem 1.4.5. The cohomology $H^*(\mathcal{X}_3, \mathbb{Q}_\ell)$ is concentrated in degrees $0, \ldots, 12$ and is given up to semisimplification by

k	$H^k(\mathcal{X}_3, \mathbb{Q}_\ell)$	k	$H^k(\mathcal{X}_3, \mathbb{Q}_\ell)$
0	1	1	0
2	$2\mathbb{L}$	3	0
4	$3\mathbb{L}^2$	5	$\varepsilon\mathbb{L}^4$
6	$\mathbb{L}^6 + \varepsilon\mathbb{L}^4 + 4\mathbb{L}^3$	7	$\varepsilon\mathbb{L}^5$
8	$\mathbb{L}^7 + \varepsilon\mathbb{L}^5 + 3\mathbb{L}^4$	9	$\mathbb{L}^7$
10	$\mathbb{L}^8 + 2\mathbb{L}^5$	11	$\mathbb{L}^8$
12	$\mathbb{L}^9 + \mathbb{L}^6$		

for some $\varepsilon \in \{0, 1\}$, presumably $\varepsilon = 0$.

Proof. We consider the spectral sequence $E_2^{p,q} = H^p(\mathcal{A}_3, \mathbb{R}^q\pi_*\mathbb{Q}_\ell) \Rightarrow H^{p+q}(\mathcal{X}_3, \mathbb{Q}_\ell)$. We have contributions to H^0, H^2, H^4, and H^6 from $H^0(\mathcal{A}_3, \mathbb{V}_{0,0,0}) = 1$, $H^2(\mathcal{A}_3, \mathbb{V}_{0,0,0}) = \mathbb{L}$, $H^4(\mathcal{A}_3, \mathbb{V}_{0,0,0}) = \mathbb{L}^2$, and $H^6(\mathcal{A}_3, \mathbb{V}_{0,0,0}) = \mathbb{L}^6 + \mathbb{L}^3$, and we have contributions to H^7 (and possibly to H^3 and H^4) given by $H^7(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = \mathbb{L}^7$ (and by $H^3(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = H^4(\mathcal{A}_3, \mathbb{V}_{1,1,0}) = \varepsilon\mathbb{L}^4$ for some $\varepsilon \in \{0,1\}$, presumably $\varepsilon = 0$). It follows that the E_2-page is given

$$\begin{array}{c|cccccccc}
6 & \mathbb{L}^3 & 0 & \mathbb{L}^4 & 0 & \mathbb{L}^5 & 0 & \mathbb{L}^9 + \mathbb{L}^6 & \\
4 & \mathbb{L}^2 & 0 & \mathbb{L}^3 & \varepsilon\mathbb{L}^5 & \varepsilon\mathbb{L}^5 + \mathbb{L}^4 & 0 & \mathbb{L}^8 + \mathbb{L}^5 & \mathbb{L}^8 \\
2 & \mathbb{L} & 0 & \mathbb{L}^2 & \varepsilon\mathbb{L}^4 & \varepsilon\mathbb{L}^4 + \mathbb{L}^3 & 0 & \mathbb{L}^7 + \mathbb{L}^4 & \mathbb{L}^7 \\
0 & 1 & 0 & \mathbb{L} & 0 & \mathbb{L}^2 & 0 & \mathbb{L}^6 + \mathbb{L}^3 & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}$$

Taking $H^k(\mathcal{X}_3, \mathbb{Q}_\ell) = \bigoplus_{p+q=k} H^p(\mathcal{A}_3, \mathbb{R}^q\pi_*\mathbb{Q}_\ell)$ yields the result. □

We cannot go much further with computations of individual cohomology groups, so we fall back to compactly supported cohomology. We obtain the following result, which is unconditional for $1 \leq n \leq 5$ on the basis of point counts, and is only conditional on the Euler characteristic $e_c(\mathcal{A}_3, \mathbb{V}_{6,6,6}) = \mathbb{S}_{\Gamma(1)}[0,10] - \mathbb{L}^9 - \mathbb{L}^8 + 1$ in the case $n = 6$ (but is very much conditional on the above predictions in the case $n \geq 9$):

Theorem 1.4.6. Assume conjectures 1.4.1 and 1.4.2. Then the compactly supported Euler characteristic $e_c(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ is Tate type for all $1 \leq n \leq 5$, and are given by:

$$\begin{aligned}
e_c(\mathcal{X}_3, \mathbb{Q}_\ell) &= \mathbb{L}^9 + 2\mathbb{L}^8 + 3\mathbb{L}^7 + 4\mathbb{L}^6 + 3\mathbb{L}^5 + 2\mathbb{L}^4 + 2\mathbb{L}^3 + 1 \\
e_c(\mathcal{X}_3^{\times 2}, \mathbb{Q}_\ell) &= \mathbb{L}^{12} + 4\mathbb{L}^{11} + 10\mathbb{L}^{10} + 20\mathbb{L}^9 + 25\mathbb{L}^8 + 24\mathbb{L}^7 + 17\mathbb{L}^6 + \mathbb{L}^5 - 8\mathbb{L}^4 - 4\mathbb{L}^3 - \mathbb{L}^2 + 4\mathbb{L} + 5 \\
e_c(\mathcal{X}_3^{\times 3}, \mathbb{Q}_\ell) &= \mathbb{L}^{15} + 7\mathbb{L}^{14} + 28\mathbb{L}^{13} + 84\mathbb{L}^{12} + 164\mathbb{L}^{11} + 237\mathbb{L}^{10} + 260\mathbb{L}^9 \\
&\quad + 164\mathbb{L}^8 - 21\mathbb{L}^7 - 171\mathbb{L}^6 - 212\mathbb{L}^5 - 107\mathbb{L}^4 + 47\mathbb{L}^3 + 99\mathbb{L}^2 + 75\mathbb{L} + 29 \\
e_c(\mathcal{X}_3^{\times 4}, \mathbb{Q}_\ell) &= \mathbb{L}^{18} + 11\mathbb{L}^{17} + 66\mathbb{L}^{16} + 286\mathbb{L}^{15} + 835\mathbb{L}^{14} + 1775\mathbb{L}^{13} + 2906\mathbb{L}^{12} + 3480\mathbb{L}^{11} + 2476\mathbb{L}^{10} \\
&\quad - 415\mathbb{L}^9 - 3846\mathbb{L}^8 - 5322\mathbb{L}^7 - 3781\mathbb{L}^6 - 597\mathbb{L}^5 + 2146\mathbb{L}^4 + 2877\mathbb{L}^3 + 1887\mathbb{L}^2 + 757\mathbb{L} + 162 \\
e_c(\mathcal{X}_3^{\times 5}, \mathbb{Q}_\ell) &= \mathbb{L}^{21} + 16\mathbb{L}^{20} + 136\mathbb{L}^{19} + 816\mathbb{L}^{18} + 3380\mathbb{L}^{17} + 10182\mathbb{L}^{16} + 23578\mathbb{L}^{15} \\
&\quad + 42433\mathbb{L}^{14} + 57157\mathbb{L}^{13} + 47250\mathbb{L}^{12} - 5213\mathbb{L}^{11} - 84003\mathbb{L}^{10} - 137082\mathbb{L}^9 - 124223\mathbb{L}^8 \\
&\quad - 52325\mathbb{L}^7 + 33070\mathbb{L}^6 + 83756\mathbb{L}^5 + 83816\mathbb{L}^4 + 53066\mathbb{L}^3 + 22340\mathbb{L}^2 + 6134\mathbb{L} + 891
\end{aligned}$$

The compactly supported Euler characteristic $e_c(\mathcal{X}_3^{\times 6}, \mathbb{Q}_\ell)$ is given by:

$$
\begin{aligned}
e_c(\mathcal{X}_3^{\times 6}, \mathbb{Q}_\ell) = &(\mathbb{L}^6 + 21\mathbb{L}^5 + 120\mathbb{L}^4 + 280\mathbb{L}^3 + 309\mathbb{L}^2 + 161\mathbb{L} + 32)\mathbb{S}_{\Gamma(1)}[0,10] \\
&+ \mathbb{L}^{24} + 22\mathbb{L}^{23} + 253\mathbb{L}^{22} + 2024\mathbb{L}^{21} + 11362\mathbb{L}^{20} + 46613\mathbb{L}^{19} \\
&+ 146665\mathbb{L}^{18} + 364262\mathbb{L}^{17} + 720246\mathbb{L}^{16} + 1084698\mathbb{L}^{15} + 1036149\mathbb{L}^{14} + 38201\mathbb{L}^{13} \\
&- 1876517\mathbb{L}^{12} - 3672164\mathbb{L}^{11} - 4024657\mathbb{L}^{10} - 2554079\mathbb{L}^9 + 101830\mathbb{L}^8 + 2028655\mathbb{L}^7 \\
&+ 2921857\mathbb{L}^6 + 2536864\mathbb{L}^5 + 1553198\mathbb{L}^4 + 687157\mathbb{L}^3 + 215631\mathbb{L}^2 + 45035\mathbb{L} + 4930
\end{aligned}
$$

where $\mathbb{S}_{\Gamma(1)}[0,10] = \mathbb{S}_{\Gamma(1)}[18] + \mathbb{L}^9 + \mathbb{L}^8$ is the 4-dimensional ℓ-adic Galois representation attached to the Saito-Kurokawa lift $\chi_{10} \in S_{0,10}(\Gamma(1))$ of the weight 18 cusp form $f_{18} = \Delta E_6 \in S_{18}(\Gamma(1))$. In particular the compactly supported Euler characteristic $e_c(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 6$.

Proof. Follows by combining 1.1.7 and 1.1.9 with 1.4.1. In this case we computed the multiplicities $m_\lambda^{j,n}$ with a SAGE program (available on request).

To argue that $e_c(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ is not Tate type if $n \geq 6$ note that $H^{24}(\mathcal{X}_3^{\times 9}, \mathbb{Q}_\ell)$ (which is not Tate type, owing to the 8-dimensional contribution $\mathbb{S}_{\Gamma(1)}[3,3,7]$ to $H^6(\mathcal{A}_3, \mathbb{V}_{9,6,3})$, which decomposes into a 1-dimensional contribution and an irreducible 7-dimensional contribution) appears as a summand in $H^{24}(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ for all $n \geq 9$ by the Künneth formula. This contribution cannot be cancelled in the Euler characteristic: since the contribution occurs in $H^i(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ for i even, any contribution leading to cancellation would have to occur in $H^i(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ for i odd. Since $H^*(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3}) = 0$ for $\lambda_1 + \lambda_2 + \lambda_3 > 0$ odd, any contribution to $H^i(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ for i odd would have to come from a contribution to $H^j(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$ for $j = 1,3,5,7,9,11$, but there are no irreducible 7-dimensional contributions in this case: the only irreducible 7-dimensional contributions come from the contributions to $H^6(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$ predicted by [49]. The remaining cases $n = 7, 8$ are checked by running the above computations further to see that the contribution $\mathbb{S}_{\Gamma(1)}[0,10]$ persists. □

Alternatively, note that $H^{26}(\mathcal{X}_3^{\times 10}, \mathbb{Q}_\ell)$ (which is not Tate type, owing to the irreducible 8-dimensional contributions $\mathbb{S}_{\Gamma(1)}[2,2,6]$ and $\mathbb{S}_{\Gamma(1)}[4,2,8]$ to $H^6(\mathcal{A}_3, \mathbb{V}_{10,8,2})$ and $H^6(\mathcal{A}_3, \mathbb{V}_{10,6,4})$ respectively, see [15, Table 1, Table 2]) appears as a summand in $H^{26}(\mathcal{X}_3^{\times n}, \mathbb{Q}_\ell)$ for all $n \geq 10$ by the Künneth formula. This contribution cannot be cancelled in the Euler characteristic by the same argument as above: the only irreducible 8-dimensional contributions come from the contribution $\mathbb{S}^{\mathrm{gen}}[\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 + 4]$ to $H^6(\mathcal{A}_3, \mathbb{V}_{\lambda_1,\lambda_2,\lambda_3})$. The remaining cases $n = 7, 8, 9$ are checked by running the above computations further to see that the contribution $\mathbb{S}_{\Gamma(1)}[0, 10]$ persists. This makes the above argument a bit less conjectural by removing the dependence on the functorial lift from G_2. That being said, since the above computations are already conditional on [15, Conjecture 7.1], we do not try to further justify the predictions of the Langlands correspondence which we have used in the above argument.

The contribution $(\mathbb{L}^6 + 21\mathbb{L}^5 + 120\mathbb{L}^4 + 280\mathbb{L}^3 + 309\mathbb{L}^2 + 161\mathbb{L} + 32)\mathbb{S}_{\Gamma(1)}[0, 10]$ to $e_c(\mathcal{X}_3^{\times 6}, \mathbb{Q}_\ell)$ comes from the following 4 contributions:

$$
\begin{aligned}
e_c(\mathcal{A}_3, \mathbb{V}_{6,6,6}) &+ (15\mathbb{L}^2 + 35\mathbb{L} + 15)e_c(\mathcal{A}_3, \mathbb{V}_{6,6,4}) \\
&+ (15\mathbb{L}^4 + 105\mathbb{L}^3 + 189\mathbb{L}^2 + 105\mathbb{L} + 15)e_c(\mathcal{A}_3, \mathbb{V}_{6,6,2}) \\
&+ (\mathbb{L}^6 + 21\mathbb{L}^5 + 105\mathbb{L}^4 + 175\mathbb{L}^3 + 105\mathbb{L}^2 + 21\mathbb{L} + 1)e_c(\mathcal{A}_3, \mathbb{V}_{6,6,0})
\end{aligned}
$$

which explains why the coefficients in the polynomial $\mathbb{L}^6 + 21\mathbb{L}^5 + 120\mathbb{L}^4 + 280\mathbb{L}^3 + 309\mathbb{L}^2 + 161\mathbb{L} + 32$ are not symmetric: it arises as the sum of 4 polynomials with symmetric coefficients of different degrees. Note that the contribution $\mathbb{S}_{\Gamma(1)}[0, 10]$ should always persist, but we cannot argue this without estimates on the multiplicities $m_\lambda^{j,n}$.

1.4.3 Point Counts for Abelian Threefolds

We now consider the point counts and consequences for arithmetic statistics which come from the above computations.

Since $e_c(\mathcal{A}_3, \mathbb{Q}_\ell) = \mathbb{L}^6 + \mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^3 + 1$ we have

$$\#\mathcal{A}_3(\mathbb{F}_q) = q^6 + q^5 + q^4 + q^3 + 1$$

Since $e_c(\mathcal{X}_3, \mathbb{Q}_\ell) = \mathbb{L}^9 + 2\mathbb{L}^8 + 3\mathbb{L}^7 + 4\mathbb{L}^6 + 3\mathbb{L}^5 + 2\mathbb{L}^4 + 2\mathbb{L}^3 + 1$ we have

$$\#\mathcal{X}_3(\mathbb{F}_q) = q^9 + 2q^8 + 3q^7 + 4q^6 + 3q^5 + 2q^4 + 2q^3 + 1$$

It follows that we have the expected value

$$\begin{aligned}\mathbb{E}(\#A_3(\mathbb{F}_q)) &= \frac{q^9+2q^8+3q^7+4q^6+3q^5+2q^4+2q^3+1}{q^6+q^5+q^4+q^3+1}\\ &= q^3+q^2+q+1-\frac{q^2+q}{q^6+q^5+q^4+q^3+1}\end{aligned}$$

For A a principally polarized Abelian threefold the Weil conjectures yield

$$\#A(\mathbb{F}_q) = q^3 + a_5 q^{\frac{5}{2}} + a_4 q^2 + a_3 q^{\frac{3}{2}} + a_2 q + a_1 q^{\frac{1}{2}} + 1$$

where a_i is a sum of n_i roots of unity for $n_i = 6, 15, 20, 15, 6$ for $i = 1, 2, 3, 4, 5$ respectively. In particular we have

$$|\#A(\mathbb{F}_q) - (q^3+1)| \leq 6q^{\frac{5}{2}} + 15q^2 + 20q^{\frac{3}{2}} + 15q + 6q^{\frac{1}{2}}$$

On the other hand by the Honda-Tate correspondence for Abelian threefolds there exists a simple Abelian threefold A over $\mathbb{F}_q$ with $\#A(\mathbb{F}_q) = q^3+q^2+q+1$, corresponding to the case $a_5 = a_3 = a_1 = 0$. In particular we have

$$\min_{[A,\lambda]\in[\mathcal{A}_3(\mathbb{F}_q)]} \#A(\mathbb{F}_q) = q^3+q^2+q+1$$

Comparing this to the computation of the expected value $E(\#A(\mathbb{F}_q))$ yields

$$\lim_{q\to\infty} |\mathbb{E}(\#A_3(\mathbb{F}_q)) - \min_{[A,\lambda]\in[\mathcal{A}_3(\mathbb{F}_q)]} \#A(\mathbb{F}_q)| = 0$$

Since $e_c(\mathcal{X}_3^{\times 2}, \mathbb{Q}_\ell) = \mathbb{L}^{12} + 4\mathbb{L}^{11} + 10\mathbb{L}^{10} + 20\mathbb{L}^9 + 25\mathbb{L}^8 + 24\mathbb{L}^7 + 17\mathbb{L}^6 + \mathbb{L}^5 - 8\mathbb{L}^4 - 4\mathbb{L}^3 - \mathbb{L}^2 + 4\mathbb{L} + 5$ we have

$$\#\mathcal{X}_3^{\times 2}(\mathbb{F}_q) = q^{12} + 4q^{11} + 10q^{10} + 20q^9 + 25q^8 + 24q^7 + 17q^6 + q^5 - 8q^4 - 4q^3 - q^2 + 4q + 5$$

It follows that we have the expected value

$$
\begin{aligned}
\mathbb{E}(\#A_3(\mathbb{F}_q)^2) &= \frac{q^{12}+4q^{11}+10q^{10}+20q^9+25q^8+24q^7+17q^6+q^5-8q^4-4q^3-q^2+4q+5}{q^6+q^5+q^4+q^3+1} \\
&= q^6+3q^5+6q^4+10q^3+6q^2+2q-2-\frac{8q^5+14q^4+12q^3+7q^2-2q-7}{q^6+q^5+q^4+q^3+1}
\end{aligned}
$$

and we have the variance

$$
\begin{aligned}
\mathrm{Var}(\#A_3(\mathbb{F}_q)) &= \mathbb{E}(\#A_3(\mathbb{F}_q)^2)-\mathbb{E}(\#A_3(\mathbb{F}_q))^2 \\
&= q^5+3q^4+6q^3+3q^2-3-\frac{6q^{11}+16q^{10}+24q^9+27q^8+17q^7-2q^5+3q^3+4q^2-4q-7}{(q^6+q^5+q^4+q^3+1)^2}
\end{aligned}
$$

We can continue in this way to obtain the first 5 terms of the moment generating function:

Corollary 1.4.7. The first 5 terms of the moment generating function $M_{\#A_3(\mathbb{F}_q)}(t)$ are rational functions in q:

$$
\begin{aligned}
1 &+ \left(q^3+q^2+q+1+\frac{-q^2-q}{q^6+q^5+q^4+q^3+1}\right)t \\
&+ \left(q^6+3q^5+6q^4+10q^3+6q^2+2q-2+\frac{-8q^5-14q^4-12q^3-7q^2+2q+7}{q^6+q^5+q^4+q^3+1}\right)\frac{t^2}{2!} \\
&+ \left(\begin{matrix} q^9+6q^8+21q^7+56q^6+81q^5 \\ +79q^4+43q^3-45q^2-119q-106 \end{matrix} + \frac{-23q^5+39q^4+110q^3+144q^2+194q+135}{q^6+q^5+q^4+q^3+1}\right)\frac{t^3}{3!} \\
&+ \left(\begin{matrix} q^{12}+10q^{11}+55q^{10}+220q^9+550q^8+950q^7+1185q^6 \\ +785q^5-499q^4-2106q^3-2576q^2-1091q+807 \end{matrix} + \frac{1478q^5+2929q^4+4176q^3+4463q^2+1848q-645}{q^6+q^5+q^4+q^3+1}\right)\frac{t^4}{4!} \\
&+ \left(\begin{matrix} q^{15}+15q^{14}+120q^{13}+680q^{12}+2565q^{11}+6817q^{10}+13515q^9+19521q^8 \\ +17184q^7-3650q^6-40833q^5-63521q^4-42593q^3+3203q^2+33402q+42708 \end{matrix} + \frac{45276q^5+71227q^4+52951q^3+19137q^2-27268q-41817}{q^6+q^5+q^4+q^3+1}\right)\frac{t^5}{5!}
\end{aligned}
$$

Chapter 2

Cohomology of Shimura Varieties

In the previous chapter we have explained how to study arithmetic statistics for principally polarized Abelian varieties over finite fields through knowledge of the Euler characteristics $e_c(\mathcal{A}_g, \mathbb{V}_\lambda)$, and ran these computations in the range where these Euler characteristics are explicitly known. In all of these computations there is a range beyond which the traces of Frobenius on the Euler characteristics $e_c(\mathcal{A}_g, \mathbb{V}_\lambda)$ is as complicated as possible, in the sense that they involve traces of Hecke operators on vector-valued Siegel modular forms of general type for $\mathrm{Sp}_{2g}(\mathbb{Z})$.

This leaves the following question: where does the knowledge of the Euler characteristics $e_c(\mathcal{A}_g, \mathbb{V}_\lambda)$ actually come from? How is this relation between traces of Frobenius and traces of Hecke operators actually established? How do these results generalize to higher level?

In this chaper we explain a general strategy for computing the cohomology of local systems on Shimura varieties in a way which relates this to the spectral theory of automorphic forms. The method which ultimately relates the traces of Frobenius on the ℓ-adic cohomology of a Shimura variety to the spectral theory of automorphic forms is the Langlands-Kottwitz method. Regrettably, explaining the necessary background in order to apply this method falls outside the scope of the present thesis; in particular, we have omitted the necessary work of analyzing terms in the Arthur-Selberg trace formula and its stabilization, particularly for GSp_4. Nevertheless, we will make an attempt to remark on how this method is used, and outline some of the main computations in later chapters; a complete treatment of these arguments for GSp_4 will have to wait for future writing.

In the remaining chapters we apply this general strategy to the groups GL_2 and GSp_4. While the computation for GL_2 is relatively simple, the computation for GSp_4 is quite involved (for example Flicker has written an entire book on the computation for PGSp_4 [33] and Weissauer has written an entire book on endoscopy for GSp_4 [121]), our goal is to summarize the main points of how the Langlands-Kottwitz method is applied while blackboxing many of the actual details from the representation theory of GSp_4, especially the various cases of the fundamental lemma and the endoscopic character identities which we need for the computation to work.

2.1 Intersection Cohomology of Shimura Varieties

We now describe a general procedure through which one can compute the intersection cohomology of local systems on Shimura varieties in terms of (discrete spectrum) automorphic representations. The intersection cohomology contains cuspidal cohomology, which one can compute in terms of cuspidal automorphic represetations. Up to understanding the structure of the discrete and cuspidal automorphic spectra, this reduces the cuspidal part of the computation to a problem of representation theory and the spectral theory of automorphic forms.

2.1.1 Shimura Varieties

We begin by recalling the definition of Shimura data, primarily following [97, Section 3] and [87, Section 1.1].

In general a Shimura variety should be a moduli space of certain Hodge structures; for example a Shimura variety of Abelian type can be understood as a moduli space of Abelian motives over $\mathbb{C}$ with certain Hodge classes and a level structure, which should admit a canonical model over a number field F (see [28], [84], [83] for general discussion). Recalling that $\mathbb{R}$-Hodge structures are equivalently representations of Deligne's torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$, one has the following definition:

Definition 2.1.1. [97, 3.1] A Shimura datum is a triple (G, X, h) where G is a connected reductive group over $\mathbb{Q}$, where X is a transitive $G(\mathbb{R})$-set, and where $h : X \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})$ is a $G(\mathbb{R})$-equivariant morphism (written $x \mapsto h_x$), such that:

(i) For all $x \in X$, the Hodge structure on $\mathfrak{g}_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ h_x : \mathbb{S} \to \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ has Hodge bidegrees in $\{(-1, 1), (0, 0), (1, -1)\}$.

(ii) For all $x \in X$ the automorphism $\mathrm{Ad}(h_x(i))$ of $G^{\mathrm{ad}}_{\mathbb{R}}$ is the Cartan involution.

(iii) For all $x \in X$ every projection of h_x onto a simple factor of G^{ad} is trivial.

The reflex field of a Shimura datum (G, X, h) is the number field F which is the field of definition of the conjugacy class of cocharacters $h_x \circ \mu_0 : \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}}$, where $\mu_0 : \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ is the cocharacter given by $z \mapsto (z, 1)$.

Let (G, X, h) be a Shimura datum. Let $\mathbb{A}_{\mathbb{Q}}^{\infty} = \widehat{\prod_p}^{\mathbb{Z}_p} \mathbb{Q}_p$ be the finite adele ring of $\mathbb{Q}$ and let $\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^{\infty} \times \mathbb{R}$ be the adele ring of $\mathbb{Q}$. For $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ a compact open subgroup one has the Shimura variety $S_{K_{\text{fin}}}$ which is a smooth quasiprojective stack (or a smooth quasiprojective variety if $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is a neat compact open subgroup) defined over the reflex field F. Fixing an embedding $\sigma : F \hookrightarrow \mathbb{C}$, the set of complex points is given by the adelic double quotient

$$S_{K_{\text{fin}}}(\mathbb{C}) = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\text{fin}}$$

Note that $X = G(\mathbb{R})/K'_{\infty}$ is a Hermitian symmetric domain where $K'_{\infty} = K_{\infty} A_G(\mathbb{R})^{\circ}$ where $K_{\infty} \subseteq G(\mathbb{R})$ is a maximal compact subgroup and $A_G(\mathbb{R})^{\circ}$ is the connected component of the identity in $A_G(\mathbb{R})$ where $A_G \subseteq Z(G)$ is a maximal $\mathbb{Q}$-split torus in the center of G.

One has the Baily-Borel compactification $\overline{S}^{\text{BB}}_{K_{\text{fin}}}$ which is a normal projective stack (or a normal projective variety if $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is a neat compact open subgroup) defined over the same reflex field F. Fixing an embedding $\sigma : F \hookrightarrow \mathbb{C}$, the set of complex points is given by the adelic double quotient

$$\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}) = G(\mathbb{Q}) \setminus \overline{X} \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\text{fin}}$$

where $\overline{X}$ is a topological space with dense open embedding $j : X \hookrightarrow \overline{X}$ such that the $G(\mathbb{Q})$-action on X extends to a continuous $G(\mathbb{Q})$-action on $\overline{X}$, with boundary components corresponding to maximal parabolic subgroups of G.

Hecke Correspondences Let $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and let $K_{\text{fin}}, K'_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be compact open subgroups such that $K'_{\text{fin}} \subseteq g K_{\text{fin}} g^{-1}$. We have a finite morphism $T_g : S_{K'_{\text{fin}}} \to S_{K_{\text{fin}}}$ (étale if K_{fin} is neat) given on complex points by

$$G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K'_{\text{fin}} \to G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\text{fin}}$$
$$G(\mathbb{Q})(x, g' K'_{\text{fin}}) \mapsto G(\mathbb{Q})(x, g' g K_{\text{fin}})$$

The finite morphism $T_g : S_{K'_{\text{fin}}} \to S_{K_{\text{fin}}}$ extends to a finite morphism $\overline{T}_g : \overline{S}^{\text{BB}}_{K'_{\text{fin}}} \to \overline{S}^{\text{BB}}_{K_{\text{fin}}}$.

As a special case of this, let $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a compact open subgroup and for $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ let $K'_{\mathrm{fin}} = gK_{\mathrm{fin}}g^{-1}$ be the corresponding compact open subgroup. We have the Hecke correspondence

$$\begin{tikzcd} & S_{K_{\mathrm{fin}} \cap K'_{\mathrm{fin}}} \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ S_{K_{\mathrm{fin}}} & & S_{K_{\mathrm{fin}}} \end{tikzcd}$$

where $\pi : S_{K_{\mathrm{fin}} \cap K'_{\mathrm{fin}}} \to S_{K_{\mathrm{fin}}}$ is the canonical projection, and where $\pi' : S_{K_{\mathrm{fin}} \cap K'_{\mathrm{fin}}} \to S_{K_{\mathrm{fin}}}$ is the canonical projection $S_{K_{\mathrm{fin}} \cap K'_{\mathrm{fin}}} \to S_{K'_{\mathrm{fin}}}$ followed by the morphism $T_g : S_{K'_{\mathrm{fin}}} \to S_{K_{\mathrm{fin}}}$.

As we now explain, these correspondences can be used to construct an action of the K_{fin}-spherical Hecke algebra on the cohomology of local systems on $S_{K_{\mathrm{fin}}}$.

Definition 2.1.2. The Hecke algebra of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ (with coefficients in a field E of characteristic 0) is the non-unital E-algebra of compactly supported locally constant functions

$$\mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})} = C_{\mathrm{c}}^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}), E)$$

with multiplication given by the convolution product

$$(f_1 * f_2)(g) = \int_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})} f_1(h^{-1}g) f_2(g) \mathrm{d}h$$

For $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ a compact open subgroup the K_{fin}-spherical Hecke algebra (with coefficients in a field E of characteristic 0) is the unital E-algebra of compactly supported locally constant K_{fin}-biinvariant functions

$$\mathcal{H}_{K_{\mathrm{fin}}} = C_{\mathrm{c}}^{\infty}(K_{\mathrm{fin}} \setminus G(\mathbb{A}_{\mathbb{Q}}^{\infty}) / K_{\mathrm{fin}}, E)$$

with multiplication given by convolution as above, and with unit $e_{K_{\mathrm{fin}}} = \mathrm{vol}(K_{\mathrm{fin}})^{-1} 1_{K_{\mathrm{fin}}}$ the normalized indicator function of K_{fin} in $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Definition 2.1.3. We say that an E-linear representation V of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is smooth if $V = \bigcup_{K_{\mathrm{fin}}} V^{K_{\mathrm{fin}}}$, where the union is taken over compact open subgroups $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

We say that an $\mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$-module V is nondegenerate if for all $v \in V$ there exists $f \in \mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ such that $f \cdot v = v$.

A smooth E-linear representation V of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ can be regarded as a $\mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$-module where $f \in \mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ acts on $v \in V$ by

$$f \cdot v = \int_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})} f(h)(h \cdot v) \mathrm{d}h$$

The resulting $\mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$-module is nondegenerate and we obtain an equivalence of categories.

$$\{\text{ smooth } E\text{-linear representations of } G(\mathbb{A}_{\mathbb{Q}}^{\infty})\} \xrightarrow{\sim} \{\text{nondegenerate } \mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}\text{-modules}\}$$

For such a nondegenerate $\mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$-module V the action of $e_{K_{\mathrm{fin}}}$ is given by the projector $V \to V^{K_{\mathrm{fin}}}$. The E-subalgebra $\mathcal{H}_{K_{\mathrm{fin}}}$ of $\mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ can be identified with $e_{K_{\mathrm{fin}}} * \mathcal{H}_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})} * e_{K_{\mathrm{fin}}}$, and is generated as an E-vector space by the functions $1_{K_{\mathrm{fin}} g K_{\mathrm{fin}}}$ for $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. In particular $V^{K_{\mathrm{fin}}}$ can be regarded as an $\mathcal{H}_{K_{\mathrm{fin}}}$-module.

Cohomology of Local Systems For $V_\lambda \in \mathrm{Rep}(G)$ a highest weight irreducible representation (defined over $\mathbb{Q}$) let $\mathbb{V}_\lambda$ be the corresponding local system of $\mathbb{Q}$-vector spaces on $S_{K_{\mathrm{fin}}}(\mathbb{C})$ given by the sheaf of local sections of the morphism

$$G(\mathbb{Q}) \setminus V_\lambda \times X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\mathrm{fin}} \to G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\mathrm{fin}}$$

where $\gamma \in G(\mathbb{Q})$ acts on $V_\lambda \times X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\mathrm{fin}}$ by $(v, x, gK_{\mathrm{fin}}) \mapsto (\gamma \cdot v, \gamma \cdot x, \gamma g K_{\mathrm{fin}})$. Note that this definition make sense for locally symmetric spaces $\Gamma \setminus X$ attached to connected reductive groups G over $\mathbb{Q}$, even when these are not Shimura varieties.

The cohomology $H^*(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ and the compactly supported cohomology $H^*_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ can be regarded as $\mathcal{H}_{K_{\mathrm{fin}}}$-modules in the following way. First, note that we have a natural action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on the limit $S(\mathbb{C}) = \varprojlim_{K_{\mathrm{fin}}} S_{K_{\mathrm{fin}}}(\mathbb{C})$, which yields a natural action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on

$$H^i(S(\mathbb{C}), \mathbb{V}_\lambda) = \varinjlim_{K_{\mathrm{fin}}} H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \qquad H^i_{\mathrm{c}}(S(\mathbb{C}), \mathbb{V}_\lambda) = \varinjlim_{K_{\mathrm{fin}}} H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

so $H^i(S(\mathbb{C}), \mathbb{V}_\lambda)$ and $H^i_c(S(\mathbb{C}), \mathbb{V}_\lambda)$ can be regarded as smooth $\mathbb{Q}$-linear representations of $G(\mathbb{A}^\infty_\mathbb{Q})$, equivalently as nondegenerate $\mathcal{H}_{G(\mathbb{A}^\infty_\mathbb{Q})}$-modules. Now by the Hochschild-Serre spectral sequence we have natural isomorphisms

$$H^*(S(\mathbb{C}), \mathbb{V}_\lambda)^{K_{\text{fin}}} = H^*(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \qquad H^*_c(S(\mathbb{C}), \mathbb{V}_\lambda)^{K_{\text{fin}}} = H^*_c(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

so $H^*(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ and $H^*_c(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ can be regarded as nondegenerate $\mathcal{H}_{K_{\text{fin}}}$-modules.

Our first goal is to understand the cohomology $H^*(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ or the compactly supported cohomology $H^*_c(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ as $\mathcal{H}_{K_{\text{fin}}}$-modules. In this situation we are allowed to use certain transcendental methods, in particular some of the structure of the cohomology can be understood in terms of the boundary of certain non-algebraic compactifications of $S_{K_{\text{fin}}}(\mathbb{C})$.

One has the Borel-Serre compactification $\overline{S}^{\text{BS}}_{K_{\text{fin}}}$ which is a smooth manifold with corners, with boundary components corresponding to parabolic subgroups of G. One has a morphism $\overline{S}^{\text{BS}}_{K_{\text{fin}}} \to \overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$ which collapses many of the Borel-Serre boundary components into a given Baily-Borel boundary component. Let $\partial\overline{S}^{\text{BS}}_{K_{\text{fin}}} = \overline{S}^{\text{BS}}_{K_{\text{fin}}} - S_{K_{\text{fin}}}(\mathbb{C})$ be the boundary of the Borel-Serre compactification of $S_{K_{\text{fin}}}(\mathbb{C})$ and let $j : S_{K_{\text{fin}}}(\mathbb{C}) \hookrightarrow \overline{S}^{\text{BS}}_{K_{\text{fin}}}$ and $i : \partial\overline{S}^{\text{BS}}_{K_{\text{fin}}} \hookrightarrow \overline{S}^{\text{BS}}_{K_{\text{fin}}}$ be the canonical inclusions. Then we have $\mathbb{R}^q j_*\mathbb{V}_\lambda = 0$ for all $q > 0$ and we have $H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) = H^i(\overline{S}^{\text{BS}}_{K_{\text{fin}}}, j_*\mathbb{V}_\lambda)$. We write $H^i(\partial\overline{S}^{\text{BS}}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ to denote $H^i(\partial\overline{S}^{\text{BS}}_{K_{\text{fin}}}, i^*j_*\mathbb{V}_\lambda)$. We have the compactly supported cohomology $H^i_c(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) = H^i(\overline{S}^{\text{BS}}_{K_{\text{fin}}}, j_!\mathbb{V}_\lambda)$. We have a short exact sequence of sheaves

$$0 \to j_!\mathbb{V}_\lambda \to j_*\mathbb{V}_\lambda \to i^*j_*\mathbb{V}_\lambda \to 0$$

which yields an $\mathcal{H}_{K_{\text{fin}}}$-equivariant long exact sequence in (compactly supported) cohomology:

$$\ldots \to H^{i-1}(\partial\overline{S}^{\text{BS}}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \xrightarrow{\delta} H^i_c(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \xrightarrow{\text{res}} H^i(\partial\overline{S}^{\text{BS}}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \to \ldots$$

Definition 2.1.4. Define the inner cohomology

$$\begin{aligned} H^i_!(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) &= \operatorname{im}(H^i_c(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)) \\ &= \ker(H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \xrightarrow{\text{res}} H^i(\overline{S}^{\text{BS}}_{K_{\text{fin}}}, \mathbb{V}_\lambda)) \end{aligned}$$

Define the Eisenstein cohomology

$$H^i_{\mathrm{Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) = \mathrm{coker}(H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \xrightarrow{\mathrm{res}} H^i(\partial \overline{S}^{\mathrm{BS}}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda))$$

Define the compactly supported Eisenstein cohomology

$$H^i_{\mathrm{c,Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) = \mathrm{im}(H^{i-1}(\partial \overline{S}^{\mathrm{BS}}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \xrightarrow{\delta} H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda))$$

It follows that we have short exact sequences

$$0 \to H^i_!(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to H^i_{\mathrm{Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to 0$$
$$0 \to H^i_{\mathrm{c,Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to H^i_!(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \to 0$$

which split the above long exact sequence in (compactly supported) cohomology into short exact sequences:

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & H^i_!(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \arrow[ur] \arrow[dr, hook, "i_!"] & & & \\
H^{i-1}(\partial \overline{S}^{\mathrm{BS}}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \arrow[rr, "\delta^{i-1}"] \arrow[dr, "\delta"'] & & H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \arrow[ur, "p_{\mathrm{c}}"] \arrow[rr] & & H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \arrow[rr, "\mathrm{res}"] \arrow[dr, "\mathrm{res}"'] & & H^i(\partial \overline{S}^{\mathrm{BS}}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \\
 & H^i_{\mathrm{c,Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \arrow[ur, hook] & & & & H^i_{\mathrm{Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \arrow[ur, hook] \arrow[dr] & \\
0 \arrow[ur] & & & & & & 0
\end{tikzcd}$$

Note that the inner cohomology, the compactly supported cohomology Eisenstein cohomology, and the Eisenstein cohomology all makes sense algebraically: after all, the forget supports morphism $H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to H^i(S_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ and hence its image and the above short exact sequences make sense algebraically. On the other hand, the relation to the cohomology of the Borel-Serre boundary does not make sense algebraically.

2.1.2 $(\mathfrak{g}, K_\infty)$-Cohomology

One of the main tools which is used to compute the cohomology of local systems on Shimura varieties is $(\mathfrak{g}, K_\infty)$-cohomology. After recalling the notions of $(\mathfrak{g}, K_\infty)$-modules and $(\mathfrak{g}, K_\infty)$-cohomology, including some crucial clarifications about the role of central characters and component groups, we will explain how this can be used to compute (compactly supported) de Rham cohomology.

$(\mathfrak{g}, K_\infty)$-**Modules** Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and let $\mathfrak{k}$ be the Lie algebra of a maximal compact subgroup $K_\infty \subseteq G(\mathbb{R})$. It is often useful to study the (often infinite-dimensional) representations of $G(\mathbb{R})$ in terms of $(\mathfrak{g}, K_\infty)$-modules, as studied by Harish-Chandra.

Definition 2.1.5. A $(\mathfrak{g}, K_\infty)$-module is a $\mathbb{C}$-vector space V along with an action $\rho_\mathfrak{g} : \mathfrak{g} \to \mathrm{End}_\mathbb{C}(V)$ (written $X \mapsto (v \mapsto X \cdot v)$) and a continuous action $\rho_{K_\infty} : K_\infty \to \mathrm{Aut}_\mathbb{C}(V)$ (written $k \mapsto (v \mapsto k \cdot v)$) such that:

(i) For every $v \in V$, for every $k \in K_\infty$, and for every $X \in \mathfrak{g}$ we have $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$;

(ii) For every $v \in V$ and for every $Y \in \mathfrak{k}$ we have $(\frac{\mathrm{d}}{\mathrm{d}t} \exp(tY))|_{t=0} = Y \cdot v$

For a $(\mathfrak{g}, K_\infty)$-module V let $V|_{K_\infty}$ be the corresponding representation of K_∞. Then by Peter-Weyl we have a decomposition into K_∞-types

$$V|_{K_\infty} = \bigoplus_{\vartheta \in \widehat{K_\infty}} m(\vartheta)\vartheta$$

where the direct sum is taken over isomorphism classes of irreducible (finite-dimensional) representations ϑ of the compact group K_∞. We say that V is admissible (a Harish-Chandra $(\mathfrak{g}, K_\infty)$-module) if for every K_∞-type $\vartheta \in \widehat{K_\infty}$ the multiplicity $m(\vartheta)$ in the above decomposition is finite. Equivalently, V is admissible if $K_\infty \cdot v$ spans a finite-dimensional subspace of V for every $v \in V$. Put another way, if $V^{(K_\infty)}$ the subspace of K_∞-finite vectors in V, then V is admissible if $V = V^{(K_\infty)}$.

Let $\mathrm{Mod}(\mathfrak{g}, K_\infty)$ be the Abelian category of admissible $(\mathfrak{g}, K_\infty)$-modules. For $V_1, V_2 \in \mathrm{Mod}(\mathfrak{g}, K_\infty)$ we have the tensor product $V_1 \otimes V_2 \in \mathrm{Mod}(\mathfrak{g}, K_\infty)$ where $X \in \mathfrak{g}$ acts on $v_1 \otimes v_2 \in V_1 \otimes V_2$ by the Leibniz rule $X \cdot (v_1 \otimes v_2) = (X \cdot v_1) \otimes v_2 + v_1 \otimes (X \cdot v_2)$ and where $k \in K_\infty$ acts on $v_1 \otimes v_2 \in V_1 \otimes V_2$ by the diagonal action $k \cdot (v_1 \otimes v_2) = (k \cdot v_1) \otimes (k \cdot v_2)$.

$(\mathfrak{g}, K_\infty)$-**Cohomology** By [20, I.2.5] the category $\mathrm{Mod}(\mathfrak{g}, K_\infty)$ has enough injectives and projectives. One can then define $(\mathfrak{g}, K_\infty)$-cohomology, which is essentially a combination of Lie algebra cohomology for $\mathfrak{g}$ and (continuous) group cohomology for K_∞, which takes into account the compatible actions of $\mathfrak{g}$ and K_∞ on Harish-Chandra $(\mathfrak{g}, K_\infty)$-modules.

Definition 2.1.6. Let V be an admissible $(\mathfrak{g}, K_\infty)$-module. Define the $(\mathfrak{g}, K_\infty)$-cohomology of V as the Ext group in $\mathrm{Mod}(\mathfrak{g}, K_\infty)$:

$$H^i(\mathfrak{g}, K_\infty; V) = \mathrm{Ext}^i_{(\mathfrak{g}, K_\infty)}(\mathbb{C}, V)$$

where $\mathbb{C} \in \mathrm{Mod}(\mathfrak{g}, K_\infty)$ is the trivial 1-dimensional $(\mathfrak{g}, K_\infty$-module. More explicitly, consider the complex

$$\mathrm{Hom}_{K_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), V) = \left(0 \to V \xrightarrow{\mathrm{d}} \mathrm{Hom}_{K_\infty}(\wedge^1(\mathfrak{g}/\mathfrak{k}), V) \xrightarrow{\mathrm{d}} \mathrm{Hom}_{K_\infty}(\wedge^2(\mathfrak{g}/\mathfrak{k}), V) \xrightarrow{\mathrm{d}} \dots\right)$$

with differential $\mathrm{Hom}_{K_\infty}(\wedge^p(\mathfrak{g}/\mathfrak{k}), V) \xrightarrow{\mathrm{d}} \mathrm{Hom}_{K_\infty}(\wedge^{p+1}(\mathfrak{g}/\mathfrak{k}), V)$ given by

$$\begin{aligned} \mathrm{d}\omega(X_0, \dots, X_p) = &\sum_{0 \le i \le p} (-1)^i X_i \omega(X_0, \dots, \widehat{X}_i, \dots, X_p) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p) \end{aligned}$$

Define the $(\mathfrak{g}, K_\infty)$-cohomology $H^i(\mathfrak{g}, K_\infty; V) = H^i(\mathrm{Hom}_{K_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), V))$.

The $(\mathfrak{g}, K_\infty)$-cohomology of an irreducible admissible $(\mathfrak{g}, K_\infty)$-module V is highly constrained by its central character. For V an irreducible admissible $(\mathfrak{g}, K_\infty)$-module, the action of $\mathfrak{g}$ on V extends to an action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ on V, which restricts to an action of the center $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ on V. The action of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ on V respects the K_∞-type decomposition $V|_{K_\infty} = \bigoplus_{\vartheta \in \widehat{K_\infty}} m(\vartheta)\vartheta$, and since V is irreducible it follows by Schur's lemma that $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ acts by scalars on V. We obtain a morphism $\omega_V : \mathcal{Z}(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ given for $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ and $v \in V$ by $z \cdot v = \omega_V(z)v$, which is called the central character of V.

Lemma 2.1.7. (Compare to [55, Section 6.1.4], [19, Lemma 5.5]) (Wigner's lemma) Let π_∞ be an irreducible admissible $(\mathfrak{g}, K_\infty)$-module and let $V_\lambda \in \mathrm{Rep}(G)$ be a finite-dimensional absolutely irreducible rational representation with highest weight λ. Then $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\lambda) = 0$ for every $i \ge 0$ unless $\omega_{\pi_\infty}(z) = \omega_{V_\lambda^\vee}(z)$ for every $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$.

Proof. For I an injective $(\mathfrak{g}, K_\infty)$-module $I \otimes V_\lambda$ is also an injective $(\mathfrak{g}, K_\infty)$-module, so an injective resolution $I^\bullet$ of π_∞ in $\mathrm{Mod}(\mathfrak{g}, K_\infty)$ yields an injective resolution $I^\bullet \otimes V_\lambda$ of $\pi_\infty \otimes V_\lambda$ in $\mathrm{Mod}(\mathfrak{g}, K_\infty)$, and we obtain an isomorphism

$$H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\lambda) \simeq \mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$$

Every $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ induces an endomorphism of V_λ and hence induces an endomorphism z' of $\mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$ by functoriality of Ext. Similarly every $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ induces an endomorphism of π_∞ and hence induces an endomorphism z'' of $\mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$ by functoriality of Ext. Now we show that $z' = z''$ by induction on the cohomological degree $i \geq 0$.

For the base case $i = 0$ this is clear by the definition $\mathrm{Ext}^0_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty) = \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$: for every $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$, for every $\phi \in \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$, and for every $v \in V_\lambda$ we have $z'\phi(v) = \phi(zv) = z''\phi(v)$.

For the induction step consider an embedding of π_∞ into an injective $(\mathfrak{g}, K_\infty)$-module I; by the corresponding short exact sequence $0 \to \pi_\infty \to I \to I/\pi_\infty \to 0$ we obtain an isomorphism

$$\mathrm{Ext}^{i-1}_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, I/\pi_\infty) \simeq \mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$$

for every $i \geq 1$. By the induction hypothesis we know that $z' = z''$ on $\mathrm{Ext}^{i-1}_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, I/\pi_\infty)$, so it follows that $z' = z''$ on $\mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$.

Now if $\omega_{\pi_\infty} \neq \omega_{\lambda^\vee}$ then there exists $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ such that $\omega_{\lambda^\vee}(z) = 0$ and $\omega_{\pi_\infty}(z) = 1$, which implies $z' = 0$ and $z'' = 1$ on $\mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$ for every $i \geq 0$. But since $z' = z''$ this implies that the identity is identically zero on $\mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty)$ for every $i \geq 0$, that is $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\lambda) \simeq \mathrm{Ext}^i_{(\mathfrak{g},K_\infty)}(V_\lambda^\vee, \pi_\infty) = 0$ for every $i \geq 0$. It follows that $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\lambda) = 0$ for every $i \geq 0$ unless $\omega_{\pi_\infty}(z) = \omega_{\lambda^\vee}(z)$ for every $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$. $\square$

As illustrated by Wigner's lemma, we will need to be a bit careful with matching of central characters in order to obtain nontrivial $(\mathfrak{g}, K_\infty)$-cohomology. By Harish-Chandra [?], for a given central character $\omega : \mathcal{Z}(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ there exists only finitely many isomorphism classes of irreducible admissible $(\mathfrak{g}, K_\infty)$-modules π_∞ with central character $\omega_{\pi_\infty} = \omega$. In particular for a given highest weight λ there exists only finitely many isomorphism classes of irreducible admissible $(\mathfrak{g}, K_\infty)$-modules π_∞ such that $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\lambda) \neq 0$ for some $i \geq 0$.

By Vogan-Zuckerman [119] there is a complete classification of cohomological irreducible admissible $(\mathfrak{g}, K_\infty)$-modules: one has an explicit family of irreducible admissible $(\mathfrak{g}, K_\infty)$-modules $A_\mathfrak{q}(\lambda)$ such that $H^i(\mathfrak{g}, K_\infty; A_\mathfrak{q}(\lambda) \otimes$

$V_\lambda)$ is nonzero for some $i \geq 0$, and every cohomological irreducible admissible $(\mathfrak{g}, K_\infty)$-module is isomorphic to some $A_\mathfrak{q}(\lambda)$. We will recall particular cases of this classification later.

Central Characters and Component Groups In general it is useful to extend the definition of $(\mathfrak{g}, K_\infty)$-modules and their cohomology to situations where $K_\infty \subseteq G(\mathbb{R})$ is not necessarily a maximal compact subgroup, but rather compact modulo center. For simplicity of discussion we will consider the following situation which is relevant to later computations: let G be a connected reductive group over $\mathbb{Q}$ with a specified character $c : G \to \mathbb{G}_\mathrm{m}$ with kernel $G^1 = \ker(c)$ yielding an exact sequence

$$0 \to G^1 \to G \xrightarrow{c} \mathbb{G}_\mathrm{m} \to 0$$

Assume that G splits over $\mathbb{Q}$ and that $A_G = Z(G) \simeq \mathrm{GL}_1$ is a $\mathbb{Q}$-split maximal torus in the center of G so that $A_G(\mathbb{R}) = \mathbb{R}^\times$ and $A_G(\mathbb{R})^\circ = \mathbb{R}_{>0}$ with component group $A_G(\mathbb{R})/A_G(\mathbb{R})^\circ = \{\pm 1\}$. Consider the subgroups $G^1(\mathbb{R}) = \{g \in G(\mathbb{R}) | c(g) = 1\}$ and $G^\pm(\mathbb{R}) = \{g \in G(\mathbb{R}) | c(g) = \pm 1\}$ of $G(\mathbb{R})$. Fix a maximal compact subgroup $K_\infty \subseteq G^1(\mathbb{R})$ and consider the subgroups

$$K_\infty^\pm = \{\pm 1\} K_\infty \subseteq G^\pm(\mathbb{R}) \subseteq G(\mathbb{R}) \qquad K_\infty^{>0} = A_G(\mathbb{R})^\circ K_\infty \subseteq G(\mathbb{R}) \qquad K_\infty^{\neq 0} = A_G(\mathbb{R}) K_\infty \subseteq G(\mathbb{R})$$

so that $K_\infty^\pm$ is a maximal compact subgroup of $G^\pm(\mathbb{R})$ and of $G(\mathbb{R})$, and we have the quotients

$$K_\infty^\pm / K_\infty = K_\infty^{\neq 0} / K_\infty^{>0} = A_G(\mathbb{R})/A_G(\mathbb{R})^\circ = \{\pm 1\} \qquad K_\infty^{>0}/K_\infty = K_\infty^{\neq 0}/K_\infty^\pm = A_G(\mathbb{R})^\circ = \mathbb{R}_{>0}$$

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and let $\mathfrak{g}^1 = \mathfrak{a}_G \setminus \mathfrak{g}$ be the Lie algebra of $G^1(\mathbb{R})$ and of $G^\pm(\mathbb{R})$. For $\mathfrak{g}^1 = \mathfrak{a}_G \setminus \mathfrak{g}$ the relevant groups are K_∞ (regarded as a maximal compact subgroup of $G^1(\mathbb{R})$) and $K_\infty^\pm$ (regarded as a maximal compact subgroup of $G^\pm(\mathbb{R})$). An $(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty)$-module can be regarded as an $(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty^\pm)$-module where the action of K_∞ is extended trivially to an action of $K_\infty^\pm$, and an $(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty^\pm)$-module can be regarded as the data of an $(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty)$-module along with an action of the component group $K_\infty^\pm/K_\infty = \{\pm 1\}$. By Hochschild-Serre we have

$$H^i(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty^\pm; V) = H^i(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty; V)^{K_\infty^\pm / K_\infty}$$

For $\mathfrak{g}$ the relevant group is $K_\infty^\pm$ (regarded as a maximal compact subgroup of $G(\mathbb{R})$). A $(\mathfrak{g}, K_\infty^\pm)$-module can be regarded as a $(\mathfrak{g}, K_\infty^{\neq 0})$-module where the action of $K_\infty^\pm$ is extended trivially to an action of $K_\infty^{\neq 0}$, and a $(\mathfrak{g}, K_\infty^{\neq 0})$-module can be regarded as the data of a $(\mathfrak{g}, K_\infty^\pm)$-module along with an action of the group $K_\infty^{\neq 0}/K_\infty^\pm = \mathbb{R}_{>0}$. A $(\mathfrak{g}, K_\infty^{\neq 0})$-module on which the group $K_\infty^{\neq 0}/K_\infty^\pm = \mathbb{R}_{>0}$ acts nontrivially cannot have nontrivial cohomology by Wigner's lemma; otherwise we have

$$H^i(\mathfrak{g}, K_\infty^{\neq 0}; V) = H^i(\mathfrak{g}, K_\infty^\pm; V) = H^i(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty^\pm; V)$$

It is not so important that the group K_∞ is a maximal compact subgroup, only that it is compact. A $(\mathfrak{g}, K_\infty)$-module can be regarded as a $(\mathfrak{g}, K_\infty^\pm)$-module where the action of K_∞ is extended trivially to an action of $K_\infty^\pm$, and a $(\mathfrak{g}, K_\infty^\pm)$-module can be regarded as the data of a $(\mathfrak{g}, K_\infty)$-module along with an action of the group $K_\infty^\pm/K_\infty = \{\pm 1\}$. By Hochschild-Serre we have

$$H^i(\mathfrak{g}, K_\infty^\pm; V) = H^i(\mathfrak{g}, K_\infty; V)^{K_\infty^\pm/K_\infty} = H^i(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty; V)^{K_\infty^\pm/K_\infty}$$

Similarly a $(\mathfrak{g}, K_\infty^{>0})$-module can be regarded as a $(\mathfrak{g}, K_\infty)$-module where the action of K_∞ is extended trivially to an action of $K_\infty^{>0}$, and a $(\mathfrak{g}, K_\infty^{>0})$-module can be regarded as the data of a $(\mathfrak{g}, K_\infty)$-module along with an action of the group $K_\infty^{\neq 0}/K_\infty^\pm = \mathbb{R}_{>0}$. A $(\mathfrak{g}, K_\infty^{>0})$-module on which the group $K_\infty^{>0}/K_\infty = \mathbb{R}_{>0}$ acts nontrivially cannot have nontrivial cohomology by Wigner's lemma; otherwise we have

$$H^i(\mathfrak{g}, K_\infty^{>0}; V) = H^i(\mathfrak{g}, K_\infty; V) = H^i(\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty; V)$$

One should keep in mind the example where $G = \mathrm{GSp}_{2n}$ and $c : \mathrm{GSp}_{2n} \to \mathbb{G}_\mathrm{m}$ is the similitude character so that $G^1(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R})$ and $G^\pm(\mathbb{R}) = \mathrm{Sp}_{2n}^\pm(\mathbb{R})$ with $K_\infty \simeq \mathrm{U}(n)$ (if $n = 1$ then $G = \mathrm{GL}_2$ and $c : \mathrm{GL}_2 \to \mathbb{G}_\mathrm{m}$ is the determinant character so that $G^1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and $G^\pm(\mathbb{R}) = \mathrm{SL}_2^\pm(\mathbb{R})$ with $K_\infty \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$ and $K_\infty^\pm \simeq \mathrm{O}(2)$).

De Rham Cohomology As a first application of $(\mathfrak{g}, K_\infty)$-cohomology to the cohomology of local systems on Shimura varieties, we consider one way in which the (compactly supported) cohomology of local systems can be computed in terms of $(\mathfrak{g}, K_\infty)$-cohomology, due to Borel.

Let (G, X, h) be a Shimura datum, so that $X = A_G(\mathbb{R})^\circ \setminus G(\mathbb{R})/K_\infty$.

Definition 2.1.8. Let $K_\infty \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ be a neat compact open subgroup. For $\mathbb{V}$ a local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$ regarded as a $\mathbb{C}$-vector bundle with flat connection $(\mathcal{V}, \nabla)$ on $S_{K_{\text{fin}}}(\mathbb{C})$, let $\Omega^\bullet(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$ be the de Rham complex where $\Omega^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = \Gamma(S_{K_{\text{fin}}}(\mathbb{C}), \Omega^i_{S_{K_{\text{fin}}}(\mathbb{C})} \otimes \mathcal{V})$, with differentials induced by ∇. Define the de Rham cohomology

$$H^i_{\text{dR}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = H^i(\Omega^\bullet(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}))$$

When K_{fin} is not neat we fix a neat compact open subgroup $\widetilde{K}_{\text{fin}} \subseteq K_{\text{fin}}$ of finite index and define the de Rham cohomology

$$H^i_{\text{dR}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = H^i(\Omega^\bullet(S_{\widetilde{K}_{\text{fin}}}(\mathbb{C}), \mathbb{V}))^{K_{\text{fin}}/\widetilde{K}_{\text{fin}}}$$

By standard results in mixed Hodge theory the de Rham cohomology $H^i_{\text{dR}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$ is a finite-dimensional $\mathbb{C}$-vector space and the complex structure on $S_{K_{\text{fin}}}(\mathbb{C})$ yields a Hodge decomposition

$$H^i_{\text{dR}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = \bigoplus_{p+q \geq i} H^{p,q}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$$

Now the main observation is that the de Rham cohomology $H^i_{\text{dR}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$ can be computed in terms of $(\mathfrak{g}, K'_\infty)$-cohomology as follows. Recalling that $S_{K_{\text{fin}}}(\mathbb{C}) = G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}K_\infty$, we consider the $(\mathfrak{g}, K'_\infty)$-cohomology of the space of smooth functions on $G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})$ transforming in a particular way under $A_G(\mathbb{R})^\circ$. Let $\omega : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$ be a (quasi)character. Consider the $\mathbb{C}$-vector space

$$C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega) = \left\{ \begin{array}{c} \text{smooth functions } \phi : G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C} \text{ such that} \\ \phi(ag) = \omega(a)\phi(g) \text{ for every } a \in A_G(\mathbb{R})^\circ \text{ and } g \in G(\mathbb{A}_\mathbb{Q}) \end{array} \right\}$$

For a compact open subgroup $K_{\text{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ let $C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}, \omega)$ be the subspace of K_{fin}-invariant functions in $C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$, so that

$$C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega) \simeq \varinjlim_{K_{\text{fin}}} C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}, \omega)$$

where the colimit is taken over compact open subgroups $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. In particular, we can regard $C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{fin}}, \omega)$ as an $\mathcal{H}_{K_{\mathrm{fin}}} \times (\mathfrak{g}, K'_{\infty})$-module.

Now let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and $A_G(\mathbb{R})^{\circ}$-character $\omega_\lambda : A_G(\mathbb{R})^{\circ} \to \mathbb{C}^{\times}$. Regarding $C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{fin}}, \omega_\lambda^{-1})$ as an $\mathcal{H}_{K_{\mathrm{fin}}} \times (\mathfrak{g}, K'_{\infty})$-module, we can regard $C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda$ as an $\mathcal{H}_{K_{\mathrm{fin}}} \times (\mathfrak{g}, K'_{\infty})$-module, or as an $\mathcal{H}_{K_{\mathrm{fin}}} \times (\mathfrak{a}_G \setminus \mathfrak{g}, K_{\infty})$-module since the $A_G(\mathbb{R})^{\circ}$-characters of $C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{fin}}, \omega_\lambda^{-1})$ and V_λ are inverse so that $\mathfrak{a}_G$ acts trivially. Then we have the following:

Theorem 2.1.9. (Borel) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and $A_G(\mathbb{R})^{\circ}$-character $\omega_\lambda : A_G(\mathbb{R})^{\circ} \to \mathbb{C}^{\times}$, and let $\mathbb{V}_\lambda$ be the corresponding local system of $\mathbb{C}$-vector spaces on $S_{K_{\mathrm{fin}}}(\mathbb{C})$. We have an isomorphism of complexes of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$\mathrm{Hom}_{K'_{\infty}}(\wedge^{\bullet}(\mathfrak{g}/\mathfrak{k}'), C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda) \xrightarrow{\sim} \Omega^{\bullet}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

In particular we have an isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq H^i(\mathfrak{g}, K'_{\infty}; C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda)$$

To spell out the isomorphism described in the above theorem, recall that we have a canonical projection $G(\mathbb{R}) \to A_G(\mathbb{R})^{\circ} \setminus G(\mathbb{R})/K_{\infty} = X$. Let $x \in X$ be the image of the identity id $\in G(\mathbb{R})$. Then the differential yields an identification $T_{X,x} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{k}'$. Consider an element

$$\omega \in \mathrm{Hom}_{K'_{\infty}}(\wedge^{i}(\mathfrak{g}/\mathfrak{k}'), C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1}) \otimes V_\lambda)$$

which can be evaluated on an element $(X_1, \ldots, X_i) \in \wedge^i(\mathfrak{g}/\mathfrak{k}')$ to yield an element $\omega(X_0, \ldots, X_{i-1}) \in C^{\infty}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1}) \otimes V_\lambda$. To produce the corresponding element

$$\omega_{\mathrm{dR}} \in \Omega^{\bullet}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

choose a point $(y, g_{\mathrm{fin}}) \in X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and choose an element $(g_\infty, g_{\mathrm{fin}}) \in G(\mathbb{R}) \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $g_\infty x = y$. Consider an element $(Y_1, \ldots, Y_i) \in \wedge^i T_{X,y}$ which is sent to the element $(X_1, \ldots, X_i) \in \wedge^i T_{X,x}$ by the differential of the left translation by g_∞. Now if we define

$$\omega_{\mathrm{dR}}(Y_1,\dots,Y_i)(y,g_{\mathrm{fin}}) = g_\infty^{-1}(\omega(X_1,\dots,X_i)(g_\infty,g_{\mathrm{fin}}))$$

then one checks that the assignment $\omega \mapsto \omega_{\mathrm{dR}}$ yields the desired isomorphism of complexes of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules.

Poincare dually, we have the following:

Theorem 2.1.10. (Borel) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and $A_G(\mathbb{R})^\circ$-character $\omega_\lambda : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$, and let $\mathbb{V}_\lambda$ be the corresponding local system of $\mathbb{C}$-vector spaces on $S_{K_{\mathrm{fin}}}(\mathbb{C})$. We have an isomorphism of complexes of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$\mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), C_{\mathrm{c}}^\infty(G(\mathbb{Q})\setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}},\omega_\lambda^{-1})\otimes V_\lambda) \xrightarrow{\sim} \Omega_{\mathrm{c}}^\bullet(S_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$$

In particular we have an isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$H_{\mathrm{c}}^i(S_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \simeq H^i(\mathfrak{g},K'_\infty; C_{\mathrm{c}}^\infty(G(\mathbb{Q})\setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}},\omega_\lambda^{-1})\otimes V_\lambda)$$

The above isomorphisms are compatible with cup products and Poincare duality. Suppose that we have two elements

$$\omega \in \mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), C^\infty(G(\mathbb{Q})\setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}},\omega_\lambda^{-1})\otimes V_\lambda)$$
$$\omega_{\mathrm{c}} \in \mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), C_{\mathrm{c}}^\infty(G(\mathbb{Q})\setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}},\omega_{\lambda^\vee}^{-1})\otimes V_{\lambda^\vee})$$

representing classes $[\omega] \in H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ and $[\omega_{\mathrm{c}}] \in H_{\mathrm{c}}^{d-i}(S_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ of complementary degree. Then the cup product $[\omega] \smile [\omega_{\mathrm{c}}]$ is given by

$$[\omega] \smile [\omega_{\mathrm{c}}] = \int_{S_{K_{\mathrm{fin}}}(\mathbb{C})} \omega \wedge \omega_{\mathrm{c}}$$

as an element of $H_{\mathrm{c}}^d(S_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{C})$.

Note that all of the above statements remain true if $(\mathfrak{g}, K'_\infty)$-cohomology is replaced by $(\mathfrak{a}_G\backslash\mathfrak{g}, K_\infty)$-cohomology, or if $C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega_\lambda^{-1})$ is replaced by $C^\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ provided the highest weight λ is normalized so that $\omega_\lambda = 1$.

2.1.3 Automorphic Representations and L^2-Cohomology

Unfortunately, the spaces $C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}, \omega_\lambda^{-1})$ and $C_c^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}, \omega_\lambda^{-1})$ are much too large for the above theorem to be useful for explicit computations. The goal of the next few sections is to expain how these spaces can be replaced by much more manageable spaces of automorphic forms which compute the same cohomology.

Automorphic Representations Let G be a connected reductive group over $\mathbb{Q}$, let $K_\infty \subseteq G(\mathbb{R})$ be a maximal compact subgroup, and let $A_G(\mathbb{R})^\circ$ be the connected component of the identity in $A_G(\mathbb{R})$ where $A_G \subseteq Z(G)$ is a maximal $\mathbb{Q}$-split torus in the center of G. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$, let $\mathfrak{a}_G$ be the Lie algebra of $A_G(\mathbb{R})$, and let $\mathfrak{k}$ be the Lie algebra of K_∞. Recalling that the adelic quotient $G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})$ has finite volume, one has the following definition:

Definition 2.1.11. Let $\omega : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$ be a (quasi)character. Fix a (normalized) Haar measure $\mathrm{d}g$ on the adelic quotient $G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})$ and consider the Hilbert space

$$L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega) = \left\{ \begin{array}{c} \text{functions } \phi : G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C} \text{ such that} \\ \phi(ag) = \omega(a)\phi(g) \text{ for every } a \in A_G(\mathbb{R})^\circ \text{ and } g \in G(\mathbb{A}_\mathbb{Q}) \\ \text{such that } \phi\omega^{-1} : G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C} \text{ is measurable} \\ \text{and such that } \int_{G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})} |(\phi\omega^{-1})(g)|^2 \mathrm{d}g < \infty \end{array} \right\}_{/\simeq}$$

of measurable square-integrable functions modulo almost-everywhere equality of functions. The Hilbert space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ is regarded as a representation of $G(\mathbb{A}_\mathbb{Q})$ by the right regular representation R, where $g \in G(\mathbb{A}_\mathbb{Q})$ acts on functions $\phi \in L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ by $(R(g)\phi)(x) = \phi(xg)$.

An automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with $A_G(\mathbb{R})^\circ$-character ω is an (irreducible) representation π of $G(\mathbb{A}_\mathbb{Q})$ occurring in the right regular representation R of $G(\mathbb{A}_\mathbb{Q})$ on $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$.

The right regular representation of $G(\mathbb{A}_\mathbb{Q})$ on the Hilbert space $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ decomposes into a discrete part and a continuous part

$$L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega) = L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega) \oplus L^2_{\mathrm{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$$

where $L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ is the closure of the direct sum of all irreducible closed subspaces occurring in $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$, which decomposes into a Hilbert direct sum of (irreducible) automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ with multiplicities, and where $L^2_{\mathrm{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ is the orthogonal complement of $L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ in $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ with respect to the L^2-inner product, which decomposes into Hilbert direct sum of direct integrals of irreducible representations of $G(\mathbb{A}_\mathbb{Q})$ with multiplicities and is related to automorphic Eisenstein series. Note that the word "occurs" in the above definition hides some subtleties in view of this decomposition: for π occurring in the discrete part we mean that such a representation occurs as a subquotient of the Hilbert direct sum decomposition into irreducible representations of $G(\mathbb{A}_\mathbb{Q})$, whereas for π occurring in the continuous part we mean that such a representation occurs in the sense of direct integrals.

By Flath [32] an (irreducible) automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ decomposes as a tensor product $\pi = \pi_{\mathrm{fin}} \otimes \pi_\infty$ where π_{fin} is an irreducible smooth representation of $G(\mathbb{A}_\mathbb{Q}^\infty)$ (equivalently a nondegenerate $\mathcal{H}_{G(\mathbb{A}_\mathbb{Q}^\infty)}$-module), and where π_∞ is regarded as an irreducible admissible $(\mathfrak{g}, K_\infty)$-module (after passing to subspaces of K_∞-finite vectors). Moreover, π_{fin} decomposes as a restricted tensor product $\pi_{\mathrm{fin}} = \bigotimes_p \pi_p$ where π_p is an irreducible smooth representation of $G(\mathbb{Q}_p)$, which is canonically independent of the choice of smooth model of G over $\mathbb{Z}[\frac{1}{N}]$ defining the restricted product decomposition $G(\mathbb{A}_\mathbb{Q}^\infty) = \widehat{\prod}_p^{G(\mathbb{Z}_p)} G(\mathbb{Q}_p)$.

For a compact open subgroup $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ we can regard the Hilbert subspace $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega)$ of K_{fin}-invariant functions in $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$, as an $\mathcal{H}_{K_{\mathrm{fin}}} \times (\mathfrak{g}, K_\infty)$-module. The same is true for the subspaces $L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega)$ and $L^2_{\mathrm{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega)$.

We say that an (irreducible) automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ with $A_G(\mathbb{R})^\circ$-character ω is K_{fin}-spherical if π occurs in $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega)$, that is $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$. Any (irreducible) automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ is K_{fin}-spherical for some (sufficiently small) compact open subgroup $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$, noting that π_{fin} decomposes as a restricted tensor product $\pi_{\mathrm{fin}} = \bigotimes_p \pi_p$ where π_p is $K_p = G(\mathbb{Z}_p)$-spherical (that is, $\pi_p^{K_p} \neq 0$) for almost all primes p.

If $\omega = 1$ is trivial then we can simply consider the Hilbert space

$$L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = \left\{ \begin{matrix} \text{measurable functions } \phi : G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C} \\ \text{such that } \int_{G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})} |(\phi\omega^{-1})(g)|^2 \mathrm{d}g < \infty \end{matrix} \right\}_{/\simeq}$$

of measurable square-integrable functions modulo almost-everywhere equality of functions. A unitary automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ is an (irreducible) representation π of $G(\mathbb{A}_\mathbb{Q})$ occurring in the right regular representation R of $G(\mathbb{A}_\mathbb{Q})$ on $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$.

The right regular representation of $G(\mathbb{A}_\mathbb{Q})$ on the Hilbert space $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ decomposes into a discrete part and a continuous part

$$L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \oplus L^2_{\text{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$$

where $L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ and $L^2_{\text{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ are as before. For a compact open subgroup $K_{\text{fin}} \subseteq G(\mathbb{A}^\infty_\mathbb{Q})$ we can regard the Hilbert subspace $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}})$ of K_{fin}-invariant functions in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}})$ as an $\mathcal{H}_{K_{\text{fin}}} \times (\mathfrak{g}, K_\infty)$-module, or as an $\mathcal{H}_{K_{\text{fin}}} \times (\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty)$-module since $\mathfrak{a}_G$ acts trivially. The same is true for the subspaces $L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}})$ and $L^2_{\text{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}})$ as before.

Note that since $G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})$ has finite volume we have $\omega^{-1} \in L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$; in particular if $\phi \in L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega)$ then $\phi\omega^{-1} \in L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$, and if π is an automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with central character ω then its character twist $\pi' = \pi\omega^{-1}$ is a unitary automorphic representation of $G(\mathbb{A}_\mathbb{Q})$.

L^2-Cohomology We now consider the following situation. Let (G, X, h) be a Shimura datum so that G is a connected reductive group over $\mathbb{Q}$ and $X = A_G(\mathbb{R})^\circ \setminus G(\mathbb{R})/K_\infty$ is a Hermitian symmetric domain where $K_\infty \subseteq G(\mathbb{R})$ is a maximal compact subgroup and $A_G(\mathbb{R})^\circ$ is the connected component of the identity in $A_G(\mathbb{R})$ where $A_G \subseteq Z(G)$ is a maximal $\mathbb{Q}$-split torus in the center of G. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$, let $\mathfrak{a}_G$ be the Lie algebra of $A_G(\mathbb{R})$, let $\mathfrak{k}$ be the Lie algebra of K_∞, and let $\mathfrak{k}' = \mathfrak{a}_G \oplus \mathfrak{k}$ be the Lie algebra of $K'_\infty = A_G(\mathbb{R})^\circ K_\infty$.

A choice of complete $G(\mathbb{R})$-invariant Riemannian metric on X induces a complete Riemannian metric with negative curvature on $S_{K_{\text{fin}}}(\mathbb{C})$ when $K_{\text{fin}} \subseteq G(\mathbb{A}^\infty_\mathbb{Q})$ is a neat compact open subgroup.

Definition 2.1.12. Let $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is a neat compact open subgroup. For $\mathbb{V}$ a local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$ let $\Omega^{\bullet}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$ be the complex of $\mathbb{C}$-vector spaces where $\Omega^{i}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$ is the $\mathbb{C}$-vector space of square-integrable differential forms on $S_{K_{\text{fin}}}(\mathbb{C})$

$$\Omega^{i}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = \{\omega \in \Omega^{i}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) \mid \textstyle\int_{S_{K_{\text{fin}}}(\mathbb{C})} \omega \wedge *\omega < \infty \text{ and } \int_{S_{K_{\text{fin}}}(\mathbb{C})} \mathrm{d}\omega \wedge *\mathrm{d}\omega < \infty\}$$

with differentials induced by the de Rham differentials. Define the L^2-cohomology

$$H^{i}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = H^{i}(\Omega^{\bullet}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}))$$

When K_{fin} is not neat we fix a neat compact open subgroup $\widetilde{K}_{\text{fin}} \subseteq K_{\text{fin}}$ of finite index and define the L^2-cohomology

$$H^{i}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = H^{i}(\Omega^{\bullet}_{L^2}(S_{\widetilde{K}_{\text{fin}}}(\mathbb{C}), \mathbb{V}))^{K_{\text{fin}}/\widetilde{K}_{\text{fin}}}$$

By Borel-Casselman the L^2-cohomology $H^{*}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$ is a finite dimensional $\mathbb{C}$-vector space and the complex structure on $S_{K_{\text{fin}}}(\mathbb{C})$ yields a Hodge decomposition

$$H^{i}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = \bigoplus_{p+q=i} H^{p,q}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V})$$

Now let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and $A_G(\mathbb{R})^\circ$-character $\omega_\lambda : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$. Regarding $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\text{fin}}, \omega_\lambda^{-1})$ as an $\mathcal{H}_{K_{\text{fin}}} \times (\mathfrak{g}, K'_\infty)$-module, we can regard $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\text{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda$ as an $\mathcal{H}_{K_{\text{fin}}} \times (\mathfrak{g}, K'_\infty)$-module or as an $\mathcal{H}_{K_{\text{fin}}} \times (\mathfrak{a}_G \setminus \mathfrak{g}, K_\infty)$-module since the $A_G(\mathbb{R})^\circ$-characters of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\text{fin}}, \omega_\lambda^{-1})$ and V_λ are inverse so that $\mathfrak{a}_G$ acts trivially. Now we have the following absolutely crucial theorem:

Theorem 2.1.13. (Borel-Casselman) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and $A_G(\mathbb{R})^\circ$-character $\omega_\lambda : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$, and let $\mathbb{V}_\lambda$ be the corresponding local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$. We have an isomorphism of complexes of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$\Omega^{\bullet}_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \mathrm{Hom}_{K_\infty}(\wedge^{\bullet}(\mathfrak{g}/\mathfrak{k}'), L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\text{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda)$$

In particular we have an isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$H^i_{L^2}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq H^i(\mathfrak{g}, K'_\infty; L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda)$$

Implicit in the above theorem is the statement that the continuous part does not contribute nontrivially to $(\mathfrak{g}, K'_\infty)$-cohomology: for $\omega \in \mathrm{Hom}_{K'_\infty}(\wedge^p(\mathfrak{g}/\mathfrak{k}'), L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega^{-1}) \otimes V_\lambda)$ consider the decomposition $\omega = \omega_{\mathrm{disc}} + \omega_{\mathrm{cont}}$ where

$$\omega_{\mathrm{disc}} \in \mathrm{Hom}_{K'_\infty}(\wedge^p(\mathfrak{g}/\mathfrak{k}'), L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda)$$
$$\omega_{\mathrm{cont}} \in \mathrm{Hom}_{K'_\infty}(\wedge^p(\mathfrak{g}/\mathfrak{k}'), L^2_{\mathrm{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda)$$

Then by [19, Lemma 5.5] we have $[\omega_{\mathrm{cont}}] = 0$. We record the above theorems in the following definition:

Definition 2.1.14. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and $A_G(\mathbb{R})^\circ$-character $\omega_\lambda : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$. Define the discrete cohomology

$$\begin{aligned} H^i_{\mathrm{disc}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) &= H^i(\mathfrak{g}, K'_\infty; L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda) \\ &= H^i(\mathfrak{g}, K'_\infty; L^2_{\mathrm{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega_\lambda^{-1}) \otimes V_\lambda)^{K_{\mathrm{fin}}} \end{aligned}$$

In view of the above theorem, we have an isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$H^i_{L^2}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq H^i_{\mathrm{disc}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

the emphasis being that $H^i_{L^2}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ is "geometric" in nature, whereas $H^i_{\mathrm{disc}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ is "spectral" in nature. The spectral nature is the following: recall that we have a Hilbert direct sum decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}_\mathbb{Q}), \omega)} m_{\mathrm{disc}}(\pi)\pi$$

where the first direct sum is taken over unitary central characters and where the second direct sum is taken over the set of automorphic representations $\pi \in \Pi_{\mathrm{disc}}(G, \omega)$ with central character ω, occurring in $L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus$

$G(\mathbb{A}_{\mathbb{Q}}))$ with multiplicity $m_{\text{disc}}(\pi)$. Then it follows that we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$H^i_{\text{disc}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}_{\mathbb{Q}}), \omega)} m_{\text{disc}}(\pi) \pi_{\text{fin}}^{K_{\text{fin}}} \otimes H^i(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

As we will also discuss later, $L^2_{\text{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1})$ decomposes further into a cuspidal part and a residual part

$$L^2_{\text{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1}) = L^2_{\text{cusp}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1}) \oplus L^2_{\text{res}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1})$$

where $L^2_{\text{cusp}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1})$ decomposes into a Hilbert direct sum of (irreducible) cuspidal automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ with finite multiplicities, and where $L^2_{\text{res}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1})$ is the orthogonal complement of $L^2_{\text{cusp}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1}))$ in $L^2_{\text{disc}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1})$ with respect to the L^2-inner product and is related to residues of automorphic Eisenstein series. We record the following definition:

Definition 2.1.15. Define the cuspidal cohomology

$$\begin{aligned} H^i_{\text{cusp}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) &= H^i(\mathfrak{g}, K'_\infty; L^2_{\text{cusp}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_{\text{fin}}, \omega_\lambda^{-1}) \otimes V_\lambda) \\ &= H^i(\mathfrak{g}, K'_\infty; L^2_{\text{cusp}}(G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}), \omega_\lambda^{-1}) \otimes V_\lambda)^{K_{\text{fin}}} \end{aligned}$$

Again we have a Hilbert direct sum decomposition

$$L^2_{\text{cusp}}(G(\mathbb{Q}) A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}})) \simeq \bigoplus_{\omega} \bigoplus_{\pi \in \Pi_{\text{cusp}}(G(\mathbb{A}_{\mathbb{Q}}), \omega)} m(\pi) \pi$$

where the first direct sum is taken over unitary central characters and where the second direct sum is taken over the set of automorphic representations $\pi \in \Pi_{\text{cusp}}(G, \omega)$ with central character ω, occurring in $L^2_{\text{cusp}}(G(\mathbb{Q}) A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}}))$ with multiplicity $m_{\text{cusp}}(\pi)$. Then it follows that we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$H^i_{\text{disc}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}_{\mathbb{Q}}), \omega)} m_{\text{cusp}}(\pi) \pi_{\text{fin}}^{K_{\text{fin}}} \otimes H^i(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

For now we record the following useful theorem relating the discrete cohomology and the cuspidal cohomology to (compactly supported) cohomology:

Theorem 2.1.16. (Borel [18]) We have a canonical morphisms of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$H^i_{\text{cusp}}(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \to H^i_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \to H^i_{\text{disc}}(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \to H^i(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$$

The composition $H^i_{\text{cusp}}(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \to H^i_!(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ is injective, and is an isomorphism for $G = \mathrm{GL}_2$ and $G = \mathrm{GSp}_4$.

Note that all of the above statements remain true if $(\mathfrak{g}, K'_\infty)$-cohomology is replaced by $(\mathfrak{a}_G\backslash\mathfrak{g}, K_\infty)$-cohomology, or if $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}), \omega_\lambda^{-1})$ is replaced by $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ provided the highest weight λ is normalized so that $\omega_\lambda = 1$.

When $S_{K_{\text{fin}}}(\mathbb{C})$ is compact (that is when G^{der} is $\mathbb{Q}$-anisotropic) the theorem of Borel-Casselman reduces to Matsushima's formula, where the discussion simplifies dramatically. In this situation, the L^2-cohomology $H^i_{L^2}(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ agrees with both the cohomology $H^i(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ and the compactly supported cohomology $H^i_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$. In general when $S_{K_{\text{fin}}}(\mathbb{C})$ is non-compact the L^2-cohomology is not so directly related to either of these, and instead is directly related to the intersection cohomology of the Baily-Borel compactification $\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$.

2.1.4 Intersection Cohomology and Zucker's Conjecture

As defined, neither the L^2-cohomology nor the $(\mathfrak{g}, K'_\infty)$-cohomology of the automorphic discrete spectrum make sense algebraically: they are analytic constructions which a priori have nothing to do with the algebraic structure on $S_{K_{\text{fin}}}$. On the other hand, the construction of L^2-cohomology is closely related to intersection cohomology, and this is something which makes sense algebraically. Let us quickly recall this story.

Intersection Cohomology We refer specifically to [10, Section 2.1]. Let $\mathcal{D}^{\text{b}}_{\text{c}}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{C})$ be the bounded derived category of complexes of $\mathbb{C}$-vector spaces on $\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$ which are constructible with respect to the stratification by boundary components. Recall that a complex $\mathcal{K}^\bullet \in \mathcal{D}^{\text{b}}_{\text{c}}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{C})$ of $\mathbb{C}$-vector spaces is perverse if:

- (Support) We have $\dim_\mathbb{C}\mathrm{supp}(H^k(i_x^*\mathcal{K}^\bullet)) \leq -k$ for all $k \in \mathbb{Z}$;
- (Cosupport) We have $\dim_\mathbb{C}\mathrm{supp}(H^k(i_x^!\mathcal{K}^\bullet)) \leq k$ for all $k \in \mathbb{Z}$.

Let $\mathrm{Perv}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{C})$ be the Abelian category of complex perverse sheaves on $\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$, the heart of the perverse t-structure on $\mathcal{D}^{\text{b}}_{\text{c}}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{C})$.

For $\mathbb{V}$ a local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$ let $\mathbb{V}[d] \in \text{Perv}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C})$ be the corresponding perverse sheaf of $\mathbb{C}$-vector spaces. Let $\text{IC}^\bullet(\mathbb{V}) \in \mathcal{D}^{\text{b}}_{\text{c}}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C})$ be the intersection cohomology complex uniquely characterized by the following properties:

- (Support) We have $\dim_{\mathbb{C}}\text{supp}(H^k(i_x^*\text{IC}^\bullet(\mathbb{V}))) < d - k$ for all $0 \neq k \in \mathbb{Z}$;
- (Cosupport) We have $\dim_{\mathbb{C}}\text{supp}(H^k(i_x^!\text{IC}^\bullet(\mathbb{V}))) < k - d$ for all $2d \neq k \in \mathbb{Z}$;
- (Stratification) For $U \subseteq \overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$ a smooth dense open subvariety of $\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$, $\text{IC}^\bullet(\mathbb{V})|_U$ is quasi-isomorphic to a constant sheaf on U.

In particular we have $\text{IC}^\bullet(\mathbb{V})[d] \in \text{Perv}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}, \mathbb{C})$. The open immersion $j : S_{K_{\text{fin}}}(\mathbb{C}) \hookrightarrow \overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$ induces the perverse extension functors ${}^{\text{p}}j_*, {}^{\text{p}}j_! : \text{Perv}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C}) \to \text{Perv}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C})$ given by ${}^{\text{p}}j_* = {}^{\text{p}}H^0\mathbb{R}j_*$ and ${}^{\text{p}}j_! = {}^{\text{p}}H^0\mathbb{R}j_!$. Since for $\mathcal{K}^\bullet \in \text{Perv}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C})$ we have ${}^{\text{p}}H^k(j_!\mathcal{K}^\bullet) = 0$ for all $k > 0$ and ${}^{\text{p}}H^k(j_!\mathcal{K}^\bullet) = 0$ for all $k < 0$ the natural transformation $j_! \to j_*$ induces a natural transformation ${}^{\text{p}}j_! \to {}^{\text{p}}j_*$ and the intermediate extension functor $j_{!*} : \text{Perv}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C}) \to \text{Perv}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C})$ given for $\mathcal{K}^\bullet \in \text{Perv}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{C})$ by $j_{!*}\mathcal{K}^\bullet = \text{im}({}^{\text{p}}j_!\mathcal{K}^\bullet \to {}^{\text{p}}j_*\mathcal{K}^\bullet)$. Then we have an isomorphism of perverse sheaves $\text{IC}^\bullet(\mathbb{V})[d] \simeq j_{!*}(\mathbb{V}[d])$

Definition 2.1.17. Let $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a neat compact open subgroup. Let $\mathbb{V}$ be a local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$. Consider the intersection complex on $\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$ given by the intermediate extension

$$\text{IC}^\bullet(\mathbb{V}) = (j_{!*}(\mathbb{V}[d]))[-d]$$

and define the intersection cohomology

$$IH^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = \mathbb{H}^i(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), \text{IC}^\bullet(\mathbb{V})) = \mathbb{H}^i(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), j_{!*}\mathbb{V})$$

When K_{fin} is not neat we fix a neat compact open subgroup $\widetilde{K}_{\text{fin}} \subseteq K_{\text{fin}}$ of finite index and define the intersection cohomology

$$IH^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}) = \mathbb{H}^i(\overline{S}^{\text{BB}}_{\widetilde{K}_{\text{fin}}}(\mathbb{C}), \text{IC}^\bullet(\mathbb{V}))^{K_{\text{fin}}/\widetilde{K}_{\text{fin}}}$$

When $S_{K_{\mathrm{fin}}}$ is compact (that is when G^{der} is $\mathbb{Q}$-anisotropic) the intersection cohomology $IH^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ agrees with both the cohomology $H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ and the compactly supported cohomology $H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$, but in general the relation between these is complicated.

Zucker's Conjecture The miracle is that for Shimura varieties, the intersection cohomology of the Baily-Borel compactification coincides with the L^2-cohomology; this is the statement of Zucker's conjecture, proved independently by Looijenga and Saper-Stern:

Theorem 2.1.18. (Looijenga [80], Saper-Stern [102]) For $\mathbb{V}$ a local system of $\mathbb{C}$-vector spaces on $S_{K_{\mathrm{fin}}}(\mathbb{C})$ we have an isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$IH^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}) \simeq H^i_{L^2}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V})$$

To give some indication of how this result is proved, let $\Omega^\bullet_{L^2}(\mathbb{V})$ be the complex of sheaves of $\mathbb{C}$-vector spaces on $\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C})$ where $\Omega^i_{L^2}(\mathbb{V})$ is the sheafification of the presheaf of $\mathbb{C}$-vector spaces whose sections over an open analytic subvariety $U \subseteq \overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C})$ are given by square-integrable differential forms on $U \cap S_{K_{\mathrm{fin}}}(\mathbb{C})$

$$\Omega^i_{L^2}(\mathbb{V})(U) = \{\omega \in \Omega^i(U \cap S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}) \mid \textstyle\int_{U\cap S_{K_{\mathrm{fin}}}(\mathbb{C})} \omega \wedge *\omega < \infty \text{ and } \int_{U\cap S_{K_{\mathrm{fin}}}(\mathbb{C})} \mathrm{d}\omega \wedge *\mathrm{d}\omega < \infty\}$$

with differentials induced by the de Rham differentials.

The sheaves $\Omega^i_{L^2}(\mathbb{V})$ are fine, so for $\Omega^\bullet_{L^2}(\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}) = H^0(\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C}), \Omega^\bullet_{L^2}(\mathbb{V}))$ the complex of global sections of $\Omega^\bullet_{L^2}(\mathbb{V})$ the hypercohomology is given

$$\mathbb{H}^i(\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C}), \Omega^\bullet_{L^2}(\mathbb{V})) \simeq H^i(\Omega^\bullet_{L^2}(\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V})) \simeq H^i_{L^2}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V})$$

By Looijenga $\Omega^\bullet_{L^2}(\mathbb{V})$ is perverse and the restriction $\Omega^\bullet_{L^2}(\mathbb{V})|_{S_{K_{\mathrm{fin}}}(\mathbb{C})}$ is a fine resolution of the sheaf of sections of $\mathbb{V}$, so by Goresky-MacPherson's characterization of intersection cohomology we have a quasi-isomorphism of perverse sheaves $\Omega^\bullet_{L^2}(\mathbb{V}) \simeq IC^\bullet(\mathbb{V})$ on $\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}}}(\mathbb{C})$ which induces the isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules which is predicted by Zucker's conjecture.

It is worth pointing out that the relation between intersection cohomology and L^2-cohomology is not a completely general phenomenon (indeed there are conterexamples outside the setting of Shimura varieties), and it is a minor miracle that this relation holds in the setting of Shimura varieties.

2.2 Eisenstein Cohomology of Shimura Varieties

We have seen how the intersection cohomology and cuspidal cohomology of local systems on Shimura varieties can be understood in terms of the discrete and cuspidal automorphic spectra, and when the Shimura variety is compact this is the end of the story. When the Shimura variety is not compact, as is the case for the examples considered in later chapters, the remaining Eisenstein cohomology is related to the continuous automorphic spectrum, albeit not so directly: one cannot simply take the $(\mathfrak{g}, K'_\infty)$-cohomology the continuous automorphic spectrum, as theorems of Borel-Casselman show that this is trivial. Nevertheless, the structure of the continuous automorphic spectrum as described by automorphic Eisenstein series still plays a crucial role: the Franke-Schwermer filtration on certain spaces of automorphic forms, defined in terms of cuspidal support and iterated residues of automorphic Eisenstein series, gives rise to a spectral sequence in $(\mathfrak{g}, K'_\infty)$-cohomology which computes the relevant Eisenstein cohomology.

It is worth remarking that one could instead approach the problem of computing Eisenstein cohomology by working directly with the boundary of the Borel-Serre compactification, which is the approach taken in work of Harder. There are several reasons why we do not take this approach. First of all, while this approach is adequate for the purposes of computing the cohomology of local systems topologically, it is somewhat awkward to justify the resulting Galois action on Eisenstein cohomology in the ℓ-adic setting because of the non-algebraic nature of the Borel-Serre compactification, although results of Pink allow one to do this rigorously. Perhaps more importantly, this approach to computing Eisenstein cohomology requires a detailed understanding of the restriction of cohomology to the Borel-Serre boundary as well as related connecting homomorphisms. As we will see later, the same problem arises when trying to understand certain connecting homomorphisms in the long exact sequence in $(\mathfrak{g}, K'_\infty)$-cohomology associated to parts of the Franke-Schwermer filtration. In any event, more problems arise when trying to compute Eisenstein cohomology using the Borel-Serre compactification especially in the case of arbitrarily large level: while the behavior of boundary and connecting homomorphisms can be understood more or less by hand in the case of small level, the number of boundary components grows rapidly as the level becomes large, and it is hard to imagine one could handle such computations in general.

2.2.1 Automorphic Forms and Cuspidal Support

In order to discuss automorphic Eisenstein series, we need to define the spaces of automorphic forms in which they live.

Normalization of Measures Let G be a connected reductive group over $\mathbb{Q}$ and let A_G be the maximal $\mathbb{Q}$-split torus in the center of G with Lie algebra $\mathfrak{a}_G$ with dual $\mathfrak{a}_G^\vee = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X^*(G)$ is the $\mathbb{Z}$-module of $\mathbb{Q}$-rational characters of G.

Consider the Harish-Chandra height function $H_G : G(\mathbb{A}_\mathbb{Q}) \to \mathfrak{a}_G$ given by $\langle \chi, H_G(g) \rangle = |\log(\chi(g))|$ for $g \in G(\mathbb{A}_\mathbb{Q})$ and $\chi \in X^*(G)$, and let $G(\mathbb{A}_\mathbb{Q})^1 = \{g \in G(\mathbb{A}_\mathbb{Q}) | H_G(g) = 0\}$ so that $G(\mathbb{A}_\mathbb{Q}) = G(\mathbb{A}_\mathbb{Q})^1 \times A_G(\mathbb{R})^\circ$. For π an irreducible unitary representation of $G(\mathbb{A}_\mathbb{Q})$ and for $s \in i\mathfrak{a}_G^\vee$ the representation $\pi \otimes e^{\langle s, H_G(g) \rangle}$ is an irreducible unitary representation of $G(\mathbb{A}_\mathbb{Q})$, and the irreducible unitary representations of $G(\mathbb{A}_\mathbb{Q})^1$ correspond bijectively to $i\mathfrak{a}_G^\vee$-orbits of irreducible unitary representations of $G(\mathbb{A}_\mathbb{Q})$ under this action.

Fix a minimal parabolic $\mathbb{Q}$-subgroup P_0 of G with Levi decomposition $P_0 = M_0 N_0$, and fix a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A}_\mathbb{Q})$ such that $K_v \subseteq G(\mathbb{Q}_v)$ is hyperspecial for all but finitely many places v of $\mathbb{Q}$ which is P_0-good in the sense that we have a decomposition $G(\mathbb{A}_\mathbb{Q}) = P_0(\mathbb{A}_\mathbb{Q})K$.

For a standard parabolic $\mathbb{Q}$-subgroup P of G with Levi decomposition $P = MN$ and central subgroup $A_P = A_M$ with Lie algebra $\mathfrak{a}_P = \mathfrak{a}_M$ with dual $\mathfrak{a}_P^\vee = \mathfrak{a}_M^\vee = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X^*(M)$ is the $\mathbb{Z}$-module of $\mathbb{Q}$-rational characters of M.

Consider the Harish-Chandra height function $H_P : G(\mathbb{A}_\mathbb{Q}) \to \mathfrak{a}_P$ given by $H_P(nmk) = H_M(m)$ for $n \in N(\mathbb{A}_\mathbb{Q})$, $m \in M(\mathbb{A}_\mathbb{Q})$, and $k \in K$, and let $M(\mathbb{A}_\mathbb{Q})^1 = \{m \in M(\mathbb{A}_\mathbb{Q}) | H_P(m) = 0\}$ so that $M(\mathbb{A}_\mathbb{Q}) = M(\mathbb{A}_\mathbb{Q})^1 \times A_P(\mathbb{R})^\circ$. Then we have a decomposition

$$G(\mathbb{A}_\mathbb{Q}) = P(\mathbb{A}_\mathbb{Q})K = M(\mathbb{A}_\mathbb{Q})N(\mathbb{A}_\mathbb{Q})K = M(\mathbb{A}_\mathbb{Q})^1 A_P(\mathbb{R})^\circ N(\mathbb{A}_\mathbb{Q})K$$

Fix a Haar measure $\mathrm{d}g$ on $G(\mathbb{A}_\mathbb{Q})$, a (left) Haar measure on $P(\mathbb{A}_\mathbb{Q})$, a Haar measure $\mathrm{d}k$ on K normalized by $\int_K \mathrm{d}k = 1$, a Haar measure $\mathrm{d}m$ on $M(\mathbb{A}_\mathbb{Q})$, a (left) Haar measure $\mathrm{d}n$ on $N(\mathbb{A}_\mathbb{Q})$ normalized by $\int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \mathrm{d}n =$

1, and a Haar measure $\mathrm{d}a$ on $A_P(\mathbb{R})^\circ$, so that for $f \in C_c^\infty(G(\mathbb{A}_\mathbb{Q}))$ we have

$$\begin{aligned}\int_{G(\mathbb{A}_\mathbb{Q})} f(g)\mathrm{d}g &= \int_K \int_{P(\mathbb{A}_\mathbb{Q})} f(pk)\mathrm{d}p\mathrm{d}k \\ &= \int_K \int_{M(\mathbb{A}_\mathbb{Q})} \int_{N(\mathbb{A}_\mathbb{Q})} f(mnk)\mathrm{d}n\mathrm{d}m\mathrm{d}k \\ &= \int_K \int_{M(\mathbb{A}_\mathbb{Q})} \int_{A_P(\mathbb{R})^\circ} \int_{N(\mathbb{A}_\mathbb{Q})} f(mank)\mathrm{d}n\mathrm{d}a\mathrm{d}m\mathrm{d}k\end{aligned}$$

Fix a Haar measure $\mathrm{d}A$ on $\mathfrak{a}_P$ corresponding to the Haar measure $\mathrm{d}a$ on $A_P(\mathbb{R})^\circ$ under the exponential morphism, and let $\mathrm{d}s$ be the Haar measure on $i\mathfrak{a}_P^\vee$ dual to $\mathrm{d}A$ so that for $f \in C_c^\infty(\mathfrak{a}_P)$ we have $\int_{i\mathfrak{a}_P^\vee} \int_{\mathfrak{a}_P} f(A)e^{-\langle s,A\rangle}\mathrm{d}A\mathrm{d}s = f(0)$.

For another standard parabolic $\mathbb{Q}$-subgroup P' of G with Levi decomposition $P' = M'N'$, if $P \subseteq P'$ we have $\mathbb{Q}$-rational embeddings $A_{P'} \subseteq A_P \subseteq M \subseteq M'$. The restriction morphism $X^*(M') \to X^*(M)$ is injective and yields a linear injection $\mathfrak{a}_{P'}^\vee \hookrightarrow \mathfrak{a}_P^\vee$ and a linear surjection $\mathfrak{a}_P \to \mathfrak{a}_{P'}$ with kernel $\mathfrak{a}_P^{P'}$. The restriction morphism $X^*(A_P) \to X^*(A_{P'})$ is surjective and yields a linear surjection $\mathfrak{a}_P^\vee \to \mathfrak{a}_{P'}^\vee$ and a linear injection $\mathfrak{a}_{P'} \hookrightarrow \mathfrak{a}_P$. The corresponding split exact sequences yield decompositions $\mathfrak{a}_P = \mathfrak{a}_{P'} \oplus \mathfrak{a}_P^{P'}$ and $\mathfrak{a}_P^\vee = \mathfrak{a}_{P'}^\vee \oplus (\mathfrak{a}_P^{P'})^\vee$.

Let Φ_P be the set of roots for (P, A_P) yielding the root space decomposition $\mathfrak{n} = \bigoplus_{\alpha\in\Phi_P} \mathfrak{n}_\alpha$, and let $\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P} \dim(\mathfrak{n}_\alpha)\alpha \in (\mathfrak{a}_P^G)^\vee$. Let $\Phi_P^+ \subseteq \Phi_P$ be the subset of positive roots, regarded as a subset $X^*(A_P)$, or regarded as a subset of $\mathfrak{a}_P^\vee = X^*(A_P) \otimes_\mathbb{Z} \mathbb{R}$ contained in $(\mathfrak{a}_P^G)^\vee$. Let $(\mathfrak{a}_P^G)^{\vee+}$ be the corresponding open positive Weyl chamber with closure $\overline{(\mathfrak{a}_P^G)^{\vee+}}$.

Automorphic Forms As we have seen previously, the spaces $C^\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ of smooth functions can be used to compute the de Rham cohomology of local systems on Shimura varieties in terms of $(\mathfrak{g}, K'_\infty)$-cohomology, but these spaces are far too large to be useful for explicit computations. On the other hand the spaces $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ of square-integrable automorphic forms can be used to compute the L^2-cohomology and intersection cohomology of local systems on Shimura varieties in terms of $(\mathfrak{g}, K'_\infty)$-cohomology, but aside from the case of compact Shimura varieties these spaces fall short of describing all of (compactly supported) cohomology.

However there are more general spaces of automorphic forms which are small enough to be described explicitly, but which are large enough so as to describe the (compactly supported) cohomology of local systems on Shimura vareities in terms of $(\mathfrak{g}, K'_\infty)$-cohomology. We consider the following notion of automorphic forms, which weakens the previously considered square-integrability condition while still requiring a certain uniform moderate growth condition:

Definition 2.2.1. An automorphic form on $G(\mathbb{A}_\mathbb{Q})$ is a smooth function $\phi : G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ such that:

(i) ϕ is left $G(\mathbb{Q})$-invariant and left $A_G(\mathbb{R})^\circ$-invariant;

(ii) ϕ is $\mathcal{Z}_\infty$-finite, that is ϕ is annihilated by an ideal of finite codimension in the center $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$;

(iii) ϕ is right K-finite, that is the span of all right K-translates $\phi_k(g) = \phi(gk)$ for $k \in K$ is finite-dimensional;

(iv) ϕ has uniform moderate growth, that is there exists $r \in \mathbb{R}_{>0}$ such that for all $D \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ there exists a constant $C_D \in \mathbb{R}_{>0}$ such that $|D\phi(g)| \leq C_D||g||^r$ for all $g \in G(\mathbb{A}_\mathbb{Q})$.

Let $\mathcal{A}(G) = \mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be the space of automorphic forms on $G(\mathbb{A}_\mathbb{Q})$ regarded as a $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-module, which is the dense subspace of smooth K_∞-finite vectors in the Hilbert space $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$.

For $\mathcal{J}$ an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ let $\mathcal{A}_\mathcal{J}(G) = \mathcal{A}_\mathcal{J}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of $\mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ consisting of automorphic forms on $G(\mathbb{A}_\mathbb{Q})$ annihilated by $\mathcal{J}$. Every automorphic form $\phi \in \mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ belongs to $\mathcal{A}_\mathcal{J}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ for some ideal $\mathcal{J}$ of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ by $\mathcal{Z}_\infty$-finiteness.

For $\lambda \in \mathfrak{h}^\vee$ a dominant highest weight and for V_λ the corresponding finite-dimensional irreducible representation of $G(\mathbb{C})$ of highest weight λ, let $\mathcal{J}_\lambda$ be the annihilator of $V_\lambda^\vee$ which is an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$. Let $\mathcal{A}_\lambda(G) = \mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of $\mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ consisting of automorphic forms on $G(\mathbb{A}_\mathbb{Q})$ annihilated by $\mathcal{J}_\lambda$.

For (G, X, h) a Shimura datum, these spaces of automorphic forms are exactly those which contribute to the cohomology of V_λ:

Theorem 2.2.2. (Franke) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ with trivial $A_G(\mathbb{R})^\circ$-character, and let $\mathbb{V}_\lambda$ be the corresponding local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$. We have an isomorphism of complexes of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$\Omega^\bullet(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \mathrm{Hom}_{K_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}) \otimes V_\lambda)$$

In particular we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq H^i(\mathfrak{g}, K'_\infty; \mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\text{fin}}) \otimes V_\lambda)$$

Cuspidal Support The spaces $\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ are still much too large to be described all at once, despite being a massive improvement over the spaces . The next step is to decompose the spaces $\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ according to cuspidal support of automorphic forms.

Recall that the constant term of an automorphic form $\phi \in \mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ along a parabolic $\mathbb{Q}$-subgroup P of G with Levi decomposition $P = MN$ is given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\phi_P(g) = \int_{N(\mathbb{Q})\setminus N(\mathbb{A}_\mathbb{Q})} \phi(ng)\mathrm{d}n$$

taken with respect to a normalized Haar measure $\mathrm{d}n$ on $N(\mathbb{A}_\mathbb{Q})$. Note that if $\phi_P = 0$ for all parabolic $\mathbb{Q}$-subgroups P of G then $\phi = 0$.

Definition 2.2.3. We say that an automorphic form $\phi \in \mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is cuspidal if $\phi_P = 0$ for all proper parabolic $\mathbb{Q}$-subgroups P of G.

Let $\mathcal{A}_{\text{cusp}}(G) = \mathcal{A}_{\text{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be the space of cuspidal automorphic forms on $G(\mathbb{A}_\mathbb{Q})$, regarded as a $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-module, which is the dense subspace of smooth K_∞-finite vectors in the Hilbert space $L^2_{\text{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$.

Definition 2.2.4. We say that $\phi \in \mathcal{A}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is negligible along a parabolic $\mathbb{Q}$-subgroup P of G with Levi quotient M if for all $g \in G(\mathbb{A}_\mathbb{Q})$ the function $\phi_P(\cdot g) : M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ given by $m \mapsto \phi_P(mg)$ is orthogonal to $\mathcal{A}_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}))$.

We say that two parabolic $\mathbb{Q}$-subgroups P and P' of G are associate if the Levi quotients M and M' are conjugate by an element of $G(\mathbb{Q})$. Let $\mathcal{C}$ be the set of associate classes of parabolic $\mathbb{Q}$-subgroups of G.

For $[P] \in \mathcal{C}$ let $\mathcal{A}_{\lambda,[P]}(G) = \mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of $\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ consisting of automorphic forms on $G(\mathbb{A}_\mathbb{Q})$ with negligible constant term along every parabolic $\mathbb{Q}$-subgroup of G not in the associate class $[P]$. By Langlands we obtain a decomposition of $\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ according to parabolic support:

Theorem 2.2.5. (Langlands) We have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}})) \simeq \bigoplus_{[P]\in\mathcal{C}} \mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}}))$$

For $[P] \in \mathcal{C}$ let $\Phi_{\lambda,[P]}$ be the set of associate classes of cuspidal automorphic representations of Levi quotients of parabolics in $[P]$ with infinitesimal characters matching the infinitesimal character of $V_\lambda^\vee$.

Let $\mathcal{A}_{\lambda,[P],\varphi}(G) = \mathcal{A}_{\lambda,[P],\varphi}(G(\mathbb{Q})A_G(\mathbb{R})^\circ\setminus G(\mathbb{A}_{\mathbb{Q}}))$ be the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})\times(\mathfrak{g}, K_\infty)$-submodule of $\mathcal{A}_{\lambda,[P]}(G) = \mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ\setminus G(\mathbb{A}_{\mathbb{Q}}))$ spanned by all residues and derivatives of automorphic Eisenstein series with cuspidal support φ evaluated at evaluated at points s_0 in the closure of the positive Weyl chamber defined by P, which we will describe in detail below. By Franke-Schwermer we obtain a finer decomposition of $\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ\setminus G(\mathbb{A}_{\mathbb{Q}}))$ according to cuspidal support:

Theorem 2.2.6. (Franke-Schwermer) We have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}})) \simeq \bigoplus_{[P]\in\mathcal{C}} \bigoplus_{\varphi\in\Phi_{\lambda,[P]}} \mathcal{A}_{\lambda,[P],\varphi}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}}))$$

The next step is to describe the spaces $\mathcal{A}_{\lambda,[P],\varphi}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}}))$ concretely in terms of automorphic Eisenstein series.

Automorphic Eisenstein Series We begin by reviewing some results about automorphic Eisenstein series, with definitions suited for the discussion of Paley-Wiener sections and Poincare series, whose L^2-inner product will be used to analyze the spectral decomposition of $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_{\mathbb{Q}}))$.

Let P be a proper parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P = MN$. For a unitary cuspidal automorphic representation π of $M(\mathbb{A}_{\mathbb{Q}})$ consider the π-isotypic subspaces

$$L^2_{\text{cusp}}(M)_\pi = L^2_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_{\mathbb{Q}}))_\pi \subseteq L^2_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_{\mathbb{Q}}))$$
$$\mathcal{A}_{\text{cusp}}(M)_\pi = \mathcal{A}_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_{\mathbb{Q}}))_\pi \subseteq \mathcal{A}_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_{\mathbb{Q}}))$$

For $s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$ let $\pi_s = e^{\langle H_P(\cdot), s+\rho_P\rangle}\pi$ and consider the π-isotypic subspaces

$$L^2(M)_{\pi_s} = e^{\langle H_P(\cdot), s+\rho_P\rangle} L^2(M)_\pi$$
$$\mathcal{A}_{\text{cusp}}(M)_{\pi_s} = e^{\langle H_P(\cdot), s+\rho_P\rangle} \mathcal{A}_{\text{cusp}}(M)_\pi$$

We consider the normalized parabolic induction

$$L^2_{\text{cusp}}(P \setminus G)_{\pi_s} = \mathcal{I}_{P(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(L^2_{\text{cusp}}(M)_{\pi_s}) = \text{Ind}_{P(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(L^2_{\text{cusp}}(M)_{\pi_{s+\rho_P}})$$

regarded as a $G(\mathbb{A}_{\mathbb{Q}})$-module, which we can view as acting on the Hilbert space

$$L^2_{\text{cusp}}(P \setminus G)_{\pi_s} = \left\{ \begin{array}{c} L^2 \text{ functions } \phi : N(\mathbb{A}_{\mathbb{Q}})M(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C} \text{ such that} \\ \phi(ag) = e^{\langle H_P(a), s+\rho_P\rangle}\phi(g) \text{ for all } a \in A_P(\mathbb{R})^\circ, g \in G(\mathbb{A}_{\mathbb{Q}}) \\ \text{and } \phi(\cdot g) \in L^2_{\text{cusp}}(M)_{\pi_s} \text{ for all } g \in G(\mathbb{A}_{\mathbb{Q}}) \end{array} \right\}$$

by the right regular action where $g \in G(\mathbb{A}_{\mathbb{Q}})$ acts on functions $\phi_s \in L^2_{\text{cusp}}(P \setminus G)_{\pi_s}$ by $(\mathcal{I}_P^G(\pi_s, g)\phi_s)(x) = \phi_s(xg)$. We consider the normalized K-finite parabolic induction

$$\mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s} = \mathcal{I}_{K \cap P(\mathbb{A}_{\mathbb{Q}})}^{K}(\mathcal{A}_{\text{cusp}}(M)_{\pi_s}) = \text{Ind}_{K \cap P(\mathbb{A}_{\mathbb{Q}})}^{K}(\mathcal{A}_{\text{cusp}}(M)_{\pi_{s+\rho_P}})$$

regarded as a $G(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{g}, K_\infty)$-module, which we can view as acting on the space

$$\mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s} = \left\{ \begin{array}{c} \text{smooth } K\text{-finite functions } \phi : N(\mathbb{A}_{\mathbb{Q}})M(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C} \text{ such that} \\ \phi(ag) = e^{\langle H_P(a), s+\rho_P\rangle}\phi(g) \text{ for all } a \in A_P(\mathbb{R})^\circ, g \in G(\mathbb{A}_{\mathbb{Q}}) \\ \text{and } \phi(\cdot k) \in \mathcal{A}_{\text{cusp}}(M)_{\pi_s} \text{ for all } k \in K \end{array} \right\}$$

by the right regular action where $f \in \mathcal{H}(K \backslash G(\mathbb{A}_{\mathbb{Q}})/K)$ acts on functions $\phi_s \in \mathcal{A}_{\text{cusp}}(P \backslash G)_{\pi_s}$ by $(\mathcal{I}_P^G(\pi_s, f)\phi_s)(g) = (R(f)\phi_s)(g)$.

Letting $s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$ vary we obtain an ind-$\mathbb{C}$-vector bundle $\mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_\bullet} \to (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$; for any finite set $\mathfrak{F}$ of K-types, the union $\bigcup_{\kappa \in \mathfrak{F}} \mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s}^\kappa$ of the κ-isotypic subspaces $\mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s}^\kappa$ in $\mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s}$ form a $\mathbb{C}$-vector bundle $\mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_\bullet}^{\mathfrak{F}} \to (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$ of finite rank.

Let $W_{P,\pi}$ be the space of Paley-Wiener sections for $\mathcal{A}_{\mathrm{cusp}}(P \setminus G)_{\pi_\bullet} \to (\mathfrak{a}_P^G)^\vee_\mathbb{C}$, consisting of sections $\phi : (\mathfrak{a}_P^G)^\vee_\mathbb{C} \to \mathcal{A}_{\mathrm{cusp}}(P \setminus G)^{\mathfrak{F}}_{\pi_\bullet}$ for some finite set $\mathfrak{F}$ of K-types, whose Fourier transform, given for $g \in G(\mathbb{A}_\mathbb{Q})$ with $g = namk$ for $n \in N(\mathbb{A}_\mathbb{Q})$, $a \in A_P(\mathbb{R})^\circ$, $m \in M(\mathbb{A}_\mathbb{Q})^1$, $k \in K$ by

$$\widehat{\phi}(g) = \int_{s_0 + i(\mathfrak{a}_P^G)^\vee} \phi_s(g)\mathrm{d}s = \int_{s_0 + i(\mathfrak{a}_P^G)^\vee} \phi(mk)e^{\langle H_P(a), s+\rho_P\rangle}\mathrm{d}s$$

is smooth and compactly supported modulo $A_G(\mathbb{R})^\circ$ in $a \in A_P(\mathbb{R})^\circ$.

Definition 2.2.7. For $\phi \in W_{P,\pi}$ define the Poincare series θ_ϕ given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\theta_\phi(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \widehat{\phi}(\gamma g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{s_0 + i(\mathfrak{a}_P^G)^\vee} \phi_s(\gamma g)\mathrm{d}s$$

For $\phi \in W_{P,\pi}$ and $s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C}$ define the automorphic Eisenstein series $\mathrm{Eis}_P^G(\phi_s)$ given for $g \in G(\mathbb{A}_\mathbb{Q})$.

$$\mathrm{Eis}_P^G(\phi_s)(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_s(\gamma g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g)e^{\langle H_P(\gamma g), s+\rho_P\rangle}$$

For $\phi \in W_{P,\pi}$ the Poincare series $\theta_\phi(g)$ is absolutely convergent for every $g \in G(\mathbb{A}_\mathbb{Q})$, and we have the following:

Proposition 2.2.8. (Langlands [74, Lemma 4.1])

(i) For $\phi \in W_{P,\pi}$ and $s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C}$ the automorphic Eisenstein series $\mathrm{Eis}_P^G(\phi_s)(g)$ converges absolutely and uniformly on compact subsets of $\{s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C} | \Re(s) \in \rho_P + (\mathfrak{a}_P^G)^{\vee +}\}$.

(ii) For $\phi \in W_{P,\pi}$ and $s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C}$ such that $\Re(s) \in \rho_P + (\mathfrak{a}_P^G)^{\vee +}$ the automorphic Eisenstein series defines an automorphic form $\mathrm{Eis}_P^G(\phi_s) \in \mathcal{A}(G)$ where $f \in \mathcal{H}(K \setminus G(\mathbb{A}_\mathbb{Q})/K)$ acts by

$$\mathrm{Eis}_P^G(\mathcal{I}_P^G(\pi_s, f)\phi_s) = R(f)\mathrm{Eis}_P^G(\phi_s)$$

(iii) For $\phi \in W_{P,\pi}$ and for $s_0 \in \rho_P + (\mathfrak{a}_P^G)^{\vee +}$ we have

$$\theta_\phi = \int_{s_0 + i(\mathfrak{a}_P^G)^\vee} \mathrm{Eis}_P^G(\phi_s)\mathrm{d}s$$

Let $W_G = W(G, M_0)$ be the Weyl group of G and for P a standard parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P = MN$ let $W_M = W(M, M_0)$ be the Weyl group of M and let W^M be the set of cosets $wW_M \subseteq W_G$ such that M^w is a standard Levi $\mathbb{Q}$-subgroup of G. For such a coset let $P^w = M^w N^w$ be the corresponding standard parabolic $\mathbb{Q}$-subgroup of G. and consider the intertwining operator $M(w, \pi_s)$ given for $\phi \in \mathcal{A}_{\mathrm{cusp}}(P \setminus G)_{\pi_s}$ and $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$(M(w,\pi_s)\phi)(g) = \int_{(N^w(\mathbb{A}_\mathbb{Q}) \cap \widetilde{w} N(\mathbb{A}_\mathbb{Q}) \widetilde{w}^{-1}) \setminus N^w(\mathbb{A}_\mathbb{Q})} \phi(\widetilde{w}^{-1} n g) \mathrm{d}n$$

where $\widetilde{w} \in G(\mathbb{Q})$ is a representative of w. Then by definition we have $\mathrm{Eis}(\mathcal{I}_P^G(s,f)\phi_s)(g) = R(f)\mathrm{Eis}_P^G(\phi_s)(g)$ and $M(w,s)\mathcal{I}_P^G(s,f) = \mathcal{I}_{P'}^G(ws,f)M(w,s)$.

Proposition 2.2.9. (Langlands, Moeglin-Waldspurger [85, II.1.6, II.1.7])

(i) $s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C}$ the intertwining operator $M(w, \pi_s)$ converges absolutely and uniformly on compact subsets of $\{s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C} | \Re(s) \in \rho_P + (\mathfrak{a}_P^G)^{\vee +}\}$.

(ii) For $s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C}$ such that $\Re(s) \in \rho_P + (\mathfrak{a}_P^G)^{\vee+}$ the intertwining operator defines a morphism of $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-modules

$$M(w,\pi_s) : \mathcal{A}_{\mathrm{cusp}}(P \setminus G)_{\pi_s} \to \mathcal{A}_{\mathrm{cusp}}(P^w \setminus G)_{\pi_s^w}$$

(iii) The constant term of $\mathrm{Eis}_P^G(\phi_s)$ along a standard parabolic $\mathbb{Q}$-subgroup $P' = M'N'$ of G is given in terms of the intertwining operators $M(w, \pi_s)$ by

$$\mathrm{Eis}_P^G(\phi_s)_{P'} = \sum_{\substack{w \in W_G/W_M \\ M^w \subseteq M'}} \mathrm{Eis}_{P^w}^{P'}(M(w,\pi_s)\phi_s)$$

where the partial Eisenstein series $\mathrm{Eis}_P^{P'}(\phi_s)$ is given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\mathrm{Eis}_P^{P'}(\phi_s)(g) = \sum_{\gamma \in P(\mathbb{Q}) \setminus P'(\mathbb{Q})} \phi_s(\gamma g) = \sum_{\gamma \in P(\mathbb{Q}) \setminus P'(\mathbb{Q})} \phi(\gamma g) e^{\langle H_P(\gamma g), s + \rho_P \rangle}$$

(iv) We have the adjoint operator $M(w,\pi_s)^* = M(w^{-1},\pi^w_{-w(\overline{s})})$ such that for $\phi \in \mathcal{A}_{\mathrm{cusp}}(P \setminus G)_{\pi_s}$ and $\phi \in \mathcal{A}_{\mathrm{cusp}}(P^w \setminus G)_{\pi^w_{-w(\overline{s})}}$ we have the adjunction formula

$$\langle M(w,\pi_s)\phi, \phi' \rangle = \langle \phi, M(w^{-1},\pi^w_{-w(\overline{s})})\phi' \rangle$$

where for $\phi_s \in \mathcal{A}(P \setminus G)_{\pi_s}$ and $\phi'_{-\overline{s}} \in \mathcal{A}(P \setminus G)_{\pi_{-\overline{s}}}$ the pairing is given

$$\langle \phi_s, \phi'_{-\overline{s}} \rangle = \int_K \int_{M(\mathbb{Q})\setminus M(\mathbb{A}_\mathbb{Q})^1} \phi_s(mk)\overline{\phi'_{-\overline{s}}(mk)}\mathrm{d}m\mathrm{d}k$$

We will need the following expression for the L^2-inner product of Poincare series:

Proposition 2.2.10. (Langlands, Moeglin-Waldspurger [85, II.2.1]) For $\phi \in W_{P,\pi}$ and $\phi' \in W_{P',\pi'}$ the L^2-inner product of the Poincare series $\theta_\phi, \theta_{\phi'} \in L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is given by

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \begin{cases} \displaystyle\int_{s_0 + i(\mathfrak{a}_P^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s & [P,\pi] = [P',\pi'] \\ 0 & \text{otherwise} \end{cases}$$

where $s_0 \in (\mathfrak{a}_P^G)^\vee$ is any point with $\alpha^\vee(s_0 + \rho_P) > 0$ for every $\alpha \in \Phi_P^+$, and where

$$A(\phi,\phi')(\pi_s) = \sum_{\substack{w \in W_G/W_M \\ M^w = M'}} \langle M(w,\pi_s)\phi_s, \phi'_{-w(\overline{s})} \rangle$$

So far we have only considered Eisenstein series and intertwining operators in the region of convergence $\{s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C} | \Re(s) \in \rho_P + (\mathfrak{a}_P^G)^{\vee +}\}$. Now we recall the fundamental result of Langlands regarding the analytic continuation and functonal equations of Eisenstein series:

Proposition 2.2.11. (Langlands, Moeglin-Waldspurger [85, IV.1]) For $\phi \in W_{P,\pi}$ the Eisenstein series $\mathrm{Eis}_P^G(\phi_s)$ and the intertwining operator $M(w,\pi_s)$ admit a meromorphic continuation to all $s \in (\mathfrak{a}_P^G)^\vee_\mathbb{C}$, where the automorphic Eisenstein series defines an automorphic form $\mathrm{Eis}_P^G(\phi_s) \in \mathcal{A}(G)$ where $f \in \mathcal{H}(K \setminus G(\mathbb{A}_\mathbb{Q})/K)$ acts by

$$\mathrm{Eis}_P^G(\mathcal{I}_P^G(\pi_s,f)\phi_s) = R(f)\mathrm{Eis}_P^G(\phi_s)$$

and the intertwining operator defines a morphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$M(w, \pi_s) : \mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s} \to \mathcal{A}_{\text{cusp}}(P^w \setminus G)_{\pi_s^w}$$

whenever $s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^{\vee}$ is not a pole. Moreover we have the functional equations given as an identity of meromorphic functions

$$\begin{aligned} \operatorname{Eis}_{P^w}^G(M(w, \pi_s)\phi_s) &= \operatorname{Eis}_P^G(\phi_s) \qquad w \in W_M \\ M(w', \pi_s^w) M(w, \pi_s) &= M(w'w, \pi_s) \qquad w \in W_M, w' \in W_{M^w} \end{aligned}$$

Poles of Automorphic Eisenstein Series We now want to understand the poles of such automorphic Eisenstein series. We follow [69, Section 5.2] and refer to [73] for further discussion.

For $w \in W_0^G / W_0^M$ such that $M^w \subseteq M'$ consider the adjoint representation $\rho_w : {}^L M \to \mathrm{GL}(\widehat{\mathfrak{n}} / (w^{-1}(\widehat{\mathfrak{n}}) \cap \widehat{\mathfrak{n}}))$ and the associated automorphic L-function and ϵ-function

$$\Lambda(s, \pi, \rho_w) = \Lambda(s, \rho_w \circ \varphi_\pi) \qquad \epsilon(s, \pi, \rho_w) = \epsilon(s, \rho_w \circ \varphi_\pi)$$

where $\varphi_\pi : L_{\mathbb{Q}} \to {}^L M$ is the Langlands parameter for π. We use the notation Λ to emphasize that these are completed L-functions which include Archimedean factors. Consider the Langlands normalization factor

$$r(w, \pi_s) = \frac{\Lambda(0, \pi_s, \rho_w)}{\Lambda(1, \pi_s, \rho_w)\epsilon(0, \pi_s, \rho_w)}$$

Suppose for simplicity of discussion that $m_{\text{cusp}}(\pi) = 1$ so that $\mathcal{A}_{\text{cusp}}(M) \simeq \pi$. Consider the restricted tensor product decomposition $\pi = \bigotimes_v \pi_v$ and fix an associated isomorphism

$$\Phi_\pi : \mathcal{A}_{\text{cusp}}(P \setminus G)_\pi \xrightarrow{\sim} \bigotimes_v \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_v)$$

For $s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^{\vee}$ we obtain an associated isomorphism

$$\Phi_{\pi_s} : \mathcal{A}_{\text{cusp}}(P \setminus G)_{\pi_s} \xrightarrow{\sim} \bigotimes_v \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_{v,s})$$

Consider the local intertwining operator $M(w, \pi_{v,s})$ given for $\phi_v \in \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_{v,s})$ and $g \in G(\mathbb{Q}_v)$ by

$$M(\widetilde{w}, \pi_{v,s})\phi_v(g) = \int_{(N^w(\mathbb{Q}_v)\cap \widetilde{w}N(\mathbb{Q}_v)\widetilde{w}^{-1})\backslash N^w(\mathbb{Q}_v)} \phi_v(\widetilde{w}^{-1}ng)\mathrm{d}n$$

where $\widetilde{w} \in G(\mathbb{Q})$ is a representative for w. Now we have a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_{\mathrm{cusp}}(P \setminus G)_{\pi_s} \arrow[r, "M(w,\pi_s)"] \arrow[d, "\Phi_{\pi_s}"'] & \mathcal{A}_{\mathrm{cusp}}(P^w \setminus G)_{\pi_s^w} \arrow[d, "\Phi_{\pi_s^w}"] \\
\bigotimes_v \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_{v,s}) \arrow[r, "\bigotimes_v M(\widetilde{w},\pi_{v,s})"] & \bigotimes_v \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_{v,s}^w)
\end{tikzcd}$$

Fixing a nontrivial character $\psi = \bigotimes_v \psi_v : \mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^\times$ we have an Euler product decomposition

$$r(w, \pi_s) = \prod_v r(w, \pi_{v,s}, \psi_v) \qquad r(w, \pi_{v,s}, \psi_v) = \frac{L(0, \pi_{v,s}, \rho_w)}{L(1, \pi_{v,s}, \rho_w)\epsilon(0, \pi_{v,s}, \rho_w, \psi_v)}$$

By [69, Proposition 5.2] (especially by [108, Theorem 7.9] in the case where π_v is tempered) the normalized local intertwining operator $N(\widetilde{w}, \pi_{v,s}, \psi_v) = r(w, \pi_{v,s}, \psi_v)^{-1}M(\widetilde{w}, \pi_{v,s})$ is holomorphic on $\{s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee | \alpha^\vee(\Re(s)) \geq 0 \text{ for all } \alpha \in \Phi_P^+ - w^{-1}\Phi_{P^w}^+\}$. For all but finitely many non-Archimedean places v of $\mathbb{Q}$ there exists a K_v-spherical vector $\phi_{\pi_{v,s}}^{\mathrm{sph}} \in \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_{v,s})$ such that $N(\widetilde{w}, \pi_{v,s}, \psi_v)\phi_{\pi_{v,s}}^{\mathrm{sph}} = \phi_{\pi_{v,s}^w}^{\mathrm{sph}}$ where ψ_v is order 0.

For $\phi \in W_{P,\pi}$ we may assume that $\Phi_{\pi_s}(\phi_s) = \bigotimes_v \phi_{v,s}$ with $\phi_{v,s} \in \mathcal{I}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi_{v,s})$. Let S be a finite set of places of $\mathbb{Q}$ such that $\phi_{v,s} = \phi_{\pi_{v,s}}^{\mathrm{sph}}$ so that $N(\widetilde{w}, \pi_{v,s}, \psi_v)\phi_{\pi_{v,s}}^{\mathrm{sph}} = \phi_{\pi_{v,s}^w}^{\mathrm{sph}}$ where ψ_v is order 0 for all $v \notin S$. Then we have

$$\begin{aligned}
M(w, \pi_s)\phi_s &= \Phi_{\pi_s^w}^{-1}\Big(\bigotimes_{v\in S} r(w, \pi_{v,s}, \psi_v)N(\widetilde{w}, \pi_{v,s}, \psi_v)\phi_{v,s} \otimes \bigotimes_{v\notin S} r(w, \pi_{v,s}, \psi_v)\phi_{\pi_{v,s}^w}^{\mathrm{sph}}\Big) \\
&= r(w, \pi_s)\Phi_{\pi_s^w}^{-1}\Big(\bigotimes_{v\in S} N(\widetilde{w}, \pi_{v,s}, \psi_v)\phi_{v,s} \otimes \bigotimes_{v\notin S} \phi_{\pi_{\pi_{v,s}}^w}^{\mathrm{sph}}\Big) \\
&= r(w, \pi_s)N(w, \pi_s)\phi_s
\end{aligned}$$

where $N(w, \pi_s)$ is the normalized intertwiner

$$N(w, \pi_s)\phi_s = \Phi_{\pi_s^w}^{-1}\Big(\bigotimes_{v\in S} N(\widetilde{w}, \pi_{v,s}, \psi_v)\phi_{v,s} \otimes \bigotimes_{v\notin S} \phi_{\pi_{\pi_{v,s}}^w}^{\mathrm{sph}}\Big)$$

Since $\bigotimes_{v\in S} N(\widetilde{w},\pi_{v,s},\psi_v)$ is holomorphic on $\{s \in (\mathfrak{a}_P^G)^\vee_{\mathbb{C}} | \alpha^\vee(\Re(s)) \geq 0 \text{ for all } \alpha \in \Phi_P^+ - w^{-1}\Phi_{Pw}^+\}$ it follows that the poles of $M(w,\pi_s)$ are exactly the poles of $r(w,\pi_s)$ in this region.

By Shahidi's nonvanishing theorem [107, Theorem 5.1] the denominator of $r(w,\pi_s)$ does not vanish in this region, so the poles of $r(w,\pi_s)$ are exactly the poles of the numerator of $r(w,\pi_s)$ in this region. This is precisely what we will use to analyze the poles of the intertwining operators, and hence the poles of the automorphic Eisenstein series by the constant term formula.

Example 2.2.12. (Gindikin-Karpelevich [73]) Let $P_0 = M_0N_0$ be a minimal parabolic $\mathbb{Q}$-subgroup of G. Let $\chi = \bigotimes_v \chi_v : M_0(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}^\times$ be a unitary character. For $\phi \in W_{P_0,\chi}$ and $s \in (\mathfrak{a}_{P_0}^G)^\vee_{\mathbb{C}}$ consider the Eisenstein series given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\mathrm{Eis}_{P_0}^G(\phi_s)(g) = \sum_{\gamma\in P_0(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_s(\gamma g) = \sum_{\gamma\in P_0(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g) e^{\langle H_{P_0}(\gamma g), s+\rho_{P_0}\rangle}$$

For $w \in W_G$ consider the intertwining operator $M(w,\chi_s) = \bigotimes_v M(\widetilde{w},\chi_{v,s})$ given for $\phi = \bigotimes_v \phi_v \in W_{P_0,\chi}$ and $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$(M(w,\chi_s)\phi_s)(g) = \int_{(N_0^w(\mathbb{A}_\mathbb{Q})\cap \widetilde{w}N_0(\mathbb{A}_\mathbb{Q})\widetilde{w}^{-1})\backslash N_0^w(\mathbb{A}_\mathbb{Q})} \phi_s(\widetilde{w}^{-1}ng)\mathrm{d}n$$

where $M(\widetilde{w},\chi_s)$ is the local intertwining operator given for $g_v \in G(\mathbb{Q}_v)$ by

$$(M(\widetilde{w},\chi_{v,s})\phi_{v,s})(g_v) = \int_{(N_0^w(\mathbb{Q}_v)\cap \widetilde{w}N_0(\mathbb{Q}_v)\widetilde{w}^{-1})\backslash N_0^w(\mathbb{Q}_v)} \phi_{v,s}(\widetilde{w}^{-1}n_v g_v)\mathrm{d}n_v$$

Let $\psi = \bigotimes_v \psi_v : \mathbb{Q}\setminus \mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ be an additive Hecke character. Let S be a finite set of places of $\mathbb{Q}$ including ∞ such that for all $v \notin S$ the characters χ_v and ψ_v are unramified, and $\phi_{v,s} = \phi_{v,s}^{\mathrm{sph}}$ is the unique K_v-spherical function normalized by $\phi_{v,s}^{\mathrm{sph}}(e_v) = 1$. By Gindikin-Karpelevich, for all $v \notin S$ we have

$$M(\widetilde{w},\chi_{v,s})\phi_{v,s}^{\mathrm{sph}} = \prod_{\substack{\alpha\in\Phi_G^+ \\ w(\alpha)\in\Phi_G^-}} \frac{\Lambda(\langle s,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)}{\Lambda(\langle s,\alpha^\vee\rangle+1, \chi_v\circ\alpha^\vee)} \phi_{v,ws}^{\mathrm{sph}}$$

Consider the Langlands normalization factor $r(w,\chi_s)=\prod_v r(\widetilde{w},\chi_{v,s},\psi_v)$ given by

$$r(w,\chi_s)=\prod_{\substack{\alpha\in\Phi_G^+\\ w(\alpha)\in\Phi_G^-}}\frac{\Lambda(\langle s,\alpha^\vee\rangle,\chi\circ\alpha^\vee)}{\Lambda(\langle s,\alpha^\vee\rangle+1,\chi\circ\alpha^\vee)\epsilon(\langle s,\alpha^\vee\rangle,\chi\circ\alpha^\vee)}$$

where $r(\widetilde{w},\chi_{v,s},\psi_v)$ is the local Langlands normalization factor given by

$$r(\widetilde{w},\chi_{v,s},\psi_v)=\prod_{\substack{\alpha\in\Phi_G^+\\ w(\alpha)\in\Phi_G^-}}\frac{L(\langle s,\alpha^\vee\rangle,\chi_v\circ\alpha^\vee)}{L(\langle s,\alpha^\vee\rangle+1,\chi_v\circ\alpha^\vee)\epsilon(\langle s,\alpha^\vee\rangle,\chi_v\circ\alpha^\vee,\psi_v)}$$

We will use these formulas extensively in later sections.

2.2.2 Franke-Schwermer Filtration

We now recall the notion of associate classes of cuspidal automorphic representations which gives rise to the decomposition of $\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ\setminus G(\mathbb{A}_\mathbb{Q}))$ according to cuspidal support, and which leads to the definition of the Franke-Schwermer filtration.

Definition 2.2.13. Let $[P]\in\mathcal{C}$ be an associate class of parabolic $\mathbb{Q}$-subgroups of G. An associate class $\varphi=\{\varphi_P\}_{P\in[P]}$ of cuspidal automorphic representations of Levi quotients of parabolics in $[P]$ with infinitesimal characters annihilated by $\mathcal{J}_\lambda$ consists of, for each $P\in[P]$ with Levi quotient M, a finite set φ_P of cuspidal automorphic representations π of $M(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\pi : A_P(\mathbb{A}_\mathbb{Q})\to\mathbb{C}^\times$ trivial on $A_P(\mathbb{Q})$ occurring in $L^2_{\text{cusp}}(M(\mathbb{Q})\setminus M(\mathbb{A}_\mathbb{Q}),\omega_\pi)$ satisfying:

- For $P,P'\in[P]$, for $\pi\in\varphi_P$, and for $g\in G(\mathbb{Q})$ such that $\mathrm{Int}(g)M'=M$, we have $\pi'=\mathrm{Int}(g)^*\pi\in\varphi_{P'}$.
- For $\varphi'=\{\varphi'_P\}_{P\in[P]}$ satisfying the above condition such that $\varphi'_P\subseteq\varphi_P$ for all $P\in[P]$, we have $\varphi'_P=\varphi_P$ for all $P\in[P]$.
- For ξ_φ the $W(\mathfrak{h},\mathfrak{g}_\mathbb{C})$-orbit in $\mathfrak{h}^\vee$ containing the $W(\mathfrak{h},\mathfrak{m}_\mathbb{C})$-orbit ξ_π in $\mathfrak{h}^\vee$ of the infinitesimal charcter of π_∞ for $\pi\in\varphi_P$ and $P\in[P]$ (which is independent of $\pi\in\varphi_P$ and $P\in[P]$ by the above condition), we have that ξ_φ is annihilated by $\mathcal{J}_\lambda$.

For an associate class $[P]\in\mathcal{C}$ let $\Phi_{\lambda,[P]}$ be the set of associate classes of cuspidal automorphic representations of Levi quotients of parabolics in $[P]$ with infinitesimal characters annihilated by $\mathcal{J}_\lambda$. For $P\in[P]$ with Levi quotient

M a cuspidal automorphic representation π of $M(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\pi : A_P(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}^\times$ trivial on $A_P(\mathbb{Q})$ occurring in $L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)$ determines a unique associate class $\varphi = \{\varphi_P\}_{P \in [P]} \in \Phi_{\lambda,[P]}$ by the above properties.

For $[P] \in \mathcal{C}$ and for $\varphi = \{\varphi_P\}_{P\in[P]} \in \Phi_{\lambda,[P]}$ let $\mathcal{A}_{\lambda,[P],\varphi}(G) = \mathcal{A}_{\lambda,[P],\varphi}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of $\mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ given by

$$\mathcal{A}_{\lambda,[P],\varphi}(G) = \left\{ \begin{matrix} \text{functions } \phi \in \mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \text{ with constant term} \\ \phi_P \in \bigoplus_{\pi\in\varphi_P} L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)_\pi \otimes \mathrm{Sym}((\mathfrak{a}_P^G)^\vee_\mathbb{C}) \text{ for all } P \in [P] \end{matrix} \right\}$$

where $L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)_\pi$ is the π-isotypic subspace of $L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)$ for $P \in [P]$ with Levi quotient M, and where $L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)_\pi \otimes \mathrm{Sym}((\mathfrak{a}_P^G)^\vee_\mathbb{C})$ is identified with the space of functions $\phi : M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ represented by finite sums $\phi(g) = \sum_i \phi_i(g)p_i(H_P(g))$ where p_i is a polynomial function on $\mathfrak{a}^G_{P,\mathbb{C}}$ and $\phi_i \in L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)_\pi$.

We now describe these $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-modules concretely in terms of automorphic Eisenstein series. Let P be a proper parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P = MN$, let π be a cuspidal automorphic representation of $M(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\pi : A_P(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}^\times$ trivial on $A_P(\mathbb{Q})$, occurring in $L^2_{\mathrm{disc}}(M(\mathbb{Q})\setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)$. Let $\mathrm{d}\omega_\pi \in (\mathfrak{a}_P^G)^\vee$ be the differential of the restriction of ω_π to $A_P(\mathbb{R})^\circ/A_G(\mathbb{R})^\circ$ and consider the unitarization $\pi' = e^{-\langle H_P(\cdot),\mathrm{d}\omega_\pi\rangle}\pi$ occurring in $L^2_{\mathrm{disc}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}))$. For $s \in \mathfrak{a}^\vee_{P,\mathbb{C}}$ consider the normalized parabolic induction $\mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P(\mathbb{A}_\mathbb{Q})}(\pi', s)^{m_{\mathrm{disc}}(\pi')} = \mathrm{Ind}^{G(\mathbb{A}_\mathbb{Q})}_{P(\mathbb{A}_\mathbb{Q})}(e^{\langle H_P(\cdot), s+\rho_P\rangle}\pi')^{m_{\mathrm{disc}}(\pi')}$ regarded as a $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K'_\infty)$-module acting on the Hilbert space

$$W_{P,\pi'} = \left\{ \begin{matrix} \text{smooth } K\text{-finite functions } \phi : M(\mathbb{Q})N(\mathbb{A}_\mathbb{Q})A_P(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C} \\ \text{such that } \phi(\cdot g) \in L^2_{\mathrm{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}))_{\pi'} \text{ for all } g \in G(\mathbb{A}_\mathbb{Q}) \end{matrix} \right\}$$

where $L^2_{\mathrm{disc}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}))_{\pi'}$ is the π'-isotypic subspace of $L^2_{\mathrm{disc}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}))$.

For $\phi \in W_{P,\pi'}$ and $s \in \mathfrak{a}^\vee_{P,\mathbb{C}}$ recall the automorphic Eisenstein series $\mathrm{Eis}^G_P(\phi_s)$ given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\mathrm{Eis}^G_P(\phi_s)(g) = \sum_{\gamma\in P(\mathbb{Q})\setminus G(\mathbb{Q})} \phi_s(\gamma g) = \sum_{\gamma\in P(\mathbb{Q})\setminus G(\mathbb{Q})} \phi(\gamma g)e^{\langle H_P(\gamma g), s+\rho_P\rangle}$$

which admits an analytic continuation to a meromorphic function in $s \in \mathfrak{a}_{P,\mathbb{C}}^\vee$ with poles along root hyperplanes. There exists a polynomial function h_0 on $\mathfrak{a}_P^\vee$ such that for all $\phi \in W_{P,\pi'}$ the function $h_0(s)\mathrm{Eis}_P^G(\phi_s)$ is holomorphic in a neighborhood of $s_0 = \mathrm{d}\omega_\pi \in (\mathfrak{a}_P^G)^\vee$. After choosing Cartesian coordinates $z_1(s), \ldots, z_r(s)$ on $\mathfrak{a}_P^\vee$ we have the Taylor expansion

$$h_0(s)\mathrm{Eis}_P^G(\phi_s) = \sum_{i_1,\ldots,i_r \geq 0} a_{i_1,\ldots,i_r}(\phi) z_1(s-s_0)^{i_1} \ldots z_r(s-s_0)^{i_r}$$

Then $\mathcal{A}_{\lambda,[P],\varphi}(G) = \mathcal{A}_{\lambda,[P],\varphi}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is equivalently the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of $\mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ spanned by the Taylor coefficients $a_{i_1,\ldots,i_r}(\phi)$ for $i_1, \ldots, i_r \geq 0$ and $\phi \in W_{P,\pi'}$; this does not depend on the choice of the polynomial function h_0 on $\mathfrak{a}_P^\vee$, and by the functional equations for $\mathrm{Eis}_P^G(\phi_s)$ this does not depend on the choice of $P \in [P]$ and $\pi \in \varphi_P$. In other words this is spanned by all residues and derivatives of automorphic Eisenstein series with cuspidal support φ evaluated at points s_0 in the closure of the positive Weyl chamber defined by P.

To give some idea of this equivalence let P' be another parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P' = M'N'$ and consider the constant term $\mathrm{Eis}_P^G(\phi, s)_{P'}$ given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\mathrm{Eis}_P^G(\phi_s)_{P'}(g) = \int_{N'(\mathbb{Q})\setminus N'(\mathbb{A}_\mathbb{Q})} \mathrm{Eis}_P^G(\phi_s)(ng)\mathrm{d}g$$

Then $\mathrm{Eis}_P^G(\phi_s)$ has a pole at $s = s_0$ precisely if there exists a parabolic $\mathbb{Q}$-subgroup P' of G such that the constant term $\mathrm{Eis}_P^G(\phi_s)_{P'}$ has a pole at $s = s_0$. By Langlands the constant term is given for $g \in G(\mathbb{A}_\mathbb{Q})$ by

$$\mathrm{Eis}_P^G(\phi_s)_{P'}(g) = \sum_{w \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})} (M(w,s)\phi)(g) e^{\langle H_{P'}(g), ws + \rho_{P'} \rangle}$$

and it follows that $D(h_0(s)\mathrm{Eis}(\phi, s))|_{s=s_0} \in \mathcal{A}_{\lambda,[P],\varphi}(G)$. For the converse, one needs to show that any function $f \in \mathcal{A}_{\lambda,[P],\varphi}(G)$, that is any function $f \in \mathcal{A}_{\lambda,[P]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ with constant term $f_P \in \bigoplus_{\pi \in \varphi_P} L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \setminus M(\mathbb{A}_\mathbb{Q}), \omega_\pi)_\pi \otimes \mathrm{Sym}((\mathfrak{a}_P^G)^\vee_\mathbb{C})$ for all $P \in [P]$, can be written as a linear combination of residues and derivatives of automorphic Eisenstein series with cuspidal support φ evaluated at points s_0 in the closure of the positive Weyl chamber defined by P.

The symmetric algebra $\mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee) = \bigoplus_{n\geq 0} \mathrm{Sym}^n((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ on $(\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$ is regarded as a $P(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{p}, K_\infty^P)$-module as follows. Regarding $\mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ as the algebra of polynomial functions on $\mathfrak{a}_{P,\mathbb{C}}^G$, an element $X \in \mathfrak{a}_{P,\mathbb{C}}^G$ acts on $P \in \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ by translation, which defines the structure of a $P(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{p}, K_\infty^P)$-module with $P(\mathbb{A}_{\mathbb{Q}}^\infty)$ acting trivially.

Regarding $\mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ as the algebra of polynomial differential operators on $(\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$, the tensor product $W_{P,\pi_0} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ is regarded as a $G(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{g}, K_\infty)$-module as follows. Let $\mathcal{S}$ be the space of functions $f(g,s)$ on $G(\mathbb{A}_{\mathbb{Q}}) \times (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$ which are smooth and compactly supported in $g \in G(\mathbb{A}_{\mathbb{Q}})$ and which are holomorphic in $s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$, and let $\mathcal{D}(\mathcal{S})$ be the space of distributions on $\mathcal{S}$ which are compactly supported in $s \in (\mathfrak{a}_P^G)_{\mathbb{C}}^\vee$. The tensor product $W_{P,\pi'} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ is regarded as a subspace of $\mathcal{D}(\mathcal{S})$ where the simple tensor $\phi \otimes D \in W_{P,\pi'} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$ is identified with the distribution given for $f \in \mathcal{S}$ by

$$(\phi \otimes D)(f) = D\Big(\int_{G(\mathbb{A}_{\mathbb{Q}})} \phi(g) f(g,\cdot)\mathrm{d}g\Big)$$

where an element $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$ acts by $(g(\phi\otimes D))(x,s) = (e^{\langle H_P(xg)-H_P(x),s\rangle}(\phi\otimes D))(xg,s)$, where an element $X \in \mathfrak{g}$ acts by $(X(\phi\otimes D))(g,s) = ((X\phi)\otimes D)(g,s) + (\langle XH_P(g), s\rangle(\phi\otimes D))(g,s)$, and where an element $k \in K_\infty$ acts by $(k(\phi\otimes D))(g,s) = (\phi\otimes D)(gk,s)$. The expressions for these actions can be written as finite sums of simple tensors in $W_{P,\pi'} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee)$, and hence define the structure of a $G(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{g}, K_\infty)$-module.

Now the above construction yields surjective morphism of $G(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{g}, K_\infty)$-modules

$$\begin{aligned}\mathcal{E}_{\lambda,[P],\varphi} : W_{P,\pi'} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee) &\to \mathcal{A}_{\lambda,[P],\varphi}(G)\\ \phi\otimes D &\mapsto D(h_0(s)\mathrm{Eis}_P^G(\phi_s))|_{s=s_0}\end{aligned}$$

which does not depend on the choice of the polynomial function h_0 on $\mathfrak{a}_P^\vee$, recalling that for all $\phi \in W_{P,\pi'}$ the function $h_0(s)\mathrm{Eis}_P^G(\phi_s)$ is holomorphic in a neighborhood of $s_0 = \mathrm{d}\omega_\pi \in (\mathfrak{a}_P^G)^\vee$. If for all $\phi \in W_{P,\pi'}$ the function $\mathrm{Eis}_P^G(\phi_s)$ is holomorphic at $s_0 = \mathrm{d}\omega_\pi \in (\mathfrak{a}_P^G)^\vee$, then we can take $h_0 = 1$ and we obtain an isomoprhism of $G(\mathbb{A}_{\mathbb{Q}}^\infty) \times (\mathfrak{g}, K_\infty)$-modules

$$\begin{aligned}\mathcal{E}_{\lambda,[P],\varphi} : W_{P,\pi'} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee) &\xrightarrow{\sim} \mathcal{A}_{\lambda,[P],\varphi}(G)\\ \phi\otimes D &\mapsto D(\mathrm{Eis}_P^G(\phi_s))|_{s=s_0}\end{aligned}$$

Now we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$W_{P,\pi'} \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^{\vee}) \simeq \mathcal{I}_{P(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^{\vee}), \mathrm{d}\omega_\pi)^{m_{\mathrm{disc}}(\pi')}$$

so the above construction realizes the normalized parabolic induction $\mathcal{I}_{P(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^{\vee}), \mathrm{d}\omega_\pi)^{m_{\mathrm{disc}}(\pi')}$ in $\mathcal{A}_{\lambda,[P],\varphi}(G)$. This leads to the definition of the Franke-Schwermer filtration on $\mathcal{A}_{\lambda,[P],\varphi}(G)$.

Let $\mathcal{M}_{\lambda,[P],\varphi}(G)$ be the set of tuples (P', π', ν, s_0) such that:

- $P' \in [P]$ is a standard parabolic $\mathbb{Q}$-subgroup of G (with Levi quotient M') containing a parabolic $\mathbb{Q}$-subgroup $P \in [P]$;
- $\pi' \in \Pi_{\mathrm{disc}}(M'(\mathbb{A}_{\mathbb{Q}}))$ is a unitary discrete spectrum automorphic representation of $M'(\mathbb{A}_{\mathbb{Q}})$ (occurring in $L^2_{\mathrm{disc}}(M'(\mathbb{Q})A_{P'}(\mathbb{R})^\circ \setminus M'(\mathbb{A}_{\mathbb{Q}}))$ with multiplicity $m_{\mathrm{disc}}(\pi')$) which is the iterated residue at $\nu \in (\mathfrak{a}_P^{P'})_{\mathbb{C}}^{\vee}$ of Eisenstein series attached to $\pi \in \varphi_P$;
- $s_0 \in \mathfrak{a}_{P',\mathbb{C}}^{\vee}$ is a point with $\Re(s_0) \in \overline{(\mathfrak{a}_{P'}^G)^{\vee +}}$ such that $e^{\langle H_{P'}(\cdot), s_0\rangle}\pi'$ has cuspidal support in φ; in particular $s_0 + \nu + \xi_\varphi$ is annihilated by $\mathcal{J}_\lambda$.

Let $\mathcal{M}^m_{\lambda,[P],\varphi}(G)$ be the set of tuples (P', π', ν, s_0) where P' has relative rank m. We regard $\mathcal{M}^m_{\lambda,[P],\varphi}(G)$ as a groupoid with objects (P', π', ν, s_0) and morphisms $(P_1', \pi_1', s_1) \to (P_2', \pi_2', s_2)$ corresponding to $w \in W^{P_1'}$ such that $w(M_1') = M_2'$ and $w(\pi_1') = \pi_2'$ and $w(s_1) = s_2$ (the $\nu \in (\mathfrak{a}_P^{P'})_{\mathbb{C}}^{\vee}$ is in some sense redundant information, but we include it as part of the definition). We have a functor

$$\begin{aligned} \mathcal{M}^m_{\lambda,[P],\varphi}(G) &\to \mathrm{Mod}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)) \\ (P', \pi', \nu, s_0) &\mapsto \mathcal{I}_{P'(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}_{P'}^G)_{\mathbb{C}}^{\vee}), s_0)^{m_{\mathrm{disc}}(\pi')} \end{aligned}$$

Let $S_{\lambda,[P],\varphi} = \{\iota(s_0) \in \mathfrak{a}_{P_0}^{\vee} | (P', \pi', \nu, s_0) \in \mathcal{M}_{\lambda,[P],\varphi}(G)\}$ be the finite subset of $\mathfrak{a}_{P_0}^{\vee}$ defined by the embedding $\iota : (\mathfrak{a}_{P'}^G)^{\vee} \hookrightarrow \mathfrak{a}_{P_0}^{\vee}$, the support of the $s_0 \in \mathfrak{a}_{P',\mathbb{C}}^{\vee}$ in the indexing set $\mathcal{M}_{\lambda,[P],\varphi}(G)$.

Choose a function $T_{\lambda,[P],\varphi} : S_{\lambda,[P],\varphi} \to \mathbb{Z}$ such that $T_{\lambda,[P],\varphi}(s_1) > T_{\lambda,[P],\varphi}(s_2)$ if $s_1 \succ s_2$, that is if $s_1 \neq s_2$ and $s_1 - s_2 \in \overline{\mathfrak{a}_{P_0}^{\vee -}}$. Now we have the following absolutely crucial theorem:

Theorem 2.2.14. (Franke-Schwermer [35, Theorem 14], [**?** , Theorem 4]) We have a finite decreasing filtration of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_{\infty})$-modules

$$\ldots \subseteq \mathcal{A}_{\lambda,[P],\varphi}^{i+1}(G) \subseteq \mathcal{A}_{\lambda,[P],\varphi}^{i}(G) \subseteq \ldots \subseteq \mathcal{A}_{\lambda,[P],\varphi}(G)$$

such that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_{\infty})$-modules

$$\mathcal{A}_{\lambda,[P],\varphi}^{i}(G)/\mathcal{A}_{\lambda,[P],\varphi}^{i+1}(G) \simeq \bigoplus_{0 \leq j \leq \mathrm{rank}(P)} \varinjlim_{\substack{(P',\pi',\nu,s_0) \in \mathcal{M}_{\lambda,[P],\varphi}^{j}(G) \\ T_{\lambda,[P],\varphi}(\iota(s_0))=i}} \mathcal{I}_{P'(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}_{P'}^{G})_{\mathbb{C}}^{\vee}), s_0)^{m_{\mathrm{disc}}(\pi')}$$

where the colimit is taken over the full subcategory of $\mathcal{M}_{\lambda,[P],\varphi}^{m}(G)$ defined by $T_{\lambda,[P],\varphi}$. Different choices of $T_{\lambda,[P],\varphi}$ give rise to different filtrations either with the same quotients or with consecutive quotients replaced by their direct sum. We can choose $T_{\lambda,[P],\varphi}$ so that the length of the filtration is minimal.

Example 2.2.15. We describe the Franke-Schwermer filtration for maximal parabolics, following [47]. Let P_1 be a standard maximal proper parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P_1 = M_1 N_1$. We have a unique simple root $\alpha_1 \in \Phi_G$ which is not a root in Φ_{M_1}, and the space $(\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee}$ is 1-dimensional with basis $\beta_1 = \langle \rho_{P_1}, \alpha_1^{\vee} \rangle^{-1} \rho_{P_1}$ with open positive Weyl chamber $(\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee +} = \{s\beta_1 \in (\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee} | \Re(s) > 0\}$. Let $w_0 \in W_{M_1}$ be the unique nontrivial element and let $P_2 = P_1^{w_0}$ be the conjugate of P_1 with Levi decomposition $P_2 = M_2 N_2$; if P_1 is self-associate we have $[P_1] = \{P_1\}$, otherwise $[P_1] = \{P_1, P_2\}$. We have a unique simple root $\alpha_2 \in \Phi_G$ which is not a root in Φ_{M_2}, and the space $(\mathfrak{a}_{P_2}^{G})_{\mathbb{C}}^{\vee}$ is 1-dimensional with basis $\beta_2 = \langle \rho_{P_2}, \alpha_2^{\vee} \rangle^{-1} \rho_{P_2}$ with open positive Weyl chamber $(\mathfrak{a}_{P_2}^{G})_{\mathbb{C}}^{\vee +} = \{s\beta_2 \in (\mathfrak{a}_{P_2}^{G})_{\mathbb{C}}^{\vee} | \Re(s) > 0\}$.

Let π_1 be a cuspidal automorphic representation of $M_1(\mathbb{A}_{\mathbb{Q}})$ and consider the unitarization $\pi_1' = e^{-\langle H_{P_1}(\cdot), s_0\beta_1 \rangle} \pi_1$ where $s_0 \in \mathbb{R}$. Let $\pi_2 = \pi_1^{w_0}$ be the conjugate of π_1 and consider the unitarization $\pi_2' = e^{-\langle H_{P_2}(\cdot), s_0\beta_2 \rangle} \pi_2$.

For $\phi \in W_{P_1,\pi_1'}$ and $s\beta_1 \in (\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee}$ consider the Eisenstein series $\mathrm{Eis}_{P_1}^{G}(\phi_{s\beta_1})$ which has a finite number of simple poles in the real interval $\{s\beta_1 \in (\mathfrak{a}_{P_1}^{G})^{\vee} | 0 < s \leq \langle \rho_{P_1}, \alpha_1^{\vee} \rangle\}$ and all other poles in the region $\{s\beta_1 \in (\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee} | \Re(s) < 0\}$. Consider the space of residual Eisenstein series

$$\mathcal{L}_{\lambda,[P_1],\varphi}(G) = \{\mathrm{Res}_{s=s_0} \mathrm{Eis}_{P_1}^{G}(\phi_{s\beta_1}) | f \in W_{P_1,\pi_1'}\}$$

Then $\mathcal{L}_{\lambda,[P_1],\varphi}(G)$ is a $G(\mathbb{A}_{\mathbb{Q}}^{\infty})\times(\mathfrak{g},K_\infty)$-submodule of $\mathcal{A}_{\lambda,[P_1],\varphi}(G)$ which is nontrivial precisely if the Eisenstein series $\mathrm{Eis}_{P_1}^{G}(\phi_{s\beta_1})$ has a pole at $s=s_0$. By Langlands the automorphic forms in $\mathcal{L}_{\lambda,[P_1],\varphi}(G)$ are square-integrable.

Now by [47, Theorem 3.1] we have a decreasing filtration of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})\times(\mathfrak{g},K_\infty)$-modules

$$0\subseteq\mathcal{L}_{\lambda,[P_1],\varphi}(G)\subseteq\mathcal{A}_{\lambda,[P_1],\varphi}(G)$$

where the quotient $\mathcal{A}_{\lambda,[P_1],\varphi}(G)/\mathcal{L}_{\lambda,[P_1],\varphi}(G)$ is nontrivial, and we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})\times(\mathfrak{g},K_\infty)$-modules

$$\mathcal{A}_{\lambda,[P_1],\varphi}(G)/\mathcal{L}_{\lambda,[P_1],\varphi}(G)\simeq\begin{cases}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi_1',0)\otimes\mathrm{Sym}((\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee}))^{+} & s_0=0\text{ and }P_2=P_1\text{ and }\pi_2'\simeq\pi_1'\\ \mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi_1',s_0\beta_1)\otimes\mathrm{Sym}((\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee}) & \text{otherwise}\end{cases}$$

where $(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi_1',0)\otimes\mathrm{Sym}((\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee}))^{+}$ is the $+1$-eigenspace of the self-intertwining operator $M(w_0,0)$ acting on $\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi_1',0)\otimes\mathrm{Sym}((\mathfrak{a}_{P_1}^{G})_{\mathbb{C}}^{\vee})$.

To see this, note that we have $(G,\mathcal{L}_{\lambda,[P_1],\varphi}(G),0)\in\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ precisely if $\mathcal{L}_{\lambda,[P_1],\varphi}(G)\neq 0$, and we have $(P_1,\pi_1',s_0\beta_1)\in\mathcal{M}_{\lambda,[P_1],\varphi}(G)$, but we only have $(P_2,\pi_2',-s_0\beta_2)\in\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ if $s_0=0$ since $-s_0\beta_2\notin\overline{(\mathfrak{a}_{P_2}^{G})^{\vee+}}$ if $s_0>0$. We have the following four cases for $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$:

$$\mathcal{M}_{\lambda,[P_1],\varphi}(G)=\begin{cases}\{(G,\mathcal{L}_{\lambda,[P_1],\varphi}(G),0),(P_1,\pi_1',s_0\beta_1)\} & \mathcal{L}_{\lambda,[P_1],\varphi}(G)\neq 0\text{ so that }s_0>0\text{ and }P_2=P_1\text{ and }\pi_2'\simeq\pi_1'\\ \{(P_1,\pi_1',s_0\beta_1)\} & \mathcal{L}_{\lambda,[P_1],\varphi}(G)=0\text{ and }s_0>0\\ \{(P_1,\pi_1',0)\} & s_0=0\text{ and }P_2=P_1\text{ and }\pi_2'\simeq\pi_1'\text{ so that }\mathcal{L}_{\lambda,[P_1],\varphi}(G)=0\\ \{(P_1,\pi_1',0),(P_2,\pi_2',0)\} & s_0=0\text{ and }P_2\neq P_1\text{ or }\pi_2'\not\simeq\pi_1'\text{ so that }\mathcal{L}_{\lambda,[P_1],\varphi}(G)=0\end{cases}$$

In the first and second cases the only morphisms in $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ are the identity, in the third case the only morphisms in $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ are the identity and w_0, and in the fourth case the only morphisms in $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ are the identities and w_0 and w_0^{-1}.

In the first case we have $S_{\lambda,[P_1],\varphi}=\{0,s_0\beta_1\}$, in the second case we have $S_{\lambda,[P_1],\varphi}=\{s_0\beta_1\}$, and in the third and fourth cases we have $S_{\lambda,[P_1],\varphi}=\{0\}$. Define $T_{\lambda,[P_1],\varphi}:S_{\lambda,[P_1],\varphi}\to\mathbb{Z}$ such that $T_{\lambda,[P_1],\varphi}(s_0\beta_1)=0$ for $s_0>0$, such that $T_{\lambda,[P_1],\varphi}(0)=1$ in the first case, and such that $T_{\lambda,[P_1],\varphi}(0)=0$ in the third and fourth cases.

In the first case the groupoid $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ is discrete and $T_{\lambda,[P_1],\varphi}$ has range $\{0,1\}$ so the Franke-Schwermer filtration is given by

$$\mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \simeq \mathcal{M}_{\lambda,[P_1],\varphi}(G, \mathcal{L}_{\lambda,[P_1],\varphi}(G), 0) \simeq \mathcal{L}_{\lambda,[P_1],\varphi}(G)$$
$$\mathcal{A}^0_{\lambda,[P_1],\varphi}(G)/\mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \simeq \mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', s_0\beta_1) \simeq \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', s_0\beta_1) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C})$$

In the second case the groupoid $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ is discrete and $T_{\lambda,[P_1],\varphi}$ has range $\{0\}$ so the Franke-Schwermer filtration is given by

$$\mathcal{A}^0_{\lambda,[P_1],\varphi}(G) \simeq \mathcal{M}_{\lambda,[P_1],\varphi}(P, \pi_1', s_0\beta_1) \simeq \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', s_0\beta_1) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C})$$

In the third case the groupoid $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ has a nontrivial morphism w_0 and $T_{\lambda,[P_1],\varphi}$ has range $\{0\}$ so the Franke-Schwermer filtration is given by the coequalizer

$$\mathcal{A}^0_{\lambda,[P_1],\varphi}(G) \simeq \mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0)/W \simeq (\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', 0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C}))^+$$

where W is the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of $\mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0) \simeq \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', 0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C})$ generated by $x - M(w_0, 0)x$ for $x \in \mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0)$. Now by the functional equations for standard intertwining operators $M(w_0, 0)^2$ is the identity and W is the -1-eigenspace hence $\mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0)/W \simeq (\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', 0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C}))^+$ is the $+1$-eigenspace.

In the fourth case the groupoid $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ has nontrivial morphisms w_0 and w_0^{-1} and $T_{\lambda,[P_1],\varphi}$ has range $\{0\}$ so the Franke-Schwermer filtration is given by the coequalizer

$$\mathcal{A}^0_{\lambda,[P_1],\varphi}(G) \simeq (\mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0) \oplus \mathcal{M}_{\lambda,[P_1],\varphi}(P_2, \pi_2', 0))/W \simeq \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', 0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C})$$

where W is the $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-submodule of the direct sum $\mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0) \oplus \mathcal{M}_{\lambda,[P_1],\varphi}(P_2, \pi_2', 0) \simeq (\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', s_0\beta_1) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C}))^{\oplus 2}$ generated by $x - M(w_0, 0)x$ for $x \in \mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0)$ hence $\mathcal{M}_{\lambda,[P_1],\varphi}(P_1, \pi_1', 0) \simeq \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_1', 0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_1}^G)^\vee_\mathbb{C})$.

Cohomology of Franke-Schwermer Filtration We now need to understand the cohomology of the induced modules appearing in the successive quotients of the Franke filtration.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ contained in $\mathfrak{m}_{\mathbb{C}}$ and choose an ordering on the roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ which makes $\mathfrak{p}_{\mathbb{C}}$ standard. Let $W = W(\mathfrak{h}, \mathfrak{g}_{\mathbb{C}})$ be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ and consider the subset

$$W^P = \{w \in W \mid w^{-1}\alpha > 0 \text{ for all } \alpha \in \Phi_M^+\}$$

which is a set of representatives for the quotient W/W_P. For $\rho = \frac{1}{2}\sum_{\alpha \in \Phi_G^+} \alpha$ the half sum of positive roots and for $\lambda \in \mathfrak{h}^\vee$ a dominant weight with highest weight $\mathfrak{g}_{\mathbb{C}}$-module V_λ consider the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ and the corresponding highest weight $\mathfrak{g}_{\mathbb{C}}$-module $V_{w\cdot\lambda}$. By Kostant we have an isomorphism of $\mathfrak{m}_{\mathbb{C}}$-modules

$$H^k(\mathfrak{n}, V_\lambda) = \bigoplus_{\substack{w \in W^P \\ \ell(w) = k}} V_{w\cdot\lambda}$$

Now we have the following crucial result of Borel-Wallach which we will use repeatedly in later sections:

Theorem 2.2.16. (Borel-Wallach [20, III Theorem 3.3]) Let $\lambda \in \mathfrak{h}^\vee$ be a dominant weight and let V_λ be the corresponding irreducible $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ.

Suppose that $H^*(\mathfrak{g}, K'_\infty; \mathcal{I}_{P(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee), s_0) \otimes V_\lambda) \neq 0$. Then there exists a unique $w \in W^P$ yielding an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-modules

$$H^k(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}_P^G)_{\mathbb{C}}^\vee), s_0) \otimes V_\lambda) \simeq \mathcal{I}_{P(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi'_{\mathrm{fin}}) \otimes H^{k-\ell(w)}(\mathfrak{m}, K'^P_\infty; \pi'_\infty \otimes V_{w\cdot\lambda})$$

Theorem 2.2.17. Let $\lambda \in \mathfrak{h}^\vee$ be a dominant weight and let V_λ be the corresponding irreducible $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ. We have a spectral sequence

$$E_1^{p,q} = \bigoplus_{[P] \in \mathcal{C}} \bigoplus_{\varphi \in \Phi_{\lambda,[P]}} \bigoplus_{(P',\pi',\nu,s_0) \in \mathcal{M}^p_{\lambda,[P],\varphi}(G)} \mathcal{I}_{P(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi'_{\mathrm{fin}})^{K_{\mathrm{fin}}} \otimes H^{p+q-\ell(w)}(\mathfrak{g}, K'_\infty; \pi'_\infty \otimes V_{w\cdot\lambda})$$
$$\Rightarrow H^{p+q}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

If λ is a regular highest weight, then the spectral sequence degenerates at the E_1-page.

Proof. We consider the spectral sequence for the Franke-Schwermer filtration:

$$E_1^{p,q} = H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{A}^p_{\lambda,[P],\varphi}(G)/\mathcal{A}^{p+1}_{\lambda,[P],\varphi}(G) \otimes V_\lambda) \Rightarrow H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P],\varphi}(G) \otimes V_\lambda)$$

By Franke the E_1-page of this spectral sequence is given by

$$\begin{aligned} E_1^{p,q} &\simeq \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^p_{\lambda,[P],\varphi}(G)} H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P'(\mathbb{A}_\mathbb{Q})}(\pi' \otimes \mathrm{Sym}((\mathfrak{a}^G_{P'})^\vee_\mathbb{C}), s_0) \otimes V_\lambda) \\ &\simeq \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^p_{\lambda,[P],\varphi}(G)} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P(\mathbb{A}^\infty_\mathbb{Q})}(\pi'_{\mathrm{fin}}) \otimes H^{p+q-\ell(w)}(\mathfrak{g}, K'_\infty; \pi'_\infty \otimes V_{w\cdot\lambda}) \end{aligned}$$

Then it follows that we have a spectral sequence

$$E_1^{p,q} = \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^p_{\lambda,[P],\varphi}(G)} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P(\mathbb{A}^\infty_\mathbb{Q})}(\pi'_{\mathrm{fin}}) \otimes H^{p+q-\ell(w)}(\mathfrak{g}, K'_\infty; \pi'_\infty \otimes V_{w\cdot\lambda}) \Rightarrow H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P],\varphi}(G) \otimes V_\lambda)$$

Now by Franke-Schwermer we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = \bigoplus_{[P]\in\mathcal{C}} \bigoplus_{\varphi\in\Phi_{\lambda,[P]}} \mathcal{A}_{\lambda,[P],\varphi}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$$

Then it follows that we have a spectral sequence

$$\begin{aligned} E_1^{p,q} = \bigoplus_{[P]\in\mathcal{C}} \bigoplus_{\varphi\in\Phi_{\lambda,[P]}} \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^p_{\lambda,[P],\varphi}(G)} & \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P(\mathbb{A}^\infty_\mathbb{Q})}(\pi'_{\mathrm{fin}}) \otimes H^{p+q-\ell(w)}(\mathfrak{g}, K'_\infty; \pi'_\infty \otimes V_{w\cdot\lambda}) \\ & \Rightarrow H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{A}_\lambda(G) \otimes V_\lambda) \end{aligned}$$

By Franke we have an isomorphism of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq H^i(\mathfrak{g}, K'_\infty; \mathcal{A}_\lambda(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})/K_{\mathrm{fin}}) \otimes V_\lambda)$$

so the first claim follows by taking K_{fin}-invariants in the above spectral sequence. If λ is a regular highest weight, then the second claim follows by examination of the construction of the Franke-Schwermer filtration: in this case the corresponding automorphic Eisenstein series are always holomorphic at the evaluation point $s_0 = \mathrm{d}\omega_\pi \in (\mathfrak{a}^G_P)^\vee$, in particular the Franke-Schwermer filtration is trivial and the differentials in the above spectral sequence must all vanish. □

2.3 ℓ-adic Cohomology of Shimura Varieties

The main issue which remains to be addressed is how the Galois action on the ℓ-adic cohomology of local systems on Shimura varieties is actually computed, and through what mechanism it can be related to the spectral theory of automorphic forms.

The key insight due to Kottwitz is that one can express the trace of Frobenius on the compactly supported ℓ-adic cohomology (or on the ℓ-adic intersection cohomology) in terms of volumes and (twisted) orbital integrals for certain carefully chosen test functions, and that the resulting expression can be compared with the geometric side of the Arthur-Selberg trace formula applied to these same test functions. By equating this with the spectral side of the Arthur-Selberg trace formula, one obtains the desired expression for the traces of Frobenius in terms of the Satake parameters of certain automorphic representations.

Two main difficulties arise. First, the Arthur-Selberg trace formula only involves volumes and orbital integrals, but does not involve the twisted orbital integrals appearing in the formula of Kottwitz; one must apply a version of the twisted fundamental lemma in order to express these twisted orbital integrals in terms of certain stable orbital integrals. Second, only the stabilization of the Arthur-Selberg trace formula can be compared with the resulting expression involving stable orbital integrals. These difficulties have largely been overcome by work of Kottwitz (some of which remains unpublished, for example [70]), but it remains challenging to evaluate all the terms in the Arthur-Selberg trace formula and its stabilization in order to apply this strategy to explicit examples with any amount of specificity.

The later parts of this thesis attempt to execute some fragment of this strategy, with a great deal of blackboxing involved. The main issue is one of exposition: most of the details of this strategy have presently been worked out for the examples which appear later in the thesis, but a proper exposition of the necessary background and a detailed execution of this strategy would nearly double the length of what is already a massive document. We hope to address this properly in future writing. On the other hand, it would be negligent to completely omit certain remarks about how the Langlands-Kottwitz method and the Arthur-Selberg trace formula is actually being used in the present work, and so we will try to give an extremely brief sketch of this, with pointers to the relevant literature.

2.3.1 Integral Models of Shimura Varieties

For (G, X, h) a Shimura datum and for $V_\lambda \in \mathrm{Rep}(G)$ an absolutely irreducible rational representation, the corresponding local system of $\mathbb{Q}_\ell$-vector spaces $\mathbb{V}_\lambda \otimes \mathbb{Q}_\ell$ on $S_{K_{\text{fin}}}(\mathbb{C})$ is the inverse image of an ℓ-adic local system $\mathbb{V}_\lambda$ on $S_{K_{\text{fin}}}$. Although we will not recall the precise definition of these ℓ-adic local systems (which appears for example in work of Pink [97]), we note that if $S_{K_{\text{fin}}}$ is a PEL Shimura variety (that is a moduli space of Abelian varieties with specified polarization, endomorphism, and level structure) then these ℓ-adic local systems $\mathbb{V}_\lambda$ can be constructed explicitly in terms of a basic ℓ-adic local system $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}_\ell$ where $\pi : U_{K_{\text{fin}}} \to S_{K_{\text{fin}}}$ is the universal family of Abelian varieties, by the same construction of Schur functors used to construct the highest weight irreducible representations V_λ in terms of basic representations V.

Let Σ be a finite set of primes of $\mathbb{Q}$ away from which G is unramified (and away from which K_{fin} and K'_{fin} are unramified). By work of Kai-Wen Lan [72] on integral models of PEL Shimura varieties we may assume the following conditions (see [87, Section 1.3] for a more precise set of conditions and discussion of integral canonical models). There exists a smooth quasiprojective scheme $\mathcal{S}_{K_{\text{fin}}}$ over $\mathcal{O}_F[\frac{1}{\Sigma}]$ with generic fiber $S_{K_{\text{fin}}}$, and there exists a normal projective scheme $\overline{\mathcal{S}}^{\text{BB}}_{K_{\text{fin}}}$ over $\mathcal{O}_F[\frac{1}{\Sigma}]$ with generic fiber $\overline{S}^{\text{BB}}_{K_{\text{fin}}}$ and dense open embedding $j : \mathcal{S}_{K_{\text{fin}}} \hookrightarrow \overline{\mathcal{S}}^{\text{BB}}_{K_{\text{fin}}}$. For $g \in G(\mathbb{A}^\infty_\mathbb{Q})$ and for $K_{\text{fin}}, K'_{\text{fin}} \subseteq G(\mathbb{A}^\infty_\mathbb{Q})$ compact open subgroups such that $K'_{\text{fin}} \subseteq gK_{\text{fin}}g^{-1}$ the finite morphisms T_g and $\overline{T}_g$ extend to these integral models. We have an ℓ-adic sheaf $\mathbb{V}_\lambda$ on $\mathcal{S}_{K_{\text{fin}}}$ which is isomorphic to the ℓ-adic sheaf $\mathbb{V}_\lambda$ on the generic fiber. Moreover the ℓ-adic sheaf $\mathbb{V}_\lambda$ on $\mathcal{S}_{K_{\text{fin}}}$ is pure of the same weight as the absolutely irreducible rational representation $V_\lambda \in \mathrm{Rep}(G)$.

We are then interested in the ℓ-adic cohomology $H^*(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ or the compactly supported ℓ-adic cohomology $H^*_c(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$, meaning one of the following:

- For $\mathcal{S}_{K_{\text{fin}}}$ over $\overline{\mathbb{F}}_q$ (away from the finite set of bad primes Σ), the ℓ-adic cohomology $H^*(\mathcal{S}_{K_{\text{fin}},\overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$ or the compactly supported ℓ-adic cohomology $H^*_c(\mathcal{S}_{K_{\text{fin}},\overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$, regarded as a $\mathbb{Q}_\ell$-linear representation of $\mathcal{H}_{K_{\text{fin}}} \times \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.
- For $\mathcal{S}_{K_{\text{fin}}}$ over $\overline{F}$, the ℓ-adic cohomology $H^*(\mathcal{S}_{K_{\text{fin}},\overline{F}}, \mathbb{V}_\lambda)$ or the compactly supported ℓ-adic cohomology $H^*_c(\mathcal{S}_{K_{\text{fin}},\overline{F}}, \mathbb{V}_\lambda)$, regarded as a $\mathbb{Q}_\ell$-linear representation of $\mathcal{H}_{K_{\text{fin}}} \times \mathrm{Gal}(\overline{F}/F)$.

The action of $\mathcal{H}_{K_{\text{fin}}}$ in either case is constructed in the same way as before, noting that the Hecke correspondences are algebraic and extend to integral models by the above assumptions.

Definition 2.3.1. Define the inner cohomology

$$H^i_!(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \mathrm{im}(H^i_{\mathrm{c}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda))$$

Define the Eisenstein cohomology

$$H^i_{\mathrm{Eis}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \mathrm{coker}(H^i_!(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \hookrightarrow H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda))$$

Define the compactly supported Eisenstein cohomology

$$H^i_{\mathrm{c,Eis}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \ker(H^i_{\mathrm{c}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \twoheadrightarrow H^i_!(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda))$$

It follows that we have short exact sequences

$$0 \to H^i_!(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to H^i_{\mathrm{Eis}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to 0$$
$$0 \to H^i_{\mathrm{c,Eis}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to H^i_{\mathrm{c}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to H^i_!(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \to 0$$

2.3.2 ℓ-adic Intersection Cohomology

The construction of intersection cohomology makes sense algebraically, in particular we can make sense of the ℓ-adic intersection cohomology of $\overline{S}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{F}}$ or of $\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_q}$. We refer specifically to [10, Section 2.2] of [64, III.1] for the basic definitions; they are essentially identical to the previous discussion of intersection cohomology.

Let E be a finite extension of $\mathbb{Q}_\ell$ (or let $E = \overline{\mathbb{Q}}_\ell$). Writing $\mathcal{S}_{K_{\mathrm{fin}}}$ to mean either $\mathcal{S}_{K_{\mathrm{fin}},\overline{F}}$ or $\mathcal{S}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_q}$ and writing $\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}$ to mean either $\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{F}}$ or $\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_q}$, let $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, E)$ be the bounded derived category of complexes of ℓ-adic sheaves which are constructible with respect to the stratification by boundary components. Let $\mathrm{Perv}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, E)$ be the Abelian category of complex perverse sheaves, the heart of the perverse t-structure on $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, E)$.

For $\mathbb{V}$ an ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$ we can construct the ℓ-adic intersection complex $\mathrm{IC}^\bullet(\mathbb{V})[d] \in \mathrm{Perv}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, E)$ exactly as before, in particular we have an isomorphism of ℓ-adic perverse sheaves $\mathrm{IC}^\bullet(\mathbb{V})[d] \simeq j_{!*}(\mathbb{V}[d])$ (see [64, III.5] for further discussion).

Definition 2.3.2. Let $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a neat compact open subgroup. Let $\mathbb{V}$ be an ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$. Consider the ℓ-adic intersection complex on $\overline{\mathcal{S}}_{K_{\text{fin}}}^{\text{BB}}$ given by the intermediate extension

$$\mathrm{IC}^{\bullet}(\mathbb{V}) = (j_{!*}(\mathbb{V}[d]))[-d]$$

and define the ℓ-adic intersection cohomology

$$IH^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}) = \mathbb{H}^i(\overline{\mathcal{S}}_{K_{\text{fin}}}^{\text{BB}}, \mathrm{IC}^{\bullet}(\mathbb{V}))$$

When K_{fin} is not neat we fix a neat compact open subgroup $\widetilde{K}_{\text{fin}} \subseteq K_{\text{fin}}$ of finite index and define the ℓ-adic intersection cohomology

$$IH^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}) = \mathbb{H}^i(\overline{\mathcal{S}}_{\widetilde{K}_{\text{fin}}}^{\text{BB}}, \mathrm{IC}^{\bullet}(\mathbb{V}))^{K_{\text{fin}}/\widetilde{K}_{\text{fin}}}$$

Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ. Fixing an embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$IH^i(S_{K_{\text{fin}}}, \mathbb{V}_\lambda) \otimes_\iota \mathbb{C} \simeq IH^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

where $\mathbb{V}_\lambda$ is regarded as an ℓ-adic local system on the left and $\mathbb{V}_\lambda$ is regarded as a local system of $\mathbb{C}$-vector spaces on the right.

Writing Gal to mean either $\mathrm{Gal}(\overline{F}/F)$ or $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, we obtain an action of $\mathcal{H}_{K_{\text{fin}}}$ on $IH^i(S_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ which commutes with the action of Gal.

By Gabber's purity theorem (see for example [64, Theorem 10.1, Corollary 10.2]) the eigenvalues of Frobenius acting on $IH^i(S_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ are algebraic integers whose conjugates all have the same absolute value $p^{i/2}$. If the alternating sum of traces of Frobenius is not too complicated (for example if there is no cancellation of individual cohomology groups) then we can use purity along with the Hodge numbers coming from the Vogan-Zuckerman classification to resolve the alternating sum of traces into traces on individual cohomology groups.

2.3.3 Weighted ℓ-adic Cohomology

In general it is useful to speak of the weight filtration on ℓ-adic cohomology. Now there is a subtle problem: while the objects of $\mathrm{Perv}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ admit weight filtrations (defined in terms of Frobenius weights), the objects of $\mathrm{Perv}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{F}}, \overline{\mathbb{Q}}_\ell)$ in general do not. We quickly summarize the strategy due to Morel [89] and Huber [58] which resolves this issue, and then summarize some results of Morel [86] and Nair [92] which give an ℓ-adic interpretation of Eisenstein cohomology in terms of weight truncations.

The first issue that arises is the following: for X an F-scheme and for $\mathcal{E} \in \mathrm{Sh}_{\mathrm{c}}(X, E)$ there need not exist $\mathcal{E}' \in \mathrm{Sh}_{\mathrm{c}}(\mathcal{X}, E)$ extending $\mathcal{E}$ for $\mathcal{X}$ an A-scheme flat and of finite type over a $\mathbb{Z}$-subalgebra $A \subseteq F$ integral and of finite type over $\mathbb{Z}$ with $F = \mathrm{Frac}(A)$.

Example 2.3.3. Let F be a number field and let S be a finite set of primes of F including all primes of F over ℓ. Let $\Gamma_F = \mathrm{Gal}(\overline{F}/F) = \pi_1^{\mathrm{et}}(\mathrm{Spec}(F), \overline{\eta})$ be the absolute Galois group of F and let $\Gamma_{F,S} = \mathrm{Gal}(F_S/F) = \pi_1^{\mathrm{et}}(\mathrm{Spec}(\mathcal{O}_F) - S, \overline{\eta})$ be the Galois group of the maximal extension of F unramified away from S. Let $\mathcal{O}_{F,S}^\times$ be the group of S-units and let $\mathrm{Cl}_{F,S}$ be the S-class group of F. On one hand we have a short exact sequence

$$0 \to \mathcal{O}_{F,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \to H^1_{\mathrm{cont}}(\Gamma_{F,S}, \mu_{\ell^n}) \to \mathrm{Cl}_{F,S}[\ell^n] \to 0$$

and since $\mathcal{O}_{F,S}^\times$ is finitely generated and $\mathrm{Cl}_{F,S}$ is finite by Dirichlet's unit theorem, we have an isomorphism $H^1_{\mathrm{cont}}(\Gamma_{F,S}, \mathbb{Z}_\ell(1)) \simeq \mathcal{O}_{F,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$.

On the other hand by Kummer theory we have an isomorphism $H^1_{\mathrm{cont}}(\Gamma_F, \mathbb{Z}_\ell(1)) \simeq \varprojlim_n F^\times/F^{\times \ell^n}$. Since $F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \varinjlim_S \mathcal{O}_{F,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \subsetneq \varprojlim_n F^\times/F^{\times \ell^n}$ it follows that $H^1_{\mathrm{cont}}(\Gamma_{F,S}, \mathbb{Q}_\ell(1)) \subsetneq H^1_{\mathrm{cont}}(\Gamma_F, \mathbb{Q}_\ell(1))$. In particular, there exists a nontrivial extension $E \in \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_F)}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)) = H^1_{\mathrm{cont}}(\Gamma_F, \mathbb{Q}_\ell(1))$ such that every $E' \in \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_{F,S})}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)) = H^1_{\mathrm{cont}}(\Gamma_{F,S}, \mathbb{Q}_\ell(1))$ does not extend E for every S. In other words, there exists $\mathcal{E} \in \mathrm{Sh}_{\mathrm{c}}(\mathrm{Spec}(F), \mathbb{Q}_\ell)$ such that every $\mathcal{E}' \in \mathrm{Sh}_{\mathrm{c}}(\mathrm{Spec}(\mathcal{O}_{F,S}), \mathbb{Q}_\ell)$ does not extend $\mathcal{E}$ for every S.

To resolve this issue, Morel uses the following notion of horizontal constructible sheaves, defined in work of Huber. Let $\mathcal{U}$ be the partially ordered set of $\mathbb{Z}$-subalgebras $A \subseteq F$ integral and of finite type over $\mathbb{Z}$ with $F = \mathrm{Frac}(A)$. For $A \in \mathcal{U}$ we say that an A-scheme $\mathcal{X}$ is horizontal if $\mathcal{X}$ is flat and of finite type over A. For X an F-scheme let $\mathcal{U}(X)$ be the category whose objects are triples $(A, \mathcal{X}, u)$ where $A \in \mathcal{U}$, where $\mathcal{X}$ is a horizontal

A-scheme, and where $u : X \xrightarrow{\sim} \mathcal{X} \otimes_A F$ is an isomorphism of F-schemes, and whose morphisms are pairs $(\iota, f) : (A, \mathcal{X}, u) \to (A', \mathcal{X}', u')$ where $\iota : A \hookrightarrow A'$ is an inclusion and where $f : \mathcal{X}' \to \mathcal{X} \otimes_A A'$ is an open embedding such that $u' = u \circ f$.

Note that we have a canonical isomorphism $X \xrightarrow{\sim} \varinjlim_{(A,\mathcal{X},u)\in\mathcal{U}(X)} \mathcal{X} \otimes_A F$, and that every morphism $(\iota, f) : (A, \mathcal{X}, u) \to (A', \mathcal{X}', u')$ in $\mathcal{U}(X)$ induces an exact functor $f^* : \mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X}, E) \to \mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X}', E)$. Now Huber and Morel define the following:

Definition 2.3.4. For X an F-scheme let $\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ be the triangulated the category of bounded complexes of sheaves of E-modules on X with horizontal constructible cohomology sheaves, given by the 2-colimit

$$\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E) = \varinjlim_{(A,\mathcal{X},u)\in\mathcal{U}(X)} \mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X}, E)$$

The triangulated category $\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ has a canonical t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with heart $\mathrm{Sh}_{\mathrm{h}}(X, E)$ the Abelian category of horizontal constructible sheaves of E-modules on X. The triangulated category $\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ also has a perverse t-structure $({}^{\mathrm{p}}\mathcal{D}^{\leq 0}, {}^{\mathrm{p}}\mathcal{D}^{\geq 0})$ with heart $\mathrm{Perv}_{\mathrm{h}}(X, E)$ the Abelian category of horizontal perverse sheaves of E-modules on X.

If $A \in \mathcal{U}$, then the residue fields of closed points of $\mathrm{Spec}(A)$ are finite since A is a $\mathbb{Z}$-algebra of finite type. We exploit this to define (punctual) weights as follows. We say that $\mathcal{E} \in \mathrm{Sh}_{\mathrm{h}}(X, E)$ is punctually pure of weight w if there exists $(A, \mathcal{X}, u) \in \mathcal{U}(X)$ and $\mathcal{E}' \in \mathrm{Sh}_{\mathrm{c}}(\mathcal{X}, E)$ extending $\mathcal{E}$ such that for every closed point x of $\mathrm{Spec}(A)$ the restriction $\mathcal{E}'|_{\mathcal{X}_x}$ is punctually pure of weight w. We say that $\mathcal{E} \in \mathrm{Sh}_{\mathrm{h}}(X, E)$ is mixed if there exists an increasing filtration $w_\bullet$ on $\mathcal{E}$ such that the associated graded $\mathrm{Gr}^w_n \mathcal{E} = w_n\mathcal{E}/w_{n-1}\mathcal{E}$ is punctually pure of some weight for every $n \in \mathbb{Z}$.

Definition 2.3.5. For X an F-scheme let $\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ be the full subcategory of $\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ whose objects are bounded complexes of sheaves of E-modules on X with mixed horizontal constructible cohomology sheaves.

The triangulated category $\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ has a canonical t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ inherited from $\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ with heart $\mathrm{Sh}_{\mathrm{m}}(X, E)$ the Abelian category of mixed horizontal constructible sheaves of E-modules on X. The triangulated category $\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ also has a perverse t-structure $({}^{\mathrm{p}}\mathcal{D}^{\leq 0}, {}^{\mathrm{p}}\mathcal{D}^{\geq 0})$ inherited from $\mathcal{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ with heart $\mathrm{Perv}_{\mathrm{m}}(X, E)$ the Abelian category of mixed horizontal perverse sheaves of E-modules on X.

Definition 2.3.6. A weight filtration on $K \in \mathrm{Perv}_{\mathrm{m}}(X, E)$ is a separated exhaustive increasing filtration $W_\bullet$ on K such that the associated graded $\mathrm{Gr}_n^W K = W_n K / W_{n-1} K$ is pure of weight n for every $n \in \mathbb{Z}$.

Since the Abelian category $\mathrm{Perv}_{\mathrm{m}}(X, E)$ is Noetherian and Artinian, such a weight filtration $W_\bullet$ is automatically finite. Since morphisms in $\mathrm{Perv}_{\mathrm{m}}(X, E)$ are strictly compatible with weight filtrations, such a weight filtration $W_\bullet$ is automatically unique.

We now encounter the second issue: for X an F-scheme and for $K \in \mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ there need not exist a weight filtration $W_\bullet$ on K in the above sense: a morphism $K \to K'$ in $\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ where K has weights $\leq w$ and K' has weights $> w$ need not be trivial.

Example 2.3.7. Let F be a number field with r_1 real places and r_2 complex places, and let S be a finite set of primes of F including all primes of F over ℓ. Let $\Gamma_{F,S} = \mathrm{Gal}(F_S/F) = \pi_1^{\mathrm{et}}(\mathrm{Spec}(\mathcal{O}_F) - S, \overline{\eta})$ be the Galois group of the maximal extension of F unramified away from S. Then for every $n \in \mathbb{Z}$ we have $\dim_{\mathbb{Q}_\ell} H^1_{\mathrm{cont}}(\Gamma_{F,S}, \mathbb{Q}_\ell(2n+1)) \geq r_1 + r_2$. In particular for every $n \in \mathbb{Z}$ there exists a nontrivial extension $E \in \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_{F,S})}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(2n+1)) = H^1_{\mathrm{cont}}(\Gamma_{F,S}, \mathbb{Q}_\ell(2n+1))$ corresponding to a mixed ℓ-adic sheaf $\mathcal{E} \in \mathrm{Sh}_{\mathrm{m}}(\mathrm{Spec}(\mathcal{O}_{F,S}), \mathbb{Q}_\ell)$ since $\mathbb{Q}_\ell(0)$ and $\mathbb{Q}_\ell(2n+1)$ are pure of weights 0 and $-2(2n+1)$ respectively. But for $n < 0$ there does not exist a weight filtration $W_\bullet$ on $\mathcal{E}$ since W_0 would yield a splitting of the short exact sequence $0 \to \mathbb{Q}_\ell(2n+1) \to E \to \mathbb{Q}_\ell(0) \to 0$.

To resolve this issue, Morel defines the following:

Definition 2.3.8. For X an F-scheme let $\mathrm{Perv}_{\mathrm{mf}}(X, E)$ be the full subcategory of $\mathrm{Perv}_{\mathrm{m}}(X, E)$ whose objects are mixed horizontal perverse sheaves of E-modules on X admitting a weight filtration.

Morel shows that that the category $\mathrm{Perv}_{\mathrm{mf}}(X, E)$ and its derived category $\mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(X, E)$ admit a four functors formalism: given a morphism of F-schemes $f : X \to Y$ one has functors $\mathbb{R}f_*, \mathbb{R}f_! : \mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(X, E) \to \mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(Y, E)$ and $f^*, f^! : \mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(Y, E) \to \mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(X, E)$. Given an F-scheme X over a number field F with embedding $F \hookrightarrow \mathbb{C}$ we have a forgetful functor $\mathrm{Perv}_{\mathrm{mf}}(X, E) \to \mathrm{Perv}(X(\mathbb{C}), E)$ which factors through $\mathrm{Perv}(X, E)$.

As the objects of $\mathrm{Perv}_{\mathrm{mf}}(X, E)$ carry weight filtrations, one can define certain weight truncation functors and a weighted t-structure. For every object $K \in \mathrm{Perv}_{\mathrm{mf}}(X, E)$ and for every integer $a \in \mathbb{Z}$ we have a canonical short exact sequence

$$0 \to W_{\leq a} K \to K \to W_{\geq a+1} K \to 0$$

where $W_{\leq a} \in \mathrm{Perv}_{\mathrm{mf}}(X, E)$ is the maximal subobject of K with weights $\leq a$ and where $W_{\geq a+1}K \in \mathrm{Perv}_{\mathrm{mf}}(X, E)$ is the maximal quotient of K with weights $\geq a + 1$. Since morphisms in $\mathrm{Perv}_{\mathrm{mf}}(X, E)$ are strictly compatible with weight filtrations we obtain functors $W_{\leq a}, W_{\geq a+1} : \mathrm{Perv}_{\mathrm{mf}}(X, E) \to \mathrm{Perv}_{\mathrm{mf}}(X, E)$ which extend to functors $W_{\leq a}, W_{\geq a+1} : \mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(X, E) \to \mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(X, E)$. The triangulated category $\mathcal{D}^{\mathrm{b}}\mathrm{Perv}_{\mathrm{mf}}(X, E)$ has a weighted t-structure $({}^{W}\mathcal{D}^{\leq a}, {}^{W}\mathcal{D}^{\geq a+1})$ with trivial heart.

We refer to [86] and [89] for the construction and properties of this weighted t-structure in the case of mixed ℓ-adic complexes, and refer to [92, Section 2] for further discussion especially in the case of mixed Hodge modules.

We now apply these weight truncation functors in the setting of Shimura varieties, following [86] and [92].

Proposition 2.3.9. [92, Proposition 2.4.2] Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. Suppose that $\mathbb{V}_\lambda$ is pure of weight w and that $\mathcal{S}_{K_{\mathrm{fin}}}$ has dimension d.

(i) If $a \leq d$ then $\mathbb{H}^i(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, W_{\geq a+w}\mathbb{R}j_!\mathbb{V}_\lambda[d])$ has weights $\leq i + d + w$.

(ii) If $a \geq d$ then $\mathbb{H}^i(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, W_{\leq a+w}\mathbb{R}j_*\mathbb{V}_\lambda[d])$ has weights $\geq i + d + w$.

(iii) We have an isomorphism $\mathbb{H}^i(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, W_{\leq d+w}W_{\geq d+w}\mathbb{R}j_*\mathbb{V}_\lambda) \simeq IH^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$.

Theorem 2.3.10. [86, Section 3] Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. Suppose that $\mathbb{V}_\lambda$ is pure of weight w and that $\mathcal{S}_{K_{\mathrm{fin}}}$ has dimension d. We have a second-quadrant spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, W_{\leq d+w-p}W_{\geq d+w-p}\mathbb{R}j_*\mathbb{V}_\lambda) \Rightarrow H^{p+q}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$$

with $p = 0$ column given by $E_1^{0,q} = IH^q(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \mathbb{H}^q(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, \mathrm{IC}^\bullet(\mathbb{V}_\lambda))$. If λ is a regular highest weight, then the spectral sequence degenerates at the E_1-page.

While the entries $\mathbb{H}^{p+q}(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}, W_{\leq d+w-p}W_{\geq d+w-p}\mathbb{R}j_*\mathbb{V}_\lambda)$ of the above spectral sequence are all pure, the cohomology $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is in general mixed. Nair [92, Theorem 1.1] uses the above spectral sequence to show that the pure subquotients of $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ are all subquotients of the intersection cohomology $IH^j(X, \mathbb{V}) = \mathbb{H}^j(\overline{X}^{\mathrm{BB}}, \mathrm{IC}^\bullet(\mathbb{V}))$ of minimal compactifications $\overline{X}^{\mathrm{BB}}$ of strata X in the minimal compactification $\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}}}$ of $\mathcal{S}_{K_{\mathrm{fin}}}$. In particular, the discrepancy between the cohomology $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ and the intersection cohomology

$IH^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \mathbb{H}^i(\overline{\mathcal{S}}^{\text{BB}}_{K_{\text{fin}}}, \text{IC}^\bullet(\mathbb{V}_\lambda))$ can be understood in terms of the intersection cohomology of boundary components.

Now we have two ways to compute the cohomology of local systems on Shimura varieties: we can compute $H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ in terms of the $(\mathfrak{g}, K'_\infty)$-cohmology $H^i(\mathfrak{g}, K'_\infty; \mathcal{A}_\lambda(G) \otimes V_\lambda)$ using Eisenstein cohomology, and we can compute $H^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ in terms of weight truncations as above. Nair shows that these two strategies are related:

Theorem 2.3.11. [92, Theorem 1.2, Theorem 1.3] Let $V_\lambda \in \text{Rep}(G)$ be an absolutely irreducible rational representation with highest weight λ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$, or the corresponding local system of $\mathbb{C}$-vector spaces on $S_{K_{\text{fin}}}(\mathbb{C})$. Fix an embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. There exists a finite decreasing filtration of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\ldots \subseteq \mathcal{A}^{i+1}_\lambda(G) \subseteq \mathcal{A}^i_\lambda(G) \subseteq \ldots \subseteq \mathcal{A}_\lambda(G)$$

such that the associated second-quadrant spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{A}^p_\lambda(G)/\mathcal{A}^{p+1}_\lambda(G) \otimes V_\lambda) \Rightarrow H^{p+q}(\mathfrak{g}, K'_\infty; \mathcal{A}_\lambda(G) \otimes V_\lambda) \simeq H^{p+q}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

is isomorphic to the spectral sequence

$$E_1^{p,q} = H^{p+q}(\overline{\mathcal{S}}^{\text{BB}}_{K_{\text{fin}}}, W_{\leq d+w-p} W_{\geq d+w-p} \mathbb{R} j_* \mathbb{V}_\lambda) \otimes_\iota \mathbb{C} \Rightarrow H^{p+q}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \otimes_\iota \mathbb{C} \simeq H^{p+q}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$$

with $p = 0$ column given by $E_1^{0,q} = H^q(\mathfrak{g}, K'_\infty; L^2_\lambda(G) \otimes V_\lambda)$. If λ is a regular highest weight, then the spectral sequence degenerates at the E_1-page.

2.3.4 The Langlands-Kottwitz Method

The Arthur-Selberg Trace Formula It should be emphasized that the property of being automorphic is extremely rigid: if one were to arbitrarily choose an irreducible admissible representation π_v of $G(\mathbb{Q}_v)$ for each place v of $\mathbb{Q}$, the corresponding representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_\mathbb{Q})$ is almost never automorphic. In more explicit terms, suppose that G is a connected quasisplit reductive group over $\mathbb{Q}$ so that we have a minimal parabolic $\mathbb{Q}$-subgroup P_0 of G with Levi decomposition $P_0 = M_0 N_0$ where M_0 is a maximal torus which splits over a finite extension

$F/\mathbb{Q}$. If $\pi = \bigotimes_v \pi_v$ is automorphic, then π is unramified outside a finite set S of primes, meaning G_p splits over an unramified extension $F_\wp/\mathbb{Q}_p$ and the trivial 1-dimensional representation of G_p occurs in $\pi_p|_{K_p}$ for some compact open subgroup $K_p \subseteq G(\mathbb{Q}_p)$ for all primes $p \notin S$. By the Satake isomorphism such an unramified representation π_p of $G(\mathbb{Q}_p)$ occurs as the unique irreducible K_p-unramified subquotient $\pi(\chi_p)$ of the normalized parabolic induction $\mathcal{I}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi_p) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\delta_{P_0(\mathbb{Q}_p)}^{1/2}\chi_p)$ of an unramified character $\chi_p : M_0(\mathbb{Q}_p) \to \mathbb{C}^\times$, meaning χ_p is trivial on $M_0(\mathbb{Q}_p) \cap K_p$. Let $c_p(\pi) = c(\pi_p)$ be the corresponding semisimple conjugacy class in the local L-group ${}^L G_p = \widehat{G} \rtimes \mathrm{Gal}(F_\wp/\mathbb{Q}_p)$; the action of $\mathrm{Gal}(F_\wp/\mathbb{Q}_p)$ on the complex dual group $\widehat{G}$ factors through the inertia subgroup $I \subseteq \mathrm{Gal}(F_\wp/\mathbb{Q}_p)$, and the conjugacy class projects to the Frobenius element Frob_p in $\mathrm{Gal}(F_\wp/\mathbb{Q}_p)/I$. Let $c^S(\pi) = \{c_p(\pi) | p \notin S\}$ be the corresponding family of semisimple conjugacy classes in the L-group ${}^L G = \widehat{G} \rtimes \mathrm{Gal}(F/\mathbb{Q})$, the Satake parameters of π. This is a family of diagonal matrices, that is a family of tuples of complex numbers, and it is this data which is extremelt rigid: if one were to arbitrarily choose a family of diagonal matrices $c^S = \{c_p\}_{p \notin S}$, such a family almost never arises as the Satake parameters of an automorphic representation of $G(\mathbb{A}_\mathbb{Q})$. The families of tuples of complex numbers that arise in this way are of great arithmetic significance, for instance they are the numerical source of the traces of Frobenius on the cohomology of Shimura varieties.

The automorphic spectrum $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is an incredibly delicate object, and from the outset it is not clear how one would go about understanding its structure. At present, the only way this can really be done is through the use of the Arthur-Selberg trace formula. To say anything meaningful about this would require enormous digression: it is immensely difficult to write the trace formula in a way that even makes sense, not least because its naïve expression involves divergent terms and operators which are not obviously trace class. In the case of compact quotients one does not encounter so many issues, and it is worth quickly recalling this situation:

Example 2.3.12. Let G be a connected reductive group over $\mathbb{Q}$ which is $\mathbb{Q}$-anisotropic so that $G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})$ has finite volume. Then $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})) = L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q}))$ and we have a Hilbert direct sum decomposition

$$L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\pi \in \Pi(G(\mathbb{A}_\mathbb{Q}))} m(\pi)\pi$$

taken over admissible unitary representations π of $G(\mathbb{A}_\mathbb{Q})$ with finite multiplicites. For $f \in C_c^\infty(G(\mathbb{A}_\mathbb{Q}))$ a test function the operator $R(f)$ on $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}))$ is trace class, and we are interested in the trace

$$\mathrm{tr}(R(f)) = \sum_{\pi \in \Pi(G(\mathbb{A}_\mathbb{Q}))} m(\pi)\mathrm{tr}(\pi(f))$$

which is the spectral side of the trace formula. To relate this to something more geometric, we write

$$\begin{aligned}(R(f)\phi)(x) &= \int_{G(\mathbb{A}_\mathbb{Q})} f(x)\phi(gx)\mathrm{d}x = \int_{G(\mathbb{A}_\mathbb{Q})} f(g^{-1}x)\phi(x)\mathrm{d}x = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} \sum_{\gamma \in G(\mathbb{Q})} f(g^{-1}\gamma x)\phi(\gamma x) \\ &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} \Big(\sum_{\gamma \in G(\mathbb{Q})} f(g^{-1}\gamma x) \Big) \phi(x)\mathrm{d}x\end{aligned}$$

so the operator $R(f)$ is an integral operator defined by the kernel $K_f(g,x) = \sum_{\gamma \in G(\mathbb{Q})} f(g^{-1}\gamma x)$. Now one shows that $K_f(g,x)$ is continuous hence square-integrable, and that $R(f)$ is of trace class. Now we have

$$\begin{aligned}\mathrm{tr}(R(f)) &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} K_f(x,x)\mathrm{d}x = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma x)\mathrm{d}x \\ &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} \sum_{[\gamma] \in [G(\mathbb{Q})]} \sum_{\delta \in G(\mathbb{Q})_\gamma \backslash G(\mathbb{Q})} f(x^{-1}\delta^{-1}\gamma\delta x)\mathrm{d}x = \sum_{[\gamma] \in [G(\mathbb{Q})]} \int_{G(\mathbb{Q})_\gamma \backslash G(\mathbb{A}_\mathbb{Q})} f(x^{-1}\gamma x)\mathrm{d}x \\ &= \sum_{[\gamma] \in [G(\mathbb{Q})]} \mathrm{vol}(G(\mathbb{Q})_\gamma \setminus G(\mathbb{A}_\mathbb{Q})_\gamma) \int_{G(\mathbb{A}_\mathbb{Q})_\gamma \backslash G(\mathbb{A}_\mathbb{Q})} f(x^{-1}\gamma x)\mathrm{d}x\end{aligned}$$

so the trace $\mathrm{tr}(R(f))$ can be written as a sum over conjugacy classes $[\gamma] \in [G(\mathbb{Q})]$ of Tamagawa numbers $\tau(G_\gamma) = \mathrm{vol}(G(\mathbb{Q})_\gamma \setminus G(\mathbb{A}_\mathbb{Q})_\gamma)$ and orbital integrals $\mathrm{O}_\gamma(f) = \int_{G(\mathbb{A}_\mathbb{Q})_\gamma \backslash G(\mathbb{A}_\mathbb{Q})} f(x^{-1}\gamma x)\mathrm{d}x$, which is the geometric side of the trace formula. Equating these two expressions yields the trace formula

$$J_{\mathrm{geom}}(f) = \sum_{[\gamma] \in [G(\mathbb{Q})]} \tau(G_\gamma)\mathrm{O}_\gamma(f) = \sum_{\pi \in \Pi(G(\mathbb{A}_\mathbb{Q}))} m(\pi)\mathrm{tr}(\pi(f)) = J_{\mathrm{spec}}(f)$$

More generally, for G a connected reductive group over $\mathbb{Q}$, the regular elliptic part of Arthur's trace formula provides an equality of distributions on $G(\mathbb{A}_\mathbb{Q})$: for a test function $f \in C_c^\infty(A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ we have an equality

$$J_{\text{geom}}(f) = J_{\text{spec}}(f)$$

where the geometric side $J_{\text{geom}}(f)$ is expressed in terms of orbital integrals, and the spectral side $J_{\text{spec}}(f)$ is expressed in terms the spectral content of the automorphic spectrum $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$.

Still more generally, Arthur's invariant trace formula provides an equality of distributions on $G(\mathbb{A}_\mathbb{Q})$: for a test function $f \in C_c^\infty(A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ we have an equality

$$I_{\text{geom}}(f) = J_{\text{geom}}(f) + \left(\begin{smallmatrix}\text{supplemental}\\ \text{geometric terms}\end{smallmatrix}\right) = J_{\text{spec}}(f) + \left(\begin{smallmatrix}\text{supplemental}\\ \text{spectral terms}\end{smallmatrix}\right) = I_{\text{spec}}(f)$$

Loosely speaking, Arthur's invariant trace formula provides an equality of distributions on $G(\mathbb{A}_\mathbb{Q})$

$$I_{\text{geom}}(f) = \sum_M \frac{\#W_M}{\#W_G} \sum_{[\gamma]\in[M(\mathbb{Q})]} a^M(\gamma) I_M(\gamma, f) = \sum_M \frac{\#W_M}{\#W_G} \int_{\Pi(M(\mathbb{A}_\mathbb{Q}))} a^M(\pi) I_M(\pi, f) \mathrm{d}\pi = I_{\text{spec}}(f)$$

The geometric side is a sum over a finite set of Levi $\mathbb{Q}$-subgroups M of G and a sum over conjugacy classes $[\gamma] \in [M(\mathbb{Q})]$, with $a^M(\gamma)$ related to the Tamagawa number $\tau(M_\gamma) = \mathrm{vol}(M(\mathbb{Q})_\gamma \setminus M(\mathbb{A}_\mathbb{Q})_\gamma)$ and with $I_M(\gamma, f)$ related to the orbital integral $\mathrm{O}_\gamma(f) = \int_{M(\mathbb{A}_\mathbb{Q})_\gamma \setminus M(\mathbb{A}_\mathbb{Q})} f(x^{-1}\gamma x)\mathrm{d}x$. The spectral side is a sum over a finite set of Levi $\mathbb{Q}$-subgroups M of G and an integral over irreducible unitary representations $\pi \in \Pi(M(\mathbb{A}_\mathbb{Q}))$, with $a^M(\pi)$ related to the multiplicity of π in $L^2(M(\mathbb{Q})A_M(\mathbb{R})^\circ \setminus M(\mathbb{A}_\mathbb{Q}))$ and with $I_M(\pi, f)$ related to the trace $\mathrm{tr}(\pi(f)) = \int_{M(\mathbb{A}_\mathbb{Q})} f(x)\pi(x)\mathrm{d}x$. We refer to [7, Section 19, Section 21, Section 23], as well as [3], [4] for further discussion.

In order to write the spectral side of the trace formula in a usable way, one needs to understand global results on the automorphic discrete spectrum and endoscopy for the group G. However these results are themselves proved using the trace formula, the analysis of which begins on the geometric side which is expressed in terms of volumes and orbital integrals, as the spectral side begins life as a black box. During this process, one must consider not just regular elliptic terms, but all supplemental terms on both the geometric and spectral side of the trace formula: one cannot easily separate the regular elliptic terms from the supplemental terms on the spectral side prior to establishing structural results about the automorphic discrete spectrum of G.

Endoscopy and Stabilization A further complication is the stabilization of the trace formula itself. Indeed, let $\mathrm{TF}^G = \mathrm{TF}^G_{\mathrm{reg,ell}} +$ (supplemental terms) be the distribution of Arthur's invariant trace formula. Then the distribution TF^G, although invariant, is not stable: on the geometric side, the orbital integrals behave well under conjugacy but may not behave well under stable conjugacy; on the spectral side, the multiplicities of automorphic representations may behave irregularly. The main obstruction to stability comes from the theory theory of endoscopy.

To explain this issue in more detail, we review some background around local endoscopy, which will be especially useful for later applications. Let F be a local field of characteristic 0. Let G be a quasisplit connected reductive group over F with root datum $\Phi(G) = (X^*(M_0), \Phi, X_*(M_0), \Phi)$ and dual root datum $\Phi^\vee(G) = (X_*(M_0), \Phi^\vee, X^*(M_0), \Phi^\vee)$. An L-datum for G is a tuple $(\widehat{G}, \rho_G, \eta_G)$ where $\widehat{G}$ is a connected reductive group over $\mathbb{C}$ with root datum $\Phi(\widehat{G}) = (X^*(\widehat{M_0}), \Phi^\vee, X_*(\widehat{M_0}), \Phi)$, where $\rho_G : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(\widehat{G})$ is an L-action, and where $\eta_G : \Phi^\vee(G) \xrightarrow{\sim} \Phi(\widehat{G})$ is a $\mathrm{Gal}(\overline{F}/F)$-equivariant bijection of root data. An L-datum for G defines an L-group ${}^LG = \widehat{G} \rtimes W_F$ where W_F acts by $\rho_G : W_F \to \mathrm{Aut}(\widehat{G})$ through the quotient $W_F \to \mathrm{Gal}(\overline{F}/F)$ such that $\rho_G : W_F \to \mathrm{Aut}(\widehat{G})$ is a splitting of the exact sequence $0 \to \widehat{G} \to {}^LG \to W_F \to 0$.

Definition 2.3.13. A standard endoscopic datum for G is a tuple $(H, \mathcal{H}, s, \eta)$ where:

- H is a quasisplit reductive group over F with L-datum $(\widehat{H}, \rho_H, \eta_H)$;
- $\mathcal{H} = \widehat{H} \rtimes W_F$ is a split extension of $\widehat{H}$ by W_F such that the splitting $\rho_{\mathcal{H}} : W_F \to \mathrm{Aut}(\widehat{H})$ of the exact sequence $0 \to \widehat{H} \to \mathcal{H} \to W_F \to 0$ coincides with $\rho_H : W_F \to \mathrm{Aut}(\widehat{H})$;
- $s \in \widehat{G}$ is a semisimple element;
- $\eta : \mathcal{H} \to {}^LG$ is an L-homomorphism such that $\eta_0 = \eta|_{\widehat{H}} : \widehat{H} \xrightarrow{\sim} Z_{\widehat{G}}(s)^\circ$ defines an isomorphism between $\widehat{H}$ and the connected component of the $\widehat{G}$-centralizer of s and $\mathrm{Inn}(s) \circ \eta \simeq a \otimes \eta$ where $a : W_F \to Z(\widehat{G})$ is a trivial 1-cocycle and $(a \otimes \eta)(h) = a(w)\eta(h)$ for every $h \in \mathcal{H}$ with image $w \in W_F$.

The trivial endoscopic datum is the tuple $(G, {}^LG, 1, \mathrm{id})$. We will be particularly interested in the so called elliptic endoscopic data:

Definition 2.3.14. A standard endoscopic datum $(H, \mathcal{H}, s, \eta)$ for G is called elliptic if $\eta(Z(\widehat{H})^{\mathrm{Gal}(\overline{F}/F)})^\circ \subseteq Z(\widehat{G})$, that is if $\xi(\mathcal{H})$ is not contained in a proper parabolic F-subgroup of ${}^LG$.

There is a natural notion of equivalence classes of standard endoscopic data; let $\mathcal{E}(G)$ be the set of equivalence classes of standard elliptic endoscopic data for G. We will often write elliptic endoscopic data in the form (H, s, η_0)

where H and s are as above, and $\eta_0 : \widehat{H} \to \widehat{G}$ is a morphism which is later extended to an L-homomorphism $\eta : {}^L H \to {}^L G$. Indeed the group $\mathcal{H}$ in the above definition plays the role of the group ${}^L H$.

Definition 2.3.15. A semisimple element $\gamma_H \in H(\overline{F})$ is called strongly regular if the $H(\overline{F})$-centralizer $H(\overline{F})_{\gamma_H} = Z_{H(\overline{F})}(\gamma_H)$ of γ_H is a torus.

We have a canonical $\mathrm{Gal}(\overline{F}/F)$-invariant morphism $\mathcal{A}_{H/G}$ from the set of semisimple conjugacy classes of $H(\overline{F})$ and the set of semisimple conjugacy classes of $G(\overline{F})$.

Definition 2.3.16. A semisimple element $\gamma_H \in H(\overline{F})$ is called strongly G-regular if the image of its conjugacy class under $\mathcal{A}_{H/G}$ consists of strongly regular elements in $G(\overline{F})$. Two strongly regular semisimple elements $\gamma_H, \gamma'_H \in H(F)$ are called stably conjugate if they are conjugate in $H(\overline{F})$.

The stable conjugacy class of a strongly regular semisimple element $\gamma_H \in H(F)$ is a disjoint union of finitely many $H(F)$-conjugacy classes. A strongly G-regular semisimple element $\gamma_H \in H(F)$ is called an image of $\gamma_G \in G(F)$ if $\gamma_G \in \mathcal{A}_{H/G}(\mathrm{Inn}(\gamma_H))$, that is if $\gamma_G \in G(F)$ is a strongly regular semisimple element in the image of the $H(\overline{F})$-conjugacy class of γ_H under $\mathcal{A}_{H/G}$.

On the geometric side of the trace formula, the failure of stability involves the irregular behavior of orbital integrals over stable conjugacy classes. In the context of local endoscopy, the main problem is the comparison of orbital integrals on G with stable orbital integrals on elliptic endoscopic groups H of G. For $f^H \in C_c^\infty(H(F))$ and for $\gamma_H \in H(F)$ we consider the orbital integral

$$\mathrm{O}_{\gamma_H}(f^H) = \int_{H(F)_{\gamma_H} \backslash H(F)} f(h^{-1}\delta h)\mathrm{d}h$$

of the test function f^H over the conjugacy class of γ_H. We consider the stable orbital integral

$$\mathrm{SO}_{\gamma_H}(f^H) = \sum_{\gamma'_H \sim \gamma_H} \mathrm{O}_{\gamma'_H}(f^H)$$

where the sum is taken over a set of representatives for the stable conjugacy class of γ_H.

For a strongly G-regular semisimple element $\gamma_H \in H(F)$ and a strongly regular semisimple element $\gamma_G \in G(F)$ we consider the Langlands-Shelstad transfer factor $\Delta(\gamma_H, \gamma_G) \in \mathbb{C}$ which depends only on the stable conjugacy class of γ_H and the conjugacy class of γ_G, nonzero only if γ_H is an image of γ_G. Up to subtle issues of signs and the normalization of the transfer factors $\Delta(\gamma_H, \gamma_G)$, we say that a pair of test functions $(f^G, f^H) \in C_c^\infty(G(F)) \times C_c^\infty(H(F))$ satisfies the matching condition for standard endoscopy if

$$\mathrm{SO}_{\gamma_H}(f^H) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G) \mathrm{O}_{\gamma_G}(f^G)$$

for every strongly G-regular semisimple element $\gamma_H \in H(F)$ where the sum is taken over representatives $\gamma_G \in G(F)$ for the $G(F)$-conjugacy classes of strongly regular semisimple elements.

By the fundamental lemma for standard endoscopy, for every $f^G \in C_c^\infty(G(F))$ there exists $f^H \in C_c^\infty(H(F))$ such that $(f^G, f^H) \in C_c^\infty(G(F)) \times C_c^\infty(H(F))$ satisfies the matching condition for standard endoscopy. We say f^H is a transfer of f^G.

On the spectral side of the trace formula, the failure of stability involves the irregular behavior of multiplicites of certain representations which are endoscopic lifts.

The endoscopic lift of a stably invariant distribution $\Theta^H : C_c^\infty(H(F)) \to \mathbb{C}$ is the stably invariant distribution $\Theta^G : C_c^\infty(G(F)) \to \mathbb{C}$ given by $\Theta^G(f^G) = \Theta^H(f^H)$ where $(f^G, f^H) \in C_c^\infty(G(F)) \times C_c^\infty(H(F))$ satisfies the matching condition for standard endoscopy so that f^H is a transfer of f^G. For an irreducible admissible representation π^H of $H(F)$ with stably invariant distribution $\Theta^H_{\pi^H} : C_c^\infty(H(F)) \to \mathbb{C}$ the endoscopic lift $\Theta^G_{\pi^H} : C_c^\infty(G(F)) \to \mathbb{C}$ can be written $\Theta^G_{\pi^H} = \sum_\pi m(\pi^H, \pi)\Theta^G_\pi$ where the sum is taken over irreducible admissible representations π of $G(F)$ with stably invariant distribution $\Theta^G_\pi : C_c^\infty(G(F)) \to \mathbb{C}$ where the $m(\pi^H, \pi)$ are finite multiplicities with only finitely many nonzero. The endoscopic lift defines a morphism of Grothendieck groups $r : K_0(H(F)) \to K_0(G(F))$ such that for an irreducible admissible representation π^H of $H(F)$ we have $\Theta^G_{\pi^H} = \Theta^G_{r(\pi^H)}$. The irreducible admissible representations π of $G(F)$ with occurring in the endoscopic lift form the local L-packet $\Pi(\pi^H) = \{\pi \in \mathrm{Irr}(G(F)) | m(\pi^H, \pi) \neq 0\}$.

In the global situation, the orbital integrals over members of a stable conjugacy classes are unstable: they are in general not constant over members of the stable conjugacy class. Likewise, the multiplicities of members of a global L-packet $\Pi(\pi^H) = \{\pi = \bigotimes_v \pi_v | \pi_v \in \Pi(\pi_v^H)\}$ are unstable: they are in general not constant over members of the global L-packet.

Definition 2.3.17. For $(H, s, \eta_0) \in \mathcal{E}(G)$ an elliptic endoscopic datum fix an L-homomorphism $\eta : {}^L H \to {}^L G$ extending $\eta_0 : \widehat{H} \to \widehat{G}$. Define the constant

$$\iota(G, H) = \frac{\tau(G)/\tau(H)}{\#\Lambda(H, s, \eta)}$$

where $\Lambda(H, s, \eta) = \mathrm{Aut}(H, s, \eta)/H^{\mathrm{ad}}(\mathbb{Q})$.

By Kottwitz one has the following result on the stabilization of the trace formula:

Theorem 2.3.18. (Kottwitz) Let G be a connected reductive group over $\mathbb{Q}$ and let $f^G \in C_c^\infty(A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ be a test function. Suppose that $f_\infty^G \in C_c^\infty(A_G(\mathbb{R})^\circ \setminus G(\mathbb{R}))$ is stable cuspidal, and suppose that for every elliptic endoscopic datum $(H, s, \eta_0) \in \mathcal{E}(G)$ there exists a transfer $f^H : C_c^\infty(A_H(\mathbb{R})^\circ \setminus H(\mathbb{A}_\mathbb{Q}))$ of $f^G \in C_c^\infty(A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ satisfying the matching condition for standard endoscopy. Then we have an equality

$$\mathrm{TF}^G(f^G) = \sum_{(H,s,\eta_0)\in\mathcal{E}(G)} \iota(G, H)\mathrm{STF}^H(f^H)$$

where STF^H is a stable distribution on $H(\mathbb{A}_\mathbb{Q})$.

In general one proves that STF^H is a stable distribution rather indirectly by proving the stabilization identities for all supplemental terms before they can be deduced for the remaining regular elliptic terms, since one cannot work with these regular elliptic terms directly prior to establishing results about endoscopy for G. The test functions which are relevant to the application of the Langlands-Kottwitz method will turn out to satisfy the condition that $f_\infty^G \in C_c^\infty(A_G(\mathbb{R})^\circ \setminus G(\mathbb{R}))$ is stable cuspidal, so this assumption is not restrictive for our purposes. We refer to [7, Section 27, Section 29] for further discussion of the stable trace formula.

The Langlands-Kottwitz Method We consider the following situation, following [87, Section 1.5] and [70]. Let (G, X, h) be a Shimura datum with reflex field F, and assume that G is not an orthogonal group. Let $g \in G(\mathbb{A}_\mathbb{Q}^\infty)$, and let $K_{\mathrm{fin}}, K'_{\mathrm{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ be compact open subgroups such that $K'_{\mathrm{fin}} \subseteq gK_{\mathrm{fin}}g^{-1}$. Suppose that K_{fin} and K'_{fin} are unramified at p, that is $K_{\mathrm{fin}} = K^p K_p$ where $K^p \subseteq G(\mathbb{A}_\mathbb{Q}^{p,\infty})$ is a compact open subgroup and $K_p = G(\mathbb{Z}_p)$, and similarly for K'_{fin}. Let $\Phi : \mathcal{S}_{K_{\mathrm{fin}},\mathbb{F}_q} \to \mathcal{S}_{K_{\mathrm{fin}},\mathbb{F}_q}$ be the absolute Frobenius. For $j \geq 1$ and for $V_\lambda \in \mathrm{Rep}(G)$

consider the cohomological correspondence

$$u_j : (\Phi^j T_g)^* \mathbb{V}_\lambda \to T_1^! \mathbb{V}_\lambda$$

with support in $(\Phi^j T_g, T_1)$. We would like to compute the trace of this cohomological correspondence in terms which can eventually be compared to the Arthur-Selberg trace formula, that is in terms of twisted orbital integrals at p and orbital integrals away from p and ∞.

Away from p and ∞ we have the following. Recall that the centralizer of γ in $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$ is given by $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})_\gamma = \{x \in G(\mathbb{A}_{\mathbb{Q}}^{p,\infty}) | x\gamma = \gamma x\}$. We say that γ' is $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$-conjugate to γ if there exists $x \in G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$ such that $\gamma' = x^{-1}\gamma x$. Consider the test function

$$f^{p,\infty} = \frac{1}{\mathrm{vol}(K'^p)} 1_{gK^p} \in C_{\mathrm{c}}^\infty(G(\mathbb{A}_{\mathbb{Z}}^{p,\infty}) \setminus G(\mathbb{A}_{\mathbb{Q}}^{p,\infty}) / G(\mathbb{A}_{\mathbb{Z}}^{p,\infty}))$$

and consider the orbital integral

$$\mathrm{O}_\gamma(f^{p,\infty}) = \int_{G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})_\gamma \setminus G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})} f^{p,\infty}(g^{-1}\gamma g) \mathrm{d}g$$

of the test function $f^{p,\infty}$ over the conjugacy class of γ. This is the orbital integral which will appear in the Kottwitz fixed point formula, although later we will want more flexibility with the choice of test function: for example we will want to choose $f^{p,\infty} \in C_{\mathrm{c}}^\infty(G(\mathbb{A}_{\mathbb{Z}}^{p,\infty}) \setminus G(\mathbb{A}_{\mathbb{Q}}^{p,\infty}) / G(\mathbb{A}_{\mathbb{Z}}^{p,\infty}))$ which project onto prescribed packets of automorphic representations in order to isolate the contributions of these packets to intersection cohomology.

At p we have the following. Fix an embedding $F \hookrightarrow \overline{\mathbb{Q}}_p$ which determines a place $\wp$ of F over p. For $j \geq 1$ let L be the unramified extension of degree $j = [L : F_\wp]$ of $F_\wp$ in $\overline{\mathbb{Q}}_p$, let $r = [L : \mathbb{Q}_p]$, let ϖ_L be a uniformizer of L and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ be the element lifting the arithmetic Frobenius $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Let $\delta \in G(L)$ and consider the norm $N : G(L) \to G(\mathbb{Q}_p)$ given by $N\delta = \delta\sigma(\delta)\ldots\sigma^{r-1}(\delta) \in G(L)$. Recall that the σ-centralizer of δ in $G(L)$ is given by $G(L)_\delta^\sigma = \{x \in G(L) | x\delta = \delta\sigma(x)\}$. We say that δ' is σ-conjugate to δ in $G(L)$ if there exists $x \in G(L)$ such that $\delta' = x^{-1}\delta\sigma(x)$.

By definition of the reflex field F the conjugacy class of cocharacters $h_x \circ \mu_0 : \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}}$ for $x \in X$ is defined over F. Choose a cocharacter μ in this conjugacy class which factors through a maximal split torus of G

over $\mathcal{O}_L$ and consider the test function

$$\phi_j = 1_{G(\mathcal{O}_L)\mu(\varpi_L^{-1})G(\mathcal{O}_L)} \in C_{\mathrm{c}}^{\infty}(G(\mathcal{O}_L) \setminus G(L)/G(\mathcal{O}_L))$$

and consider the twisted orbital integral

$$\mathrm{TO}_{\delta}^{\sigma}(\phi_j) = \int_{G(L)_{\delta}^{\sigma} \setminus G(L)} \phi_j(g^{-1}\delta\sigma(g))\mathrm{d}g$$

of the test function ϕ_j over the σ-conjugacy class of δ. This is the twisted orbital integral which will appear in the Kottwitz fixed point formula.

Consider the set of triples $(\gamma_0; \gamma, \delta) \in G(\mathbb{Q}) \times G(\mathbb{A}_{\mathbb{Q}}^{p,\infty}) \times G(L)$ such that γ_0 is semisimple and elliptic in $G(\mathbb{R})$ (that is there exists an elliptic maximal torus T of $G_{\mathbb{R}}$ such that $\gamma_0 \in T(\mathbb{R})$), such that for every place $v \neq p, \infty$ of $\mathbb{Q}$ the local component γ_v of γ at v is $G(\overline{\mathbb{Q}}_v)$-conjugate to γ_0, such that $N\delta$ is $G(\overline{\mathbb{Q}}_p)$-conjugate to γ_0, and such that the image of the σ-conjugacy class of δ under the morphism $B(G_{\mathbb{Q}_p}) \to X^*(Z(\widehat{G})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})$ is the restriction of $-\mu$ to $Z(\widehat{G})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$.

We say that two triples $(\gamma_0; \gamma, \delta)$ and $(\gamma_0'; \gamma', \delta')$ are equivalent if γ_0 is $G(\overline{\mathbb{Q}})$-conjugate to γ_0', if γ is $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$-conjugate to γ', and if δ is σ-conjugate to δ' in $G(L)$. For such a triple $(\gamma_0; \gamma, \delta)$, Kottwitz defines a group $\mathfrak{K}(I_0/\mathbb{Q})$ and an element $\alpha(\gamma_0; \gamma, \delta) \in \mathrm{Hom}(\mathfrak{K}(I_0/\mathbb{Q}), \mathbb{C}^{\times})$ which depends only on the equivalence class of $(\gamma_0; \gamma, \delta)$. For every place $v \neq p, \infty$ of $\mathbb{Q}$ let $I(v)$ be the centralizer of γ_v in $G_{\mathbb{Q}_v}$ (so that $I(v)(\mathbb{Q}_v) = G(\mathbb{Q}_v)_{\gamma}$) which is an inner form of I_0 over $\mathbb{Q}_v$ since γ_v is $G(\overline{\mathbb{Q}}_v)$-conjugate to γ_0. Let $I(p)$ be the σ-centralizer of δ in G_L (so that $I(p)(\mathbb{Q}_p) = G(L)_{\delta}^{\sigma}$) which is an inner form of I_0 over $\mathbb{Q}_p$ since $N\delta$ is $G(\overline{\mathbb{Q}}_p)$-conjugate to γ_0. Kottwitz defines $I(\infty)$ which is an inner form of I_0 over $\mathbb{R}$ such that $A_G(\mathbb{R}) \setminus I(\infty)(\mathbb{R})$ is anisotropic and $\mathfrak{A} \setminus I(\infty)(\mathbb{R})$ is finite. Kottwitz shows that if $\alpha(\gamma_0; \gamma, \delta) = 1$ then there exists an inner form I of I_0 over $\mathbb{Q}$ such that for every place v of $\mathbb{Q}$ we have an isomorphism $I_{\mathbb{Q}_v} \simeq I(v)$.

Let $C_{G,j}$ be the set of equivalence classes of triples $(\gamma_0; \gamma, \delta)$ such that $\alpha(\gamma_0; \gamma, \delta) = 1$. For $(\gamma_0; \gamma, \delta) \in C_{G,j}$ define the constant

$$c(\gamma_0; \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q}) \setminus I(\mathbb{A}_{\mathbb{Q}}^{p,\infty}))|\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, G))|$$

We have chosen Haar measures on $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$, $G(\mathbb{Q}_p)$, and $G(L)$ normalized so that $\mathrm{vol}(K^p) = 1$, $\mathrm{vol}(G(\mathbb{Z}_p)) = 1$, and $\mathrm{vol}(G(\mathcal{O}_L)) = 1$, and consider the Haar measures on $I(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$ and $I(\mathbb{Q}_p)$ normalized so that $\mathrm{vol}(K^p) \in \mathbb{Q}$ and $\mathrm{vol}(K_p) \in \mathbb{Q}$ for every compact open subgroup $K^p \subseteq I(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$ and $K_p \subseteq I(\mathbb{Q}_p)$, and use the inner twistings to transport these measures to $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})_\gamma$ and $G(L)_\delta^\sigma$. Now the theorem of Kottwitz asserts the following:

Proposition 2.3.19. (Kottwitz fixed point formula, [70, 3.1]) For $j \geq 1$ and for $V_\lambda \in \mathrm{Rep}(G)$ let $T(j,g)$ be the sum over the set of fixed points in $\mathcal{S}_{K'_{\mathrm{fin}}}(\overline{\mathbb{F}}_q)$ of the correspondence $(\Phi^j T_g, T_1)$ of the naïve local terms of the cohomological correspondence $u_j : (\Phi^j T_g)^* \mathbb{V}_\lambda \to T_1^! \mathbb{V}_\lambda$ on $H_c^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$. Then we have

$$T(j,g) = \sum_{(\gamma_0;\gamma,\delta) \in C_{G,j}} c(\gamma_0;\gamma,\delta) \mathrm{O}_\gamma(f^{p,\infty}) \mathrm{TO}_\delta^\sigma(\phi_j) \mathrm{tr}(V_\lambda(\gamma_0))$$

For some idea of this equality, note that $\mathrm{vol}(I(\mathbb{Q}) \setminus I(\mathbb{A}_{\mathbb{Q}}^\infty)) \mathrm{O}_\gamma(f^p) \mathrm{TO}_\delta^\sigma(\phi_j)$ is the number of fixed points of the correspondence $f^p \mathrm{Frob}_\wp^j$ isogenous to a given polarized virtual Abelian variety (A, λ, ι), since giving an object $(A', \lambda', \iota', \overline{\eta}')$ with an isogeny to (A, λ, ι) amounts to giving a lattice in $H_1(A_{\overline{\mathbb{F}}_q}, \mathbb{A}_{\mathbb{Q}}^{p,\infty})$ and a lattice in the isocrystal associated to A satisfying certain conditions, and the orbital integrals $\mathrm{O}_\gamma(f^p)$ and twisted orbital integrals $\mathrm{TO}_\delta^\sigma(\phi_j)$ count such lattices.

Now by Deligne's conjecture, which is a theorem of Pink in the present situation [**?**], the fixed points of the correspondence $(\Phi^j T_g, T_1)$ are all isolated fixed points, and the trace of the cohomological correspondence $u_j : (\Phi^j T_g)^* \mathbb{V}_\lambda \to T_1^! \mathbb{V}_\lambda$ on $H_c^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$ is the sum over these fixed points of the naïve local terms. We obtain the following:

Proposition 2.3.20. For $j \gg 0$ sufficiently large and for $V_\lambda \in \mathrm{Rep}(G)$ the trace of the cohomological correspondence $u_j : (\Phi^j T_g)^* \mathbb{V}_\lambda \to T_1^! \mathbb{V}_\lambda$ on $H_c^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$ is given by

$$\mathrm{tr}(u_j | H_c^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)) = T(j,g)$$

If $g = 1$ and $K_{\mathrm{fin}} = K'_{\mathrm{fin}}$ then this is true for all $j \geq 1$.

Since we have not carefully defined everything, it is perhaps helpful to consider a special case of the above result where we can at least define everything more carefully, which hopefully adds some clarity to the situation.

Example 2.3.21. [1, Section 2] For $j \geq 1$ and $q = p^j$ a prime power let $[A, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ be a principally polarized Abelian variety of dimension g over $\mathbb{F}_q$. Consider the isogeny groupoid $\mathcal{I}([A, \lambda], \mathbb{F}_q)$ with underlying set of objects given by the isogeny class $I([A,\lambda],\mathbb{F}_q) = \{[A',\lambda'] \in \mathcal{A}_g(\mathbb{F}_q) | A' \text{ isogenous to } A\}$ and with the same automorphism groups as the groupoid $\mathcal{A}_g(\mathbb{F}_q)$. Consider the groupoid cardinality

$$\#\mathcal{I}([A,\lambda],\mathbb{F}_q) = \sum_{[A',\lambda'] \in [\mathcal{I}([A,\lambda],\mathbb{F}_q)]} \frac{1}{\#\mathrm{Aut}_{\mathbb{F}_q}(A',\lambda')}$$

Let $f_{A/\mathbb{F}_q}(t) \in \mathbb{Z}[t]$ be the characteristic polynomial of Frobenius on A of degree $2g$.

For each prime $\ell \neq p$ the ℓ-adic cohomology $H^1(A_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-module of rank $2g$ with symplectic pairing $\langle\cdot,\cdot\rangle_\lambda$ induced by the polarization. The Frobenius endomorphism $\phi_{A/\mathbb{F}_q}$ induces an element $\gamma_{A/\mathbb{F}_q,\ell} \in \mathrm{GSp}(H^1(A_{\overline{\mathbb{F}}_q},\mathbb{Z}_\ell),\langle\cdot,\cdot\rangle_\lambda) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ defined up to conjugacy. We have an equality of characteristic polynomials $f_{\gamma_{A/\mathbb{F}_q,\ell}}(t) = f_{A/\mathbb{F}_q}(t)$.

For the prime $\ell = p$ the crystalline cohomology $H^1_{\mathrm{cris}}(A,\mathbb{Q}_q)$ is a free $\mathbb{Q}_q$-module of rank $2g$ with symplectic pairing $\langle\cdot,\cdot\rangle_\lambda$ induced by the polarization, with integral structure $H^1_{\mathrm{cris}}(A,\mathbb{Z}_q)$ and σ-linear endomorphism F. The Frobenius endomorphism $\phi_{A/\mathbb{F}_q}$ induces the endomorphism F^j of $H^1_{\mathrm{cris}}(A,\mathbb{Q}_q)$ and induces an element $\delta_{A/\mathbb{F}_q} \in \mathrm{GSp}(H^1_{\mathrm{cris}}(A,\mathbb{Q}_q),\langle\cdot,\cdot\rangle_\lambda) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_q)$ with multiplier $\mathrm{sim}(\delta_{A/\mathbb{F}_q}) = p$ defined up to σ-conjugacy.

Let $G = \mathrm{GSp}_{2g}$ and let $T = T_{[A,\lambda]}$ represent the automorphism group of $[A,\lambda]$ in the $\mathbb{Q}$-isogeny category of Abelian varieties over $\mathbb{F}_q$. Explicitly, for the Rosati involution $a \mapsto a^\dagger$ on $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ we have

$$T_{[A,\lambda]}(R) = \{\alpha \in (\mathrm{End}(A) \otimes_{\mathbb{Z}} R)^\times | \alpha\alpha^\dagger \in R^\times\}$$

By Tate's theorem, for each prime $\ell \neq p$ we have that $T_{[A,\lambda]}(\mathbb{Q}_\ell) = G(\mathbb{Q}_\ell)_{\gamma_{A/\mathbb{F}_q,\ell}}$ is the centralizer of $\gamma_{A/\mathbb{F}_q,\ell}$ in $G(\mathbb{Q}_\ell)$, and for the prime $\ell = p$ we have that $T_{[A,\lambda]}(\mathbb{Q}_q) = G(\mathbb{Q}_p)^\sigma_{\delta_{A/\mathbb{F}_q}}$ is the σ-centralizer of $\delta_{A/\mathbb{F}_q}$ in $G(\mathbb{Q}_q)$.

We can rephrase this in terms of twisted orbital integrals at p and orbital integrals away from p and ∞. Away from p and ∞ consider the test function

$$f^{p,\infty} = 1_{G(\mathbb{A}^{p,\infty}_{\mathbb{Q}})} \in C^\infty_{\mathrm{c}}(G(\mathbb{A}^{p,\infty}_{\mathbb{Z}}) \setminus G(\mathbb{A}^{p,\infty}_{\mathbb{Q}})/G(\mathbb{A}^{p,\infty}_{\mathbb{Z}}))$$

and consider the orbital integral

$$\mathrm{O}_{\gamma_{A/\mathbb{F}_q}}(f^{p,\infty}) = \int_{G(\mathbb{A}^{p,\infty}_{\mathbb{Q}})_{\gamma_{A/\mathbb{F}_q}} \setminus G(\mathbb{A}^{p,\infty}_{\mathbb{Q}})} 1_{G(\mathbb{A}^{p,\infty}_{\mathbb{Z}})}(g^{-1}\gamma_{A/\mathbb{F}_q} g)\mathrm{d}g$$

of the test function $f^{p,\infty}$ over the conjugacy class of $\gamma_{A/\mathbb{F}_q} = (\gamma_{A/\mathbb{F}_q,\ell})_{\ell\neq p}$. At p consider the test function

$$\phi_j = 1_{G(\mathbb{Z}_q)\mathrm{diag}(p,\ldots,p,1,\ldots,1)G(\mathbb{Z}_q)} \in C_{\mathrm{c}}^{\infty}(G(\mathbb{Z}_q)\setminus G(\mathbb{Q}_q)/G(\mathbb{Z}_q))$$

and consider the twisted orbital integral

$$\mathrm{TO}^{\sigma}_{\delta_{A/\mathbb{F}_q}}(\phi_j) = \int_{G(\mathbb{Q}_q)^{\sigma}_{\delta_{A/\mathbb{F}_q}}\setminus G(\mathbb{Q}_q)} 1_{G(\mathbb{Z}_q)\mathrm{diag}(p,\ldots,p,1,\ldots,1)G(\mathbb{Z}_q)}(g^{-1}\delta_{X/\mathbb{F}_q}\sigma(g))\mathrm{d}g$$

of the test function ϕ_j over the σ-conjugacy class of $\delta_{A/\mathbb{F}_q}$. We have chosen Haar measures on $G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$, $G(\mathbb{Q}_p)$, $G(\mathbb{Q}_q)$ normalized so that $\mathrm{vol}(G(\mathbb{A}_{\mathbb{Z}}^{p,\infty})) = 1$, $\mathrm{vol}(G(\mathbb{Z}_p)) = 1$, and $\mathrm{vol}(G(\mathbb{Z}_q)) = 1$. Now the theorem of Kottwitz asserts the following: the groupoid cardinality of $\mathcal{I}([A,\lambda],\mathbb{F}_q)$ is given by

$$\#\mathcal{I}([A,\lambda],\mathbb{F}_q) = \mathrm{vol}(T_{[A,\lambda]}(\mathbb{Q})\setminus T_{[A,\lambda]}(\mathbb{A}_{\mathbb{Q}}^{\infty}))\mathrm{O}_{\gamma_{A/\mathbb{F}_q}}(f^{p,\infty})\mathrm{TO}^{\sigma}_{\delta_{A/\mathbb{F}_q}}(\phi_j)$$

Using Honda-Tate theory we have an element $\gamma_{A/\mathbb{F}_q,0} \in G(\mathbb{Q})$ defined up to $G(\overline{\mathbb{Q}})$-conjugacy such that $\gamma_{A/\mathbb{F}_q,0}$ and $\gamma_{A/\mathbb{F}_q,\ell}$ are conjugate in $G(\overline{\mathbb{Q}}_\ell)$ for each prime $\ell \neq p$ and such that $\gamma_{A/\mathbb{F}_q,0}$ and $N\delta_{A/\mathbb{F}_q}$ are conjugate in $G(\overline{\mathbb{Q}}_q)$ where $N : G(\mathbb{Q}_q) \to G(\mathbb{Q}_q)$ is given by $N(g) = g\sigma(g)\ldots\sigma^{e-1}(g)$. By adjusting $\delta_{A/\mathbb{F}_q}$ in its twisted conjugacy class we may assume that $N\delta_{A/\mathbb{F}_q} \in G(\mathbb{Q}_p) \subseteq G(\mathbb{Q}_q)$. In particular we can replace $\gamma_{A/\mathbb{F}_q}$ by the global object $\gamma_{A/\mathbb{F}_q,0}$. This $\gamma_0 = \gamma_{A/\mathbb{F}_q,0} \in G(\mathbb{Q})$ and these $\gamma = \gamma_{A/\mathbb{F}_q} \in G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$ and $\delta = \delta_{A/\mathbb{F}_q} \in G(\mathbb{Q}_q)$ define triples $(\gamma_0;\gamma,\delta)$ which are precisely those which are indexed in the Kottwitz fixed point formula.

Comparison with the Stable Trace Formula We would now like to compare the Kottwitz fixed point formula to the stabilization of Arthur's trace formula. On one hand the Kottwitz fixed point formula is an expression involving orbital integrals away from p and twisted orbital integrals at p:

$$T(j,g) = \sum_{(\gamma_0;\gamma,\delta)\in C_{G,j}} c(\gamma_0;\gamma,\delta)\mathrm{O}_{\gamma}(f^{p,\infty})\mathrm{TO}^{\sigma}_{\delta}(\phi_j)\mathrm{tr}(V_\lambda(\gamma_0))$$

On the other hand the stabilization of the regular elliptic part of Arthur's trace formula is an expression involving stable orbital integrals

$$\sum_{(H,s,\eta_0)\in\mathcal{E}(G)} \iota(G,H)\mathrm{STF}^H_{\mathrm{reg,ell}}(f^H) = \sum_{(H,s,\eta_0)\in\mathcal{E}(G)} \iota(G,H)\tau(H)\sum_{\gamma_H}\mathrm{SO}_{\gamma_H}(f^H)$$

where the test functions f^H are transfers of a test function f^G such that f^G_∞ is stable cuspidal.

In order to make this comparison, one needs two versions of the fundamental lemma, following [87, Section 5.3]. Let G be a connected reductive group over $\mathbb{Q}$. For $(H,s,\eta_0)\in\mathcal{E}(G)$ an elliptic endoscopic datum fix an L-homomorphism $\eta : {}^LH \to {}^LG$ extending $\eta_0 : \widehat{H} \to \widehat{G}$.

The fundamental lemma states that for every finite place p of $\mathbb{Q}$ where G and H are unramified, if η is unramified at p and if $b : C^\infty_{\mathrm{c}}(G(\mathbb{Z}_p)\setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p)) \to C^\infty_{\mathrm{c}}(H(\mathbb{Z}_p)\setminus H(\mathbb{Q}_p)/H(\mathbb{Z}_p))$ is the morphism of local Hecke algebras induced by η, then for every $f^G_p \in C^\infty_{\mathrm{c}}(G(\mathbb{Z}_p)\setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p))$ and for every semisimple (G,H)-regular $\gamma_H \in H(\mathbb{Q}_p)$ we have

$$\mathrm{SO}_{\gamma_H}(b(f^G_p)) = \sum_\gamma \Delta_p(\gamma_H,\gamma)e(G_\gamma)\mathrm{O}_\gamma(f^G_p)$$

where the sum is taken over the set of conjugacy classes γ in $G(\mathbb{Q}_v)$ which are images of γ_H (if γ_H has no image in $G(\mathbb{Q}_v)$ we have $\mathrm{SO}_{\gamma_H}(b(f^G_p)) = 0$), where G_γ is the centralizer of γ in G, and where $e(G_\gamma)$ is a certain sign defined by Kottwitz. We say that $f^H_p = b(f^G_p) \in C^\infty_{\mathrm{c}}(H(\mathbb{Z}_p)\setminus H(\mathbb{Q}_p)/H(\mathbb{Z}_p))$ is a transfer of f^G_p. The fundamental lemma can also be stated in the same way for the Archimedean place of $\mathbb{Q}$, where it is a theorem of Shelstad.

The twisted fundamental lemma states that for every finite place p of $\mathbb{Q}$ where G and H are unramified, if η is unramified at p and if $b : C^\infty_{\mathrm{c}}(G(\mathcal{O}_L)\setminus G(L)/G(\mathcal{O}_L)) \to C^\infty_{\mathrm{c}}(H(\mathbb{Z}_p)\setminus H(\mathbb{Q}_p)/H(\mathbb{Z}_p))$ is the morphism of local Hecke algebras induced by η where L is the unramified extension of degree $j \geq 1$ of $E_\wp$ in $\overline{\mathbb{Q}}_p$, then for every $\phi^G_j \in C^\infty_{\mathrm{c}}(G(\mathcal{O}_L)\setminus G(L)/G(\mathcal{O}_L))$ and for every semisimple (G,H)-regular $\gamma_H \in H(\mathbb{Q}_v)$ we have

$$\mathrm{SO}_{\gamma_H}(b(\phi^G_j)) = \sum_\delta \langle \alpha_p(\gamma_0;\delta), s\rangle \Delta_p(\gamma_H,\gamma_0)e(G^\sigma_\delta)\mathrm{TO}^\sigma_\delta(\phi_j)$$

where the sum is taken over the set of σ-conjugacy classes δ in $G(L)$ such that $N\delta$ is $G(\overline{\mathbb{Q}}_p)$-conjugate to an image $\gamma_0 \in G(\mathbb{Q}_p)$ of γ_H, and where G^σ_δ is the σ-centralizer of δ in $\mathrm{Res}_{L/\mathbb{Q}_p} G_L$, and where $\alpha_p(\gamma_0; \delta)$ is defined by Kottwitz. We say that $\phi^H_j = b(\phi^G_j) \in C^\infty_{\mathrm{c}}(H(\mathbb{Z}_p) \setminus H(\mathbb{Q}_p)/H(\mathbb{Z}_p))$ is a transfer of ϕ^G_j.

Proposition 2.3.22. (Kottwitz stabilization, [70, 7.2]) For $j \geq 1$ and for $V_\lambda \in \mathrm{Rep}(G)$ let $T(j, g)$ be the sum over the set of fixed points in $\mathcal{S}_{K'_{\mathrm{fin}}}(\overline{\mathbb{F}}_q)$ of the correspondence $(\Phi^j T_g, T_1)$ of the naïve local terms of the cohomological correspondence $u_j : (\Phi^j T_g)^* \mathbb{V}_\lambda \to T_1^! \mathbb{V}_\lambda$ on $H^*_{\mathrm{c}}(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$. Then we have

$$T(j, g) = \sum_{(H, s, \eta_0) \in \mathcal{E}(G)} \iota(G, H) \mathrm{STF}^H_{\text{reg,ell}}(f^H)$$

where the test functions $f^H = f^{p,\infty}_H \phi^H_j f^H_\infty$ (depending on j) are given as follows:

- Away from p and ∞, let $f^{p,\infty}_H = b^H(f^{p,\infty}_G) \in C^\infty_{\mathrm{c}}(H(\mathbb{A}^{p,\infty}_{\mathbb{Q}}))$ be a transfer of $f^{p,\infty} = f^{p,\infty}_G \in C^\infty_{\mathrm{c}}(G(\mathbb{A}^{p,\infty}_{\mathbb{Q}}))$;
- At p, let $\phi^H_j = b^H_j(\phi^G_j) \in C^\infty_{\mathrm{c}}(H(\mathbb{Q}_p))$ be a twisted transfer of $\phi_j = \phi^G_j \in C^\infty_{\mathrm{c}}(G(L))$;
- At ∞, we define the test function

  $$f^H_\infty = (-1)^d \langle \mu_0, s \rangle \sum_{\varphi_H \in \Phi_H(\varphi)} \det(\omega_*(\varphi_H)) f_{\varphi_H}$$

  where $d = \dim(S_{K_{\mathrm{fin}}})$ is the dimension of the Shimura variety, where $\mu_0 : \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}}$ is the minescule cocharacter determined by the Shimura datum, where $\Phi_H(\varphi)$ is the set of elliptic L-parameters $\varphi_H : W_{\mathbb{R}} \to {}^L H$ such that $\eta \circ \varphi_H : W_{\mathbb{R}} \to {}^L G$ is equivalent to the elliptic L-parameter $\varphi : W_{\mathbb{R}} \to {}^L G$ defined by V_λ, where Ω_* is a certain subset of the Weyl group of G yielding a bijection $\Phi_H(\varphi) \xrightarrow{\sim} \Omega_*$ given by $\varphi_H \mapsto \omega_*(\varphi_H)$, and where for $\varphi_H \in \Phi_H(\varphi)$ we define the average of pseudo-coefficients of representations in the L-packet $\Pi(\varphi_H)$:

  $$f_{\varphi_H} = \frac{1}{\#\Pi(\varphi_H)} \sum_{\pi \in \Pi(\varphi_H)} f_\pi$$

Ultimately, one has the following stabilization of the Kottwitz fixed point formula, relating it to the regular elliptic part of the stable trace formula:

Theorem 2.3.23. For $j \gg 0$ sufficiently large and for $V_\lambda \in \mathrm{Rep}(G)$ the trace of the cohomological correspondence $u_j : (\Phi^j T_g)^* \mathbb{V}_\lambda \to T_1^! \mathbb{V}_\lambda$ on $H_c^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$ is given by

$$\mathrm{tr}(u_j | H_c^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)) = \sum_{(H,s,\eta_0) \in \mathcal{E}(G)} \iota(G, H) \mathrm{STF}^H_{\mathrm{reg,ell}}(f^H)$$

If $g = 1$ and $K_{\mathrm{fin}} = K'_{\mathrm{fin}}$ then this is true for all $j \geq 1$.

A more complicated version of the Kottwitz fixed point formula and its stabilization exists for intersection cohomology, following [87, Section 6.3]. The additional difficulty lies in the fact that in considering the naïve local terms of the cohomological correspondence $\overline{u}_j : (\Phi^j T_g)^* \mathrm{IC}^\bullet(\mathbb{V}_\lambda) \to T_1^! \mathrm{IC}^\bullet(\mathbb{V}_\lambda)$ on $IH^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$ one must consider not only the the fixed points in $\mathcal{S}_{K'_{\mathrm{fin}}}(\overline{\mathbb{F}}_q)$ (where one uses the Kottwitz fixed point formula and its stabilization, corresponding to the regular elliptic terms of the stable trace formula), but also the fixed points in the boundary strata of $\overline{\mathcal{S}}^{\mathrm{BB}}_{K'_{\mathrm{fin}}}(\overline{\mathbb{F}}_q)$ (where one essentially combines the Kottwitz fixed point formula for Levi quotients with the topological trace formula of Goresky-Kottwitz-MacPherson by parabolic induction, corresponding to the supplementary terms of the stable trace formula; this is explained in [87, Section 1.7]):

Theorem 2.3.24. For $j \gg 0$ sufficiently large and for $V_\lambda \in \mathrm{Rep}(G)$ the trace of the cohomological correspondence $\overline{u}_j : (\Phi^j \overline{T}_g)^* \mathrm{IC}^\bullet(\mathbb{V}_\lambda) \to \overline{T}_1^! \mathrm{IC}^\bullet(\mathbb{V}_\lambda)$ on $IH^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)$ is given by

$$\mathrm{tr}(\overline{u}_j | IH^*(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)) = \sum_{(H,s,\eta_0) \in \mathcal{E}(G)} \iota(G, H) \mathrm{STF}^H(f^H)$$

If $g = 1$ and $K_{\mathrm{fin}} = K'_{\mathrm{fin}}$ then this is true for all $j \geq 1$.

The upshot of the above discussion is the following. For $V_\lambda \in \mathrm{Rep}(G)$ define elements $H^*_{c,\lambda}$ and IH^*_λ of the Grothendieck group of ℓ-adic representations of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}(\overline{F}/F)$ by the alternating sums

$$H^*_{c,\lambda} = \sum_{i \geq 0} (-1)^i [H^i_c(\mathcal{S}_{K_{\mathrm{fin}}, \overline{F}}, \mathbb{V}_\lambda)]$$
$$IH^*_\lambda = \sum_{i \geq 0} (-1)^i [IH^i(\mathcal{S}_{K_{\mathrm{fin}}, \overline{F}}, \mathbb{V}_\lambda)]$$

Then for all $j \geq 1$ we have

$$\begin{aligned}
\operatorname{tr}(u_j | H^*_{\mathrm{c}}(\mathcal{S}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)) &= \operatorname{tr}(\mathrm{Frob}^j_\wp f^\infty | H^*_{\mathrm{c},\lambda}) \\
\operatorname{tr}(\overline{u}_j | IH^*(\mathcal{S}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)) &= \operatorname{tr}(\mathrm{Frob}^j_\wp f^\infty | IH^*_\lambda)
\end{aligned}$$

where $\mathrm{Frob}_\wp \in \mathrm{Gal}(\overline{F}/F)$ is a lift of the geometric Frobenius at $\wp$ a prime of F over p, and where $f^\infty = \frac{1}{\mathrm{vol}(K_{\mathrm{fin}})} 1_{K_{\mathrm{fin}} g K_{\mathrm{fin}}}$. In particular for $j \gg 0$ sufficiently large and for every $f^\infty \in \mathcal{H}_{K_{\mathrm{fin}}}$ with factorization $f^\infty = f^{p,\infty} 1_{G(\mathbb{Z}_p)}$ we have

$$\begin{aligned}
\operatorname{tr}(\mathrm{Frob}^j_\wp f^\infty | H^*_{\mathrm{c},\lambda}) &= \sum_{(H,s,\eta_0) \in \mathcal{E}(G)} \iota(G,H) \mathrm{STF}^H_{\mathrm{reg,ell}}(f^H) \\
\operatorname{tr}(\mathrm{Frob}^j_\wp f^\infty | IH^*_\lambda) &= \sum_{(H,s,\eta_0) \in \mathcal{E}(G)} \iota(G,H) \mathrm{STF}^H(f^H)
\end{aligned}$$

In many ways the intersection cohomology is simpler than the compactly supported cohomology: although both can be written in terms of each other modulo boundary contributions, the decomposition of intersection cohomology according to Arthur's conjectures is much simpler than the decomposition of compactly supported cohomology according to the Franke-Schwermer decomposition of larger spaces of automorphic forms. On the other hand, the trace of Frobneius on compactly supported cohomology will ultimately be expressed in terms of the regular elliptic part of the stable trace formula, whereas the expression for the trace of Frobenius on intersection cohomology will also involve supplemental terms of the stable trace formula.

Finally, it should be emphasized that the results of Kottwitz have since been greatly generalized. First, the original results of Kottwitz apply only to certain Shimura varieties of PEL type, and excludes certain kinds of reductive groups (they apply to certain groups of types A and C, but not to certain orthogonal groups of type D). After all, the main content involves expressing the relevant point counts over finite fields in group theoretic terms, and for Shimura varieties with moduli interpretations in terms of Abelian varieties with additional structure as for PEL Shimura varieties, this can be done using Honda-Tate theory. More recently, work of Kisin has extended these results to the case of Abelian type Shimura vareities. Second, the original results of Kottwitz apply only to those primes where the Shimura variety has good reduction, which amounts to the assumption that K_{fin} and K'_{fin} are unramified at p. Of course, one also wants to understand what happens at the primes where the Shimura variety has bad reduction, for example one would like to compute the complete Hasse-Weil zeta function of the Shimura variety rather than just the partial Hasse-Weil zeta function. To do this, one is required to construct more general test functions which compute the relevant point counts over finite fields in the case of bad reduction. Work of Haines-Richarz [54] achieves exactly this in the case of parahoric level structure, which is particularly relevant to the examples considered later in this thesis. We hope to explore this in future writing.

2.3.5 Arthur's Conjectures

Informed by the phenomenon of endoscopy, Arthur has formulated conjectures which give a description of the decomposition of the automorphic discrete spectrum $L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ along with precise multiplicity formulas. We quickly review some of the content of these conjectures in the case when G is a connected split reductive group over $\mathbb{Q}$ for simplicity of discussion, following [39, Section 4].

It is helpful to state Arthur's conjectures with the help of the conjectural global L-group:

Conjecture 2.3.25. There exists a topological group $L_\mathbb{Q}$, the global L-group of $\mathbb{Q}$, satisfying the following properties:

(i) The connected component $L^0_\mathbb{Q}$ of the identity in $L_\mathbb{Q}$ is a compact topological group, and the component group $\pi_0(L_\mathbb{Q}) = L_\mathbb{Q}/L^0_\mathbb{Q}$ of $L_\mathbb{Q}$ is isomorphic to the global Weil group $W_\mathbb{Q}$ yielding an exact sequence of topological groups

$$0 \to L^0_\mathbb{Q} \to L_\mathbb{Q} \to W_\mathbb{Q} \to 0$$

(ii) For every place v of $\mathbb{Q}$ we have a natural conjugacy class of embeddings $L_{\mathbb{Q}_v} \hookrightarrow L_\mathbb{Q}$ where $L_{\mathbb{Q}_v}$ is the local L-group of $\mathbb{Q}_v$ given by

$$L_{\mathbb{Q}_v} = \begin{cases} W_\mathbb{R} & v = \infty \\ WD_{\mathbb{Q}_p} & v = p \end{cases}$$

where $W_{\mathbb{Q}_v}$ is the local Weil group of $\mathbb{Q}_v$ and where $WD_{\mathbb{Q}_v} = W_{\mathbb{Q}_v} \times \mathrm{SU}_2(\mathbb{C})$ is the local Weil-Deligne group of $W_{\mathbb{Q}_v}$.

(iii) There exists a natural bijection (the global Langlands correspondence)

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{irreducible representations}\\ \phi : L_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{cuspidal automorphic}\\ \text{representations } \pi \text{ of } \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})\end{array}\right\}$$

which, for every place v of $\mathbb{Q}$, is compatible with the natural bijection (the local Langlands correspondence)

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{irreducible representations}\\ \phi : L_{\mathbb{Q}_v} \to \mathrm{GL}_n(\mathbb{C})\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{equivalence classes of}\\ \text{irreducible admissible}\\ \text{representations } \pi_v \text{ of } \mathrm{GL}_n(\mathbb{Q}_v)\end{array}\right\}$$

The local Langlands correspondence in the above should be taken to be the local Langlands correspondence constructed by Langlands for $v = \infty$ and by Harris-Taylor for $v = p$. The global Langlands correspondence in the above, like the local Langlands correspondence it should be compatible with, should be characterized by various standard compatibilities between L-factors and ϵ-factors, compatibility with class field theory, and so on. More generally, one should have the following:

Conjecture 2.3.26. Let G be a connected split reductive group over $\mathbb{Q}$. There exists a natural morphism

$$\left\{\begin{array}{c}\widehat{G} - \text{conjugacy classes of}\\ \text{global L-parameters } \phi : L_{\mathbb{Q}} \to \widehat{G}\end{array}\right\} \to \left\{\begin{array}{c}\text{equivalence classes of}\\ \text{irreducible automorphic}\\ \text{representations } \pi \text{ of } G(\mathbb{A}_{\mathbb{Q}})\end{array}\right\}$$

which, for every place v of $\mathbb{Q}$, is compatible with the natural morphism (the local Langlands correspondence)

$$\left\{\begin{array}{c}\widehat{G} - \text{conjugacy classes of}\\ \text{local L-parameters } \phi : L_{\mathbb{Q}_v} \to \widehat{G}\end{array}\right\} \to \left\{\begin{array}{c}\text{equivalence classes of}\\ \text{irreducible admissible}\\ \text{representations } \pi_v \text{ of } G(\mathbb{Q}_v)\end{array}\right\}$$

The local Langlands correspondence in the above should be taken to be the local Langlands correspondence constructed by Langlands for $v = \infty$ and more recently by Fargues-Scholze for $v = p$. Again the global Langlands correspondence in the above, like the local Langlands correspondence it should be compatible with, should be characterized by various standard compatibilities between L-factors and ϵ-factors.

Now let G be a connected split reductive group over $\mathbb{Q}$. Arthur's conjectures describe a decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_\omega \bigoplus_{\psi\in\Psi_{\mathrm{disc}}(G,\omega)} \mathcal{A}_\psi$$

where each $\mathcal{A}_\psi$ is a near-equivalence class of discrete spectrum automorphic representations of $G(\mathbb{A}_\mathbb{Q})$, where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and the second direct sum is taken over a set $\Psi_{\mathrm{disc}}(G,\omega)$ of equivalence classes of global A-parameters $\psi : L_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ which are admissible (so that $\psi(L_\mathbb{Q})$ is bounded in $\widehat{G}$) and discrete (so that the centralizer group $S_\psi = Z_{\widehat{G}}(\psi)/Z(\widehat{G})$ is finite); these are formal unorderd isobaric sums $\psi = \boxplus_i \mu_i \boxtimes \nu_{d_i}$ where π_i is an ω-self dual unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_\mathbb{Q})$ and ν_{d_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d_i. The usage of the conjectural global L-group $L_\mathbb{Q}$ can be avoided as follows: for each $\psi \in \Psi_{\mathrm{disc}}(G,\omega)$ one can define a complex reductive group $\mathcal{L}_\psi$ and an A-parameter $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$, so that all mentions of the global L-group $L_\mathbb{Q}$ and its irreducible representations are replaced by the groups $\mathcal{L}_\psi$ and cuspidal automorphic representations.

For each place v of $\mathbb{Q}$ the global A-parameter $\psi : L_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ gives rise to a local A-parameter $\psi_v : L_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ with component group $S_{\psi_v} = \pi_0(Z_{\widehat{G}}(\psi_v)/Z(\widehat{G}))$. For each character $\eta_v \in S_{\psi_v}^\vee$ we have a unitarizable finite length representation π_{η_v} of $G(\mathbb{Q}_v)$ which defines the local L-packet

$$\Pi(\psi_v) = \{\pi_{\eta_v} | \eta_v \in S_{\psi_v}^\vee\}$$

of admissible representations of $G(\mathbb{Q}_v)$. For almost all places v of $\mathbb{Q}$, the representation π_v^+ corresponding to the trivial character of S_{ψ_v} is the irreducible unramified representation with Satake parameter $c(\psi_v) = \psi_v(\mathrm{Frob}_v \times \mathrm{diag}(q_v^{1/2}, q_v^{-1/2}))$. The local L-packets define the global A-packet

$$\Pi(\psi) = \{\pi = \bigotimes_v \pi_v | \pi_v \in \Pi(\psi_v), \pi_v = \pi_v^+ \text{ for almost all places } v \text{ of } \mathbb{Q}\}$$

of near-equivalent representations of $G(\mathbb{A}_\mathbb{Q})$ indexed by the characters of the compact group $\mathcal{S}_\psi = \prod_v S_{\psi_v}$. For $\eta = \bigotimes_v \eta_v \in \mathcal{S}_\psi^\vee$ we define $\pi_\eta = \bigotimes_v \pi_{\eta_v}$, then since $\eta_v = 1_{S_{\psi_v}}$ for almost all places v of $\mathbb{Q}$ we have $\pi_\eta \in \Pi(\psi)$.

The multiplicity of π_η involves a quadratic character $\epsilon_\psi \in S_\psi^\vee$ attached to ψ in the following way. Consider the adjoint action of $S_\psi \times L_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C})$ on $\widehat{\mathfrak{g}}$ defined by ψ and decompose this into a direct sum of irreducible representations of the form $\eta_i \otimes \rho_i \otimes \nu_{d_i}$. Note $\bigoplus_i \eta_i \otimes \rho_i \otimes \nu_{d_i}$ is an orthogonal representation since the adjoint representation admits a nondegenerate invariant symmetric bilinear form. We consider the set $\mathcal{T}_\psi$ of those direct summands $\eta_i \otimes \rho_i \otimes \nu_{n_i}$ which are orthogonal, where d_i is even so that ν_{d_i} is symplectic, and where ρ_i is symplectic with $\epsilon(\frac{1}{2}, \rho_i) = -1$. These conditions imply η_i is orthogonal so that $\det(\eta_i) \in S_\psi^\vee$ is a quadratic character. Define the quadratic character

$$\epsilon_\psi = \prod_{\eta_i \otimes \rho_i \otimes \nu_{d_i} \in \mathcal{T}_\psi} \det(\eta_i)$$

Now for $\eta \in \mathcal{S}_\psi^\vee$ a character consider the canonical morphism $S_\psi \to \mathcal{S}_\psi$ and the corresponding character $\eta \in S_\psi^\vee$. Define the multiplicity

$$m_\eta = \frac{1}{\# S_\psi} \sum_{s \in S_\psi} \epsilon_\psi(s) \eta(s)$$

Then Arthur conjectures that the near-equivalence class $\mathcal{A}_\psi$ is given

$$\mathcal{A}_\psi \simeq \bigoplus_\eta m_\eta \pi_\eta$$

We can write this in the following way: for $\psi \in \Psi_{\mathrm{disc}}(G, \omega)$ we have a morphism $\Pi(\psi) \to \mathcal{S}_\psi^\vee$ sending $\pi \in \Pi(\psi)$ to a character $\langle \cdot, \pi \rangle \in \mathcal{S}_\psi^\vee$; consider the canonical morphism $S_\psi \to \mathcal{S}_\psi$ and the corresponding character $\langle \cdot, \pi \rangle \in S_\psi^\vee$. Writing $m_{\mathrm{disc}}(\pi) = m_\eta$ for the multiplicity of $\pi = \pi_\eta \in \Pi(\psi)$ corresponding to a character $\eta \in \mathcal{S}_\psi^\vee$, we obtain a decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q}) A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_\omega \bigoplus_{\psi \in \Psi_{\mathrm{disc}}(G,\omega)} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = \epsilon_\psi}} m_{\mathrm{disc}}(\pi) \pi$$

As a consequence we note the following decompositions of intersection and cuspidal cohomology. Let (G, X, h) be a Shimura datum and let $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ be a compact open subgroup. Then we have an isomorphism of

$\mathcal{H}_{K_{\text{fin}}}$-modules

$$H^i_{\text{disc}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) = \bigoplus_{\omega} \bigoplus_{\psi \in \Psi_{\text{disc}}(G(\mathbb{A}_\mathbb{Q}), \omega)} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = \epsilon_\psi}} m_{\text{disc}}(\pi) \pi_{\text{fin}}^{K_{\text{fin}}} \otimes H^i(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

In particular, the problem of computing L^2-cohomology is reduced to the problem of understanding the structure of $L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ as described by Arthur's conjectures, and the problem of determining which unitary $(\mathfrak{g}, K'_\infty)$-modules $A_\mathfrak{q}(\lambda)$ are the Archimedean components of which automorphic representations of $G(\mathbb{A}_\mathbb{Q})$.

In the ℓ-adic setting, we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$-modules

$$IH^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \bigoplus_{\omega} \bigoplus_{\psi \in \Psi_{\text{disc}}(G(\mathbb{A}_\mathbb{Q}), \omega)} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = \epsilon_\psi}} m_{\text{disc}}(\pi) \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$$

and it is precisely the Langlands-Kottwitz method which allows us to determine the ℓ-adic Galois representations ρ_π appearing in this decomposition.

Chapter 3

Cohomology of Modular Curves

3.1 Classical and Adelic Modular Curves

Shimura Datum Let $G = \mathrm{GL}_2$ be the general linear group over $\mathbb{Q}$. We have the determinant character $\det : G \to \mathbb{G}_\mathrm{m}$ whose kernel $G^1 = \mathrm{SL}_2$ is the special linear group, the derived group of G.

Consider the maximal torus T of G given by

$$\begin{aligned} T &= \{\mathrm{diag}(t_1, t_2) | t_1, t_2 \in \mathrm{GL}_1\} \\ &= \{\mathrm{diag}(t_1, t/t_1) | t_1, t \in \mathrm{GL}_1\} \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \end{aligned}$$

Since G is already $\mathbb{Q}$-split, $T \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ is a $\mathbb{Q}$-split maximal torus and $A_G = Z(G) \simeq \mathrm{GL}_1$ is a $\mathbb{Q}$-split maximal torus in the center of G. In particular, $A_G(\mathbb{R})^\circ \simeq \mathbb{R}_{>0}$. In this case the Langlands dual group is simply $\widehat{G} = \mathrm{GL}_2(\mathbb{C})$.

We now consider the Hermitian locally symmetric space associated to $G = \mathrm{GL}_2$, and the associated Shimura datum. Consider the element

$$I_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$$

The centralizer K_∞ of I_0 in $\mathrm{SL}_2(\mathbb{R})$ is a connected component of a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$, and is isomorphic to $\mathrm{SO}(2) \simeq \mathrm{U}(1)$. The centralizer K'_∞ of I_0 in $\mathrm{GL}_2(\mathbb{R})$ is connected but not compact, and is isomorphic to $\mathrm{SO}(2)\mathbb{R}_{>0} \simeq \mathrm{U}(1)\mathbb{R}_{>0}$. The corresponding symmetric space $X = X^+ \amalg X^- = G(\mathbb{R})/K'_\infty =$

$A_G(\mathbb{R})^\circ \setminus G(\mathbb{R})/K_\infty$ is a Hermitian symmetric domain and is identified with the space of 1-dimensional linear subspaces of $\mathbb{R}^2$ on which the skew-symmetric bilinear form $(v_1, v_2) \mapsto v_1^\top \mathrm{adiag}(i, i) v_2$ is positive or negative definite, which is identified with the double half space

$$\mathfrak{H}^\pm = \mathfrak{H} \amalg \overline{\mathfrak{H}} = \{\tau \in \mathbb{C} | \Im(\tau) \neq 0\}$$

Let $x_0 \in X$ be the subspace generated by the standard basis vector $e_2 \in \mathbb{R}^2$. Then $G(\mathbb{R})$ acts transitively on X. Consider the $G(\mathbb{R})$-equivariant morphism $h : X \to \mathrm{Hom}_\mathbb{R}(\mathbb{S}, G_\mathbb{R})$ determined by

$$h_0 = h(x_0) = (z \mapsto \mathrm{diag}(z, 1))$$

Then (G, X, h) is a Shimura datum in the sense of Deligne and Pink.

Shimura Varieties and Connected Components Let $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ be a compact open subgroup. Let $S_{K_{\mathrm{fin}}}$ be the corresponding Shimura variety which is a smooth quasiprojective variety over $\mathbb{Q}$ if K_{fin} is neat, and considered as a stack otherwise, with complex points given by

$$S_{K_{\mathrm{fin}}}(\mathbb{C}) = G(\mathbb{Q}) \setminus \mathfrak{H}^\pm \times G(\mathbb{A}_\mathbb{Q}^\infty)/K_{\mathrm{fin}}$$

The Shimura variety $S_{K_{\mathrm{fin}}}$ is connected but in general not geometrically connected: the set of connected components of $S_{K_{\mathrm{fin}}}(\mathbb{C})$ is given by

$$\begin{aligned}\pi_0(S_{K_{\mathrm{fin}}}(\mathbb{C})) &= G(\mathbb{Q}) \setminus \pi_0(X) \times G(\mathbb{A}_\mathbb{Q}^\infty)/K_{\mathrm{fin}} \\ &\simeq \mathbb{Q}_{>0} \setminus \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}^\infty)/\det(K_{\mathrm{fin}}) \\ &\simeq \widehat{\mathbb{Z}}^\times/\det(K_{\mathrm{fin}})\end{aligned}$$

where the first isomorphism is given by the determinant and by $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q}^\infty) = \mathrm{SL}_2(\mathbb{Q})K_{\mathrm{fin}}$ by strong approximation, and the second isomorphism is given by the decomposition $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q}^\infty) = \mathbb{Q}_{>0}\widehat{\mathbb{Z}}^\times$ which induces a decomposition

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty) = \coprod_{a \in \widehat{\mathbb{Z}}^\times/\det(K_{\mathrm{fin}})} \mathrm{GL}_2^+(\mathbb{Q})\mathrm{diag}(1, a)K_{\mathrm{fin}}$$

For $a \in \widehat{\mathbb{Z}}^\times/\det(K_{\text{fin}})$ consider the morphism $\mathfrak{H} \to S_{K_{\text{fin}}}(\mathbb{C})$ sending $z \in \mathfrak{H}$ to the point represented by $(z, \operatorname{diag}(1,a))$, then we obtain an embedding of classical modular curves $\Gamma_a \setminus \mathfrak{H} \hookrightarrow S_{K_{\text{fin}}}(\mathbb{C})$ where

$$\Gamma_a = \operatorname{diag}(1,a)K_{\text{fin}}\operatorname{diag}(1,a^{-1}) \cap \mathrm{GL}_2^+(\mathbb{Q})$$

In particular $S_{K_{\text{fin}}}(\mathbb{C})$ is a disjoint union of classical modular curves $\Gamma_a \setminus \mathfrak{H}$.

Moduli Problems and Level Structures The Shimura variety $S_{K_{\text{fin}}}$ is defined over $\mathbb{Q}$ (in fact over $\mathbb{Z}[\frac{1}{N}]$ for $K_{\text{fin}} = \prod_{p|N} K_p \times \prod_{p\nmid N} \mathrm{GL}_2(\mathbb{Z}_p)$) by the moduli functor

$$\begin{aligned} \mathcal{S}_{K_{\text{fin}}} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{ \begin{matrix} \text{Tuples } (E,\eta) \text{ where } E/S \text{ is an elliptic curve} \\ \text{and } \eta \text{ is a } K_{\text{fin}}\text{-level structure} \end{matrix} \right\}_{/\simeq} \end{aligned}$$

where the K_{fin}-level structure is a K_{fin}-conjugacy class of isomorphism $\eta : (\mathbb{A}_{\mathbb{Q}}^\infty)^2 \xrightarrow{\sim} H_1(E(\mathbb{C}), \mathbb{A}_{\mathbb{Q}}^\infty)$, and $(E_1,\eta_1) \simeq (E_2,\eta_2)$ are equivalent precisely if there exists an isogeny $\phi : E_1 \to E_2$ such that $\phi_* \circ \eta_1 = \eta_2$. The moduli functor $\mathcal{S}_{K_{\text{fin}}} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Grpd}$ is defined similarly. Each connected component of $\mathcal{S}_{K_{\text{fin}}}$ is defined over $\mathbb{Q}^{\text{ab}}$ (in fact over $\mathbb{Z}[\frac{1}{N}, \mu_N]$) and the induced action of $\mathrm{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) = \widehat{\mathbb{Z}}^\times$ on the set of geometric connected components $\pi_0(\mathcal{S}_{K_{\text{fin}}}) = \widehat{\mathbb{Z}}^\times/\det(K_{\text{fin}})$ is given by the usual multiplication in $\widehat{\mathbb{Z}}^\times$ by class field theory.

We now collect some running examples of moduli problems and level structures for modular curves, following [62].

Example 3.1.1. Let $\Gamma(N)$ denote the inverse image of the identity under the reduction morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$$

The corresponding quotient $\Gamma(N) \setminus \mathfrak{H}$ of the upper half plane $\mathfrak{H}$ is the classical modular curve of full level N over $\mathbb{C}$. Let $K(N)$ denote the inverse image of the identity under the reduction morphism $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$:

$$K(N) = \prod_{p|N} \mathscr{K}_p^+ \times \prod_{p\nmid N} \mathrm{GL}_2(\mathbb{Z}_p) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$$

Let $K'(N)$ be the following modified congruence subgroup:

$$K'(N) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix}\right) \mod N\}$$

A $\Gamma(N)$-structure on an elliptic curve E/S (in the sense of Katz-Mazur [62, Section 3.1]) is a group homomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N](S)$ which is a generator of $E[N]$ in the sense that we have an equality of effective Cartier divisors

$$E[N] = \sum_{a_1, a_2 \in \mathbb{Z}/N\mathbb{Z}} [\phi(a_1, a_2)]$$

The points $x_1 = \phi(1, 0)$ and $x_2 = \phi(0, 1)$ of $E[N](S)$ are the corresponding Drinfeld basis. When N is invertible on S a $\Gamma(N)$-structure on an elliptic curve E/S is equivalently a pair of linearly independent generators $x_1, x_2 \in E[N](S)$.

The group $K(N) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$ corresponds to the moduli problem

$$\begin{aligned} \mathcal{S}_{K(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{ \begin{matrix} \text{Tuples } (E, x_1, x_2) \text{ where } E/S \text{ is an elliptic curve} \\ x_1, x_2 \in E[N](S) \text{ are points of exact order } N \\ \text{such that } E[N] \text{ is generated by } x_1 \text{ and } x_2 \end{matrix} \right\}_{/\simeq} \end{aligned}$$

which is representable by a scheme for $N \geq 3$ (and representable by a Deligne-Mumford stack for $N = 1, 2$, where the moduli functor $\mathcal{S}_{K(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Grpd}$ is defined similarly).

Recall that we have the Weil pairing $e_N : E[N] \times E[N] \to \mu_N$, and the points $x_1, x_2 \in E[N](S)$ determine an element $\zeta = e_N(x_1, x_2) \in \mu_N(S)$, and hence a morphism $e : \mathcal{S}_{K(N)} \to \mu_N$. The moduli space $\mathcal{S}_{K(N)}$ is connected but not geometrically connected: its geometric connected components are the fibers of $e : \mathcal{S}_{K(N)} \to \mu_N$

after base change to $\mathbb{Z}[\frac{1}{N}, \mu_N]$. Fixing a primitive N-th root of unity $\zeta \in \mu_N(\mathbb{Z}[\frac{1}{N}, \mu_N])$, for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ we have the moduli problem

$$\begin{aligned}\mathcal{S}_{K(N),a} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N},\mu_N]} &\to \mathrm{Set} \\ S &\mapsto \left\{\begin{matrix}\text{Tuples } (E, x_1, x_2) \text{ where } E/S \text{ is an elliptic curve} \\ x_1, x_2 \in E[N](S) \text{ are points of exact order } N \\ \text{such that } E[N] \text{ is generated by } x_1 \text{ and } x_2 \\ \text{and } e_N(x_1, x_2) = \zeta^a\end{matrix}\right\}_{/\simeq}\end{aligned}$$

which is representable by a scheme for $N \geq 3$ (and representable by a Deligne-Mumford stack for $N = 1, 2$, where the moduli functor $\mathcal{S}_{K(N),a} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N},\mu_N]} \to \mathrm{Grpd}$ is defined similarly). In particular we have

$$S_{K(N)}(\mathbb{C}) = \coprod_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} S_{K(N),a}(\mathbb{C}) \qquad S_{K(N),a}(\mathbb{C}) = \Gamma(N)_a \setminus \mathfrak{H}$$

The group $K'(N) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix}1 & 0 \\ 0 & *\end{smallmatrix}\right) \mod N\}$ corresponds to the moduli problem

$$\begin{aligned}\mathcal{S}_{K'(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{\begin{matrix}\text{Tuples } (E, \phi, x) \text{ where } E/S \text{ is an elliptic curve} \\ \phi : E \to E' \text{ is a cyclic } N\text{-isogeny over } S \text{ and} \\ x \in E[N](S) \text{ is a point of exact order } N \\ \text{such that } E[N] \text{ is generated by } x \text{ and } \ker(\phi)\end{matrix}\right\}_{/\simeq}\end{aligned}$$

which is representable by a scheme for $N \geq 3$ (and representable by a Deligne-Mumford stack for $N = 1, 2$ where the moduli functor $\mathcal{S}_{K'(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Grpd}$ is defined similarly). The moduli space $\mathcal{S}_{K'(N)}$ is geometrically connected, in particular we have

$$S_{K'(N)}(\mathbb{C}) = \Gamma(N) \setminus \mathfrak{H}$$

Example 3.1.2. Let $\Gamma_1(N)$ denote the inverse image of the unipotent subgroup under the reduction morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma_1(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$$

The corresponding quotient $\Gamma_1(N) \setminus \mathfrak{H}$ of the upper half plane $\mathfrak{H}$ is the classical modular curve of level $\Gamma_1(N)$ over $\mathbb{C}$. Let $K_1(N)$ denote the subgroup

$$K_1(N) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \mod N\}$$

A $\Gamma_1(N)$-structure on an elliptic curve E/S (in the sense of Katz-Mazur [62, Section 3.2]) is a group homomorphism $\phi : \mathbb{Z}/N\mathbb{Z} \to E[N](S)$ such that the effective Cartier divisor

$$\sum_{a \in \mathbb{Z}/N\mathbb{Z}} [\phi(a)]$$

is a subgroup scheme of E. The point $x = \phi(1)$ of $E[N](S)$ is the corresponding point of exact order N. When N is invertible on S a $\Gamma_1(N)$-structure on an elliptic curve E/S is equivalently a point $x \in E[N](S)$ of exact order N.

The group $K_1(N) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \mod N\}$ corresponds to the moduli problem

$$\begin{aligned} \mathcal{S}_{K_1(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{ \begin{matrix} \text{Tuples } (E, x_1, x_2) \text{ where } E/S \text{ is an elliptic curve} \\ x_1 \in E[N](S) \text{ is a point of exact order } N \end{matrix} \right\}_{/\simeq} \end{aligned}$$

which is representable by a scheme for $N \geq 4$ (and representable by a Deligne-Mumford stack for $N = 1, 2, 3$, where the moduli functor $S_{K_1(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Grpd}$ is defined similarly). A similar issue of connected components exists as in the previous example, and a similar discussion applies to the inverse image of the unipotent subgroup under the reduction morphism $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$:

$$K_1(N) = \prod_{p|N} \mathscr{I}_p^+ \times \prod_{p\nmid N} \mathrm{GL}_2(\mathbb{Z}_p) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod N\}$$

Example 3.1.3. Let $\Gamma_0(N)$ denote the inverse image of the Borel subgroup under the reduction morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N\}$$

The corresponding quotient $\Gamma_0(N) \setminus \mathfrak{H}$ of the upper half plane $\mathfrak{H}$ is the classical modular curve of level $\Gamma_0(N)$ over $\mathbb{C}$.

Let $K_0(N)$ denote the inverse image of the Borel subgroup under the reduction morphism $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$:

$$K_0(N) = \prod_{p|N} \mathscr{I}_p \times \prod_{p\nmid N} \mathrm{GL}_2(\mathbb{Z}_p) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N\}$$

A $\Gamma_0(N)$-structure on an elliptic curve E/S (in the sense of Katz-Mazur [62, Section 3.4]) is an N-isogeny $\phi : E \to E'$ which is cyclic in the sense that $\ker(\phi)$ admits a generator (fppf locally on S). Equivalently, a $\Gamma_0(N)$-structure on an elliptic curve E/S is a finite flat subgroup scheme $H \subseteq E[N]$ which is locally free of rank N and which is cyclic in the sense that H admits a generator (fppf locally on S). When N is invertible on S a $\Gamma_0(N)$-structure on an elliptic curve E/S is a rank N finite flat subgroup scheme $H \subseteq E[N]$ (automatically locally free and cyclic, and isotropic with respect to the Weil pairing).

The group $K_0(N) = \{\gamma \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N\}$ corresponds to the moduli problem

$$\begin{aligned} \mathcal{S}_{K_0(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Grpd} \\ S &\mapsto \left\{ \begin{array}{c} \text{Tuples } (E, H) \text{ where } E/S \text{ is an elliptic curve} \\ H \subseteq E[N] \text{ is a rank } N \text{ finite flat subgroup scheme} \end{array} \right\} \end{aligned}$$

which is representable by a Deligne-Mumford stack. The moduli space $\mathcal{S}_{K_0(N)}$ is geometrically connected, in particular we have

$$S_{K_0(N)}(\mathbb{C}) = \Gamma_0(N) \setminus \mathfrak{H}$$

Let $\mathcal{A}_1[N] = \mathcal{S}_{K(N)}$, let $\mathcal{A}_1[\Gamma_1(N)] = \mathcal{S}_{K_1(N)}$, and let $\mathcal{A}_1[\Gamma_0(N)] = \mathcal{S}_{K_0(N)}$ denote the moduli stacks constructed in the above examples. In particular let $\mathcal{A}_1$ denote any of these in the case $N = 1$: this is the moduli stack of elliptic curves, corresponding to the hyperspecial maximal compact subgroup $K_{\text{fin}} = \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

Local Systems and Modular Forms We now recall the basic local systems on modular curves and their relation to modular forms. Recall that we have the maximal torus $T = \{\mathrm{diag}(t_1, t/t_1) | t_1, t \in \mathrm{GL}_1\} \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$. We identify elements of the character lattice $X^*(T)$ with pairs of integers $\lambda = (\lambda_1; c) \in \mathbb{Z}^2$ with $c \equiv \lambda_1 \mod 2$, corresponding to the character

$$\mathrm{diag}(t_1, t/t_1) \mapsto t_1^{\lambda_1} t^{\frac{c-\lambda_1}{2}}$$

The (finite dimensional) absolutely irreducible rational representations of GL_2 are parameterized by dominant highest weights $\lambda \in X^*(T)^+ \subseteq X^*(T)$. We identify elements of the subset $X^*(T)^+$ with pairs of integers $\lambda = (\lambda_1; c) \in \mathbb{Z}^2$ with $c \equiv \lambda_1 \mod 2$ and $\lambda_1 \geq 0$. For $\lambda \in X^*(T)^+$ let V_λ be the corresponding irreducible representation of GL_2. For $\lambda = (\lambda_1; c) \in X^*(T)^+$ let $\lambda^\vee = (\lambda_1; -c) \in X^*(T)^+$ so that $V_{\lambda^\vee} = V_\lambda^\vee$ is the contragredient representation. Noting that the determinant character $\det : \mathrm{diag}(t_1, t/t_1) \mapsto t$ corresponds to the highest weight $\lambda = (0; 2)$, we have an isomorphism

$$V_\lambda \xrightarrow{\sim} V_\lambda^\vee \otimes \det{}^{2c}$$

For an integer $\lambda_1 \geq 0$ let V_{λ_1} be the irreducible representation of GL_2 with highest weight $\lambda = (\lambda_1; 0)$, and let $V_{\lambda_1}(\det^{\frac{\lambda_1}{2}})$ be the irreducible representation of GL_2 with highest weight $\lambda = (\lambda_1; \lambda_1)$. Then we have an isomorphism $V_{\lambda_1}(\det^{\frac{\lambda_1}{2}}) = V_{\lambda_1} \otimes \det^{\frac{\lambda_1}{2}}$, in particular V_{λ_1} is self-dual.

Since the above Shimura varieties are moduli spaces of elliptic curves, we can consider local systems of geometric origin coming from the cohomology of the universal family of elliptic curves.

Let $\pi : U_{K_{\text{fin}}}(\mathbb{C}) \to S_{K_{\text{fin}}}(\mathbb{C})$ be the universal family of elliptic curves which gives rise to a local system of $\mathbb{Q}$-vector spaces $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}$ of rank 2 on $S_{K_{\text{fin}}}(\mathbb{C})$ whose fiber over a point $[E] \in S_{K_{\text{fin}}}(\mathbb{C})$ is the singular cohomology $H^1(E(\mathbb{C}), \mathbb{Q})$. For an integer $\lambda_1 \geq 0$ consider the local system of $\mathbb{Q}$-vector spaces $\mathbb{V}_{\lambda_1} = \mathrm{Sym}^{\lambda_1}(\mathbb{V})$ on $S_{K_{\text{fin}}}(\mathbb{C})$ whose fiber over a point $[E] \in S_{K_{\text{fin}}}(\mathbb{C})$ is the singular cohomology $\mathrm{Sym}^{\lambda} H^1(E(\mathbb{C}), \mathbb{Q})$.

We are then interested in the cohomology $H^*(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1})$ or the compactly supported cohomology $H^*_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1})$ as an $\mathcal{H}_{K_{\text{fin}}}$-module. In this case by Drinfeld-Manin we have isomorphisms of $\mathcal{H}_{K_{\text{fin}}}$-modules

$$
\begin{aligned}
H^i(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) &= H^i_!(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) \oplus H^i_{\text{Eis}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) \\
H^i_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) &= H^i_!(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) \oplus H^i_{\text{c,Eis}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1})
\end{aligned}
$$

The inner cohomology $H^i_!(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1})$ is identified with the cuspidal cohomology $H^i_{\text{cusp}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1})$, which will turn out to be concentrated in degree 1.

One can also consider the Hodge line bundle $\omega = \pi_*\Omega^1_{U_{K_{\text{fin}}}/S_{K_{\text{fin}}}}$ on $S_{K_{\text{fin}}}(\mathbb{C})$ whose fiber over a point $[E] \in \mathcal{S}_{K_{\text{fin}}}(\mathbb{C})$ is the cohomology $H^0(E, \Omega^1_E)$. For an integer $k \in \mathbb{Z}$ let $\mathcal{V}_k = \omega^{\otimes k}$ be the k-th tensor power of the Hodge line bundle on $S_{K_{\text{fin}}})\mathbb{C})$, which extends to the Baily-Borel compactification $\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C})$ of $S_{K_{\text{fin}}}(\mathbb{C})$, whose sections are modular forms of weight k:

Definition 3.1.4. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$ be a congruence subgroup. Consider the action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ on $\tau \in \mathfrak{H}$ given by $g \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$, and consider the factor of automorphy $j(g, \tau) = c\tau + d$. For an integer $k \in \mathbb{Z}$ consider the right action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space $C^\infty(\mathfrak{H})$ of smooth $\mathbb{C}$-valued functions on $\mathfrak{H}$ given by

$$(f|_k g)(\tau) = \det(g)^{k/2} j(g, \tau)^{-k} f(g \cdot \tau)$$

A modular form of weight k for Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ which is holomorphic at the cusps of Γ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

A cusp form of weight k for Γ is a modular form of weight k for Γ vanishing at the cusps of Γ, that is $\lim_{t\to\infty}(f|_k g)(it) = 0$ for all $g \in \mathrm{SL}_2(\mathbb{Q})$.

Let $M_k(\Gamma)$ be the $\mathbb{C}$-vector space of modular forms of weight k for Γ, and let $S_k(\Gamma)$ be the $\mathbb{C}$-vector subspace of cusp forms of weight k for Γ.

Let $M_k(K_{\text{fin}}) = \bigoplus_{a\in\widehat{\mathbb{Z}}^\times/\det(K_{\text{fin}})} M_k(\Gamma_a)$ and $S_k(K_{\text{fin}}) = \bigoplus_{a\in\widehat{\mathbb{Z}}^\times/\det(K_{\text{fin}})} S_k(\Gamma_a)$ be the corresponding spaces of modular forms and cusp forms taking into account connected components. Letting $D \subseteq \overline{S}^{\text{BB}}_{K_{\text{fin}}}$ be the divisor of cusps in the Baily-Borel compactification of $\mathcal{S}_{K_{\text{fin}}}$, we have identifications

$$M_k(K_{\text{fin}}) = H^0(\overline{S}_{K_{\text{fin}}}(\mathbb{C}), \mathcal{V}_k) \qquad S_k(K_{\text{fin}}) = H^0(\overline{S}_{K_{\text{fin}}}(\mathbb{C}), \mathcal{V}_k(-D))$$

By Faltings-Chai [31] the cohomology groups $H^i_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ carry a Hodge filtration with

$$\begin{aligned}\mathrm{Gr}^0_F H^i_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) &= H^i(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), \mathcal{V}_{-\lambda_1}(-D))\\ \mathrm{Gr}^{\lambda_1+1}_F H^i_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1}) &= H^{i-1}(\overline{S}^{\text{BB}}_{K_{\text{fin}}}(\mathbb{C}), \mathcal{V}_{\lambda_1+2}(-D))\end{aligned}$$

In particular one finds cusp forms of weight $\lambda_1 + 2$ in $H^1_{\text{c}}(S_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1})$.

The same construction applies to the construction of ℓ-adic local systems on $\mathcal{S}_{K_{\text{fin}}}$. Writing $\mathcal{S}_{K_{\text{fin}}}$ to mean either $\mathcal{S}_{K_{\text{fin}},\overline{\mathbb{Q}}}$ or $\mathcal{S}_{K_{\text{fin}},\overline{\mathbb{F}}_p}$ as before, let $\pi : \mathcal{U}_{K_{\text{fin}}} \to \mathcal{S}_{K_{\text{fin}}}$ be the universal family of elliptic curves and consider the ℓ-adic local system $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}_\ell$ of rank 2 on $\mathcal{S}_{K_{\text{fin}}}$ whose fiber over a point $[E] \in \mathcal{S}_{K_{\text{fin}}}$ is the ℓ-adic cohomology $H^1(E, \mathbb{Q}_\ell)$ (either over $\overline{\mathbb{Q}}$ or over $\overline{\mathbb{F}}_p$), which is pure of weight 1. For an integer $\lambda_1 \geq 0$ consider the ℓ-adic local system $\mathbb{V}_{\lambda_1} = \mathrm{Sym}^{\lambda_1}(\mathbb{V})$ on $\mathcal{S}_{K_{\text{fin}}}$ whose fiber over a point $[E] \in \mathcal{S}_{K_{\text{fin}}}$ is the ℓ-adic cohomology $\mathrm{Sym}^\lambda H^1(E, \mathbb{Q}_\ell)$ (either over $\overline{\mathbb{Q}}$ or over $\overline{\mathbb{F}}_p$), which is pure of weight λ_1.

We are then interested in the ℓ-adic $H^*(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1})$ or the compactly supported ℓ-adic cohomology $H^*_{\text{c}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1})$ (either over $\overline{\mathbb{Q}}$ or over $\overline{\mathbb{F}}_p$). In this case by Drinfeld-Manin we have isomorphisms of $\mathcal{H}_{K_{\text{fin}}} \times \mathrm{Gal}$-modules

$$\begin{aligned}H^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1}) &= H^i_!(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1}) \oplus H^i_{\text{Eis}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1})\\ H^i_{\text{c}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1}) &= H^i_!(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1}) \oplus H^i_{\text{c,Eis}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1})\end{aligned}$$

The inner cohomology $H^i_!(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1})$ is identified with the cuspidal cohomology $H^i_{\text{cusp}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1})$, which will turn out to be concentrated in degree 1.

In both of the above situations, the local systems $\mathbb{V}_{\lambda_1}$ correspond to the irreducible representation $V_{\lambda_1}(\det^{\frac{\lambda_1}{2}})$ of GL_2 with highest weight $\lambda = (\lambda_1; \lambda_1)$. On the other hand when computing cuspidal cohomology or Eisenstein cohomology in terms of $(\mathfrak{g}, K'_\infty)$-cohomology we will need to use the irreducible representation V_{λ_1} of GL_2 with highest weight $\lambda = (\lambda_1; 0)$, corresponding to the half Tate twisted local system $\mathbb{V}_{\lambda_1}(-\frac{\lambda_1}{2})$. The discrepancy involving the determinant character accounts for the difference between the unitary normalization and cohomological normalization of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, and will account for the additional Tate twist which appears in the later discussion of the Langlands-Kottwitz method. We will often use the abbreviated notation V_λ and $\mathbb{V}_\lambda$ for either of these representations and local systems, being careful to disambiguate when their meaning is not clear.

3.2 Induced Representations and Discrete Series for $\mathrm{GL}_2(\mathbb{R})$

We now recall some structural facts related to the group $\mathrm{GL}_2(\mathbb{R})$, especially the construction of discrete series representations and induced representations, and the Vogan-Zuckerman classification of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{R})$ with nonzero $(\mathfrak{g}, K'_\infty)$-cohomology, which will be used throughout the rest of this chapter.

Lie Algebras and Compact Subgroups Let $\mathfrak{g} = \mathfrak{gl}_2$ be the Lie algebra of $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$, and let $\mathfrak{g}^1 = \mathfrak{sl}_2$ be the Lie algebra of $G^1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.

Let K_∞ be the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$ given by

$$K_\infty = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})\}$$

We have an isomorphism $K_\infty \xrightarrow{\sim} \mathrm{U}(1)$ given by $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mapsto a + bi$, that is $K_\infty \simeq \mathrm{SO}(2)$. Let $\mathfrak{k}$ be the Lie algebra of K_∞, that is the Cartan subalgebra $\mathfrak{h}$ corresponding to the compact torus

$$T_c = \{\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) | \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

Recall that the weights of $\mathfrak{sl}_2$ are elements of the space $\mathfrak{h}_\mathbb{C}^\vee = \mathrm{Hom}_\mathbb{C}(\mathfrak{h}_\mathbb{C}, \mathbb{C})$; an element $\lambda \in \mathfrak{h}_\mathbb{C}^\vee$ is identified with a complex number $\lambda \in \mathbb{C}$. Let $\mathfrak{h}^\vee \subseteq \mathfrak{h}_\mathbb{C}^\vee$ be the subset where an element $\lambda \in \mathfrak{h}^\vee$ is identified with a real number $\lambda \in \mathbb{R}^2$. We say that $\lambda \in \mathfrak{h}^\vee$ is analytically integral if λ is identified with an integer $\lambda \in \mathbb{Z}$. Under this identification, an integer $k \in \mathbb{Z}$ corresponds to the derivative of the character ψ_k of T_c given by

$$\psi_k : \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \mapsto e^{ik\theta}$$

Every analytically integral weight $\lambda \in \mathfrak{h}^\vee$ with $\lambda \geq 0$ corresponds to a K_∞-type V_λ, that is an equivalence class of irreducible representation of the compact group $K_\infty \simeq \mathrm{SO}(2)$. Such a K_∞-type V_λ has weights $\lambda - j$ for $j \in \{0, 1, \ldots, \lambda\}$ each with multiplicity 1; in particular V_λ has dimension $\lambda + 1$ with highest weight λ.

Characters and Roots Recall that we have the maximal torus

$$T = \{\mathrm{diag}(t_1, t_2) | t_1, t_2 \in \mathrm{GL}_1\}$$

which is identified with the Levi quotient of the Borel minimal parabolic $\mathbb{Q}$-subgroup P_0 of upper triangular matrices in $G = \mathrm{GL}_2$.

We have the elementary characters $e_i : T \to \mathbb{G}_\mathrm{m}$ given by

$$e_1(\mathrm{diag}(t_1, t_2)) = t_1 \qquad e_2(\mathrm{diag}(t_1, t_2)) = t_2$$

We have the elementary cocharacters $f_i : \mathbb{G}_\mathrm{m} \to T$ given by

$$f_1(t) = \mathrm{diag}(t, 1) \qquad f_2(t) = \mathrm{diag}(1, t)$$

so that $e_i \circ f_j = \delta_{i,j}$. We have the character lattice $X^*(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and we have the cocharacter lattice $X_*(T) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$. We have the roots $\pm\alpha = \pm(e_1 - e_2)$ and we have the coroots $\pm\alpha^\vee = \pm(f_1 - f_2)$, and we have the fundamental weight $\omega = \frac{1}{2}(e_1 - e_2) = \frac{1}{2}\alpha$ defined by $2\frac{\langle\omega,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 1$, which in this case coincides with the half sum of positive roots $\rho_{P_0} = \frac{1}{2}\alpha = \omega$.

For $z_1, z_2 \in \mathbb{C}$ consider the unramified character $\chi_{z_1,z_2} : M_0 \to \mathbb{G}_\mathrm{m}$ given by

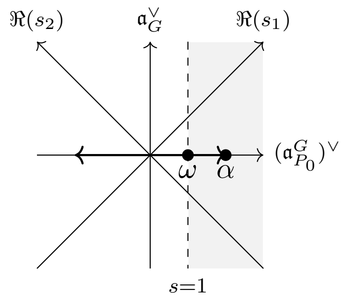
$$\chi_{z_1,z_2}(\mathrm{diag}(t_1, t_2)) = |t_1|^{z_1} |t_2|^{z_2}$$

so that $(\chi_{z_1,z_2} \circ \alpha^\vee)(t) = |t|^{z_1 - z_2} = |t|^s$ for $s = z_1 - z_2$. In other words $\chi_{z_1,z_2} = \nu^{z_1} \boxtimes \nu^{z_2}$ where $\nu = |\cdot|$ is the norm character.

Recall that $\mathfrak{a}_{P_0}^\vee = \mathfrak{a}_G^\vee \oplus (\mathfrak{a}_{P_0}^G)^\vee$ where

$$\mathfrak{a}_{P_0}^\vee = \{(z_1, z_2) \in \mathbb{R}^2\} \simeq \mathbb{R}^2$$
$$\mathfrak{a}_G^\vee = \{(z_1, z_2) \in \mathbb{R}^2 | z_1 - z_2 = 0\} = \{(s, s) \in \mathbb{R}^2\} \simeq \mathbb{R}$$
$$(\mathfrak{a}_{P_0}^G)^\vee = \{(z_1, z_2) \in \mathbb{R}^2 | z_1 + z_2 = 0\} = \{(s, -s) \in \mathbb{R}^2\} \simeq \mathbb{R}$$

with $s = z_1 - z_2$ providing the coordinate for $(\mathfrak{a}_{P_0}^G)^\vee$.



Remark 3.2.1. There is an another choice of convention for the above. Recall that we have the maximal torus

$$T = \{\mathrm{diag}(t_1, t/t_1) | t_1, t \in \mathrm{GL}_1\}$$

We have the elementary characters $e_i : T \to \mathbb{G}_\mathrm{m}$ given by

$$e_1(\mathrm{diag}(t_1, t/t_1)) = t_1 \qquad e_0(\mathrm{diag}(t_1, t/t_1)) = t$$

We have the elementary cocharacters $f_i : \mathbb{G}_\mathrm{m} \to T$ given by

$$f_1(t) = \mathrm{diag}(t, 1/t) \qquad f_0(t) = \mathrm{diag}(1, t)$$

so that $e_i \circ f_j = \delta_{i,j}$. We have the character lattice $X^*(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_0$ and we have the cocharacter lattice $X_*(T) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_0$. We have the roots $\pm\alpha = \pm(2e_1 - e_0)$ and we have the coroots $\pm\alpha^\vee = \pm f_1$, and we have the fundamental weight $\omega = \frac{1}{2}(2e_1 - e_0) = \frac{1}{2}\alpha$ defined by $2\frac{\langle\omega,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 1$, which in this case coincides with the half sum of positive roots $\rho_{P_0} = \frac{1}{2}\alpha = \omega$.

Parabolic Induction In this case there is only one standard parabolic $\mathbb{Q}$-subgroup of $G = \mathrm{GL}_2$ we need to consider. We have the Borel parabolic $\mathbb{Q}$-subgroup $P_0 = M_0 N_0$ of upper triangular matrices with Levi quotient $M_0 = T$ a maximal torus and unipotent $\mathbb{Q}$-subgroup $N_0 = U$ given by

$$P_0 = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \cap G \qquad M_0 = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) \cap G \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \qquad N_0 = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \cap G$$

Write $M_0 = \{\mathrm{diag}(t_1, t_2) | t_1, t_2 \in \mathrm{GL}_1\}$. For characters χ_1, χ_2 of GL_1 we have a character $\pi = \chi_1 \boxtimes \chi_2$ of $T \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ given by

$$\pi(\mathrm{diag}(t_1, t_2)) = \chi_1(t_1)\chi_2(t_2)$$

with central character $\omega = \chi_1\chi_2$. If the central character ω is fixed we can write $\chi_2 = \chi_1^{-1}\omega$ so that $\pi = \chi_1 \boxtimes \chi_1^{-1}\omega$. We have the norm character δ_{P_0} of P_0 whose restriction to M_0 is given by

$$\delta_{P_0}(\mathrm{diag}(t_1, t_2)) = |t_1||t_2|^{-1}$$

For $s \in \mathbb{C}$ we have the unramified character

$$e^{\langle H_{P_0}(\cdot), s\rangle}(\mathrm{diag}(t_1, t_2)) = |t_1|^s |t_2|^{-s}$$

Let $\pi = \chi_1 \boxtimes \chi_2 : M_0(\mathbb{R}) \to \mathbb{C}^\times$ be a (continuous) character regarded as a character of $P_0(\mathbb{R})$. Consider the Borel parabolic induction

$$\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \left\{ \begin{array}{c} \text{smooth functions } \phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \text{ such that} \\ \phi(pg) = \chi_1(t_1)\chi_2(t_2)\phi(g) \text{ for every } g \in \mathrm{GL}_2(\mathbb{R}) \\ \text{and } p \in P_0(\mathbb{R}) \text{ with } p \equiv \mathrm{diag}(t_1, t_2) \in M_0(\mathbb{R}) \end{array} \right\}$$

Recalling the norm character $e^{\langle H_{P_0}(\cdot), s+\rho_{P_0}\rangle}(\mathrm{diag}(t_1, t_2)) = |t_1|^{s+\frac{1}{2}}|t_2|^{-s-\frac{1}{2}}$ we consider the family of normalized Borel parabolic inductions

$$\mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi, s) = \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(e^{\langle H_{P_0}(\cdot), s+\rho_{P_0}\rangle}\pi)$$

In particular the norm character $\delta_{P_0(\mathbb{R})}^{1/2} = e^{\langle H_{P_0}(\cdot), \rho_{P_0} \rangle}$ defines the normalized Borel parabolic induction

$$\chi_1 \times \chi_2 = \mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\delta_{P_0(\mathbb{R})}^{1/2}\pi)$$

Let $W = \{1, w_0\}$ be the corresponding Weyl group, where w_0 is a simple reflection. Each Weyl group element will give rise to intertwining operators between the above Borel parabolic inductions.

Remark 3.2.2. For characters χ_1, χ_2 of GL_1 we have a character $\pi = \chi_1 \otimes \chi$ of $M_0 \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ given by

$$\pi(\mathrm{diag}(t_1, t/t_1)) = \chi_1(t_1)\chi(t)$$

with central character $\omega = \chi$. In other words, $\pi = \chi_1 \boxtimes \chi_1^{-1}\chi$ in the notation of the previous convention. We have the norm character δ_{P_0} of P_0 whose restriction to M_0 is given by

$$\delta_{P_0}(\mathrm{diag}(t_1, t/t_1)) = |t_1|^2|t|^{-1}$$

We will sometimes switch to using this convention in later sections, as it is closer to the convention used for GSp_4. The differing notations $\boxtimes$ and $\otimes$ should hopefully clarify which of these two conventions is implicitly being used when writing characters of $M_0 \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$.

All such parabolic inductions are regarded as a representation of $\mathrm{GL}_2(\mathbb{R})$ by the right translation action, or regarded as an admissible $(\mathfrak{gl}_2, K_\infty)$-module after passing to the subspace of K_∞-finite vectors (which we abusively denote by the same notation). We can restrict these to representations of $\mathrm{SL}_2(\mathbb{R})$, or to $(\mathfrak{sl}_2, K_\infty)$-modules. The K_∞-type decompositions are given as follows.

Recalling that $K_\infty = \{\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) | \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ and that $K_\infty^{M_0} = K_\infty \cap M_0(\mathbb{R})$ is cyclic of order 2 generated by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we have that $\pi_c = \pi|_{T_c}$ is determined by an integer modulo 2: we have $m \in \mathbb{Z}/2\mathbb{Z}$ such that $\pi_c(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) = (-1)^m$. For $k \in \mathbb{Z}$ such that $k \equiv m \mod 2$ let $\psi_k \in \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi)$ be the function given by

$$\psi_k(\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)) = e^{ik\theta}$$

Then $\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi)$ admits a K_∞-type decomposition

$$\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi)|_{K_\infty} = \bigoplus_{\substack{k\in\mathbb{Z} \\ k\equiv m \mod 2}} \mathbb{C}\psi_k$$

The finite dimensional representation V_λ admits a K_∞-type decomposition

$$V_\lambda|_{K_\infty} = \bigoplus_{\substack{k\in\mathbb{Z} \\ -n\leq k\leq n \\ k\equiv n \mod 2}} \mathbb{C}\psi_k$$

In particular for an analytically integral dominant weight $\lambda \in \mathfrak{h}^\vee$ we have an embedding $V_\lambda \hookrightarrow \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0})$ and $\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0})$ is reducible; the corresponding quotient leads to the construction of discrete series representations, as we now recall.

Holomorphic Discrete Series The Harish-Chandra classification of discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ and of $\mathrm{GL}_2(\mathbb{R})$ is particularly simple, and we will recall the results directly.

For $\mathrm{SL}_2(\mathbb{R})$ we have the following discrete series representations:

(i) (Holomorphic discrete series) For an integer $\lambda_1 \geq 0$ we have a corresponding analytically integral weight $\lambda_1 \in \mathfrak{h}^\vee$ and a corresponding analytically integral nonsingular weight $\lambda_1 + 1 \in \mathfrak{h}^\vee$. Then we have the holomorphic discrete series representation $\mathcal{D}^+_{\lambda_1+1} = \mathcal{D}_{\lambda_1+1}$ of $\mathrm{SL}_2(\mathbb{R})$ with Harish-Chandra parameter (infinitesimal character) $\lambda_1 + 1$ and Blattner parameter (minimal K_∞-type) $\Lambda = \lambda_1 + 2$.

(ii) (Antiholomorphic discrete series) For an integer $\lambda_1 \geq 0$ we have a corresponding analytically integral weight $-\lambda_1 \in \mathfrak{h}^\vee$ and a corresponding analytically integral nonsingular weight $-\lambda_1 - 1 \in \mathfrak{h}^\vee$. Then we have the antiholomorphic discrete series representation $\mathcal{D}^-_{\lambda_1+1} = \mathcal{D}_{-\lambda_1-1}$ of $\mathrm{SL}_2(\mathbb{R})$ with Harish-Chandra parameter (infinitesimal character) $-\lambda_1 - 1$ and Blattner parameter (minimal K_∞-type) $\Lambda = -\lambda_1 - 2$.

The (anti)holomorphic discrete series representations $\mathcal{D}^{\pm}_{\lambda_1+1}$ both occur as infinite-dimensional subrepresentations of the normalized Borel parabolic induction

$$\mathcal{I}^{\mathrm{SL}_2(\mathbb{R})}_{P_0(\mathbb{R})}(\mathrm{sign}^{\lambda_1}\nu^{\lambda_1+1}) = \mathrm{Ind}^{\mathrm{SL}_2(\mathbb{R})}_{P_0(\mathbb{R})}(\mathrm{sign}^{\lambda_1}\nu^{\lambda_1+2}) = \left\{\begin{array}{c}\text{smooth functions } \phi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C} \text{ such that} \\ \phi(pg) = \mathrm{sign}^{\lambda_1}(t)|t|^{\lambda_1+2}\phi(g) \text{ for every } g \in \mathrm{SL}_2(\mathbb{R}) \\ \text{and } p \in P_0(\mathbb{R}) \text{ with } p \equiv \mathrm{diag}(t, 1/t) \in M_0(\mathbb{R})\end{array}\right\}$$

We can regard such (anti)holomorphic discrete series as representations of $\mathrm{SL}_2(\mathbb{R})$, or as $(\mathfrak{sl}_2, K_\infty)$-modules after passing to spaces of K_∞-finite vectors.

For $\mathrm{GL}_2(\mathbb{R})$ we have the following discrete series representations. For an integer $\lambda_1 \geq 0$ and a central character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ with $\omega(-1) = (-1)^{\lambda_1}$ we have the holomorphic discrete series representation $\mathcal{D}_{\lambda_1+1}(\omega)$ of $\mathrm{GL}_2(\mathbb{R})$ with Harish-Chandra parameter (infinitesimal character) $\lambda_1 + 1$ and Blattner parameter (minimal K_∞-type) $\Lambda = \lambda_1 + 2$, occurring as an infinite-dimensional subrepresentation of the normalized Borel parabolic induction

$$\mathcal{I}^{\mathrm{GL}_2(\mathbb{R})}_{P_0(\mathbb{R})}(\chi_1 \boxtimes \chi_2) = \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{R})}_{P_0(\mathbb{R})}(\chi_1\nu^{\frac{1}{2}} \boxtimes \chi_2\nu^{-\frac{1}{2}}) = \left\{\begin{array}{c}\text{smooth functions } \phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \text{ such that} \\ \phi(pg) = |t_1/t_2|^{\frac{1}{2}}\chi_1(t_1)\chi_2(t_2)\phi(g) \text{ for every } g \in \mathrm{GL}_2(\mathbb{R}) \\ \text{and } p \in P_0(\mathbb{R}) \text{ with } p \equiv \mathrm{diag}(t_1, t_2) \in M_0(\mathbb{R})\end{array}\right\}$$

where $\chi_1, \chi_2 : \mathbb{R}^\times \to \mathbb{C}^\times$ are (continuous) characters such that $\chi_1\chi_2 = \omega$ and $\chi_1\chi_2^{-1} = \mathrm{sign}^{\lambda_1}\nu^{\lambda_1+1}$. When the central character ω is trivial we write $\mathcal{D}_{\lambda_1+1}$ for the corresponding holomorphic discrete series representation of $\mathrm{GL}_2(\mathbb{R})$. We can regard such discrete series as representations of $\mathrm{GL}_2(\mathbb{R})$, or as $(\mathfrak{gl}_2, K_\infty)$-modules after passing to spaces of K_∞-finite vectors. Here we may choose to work with $K_\infty \simeq \mathrm{SO}(2)$ as above (which is insensitive to the central character ω), or with $K'_\infty = \mathbb{R}_{>0}K_\infty$ (which is insensitive to the parity of the central character ω), or with $K''_\infty = \mathbb{R}^\times K_\infty$ (which is sensitive to the central character ω).

Restricting from $\mathrm{GL}_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})$ yields an isomorphism

$$\mathcal{D}_{\lambda_1+1}(\omega)|_{\mathrm{SL}_2(\mathbb{R})} = \mathcal{D}^+_{\lambda_1+1} \oplus \mathcal{D}^-_{\lambda_1+1}$$

One also has the limit discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ (denoted $\mathcal{D}^+_0$ and $\mathcal{D}^-_0$) and of $\mathrm{GL}_2(\mathbb{R})$ (denoted $\mathcal{D}_0(\omega)$), which we will not need.

As the above discrete series representations occur as infinite-dimensional subrepresentation of the normalized Borel parabolic induction, their construction can be understood in terms of local intertwining operators. Recall the Weyl group $W = \{1, w_0\}$ where the simple reflection $w_0 \in W$ is represented by $\widetilde{w}_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in W$.

For $\phi_s \in \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi^{w_0}\delta_{P_0(\mathbb{R})}^s)$ consider the local intertwining operator

$$M(\widetilde{w}_0^{-1}, \pi_s^{w_0}) : \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi^{w_0}\delta_{P_0(\mathbb{R})}^s) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi\delta_{P_0(\mathbb{R})}^{1-s})$$
$$(M(\widetilde{w}_0^{-1}, \pi_s^{w_0})\phi_s)(g) = \int_{N_0(\mathbb{R})} \phi_s(\widetilde{w}_0 ng)\mathrm{d}n$$

which admits a meromorphic continuation to all $s \in \mathbb{C}$. At those $s_0 \in \mathbb{C}$ where the local intertwining operator has a pole, there exists an integer $m \geq 0$ yielding a nonzero regularized local intertwining operator

$$M^{\mathrm{reg}}(w_0^{-1}, \pi_{s_0}^{w_0}) = (s - s_0)^m M(w_0^{-1}, \pi_s^{w_0})|_{s=s_0} : \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi^{w_0}\rho_{P_0}^{s_0}) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi\rho_{P_0}^{1-s_0})$$

Likewise for $\phi_s \in \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi\delta_{P_0(\mathbb{R})}^s)$ consider the local intertwining operator

$$M(\widetilde{w}_0, \pi_s) : \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi\delta_{P_0(\mathbb{R})}^s) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi^{w_0}\delta_{P_0(\mathbb{R})}^{1-s})$$
$$(M(\widetilde{w}_0, \pi_s)\phi_s)(g) = \int_{N_0(\mathbb{R})} \phi_s(\widetilde{w}_0^{-1} ng)\mathrm{d}n$$

which admits a meromorphic continuation to all $s \in \mathbb{C}$. At those $s_0 \in \mathbb{C}$ where the local intertwining operator has a pole, there exists an integer $m \geq 0$ yielding a nonzero regularized local intertwining operator

$$M^{\mathrm{reg}}(w_0, \pi_{s_0}) = (s - s_0)^m M(w_0, \pi_s)|_{s=s_0} : \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi\rho_{P_0}^{1-s_0}) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi^{w_0}\rho_{P_0}^{s_0})$$

The induced representation $\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi)$ is reducible if the intertwining operator $M(\widetilde{w}_0^{-1}, \pi)$ is not an isomorphism, which happens precisely if $\pi = \lambda$ or $\pi = \lambda^{w_0}\delta_{P_0(\mathbb{R})}$ for some analytically integral dominant element $\lambda \in \mathfrak{h}^\vee$.

For an analytically integral dominant weight $\lambda \in \mathfrak{h}^\vee$ write $\lambda = \lambda_1\omega + c\mathrm{det}$ with $\lambda_1 \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{Z}$ such that $\lambda_1 \equiv c \mod 2$. Then we have two nonzero regularized local intertwining operators

$$M^{\mathrm{reg}}(\widetilde{w}_0^{-1}, \lambda^{w_0}) : \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda\delta_{P_0(\mathbb{R})})$$
$$M^{\mathrm{reg}}(\widetilde{w}_0, \lambda\delta_{P_0(\mathbb{R})}) : \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda\delta_{P_0(\mathbb{R})}) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0})$$

In the first case we have $\ker(M^{\mathrm{reg}}(\widetilde{w}_0^{-1}, \lambda^{w_0})) = V_\lambda$ which induces an isomorphism

$$\mathcal{D}_{\lambda_1+1}(\det{}^c) \xrightarrow{\sim} \mathcal{D}_{\lambda_1+1}(\det{}^c)^\vee \otimes \det{}^{2c}$$

In the second case we have $\ker(M^{\mathrm{reg}}(\widetilde{w}_0, \lambda\delta_{P_0(\mathbb{R})})) = \mathcal{D}_{\lambda_1+1}^\vee \otimes \det{}^{2c}$ which induces an isomorphism

$$V_{\lambda_1}(\det{}^c) = V_\lambda \xrightarrow{\sim} V_\lambda^\vee \otimes \det{}^{2c}$$

In particular we have a short exact sequence of $(\mathfrak{gl}_2, K_\infty)$-modules

$$0 \to V_\lambda \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \to \mathcal{D}_{\lambda_1+1}(\det{}^c) \to 0$$

Taking duals and tensoring with $\det^{2c}$ we have a short exact sequence of $(\mathfrak{gl}_2, K_\infty)$-modules

$$0 \to \mathcal{D}_{\lambda_1+1}(\det{}^c)^\vee \otimes \det{}^{2c} \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0})^\vee \otimes \det{}^{2c} \to V_\lambda^\vee \otimes \det{}^{2c} \to 0$$

Since $\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0})^\vee \otimes \det^{2c} \simeq \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda\delta_{P_0(\mathbb{R})})$, we obtain a short exact sequence of $(\mathfrak{gl}_2, K_\infty)$-modules

$$0 \to \mathcal{D}_{\lambda_1+1}(\det{}^c) \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda\delta_{P_0(\mathbb{R})}) \to V_\lambda \to 0$$

The $(\mathfrak{gl}_2, K_\infty)$-module $\mathcal{D}_{\lambda_1+1}(\det^c)$ admits a decomposition into K_∞-types

$$\mathcal{D}_{\lambda_1+1}(\det{}^c)|_{K_\infty} = \bigoplus_{\substack{k\in\mathbb{Z} \\ k\geq\lambda_1+2 \\ k\equiv\lambda_1 \mod 2}} \mathbb{C}\psi_k \oplus \bigoplus_{\substack{k\in\mathbb{Z} \\ k\leq-\lambda_1-2 \\ k\equiv\lambda_1 \mod 2}} \mathbb{C}\psi_k$$

Restricting to $\mathrm{SL}_2(\mathbb{R})$ we have a decomposition $\mathcal{D}_{\lambda_1+1}(\det^c)|_{\mathrm{SL}_2(\mathbb{R})} = \mathcal{D}^+_{\lambda_1+1} \oplus \mathcal{D}^-_{\lambda_1+1}$ where $\mathcal{D}^+_{\lambda_1+1}$ and $\mathcal{D}^-_{\lambda_1+1}$ admit K_∞-type decompositions

$$\mathcal{D}^+_{\lambda_1+1} = \bigoplus_{\substack{k\in\mathbb{Z} \\ k\geq \lambda_1+2 \\ k\equiv\lambda_1 \mod 2}} \mathbb{C}\psi_k \qquad \mathcal{D}^-_{\lambda_1+1} = \bigoplus_{\substack{k\in\mathbb{Z} \\ k\leq -\lambda_1-2 \\ k\equiv\lambda_1 \mod 2}} \mathbb{C}\psi_k$$

Cohomology The nontrivial extension of $(\mathfrak{g}, K'_\infty)$-modules

$$0 \to V_\lambda \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \to \mathcal{D}_{\lambda_1+1}(\det^c) \to 0$$

determines a nontrivial class in $\mathrm{Ext}^1_{(\mathfrak{g},K'_\infty)}(\mathcal{D}_{\lambda_1+1}(\det^c), V_\lambda) = H^1(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda_1+1}(\det^c) \otimes V_{\lambda^\vee})$, and indeed this group is nontrivial. We will compute the $(\mathfrak{g}, K'_\infty)$-cohomology of $\mathcal{D}_{\lambda+1}$ by computing the $(\mathfrak{g}, K'_\infty)$-cohomology of the Borel parabolic induction $\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0})$ and by considering the corresponding long exact sequence in $(\mathfrak{g}, K'_\infty)$-cohomology.

An element $\omega \in \mathrm{Hom}_{K'_\infty}(\wedge^p(\mathfrak{g}/\mathfrak{k}'), \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda)$ sends an element $v_1 \wedge \ldots \wedge v_p \in \wedge^p(\mathfrak{g}/\mathfrak{k}')$ to an element $\omega(v_1 \wedge \ldots \wedge v_p) \in \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda$ such that $\omega(\mathrm{Ad}(k)v_1 \wedge \ldots \wedge \mathrm{Ad}(k)v_p) = k\omega(v_1 \wedge \ldots \wedge v_p)$ for every $k \in K'_\infty$.

Write $\omega(v_1 \wedge \ldots \wedge v_p) = \sum_i f_i \otimes g_i$ where $f_i \in \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi)$ and $g_i \in V_\lambda$. Evaluating at the identity yields an element $\omega(v_1 \wedge \ldots \wedge v_p)(\mathrm{id}) = \sum_i f_i(\mathrm{id}) \otimes g_i \in \mathbb{C}\pi \otimes V_\lambda$. By K'_∞-invariance we have that ω is determined by this evaluation at the identity, and we obtain an isomorphism of complexes

$$\mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda) \xrightarrow{\sim} \mathrm{Hom}_{K'^{M_0}_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \pi \otimes V_\lambda)$$
$$(v_1 \wedge \ldots \wedge v_p \mapsto \omega(v_1 \wedge \ldots \wedge v_p)) \mapsto (v_1 \wedge \ldots \wedge v_p \mapsto \omega(v_1 \wedge \ldots \wedge v_p)(\mathrm{id}))$$

Now by the direct sum decomposition $\mathfrak{gl}_2 = \mathfrak{p}_0 \oplus \mathfrak{k}' = \mathfrak{m}_0 \oplus \mathfrak{n}_0 \oplus \mathfrak{k}'$ we obtain an isomorphism of complexes

$$\mathrm{Hom}_{K'^{M_0}_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \pi \otimes V_\lambda) = \mathrm{Tot}(\mathrm{Hom}_{K'^{M_0}_\infty}(\wedge^\bullet(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_{\mathbb{C}}(\wedge^\bullet(\mathfrak{n}_0), V_\lambda)))$$

Passing to $(\mathfrak{g}, K'_\infty)$-cohomology yields Delorme's isomorphism

$$\begin{aligned} &H^i(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda) = H^i(\mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda)) \\ &\xrightarrow{\sim} H^i(\mathrm{Tot}(\mathrm{Hom}_{K'^{M_0}_\infty}(\wedge^\bullet(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_{\mathbb{C}}(\wedge^\bullet(\mathfrak{n}_0), V_\lambda)))) = \bigoplus_{p+q=i} H^p(\mathfrak{m}_0, K'^{M_0}_\infty; \pi \otimes H^q(\mathfrak{n}_0, V_\lambda)) \end{aligned}$$

To compute $H^q(\mathfrak{n}_0, V_\lambda) = H^q(\mathrm{Hom}_{\mathbb{C}}(\wedge^\bullet(\mathfrak{n}_0), V_\lambda))$ we appeal to Kostant's theorem, which can be made explicit as follows. Recall that $\mathfrak{n}_0$ is generated by $E_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and that V_λ admits a weight space decomposition

$$V_\lambda = \bigoplus_{\substack{k \in \mathbb{Z} \\ |k| \leq n \\ k \equiv n \mod 2}} \mathbb{C} e_k$$

where the torus $M_0^1 = \{\mathrm{diag}(t, 1/t)\}$ acts on e_k by $\rho_\lambda(\mathrm{diag}(t, 1/t))(e_k) = t^k e_k$, and where the Lie algebra $\mathfrak{n}_0$ acts on e_k by $\mathrm{d}\rho_\lambda(E_+)(e_k) = \frac{n-k}{2} e_{k+2}$. Then we have an isomorphism of complexes

$$\mathrm{Hom}_{\mathbb{C}}(\wedge^\bullet(\mathfrak{n}_0), V_\lambda) = \Bigg(\bigoplus_{\substack{k \in \mathbb{Z} \\ |k| \leq n \\ k \equiv n \mod 2}} \mathbb{C} e_k \xrightarrow{d} \bigoplus_{\substack{k \in \mathbb{Z} \\ |k| \leq n \\ k \equiv n \mod 2}} \mathbb{C} E_+^\vee \otimes e_k \Bigg)$$

with differential $d(e_k) = \frac{n-k}{2} E_+^\vee \otimes e_{k+2}$. We obtain a direct sum of complexes

$$\mathrm{Hom}_{\mathbb{C}}(\wedge^\bullet(\mathfrak{n}_0), V_\lambda) = \mathbb{H}^\bullet(\mathfrak{n}_0, V_\lambda) \oplus \mathbb{A}^\bullet(\mathfrak{n}_0, V_\lambda)$$

where $\mathbb{A}^\bullet(\mathfrak{n}_0, V_\lambda)$ is acyclic and $\mathbb{H}^\bullet(\mathfrak{n}_0, V_\lambda)$ is given by

$$\mathbb{H}^\bullet(\mathfrak{n}_0, V_\lambda) = \left(\mathbb{C} e_n \xrightarrow{0} \mathbb{C} E_+^\vee \otimes e_{-n} \right)$$

Noting that the torus $T = \{\mathrm{diag}(t_1, t_2)\}$ acts on $\mathbb{H}^0(\mathfrak{n}_0, V_\lambda) = \mathbb{C}e_n$ by the character λ and acts on $\mathbb{H}^1(\mathfrak{n}_0, V_\lambda) = \mathbb{C}E_+^\vee \otimes e_{-n}$ by the character $w_0 \cdot \lambda = \lambda^{w_0} - 2\rho_{P_0}$, we obtain an isomorphism

$$H^q(\mathfrak{n}_0, V_\lambda) = \bigoplus_{\substack{w\in W\\ \ell(w)=q}} V_{w\cdot\lambda} = \begin{cases} \mathbb{C}e_n & q = 0 \\ \mathbb{C}E_+^\vee \otimes e_{-n} & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now to compute $H^p(\mathfrak{m}_0, K_\infty'^{M_0}; \pi \otimes H^q(\mathfrak{n}_0, V_\lambda))$ we only need to consider the horizontal differentials in the double complex $\mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^\bullet(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_\mathbb{C}(\wedge^\bullet(\mathfrak{n}_0), V_\lambda)) = \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^\bullet(\mathfrak{m}_0/\mathfrak{k}'), H^\bullet(\mathfrak{n}_0, V_\lambda))$:

$$\begin{array}{c|ccc} 1 & \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^0(\mathfrak{m}_0/\mathfrak{k}'), \mathbb{C}\pi \otimes H^1(\mathfrak{n}_0, V_\lambda)) & \xrightarrow{\mathrm{d}} & \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^1(\mathfrak{m}_0/\mathfrak{k}'), \mathbb{C}\pi \otimes H^1(\mathfrak{n}_0, V_\lambda)) \\ & \uparrow 0 & & \uparrow 0 \\ 0 & \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^0(\mathfrak{m}_0/\mathfrak{k}'), \mathbb{C}\pi \otimes H^0(\mathfrak{n}_0, V_\lambda)) & \xrightarrow{\mathrm{d}} & \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^1(\mathfrak{m}_0/\mathfrak{k}'), \mathbb{C}\pi \otimes H^0(\mathfrak{n}_0, V_\lambda)) \\ \hline & 0 & & 1 \end{array}$$

In the case $q = 0$ we consider the complex

$$\left(\mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^0(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_\mathbb{C}(\wedge^0(\mathfrak{n}_0), V_\lambda)) \to \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^1(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_\mathbb{C}(\wedge^0(\mathfrak{n}_0), V_\lambda))\right)$$

with differential given by multiplication by $\mathrm{d}\pi(H) + \mathrm{d}\lambda(H)$. The cohomology of this complex is trivial unless $\omega_\pi^{-1} = \lambda$. In the case $q = 1$ we consider the complex

$$\left(\mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^0(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_\mathbb{C}(\wedge^1(\mathfrak{n}_0), V_\lambda)) \to \mathrm{Hom}_{K_\infty'^{M_0}}(\wedge^1(\mathfrak{m}_0/\mathfrak{k}'), \mathrm{Hom}_\mathbb{C}(\wedge^1(\mathfrak{n}_0), V_\lambda))\right)$$

with differential given by multiplication by $\mathrm{d}\pi(H) + \mathrm{d}(w_0 \cdot \lambda)(H)$. The cohomology of this complex is trivial unless $\omega_\pi^{-1} = w_0 \cdot \lambda$.

Proposition 3.2.3. Let $\pi = \chi_1 \boxtimes \chi_2$ be a character of $M_0(\mathbb{R})$. If there exists an element $w \in W$ such that $\omega_\pi^{-1} = w \cdot \lambda$ for some analytically integral dominant element $\lambda \in \mathfrak{h}^\vee$, then we have an isomorphism

$$H^\bullet(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda) \simeq H^{\ell(w)}(\mathfrak{n}_0, V_\lambda) \otimes \wedge^\bullet(\mathfrak{m}_0/\mathfrak{k}')^\vee$$

If no such element $w \in W$ exists, then we have $H^\bullet(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) \otimes V_\lambda) = 0$.

Proposition 3.2.4. (i) For $\lambda = \lambda_1\omega + c\det$ dominant, we have

$$H^i(\mathfrak{g}, K'_\infty; V_\lambda \otimes V_{\lambda^\vee}) = H^i(\mathfrak{g}, K'_\infty; \mathbb{C}) = \begin{cases} \mathbb{C} & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

(ii) For $\lambda = \lambda_1\omega + c\det$ dominant, we have

$$H^i(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) = \begin{cases} \mathbb{C}^2 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We have the following:

(i) By Wigner's lemma we have $H^i(\mathfrak{g}, K'_\infty; V_\lambda) = 0$ unless $n = 0$, that is unless V_λ is 1-dimensional, in which case by Clebsch-Gordon we have

$$H^i(\mathfrak{g}, K'_\infty; V_\lambda) = H^i(\mathfrak{g}, K'_\infty; \mathbb{C}) = \begin{cases} \mathbb{C} & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

The claim follows since for $\lambda = \lambda_1\omega + c\det$ dominant $V_\lambda \otimes V_{\lambda^\vee}$ has a single 1-dimensional summand.

(ii) We tensor the short exact sequence $0 \to V_\lambda \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \to \mathcal{D}_{\lambda+1} \to 0$ with $V_{\lambda^\vee}$ to obtain a short exact sequence

$$0 \to V_\lambda \otimes V_{\lambda^\vee} \to \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \otimes V_{\lambda^\vee} \to \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee} \to 0$$

and consider the corresponding long exact sequence in $(\mathfrak{g}, K'_\infty)$-cohomology

$$\begin{aligned}
0 \to \underbrace{H^0(\mathfrak{g}, K'_\infty; V_\lambda \otimes V_{\lambda^\vee})}_{=\mathbb{C}} &\to \underbrace{\cancel{H^0(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \otimes V_{\lambda^\vee})}}_{=0} \to H^0(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) \\
\xrightarrow{\delta} \underbrace{\cancel{H^1(\mathfrak{g}, K'_\infty; V_\lambda \otimes V_{\lambda^\vee})}}_{=0} &\to \underbrace{H^1(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \otimes V_{\lambda^\vee})}_{=\mathbb{C}} \to H^1(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) \\
\xrightarrow{\delta} \underbrace{H^2(\mathfrak{g}, K'_\infty; V_\lambda \otimes V_{\lambda^\vee})}_{=\mathbb{C}} &\to \underbrace{\cancel{H^2(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \otimes V_{\lambda^\vee})}}_{=0} \to H^2(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) \to 0
\end{aligned}$$

so it follows that $H^0(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) = H^2(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) = 0$ and we have a short exact sequence

$$0 \to \underbrace{H^1(\mathfrak{g}, K'_\infty; \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\lambda^{w_0}) \otimes V_{\lambda^\vee})}_{=\mathbb{C}} \to H^1(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1} \otimes V_{\lambda^\vee}) \to \underbrace{H^2(\mathfrak{g}, K'_\infty; V_\lambda \otimes V_{\lambda^\vee})}_{=\mathbb{C}} \to 0$$

so it follows that $H^1(\mathfrak{g}, K'_\infty; \mathcal{D}_\lambda \otimes V_{\lambda^\vee}) \simeq \mathbb{C}^2$. $\square$

Note that the second claim can be seen more directly: in this situation the $(\mathfrak{g}, K'_\infty)$-cohomology complex $\mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \mathcal{D}_{\lambda+1} \otimes V_\lambda)$ simplifies to

$$\mathrm{Hom}_{K'_\infty}(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}'), \mathcal{D}_{\lambda+1} \otimes V_\lambda) = \Big(0 \to \mathrm{Hom}_{K'_\infty}(\wedge^1(\mathfrak{g}/\mathfrak{k}'), \mathcal{D}_{\lambda+1} \otimes V_\lambda) \to 0\Big)$$

and $\mathrm{Hom}_{K'_\infty}(\wedge^1(\mathfrak{g}/\mathfrak{k}'), \mathcal{D}_{\lambda+1} \otimes V_\lambda) \simeq \mathbb{C}^2$. Recalling that the corresponding $(\mathfrak{sl}_2, K_\infty)$-module $\mathcal{D}_{\lambda+1}$ decomposes as a direct sum $\mathcal{D}_{\lambda+1} = \mathcal{D}^-_{\lambda+1} \oplus \mathcal{D}^+_{\lambda+1}$, we have

$$H^i(\mathfrak{sl}_2, K_\infty; \mathcal{D}^\pm_\lambda \otimes V_{\lambda^\vee}) = \begin{cases} \mathbb{C} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Vogan-Zuckerman Classification Having constructed the discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ and computed their cohomology, we quickly recall how this is commensurate with the Vogan-Zuckerman classification of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{R})$ with nonzero cohomology.

Let T^1_{c} be a maximal torus in the maximal compact subgroup $K_\infty \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$ of $G^1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ so that the centralizer T_{c} of T^1_{c} is a maximal torus in $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$. Let $\mu : \mathbb{G}_{\mathrm{m}} \to T^1_{\mathrm{c}}$ be a cocharacter defined over $\mathbb{R}$, let $Z_\mu \subseteq G(\mathbb{R})$ be the corresponding centralizer, and let $Q_\mu \subseteq G(\mathbb{C})$ be the corresponding parabolic subgroup with Lie $\mathbb{C}$-algebra $\mathfrak{q}$. Let $\lambda \in X^*(T_{\mathbb{C}})$ be a highest weight for $G(\mathbb{C})$ which is det-self-dual, and suppose that the highest weight λ is trivial on the semisimple part Z^1_μ of $Z_\mu \subseteq G(\mathbb{R})$, that is λ extends to a character $\lambda : Q_\mu \to \mathbb{C}^\times$. We have two cases depending on μ, in each case obtaining a constraint on λ:

- (μ regular) We have $Z_\mu = M_0$ and $Q_\mu = P_0$ with Lie $\mathbb{C}$-algebra $\mathfrak{q} = \mathfrak{p}_0$ and $A_{\mathfrak{q}}(\lambda)$ is a tempered $(\mathfrak{g}, K'_\infty)$-module. The set of regular cocharacters $X_*(T^1)^0_{\mathbb{R}}$ is the complement of finitely many root hyperplanes and is a disjoint union of connected components parameterized by the Weyl group W. The representations $A_{\mathfrak{q}}(\lambda)$

are locally constant in $\chi \in X_*(T^1)^0_{\mathbb{R}}$, and after choosing a basepoint $[\mu_0] \in \pi_0(X_*(T^1)^0_{\mathbb{R}})$ we obtain a family of representations $\{A_{w\mu_0}(\lambda)\}_{w\in W}$ which by Vogan-Zuckerman are pairwise non-isomorphic and are the Harish-Chandra modules corresponding to discrete series representations of $G(\mathbb{R})$. Here we have no constraint on λ and one possibility:

(i) $A_{\mathfrak{q}}(\lambda) = \mathcal{D}_{\lambda+1}$ is the holomorphic discrete series representation (underlying holomorphic modular forms of weight $\lambda+2$) where $H^1(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1}\otimes V_\lambda)$ is 2-dimensional with Hodge numbers $(\lambda+1, 0)$ and $(0, \lambda+1)$:

$$H^1(\mathfrak{g}, K'_\infty; \mathcal{D}_{\lambda+1}\otimes V_\lambda) = \begin{array}{c} H^1(\mathfrak{sl}_2, K_\infty; \mathcal{D}_{\lambda+1}\otimes V_\lambda) \\ \oplus H^1(\mathfrak{sl}_2, K_\infty; \mathcal{D}_{-\lambda-1}\otimes V_\lambda) \end{array} = \mathbb{C}\oplus\mathbb{C}$$

- (μ trivial) We have $Z_\mu = G$ and $Q_\mu = G$ with Lie $\mathbb{C}$-algebra $\mathfrak{q} = \mathfrak{g}$. Here we must have $\lambda = 0$ and one possibility:

(ii) $A_{\mathfrak{q}}(0) = \chi_\infty \in \{1, \mathrm{sign}\}$ is a character where $H^0(\mathfrak{g}, K'_\infty; \mathbb{C})$ and $H^2(\mathfrak{g}, K'_\infty; \mathbb{C})$, are 1-dimensional with Hodge numbers $(0, 0)$ and $(1, 1)$ respectively:

$$H^0(\mathfrak{g}, K'_\infty; \chi_\infty\otimes V_\lambda) = H^0(\mathfrak{sl}_2, K_\infty; \chi_\infty\otimes V_\lambda) = \mathbb{C}$$
$$H^2(\mathfrak{g}, K'_\infty; \chi_\infty\otimes V_\lambda) = H^2(\mathfrak{sl}_2, K_\infty; \chi_\infty\otimes V_\lambda) = \mathbb{C}$$

In particular, the irreducible admissible representations of $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$ with nonzero $(\mathfrak{g}, K'_\infty)$-cohomology are either discrete series representations (with cohomology concentrated in middle degree), or 1-dimensional representations (which are Langlands quotients associated to the minimal parabolic $\mathbb{Q}$-subgroup of G) which have cohomology as far from middle degree as possible.

Finally we summarize the constraints on the possible Hodge numbers appearing in intersection cohomology provided by the above classification:

- ($\lambda > 0$) The only Hodge numbers appearing in $IH^i(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ are given by

H^1	$(\lambda{+}1,0)$	$\cdots$	$(0,\lambda{+}1)$

- ($\lambda = 0$) The only Hodge numbers appearing in $IH^i(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ are given by

H^2	(1,1)
H^1	(1,0) (0,1)
H^0	(0,0)

In particular for $\lambda > 1$ the cohomology must be concentrated in middle degree (and in this case intersection cohomology, cuspidal cohomology, and inner cohomology will all agree), while for $\lambda = 0$ one simply finds the expected Hodge diamond of a smooth projective curve.

3.3 Eisenstein Cohomology for GL_2

The goal of this section is to compute the Eisenstein cohomology of local systems on modular curves. To that end, we review the structure of automorphic Eisenstein series for GL_2, and describe the Franke-Schwermer filtration on spaces of automorphic forms for GL_2 in terms of the poles of such automorphic Eisenstein series, and then compute the relevant $(\mathfrak{g}, K'_\infty)$-cohomology.

3.3.1 Eisenstein Series for GL_2

We review the spectral decomposition of $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ for $G = \mathrm{GL}_2$: the continuous spectrum $L^2_{\mathrm{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is described in terms of automorphic Eisenstein series, while the residual spectrum $L^2_{\mathrm{res}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is described in terms of the poles of such automorphic Eisenstein series.

Intertwining Operators and Normalization Factors To compute the Langlands normalization factor for the minimal parabolic $\mathbb{Q}$-subgroup we simply apply the Gindikin-Karepelevich formula:

Proposition 3.3.1. Let $\pi : M_0(\mathbb{Q})A_{P_0}(\mathbb{R})^\circ \setminus M_0(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}^\times$ be the character given by $\pi = \chi_1 \boxtimes \chi_2$ for unitary Hecke characters $\chi_1, \chi_2 : \mathbb{Q}^\times\mathbb{R}_{>0} \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ with $\chi_1\chi_2 = \omega$. For the simple reflection w_0 we have the Langlands normalization factor

$$r(w_0, \pi_s) = \frac{\Lambda(s, \chi_1^2\omega^{-1})}{\Lambda(s+1, \chi_1^2\omega^{-1})\epsilon(s, \chi_1^2\omega^{-1})}$$

Proof. Following the first convention, the coroot $\alpha^\vee = f_1 - f_2$ satisfies $\langle s, \alpha^\vee \rangle = s_1 - s_2 = s$ and $\pi \circ \alpha^\vee = \chi_1 \chi_2^{-1} = \chi_1^2 \omega^{-1}$, so by the Gindikin-Karepelevich formula we have

$$\begin{aligned} r(w_0, \pi_s) &= \frac{\Lambda(\langle s, \alpha^\vee \rangle, \pi \circ \alpha^\vee)}{\Lambda(\langle s, \alpha^\vee \rangle + 1, \pi \circ \alpha^\vee) \epsilon(\langle s, \alpha^\vee \rangle, \pi \circ \alpha^\vee)} \\ &= \frac{\Lambda(s_1 - s_2, \chi_1 \chi_2^{-1})}{\Lambda(s_1 - s_2 + 1, \chi_1 \chi_2^{-1}) \epsilon(s_1 - s_2, \chi_1 \chi_2^{-1})} \\ &= \frac{\Lambda(s, \chi_1^2 \omega^{-1})}{\Lambda(s+1, \chi_1^2 \omega^{-1}) \epsilon(s, \chi_1^2 \omega^{-1})} \end{aligned}$$

□

Now recall from the discussion of normalized intertwining operators that the poles of $M(w_0, \pi_s)$ in the region $\Re(s) \geq 0$ are exactly the poles of the Langlands normalization factor $r(w_0, \pi_s)$ in the same region. We obtain the following:

Proposition 3.3.2. We have the following singularity of $M(w_0, \pi_s)$ intersecting the positive closed Weyl chamber $\overline{(\mathfrak{a}_{P_0}^G)^{\vee +}}$: for $\pi = \chi_1 \boxtimes \chi_2 = \chi \boxtimes \chi_1^{-1} \omega$, if $\chi_1 = \chi_2$ (that is if $\chi_1^2 = \omega$) we have the singularity

$$\mathfrak{S}_\omega = \chi_1 \nu^{1/2} \boxtimes \chi_1 \nu^{-1/2}$$

Proof. In the region $\Re(s) \geq 0$ the Langlands normalization factor $r(w_0, \pi_s) = \frac{\Lambda(s, \chi_1^2 \omega^{-1})}{\Lambda(s+1, \chi_1^2 \omega^{-1}) \epsilon(s, \chi_1^2 \omega^{-1})}$ has a pole precisely if the numerator $\Lambda(s, \chi^2 \omega^{-1})$ has a pole, which can only happen if $\chi_1^2 = \omega$ in which case we have a simple pole at $s = 1$, corresponding to the point $(s_1, s_2) = (\frac{1}{2}, -\frac{1}{2})$. □

Residues of Eisenstein Series and the Residual Spectrum for GL_2 We now need to compute the residue of the automorphic Eisenstein series $\mathrm{Eis}_{P_0}^G(\phi_s)$, taking into account the possible singularities intersecting the positive closed Weyl chamber $\overline{(\mathfrak{a}_{P_0}^G)^{\vee +}}$. To that end we compute the L^2-inner products of Poincare series by moving contours to the unitary axis $s \in i(\mathfrak{a}_{P_0}^G)^\vee$:

Proposition 3.3.3. For $\phi \in W_{P_0, \pi}$ and $\phi' \in \bigoplus_{(P', \pi') \in [P_0, \pi]} W_{P', \pi'}$ the L^2-inner product $\langle \theta_\phi, \theta_{\phi'} \rangle$ is given by

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi, \phi')(\pi_s) \mathrm{d}s + c \langle N(w_0, \mathfrak{S}_\chi) \phi_1, \phi_1' \rangle$$

where c is a nonzero constant.

Proof. Let $\phi \in W_{P_0,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_0,\pi]} W_{P',\pi'}$ where $\pi = \chi_1 \boxtimes \chi_2$ as before. Fixing $s_0 \in (\mathfrak{a}_{P_0}^G)^\vee$ sufficiently positive we want to compute $\int_{s_0+i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s$. To that end we consider the contour integral $\lim_{t\to\infty} \oint_{C_t} A(\phi,\phi')(\pi_s)\mathrm{d}s$, where the contour integrals at infinity vanish by estimates on the intertwining operators $M(w,\pi_s)$ and by the rapid decay of ϕ and ϕ' in $\Im(s)$ since they are Paley-Wiener. It follows by the residue theorem that

$$\int_{s_0+i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s = \mathrm{Res}_{\mathfrak{S}_\chi} A(\phi,\phi')(\pi_s)$$

Now recall that the Langlands normalization factor

$$r(w_0,\pi_s) = \frac{\Lambda(s,\chi_1^2\omega^{-1})}{\Lambda(s+1,\chi_1^2\omega^{-1})\epsilon(s,\chi_1^2\omega^{-1})}$$

has a pole at $s=1$ if $\chi_1 = \chi_2$ where it is given

$$r(w_0,\pi_s) = \frac{Z(s)}{Z(s+1)}$$

Then we have the residue

$$\begin{aligned}
\mathrm{Res}_{\mathfrak{S}_\chi} A(\phi,\phi')(\pi_s) &= \mathrm{Res}_{s=1} r(w_0,\pi_s)\langle N(w_0,\mathfrak{S}_\chi)\phi_s,\phi'_s\rangle \\
&= \mathrm{Res}_{s=1}\frac{Z(s)}{Z(s+1)}\langle N(w_0,\mathfrak{S}_\chi)\phi_s,\phi'_s\rangle \\
&= c\langle N(w_0,\mathfrak{S}_\chi)\phi_1,\phi'_1\rangle
\end{aligned}$$

where $c = \mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{6}{\pi}$, so it follows that the L^2-inner product is given

$$\begin{aligned}
\langle \theta_\phi,\theta_{\phi'}\rangle &= \int_{s_0+i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s \\
&= \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s + c\langle N(w_0,\mathfrak{S}_\chi)\phi_1,\phi'_1\rangle
\end{aligned}$$

□

The same computation applies when $\mathbb{Q}$ is replaced by any number field F, in which case the nonzero constant c instead involves $\mathrm{Res}_{s=1}\frac{Z_F(s)}{Z_F(2)}$. We obtain the following corollary:

Corollary 3.3.4. The Eisenstein series and their residues for $G = \mathrm{GL}_2$ are given as follows: we have the Eisenstein series

$$\mathrm{Eis}_{P_0}^G(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \boxtimes \chi_1^{-1}\omega\nu^{-s})) \qquad s \in i(\mathfrak{a}_{P_0}^G)^\vee$$

and for $\chi_1^2 = \omega$ we have a pole at $o = \chi_1\nu^{1/2} \boxtimes \chi_1\nu^{-1/2}$ (that is at $s = 0$) with

$$\langle \mathrm{Res}_{s=0}\mathrm{Eis}_{P_0}^G(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \boxtimes \chi_1\nu^{-s}))\rangle \simeq \chi_1 \circ \det$$

It follows that we have a decomposition

$$L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \oplus L^2_{\mathrm{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$$

where the continuous spectrum is given

$$L^2_{\mathrm{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = \bigoplus_{\omega}\bigoplus_{\chi_1}\int_{i(\mathfrak{a}_{P_0}^G)^\vee}^{\oplus} \mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \boxtimes \chi_1^{-1}\omega\nu^{-s})\mathrm{d}s$$

We have a further decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = L^2_{\mathrm{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \oplus L^2_{\mathrm{res}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$$

where the residual spectrum is given

$$L^2_{\mathrm{res}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = \bigoplus_{\omega}\bigoplus_{\substack{\chi \\ \chi^2=\omega}} \chi \circ \det$$

where the outer direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and the inner direct sum is taken over Hecke characters $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ with $\chi^2 = \omega$.

3.3.2 Eisenstein Cohomology for GL_2

In this section we compute the Eisenstein cohomology as an $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module. First, we recall some facts about the poles of Eisenstein series, and the evaluation points and infinitesimal characters which will enter into the description of the Franke filtration for the Borel parabolic subgroup.

We begin by restating the results of the previous section on the location of poles of Eisenstein series:

Proposition 3.3.5. The automorphic Eisenstein series $\mathrm{Eis}_{P_0}^G(\phi_s)$ attached to a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ has a pole at $s = \nu \in \overline{(\mathfrak{a}_{P_0}^G)^{\vee +}}$ precisely if $\nu = \rho_{P_0}$ and $\chi_1 = 1$ in which case we have a simple pole at $s = 1$ and the space spanned by the residues $\mathrm{Res}_{s=1}\mathrm{Eis}_{P_1}^G(\phi_s)$ is isomorphic to the 1-dimensional representation $\chi \circ \det$.

We now record the following result on infinitesimal characters coming from the action of the Weyl group:

Proposition 3.3.6. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$ and let $\mathcal{J}_\lambda$ be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ annihilating $V_\lambda^\vee$. Let $\varphi = \{\varphi_P\}_{P \in [P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then the infinitesimal character $\xi \in \mathfrak{a}_{P'}^{\vee\perp}$ and the corresponding $s_0 \in \mathfrak{a}_{P'}^\vee$ such that $s_0 + \xi$ is annihilated by $\mathcal{J}_\lambda$ are given by:

- For $P' = P_0$ we have s_0 in the Weyl orbit of $\lambda + \rho_{P_0} = (\lambda_1 + 1; 0)$ and $\xi = 0$.
- For $P' = G$ we have $s_0 = 0$ and ξ is in the Weyl orbit of $s + \rho_{P_0}$.

The Franke-Schwermer Filtration We now describe the structure of the Franke-Schwermer filtration. As expected, the bottom piece is given by the Langlands quotient of normalized Borel parabolic induction, and the top piece is given by normalized Borel parabolic induction.

Proposition 3.3.7. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$. Let $\varphi = \{\varphi_P\}_{P \in [P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then the Franke-Schwermer filtration on $\mathcal{A}_{\lambda,[P_0],\varphi}(G)$ is given by

$$\mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \subseteq \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) = \mathcal{A}_{\lambda,[P_0],\varphi}(G)$$

where $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial precisely if $\lambda_1 = 0$ and $\chi_1 = 1$, in which case we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \simeq \chi \circ \det$$

In any case have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)/\mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \simeq \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_0(\mathbb{A}_\mathbb{Q})}(\chi_1 \otimes \chi, \lambda_1 + 1) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_0})^\vee_\mathbb{C})$$

Proof. Let $\mathcal{M}_{\lambda,[P_0],\varphi}(G)$ be the set of tuples (P', π', ν, s_0) where $P' \in [P_0]$ is a standard parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P' = M'N'$ containing an element of the associate class $[P_0]$, where π' is a discrete spectrum automorphic representation of $M'(\mathbb{A}_\mathbb{Q})$ with cuspidal support π obtained as the residue at $\nu \in (\mathfrak{a}^{P'}_{P_0})^\vee_\mathbb{C}$ of the Eisenstein series attached to $\pi \in \varphi_{P_0}$, and where $s_0 \in \mathfrak{a}^\vee_{P',\mathbb{C}}$ is a point such that $\Re(s_0) \in \overline{(\mathfrak{a}^G_{P'})^{\vee+}}$ is such that $s_0 + \nu + \xi$ is annihilated by $\mathcal{J}_\lambda$. For $m \in \mathbb{Z}$ let $\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)$ be the subset of tuples (P', π', ν, s_0) such that $T(s_0) = m$, where $T : \overline{\mathfrak{a}^{\vee+}_{P'}} \to \mathbb{Z}$ will be fixed at the end of the proof. Then we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K'_\infty)$-modules

$$\mathcal{A}^m_{\lambda,[P_0],\varphi}(G)/\mathcal{A}^{m+1}_{\lambda,[P_0],\varphi}(G) \simeq \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)} \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P'(\mathbb{A}_\mathbb{Q})}(\pi', s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_0})^\vee_\mathbb{C})$$

Now we have the following:

- For $P' = P_0$ we have $\pi' = \chi_1 \boxtimes \chi$ hence $\nu = 0 \in (\mathfrak{a}^{P_0}_{P_0})^\vee_\mathbb{C} = 0$. By 3.3.6 such ν can only be obtained for $s_0 = \pm(\lambda_1 + 1)$ and $\xi = 0$. It follows that

$$\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)_{P_0} = \begin{cases} (P_0, \chi_1 \otimes \chi, 0, s_0) \\ \qquad \text{if } T(\lambda) = m \text{ and } s_0 = \lambda_1 + 1 \text{ and } \xi = 0 \\ 0 \qquad \text{otherwise} \end{cases}$$

- For $P' = G$ since π' is a residual representation of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$, by 3.3.5 we have $\pi' = \chi \circ \det$ hence $\chi_1 = 1$ and $\xi = \rho_{P_0}$. By 3.3.6 such ξ can only be obtained for $\lambda_1 = 0$ and $s_0 = 0$. It follows that

$$\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)_G = \begin{cases} (G, \chi \circ \det, \rho_{P_0}, (0,0)) \\ \qquad \text{if } T(0) = m \text{ and } \lambda_1 = 0 \text{ and } \chi_1 = 1 \\ 0 \qquad \text{otherwise} \end{cases}$$

Now the result follows by taking the filtration defined by $T(0) = 1$ and $T(\lambda+1) = 0$. □

Cohomology of Franke-Schwermer Filtration Recall that the Levi quotient $M_0(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$ admits a decomposition $M_0(\mathbb{R}) = M_0^{\mathrm{ss}}(\mathbb{R}) \times A_{P_0}(\mathbb{R})^\circ$ where $M_0^{\mathrm{ss}}(\mathbb{R}) = \{\pm 1\} \times \{\pm 1\}$ is semisimple with Lie algebra $\mathfrak{m}_0^{\mathrm{ss}} = 0$ and $A_{P_0}(\mathbb{R})^\circ = \mathbb{R}^2_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_0} with Lie algebra $\mathfrak{a}_{P_0} = \mathbb{R}^2$. Recalling that $K'_\infty = A_G(\mathbb{R})^\circ K_\infty \simeq \mathbb{R}_{>0}\mathrm{SO}(2)$, for $K'^{M_0}_\infty$ the image of $K'^{P_0}_\infty$ under the canonical projection $P_0(\mathbb{R}) \to M_0(\mathbb{R})$ we have $K'^{M_0}_\infty = \mathbb{R}^\times$, and for $K'^{M_0^{\mathrm{ss}}}_\infty$ the image of $K'^{P_0}_\infty$ under the canonical projection $P_0(\mathbb{R}) \to M_0^{\mathrm{ss}}(\mathbb{R})$ we have $K'^{M_0}_\infty = \{\pm 1\}$.

Proposition 3.3.8. For $\epsilon_1, \epsilon \in \{0,1\}$ we have

$$H^q(\mathfrak{m}_0^{\mathrm{ss}}, K'^{M_0^{\mathrm{ss}}}_\infty; \pi_\infty \otimes (\mathrm{sign}^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon})) \simeq \begin{cases} \mathbb{C} & q = 0, \pi_\infty \simeq \mathrm{sign}^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon} \\ 0 & \text{otherwise} \end{cases}$$

Proof. For $G = \mathrm{SL}_2$ recall that the Levi quotient $M_0(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R})$ admits a decomposition $M_0(\mathbb{R}) = M_0^{\mathrm{ss}}(\mathbb{R}) \times A_{P_0}(\mathbb{R})^\circ$ where $M_0^{\mathrm{ss}}(\mathbb{R}) = \{\pm 1\}$ is semisimple with Lie algebra $\mathfrak{m}_0^{\mathrm{ss}} = 0$ and $A_{P_0}(\mathbb{R})^\circ = \mathbb{R}_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_0} with Lie algebra $\mathfrak{a}_{P_0} = \mathbb{R}$. Recalling that $K_\infty \simeq \mathrm{SO}(2)$, for $K^{M_0}_\infty$ the image of $K^{P_0}_\infty$ under the canonical projection $P_0(\mathbb{R}) \to M_0(\mathbb{R})$ we have $K^{M_0}_\infty = \{\pm 1\}$, and for $K^{M_0^{\mathrm{ss}}}_\infty$ the image of $K^{P_0}_\infty$ under the canonical projection $P_0(\mathbb{R}) \to M_0^{\mathrm{ss}}(\mathbb{R})$ we have $K^{M_0}_\infty = \{\pm 1\}$. For $\epsilon_1 \in \{0,1\}$ we have

$$H^q(\mathfrak{m}_0^{\mathrm{ss}}, K^{M_0^{\mathrm{ss}}}_\infty; \pi_\infty \otimes \mathrm{sign}^{\epsilon_1}) \simeq \begin{cases} \mathbb{C} & q = 0, \pi_\infty \simeq \mathrm{sign}^{\epsilon_1} \\ 0 & \text{otherwise} \end{cases}$$

The result follows from this, noting that the $(\mathfrak{m}_0^{\mathrm{ss}}, K_\infty'^{M_0^{\mathrm{ss}}})$-cohomology is independent of the character sign^ϵ on the second factor $\{\pm 1\}$ of $M_0^{\mathrm{ss}}(\mathbb{R})$, as the factor $\mathbb{R}_{>0}$ of K'_∞ intersects this factor only at the identity. $\square$

Now there are two pieces of the Franke-Schwermer filtration whose $(\mathfrak{g}, K'_\infty)$-cohomology we need to compute: we need to compute $H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)$ as well as $H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))$ in the case where $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial.

Proposition 3.3.9. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$. Let $\varphi = \{\varphi_P\}_{P \in [P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) = \begin{cases} \mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \lambda_1 + 1) & q = 1, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial, that is precisely if $\lambda_1 = 0$ and $\chi_1 = 1$, then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = \begin{cases} \chi_{\mathrm{fin}} \circ \det & q = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the first claim we have

$$\begin{aligned} &H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \\ &\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, s_0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)^\vee_\mathbb{C}) \otimes V_\lambda) \\ &\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)^\vee_\mathbb{C}) \otimes V_\lambda) \otimes \mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, s_0) \end{aligned}$$

By Borel-Wallach [20, III Theorem 3.3] there exists a unique $w \in W$ such that $\pi_\infty \otimes \mathbb{C}_{s_0 + \rho_{P_0}}$ has nonzero $(\mathfrak{m}_0, K_\infty'^{M_0})$-cohomology with respect to $\mathrm{Sym}((\mathfrak{a}_{P_0}^G)^\vee_\mathbb{C}) \otimes V_{w \cdot \lambda}$: we have $w = w_0$ in the case $s_0 = \lambda_1 + 1$ and

$\xi = 0$. Now recalling that $M_0(\mathbb{R}) = M_0^{\text{ss}}(\mathbb{R}) \times A_{P_0}(\mathbb{R})^\circ$ and $\mathfrak{m}_0 = \mathfrak{m}_0^{\text{ss}} \oplus \mathfrak{a}_{P_0}$, by 3.3.8 we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty, \lambda_1 + 1) \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)^\vee_{\mathbb{C}}) \otimes V_\lambda) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_0, K'^{M_0}_\infty; \pi_\infty \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)^\vee_{\mathbb{C}}) \otimes \mathbb{C}_{\lambda + 2\rho_{P_0}} \otimes V_{w\cdot\lambda}) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_0^{\text{ss}}, K'^{M_0^{\text{ss}}}_\infty; \pi_\infty \otimes V_{w\cdot\lambda}) \\
&\simeq \begin{cases} \mathbb{C} & q = 1, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

For the second claim suppose that $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial, so that we are in the case $\lambda = 0, \xi = 1$, and there exists a section ϕ of the normalized parabolic induction $\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, s)$ such that the automorphic Eisenstein series $\mathrm{Eis}_{P_0}^G(\phi_s)$ has pole at $s = s_0 = 0$. Then we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \chi \circ \det) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \chi_\infty \circ \det) \otimes \chi_{\text{fin}} \circ \det
\end{aligned}$$

Now by Borel-Wallach the $(\mathfrak{g}, K'_\infty)$-cohomology of the 1-dimensional representation $\chi_\infty \circ \det$ is given

$$H^q(\mathfrak{g}, K'_\infty; \chi_\infty \circ \mathrm{sim}) = \begin{cases} \mathbb{C} & q = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

The result follows. $\square$

Eisenstein Cohomology Having computed the $(\mathfrak{g}, K'_\infty)$-cohomology of the pieces of the Franke-Schwermer filtration, we are now in the position to compute Eisenstein cohomology. Up to indeterminacies regarding the behavior of certain connecting morphisms in the case where $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial, we have the following result:

Theorem 3.3.10. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$. Let $\varphi = \{\varphi_P\}_{P\in[P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$.

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is trivial and if $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$ then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}_\mathbb{Q}^\infty)}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \lambda_1 + 1) & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial ($\lambda_1 = 0$ and $\chi_1 = 1$), then (with assumption the 3.3.11 on connecting morphisms) we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G)) \simeq \begin{cases} \chi_{\mathrm{fin}} \circ \det & q = 0 \\ \mathcal{K}(\chi) & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}(\chi)$ is the $G(\mathbb{A}_\mathbb{Q}^\infty)$-module

$$\mathcal{K}(\chi) \simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}_\mathbb{Q}^\infty)}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}(1 \otimes \chi_{\mathrm{fin}}, 1) \to \chi_{\mathrm{fin}} \circ \det\Big)$$

Otherwise, $H^*(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = 0$.

Proof. By definition $H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$, and if $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is trivial we clearly have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)$$

If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$ then by 3.3.9 we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) & q = 1 \\ 0 \quad \text{otherwise} \end{cases}$$
$$\simeq \begin{cases} \mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda_1 + 1) & q = 1 \\ 0 \quad \text{otherwise} \end{cases}$$

If $\chi_{1,\infty} \neq \mathrm{sign}^{\lambda}$ we have $H^*(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = 0$.

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial consider the short exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$0 \to \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \to \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \to \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \to 0$$

which by 3.3.9 gives rise to a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned}
0 &\to \underbrace{H^0(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \det} \to H^0(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^0(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^1(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^1(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{H^1(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}_{\simeq \mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}}, \lambda_1 + 1)} \\
&\to \underbrace{H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \det} \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^2(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \to 0
\end{aligned}$$

Now we make the following assumption on connecting morphisms:

Assumption 3.3.11. If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial as above, then connecting morphism

$$H^1(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}) \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))$$

is surjective, so the morphism $H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G))$ is zero.

Granting this, it follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \simeq \begin{cases} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) & q = 0 \\ \mathcal{K}(\chi) & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\simeq \begin{cases} \chi_{\text{fin}} \circ \det & q = 0 \\ \mathcal{K}(\chi) & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\begin{aligned} \mathcal{K}(\chi) &\simeq \ker\Big(H^1(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}) \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))\Big) \\ &\simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\text{fin}}, 1) \to \chi_{\text{fin}} \circ \det\Big) \end{aligned}$$

The result follows. $\square$

The behavior of the connecting morphism $H^1(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}) \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))$ is subtle; its behavior can be determined in this case by computing the behavior of certain modular symbols.

The $\mathcal{H}_{K_{\text{fin}}}$-modules in the above theorem will be paired with 1-dimensional Gal-modules. For an integer $n \in \mathbb{Z}$ and for $\chi = \chi_{\text{fin}} \otimes \chi_\infty$ a (finite order) character of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$ let $\mathbb{L}^n_\chi = \rho_\chi(-n)$ be the 1-dimensional ℓ-adic Gal-module attached to χ twisted by the n-th power of the ℓ-adic cyclotomic character, with

$$\mathrm{tr}(\mathrm{Frob}_p^j | \mathbb{L}^n_\chi) = p^{nj} c(\chi_p)^j = p^{nj} \chi(p)^j$$

Now we have the following result, which is conditional on the assumption 3.3.11 in the case $\lambda_1 = 0$:

Theorem 3.3.12. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$. Then (with the assumption 3.3.11 on connecting morphisms in the case $\lambda_1 = 0$) the Eisenstein cohomology $H^*_{\text{Eis}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees

$0, 1$ and given as an $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$-module by

$$H^0_{\text{Eis},[P_0]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus\limits_{\chi} (\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes \mathbb{L}^0_\chi & \lambda_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^1_{\text{Eis},[P_0]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus\limits_{\chi} \mathcal{K}(\chi)^{K_{\text{fin}}} \boxtimes \mathbb{L}^1_\chi & \\ \oplus \bigoplus\limits_{\substack{\chi_1,\chi \\ \chi_1 \neq 1 \\ \chi_{1,\infty}=1}} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\text{fin}} \otimes \chi_{\text{fin}}, 1)^{K_{\text{fin}}} \boxtimes \mathbb{L}^1_{\chi_1\chi} & \lambda_1 = 0 \\ \bigoplus\limits_{\substack{\chi_1,\chi \\ \chi_{1,\infty}=\text{sign}^{\lambda_1}}} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\text{fin}} \otimes \chi_{\text{fin}}, \lambda_1 + 1)^{K_{\text{fin}}} \boxtimes \mathbb{L}^{\lambda+1}_{\chi_1\chi} & \lambda_1 > 0 \end{cases}$$

where $\mathcal{K}(\chi)$ is given by

$$\mathcal{K}(\chi) = \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \chi_{\text{fin}}, 1) \to \chi_{\text{fin}} \circ \det\Big)$$

Proof. The result follows by taking the direct sum over associate classes of unitary (cuspidal) automorphic representations $\pi = \chi_1 \boxtimes \chi$ of $M_0(\mathbb{A}_\mathbb{Q}) = \text{GL}_1(\mathbb{A}_\mathbb{Q}) \times \text{GL}_1(\mathbb{A}_\mathbb{Q})$ of the contributions to 3.3.10. The Galois actions can be obtained from the parabolic terms in the GL_2 trace formula.

By Pink the Tate twists are given as follows. For $\lambda = n\omega$ and $d(\lambda) = \frac{n}{2}$ the Galois action must be twisted by $\mathbb{L}^{d(\lambda)-\langle\mu, w\cdot\lambda\rangle}$ where $\mu : \text{GL}_1 \to \text{GL}_2$ is the cocharacter given by $t \mapsto \text{diag}(t, 1)$. Since $\omega(\text{diag}(t, 1)) = t^{1/2}$ we have $\langle\mu, w \cdot \lambda\rangle = \frac{n^w}{2}$ for $w \in W$ and we obtain the following Tate twists:

w	n^w	$\langle\mu, w \cdot \lambda\rangle$	$d(\lambda) - \langle\mu, w \cdot \lambda\rangle$
1	n	n	0
w_0	$-n-2$	$-n-1$	$n+1$

Recalling that n is the integer $\lambda_1 \geq 0$, this gives the Tate twists in the theorem. □

One can also state a Poincare dual version of the above theorem for compactly supported Eisenstein cohomology by modifying the above Tate twists.

3.4 Intersection and Cuspidal Cohomology for GL_2

The goal of this section is to compute the intersetion and cuspidal cohomology of local systems on modular curves. To that end, we briefely explain how the Langlands-Kottwitz method is used to justify the Galois action on ℓ-adic cohomology, and then compute the relevant $(\mathfrak{g}, K'_\infty)$-cohomology.

Langlands-Kottwitz Method Although there are simpler ways to explain the Galois action on the cohomology of modular curves, one method which generalizes nicely to higher dimensional Shimura varieties is the Langlands-Kottwitz method.

First recall that for $G = \mathrm{GL}_2$ over $\mathbb{Q}$, Arthur's conjectures describe (rather trivially in this case) a decomposition

$$L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_\omega \bigoplus_{\psi \in \Psi_{\text{disc}}(G,\omega)} \mathcal{A}_\psi$$

where each $\mathcal{A}_\psi$ is a near-equivalence class of discrete spectrum automorphic representations of $G(\mathbb{A}_\mathbb{Q})$, where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and the second direct sum is taken over a set $\Psi_{\text{disc}}(G, \omega)$ of equivalence classes of admissible discrete global A-parameters $\psi : L_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$; these are formal unorderd isobaric sums $\psi = \boxplus_i \mu_i \boxtimes \nu_{d_i}$ where μ_i is an ω-self dual unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_\mathbb{Q})$ and ν_{d_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d_i, and in this case we require $\sum_i n_i d_i = 2$. This gives two possible shapes of global A-parameters:

$$\mu \boxtimes 1 \qquad \chi \boxtimes \nu_2$$

Let $\Psi_{\text{disc,gen}}(G, \omega)$ and $\Psi_{\text{disc,1dim}}(G, \omega)$ be the corresponding sets of A-parameters. By strong multiplicity 1, for every $\psi \in \Psi_{\text{disc}}(G, \omega)$ the packet $\Pi(\psi) = \{\pi\}$ has size 1 and its unique member has multiplicity $m(\pi) = 1$. Moreover the member of this packet is simply the representation appearing in the A-parameter: for $\psi = \mu \boxtimes 1 \in \Psi_{\text{disc,gen}}(G, \omega)$ we have $\pi = \mu$, for $\psi = \chi \boxtimes \nu_2 \in \Psi_{\text{disc,1dim}}(G, \omega)$ we have $\pi = \chi \circ \det$. We obtain a decomposition

$$L^2_{\text{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_\omega \bigoplus_{\substack{\psi \in \Psi_{\text{disc,gen}}(G,\omega) \\ (\psi = \mu \boxtimes 1)}} \mu \oplus \bigoplus_\omega \bigoplus_{\substack{\psi \in \Psi_{\text{disc,1dim}}(G,\omega) \\ (\psi = \chi \boxtimes 1)}} \chi \circ \det = \bigoplus_\pi \pi \oplus \bigoplus_\chi \chi \circ \det$$

In this case the global A-parameters see the difference between the cuspidal automorphic representations π and the non-cuspidal 1-dimensional representations $\chi \circ \det$. We obtain a decomposition

$$L^2_{\mathrm{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_\omega \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc,gen}}(G,\omega) \\ (\psi=\mu\boxtimes 1)}} \mu = \bigoplus_\pi \pi$$

Let $K_{\mathrm{fin}} \subseteq G(\mathbb{A}^\infty_\mathbb{Q})$ be a compact open subgroup, and let p be a prime such that $K_p = G(\mathbb{Z}_p)$ is hyperspecial, so that $\mathcal{S}_{K_{\mathrm{fin}}}$ has good reduction at p. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. Consider the ℓ-adic intersection cohomology

$$IH^*_\lambda = \sum_{i \geq 0} (-1)^i [IH^i(\overline{\mathcal{S}}^{\mathrm{BB}}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_p}, \mathbb{V}_\lambda)]$$

as an element of the Grothendieck group of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-modules. The Langlands-Kottwitz method in this case says that for all $j \geq 1$ one has an equality

$$\mathrm{tr}(\mathrm{Frob}_p^j f^G | IH^*_\lambda) = \mathrm{STF}^G(f^G)$$

where $f^G \in C^\infty_{\mathrm{c}}(G(\mathbb{A}_\mathbb{Q}))$ is a certain explicit test function.

In this case the Arthur-Selberg trace formula is already stable: we have $\mathrm{STF}^G(f^G) = \mathrm{TF}^G(f^G)$. In view of the above decomposition, the stable trace formula $\mathrm{STF}^G(f^G)$ takes the form

$$\mathrm{STF}^G(f^G) = \sum_\pi \prod_v \mathrm{tr}(\pi_v(f^G_v)) + \sum_\chi \prod_v \mathrm{tr}(\chi_v \circ \det(f^G_v))$$

where π and χ are as in the description of the automorphic discrete spectrum, and where $f^G \in C^\infty_{\mathrm{c}}(G(\mathbb{A}_\mathbb{Q}))$ is a certain carefully chosen test function described by Kottwitz. Let us say a word about this test function, all too briefly.

For π_p an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ which is unramified, that is a subquotient of the normalized parabolic induction $\chi_{1,p} \times \chi_{2,p} = \mathcal{I}^{G(\mathbb{Q}_p)}_{P_0(\mathbb{Q}_p)}(\chi_{1,p} \boxtimes \chi_{2,p})$ for unramified characters $\chi_{1,p}, \chi_{2,p} : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, corresponding to a conjugacy class $c(\pi_p) \times \mathrm{Frob}_p$ in ${}^LG_p = \mathrm{GL}_2(\mathbb{C}) \rtimes \langle \mathrm{Frob}_p \rangle$, write $c(\pi_p) =$

$\mathrm{diag}(c_1(\pi_p), c_2(\pi_p))$ for the Satake parameters of π_p where $c_i(\pi_p) = \chi_{i,p}(p) \in \mathbb{C}^\times$. Consider the test function at p given by

$$f_p^G = 1_{K_{p^j}\mu(p^{-1})K_{p^j}} \qquad K_{p^j} = \mathrm{GL}_2(\mathbb{Z}_{p^j}) \subseteq \mathrm{GL}_2(\mathbb{Q}_{p^j})$$

Then by Kottwitz we have the trace

$$\mathrm{tr}(\pi_p(f_p^G)) = p^{\frac{1}{2}j}(c_1(\pi_p)^j + c_2(\pi_p)^j)$$

Away from p and ∞, we can choose test functions $f_G^{p,\infty} \in C_c^\infty(K^{p,\infty} \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^{p,\infty})$ which project onto individual (packets of) representations $\{\pi_{\mathrm{fin}}\}$ so as to isolate their contirbutions to cohomology. At ∞, we choose the test function $f_\infty^G = -f_{\mathcal{D}_{\lambda_1+1}}$ which is minus the pseudocoefficient of the discrete series representations $\mathcal{D}_{\lambda_1+1}$ of $G(\mathbb{R})$.

We should remark that the usual choice of ℓ-adic local systems on $\mathcal{S}_{K_{\mathrm{fin}}}$ constructed previously in terms of the cohomology of the universal family of elliptic curves, involves the highest weight $\lambda = (\lambda_1; \lambda_1)$. For these ℓ-adic local systems, the above formulas for the trace of Frobenius should be multiplied by an additional factor of $p^{\frac{\lambda_1}{2}j}$, corresponding to a half Tate twist.

Intersection and Cuspidal Cohomology To state the result, we need to introduce the Hecke modules and ℓ-adic Galois representations which appear.

For an integer $k \geq 2$ let $\mathbb{S}_{K_{\mathrm{fin}}}[k]$ be the $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module attached to the space of cuspidal automorphic representations $\pi = \pi_{\mathrm{fin}} \otimes \pi_\infty$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$ and $\pi_\infty = \mathcal{D}_{k-1}(\det^{k-1})$ a holomorphic discrete series representation. We have a decomposition

$$\mathbb{S}_{K_{\mathrm{fin}}}[k] = \bigoplus_\pi \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_\pi$$

where ρ_π is the irreducible 2-dimensional ℓ-adic Gal-module attached to π, with

$$\mathrm{tr}(\mathrm{Frob}_p^j|\rho_\pi) = p^{\frac{k-1}{2}j}(c_1(\pi_p)^j + c_2(\pi_p)^j)$$

For an integer $n \in \mathbb{Z}$ and for $\chi = \chi_{\mathrm{fin}} \otimes \chi_\infty$ a (finite order) character of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ let $\mathbb{L}_\chi^n = \rho_\chi(-n)$ be the 1-dimensional ℓ-adic Gal-module attached to χ twisted by the n-th power of the ℓ-adic cyclotomic character, with

$$\mathrm{tr}(\mathrm{Frob}_p^j|\mathbb{L}_\chi^n) = p^{nj}c(\chi_p)^j = p^{nj}\chi(p)^j$$

Now we have the following result:

Theorem 3.4.1. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; 0)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. The ℓ-adic intersection cohomology $IH^*(S_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees 0, 1, 2 and given as an $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module by

$$IH^0(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_\chi (\chi_{\mathrm{fin}} \circ \det)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}_\chi^0 & \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$IH^1(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \mathbb{S}_{K_{\mathrm{fin}}}[\lambda + 2] = \bigoplus_\pi \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_\pi$$

$$IH^2(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_\chi (\chi_{\mathrm{fin}} \circ \det)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}_\chi^1 & \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

In particular the ℓ-adic cuspidal cohomology $H^*_{\mathrm{cusp}}(S_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degree 1 and given as an $\mathcal{H}_{K_{\mathrm{fin}}} \times$ Gal-module by

$$H^1_{\mathrm{cusp}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \mathbb{S}_{K_{\mathrm{fin}}}[\lambda + 2] = \bigoplus_\pi \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_\pi$$

Proof. We have the following:

(i) For $\pi = \pi_{\mathrm{fin}} \otimes \pi_\infty$ a cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$ and with π_∞ a cohomological $(\mathfrak{g}, K'_\infty)$-module with central and infinitesimal characters determined by those of V_λ, we must have that $\pi_\infty = \mathcal{D}_{\lambda_1+1}$ is a holomorphic discrete series representation.

Let $f_\infty^G = -f_{\mathcal{D}_{\lambda+1}}$ be minus the pseudocoefficient of $\mathcal{D}_{\lambda_1+1}$ so that $\mathrm{tr}(\pi_\infty(f_\infty^G)) = -1$. Let $f_G^{p,\infty} \in C_c^\infty(K^{p,\infty} \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^{p,\infty})$ be a correspondence projecting onto $\{\pi_{\text{fin}}\}$. By Kottwitz we have the trace

$$\begin{aligned}\mathrm{tr}(\mathrm{Frob}_p^j | IH^*_{\lambda,\{\pi_{\text{fin}}\}}) &= \mathrm{tr}(\pi_\infty(f_\infty^G))\mathrm{tr}(\pi_{\text{fin}}(f_G^{p,\infty}))p^{\frac{\lambda_1+1}{2}j}(c_1(\pi_p)^j + c_2(\pi_p)^j) \\ &= -p^{\frac{\lambda_1+1}{2}j}(c_1(\pi_p)^j + c_2(\pi_p)^j)\end{aligned}$$

hence $IH^*_{\lambda,\{\pi_{\text{fin}}\}} = \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$ as an $\mathcal{H}_{K_{\text{fin}}} \times$ Gal-module.

(ii) For $\pi = \pi_{\text{fin}} \otimes \pi_\infty$ a residual automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with $\pi_{\text{fin}}^{K_{\text{fin}}} \neq 0$ and with π_∞ a cohomological $(\mathfrak{g}, K'_\infty)$-module with central and infinitesimal characters determined by those of V_λ, we must have that $\lambda_1 = 0$ and $\pi_\infty \in \{1, \mathrm{sign}\}$ is a 1-dimensional representation.

Let $f_\infty^G = -f_{\mathcal{D}_{\lambda+1}}$ be minus the pseudocoefficient of $\mathcal{D}_{\lambda+1}$ so that $\mathrm{tr}(\pi_\infty(f_\infty^G)) = 1$. Let $f_G^{p,\infty} \in C_c^\infty(K^{p,\infty} \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^{p,\infty})$ be a correspondence projecting onto $\{\pi_{\text{fin}}\}$. By Kottwitz we have the trace

$$\begin{aligned}\mathrm{tr}(\mathrm{Frob}_p^j | IH^*_{\{\pi_{\text{fin}}\}}) &= \mathrm{tr}(\pi_\infty(f_\infty^G))\mathrm{tr}(\pi_{\text{fin}}(f_G^{p,\infty}))p^{\frac{\lambda_1+1}{2}j}(c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j}) \\ &= p^{\frac{\lambda_1+1}{2}j}(c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j})\end{aligned}$$

hence $IH^*_{\{\pi_{\text{fin}}\}} = (\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes (\mathbb{L}_\chi^0 \oplus \mathbb{L}_\chi^1)$ as an $\mathcal{H}_{K_{\text{fin}}} \times$ Gal-module.

By Gabber's purity theorem the cuspidal contributions $\pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$ are concentrated in degree 1, while the residual contributions $(\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes (\mathbb{L}_\chi^0 \oplus \mathbb{L}_\chi^1)$ are concentrated in degrees 0 and 2, with $(\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes \mathbb{L}_\chi^0$ concentrated in degree 0 and $(\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes \mathbb{L}_\chi^1$ concentrated in degree 2. It follows that we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}} \times$ Gal-modules

$$IH^0(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_\chi (\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes \mathbb{L}_\chi^0 & \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$IH^1(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \mathbb{S}_{K_{\text{fin}}}[\lambda + 2] = \bigoplus_\pi \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$$

$$IH^2(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_\chi (\chi_{\text{fin}} \circ \det)^{K_{\text{fin}}} \boxtimes \mathbb{L}_\chi^1 & \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

and since the only cuspidal contributions are concentrated in degree 1 it follows that we have an isomorphism of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$-modules $H^1_{\text{cusp}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = IH^1(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ and the result follows. $\square$

3.5 Example: Cohomology of Local Systems on $\mathcal{A}_1$

Let $\mathcal{A}_1$ be the moduli stack of elliptic curves. We revisit earlier results and compute the cohomology $H^*(\mathcal{A}_1, \mathbb{V}_\lambda)$ as a Gal-module.

Theorem 3.5.1. Let $V_\lambda \in \text{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; \lambda_1)$ with λ_1 even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_1$. The compactly supported cohomology $H^*_c(\mathcal{A}_1, \mathbb{V}_\lambda)$ is supported in degrees 1 and 2 and is given as a Gal-module by

$$\begin{aligned} H^1_c(\mathcal{A}_1, \mathbb{V}_\lambda) &= \mathbb{S}_{\Gamma(1)}[\lambda_1 + 2] \\ &\oplus \begin{cases} \mathbb{L}^0 & \lambda_1 > 0 \text{ even} \\ 0 & \text{otherwise} \end{cases} \\ H^2_c(\mathcal{A}_1, \mathbb{V}_\lambda) &= \begin{cases} \mathbb{L}^1 & \lambda_1 = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Proof. Recall that we have $H^i_c(\mathcal{A}_1, \mathbb{V}_\lambda) \simeq H^i_{\text{cusp}}(\mathcal{A}_1, \mathbb{V}_\lambda) \oplus H^i_{\text{c,Eis}}(\mathcal{A}_1, \mathbb{V}_\lambda)$, and Poincare dually we have $H^i(\mathcal{A}_1, \mathbb{V}_\lambda) \simeq H^i_{\text{cusp}}(\mathcal{A}_1, \mathbb{V}_\lambda) \oplus H^i_{\text{Eis}}(\mathcal{A}_1, \mathbb{V}_\lambda)$. The description of cuspidal cohomology follows immediately from 3.4.1: the cuspidal cohomology is concentrated in degree 1 and given as a Gal-module by

$$H^1_{\text{cusp}}(\mathcal{A}_1, \mathbb{V}_\lambda) = \mathbb{S}_{\Gamma(1)}[\lambda_1 + 2]$$

The description of Eisenstein cohomology follows immediately from 3.3.12: the Eisenstein cohomology is concentrated in degrees 0 and 1 and given as a Gal-module by

$$H^0_{\mathrm{Eis}}(\mathcal{A}_1, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^0 & \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^1_{\mathrm{Eis}}(\mathcal{A}_1, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^{\lambda+1} & \lambda > 0 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

The result follows by Poincare duality. □

Congruences The 2-dimensional ℓ-adic Galois representations attached to automorphic representations for $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of general type are irreducible. On the other hand, they need not remain irreducible after reduction modulo a prime $\mathfrak{l}$. When this happens, we obtain congruences for the Hecke eigenvalues of such automorphic representations. Such congruences, and the divisivility of L-values which controls them, are well known. We recall some results around these, particularly in the case of level 1.

Let $f \in S_k(\Gamma(1))$ be a cuspidal Hecke eigenform of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. let p be a prime, and let $(\alpha_{0,p}(f), \alpha_{p,1}(f)) \in \mathbb{C}^2$ be the Satake parameters of f at p. For $q = p^n$ a power of p consider the Hecke eigenvalues

$$\lambda_q(f) = \alpha_{p,0}(f)^n + (\alpha_{p,0}(f)\alpha_{p,1}(f))^n$$

Let $\mathbb{Q}(f)$ be the number field generated by the Hecke eigenvalues $\lambda_p(f)$ for primes p.

Now one has certain congruences between the Hecke eigenvalues $\lambda_p(f)$ and the Hecke eigenvalues of Eisenstein series modulo certain congruence primes dividing certain expressions involving L-values which are related to the constant terms of Eisenstein series, so that these Eisenstein series behave as cusp forms modulo such congruence primes.

The constant terms of Eisenstein series are related to the expression

$$\frac{\zeta(\lambda_1+1)}{\zeta(\lambda_1+2)}$$

and we are interested the denominators of $\frac{\zeta(\lambda_1+2)}{(2\pi i)^{\lambda_1+2}}$ where we have divided by the Deligne period $(2\pi i)^k$ of $\mathbb{Q}(k)$ in order to obtain an element of $\mathbb{Q}$. Let $\mathfrak{Den}(\frac{(2\pi i)^{\lambda_1+2}}{\zeta(\lambda_1+2)})$ be the corresponding fractional ideal of $\mathbb{Z}$. Then we have the following:

Proposition 3.5.2. Let $\lambda_1 \geq 0$ be an integer. Suppose that $\ell > \lambda_1 + 2$ is a prime such that

$$\ell^n \mid \mathfrak{Den}\Big(\frac{(2\pi i)^{\lambda_1+2}}{\zeta(\lambda_1+2)}\Big)$$

Then there exists a normalized cuspidal Hecke eigenform $f \in S_{\lambda_1+2}(\Gamma(1))$ of weight $\lambda_1 + 2$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ such that

$$\lambda_p(f) \equiv p^{\lambda_1+1} + 1 \mod \mathfrak{l}^n$$

for every prime p and for every prime $\mathfrak{l}$ of $\mathbb{Q}(f)$ above the prime ℓ of $\mathbb{Q}$.

The first example of such a congruence was discovered by Ramanujan:

Example 3.5.3. Let $\lambda_1 = 10$. In this case the prime $\ell = 691$ divides $\frac{\zeta(12)}{(2\pi i)^{12}} = \frac{691}{2615348736000}$. Then there exists a normalized cuspidal Hecke eigenform $f \in S_{12}(\Gamma(1))$ of weight 12 for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ such that

$$\lambda_p(f) \equiv p^{11} + 1 \mod 691$$

for every prime p. Indeed $f = \Delta$ is the discriminant cusp form

$$\Delta = q \prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1} \tau(n)q^n$$

and the congruence $\tau(p) \equiv p^{11} + 1 \mod 691$ for every prime p is Ramanujan's congruence.

Such congruences, which are controlled by the divisibility of L-values, correspond to the reducibility of Galois representations modulo ℓ (up to semisimplification). The (conjectural) situation can be summarized by the following diagram:

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Denominator}\\ \mathrm{ord}_\ell\left(\frac{\zeta(k)}{(2\pi i)^k}\right) > 0\end{array}} \arrow[d, bend right, "\text{Eisenstein series}"'] \\
\boxed{\begin{array}{c}\text{Congruence}\\ \lambda_q(f) \equiv 1 + q^{k-1} \mod \mathfrak{l}\end{array}} \arrow[u, bend right, "\text{Bloch-Kato}"'] \arrow[d, bend right, "\substack{\text{Chebotarev density}\\ \text{Brauer-Nesbitt}}"'] \\
\boxed{\begin{array}{c}\text{Reducibility}\\ \overline{\rho}_f^{\mathrm{ss}} = \overline{1} \oplus \overline{\chi}^{k-1}\end{array}} \arrow[u, bend right, "\substack{\text{Deligne}\\ \text{Fontaine}}"']
\end{tikzcd}
$$

3.6 Example: Cohomology of Local Systems on $\mathcal{A}_1[2]$

Let $\mathcal{A}_1[2]$ be the moduli stack of elliptic curves with full level 2 structure. The group $\mathrm{GL}_2(\mathbb{F}_2) \simeq \mathrm{SL}_2(\mathbb{F}_2)$ acts on $\mathcal{A}_1[2]$. We compute the cohomology $H^*(\mathcal{A}_1[2], \mathbb{V}_\lambda)$ as a $\mathrm{GL}_2(\mathbb{F}_2) \times \mathrm{Gal}$-module.

We have isomorphisms $\mathrm{GL}_2(\mathbb{F}_2) \simeq \mathrm{SL}_2(\mathbb{F}_2) \simeq S_3$ so that the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_2)$ can be labeled by partitions of 3. We have

Representation of $\mathrm{GL}_2(\mathbb{F}_2)$	θ_0	θ_1	θ_2
Representation of S_3	V_3	V_{1^3}	$V_{2,1}$
Dimension	1	1	2

in particular $\theta_0 = 1_{\mathrm{GL}_2(\mathbb{F}_2)}$ is the trivial representation, θ_1 is the sign representation, and $\theta_2 = \mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_2)}$ is the Steinberg representation.

The local components of cusp forms for $\Gamma(2)$ are particularly simple. First, recall that we have a Hecke-equivariant isomorphism

$$S_k(\Gamma(2)) \simeq S_k(\Gamma_0(4)) \simeq S_k^{\mathrm{new}}(\Gamma_0(4)) \oplus 2S_k^{\mathrm{new}}(\Gamma_0(2)) \oplus 3S_k(\Gamma(1))$$

For f a cusp form of weight $k \geq 2$ for $\Gamma(2)$ let $\pi = \bigotimes_v \pi_v$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f. At the Archmedean place we have the local component $\pi_\infty = \mathcal{D}_{k-1}$ the holomorphic discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with Harish-Chandra parameter $k-1$ and trivial central character. At the finite places $p \neq 2$ we have the local component $\pi_p = \chi_p \times \chi_p^{-1}$ an unramified principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for χ_p an unramified character of $\mathbb{Q}_p^\times$ with $\chi_p^2 \neq |\cdot|_p^{\pm 1}$. At the place $v = 2$ the local component π_2 is one of the following (see [100, Lemma 5.22]):

- The unramified principal series representation $\pi_2 = \chi_2 \times \chi_2^{-1}$ for χ_2 an unramified character of $\mathbb{Q}_2^\times$ with $\chi_2^2 \neq |\cdot|_2^{\pm 1}$. This occurs precisely if f is a newform for $\Gamma(1)$, in which case $\epsilon(\frac{1}{2}, \pi_2) = 1$.
- The Steinberg representation $\pi_2 = \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$. This occurs precisely if f is a newform for $\Gamma_0(2)$ with Atkin-Lehner eigenvalue $\epsilon(\frac{1}{2}, \pi_2) = -1$.
- The twisted Steinberg representation $\pi_2 = \xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$ where ξ is the unique unramified quadratic character of $\mathbb{Q}_2^\times$. This occurs precisely if f is a newform for $\Gamma_0(2)$ with Atkin-Lehner eigenvalue $\epsilon(\frac{1}{2}, \pi_2) = 1$.
- The unique depth 0 supercuspidal representation $\pi_2 = \theta$ of $\mathrm{GL}_2(\mathbb{Q}_2)$ with trivial central character. This occurs precisely if f is a newform for $\Gamma_0(4)$, in which case $\epsilon(\frac{1}{2}, \pi_2) = -1$ (see [100, Lemma 5.21]).

Under the isomorphism $\mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$ the hyperspecial parahoric restriction of these local components is given $r_{\mathscr{K}_2}(\chi_2 \times \chi_2^{-1}) = V_3 \oplus V_{2,1}$, $r_{\mathscr{K}_2}(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}) = r_{\mathscr{K}_2}(\xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}) = V_{2,1}$ (the standard representation), and $r_{\mathscr{K}_2}(\theta_2) = V_{1^3}$ (the sign representation). In particular we have an isomorphism of $\mathrm{GL}_2(\mathbb{F}_2) \times \mathrm{Gal}$-modules

$$\mathbb{S}_{\Gamma(2)}[k] \simeq V_{1^3} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[k] \oplus V_{2,1} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[k] \oplus (V_3 \oplus V_{2,1}) \boxtimes \mathbb{S}_{\Gamma(1)}[k]$$

Now we have the following:

Theorem 3.6.1. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1; \lambda_1)$ with λ_1 even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_1[2]$. The compactly supported cohomology $H^*_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)$ is concentrated in degrees 1 and 2 and is given as a $\mathrm{GL}_2(\mathbb{F}_2)\times\mathrm{Gal}$-module by

$$
\begin{aligned}
H^1_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda) &= V_{1^3}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+2]\oplus V_{2,1}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+2]\oplus(V_3\oplus V_{2,1})\boxtimes\mathbb{S}_{\Gamma(1)}[\lambda_1+2]\\
&\oplus\begin{cases} V_{2,1}\boxtimes\mathbb{L}^0 & \lambda_1=0\\ (V_3\oplus V_{2,1})\boxtimes\mathbb{L}^0 & \lambda_1>0 \text{ even}\end{cases}\\
H^2_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda) &= \begin{cases} V_3\boxtimes\mathbb{L}^1 & \lambda_1=0\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

In particular the compactly supported cohomology $H^*_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)$ is given as a Gal-module by

$$
\begin{aligned}
H^1_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda) &= \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+2]\oplus 2\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+2]\oplus\mathbb{S}_{\Gamma(1)}[\lambda_1+2]\\
&\oplus\begin{cases} 2\mathbb{L}^0 & \lambda_1=0\\ 3\mathbb{L}^0 & \lambda_1>0 \text{ even}\end{cases}\\
H^2_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda) &= \begin{cases} \mathbb{L}^1 & \lambda_1=0\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

Proof. Recall that we have $H^i_{\mathrm{c}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)\simeq H^i_{\mathrm{cusp}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)\oplus H^i_{\mathrm{c,Eis}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)$, and Poincare dually we have $H^i(\mathcal{A}_1[2],\mathbb{V}_\lambda)\simeq H^i_{\mathrm{cusp}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)\oplus H^i_{\mathrm{Eis}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)$. The description of cuspidal cohomology follows from 3.4.1 and from the above remarks on local components of cusp forms for $\Gamma(2)$ and their parahoric restrictions. The description of Eisenstein cohomology follows from 3.3.12 and from the following observations. Recall that for $\Gamma(2)$ we have precisely 3 Eisenstein series for weights $k\geq 4$ even. We have precisely 2 Eisenstein series for weight $k=2$. For $\lambda_1>0$ even the Eisenstein cohomology is concentrated in degree 1 and given as a $\mathrm{GL}_2(\mathbb{F}_2)\times\mathrm{Gal}$-module by

$$
H^1_{\mathrm{Eis}}(\mathcal{A}_1[2],\mathbb{V}_\lambda)=\mathcal{I}^{G(\mathbb{A}^\infty_{\mathbb{Q}})}_{P_0(\mathbb{A}^\infty_{\mathbb{Q}})}(1\boxtimes 1,\lambda+1)^{K(2)}=V_3\oplus V_{2,1}
$$

which has dimension 3 as expected. For $\lambda_1 = 0$ the Eisenstein cohomology iss concentrated in degrees 0 and 1 and given as a $\mathrm{GL}_2(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$
\begin{aligned}
H^0_{\mathrm{Eis}}(\mathcal{A}_1[2], \mathbb{V}_\lambda) &= (1 \circ \det)^{K(2)} = V_3 \\
H^1_{\mathrm{Eis}}(\mathcal{A}_1[2], \mathbb{V}_\lambda) &= \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \boxtimes 1, 1) \to 1 \circ \det\Big)^{K(2)} = V_{2,1}
\end{aligned}
$$

which have dimensions 1 and 2 as expected. The result follows by Poincare duality. $\square$

Chapter 4

Cohomology of Siegel Modular Threefolds

4.1 Classical and Adelic Siegel Modular Threefolds

Shimura Datum Let $G = \mathrm{GSp}_4$ be the reductive group of symplectic similitudes over $\mathbb{Q}$:

$$G = \{(g, c) \in \mathrm{GL}_4 \times \mathrm{GL}_1 \mid g^\top J g = cJ\} \qquad J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

We have the similitude character $\mathrm{sim} : G \to \mathrm{GL}_1$ given by $(g, c) \mapsto c$ whose kernel $G^1 = \mathrm{Sp}_4$ is the symplectic group, the derived group of G.

Consider the maximal torus T of G given by

$$\begin{aligned} T &= \{\mathrm{diag}(t_1, t_2, t_3, t_4) | t_1, t_2, t_3, t_4 \in \mathbb{G}_\mathrm{m}, t_1 t_4 = t_2 t_3\} \\ &= \{\mathrm{diag}(t_1, t_2, t/t_2, t/t_1) | t_1, t_2, t \in \mathbb{G}_\mathrm{m}\} \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \end{aligned}$$

Since G is already $\mathbb{Q}$-split, $T \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \simeq \mathrm{GL}_1$ is a $\mathbb{Q}$-split maximal torus and $A_G = Z(G) \simeq \mathrm{GL}_1$ is a $\mathbb{Q}$-split maximal torus in the center of G. In particular, $A_G(\mathbb{R})^\circ \simeq \mathbb{R}_{>0}$. In this case we have the Langlands dual group $\widehat{G} = \mathrm{GSpin}_5(\mathbb{C})$ the spin similitude group, and an exceptional isomorphism $\mathrm{GSpin}_5(\mathbb{C}) \simeq \mathrm{GSp}_4(\mathbb{C})$.

We now consider the locally symmetric space associated to G, and the associated Shimura datum. Consider the element

$$I_0 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R})$$

The centralizer K_∞ of I_0 in $G^1(\mathbb{R}) = \mathrm{Sp}_4(\mathbb{R})$ is the connected component of a maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$, and is isomorphic to $\mathrm{SU}(2)$. The centralizer K'_∞ of I_0 in $G(\mathbb{R}) = \mathrm{GSp}_4(\mathbb{R})$ is connected, and is isomorphic to $A_G(\mathbb{R})^\circ K_\infty = \mathbb{R}_{>0}\mathrm{U}(2)$. The symmetric space $X = X^+ \amalg X^- = G(\mathbb{R})/K'_\infty = A_G(\mathbb{R})^\circ \backslash G(\mathbb{R})/K_\infty$ is a Hermitian symmetric domain and is identified with the space of 2-dimensional linear subspaces of $\mathbb{R}^4$ on which the skew-symmetric bilinear form $(v_1, v_2) \mapsto v_1^\top \mathrm{adiag}(i,i,i,i)v_2$ is positive or negative definite, which is identified with the Siegel double half space

$$\mathfrak{H}_2^\pm = \mathfrak{H}_2^+ \amalg \mathfrak{H}_2^- = \{\tau \in \mathrm{M}_2(\mathbb{C}) | \tau \text{ symmetric}, \Im(\tau) \text{ positive or negative definite}\}$$

Let $x_0 \in X$ be the subspace generated by the standard basis vectors $e_3, e_4 \in \mathbb{R}^4$. Then $G(\mathbb{R})$ acts transitively on X by the injection $\mathrm{GSp}_4(\mathbb{R}) \subseteq \mathrm{GL}_4(\mathbb{R})$. Consider the $G(\mathbb{R})$-equivariant morphism $h : X \to \mathrm{Hom}_\mathbb{R}(\mathbb{S}, G_\mathbb{R})$ given by

$$h_0 = h(x_0) = (z \mapsto \mathrm{diag}(z, z, 1, 1))$$

Then (G, X, h) is a Shimura datum in the sense of Deligne and Pink.

Shimura Varieties and Connected Components Let $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$ be a compact open subgroup. Let $S_{K_{\mathrm{fin}}}$ be the corresponding Shimura variety which is a quasiprojective variety over $\mathbb{Q}$ (smooth over $\mathbb{Q}$ if K_{fin} is neat, and considered as a stack otherwise) with set of complex points given by

$$S_{K_{\mathrm{fin}}}(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathfrak{H}_2^\pm \times G(\mathbb{A}_\mathbb{Q}^\infty)/K_{\mathrm{fin}}$$

The Shimura variety $S_{K_{\text{fin}}}$ is connected but in general not geometrically connected: the set of connected components of $S_{K_{\text{fin}}}(\mathbb{C})$ is given by

$$\begin{aligned}\pi_0(S_{K_{\text{fin}}}(\mathbb{C})) &= \mathrm{GSp}_4(\mathbb{Q}) \setminus \{\pm 1\} \times \mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_{\text{fin}} \\ &\simeq \mathbb{Q}_{>0} \setminus \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}^{\infty})/\det(K_{\text{fin}}) \\ &\simeq \widehat{\mathbb{Z}}^{\times}/\det(K_{\text{fin}})\end{aligned}$$

where the first isomorphism is given by the determinant and by $\mathrm{Sp}_4(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \mathrm{Sp}_4(\mathbb{Q})K_{\text{fin}}$ by strong approximation, and the second isomorphism is given by the decomposition $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \mathbb{Q}_{>0}\widehat{\mathbb{Z}}^{\times}$ which induces a decomposition

$$\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \coprod_{a \in \widehat{\mathbb{Z}}^{\times}/\det(K_{\text{fin}})} \mathrm{GSp}_4^{+}(\mathbb{Q})\mathrm{diag}(1,a)K_{\text{fin}}$$

For $a \in \widehat{\mathbb{Z}}^{\times}/\det(K_{\text{fin}})$ consider the morphism $\mathfrak{H}_2 \to S_{K_{\text{fin}}}(\mathbb{C})$ sending $z \in \mathfrak{H}_2$ to the point represented by $(z, \mathrm{diag}(1,1,a,a))$, then we obtain an embedding of classical Siegel modular threefolds $\Gamma_a \setminus \mathfrak{H}_2 \hookrightarrow S_{K_{\text{fin}}}(\mathbb{C})$ where

$$\Gamma_a = \mathrm{diag}(1,1,a,a)K_{\text{fin}}\mathrm{diag}(1,1,a^{-1},a^{-1}) \cap \mathrm{GSp}_4^{+}(\mathbb{Q})$$

In particular $S_{K_{\text{fin}}}(\mathbb{C})$ is a disjoint union of classical Siegel modular threefolds $\Gamma_a \setminus \mathfrak{H}_2$.

Moduli Problems and Level Structures The Shimura variety $S_{K_{\text{fin}}}$ is defined over $\mathbb{Q}$ (in fact over $\mathbb{Z}[\frac{1}{N}]$ for $K_{\text{fin}} = \prod_{p|N} K_p \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p)$) by the moduli functor

$$\begin{aligned} S_{K_{\text{fin}}} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{ \begin{matrix} \text{Tuples } (A,\lambda,\eta) \text{ where } A/S \text{ is an Abelian surface} \\ \lambda \text{ is a polarization and } \eta \text{ is a } K_{\text{fin}}\text{-level structure} \end{matrix} \right\}_{/\simeq} \end{aligned}$$

where the K_{fin}-level structure is a K_{fin}-conjugacy class of isomorphism $\eta : (\mathbb{A}_{\mathbb{Q}}^{\infty})^4 \xrightarrow{\sim} H_1(A(\mathbb{C}), \mathbb{A}_{\mathbb{Q}}^{\infty})$, and $(A_1,\lambda_1,\eta_1) \simeq (A_2,\lambda_2,\eta_2)$ are equivalent precisely if there exists an isogeny $\phi : A_1 \to A_2$ such that $\phi_* \circ \eta_1 = \eta_2$. The moduli functor $S_{K_{\text{fin}}} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Grpd}$ is defined similarly. Each connected component of $S_{K_{\text{fin}}}$ is defined over $\mathbb{Q}^{\mathrm{ab}}$ (in fact over $\mathbb{Z}[\frac{1}{N}, \mu_N]$), and the induced action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \widehat{\mathbb{Z}}^\times$ on the set of geometric connected components $\pi_0(S_{K_{\mathrm{fin}}}) = \widehat{\mathbb{Z}}^\times/\det(K_{\mathrm{fin}})$ is given by the usual multiplication in $\widehat{\mathbb{Z}}^\times$ by class field theory.

We now collect some running examples of moduli problems and level structures for modular curves, following [115].

Example 4.1.1. Let $\Gamma(N)$ denote the inverse image of the identity under the reduction morphism $\mathrm{Sp}_4(\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma(N) = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \equiv N \right\}$$

The corresponding quotient $\Gamma(N) \setminus \mathfrak{H}_2$ of the Siegel upper half plane $\mathfrak{H}_2$ is the classical Siegel modular threefold of full level N over $\mathbb{C}$. Let $K(N)$ denote the inverse image of the identity under the reduction morphism $\mathrm{GSp}_4(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$K(N) = \prod_{p|N} \mathscr{K}_p^+ \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \equiv N \right\}$$

The group $K(N)$ corresponds to the moduli problem

$$\begin{aligned} \mathcal{S}_{K(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{ \begin{matrix} \text{Tuples } (A, \lambda, x_1, x_2, x_3, x_4) \text{ where } A/S \text{ is an Abelian surface} \\ \lambda \text{ is a principal polarization of } A/S \text{ and} \\ x_1, x_2, x_3, x_4 \in A[N](S) \text{ are points of exact order } N \\ \text{such that } A[N] \text{ is generated by } x_1, x_2, x_3, x_4 \end{matrix} \right\}_{/\simeq} \end{aligned}$$

which is representable by a scheme for $N \geq 3$ (and representable by a Deligne-Mumford stack for $N = 1, 2$, where the moduli functor $\mathcal{S}_{K(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Grpd}$ is defined similarly). A similar issue of connected components exists as in the case of modular curves, and a similar discussion applies to the modified congruence subgroup

$$K'(N) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & \\ & & & * \end{pmatrix} \equiv N \right\}$$

Example 4.1.2. Let $\Gamma_0^{P_1}(N)$ denote the inverse image of the Siegel parabolic subgroup under the reduction morphism $\mathrm{Sp}_4(\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma_0^{P_1}(N) = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \mod N \right\}$$

The corresponding quotient $\Gamma_0^{P_1}(N) \setminus \mathfrak{H}_2$ of the Siegel upper half plane $\mathfrak{H}_2$ is the classical Siegel modular threefold of (Siegel) level $\Gamma_0(N)$ over $\mathbb{C}$. Let $K_0^{P_1}(N)$ denote the inverse image of the Siegel parabolic subgroup under the reduction morphism $\mathrm{GSp}_4(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$K_0^{P_1}(N) = \prod_{p|N} \mathscr{I}_p^{P_1} \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \mod N \right\}$$

The group $K_0^{P_1}(N)$ corresponds to the moduli problem

$$\begin{aligned} \mathcal{S}_{K_0^{P_1}(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Grpd} \\ S &\mapsto \left\{ \begin{array}{c} \text{Tuples } (A, \lambda, H_2) \text{ where } A/S \text{ is an Abelian surface} \\ \lambda \text{ is a principal polarization of } A/S \text{ and} \\ H_2 \subseteq A[N] \text{ is a rank } N^2 \text{ finite flat subgroup scheme} \\ \text{totally isotropic with respect to the } \lambda\text{-Weil pairing} \end{array} \right\} \end{aligned}$$

which is representable by a Deligne-Mumford stack. The moduli space $\mathcal{S}_{K_0^{P_1}(N)}(\mathbb{C})$ is geometrically connected, in particular we have

$$\mathcal{S}_{K_0^{P_1}(N)}(\mathbb{C}) = \Gamma_0^{P_1}(N) \setminus \mathfrak{H}_2$$

A similar discussion applies to the congruence subgroup

$$K_1^{P_1}(N) = \prod_{p|N} \mathscr{I}_p^{P_1+} \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \mod N \right\}$$

or to the modified congruence subgroup

$$K_1^{P_1}(N) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & * & \\ & & & * \end{pmatrix} \mod N \right\}$$

where similar issues of connected components arise.

Example 4.1.3. Let $\Gamma_0^{P_2}(N)$ denote the inverse image of the Siegel parabolic subgroup under the reduction morphism $\mathrm{Sp}_4(\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma_0^{P_1}(N) = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \mod N \right\}$$

The corresponding quotient $\Gamma_0^{P_2}(N) \setminus \mathfrak{H}_2$ of the Siegel upper half plane $\mathfrak{H}_2$ is the classical Siegel modular threefold of (Klingen) level $\Gamma_0(N)$ over $\mathbb{C}$. Let $K_0^{P_2}(N)$ denote the inverse image of the Siegel parabolic subgroup under the reduction morphism $\mathrm{GSp}_4(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$K_0^{P_2}(N) = \prod_{p|N} \mathscr{I}_p^{P_2} \times \prod_{p \nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \mod N \right\}$$

The group $K_0^{P_2}(N)$ corresponds to the moduli problem

$$\begin{aligned} \mathcal{S}_{K_0^{P_2}(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} &\to \mathrm{Set} \\ S &\mapsto \left\{ \begin{matrix} \text{Tuples } (A, \lambda, H_1) \text{ where } A/S \text{ is an Abelian surface} \\ \lambda \text{ is a principal polarization of } A/S \text{ and} \\ H_1 \subseteq A[N] \text{ is a rank } N \text{ finite flat subgroup scheme} \end{matrix} \right\}_{/\simeq} \end{aligned}$$

which is representable by a Deligne-Mumford stack. Note that such a finite flat subgroup scheme $H_1 \subseteq A[N]$ is automatically totally isotropic with respect to the λ-Weil pairing. The moduli space $\mathcal{S}_{K_0^{P_2}(N)}(\mathbb{C})$ is geometrically connected, in particular we have

$$\mathcal{S}_{K_0^{P_2}(N)}(\mathbb{C}) = \Gamma_0^{P_2}(N) \setminus \mathfrak{H}_2$$

A similar discussion applies to the congruence subgroup

$$K_1^{P_2}(N) = \prod_{p|N} \mathscr{I}_p^{P_2+} \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \mod N \right\}$$

or to the modified congruence subgroup

$$K_1^{P_2}(N) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & * & * \\ & & & * \end{pmatrix} \mod N \right\}$$

where similar issues of connected components arise.

Example 4.1.4. Let $\Gamma_0^{P_0}(N)$ denote the inverse image of the Borel parabolic subgroup under the reduction morphism $\mathrm{Sp}_4(\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$\Gamma_0^{P_0}(N) = \left\{ \gamma \in \mathrm{Sp}_4(\mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \mod N \right\}$$

The corresponding quotient $\Gamma_0^{P_0}(N) \setminus \mathfrak{H}_2$ of the Siegel upper half plane $\mathfrak{H}_2$ is the classical Siegel modular threefold of (Borel) level $\Gamma_0(N)$ over $\mathbb{C}$. Let $K_0^{P_0}(N)$ denote the inverse image of the Siegel parabolic subgroup under the reduction morphism $\mathrm{GSp}_4(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_4(\mathbb{Z}/N\mathbb{Z})$:

$$K_0^{P_0}(N) = \prod_{p|N} \mathscr{I}_p^{P_0} \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \mod N \right\}$$

The group $K_0^{P_0}(N)$ corresponds to the moduli problem

$$\mathcal{S}_{K_0^{P_0}(N)} : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Set}$$
$$S \mapsto \left\{ \begin{array}{c} \text{Tuples } (A, \lambda, H_1) \text{ where } A/S \text{ is an Abelian surface} \\ \lambda \text{ is a principal polarization of } A/S \text{ and} \\ H_2 \subseteq A[N] \text{ is a rank } N^2 \text{ finite flat subgroup scheme} \\ \text{totally isotropic with respect to the } \lambda\text{-Weil pairing} \\ H_1 \subseteq H_2 \text{ is a rank } N \text{ finite flat subgroup scheme} \end{array} \right\}_{/\simeq}$$

which is representable by a Deligne-Mumford stack. Note that such a finite flat subgroup scheme $H_1 \subseteq A[N]$ is automatically totally isotropic with respect to the λ-Weil pairing. The moduli space $\mathcal{S}_{K_0^{P_0}(N)}(\mathbb{C})$ is geometrically connected, in particular we have

$$\mathcal{S}_{K_0^{P_0}(N)}(\mathbb{C}) = \Gamma_0^{P_0}(N) \setminus \mathfrak{H}_2$$

Note that we have canonical morphisms $\mathcal{S}_{K_0^{P_0}(N)} \to \mathcal{S}_{K_0^{P_i}(N)}$ for $i = 1, 2$ forgetting either H_1 or H_2. A similar discussion applies to the congruence subgroup

$$K_1^{P_0}(N) = \prod_{p|N} \mathscr{I}_p^{P_0+} \times \prod_{p\nmid N} \mathrm{GSp}_4(\mathbb{Z}_p) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \mod N \right\}$$

or to the modified congruence subgroup

$$K_1^{P_0}(N) = \left\{ \gamma \in \mathrm{GSp}_4(\widehat{\mathbb{Z}}) | \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & * & * \\ & & & * \end{pmatrix} \mod N \right\}$$

where similar issues of connected components arise.

Let $\mathcal{A}_2[N] = \mathcal{S}_{K(N)}$, let $\mathcal{A}_2[\Gamma_1^P(N)] = \mathcal{S}_{K_1^P(N)}$, and let $\mathcal{A}_2[\Gamma_0^P(N)] = \mathcal{S}_{K_0(N)}$ denote the moduli stacks constructed in the above examples. In particular let $\mathcal{A}_2$ denote any of these in the case $N = 1$: this is the moduli stack of principally polarized Abelian surfaces, corresponding to the hyperspecial maximal compact subgroup $K_{\text{fin}} = \mathrm{GSp}_4(\widehat{\mathbb{Z}})$.

Local Systems We now recall the basic local systems on Siegel modular threefolds and their relation to Siegel modular forms. Recall that we have the maximal torus $T = \{\mathrm{diag}(t_1, t_2, t/t_2, t/t_1) | t_1, t_2, t \in \mathrm{GL}_1\} \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1$. We identify elements of the character lattice $X^*(T)$ with triples of integers $\lambda = (\lambda_1, \lambda_2; c) \in \mathbb{Z}^3$ with $c \equiv \lambda_1 + \lambda_2 \mod 2$, corresponding to the character

$$\mathrm{diag}(t_1, t_2, t/t_2, t/t_1) \mapsto t_1^{\lambda_1} t_2^{\lambda_2} t^{\frac{c-\lambda_1-\lambda_2}{2}}$$

The (finite dimensional) absolutely irreducible rational representations of GSp_4 are parameterized by dominant highest weights $\lambda \in X^*(T)^+ \subseteq X^*(T)$. We identify elements of the subset $X^*(T)^+$ with triples of integers

$\lambda = (\lambda_1, \lambda_2; c) \in \mathbb{Z}^3$ with $c \equiv \lambda_1 + \lambda_2 \mod 2$ and $\lambda_1 \geq \lambda_2 \geq 0$. For $\lambda \in X^*(T)^+$ let V_λ be the corresponding irreducible representation of GSp_4. For $\lambda = (\lambda_1, \lambda_2; c) \in X^*(T)^+$ let $\lambda^\vee = (\lambda_1, \lambda_2; -c) \in X^*(T)^+$ so that $V_{\lambda^\vee} = V_\lambda^\vee$ is the contragredient representation. Noting that the similitude character $\mathrm{sim} : \mathrm{diag}(t_1, t_2, t/t_2, t/t_1) \mapsto t$ corresponds to the highest weight $\lambda = (0, 0; 2)$, we have an isomorphism

$$V_\lambda \xrightarrow{\sim} V_\lambda^\vee \otimes \mathrm{sim}^{2c}$$

For integers $\lambda_1 \geq \lambda_2 \geq 0$ let V_{λ_1,λ_2} be the irreducible representation of GSp_4 with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $V_{\lambda_1,\lambda_2}(\mathrm{sim}^{\frac{\lambda_1+\lambda_2}{2}})$ be the irreducible representation of GSp_4 with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$. Then we have an isomorphism $V_{\lambda_1,\lambda_2}(\mathrm{sim}^{\frac{\lambda_1+\lambda_2}{2}}) = V_{\lambda_1,\lambda_2} \otimes \mathrm{sim}^{\frac{\lambda_1+\lambda_2}{2}}$, in particular V_{λ_1,λ_2} is self-dual.

Since the above Shimura varieties are moduli spaces of elliptic curves, we can consider local systems of geometric origin coming from the cohomology of the universal family of elliptic curves.

Let $\pi : U_{K_{\mathrm{fin}}}(\mathbb{C}) \to S_{K_{\mathrm{fin}}}(\mathbb{C})$ be the universal family of Abelian surfaces which gives rise to a local system of $\mathbb{Q}$-vector spaces $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}$ of rank 4 on $S_{K_{\mathrm{fin}}}(\mathbb{C})$ whose fiber over a point $[A] \in S_{K_{\mathrm{fin}}}(\mathbb{C})$ is the singular cohomology $H^1(A(\mathbb{C}), \mathbb{Q})$. For integers $\lambda_1 \geq \lambda_2 \geq 0$ consider the local system of $\mathbb{Q}$-vector spaces $\mathbb{V}_{\lambda_1,\lambda_2}$ cut out by Schur functors from $\mathrm{Sym}^{\lambda_1-\lambda_2}(\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_2}(\wedge^2\mathbb{V})$ on $S_{K_{\mathrm{fin}}}(\mathbb{C})$.

We are then interested in the cohomology $H^*(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2})$ or the compactly supported cohomology $H^*_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2})$ as an $\mathcal{H}_{K_{\mathrm{fin}}}$-module. We have short exact sequences of $\mathcal{H}_{K_{\mathrm{fin}}}$-modules

$$0 \to H^i_!(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) \to H^i(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) \to H^i_{\mathrm{Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) \to 0$$
$$0 \to H^i_{\mathrm{c,Eis}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) \to H^i_{\mathrm{c}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) \to H^i_!(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) \to 0$$

The inner cohomology $H^i_!(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ is identified with the cuspidal cohomology $H^i_{\mathrm{cusp}}(S_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$, which will turn out to be concentrated in degrees 2, 3, 4.

One can also consider the Hodge vector bundle $\omega = \pi_*\Omega^1_{\mathcal{E}_{K_{\mathrm{fin}}}/\mathcal{S}_{K_{\mathrm{fin}}}}$ on $\mathcal{S}_{K_{\mathrm{fin}}}$ whose fiber over a point $[A] \in \mathcal{S}_{K_{\mathrm{fin}}}$ is the cohomology $H^0(A, \Omega^1_A)$. For integers $k_1, k_2 \in \mathbb{Z}$ let $\mathcal{V}_{k_1,k_2}$ be the vector bundle on $\mathcal{S}_{K_{\mathrm{fin}}}$, which extend to a toroidal compactification $\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}$ of $\mathcal{S}_{K_{\mathrm{fin}}}$, whose sections are vector-valued Siegel modular forms of weight (k_1, k_2):

Definition 4.1.5. Let $\Gamma \subseteq \mathrm{Sp}_4(\mathbb{Q})$ be a congruence subgroup. Consider the action of $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_4^+(\mathbb{R})$ on $\tau \in \mathfrak{H}_2$ given by $g \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$, and consider the factor of automorphy $j(g, \tau) = C\tau + D$. For integers $k_1, k_2 \in \mathbb{Z}$ let ρ_{k_1,k_2} be the representation $\mathrm{Sym}^{k_1} \otimes \det^{k_2}$ of $\mathrm{GL}_2(\mathbb{C})$ and consider the right action of $\mathrm{GSp}_4^+(\mathbb{R})$ on the space $C^\infty_{k_1}(\mathfrak{H})$ of smooth $\mathrm{Sym}^{k_1}(\mathbb{C}^2)$-valued functions on $\mathfrak{H}_2$ given by

$$(f|_{k_1,k_2} g)(\tau) = \mathrm{sim}(g)^{k_1/2+k_2} \rho_{k_1,k_2}(j(g,\tau)) f(g \cdot \tau)$$

A (vector-valued) Siegel modular form of weight (k_1, k_2) for Γ is a holomorphic function $f : \mathfrak{H}_2 \to \mathrm{Sym}^{k_1}(\mathbb{C}^2)$ such that $f|_{k_1,k_2}\gamma = f$ for all $\gamma \in \Gamma$.

A (vector-valued) Siegel cusp form of weight (k_1, k_2) for Γ is a (vector-valued) Siegel modular form of weight (k_1, k_2) for Γ such that $\lim_{t\to\infty}(f|_{k_1,k_2} g)\left(\begin{smallmatrix} it & 0 \\ 0 & \tau \end{smallmatrix}\right) = 0$ for all $\tau \in \mathfrak{H}$ and all $g \in \mathrm{Sp}_4(\mathbb{Q})$.

Let $M_{k_1,k_2}(\Gamma)$ be the $\mathbb{C}$-vector space of (vector-valued) Siegel modular forms of weight (k_1, k_2) for Γ, and let $S_{k_1,k_2}(\Gamma)$ be the $\mathbb{C}$-vector subspace of (vector-valued) Siegel cusp forms of weight (k_1, k_2) for Γ.

Let $M_{k_1,k_2}(K_{\mathrm{fin}}) = \bigoplus_{a\in\widehat{\mathbb{Z}}^\times/\det(K_{\mathrm{fin}})} M_{k_1,k_2}(\Gamma_a)$ and $S_{k_1,k_2}(K_{\mathrm{fin}}) = \bigoplus_{a\in\widehat{\mathbb{Z}}^\times/\det(K_{\mathrm{fin}})} S_{k_1,k_2}(\Gamma_a)$ be the corresponding spaces of modular forms and cusp forms taking into account connected components. Letting $D \subseteq \overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}$ be the divisor of cusps in a toroidal compactification of $\mathcal{S}_{K_{\mathrm{fin}}}$, we have identifications

$$M_{k_1,k_2}(K_{\mathrm{fin}}) = H^0(\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}(\mathbb{C}), \mathcal{V}_{k_1,k_2}) \qquad S_{k_1,k_2}(K_{\mathrm{fin}}) = H^0(\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}(\mathbb{C}), \mathcal{V}_{k_1,k_2}(-D))$$

By Faltings-Chai [31] the cohomology groups $H^i_c(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2})$ carry a Hodge filtration with

$$\begin{aligned}
\mathrm{Gr}^0_F H^i_c(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) &= H^i(\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}(\mathbb{C}), \mathcal{V}_{\lambda_1-\lambda_2,-\lambda_1}(-D)) \\
\mathrm{Gr}^{\lambda_2+1}_F H^i_c(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) &= H^{i-1}(\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}(\mathbb{C}), \mathcal{V}_{\lambda_1+\lambda_2+2,-\lambda_1}(-D)) \\
\mathrm{Gr}^{\lambda_1+2}_F H^i_c(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) &= H^{i-2}(\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}(\mathbb{C}), \mathcal{V}_{\lambda_1+\lambda_2+2,1-\lambda_2}(-D)) \\
\mathrm{Gr}^{\lambda_1+\lambda_2+3}_F H^i_c(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_{\lambda_1,\lambda_2}) &= H^{i-3}(\overline{\mathcal{S}}^\Sigma_{K_{\mathrm{fin}}}(\mathbb{C}), \mathcal{V}_{\lambda_1-\lambda_2,\lambda_2+3}(-D))
\end{aligned}$$

In particular one finds (vector-valued) Siegel cusp forms of weight $(\lambda_1 - \lambda_2, \lambda_2 + 3)$ in $H^i_c(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2})$.

The same construction applies to the construction of ℓ-adic local systems on $\mathcal{S}_{K_{\mathrm{fin}}}$. Writing $\mathcal{S}_{K_{\mathrm{fin}}}$ to mean either $\mathcal{S}_{K_{\mathrm{fin}},\overline{\mathbb{Q}}}$ or $\mathcal{S}_{K_{\mathrm{fin}},\overline{\mathbb{F}}_p}$ as before, let $\pi : \mathcal{U}_{K_{\mathrm{fin}}} \to \mathcal{S}_{K_{\mathrm{fin}}}$ be the universal family of Abelian surfaces and consider the

ℓ-adic local system $\mathbb{V} = \mathbb{R}^1\pi_*\mathbb{Q}_\ell$ of rank 4 on $\mathcal{S}_{K_{\text{fin}}}$ whose fiber over a point $[E] \in \mathcal{S}_{K_{\text{fin}}}$ is the ℓ-adic cohomology $H^1(A, \mathbb{Q}_\ell)$ (either over $\overline{\mathbb{Q}}$ or over $\overline{\mathbb{F}}_p$), which is pure of weight 1. For integers $\lambda_1 \geq \lambda_2 \geq 0$ consider the ℓ-adic local system $\mathbb{V}_{\lambda_1,\lambda_2}$ cut out by Schur functors from $\mathrm{Sym}^{\lambda_1-\lambda_2}(\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_2}(\wedge^2\mathbb{V})$ on $\mathcal{S}_{K_{\text{fin}}}$, which is pure of weight $\lambda_1+\lambda_2$.

We are then interested in the ℓ-adic $H^*(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2})$ or the compactly supported ℓ-adic cohomology $H^*_{\mathrm{c}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2})$ (either over $\overline{\mathbb{Q}}$ or over $\overline{\mathbb{F}}_p$). We have short exact sequences of $\mathcal{H}_{K_{\text{fin}}} \times \mathrm{Gal}$-modules

$$0 \to H^i_!(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2}) \to H^i(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2}) \to H^*_{\mathrm{Eis}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2}) \to 0$$
$$0 \to H^*_{\mathrm{c,Eis}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2}) \to H^*_{\mathrm{c}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2}) \to H^*_!(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2}) \to 0$$

The inner cohomology $H^i_!(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2})$ is identified with the cuspidal cohomology $H^i_{\mathrm{cusp}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_{\lambda_1,\lambda_2})$, which will turn out to be concentrated in degrees 2, 3, 4.

In both of the above situations, the local systems $\mathbb{V}_{\lambda_1,\lambda_2}$ correspond to the irreducible representation $V_{\lambda_1,\lambda_2}(\det^{\frac{\lambda_1+\lambda_2}{2}})$ of GSp_4 with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1+\lambda_2)$. On the other hand when computing cuspidal cohomology or Eisenstein cohomology in terms of $(\mathfrak{g}, K'_\infty)$-cohomology we will need to use the irreducible representation we will need to use the irreducible representation V_{λ_1,λ_2} of GSp_4 with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, corresponding to the half Tate twisted local system $\mathbb{V}_{\lambda_1,\lambda_2}(-\frac{\lambda_1+\lambda_2}{2})$. The discrepancy involving the similitude character accounts for the difference between the unitary normalization and cohomological normalization of automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, and will account for the additional Tate twist which appears in the later discussion of the Langlands-Kottwitz method. We will often use the abbreviated notation V_λ and $\mathbb{V}_\lambda$ for either of these representations and local systems, being careful to disambiguate when their meaning is not clear.

4.2 Discrete Series and Induced Representations for $\mathrm{GSp}_4(\mathbb{R})$

We recall some structural facts related to the group $\mathrm{GSp}_4(\mathbb{R})$, especially the construction of discrete series representations and induced representations, and the Vogan-Zuckerman classification of irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{R})$ with nonzero $(\mathfrak{g}, K'_\infty)$-cohomology.

Lie Algebras and Compact Subgroups Let $\mathfrak{g} = \mathfrak{gsp}_4$ be the Lie algebra of $G(\mathbb{R}) = \mathrm{GSp}_4(\mathbb{R})$:

$$\mathfrak{gsp}_4 = \{(G, C) \in \mathfrak{gl}_4 \times \mathfrak{gl}_1 \mid JG + G^\top J = CJ\}$$

We have the similitude character $\mathrm{sim} : \mathfrak{gsp}_4 \to \mathfrak{gl}_1$ given by $(G, C) \mapsto C$ whose kernel $\mathfrak{g}^1 = \mathfrak{sp}_4$ is the Lie algebra of $G^1(\mathbb{R}) = \mathrm{Sp}_4(\mathbb{R})$.

Let K_∞ be the maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$ given by

$$K_\infty = \{\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{R})\}$$

We have an isomorphism $K_\infty \xrightarrow{\sim} \mathrm{U}(2)$ given by $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mapsto A + Bi$.

Let $\mathfrak{k}$ be the Lie algebra of K_∞, and let $\mathfrak{h} \subseteq \mathfrak{k}$ be the Cartan subalgebra corresponding to the compact torus

$$T_c = \left\{ \left(\begin{smallmatrix} \cos(\theta_1) & & & \sin(\theta_1) \\ & \cos(\theta_2) & \sin(\theta_2) & \\ & -\sin(\theta_2) & \cos(\theta_2) & \\ -\sin(\theta_1) & & & \cos(\theta_1) \end{smallmatrix} \right) \middle| \theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

Recall that the weights of $\mathfrak{sp}_4$ are elements of the space $\mathfrak{h}_{\mathbb{C}}^\vee = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}, \mathbb{C})$; an element $\lambda \in \mathfrak{h}_{\mathbb{C}}^\vee$ is identified with a pair of complex numbers $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. Let $\mathfrak{h}^\vee \subseteq \mathfrak{h}_{\mathbb{C}}^\vee$ be the subset where an element $\lambda \in \mathfrak{h}^\vee$ is identified with a pair of real numbers $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. We say that $\lambda \in \mathfrak{h}^\vee$ is analytically integral if $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ is identified with a pair of integers. Under this identification, a pair of integers $(k_1, k_2) \in \mathbb{Z}^2$ corresponds to the derivative of the character ψ_{k_1,k_2} of T_c given by

$$\psi_{k_1,k_2} : \left(\begin{smallmatrix} \cos(\theta_1) & & & \sin(\theta_1) \\ & \cos(\theta_2) & \sin(\theta_2) & \\ & -\sin(\theta_2) & \cos(\theta_2) & \\ -\sin(\theta_1) & & & \cos(\theta_1) \end{smallmatrix} \right) \mapsto e^{ik_1\theta_1 + ik_2\theta_2}$$

Every analytically integral weight $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{h}^\vee$ with $\lambda_1 \geq \lambda_2$ corresponds to a K_∞-type $V_\lambda = V_{\lambda_1,\lambda_2}$, that is an equivalence class of irreducible representation of the compact group $K_\infty \simeq \mathrm{U}(2)$. Such a K_∞-type V_{λ_1,λ_2} has weights $(\lambda_1 - j, \lambda_2 + j)$ for $j \in \{0, 1, \ldots, \lambda_1 - \lambda_2\}$ each with multiplicity 1; in particular V_{λ_1,λ_2} has dimension $\lambda_1 - \lambda_2 + 1$ with highest weight (λ_1, λ_2).

Characters and Roots Recall that we have the maximal torus

$$T = \{\mathrm{diag}(t_1, t_2, t/t_2, t/t_1) | t_1, t_2, t \in \mathrm{GL}_1\}$$

which is identified with the Levi quotient of the Borel minimal parabolic $\mathbb{Q}$-subgroup P_0 of upper triangular matrices in $G = \mathrm{GSp}_4$.

We have the elementary characters $e_i : T \to \mathbb{G}_{\mathrm{m}}$ given by

$$e_1(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = t_1 \qquad e_2(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = t_2 \qquad e_0(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = t$$

We have the elementary cocharacters $f_i : \mathbb{G}_{\mathrm{m}} \to T$ given by

$$f_1(t) = \mathrm{diag}(t, 1, 1, 1/t) \qquad f_2(t) = \mathrm{diag}(1, t, 1/t, 1) \qquad f_0(t) = \mathrm{diag}(1, 1, t, t)$$

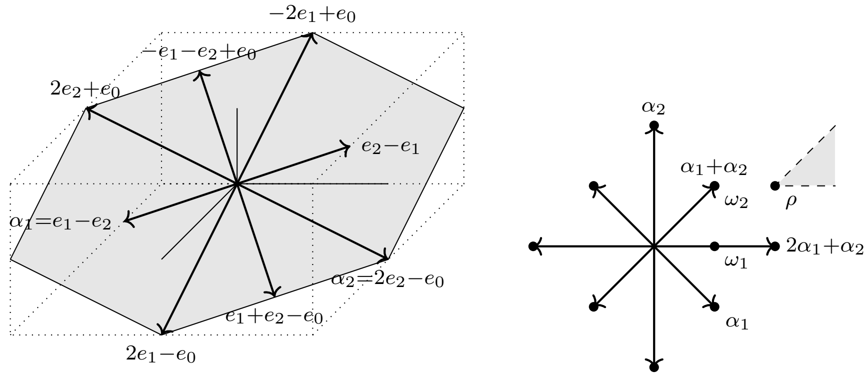
We have the character lattice $X^*(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_0$ and we have the cocharacter lattice $X_*(T) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}f_0$. We have the roots

$$\pm (e_1 - e_2) \qquad \pm (e_1 + e_2 - e_0)$$
$$\pm (2e_1 - e_0) \qquad \pm (2e_2 - e_0)$$

with simple roots $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2 - e_0$, and we have the coroots

$$\pm (e_1 - e_2)^\vee = \pm(f_1 - f_2) \qquad \pm (e_1 + e_2 - e_0)^\vee = \pm(f_1 + f_2)$$
$$\pm (2e_1 - e_0)^\vee = \pm f_1 \qquad \pm (2e_2 - e_0) = \pm f_2$$

and we have the fundamental weights $\omega_1 = e_1 - \frac{1}{2}e_0$ and $\omega_2 = e_1 + e_2 - e_0$ defined by $2\frac{\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}$. We have the half sum of positive roots $\rho_{P_0} = 2e_1 + e_2 - \frac{3}{2}e_0$.



For $z_1, z_2, z \in \mathbb{C}$ consider the unramified character $\chi_{z_1,z_2,z} : T \to \mathbb{G}_\mathrm{m}$ given by

$$\chi_{z_1,z_2,z}(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = |t_1|^{z_1}|t_2|^{z_2}|t|^{z}$$

so that $(\chi_{z_1,z_2,z} \circ \alpha_1^\vee)(t) = |t|^{z_1-z_2}$ and $(\chi_{z_1,z_2,z} \circ \alpha_2^\vee)(t) = |t|^{2z_2-z}$. In other words $\chi_{z_1,z_2,z} = \nu^{z_1} \otimes \nu^{z_2} \otimes \nu^{z}$ where $\nu = |\cdot|$ is the norm character.

Recall that $\mathfrak{a}_{P_0}^\vee = \mathfrak{a}_G^\vee \oplus (\mathfrak{a}_{P_0}^G)^\vee$ where

$$\begin{aligned}
\mathfrak{a}_{P_0}^\vee &= \{(z_1, z_2, z) \in \mathbb{R}^3\} \simeq \mathbb{R}^3 \\
\mathfrak{a}_G^\vee &= \{(z_1, z_2, z) \in \mathbb{R}^3 | z_1 - z_2 = 0 \text{ and } 2z_2 - z = 0\} \simeq \mathbb{R} \\
(\mathfrak{a}_{P_0}^G)^\vee &= \{(z_1, z_2, z) \in \mathbb{R}^3 | z_1 + z_2 + 2z = 0\} \simeq \mathbb{R}^2
\end{aligned}$$

with $s_1 = z_1 - z_2$ and $s_2 = 2z_2 - z$ providing coordinates for $(\mathfrak{a}_{P_0}^G)_\mathbb{C}^\vee$.

Parabolic Induction We consider the following standard parabolic $\mathbb{Q}$-subgroups of $G = \mathrm{GSp}_4$ and the corresponding parabolic inductions:

- We have the Borel parabolic $\mathbb{Q}$-subgroup $P_0 = M_0N_0$ with Levi quotient $M_0 = T$ and unipotent $\mathbb{Q}$-subgroup $N_0 = U$ given by

$$P_0 = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \cap G \qquad M_0 = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} \cap G \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \qquad N_0 = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \cap G$$

Write $M_0 = \{\mathrm{diag}(t_1, t_2, t/t_2, t/t_1) | t_1, t_2, t \in \mathrm{GL}_1\}$. For characters χ_1, χ_2, χ of GL_1 we have a character $\pi = \chi_1 \otimes \chi_2 \otimes \chi$ of M_0 given by

$$\pi(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = \chi_1(t_1)\chi_2(t_2)\chi(t)$$

We have the norm character δ_{P_0} of P_0 whose restriction to M_0 is given by

$$\delta_{P_0}(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = |t_1|^4 |t_2|^2 |t|^{-3}$$

For $(s_1, s_2) \in \mathbb{C}^2$ we have the unramified character

$$e^{\langle H_{P_0}(\cdot),(s_1,s_2)\rangle}(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = |t_1|^{s_1} |t_2|^{s_2} |t|^{-\frac{s_1+s_2}{2}}$$

Let $\pi = \chi_1 \otimes \chi_2 \otimes \chi : M_0(\mathbb{R}) \to \mathbb{C}^\times$ be a (continuous) character regarded as a character of $P_0(\mathbb{R})$. Consider the Borel parabolic induction

$$\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \left\{ \begin{array}{c} \text{smooth functions } \phi : G(\mathbb{R}) \to \mathbb{C} \text{ such that} \\ \phi(pg) = \chi_1(t_1)\chi_2(t_2)\chi(t)\phi(g) \text{ for every } g \in G(\mathbb{R}) \\ \text{and } p \in P_0(\mathbb{R}) \text{ with } p \equiv \mathrm{diag}(t_1, t_2, t/t_2, t/t_1) \in M_0(\mathbb{R}) \end{array} \right\}$$

Recalling the character $e^{\langle H_{P_0}(\cdot),(s_1,s_2)+\rho_{P_0}\rangle}(\mathrm{diag}(t_1, t_2, t/t_2, t/t_1)) = |t_1|^{s_1+2}|t_2|^{s_2+1}|t|^{-\frac{s_1+s_2+3}{2}}$ we consider the family of normalized Borel parabolic inductions

$$\mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi, s) = \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(e^{\langle H_{P_0}(\cdot),(s_1,s_2)+\rho_{P_0}\rangle}\pi)$$

In particular the norm character $\delta_{P_0(\mathbb{R})}^{1/2} = e^{\langle H_{P_0}(\cdot),\rho_{P_0}\rangle}$ defines the normalized Borel parabolic induction

$$\chi_1 \times \chi_2 \rtimes \chi = \mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\delta_{P_0(\mathbb{R})}^{1/2}\pi)$$

Let $W = \{1, s_1, s_2, s_{12}, s_{21}, s_{121}, s_{212}, s_{1212}\}$. Each Weyl group element will give rise to intertwining operators between the above Borel parabolic inductions.

- We have the Siegel parabolic $\mathbb{Q}$-subgroup $P_1 = M_1 N_1$ with Levi quotient M_1 and unipotent $\mathbb{Q}$-subgroup N_1 given by

$$P_1 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \cap G \qquad M_1 = \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix} \cap G \simeq \mathrm{GL}_2 \times \mathrm{GL}_1 \qquad N_1 = \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \cap G$$

  Write $M_1 = \{\mathrm{diag}(A, tA^{\top -1}) | A \in \mathrm{GL}_2, t \in \mathrm{GL}_1\}$. For μ a representation of GL_2 and for a character χ of GL_1 we have a representation $\pi = \mu \otimes \chi$ of M_1 given by

$$\pi(\mathrm{diag}(A, tA^{\top -1})) = \mu(A)\chi(t)$$

  We have the norm character δ_{P_1} of P_1 whose restriction to M_1 is given by

$$\delta_{P_1}(\mathrm{diag}(A, tA^{\top -1})) = |\det(A)|^3 |t|^{-3}$$

  For $s \in \mathbb{C}$ we have the unramified character

$$e^{\langle H_{P_1}(\cdot), s \rangle}(\mathrm{diag}(A, tA^{\top -1})) = |\det(A)|^s |t|^{-s}$$

  Let $\pi = \mu \otimes \chi$ be an irreducible admissible representation of $M_1(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$ on a Hilbert space H_π, regarded as an irreducible admissible representation of $P_1(\mathbb{R})$. Consider the Siegel parabolic induction

$$\mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \left\{ \begin{array}{c} \text{smooth functions } \phi : G(\mathbb{R}) \to H_\pi \text{ such that} \\ \phi(pg) = \pi(A)\chi(t)\phi(g) \text{ for every } g \in G(\mathbb{R}) \\ \text{and } p \in P_1(\mathbb{R}) \text{ with } p \equiv \mathrm{diag}(A, tA^{\top -1}) \in M_1(\mathbb{R}) \end{array} \right\}$$

  Recalling the character $e^{\langle H_{P_1}(\cdot), 2 + \rho_{P_1} \rangle}(\mathrm{diag}(A, tA^{\top -1})) = |\det(A)|^{s+\frac{3}{2}} |t|^{-s-\frac{3}{2}}$ we consider the family of normalized Siegel parabolic inductions

$$\mathcal{I}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\pi, s) = \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(e^{\langle H_{P_1}(\cdot), s + \rho_{P_0} \rangle} \pi)$$

In particular the norm character $\delta_{P_1(\mathbb{R})}^{1/2} = e^{\langle H_{P_1}(\cdot), \rho_{P_1}\rangle}$ defines the normalized Siegel parabolic induction

$$\pi \rtimes \chi = \mathcal{I}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\delta_{P_1(\mathbb{R})}^{1/2}\pi)$$

Let $W_{P_1} = \{1, s_1\}$ and let $W^{P_1} = W/W_{P_1} = \{1, s_2, s_{21}, s_{212}\}$. Each Weyl group element will give rise to intertwining operators between the above Siegel parabolic inductions.

- We have the Klingen parabolic $\mathbb{Q}$-subgroup P_2 with Levi quotient M_2 and unipotent $\mathbb{Q}$-subgroup N_2 given by

$$P_2 = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \cap G \qquad M_2 = \begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix} \cap G \simeq \mathrm{GL}_1 \times \mathrm{GL}_2 \qquad N_2 = \begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \cap G$$

Write $M_2 = \{\mathrm{diag}(t, A, t/\det(A)) | t \in \mathrm{GL}_1, A \in \mathrm{GL}_2\}$. For a character χ of GL_1 and for μ a representation of GL_2 we have a representation $\pi = \chi \otimes \mu$ of $M_2 \simeq \mathrm{GL}_1 \times \mathrm{GL}_2$ given by

$$\pi(\mathrm{diag}(t, A, t/\det(A))) = \chi(t)\mu(A)$$

We have the norm character δ_{P_2} of P_2 whose restriction to M_2 is given by

$$\delta_{P_2}(\mathrm{diag}(t, A, t/\det(A))) = |t|^4 |\det(A)|^{-2}$$

For $s \in \mathbb{C}$ we have the unramified character

$$e^{\langle H_{P_2}(\cdot), s\rangle}(\mathrm{diag}(t, A, t/\det(A))) = |t|^s |\det(A)|^{-\frac{s}{2}}$$

Let $\pi = \chi \otimes \mu$ be an irreducible admissible representation of $M_2(\mathbb{R}) \simeq \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ on a Hilbert space H_π, regarded as an irreducible admissible representation of $P_2(\mathbb{R})$. Consider the Klingen parabolic induction

$$\mathrm{Ind}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \left\{ \begin{array}{c} \text{smooth functions } \phi : G(\mathbb{R}) \to H_\pi \text{ such that} \\ \phi(pg) = \chi(t)\pi(A)\phi(g) \text{ for every } g \in G(\mathbb{R}) \\ \text{and } p \in P_2(\mathbb{R}) \text{ with } p \equiv \mathrm{diag}(t, A, t/\det(A)) \in M_2(\mathbb{R}) \end{array} \right\}$$

Recalling the character $e^{\langle H_{P_2}(\cdot),2+\rho_{P_2}\rangle}(\mathrm{diag}(t,A,t/\det(A))) = |t|^{s+2}|\det(A)|^{-\frac{s}{2}-1}$ we consider the family of normalized Klingen parabolic inductions

$$\mathcal{I}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\pi,s) = \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(e^{\langle H_{P_1}(\cdot),s+\rho_{P_0}\rangle}\pi)$$

In particular the norm character $\delta_{P_2(\mathbb{R})}^{1/2} = e^{\langle H_{P_2}(\cdot),\rho_{P_2}\rangle}$ defines the normalized Klingen parabolic induction

$$\pi \rtimes \chi = \mathcal{I}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\pi) = \mathrm{Ind}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\delta_{P_2(\mathbb{R})}^{1/2}\pi)$$

Let $W_{P_2} = \{1,s_2\}$ and let $W^{P_2} = W/W_{P_2} = \{1,s_1,s_{12},s_{121}\}$. Each Weil group element will give rise to intertwining operators between the above Klingen parabolic inductions.

All such parabolic inductions are regarded as representations of $\mathrm{GSp}_4(\mathbb{R})$ by the right translation action, or regarded as admissible $(\mathfrak{gsp}_4, K_\infty)$-modules after passing to the subspace of K_∞-finite vectors (which we abusively denote by the same notation). We can restrict these to representations of $\mathrm{Sp}_4(\mathbb{R})$, or to $(\mathfrak{sp}_4, K_\infty)$-modules. The decomposition into K_∞-types for each of these parabolically induced representations appears for example in [90, Lemma 6.1].

Langlands Quotients We will need to consider certain Langlands quotients with nontrivial $(\mathfrak{g}, K_\infty')$-cohomology, particularly those associated with the maximal parabolic $\mathbb{Q}$-subgroups of GSp_4.

We have the Langlands quotient $\pi_\infty^{(1)+} = L(\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2})$ which is the unique irreducible quotient of the normalized Siegel parabolic induction $\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}$:

$$\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2} = \mathcal{I}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\lambda_1+\lambda_2+3}\otimes 1, \tfrac{1}{2}) \twoheadrightarrow \mathcal{J}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\lambda_1+\lambda_2+3}\otimes 1, \tfrac{1}{2}) = \pi_\infty^{(1)+}$$

We have the Langlands quotient $\pi_\infty^{(1)-} = L(\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}\mathrm{sign})$ which is the unique irreducible quotient of the normalized Siegel parabolic induction $\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}\mathrm{sign}$:

$$\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}\mathrm{sign} = \mathcal{I}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\lambda_1+\lambda_2+3}\otimes \mathrm{sign}, \tfrac{1}{2}) \twoheadrightarrow \mathcal{J}_{P_1(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\lambda_1+\lambda_2+3}\otimes \mathrm{sign}, \tfrac{1}{2}) = \pi_\infty^{(1)-}$$

The Langlands quotients $\pi_\infty^{(1)+}$ and $\pi_\infty^{(1)-}$ are related by character twist: we have $\pi_\infty^{(1)-} = \pi_\infty^{(1)+}\otimes\mathrm{sign}$.

We have the Langlands quotient $\pi_\infty^{(2)} = L(\nu\mathrm{sign} \rtimes \nu^{-1/2}\mathcal{D}^{\mathrm{H}}_{\lambda_1+2})$ is the unique irreducible quotient of the normalized Klingen parabolic induction $\nu\mathrm{sign} \rtimes \nu^{-1/2}\mathcal{D}^{\mathrm{H}}_{\lambda_1+2}$:

$$\nu\mathrm{sign} \rtimes \nu^{-1/2}\mathcal{D}^{\mathrm{H}}_{\lambda_1+2} = \mathcal{I}^{G(\mathbb{R})}_{P_2(\mathbb{R})}(\mathrm{sign} \otimes \mathcal{D}^{\mathrm{H}}_{\lambda_1+2}, 1) \twoheadrightarrow \mathcal{J}^{G(\mathbb{R})}_{P_2(\mathbb{R})}(\mathrm{sign} \otimes \mathcal{D}^{\mathrm{H}}_{\lambda_1+2}, 1) = \pi_\infty^{(2)}$$

The 1-dimensional representations of $\mathrm{GSp}_4(\mathbb{R})$ are also Langlands quotients associated with the minimal parabolic $\mathbb{Q}$-subgroup of GSp_4.

Discrete Series Representations We recall the Harish-Chandra classification of discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ and of $\mathrm{GSp}_4(\mathbb{R})$, following [104, Section 2.2].

Let $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$ be the set of roots of $\mathfrak{sp}_4$. Under the above identification we have $\pm\alpha_1 = \pm(1,-1)$, $\pm\alpha_2 = \pm(0,2)$, $\pm(\alpha_1+\alpha_2) = \pm(1,1)$, $\pm(2\alpha_1+\alpha_2) = \pm(2,0)$. Among these, the roots $\pm\alpha_1$ are compact roots, while the remaining roots $\pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)$ are non-compact roots.

Let $W = \{1, s_1, s_2, s_{12}, s_{21}, s_{121}, s_{212}, s_{1212} = s_{2121}\}$ be the Weyl group of this root system whose elements are reflections along hyperplanes perpendicular to the roots. Let $W_K = \{1, s_1\}$ be the compact Weyl group whose elements are reflections along the hyperplane perpendicular to the compact roots $\pm\alpha_1$.

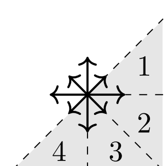
Recall that an element $\lambda \in \mathfrak{h}^\vee$ is nonsingular if $\langle\lambda,\alpha\rangle \neq 0$ for every root $\alpha \in \Phi$, that is λ is not contained in a root hyperplane. Every nonsingular element $\lambda \in \mathfrak{h}^\vee$ determines a system of positive roots

$$\Phi^+_\lambda = \{\alpha \in \Delta | \langle\lambda,\alpha\rangle > 0\}$$

and determines a half sum of positive roots

$$\rho^{\mathrm{nc}}_\lambda = \frac{1}{2}\sum_{\substack{\alpha\in\Phi^+_\lambda \\ \text{noncompact}}} \alpha \qquad \rho^{\mathrm{c}}_\lambda = \frac{1}{2}\sum_{\substack{\alpha\in\Phi^+_\lambda \\ \text{compact}}} \alpha \qquad \rho_\lambda = \rho^{\mathrm{nc}}_\lambda + \rho^{\mathrm{c}}_\lambda = \frac{1}{2}\sum_{\alpha\in\in\Phi^+_\lambda} \alpha$$

Up to the action of the compact Weyl group W_K we may consider the following four regions:



In each of the four regions the values of $\rho_\lambda^{\mathrm{nc}}$ and $\rho_\lambda^{\mathrm{c}}$ and their sum $\rho_\lambda = \rho_\lambda^{\mathrm{nc}} + \rho_\lambda^{\mathrm{c}}$ and difference $\rho_\lambda^{\mathrm{nc}} - \rho_\lambda^{\mathrm{c}}$ are given by

#	$\rho_\lambda^{\mathrm{nc}}$	$\rho_\lambda^{\mathrm{c}}$	$\rho_\lambda = \rho_\lambda^{\mathrm{nc}} + \rho_\lambda^{\mathrm{c}}$	$\rho_\lambda^{\mathrm{nc}} - \rho_\lambda^{\mathrm{c}}$
1	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(2, 1)$	$(1, 2)$
2	$(\frac{3}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(2, -1)$	$(1, 0)$
3	$(\frac{1}{2}, -\frac{3}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(1, -2)$	$(0, -1)$
4	$(-\frac{3}{2}, -\frac{3}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-1, -2)$	$(-2, -1)$

Now by Harish-Chandra we have the following:

Proposition 4.2.1. ([66, Theorem 9.20, Theorem 12.21]) Every analytically integral nonsingular weight $\lambda \in \mathfrak{h}^\vee$ determines a discrete series representation $\mathcal{D}_\lambda$ of $\mathrm{Sp}_4(\mathbb{R})$ with infinitesimal character λ, and every discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ is of this form. Two discrete series representations $\mathcal{D}_\lambda$ and $\mathcal{D}_{\lambda'}$ are equivalent precisely if $\lambda = w\lambda'$ for some $w \in W_K$.

Note that every analytically integral weight $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{h}^\vee$ with $\lambda_1 \geq \lambda_2$ gives rise to an analytically integral nonsingular weight $\lambda + \rho_\lambda \in \mathfrak{h}^\vee$, and every analytically integral nonsingular weight is of this form. The corresponding discrete series representation $\mathcal{D}_{\lambda+\rho_\lambda}$ of $\mathrm{Sp}_4(\mathbb{R})$ has Harish-Chandra parameter $\lambda + \rho_\lambda$ and Blattner parameter $\Lambda = \lambda + 2\rho_\lambda^{\mathrm{nc}}$, as $\mathcal{D}_\lambda$ has Harish-Chandra parameter λ and Blattner parameter $\Lambda = \lambda + \rho_\lambda^{\mathrm{nc}} - \rho_\lambda^{\mathrm{c}}$. Note that the Harish-Chandra parameter corresponds to the infinitesimal character, while the Blattner parameter corresponds to the minimal K_∞-type.

For $\mathrm{Sp}_4(\mathbb{R})$ we have the following discrete series representations:

(i) (Holomorphic discrete series) For integers $\lambda_1 \geq \lambda_2 \geq 0$ we have a corresponding analytically integral weight $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{h}^\vee$ and a corresponding analytically integral nonsingular weight $\lambda + \rho_\lambda = (\lambda_1 + 2, \lambda_2 + 1) \in \mathfrak{h}^\vee$. Then we have the holomorphic discrete series representation $\mathcal{D}_{\lambda_1+2,\lambda_2+1}$ of $\mathrm{Sp}_4(\mathbb{R})$ with Harish-Chandra parameter $(\lambda_1 + 2, \lambda_2 + 1)$ and Blattner parameter $\Lambda = (\lambda_1 + 3, \lambda_2 + 3)$.

(ii) (Antiholomorphic discrete series) For integers $\lambda_1 \geq \lambda_2 \geq 0$ we have a corresponding analytically integral weight $\lambda = (-\lambda_2, -\lambda_1) \in \mathfrak{h}^\vee$ and a corresponding analytically integral nonsingular weight $\lambda + \rho_\lambda = (-\lambda_2 - 1, -\lambda_1 - 2) \in \mathfrak{h}^\vee$. Then we have the antiholomorphic discrete series representation $\mathcal{D}_{-\lambda_2-1,-\lambda_1-2}$ of $\mathrm{Sp}_4(\mathbb{R})$ with Harish-Chandra parameter $(-\lambda_2 - 1, -\lambda_1 - 2)$ and Blattner parameter $\Lambda = (-\lambda_2 - 3, -\lambda_1 - 3)$.

(iii) (Large generic discrete series) For integers $\lambda_1 \geq \lambda_2 \geq 0$ we have a corresponding analytically integral weight $\lambda = (\lambda_1, -\lambda_2) \in \mathfrak{h}^\vee$ and a corresponding analytically integral nonsingular weight $\lambda + \rho_\lambda = (\lambda_1 + 2, -\lambda_2 - 1) \in \mathfrak{h}^\vee$. Then we have the large generic discrete series representation $\mathcal{D}_{\lambda_1+2,-\lambda_2-1}$ of $\mathrm{Sp}_4(\mathbb{R})$ with Harish-Chandra parameter $(\lambda_1+2, -\lambda_2-1)$ and Blattner parameter $\Lambda = (\lambda_1+3, -\lambda_2-1)$.

(iv) (Large generic discrete series) For integers $\lambda_1 \geq \lambda_2 \geq 0$ we have a corresponding analytically integral weight $\lambda = (\lambda_2, -\lambda_1) \in \mathfrak{h}^\vee$ and a corresponding analytically integral nonsingular weight $\lambda + \rho_\lambda = (\lambda_2 + 1, -\lambda_1 - 2) \in \mathfrak{h}^\vee$ Then we have the large generic discrete series representation $\mathcal{D}_{\lambda_2+1,-\lambda_1-2}$ of $\mathrm{Sp}_4(\mathbb{R})$ with Harish-Chandra parameter $(\lambda_2+1, -\lambda_1-2)$ and Blattner parameter $\Lambda = (\lambda_2+1, -\lambda_1-3)$.

These discrete series representations all occur as infinite-dimensional subrepresentations normalized parabolic inductions. We can regard such discrete as representations of $\mathrm{Sp}_4(\mathbb{R})$, or as $(\mathfrak{sp}_4, K_\infty)$-modules after passing to spaces of K_∞-finite vectors.

For $\mathrm{GSp}_4(\mathbb{R})$ we have the following discrete series representations:

(i) (Holomorphic discrete series) For integers $\lambda_1 \geq \lambda_2 \geq 0$ and a central character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ with $\omega(-1) = (-1)^{\lambda_1+\lambda_2}$ we have the holomorphic discrete series representation $\mathcal{D}_{\lambda_1+2,\lambda_2+1}(\omega)$ of $\mathrm{GSp}_4(\mathbb{R})$ with Harish-Chandra parameter $(\lambda_1 + 2, \lambda_2 + 1)$ and Blattner parameter $\Lambda = (\lambda_1 + 3, \lambda_2 + 3)$.

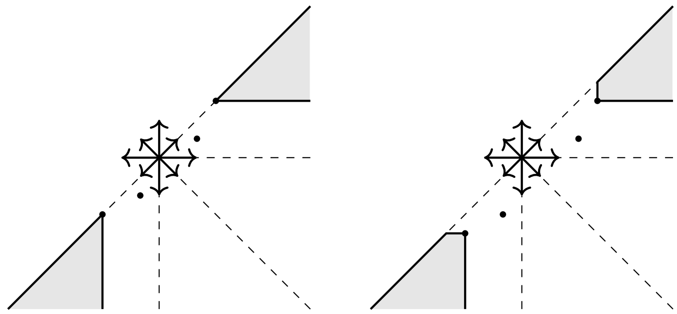
(ii) (Large generic discrete series) For integers $\lambda_1 \geq \lambda_2 \geq 0$ and a central character $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ with $\omega(-1) = (-1)^{\lambda_1+\lambda_2}$ we have the large generic discrete series representation $\mathcal{D}_{\lambda_1+2,-\lambda_2-1}(\omega)$ of $\mathrm{GSp}_4(\mathbb{R})$ with Harish-Chandra parameter $(\lambda_1 + 2, -\lambda_2 - 1)$ and Blattner parameter $\Lambda = (\lambda_1 + 3, -\lambda_2 - 1)$.

When the central character ω is trivial we write $\mathcal{D}_{\lambda_1+2,\lambda_2+1}$ and $\mathcal{D}_{\lambda_1+2,-\lambda_2-1}$ for the corresponding holomorphic discrete series and large generic discrete series representations of $\mathrm{GSp}_4(\mathbb{R})$. We can regard such discrete series as representations of $\mathrm{GSp}_4(\mathbb{R})$, or as $(\mathfrak{gsp}_4, K_\infty)$-modules after passing to spaces of K_∞-finite vectors. Here we may choose to work with $K_\infty \simeq \mathrm{U}(2)$ as above (which is insensitive to the central character ω), or with $K'_\infty = \mathbb{R}_{>0} K_\infty$ (which is insensitive to the parity of the central character ω), or with $K''_\infty = \mathbb{R}^\times K_\infty$ (which is sensitive to the central character ω).

For the holomorphic discrete series representation $\mathcal{D}_{\lambda_1+2,\lambda_2+1}(\omega)$, restricting from $\mathrm{GSp}_4(\mathbb{R})$ to $\mathrm{Sp}_4(\mathbb{R})$ yields an isomorphism

$$\mathcal{D}_{\lambda_1+2,\lambda_2+1}(\omega)|_{\mathrm{Sp}_4(\mathbb{R})} = \mathcal{D}_{\lambda_1+2,\lambda_2+1} \oplus \mathcal{D}_{-\lambda_2-1,-\lambda_1-2}$$

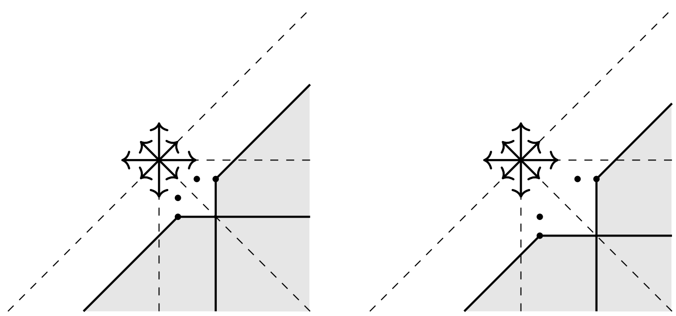
We obtain the following K_∞-type regions, illustrated for $\lambda_1 = \lambda_2 = 0$ and $\lambda_1 = \lambda_2 = 1$:



For the large generic discrete series representation $\mathcal{D}_{\lambda_1+2,-\lambda_2-1}(\omega)$, restricting from $\mathrm{GSp}_4(\mathbb{R})$ to $\mathrm{Sp}_4(\mathbb{R})$ yields an isomorphism

$$\mathcal{D}_{\lambda_1+2,-\lambda_2-1}(\omega)|_{\mathrm{Sp}_4(\mathbb{R})} = \mathcal{D}_{\lambda_1+2,-\lambda_2-1} \oplus \mathcal{D}_{\lambda_2+1,-\lambda_1-2}$$

We obtain the following K_∞-type regions, illustrated for $\lambda_1 = \lambda_2 = 0$ and $\lambda_1 = \lambda_2 = 1$:



One also has the limit discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ and of $\mathrm{GSp}_4(\mathbb{R})$, which we will not need; their structure is recalled in [104, Section 2.3].

By Harish-Chandra the discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ occur as infinite-dimensional subrepresentations of normalized parabolic inductions. The construction of the above discrete series representations and Langlands quotients from normalized parabolic inductions involves the analysis of intertwining operators between families of normalized parabolic inductions. The relevant analysis of intertwining operators is given by [90, Lemma 7.2, Lemma 7.3]. A complete list of irreducible admissible $(\mathfrak{sp}_4, K_\infty)$-modules appears in [90, Section 8], and the composition series of such representations with integral infinitesimal characters appears in [90, Section 9, Section 10, Section 11].

Vogan-Zuckerman Classification We quickly recall the Vogan-Zuckerman classification of irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{R})$ with nonzero cohomology.

Let T_{c}^1 be a maximal torus in the maximal compact subgroup $K_\infty = \mathrm{U}(2)$ of $G^1(\mathbb{R}) = \mathrm{Sp}_4(\mathbb{R})$ so that the centralizer T_{c} of T_{c}^1 is a maximal torus in $G(\mathbb{R}) = \mathrm{GSp}_4(\mathbb{R})$. Let $\mu : \mathbb{G}_{\mathrm{m}} \to T_{\mathrm{c}}^1$ be a cocharacter over $\mathbb{R}$, let $Z_\mu \subseteq G(\mathbb{R})$ be the corresponding centralizer, and let $Q_\mu \subseteq G(\mathbb{C})$ be the corresponding parabolic subgroup with Lie $\mathbb{C}$-algebra $\mathfrak{q}$. Let $\lambda \in X^*(T_{\mathbb{C}})$ be a highest weight for $G(\mathbb{C})$ which is sim-self-dual, and suppose that the highest weight λ is trivial on the semisimple part Z_μ^1 of $Z_\mu \subseteq G(\mathbb{R})$, that is λ extends to a character $\lambda : Q_\mu \to \mathbb{C}^\times$. We have four cases depending on μ, in each case obtaining a constraint on λ:

- (μ regular) We have $Z_\mu = M_0$ and $Q_\mu = P_0$ with Lie $\mathbb{C}$-algebra $\mathfrak{q} = \mathfrak{p}_0$ and $A_{\mathfrak{q}}(\lambda)$ is a tempered $(\mathfrak{g}, K'_\infty)$-module. The set of regular cocharacters $X_*(T^1)^0_{\mathbb{R}}$ is the complement of finitely many root hyperplanes and is a disjoint union of connected components parameterized by the Weyl group W. The representations $A_{\mathfrak{q}}(\lambda)$ are locally constant in $\chi \in X_*(T^1)^0_{\mathbb{R}}$, and after choosing a basepoint $[\mu_0] \in \pi_0(X_*(T^1)^0_{\mathbb{R}})$ we obtain a family of representations $\{A_{w\mu_0}(\lambda)\}_{w \in W_K \backslash W}$ which by Vogan-Zuckerman are pairwise non-isomorphic and are the Harish-Chandra modules corresponding to discrete series representations of $G(\mathbb{R})$. Here we have no constraint on λ and one of two possibilities:

  (i) $A_{\mathfrak{q}}(\lambda) = \pi_\infty^{\mathrm{H}}$ is the holomorphic discrete series representation

  $$\pi_\infty^{\mathrm{H}} = \mathcal{D}_{\lambda_1+2,\lambda_2+1}$$

  (underlying holomorphic vector-valued Siegel modular forms of weight $(\lambda_1 - \lambda_2, \lambda_2 + 3)$) where $H^3(\mathfrak{g}, K'_\infty; \pi_\infty^{\mathrm{H}} \otimes V_\lambda)$ is 2-dimensional with Hodge numbers $(\lambda_1 + \lambda_2 + 3, 0)$ and $(0, \lambda_1 + \lambda_2 + 3)$:

  $$H^3(\mathfrak{g}, K'_\infty; \pi_\infty^{\mathrm{H}} \otimes V_\lambda) = \begin{matrix} H^3(\mathfrak{sp}_4, K_\infty; \mathcal{D}_{\lambda_1+2,\lambda_2+1} \otimes V_\lambda) \\ \oplus H^3(\mathfrak{sp}_4, K_\infty; \mathcal{D}_{-\lambda_2-1,-\lambda_1-2} \otimes V_\lambda) \end{matrix} = \mathbb{C} \oplus \mathbb{C}$$

  (ii) $A_{\mathfrak{p}}(\lambda) = \pi_\infty^{\mathrm{W}}$ is the large generic discrete series representation

  $$\pi_\infty^{\mathrm{W}} = \mathcal{D}_{\lambda_1+2,-\lambda_2-1}$$

(admitting a Whittaker model) where $H^3(\mathfrak{g}, K'_\infty; \pi_\infty^{\mathrm{W}} \otimes V_\lambda)$ is 2-dimensional with Hodge numbers $(\lambda_1+2, \lambda_2+1)$ and $(\lambda_2+1, \lambda_1+2)$:

$$H^3(\mathfrak{g}, K'_\infty; \pi_\infty^{\mathrm{W}} \otimes V_\lambda) = \begin{matrix} H^3(\mathfrak{sp}_4, K_\infty; \mathcal{D}_{\lambda_1+2,-\lambda_2-1} \otimes V_\lambda) \\ \oplus H^3(\mathfrak{sp}_4, K_\infty; \mathcal{D}_{\lambda_2+1,-\lambda_1-2} \otimes V_\lambda) \end{matrix} = \mathbb{C} \oplus \mathbb{C}$$

- (μ singular, M_1-regular) We have $Z_\mu = M_1$ and $Q_\mu = P_1$ with Lie $\mathbb{C}$-algebra $\mathfrak{q} = \mathfrak{p}_1$, and we must have $\lambda_1 = \lambda_2$ so that $A_\mathfrak{q}(\lambda)$ is a non-tempered $(\mathfrak{g}, K'_\infty)$-module. The set of singular but M_1-regular cocharacters $X_*(T^1_{M_1})^0_\mathbb{R}$ is the complement of the trivial cocharacter and is a disjoint union of connected components parameterized by the Weyl group $W_{M_1} = \{1, s_2\}$. The representations $A_1(\lambda)$ are locally constant in $\mu \in X_*(T^1_{M_1})^0_\mathbb{R}$, and after choosing a basepoint $[\mu_0] \in \pi_0(X_*(T^1_{M_1})^0_\mathbb{R})$ we obtain a family of representations $\{A_{w\mu_0}(\lambda)\}_{w \in W_K \backslash W_{M_1}}$ which by Vogan-Zuckerman are pairwise non-isomorphic and are Harish-Chandra modules corresponding non-tempered Langlands quotients. Here we must have $\lambda_1 = \lambda_2$ and one possibility:

  (iii) $A_\mathfrak{q}(\lambda) = \pi_\infty^{(1)\pm}$ is a non-tempered Langlands quotient for the Siegel parabolic P_1 (these are the representations $\pi^{2\pm}$ in the notation of [114] and [96]):

  $$\pi_\infty^{(1)+} = L(\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}) \qquad \pi_\infty^{(1)-} = L(\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}\mathrm{sign})$$

  where $H^2(\mathfrak{g}, K'_\infty; \pi_\infty^{(1)\pm} \otimes V_\lambda)$ and $H^4(\mathfrak{g}, K'_\infty; \pi_\infty^{(1)\pm} \otimes V_\lambda)$ are 1-dimensional with Hodge numbers $(\lambda_1+1, \lambda_2+1)$ and $(\lambda_1+2, \lambda_2+2)$ respectively:

  $$H^2(\mathfrak{g}, K'_\infty; \pi_\infty^{(1)\pm} \otimes V_\lambda) = \mathbb{C}$$
  $$H^4(\mathfrak{g}, K'_\infty; \pi_\infty^{(1)\pm} \otimes V_\lambda) = \mathbb{C}$$

- (μ singular, M_2-regular) We have $Z_\mu = M_2$ and $Q_\mu = P_2$ with Lie $\mathbb{C}$-algebra $\mathfrak{q} = \mathfrak{p}_2$, and we must have $\lambda_2 = 0$ so that $A_\mathfrak{q}(\lambda)$ is a non-tempered $(\mathfrak{g}, K'_\infty)$-module. The set of singular but M_2-regular cocharacters $X_*(T^1_{M_2})^0_\mathbb{R}$ is the complement of the trivial cocharacter and is a disjoint union of connected components parameterized by the Weyl group $W_{M_2} = \{1, s_1\}$. The representations $A_\mathfrak{q}(\lambda)$ are locally constant in $\mu \in X_*(T^1_{M_2})^0_\mathbb{R}$, and after choosing a basepoint $[\mu_0] \in \pi_0(X_*(T^1_{M_2})^0_\mathbb{R})$ we obtain a family of representations

$\{A_{w\mu_0}(\lambda)\}_{w\in W_K\backslash W_{M_2}}$ which by Vogan-Zuckerman are non-tempered Langlands quotients. Here we must have $\lambda_2 = 0$ and one possibility:

(iv) $A_{\mathfrak{p}}(\lambda) = \pi_\infty^{(2)}$ is a non-tempered Langlands quotient for the Klingen parabolic P_2 (this is the representation π^1 in the notation of [114] and [96]):

$$\pi_\infty^{(2)} = L(\nu\mathrm{sign} \rtimes \nu^{-1/2}\mathcal{D}^{\mathrm{H}}_{\lambda_1+2})$$

where $H^2(\mathfrak{g}, K'_\infty; \pi_\infty^{(2)} \otimes V_\lambda)$ and $H^4(\mathfrak{g}, K'_\infty; \pi_\infty^{(2)} \otimes V_\lambda)$ are 2-dimensional with Hodge numbers (λ_1+2, λ_2) and (λ_1, λ_2+2), respectively $(\lambda_1+3, \lambda_2+1)$ and $(\lambda_1+1, \lambda_2+3)$:

$$H^2(\mathfrak{g}, K'_\infty; \pi_\infty^{(2)} \otimes V_\lambda) = \mathbb{C}$$
$$H^4(\mathfrak{g}, K'_\infty; \pi_\infty^{(2)} \otimes V_\lambda) = \mathbb{C}$$

- (μ trivial) We have $Z_\mu = G$ and $Q_\mu = G$ with Lie $\mathbb{C}$-algebra $\mathfrak{q} = \mathfrak{g}$. Here we must have $\lambda_1 = \lambda_2 = 0$ and one possibility:

(v) $A_{\mathfrak{q}}(0) = \chi_\infty \in \{1, \mathrm{sign}\}$ is a character where $H^0(\mathfrak{g}, K'_\infty; \mathbb{C})$, $H^2(\mathfrak{g}, K'_\infty; \mathbb{C})$, $H^4(\mathfrak{g}, K'_\infty; \mathbb{C})$, and $H^6(\mathfrak{g}, K'_\infty; \mathbb{C})$ are 1-dimensional with Hodge numbers $(0,0)$, $(1,1)$, $(2,2)$, and $(3,3)$ respectively:

$$H^0(\mathfrak{g}, K'_\infty; \chi_\infty \otimes V_\lambda) = H^0(\mathfrak{sp}_4, K_\infty; \chi_\infty \otimes V_\lambda) = \mathbb{C}$$
$$H^2(\mathfrak{g}, K'_\infty; \chi_\infty \otimes V_\lambda) = H^2(\mathfrak{sp}_4, K_\infty; \chi_\infty \otimes V_\lambda) = \mathbb{C}$$
$$H^4(\mathfrak{g}, K'_\infty; \chi_\infty \otimes V_\lambda) = H^4(\mathfrak{sp}_4, K_\infty; \chi_\infty \otimes V_\lambda) = \mathbb{C}$$
$$H^6(\mathfrak{g}, K'_\infty; \chi_\infty \otimes V_\lambda) = H^6(\mathfrak{sp}_4, K_\infty; \chi_\infty \otimes V_\lambda) = \mathbb{C}$$

In particular, the irreducible admissible representations of $G(\mathbb{R}) = \mathrm{GSp}_4(\mathbb{R})$ with nonzero $(\mathfrak{g}, K'_\infty)$-cohomology are either discrete series representations (with cohomology concentrated in middle degree), or Langlands quotients of parabolic inductions (with cohomology concentrated away from middle degree). The Langlands quotients with cohomology closest to middle degree are associated with the maximal parabolic $\mathbb{Q}$-subgroups of G, while the 1-dimensional representations (which are Langlands quotients associated with the minimal parabolic $\mathbb{Q}$-subgroup of G) have cohomology as far from middle degree as possible.

Finally we summarize the constraints on the possible Hodge numbers appearing in intersection cohomology provided by the above classification:

- $(\lambda_1 > \lambda_2 > 0)$ The only Hodge numbers appearing in $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ are given by

$$\begin{array}{lccccccc} H^3 & (\lambda_1+\lambda_2+3,0) & \cdots & (\lambda_1+2,\lambda_2+1) & \cdots & (\lambda_2+1,\lambda_1+2) & \cdots & (0,\lambda_1+\lambda_2+3) \end{array}$$

- $(\lambda_1 = \lambda_2 > 0)$ The only Hodge numbers appearing in $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ are given by

$$\begin{array}{lcccccc} H^4 & & & \multicolumn{2}{c}{(\lambda_1+2,\lambda_1+2)} & & \\ H^3 & (2\lambda_1+3,0) & \cdots & (\lambda_1+2,\lambda_1+1) & (\lambda_1+1,\lambda_1+2) & \cdots & (0,2\lambda_1+3) \\ H^2 & & & \multicolumn{2}{c}{(\lambda_1+1,\lambda_1+1)} & & \end{array}$$

- $(\lambda_1 > \lambda_2 = 0)$ The only Hodge numbers appearing in $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ are given by

$$\begin{array}{lccccccc} H^4 & & (\lambda_1+3,1) & & & & (1,\lambda_1+3) & \\ H^3 & (\lambda_1+3,0) & & (\lambda_1+2,1) & \cdots & (1,\lambda_1+2) & & (0,\lambda_1+3) \\ H^2 & & (\lambda_1+2,0) & & & & (0,\lambda_1+2) & \end{array}$$

- $(\lambda_1 = \lambda_2 = 0)$ The only Hodge numbers appearing in $H^i(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ are given by

$$\begin{array}{lccccccc} H^6 & & & & (3,3) & & & \\ & & & & & & & \\ H^4 & & (3,1) & & (2,2) & & (1,3) & \\ H^3 & (3,0) & & (2,1) & & (1,2) & & (0,3) \\ H^2 & & (2,0) & & (1,1) & & (0,2) & \\ & & & & & & & \\ H^0 & & & & (0,0) & & & \end{array}$$

In particular for $\lambda_1 > \lambda_2 > 0$ the intersection cohomology must be concentrated in middle degree, for $\lambda_1 = \lambda_2 > 0$ or $\lambda_1 > \lambda_2 = 0$ the intersection cohomology must be concentrated in degrees 2, 3, 4, and for $\lambda = 0$ one simply finds the expected Hodge diamond of a smooth projective threefold. In each of these cases notice that H^1 and H^5 must vanish.

4.3 Eisenstein Cohomology for GSp_4

The goal of this section is to compute the Eisenstein cohomology of local systems on Siegel modular threefolds. To that end, we review the structure of automorphic Eisenstein series for GSp_4, and describe the Franke-Schwermer filtration on spaces of automorphic forms for GSp_4 in terms of the poles of such automorphic Eisenstein series, and then compute the relevant $(\mathfrak{g}, K'_\infty)$-cohomology.

4.3.1 Eisenstein Series for GSp_4

In this section we will consider the spectral decomposition of $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ for $G = \mathrm{GSp}_4$: the continuous spectrum $L^2_{\mathrm{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is described in terms of automorphic Eisenstein series, while the residual spectrum $L^2_{\mathrm{res}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ is described in terms of the poles of such automorphic Eisenstein series.

Intertwining Operators and Normalization Factors To compute the Langlands normalization factors for the minimal parabolic $\mathbb{Q}$-subgroup we simply apply the Gindikin-Karepelevich formula as we have for GL_2, although now the Weil group combinatorics is more involved. To compute the Langlands normalization factors for the maximal parabolic $\mathbb{Q}$-subgroups P_1 and P_2 it suffices to understand the adjoint action of ${}^LM_i$ on ${}^L\mathfrak{n}_i$.

The main result is the following:

Proposition 4.3.1. (Compare to [69, Section 5.1]) We have the following Langlands normalization factors $r(w, \pi_s)$:

- ($P = P_1$) For $\pi = \mu \otimes \chi$ and for the reflection w_{212} we have the normalization factor

$$r(w_2, \pi_s) = \frac{\Lambda(s, \mu)\Lambda(2s, \omega_\mu)}{\Lambda(s+1, \mu)\Lambda(2s+1, \omega_\mu)\epsilon(s, \mu)\epsilon(2s, \omega_\mu)}$$

- ($P = P_2$) For $\pi = \chi \otimes \mu$ and for the reflection w_{121} we have the normalization factor

$$r(w_1, \pi_s) = \frac{\Lambda(s, \mathrm{Ad}(\mu) \times \chi)}{\Lambda(s, \mathrm{Ad}(\mu) \times \chi)\epsilon(s, \mathrm{Ad}(\mu) \times \chi)}$$

- ($P = P_0$) For $\pi = \chi_1 \otimes \chi_2 \otimes \chi$ and simple reflections w_i corresponding to α_i we have the normalization factors

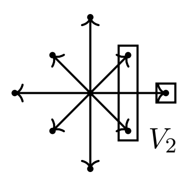
$$
\begin{aligned}
r(w_1, \pi_s) &= \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\epsilon(s_1-s_2,\chi_1\chi_2^{-1})}\\
r(w_2, \pi_s) &= \frac{\Lambda(s_2,\chi_2)}{\Lambda(s_2+1,\chi_2)\epsilon(s_2,\chi_2)}\\
r(w_{12}, \pi_s) &= \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)}\\
r(w_{21}, \pi_s) &= \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})\Lambda(s_1,\chi_1)}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\Lambda(s_1+1,\chi_1)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})\epsilon(s_1,\chi_1)}\\
r(w_{121}, \pi_s) &= \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})\Lambda(s_1,\chi_1)\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\Lambda(s_1+1,\chi_1)\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})\epsilon(s_1,\chi_1)\epsilon(s_1+s_2,\chi_1\chi_2)}\\
r(w_{212}, \pi_s) &= \frac{\Lambda(s_2,\chi_2)L(s_1+s_2,\chi_1\chi_2)\Lambda(s_1,\chi_1)}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\Lambda(s_1,\chi_1)\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)\epsilon(s_1,\chi_1)}\\
r(w_{1212}, \pi_s) &= \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)\Lambda(s_1,\chi_1)\Lambda(s_1-s_2,\chi_1\chi_2^{-1})}{\Lambda(s_2+1,\chi_2)L(s_1+s_2+1,\chi_1\chi_2)\Lambda(s_1,\chi_1)L(s_1-s_2+1,\chi_1\chi_2^{-1})\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)\epsilon(s_1,\chi_1)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})}
\end{aligned}
$$

Proof. We have the following:

- ($P = P_1$) We have the Levi decomposition $P_1 = M_1 N_1$ where $M_1 = \mathrm{GL}_2 \times \mathrm{GL}_1$ and N_1 is a length 2 unipotent algebraic group $N_1' \subseteq N_1$ with $N_1/N_1' = V_2$ the standard representation and $N_1' = \det$ the determinant representation, so the adjoint action of ${}^L M_1$ on ${}^L\mathfrak{n}_1$ is given

$$
{}^L\mathfrak{n}_1 = R_1^\vee \oplus R_2^\vee = \rho_2 \oplus \wedge^2 \rho_2
$$

where ρ_2 is the standard representation of ${}^L M_1 = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.



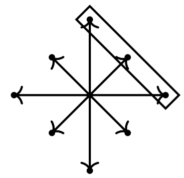
It follows that the Langlands normalizing factor is given

$$
r(w_{212}, \pi_s) = \prod_{1 \le j \le 2} \frac{\Lambda(js, \pi, R_j^\vee)}{\Lambda(js+1, \pi, R_j^\vee)\epsilon(js+1, \pi, R_j^\vee)} = \frac{\Lambda(s, \mu)\Lambda(2s, \omega_\mu)}{\Lambda(s+1, \mu)\Lambda(2s+1, \omega_\mu)\epsilon(s, \mu)\epsilon(2s, \omega_\mu)}
$$

- ($P = P_2$) We have the Levi decomposition $P_2 = M_2 N_2$ where $M_2 = \mathrm{GL}_1 \times \mathrm{GL}_2$ and N_2 is a length 1 unipotent algebraic group with $N_2 = \mathrm{Ad}^2(V_2) = \mathrm{Sym}^2(V_2) \otimes \det^{-1}$ the adjoint square representation, so the adjoint action of ${}^LM_2$ on ${}^L\mathfrak{n}_2$ is given

$$ {}^L\mathfrak{n}_2 = R^\vee = \mathrm{Ad}^2(\rho_2) $$

where ρ_2 is the standard representation of ${}^LM_1 = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.



It follows that the Langlands normalizing factor is given

$$ r(w_{121}, \pi_s) = \frac{\Lambda(s, \pi, R^\vee)}{\Lambda(s+1, \pi, R^\vee)\epsilon(s+1, \pi, R^\vee)} = \frac{\Lambda(s, \mathrm{Ad}(\mu) \times \chi)}{\Lambda(s, \mathrm{Ad}(\mu) \times \chi)\epsilon(s, \mathrm{Ad}(\mu) \times \chi)} $$

- ($P = P_0$) Recall that by the Gindikin-Karpelevich formula we have

$$ r(w, \pi_s) = \prod_{\substack{\alpha \in \Phi_G^+ \\ w(\alpha) \in \Phi_G^-}} \frac{\Lambda(\langle s, \alpha^\vee \rangle, \pi \circ \alpha^\vee)}{\Lambda(\langle s, \alpha^\vee \rangle + 1, \pi \circ \alpha^\vee)\epsilon(\langle s, \alpha^\vee \rangle, \pi \circ \alpha^\vee)} $$

The coroot $\alpha_1^\vee = f_1 - f_2$ satisfies $\langle s, \alpha_1^\vee \rangle = s_1 - s_2$ and $\pi \circ \alpha_1^\vee = \chi_1 \chi_2^{-1}$, and the only positive root $\alpha \in \Phi_G^+$ such that $w_1(\alpha) \in \Phi_G^-$ is a negative root is $\alpha = \alpha_1$, so we have

$$ r(w_1, \pi_s) = \tfrac{\Lambda(s_1 - s_2, \chi_1 \chi_2^{-1})}{\Lambda(s_1 - s_2 + 1, \chi_1 \chi_2^{-1})\epsilon(s_1 - s_2, \chi_1 \chi_2^{-1})} $$

The coroot $\alpha_2^\vee = f_2$ satisfies $\langle s, \alpha_2^\vee \rangle = s_2$ and $\pi \circ \alpha_2^\vee = \chi_2$, and the only positive root $\alpha \in \Phi_G^+$ such that $w_2(\alpha) \in \Phi_G^-$ is a negative root is $\alpha = \alpha_2$, so we have

$$ r(w_2, \pi_s) = \tfrac{\Lambda(s_2, \chi_2)}{\Lambda(s_2 + 1, \chi_2)\epsilon(s_2, \chi_2)} $$

For $w = s_{i_1} \dots s_{i_\ell}$ a reduced expression of simple reflections we have the normalization factor

$$ r(w, \pi_s) = \prod_{1 \le j \le \ell} r(w_{i_j}, {\pi_s}^{w_{i_{j+1}} \dots w_{i_\ell}}) $$

Since $r(w_1,\pi_s^{w_2}) = \frac{\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)}$ we have

$$\begin{aligned} r(w_{12},\pi_s) &= r(w_2,\pi_s)r(w_1,\pi_s^{w_2}) \\ &= \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)} \end{aligned}$$

Since $r(w_2,\pi_s^{w_1}) = \frac{\Lambda(s_1,\chi_1)}{\Lambda(s_1+1,\chi_1)\epsilon(s_1,\chi_1)}$ we have

$$\begin{aligned} r(w_{21},\pi_s) &= r(w_1,\pi_s)r(w_2,\pi_s^{w_1}) \\ &= \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})\Lambda(s_1,\chi_1)}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\Lambda(s_1+1,\chi_1)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})\epsilon(s_1,\chi_1)} \end{aligned}$$

Since $r(w_1,\pi_s^{w_{21}}) = \frac{\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)}$ we have

$$\begin{aligned} r(w_{121},\pi_s) &= r(w_1,\pi_s)r(w_2,\pi_s^{w_1})r(w_1,\pi_s^{w_{21}}) \\ &= \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})\Lambda(s_1,\chi_1)\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\Lambda(s_1+1,\chi_1)\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})\epsilon(s_1,\chi_1)\epsilon(s_1+s_2,\chi_1\chi_2)} \end{aligned}$$

Since $r(w_2,\pi_s^{w_{12}}) = \frac{\Lambda(s_1,\chi_1)}{\Lambda(s_1+1,\chi_1)\epsilon(s_1,\chi_1)}$ we have

$$\begin{aligned} r(w_{212},\pi_s) &= r(w_2,\pi_s)r(w_1,\pi_s^{w_2})r(w_2,\pi_s^{w_{12}}) \\ &= \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)\Lambda(s_1,\chi_1)}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\Lambda(s_1,\chi_1)\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)\epsilon(s_1,\chi_1)} \end{aligned}$$

Since $r(w_1,\pi_s^{w_{212}}) = \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\epsilon(s_1-s_2,\chi_1\chi_2^{-1})}$ we have

$$\begin{aligned} r(w_{1212},\pi_s) &= r(w_2,\pi_s)r(w_1,\pi_s^{w_2})r(w_2,\pi_s^{w_{12}})r(w_1,\pi_s^{w_{212}}) \\ &= \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)\Lambda(s_1,\chi_1)\Lambda(s_1-s_2,\chi_1\chi_2^{-1})}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\Lambda(s_1,\chi_1)\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)\epsilon(s_1,\chi_1)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})} \end{aligned}$$

Note that since $r(w_2,\pi_s^{w_{121}}) = \frac{L(s_2,\chi_2)}{L(s_2+1,\chi_2)\epsilon(s_2,\chi_2)}$ we equivalently have

$$\begin{aligned} r(w_{2121},\pi_s) &= r(w_1,\pi_s)r(w_2,\pi_s^{w_1})r(w_1,\pi_s^{w_{21}})r(w_2,\pi_s^{w_{121}}) \\ &= \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)\Lambda(s_1,\chi_1)\Lambda(s_1-s_2,\chi_1\chi_2^{-1})}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\Lambda(s_1,\chi_1)\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)\epsilon(s_1,\chi_1)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})} \end{aligned}$$

as expected. $\square$

Now recall from the discussion of normalized intertwining operators that the poles of $M(w,\pi_s)$ intersecting the positive closed Weyl chamber $\overline{(\mathfrak{a}_P^G)^{\vee+}}$ are exactly the poles of the Langlands normalization factor $r(w,\pi_s)$ in the same region. We obtain the following:

Proposition 4.3.2. (Compare to [69, Proposition 5.4]) We have the following singularities of $M(w,\pi_s)$ intersecting the positive closed Weyl chamber $\overline{(\mathfrak{a}_P^G)^{\vee+}}$:

- ($P=P_1$) For $\pi=\mu\otimes\chi$, if μ is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\mu=1$ and with nonvanishing central L-value $\Lambda(\frac{1}{2},\mu)\neq 0$, then we have the singularity

$$\mathfrak{S}_{\mu,\chi}=\mu\nu\otimes\chi\nu^{-1}$$

- ($P=P_2$) For $\pi=\chi\otimes\mu$, if $\mu=\mathcal{AI}_\mathbb{Q}^F(\theta)$ is the automorphic induction of the unitary Hecke character $\theta:F^\times\setminus\mathbb{A}_F^\times\to\mathbb{C}^\times$ corresponding to a quadratic extension $F/\mathbb{Q}$ and $\chi=\omega_{F/\mathbb{Q}}$, then we have the singularity

$$\mathfrak{S}_{(F/\mathbb{Q},\theta)}=\omega_{F/\mathbb{Q}}\nu\otimes\mathcal{AI}_\mathbb{Q}^F(\theta)\nu^{-1/2}$$

- ($P=P_0$) For $\pi=\chi_1\otimes\chi_2\otimes\chi$ we have the following singularities:

  (i) If $\chi_1=\chi_2$ we have the singular hyperplane $\{s_1=1+s_2\}$ yielding the region

$$\mathfrak{S}_1=\{(1+s,s)\in(\mathfrak{a}_{P_0}^G)^\vee\}=o_1+(\mathfrak{a}_{P_0}^G)_1^\vee \qquad o_1=(\tfrac{1}{2},-\tfrac{1}{2})$$

  (ii) If $\chi_2=1$ we have the singular hyperplane $\{s_2=1\}$ yielding the region

$$\mathfrak{S}_2=\{(s,1)\in(\mathfrak{a}_{P_0}^G)^\vee\}=o_2+(\mathfrak{a}_{P_0}^G)_2^\vee \qquad o_2=(0,1) \qquad (\mathfrak{a}_{P_0}^G)_2^\vee=\{(s,0)\in(\mathfrak{a}_{P_0}^G)^\vee\}$$

  (iii) If $\chi_1=\chi_2^{-1}$ we have the singular hyperplane $\{s_1=1-s_2\}$ yielding the region

$$\mathfrak{S}_3=\{(1-s,s)\in(\mathfrak{a}_{P_0}^G)^\vee\}=o_3+(\mathfrak{a}_{P_0}^G)_3^\vee \qquad o_3=(\tfrac{1}{2},\tfrac{1}{2}) \qquad (\mathfrak{a}_{P_0}^G)_3^\vee=\{(s,-s)\in(\mathfrak{a}_{P_0}^G)^\vee\}$$

(iv) If $\chi_1 = 1$ we have the singular hyperplane $\{s_1 = 1\}$ yielding the region

$$\mathfrak{S}_4 = \{(1, s) \in (\mathfrak{a}_{P_0}^G)^\vee\} = o_4 + (\mathfrak{a}_{P_0}^G)_4^\vee \qquad o_4 = (1, 0) \qquad (\mathfrak{a}_{P_0}^G)_4^\vee = \{(0, s) \in (\mathfrak{a}_{P_0}^G)^\vee\}$$

(v) If $\chi_1 = \chi_2 = \omega_{F/\mathbb{Q}}$ corresponding to a quadratic extension $F/\mathbb{Q}$ we have the singularity

$$\mathfrak{S}_{F/\mathbb{Q},\chi} = \omega_{F/\mathbb{Q}}\nu \otimes \omega_{F/\mathbb{Q}} \otimes \chi\nu^{-1/2}$$

(vi) If $\chi_1 = \chi_2 = 1$ we have the singularity

$$\mathfrak{S}_\chi = \nu^2 \otimes \nu \otimes \chi\nu^{-3/2}$$

Proof. • ($P = P_1$) The L-function $\Lambda(s, \mu)$ is entire, and the L-function $\Lambda(2s, \omega_\mu)$ has a pole in the region $\Re(s) \geq 0$ precisely if $\omega_\mu = 1$ in which case $\Lambda(2s, \omega_\mu) = Z(2s)$ has a simple pole at $s = \frac{1}{2}$. This pole can only be cancelled if $\Lambda(\frac{1}{2}, \mu) = 0$, whence the condition $\omega_\mu = 1$ and $\Lambda(\frac{1}{2}, \mu) \neq 0$.

• ($P = P_2$) In the case $P = P_2$ we have the following. If $\mu \otimes \omega \neq \mu$ for any nontrivial unitary Hecke character $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ then by [44, Theorem 9.3] the adjoint L-function $\Lambda(s, \mathrm{Ad}^2(\mu) \times \chi)$ is entire. If $\mu \otimes \omega_{F/\mathbb{Q}} \simeq \mu$ for some quadratic extension $F/\mathbb{Q}$ then $\mu = \mathcal{AI}_\mathbb{Q}^F(\theta)$ is the automorphic induction of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ corresponding to $F/\mathbb{Q}$ and we have

$$\Lambda(s, \mathrm{Ad}^2(\mu) \times \chi) = \Lambda_F(s, \theta\sigma(\theta)^{-1}\chi(\mathrm{Nm}_{F/\mathbb{Q}}))\Lambda(s, \omega_{F/\mathbb{Q}}\chi)$$

Now the L-function $\Lambda(s, \omega_{F/\mathbb{Q}}\chi)$ has a pole in the region $\Re(s) \geq 0$ precisely if $\chi = \omega_{F/\mathbb{Q}}$ in which case $\Lambda(s, \omega_{F/\mathbb{Q}}\chi) = Z(s)$ has a simple pole at $s = 1$. In this case the L-function $\Lambda_F(s, \theta\sigma(\theta)^{-1}\chi(\mathrm{Nm}_{F/\mathbb{Q}}))$ is entire in the region $\Re(s) \geq 0$, so the L-function $\Lambda(s, \mathrm{Ad}^2(\mu) \times \chi)$ has a pole in the region $\Re(s) \geq 0$ precisely if $\chi = \omega_{F/\mathbb{Q}}$ in which case $\Lambda(s, \mathrm{Ad}^2(\mu) \times \omega_{F/\mathbb{Q}}) = \Lambda_F(s, \theta\sigma(\theta)^{-1}\omega_{F/\mathbb{Q}}(\mathrm{Nm}_{F/\mathbb{Q}}))Z(s)$ has a simple pole at $s = 1$.

On the other hand if the L-function $\Lambda(s, \mathrm{Ad}^2(\mu) \times \chi)$ has a pole in the region $\Re(s) \geq 0$ then $\mu = \mathcal{AI}_\mathbb{Q}^F(\theta)$ is the automorphic induction of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ corresponding to a quadratic extension $F/\mathbb{Q}$ and $\theta\sigma(\theta)^{-1}$ is a nontrivial quadratic character, and we have three distinct automorphic induction data $(F/\mathbb{Q},\theta)$, $(F'/\mathbb{Q},\theta')$, $(F''/\mathbb{Q},\theta'')$ so that $\mu = \mathcal{AI}_{\mathbb{Q}}^{F}(\theta) = \mathcal{AI}_{\mathbb{Q}}^{F'}(\theta') = \mathcal{AI}_{\mathbb{Q}}^{F''}(\theta'')$ and we have

$$\Lambda(s, \mathrm{Ad}^2(\mu)\times\chi) = \Lambda(s, \omega_{F/\mathbb{Q}}\chi)\Lambda(s, \omega_{F'/\mathbb{Q}}\chi)L(s, \omega_{F''/\mathbb{Q}}\chi)$$

which has a simple pole at $s = 1$ precisely if $\chi \in \{\omega_{F/\mathbb{Q}}, \omega_{F'/\mathbb{Q}}, \omega_{F''/\mathbb{Q}}\}$, whence the condition $\mu = \mathcal{AI}_{\mathbb{Q}}^{F}(\theta)$ and $\chi = \omega_{F/\mathbb{Q}}$.

- ($P = P_0$) The normalization factor $r(w_1, \pi_s) = \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\epsilon(s_1-s_2,\chi_1\chi_2^{-1})}$ (and hence the normalization factors $r(w_{21},\pi_s)$, $r(w_{121},\pi_s)$, and $r(w_{1212},\pi_s)$) have a pole along the singular hyperplane $\{s_1 = 1+s_2\}$ precisely if $\chi_1 = \chi_2$.

  The normalization factor $r(w_2,\pi_s) = \frac{\Lambda(s_2,\chi_2)}{\Lambda(s_2+1,\chi_2)\epsilon(s_2,\chi_2)}$ (and hence the normalization factors $r(w_{12},\pi_s)$, $r(w_{212},\pi_s)$, and $r(w_{1212},\pi_s)$) have a pole along the singular hyperplane $\{s_2 = 1\}$ precisely if $\chi_2 = 1$.

  The normalization factor $r(w_{12},\pi_s) = \frac{\Lambda(s_2,\chi_2)\Lambda(s_1+s_2,\chi_1\chi_2)}{\Lambda(s_2+1,\chi_2)\Lambda(s_1+s_2+1,\chi_1\chi_2)\epsilon(s_2,\chi_2)\epsilon(s_1+s_2,\chi_1\chi_2)}$ (and hence the normalization factors $r(w_{121},\pi_s)$ and $r(w_{1212},\pi_s)$) have a pole along the singular hyperplane $\{s_1 = 1-s_2\}$ precisely if $\chi_1 = \chi_2^{-1}$.

  The normalization factor $r(w_{21},\pi_s) = \frac{\Lambda(s_1-s_2,\chi_1\chi_2^{-1})\Lambda(s_1,\chi_1)}{\Lambda(s_1-s_2+1,\chi_1\chi_2^{-1})\Lambda(s_1+1,\chi_1)\epsilon(s_1-s_2,\chi_1\chi_2^{-1})\epsilon(s_1,\chi_1)}$ (and hence the normalization factors $r(w_{212},\pi_s)$ and $r(w_{1212},\pi_s)$) have a pole along the singular hyperplane $\{s_1 = 1\}$ precisely if $\chi_1 = 1$. □

Residues of Eisenstein Series and the Residual Spectrum for GSp_4 We now need to compute the residues of the automorphic Eisenstein series $\mathrm{Eis}_P^G(\phi_s)$, taking into account the possible singularities intersecting the positive closed Weyl chamber $\overline{(\mathfrak{a}_{P_0}^G)^{\vee+}}$. Again there are three cases to consider, one for each standard parabolic $\mathbb{Q}$-subgroup; in each case we compute the L^2-inner products of Poincare series by moving the contours to the unitary axis $s \in i(\mathfrak{a}_P^G)^\vee$. For the two maximal parabolic $\mathbb{Q}$-subgroups P_1 and P_2 the computation is similar to the case of GL_2: we have only a single complex parameter $s \in (\mathfrak{a}_{P_i}^G)^\vee$ and we pick up a single residual term after the contour crosses a simple pole. For the minimal parabolic $\mathbb{Q}$-subgroup P_0 the computation is more complicated: we now have two complex parameters $(s_1,s_2) \in (\mathfrak{a}_{P_0}^G)^\vee$ and we pick up residual terms after the contour crosses poles at shifts of root hyperplanes.

To that end we quickly summarize the arguments given by Konno in [69, Section 6]; they are similar to arguments given by Kim in [65] for the group Sp_4, and to arguments developed by Langlands in [74] and by Moeglin-Waldspurger in [85] more generally. Since G is (quasi)-split of $\mathbb{Q}$-rank 2, the contour integrals at infinity vanish without needing to introduce additional cutoff integrals as Langlands and Moeglin-Waldspurger do for more general G; for this reason the estimates on intertwining operators explained by Konno are sufficient to guarantee that the contour integrals at infinity vanish in order for this argument to work without introducing further complications. The main contour shifting argument is summarized as follows:

Proposition 4.3.3. (Compare to [69, Proposition 6.1, Proposition 6.2]) For $\phi \in W_{P,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P,\pi]} W_{P',\pi'}$ the L^2-inner products $\langle \theta_\phi, \theta_{\phi'}\rangle$ are given as follows:

- $(P = P_1)$ For $\phi \in W_{P_1,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_1,\pi]} W_{P',\pi'}$ we have

$$\langle \theta_\phi, \theta_{\phi'}\rangle = \int_{i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s + c\frac{\Lambda(\frac{1}{2},\mu)}{\Lambda(\frac{3}{2},\mu)\epsilon(\frac{1}{2},\mu)}\langle N(w_{212},\mathfrak{S}_{\mu,\chi})\phi_{1/2},\phi'_{1/2}\rangle$$

  where $c = \frac{1}{\sqrt{2}}\mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{3\sqrt{2}}{\pi}$ is a nonzero constant.
- $(P = P_2)$ For $\phi \in W_{P_2,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_2,\pi]} W_{P',\pi'}$ we have

$$\langle \theta_\phi, \theta_{\phi'}\rangle = \int_{i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s + c\frac{\Lambda_F(1,\theta\sigma(\theta)^{-1})}{\Lambda_F(2,\theta\sigma(\theta)^{-1})\epsilon_F(1,\theta\sigma(\theta)^{-1})}\langle N(w_{121},\mathfrak{S}_{(F/\mathbb{Q},\theta)})\phi_1,\phi'_1\rangle$$

  where $c = \mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{6}{\pi}$ is a nonzero constant.
- $(P = P_0)$ For $\phi \in W_{P_0,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_0,\pi]} W_{P',\pi'}$ we have

$$\begin{aligned}\langle \theta_\phi, \theta_{\phi'}\rangle &= \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s + \sum_{1\le j\le 3}\int_{o_j+(\mathfrak{a}_{P_0}^G)_j^\vee} \mathrm{Res}_{\mathfrak{S}_j} A(\phi,\phi')(\pi_s)\mathrm{d}s\\ &+ \lim_{\epsilon\to 0}\int_{o_4+i(\mathfrak{a}_{P_0}^G)_4^\vee}\frac{1}{2}\Big(\mathrm{Res}_{\mathfrak{S}_4}A(\phi,\phi')(\pi_{s+\epsilon}) + \mathrm{Res}_{\mathfrak{S}_4}A(\phi,\phi')(\pi_{s-\epsilon})\Big)\mathrm{d}s\\ &+ c^2\langle M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{121}})N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{21}})M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_1})N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi})\phi_{(1,0)},\phi'_{(1,0)}\rangle\\ &+ c^2\langle N(w_2,\mathfrak{S}_\chi^{w_{121}})M(w_{12},\mathfrak{S}_\chi^{w_1})N(w_1,\mathfrak{S}_\chi)\phi_{\rho_{P_0}},\phi'_{\rho_{P_0}}\rangle\end{aligned}$$

  where $c = \mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{6}{\pi}$ is a nonzero constant.

Proof. We have the following:

- $(P=P_1)$ Let $\phi \in W_{P_1,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_1,\pi]} W_{P',\pi'}$ where $\pi = \mu \otimes \chi$ as before. Fixing $s_0 \in (\mathfrak{a}_{P_1}^G)^\vee$ sufficiently positive we want to compute $\int_{s_0+i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s$. To that end we consider the contour integral $\lim_{t\to\infty} \oint_{C_t} A(\phi,\phi')(\pi_s)\mathrm{d}s$, and as explained by [69, Section 6.1] the contour integrals at infinity vanish by estimates on the intertwining operators $M(w,\pi_s)$ and by the rapid decay of ϕ and ϕ' in $\Im(s)$ since they are Paley-Wiener. It follows by the residue theorem that

$$\int_{s_0+i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s = \mathrm{Res}_{\mathfrak{S}_{\mu,\chi}} A(\phi,\phi')(\pi_s)$$

Now recall that the Langlands normalization factor

$$r(w_{212},\pi_s) = \frac{\Lambda(s,\mu)\Lambda(2s,\omega_\mu)}{\Lambda(s+1,\mu)\Lambda(2s+1,\omega_\mu)\epsilon(s,\mu)\epsilon(2s,\omega_\mu)}$$

has a pole at $s=\frac{1}{2}$ for μ a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\mu = 1$ and nonvanishing central L-value $\Lambda(\frac{1}{2},\mu) \neq 0$ where it is given

$$r(w_{212},\pi_s) = \frac{\Lambda(s,\mu)Z(2s)}{\Lambda(s+1,\mu)Z(2s+1)\epsilon(s,\mu)}$$

Then we have the residue

$$\begin{aligned}
\mathrm{Res}_{\mathfrak{S}_{\mu,\chi}} A(\phi,\phi')(\pi_s) &= \mathrm{Res}_{s=\frac{1}{2}}\Big(\langle \phi_s,\phi'_s\rangle + r(w_{212},\pi_s)\langle N(w_{212},\mathfrak{S}_{\mu,\chi})\phi_s,\phi'_s\rangle\Big) \\
&= \mathrm{Res}_{s=\frac{1}{2}} \frac{\Lambda(s,\mu)Z(2s)}{\Lambda(s+1,\mu)Z(2s+1)\epsilon(s,\mu)}\langle N(w_{212},\mathfrak{S}_{\mu,\chi})\phi_s,\phi'_s\rangle \\
&= c\frac{\Lambda(\frac{1}{2},\mu)}{\Lambda(\frac{3}{2},\mu)\epsilon(\frac{1}{2},\mu)}\langle N(w_{212},\mathfrak{S}_{\mu,\chi})\phi_{1/2},\phi'_{1/2}\rangle
\end{aligned}$$

where $c = \frac{1}{\sqrt{2}}\mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{3\sqrt{2}}{\pi}$, so it follows that the L^2-inner product is given

$$\begin{aligned}
\langle \theta_\phi,\theta_{\phi'}\rangle &= \int_{s_0+i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s \\
&= \int_{i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s + c\frac{\Lambda(\frac{1}{2},\mu)}{\Lambda(\frac{3}{2},\mu)\epsilon(\frac{1}{2},\mu)}\langle N(w_{212},\mathfrak{S}_{\mu,\chi})\phi_{1/2},\phi'_{1/2}\rangle
\end{aligned}$$

- $(P=P_2)$ Let $\phi \in W_{P_2,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_2,\pi]} W_{P',\pi'}$ where $\pi = \chi \otimes \mu$ as before. Fixing $s_0 \in (\mathfrak{a}_{P_2}^G)^\vee$ sufficiently positive we want to compute $\int_{s_0+i(\mathfrak{a}_{P_2}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_2}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s$. To that end we consider the contour integral $\lim_{t\to\infty}\oint_{C_t} A(\phi,\phi')(\pi_s)\mathrm{d}s$, and as explained by [69, Section 6.1] the contour integrals at infinity vanishing by estimates on the intertwining operators $M(w,\pi_s)$ and by the rapid decay of ϕ and ϕ' in $\Im(s)$ since they are Paley-Wiener. It follows by the residue theorem that

$$\int_{s_0+i(\mathfrak{a}_{P_2}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_2}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s = \mathrm{Res}_{\mathfrak{S}_{(F/\mathbb{Q},\theta)}} A(\phi,\phi')(\pi_s)$$

Now recall that the Langlands normalization factor

$$r(w_{121},\pi_s) = \frac{\Lambda(s,\mathrm{Ad}(\mu)\times\chi)}{\Lambda(s,\mathrm{Ad}(\mu)\times\chi)\epsilon(s,\mathrm{Ad}(\mu)\times\chi)}$$

has a pole at $s = 1$ for $\mu = \mathcal{AI}_{\mathbb{Q}}^F(\theta)$ the automorphic induction of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ corresponding to a quadratic extension $F/\mathbb{Q}$ and $\chi = \omega_{F/\mathbb{Q}}$ where it is given

$$r(w_{121},\pi_s) = \frac{\Lambda_F(s,\theta\sigma(\theta)^{-1})Z(s)}{\Lambda_F(s+1,\theta\sigma(\theta)^{-1})Z(s+1)\epsilon_F(s,\theta\sigma(\theta)^{-1})}$$

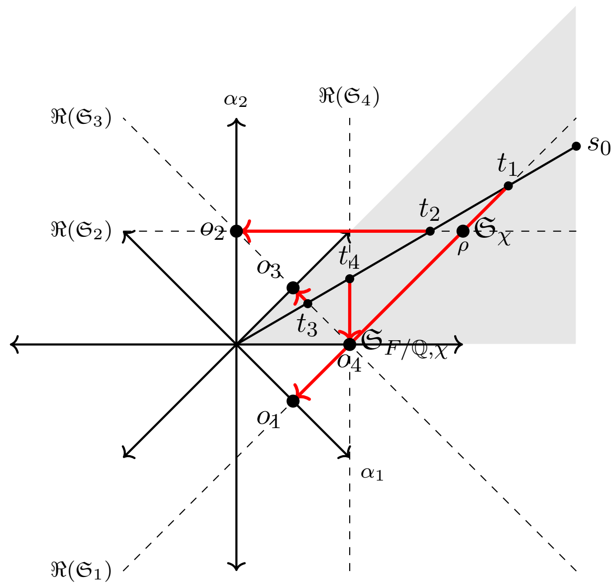
Then we have the residue

$$\begin{aligned}
\mathrm{Res}_{\mathfrak{S}_{(F/\mathbb{Q},\theta)}} A(\phi,\phi')(\pi_s) &= \mathrm{Res}_{s=1} r(w_{121},\pi_s)\langle N(w_{121},\mathfrak{S}_{(F/\mathbb{Q},\theta)})\phi_s,\phi_s'\rangle \\
&= \mathrm{Res}_{s=1}\frac{\Lambda_F(s,\theta\sigma(\theta)^{-1})Z(s)}{\Lambda_F(s+1,\theta\sigma(\theta)^{-1})Z(s+1)\epsilon_F(s,\theta\sigma(\theta)^{-1})}\langle N(w_{121},\mathfrak{S}_{(F/\mathbb{Q},\theta)})\phi_s,\phi_s'\rangle \\
&= c\frac{\Lambda_F(1,\theta\sigma(\theta)^{-1})}{\Lambda_F(2,\theta\sigma(\theta)^{-1})\epsilon_F(1,\theta\sigma(\theta)^{-1})}\langle N(w_{121},\mathfrak{S}_{(F/\mathbb{Q},\theta)})\phi_1,\phi_1'\rangle
\end{aligned}$$

where $c = \mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{6}{\pi}$, so it follows that the L^2-inner product is given

$$\begin{aligned}
\langle\theta_\phi,\theta_{\phi'}\rangle &= \int_{s_0+i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s \\
&= \int_{i(\mathfrak{a}_{P_1}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s + c\frac{\Lambda_F(1,\theta\sigma(\theta)^{-1})}{\Lambda_F(2,\theta\sigma(\theta)^{-1})\epsilon_F(1,\theta\sigma(\theta)^{-1})}\langle N(w_{121},\mathfrak{S}_{(F/\mathbb{Q},\theta)})\phi_1,\phi_1'\rangle
\end{aligned}$$

- Let $\phi \in W_{P_0,\pi}$ and $\phi' \in \bigoplus_{(P',\pi')\in[P_0,\pi]} W_{P',\pi'}$ where $\pi = \chi_1 \otimes \chi_2 \otimes \chi$ as before. Fixing $s_0 \in (\mathfrak{a}_{P_0}^G)^\vee$ sufficiently positive let γ be the straight path from s_0 to 0 in $(\mathfrak{a}_{P_0}^G)^\vee$ and let $t_1, t_2, t_3, t_4 \in (\mathfrak{a}_{P_0}^G)^\vee$ be the intersections of γ with the singularities $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4$ respectively.



We compute the difference $\int_{s_0+i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s$ as a (double) countour integral $\lim_{t\to\infty} \oint_{C_t} A(\phi,\phi')(\pi_s)\mathrm{d}s$ with the contour integrals at infinity vanishing by standard estimates on the intertwining operators $M(w,\pi_s)$ and by the rapid decay of ϕ and ϕ' in $\Im(s)$ since they are Paley-Wiener. It follows by the residue theorem that

$$\int_{s_0+i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s = \sum_{1\le j\le 4} \int_{t_j+i(\mathfrak{a}_{P_0}^G)_j^\vee} \mathrm{Res}_{\mathfrak{S}_j} A(\phi,\phi')(\pi_s)$$

In order to obtain a $G(\mathbb{A}_\mathbb{Q})$-invariant expression we need to move the integration regions $t_j + i(\mathfrak{a}_{P_0}^G)_j^\vee$ to the integration regions $o_j + i(\mathfrak{a}_{P_0}^G)_j^\vee$ along the planes $\Re(\mathfrak{S}_j) = 0$ (shown in red). This gives the following contributions:

  - Upon moving t_1 to o_1 we pick up the residues $\mathrm{Res}_{\mathfrak{S}_\chi} A(\phi,\phi')(\pi_s)$ and $\mathrm{Res}_{\mathfrak{S}_{F/\mathbb{Q},\chi}} A(\phi,\phi')(\pi_s)$ and the residue $\int_{t_1+i(\mathfrak{a}_{P_0}^G)_1^\vee} \mathrm{Res}_{\mathfrak{S}_1} A(\phi,\phi')(\pi_s)$ becomes $\int_{o_1+i(\mathfrak{a}_{P_0}^G)_1^\vee} \mathrm{Res}_{\mathfrak{S}_1} A(\phi,\phi')(\pi_s)$.
  - Upon moving t_2 to o_2 the residue $\int_{t_2+i(\mathfrak{a}_{P_0}^G)_2^\vee} \mathrm{Res}_{\mathfrak{S}_2} A(\phi,\phi')(\pi_s)$ becomes $\int_{o_2+i(\mathfrak{a}_{P_0}^G)_2^\vee} \mathrm{Res}_{\mathfrak{S}_2} A(\phi,\phi')(\pi_s)$.
  - Upon moving t_3 to o_3 the residue $\int_{t_3+i(\mathfrak{a}_{P_0}^G)_3^\vee} \mathrm{Res}_{\mathfrak{S}_3} A(\phi,\phi')(\pi_s)$ becomes $\int_{o_3+i(\mathfrak{a}_{P_0}^G)_3^\vee} \mathrm{Res}_{\mathfrak{S}_3} A(\phi,\phi')(\pi_s)$.

- Upon moving t_4 to o_4 the residue $\int_{t_4+i(\mathfrak{a}_{P_0}^G)_4^\vee} \mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_s)$ is replaced by the principal value $\lim_{\epsilon\to 0}\int_{o_4+i(\mathfrak{a}_{P_0}^G)_4^\vee} \frac{1}{2}(\mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_{s+\epsilon}) + \mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_{s-\epsilon}))\mathrm{d}s$.

The same estimates of contour integrals apply and it follows by the residue theorem that

$$\begin{aligned}
&\int_{s_0+i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s - \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi,\phi')(\pi_s)\mathrm{d}s \\
&= \sum_{1\le j\le 3}\int_{t_j+i(\mathfrak{a}_{P_0}^G)_j^\vee} \mathrm{Res}_{\mathfrak{S}_j} A(\phi,\phi')(\pi_s) \\
&+ \lim_{\epsilon\to 0}\int_{o_4+i(\mathfrak{a}_{P_0}^G)_4^\vee} \frac{1}{2}\Big(\mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_{s+\epsilon}) + \mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_{s-\epsilon})\Big)\mathrm{d}s \\
&+ \mathrm{Res}_{\mathfrak{S}_{F/\mathbb{Q},\chi}} A(\phi,\phi')(\pi_s) + \mathrm{Res}_{\mathfrak{S}_\chi} A(\phi,\phi')(\pi_s)
\end{aligned}$$

Then we have the residue

$$\begin{aligned}
&\mathrm{Res}_{\mathfrak{S}_{F/\mathbb{Q},\chi}} A(\phi,\phi')(\pi_s) \\
&= \mathrm{Res}_{(s_1,s_2)=(1,0)} r(w_{1212},\mathfrak{S}_{F/\mathbb{Q},\chi})\langle N(w_{1212},\mathfrak{S}_{F/\mathbb{Q},\chi})\phi_s,\phi'_s\rangle \\
&= \mathrm{Res}_{(s_1,s_2)=(1,0)} r(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{21}}) r(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi})\langle M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{121}}) N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{21}}) M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_1}) N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi})\phi_s,\phi'_s\rangle \\
&= \mathrm{Res}_{(s_1,s_2)=(1,0)} \frac{Z(s_1)^2}{Z(s_1+1)^2}\langle M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{121}}) N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{21}}) M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_1}) N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi})\phi_s,\phi'_s\rangle \\
&= c^2\langle M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{212}}) N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{21}}) M(w_2,\mathfrak{S}_{F/\mathbb{Q},\chi}^{w_1}) N(w_1,\mathfrak{S}_{F/\mathbb{Q},\chi})\phi_{(1,0)},\phi'_{(1,0)}\rangle
\end{aligned}$$

and we have the residue

$$\begin{aligned}
&\mathrm{Res}_{\mathfrak{S}_\chi} A(\phi,\phi')(\pi_s) \\
&= \mathrm{Res}_{(s_1,s_2)=\rho_{P_0}} r(w_{1212},\mathfrak{S}_\chi)\langle N(w_{1212},\mathfrak{S}_\chi)\phi_s,\phi'_s\rangle \\
&= \mathrm{Res}_{(s_1,s_2)=\rho_{P_0}} r(w_2,\mathfrak{S}_\chi^{w_{121}}) r(w_1,\mathfrak{S}_\chi)\langle N(w_2,\mathfrak{S}_\chi^{w_{121}}) M(w_{12},\mathfrak{S}_\chi^{w_1}) N(w_1,\mathfrak{S}_\chi)\phi_s,\phi'_s\rangle \\
&= \mathrm{Res}_{(s_1,s_2)=\rho_{P_0}} \frac{Z(s_1-s_2)Z(s_2)}{Z(s_1-s_2+1)Z(s_2+1)}\langle N(w_2,\mathfrak{S}_\chi^{w_{121}}) M(w_{12},\mathfrak{S}_\chi^{w_1}) N(w_1,\mathfrak{S}_\chi)\phi_s,\phi'_s\rangle \\
&= c^2\langle N(w_2,\mathfrak{S}_\chi^{w_{121}}) M(w_{12},\mathfrak{S}_\chi^{w_1}) N(w_1,\mathfrak{S}_\chi)\phi_{\rho_{P_0}},\phi'_{\rho_{P_0}}\rangle
\end{aligned}$$

where $c = \mathrm{Res}_{s=1}\frac{Z(s)}{Z(2)} = \frac{6}{\pi}$, so it follows that the L^2-inner product is given

$$\begin{aligned}\langle \theta_\phi, \theta_{\phi'} \rangle &= \int_{s_0 + i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi, \phi')(\pi_s)\mathrm{d}s \\ &= \int_{i(\mathfrak{a}_{P_0}^G)^\vee} A(\phi, \phi')(\pi_s)\mathrm{d}s + \sum_{1\le j \le 3} \int_{o_j + i(\mathfrak{a}_{P_0}^G)_j^\vee} \mathrm{Res}_{\mathfrak{S}_j} A(\phi, \phi')(\pi_s)\mathrm{d}s \\ &+ \lim_{\epsilon \to 0} \int_{o_4 + i(\mathfrak{a}_{P_0}^G)_4^\vee} \frac{1}{2}\Big(\mathrm{Res}_{\mathfrak{S}_4} A(\phi, \phi')(\pi_{s+\epsilon}) + \mathrm{Res}_{\mathfrak{S}_4} A(\phi, \phi')(\pi_{s-\epsilon})\Big)\mathrm{d}s \\ &+ c^2 \langle M(w_2, \mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{212}}) N(w_1, \mathfrak{S}_{F/\mathbb{Q},\chi}^{w_{21}}) M(w_2, \mathfrak{S}_{F/\mathbb{Q},\chi}^{w_1}) N(w_1, \mathfrak{S}_{F/\mathbb{Q},\chi}) \phi_{(1,0)}, \phi'_{(1,0)} \rangle \\ &+ c^2 \langle N(w_2, \mathfrak{S}_\chi^{w_{121}}) M(w_{12}, \mathfrak{S}_\chi^{w_1}) N(w_1, \mathfrak{S}_\chi) \phi_{\rho_{P_0}}, \phi'_{\rho_{P_0}} \rangle \qquad \square\end{aligned}$$

The same computation applies when $\mathbb{Q}$ is replaced by any number field F (as in [69]), in which case the nonzero constants c instead involve $\mathrm{Res}_{s=1} Z_F(s)/Z_F(2)$. We obtain the following corollary:

Corollary 4.3.4. The Eisenstein series and their residues for $G = \mathrm{GSp}_4$ over $\mathbb{Q}$ are given as follows:

- At $o = (0,0)$ we have the cuspidal Eisenstein series

$$\mathrm{Eis}_P^G(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi_s)) \qquad s \in i(\mathfrak{a}_P^G)^\vee$$

- At $o_1 = \chi_1\nu^{1/2} \otimes \chi_1\nu^{-1/2} \otimes \chi$ and $o_3 = \chi_1\nu^{1/2} \otimes \chi_1^{-1}\nu^{1/2} \otimes \chi_1\chi\nu^{-1/2}$ we have the Siegel-Eisenstein series

$$\mathrm{Eis}_{P_1}^G(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \circ \det \otimes \chi\nu^{-s})) \qquad s \in i(\mathfrak{a}_{P_1}^G)^\vee$$

- At $o_2 = \chi_1 \otimes \nu \otimes \chi\nu^{-1/2}$ and $o_4 = \nu \otimes \chi_1 \otimes \chi\nu^{-1/2}$ we have the Klingen Eisenstein series

$$\mathrm{Eis}_{P_2}^G(\mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \otimes \chi\nu^{-s/2} \circ \det)) \qquad s \in i(\mathfrak{a}_{P_2}^G)^\vee$$

- At $\mathfrak{S}_{\mu,\chi} = \mu\nu^{1/2} \otimes \chi\nu^{-1/2}$ we have the Langlands quotient

$$\mathcal{J}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\mu\nu \otimes \chi\nu^{-1})$$

of the normalized parabolic induction $\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\mu\nu \otimes \chi\nu^{-1})$.

- At $\mathfrak{S}_{(F/\mathbb{Q},\theta)} = \omega_{F/\mathbb{Q}}\nu \otimes \pi(\theta)\nu^{-1/2}$ we have the Langlands quotient

$$\mathcal{J}_{P_2(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu \otimes \pi(\theta)\nu^{-1/2})$$

  of the normalized parabolic induction $\mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu \otimes \pi(\theta)\nu^{-1/2})$.

- At $\mathfrak{S}_{F/\mathbb{Q},\chi}$ we have the Langlands quotient

$$\mathcal{J}_{P_2(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu \otimes \pi(\chi \circ \mathrm{Nm}_{F/\mathbb{Q}})\nu^{-1/2})$$

  of the normalized parabolic induction $\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu \otimes \omega_{F/\mathbb{Q}} \otimes \chi\nu^{-1/2})$, or of the normalized parabolic induction $\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu^{1/2} \circ \det \otimes \chi\nu^{-1/2})$ (a degenerate principal series representation).

- At $\mathfrak{S}_\chi$ we have the 1-dimensional representation $\chi \circ \mathrm{sim}$.

It follows that we have a decomposition

$$L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ G(\mathbb{A}_\mathbb{Q})) \oplus L^2_{\mathrm{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$$

where the continuous spectrum is given

$$\begin{aligned} L^2_{\mathrm{cont}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = &\bigoplus_{[P,\pi]} \int_{i(\mathfrak{a}_P^G)^\vee} \mathcal{I}_{P(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, s)\mathrm{d}s \\ &\oplus \bigoplus_{\chi_1,\chi} \int_{i(\mathfrak{a}_{P_1}^G)^\vee} \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \circ \det \otimes \chi\nu^{-s})\mathrm{d}s \\ &\oplus \bigoplus_{\chi_1,\chi} \int_{i(\mathfrak{a}_{P_2}^G)^\vee} \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\chi_1\nu^s \otimes \chi\nu^{-s/2} \circ \det)\mathrm{d}s \end{aligned}$$

where the first direct sum is taken over cuspidal pairs $[M, \pi]$ where M is the Levi quotient of a parabolic subgroup P and where where π is a cuspidal automorphic representation of $M(\mathbb{A}_\mathbb{Q})$. We have a further decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = L^2_{\mathrm{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \oplus L^2_{\mathrm{res}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$$

where the residual spectrum is given

$$
\begin{aligned}
L^2_{\mathrm{res}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = & \bigoplus_{\substack{\mu,\chi \\ \omega_\mu=\chi^2=1 \\ L(\frac{1}{2},\mu)\neq 0}} \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\mu\nu \otimes \chi\nu^{-1}) \\
& \oplus \bigoplus_{(F/\mathbb{Q},\theta)} \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu \otimes \pi(\theta)\nu^{-1/2}) \\
& \oplus \bigoplus_{F/\mathbb{Q},\chi} \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\omega_{F/\mathbb{Q}}\nu \otimes \pi(\chi \circ \mathrm{Nm}_{F/\mathbb{Q}})\nu^{-1/2}) \\
& \oplus \bigoplus_{\chi} \chi \circ \mathrm{sim}
\end{aligned}
$$

4.3.2 Siegel Eisenstein Cohomology

In this section we compute Siegel Eisenstein cohomology as a $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module. First, we recall some facts about the poles of Siegel Eisenstein series, and the evaluation points and infinitesimal characters which will enter into the description of the Franke filtration for the Siegel parabolic subgroup.

We begin by restating the results of the previous section on the locations of poles of Siegel Eisenstein series:

Proposition 4.3.5. (Compare to [48, Proposition 3.4]) The automorphic Eisenstein series $\mathrm{Eis}^G_{P_1}(\phi_s)$ attached to a unitary cuspidal automorphic representation $\pi = \mu \otimes \chi$ of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ has a pole at $s = \nu \in \overline{\mathfrak{a}^{\vee +}_{P_1}}$ precisely if $\nu = (\frac{1}{2}, \frac{1}{2})$ and $\omega_\mu = 1$ and $\Lambda(\frac{1}{2}, \mu) \neq 0$, in which case we have a simple pole at $s = \frac{1}{2}$ and the space spanned by the residues $\mathrm{Res}_{s=\frac{1}{2}} \mathrm{Eis}^G_{P_1}(\phi_s)$ is isomorphic to the Langlands quotient $\mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\pi, \frac{1}{2})$.

We now record the following result on infinitesimal characters coming from the action of the Weyl group:

Proposition 4.3.6. (Compare to [48, Lemma 3.5]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $\mathcal{J}_\lambda$ be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ annihilating $V^\vee_\lambda$. Let $\varphi = \{\varphi_P\}_{P\in[P_1]} \in \Phi_{\lambda,[P_1]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \mu \otimes \chi \in \varphi_{P_1}$ of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then the infinitesimal character $\xi \in \mathfrak{a}^{\vee\perp}_{P_1}$ and the corresponding $s_0 \in \mathfrak{a}^\vee_{P_1}$ such that $s_0 + \xi$ is annihilated by $\mathcal{J}_\lambda$ are given by $s_0 = \pm(\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$ and $\xi = (\frac{\lambda_1-\lambda_2+1}{2}, -\frac{\lambda_1-\lambda_2+1}{2})$, or $s_0 = \pm(\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2})$ and $\xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$.

The Franke-Schwermer Filtration We now describe the Franke filtration for the Siegel parabolic subgroup. As expected, the bottom piece is given by the Langlands quotient of normalized Siegel parabolic induction, and the top piece is given by normalized Siegel parabolic induction.

Proposition 4.3.7. (Compare to [48, Theorem 3.6]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $\mathcal{J}_\lambda$ be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ annihilating $V_\lambda^\vee$. Let $\varphi = \{\varphi_P\}_{P\in[P_1]} \in \Phi_{\lambda,[P_1]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \mu \otimes \chi \in \varphi_{P_1}$ of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ with infinitesimal character $\xi \in \mathfrak{a}_{P_1}^{\vee\perp}$. Let $s_0 \in \mathfrak{a}_{P_1}^\vee$ such that $s_0 + \xi$ is annihilated by $\mathcal{J}_\lambda$. Then the Franke-Schwermer filtration on $\mathcal{A}_{\lambda,[P_1],\varphi}(G)$ is given by

$$\mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \subseteq \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) = \mathcal{A}_{\lambda,[P_1],\varphi}(G)$$

where $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial precisely if $\lambda_1 = \lambda_2$, $\xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$, and there exists a section ϕ of the normalized parabolic induction $\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, s)$ such that the automorphic Eisenstein series $\mathrm{Eis}_{P_1}^G(\phi_s)$ has a pole at $s = s_0 = (\frac{1}{2}, \frac{1}{2})$, in which case we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \simeq \mathcal{J}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, \tfrac{1}{2})$$

In any case we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)/\mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \simeq \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, s_0) \otimes \mathrm{Sym}(\mathfrak{a}_{P_1,\mathbb{C}}^\vee)$$

where $s_0 = (\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$ in the case $\xi = (\frac{\lambda_1-\lambda_2+1}{2}, -\frac{\lambda_1-\lambda_2+1}{2})$, and $s_0 = (\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2})$ in the case $\xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$.

Proof. Let $\mathcal{M}_{\lambda,[P_1],\varphi}(G)$ be the set of tuples (P', π', ν, s_0) where $P' \in [P_1]$ is a standard parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P' = M'N'$ containing an element of the associate class $[P_1]$, where π' is a discrete spectrum automorphic representation of $M'(\mathbb{A}_\mathbb{Q})$ with cuspidal support π obtained as the iterated residue at $\nu \in (\mathfrak{a}_{P_1}^{P'})^\vee_\mathbb{C}$ of the automorphic Eisenstein series attached to $\pi \in \varphi_{P_1}$, and where $s_0 \in \mathfrak{a}_{P',\mathbb{C}}^\vee$ is a point with $\Re(s_0) \in \overline{(\mathfrak{a}_{P'}^G)^{\vee+}}$ such that $s_0 + \nu + \xi$ is annihilated by $\mathcal{J}_\lambda$. For $m \in \mathbb{Z}$ let $\mathcal{M}^m_{\lambda,[P_1],\varphi}(G)$ be the subset of tuples (P', π', ν, s_0) such that $T(s_0) = m$, where $T : \overline{\mathfrak{a}_{P'}^{\vee+}} \to \mathbb{Z}$ is fixed at the end of the proof. Then we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^m_{\lambda,[P_1],\varphi}(G)/\mathcal{A}^{m+1}_{\lambda,[P_1],\varphi}(G) \simeq \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^m_{\lambda,[P_1],\varphi}(G)} \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P'(\mathbb{A}_\mathbb{Q})}(\pi',s_0)\otimes \mathrm{Sym}((\mathfrak{a}^G_{P'})^\vee_\mathbb{C})$$

Now we have the following:

- For $P' = P_1$ we have $\pi' = \mu \otimes \chi$ hence $\nu = (0,0) \in (\mathfrak{a}^{P'}_{P_1})^\vee_\mathbb{C}$. By 4.3.6 such ν can only be obtained for $s_0 = \pm(\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$ and $\xi = (\frac{\lambda_1-\lambda_2+1}{2}, -\frac{\lambda_1-\lambda_2+1}{2})$, or $s_0 = \pm(\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2})$ and $\xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$. It follows that

$$\mathcal{M}^m_{\lambda,[P_1],\varphi}(G)_{P_1} = \begin{cases} (P_1, \mu\otimes\chi, (0,0), s_0) \\ \qquad \text{if } T(\lambda) = m \text{ and } s_0 = \begin{cases} (\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2}) & \xi = (\frac{\lambda_1-\lambda_2+1}{2}, -\frac{\lambda_1-\lambda_2+1}{2}) \\ (\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2}) & \xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2}) \end{cases} \\ 0 \qquad \text{otherwise} \end{cases}$$

- For $P' = G$ since π' is a residual representation of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$, by 4.3.5 we have $\pi' \simeq \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\pi, (\frac{1}{2}, \frac{1}{2}))$ hence $\nu = (\frac{1}{2}, \frac{1}{2}) \in (\mathfrak{a}^{P'}_{P_1})^\vee_\mathbb{C}$. By 4.3.6 such ν can only be obtained for $\lambda_1 = \lambda_2$ and $s = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$. It follows that

$$\mathcal{M}^m_{\lambda,[P_1],\varphi}(G)_G = \begin{cases} (G, \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\pi, (\frac{1}{2}, \frac{1}{2})), (\frac{1}{2}, \frac{1}{2}), (0,0)) \\ \qquad \text{if } T(0) = m \text{ and } \lambda_1 = \lambda_2 \text{ and } \omega_\mu = 1 \text{ and } \Lambda(\mu, \frac{1}{2}) \neq 0 \\ 0 \qquad \text{otherwise} \end{cases}$$

The result follows by taking the filtration defined by $T(0) = 2$ and $T(\lambda) = 1$ for $\lambda \neq 0$. $\square$

Cohomology of the Franke-Schwermer Filtration Recall that the Levi quotient $M_1(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$ admits a decomposition $M_1(\mathbb{R}) = M_1^{\mathrm{ss}}(\mathbb{R}) \times A_{P_1}(\mathbb{R})^\circ$ where $M_1^{\mathrm{ss}}(\mathbb{R}) = \mathrm{SL}_2^{\pm}(\mathbb{R}) \times \{\pm 1\}$ is semisimple with Lie algebra $\mathfrak{m}_1^{\mathrm{ss}} = \mathfrak{sl}_2$ and $A_{P_1}(\mathbb{R})^\circ = \mathbb{R}^2_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_1} with Lie algebra $\mathfrak{a}_{P_1} = \mathbb{R}^2$. Recalling that $K'_\infty = \mathbb{R}_{>0}\mathrm{U}(2)$, for $K'^{M_1}_\infty$ the image of $K'_\infty \cap P_1(\mathbb{R})$ under the canonical projection $P_1(\mathbb{R}) \to M_1(\mathbb{R})$ we have $K'^{M_1}_\infty = \mathbb{R}_{>0}\mathrm{O}(2)$, and for $K'^{M_1^{\mathrm{ss}}}_\infty$ the image of $K'_\infty \cap P_1(\mathbb{R})$

under the canonical projection $P_1(\mathbb{R}) \to M_1^{\mathrm{ss}}(\mathbb{R})$ we have $K_\infty'^{M_1^{\mathrm{ss}}} = \mathrm{O}(2)$. Let $V^\epsilon = V \otimes \mathrm{sign}^{\epsilon_1}$ be the standard 2-dimensional representation of $\mathrm{SL}_2^\pm(\mathbb{R})$ with character $\mathrm{sign}^{\epsilon_1}$ and for $k \geq 0$ let $V_k^{\epsilon_1} = \mathrm{Sym}^k(V) \otimes \mathrm{sign}^{\epsilon_1}$ be the irreducible $k+1$-dimensional representation of $\mathrm{SL}_2^\pm(\mathbb{R})$ with character $\mathrm{sign}^{\epsilon_1}$. Let $\mathcal{D}_k$ be the discrete series representation of $\mathrm{SL}_2^\pm(\mathbb{R})$ with minimal $\mathrm{O}(2)$-type $k+1$.

Proposition 4.3.8. (Compare to [**?** , Lemma 4.1]) For $\epsilon_1, \epsilon \in \{0, 1\}$ we have

$$H^q(\mathfrak{m}_1^{\mathrm{ss}}, K_\infty'^{M_1^{\mathrm{ss}}}; \pi_\infty \otimes (V_0^{\epsilon_1} \otimes \mathrm{sign}^\epsilon)) \simeq \begin{cases} \mathbb{C} & \begin{cases} q = 0, \pi_\infty \simeq V_0^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon'} \\ q = 1, \pi_\infty \simeq \mathcal{D}_1 \otimes \mathrm{sign}^{\epsilon'} \\ q = 2, \pi_\infty \simeq V_0^{\epsilon_1+1} \otimes \mathrm{sign}^{\epsilon'} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

and for $k \geq 1$ we have

$$H^q(\mathfrak{m}_1^{\mathrm{ss}}, K_\infty'^{M_1^{\mathrm{ss}}}; \pi_\infty \otimes (V_k^{\epsilon_1} \otimes \mathrm{sign}^\epsilon)) \simeq \begin{cases} \mathbb{C} & q = 1, \pi_\infty \simeq \mathcal{D}_{k+1} \otimes \mathrm{sign}^{\epsilon'} \\ 0 & \text{otherwise} \end{cases}$$

Proof. For $G = \mathrm{Sp}_4$ recall that the Levi quotient $M_1(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$ admits a decomposition $M_1(\mathbb{R}) = M_1^{\mathrm{ss}}(\mathbb{R}) \times A_{P_1}(\mathbb{R})^\circ$ where $M_1^{\mathrm{ss}}(\mathbb{R}) = \mathrm{SL}_2^\pm(\mathbb{R}) \times \{\pm 1\}$ is semisimple with Lie algebra $\mathfrak{m}_1^{\mathrm{ss}} = \mathfrak{sl}_2$ and $A_{P_1}(\mathbb{R})^\circ = \mathbb{R}_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_1} with Lie algebra $\mathfrak{a}_{P_1} = \mathbb{R}$. Recalling that $K_\infty = \mathrm{U}(2)$, for $K_\infty^{M_1}$ the image of $K_\infty \cap P_1(\mathbb{R})$ under the canonical projection $P_1(\mathbb{R}) \to M_1(\mathbb{R})$ we have $K_\infty^{M_1} = \mathrm{O}(2)$, and for $K_\infty'^{M_1^{\mathrm{ss}}}$ the image of $K_\infty \cap P_1(\mathbb{R})$ under the canonical projection $P_1(\mathbb{R}) \to M_1^{\mathrm{ss}}(\mathbb{R})$ we have $K_\infty^{M_1^{\mathrm{ss}}} = \mathrm{O}(2)$. By [48, Lemma 4.1] for $\epsilon_1 \in \{0, 1\}$ we have

$$H^q(\mathfrak{m}_1^{\mathrm{ss}}, K_\infty^{M_1^{\mathrm{ss}}}; \pi_\infty \otimes V_0^{\epsilon_1}) \simeq \begin{cases} \mathbb{C} & \begin{cases} q = 0, \pi_\infty \simeq V_0^{\epsilon_1} \\ q = 1, \pi_\infty \simeq \mathcal{D}_1 \\ q = 2, \pi_\infty \simeq V_0^{\epsilon_1+1} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

and for $k \geq 1$ we have

$$H^q(\mathfrak{m}_1^{\mathrm{ss}}, K_\infty^{M_1^{\mathrm{ss}}}; \pi_\infty \otimes V_k^{\epsilon_1}) \simeq \begin{cases} \mathbb{C} & q = 1, \pi_\infty \simeq \mathcal{D}_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

The result follows from this, noting that the $(\mathfrak{m}_1^{\mathrm{ss}}, K_\infty'^{M_1^{\mathrm{ss}}})$-cohomology is independent of the character sign^ϵ on the factor $\{\pm 1\}$ of $M_1^{\mathrm{ss}}(\mathbb{R})$, as the factor $\mathbb{R}_{>0}$ of K_∞' intersects this factor only at the identity. $\square$

Note that $\mathcal{D}_k|_{\mathrm{SL}_2(\mathbb{R})} \simeq \mathcal{D}_k^+ \oplus \mathcal{D}_k^-$ where $\mathcal{D}_k^+$ is the holomorphic discrete series representation of $\mathrm{SL}_2(\mathbb{R})$ with minimal $\mathrm{SO}(2)$-type $k+1$ and $\mathcal{D}_k^-$ is the antiholomorphic discrete series representation of $\mathrm{SL}_2(\mathbb{R})$ with maximal $\mathrm{SO}(2)$-type $-k-1$; the $(\mathfrak{sl}_2, \mathrm{O}(2))$-cohomology of this representation has dimension 1, whereas the $(\mathfrak{sl}_2, \mathrm{SO}(2))$-cohomology of this representation has dimension 2, as we will see in the case of the Klingen parabolic subgroup.

Now there are two pieces of the Franke-Schwermer filtration whose $(\mathfrak{g}, K_\infty')$-cohomology we need to compute: we need to compute $H^q(\mathfrak{g}, K_\infty'; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda)$ as well as $H^q(\mathfrak{g}, K_\infty'; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)$ in the case where $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial.

Proposition 4.3.9. (Compare to [48, Proposition 4.2]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P \in [P_1]} \in \Phi_{\lambda,[P_1]}(G)$ be the associate class of a cuspidal automorphic representation $\pi \otimes \chi \in \varphi_{P_1}$ of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K_\infty'; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda) \simeq \begin{cases} \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \frac{\lambda_1 - \lambda_2 + 1}{2}) & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3} \\ \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \frac{\lambda_1 + \lambda_2 + 3}{2}) & q = 4, \mu_\infty = \mathcal{D}_{\lambda_1-\lambda_2+1} \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial, that is precisely if $\lambda_1 = \lambda_2$ and $\mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}$ with $\omega_\mu = 1$ and $\Lambda(\frac{1}{2},\mu) \neq 0$, then we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \frac{1}{2}) & q = 2,4 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the first claim we have

$$\begin{aligned} &H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda) \\ &\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\pi, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_\lambda) \\ &\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\pi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_\lambda) \otimes \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, s_0) \end{aligned}$$

By Borel-Wallach [20, III Theorem 3.3] there exists a unique $w \in W^{P_1}$ such that $\pi_\infty \otimes \mathbb{C}_{s_0+\rho_{P_1}}$ has nonzero $(\mathfrak{m}_1, K'^{M_1}_\infty)$-cohomology with respect to $\mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_{w\cdot\lambda}$: we have $w = s_{21}$ in the case $s_0 = (\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2})$ and $\xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$, and we have $w = s_{212}$ in the case $s_0 = (\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$ and $\xi = (\frac{\lambda_1-\lambda_2+1}{2}, -\frac{\lambda_1-\lambda_2+1}{2})$. Now in both cases, recalling that $M_1(\mathbb{R}) = M_1^{\mathrm{ss}}(\mathbb{R}) \times A_{P_1}(\mathbb{R})^\circ$ and $\mathfrak{m}_1 = \mathfrak{m}_1^{\mathrm{ss}} \oplus \mathfrak{a}_{P_1}$, by 4.3.8 we have

$$\begin{aligned} &H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\pi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_\lambda) \\ &\simeq H^{q-\ell(w)}(\mathfrak{m}_1, K'^{M_1}_\infty; \pi_\infty \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes \mathbb{C}_{s_0+\rho_{P_1}} \otimes V_{w\cdot\lambda}) \\ &\simeq H^{q-\ell(w)}(\mathfrak{m}_1^{\mathrm{ss}}, K'^{M_1^{\mathrm{ss}}}_\infty; \pi_\infty \otimes V_{w\cdot\lambda}) \\ &\simeq \begin{cases} \mathbb{C} & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3} \\ \mathbb{C} & q = 4, \mu_\infty = \mathcal{D}_{\lambda_1-\lambda_2+1} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For the second claim suppose that $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial, so that we are in the case $\lambda_1 = \lambda_2, \xi = (\frac{\lambda_1+\lambda_2+3}{2}, -\frac{\lambda_1+\lambda_2+3}{2})$, and there exists a section ϕ of the normalized parabolic induction $\mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\pi, s)$ such that the automorphic Eisenstein series $\mathrm{Eis}_{P_1}^G(\phi_s)$ has a pole at $s = s_0 = (\frac{1}{2}, \frac{1}{2})$. Then by 4.3.8 we have

$$\begin{aligned} & H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \\ & \simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\pi, \tfrac{1}{2}) \otimes V_\lambda) \\ & \simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{J}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\pi_\infty, \tfrac{1}{2}) \otimes V_\lambda) \otimes \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \end{aligned}$$

Now by Borel-Wallach [20, VI Lemma 1.5, Theorem 1.7] the $(\mathfrak{g}, K'_\infty)$-cohomology of the Langlands quotient $\mathcal{J}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\pi_\infty, \frac{1}{2})$ with respect to V_λ is given

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{J}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\pi_\infty, \tfrac{1}{2}) \otimes V_\lambda) = \begin{cases} \mathbb{C} & q = 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

The result follows. □

Siegel Eisenstein Cohomology Having computed the $(\mathfrak{g}, K'_\infty)$-cohomology of the pieces of the Franke-Schwermer filtration, we are now in the position to compute Eisenstein cohomology. Up to indeterminacies regarding the behavior of certain connecting morphisms in the case where $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial, we have the following result:

Theorem 4.3.10. (Compare to [48, Theorem 5.1]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P \in [P_1]} \in \Phi_{\lambda,[P_1]}(G)$ be the associate class of a cuspidal automorphic representation $\pi = \mu \otimes \chi \in \varphi_{P_1}$ of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$.

If $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is trivial then we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \frac{\lambda_1 - \lambda_2 + 1}{2}) & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1 + \lambda_2 + 3} \\ \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \frac{\lambda_1 + \lambda_2 + 3}{2}) & q = 4, \mu_\infty = \mathcal{D}_{\lambda_1 - \lambda_2 + 1} \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial, that is precisely if $\lambda_1 = \lambda_2$ and $\mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}$ with $\omega_\mu = 1$ and $\Lambda(\frac{1}{2}, \mu) \neq 0$, then (with the assumption 4.3.11 on connecting morphisms) we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \frac{1}{2}) & q = 2 \\ \mathcal{K}^3(\mu, \chi) & q = 3 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}^3(\mu, \chi)$ is the $G(\mathbb{A}^\infty_\mathbb{Q})$-module

$$\mathcal{K}^3(\mu, \chi) \simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)$$

Proof. By definition $H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) = H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)$, and if $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is trivial we clearly have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \simeq H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda)$$

It follows that we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$\begin{aligned} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) &\simeq \begin{cases} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) & q = 3, 4 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \frac{\lambda_1-\lambda_2+1}{2}) & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3} \\ \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \frac{\lambda_1+\lambda_2+3}{2}) & q = 4, \mu_\infty = \mathcal{D}_{\lambda_1-\lambda_2+1} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

If $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial consider short exact sequence of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K'_\infty)$-modules

$$0 \to \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \to \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \to \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \to 0$$

which gives rise to a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$
\begin{aligned}
0 &\to \underbrace{H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)}_{\simeq \mathcal{J}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \frac{1}{2})} \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^2(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda)}}_{=0} \\
&\to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)}}_{=0} \to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \to \underbrace{H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \frac{1}{2})} \\
&\to \underbrace{H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)}_{\simeq \mathcal{J}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \frac{1}{2})} \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda)}}_{=0} \to 0
\end{aligned}
$$

Now we make the following assumption on connecting morphisms:

Assumption 4.3.11. If $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial as above, then the connecting morphism

$$H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)$$

is surjective, so the morphism $H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)$ is zero.

Granting this, it follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$
\begin{aligned}
H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) &\simeq \begin{cases} H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda) & q = 2 \\ \mathcal{K}^3(\mu, \chi) & q = 3 \\ 0 & \text{otherwise} \end{cases} \\
&\simeq \begin{cases} \mathcal{J}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \frac{1}{2}) & q = 2 \\ \mathcal{K}^3(\mu, \chi) & q = 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

where $\mathcal{K}^3(\mu, \chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$
\begin{aligned}
\mathcal{K}^3(\mu, \chi) &\simeq \ker\Big(H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)\Big) \\
&\simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)
\end{aligned}
$$

The result follows. $\square$

The behavior of the connecting morphism $H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)} \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_1],\varphi}(G) \otimes V_\lambda)$ is subtle; its behavior can be determined in this case by computing the behavior of certain modular symbols.

The $\mathcal{H}_{K_{\text{fin}}}$-modules in the above theorem will be paired with 1-dimensional Gal-modules. For an integer $n \in \mathbb{Z}$ and for $\chi = \chi_{\text{fin}} \otimes \chi_\infty$ a (finite order) character of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ let $\mathbb{L}^n_\chi = \rho_\chi(-n)$ be the 1-dimensional ℓ-adic Gal-module attached to χ twisted by the n-th power of the ℓ-adic cyclotomic character, with

$$\mathrm{tr}(\mathrm{Frob}_p^j | \mathbb{L}^n_\chi) = p^{nj} c(\chi_p)^j = p^{nj}\chi(p)^j$$

Now we have the following result, which is conditional on the assumption 4.3.11 in the case $\lambda_1 = \lambda_2$:

Theorem 4.3.12. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$. Then (with the assumption 4.3.11 on connecting morphisms in the case $\lambda_1 = \lambda_2$) the Siegel Eisenstein cohomology $H^*_{\text{Eis},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4$ and given as an $\mathcal{H}_{K_{\text{fin}}} \times$ Gal-module by

$$H^2_{\text{Eis},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \displaystyle\bigoplus_{\substack{\pi=\mu\otimes\chi \\ \mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3} \\ \omega_\mu=\chi^2=1, \Lambda(\frac{1}{2},\mu)\neq 0}} \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, \tfrac{1}{2})^{K_{\text{fin}}} \boxtimes \mathbb{L}^{\lambda_2+1}_\chi & \lambda_1 = \lambda_2 \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\text{Eis},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \displaystyle\bigoplus_{\substack{\pi=\mu\otimes\chi \\ \mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}}} \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, \tfrac{1}{2}) \to \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, \tfrac{1}{2})\Big)^{K_{\text{fin}}} \boxtimes \mathbb{L}^{\lambda_1+2}_{\omega_\mu\chi} & \lambda_1 = \lambda_2 \\ \displaystyle\bigoplus_{\substack{\pi=\mu\otimes\chi \\ \mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}}} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, \tfrac{\lambda_1-\lambda_2+1}{2})^{K_{\text{fin}}} \boxtimes \mathbb{L}^{\lambda_1+2}_{\omega_\mu\chi} & \text{otherwise} \end{cases}$$

$$H^4_{\text{Eis},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) = \bigoplus_{\substack{\pi=\mu\otimes\chi \\ \mu_\infty = \mathcal{D}_{\lambda_1-\lambda_2+1}}} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2})^{K_{\text{fin}}} \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3}_{\omega_\mu\chi}$$

Proof. The result follows by taking the direct sum over associate classes of (unitary) cuspidal automorphic representations $\pi = \mu \otimes \chi$ of $M_1(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ of the contributions to 4.3.10. The Galois action can be obtained from the Siegel parabolic terms in the GSp_4 trace formula.

Alternatively, by Pink the Tate twists are given as follows. For $\lambda = n_1\omega + n_2\omega_2$ and $d(\lambda) = \frac{n_1}{2} + n_2$ the Galois action must be twisted by $\mathbb{L}^{d(\lambda)-\langle\mu, w\cdot\lambda\rangle}$ where $\mu : \mathrm{GL}_1 \to \mathrm{GSp}_4$ is the cocharacter given by $t \mapsto \mathrm{diag}(t, t, 1, 1)$. Since $\omega_1(\mathrm{diag}(t, t, 1, 1)) = t^{1/2}$ and $\omega_2(\mathrm{diag}(t, t, 1, 1)) = t$ we have $\langle\mu, w\cdot\lambda\rangle = \frac{n_1^w}{2} + n_2^w$ for $w \in W$ and we obtain the following Tate twists:

w	n_1^w	n_2^w	$\langle\mu, w\cdot\lambda\rangle$	$d(\lambda) - \langle\mu, w\cdot\lambda\rangle$
1	n_1	n_2	$\frac{n_1}{2} + n_2$	0
s_2	$n_1 + 2n_2 + 2$	$-n_2 - 2$	$\frac{n_1}{2} - 1$	$n_2 + 1$
s_{21}	$n_1 + 2n_2 + 2$	$-n_1 - n_2 - 3$	$-\frac{n_1}{2} - 2$	$n_1 + n_2 + 2$
s_{212}	n_1	$-n_1 - n_2 - 3$	$-\frac{n_1}{2} - n_2 - 3$	$n_1 + 2n_2 + 3$

Recalling that $n_1 = \lambda_1 - \lambda_2$ and $n_2 = \lambda_2$, this gives the Tate twists in the theorem. $\square$

One can also state a Poincare dual version of the above theorem for compactly supported Siegel Eisenstein cohomology by modifying the above Tate twists.

Example 4.3.13. Let $K_{\text{fin}} = G(\widehat{\mathbb{Z}})$ so that $\mathcal{S}_{K_{\text{fin}}} = \mathcal{A}_2$ is the moduli stack of principally polarized Abelian surfaces. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2$.

For an integer $k \geq 2$ let $s_{\Gamma(1)}[k]$ be the dimension of the space of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Let $s_{\Gamma(1)}[k]_{L(\frac{1}{2})=0}$ be the dimension of the subspace of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ with vanishing central L-value $L(f, \frac{k}{2}) = 0$. Let $s_{\Gamma(1)}[k]_{L(\frac{1}{2})\neq 0}$ be the dimension of the subspace of cusp forms of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ with nonvanishing central L-value $L(f, \frac{k}{2}) \neq 0$. Note that by the functional equation, $\mathrm{ord}_{s=\frac{k}{2}} L(f, s)$ is odd for $k \equiv 2 \mod 4$ (in which case we have $s_{\Gamma(1)}[k]_{L(\frac{1}{2})=0} = s_{\Gamma(1)}[k]$) and is even for $k \equiv 0 \mod 4$ (in which case we have $0 \leq s_{\Gamma(1)}[k]_{L(\frac{1}{2})\neq 0} \leq s_{\Gamma(1)}[k]$). By Maeda's conjecture (which is special to the case of level 1), for $k \equiv 0 \mod 4$ we should have $s_{\Gamma(1)}[k]_{L(\frac{1}{2})=0} = 0$.

The Siegel Eisenstein cohomology is concentrated in degrees 2, 3, 4 and given by

$$H^2_{\mathrm{Eis},[P_1]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_1+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{Eis},[P_1]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})= 0}\mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_1+2} & \text{otherwise} \end{cases}$$

$$H^4_{\mathrm{Eis},[P_1]}(\mathcal{A}_2, \mathbb{V}_\lambda) = s_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_1+\lambda_2+3}$$

4.3.3 Klingen Eisenstein Cohomology

In this section we compute Klingen Eisenstein cohomology as a $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module. First, we recall some facts about the poles of Klingen Eisenstein series, and the evaluation points and infinitesimal characters which will enter into the description of the Franke filtration for the Klingen parabolic subgroup.

We begin by restating the results of the previous section on the locations of poles of Siegel Eisenstein series:

Proposition 4.3.14. (Compare to [48, Proposition 3.4]) The automorphic Eisenstein series $\mathrm{Eis}^G_{P_2}(\phi_s)$ attached to a unitary cuspidal automorphic representation $\pi = \chi \otimes \mu$ of $M_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ has a pole at $s = \nu \in \overline{\mathfrak{a}^{\vee+}_{P_2}}$ precisely if $\nu = (1,0)$ and $\mu = \mathcal{AI}^F_\mathbb{Q}(\theta)$ is the automorphic induction of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}^\times_F \to \mathbb{C}^\times$ corresponding to an imaginary quadratic extension $F/\mathbb{Q}$, in which case we have a simple pole at $s = 1$ and the space spanned by the residues $\mathrm{Res}_{s=1}\mathrm{Eis}^G_{P_2}(\phi_s)$ is isomorphic to the Langlands quotient $\mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, 1)$.

We now record the following result on infinitesimal characters coming from the action of the Weyl group:

Proposition 4.3.15. (Compare to [48, Lemma 3.5]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $\mathcal{J}_\lambda$ be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ annihilating $V^\vee_\lambda$. Let $\varphi = \{\varphi_P\}_{P\in[P_2]} \in \Phi_{\lambda,[P_2]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi \otimes \mu \in \varphi_{P_2}$ of $M_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Then the infinitesimal character $\xi \in \mathfrak{a}^{\vee\perp}_{P_2}$ and the corresponding $s_0 \in \mathfrak{a}^\vee_{P_2}$ such that $s_0 + \xi$ is annihilated by $\mathcal{J}_\lambda$ are given by $s_0 = \pm(\lambda_1+2, 0)$ and $\xi = (0, \lambda_2+1)$, or $s_0 = \pm(\lambda_2+1, 0)$ and $\xi = (0, \lambda_1+2)$.

The Franke-Schwermer Filtration We now describe the Franke filtration for the Klingen parabolic subgroup. As expected, the bottom piece is given by the Langlands quotient of normalized Klingen parabolic induction, and the top piece is given by normalized Klingen parabolic induction.

Proposition 4.3.16. (Compare to [48, Theorem 3.6]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $\mathcal{J}_\lambda$ be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ annihilating $V_\lambda^\vee$. Let $\varphi = \{\varphi_P\}_{P\in[P_2]} \in \Phi_{\lambda,[P_2]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi \otimes \mu \in \varphi_{P_2}$ of $M_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with infinitesimal character $\xi \in \mathfrak{a}_{P_2}^{\vee\perp}$. Let $s_0 \in \mathfrak{a}_{P_2}^\vee$ such that $s_0 + \xi$ is annihilated by $\mathcal{J}_\lambda$. Then the Franke-Schwermer filtration on $\mathcal{A}_{\lambda,[P_2],\varphi}(G)$ is given by

$$\mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \subseteq \mathcal{A}^1_{\lambda,[P_2],\varphi}(G) = \mathcal{A}_{\lambda,[P_2],\varphi}(G)$$

where $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial precisely if $\lambda_2 = 0, \xi = (0, \lambda_1 + 2)$, and there exists a section ϕ of the normalized parabolic induction $\mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, s)$ such that the automorphic Eisenstein series $\mathrm{Eis}^G_{P_2}(\phi_s)$ has a pole at $s = s_0 = (1, 0)$, in which case we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \simeq \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, 1)$$

In any case we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)/\mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \simeq \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C})$$

where $s_0 = (\lambda_1 + 2, 0)$ in the case $\xi = (0, \lambda_2 + 1)$, and $s_0 = (\lambda_2 + 1, 0)$ in the case $\xi = (0, \lambda_1 + 2)$.

Proof. Let $\mathcal{M}_{\lambda,[P_2],\varphi}(G)$ be the set of tuples (P', π', ν, s) where $P' \in [P_2]$ is a standard parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P' = M'N'$ containing an element of the associate class $[P_2]$, where π' is a discrete spectrum automorphic representation of $M'(\mathbb{A}_\mathbb{Q})$ with cuspidal support π obtained as the iterated residue at $\nu \in (\mathfrak{a}^{P'}_{P_2})^\vee_\mathbb{C}$ of the Eisenstein series attached to $\pi \in \phi_{P_2}$, and where $s_0 \in \mathfrak{a}^\vee_{P',\mathbb{C}}$ is a point with $\Re(s_0) \in \overline{(\mathfrak{a}^G_{P_2})^{\vee+}}$ such that $s_0 + \nu + \xi$ is annihilated by $\mathcal{J}_\lambda$. For $m \in \mathbb{Z}$ let $\mathcal{M}^m_{\lambda,[P_2],\varphi}(G)$ be the subset of tuples (P', π', ν, s_0) such that $T(s_0) = m$, where $T : \overline{\mathfrak{a}^{\vee+}_{P'}} \to \mathbb{Z}$ is fixed at the end of the proof. Then we have an isomorphism of

$G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_{\infty})$-modules

$$\mathcal{A}^m_{\lambda,[P_2],\varphi}(G)/\mathcal{A}^{m+1}_{\lambda,[P_2],\varphi}(G) \simeq \bigoplus_{(P',\pi',\nu,s_0)\in\mathcal{M}^m_{\lambda,[P_2],\varphi}(G)} \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}})}_{P'(\mathbb{A}_{\mathbb{Q}})}(\pi', s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P'})^{\vee}_{\mathbb{C}})$$

Now we have the following:

- For $P' = P_2$ we have $\pi = \chi \otimes \mu$ hence $\nu = (0,0) \in (\mathfrak{a}^{P'}_{P_2})^{\vee}_{\mathbb{C}}$. By 4.3.15 such ν can only be obtained for $s_0 = \pm(\lambda_1+2, 0)$ and $\xi = (0, \lambda_2+1)$, or $s_0 = \pm(\lambda_2+1, 0)$ and $\xi = (0, \lambda_1+2)$. It follows that

$$\mathcal{M}^m_{\lambda,[P_2],\varphi}(G)_{P_2} = \begin{cases} (P_2, \chi\otimes\mu, (0,0), s_0) \\ \qquad \text{if } T(\lambda) = m \text{ and } s_0 = \begin{cases} (\lambda_1+2,0) & \xi = (0,\lambda_2+1) \\ (\lambda_2+1,0) & \xi = (0,\lambda_1+2) \end{cases} \\ 0 \qquad \text{otherwise} \end{cases}$$

- For $P' = G$ since π' is a residual representation of $M_2(\mathbb{A}_{\mathbb{Q}}) \simeq \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, by 4.3.14 we have $\pi' \simeq \mathcal{J}^{G(\mathbb{A}_{\mathbb{Q}})}_{P_2(\mathbb{A}_{\mathbb{Q}})}(\pi, (1,0))$ hence $\nu = (1,0) \in (\mathfrak{a}^{P'}_{P_1})^{\vee}_{\mathbb{C}}$. By 4.3.15 such ν can only be obtained for $\lambda_2 = 0$ and $s = (0, \lambda_1+2)$. It follows that

$$\mathcal{M}^m_{\lambda,[P_2],\varphi}(G)_G = \begin{cases} (G, \mathcal{J}^{G(\mathbb{A}_{\mathbb{Q}})}_{P_2(\mathbb{A}_{\mathbb{Q}})}(\pi,(1,0)), (1,0), (0,0)) \\ \qquad \text{if } T(0) = m \text{ and } \lambda_2 = 0 \text{ and } \mu = \mathcal{AI}^F_{\mathbb{Q}}(\theta) \\ 0 \qquad \text{otherwise} \end{cases}$$

The result follows by taking the filtration defined by $T(0) = 2$ and $T(\lambda) = 1$ for $\lambda \neq 0$. $\square$

Cohomology of the Franke-Schwermer Filtration Recall that the Levi quotient $M_2(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ admits a decomposition $M_2(\mathbb{R}) = M_2^{\mathrm{ss}}(\mathbb{R}) \times A_{P_2}(\mathbb{R})^{\circ}$ where $M_2^{\mathrm{ss}}(\mathbb{R}) = \{\pm1\} \times \mathrm{SL}_2(\mathbb{R}) \times \{\pm1\}$ is semisimple with Lie algebra $\mathfrak{m}_2^{\mathrm{ss}} = \mathfrak{sl}_2$ and $A_{P_2}(\mathbb{R})^{\circ} = \mathbb{R}^2_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_2} with Lie algebra $\mathfrak{a}_{P_2} = \mathbb{R}^2$. Recalling that $K'_{\infty} = \mathbb{R}_{>0}\mathrm{U}(2)$, for $K'^{M_2}_{\infty}$ the image of $K'_{\infty} \cap P_2(\mathbb{R})$ under the canonical projection $P_2(\mathbb{R}) \to M_2(\mathbb{R})$ we have $K'^{M_2}_{\infty} = \mathbb{R}_{>0}\mathrm{SO}(2)$, and for $K'^{M_2^{\mathrm{ss}}}_{\infty}$ the image of $K'_{\infty} \cap P_2(\mathbb{R})$ under the canonical projection $P_2(\mathbb{R}) \to M_2^{\mathrm{ss}}(\mathbb{R})$ we have $K'^{M_2^{\mathrm{ss}}}_{\infty} = \mathrm{SO}(2)$. Let $V^{\epsilon_1} = V \otimes \mathrm{sign}^{\epsilon_1}$

be the tensor product of the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{R})$ with the character $\mathrm{sign}^{\epsilon_1}$ and for $k \geq 0$ let $V_k^{\epsilon_1} = \mathrm{Sym}^k(V) \otimes \mathrm{sign}^{\epsilon_1}$ be the tensor product of the irreducible $k+1$-dimensional representation of $\mathrm{SL}_2(\mathbb{R})$ with the character $\mathrm{sign}^{\epsilon_1}$. Let $\mathcal{D}_k = \mathcal{D}_k^+ \oplus \mathcal{D}_k^-$ be the direct sum of discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ with minimal $\mathrm{SO}(2)$-types $k+1$ and $-k-1$, that is the restriction of the holomorphic discrete series representation $\mathcal{D}_k$ of $\mathrm{GL}_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})$.

Proposition 4.3.17. (Compare to [? , Lemma 4.1]) For $\epsilon_1, \epsilon \in \{0,1\}$ we have

$$H^q(\mathfrak{m}_2^{\mathrm{ss}}, K_\infty'^{M_2^{\mathrm{ss}}}; \pi_\infty \otimes (V_0^{\epsilon_1} \otimes \mathrm{sign}^\epsilon)) \simeq \begin{cases} \mathbb{C} & q = 0, 2, \pi_\infty \simeq V_0^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon'} \\ \mathbb{C}^2 & q = 1, \pi_\infty \simeq \mathcal{D}_1 \otimes \mathrm{sign}^{\epsilon'} \\ 0 & \text{otherwise} \end{cases}$$

and for $k \geq 1$ we have

$$H^q(\mathfrak{m}_2^{\mathrm{ss}}, K_\infty'^{M_2^{\mathrm{ss}}}; \pi_\infty \otimes (V_k^{\epsilon_1} \otimes \mathrm{sign}^\epsilon)) \simeq \begin{cases} \mathbb{C}^2 & q = 1, \pi_\infty \simeq \mathcal{D}_k \otimes \mathrm{sign}^{\epsilon'} \\ 0 & \text{otherwise} \end{cases}$$

Proof. For $G = \mathrm{Sp}_4$ recall that the Levi quotient $M_2(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ admits a decomposition $M_2(\mathbb{R}) = M_2^{\mathrm{ss}}(\mathbb{R}) \times A_{P_2}(\mathbb{R})^\circ$ where $M_2^{\mathrm{ss}}(\mathbb{R}) = \{\pm 1\} \times \mathrm{SL}_2(\mathbb{R})$ is semisimple with Lie algebra $\mathfrak{m}_2^{\mathrm{ss}} = \mathfrak{sl}_2$ and $A_{P_2}(\mathbb{R})^\circ = \mathbb{R}_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_2} with Lie algebra $\mathfrak{a}_{P_2} = \mathbb{R}$. Recalling that $K_\infty = \mathrm{U}(2)$, for $K_\infty^{M_2}$ the image of $K_\infty \cap P_2(\mathbb{R})$ under the canonical projection $P_2(\mathbb{R}) \to M_2(\mathbb{R})$ we have $K_\infty^{M_2} = \mathrm{SO}(2)$, and for $K_\infty^{M_2^{\mathrm{ss}}}$ the image of $K_\infty \cap P_2(\mathbb{R})$ under the canonical projection $P_2(\mathbb{R}) \to M_2^{\mathrm{ss}}(\mathbb{R})$ we have $K_\infty^{M_2^{\mathrm{ss}}} = \mathrm{SO}(2)$. By [48, Lemma 4.1], for $\epsilon_1 \in \{0,1\}$ we have

$$H^q(\mathfrak{m}_2^{\mathrm{ss}}, K_\infty^{M_2^{\mathrm{ss}}}; \pi_\infty \otimes V_0^{\epsilon_1}) \simeq \begin{cases} \mathbb{C} & q = 0, 2, \pi_\infty \simeq V_0^{\epsilon_1} \\ \mathbb{C}^2 & q = 1, \pi_\infty \simeq \mathcal{D}_1 \\ 0 & \text{otherwise} \end{cases}$$

and for $k \geq 1$ we have

$$H^q(\mathfrak{m}_2^{\text{ss}}, K_\infty^{M_2^{\text{ss}}}; \pi_\infty \otimes V_k^{\epsilon_1}) \simeq \begin{cases} \mathbb{C}^2 & q = 1, \pi_\infty \simeq \mathcal{D}_k \\ 0 & \text{otherwise} \end{cases}$$

The result follows from this, noting that the $(\mathfrak{m}_2^{\text{ss}}, K_\infty'^{M_2^{\text{ss}}})$-cohomology is independent of the character sign^ϵ on the factor $\{\pm 1\}$ of $M_2^{\text{ss}}(\mathbb{R})$, as the factor $\mathbb{R}_{>0}$ of K'_∞ intersects this factor only at the identity. $\square$

Now there are two pieces of the Franke-Schwermer filtration whose $(\mathfrak{g}, K'_\infty)$-cohomology we need to compute: we need to compute $H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)} \otimes V_\lambda)$ as well as $H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda)$ in the case where $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial.

Proposition 4.3.18. (Compare to [48, Proposition 4.3]) Let $V_\lambda \in \text{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P \in [P_2]} \in \Phi_{\lambda,[P_2]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi \otimes \mu \in \varphi_{P_2}$ of $M_2(\mathbb{A}_\mathbb{Q}) = \text{GL}_1(\mathbb{A}_\mathbb{Q}) \times \text{GL}_2(\mathbb{A}_\mathbb{Q})$. Then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)} \otimes V_\lambda) \simeq \begin{cases} \mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\text{fin}}, \lambda_2 + 1) & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1+2} \\ \mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\text{fin}}, \lambda_1 + 2) & q = 4, \mu_\infty = \mathcal{D}_{\lambda_2+1} \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial, that is precisely if $\lambda_2 = 0$, $\mu_\infty = \mathcal{D}_{\lambda_1+2}$, and $\mu = \mathcal{AI}_\mathbb{Q}^F(\theta)$ is the automorphic induction of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ corresponding to an imaginary quadratic extension $F/\mathbb{Q}$, then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathbb{C}^2 \otimes \mathcal{J}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\text{fin}}, 1) & q = 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the first claim we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)} \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_2(\mathbb{R})}(\pi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_\lambda) \otimes \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, s_0)
\end{aligned}$$

By Borel-Wallach [20, III Theorem 3.3] there exists a unique $w \in W^{P_2}$ such that $\pi_\infty \otimes \mathbb{C}_{s_0+\rho_{P_2}}$ has nonzero $(\mathfrak{m}_2, K'^{M_2}_\infty)$-cohomology with respect to $\mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_{w\cdot\lambda}$: we have $w = s_{12}$ in the case $s_0 = (\lambda_2+1, 0)$ and $\xi = (0, \lambda_1+2)$, and we have $w = s_{121}$ in the case $s_0 = (\lambda_1+2, 0)$ and $\xi = (0, \lambda_2+1)$. Now in both cases, recalling that $M_2(\mathbb{R}) = M_2^{\mathrm{ss}}(\mathbb{R}) \times A_{P_2}(\mathbb{R})^\circ$ and $\mathfrak{m}_2 = \mathfrak{m}_2^{\mathrm{ss}} \oplus \mathfrak{a}_{P_2}$, by 4.3.17 we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_2(\mathbb{R})}(\pi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_\lambda) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_2, K'^{M_2}_\infty; \pi_\infty \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes \mathbb{C}_{s_0+\rho_{P_2}} \otimes V_{w\cdot\lambda}) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_2^{\mathrm{ss}}, K'^{M_2^{\mathrm{ss}}}_\infty; \pi_\infty \otimes V_{w\cdot\lambda}) \\
&\simeq \begin{cases} \mathbb{C}^2 & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1+2} \\ \mathbb{C}^2 & q = 4, \mu_\infty = \mathcal{D}_{\lambda_2+1} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

For the second claim suppose that $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial, so that we are in the case $\lambda_2 = 0, \xi = (0, \lambda_1+2)$, and there exists a section ϕ of the normalized parabolic induction $\mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, s)$ such that the automorphic Eisenstein series $\mathrm{Eis}^G_{P_2}(\phi_s)$ has a pole at $s = s_0 = (1, 0)$. Then by 4.3.17 we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{J}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\pi, 1) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{J}^{G(\mathbb{R})}_{P_2(\mathbb{R})}(\pi_\infty, 1) \otimes V_\lambda) \otimes \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, 1)
\end{aligned}$$

Now by Borel-Wallach [20, VI Lemma 1.5, Theorem 1.7] the $(\mathfrak{g}, K'_\infty)$-cohomology of the Langlands quotient $\mathcal{J}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty, 1)$ with respect to V_λ is given

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{J}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty, 1) \otimes V_\lambda) = \begin{cases} \mathbb{C}^2 & q = 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

The result follows. $\square$

Klingen Eisenstein Cohomology Having computed the $(\mathfrak{g}, K'_\infty)$-cohomology of the pieces of the Franke-Schwermer filtration, we are now in the position to compute Eisenstein cohomology. Up to indeterminacies regarding the behavior of certain connecting morphisms in the case where $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial, we have the following result:

Theorem 4.3.19. (Compare to [48, Theorem 5.1]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P \in [P_2]} \in \Phi_{\lambda,[P_2]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi \otimes \mu \in \varphi_{P_2}$ of $M_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

If $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is trivial then (with the assumption 4.3.20 on connecting morphisms) we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \lambda_2 + 1) & q = 3, \mu_\infty = \mathcal{D}_{\lambda_1+2} \\ \mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \lambda_1 + 2) & q = 4, \mu_\infty = \mathcal{D}_{\lambda_2+1} \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial, that is precisely if $\lambda_2 = 0$ and $\mu = \mathcal{AI}_\mathbb{Q}^F(\theta)$ is the automorphic induction of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ corresponding to an imaginary quadratic extension $F/\mathbb{Q}$, then (with assumptions on connecting morphisms) we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathbb{C}^2 \otimes \mathcal{J}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, 1) & q = 2 \\ \mathcal{K}^3(\chi, \mu) & q = 3 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}^3(\chi,\mu)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\mathcal{K}^3(\chi,\mu) \simeq \ker\Big(\mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},1) \to \mathbb{C}^2 \otimes \mathcal{J}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},1)\Big)$$

Proof. By definition $H^q(\mathfrak{g},K'_\infty;\mathcal{A}_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) = H^q(\mathfrak{g},K'_\infty;\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda)$, and if $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is trivial we clearly have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g},K'_\infty;\mathcal{A}_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) \simeq H^q(\mathfrak{g},K'_\infty;\frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)}\otimes V_\lambda)$$

It follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned} H^q(\mathfrak{g},K'_\infty;\mathcal{A}_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) &\simeq \begin{cases} H^q(\mathfrak{g},K'_\infty;\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) & q=3,4 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},\lambda_2+1) & q=3, \mu_\infty=\mathcal{D}_{\lambda_1+2} \\ \mathbb{C}^2 \otimes \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},\lambda_1+2) & q=4, \mu_\infty=\mathcal{D}_{\lambda_2+1} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

If $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial consider short exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})\times(\mathfrak{g},K'_\infty)$-modules

$$0 \to \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \to \mathcal{A}^1_{\lambda,[P_2],\varphi}(G) \to \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)} \to 0$$

which gives rise to a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned} 0 &\to \underbrace{H^2(\mathfrak{g},K'_\infty;\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda)}_{\simeq \mathbb{C}^2\otimes \mathcal{J}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},1)} \to H^2(\mathfrak{g},K'_\infty;\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) \to \underbrace{\cancel{H^2(\mathfrak{g},K'_\infty;\frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)}\otimes V_\lambda)}}_{=0} \\ &\to \underbrace{\cancel{H^3(\mathfrak{g},K'_\infty;\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda)}}_{=0} \to H^3(\mathfrak{g},K'_\infty;\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) \to \underbrace{H^3(\mathfrak{g},K'_\infty;\frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)}\otimes V_\lambda)}_{\simeq \mathbb{C}^2\otimes \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},1)} \\ &\to \underbrace{H^4(\mathfrak{g},K'_\infty;\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda)}_{\simeq \mathbb{C}^2\otimes \mathcal{J}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}},1)} \to H^4(\mathfrak{g},K'_\infty;\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)\otimes V_\lambda) \to \underbrace{\cancel{H^4(\mathfrak{g},K'_\infty;\frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)}\otimes V_\lambda)}}_{=0} \to 0 \end{aligned}$$

Now we make the following assumption on connecting morphisms:

Assumption 4.3.20. If $\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)$ is nontrivial, then the connecting morphism

$$H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)} \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda)$$

is surjective, the morphism $H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda)$ is zero.

Granting this, it follows that we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$\begin{aligned} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) &\simeq \begin{cases} H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda) & q = 2 \\ \mathcal{K}^3(\chi, \mu) & q = 3 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \mathbb{C}^2 \otimes \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, 1) & q = 2 \\ \mathcal{K}^3(\chi, \mu) & q = 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\mathcal{K}^3(\chi, \mu)$ is the $G(\mathbb{A}^\infty_\mathbb{Q})$-module

$$\begin{aligned} \mathcal{K}^3(\chi, \mu) &\simeq \ker\Big(H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)} \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda)\Big) \\ &\simeq \ker\Big(\mathbb{C}^2 \otimes \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, 1) \to \mathbb{C}^2 \otimes \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\text{fin}}, 1)\Big) \end{aligned}$$

The result follows. □

The behavior of the connecting morphism $H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_2],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_2],\varphi}(G)} \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_2],\varphi}(G) \otimes V_\lambda)$ is again subtle; its behavior can be determined in this case by computing the behavior of certain modular symbols.

The factors of $\mathbb{C}^2$ will disappear in the following theorem, as the $\mathcal{H}_{K_{\text{fin}}}$-modules in the above theorem will be paired with 2-dimensional Gal-modules. For an integer $n \in \mathbb{Z}$ and for $\pi = \chi \otimes \mu$ a cuspidal automorphic representation of $M_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ where $\mu = \mu_{\text{fin}} \otimes \mu_\infty$ is a cuspidal automorphic representatioon of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $\mu_\infty = \mathcal{D}_{k-1}(\det^{k-1})$ a holomorphic discrete series representation, let $\rho_\mu \mathbb{L}^n_\chi = \rho_\mu \otimes \rho_\chi(-n)$ denote the 2-dimensional ℓ-adic Galois representation attached to μ, twisted by the 1-dimensional ℓ-adic Gal-module attached to χ, twisted by the n-th power of the ℓ-adic cyclotomic character, with

$$\mathrm{tr}(\mathrm{Frob}_p^j|\rho_\mu\mathbb{L}_\chi^n) = p^{\frac{k-1}{2}j}(c_1(\mu_p)^j + c_2(\mu_p)^j)p^{nj}c(\chi_p)^j = p^{(\frac{k-1}{2}+n)j}\chi(p)^j(c_1(\mu_p)^j + c_2(\mu_p)^j)$$

Now we have the following result:

Theorem 4.3.21. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. Then (with the assumption 4.3.20 on connecting morphisms in the case $\lambda_2 = 0$) the Klingen Eisenstein cohomology $H^*_{\mathrm{Eis},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees 2, 3, 4 and given as an $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module by

$$H^2_{\mathrm{Eis},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus\limits_{\substack{\pi=\chi\otimes\mu \\ \mu_\infty=\mathcal{D}_{\lambda_1+2} \\ \mu=\mathcal{AI}_{\mathbb{Q}}^F(\theta)}} \mathcal{J}_{P_2(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi_{\mathrm{fin}}, 1)^{K_{\mathrm{fin}}} \boxtimes \rho_\mu\mathbb{L}_\chi^0 & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{Eis},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus\limits_{\substack{\pi=\chi\otimes\mu \\ \mu_\infty=\mathcal{D}_{\lambda_1+2}}} \ker\Big(\mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi_{\mathrm{fin}}, 1) \to \mathcal{J}_{P_2(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi_{\mathrm{fin}}, 1)\Big)^{K_{\mathrm{fin}}} \boxtimes \rho_\mu\mathbb{L}_\chi^1 & \lambda_2 = 0 \\ \bigoplus\limits_{\substack{\pi=\chi\otimes\mu \\ \mu_\infty=\mathcal{D}_{\lambda_1+2}}} \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi_{\mathrm{fin}}, \lambda_2 + 1)^{K_{\mathrm{fin}}} \boxtimes \rho_\mu\mathbb{L}_\chi^{\lambda_2+1} & \text{otherwise} \end{cases}$$

$$H^4_{\mathrm{Eis},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \bigoplus_{\substack{\pi=\chi\otimes\mu \\ \mu_\infty=\mathcal{D}_{\lambda_2+1}}} \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^\infty)}^{G(\mathbb{A}_{\mathbb{Q}}^\infty)}(\pi_{\mathrm{fin}}, \lambda_1 + 2)^{K_{\mathrm{fin}}} \boxtimes \rho_\mu\mathbb{L}_\chi^{\lambda_1+2}$$

Proof. The result follows by taking the direct sum over associate classes of unitary cuspidal automorphic representations $\pi = \chi \otimes \mu$ of $M_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ of the contributions to 4.3.19. The Galois action can be obtained from the Klingen parabolic terms in the GSp_4 trace formula.

By Pink the Tate twists are given as follows. For $\lambda = n_1\omega_1 + n_2\omega_2$ and $d(\lambda) = \frac{n_1}{2} + n_2$ the Galois action must be twisted by $\mathbb{L}^{d(\lambda)-\langle\mu, w\cdot\lambda\rangle}$ where $\mu : \mathrm{GL}_1 \to \mathrm{GSp}_4$ is the cocharacter given by $t \mapsto \mathrm{diag}(t, t, 1, 1)$. Since $\omega_1(\mathrm{diag}(t, t, 1, 1)) = t^{1/2}$ and $\omega_2(\mathrm{diag}(t, t, 1, 1)) = t$ we have $\langle\mu, w \cdot \lambda\rangle = \frac{n_1^w}{2} + n_2^w$ for $w \in W$ and we obtain the following Tate twists:

w	n_1^w	n_2^w	$\langle \mu, w \cdot \lambda \rangle$	$d(\lambda) - \langle \mu, w \cdot \lambda \rangle$
1	n_1	n_2	$\frac{n_1}{2} + n_2$	0
s_1	$-n_1 - 2$	$n_1 + n_2 + 1$	$\frac{n_1}{2} + n_2$	0
s_{12}	$-n_1 - 2n_2 - 4$	$n_1 + n_2 + 1$	$\frac{n_1}{2} - 1$	$n_2 + 1$
s_{121}	$-n_1 - 2n_2 - 4$	n_2	$-\frac{n_1}{2} - 2$	$n_1 + n_2 + 2$

Recalling that $n_1 = \lambda_1 - \lambda_2$ and $n_2 = \lambda_2$, this gives the Tate twists in the theorem. $\square$

One can also state a Poincare dual version of the above theorem for compactly supported Eisenstein cohomology by modifying the above Tate twists.

The automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ contributing to the irregular behavior of Klingen Eisenstein cohomology are the automorphic inductions $\mu = \mathcal{AI}_\mathbb{Q}^F(\theta)$ of the unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ corresponding to an imaginary quadratic extension $F/\mathbb{Q}$, which classically correspond to CM cusp forms. Recall that a cusp form $f = \sum_{n \geq 1} a_n q^n$ of weight $k \geq 1$ and level N admits a self-twist by a nontrivial primitive Dirichlet character ϵ if $a_p = \epsilon(p) a_p$ for all but finitely many primes p (this can only occur if ϵ is quadratic and $a_p = 0$ for all primes $p \nmid N$ with $\chi(p) = -1$). The character ϵ is then the Kronecker character $\omega_{F/\mathbb{Q}} = (\frac{D}{\bullet})$ of a quadratic extension $F = \mathbb{Q}(\sqrt{D})$; when $F/\mathbb{Q}$ is imaginary quadratic we say that f has CM by F, and when $F/\mathbb{Q}$ is real quadratic we say that f has RM by F (this can only occur if $k = 1$ and the associated projective Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ has dihedral image).

Such CM cusp forms can be constructed explicitly. Fixing a weight $k \geq 2$, an imaginary quadratic extension $F/\mathbb{Q}$ with Kronecker character $\omega_{F/\mathbb{Q}}$, and a Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ of conductor $\mathfrak{f} \subseteq \mathcal{O}_F$, that is a character $\theta : \mathcal{I}(\mathfrak{f}) \to \mathbb{C}^\times$ from the group of fractional ideals of $\mathcal{O}_F$ coprime to $\mathfrak{f}$ (so that $\theta(\alpha\mathcal{O}_F) = \alpha^{k-1}$ for $\alpha \equiv 1 \mod \mathfrak{f}$), consider the Dirichlet character δ given by $\delta(n) = \theta(n\mathcal{O}_F)/n^{k-1}$ for $n \in \mathbb{Z}$ coprime to $\mathfrak{f}$. Then we obtain a CM cusp form of level $N = |D|N(\mathfrak{f})$ and Nebentypus $\chi = \omega_{F/\mathbb{Q}}\delta$ given by

$$f = \sum_{\mathfrak{a}} \theta(\mathfrak{a}) q^{N(\mathfrak{a})} \in S_k(\Gamma_0(N), \chi)$$

where the sum is taken over integral ideals $\mathfrak{a}$ of $\mathcal{O}_F$ coprime to $\mathfrak{f}$.

Example 4.3.22. Let $K_{\text{fin}} = G(\widehat{\mathbb{Z}})$ so that $\mathcal{S}_{K_{\text{fin}}} = \mathcal{A}_2$ is the moduli stack of principally polarized Abelian surfaces. Let $V_\lambda \in \operatorname{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2$.

The compactly supported Klingen Eisenstein cohomology is concentrated in degrees 2 and 3 and given by

$$H^3_{\text{Eis},[P_2]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \mathbb{S}_{\Gamma(1)}[\lambda_1 + 3]\mathbb{L}^{\lambda_2+1}$$
$$H^4_{\text{Eis},[P_2]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \mathbb{S}_{\Gamma(1)}[\lambda_2 + 2]\mathbb{L}^{\lambda_1+2}$$

Indeed there are no CM cusp forms for $\Gamma(1)$: the automorphic representations attached to cusp forms for $\Gamma(1)$ are everywhere unramified, but the automorphic induction of an algebraic Hecke character for an imaginary quadratic extension $F/\mathbb{Q}$ cannot be everywhere unramified, since $F/\mathbb{Q}$ cannot be everywhere unramified.

4.3.4 Borel Eisenstein Cohomology

In this section we compute Borel Eisenstein cohomology as a $\mathcal{H}_{K_{\text{fin}}} \times \mathrm{Gal}$-module. First, we recall some facts about the poles of Borel Eisenstein series, and the evaluation points and infinitesimal characters which will enter into the description of the Franke filtration for the Borel parabolic subgroup.

We begin by restating the results of the previous section on the locations of poles of Siegel Eisenstein series:

Proposition 4.3.23. The space $\mathcal{A}_{\lambda,[P_0],\varphi}(G)$ contains no irreducible constituent of $\Pi_{\text{disc}}(G(\mathbb{A}_\mathbb{Q}))$ unless $\lambda = 0$ and $1 \otimes 1 \otimes \chi \in \varphi_{P_0}$, in which case the only irreducible constituent of $\Pi_{\text{disc}}(G(\mathbb{A}_\mathbb{Q}))$ belonging to $\mathcal{A}_{\lambda,[P_0],\varphi}(G)$ is the 1-dimensional representation $\chi \circ \operatorname{sim}$.

We now record the following result on infinitesimal characters coming from the action of the Weyl group:

Proposition 4.3.24. (Compare to [48, Lemma 3.1]) Let $V_\lambda \in \operatorname{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $\mathcal{J}_\lambda$ be the ideal of finite codimension in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ annihilating $V_\lambda^\vee$. Let $\varphi = \{\varphi_P\}_{P \in [P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi_2 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then the infinitesimal character $\xi \in \mathfrak{a}_{P'}^{\vee\perp}$ and the corresponding $s_0 \in \mathfrak{a}_{P'}^\vee$ such that $s_0 + \xi$ is annihilated by $\mathcal{J}_\lambda$ are given by:

- For $P' = P_0$ we have s_0 in the Weyl orbit of $\lambda + \rho_{P_0}$ and $\xi = 0$;

- For $P' = P_1$ we have either $s_0 = \pm(\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$ and $\xi = (\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2})$, or $s_0 = \pm(\frac{\lambda_1-\lambda_2+1}{2}, \frac{\lambda_1-\lambda_2+1}{2})$ and $\xi = (\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$;
- For $P' = P_2$ we have either $s_0 = \pm(\lambda_1 + 2, 0)$ and $\xi = (0, \lambda_2 + 1)$, or $s_0 = \pm(\lambda_2 + 1, 0)$ and $\xi = (0, \lambda_1 + 2)$;
- For $P' = G$ we have $s_0 = 0$ and ξ in the Weyl orbit of ρ_{P_0}.

The Franke-Schwermer Filtration We now describe the Franke filtration for the Borel parabolic subgroup. As expected, the bottom piece is given by the Langlands quotient of normalized Borel parabolic induction, which is a 1-dimensional representation, the middle pieces are given by normalized Siegel and Klingen parabolic induction, and the top piece is given by normalized Borel parabolic induction.

Proposition 4.3.25. (Compare to [48, Theorem 3.3]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P\in[P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi_2 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. The Franke-Schwermer filtration on $\mathcal{A}_{\lambda,[P_0],\varphi}(G)$ is given by

$$\mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \subseteq \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \subseteq \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) = \mathcal{A}_{\lambda,[P_0],\varphi}(G)$$

where $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial precisely if $\lambda_1 = \lambda_2 = 0$ and $\chi_1 = \chi_2 = 1$, in which case we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \simeq \chi \circ \mathrm{sim}$$

where $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial precisely if $\lambda_1 = \lambda_2$ and $\chi_1 = \chi_2$, or $\lambda_2 = 0$ and $\chi_2 = 1$, in which case we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)/\mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\chi_1 \circ \det \otimes \chi, \lambda_1 + \frac{3}{2}) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \\ \qquad \text{if } \lambda_1 = \lambda_2 \text{ and } \chi_1 = \chi_2 \text{ but } \lambda_1 \neq 0 \text{ or } \chi_1 \neq 1 \\ \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\chi_1 \otimes \chi \circ \det, \lambda_1 + 2) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \\ \qquad \text{if } \lambda_2 = 0 \text{ and } \chi_2 = 1 \text{ but } \lambda_1 \neq 0 \text{ or } \chi_1 \neq 1 \\ \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(1 \circ \det \otimes \chi, \frac{3}{2}) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \\ \oplus \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(1 \otimes \chi \circ \det, 2) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \\ \qquad \text{if } \lambda_1 = \lambda_2 = 0 \text{ and } \chi_1 = \chi_2 = 1 \end{cases}$$

and where $\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)$ is nontrivial and we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)/\mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \simeq \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_0(\mathbb{A}_\mathbb{Q})}(\chi_1 \otimes \chi_2 \otimes \chi, \lambda + \rho_{P_0}) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_0})^\vee_\mathbb{C})$$

Proof. Let $\mathcal{M}_{\lambda,[P_0],\varphi}(G)$ be the set of tuples (P', π', ν, s_0) where $P' \in [P_0]$ is a standard parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P' = M'N'$ containing an element of the associate class $[P_0]$, where π' is a discrete spectrum automorphic representation of $M'(\mathbb{A}_\mathbb{Q})$ with cuspidal support π obtained as the iterated residue at $\nu \in (\mathfrak{a}^{P'}_{P_0})^\vee_\mathbb{C}$ of the Eisenstein series attached to $\pi \in \varphi_{P_0}$, and where $s_0 \in \mathfrak{a}^\vee_{P',\mathbb{C}}$ is a point with $\Re(s_0) \in \overline{(\mathfrak{a}^G_{P'})^{\vee+}}$ such that $s_0 + \nu$ is annihilated by $\mathcal{J}_\lambda$. For $m \in \mathbb{Z}$ let $\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)$ be the subset of tuples such that $T(s_0) = m$, where $T : \overline{\mathfrak{a}^{\vee+}_{P'}} \to \mathbb{Z}$ is fixed at the end of the proof. Then by Franke we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$\mathcal{A}^m_{\lambda,[P_0],\varphi}(G)/\mathcal{A}^{m+1}_{\lambda,[P_0],\varphi}(G) \simeq \bigoplus_{(P',\pi',\nu,s_0) \in \mathcal{M}^m_{\lambda,[P_0],\varphi}(G)} \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P'(\mathbb{A}_\mathbb{Q})}(\pi', s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P'})^\vee_\mathbb{C})$$

Now we have the following:

- For $P' = P_0$ we have $\pi = \chi_1 \otimes \chi_2 \otimes \chi$ hence $\nu = (0,0) \in (\mathfrak{a}_{P_0}^{P_0})^\vee_{\mathbb{C}} = 0$. By 4.3.24 such ν can only be obtained for $s_0 = (\lambda_1 + 2, \lambda_2 + 1)$. It follows that

$$\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)_{P_0} = \begin{cases} (P_0, \chi_1 \otimes \chi_2 \otimes \chi, (0,0), (\lambda_1 + 2, \lambda_2 + 1)) \\ \qquad \text{if } T(\lambda_1 + 2, \lambda_2 + 1) = m \\ 0 \qquad \text{otherwise} \end{cases}$$

- For $P' = P_1$ since π' is a residual representation of $M_1(\mathbb{A}_{\mathbb{Q}}) \simeq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$ we have $\pi \simeq \chi_1 \circ \det \otimes \chi$ hence $\chi_1 = \chi_2$ and $\xi = (\frac{1}{2}, -\frac{1}{2}) \in (\mathfrak{a}_{P_0}^{P_1})^\vee_{\mathbb{C}}$. By 4.3.24 such ξ can only be obtained for $\lambda_1 = \lambda_2$ and $s_0 = (\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2})$. It follows that

$$\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)_{P_1} = \begin{cases} (P_1, \chi_1 \circ \det \otimes \chi, (\frac{1}{2}, -\frac{1}{2}), (\lambda_1 + \frac{3}{2}, \lambda_1 + \frac{3}{2})) \\ \qquad \text{if } T(\lambda_1 + \frac{3}{2}, \lambda_1 + \frac{3}{2}) = m \text{ and } \lambda_1 = \lambda_2 \text{ and } \chi_1 = \chi_2 \\ 0 \qquad \text{otherwise} \end{cases}$$

- For $P' = P_2$ since π' is a residual representation of $M_2(\mathbb{A}_{\mathbb{Q}}) \simeq \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ we have $\pi' \simeq \chi_1 \otimes \chi \circ \det$ hence $\chi_2 = 1$ and $\xi = (0,1) \in (\mathfrak{a}_{P_0}^{P_2})^\vee_{\mathbb{C}}$. By 4.3.24 such ξ can only be obtained for $\lambda_2 = 0$ and $s_0 = (\lambda_1 + 2, 0)$. It follows that

$$\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)_{P_2} = \begin{cases} (P_2, \chi_1 \otimes \chi \circ \det, (0,1), (\lambda_1 + 2, 0)) \\ \qquad \text{if } T(\lambda_1 + 2, 0) = m \text{ and } \lambda_2 = 0 \text{ and } \chi_2 = 1 \\ 0 \qquad \text{otherwise} \end{cases}$$

- For $P' = G$ since π' is a residual representation of the trivial group, by 4.3.23 we have $\pi' = \chi \circ \mathrm{sim}$ hence $\chi_1 = \chi_2 = 1$ and $\xi = (2,1)$. By 4.3.24 such ξ can only be obtained for $\lambda_1 = \lambda_2 = 0$ and $s_0 = (0,0)$. It follows that

$$\mathcal{M}^m_{\lambda,[P_0],\varphi}(G)_G = \begin{cases} (G, \chi \circ \mathrm{sim}, (2,1), (0,0)) \\ \qquad \text{if } T(0,0) = m \text{ and } \lambda_1 = \lambda_2 = 0 \text{ and } \chi_1 = \chi_2 = 1 \\ 0 \qquad \text{otherwise} \end{cases}$$

Now the result follows by taking the filtration defined by $T(0,0) = 2, T(\frac{\lambda_1+\lambda_2+3}{2}, \frac{\lambda_1+\lambda_2+3}{2}) = T(\lambda_1+2, 0) = 1$, and $T(\lambda_1 + 2, \lambda_2 + 1) = 0$. □

Cohomology of Franke-Schwermer Filtration Recall that the Levi quotient $M_0(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$ admits a decomposition $M_0(\mathbb{R}) = M_0^{\mathrm{ss}}(\mathbb{R}) \times A_{P_0}(\mathbb{R})^\circ$ where $M_0^{\mathrm{ss}}(\mathbb{R}) = \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\}$ is semisimple with Lie algebra $\mathfrak{m}_0^{\mathrm{ss}} = 0$ and $A_{P_0}(\mathbb{R})^\circ = \mathbb{R}^3_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_0} with Lie algebra $\mathfrak{a}_{P_0} = \mathbb{R}^3$. Recalling that $K'_\infty = \mathbb{R}_{>0}\mathrm{U}(2)$, for $K'^{M_0}_\infty$ the image of $K'_\infty \cap P_0(\mathbb{R})$ under the canonical projection $P_0(\mathbb{R}) \to M_0(\mathbb{R})$ we have $K'^{M_0}_\infty = (\{\pm 1\} \times \{\pm 1\})\mathbb{R}_{>0}$, and for $K'^{M_0^{\mathrm{ss}}}_\infty$ the image of $K'_\infty \cap P_0(\mathbb{R})$ under the canonical projection $P_0(\mathbb{R}) \to M_0^{\mathrm{ss}}(\mathbb{R})$ we have $K'^{M_0^{\mathrm{ss}}}_\infty = \{\pm 1\} \times \{\pm 1\}$.

Proposition 4.3.26. (Compare to [48, Lemma 4.1]) For $\epsilon_1, \epsilon_2, \epsilon \in \{0,1\}$ we have

$$H^q(\mathfrak{m}_0^{\mathrm{ss}}, K'^{M_0^{\mathrm{ss}}}_\infty; \pi_\infty \otimes (\mathrm{sign}^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon_2} \otimes \mathrm{sign}^{\epsilon})) \simeq \begin{cases} \mathbb{C} & q = 0, \pi_\infty \simeq \mathrm{sign}^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon_2} \otimes \mathrm{sign}^{\epsilon'} \\ 0 & \text{otherwise} \end{cases}$$

Proof. For $G = \mathrm{Sp}_4$ recall that the Levi quotient $M_0(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$ admits a decomposition $M_0(\mathbb{R}) = M_0^{\mathrm{ss}}(\mathbb{R}) \times A_{P_0}(\mathbb{R})^\circ$ where $M_0^{\mathrm{ss}}(\mathbb{R}) = \{\pm 1\} \times \{\pm 1\}$ is semisimple with Lie algebra $\mathfrak{m}_0^{\mathrm{ss}} = 0$ and $A_{P_0}(\mathbb{R})^\circ = \mathbb{R}^2_{>0}$ is the connected component of the maximal central $\mathbb{Q}$-split torus A_{P_0} with Lie algebra $\mathfrak{a}_{P_0} = \mathbb{R}^2$. Recalling that $K_\infty = \mathrm{U}(2)$, for $K^{M_0}_\infty$ the image of $K_\infty \cap P_0(\mathbb{R})$ under the canonical projection $P_0(\mathbb{R}) \to M_0(\mathbb{R})$ we have $K^{M_0}_\infty = \{\pm 1\} \times \{\pm 1\}$, and for $K^{M_0^{\mathrm{ss}}}_\infty$ the image of $K_\infty \cap P_0(\mathbb{R})$ under the canonical projection $P_0(\mathbb{R}) \to M_0^{\mathrm{ss}}(\mathbb{R})$ we have $K^{M_0^{\mathrm{ss}}}_\infty = \{\pm 1\} \times \{\pm 1\}$. By [? , Lemma 4.1], for $\epsilon_1, \epsilon_2 \in \{0,1\}$ we have

$$H^q(\mathfrak{m}_0^{\mathrm{ss}}, K^{M_0^{\mathrm{ss}}}_\infty; \pi_\infty \otimes (\mathrm{sign}^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon_2})) \simeq \begin{cases} \mathbb{C} & q = 0, \pi_\infty \simeq \mathrm{sign}^{\epsilon_1} \otimes \mathrm{sign}^{\epsilon_2} \\ 0 & \text{otherwise} \end{cases}$$

The result follows from this, noting that the $(\mathfrak{m}_0^{\mathrm{ss}}, K_\infty'^{M_0^{\mathrm{ss}}})$-cohomology is independent of the character sign^ϵ on the third factor $\{\pm 1\}$ of $M_0^{\mathrm{ss}}(\mathbb{R})$, as the factor $\mathbb{R}_{>0}$ of K'_∞ intersects this factor only at the identity. □

Now there are three pieces of the Franke-Schwermer filtration whose $(\mathfrak{g}, K'_\infty)$-cohomology we need to compute: we need to compute $H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)$, we need to compute $H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)$ in the case where $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial, and we need to compute $H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$ in the case where $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial.

Proposition 4.3.27. (Compare to [48, Proposition 4.4, Proposition 4.5, Proposition 4.6]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P\in[P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi_2 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. Then we have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) = \begin{cases} \mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) & q = 4, \chi_{1,\infty} \otimes \chi_{2,\infty} = \mathrm{sign}^{\lambda_1} \otimes \mathrm{sign}^{\lambda_2} \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial but $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is trivial we have the following cases:

- ($\lambda_1 = \lambda_2$ and $\chi_1 = \chi_2$ but $\lambda_2 > 0$ or $\chi_2 \neq 1$) We have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) = \begin{cases} \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{\lambda_1+\lambda_2+3}{2}) & q = 3, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1+1} \\ \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{\lambda_1+\lambda_2+3}{2}) & q = 5, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ 0 & \text{otherwise} \end{cases}$$

- ($\lambda_2 = 0$ and $\chi_2 = 1$ but $\lambda_1 > 0$ or $\chi_1 \neq 1$) We have an isomorphism of $G(\mathbb{A}_\mathbb{Q}^\infty)$-modules

$$H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) = \begin{cases} \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1 + 2) & q = 3, 5, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ 0 & \text{otherwise} \end{cases}$$

- ($\lambda_1 = \lambda_2 = 0$ and $\chi_1 = \chi_2 = 1$) We have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) = \begin{cases} \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, 2) & q = 3 \\ \mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{3}{2}) & \\ \oplus \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, 2) & q = 5 \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = \begin{cases} \chi_{\mathrm{fin}} \circ \mathrm{sim} & q = 0, 2, 4, 6 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the first claim we have

$$\begin{aligned} & H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \\ & \simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}})}^{G(\mathbb{A}_{\mathbb{Q}})}(\pi, s_0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)_{\mathbb{C}}^\vee) \otimes V_\lambda) \\ & \simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)_{\mathbb{C}}^\vee) \otimes V_\lambda) \otimes \mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, s_0) \end{aligned}$$

By Borel-Wallach [20, III Theorem 3.3] there exists a unique $w \in W$ such that $\pi_\infty \otimes \mathbb{C}_{s_0+\rho_{P_0}}$ has nonzero $(\mathfrak{m}_0, K'^{M_0}_\infty)$-cohomology with respect to $\mathrm{Sym}((\mathfrak{a}_{P_0}^G)_{\mathbb{C}}^\vee) \otimes V_{w\cdot\lambda}$: we have $w = s_{1212}$ in the case $s_0 = \lambda + \rho_{P_0}$ and $\xi = 0$. Now recalling that $M_0(\mathbb{R}) = M_0^{\mathrm{ss}}(\mathbb{R}) \times A_{P_0}(\mathbb{R})^\circ$ and $\mathfrak{m}_0 = \mathfrak{m}_0^{\mathrm{ss}} \oplus \mathfrak{a}_{P_0}$, by 4.3.26 we have

$$\begin{aligned} & H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty, \lambda + \rho_{P_0}) \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)_{\mathbb{C}}^\vee) \otimes V_\lambda) \\ & \simeq H^{q-\ell(w)}(\mathfrak{m}_0, K'^{M_0}_\infty; \pi_\infty \otimes \mathrm{Sym}((\mathfrak{a}_{P_0}^G)_{\mathbb{C}}^\vee) \otimes \mathbb{C}_{\lambda+2\rho_{P_0}} \otimes V_{w\cdot\lambda}) \\ & \simeq H^{q-\ell(w)}(\mathfrak{m}_0^{\mathrm{ss}}, K'^{M_0^{\mathrm{ss}}}_\infty; \pi_\infty \otimes V_{w\cdot\lambda}) \\ & \simeq \begin{cases} \mathbb{C} & q = 4, \chi_{1,\infty} \otimes \chi_{2,\infty} = \mathrm{sign}^{\lambda_1} \otimes \mathrm{sign}^{\lambda_2} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For the second claim suppose that $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial but $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is trivial. Then we have the following cases:

- ($\lambda_1 = \lambda_2$ and $\chi_1 = \chi_2$ but $\lambda_2 > 0$ or $\chi_2 \neq 1$) We have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_1(\mathbb{A}_\mathbb{Q})}(\chi_1 \circ \det \otimes \chi, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\chi_{1,\infty} \circ \det \otimes \chi_\infty, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_\lambda) \otimes \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, s_0)
\end{aligned}$$

  By Borel-Wallach [20, III Theorem 3.3] there exists a unique $w \in W^{P_1}$ such that $\pi_\infty \otimes \mathbb{C}_{s_0+\rho_{P_1}}$ has nonzero $(\mathfrak{m}_1, K'^{M_1}_\infty)$-cohomology with respect to $\mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_{w\cdot\lambda}$: we have $w = s_{212}$ in the case $s_0 = \frac{\lambda_1+\lambda_2+3}{2}$. Now recalling that $M_1(\mathbb{R}) = M_1^{\mathrm{ss}}(\mathbb{R}) \times A_{P_1}(\mathbb{R})^\circ$ and $\mathfrak{m}_1 = \mathfrak{m}_1^{\mathrm{ss}} \oplus \mathfrak{a}_{P_1}$, by 4.3.8 we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_1(\mathbb{R})}(\chi_{1,\infty} \circ \det \otimes \chi_\infty, \tfrac{\lambda_1+\lambda_2+3}{2}) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes V_\lambda) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_1, K'^{M_1}_\infty; (\chi_{1,\infty} \circ \det \otimes \chi_\infty) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_1})^\vee_\mathbb{C}) \otimes \mathbb{C}_{\frac{\lambda_1+\lambda_2+3}{2}+\rho_{P_1}} \otimes V_{w\cdot\lambda}) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_1^{\mathrm{ss}}, K'^{M_1^{\mathrm{ss}}}_\infty; (\chi_{1,\infty} \circ \det \otimes \chi_\infty)|_{M_1^{\mathrm{ss}}(\mathbb{R})} \otimes V_{w\cdot\lambda}) \\
&\simeq \begin{cases} \mathbb{C} & q = 3, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1+1} \\ \mathbb{C} & q = 5, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

- ($\lambda_2 = 0$ and $\chi_2 = 1$ but $\lambda_1 > 0$ or $\chi_1 \neq 1$) We have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{A}_\mathbb{Q})}_{P_2(\mathbb{A}_\mathbb{Q})}(\chi_1 \otimes \chi \circ \det, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \mathcal{I}^{G(\mathbb{R})}_{P_2(\mathbb{R})}(\chi_{1,\infty} \otimes \chi_\infty \circ \det, s_0) \otimes \mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_\lambda) \otimes \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, s_0)
\end{aligned}$$

  By Borel-Wallach [20, III Theorem 3.3] there exists a unique $w \in W^{P_2}$ such that $\pi_\infty \otimes \mathbb{C}_{s_0+\rho_{P_2}}$ has nonzero $(\mathfrak{m}_2, K'^{M_2}_\infty)$-cohomology with respect to $\mathrm{Sym}((\mathfrak{a}^G_{P_2})^\vee_\mathbb{C}) \otimes V_{w\cdot\lambda}$: we have $w = s_{121}$ in the case $s_0 = \lambda_1 + 2$.

Now recalling that $M_2(\mathbb{R}) = M_2^{\mathrm{ss}}(\mathbb{R}) \times A_{P_2}(\mathbb{R})^\circ$ and $\mathfrak{m}_2 = \mathfrak{m}_2^{\mathrm{ss}} \oplus \mathfrak{a}_{P_2}$, by 4.3.17 we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{I}_{P_2(\mathbb{R})}^{G(\mathbb{R})}(\chi_{1,\infty} \otimes \chi_\infty \circ \det, \lambda_1 + 2) \otimes \mathrm{Sym}((\mathfrak{a}_{P_2}^G)_{\mathbb{C}}^\vee) \otimes V_\lambda) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_2, K'^{M_2}_\infty; (\chi_{1,\infty} \otimes \chi_\infty \circ \det) \otimes \mathrm{Sym}((\mathfrak{a}_{P_2}^G)_{\mathbb{C}}^\vee) \otimes \mathbb{C}_{\lambda_1+2+\rho_{P_2}} \otimes V_{w\cdot\lambda}) \\
&\simeq H^{q-\ell(w)}(\mathfrak{m}_2^{\mathrm{ss}}, K'^{M_2^{\mathrm{ss}}}_\infty; (\chi_{1,\infty} \otimes \chi_\infty \circ \det)|_{M_2^{\mathrm{ss}}(\mathbb{R})} \otimes V_{w\cdot\lambda}) \\
&\simeq \begin{cases} \mathbb{C} & q = 3, 5, \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

- ($\lambda_1 = \lambda_2 = 0$ and $\chi_1 = \chi_2 = 1$) In this case we have the direct sum of the contributions from the previous two cases with $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$.

For the third claim suppose that $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial, so that we are in the case $\lambda_1 = \lambda_2 = 0, \xi = (2,1)$, and there exists a section ϕ of the normalized parabolic induction $\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q})}^{G(\mathbb{A}_\mathbb{Q})}(\pi, s)$ such that the automorphic Eisenstein series $\mathrm{Eis}_{P_0}^G(\phi_s)$ has pole at $s = s_0 = (0,0)$. Then we have

$$\begin{aligned}
&H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \chi \circ \mathrm{sim}) \\
&\simeq H^q(\mathfrak{g}, K'_\infty; \chi_\infty \circ \mathrm{sim}) \otimes \chi_{\mathrm{fin}} \circ \mathrm{sim}
\end{aligned}$$

Now by Borel-Wallach the $(\mathfrak{g}, K'_\infty)$-cohomology of the 1-dimensional representation $\chi_\infty \circ \mathrm{sim}$ is given

$$H^q(\mathfrak{g}, K'_\infty; \chi_\infty \circ \mathrm{sim}) = \begin{cases} \mathbb{C} & q = 0, 2, 4, 6 \\ 0 & \text{otherwise} \end{cases}$$

The result follows. $\square$

Borel Eisenstein Cohomology Having computed the $(\mathfrak{g}, K'_\infty)$-cohomology of the pieces of the Franke-Schwermer filtration, we are now in the position to compute Eisenstein cohomology. Up to indeterminacies regarding the behavior of certain connecting morphisms in the case where $\mathcal{A}^1_{\lambda,[P_1],\varphi}(G)$ or $\mathcal{A}^2_{\lambda,[P_1],\varphi}(G)$ is nontrivial, we have the following result:

Theorem 4.3.28. (Compare to [48, Theorem 5.4]) Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$. Let $\varphi = \{\varphi_P\}_{P\in[P_0]} \in \Phi_{\lambda,[P_0]}(G)$ be the associate class of a unitary cuspidal automorphic representation $\pi = \chi_1 \otimes \chi_2 \otimes \chi \in \varphi_{P_0}$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$.

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is trivial ($\lambda_1 > \lambda_2 > 0$) and if $\chi_{1,\infty} \otimes \chi_{2,\infty} = \mathrm{sign}^{\lambda_1} \otimes \mathrm{sign}^{\lambda_2}$ then we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial but $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is trivial we have the following cases:

- ($\lambda_1 = \lambda_2$ and $\chi_1 = \chi_2$ but $\lambda_2 > 0$ or $\chi_2 \neq 1$) If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$ then (with assumption 4.3.29 on connecting morphisms) we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{K}^4_1(\chi_1, \chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

  where $\mathcal{K}^4_1(\chi_1, \chi)$ is the $G(\mathbb{A}^\infty_\mathbb{Q})$-module

$$\mathcal{K}^4_1(\chi_1, \chi) \simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) \to \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2})\Big)$$

  If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1+1}$ then we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2}) & q = 3 \\ 0 & \text{otherwise} \end{cases}$$

- ($\lambda_2 = 0$ and $\chi_2 = 1$ but $\lambda_1 > 0$ or $\chi_1 \neq 1$) If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$ then (with assumption 4.3.30 on connecting morphisms) we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1 + 2) & q = 3 \\ \mathcal{K}_2^4(\chi_1, \chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}_2^4(\chi_1, \chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\mathcal{K}_2^4(\chi_1, \chi) \simeq \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) \to \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1 + 2)\Big)$$

If $\mathcal{A}_{\lambda,[P_0],\varphi}^2(G)$ is nontrivial ($\lambda_1 = \lambda_2 = 0$ and $\chi_1 = \chi_2 = 1$) then (with assumptions 4.3.31, 4.3.32, 4.3.33 on connecting morphisms) we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G)) \simeq \begin{cases} \chi_{\mathrm{fin}} \circ \mathrm{sim} & q = 0, 2 \\ \mathcal{K}_0^3(\chi) & q = 3 \\ \mathcal{K}_0^4(\chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}_0^3(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\mathcal{K}_0^3(\chi) \simeq \ker\Big(\mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\mathrm{fin}} \circ \det, 2) \to \chi_{\mathrm{fin}} \circ \mathrm{sim}\Big)$$

and where $\mathcal{K}_0^4(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\mathcal{K}_0^4(\chi) \simeq \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) \to \ker\Big(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\det \otimes \chi_{\mathrm{fin}}, \tfrac{3}{2}) \oplus \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\mathrm{fin}} \circ \det, 2) \to \chi_{\mathrm{fin}} \circ \mathrm{sim}\Big)\Big)$$

Otherwise, $H^*(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = 0$.

Proof. By definition $H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$, and if $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is trivial we clearly have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)$$

If $\chi_{1,\infty} \otimes \chi_{2,\infty} = \mathrm{sign}^{\lambda_1} \otimes \mathrm{sign}^{\lambda_2}$ then by 4.3.27 we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$\begin{aligned} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) &\simeq \begin{cases} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) & q = 4 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) & q = 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

If $\chi_{1,\infty} \otimes \chi_{2,\infty} \neq \mathrm{sign}^{\lambda_1} \otimes \mathrm{sign}^{\lambda_2}$ we have $H^*(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = 0$.

If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial but $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is trivial we clearly have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq H^q(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)$$

Consider the short exact sequence of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$0 \to \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \to \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \to \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \to 0$$

which gives rise to a long exact sequence of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$\begin{aligned} 0 &\to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \\ &\to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \\ &\to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \to 0 \end{aligned}$$

vanishing outside degrees $3, 4, 5$. Now we have the following cases:

- ($\lambda_1 = \lambda_2$ and $\chi_1 = \chi_2$ but $\lambda_2 > 0$ or $\chi_2 \neq 1$) If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$ then by 4.3.27 we have a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned}
0 &\to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}}_{=0} \to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \\
&\to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}}_{=0} \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0})} \\
&\to \underbrace{H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{\lambda_1+\lambda_2+3}{2})} \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^5(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \to 0
\end{aligned}$$

Now we make the following assumption on connecting morphisms:

Assumption 4.3.29. If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial but $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is trivial as above, then the connecting morphism

$$H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$$

is surjective, so the morphism $H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$ is zero.

Granting this, It follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} \mathcal{K}^4_1(\chi_1, \chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}^4_1(\chi_1, \chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\begin{aligned}
\mathcal{K}^4_1(\chi_1, \chi) &\simeq \ker\Big(H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)\Big) \\
&\simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) \to \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2})\Big)
\end{aligned}$$

If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1+1}$ then by 4.3.27 we have a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned}
0 \to \underbrace{H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{\lambda_1+\lambda_2+3}{2})} &\to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \\
\to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}}_{=0} &\to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \\
\to \underbrace{\cancel{H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}}_{=0} &\to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^5(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \to 0
\end{aligned}$$

It follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned}
H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) &\simeq \begin{cases} H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) & q = 3 \\ 0 & \text{otherwise} \end{cases} \\
&\simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{\lambda_1+\lambda_2+3}{2}) & q = 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

- ($\lambda_2 = 0$ and $\chi_2 = 1$ but $\lambda_1 > 0$ or $\chi_1 \neq 1$) If $\chi_{1,\infty} = \mathrm{sign}^{\lambda_1}$ then by 4.3.27 we have a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned}
0 \to \underbrace{H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1+2)} &\to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \\
\to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}}_{=0} &\to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda+\rho_{P_0})} \\
\to \underbrace{H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1+2)} &\to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to \underbrace{\cancel{H^5(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda)}}_{=0} \to 0
\end{aligned}$$

Now we make the following assumption on connecting morphisms:

Assumption 4.3.30. If $\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)$ is nontrivial but $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is trivial as above, then the connecting morphism

$$H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$$

is surjective, so the morphism $H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$ is zero.

Granting this, it follows that we have an isomorphism of $G(\mathbb{A}^\infty_\mathbb{Q})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) \simeq \begin{cases} H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) & q = 3 \\ \mathcal{K}^4_2(\chi_1, \chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$
$$\simeq \begin{cases} \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1 + 2) & q = 3 \\ \mathcal{K}^4_2(\chi_1, \chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}^4_2(\chi_1, \chi)$ is the $G(\mathbb{A}^\infty_\mathbb{Q})$-module

$$\begin{aligned}\mathcal{K}^4_2(\chi_1, \chi) &\simeq \ker\Big(H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \otimes V_\lambda) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)\Big) \\ &\simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \lambda + \rho_{P_0}) \to \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1 + 2)\Big)\end{aligned}$$

If $\chi_{1,\infty} \neq \mathrm{sign}^{\lambda_1}$ we have $H^*(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda) = 0$.

If $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial ($\lambda_1 = \lambda_2 = 0$ and $\chi_1 = \chi_2 = 1$) consider the short exact sequence of $G(\mathbb{A}^\infty_\mathbb{Q}) \times (\mathfrak{g}, K_\infty)$-modules

$$0 \to \mathcal{A}^2_{\lambda,[P_0],\varphi}(G) \to \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \to \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)} \to 0$$

which by 4.3.27 gives rise to a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$\begin{aligned}
0 &\to \underbrace{H^0(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \mathrm{sim}} \to H^0(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^0(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^1(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^1(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^1(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \mathrm{sim}} \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^2(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{H^3(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}_{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, 2)} \\
&\to \underbrace{H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \mathrm{sim}} \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{H^5(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}_{\substack{\simeq \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \frac{3}{2}) \\ \oplus \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, 2)}} \\
&\to \underbrace{H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \mathrm{sim}} \to H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^6(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)})}}_{=0} \to 0
\end{aligned}$$

Now we make the following assumptions on connecting morphisms:

Assumption 4.3.31. If $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial as above, then the connecting morphism

$$H^3(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)}) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))$$

is surjective, so the morphism $H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G)) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))$ is zero.

Assumption 4.3.32. If $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial as above, then the connecting morphism

$$H^5(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)}) \to H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G))$$

is surjective, so the morphism $H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G)) \to H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))$ is zero.

Granting these, it follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \simeq \begin{cases} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^2_{\lambda,[P_0],\varphi}(G)) & q = 0, 2 \\ \mathcal{K}^3(\chi) & q = 3 \\ \mathcal{K}^5(\chi) & q = 5 \\ 0 & \text{otherwise} \end{cases}$$
$$\simeq \begin{cases} \chi_{\text{fin}} \circ \text{sim} & q = 0, 2 \\ \mathcal{K}^3(\chi) & q = 3 \\ \mathcal{K}^5(\chi) & q = 5 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}^3(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\begin{aligned} \mathcal{K}^3(\chi) &\simeq \ker\Big(H^3(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)}) \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))\Big) \\ &\simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\text{fin}} \circ \det, 2) \to \chi_{\text{fin}} \circ \text{sim}\Big) \end{aligned}$$

and where $\mathcal{K}^5(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\begin{aligned} \mathcal{K}^5(\chi) &\simeq \ker\Big(H^5(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)}) \to H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))\Big) \\ &\simeq \ker\Big(\mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\det \otimes \chi_{\text{fin}}, \tfrac{3}{2}) \oplus \mathcal{I}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\text{fin}} \circ \det, 2) \to \chi_{\text{fin}} \circ \text{sim}\Big) \end{aligned}$$

Now consider the short exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times (\mathfrak{g}, K_\infty)$-modules

$$0 \to \mathcal{A}^1_{\lambda,[P_0],\varphi}(G) \to \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \to \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)} \to 0$$

which by 4.3.27 gives rise to a long exact sequence of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$
\begin{aligned}
0 &\to \underbrace{H^0(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \mathrm{sim}} \to H^0(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^0(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^1(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^1(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^1(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}_{\simeq \chi_{\mathrm{fin}} \circ \mathrm{sim}} \to H^2(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^2(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}_{\simeq \mathcal{K}^3(\chi)} \to H^3(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^3(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^4(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{H^4(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}_{\simeq \mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \lambda+\rho_{P_0})} \\
&\to \underbrace{H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}_{\simeq \mathcal{K}^5(\chi)} \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^5(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \\
&\to \underbrace{\cancel{H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))}}_{=0} \to H^6(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G)) \to \underbrace{\cancel{H^6(\mathfrak{g}, K'_\infty; \tfrac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)})}}_{=0} \to 0
\end{aligned}
$$

Now we make the following assumption on connecting morphisms:

Assumption 4.3.33. If $\mathcal{A}^2_{\lambda,[P_0],\varphi}(G)$ is nontrivial as above, then the connecting morphism

$$H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))$$

is surjective, so the morphism $H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G))$ is zero.

Granting this, it follows that we have an isomorphism of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules

$$H^q(\mathfrak{g}, K'_\infty; \mathcal{A}_{\lambda,[P_0],\varphi}(G)) \simeq \begin{cases} H^q(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G)) & q = 0,2,3 \\ \mathcal{K}_0^4(\chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\simeq \begin{cases} \chi_{\text{fin}} \circ \text{sim} & q = 0,2 \\ \mathcal{K}_0^3(\chi) & q = 3 \\ \mathcal{K}_0^4(\chi) & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{K}_0^3(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\mathcal{K}_0^3(\chi) \simeq \mathcal{K}^3(\chi) \simeq \ker\Big(\mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\text{fin}} \circ \det, 2) \to \chi_{\text{fin}} \circ \text{sim}\Big)$$

and where $\mathcal{K}_0^4(\chi)$ is the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module

$$\begin{aligned} \mathcal{K}_0^4(\chi) &\simeq \ker\Big(H^4(\mathfrak{g}, K'_\infty; \frac{\mathcal{A}^0_{\lambda,[P_0],\varphi}(G)}{\mathcal{A}^1_{\lambda,[P_0],\varphi}(G)}) \to H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^1_{\lambda,[P_0],\varphi}(G))\Big) \\ &\simeq \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}}, \rho_{P_0}) \to \mathcal{K}^5(\chi)\Big) \\ &\simeq \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\text{fin}}, \rho_{P_0}) \to \ker\Big(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\det \otimes \chi_{\text{fin}}, \tfrac{3}{2}) \oplus \mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(1 \otimes \chi_{\text{fin}} \circ \det, 2) \to \chi_{\text{fin}} \circ \text{sim}\Big)\Big) \end{aligned}$$

The result follows. $\square$

Note that three of the assumptions on connecting morphisms 4.3.29, 4.3.30, 4.3.33 are rather harmless: if any of these three assumptions failed, then we would be able to conclude the nonvanishing of some $H^5(\mathfrak{g}, K'_\infty; \mathcal{A}^0_{\lambda,[P_0],\varphi}(G) \otimes V_\lambda)$, which contradicts known vanishing results for $H^5(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$. On the other hand two of the assumptions on connecting morphisms 4.3.31, 4.3.32 are especially subtle.

The $\mathcal{H}_{K_{\text{fin}}}$-modules in the above theorem will be paired with 1-dimensional Gal-modules. For an integer $n \in \mathbb{Z}$ and for $\chi = \chi_{\text{fin}} \otimes \chi_\infty$ a (finite order) character of $\text{GL}_1(\mathbb{A}_{\mathbb{Q}})$ let $\mathbb{L}_\chi^n = \rho_\chi(-n)$ be the 1-dimensional

ℓ-adic Gal-module attached to χ twisted by the n-th power of the ℓ-adic cyclotomic character, with

$$\mathrm{tr}(\mathrm{Frob}_p^j|\mathbb{L}_\chi^n) = p^{nj}c(\chi_p)^j = p^{nj}\chi(p)^j$$

Now we have the following result, which is conditional on the assumptions on connecting morphisms 4.3.31, 4.3.32 in the case $\lambda_1 = \lambda_2 = 0$:

Theorem 4.3.34. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. Then (with the assumptions 4.3.31, 4.3.32 on connecting morphisms in the case $\lambda_1 = \lambda_2 = 0$) the Borel Eisenstein cohomology $H^*_{\mathrm{Eis},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees $0, 2, 3, 4$ and given as an $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$-module by

$$H^0_{\mathrm{Eis},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_{\chi} (\chi_{\mathrm{fin}} \circ \mathrm{sim})^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^0_\chi & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^2_{\mathrm{Eis},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_{\chi} (\chi_{\mathrm{fin}} \circ \mathrm{sim})^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^1_\chi & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{Eis},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_{\chi} \mathcal{K}^3_0(\chi)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^2_\chi & \\ \oplus \bigoplus_{\substack{\chi_1, \chi \\ \chi_1 \neq 1 \\ \chi_{1,\infty} = \mathrm{sign}}} \mathcal{I}^{G(\mathbb{A}^\infty_{\mathbb{Q}})}_{P_1(\mathbb{A}^\infty_{\mathbb{Q}})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \tfrac{3}{2})^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^3_{\chi_1^2\chi} & \\ \oplus \bigoplus_{\substack{\chi_1, \chi \\ \chi_1 \neq 1 \\ \chi_{1,\infty} = 1}} \mathcal{I}^{G(\mathbb{A}^\infty_{\mathbb{Q}})}_{P_2(\mathbb{A}^\infty_{\mathbb{Q}})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, 2)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^2_{\chi_1\chi} & \lambda_1 = \lambda_2 = 0 \\ \bigoplus_{\substack{\chi_1, \chi \\ \chi_{1,\infty} = \mathrm{sign}^{\lambda_1+1}}} \mathcal{I}^{G(\mathbb{A}^\infty_{\mathbb{Q}})}_{P_1(\mathbb{A}^\infty_{\mathbb{Q}})}(\chi_{1,\mathrm{fin}} \circ \det \otimes \chi_{\mathrm{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2})^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3}_{\chi_1^2\chi} & \lambda_1 = \lambda_2 > 0 \\ \bigoplus_{\substack{\chi_1, \chi \\ \chi_{1,\infty} = \mathrm{sign}^{\lambda_1}}} \mathcal{I}^{G(\mathbb{A}^\infty_{\mathbb{Q}})}_{P_2(\mathbb{A}^\infty_{\mathbb{Q}})}(\chi_{1,\mathrm{fin}} \otimes \chi_{\mathrm{fin}} \circ \det, \lambda_1 + 2)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^{\lambda_1+2}_{\chi_1\chi} & \lambda_1 > \lambda_2 = 0 \\ 0 & \lambda_1 > \lambda_2 > 0 \end{cases}$$

$$H^4_{\mathrm{Eis},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \begin{cases} \bigoplus_{\chi} \mathcal{K}^4_0(\chi)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^3_\chi & \\ \oplus \bigoplus_{\substack{\chi_1, \chi \\ \chi_1 \neq 1 \\ \chi_{1,\infty} = 1}} \mathcal{K}^4_1(\chi_1, \chi)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^3_{\chi_1^2\chi} & \\ \oplus \bigoplus_{\substack{\chi_1, \chi \\ \chi_1 \neq 1 \\ \chi_{1,\infty} = 1}} \mathcal{K}^4_2(\chi_1, \chi)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^3_{\chi_1\chi} & \lambda_1 = \lambda_2 = 0 \\ \bigoplus_{\substack{\chi_1, \chi \\ \chi_{1,\infty} = \mathrm{sign}^{\lambda_1}}} \mathcal{K}^4_1(\chi_1, \chi)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3}_{\chi_1^2\chi} & \lambda_1 = \lambda_2 > 0 \\ \bigoplus_{\substack{\chi_1, \chi \\ \chi_{1,\infty} = \mathrm{sign}^{\lambda_1}}} \mathcal{K}^4_2(\chi_1, \chi)^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^{\lambda_1+3}_{\chi_1\chi} & \lambda_1 > \lambda_2 = 0 \\ \bigoplus_{\substack{\chi_1, \chi_2, \chi \\ \chi_{1,\infty} = \mathrm{sign}^{\lambda_1} \\ \chi_{2,\infty} = \mathrm{sign}^{\lambda_2}}} \mathcal{I}^{G(\mathbb{A}^\infty_{\mathbb{Q}})}_{P_0(\mathbb{A}^\infty_{\mathbb{Q}})}(\chi_{1,\mathrm{fin}} \otimes \chi_{2,\mathrm{fin}} \otimes \chi_{\mathrm{fin}}, \lambda + \rho_{P_0})^{K_{\mathrm{fin}}} \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3}_{\chi_1\chi_2\chi} & \lambda_1 > \lambda_2 > 0 \end{cases}$$

where $\mathcal{K}_0^3(\chi), \mathcal{K}_0^4(\chi), \mathcal{K}_1^4(\chi_1,\chi)$, and $\mathcal{K}_2^4(\chi_1,\chi)$ are given by

$$\mathcal{K}_0^3(\chi) = \ker\Big(\mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes\chi_{\mathrm{fin}}\circ\det, 2) \to \chi_{\mathrm{fin}}\circ\mathrm{sim}\Big)$$

$$\mathcal{K}_0^4(\chi) = \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes\chi_{\mathrm{fin}}, \rho_{P_0}) \to \ker\Big(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\det\otimes\chi_{\mathrm{fin}}, \tfrac{3}{2}) \oplus \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes\chi_{\mathrm{fin}}\circ\det, 2) \to \chi_{\mathrm{fin}}\circ\mathrm{sim}\Big)\Big)$$

$$\mathcal{K}_1^4(\chi_1,\chi) = \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}}\otimes\chi_{1,\mathrm{fin}}\otimes\chi_{\mathrm{fin}}, \lambda+\rho_{P_0}) \to \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}}\circ\det\otimes\chi_{\mathrm{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2})\Big)$$

$$\mathcal{K}_2^4(\chi_1,\chi) = \ker\Big(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}}\otimes\chi_{1,\mathrm{fin}}\otimes\chi_{\mathrm{fin}}, \lambda+\rho_{P_0}) \to \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\chi_{1,\mathrm{fin}}\otimes\chi_{\mathrm{fin}}\circ\det, \lambda_1+2)\Big)$$

Proof. The result follows by taking the direct sum over associate classes of unitary cuspidal automorphic representations $\pi = \chi\otimes\mu$ of $M_0(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_1(\mathbb{A}_\mathbb{Q})\times\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})\times\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ of the contributions to 4.3.28. The Galois action can be obtained from the Borel parabolic terms in the GSp_4 trace formula.

By Pink the Tate twists are given as follows. For $\lambda = n_1\omega + n_2\omega_2$ and $d(\lambda) = \frac{n_1}{2} + n_2$ the Galois action must be twisted by $\mathbb{L}^{d(\lambda)-\langle\mu, w\cdot\lambda\rangle}$ where $\mu : \mathrm{GL}_1 \to \mathrm{GSp}_4$ is the cocharacter given by $t \mapsto \mathrm{diag}(t,t,1,1)$. Since $\omega_1(\mathrm{diag}(t,t,1,1)) = t^{1/2}$ and $\omega_2(\mathrm{diag}(t,t,1,1)) = t$ we have $\langle\mu, w\cdot\lambda\rangle = \frac{n_1^w}{2} + n_2^w$ for $w\in W$ and we obtain the following Tate twists:

w	n_1^w	n_2^w	$\langle\mu, w\cdot\lambda\rangle$	$d(\lambda)-\langle\mu, w\cdot\lambda\rangle$
1	n_1	n_2	$\frac{n_1}{2}+n_2$	0
s_1	$-n_1-2$	n_1+n_2+1	$\frac{n_1}{2}+n_2$	0
s_2	n_1+2n_2+2	$-n_2-2$	$\frac{n_1}{2}-1$	n_2+1
s_{12}	$-n_1-2n_2-4$	n_1+n_2+1	$\frac{n_1}{2}-1$	n_2+1
s_{21}	n_1+2n_2+2	$-n_1-n_2-3$	$-\frac{n_1}{2}-2$	n_1+n_2+2
s_{121}	$-n_1-2n_2-4$	n_2	$-\frac{n_1}{2}-2$	n_1+n_2+2
s_{212}	n_1	$-n_1-n_2-3$	$-\frac{n_1}{2}-n_2-3$	n_1+2n_2+3
s_{1212}	$-n_1-2$	$-n_2-2$	$-\frac{n_1}{2}-n_2-3$	n_1+2n_2+3

Recalling that $n_1 = \lambda_1-\lambda_2$ and $n_2 = \lambda_2$, this gives the Tate twists in the theorem. $\square$

Example 4.3.35. Let $K_{\mathrm{fin}} = G(\widehat{\mathbb{Z}})$ so that $\mathcal{S}_{K_{\mathrm{fin}}} = \mathcal{A}_2$ is the moduli stack of principally polarized Abelian surfaces. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1,\lambda_2;\lambda_1+\lambda_2)$ with $\lambda_1+\lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2$.

The compactly supported Borel Eisenstein cohomology is concentrated in degrees 0, 2, 3, 4 and given by

$$H^0_{\mathrm{Eis},[P_0]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^0 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^2_{\mathrm{Eis},[P_0]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^1 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{Eis},[P_0]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 = \lambda_2 > 0; \lambda_1 \text{ odd} \\ \mathbb{L}^{\lambda_1+2} & \lambda_1 > \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^4_{\mathrm{Eis},[P_0]}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 > \lambda_2 > 0; \lambda_1 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

4.4 Intersection and Cuspidal Cohomology for GSp_4

The goal of this section is to compute the intersection and inner/cuspidal cohomology of local systems on Siegel modular threefolds. To that end, we review the structure of the automorphic discrete spectrum for GSp_4 as described by Arthur's conjectures, and how the endoscopic and CAP packets can be constructed explicitly as theta lifts. We then describe the internal structure of these packets, and in each case compute the relevant $(\mathfrak{g}, K'_\infty)$-cohomology, and sketch how the Langlands Kottwitz method is used to compute the Galois action on each contribution.

4.4.1 Arthur Parameters for GSp_4

For $G = \mathrm{GSp}_4$ over $\mathbb{Q}$, Arthur's conjectures describe a decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\psi \in \Psi_{\mathrm{disc}}(G,\omega)} \mathcal{A}_\psi$$

where each $\mathcal{A}_\psi$ is a near-equivalence class of discrete spectrum automorphic representations of $G(\mathbb{A}_\mathbb{Q})$, where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and the second direct sum is taken over a set $\Psi_{\mathrm{disc}}(G,\omega)$ of equivalence classes of admissible discrete global A-parameters $\psi : L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$; these are formal unorderd isobaric sums $\psi = \boxplus_i \mu_i \boxtimes \nu_{d_i}$ where μ_i is an ω-self dual unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_{\mathbb{Q}})$ and ν_{d_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d_i, and in this case we require $\sum_i n_i d_i = 4$. This gives six possible shapes of global A-parameters:

$$\mu \boxtimes 1 \qquad (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1) \qquad (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2) \qquad \mu \boxtimes \nu_2 \qquad (\chi_1 \boxtimes \nu_2) \boxplus (\chi_2 \boxtimes \nu_2) \qquad \chi \boxtimes \nu_4$$

By [43, Remark 6.1.8] (see also [6], [**?**], [106]) the A-parameters for discrete spectrum automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ with central character ω are given as follows:

(i) (General Type) We have a set $\Psi_{\mathrm{disc,gen}}(G,\omega)$ of A-parameters of the form $\psi = \mu \boxtimes 1$ for μ a ω-self-dual unitary cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ of symplectic type, that is $\mu^\vee \otimes \omega = \mu$ and $L^S(s, \wedge^2(\mu) \otimes \omega^{-1})$ has a pole at $s = 1$; these are of general type (stable and semisimple) with $\mathcal{S}_\psi = 1$ and $\epsilon_\psi = 1$ with Satake parameters

$$c(\psi) = c(\mu) = \{\mathrm{diag}(c_1(\mu_p), c_2(\mu_p), c_3(\mu_p), c_4(\mu_p))\}_p$$

For $\omega = 1$ we have $\mathcal{L}_\psi = \mathrm{Sp}_4(\mathbb{C})$ and the morphism $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ is the projection onto the first component.

(ii) (Yoshida Type) We have a set $\Psi_{\mathrm{disc,endo}}(G,\omega)$ of A-parameters of the form $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1)$ for μ_1 and μ_2 distinct unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central characters $\omega_{\mu_1} = \omega_{\mu_2} = \omega$; these are of Yoshida type (unstable and semisimple) with $\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$ and $\epsilon_\psi = 1$ with Satake parameters

$$c(\psi) = c(\mu_1) \oplus c(\mu_2) = \{\mathrm{diag}(c_1(\mu_{1,p}), c_1(\mu_{2,p}), c_2(\mu_{2,p}), c_2(\mu_{1,p}))\}_p$$

For $\omega = 1$ we have $\mathcal{L}_\psi = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ and the morphism $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ is given

$$\psi\left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right); 1\right) = \begin{pmatrix} a_1 & & & b_1 \\ & a_2 & b_2 & \\ & c_2 & d_2 & \\ c_1 & & & d_1 \end{pmatrix}$$

(iii) (Saito-Kurokawa Type) We have a set $\Psi_{\mathrm{disc},[P_1]}(G,\omega)$ of Siegel CAP A-parameters of the form $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2)$ for μ a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\mu = \omega$ and $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a unitary Hecke character with $\chi^2 = \omega$; these are of Saito-Kurokawa type (unstable and mixed) with $\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$ and

$$\epsilon_\psi = \begin{cases} \mathrm{sign} & \varepsilon(\frac{1}{2}, \mu \otimes \chi^{-1}) = -1 \\ 1 & \varepsilon(\frac{1}{2}, \mu \otimes \chi^{-1}) = 1 \end{cases}$$

with Satake parameters

$$c(\psi) = (c(\chi) \otimes c(\nu_2)) \oplus c(\mu) = \{\mathrm{diag}(c(\chi_p)p^{\frac{1}{2}}, c_1(\mu_p), c_2(\mu_p), c(\chi_p)p^{-\frac{1}{2}})\}_p$$

For $\omega = 1$ we have $\mathcal{L}_\psi = \mathrm{SL}_2(\mathbb{C}) \times \{\pm 1\}$ and the morphism $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ is given

$$\psi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), t; 1) = \begin{pmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & t \end{pmatrix} \qquad \psi(1, 1; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \begin{pmatrix} a & & & b \\ & 1 & & \\ & & 1 & \\ c & & & d \end{pmatrix}$$

(iv) (Soudry Type) We have a set $\Psi_{\mathrm{disc},[P_2]}(G,\omega)$ of Klingen CAP A-parameters of the form $\psi = \mu \boxtimes \nu_2$ for μ an ω-self dual unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of orthogonal type, that is $\mu^\vee \otimes \omega = \mu$ and $L^S(s, \mathrm{Sym}^2(\mu) \otimes \omega^{-1})$ has a pole at $s = 1$, with central character ω_μ/ω of order 2 so that μ is the automorphic induction of a unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ of a quadratic extension $F/\mathbb{Q}$ corresponding to $\omega_\mu\omega^{-1}$ such that $\theta^c \neq \theta$ and $\theta|_{\mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times} = \omega$; these are of Soudry type (stable and mixed) with $\mathcal{S}_\psi = 1$ and $\epsilon_\psi = 1$ with Satake parameters

$$c(\psi) = c(\mu) \otimes c(\nu_2) = \{\mathrm{diag}(c_1(\mu_p)p^{\frac{1}{2}}, c_2(\mu_p)p^{\frac{1}{2}}, c_1(\mu_p)p^{-\frac{1}{2}}, c_2(\mu_p)p^{-\frac{1}{2}})\}_p$$

For $\omega = 1$ we have $\mathcal{L}_\psi = \mathrm{O}_2(\mathbb{C}) = \{g \in \mathrm{GL}_2(\mathbb{C}) | {}^\top g \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) g = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\}$ and the morphism $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ is given

$$\psi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), 1) = \begin{pmatrix} a & b & & \\ c & d & & \\ & & a & b \\ & & c & d \end{pmatrix} \qquad \psi(1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \begin{pmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{pmatrix}$$

(v) (Howe-Piatetski-Shapiro Type) We have a set $\Psi_{\mathrm{disc},[P_0]}(G,\omega)$ of Borel CAP A-parameters of the form $\psi = (\chi_1 \boxtimes \nu_2) \boxplus (\chi_2 \boxtimes \nu_2)$ for $\chi_1, \chi_2 : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ unitary Hecke characters with $\chi_1^2 = \chi_2^2 = \omega$; these are unstable L-packets of Howe-Piatetski-Shapiro type with $\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$, $\epsilon_\psi = 1$, with Satake parameters

$$\begin{aligned} c(\psi) &= (c(\chi_1) \otimes c(\nu_2)) \oplus (c(\chi_2) \otimes c(\nu_2)) \\ &= \{\mathrm{diag}(c(\chi_{1,p})p^{\frac{1}{2}}, c(\chi_{2,p})p^{\frac{1}{2}}, c(\chi_{2,p})p^{-\frac{1}{2}}, c(\chi_{1,p})p^{-\frac{1}{2}})\}_p \end{aligned}$$

For $\omega = 1$ we have $\mathcal{L}_\psi = \{\pm 1\} \times \{\pm 1\}$ and the morphism $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ is given

$$\psi(w;1) = \mathrm{diag}(\phi_1(w), \phi_2(w), \phi_2(w), \phi_1(w)) \qquad \psi(1,1; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \begin{pmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d \end{pmatrix}$$

(vi) (One-Dimensional Type) We have a set $\Psi_{\mathrm{disc,1dim}}(G,\omega)$ of A-parameters of the form $\psi = \chi \boxtimes \nu_4$ for $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a unitary Hecke character with $\chi^4 = \omega$; these are one-dimensional type with $\mathcal{S}_\psi = 1$ and $\epsilon_\psi = 1$ and Satake parameters

$$c(\psi) = c(\chi) \otimes c(\nu_4) = \{\mathrm{diag}(c(\chi_p)p^{\frac{3}{2}}, c(\chi_p)p^{\frac{1}{2}}, c(\chi_p)p^{-\frac{1}{2}}, c(\chi_p)p^{-\frac{3}{2}})\}_p$$

For $\omega = 1$ we have $\mathcal{L}_\psi = \{\pm 1\}$ and the morphism $\psi : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ identifies $\{\pm 1\}$ with the center of $\mathrm{Sp}_4(\mathbb{C})$.

Note that for μ an ω-self dual unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, either $\omega_\mu = \omega$ and $L^S(s, \wedge^2(\mu) \otimes \omega^{-1})$ has a pole at $s = 1$, or μ is the automorphic induction of a unitary Hecke character $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ of a quadratic extension $F/\mathbb{Q}$ corresponding to $\omega_\mu \omega^{-1}$ such that $\theta^c \neq \theta$ and $\theta|_{\mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times} = \omega$ and $L^S(s, \mathrm{Sym}^2(\mu) \otimes \omega^{-1})$ has a pole at $s = 1$ (see [43, Theorem 2.7.1]). The unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ appearing in the A-parameters of Yoshida type and of Saito-Kurokawa type are of this first type, while the unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ appearing in the the A-parameters of Soudry type are of this second type.

The classification of such A-parameters, along with conjectures [43, Conjecture 2.4.2] on local A-packets in terms of endoscopic transfer relations and conjectures [43, Conjecture 2.5.6] on global A-parameters and their multiplicity formula, follows by work of Arthur [6] in the case $\omega = 1$ (as well as in the case $\omega = \chi^2$ by a twisting argument, see [43, Theorem 2.6.1]), and follows by work of Gee-Taïbi [43] in general.

We quickly recall the twisting argument. For μ an ω-self dual unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ and for $\eta : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ a unitary Hecke character with $\eta^2 = \omega$, the twist $\mu \otimes \eta^{-1} := \mu \otimes (\eta \circ \det)^{-1}$ is self-dual. As the conjectures are compatible with these twists, they reduce to the case $\omega = 1$ where an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ is the same as an automorphic representation of $\mathrm{PGSp}_4(\mathbb{A}_{\mathbb{Q}}) \simeq \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})$.

Given an A-parameter $\psi \in \Psi_{\mathrm{disc}}(G, \omega)$ and a unitary Hecke character $\eta : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ we may consider the character twist $\psi \otimes \eta$ as follows:

(i) For $\psi = \mu \boxtimes 1 \in \Psi_{\mathrm{disc,gen}}(G, \omega)$ we have $\psi \otimes \eta = (\mu \otimes \eta) \boxtimes 1 \in \Psi_{\mathrm{disc,gen}}(G, \omega\eta^4)$ with central character $\omega_{\mu \otimes \eta} = \omega_\mu \eta^4 = \omega\eta^4$.

(ii) For $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1) \in \Psi_{\mathrm{disc,endo}}(G, \omega)$ we have $\psi \otimes \eta = ((\mu_1 \otimes \eta) \boxtimes 1) \boxplus ((\mu_2 \otimes \eta) \boxtimes 1) \in \Psi_{\mathrm{disc,endo}}(G, \omega\eta^2)$ with central character $\omega_{\mu_i \otimes \eta} = \omega_{\mu_i} \eta^2 = \omega\eta^2$.

(iii) For $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2) \in \Psi_{\mathrm{disc},[P_1]}(G, \omega)$ we have $\psi \otimes \eta = ((\mu \otimes \eta) \boxtimes 1) \boxplus (\chi\eta \boxtimes \nu_2) \in \Psi_{\mathrm{disc},[P_1]}(G, \omega\eta^2)$ with central character $\omega_{\mu \otimes \eta} = \omega_\mu \eta^2 = \omega\eta^2$ and $(\chi\eta)^2 = \chi^2\eta^2 = \omega\eta^2$.

(iv) For $\psi = \mu \boxtimes \nu_2 \in \Psi_{\mathrm{disc},[P_2]}(G, \omega)$ we have $\psi \otimes \eta = \mu \otimes \eta \boxtimes \nu_2 \in \Psi_{\mathrm{disc},[P_2]}(G, \omega\eta^2)$ with central character $\omega_{\mu \otimes \eta} = \omega_\mu \eta^2 = \omega\eta^2$.

(v) For $\psi = (\chi_1 \boxtimes \nu_2) \boxplus (\chi_2 \boxtimes \nu_2) \in \Psi_{\mathrm{disc},[P_0]}(G, \omega)$ we have $\psi \otimes \eta = (\chi_1\eta \boxtimes \nu_2) \boxplus (\chi_2\eta \boxtimes \nu_2) \in \Psi_{\mathrm{disc},[P_0]}(G, \omega\eta^2)$ with central character $(\chi_i\eta)^2 = \chi_i^2\eta^2 = \omega\eta^2$.

(vi) For $\psi = \chi \boxtimes \nu_4 \in \Psi_{\mathrm{disc,1dim}}(G, \omega)$ we have $\psi \otimes \eta = \chi\eta \boxtimes \nu_4 \in \Psi_{\mathrm{disc,1dim}}(G, \omega\eta^4)$ with central character $(\chi\eta)^4 = \chi^4\eta^4 = \omega\eta^4$.

In particular for $\psi \in \Psi_{\mathrm{disc},?}(G, \omega)$ with $? \in \{P_1, P_0, \mathrm{1dim}\}$ we can always twist by a suitable unitary Hecke character $\eta : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ to obtain $\psi \otimes \eta \in \Psi_{\mathrm{disc},?}(G, 1)$, in which case we have the global A-packets $\Pi(\psi) = \Pi(\psi \otimes \eta) \otimes \eta^{-1} = \{\pi \otimes (\eta \circ \mathrm{sim})^{-1} | \pi \in \Pi(\psi \otimes \eta)\}$.

Among other things, this twisting argument means we will only need to analyze the Saito-Kurokawa packets and the Howe-Piatetski-Shapiro packets in the case of trivial central character. Only the Yoshida packets and the Soudry packets will need to be analyzed without the assumption of trivial central character.

We obtain a decomposition

$$
\begin{aligned}
L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq & \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc,gen}}(G,\omega)\\(\psi=\mu\boxtimes 1)}} \bigoplus_{\pi\in\Pi(\psi)} \pi \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc,endo}}(G,\omega)\\(\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} \pi \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_1]}(G,\omega)\\(\psi=(\mu\boxtimes 1)\boxplus(\chi\boxtimes\nu_2))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=\epsilon(\frac{1}{2},\mu\otimes\chi^{-1})}} \pi \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_2]}(G,\omega)\\(\psi=\mu\boxtimes\nu_2)}} \bigoplus_{\pi\in\Pi(\psi)} \pi \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_0]}(G,\omega)\\(\psi=(\chi_1\boxtimes\nu_2)\boxplus(\chi_2\boxtimes\nu_2))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} \pi \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{1dim}}(G,\omega)\\(\psi=\chi\boxtimes\nu_4)}} \chi\circ\mathrm{sim}
\end{aligned}
$$

The group $G = \mathrm{GSp}_4$ over $\mathbb{Q}$ has a unique up to isomorphism proper elliptic endoscopic datum $(H, \mathcal{H}, s, \xi)$ where $H = \mathrm{GSO}_{2,2} = \mathrm{GL}_2 \times \mathrm{GL}_2/\mathrm{GL}_1$ where the quotient is taken with respect to the diagonal embedding $\mathrm{GL}_1 \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2$ given by $z \mapsto (\mathrm{diag}(z, z), \mathrm{diag}(z^{-1}, z^{-1}))$, where $\mathcal{H} = \widehat{H} \times W_F$ is defined by the split L-datum $(\widehat{H}, \rho_H, \eta_H)$ where $\widehat{H} = (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 = \{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) | \det(g_1) = \det(g_2)\}$ and $\rho_H = \eta_H = \mathrm{id}$, and where $\xi : \mathcal{H} \to {}^LG$ is the embedding of dual groups into the connected centralizer of $s = \mathrm{diag}(1, -1, -1, 1)$:

$$
\xi : (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 \to \mathrm{GSp}_4(\mathbb{C}) \qquad \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)\right) \mapsto \begin{pmatrix} a_1 & & & b_1 \\ & a_2 & b_2 & \\ & c_2 & d_2 & \\ c_1 & & & d_1 \end{pmatrix}
$$

We have a decomposition

$$
\begin{aligned}
L^2_{\mathrm{disc}}(H(\mathbb{Q})A_H(\mathbb{R})^\circ \setminus H(\mathbb{A}_\mathbb{Q})) \simeq & \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc,endo}}(G,\omega) \\ (\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \mu_1 \boxtimes \mu_2 \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_1]}(G,\omega) \\ (\psi=(\mu\boxtimes 1)\oplus(\chi\boxtimes\nu_2))}} \mu \boxtimes (\chi\circ\det) \\
& \oplus \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_0]}(G,\omega) \\ (\psi=(\chi_1\boxtimes\nu_2)\oplus(\chi_2\boxtimes\nu_2))}} (\chi_1\circ\det) \boxtimes (\chi_2\circ\det)
\end{aligned}
$$

corresponding to the unstable terms in the decomposition $L^2_{\mathrm{disc}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$.

Both of these decompositions will play a role in the application of the Langlands-Kottwitz method, as we now discuss. In particular, it is the cancellation between terms in the stable trace formula for G and the stable trace formula for H which is responsible for the irregular contributions to cohomology of Siegel threefolds.

4.4.2 Langlands-Kottwitz Method

It remains to explain where the Galois action is coming from. In this case, the only reasonable method for establishing this is through the Langlands-Kottwitz method. We largely follow [76, Section 5], which explains the computation in the case $\lambda_1 = \lambda_2 = 0$.

Let $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a compact open subgroup, and let p be a prime such that $K_p = G(\mathbb{Z}_p)$ is hyperspecial, so that $\mathcal{S}_{K_{\text{fin}}}$ has good reduction at p. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; 0)$, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$. Consider the ℓ-adic intersection cohomology

$$IH^*_\lambda = \sum_{i \geq 0} (-1)^i [IH^i(\overline{\mathcal{S}}^{\text{BB}}_{K_{\text{fin}}, \overline{\mathbb{F}}_p}, \mathbb{V}_\lambda)]$$

as an element of the Grothendieck group of $\mathcal{H}_{K_{\text{fin}}} \times \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-modules. The Langlands-Kottwitz method in this case says that for $j \gg 0$ sufficiently large one has an equality

$$\begin{aligned}
\mathrm{tr}(\mathrm{Frob}_p^j f^G | IH^*) &= \mathrm{STF}^G(f^G) + c(\Delta)\iota(G,H)\mathrm{STF}^H(f^H) \\
&= \mathrm{STF}^G(f^G) - \frac{1}{4}\mathrm{STF}^H(f^H)
\end{aligned}$$

where $f^G \in C_c^\infty(G(\mathbb{A}_{\mathbb{Q}}))$ and $f^H \in C_c^\infty(H(\mathbb{A}_{\mathbb{Q}}))$ are certain explicit test functions (depending on j) satisfying the matching conditions of standard endoscopy.

Remark 4.4.1. There are a few subtleties in the above which are noteworthy:

(i) The constant $\iota(G,H) = \frac{1}{4}$ amounts to a computation of Tamagawa numbers and automorphisms of the endoscopic datum. Indeed, this follows from the definition $\iota(G,H) = \frac{\tau(G)/\tau(H)}{|\Lambda(H,s,\eta)|}$ and the following three facts:

- We have $\tau(G) = 1$: since G is connected and G^{der} is simply connected we have

$$\tau(G) = \frac{\#\pi_0(Z(\widehat{G}))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}}{\#\ker^1(\mathbb{Q}, Z(\widehat{G}))}$$

and since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on $Z(\widehat{G})$ we have $\#\pi_0(Z(\widehat{G}))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = 1$ and we have $\#\ker^1(\mathbb{Q}, Z(\widehat{G})) = 1$, so it follows that $\tau(G) = 1$.

- We have $\tau(H) = 2$: since H is connected but H^{der} is not simply connected we cannot use the above formula; instead since we have a short exact sequence $0 \to \mathrm{GL}_1 \to \mathrm{GL}_2 \times \mathrm{GL}_2 \to H \to 0$ we have

$$\frac{\tau(H)\tau(\mathrm{GL}_1)}{\tau(\mathrm{GL}_2 \times \mathrm{GL}_2)} = \frac{\#\mathrm{coker}(\mathrm{Hom}(\mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_1) \to \mathrm{Hom}(\mathrm{GL}_1, \mathrm{GL}_1))}{\#\ker(\ker^1(\mathbb{Q}, \mathrm{GL}_1) \to \ker^1(\mathbb{Q}, \mathrm{GL}_2 \times \mathrm{GL}_2)}$$

and by Hilbert 90 we have $\#\mathrm{coker}(\mathrm{Hom}(\mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_1) \to \mathrm{Hom}(\mathrm{GL}_1, \mathrm{GL}_1)) = 2$ and we have $\#\ker(\ker^1(\mathbb{Q}, \mathrm{GL}_1) \to \ker^1(\mathbb{Q}, \mathrm{GL}_2 \times \mathrm{GL}_2) = 1$, so it follows that $\tau(H) = 2$.

- We have $\Lambda(H, s, \eta) = \mathrm{Aut}(H, s, \eta)/H^{\mathrm{ad}}(\mathbb{Q}) = \{1, \iota\}$ where ι exchanges the two factors of H^{ad}.

(ii) The global constant $c(\Delta) = -1$ is more subtle, its computation amounts to understanding the normalization of Langlands-Shelstad transfer factors, for which we use the conventions of Kottwitz: one has two different Langlands-Shelstad transfer factors $\Delta = \prod_v \Delta_v$ and $\Delta_{\mathbb{A}}$ related by $\Delta_{\mathbb{A}} = c(\Delta)\Delta$ for some global constant $c(\Delta) \in \mathbb{C}^\times$. In the present situation, by [76, Lemma 3.5] we have $c(\Delta) = -1$ using the conventions of Kottwitz.

(iii) The Kottwitz fixed point formula $\mathrm{tr}(u_j | H^*_{\mathrm{c}}(\mathcal{S}_{K_{\mathrm{fin}}, \overline{\mathbb{F}}_q}, \mathbb{V}_\lambda)) = T(j, g)$ holds only for $j \gg 0$ as it relates to the trace formula. In general for $G = \mathrm{GSp}_{2g}$ the "right" bound should be $j \geq g$, which in the present situation excludes only $j = 1$ from consideration. On the other hand, the identity for $j \gg 0$ uniquely determines the cohomology as an element of the Grothendieck group of ℓ-adic Galois representations.

The stabilization of the Arthur-Selberg trace formula is absolutely crucial in the present situation. As we will see in later computations, it is precisely the appearance of the nontrivial elliptic endoscopic group $H = \mathrm{GSO}_{2,2}$ of $G = \mathrm{GSp}_4$ which accounts for some Galois representations being smaller than the expected 4-dimensional irreducible Galois representations attached to the contributions of general type. In view of the above description of the automorphic discrete spectrum, the two terms $\mathrm{STF}^G(f^G)$ and $\mathrm{STF}^H(f^H)$ take the following form:

Proposition 4.4.2. • For $G = \mathrm{GSp}_4$ the stable trace formula $\mathrm{STF}^G(f^G)$ takes the form

$$\begin{aligned}
\mathrm{STF}^G(f^G) = &\sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc,gen}}(G,\omega) \\ (\psi=\mu\boxtimes 1)}} \sum_{\pi\in\Pi(\psi)} \prod_v \mathrm{tr}(\pi_v(f_v^G)) \\
&+ \sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc,endo}}(G,\omega) \\ (\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \underbrace{\sum_{\pi\in\Pi(\psi)} \prod_v \mathrm{tr}(\pi_v(f_v^G))}_{\prod_v \mathrm{tr}(\pi_v^+(f_v^G)) - \mathrm{tr}(\pi_v^-(f_v^G))} \\
&+ \sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc},[P_1]}(G,\omega) \\ (\psi=(\mu\boxtimes 1)\boxplus(\chi\boxtimes\nu_2))}} \underbrace{\sum_{\pi\in\Pi(\psi)} \epsilon(\mu,\chi) \prod_v \mathrm{tr}(\pi_v(f_v^G))}_{\prod_v \mathrm{tr}(\pi_v^+(f_v^G)) + \mathrm{tr}(\pi_v^-(f_v^G))} \\
&+ \sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc},[P_2]}(G,\omega) \\ (\psi=\mu\boxtimes\nu_2)}} \sum_{\pi\in\Pi(\psi)} \prod_v \mathrm{tr}(\pi_v(f_v^G)) \\
&+ \sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc},[P_0]}(G,\omega) \\ (\psi=(\chi_1\boxtimes\nu_2)\boxplus(\chi_2\boxtimes\nu_2),\chi=\chi_1\boxtimes\chi_2)}} \sum_{\pi\in\Pi(\psi)} \epsilon(\chi) \prod_v \mathrm{tr}(\pi_v(f_v^G)) \\
&+ \sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc,1dim}}(G,\omega) \\ (\psi=\chi\boxtimes\nu_4)}} \prod_v \mathrm{tr}(\chi_v \circ \mathrm{sim}(f_v^G))
\end{aligned}$$

• For $H = \mathrm{GSO}_{2,2} = (\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathrm{GL}_1$ the stable trace formula $\mathrm{STF}^H(f^H)$ takes the form

$$\begin{aligned}
\mathrm{STF}^H(f^H) = &\,2\sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc,endo}}(G,\omega) \\ (\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \underbrace{\sum_{\pi\in\Pi(\psi)} \prod_v \mathrm{tr}(\pi_v(f_v^H))}_{\prod_v \mathrm{tr}(\pi_v^+(f_v^G)) + \mathrm{tr}(\pi_v^-(f_v^G))} \\
&+ 2\sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc},[P_1]}(G,\omega) \\ (\psi=(\mu\boxtimes 1)\boxplus(\chi\boxtimes\nu_2))}} \underbrace{\sum_{\pi\in\Pi(\psi)} \prod_v \mathrm{tr}(\pi_v(f_v^H))}_{\prod_v \mathrm{tr}(\pi_v^\times(f_v^G)) - \mathrm{tr}(\pi_v^-(f_v^G))} \\
&+ 2\sum_{\omega} \sum_{\substack{\psi \in \Psi_{\mathrm{disc},[\mathrm{P}_0]}(G,\omega) \\ (\psi=(\chi_1\boxtimes\nu_2)\boxplus(\chi_2\boxtimes\nu_2))}} \prod_v \mathrm{tr}(\chi_v \circ \mathrm{sim}(f_v^H))
\end{aligned}$$

In underbraces we have included, for the unstable packets, expressions for the traces in the first case, and expression for the traces after application of the fundamental lemma in the second case. These expressions come from various endoscopic character identities, the details of which we do not mention here. The main point is that $f^G \in C_c^\infty(G(\mathbb{A}_\mathbb{Q}))$ and $f^H \in C_c^\infty(H(\mathbb{A}_\mathbb{Q}))$ are certain carefully chosen test function described by Kottwitz which satisfy the matching condition of standard endoscopy defined by Langlands-Shelstad. Let us say a word about these test function, all too briefly.

For π_p an irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ which is unramified, that is a subquotient of the normalized parabolic induction $\chi_{1,p} \times \chi_{2,p} \rtimes \chi_p = \mathcal{I}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi_{1,p} \otimes \chi_{2,p} \otimes \chi_p)$ for unramified characters $\chi_{1,p}, \chi_{2,p}, \chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ corresponding to a conjugacy class $c(\pi_p) \times \mathrm{Frob}_p$ in ${}^LG_p = \mathrm{GL}_2(\mathbb{C}) \rtimes \langle \mathrm{Frob}_p \rangle$, write $c_1(\pi_p) = \chi_{1,p}(p)$, $c_2(\pi_p) = \chi_{2,p}(p)$, $c_3(\pi_p) = \chi_p(p)/\chi_{2,p}(p)$, and $c_4(\pi_p) = \chi_p(p)/\chi_{1,p}(p)$.

Let (H, s, η) be an elliptic endoscopic datum. The spherical function $f_p^H \in \mathcal{H}_{K_p^H}$ is defined in terms of the L-homomorphism ${}^LH \to {}^LG \to {}^LG_{p^j}$ where $G_{p^j} = \mathrm{Res}_{\mathbb{Q}_{p^j}/\mathbb{Q}_p} G$ and where ${}^LG_{p^j} = \widehat{G}^j \rtimes \langle \mathrm{Frob}_p \rangle$. We have a morphism $\widetilde{\eta}_j : {}^LH_p \to {}^LG_{p^j}$ given by $t \mapsto (t, \ldots, t)$ and $\mathrm{Frob}_p \mapsto (s, \ldots, s)$. The diagonal morphism $G_p \to G_{p^j}$ defines an L-homomorphism ${}^LG_{p^j} \to {}^LG_p$ given by $(t_1, \ldots, t_j) \times \mathrm{Frob}_p^i \mapsto t_1 \ldots t_j \times \mathrm{Frob}_p^i$, and defines an L-homomorphism $\eta_j : {}^LH_p \to {}^LG_p$ given by $t \times \mathrm{Frob}_p^i \mapsto t^j s^i \times \mathrm{Frob}_p^i$. We have a dual morphism $\mathcal{H}_{K_{p^j}} \to \mathcal{H}_{K_{p^j}^H}$ and the function $f_p^H \in \mathcal{H}_{K_{p^j}^H}$ is defined by the relation $\mathrm{tr}(\pi_p(\widetilde{\eta}_j(t))(\phi_j)) = \mathrm{tr}(\pi_p^H(t)(f_p^H))$ where π_p^H is defined by the L-homomorphism ${}^LH_p \to {}^LG_p$ and where $\phi_j = 1_{K_{p^j}\mu(p^{-1})K_{p^j}}$ as before. Then by Kottwitz we have the trace

$$\mathrm{tr}(\pi_p(f_p^H)) = p^{\frac{3}{2}j} \mathrm{tr}(r_{-\mu}(s(c(\pi_p) \times \mathrm{Frob}_p)^j))$$

Proposition 4.4.3. For the trivial elliptic endoscopic datum $(\mathrm{GSp}_4, \mathrm{diag}(1,1,1,1), \mathrm{id})$ we have the trace

$$\mathrm{tr}(\pi_p(f_p^G)) = p^{\frac{3}{2}j}(c_1(\pi_p)^j + c_2(\pi_p)^j + c_3(\pi_p)^j + c_4(\pi_p)^j)$$

For the nontrivial elliptic endoscopic datum $(H, \mathrm{diag}(1,-1,-1,1), \mathrm{id})$ we have the trace

$$\mathrm{tr}(\pi_p(f_p^H)) = p^{\frac{3}{2}j}(c_1(\pi_p)^j - c_2(\pi_p)^j - c_3(\pi_p)^j + c_4(\pi_p)^j)$$

Away from p and ∞, we can choose test functions $f_G^{p,\infty} \in C_c^\infty(K^{p,\infty} \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^{p,\infty})$ which project onto individual (packets of) representations $\{\pi_{\mathrm{fin}}\}$ so as to isolate their contirbutions to cohomology, and $f_H^{p,\infty} \in C_c^\infty(K_H^{p,\infty} \setminus H(\mathbb{A}_\mathbb{Q}^{p,\infty})/K_H^{p,\infty})$ is a transfer of f^G (satisfying the required matching condition). At ∞, the test function f_∞^G is essentially the average of pseudocoefficients of discrete series representations of $G(\mathbb{R})$ with central and infinitesimal characters matching those of V_λ, and f_∞^H is again transfer of f^H (satisfying the required matching condition), expressed in terms of pseudocoefficients of discrete series representations of $H(\mathbb{R})$. We refer to [76] for a more detailed discussion.

We should remark that the usual choice of ℓ-adic local systems on $\mathcal{S}_{K_{\text{fin}}}$ constructed previously in terms of the cohomology of the universal family of Abelian surfaces, involves the highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$. For these ℓ-adic local systems, the above formulas for the trace of Frobenius should be multiplied by an additional factor of $p^{\frac{\lambda_1+\lambda_2}{2}j}$, corresponding to a half Tate twist.

4.4.3 Theta Correspondence

It will turn out that the packets attached to A-parameters for $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of Yoshida type, Saito-Kurokawa type, Soudry type, and Howe-Piatetski-Shapiro type can all be constructed using various theta correspondences. In this section we review some facts about these various theta correspondences which we will use in the next few sections. In brief, we will employ the following theta lifts:

- The theta lift between Mp_2 and SO_V for $\dim(W) = 2$ and $\dim(V) = 3, 5$;
- The similitude theta lift between GSp_W and GO_V for $\dim(W) = 4$ and $\dim(V) = 2, 4$;

We will mainly focus on the case $\dim(V) = 3$ which is the Shimura lift studied by Waldspurger and recalled in [38], and the case $\dim(V) = 2$ which is the similitude theta lift studied by Soudry [110]. We also refer to [39] and [41] for relevant discussion around theta lifts. The goal is simply to recall the definitions of these lifts and to summarize their basic properties without proof.

Local Shimura Correspondence We follow [38]. Let F be a local field of characteristic 0 and let $(W, \langle\cdot,\cdot\rangle)$ be a symplectic F-vector space with symplectic group Sp_W. When $F \neq \mathbb{C}$ the symplectic group $\mathrm{Sp}_W(F)$ has a unique two-fold central extension

$$0 \to \mu_2(F) \to \mathrm{Mp}_W(F) \to \mathrm{Sp}_W(F) \to 0$$

We will only consider the case where $\dim(W) = 2$ so that $\mathrm{Sp}_W(F) = \mathrm{SL}_2(F)$ and $\mathrm{Mp}_W(F) = \mathrm{Mp}_2(F)$. Let $\mathrm{Irr}(\mathrm{Mp}_2(F))$ be the set of genuine irreducible representations of $\mathrm{Mp}_2(F)$ (those not factoring through an irreducible representation of $\mathrm{SL}_2(F)$).

Let (V, q) be $2m+1$-dimensional quadratic F-vector space of discriminant 1 with orthogonal groups O_V and SO_V. We will mainly consider the case where $\dim(V) = 3, 5$. In the case $\dim(V) = 3$ there are exactly two such quadratic spaces; let V^+ be the split 3-dimensional quadratic F-vector space, and let V^- be the non-split 3-dimensional quadratic F-vector space. These can be constructed as follows: let D be a quaternion F-algebra with norm form Nm_D and let D_0 be the F-subalgebra of trace 0 elements in D. Then the 3-dimensional quadratic F-vector space $V_D = (D_0, -\mathrm{Nm}_D)$ has discriminant 1 and is isomorphic to V^+ or V^- depending on whether D is split or non-split. Every $\pi \in \mathrm{Irr}(\mathrm{SO}_V(F))$ admits two possible extensions $\pi^+, \pi^- \in \mathrm{Irr}(\mathrm{O}_V(F))$ such that $-1 \in \mathrm{O}_V(F)$ acts trivially on π^+ and nontrivially on π^-.

Since $\mathrm{SO}_{V^-}(F) = PD^\times$ is compact every $\pi \in \mathrm{Irr}(\mathrm{SO}_{V^-}(F))$ is finite-dimensional. Recall that the local Jacquet-Langlands correpsondence yields an injection

$$\mathrm{JL} : \mathrm{Irr}(\mathrm{SO}_{V^-}(F)) \hookrightarrow \mathrm{Irr}(\mathrm{SO}_{V^+}(F)) = \mathrm{Irr}(\mathrm{PGL}_2(F))$$

whose image is the subset of discrete series representations.

Let $\psi : F \to \mathbb{C}$ be a nontrivial additive character and for $a \in F^\times$ let $\psi_a(x) = \psi(ax)$. For $\widetilde{T} \subseteq \mathrm{Mp}_2(F)$ the preimage of the diagonal torus $T \subseteq \mathrm{SL}_2(F)$ we have a genuine character $\chi_\psi : \widetilde{T} \to \mathbb{C}^\times$ given by $(\mathrm{diag}(a, a^{-1}), \epsilon) \mapsto \epsilon\gamma(a, \psi)^{-1}$ where $\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$ is the quotient of Weil indices $\gamma(\psi_a), \gamma(\psi) \in \mu_8$. Consider the oscillator representation ω_ψ of $\mathrm{Mp}_2(F)$ on the space $\mathcal{S}(F)$ of Schwarz-Bruhat functions on F given by

$$\begin{aligned} \omega_\psi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right)\phi(x) &= |a|^{1/2}\chi_\psi(a)\phi(ax) \\ \omega_\psi\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right)\phi(x) &= \psi(bx^2)\phi(x) \\ \omega_\psi\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\phi(x) &= \gamma(\psi) \int_F \psi(-2tx)\phi(t)\mathrm{d}t \end{aligned}$$

where the integral is taken with respect to the measure $\mathrm{d}t$ on F which is Fourier self-dual with respect to the character $\psi_2(x) = \psi(2x)$. The representation ω_ψ is reducible and decomposes as $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$ where ω_ψ^+ is realized on the subspace of even functions and ω_ψ^- is realized on the subspace of odd functions.

For $a \in F^\times$ let $\psi_a : F \to \mathbb{C}$ be the nontrivial quadratic character given by $\psi_a(x) = \psi(ax)$. Then every nontrivial additive character of F is of this form, and we have $\omega_{\psi_{a_1}} \simeq \omega_{\psi_{a_2}}$ precisely if $a_1/a_2 \in F^{\times 2}$. For

$a \in F^\times/F^{\times 2}$ with associated quadratic character $\chi : F^\times \to \mathbb{C}^\times$ let $\omega_{\psi,\chi} = \omega_{\psi_a}$ so that $\omega_{\psi,\chi} = \omega^+_{\psi,\chi} \oplus \omega^-_{\psi,\chi}$ and define the local packet

$$\widetilde{\Pi}_\psi(\chi) = \{\widetilde{\pi}^+, \widetilde{\pi}^-\}$$

where $\widetilde{\pi}^+ = \omega^+_{\psi,\chi}$ and where $\widetilde{\pi}^- = \omega^-_{\psi,\chi}$.

Consider the Weil representation $\omega_{V,W,\psi}$ of $\mathrm{O}_V(F) \times \mathrm{Mp}_2(F)$ on the space $\mathcal{S}(V)$ of Schwarz-Bruhat functions on V given by

$$\begin{aligned}
\omega_{V,W,\psi}(h,1)\phi(v) &= \phi(h^{-1}x)\\
\omega_{V,W,\psi}(1, \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right))\phi(v) &= |a|^{\dim(V)/2}\chi_\psi(a)\phi(av)\\
\omega_{V,W,\psi}(1, \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right))\phi(v) &= \psi(bq(v))\phi(v)\\
\omega_{V,W,\psi}(1, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))\phi(v) &= \gamma(\psi \circ q)\int_V \psi(-\langle v,t\rangle)\phi(t)\mathrm{d}t\text{‘}
\end{aligned}$$

where the integral is taken with respect to the measure $\mathrm{d}t$ on V which is Fourier self-dual with respect to the character $\psi(\langle\cdot,\cdot\rangle)$ on $V \times V$ where $\langle v_1, v_2\rangle = q(v_1+v_2) - q(v_1) - q(v_2)$ is the symetric bilinear form associated to the quadratic form q.

The Weil representation $\omega_{V,W,\psi}$ of $\mathrm{O}_V(F) \times \mathrm{Mp}_2(F)$ defines a theta correspondence between $\mathrm{O}_V(F)$ and $\mathrm{Mp}_2(F)$ as follows. For $\pi \in \mathrm{Irr}(\mathrm{O}_V(F))$ the maximal π-isotypic subrepresentation of $\omega_{V,W,\psi}$ has the form $\pi \boxtimes \Theta_{V,W,\psi}(\pi)$ for some smooth representation $\Theta_{V,W,\psi}(\pi)$ of $\mathrm{Mp}_2(F)$, the big theta lift of π. The big theta lift $\Theta_{V,W,\psi}(\pi)$ is finite length, hence admissible. Let $\theta_{V,W,\psi}(\pi)$ be the maximal semisimple quotient of $\Theta_{V,W,\psi}(\pi)$, the small theta lift of π. The small theta lift $\theta_{V,W,\psi}(\pi)$ is irreducible if $\Theta_{V,W,\psi}(\pi)$ is nonzero, and the assignment $\pi \mapsto \theta_{V,W,\psi}(\pi)$ is injective on its domain. By Kudla, $\Theta_{V,W,\psi}(\pi) = \theta_{V,W,\psi}(\pi)$ is irreducible or 0 for π supercuspidal, and if $\Theta_{V,W,\psi}(\pi_1) = \Theta_{V,W,\psi}(\pi_2) \neq 0$ for π_1 and π_2 supercuspidal, then $\pi_1 = \pi_2$.

Similarly for $\widetilde{\pi} \in \mathrm{Irr}(\mathrm{Mp}_2(F))$ the maximal $\widetilde{\pi}$-isotypic quotient representation of $\omega_{V,W,\psi}$ has the form $\Theta_{V,W,\psi}(\widetilde{\pi}) \boxtimes \widetilde{\pi}$ for some smooth representation $\Theta_{V,W,\psi}(\widetilde{\pi})$ of $\mathrm{O}_V(F)$, the big theta lift of $\widetilde{\pi}$. Let $\theta_{V,W,\psi}(\widetilde{\pi})$ be the maximal semisimple quotient of $\Theta_{V,W,\psi}(\widetilde{\pi})$, the small theta lift of $\widetilde{\pi}$. Similar statements regarding the irreducibility of $\Theta_{V,W,\psi}(\widetilde{\pi})$ and $\theta_{V,W,\psi}(\widetilde{\pi})$, the injectivity of the assignment $\widetilde{\pi} \mapsto \theta_{V,W,\psi}(\widetilde{\pi})$, and its behavior on supercuspidal representations hold in this case.

In the case $\dim(V) = 3$ the local theta lift between $\mathrm{Mp}_2(F)$ and $\mathrm{O}_V(F)$ defines the local Shimura correspondence, studied by Waldspurger. The main local theorems are as follows.

For $\pi \in \mathrm{Irr}(\mathrm{SO}_V(F))$, we have $\Theta_{V,W,\psi}(\pi^\epsilon) \neq 0$ for exactly one extension $\pi^\epsilon \in \mathrm{Irr}(\mathrm{O}_V(F))$ determined by the sign

$$\epsilon = \epsilon(V)\epsilon(\tfrac{1}{2}, \pi, \psi)$$

where $\epsilon(\frac{1}{2}, \pi, \psi)$ is the standard local ϵ-factor of π defined by the doubling method. For this unique extension the big theta lift $\Theta_{V,W,\psi}(\pi^\epsilon)$ has a unique irreducible quotient representation $\theta_{V,W,\psi}(\pi)$, the small theta lift of π (note that $\Theta_{V,W,\psi}(\pi^\epsilon)$ is irreducible unless π is 1-dimensional).

For $\widetilde{\pi} \in \mathrm{Irr}(\mathrm{Mp}_2(F))$, we have $\Theta_{V,W,\psi}(\widetilde{\pi}) \neq 0$ for exactly one V determined by the sign

$$\epsilon(V) = z_\psi(\widetilde{\pi})\epsilon(\tfrac{1}{2}, \widetilde{\pi}, \psi)$$

where $\epsilon(\frac{1}{2}, \widetilde{\pi}, \psi)$ is the standard local ϵ-factor of $\widetilde{\pi}$ defined by the doubling method. For this unique V the big theta lift $\Theta_{V,W,\psi}(\widetilde{\pi})$ has a unique irreducible quotient representation $\theta_{V,W,\psi}(\widetilde{\pi})$, the small theta lift of $\widetilde{\pi}$ (note that $\Theta_{V,W,\psi}(\widetilde{\pi})$ is irreducible unless $\widetilde{\pi} \simeq \omega^+_{\psi,\chi}$).

The assignment $\widetilde{\pi} \mapsto \theta_{V,W,\psi}(\widetilde{\pi})$ defines a bijection (the local Shimura correspondence)

$$\mathrm{Sh}_\psi : \mathrm{Irr}(\mathrm{Mp}_2(F)) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{SO}_{V^+}(F)) \amalg \mathrm{Irr}(\mathrm{SO}_{V^-}(F))$$

which is compatible with standard local γ-factors, L-factors, and ϵ-factors.

Composing the local Shimura correspondence with the local Jacquet-Langlands correspondence defines a surjection (the local Waldspurger correspondence)

$$\mathrm{Wd}_\psi : \mathrm{Irr}(\mathrm{Mp}_2(F)) \to \mathrm{Irr}(\mathrm{SO}_{V^+}(F)) = \mathrm{Irr}(\mathrm{PGL}_2(F))$$

whose fibers have cardinality 1 or 2. For $\pi \in \mathrm{Irr}(\mathrm{PGL}_2(F))$ define the local Waldspurger packet

$$\widetilde{\Pi}_\psi(\pi) = \mathrm{Wd}_\psi^{-1}(\pi) = \begin{cases} \{\widetilde{\pi}^+, \widetilde{\pi}^-\} & \pi \text{ discrete series} \\ \{\widetilde{\pi}^+\} & \text{otherwise} \end{cases}$$

where $\widetilde{\pi}^+ = \theta_{W,V^+,\psi}(\pi)$ and where $\widetilde{\pi}^- = \theta_{W,V^-,\psi}(\pi^D)$ for π^D the Jacquet-Langlands transfer of π to $PD^\times$ if π is in the discrete series, otherwise $\widetilde{\pi}^- = 0$. For $a \in F^\times/F^{\times 2}$ with associated quadratic character $\chi_a : F^\times \to \mathbb{C}^\times$, we have $\widetilde{\Pi}_\psi(\pi) = \widetilde{\Pi}_{\psi_a}(\pi \otimes \chi_a)$, so we obtain a canonical partition

$$\mathrm{Irr}(\mathrm{Mp}_2(F)) = \coprod_{\pi \in \mathrm{Irr}(\mathrm{PGL}_2(F))} \widetilde{\Pi}_\psi(\pi)$$

into a disjoint union of finite subsets, whose labeling by elements of $\mathrm{Irr}(\mathrm{PGL}_2(F))$ depends on ψ. For $\widetilde{\pi} \in \widetilde{\Pi}_\psi(\pi) = \widetilde{\Pi}_{\psi_a}(\pi \otimes \chi_a)$ we have $z_{\psi_a}(\widetilde{\pi}) = z_\psi(\widetilde{\pi})\chi_a(-1)$, and the labelings $\widetilde{\pi}^\epsilon$ and $\widetilde{\pi}^{\epsilon_a}$ of $\widetilde{\pi}$ as an element of $\widetilde{\Pi}_\psi(\pi)$ and $\widetilde{\Pi}_{\psi_a}(\pi \otimes \chi_a)$ are related by

$$\epsilon_a(\widetilde{\pi})\epsilon(\widetilde{\pi}) = \epsilon(\tfrac{1}{2}, \pi \otimes \chi_a)\epsilon(\tfrac{1}{2}, \pi)\chi_a(-1)$$

In the case $\dim(V) = 5$ the local theta lift from $\mathrm{Mp}_2(F)$ to $\mathrm{SO}_5(F) \simeq \mathrm{PGSp}_4(F)$ along with the local Waldspurger correspondence will be used in the construction of the Saito-Kurokawa lift and the Howe-Piatetski-Shapiro lift.

Global Shimura Correspondence We continue to follow [38]. Let F be a number field and let $(W, \langle \cdot, \cdot \rangle)$ be a symplectic F-vector space with symplectic group Sp_W. For v a place of F consider the maximal compact subgroup $K_v = \mathrm{Sp}_W(\mathcal{O}_{F_v}) \subseteq \mathrm{Mp}_W(F_v)$ and consider the restricted direct product

$$0 \to \bigoplus_v \mu_2(F_v) \to \widehat{\prod_v}^{K_v} \mathrm{Mp}_W(F_v) \to \mathrm{Sp}_W(\mathbb{A}_F) \to 0$$

Let $Z_0 = \{(\epsilon_v)_v \in \bigoplus_v \mu_2(F_v) | \prod_v \epsilon_v = 1\}$ and consider the quotient $\mathrm{Mp}_W(\mathbb{A}_F) = (\widehat{\prod_v}^{K_v} \mathrm{Mp}_W(F_v))/Z_0$ which is a two-fold central extension

$$0 \to \mu_2(F) \to \mathrm{Mp}_W(\mathbb{A}_F) \to \mathrm{SL}_W(\mathbb{A}_F) \to 0$$

Again we will only consider the case where $\dim(W) = 2$ so that $\mathrm{Sp}_W(\mathbb{A}_F) = \mathrm{SL}_2(\mathbb{A}_F)$ and $\mathrm{Mp}_W(\mathbb{A}_F) = \mathrm{Mp}_2(\mathbb{A}_F)$. Let $\mathrm{Irr}(\mathrm{Mp}_2(\mathbb{A}_F))$ be the set of genuine irreducible representations of $\mathrm{Mp}_2(\mathbb{A}_F)$ (those not factoring through an irreducible representation of $\mathrm{SL}_2(\mathbb{A}_F)$).

Let (V, q) be a $2m+1$-dimensional quadratic F-vector space of discriminant 1 with orthogonal groups O_V and SO_V. We will mainly consider the case where $\dim(V) = 3, 5$. In the case $\dim(V) = 3$ these can be constructed as follows: let D be a quaternion F-algebra with norm form Nm_D and let D_0 be the F-subalgebra of trace 0 elements in D. Then the 3-dimensional quadratic F-vector space $V_D = (D_0, -\mathrm{Nm}_D)$ has discriminant 1, and every 3-dimensional quadratic F-vector space of discriminant 1 is obtained in this way. For Σ a finite set of places of F with $\#\Sigma$ even let sign^Σ be the automorphic character of $\mu_2(F) \setminus \mu_2(\mathbb{A}_F)$ given by

$$\mathrm{sign}^\Sigma = \Big(\bigotimes_{v \in \Sigma} \mathrm{sign}_{\mu_2(F_v)} \Big) \otimes \Big(\bigotimes_{v \notin \Sigma} 1_{\mu_2(F_v)} \Big)$$

If π is an automorphic representation of $\mathrm{SO}_V(\mathbb{A}_F)$ then $\pi \otimes \mathrm{sign}^\Sigma$ is an automorphic representation of $\mathrm{O}_V(\mathbb{A}_F)$, and every automorphic representation of $\mathrm{O}_V(\mathbb{A}_F)$ is obtained in this way.

Let $\psi = \bigotimes_v \psi_v : F \setminus \mathbb{A}_F \to \mathbb{C}$ be a nontrivial additive character and consider the global oscillator representation $\omega_\psi = \bigotimes_v \omega_{\psi_v}$ of $\mathrm{Mp}_2(\mathbb{A}_F)$ on the space $\mathcal{S}(\mathbb{A}_F)$ of adelic Schwarz-Bruhat functions on F. The representation ω_ψ is highly reducible and decomposes as

$$\omega_\psi = \bigoplus_{\Sigma} \omega_\psi^\Sigma \qquad \omega_\psi^\Sigma = \Big(\bigotimes_{v \in \Sigma} \omega_{\psi_v}^- \Big) \otimes \Big(\bigotimes_{v \notin \Sigma} \omega_{\psi_v}^+ \Big)$$

where the direct sum is taken over finite sets Σ of places of F. We have an $\mathrm{Mp}_2(\mathbb{A}_F)$-equivariant morphism $\theta_\psi : \omega_\psi \to \mathcal{A}(\mathrm{Mp}_2(\mathbb{A}_F))$ given by

$$\theta_\psi(\phi)(g) = \sum_{x \in F} \omega_\psi(g)(\phi)(x)$$

with kernel $\ker(\theta_\psi) \simeq \bigoplus_{\#\Sigma \text{ odd}} \omega_\psi^\Sigma$ and with image $\mathrm{im}(\theta_\psi) \simeq \bigoplus_{\#\Sigma \text{ even}} \omega_\psi^\Sigma$ which is contained in $L^2_{\mathrm{disc}}(\mathrm{SL}_2(F)\setminus \mathrm{Mp}_2(\mathbb{A}_F))$ and constitutes a full near-equivalence class in $L^2_{\mathrm{disc}}(\mathrm{SL}_2(F) \setminus \mathrm{Mp}_2(\mathbb{A}_F))$. Moreover ω_ψ^Σ is cuspidal precisely if $\#\Sigma > 0$.

For $a \in F^\times$ let $\psi_a : F \setminus \mathbb{A}_F \to \mathbb{C}$ be the nontrivial additive character given by $\psi_a(x) = \psi(ax)$. Then every nontrivial additive character of $F \setminus \mathbb{A}_F$ is of this form, and we have $\omega_{\psi_{a_1}} \simeq \omega_{\psi_{a_2}}$ precisely if $a_1/a_2 \in F^{\times 2}$. For $a \in F^\times/F^{\times 2}$ with associated quadratic Hecke character $\chi : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ define the global packet

$$\widetilde{\Pi}_\psi(\chi) = \{\widetilde{\pi} = \bigotimes_v \widetilde{\pi}_v | \widetilde{\pi}_v \in \widetilde{\Pi}_{\psi_v}(\chi_v), \widetilde{\pi}_v = \widetilde{\pi}_v^+ \text{ for almost all places } v \text{ of } F\}$$

which is an infinite set. Consider the submodule

$$\Theta = \sum_{a \in F^\times/F^{\times 2}} \mathrm{im}(\theta_{\psi_a}) \subseteq L^2_{\mathrm{disc}}(\mathrm{SL}_2(F) \setminus \mathrm{Mp}_2(\mathbb{A}_F))$$

Then an element $\widetilde{\pi} \in \widetilde{\Pi}_\psi(\chi)$ has discrete multiplicity $m(\widetilde{\pi}) = 1$ precisely if $\epsilon(\frac{1}{2}, \widetilde{\pi}, \psi) = 1$. That is, for $\widetilde{\pi} = \bigotimes_v \widetilde{\pi}_v \in \widetilde{\Pi}_\psi(\chi)$ and for Σ the set of places v of F such that $\widetilde{\pi}_v = \widetilde{\pi}_v^-$ we have

$$m(\widetilde{\pi}) = \frac{1}{2}\Big(1 + \epsilon(\tfrac{1}{2}, \widetilde{\pi}, \psi)\Big) = \begin{cases} 1 & \#\Sigma \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

In particular we have a decomposition

$$\Theta \simeq \bigoplus_\chi \bigoplus_{\substack{\widetilde{\pi} \in \widetilde{\Pi}_\psi(\chi) \\ \epsilon(\frac{1}{2}, \widetilde{\pi}, \psi) = 1}} \widetilde{\pi} \simeq \bigoplus_\chi \bigoplus_{\#\Sigma \text{ even}} \omega_{\psi,\chi}^\Sigma$$

Consider the global Weil representation $\omega_{V,W,\psi} = \bigotimes_v \omega_{V_v,W_v,\psi_v}$ of $\mathrm{O}_V(\mathbb{A}_F) \times \mathrm{Mp}_2(\mathbb{A}_F)$ on the space $\mathcal{S}(V(\mathbb{A}_F))$ of adelic Schwarz-Bruhat functions on $V(\mathbb{A}_F)$. We have an $\mathrm{O}_V(\mathbb{A}_F) \times \mathrm{Mp}_2(\mathbb{A}_F)$-equivariant morphism $\theta_{V,W,\psi} : \omega_{V,W,\psi} \to \mathcal{A}(\mathrm{O}_V(\mathbb{A}_F) \times \mathrm{Mp}_2(\mathbb{A}_F))$ given by

$$\theta_{V,W,\psi}(\phi)(h,g) = \sum_{x \in V(F)} \omega_{V,W,\psi}(h,g)(\phi)(x)$$

which is absolutely convergent. The functions in $\mathrm{im}(\theta_{V,W,\psi})$ can be used as integration kernels to lift automorphic forms on $\mathrm{O}_V(\mathbb{A}_F)$ to automorphic forms on $\mathrm{Mp}_2(\mathbb{A}_F)$. For an automorphic form $f \in \mathcal{A}(\mathrm{O}_V(\mathbb{A}_F))$ and an adelic Schwarz-Bruhat function $\phi \in \mathcal{S}(V(\mathbb{A}_F))$ consider the automorphic form $\theta_{V,W,\psi}(\phi, f) \in \mathcal{A}(\mathrm{Mp}_2(\mathbb{A}_F))$ given by

$$\theta_{V,W,\psi}(\phi, f)(g) = \int_{\mathrm{O}_V(F)\backslash \mathrm{O}_V(\mathbb{A}_F)} \theta_{V,W,\psi}(\phi)(h, g)\overline{f(h)}\mathrm{d}h$$

where the integral is taken with respect to the Tamagawa measure $\mathrm{d}h$ on $\mathrm{O}_V(\mathbb{A}_F)$, which converges if $f \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{O}_V(\mathbb{A}_F))$. For $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{O}_V(\mathbb{A}_F)$ define the global theta lift

$$\Theta_{V,W,\psi}(\pi) = \langle \theta_{V,W,\psi}(\phi, f) | \phi \in \mathcal{S}(V(\mathbb{A}_F)), f \in \pi \rangle \subseteq \mathcal{A}(\mathrm{Mp}_2(\mathbb{A}_F))$$

The automorphic representation $\Theta_{V,W,\psi}(\pi)$, if nonzero, has a unique irreducible quotient isomorphic to $\bigotimes_v \theta_{V_v,W_v,\psi_v}(\pi_v)$. In particular if $\Theta_{V,W,\psi}(\pi) \subseteq \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_2(\mathbb{A}_F))$ is cuspidal then we have an isomorphism $\Theta_{V,W,\psi}(\pi) \simeq \bigotimes_v \theta_{V_v,W_v,\psi_v}(\pi_v)$.

Similarly the functions in $\mathrm{im}(\theta_{V,W,\psi})$ can also be used as integration kernels to lift automorphic forms on $\mathrm{Mp}_2(\mathbb{A}_F)$ to automorphic forms on $\mathrm{O}_V(\mathbb{A}_F)$. For an automorphic form $f \in \mathcal{A}(\mathrm{Mp}_2(\mathbb{A}_F))$ and an adelic Schwarz-Bruhat function $\phi \in \mathcal{S}(V(\mathbb{A}_F))$ consider the automorphic form $\theta_{V,W,\psi}(\phi, f) \in \mathcal{A}(\mathrm{O}_V(\mathbb{A}_F))$ given by

$$\theta_{V,W,\psi}(\phi, f)(g) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{Mp}_2(\mathbb{A}_F)} \theta_{V,W,\psi}(\phi)(h, g)\overline{f(g)}\mathrm{d}g$$

where the integral is taken with respect to the Tamagawa measure $\mathrm{d}g$ on $\mathrm{Mp}_2(\mathbb{A}_F)$, which converges if $f \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_2(\mathbb{A}_F))$. For $\widetilde{\pi} = \bigotimes_v \widetilde{\pi}_v$ a cuspidal automorphic representation of $\mathrm{Mp}_2(\mathbb{A}_F)$ define the global theta lift

$$\Theta_{V,W,\psi}(\widetilde{\pi}) = \langle \theta_{V,W,\psi}(\phi, f) | \phi \in \mathcal{S}(V(\mathbb{A}_F)), f \in \widetilde{\pi} \rangle \subseteq \mathcal{A}(\mathrm{O}_V(\mathbb{A}_F))$$

The automorphic representation $\Theta_{V,W,\psi}(\widetilde{\pi})$, if nonzero, has a unique irreducible quotient isomorphic to $\bigotimes_v \theta_{V_v,W_v,\psi_v}(\widetilde{\pi}_v)$. In particular if $\Theta_{V,W,\psi}(\widetilde{\pi}) \subseteq \mathcal{A}_{\mathrm{cusp}}(\mathrm{O}_V(\mathbb{A}_F))$ is cuspidal then we have an isomorphism $\Theta_{V,W,\psi}(\widetilde{\pi}) \simeq \bigotimes_v \theta_{V_v,W_v,\psi_v}(\widetilde{\pi}_v)$.

In the case $\dim(V) = 3$ the global theta lift between $\mathrm{Mp}_2(\mathbb{A}_F)$ and $\mathrm{O}_V(\mathbb{A}_F)$ defines the local Shimura correspondence, studied by Waldspurger. the main global theorems are as follows.

For $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$ define the global Waldspurger packet

$$\widetilde{\Pi}_\psi(\pi) = \{\pi = \bigotimes_v \pi_v | \pi_v \in \widetilde{\Pi}_{\psi_v}(\mu_v), \pi_v = \pi_v^+ \text{ for almost all places } v \text{ of } F\}$$

which is a finite set of cardinality $2^{\# S_\pi}$ where S_π is the finite set of places v of F such that π_v is in the discrete series. For $a \in F^\times / F^{\times 2}$ with associated quadratic character $\chi_a : F^\times \to \mathbb{C}^\times$, we have $\widetilde{\Pi}_\psi(\pi) = \widetilde{\Pi}_{\psi_a}(\pi \otimes \chi_a)$. Consider the submodule

$$\Theta_\pi = \sum_{a \in F^\times / F^{\times 2}} \Theta_{V,W,\psi_a}(\pi \otimes \chi_a) \subseteq L^2_{\mathrm{disc}}(\mathrm{SL}_2(F) \setminus \mathrm{Mp}_2(\mathbb{A}_F))$$

which constitutes a full near-equivalence class in $L^2_{\mathrm{disc}}(\mathrm{SL}_2(F) \setminus \mathrm{Mp}_2(\mathbb{A}_F))$: every irreducible summmand in Θ_π is isomorphic to an element of the global Waldspurger packet $\widetilde{\Pi}_\psi(\pi)$, and an element $\widetilde{\pi} \in \widetilde{\Pi}_\psi(\pi)$ has discrete multiplicity $m(\widetilde{\pi}) = 1$ precisely if $\epsilon(\frac{1}{2}, \widetilde{\pi}, \psi) = \epsilon(\frac{1}{2}, \pi)$. That is, for $\widetilde{\pi} = \bigotimes_v \widetilde{\pi}_v \in \widetilde{\Pi}_\psi(\chi)$ and for Σ the set of places v of F such that $\widetilde{\pi}_v = \widetilde{\pi}_v^-$ we have

$$m(\widetilde{\pi}) = \frac{1}{2}\Big(1 + \epsilon(\tfrac{1}{2}, \pi)\epsilon(\tfrac{1}{2}, \widetilde{\pi}, \psi)\Big) = \begin{cases} 1 & \begin{array}{l} \#\Sigma \text{ even and } \epsilon(\frac{1}{2}, \pi) = 1 \\ \#\Sigma \text{ odd and } \epsilon(\frac{1}{2}, \pi) = -1 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

In particular we have a decomposition

$$\Theta_\pi \simeq \bigoplus_\pi \bigoplus_{\substack{\widetilde{\pi} \in \widetilde{\Pi}_\psi(\pi) \\ \epsilon(\frac{1}{2}, \widetilde{\pi}, \psi) = \epsilon(\frac{1}{2}, \pi)}} \widetilde{\pi}$$

The main global theorem is that every irreducible cuspidal automorphic representation of $\mathrm{Mp}_2(\mathbb{A}_F)$ is contained in some Θ_π, so that we have a decomposition

$$L^2_{\mathrm{disc}}(\mathrm{SL}_2(F) \setminus \mathrm{Mp}_2(\mathbb{A}_F)) = \bigoplus_\pi \Theta_\pi \oplus \Theta$$

where the first direct sum is taken over cuspidal automorphic representations π of $\mathrm{PGL}_2(\mathbb{A}_F)$ such that $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$ for some quadratic Hecke character $\chi : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$, equivalently such that $\epsilon(\frac{1}{2}, \pi \otimes \chi) = 1$ for some quadratic Hecke character $\chi : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$. In particular we have a decomposition

$$L^2_{\mathrm{disc}}(\mathrm{SL}_2(F) \setminus \mathrm{Mp}_2(\mathbb{A}_F)) = \bigoplus_{\pi} \bigoplus_{\substack{\widetilde{\pi} \in \widetilde{\Pi}_\psi(\pi) \\ \epsilon(\frac{1}{2}, \widetilde{\pi}, \psi) = \epsilon(\frac{1}{2}, \pi)}} \widetilde{\pi} \oplus \bigoplus_{\chi} \bigoplus_{\substack{\widetilde{\pi} \in \widetilde{\Pi}_\psi(\chi) \\ \epsilon(\frac{1}{2}, \widetilde{\pi}, \psi) = 1}} \widetilde{\pi}$$

In the case $\dim(V) = 5$ the global theta lift from $\mathrm{Mp}_2(\mathbb{A}_F)$ to $\mathrm{SO}_5(\mathbb{A}_F) \simeq \mathrm{PGSp}_4(\mathbb{A}_F)$ along with the global Waldspurger correspondence will be used in the construction of the Saito-Kurokawa lift and the Howe-Piatetski-Shapiro lift.

Local Similitude Theta Correspondence We follow [110] and [41]. Let F be a local field of characteristic 0 and let $(W, \langle \cdot, \cdot \rangle)$ be a symplectic F-vector space with symplectic group Sp_W and similitude group GSp_W with similitude character $\mathrm{sim}_W : \mathrm{GSp}_W \to \mathrm{GL}_1$. Let $W = X \oplus Y$ be a Witt decomposition and let $P_Y \subseteq \mathrm{Sp}_W$ be the parabolic subgroup stabilizing the maximal isotropic subspace $Y \subseteq W$ with Levi decomposition $P_Y = M_Y N_Y$ with Levi quotient $M_Y = \mathrm{GL}(Y)$ and unipotent radical $N_Y = \{n \in \mathrm{Hom}(X, Y) | n^\top = n\}$ where $n^\top \in \mathrm{Hom}(Y^\vee, X^\vee) \simeq \mathrm{Hom}(X, Y)$. We will only consider the case where $\dim(W) = 4$ so that $\mathrm{Sp}_W(F) = \mathrm{Sp}_4(F)$ and $\mathrm{GSp}_W(F) = \mathrm{GSp}_4(F)$, and $P_Y = P_1 \subseteq \mathrm{Sp}_4$ is the Siegel parabolic subgroup with Levi decomposition $P_1 = M_1 N_1$ with Levi quotient $M_1 = \mathrm{GL}_2$ and unipotent radical $N_1 = \mathrm{M}_2^{\mathrm{sym}}$.

Let (V, q) be a $2m$-dimensional quadratic F-vector space with orthogonal groups O_V and SO_V and similitude group GO_V with similitude character $\mathrm{sim}_V : \mathrm{GO}_V \to \mathrm{GL}_1$. We will mainly consider the case where $\dim(V) = 2$ or $\dim(V) = 4$. In the case $\dim(V) = 4$ there are exactly two such quadratic spaces; let V^+ be the split 4-dimensional quadratic F-vector space, and let V^- be the non-split 4-dimensional quadratic F-vector space. These can be constructed as follows: let D be a quaternion F-algebra with norm form Nm_D. Then the 4-dimensional quadratic F-vector space $V_D = (D, \mathrm{Nm}_D)$ is isomorphic to V^+ or V^- depending on whether D is split or non-split.

Let $\psi : F \to \mathbb{C}$ be a nontrivial additive character and consider the Weil representation $\omega_{V,W,\psi}$ of $\mathrm{O}_V(F) \times \mathrm{Sp}_4(F)$ on the space $\mathcal{S}(V(F)^2)$ of Schwarz-Bruhat functions on $(V \otimes_F X)(F) = V(F)^2$, where $P_1(F) \times$

$\mathrm{O}_V(F)$ acts by

$$\omega_{V,W,\psi}(h,1)\phi(x) = \phi(h^{-1}x) \qquad h \in \mathrm{O}_V(F)$$

$$\omega_{V,W,\psi}(1,m)\phi(x) = \chi_V(\det(m))|\det(m)|^{\dim(V)/2}\phi(m^{-1}x) \qquad m \in M_1(F)$$

$$\omega_{V,W,\psi}(1,n)\phi(x) = \psi(\langle nx, x\rangle)\phi(x) \qquad n \in N_1(F)$$

where χ_V is the quadratic character associated to $\mathrm{disc}(V) \in F^\times/F^{\times 2}$ and $\langle \cdot, \cdot \rangle$ is the induced symplectic form on $W \otimes_F V$. The Weyl group elements $w \in \mathrm{Sp}_4(F)$ act by Fourier transform.

The Weil representation $\omega_{V,W,\psi}$ of $\mathrm{O}_V(F) \times \mathrm{Sp}_4(F)$ defines a theta correspondence between $\mathrm{O}_V(F)$ and Sp_4 as follows. For $\pi \in \mathrm{Irr}(\mathrm{O}_V(F))$ the maximal π-isotypic subrepresentation of $\omega_{V,W,\psi}$ has the form $\pi \boxtimes \Theta_{V,W,\psi}(\pi)$ for some smooth representation $\Theta_{V,W,\psi}(\pi) \in \mathrm{Irr}(\mathrm{Sp}_4(F))$, the big theta lift of π. The big theta lift $\Theta_{V,W,\psi}(\pi)$ is finite length, hence admissible. Let $\theta_{V,W,\psi}(\pi)$ be the maximal semisimple quotient of $\Theta_{V,W,\psi}(\pi)$, the small theta lift of π. By Howe's conjecture (proved by Waldspurger when F has residual characteristic $p \neq 2$) the small theta lift $\theta_{V,W,\psi}(\pi)$ is irreducible if $\Theta_{V,W,\psi}(\pi)$ is nonzero, and the assignment $\pi \mapsto \theta_{V,W,\psi}(\pi)$ is injective on its domain. By Kudla, $\Theta_{V,W,\psi}(\pi) = \theta_{V,W,\psi}(\pi)$ is irreducible or 0 for π supercuspidal, and if $\Theta_{V,W,\psi}(\pi_1) = \Theta_{V,W,\psi}(\pi_2) \neq 0$ for π_1 and π_2 supercuspidal, then $\pi_1 = \pi_2$.

We consider a similitude version of the above theta correspondence. Consider the subgroup $\mathrm{GSp}_4^+ = \{g \in \mathrm{GSp}_4 | \mathrm{sim}_W(g) \in \mathrm{sim}_V(\mathrm{GO}_V)\}$ of GSp_4; when sim_V is surjective (for example in the case $\dim(V) = 4$) we have $\mathrm{GSp}_4^+ = \mathrm{GSp}_4$. Consider the group $R = \mathrm{GO}_V \times \mathrm{GSp}_4^+$ and the subgroup $R_0 = \{(g,h) \in R | \mathrm{sim}_V(h)\mathrm{sim}_W(g) = 1\}$ of R. The Weil representation $\omega_{V,W,\psi}$ of $\mathrm{O}_V(F) \times \mathrm{Sp}_4(F)$ extends to a representation $\widetilde{\omega}_{V,W,\psi}$ of $R_0(F)$ given by

$$\widetilde{\omega}_{V,W,\psi}(g,h)\phi = |\mathrm{sim}_V(h)|^{\dim(V)/2}\omega_\psi(g_1,1)(\phi \circ h^{-1}) \qquad h \in \mathrm{GO}_V(F), g \in \mathrm{GSp}_4^+(F)$$

where $g_1 = g\mathrm{diag}(\mathrm{sim}_W(g)^{-1}, 1) \in \mathrm{Sp}_4(F)$. The central elements $(t, t^{-1}) \in R_0(F)$ act by the quadratic character $\chi_V(t)^{\dim(V)/2}$. Consider the compact induction $\widetilde{\omega}^+_{V,W,\psi} = \mathrm{cInd}_{R_0(F)}^{R(F)}\widetilde{\omega}_{V,W,\psi}$ which depends only on the orbit of ψ under $\mathrm{sim}_V(\mathrm{GO}_V(F)) \subseteq F^\times$; when sim_V is surjective this is independent of ψ and we simply write $\widetilde{\omega}_{V,W}$ for the resulting representation of $R(F) = \mathrm{GO}_V(F) \times \mathrm{GSp}_4(F)$.

The extended Weil representation $\widetilde{\omega}^+_{V,W,\psi}$ of $R(F) = \mathrm{GO}_V(F) \times \mathrm{GSp}_4^+(F)$ defines a similitude theta correspondence between $\mathrm{GO}_V(F)$ and $\mathrm{GSp}_4^+(F)$ as follows. For $\pi \in \mathrm{Irr}(\mathrm{GO}_V(F))$ the maximal π-isotypic subrepresentation of $\widetilde{\omega}^+_{V,W,\psi}$ has the form $\pi \boxtimes \Theta^+_{V,W,\psi}(\pi)$ for some smooth representation $\Theta^+_{V,W,\psi}(\pi) \in \mathrm{Irr}(\mathrm{GSp}_4^+(F))$, the big similitude theta lift of π. The big similitude theta lift $\Theta^+_{V,W,\psi}(\pi)$ is finite length, hence admissible. Let $\theta^+_{V,W,\psi}(\pi)$ be the maximal semisimple quotient of $\Theta^+_{V,W,\psi}(\pi)$, the small similitide theta lift of π. By a similitude extension of Howe's conjecture (proved by Roberts when F has residual characteristic $p \neq 2$ assuming Howe's conjecture for isometry groups) the small similitude theta lift $\theta^+_{V,W,\psi}(\pi)$ is irreducible whenever $\Theta^+_{V,W,\psi}(\pi)$ is nonzero, and the assignment $\pi \mapsto \theta^+_{V,W,\psi}(\pi)$ is injective on its domain. By Gan-Takeda, $\Theta^+_{V,W,\psi}(\pi) = \theta^+_{V,W,\psi}(\pi)$ is irreducible or 0 for π supercuspidal, and if $\Theta^+_{V,W,\psi}(\pi_1) = \Theta^+_{V,W,\psi}(\pi_2) \neq 0$ for π_1 and π_2 supercuspidal, then $\pi_1 = \pi_2$.

We can extend this from $\mathrm{GSp}_4^+(F)$ to $\mathrm{GSp}_4(F)$ by defining the big similitude theta lift $\Theta_{V,W,\psi}(\pi) = \mathrm{Ind}_{\mathrm{GSp}_4^+(F)}^{\mathrm{GSp}_4(F)} \Theta^+_{V,W,\psi}(\pi)$ and the small similitude theta lift $\theta_{V,W,\psi}(\pi) = \mathrm{Ind}_{\mathrm{GSp}_4^+(F)}^{\mathrm{GSp}_4(F)} \theta^+_{V,W,\psi}(\pi)$.

In the case $\dim(V) = 2$ the local similitude theta lift from $\mathrm{GO}_V(\mathbb{A}_F)$ to $\mathrm{GSp}_4(\mathbb{A}_F)$ will be used in the construction of the Soudry lift, while in the case $\dim(V) = 4$ the local similitude theta lift from $\mathrm{GO}_V(\mathbb{A}_F)$ to $\mathrm{GSp}_4(\mathbb{A}_F)$ will be used in the construction of the Yoshida lift.

Global Similitude Theta Correspondence We continue to follow [110] and [41]. Let F be a number field and let $(W, \langle \cdot, \cdot \rangle)$ be a symplectic F-vector space with symplectic group Sp_W and similitude group GSp_W with similitude character $\mathrm{sim}_W : \mathrm{GSp}_W \to \mathrm{GL}_1$. Let $W = X \oplus Y$ be a Witt decomposition. Again we will only consider the case where $\dim(W) = 4$ so that $\mathrm{Sp}_W(\mathbb{A}_F) = \mathrm{Sp}_4(\mathbb{A}_F)$ and $\mathrm{GSp}_W(\mathbb{A}_F) = \mathrm{GSp}_4(\mathbb{A}_F)$.

Let (V, q) be a $2m$-dimensional quadratic F-vector space of discriminant 1 with orthogonal groups O_V and SO_V and similitude group GO_V with similitude character $\mathrm{sim}_V : \mathrm{GO}_V \to \mathrm{GL}_1$. Again we will mainly consider the case where $\dim(V) = 2$ or $\dim(V) = 4$. In the latter case these can be constructed as follows: let D be a quaternion F-algebra with norm form Nm_D. Then the 4-dimensional quadratic F-vector space $V_D = (D, \mathrm{Nm}_D)$ has discriminant 1, and every 4-dimensional quadratic F-vector space of discriminant 1 is obtained in this way.

Let $\psi = \bigotimes_v \psi_v : F \setminus \mathbb{A}_F \to \mathbb{C}$ be a nontrivial additive character and consider the global Weil representation $\omega_{V,W,\psi} = \bigotimes_{V_v,W_v,\psi_v}$ of $\mathrm{O}_V(\mathbb{A}_F) \times \mathrm{Sp}_4(\mathbb{A}_F)$ on the space $\mathcal{S}(V(\mathbb{A}_F)^2)$ of adelic Schwarz-Bruhat functions on $(V \otimes_F X)(\mathbb{A}_F) = V(\mathbb{A}_F)^2$. We have an $\mathrm{O}_V(\mathbb{A}_F) \times \mathrm{Sp}_4(\mathbb{A}_F)$-equivariant morphism $\theta_{V,W,\psi} : \omega_{V,W,\psi} \to$

$\mathcal{A}(\mathrm{O}_V(\mathbb{A}_F) \times \mathrm{Sp}_4(\mathbb{A}_F))$ given by

$$\theta_{V,W,\psi}(\phi)(h,g) = \sum_{x \in V(F)^n} \omega_{V,W,\psi}(h,g)(\phi)(x)$$

which is absolutely convergent. The functions in $\mathrm{im}(\theta_{V,W,\psi})$ can be used as integration kernels to lift automorphic forms on $\mathrm{O}_V(\mathbb{A}_F)$ to automorphic forms on $\mathrm{Sp}_4(\mathbb{A}_F)$ as before. For an automorphic form $f \in \mathcal{A}(\mathrm{O}_V(\mathbb{A}_F))$ and an adelic Schwarz-Bruhat function $\phi \in \mathcal{S}(V(\mathbb{A}_F)^n)$ consider the automorphic form $\theta_{V,W,\psi}(\phi, f) \in \mathcal{A}(\mathrm{Sp}_4(\mathbb{A}_F))$ given by

$$\theta_{V,W,\psi}(\phi, f)(g) = \int_{\mathrm{O}_V(F)\backslash \mathrm{O}_V(\mathbb{A}_F)} \theta_{V,W,\psi}(\phi)(h,g)\overline{f(h)}\mathrm{d}h$$

where the integral is taken with respect to the Tamagawa measure $\mathrm{d}h$ on $\mathrm{O}_V(\mathbb{A}_F)$, which converges if $f \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{O}_V(\mathbb{A}_F))$. For $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{O}_V(\mathbb{A}_F)$ define the global theta lift

$$\Theta_{V,W,\psi}(\pi) = \langle \theta_{V,W,\psi}(\phi, f) | \phi \in \mathcal{S}(V(\mathbb{A}_F)^n), f \in \pi \rangle \subseteq \mathcal{A}(\mathrm{Sp}_4(\mathbb{A}_F))$$

The automorphic representation $\Theta_{V,W,\psi}(\pi)$, if nonzero, has a unique irreducible quotient isomorphic to $\bigotimes_v \theta_{V_v,W_v,\psi_v}(\pi)$. In particular if $\Theta_{V,W,\psi}(\pi) \subseteq \mathcal{A}_{\mathrm{cusp}}(\mathrm{Sp}_4(\mathbb{A}_F))$ is cuspidal then we have an isomorphism $\Theta_{V,W,\psi}(\pi) \simeq \bigotimes_v \theta_{V_v,W_v,\psi_v}(\pi_v)$.

We consider a similitude version of the above theta correspondence. Consider the global extended Weil representation $\widetilde{\omega}^+_{V,W,\psi} = \bigotimes_v \widetilde{\omega}^+_{V_v,W_v,\psi_v}$ of $R(\mathbb{A}_F) = \mathrm{GO}_V(\mathbb{A}_F) \times \mathrm{GSp}_4^+(\mathbb{A}_F)$ on the space $\mathcal{S}(V(\mathbb{A}_F)^2)$ of adelic Schwarz-Bruhat functions on $(V \otimes_F X)(\mathbb{A}_F) = V(\mathbb{A}_F)^2$; when sim_V is surjective this is independent of ψ and we simply write $\widetilde{\omega}_{V,W} = \bigotimes_v \widetilde{\omega}_{V_v,W_v}$ for the resulting representation of $R(\mathbb{A}_F) = \mathrm{GO}_V(\mathbb{A}_F) \times \mathrm{GSp}_4(\mathbb{A}_F)$. We have a $\mathrm{GO}_V(\mathbb{A}_F) \times \mathrm{GSp}_4^+(\mathbb{A}_F)$-equivariant morphism $\theta_{V,W,\psi} : \widetilde{\omega}^+_{V,W,\psi} \to \mathcal{A}(\mathrm{GO}_V(\mathbb{A}_F) \times \mathrm{GSp}_4^+(\mathbb{A}_F))$ given by

$$\theta_{V,W,\psi}(\phi)(h,g) = \sum_{x \in V(F)^n} \widetilde{\omega}^+_{V,W,\psi}(h,g)(\phi)(x)$$

which is absolutely convergent. The functions in $\mathrm{im}(\theta_{V,W,\psi})$ can be used as integration kernels to lift automorphic forms on $\mathrm{GO}_V(\mathbb{A}_F)$ to automorphic forms on $\mathrm{GSp}_4^+(\mathbb{A}_F)$ as before. For an automorphic form $f \in \mathcal{A}(\mathrm{O}_V(\mathbb{A}_F))$

and an adelic Schwarz-Bruhat function $\phi \in \mathcal{S}(V(\mathbb{A}_F)^n)$ consider the automorphic form $\theta^+_{V,W,\psi}(\phi, f) \in \mathcal{A}(\mathrm{GSp}_4^+(\mathbb{A}_F))$ given by

$$\theta^+_{V,W,\psi}(\phi, f)(g) = \int_{\mathrm{O}_V(F)\backslash \mathrm{O}_V(\mathbb{A}_F)} \theta_{V,W,\psi}(\phi)(h_1 h, g)\overline{f(h_1 h)}\mathrm{d}h_1$$

where $h \in \mathrm{O}_V(\mathbb{A}_F)$ is an element such that $\mathrm{sim}_V(h)\mathrm{sim}_W(g) = 1$ and where the integral is taken with respect to the Tamagawa measure $\mathrm{d}h_1$ on $\mathrm{O}_V(\mathbb{A}_F)$, which converges if $f \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{O}_V(\mathbb{A}_F))$ and is independent of the choice of $h \in \mathrm{O}_V(\mathbb{A}_F)$. We can extend this to an automorphic form $\theta_{V,W,\psi}(\phi, f) \in \mathcal{A}(\mathrm{GSp}_4(\mathbb{A}_F))$ by requiring this to be left $\mathrm{GSp}_4(F)$-invariant and zero outside of $\mathrm{GSp}_4(F)\mathrm{GSp}_4^+(\mathbb{A}_F)$.

For $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{GO}_V(\mathbb{A}_F)$ define the global similitude theta lift

$$\Theta^+_{V,W,\psi}(\pi) = \langle \theta^+_{V,W,\psi}(\phi, f) | \phi \in \mathcal{S}(V(\mathbb{A}_F)^2), f \in \pi \rangle \subseteq \mathcal{A}(\mathrm{GSp}_4^+(\mathbb{A}_F))$$

The automorphic representation $\Theta^+_{V,W,\psi}(\pi)$, if nonzero, has a unique irreducible quotient isomorphic to $\bigotimes_v \theta^+_{V_v,W_v,\psi_v}(\pi_v)$. In particular if $\Theta^+_{V,W,\psi}(\pi) \subseteq \mathcal{A}_{\mathrm{cusp}}(\mathrm{GSp}_4^+(\mathbb{A}_F))$ is cuspidal then we have an isomorphism $\Theta^+_{V,W,\psi}(\pi) \simeq \bigotimes_v \theta^+_{V_v,W_v,\psi_v}(\pi_v)$. We can extend this from $\mathrm{GSp}_4^+(\mathbb{A}_F)$ to $\mathrm{GSp}_4(\mathbb{A}_F)$ by defining the global similitude theta lift

$$\Theta_{V,W,\psi}(\pi) = \langle \theta_{V,W,\psi}(\phi, f) | \phi \in \mathcal{S}(V(\mathbb{A}_F)^2), f \in \pi \rangle \subseteq \mathcal{A}(\mathrm{GSp}_4(\mathbb{A}_F))$$

The automorphic representation $\Theta_{V,W,\psi}(\pi)$, if nonzero, has a unique irreducible quotient isomorphic to $\bigotimes_v \theta_{V_v,W_v,\psi_v}(\pi_v)$. In particular if $\Theta_{V,W,\psi}(\pi) \subseteq \mathcal{A}_{\mathrm{cusp}}(\mathrm{GSp}_4(\mathbb{A}_F))$ is cuspidal then we have an isomorphism $\Theta_{V,W,\psi}(\pi) \simeq \bigotimes_v \theta_{V_v,W_v,\psi_v}(\pi_v)$.

In the case $\dim(V) = 2$ the global similitude theta lift from $\mathrm{GO}_V(\mathbb{A}_F)$ to $\mathrm{GSp}_4(\mathbb{A}_F)$ will be used in the construction of the Soudry lift, while in the case $\dim(V) = 4$ the global similitude theta lift from $\mathrm{GO}_V(\mathbb{A}_F)$ to $\mathrm{GSp}_4(\mathbb{A}_F)$ will be used in the construction of the Yoshida lift.

4.4.4 General Type Cohomology

Consider the general part of the automorphic discrete spectrum $L^2_{\text{disc,gen}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ which by Arthur's classification admits a spectral decomposition

$$L^2_{\text{disc,gen}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc,gen}}(G,\omega)\\(\psi=\mu\boxtimes 1)}} \bigoplus_{\pi\in\Pi(\psi)} \pi$$

The general part $H^*_{\text{disc,gen}}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ of the intersection cohomology $H^*_{\text{disc}}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\text{disc,gen}}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc,gen}}(G,\omega)\\(\psi=\mu\boxtimes 1)}} \bigoplus_{\pi\in\Pi(\psi)} \pi_{\text{fin}}^{K_{\text{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_{\text{fin}} \otimes V_\lambda)$$

as a representation of $\mathcal{H}_{K_{\text{fin}}}$. Similarly, the general part $H^*_{\text{disc,gen}}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)$ of the ℓ-adic intersection cohomology $H^*_{\text{disc}}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\text{disc,gen}}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc,gen}}(G,\omega)\\(\psi=\mu\boxtimes 1)}} \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$$

as a representation of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$.

To determine the structure of the representations $H^*_{\text{disc,1dim}}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)_{\{\pi_{\text{fin}}\}} = \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$ we use the trace formula:

Theorem 4.4.4. Let $V_\lambda \in \text{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$ for a compact open subgroup $K_{\text{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$. Let $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ be a unitary Hecke character and let $\psi = \mu \boxtimes 1 \in \Psi_{\text{disc,gen}}(G,\omega)$ be an A-parameter of general type, where μ is an ω-self-dual unitary cuspidal automorphic representation of $\text{GL}_4(\mathbb{A}_\mathbb{Q})$ of symplectic type.

If $\pi \in \Pi(\psi)$ contributes nontrivially in the intersection cohomology $H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$, then the contribution is given as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$ by

$$H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_\mu$$

Proof. For $\psi = \mu \boxtimes 1 \in \Psi_{\mathrm{disc,gen}}(G, \omega)$ an A-parameter of general type and for $\pi \in \Pi(\psi)$ an automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$ and with π_∞ a cohomological $(\mathfrak{g}, K'_\infty)$-module with central and infinitesimal characters determined by those of V_λ, we have that $\pi_\infty \in \{\pi_\infty^{\mathrm{W}}, \pi_\infty^{\mathrm{H}}\}$ is either the large generic discrete series representation $\pi_\infty^{\mathrm{W}} = \mathcal{D}_{\lambda_1+2,-\lambda_2-1}$ which has nonzero $(\mathfrak{g}, K'_\infty)$-cohomology concentrated in degree 3 with Hodge numbers $(\lambda_1+2, \lambda_2+1)$ and $(\lambda_2+1, \lambda_1+2)$, or the holomorphic discrete series representation $\pi_\infty^{\mathrm{H}} = \mathcal{D}_{\lambda_1+2,\lambda_2+1}$ which has nonzero $(\mathfrak{g}, K'_\infty)$-cohomology concentrated in degree 3 with Hodge numbers $(\lambda_1 + \lambda_2 + 3, 0)$ and $(0, \lambda_1 + \lambda_2 + 3)$. Such a representation π has multiplicity $m(\pi) = 1$: the packet $\Pi(\psi)$ is stable.

Choose a correspondence $f_G^{p,\infty} \in C_c^\infty(K^p \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^p)$ which is a projection onto the A-packet $\Pi(\psi_{\mathrm{fin}})$. Recall that the test function f_∞^G satisfies $\mathrm{tr}(\pi_\infty^+(f_\infty^G)) = \mathrm{tr}(\pi_\infty^-(f_\infty^G)) = -\frac{1}{2}$. Recall that the test function ϕ_j^G satisfies

$$\mathrm{tr}(\Pi(\psi_p)(\phi_j^G)) = p^{\frac{3}{2}j}\Big(c_1(\mu_p)^j + c_2(\mu_p)^j + c_3(\mu_p)^j + c_4(\mu_p)^j\Big)$$

Consider the test function $f^G = \phi_j^G f_G^{p,\infty} f_\infty^G$. Now the contribution of the A-packet $\Pi(\psi)$ to $\mathrm{STF}^G(f^G)$ is given by

$$\begin{aligned}
&\sum_{\pi \in \Pi(\psi)} m(\pi)\mathrm{tr}(\pi_\infty(f_\infty^G))\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})(f_G^{p,\infty}))p^{\frac{3}{2}j}\Big(c_1(\mu_p)^j + c_2(\mu_p)^j + c_3(\mu_p)^j + c_4(\mu_p)^j\Big) \\
&= -p^{\frac{3}{2}j}\Big(c_1(\mu_p)^j + c_2(\mu_p)^j + c_3(\mu_p)^j + c_4(\mu_p)^j\Big)
\end{aligned}$$

By stability of the packet $\Pi(\psi)$ and by matching, only the stable trace formula $\mathrm{STF}^G(f^G)$ for G contributes to $\mathrm{tr}(\mathrm{Frob}_p^j | H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}})$. It follows that we have the trace

$$\mathrm{tr}(\mathrm{Frob}_p^j | H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}}) = p^{\frac{\lambda_1+\lambda_2}{2}}\mathrm{STF}^G(f^G) = -p^{\frac{\lambda_1+\lambda_2+3}{2}j}\Big(c_1(\mu_p)^j + c_2(\mu_p)^j + c_3(\mu_p)^j + c_4(\mu_p)^j\Big)$$

which is the trace of Frob_p^j on ρ_μ whose contribution to $H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degree 3. It follows that

$$H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq \rho_\mu$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$. The result follows. $\square$

Note that the Langlands correspondence predicts that ρ_μ is an irreducible 4-dimensional ℓ-adic Galois representation attached to such a ω-self-dual unitary cuspidal automorphic representation μ of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$, and we have located this irreducible 4-dimensional Galois representation in the intersection cohomology of Siegel threefolds. As we will see in the remaining cases, the ℓ-adic Galois representations appearing in the intersection cohomology of Siegel threefolds will either be of dimension 2 or of dimension 1, and may be concentrated outside of middle degree 3.

4.4.5 Yoshida Lifts, Endoscopic Cohomology

In this section we determine the endoscopic contributions to ℓ-adic intersection cohomology and ℓ-adic inner cohomology.

Yoshida Lifts We consider the Yoshida lift corresponding to A-parameters for $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of the form $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1)$ for μ_1 and μ_2 unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central characters $\omega_{\mu_1} = \omega_{\mu_2}$ corresponding to the above embedding of dual groups which defines a global L-packet $\Pi(\psi)$ of automorphic representations π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with an equality of L-functions and epsilon-factors

$$L(s, \pi) = L(s, \mu_1)L(s, \mu_2)$$
$$\epsilon(s, \pi) = \epsilon(s, \mu_1)\epsilon(s, \mu_2)$$

For $\mu_1 = \bigotimes_v \mu_{1,v}$ and $\mu_2 = \bigotimes_v \mu_{2,v}$ where $\mu_{1,v}$ and $\mu_{2,v}$ are irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_v)$ with central characters $\omega_{\mu_{1,v}} = \omega_{\mu_{2,v}}$ we have $\Pi(\psi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets $\Pi(\psi_v)$ corresponding to the tempered local A-parameter $\psi_v : L_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$ given in terms of the local L-parameters

$\varphi_{\mu_{1,v}} = \left(\begin{smallmatrix} \alpha_{\mu_{1,v}} & \beta_{\mu_{1,v}} \\ \gamma_{\mu_{1,v}} & \delta_{\mu_{1,v}} \end{smallmatrix}\right) : L_{\mathbb{Q}_v} \to \mathrm{GL}_2(\mathbb{C})$ and $\varphi_{\mu_{2,v}} = \left(\begin{smallmatrix} \alpha_{\mu_{2,v}} & \beta_{\mu_{2,v}} \\ \gamma_{\mu_{2,v}} & \delta_{\mu_{2,v}} \end{smallmatrix}\right) : L_{\mathbb{Q}_v} \to \mathrm{GL}_2(\mathbb{C})$ by

$$(w,1) \mapsto \begin{pmatrix} \alpha_{\mu_{1,v}}(w) & & & \beta_{\mu_{1,v}}(w) \\ & \alpha_{\mu_{2,v}}(w) & \beta_{\mu_{2,v}}(w) & \\ & \gamma_{\mu_{2,v}}(w) & \delta_{\mu_{2,v}}(w) & \\ \gamma_{\mu_{1,v}}(w) & & & \delta_{\mu_{1,v}}(w) \end{pmatrix}$$

which are fibered over the characters of the centralizer group $\mathcal{S}_{\psi_v} = S_{\psi_v}/S^0_{\psi_v}Z$ given by

$$\mathcal{S}_{\psi_v} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \mu_{1,v} \text{ and } \mu_{2,v} \text{ discrete series} \\ 0 & \text{otherwise} \end{cases}$$

In other words, we have $\Pi(\psi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets

$$\Pi(\psi_v) = \begin{cases} \{\pi_v^+, \pi_v^-\} & \mu_{1,v} \text{ and } \mu_{2,v} \text{ discrete series} \\ \{\pi_v^+\} & \text{otherwise} \end{cases}$$

The local L-packet $\Pi(\psi_v)$ contains the unique unitary irreducible admissible representation π_v^+ (the basepoint) with local L-parameter $\varphi_{\psi_v} : L_{\mathbb{Q}_v} \to \mathrm{GSp}_4(\mathbb{C})$ given by $w \mapsto \psi(w, \mathrm{diag}(|w|_v^{1/2}, |w|_v^{-1/2}))$.

The global A-packet $\Pi(\psi) = \{\pi = \bigotimes \pi_v | \pi_v \in \Pi(\psi_v), \pi_v = \pi_v^+ \text{ for almost all places } v \text{ of } \mathbb{Q}\}$ contains the automorphic representation $\pi^+ = \bigotimes_v \pi_v^+$ as basepoint. By Arthur's multiplicity formula the Yoshida packets are unstable: for $\pi \in \Pi(\psi)$ we have $m(\pi) = 1$ precisely if $\epsilon(\frac{1}{2}, \pi) = 1$. That is, for $\pi = \bigotimes_v \pi_v \in \Pi(\psi)$ and for Σ the set of places v of $\mathbb{Q}$ such that π_v is non-generic we have

$$m(\pi) = \tfrac{1}{2}(1 + \epsilon(\tfrac{1}{2}, \pi)) = \begin{cases} 1 & \#\Sigma \text{ even} \\ 0 & \#\Sigma \text{ odd} \end{cases}$$

In particular $\Pi(\psi)$ contains 1 discrete element if $\#\Sigma = 0$ and contains $2^{\#\Sigma - 1}$ discrete elements otherwise.

Theta Lifts and Jacquet-Langlands We now explain how the above A-packets can be constructed by theta lifts and the Jacquet-Langlands correspondence. For v a place of $\mathbb{Q}$, for $\omega_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ a smooth character, and for

D_v a quaternion algebra over $\mathbb{Q}_v$ recall that the local Jacquet-Langlands correspondence yields a bijection

$$\left\{\begin{matrix}\text{irreducible smooth representations} \\ \pi'_v \text{ of } D_v^\times \text{ with central character } \omega_v\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{irreducible discrete series representations} \\ \pi_v \text{ of } \mathrm{GL}_2(\mathbb{Q}_v) \text{ with central character } \omega_v\end{matrix}\right\}$$

which is compatible with twists: for $\chi_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ a smooth character, if $\mathrm{JL}(\pi'_v) \simeq \pi_v$ then we have an isomorphism $\mathrm{JL}(\pi'_v \otimes (\chi_v \circ \mathrm{Nm})) \simeq \pi_v \otimes (\chi_v \circ \det)$. For $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a smooth character and for D a quaternion algebra over $\mathbb{Q}$ ramified exactly at a finite set of places Σ of $\mathbb{Q}$ recall that the global Jacquet-Langlands correspondence yields an injection

$$\left\{\begin{matrix}\text{irreducible automorphic representations } \pi' \text{ of } \mathbb{A}_D^\times \\ \text{with dimension } > 1 \text{ and central character } \omega\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{irreducible cuspidal automorphic representations} \\ \pi \text{ of } \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \text{ with central character } \omega\end{matrix}\right\}$$

which is compatible with twists: for $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a smooth character, if $\mathrm{JL}(\pi') \simeq \pi$ then we have an isomorphism $\mathrm{JL}(\pi' \otimes (\chi \circ \mathrm{Nm})) \simeq \pi \otimes (\chi \circ \det)$. The global Jacquet-Langlands correspondence is compatible with the local Jacquet-Langlands correspondence: $\mathrm{JL}(\pi') \simeq \pi$ precisely if $\pi'_v \simeq \pi_v$ for all places $v \notin \Sigma$ (where $D_v^\times \simeq \mathrm{GL}_2(\mathbb{Q}_v)$) and $\mathrm{JL}(\pi'_v) \simeq \pi_v$ for all places $v \in \Sigma$. The image of the global Jacquet-Langlands correspondence consists of those cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that π_v is in the discrete series for all places $v \in \Sigma$.

Let Σ be a finite set of places of $\mathbb{Q}$ with $\#\Sigma$ even and let D be the unique quaternion algebra over $\mathbb{Q}$ which is non-split at every place $v \in \Sigma$ and split at every place $v \notin \Sigma$. We have a short exact sequence

$$0 \to \mathbb{A}_\mathbb{Q}^\times \to \mathbb{A}_D^\times \times \mathbb{A}_D^\times \to \mathrm{GSO}(\mathbb{A}_D) \to 0$$

so that a cuspidal automorphic representation π' of $\mathrm{GSO}(\mathbb{A}_D)$ can be written $\pi' = \mu'_1 \boxtimes \mu'_2$ where μ'_1, μ'_2 are cuspidal automorphic representations of $\mathbb{A}_D^\times$ with $\omega_{\mu'_1} = \omega_{\mu'_2}$. Now we have the global similitude theta correspondence

$$\theta_{\mathrm{GSO}(\mathbb{A}_D)}^{\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})} : \mathrm{Irr}(\mathrm{GSO}(\mathbb{A}_D)) \to \mathrm{Irr}(G(\mathbb{A}_\mathbb{Q})) \cup \{0\}$$

which is given by the induction $\mathrm{Ind}_{\mathrm{GSO}(\mathbb{A}_D)}^{\mathrm{GO}(\mathbb{A}_D)}$ (which is either irreducible or has a unique irreducible constituent) followed by the global similitude theta correspondence

$$\theta_{\mathrm{GO}(\mathbb{A}_D)}^{\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})} : \mathrm{Irr}(\mathrm{GO}(\mathbb{A}_D)) \to \mathrm{Irr}(G(\mathbb{A}_\mathbb{Q})) \cup \{0\}$$

Now for μ_1 and μ_2 cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central characters $\omega_{\mu_1} = \omega_{\mu_2}$ let S be the set of places v of $\mathbb{Q}$ such that $\mu_{1,v}$ and $\mu_{2,v}$ are both in the discrete series. For $\mu_1 = \bigotimes_v \mu_{1,v}$ and $\mu_2 = \bigotimes_v \mu_{2,v}$ where $\mu_{1,v}$ and $\mu_{2,v}$ are irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_v)$ with central characters $\omega_{\mu_{1,v}} = \omega_{\mu_{2,v}}$ and for $\Sigma \subseteq S$ a finite set of places of $\mathbb{Q}$ with $\#\Sigma$ even let D be the unique quaternion algebra over $\mathbb{Q}$ which is non-split at every place $v \in \Sigma$ and split at every place $v \in \Sigma$. Let $\mu_1^D = (\bigotimes_{v\in\Sigma} \mu_{1,v}^{D_v}) \otimes (\bigotimes_{v\notin\Sigma} \mu_{1,v})$ and $\mu_2^D = (\bigotimes_{v\in\Sigma} \mu_{2,v}^{D_v}) \otimes (\bigotimes_{v\notin\Sigma} \mu_{2,v})$ be the irreducible automorphic representations of $\mathbb{A}_D^\times$ obtained from μ_1 and μ_2 by the global Jacquet-Langlands correspondence, where $\mu_{1,v}^{D_v}$ and $\mu_{2,v}^{D_v}$ are the irreducible admissible representations of $D_v^\times$ obtained from $\mu_{1,v}$ and $\mu_{2,v}$ by the local Jacquet-Langlands correspondence. Now by [120, Theorem 5.2, Corollary 5.5], as Σ runs over the subsets of S with $\#\Sigma$ even, the similitude theta lifts of the irreducible automorphic representations $\mu_1^D \boxtimes \mu_2^D = (\bigotimes_{v\in\Sigma} \mu_{1,v}^{D_v} \boxtimes \mu_{2,v}^{D_v}) \otimes (\bigotimes_{v\notin\Sigma} \mu_{1,v} \boxtimes \mu_{2,v})$ from $\mathrm{GSO}(\mathbb{A}_D)$ to $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ run through the discrete automorphic representations in the global A-packet $\Pi(\psi)$ with parameter $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1)$. Write $\pi^\Sigma(\mu_1, \mu_2) = \theta_{\mathrm{GSO}(\mathbb{A}_D)}^{\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})}(\mu_1 \boxtimes \mu_2)$ for the corresponding similitude theta lift. These theta lifts are all distinct except for the equivalence $\pi^\Sigma(\mu_1, \mu_2) \simeq \pi^\Sigma(\mu_2, \mu_1)$, and are cuspidal unless $\mu_1 \simeq \mu_2$.

For $v = p$ a finite place of $\mathbb{Q}$ the similitude theta lift produces irreducible admissible representations $\pi_p^\pm$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$ from the representations $\mu_{1,p} \boxtimes \mu_{2,p}$ of $\mathrm{GSO}_{2,2}(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_1(\mathbb{Q}_p)$. The local L-packets $\Pi(\psi_p)$ for the local A-parameter $\psi_p = (\mu_{1,p} \boxtimes 1) \boxplus (\mu_{2,p} \boxtimes 1)$ are given by $\Pi(\psi_p) = \{\pi_p^+, \pi_p^-\}$ when $\mu_{1,p}$ and $\mu_{2,p}$ are both in the discrete series, and otherwise by $\Pi(\psi_p) = \{\pi_p^+\}$ where the $\pi_p^\pm$ are defined as follows. The unique generic member π_p^+ of the local L-packet is given by the nonzero irreducible local similitude theta lift $\pi_p^+ = \theta_{\mathrm{GSO}_{2,2}(\mathbb{Q}_p)}^{\mathrm{GSp}_4(\mathbb{Q}_p)}(\mu_{1,p} \boxtimes \mu_{2,p})$. If $\mu_{1,p}$ and $\mu_{2,p}$ are both in the discrete series, let D be the non-split quaternion algebra over $\mathbb{Q}_p$ and let $\mu_{1,p}^D$ and $\mu_{2,p}^D$ be the irreducible admissible representations of $D^\times$ obtained by the local Jacquet-Langlands correspondence. The unique non-generic member π_p^- of the local L-packet is given by the nonzero irreducible local similitude theta lift $\pi_p^- = \theta_{\mathrm{GSO}(D)}^{\mathrm{GSp}_4(\mathbb{Q}_p)}(\mu_{1,p}^D \boxtimes \mu_{2,p}^D)$. These similitude theta lifts have been computed (for instance in [**?** , Theorem 8.1, Theorem 8.2]). We obtain the following description for the members of the local L-packets $\Pi(\psi_p)$ of Yoshida type, along with their corresponding epsilon-values (compare to [**?** , Table 1, Table 2, Table 3]):

Type		$\{\mu_{1,p}, \mu_{2,p}\}$	$\Pi^{\pm}(\mu_{1,p}, \mu_{2,p})$	ϵ
I		$\{\chi'_{1,p} \times \chi'_{2,p}, \chi_{1,p} \times \chi_{2,p}\}$	$\chi'_{1,p}\chi_{1,p}^{-1} \times \chi'_{2,p}\chi_{1,p}^{-1} \rtimes \chi_{1,p}$	1
II	a	$\{\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_{1,p} \times \chi_{2,p}\}$	$\chi_p\chi_{1,p}^{-1}\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \rtimes \chi_{1,p}$	1
V	a	$\{\chi_p\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta([\xi, \xi\nu] \rtimes \chi_p\nu^{-1/2})$	1
	a*		$\delta^*([\xi, \xi\nu] \rtimes \chi_p\nu^{-1/2})$	-1
VI	a	$\{\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\tau(S, \chi_p\nu^{-1/2})$	1
	b		$\tau(T, \chi_p\nu^{-1/2})$	-1
VIII	a	$\{\mu_{1,p}, \mu_{2,p}\}, \mu_{1,p} = \mu_{2,p}$	$\tau(S, \mu_{1,p})$	1
	b		$\tau(T, \mu_{1,p})$	-1
		$\{\mu_{1,p}, \mu_{2,p}\}, \mu_{1,p} \neq \mu_{2,p}$	$\theta^+(\mu_{1,p}, \mu_{2,p})$	1
			$\theta^-(\mu_{1,p}, \mu_{2,p})$	-1
X		$\{\mu_{1,p}, \chi_{1,p} \times \chi_{2,p}\}$	$\chi_{1,p}^{-1}\mu_{1,p} \rtimes \chi_{1,p}$	1
XI	a	$\{\mu_{1,p}, \chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta(\mu_{1,p}\nu \rtimes \chi_p\nu)$	1
	a*		$\delta^*(\mu_{1,p}\nu \rtimes \chi_p\nu)$	-1

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ the similitude theta lift produces irreducible admissible representations $\pi_\infty^{\pm}$ of $\mathrm{GSp}_4(\mathbb{R})$ from the representations $\mu_{1,\infty} \boxtimes \mu_{2,\infty}$ of $\mathrm{GSO}_{2,2}(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(\mathbb{R})$. The local L-packets $\Pi(\psi_\infty)$ for the local A-parameter $\psi_\infty = (\mu_{1,\infty} \boxtimes 1) \boxplus (\mu_{2,\infty} \boxtimes 1)$ are given by $\Pi(\psi_\infty) = \{\pi_\infty^+, \pi_\infty^-\}$ when $\mu_{1,\infty}$ and $\mu_{2,\infty}$ are both in the discrete series, and otherwise by $\Pi(\psi_\infty) = \{\pi_\infty^+\}$ where the $\pi_\infty^{\pm}$ are defined as follows. The unique generic member π_∞^+ of the local L-packet is given by the nonzero irreducible local similitude theta lift $\pi_\infty^+ = \theta_{\mathrm{GSO}_{2,2}(\mathbb{R})}^{\mathrm{GSp}_4(\mathbb{R})}(\mu_{1,\infty} \boxtimes \mu_{2,\infty})$. If $\mu_{1,\infty}$ and $\mu_{2,\infty}$ are both holomorphic discrete series representations, let $\mathbb{H}$ be the non-split quaternion algebra over $\mathbb{R}$ and let $\mu_{1,\infty}^{\mathbb{H}}$ and $\mu_{2,\infty}^{\mathbb{H}}$ be the irreducible admissible representations of $\mathbb{H}^\times$ obtained by the local Jacquet-Langlands correspondence. The unique non-generic member π_∞^- of the local L-packet is given by the nonzero irreducible local similitude theta lift $\pi_\infty^- = \theta_{\mathrm{GSO}(\mathbb{H})}^{\mathrm{GSp}_4(\mathbb{R})}(\mu_{1,\infty}^{\mathbb{H}} \boxtimes \mu_{2,\infty}^{\mathbb{H}})$. We obtain the following description for the members of the local L-packets $\Pi(\psi_\infty)$ of Yoshida type, along with their corresponding epsilon-values:

Type	$\{\mu_{1,\infty}, \mu_{2,\infty}\}$	$\pi^{\pm}(\mu_{1,\infty}, \mu_{2,\infty})$	$\epsilon(\pi_\infty)$
	$\{\chi'_{1,\infty} \times \chi'_{2,p}, \chi_{1,p} \times \chi_{2,p}\}$	$\chi'_{1,p}\chi_{1,p}^{-1} \times \chi'_{2,p}\chi_{1,p}^{-1} \rtimes \chi_{1,p}$	1
	$\{\chi_\infty \mathcal{D}_k, \chi_{1,\infty} \times \chi_{2,\infty}\}$	$\chi_\infty \chi_{1,\infty}^{-1} \mathcal{D}_k \rtimes \chi_{1,\infty}$	1
$(\lambda_1 + 3, -\lambda_2 - 1)$	$\{\chi_\infty \mathcal{D}_{\lambda_1+\lambda_2+3}, \chi_\infty \mathcal{D}_{\lambda_1-\lambda_2+1}\}$	$\chi_\infty \mathcal{D}_{\lambda_1+2,-\lambda_2-1}$	1
$(\lambda_1 + 3, \lambda_2 + 3)$		$\chi_\infty \mathcal{D}_{\lambda_1+2,\lambda_2+1}$	-1

Parahoric Restriction We now review results about the parahoric restriction of endoscopic lifts for GSp_4, following [100, Section 4.2]. Let $\chi_{1,p}, \chi_{2,p}, \chi'_{1,p}, \chi'_{2,p}, \chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be at most tamely ramified characters which restrict to nonzero characters $\chi_{1,p}, \chi_{2,p}, \chi'_{1,p}, \chi_{2,p}, \chi_p : \mathbb{F}_p^\times \to \mathbb{C}^\times$ which we abusively denote by the same notation. Let $\mu_{1,p}, \mu_{2,p}$ be depth 0 supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with the same central characters, with hyperspecial parahoric restrictions $\pi_{\Lambda_1}, \pi_{\Lambda_2}$ for characters $\Lambda_1, \Lambda_2 : \mathbb{F}_{p^2}^\times \to \mathbb{C}^\times$ with $\Lambda_1|_{\mathbb{F}_p^\times} = \Lambda_2|_{\mathbb{F}_p^\times}$; such a character Λ_i with $\Lambda_i|_{\mathbb{F}_p^\times} = 1$ factors over a character $\omega_{\Lambda_i} : \mathbb{F}_p^\times[p+1] \to \mathbb{C}^\times$ with $\Lambda_i(x) = \omega_{\Lambda_i}(x^{p-1})$. Let $\xi_u : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be the nontrivial unramified quadratic character and for $p > 2$ let $\xi_t : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be either one of the tamely ramified quadratic characters. For $p = 2$ let $\theta : \mathbb{F}_4^\times \to \mathbb{C}^\times$ be a primitive character, let $l_i \in \mathbb{Z}/3\mathbb{Z}$ such that $\theta^{l_i} = \Lambda_i$, and let $k_\pm = \frac{1}{2}(l_1 \pm l_2)$. By [100, Theorem 4.7] the hyperspecial parahoric restriction $r_{\mathscr{K}_p}(\Pi^+(\mu_{1,p}, \mu_{2,p}))$ is given as follows (compare to [100, Table 4.2]):

Type	$\{\mu_{1,p}, \mu_{2,p}\}$	$\Pi^+(\mu_{1,p}, \mu_{2,p})$	$r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2}, \mu_{2,2}))$	$r_{\mathscr{K}_p}(\Pi^+(\mu_{1,p}, \mu_{2,p}))$	Dimension
I	$\{\chi'_{1,p} \times \chi'_{2,p}, \chi_{1,p} \times \chi_{2,p}\}$	$\chi'_{1,p}\chi_{1,p}^{-1} \times \chi'_{2,p}\chi_{1,p}^{-1} \rtimes \chi_{1,p}$	X_1	$X_1(\chi'_{1,p}/\chi_{1,p}, \chi'_{2,p}/\chi_{1,p}, \chi'_{1,p})$	$(p+1)^2(p^2+1)$
IIa	$\{\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_{1,p} \times \chi_{2,p}\}$	$\chi_p \chi_{1,p}^{-1} \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \rtimes \chi_{1,p}$	χ_4	$\chi_4(\chi_p/\chi_{1,p}, \chi_{1,p})$	$p(p+1)(p^2+1)$
Va	$\{\chi_p \xi_u \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta([\xi_u, \xi_u \nu] \rtimes \chi_p \nu^{-1/2})$	$\theta_4 \oplus \theta_5$	$\theta_4(\chi_p) \oplus \theta_5(\chi_p)$	$p^4 + p(p^2+1)/2$
	$\{\chi_p \xi_t \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta([\xi_t, \xi_t \nu] \rtimes \chi_p \nu^{-1/2})$		$\tau_3(\chi_p)$	$p^2(p^2+1)$
VIa	$\{\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}, \chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\tau(S, \chi_p \nu^{-1/2})$	$\theta_1 \oplus \theta_5$	$\theta_1(\chi_p) \oplus \theta_5(\chi_p)$	$p^4 + p(p+1)^2/2$
VIIIa	$\{\pi_p, \pi_p\}$	$\tau(S, \pi_p)$	$\chi_8(l_1)$	$\chi_8(\Lambda_1)$	$p(p-1)(p^2+1)$
	$\{\pi_{1,p}, \pi_{2,p}\}, \pi_{1,p} \neq \pi_{2,p}$	$\theta^+(\pi_{1,p}, \pi_{2,p})$	$X_5(k_+, k_-)$	$X_5(\Lambda_1, \omega_{\Lambda_2/\Lambda_1})$	$(p-1)^2(p^2+1)$
X	$\{\chi_{1,p}\pi_p, \chi_{1,p} \times \chi_{2,p}\}$	$\chi_{1,p}^{-1}\pi_p \rtimes \chi_{1,p}$	$X_2(l_1)$	$X_2(\Lambda_1, \chi_{1,p})$	$p^4 - 1$
XIa	$\{\chi_p \pi_p, \chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta(\chi_p^{-1}\pi_p \nu^{1/2} \rtimes \chi_p \nu^{-1/2})$	$\chi_6(l_1)$	$\chi_6(\omega_{\Lambda_1}, \chi_p)$	$p(p-1)(p^2+1)$

and the hyperspecial parahoric restriction $r_{\mathscr{K}_p}(\Pi^-(\mu_{1,p}, \mu_{2,p}))$ is given as follows (compare to [100, Table 4.2]):

Type	$\{\mu_{1,p},\mu_{2,p}\}$	$\Pi^-(\mu_{1,p},\mu_{2,p})$	$r_{\mathscr{K}_2}(\Pi^-(\mu_{1,2},\mu_{2,2}))$	$r_{\mathscr{K}_p}(\Pi^-(\mu_{1,p},\mu_{2,p}))$	Dimension
Va*	$\{\chi_p\xi_u\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta^*([\xi_u,\xi_u\nu]\rtimes\chi_p\nu^{-1/2})$	θ_2	$\theta_2(\chi_p)$	$p(p-1)^2/2$
	$\{\chi_p\xi_t\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta^*([\xi_t,\xi_t\nu]\rtimes\chi_p\nu^{-1/2})$		0	0
VIb	$\{\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\tau(T,\chi_p\nu^{-1/2})$	θ_3	$\theta_3(\chi_p)$	$p(p^2+1)/2$
VIIIb	$\{\pi_p,\pi_p\}$	$\tau(T,\pi_p)$	$\chi_7(l_1)$	$\chi_7(\Lambda_1)$	$(p-1)(p^2+1)$
	$\{\pi_{1,p},\pi_{2,p}\},\pi_{1,p}\neq\pi_{2,p}$	$\theta^-(\pi_{1,p},\pi_{2,p})$	0	0	0
XIa*	$\{\chi_p\pi_p,\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta^*(\chi_p^{-1}\pi_p\nu^{1/2}\rtimes\chi_p\nu^{-1/2})$	0	0	0

Let $\lambda_0:\mathbb{F}_p^\times\to\mathbb{C}^\times$ and $\Lambda_0:\mathbb{F}_{p^2}^\times\to\mathbb{C}^\times$ be the nontrivial quadratic characters. For $\Lambda:\mathbb{F}_{p^2}^\times\to\mathbb{C}^\times$ a character with $\Lambda^{p+1}=1_{\mathrm{GL}_1(\mathbb{F}_p)}$ let $\Lambda':\mathbb{F}_{p^2}^\times\to\mathbb{C}^\times$ be a character such that $\Lambda'^{p-1}=\Lambda$. Let $\Lambda_0':\mathbb{F}_{p^2}^\times\to\mathbb{C}^\times$ be a character such that $\Lambda_0'^{p-1}=\Lambda_0$. By [100, Theorem 4.11] the paramodular restriction $r_{\mathscr{K}_p^{P_2}}(\Pi^+(\mu_{1,p},\mu_{2,p}))$ is given as follows (compare to [100, Table 4.4]):

Type	$\{\mu_{1,p},\mu_{2,p}\}$	$\Pi^+(\mu_{1,p},\mu_{2,p})$	$r_{\mathscr{K}_p^{P_2}}(\Pi^+(\mu_{1,p},\mu_{2,p}))$	Dimension
I	$\{\chi_{1,p}'\times\chi_{2,p}',\chi_{1,p}\times\chi_{2,p}\}$	$\chi_{1,p}'\chi_{1,p}^{-1}\times\chi_{2,p}'\chi_{1,p}^{-1}\rtimes\chi_{1,p}$	$\chi_{1,p}^{-1}[\chi_{1,p}\times\chi_{1,p}',\chi_{1,p}\times\chi_{2,p}']+\chi_{1,p}^{-1}[\chi_{1,p}\times\chi_{2,p}',\chi_{1,p}\times\chi_{1,p}']$	$2(p+1)^2$
IIa	$\{\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_{1,p}\times\chi_{2,p}\}$	$\chi_p\chi_{1,p}^{-1}\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\rtimes\chi_{1,p}$	$\chi_{1,p}^{-1}[\chi_{1,p}\times\chi_p,\chi_{1,p}\times\chi_p]$	$(p+1)^2$
Va	$\{\chi_p\xi_u\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta([\xi_u,\xi_u\nu]\rtimes\chi_p\nu^{-1/2})$	$\chi_p[1_{\mathrm{GL}_2(\mathbb{F}_p)},\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]+\chi_p[\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)},1_{\mathrm{GL}_2(\mathbb{F}_p)}]$	$2p$
	$\{\chi_p\xi_t\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta([\xi_t,\xi_t\nu]\rtimes\chi_p\nu^{-1/2})$	$\chi_p[1_{\mathrm{GL}_1(\mathbb{F}_p)}\times\lambda_0,1_{\mathrm{GL}_1(\mathbb{F}_p)}\times\lambda_0]_\pm$	$(p+1)^2/2$
VIa	$\{\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\tau(S,\chi_p\nu^{-1/2})$	$\chi_p[1_{\mathrm{GL}_2(\mathbb{F}_p)},\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]+\chi_p[\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)},1_{\mathrm{GL}_2(\mathbb{F}_p)}]$ $+\chi_p[\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)},\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]$	$2p$
VIIIa	$\{\pi_p,\pi_p\}$	$\tau(S,\pi_p)$	$[\pi_{\Lambda_1},1_{\mathrm{GL}_2(\mathbb{F}_p)}]+[1_{\mathrm{GL}_2(\mathbb{F}_p)},\pi_{\Lambda_1}]$	$2(p-1)$
	$\{\pi_{1,p},\pi_{2,p}\},\pi_{1,p}\neq\pi_{2,p}$	$\theta^+(\pi_{1,p},\pi_{2,p})$	0	0
X	$\{\chi_{1,p}\pi_p,\chi_{1,p}\times\chi_{2,p}\}$	$\chi_{1,p}^{-1}\pi_p\rtimes\chi_{1,p}$	0	0
XIa	$\{\chi_p\pi_p,\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta(\chi_p^{-1}\pi_p\nu^{1/2}\rtimes\chi_p\nu^{-1/2})$	0	0

and the paramodular restriction $r_{\mathscr{K}_p^{P_2}}(\Pi^-(\mu_{1,p},\mu_{2,p}))$ is given as follows (compare to [100, Table 4.4]):

Type	$\{\mu_{1,p},\mu_{2,p}\}$	$\Pi^-(\mu_{1,p},\mu_{2,p})$	$r_{\mathscr{K}_p^{P_2}}(\Pi^-(\mu_{1,p},\mu_{2,p}))$	Dimension
Va*	$\{\chi_p\xi_u\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta^*([\xi_u,\xi_u\nu]\rtimes\chi_p\nu^{-1/2})$	0	0
	$\{\chi_p\xi_t\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta^*([\xi_t,\xi_t\nu]\rtimes\chi_p\nu^{-1/2})$	$\chi_p[\pi_{\Lambda_0'},\pi_{\Lambda_0'^{-1}}]_\pm$	$(p-1)^2/2$
VIb	$\{\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)},\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\tau(T,\chi_p\nu^{-1/2})$	$\chi_p[\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)},\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]$	p^2
VIIIb	$\{\pi_p,\pi_p\}$	$\tau(T,\pi_p)$	$[\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)},\pi_{\Lambda_1}]+[\pi_{\Lambda_1},\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]$	$2p(p-1)$
	$\{\pi_{1,p},\pi_{2,p}\},\pi_{1,p}\neq\pi_{2,p}$	$\theta^-(\pi_{1,p},\pi_{2,p})$	$[\pi_{\Lambda_a},\pi_{\Lambda_b}]+[\pi_{\Lambda_b},\pi_{\Lambda_a}]$	$2(p-1)^2$
XIa*	$\{\chi_p\pi_p,\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\}$	$\delta^*(\chi_p^{-1}\pi_p\nu^{1/2}\rtimes\chi_p\nu^{-1/2})$	$\chi_p[\pi_{\Lambda_1'},\pi_{\Lambda_1'^{-1}}]$	$(p-1)^2$

where (Λ_a,Λ_b) is any pair of characters with $\Lambda_a\Lambda_b=\Lambda_1$ and $\Lambda_a\Lambda_b^p=\Lambda_2$, and where the sign is given by $\xi_t(p)=\pm1$.

Endoscopic Cohomology Consider the endoscopic part of the automorphic discrete spectrum which by Arthur's classification admits a spectral decomposition

$$L^2_{\text{disc,endo}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc,endo}}(G,\omega)\\ (\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} \pi$$

The endoscopic part $H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ of the intersection cohomology $H^*_{\text{disc}}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc,endo}}(G,\omega)\\ (\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} \pi_{\text{fin}}^{K_{\text{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

as a representation of $\mathcal{H}_{K_{\text{fin}}}$. Similarly, the endoscopic part $H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ of the ℓ-adic intersection cohomology $H^*_{\text{disc}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc,endo}}(G,\omega)\\ (\psi=(\mu_1\boxtimes 1)\boxplus(\mu_2\boxtimes 1))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$$

as a representation of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$.

To determine the structure of the representations $H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\text{fin}}\}} = \pi^{K_{\text{fin}}} \boxtimes \rho_\pi$ we use the trace formula:

Proposition 4.4.5. Let $V_\lambda \in \text{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$ for a compact open subgroup $K_{\text{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$. Let $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1) \in \Psi_{\text{disc,endo}}(G, \omega)$ be an A-parameter of Yoshida type, where μ_1 and μ_2 are unitary cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ with central characters $\omega_{\mu_1} = \omega_{\mu_2} = \omega$.

If $\pi \in \Pi(\psi)$ contributes nontrivially in the intersection cohomology $H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$, then the contribution is given as a representation of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$ by

$$H^*_{\text{disc,endo}}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\text{fin}}\}} \simeq \begin{cases} \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_{\mu_2}\mathbb{L}^{\lambda_2+1} & \pi_\infty \simeq \pi_\infty^{\text{W}} \\ \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_{\mu_1} & \pi_\infty \simeq \pi_\infty^{\text{H}} \end{cases}$$

Proof. For $\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1) \in \Psi_{\mathrm{disc,endo}}(G, \omega)$ an A-parameter of Yoshida type and for $\pi \in \Pi(\psi)$ an automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ lifted from the automorphic representation $\pi^H = \mu_1 \boxtimes \mu_2$ of $H(\mathbb{A}_\mathbb{Q})$ with $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$ and with π_∞ a cohomological $(\mathfrak{g}, K'_\infty)$-module with central and infinitesimal characters determined by those of V_λ, we have that $\pi_\infty \in \{\pi_\infty^{\mathrm{W}}, \pi_\infty^{\mathrm{H}}\}$ is either the large generic discrete series representation $\pi_\infty^{\mathrm{W}} = \mathcal{D}_{\lambda_1+2,-\lambda_2-1}$ which has nonzero $(\mathfrak{g}, K'_\infty)$-cohomology concentrated in degree 3 with Hodge numbers $(\lambda_1 + 2, \lambda_2+1)$ and $(\lambda_2+1, \lambda_1+2)$, or the holomorphic discrete series representation $\pi_\infty^{\mathrm{H}} = \mathcal{D}_{\lambda_1+2,\lambda_2+1}$ which has nonzero $(\mathfrak{g}, K'_\infty)$-cohomology concentrated in degree 3 with Hodge numbers $(\lambda_1+\lambda_2+3, 0)$ and $(0, \lambda_1+\lambda_2+3)$. Let $\pi_\infty^+ = \pi_\infty^{\mathrm{W}}$ and let $\pi_\infty^- = \pi_\infty^{\mathrm{H}}$. Let π_p^+ be the basepoint of the local A-packet $\Pi(\psi_p)$, and let π_p^- be the cuspidal member of the local A-packet $\Pi(\psi_p)$. Each component π_v of $\pi = \bigotimes_v \pi_v$ has a sign

$$\langle \pi_v^H, \pi_v \rangle = \begin{cases} 1 & \pi_v = \pi_v^+ \\ -1 & \pi_v = \pi_v^- \end{cases} \qquad \langle \pi_{\mathrm{fin}}^H, \pi_{\mathrm{fin}} \rangle = \prod_p \langle \mu_p, \pi_p \rangle$$

In terms of these signs, π has multiplicity

$$m(\pi) = \frac{1}{2}\Big(1 + \prod_v \langle \mu_v, \pi_v \rangle\Big) = \begin{cases} 1 & \langle \pi_{\mathrm{fin}}^H, \pi_{\mathrm{fin}} \rangle = 1 \text{ and } \pi_\infty = \pi_\infty^+ \\ & \text{ or } \langle \pi_{\mathrm{fin}}^H, \pi_{\mathrm{fin}} \rangle = -1 \text{ and } \pi_\infty = \pi_\infty^- \\ 0 & \text{otherwise} \end{cases}$$

Choose a correspondence $f_G^{p,\infty} \in C_c^\infty(K^p \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^p)$ which is a projection onto the A-packet $\Pi(\psi_{\mathrm{fin}})$, and choose a matching correspondence $f_H^{p,\infty}$ which is a projection onto π_{fin}^H. Consider the sets

$$\Pi(\psi_{\mathrm{fin}})^+ = \{\pi \in \Pi(\psi_{\mathrm{fin}}) | \langle \pi_{\mathrm{fin}}^H, \pi_{\mathrm{fin}} \rangle = 1\}$$
$$\Pi(\psi_{\mathrm{fin}})^- = \{\pi \in \Pi(\psi_{\mathrm{fin}}) | \langle \pi_{\mathrm{fin}}^H, \pi_{\mathrm{fin}} \rangle = -1\}$$

By matching and endoscopic character identities we have

$$\mathrm{tr}(\pi_{\mathrm{fin}}^H(f_H^{p,\infty})) = \mathrm{tr}(\Pi(\pi_{\mathrm{fin}})^+(f_G^{p,\infty})) - \mathrm{tr}(\Pi(\pi_{\mathrm{fin}})^-(f_G^{p,\infty}))$$

Recall that the test function f_∞^G satisfies $\mathrm{tr}(\pi_\infty^+(f_\infty^G)) = \mathrm{tr}(\pi_\infty^-(f_\infty^G)) = -\frac{1}{2}$, and the matching test function f_∞^H satisfies $\mathrm{tr}(\pi_\infty^H(f_\infty^H)) = -1$. Recall that the test function ϕ_j^G satisfies

$$\mathrm{tr}(\Pi(\psi_p)(\phi_p^G)) = p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j + c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)$$

and the matching test function ϕ_j^H satisfies

$$\mathrm{tr}(\pi_p^H(\phi_j^H)) = p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j - c_1(\mu_{2,p})^j - c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)$$

Consider the test functions $f^G = \phi_j^G f_G^{p,\infty} f_\infty^G$ and the matching test function $f^H = \phi_j^H f_H^{p,\infty} f_\infty^H$. Now the contribution of the A-packet $\Pi(\psi)$ to $\mathrm{STF}^G(f^G)$ is given by

$$\begin{aligned}
&\sum_{\pi\in\Pi(\psi)} m(\pi)\mathrm{tr}(\pi_\infty(f_\infty^G))\mathrm{tr}(\pi_{\mathrm{fin}}(f_G^{p,\infty}))p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j + c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)\\
&= \frac{1}{2}\Big(\mathrm{tr}(\pi_\infty^+(f_\infty^G)) + \mathrm{tr}(\pi_\infty^-(f_\infty^G))\Big)\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^+(f_G^{p,\infty})) + \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j + c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)\\
&= -\frac{1}{2}\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^+(f_G^{p,\infty})) + \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j + c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)
\end{aligned}$$

and the contribution of π^H to $\mathrm{STF}^H(f^H)$ is given by

$$\begin{aligned}
&2m(\pi^H)\mathrm{tr}(\pi_\infty^H(f_\infty^H))\mathrm{tr}(\pi_{\mathrm{fin}}^H(f_H^{p,\infty}))p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j - c_1(\mu_{2,p})^j - c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)\\
&= -2\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^+(f_G^{p,\infty})) - \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{3}{2}j}\Big(c_1(\mu_{1,p})^j - c_1(\mu_{2,p})^j - c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)
\end{aligned}$$

Then we have the trace

$$\begin{aligned}
&\mathrm{tr}(\mathrm{Frob}_p^j|H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}}) = p^{\frac{\lambda_1+\lambda_2}{2}}\Big(\mathrm{STF}^G(f^G) - \frac{1}{4}\mathrm{STF}^H(f^H)\Big)\\
&= -\frac{1}{2}\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^+(f_G^{p,\infty})) + \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{\lambda_1+\lambda_2+3}{2}j}\Big(c_1(\mu_{1,p})^j + c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)\\
&+\frac{1}{2}\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^+(f_G^{p,\infty})) - \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{\lambda_1+\lambda_2+3}{2}j}\Big(c_1(\mu_{1,p})^j - c_1(\mu_{2,p})^j - c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j\Big)\\
&= \frac{1}{2}p^{\frac{\lambda_1+\lambda_2+3}{2}j}\begin{pmatrix} -(c_1(\mu_{1,p})^j + c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j) \\ +\langle\mu_{\mathrm{fin}},\pi_{\mathrm{fin}}\rangle(c_1(\mu_{1,p})^j - c_1(\mu_{2,p})^j - c_2(\mu_{2,p})^j + c_2(\mu_{1,p})^j)\end{pmatrix}\\
&= \begin{cases} -p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j) & \langle\pi_{\mathrm{fin}}^H,\pi_{\mathrm{fin}}\rangle = 1\\ -p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_{1,p})^j + c_2(\mu_{1,p})^j) & \langle\pi_{\mathrm{fin}}^H,\pi_{\mathrm{fin}}\rangle = -1\end{cases}
\end{aligned}$$

We can write $-p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j) = -p^{(\lambda_2+1)j}p^{\frac{\lambda_1-\lambda_2+1}{2}j}(c_1(\mu_{2,p})^j + c_2(\mu_{2,p})^j)$, which is the trace of Frob_p^j on $\rho_{\mu_2}\mathbb{L}^{\lambda_2+1}$ whose contribution to $H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)$ is concentrated in degree 3. Similarly

$-p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_{1,p})^j + c_2(\mu_{1,p})^j)$ is the trace of Frob_p^j on ρ_{μ_1} whose contribution to $H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degree 3. It follows that

$$H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq \begin{cases} \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_{\mu_2}\mathbb{L}^{\lambda_2+1} & \pi_\infty \simeq \pi_\infty^{\mathrm{W}} \\ \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_{\mu_1} & \pi_\infty \simeq \pi_\infty^{\mathrm{H}} \end{cases}$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$. The result follows. $\square$

At this point we specialize the discussion to the case of square-free parahoric level structure.

Theorem 4.4.6. (compare to [100, Theorem 5.8]) Let S be a finite set of places of $\mathbb{Q}$ including ∞, and let $S_{\mathrm{fin}} = S - \{\infty\}$. Let $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^\infty)$ be a compact open subgroup of the form $K_{\mathrm{fin}} = \prod_{p \in S_{\mathrm{fin}}} \mathscr{P}_p \times \prod_{p \notin S_{\mathrm{fin}}} G(\mathbb{Z}_p)$ where $\mathscr{P}_p \subseteq G(\mathbb{Q}_p)$ is a standard parahoric subgroup. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$. Then the endoscopic cohomology $H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees 3 and is given as a representation of $\prod_{p \in S_{\mathrm{fin}}} \mathscr{P}_p / \mathscr{P}_p^+ \times \mathrm{Gal}$ by

$$H^3_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\mu_1,\mu_2} \bigoplus_{\Sigma \subseteq S} \Big(\boxtimes_{p \in S_{\mathrm{fin}}} r_{\mathscr{P}_p}(\pi^\Sigma(\mu_{1,p}, \mu_{2,p})) \Big) \boxtimes \begin{cases} \rho_{\mu_2}\mathbb{L}^{\lambda_2+1} & \infty \notin \Sigma \\ \rho_{\mu_1} & \infty \in \Sigma \end{cases}$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ unramified outside S with $\omega_\infty \in \{1, \mathrm{sign}\}$ and with ω_p tamely ramified for every place $p \in S_{\mathrm{fin}}$, where the second direct sum is taken over unitary cuspidal automorphic representations μ_1 and μ_2 of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central character $\omega_{\mu_1} = \omega_{\mu_2} = \omega$ unramified outside S with $\mu_{1,\infty} = \mathcal{D}_{\lambda_1+\lambda_2+3}$ and $\mu_{2,\infty} = \mathcal{D}_{\lambda_1-\lambda_2+1}$ and with $\mu_{1,p}$ and $\mu_{2,p}$ of depth 0 for every place $p \in S_{\mathrm{fin}}$, and where the third direct sum is taken over subsets $\Sigma \subseteq S$ such that $(-1)^{\#\Sigma} = 1$.

Proof. We have the spectral decomposition

$$H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc,endo}}(G,\omega) \\ (\psi = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1))}} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = 1}} \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, and where the second direct sum is taken over unitary cuspidal automorphic representations μ_1 and μ_2 of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_{\mu_1} = \omega_{\mu_2} = \omega$ with $\mu_{1,\infty} = \mathcal{D}_{\lambda_1+\lambda_2+3}$ and $\mu_{2,\infty} = \mathcal{D}_{\lambda_1-\lambda_2+1}$. Equivalently, we have

$$H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\mu_1,\mu_2} \bigoplus_{\Sigma \subseteq S} \pi^\Sigma(\mu_{1,\mathrm{fin}}, \mu_{2,\mathrm{fin}})^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi^\Sigma(\mu_{1,\infty}, \mu_{2,\infty}) \otimes V_\lambda)$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, where the second direct sum is taken over unitary cuspidal automorphic representations μ_1 and μ_2 of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_{\mu_1} = \omega_{\mu_2}$ with $\mu_{1,\infty} = \mathcal{D}_{\lambda_1+\lambda_2+3}$ and $\mu_{2,\infty} = \mathcal{D}_{\lambda_1-\lambda_2+1}$, and where the third direct sum is taken over subsets $\Sigma \subseteq S$ such that $(-1)^{\#\Sigma} = 1$.

Now by Vogan-Zuckerman we have $H^*(\mathfrak{g}, K'_\infty; \pi^\Sigma(\mu_{1,\infty}, \mu_{2,\infty}) \otimes V_\lambda) = 0$ unless $\omega_\infty \in \{1, \mathrm{sign}\}$. Since $K_{\mathrm{fin}} = \prod_{p \in S_{\mathrm{fin}}} \mathscr{P}_p \times \prod_{p \notin S_{\mathrm{fin}}} \mathrm{GSp}_4(\mathbb{Z}_p)$, we have $\pi^\Sigma(\mu_{1,\mathrm{fin}}, \mu_{2,\mathrm{fin}})^{K_{\mathrm{fin}}} = 0$ unless μ_1 and μ_2 are unramified outside S (in particular ω is unramified outside S), in which case we have $\pi^\Sigma(\mu_{1,\mathrm{fin}}, \mu_{2,\mathrm{fin}})^{K_{\mathrm{fin}}} \simeq \boxtimes_{p \in S_{\mathrm{fin}}} r_{\mathscr{P}_p}(\pi^\Sigma(\mu_{1,p}, \mu_{2,p}))$, and we have $r_{\mathscr{P}_p}(\pi^\Sigma(\mu_{1,p}, \mu_{2,p})) = 0$ unless $\mu_{1,p}$ and $\mu_{2,p}$ are of depth 0 (in particular ω_p is tamely ramified).

For the Galois action, we have a spectral decomposition

$$H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\mu_1,\mu_2} \bigoplus_{\Sigma \subseteq S} \Big(\boxtimes_{p \in S_{\mathrm{fin}}} r_{\mathscr{P}_p}(\pi^\Sigma(\mu_{1,p}, \mu_{2,p})) \Big) \boxtimes \rho_{\pi^\Sigma(\mu_1,\mu_2)}$$

and by 4.4.5 the ℓ-adic Galois representation $\rho_{\pi^\Sigma(\mu_1,\mu_2)}$ is given by

$$\rho_{\pi^\Sigma(\mu_1,\mu_2)} \simeq \begin{cases} \rho_{\mu_2}\mathbb{L}^{\lambda_2+1} & \pi^\Sigma(\mu_{1,\infty}, \mu_{2,\infty}) \simeq \pi^{\mathrm{W}}_\infty \\ \rho_{\mu_1} & \pi^\Sigma(\mu_{1,\infty}, \mu_{2,\infty}) \simeq \pi^{\mathrm{H}}_\infty \end{cases}$$

By Gabber's purity theorem, the Galois representations $\rho_{\mu_2}\mathbb{L}^{\lambda_2+1}$ and ρ_{μ_1}, which is pure of weight $\lambda_1 + \lambda_2 + 3$, are concentrated in degree 3 (we already knew this by Vogan-Zuckerman). The result follows, noting that $\pi^\Sigma(\mu_{1,\infty}, \mu_{2,\infty}) \simeq \pi^{\mathrm{W}}_\infty$ precisely if $\infty \notin \Sigma$ and $\pi^\Sigma(\mu_{1,\infty}, \mu_{2,\infty}) \simeq \pi^{\mathrm{H}}_\infty$ precisely if $\infty \in \Sigma$. $\square$

Note that the endoscopic intersection cohomology $H^*_{\mathrm{disc,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is isomorphic to the endoscopic cuspidal cohomology $H^*_{\mathrm{cusp,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$: the Yoshida lift $\pi = \pi^\Sigma(\mu_1, \mu_2)$ is cuspidal unless $\mu_1 \simeq \mu_2$, which is already ruled out by the assumption $\mu_{1,\infty} = \mathcal{D}_{\lambda_1+\lambda_2+3}$ and $\mu_{2,\infty} = \mathcal{D}_{\lambda_1-\lambda_2+1}$. Note also that the endoscopic inner cohomology $H^*_{!,\mathrm{endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is isomorphic to the endoscopic cuspidal cohomology $H^*_{\mathrm{cusp,endo}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$.

4.4.6 Saito-Kurokawa Lifts, Siegel-CAP Cohomology

Following [106, Section 3] and [105, Section 1] We review the Saito-Kurokawa lift.

Saito-Kurokawa Lift We consider the trivial central character Saito-Kurokawa lift corresponding to A-parameters for $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of the form $\psi = (\mu \boxtimes \nu_1) \boxplus (\chi \boxtimes \nu_2)$ for μ a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character and for $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a quadratic Hecke character corresponding to the embedding of dual groups

$$\mathrm{SL}_2(\mathbb{C}) \times \{\pm 1\} \to \mathrm{Sp}_4(\mathbb{C}) \qquad (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), t) \mapsto \begin{pmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & t \end{pmatrix}$$

which defines a global A-packet $\Pi(\psi)$ of automorphic representations π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with an equality of L-functions and epsilon-factors

$$L(s,\pi) = L(s,\mu)L(s+\tfrac{1}{2},\chi)L(s-\tfrac{1}{2},\chi)$$
$$\epsilon(s,\pi) = \epsilon(s,\mu)\epsilon(s+\tfrac{1}{2},\chi)\epsilon(s-\tfrac{1}{2},\chi)$$

For $\mu = \bigotimes_v \mu_v$ where μ_v is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_v)$ with trivial central character and for $\chi = \bigotimes_v \chi_v$ where $\chi_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ is a quadratic character we have $\Pi(\psi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets $\Pi(\psi_v)$ corresponding to the local A-parameter $\psi_v : L_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ given in terms of the local L-parameters $\varphi_{\mu_v} : L_{\mathbb{Q}_v} \to \mathrm{SL}_2(\mathbb{C})$ and $\varphi_{\chi_v} : L_{\mathbb{Q}_v} \to \{\pm 1\}$ by

$$(w,1) \mapsto \mathrm{diag}(\varphi_{\sigma_v}(w), \varphi_{\mu_v}(w), \varphi_{\sigma_v}(w)) \qquad (1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \mapsto \begin{pmatrix} a & & & b \\ & 1 & & \\ & & 1 & \\ c & & & d \end{pmatrix}$$

which are fibered over the characters of the centralizer group $\mathcal{S}_{\psi_v} = S_{\psi_v}/S^0_{\psi_v} Z$ given by

$$\mathcal{S}_{\psi_v} = \begin{cases} 0 & \mu_v \text{ principal series} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

In other words, we have $\Pi(\psi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets

$$\Pi(\psi_v) = \begin{cases} \{\pi_v^+\} & \mu_v \text{ principal series} \\ \{\pi_v^+, \pi_v^-\} & \text{otherwise} \end{cases}$$

The local L-packet $\Pi(\psi_v)$ contains the unique unitary non-tempered non-generic irreducible admissible representation π_v^+ (the basepoint) with local L-parameter $\varphi_{\psi_v} : L_{\mathbb{Q}_v} \to \mathrm{Sp}(\mathbb{C})$ given by $w \mapsto \psi(w, \mathrm{diag}(|w|_v^{1/2}, |w|_v^{-1/2}))$. For $v = p$ a finite place of $\mathbb{Q}$ the basepoint π_p^+ is the Langlands quotient of the Siegel induced representation $\sigma_p \mu_p \nu^{1/2} \rtimes \sigma_p \nu^{-1/2}$ where:

- $\pi_p^+ = \chi_{1,p}\chi_p 1_{\mathrm{GL}_2(\mathbb{Q}_p)} \rtimes \chi_{1,p}^{-1}$ (type IIb) for $\mu_p = \chi_{1,p} \times \chi_{1,p}^{-1}$ a principal series representation;
- $\pi_p^+ = L(\chi_{1,p}\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \nu^{1/2} \rtimes \chi_p \nu^{-1/2})$ (type Vb) for $\mu_p = \chi_{1,p}\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$ a twisted Steinberg representation for a character $\chi_{1,p} : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ distinct from χ_p;
- $\pi_p^+ = L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \nu^{1/2} \rtimes \chi_p \nu^{-1/2})$ (type VIc) for $\mu_p = \chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$ a twisted Steinberg representation;
- $\pi_p^+ = L(\mu_p \chi_p \nu^{1/2} \rtimes \chi_p \nu^{-1/2})$ (type XIb) for μ_p supercuspidal.

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ the basepoint π_∞^+ is the Langlands quotient of the Siegel induced representation $\sigma_\infty \mu_\infty \nu^{1/2} \rtimes \sigma_\infty \nu^{-1/2}$ where:

- $\pi_\infty^+ = \chi_{1,\infty}\chi_\infty 1_{\mathrm{GL}_2(\mathbb{R})} \rtimes \chi_{1,\infty}^{-1}$ for $\mu_\infty = \chi_{1,\infty} \times \chi_{1,\infty}^{-1}$ a principal series representation;
- $\pi_\infty^+ = L(\mathcal{D}_{\lambda_1+\lambda_2+3} \nu^{1/2} \rtimes \chi_\infty \nu^{-1/2})$ for $\mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}$ a holomorphic discrete series representation.

The global A-packet $\Pi(\psi) = \{\pi = \bigotimes_v \pi_v | \pi_v \in \Pi(\psi_v), \pi_v = \pi_v^+ \text{ for almost all places } v \text{ of } \mathbb{Q}\}$ contains the automorphic representation $\pi^+ = \bigotimes_v \pi_v^+$ an isobaric constituent of the Siegel induced representation $\mu\chi\nu^{1/2} \rtimes \chi\nu^{-1/2}$. By Arthur's multiplicity formula the Saito-Kurokawa packets are unstable: for $\pi \in \Pi(\psi)$ we have $m(\pi) = 1$ precisely if $\epsilon(\frac{1}{2}, \pi) = \epsilon(\frac{1}{2}, \mu \otimes \chi^{-1})$. That is, for $\pi = \bigotimes_v \pi_v \in \Pi(\psi)$ and for Σ the set of places v of $\mathbb{Q}$ such that π_v is non-generic we have

$$m(\pi) = \frac{1}{2}\Big(1 + \epsilon(\tfrac{1}{2}, \mu \otimes \chi^{-1})\epsilon(\tfrac{1}{2}, \pi)\Big) = \begin{cases} 1 & \begin{array}{l} \#\Sigma \text{ even and } \epsilon(\frac{1}{2}, \mu \otimes \chi^{-1}) = 1 \\ \#\Sigma \text{ odd and } \epsilon(\frac{1}{2}, \mu \otimes \chi^{-1}) = -1 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

In particular $\Pi(\psi)$ contains 1 discrete element if $\#\Sigma = 0$ and contains $2^{\#\Sigma-1}$ discrete elements otherwise.

Theta Lifts We now explain how the above A-packets can be constructed as theta lifts. Recall that for μ a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character we have the Waldspurger packet $\widetilde{\Pi}(\mu)$ of representations of $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ which are orthogonal to all global Weil representations $\widetilde{\pi}^S_\chi$. For $\mu = \bigotimes_v \mu_v$ where μ_v is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_v)$ with trivial central character we have $\widetilde{\Pi}(\mu) = \bigotimes_v \widetilde{\Pi}(\mu_v)$ for local Waldspurger packets

$$\widetilde{\Pi}(\mu_v) = \begin{cases} \{\widetilde{\mu}_v^+\} & \mu_v \text{ principal series} \\ \{\widetilde{\mu}_v^+, \widetilde{\mu}_v^-\} & \text{otherwise} \end{cases}$$

For $\widetilde{\mu}_v \in \widetilde{\Pi}(\mu)$ we have the local sign $\epsilon(\widetilde{\mu}_v) \in \{\pm 1\}$, and in order for $\widetilde{\mu} = \bigotimes_v \widetilde{\mu}_v$ with global sign $\epsilon(\widetilde{\mu}) = \prod_v \epsilon(\widetilde{\mu}_v)$ to define an element of $\widetilde{\Pi}(\mu)$ we must have $\epsilon(\widetilde{\mu}) = \epsilon(\frac{1}{2}, \mu)$, otherwise $\widetilde{\mu}$ does not define a representation of $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$. Now we have the global theta correspondence

$$\theta^{\mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})}_{\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})} : \mathrm{Irr}(\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})) \to \mathrm{Irr}(\mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})) \cup \{0\}$$

Now by [106, Lemma 3.1], as $\widetilde{\mu}$ runs through the Waldspurger packet $\widetilde{\Pi}(\mu)$, the theta lifts of $\widetilde{\mu}$ from $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ to $\mathrm{SO}_5(\mathbb{A}_\mathbb{Q}) \simeq \mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})$ run through the discrete automorphic representations in the global A-packet $\Pi(\psi)$ with parameter $\psi = (\mu \boxtimes 1) \boxplus (1 \boxtimes \nu_2)$. The general case is obtained by twisting: for $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a unitary Hecke character, for μ a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\mu = \omega$, and for $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a unitary Hecke character with $\chi^2 = \omega$, as $\widetilde{\mu}$ runs through the Waldspurger packet $\widetilde{\Pi}(\mu \otimes \chi^{-1})$, the twists by χ of the theta lifts of $\widetilde{\mu}$ from $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ to $\mathrm{SO}_5(\mathbb{A}_\mathbb{Q}) \simeq \mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})$ run through the discrete automorphic representations in the global A-packet $\Pi(\psi)$ with parameter $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2)$.

For $v = p$ a finite place of $\mathbb{Q}$ the theta lift produces irreducible admissible representations $\pi_p^\pm$ of $\mathrm{SO}_5(\mathbb{Q}_p) \simeq \mathrm{PGSp}_4(\mathbb{Q}_p)$ from the representations $\widetilde{\mu}_p^\pm$ of $\mathrm{Mp}_2(\mathbb{Q}_p)$ corresponding to μ_p. By [106, Lemma 3.1] the local L-packets $\Pi(\psi_p)$ for the local A-parameter $\psi_p = (\mu_p \boxtimes 1) \boxplus (1 \boxtimes \nu_2)$ are given by $\Pi(\psi_p) = \{\pi_p^+\}$ when μ_p is a principal series representation, and otherwise by $\Pi(\psi_p) = \{\pi_p^+, \pi_p^-\}$ where $\pi_p^\pm$ are the theta lifts of the representations $\widetilde{\mu}_p^\pm$ in the Waldspurger packet $\widetilde{\Pi}(\mu_p)$. The general case is obtained by twisting: for $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ a character, for μ_p an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character $\omega_{\mu_p} = \omega_p$, and for $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ a character with $\chi_p^2 = \omega_p$, the local L-packets $\Pi(\psi_p)$ for the local A-parameter $\psi_p = (\mu_p \boxtimes 1) \boxplus (\chi_p \boxtimes \nu_2)$ are given by $\Pi(\psi_p) = \{\pi_p^+\}$ when $\mu_p \otimes \chi_p^{-1}$ is a principal series representation, and otherwise by $\Pi(\psi_p) = \{\pi_p^+, \pi_p^-\}$ where $\pi_p^\pm$ are twists by χ_p of the theta lifts of the representations $\widetilde{\mu}_p^\pm$ in the Waldspurger packet $\widetilde{\Pi}(\mu_p \otimes \chi_p^{-1})$. These theta lifts have been computed (for instance in [103, Table 2]). We obtain the following description for the members of the local L-packets $\Pi(\psi_p)$ of Saito-Kurokawa type, along with their corresponding L-parameters and epsilon-values (compare to [106, Table 2]):

Type		μ_p	π_p	φ_p	$\epsilon(\pi_p)$
II	b	$\chi_{1,p} \times \chi_{1,p}^{-1}$	$\chi_{1,p}\chi_p 1_{\mathrm{GL}_2(\mathbb{Q}_p)} \rtimes \chi_{1,p}^{-1}$	$\chi_{1,p} \oplus \chi_{1,p}^{-1} \oplus \chi_p\varphi_1$	1
V	a*	$\chi_{1,p}\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*([\chi_{1,p}\chi_p, \chi_{1,p}\chi_p\nu], \chi_p\nu^{-1/2})$	$\chi_{1,p}\varphi_{\mathrm{St}} \oplus \chi_p\varphi_{\mathrm{St}}$	-1
	b	$\chi_{1,p} \neq \chi_p$	$L(\chi_{1,p}\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \chi_p\nu^{-1/2})$	$\chi_{1,p}\varphi_{\mathrm{St}} \oplus \chi_p\varphi_1$	1
VI	b	$\chi_p\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\tau(T, \chi_p\nu^{-1/2})$	$\chi_p\varphi_{\mathrm{St}} \oplus \chi_p\varphi_{\mathrm{St}}$	-1
	c		$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \chi_p\nu^{-1/2})$	$\chi_p\varphi_{\mathrm{St}} \oplus \chi_p\varphi_1$	1
XI	a*	Supercuspidal	$\delta^*(\mu_p\chi_p\nu^{1/2} \rtimes \chi_p\nu^{-1/2})$	$\phi \oplus \chi_p\varphi_{\mathrm{St}}$	-1
	b		$L(\mu_p\chi_p\nu^{1/2} \rtimes \chi_p\nu^{-1/2})$	$\phi \oplus \chi_p\varphi_1$	1

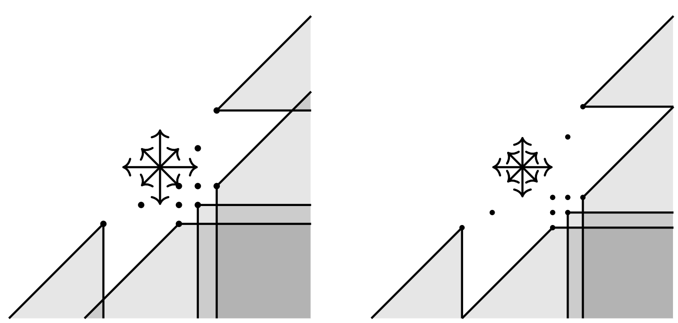
where φ_1 is the L-parameter of the trivial representation $1_{\mathrm{GL}_2(\mathbb{Q}_p)}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, where φ_{St} is the L-parameter of the Steinberg representation $\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, and where ϕ is the L-parameter of a supercuspidal representation μ_p of $\mathrm{GL}_2(\mathbb{Q}_p)$. The general case is obtained by twisting as above.

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ the theta lift produces irreducible admissible representations $\pi_\infty^\pm$ of $\mathrm{SO}_5(\mathbb{R}) \simeq \mathrm{PGSp}_4(\mathbb{R})$ from the representations $\widetilde{\mu}_\infty^\pm$ of $\mathrm{Mp}_2(\mathbb{R})$ corresponding to μ_∞. By [106, Lemma 3.1] the local L-packets $\Pi(\psi_\infty)$ for the local A-parameter $\psi_\infty = (\mu_\infty \boxtimes 1) \boxplus (1 \boxtimes \nu_2)$ are given by $\Pi(\psi_\infty) = \{\pi_\infty^+\}$ when μ_∞ is a principal series representation, and otherwise by $\Pi(\psi_\infty) = \{\pi_\infty^+, \pi_\infty^-\}$ where $\pi_\infty^\pm$ are the theta lifts of the representations $\widetilde{\mu}_\infty^\pm$ in the Waldspurger packet $\widetilde{\Pi}(\mu_\infty)$. The general case is obtained by twisting: for $\omega_\infty : \mathbb{R}^\times \to \mathbb{C}^\times$ a character, for μ_∞ an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{R})$ with central character $\omega_{\mu_\infty} = \omega_\infty$, and for $\chi_\infty : \mathbb{R}^\times \to \mathbb{C}^\times$ a character with $\chi_\infty^2 = \omega_\infty$, the local L-packets $\Pi(\psi_\infty)$ for the local A-parameter $\psi_\infty = (\mu_\infty \boxtimes 1) \boxplus (\chi_\infty \boxtimes \nu_2)$ are given by $\Pi(\psi_\infty) = \{\pi_\infty^+\}$ when $\mu_\infty \otimes \chi_\infty^{-1}$ is a principal series representation, and otherwise by $\Pi(\psi_\infty) = \{\pi_\infty^+, \pi_\infty^-\}$ where $\pi_\infty^\pm$ are twists by χ_∞ of the theta lifts of the representations $\widetilde{\mu}_\infty^\pm$ in the Waldspurger packet $\widetilde{\Pi}(\mu_\infty \otimes \chi_\infty^{-1})$. These theta lifts have been computed (for instance in [103, Table 2]). We obtain the following description for the members of the local L-packets $\Pi(\psi_\infty)$ of Saito-Kurokawa type, along with their corresponding L-parameters and epsilon-values (compare to [106, Table 2]):

Type	μ_∞	π_∞	φ_∞	$\epsilon(\pi_\infty)$
	$\chi_{1,\infty} \times \chi_{1,\infty}^{-1}$	$\chi_{1,\infty}\chi_\infty 1_{\mathrm{GL}_2(\mathbb{R})} \rtimes \chi_{1,\infty}^{-1}$	$\chi_{1,\infty} \oplus \chi_{1,\infty}^{-1} \oplus \chi_\infty\varphi_1$	1
$(\lambda_1+3, \lambda_2+3)$	$\mathcal{D}_{\lambda_1+\lambda_2+3}, \lambda_1 = \lambda_2 \geq 0$	$\mathcal{D}_{\lambda_1+2,\lambda_2+1}$	$\varphi_{\mathcal{D}_{\lambda_1+\lambda_2+3}} \oplus \chi_\infty\varphi_{\mathcal{D}_1}$	-1
$(\lambda_1+2, -\lambda_2-2)$		$L(\mathcal{D}_{\lambda_1+\lambda_2+3}\nu^{1/2} \rtimes \chi_\infty\nu^{-1/2})$	$\varphi_{\mathcal{D}_{\lambda_1+\lambda_2+3}} \oplus \chi_\infty\varphi_1$	1

where φ_1 is the L-parameter of the trivial representation $1_{\mathrm{GL}_2(\mathbb{R})}$ of $\mathrm{GL}_2(\mathbb{R})$ and $\varphi_{\mathcal{D}_k}$ is the L-parameter of the discrete series representation $\mathcal{D}_k$ of $\mathrm{GL}_2(\mathbb{R})$ with minimal $\mathrm{O}(2)$-type $k+1$ and central character sign^{k+1}. The general case is obtained by twisting as above.

The K_∞-types in the table are obtained as follows. For $\chi_\infty : \mathbb{R}^\times \to \mathbb{C}^\times$ a quadratic character the representation $\mathcal{D}_{\lambda_1+\lambda_2+3}\nu^{1/2} \rtimes \chi_\infty\nu^{-1/2}$ has two irreducible constituents: $\mathcal{D}_{\lambda_1+2,-\lambda_2-1}$ the large generic discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with minimal K_∞-type $(\lambda_1+3, -\lambda_2-1)$ occurring with multiplicity 1, and $L(\mathcal{D}_{\lambda_1+\lambda_2+3}\nu^{1/2} \rtimes \chi_\infty\nu^{-1/2})$ the Langlands quotient of $\mathcal{D}_{\lambda_1+\lambda_2+3}\nu^{1/2} \rtimes \chi_\infty\nu^{-1/2}$ with minimal K_∞-type $(-\lambda_1-2, -\lambda_2-2)$ occurring with multiplicity 1. We obtain the following K_∞-type regions for the representations in the above table, illustrated for $\lambda_1 = \lambda_2 = 0$ and $\lambda_1 = \lambda_2 = 1$:



Parahoric Restriction We now review results about parahoric restriction of Saito-Kurokawa lifts, following [100, Section 5.3].

From now on we will parameterize Saito-Kurokawa lifts in the following way. Let μ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character. Let S be a finite set of places of $\mathbb{Q}$ containing ∞ such that μ_v is spherical for all $v \notin S$. Let $\Sigma \subseteq S$ be a subset of S and let μ^Σ be the non-cuspidal irreducible subquotient of the parabolically induced representation $\nu^{1/2} \times \nu^{-1/2}$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that μ_v^Σ is locally in the discrete series exactly at the places $v \in \Sigma$; we have $\mu_v^\Sigma = \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_v)}$ for $v \in \Sigma$ and $\mu_v^\Sigma = 1_{\mathrm{GL}_2(\mathbb{Q}_v)}$ for $v \notin \Sigma$, where $\mathrm{St}_{\mathrm{GL}_2(\mathbb{R})} = \mathcal{D}_1$. We assume that μ_v is in the discrete series for all $v \in \Sigma$. Then the Saito-Kurokawa lift yields an irreducible admissible representation $\pi = \pi(\mu, \mu^\Sigma)$ of $G(\mathbb{A}_\mathbb{Q})$ with trivial central character such that $L(s, \pi_v) = L(s, \mu_v)L(s, \mu_v^\Sigma)$ for almost every place v of $\mathbb{Q}$. In fact, we have $L(s, \pi_v) = L(s, \mu_v)L(s, \mu_v^\Sigma)$ and $\epsilon(s, \pi_v) = \epsilon(s, \mu_v)\epsilon(s, \mu_v^\Sigma)$ for all places v of $\mathbb{Q}$. This lift is local in the sense that $\pi_v = \pi(\mu_v, \mu_v^\Sigma)$ depends only on μ_v and μ_v^Σ; every local factor π_v is unitary and non-generic.

Let $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be an at most tamely ramified character which restricts to a nonzero character $\chi_p : \mathbb{F}_p^\times \to \mathbb{C}^\times$ which we abusely denote by the same notation. Let μ_p be a depth 0 supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, with hyperspecial parahoric restriction π_Λ for a character $\Lambda : \mathbb{F}_{p^2}^\times \to \mathbb{C}^\times$; such a character Λ with $\Lambda|_{\mathbb{F}_p^\times} = 1$ factors over a character $\omega_\Lambda : \mathbb{F}_p^\times[p+1] \to \mathbb{C}^\times$ with $\Lambda(x) = \omega_\Lambda(x^{p-1})$. Let $\chi_u : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be the nontrivial unramified quadratic character and for $p > 2$ let $\chi_t : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be either one of the tamely ramified quadratic characters. For $p = 2$ let $\theta : \mathbb{F}_4^\times \to \mathbb{C}^\times$ be a primitive character, and let $l \in \mathbb{Z}/3\mathbb{Z}$ such that $\theta^l = \Lambda$. By [100, Theorem 5.2] the hyperspecial parahoric restriction $r_{\mathscr{K}_p}(\pi(\mu_p, \mu_p^\Sigma))$ is given as follows (compare to [100, Table 5.1]):

Type		μ_p	μ_p^Σ	$\pi_p(\mu_p, \mu_p^\Sigma)$	$r_{\mathscr{K}_2}(\pi(\mu_2, \mu_2^\Sigma))$	$r_{\mathscr{K}_p}(\pi(\mu_p, \mu_p^\Sigma))$	Dimension
II	b	$\chi_p \times \chi_p^{-1}$	$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\chi_p 1_{\mathrm{GL}_2(\mathbb{Q}_p)} \rtimes \chi_p^{-1}$	χ_3	$\chi_3(\chi_p, \chi_p^{-1})$	$(p+1)(p^2+1)$
V	a*	$\xi_u \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*([\xi_u, \xi_u\nu], \nu^{-1/2})$	θ_2	$\theta_2(1)$	$\frac{1}{2}(p^2-p)$
		$\xi_t \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*([\xi_t, \xi_t\nu], \nu^{-1/2})$	0	0	0
	b	$\xi_u \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \nu^{-1/2})$	θ_1	$\theta_1(1)$	$\frac{1}{2}p(p+1)^2$
		$\xi_t \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \nu^{-1/2})$		$\tau_2(1)$	$p(p^2+1)$
VI	b	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\tau(T, \nu^{-1/2})$	θ_3	$\theta_3(1)$	$\frac{1}{2}p(p^2+1)$
	c		$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \nu^{-1/2})$	θ_4	$\theta_4(1)$	$\frac{1}{2}p(p^2+1)$
XI	a*	Supercuspidal	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*(\mu_p\nu^{1/2} \rtimes \nu^{-1/2})$	0	0	0
	b		$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\mu_p\nu^{1/2} \rtimes \nu^{-1/2})$	$\chi_5(l)$	$\chi_5(\omega_\Lambda, 1)$	$(p-1)(p^2+1)$

Let $\lambda_0 : \mathbb{F}_p^\times \to \mathbb{C}^\times$ and $\Lambda_0 : \mathbb{F}_{p^2}^\times \to \mathbb{C}^\times$ be the nontrivial quadratic characters. For $\Lambda : \mathbb{F}_{p^2}^\times \to \mathbb{C}^\times$ a character with $\Lambda^{p+1} = 1$ let $\Lambda' : \mathbb{F}_{p^2}^\times \to \mathbb{C}^\times$ be a character such that $\Lambda'^{p-1} = \Lambda$. Let $\Lambda'_0 : \mathbb{F}_{p^2}^\times \to \mathbb{C}^\times$ be a character such that $\Lambda_0'^{p-1} = \Lambda_0$. By [100, Theorem 5.2] the paramodular restriction $r_{\mathscr{K}_p^{P_2}}(\pi(\mu_p, \mu_p^\Sigma))$ is given as follows (compare to [100, Table 5.1]):

Type		μ_p	μ_p^Σ	$\pi_p(\mu_p, \mu_p^\Sigma)$	$r_{\mathscr{K}_p^{P_2}}(\pi(\mu_p, \mu_p^\Sigma))$	Dimension
II	b	$\chi_p \times \chi_p^{-1}$	$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\chi_p 1_{\mathrm{GL}_2(\mathbb{Q}_p)} \rtimes \chi_p^{-1}$	$[1_{\mathrm{GL}_1(\mathbb{F}_p)} \times \chi_p, 1_{\mathrm{GL}_2(\mathbb{F}_p)} \times \chi_p^{-1}]$	$(p+1)^2$
V	a*	$\xi_u \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*([\xi_u, \xi_u\nu], \nu^{-1/2})$	0	0
		$\xi_t \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*([\xi_t, \xi_t\nu], \nu^{-1/2})$	$[\pi_{\Lambda_0'}, \pi_{\Lambda_0'^{-1}}]_\pm$	$(p-1)^2/2$
	b	$\xi_u \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \nu^{1/2} \rtimes \nu^{-1/2})$	$[1_{\mathrm{GL}_2(\mathbb{F}_p)}, 1_{\mathrm{GL}_2(\mathbb{F}_p)}] + [\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}, \mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]$	p^2+1
		$\xi_t \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\chi_p \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \nu^{1/2} \rtimes \nu^{-1/2})$	$[1_{\mathrm{GL}_1(\mathbb{F}_p)} \times \lambda_0, 1_{\mathrm{GL}_1(\mathbb{F}_p)} \times \lambda_0]_\mp$	$(p+1)^2/2$
VI	b	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\tau(T, \nu^{-1/2})$	$[\mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}, \mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_p)}]$	p^2
	c		$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)} \nu^{1/2} \rtimes \nu^{-1/2})$	$[1_{\mathrm{GL}_2(\mathbb{F}_p)}, 1_{\mathrm{GL}_2(\mathbb{F}_p)}]$	1
XI	a*	Supercuspidal	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$\delta^*(\mu_p \nu^{1/2} \rtimes \nu^{-1/2})$	$[\pi_{\Lambda'}, \pi_{\Lambda'^{-1}}]$	$(p-1)^2$
	b		$1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$L(\mu_p \nu^{1/2} \rtimes \nu^{-1/2})$	0	0

where the sign is given by $\xi_t(p) = \pm 1$.

The general case is obtained by twisting as above. The parahoric restrictions $r_{\mathscr{P}_p}(\pi(\mu_p, \mu_p^\Sigma))$ for $\mathscr{P}_p \subseteq \mathrm{GSp}_4(\mathbb{Q}_p)$ a standard parahoric subgroup are obtained from the hyperspecial parahoric restrictions $r_{\mathscr{K}_p}(\pi(\mu_p, \mu_p^\Sigma))$ by parabolic restriction.

Siegel-CAP Cohomology Consider the Siegel-CAP part of the automorphic discrete spectrum which by Arthur's classification admits a spectral decomposition

$$L^2_{\mathrm{disc},[P_1]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc},[P_1]}(G,\omega) \\ (\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2))}} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = \epsilon(\frac{1}{2}, \mu \otimes \chi^{-1})}} \pi$$

The Siegel-CAP part $H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ of the intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc},[P_1]}(G,\omega) \\ (\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2))}} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = \epsilon(\frac{1}{2}, \mu \otimes \chi^{-1})}} \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}}$. Similarly, the Siegel-CAP part $H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ of the ℓ-adic intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc},[P_1]}(G,\omega) \\ (\psi = (\mu\boxtimes 1)\boxplus(\chi\boxtimes\nu_2))}} \bigoplus_{\substack{\pi\in\Pi(\psi) \\ \langle\cdot,\pi\rangle = \epsilon(\frac{1}{2},\mu\otimes\chi^{-1})}} \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_\pi$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$.

To determine the structure of the representations $H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} = \pi^{K_{\mathrm{fin}}} \boxtimes \rho_\pi$ we use the trace formula:

Theorem 4.4.7. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$ for a compact open subgroup $K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^\infty)$. Let $\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2) \in \Psi_{\mathrm{disc},[P_1]}(G, \omega)$ be an A-parameter of Saito-Kurokawa type, where μ is a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central character $\omega_\mu = \omega$ and where $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ is a unitary Hecke character with $\chi^2 = \omega$.

If $\pi \in \Pi(\psi)$ contributes nontrivially in the intersection cohomology $H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$, then the contribution occurs only for $\lambda_1 = \lambda_2$ and is given as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$ by

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq \begin{cases} \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes (\mathbb{L}_\chi^{\lambda_2+1} \oplus \mathbb{L}_\chi^{\lambda_1+2}) & \pi_\infty \simeq \pi_\infty^{(1)\pm} \\ \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes \rho_\mu & \pi_\infty \simeq \pi_\infty^{\mathrm{H}} \end{cases}$$

Proof. For $\psi = (\mu \boxtimes 1) \boxplus (\chi \otimes \nu_2) \in \Psi_{\mathrm{disc},[P_1]}(G, \omega)$ an A-parameter of Saito-Kurokawa type and for $\pi \in \Pi(\psi)$ an automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ lifted from the automorphic representation $\pi^H = \mu \boxtimes (\chi \circ \det)$ of $H(\mathbb{A}_{\mathbb{Q}})$ with $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$ and with π_∞ a cohomological $(\mathfrak{g}, K'_\infty)$-module with central and infinitesimal characters determined by those of V_λ, we have that $\lambda_1 = \lambda_2$ and we have that $\pi_\infty \in \{\pi_\infty^{\mathrm{H}}, \pi_\infty^{(1)\pm}\}$ is either one of the Langlands quotients $\pi_\infty^{(1)+} = L(\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2})$ or $\pi_\infty^{(1)-} = L(\nu^{1/2}\mathcal{D}_{\lambda_1+\lambda_2+3} \rtimes \nu^{-1/2}\mathrm{sign})$ which have nonzero $(\mathfrak{g}, K'_\infty)$ concentrated in degrees 2 and 4 with Hodge numbers $(\lambda_1+1, \lambda_2+1)$ and $(\lambda_1+2, \lambda_2+2)$, or the holomorphic discrete series representation $\pi_\infty^{\mathrm{H}} = \mathcal{D}_{\lambda_1+2,\lambda_2+1}$ which has nonzero $(\mathfrak{g}, K'_\infty)$-cohomology concentrated in degree 3 with Hodge numbers $(\lambda_1+\lambda_2+3, 0)$ and $(0, \lambda_1+\lambda_2+3)$. Let $\pi_\infty^\times = \pi_\infty^{(1)+}$ if $\omega_{\mu_\infty} = 1$ and $\pi_\infty^\times = \pi_\infty^{(1)-}$ if $\omega_{\mu_\infty} = \mathrm{sign}$, and let $\pi_\infty^- = \pi_\infty^{\mathrm{H}}$. Let $\pi_p^\times$ be the basepoint of the local L-packet $\Pi(\psi_p)$, and let

π_p^- be the cuspidal member of the local L-packet $\Pi(\psi_p)$. Each component π_v of $\pi = \bigotimes_v \pi_v$ has a sign

$$\langle \pi_v^H, \pi_v \rangle = \begin{cases} 1 & \pi_v = \pi_v^\times \\ -1 & \pi_v = \pi_v^- \end{cases} \qquad \langle \pi_{\text{fin}}^H, \pi_{\text{fin}} \rangle = \prod_p \langle \mu_p, \pi_p \rangle$$

In terms of these signs, π has multiplicity

$$m(\pi) = \frac{1}{2}\Big(1 + \epsilon(\tfrac{1}{2}, \mu \otimes \chi^{-1}) \prod_v \langle \pi_v^H, \pi_v \rangle \Big) = \begin{cases} 1 & \langle \pi_{\text{fin}}^H, \pi_{\text{fin}} \rangle = \epsilon(\tfrac{1}{2}, \mu \otimes \chi^{-1}) \text{ and } \pi_\infty = \pi_\infty^\times \\ & \text{ or } \langle \pi_{\text{fin}}^H, \pi_{\text{fin}} \rangle = -\epsilon(\tfrac{1}{2}, \mu \otimes \chi^{-1}) \text{ and } \pi_\infty = \pi_\infty^- \\ 0 & \text{otherwise} \end{cases}$$

Choose a correspondence $f_G^{p,\infty} \in C_c^\infty(K^p \setminus G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})/K^p)$ which is a projection onto the A-packet $\Pi(\psi_{\text{fin}})$, and choose a matching correspondence $f_H^{p,\infty}$ which is a projection onto π_{fin}^H. Consider the sets

$$\Pi(\psi_{\text{fin}})^\times = \{\pi \in \Pi(\psi_{\text{fin}}) | \langle \pi_{\text{fin}}^H, \pi_{\text{fin}} \rangle = 1\}$$
$$\Pi(\psi_{\text{fin}})^- = \{\pi \in \Pi(\psi_{\text{fin}}) | \langle \pi_{\text{fin}}^H, \pi_{\text{fin}} \rangle = -1\}$$

By matching and endoscopic character identities we have

$$\operatorname{tr}(\pi_{\text{fin}}^H(f_H^{p,\infty})) = \operatorname{tr}(\Pi(\psi_{\text{fin}})^\times(f_G^{p,\infty})) + \operatorname{tr}(\Pi(\psi_{\text{fin}})^-(f_G^{p,\infty}))$$

Recall that the test function f_∞^G satisfies $\operatorname{tr}(\pi_\infty^-(f_\infty^G)) = -\frac{1}{2}$ and $\operatorname{tr}(\pi_\infty^\times(f_\infty^G)) = \frac{1}{2}$, and the matching test function f_∞^H satisfies $\operatorname{tr}(\pi_\infty^H(f_\infty^H)) = -1$. Recall that the test function ϕ_j^G satisfies

$$\operatorname{tr}(\Pi(\psi_p)(\phi_p^G)) = p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j + c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)$$

and the matching test function ϕ_j^H satisfies

$$\operatorname{tr}(\pi_p^H(\phi_j^H)) = p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} - c_1(\mu_p)^j - c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)$$

Consider the test functions $f^G = \phi_j^G f_G^{p,\infty} f_\infty^G$ and the matching test function $f^H = \phi_j^H f_H^{p,\infty} f_\infty^H$. Now the contribution of the A-packet $\Pi(\psi)$ to $\mathrm{STF}^G(f^G)$ is given by

$$\begin{aligned}
&\sum_{\pi\in\Pi(\psi)} m(\pi)\mathrm{tr}(\pi_\infty(f_\infty^G))\mathrm{tr}(\pi_{\mathrm{fin}}(f_G^{p,\infty}))p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j + c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)\\
&= \tfrac{\epsilon(\frac{1}{2},\mu\otimes\chi^{-1})}{2}\Big(\mathrm{tr}(\pi_\infty^\times(f_\infty^G)) - \mathrm{tr}(\pi_\infty^-(f_\infty^G))\Big)\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^\times(f_G^{p,\infty})) - \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)\\
&\cdot p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j + c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)\\
&= \tfrac{\epsilon(\frac{1}{2},\mu\otimes\chi^{-1})}{2}\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^\times(f_G^{p,\infty})) - \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j + c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)
\end{aligned}$$

and the contribution of π^H to $\mathrm{STF}^H(f^H)$ is given by

$$\begin{aligned}
&2m(\pi^H)\mathrm{tr}(\pi_\infty^H(f_\infty^H))\mathrm{tr}(\pi_{\mathrm{fin}}^H(f_H^{p,\infty}))p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} - c_1(\mu_p)^j - c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)\\
&= -2\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^\times(f_G^{p,\infty})) + \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} - c_1(\mu_p)^j - c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)
\end{aligned}$$

It follows that we have the trace

$$\begin{aligned}
&\mathrm{tr}(\mathrm{Frob}_p^j | H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}}) = p^{\frac{\lambda_1+\lambda_2}{2}}\Big(\mathrm{STF}^G(f^G) - \frac{1}{4}\mathrm{STF}^H(f^H)\Big)\\
&= \tfrac{\epsilon(\frac{1}{2},\mu\otimes\chi^{-1})}{2}\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^\times(f_G^{p,\infty})) - \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{\lambda_1+\lambda_2+3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j + c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)\\
&+ \frac{1}{2}\Big(\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^\times(f_G^{p,\infty})) + \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})^-(f_G^{p,\infty}))\Big)p^{\frac{\lambda_1+\lambda_2+3}{2}j}\Big(c(\chi_p)^j p^{\frac{1}{2}j} - c_1(\mu_p)^j - c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j}\Big)\\
&= \frac{1}{2}p^{\frac{\lambda_1+\lambda_2+3}{2}j}\begin{pmatrix}\epsilon(\frac{1}{2},\mu\otimes\chi^{-1})(c(\chi_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j + c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j})\\ +\langle\pi_{\mathrm{fin}}^H,\pi_{\mathrm{fin}}\rangle(c(\chi_p)^j p^{\frac{1}{2}j} - c_1(\mu_p)^j - c_2(\mu_p)^j + c(\chi_p)^j p^{-\frac{1}{2}j})\end{pmatrix}\\
&= \begin{cases} p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j}) & \langle\pi_{\mathrm{fin}}^H,\pi_{\mathrm{fin}}\rangle = \epsilon(\frac{1}{2},\mu\otimes\chi^{-1})\\ -p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_p)^j + c_2(\mu_p)^j) & \langle\pi_{\mathrm{fin}}^H,\pi_{\mathrm{fin}}\rangle = -\epsilon(\frac{1}{2},\mu\otimes\chi^{-1})\end{cases}
\end{aligned}$$

Since $\lambda_1 = \lambda_2$ we can write $p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j}) = c(\chi_p)^j p^{(\lambda_1+2)j} + c(\chi_p)^j p^{(\lambda_2+1)j}$, which is the trace of Frob_p^j on $\mathbb{L}_\chi^{\lambda_2+1}\oplus\mathbb{L}_\chi^{\lambda_1+2}$ whose contribution to $H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)$ is concentrated in degrees 2 and 4. Similarly, $-p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_p)^j + c_2(\mu_p)^j)$ is the trace of Frob_p^j on ρ_μ whose contribution to $H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)$ is concentrated in degree 3. It follows that

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq \begin{cases}\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}}\boxtimes(\mathbb{L}_\chi^{\lambda_2+1}\oplus\mathbb{L}_\chi^{\lambda_1+2}) & \pi_\infty\simeq\pi_\infty^{(1)\pm}\\ \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}}\boxtimes\rho_\mu & \pi_\infty\simeq\pi_\infty^{\mathrm{H}}\end{cases}$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}}\times\mathrm{Gal}$. The result follows. □

At this point we specialize the discussion to the case of square-free parahoric level structure.

Theorem 4.4.8. (compare to [100, Theorem 5.4]) Let S be a finite set of places of $\mathbb{Q}$ including ∞, and let $S_{\text{fin}} = S - \{\infty\}$. Let $K_{\text{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a compact open subgroup of the form $K_{\text{fin}} = \prod_{p \in S_{\text{fin}}} \mathscr{P}_p \times \prod_{p \notin S_{\text{fin}}} G(\mathbb{Z}_p)$ where $\mathscr{P}_p \subseteq G(\mathbb{Q}_p)$ is a standard parahoric subgroup. Let $V_\lambda \in \operatorname{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$. Assume that $\lambda_1 = \lambda_2$.

Then the Siegel-CAP intersection cohomology $H^*_{\text{disc},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4$ and is given as a representation of $\prod_{p \in S_{\text{fin}}} \mathscr{P}_p / \mathscr{P}_p^+ \times \text{Gal}$ by

$$H^2_{\text{disc},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\mu} \bigoplus_{\substack{\Sigma \subseteq S \\ \infty \notin \Sigma}} \Big(\boxtimes_{p \in S_{\text{fin}}} r_{\mathscr{P}_p}(\omega_p \pi(\mu_p, \mu_p^{\Sigma})) \Big) \boxtimes \mathbb{L}_{\omega}^{\lambda_2 + 1}$$

$$H^3_{\text{disc},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\mu} \bigoplus_{\substack{\Sigma \subseteq S \\ \infty \in \Sigma}} \Big(\boxtimes_{p \in S_{\text{fin}}} r_{\mathscr{P}_p}(\omega_p \pi(\mu_p, \mu_p^{\Sigma})) \Big) \boxtimes \rho_{\omega\mu}$$

$$H^4_{\text{disc},[P_1]}(\mathcal{S}_{K_{\text{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\mu} \bigoplus_{\substack{\Sigma \subseteq S \\ \infty \notin \Sigma}} \Big(\boxtimes_{p \in S_{\text{fin}}} r_{\mathscr{P}_p}(\omega_p \pi(\mu_p, \mu_p^{\Sigma})) \Big) \boxtimes \mathbb{L}_{\omega}^{\lambda_1 + 2}$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ unramified outside S with $\omega_\infty \in \{1, \text{sign}\}$ and with ω_p tamely ramified for every place $p \in S_{\text{fin}}$, where the second direct sum is taken over unitary cuspidal automorphic representations μ of $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character unramified outside S with $\mu_\infty = \mathcal{D}_{\lambda_1 + \lambda_2 + 3}$ and with μ_p of depth 0 for every place $p \in S_{\text{fin}}$, and where the third direct sum is taken over subsets $\Sigma \subseteq S$ with μ_v a discrete series representation for every place $v \in \Sigma$ such that $(-1)^{\#\Sigma} = \epsilon(\frac{1}{2}, \mu)$.

Proof. We have the spectral decomposition

$$H^*_{\text{disc},[P_1]}(\mathcal{S}_{K_{\text{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\text{disc},[P_1]}(G,\omega) \\ (\psi = (\mu \boxtimes 1) \boxplus (\chi \boxtimes \nu_2))}} \bigoplus_{\substack{\pi \in \Pi(\psi) \\ \langle \cdot, \pi \rangle = \epsilon(\frac{1}{2}, \mu \otimes \chi^{-1})}} \pi_{\text{fin}}^{K_{\text{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, and where the second direct sum is taken over unitary cuspidal automorphic representations μ of $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with central character $\omega_\mu = \omega$ with $\mu_\infty = \mathcal{D}_{\lambda_1 + \lambda_2 + 3}$, and over unitary Hecke characters $\chi : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ with $\chi^2 = \omega$. Twisting by χ^{-1}, we may reduce to the case of trivial central character, in which case we have

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_1]}(G,1)\\ (\psi=(\mu\boxtimes 1)\boxplus(1\boxtimes\nu_2))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=\epsilon(\frac{1}{2},\mu)}} (\omega_{\mathrm{fin}}\pi_{\mathrm{fin}})^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g},K'_\infty;\omega_\infty\pi_\infty\otimes V_\lambda)$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, and where the second direct sum is taken over unitary cuspidal automorphic representations μ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character with $\mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}$. Equivalently, we have

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}),\mathbb{V}_\lambda) \simeq \bigoplus_{\omega}\bigoplus_{\mu}\bigoplus_{\Sigma\subseteq S}(\omega_{\mathrm{fin}}\pi(\mu_{\mathrm{fin}},\mu_{\mathrm{fin}}^\Sigma))^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g},K'_\infty;\omega_\infty\pi(\mu_\infty,\mu_\infty^\Sigma)\otimes V_\lambda)$$

where the first direct sum is taken over unitary Hecke characters $\omega : \mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, where the second direct sum is taken over unitary cuspidal automorphic representations μ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character with $\mu_\infty = \mathcal{D}_{\lambda_1+\lambda_2+3}$, and where the third direct sum is taken over subsets $\Sigma \subseteq S$ with μ_v a discrete series representation for every place $v \in \Sigma$ such that $(-1)^{\#\Sigma} = \epsilon(\frac{1}{2},\mu)$.

Now by Vogan-Zuckerman we have $H^*(\mathfrak{g},K'_\infty;\omega_\infty\pi(\mu_\infty,\mu_\infty^\Sigma)\otimes V_\lambda) = 0$ unless $\omega_\infty \in \{1,\mathrm{sign}\}$. Since $K_{\mathrm{fin}} = \prod_{p\in S_{\mathrm{fin}}}\mathscr{P}_p \times \prod_{p\notin S_{\mathrm{fin}}}\mathrm{GSp}_4(\mathbb{Z}_p)$, we have $(\omega_{\mathrm{fin}}\pi(\mu_{\mathrm{fin}},\mu_{\mathrm{fin}}^\Sigma))^{K_{\mathrm{fin}}} = 0$ unless ω and μ are unramified outside S, in which case we have $(\omega_{\mathrm{fin}}\pi(\mu_{\mathrm{fin}},\mu_{\mathrm{fin}}^\Sigma))^{K_{\mathrm{fin}}} \simeq \boxtimes_{p\in S_{\mathrm{fin}}} r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\Sigma))$, and we have $r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\Sigma)) = 0$ unless ω_p is tamely ramified and μ_p is of depth 0.

For the Galois action, we have a spectral decomposition

$$H^*_{\mathrm{disc},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda) \simeq \bigoplus_{\omega}\bigoplus_{\mu}\bigoplus_{\Sigma\subseteq S}\Big(\boxtimes_{p\in S_{\mathrm{fin}}} r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\Sigma))\Big)\boxtimes\rho_{\omega\pi(\mu,\mu^\Sigma)}$$

and by 4.4.7 the ℓ-adic Galois representation $\rho_{\omega\pi(\mu,\mu^\Sigma)}$ is given by

$$\rho_{\omega\pi(\mu,\mu^\Sigma)} \simeq \begin{cases} \mathbb{L}_\omega^{\lambda_2+1}\oplus\mathbb{L}_\omega^{\lambda_1+2} & \pi(\mu_\infty,\mu_\infty^\Sigma)\simeq\pi_\infty^{(1)\pm} \\ \rho_{\omega\mu} & \pi(\mu_\infty,\mu_\infty^\Sigma)\simeq\pi_\infty^{\mathrm{H}} \end{cases}$$

By Gabber's purity theorem, the Galois representation $\mathbb{L}_\omega^{\lambda_2+1}$, which is pure of weight $\lambda_1+\lambda_2+2$, is concentrated in degree 2, and the Galois representation $\mathbb{L}_\omega^{\lambda_1+2}$, which is pure of weight $\lambda_1+\lambda_2+4$, is concentrated in degree 4.

Similarly, the Galois representation $\rho_{\omega\mu}$, which is pure of weight $\lambda_1+\lambda_2+3$, is concentrated in degree 3. The result follows, noting that $\pi(\mu_\infty,\mu_\infty^\Sigma)\simeq\pi_\infty^{(1)\pm}$ precisely if $\infty\notin\Sigma$ and $\pi(\mu_\infty,\mu_\infty^\Sigma)\simeq\pi_\infty^{\mathrm{H}}$ precisely if $\infty\in\Sigma$. $\square$

Theorem 4.4.9. Let S be a finite set of places of $\mathbb{Q}$ including ∞, and let $S_{\text{fin}}=S-\{\infty\}$. Let $K_{\text{fin}}\subseteq G(\mathbb{A}_{\mathbb{Q}}^\infty)$ be a compact open subgroup of the form $K_{\text{fin}}=\prod_{p\in S_{\text{fin}}}\mathscr{P}_p\times\prod_{p\notin S_{\text{fin}}}G(\mathbb{Z}_p)$ where $\mathscr{P}_p\subseteq G(\mathbb{Q}_p)$ is a standard parahoric subgroup. Let $V_\lambda\in\mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda=(\lambda_1,\lambda_2;\lambda_1+\lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$. Assume that $\lambda_1=\lambda_2$.

Then the Siegel-CAP cuspidal cohomology $H^*_{\text{cusp},[P_1]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)$ is concentrated in degrees $2,3,4$ and is given as a representation of $\prod_{p\in S_{\text{fin}}}\mathscr{P}_p/\mathscr{P}_p^+\times\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by

$$\begin{aligned}
H^2_{\text{cusp},[P_1]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)&\simeq\bigoplus_{\omega}\bigoplus_{\mu}\bigoplus_{\substack{\emptyset\neq\Sigma\subseteq S\\ \infty\notin\Sigma}}\Big(\bigboxtimes_{p\in S_{\text{fin}}}r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\Sigma))\Big)\boxtimes\mathbb{L}_\omega^{\lambda_2+1}\\
&\quad\oplus\bigoplus_{\omega}\bigoplus_{\substack{\mu\\ L(\frac12,\mu)=0}}\Big(\bigboxtimes_{p\in S_{\text{fin}}}r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\emptyset))\Big)\boxtimes\mathbb{L}_\omega^{\lambda_2+1}\\
H^3_{\text{cusp},[P_1]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)&\simeq\bigoplus_{\omega}\bigoplus_{\mu}\bigoplus_{\substack{\Sigma\subseteq S\\ \infty\in\Sigma}}\Big(\bigboxtimes_{p\in S_{\text{fin}}}r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\Sigma))\Big)\boxtimes\rho_{\omega\mu}\\
H^4_{\text{cusp},[P_1]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)&\simeq\bigoplus_{\omega}\bigoplus_{\mu}\bigoplus_{\substack{\emptyset\neq\Sigma\subseteq S\\ \infty\notin\Sigma}}\Big(\bigboxtimes_{p\in S_{\text{fin}}}r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\Sigma))\Big)\boxtimes\mathbb{L}_\omega^{\lambda_1+2}\\
&\quad\oplus\bigoplus_{\omega}\bigoplus_{\substack{\mu\\ L(\frac12,\mu)=0}}\Big(\bigboxtimes_{p\in S_{\text{fin}}}r_{\mathscr{P}_p}(\omega_p\pi(\mu_p,\mu_p^\emptyset))\Big)\boxtimes\mathbb{L}_\omega^{\lambda_1+2}
\end{aligned}$$

where the first direct sum is taken over unitary Hecke characters $\omega:\mathbb{Q}^\times\setminus\mathbb{A}_{\mathbb{Q}}^\times\to\mathbb{C}^\times$ unramified outside S with $\omega_\infty\in\{1,\mathrm{sign}\}$ and with ω_p tamely ramified for every place $p\in S_{\text{fin}}$, where the second direct sum is taken over unitary cuspidal automorphic representations μ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character unramified outside S with $\mu_\infty=\mathcal{D}_{\lambda_1+\lambda_2+3}$ and with μ_p of depth 0 for every place $p\in S_{\text{fin}}$, and where the third direct sum is taken over subsets $\Sigma\subseteq S$ with μ_v a discrete series representation for every place $v\in\Sigma$ such that $(-1)^{\#\Sigma}=\epsilon(\frac12,\mu)$.

Proof. The result follows from 4.4.8 noting that the Saito-Kurokawa lift $\pi=\pi(\mu,\mu^\Sigma)$ is cuspidal unless $\Sigma=\emptyset$ and $L(\frac12,\mu)\neq 0$, so the third direct sum is taken over nonempty subsets $\emptyset\neq\Sigma\subseteq S$ as above, or in the case $\Sigma=\emptyset$ the second direct sum is taken over unitary cuspidal automorphic representations μ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character as above with $L(\frac12,\mu)=0$. $\square$

Note that the Siegel-CAP inner cohomology $H^*_{!,P_1}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)$ is isomorphic to the Siegel-CAP cuspidal cohomology $H^*_{\mathrm{cusp},[P_1]}(\mathcal{S}_{K_{\mathrm{fin}}},\mathbb{V}_\lambda)$.

4.4.7 Soudry Lifts, Klingen-CAP Cohomology

We review the Soudry lift, following [106, Section 4] and [105, Section 1] specialized to the case $F=\mathbb{Q}$.

Soudry Lifts We consider the trivial central character Soudry lift corresponding to the A-parameters for $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of the form $\psi=\mu\boxtimes\nu_2$ for μ a self-dual unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with nontrivial central character ω_μ coming from the embedding of dual groups

$$\mathrm{O}_2(\mathbb{C})\to\mathrm{Sp}_4(\mathbb{C})\qquad A\mapsto\mathrm{diag}(A,A)$$

For such an A-parameter ψ we have a corresponding L-packet $\Pi(\psi)=\Pi(\mu)$ of automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with an equality of L-functions

$$L(s,\Pi(\mu))=L(s+\tfrac{1}{2},\mu)L(s-\tfrac{1}{2},\mu)$$

The central character ω_μ of μ determines a quadratic extension F of $\mathbb{Q}$ and a quadratic Hecke character $\theta:F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$ such that $\mu=\mathcal{AI}_\mathbb{Q}^F(\theta)$ is the automorphic induction of θ from F to $\mathbb{Q}$.

For $\mu=\bigotimes_v\mu_v$ where μ_v is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_v)$ we have $\Pi(\mu)=\bigotimes_v\Pi(\mu_v)$ for local L-packets $\Pi(\psi_v)=\Pi(\mu_v)$ corresponding to the local A-parameter $\psi_v:L_{\mathbb{Q}_v}\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{Sp}_4(\mathbb{C})$ given in terms of the L-parameter $\varphi_{\mu_v}:L_{\mathbb{Q}_v}\to\mathrm{O}_2(\mathbb{C})$ by

$$(w,1)\mapsto\mathrm{diag}(\varphi_{\mu_v}(w),\varphi_{\mu_v}(w))\qquad(1,\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))\mapsto\begin{pmatrix}a&&b&\\&a&&b\\c&&d&\\&c&&d\end{pmatrix}$$

which are fibered over characters of the centralizer group $\mathcal{S}_{\psi_v}=S_{\psi_v}/S^0_{\psi_v}Z$ given as follows: for $\theta=\bigotimes_\mathfrak{v}\theta_\mathfrak{v}$ where $\theta_\mathfrak{v}:F_\mathfrak{v}^\times\to\mathbb{C}^\times$ is a quadratic character, if v does not split in F and $\mathfrak{v}$ is the unique place of F above v then $\mu_v=\mathrm{Ind}_{\mathrm{GL}_1(\mathbb{Q}_v)}^{\mathrm{GL}_1(F_\mathfrak{v})}\theta_\mathfrak{v}$, and if v splits in F and $\mathfrak{v}_1,\mathfrak{v}_2$ are the places of F above v then $\theta_\mathfrak{v}:=(\theta_{\mathfrak{v}_1},\theta_{\mathfrak{v}_2})$ is a pair of characters $\theta_{\mathfrak{v}_1}, \theta_{\mathfrak{v}_2} : F_v^\times \to \mathbb{C}^\times$ and $\mu_v = \theta_{\mathfrak{v}_1} \times \theta_{\mathfrak{v}_2}$ is a principal series representation. Then we have

$$\mathcal{S}_{\psi_v} = \begin{cases} 0 & \theta_{\mathfrak{v}} \text{ not Galois invariant} \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

In other words, we have $\Pi(\psi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets

$$\Pi(\psi_v) = \begin{cases} \{\pi_v^+\} & \theta_{\mathfrak{v}} \text{ not Galois invariant} \\ \{\pi_v^+, \pi_v^-\} & \text{otherwise} \end{cases}$$

The local L-packet $\Pi(\psi_v)$ contains the unique unitary non-tempered non-generic irreducible admissible representation π_v^+ (the basepoint) with local L-parameter $\varphi_{\psi_v} : L_{\mathbb{Q}_v} \to \mathrm{Sp}_4(\mathbb{C})$, $w \mapsto \psi(w, \mathrm{diag}(\nu^{1/2}, \nu^{-1/2}))$. For $v = p$ a finite place of $\mathbb{Q}$ the basepoint π_p^+ is the Langlands quotient of the Klingen induced representation $\omega_{\mu_p}\nu \rtimes \mu_p\nu^{-1/2}$ where:

- $\pi_p^+ = \theta_{\mathfrak{p}_1}\theta_{\mathfrak{p}_2}^{-1} \rtimes \theta_{\mathfrak{p}_2} 1_{\mathrm{GL}_2(\mathbb{Q}_p)}$ (type IIIb) for $\omega_{\mu_p} = 1$ and $\theta_{\mathfrak{p}} = (\theta_{\mathfrak{p}_1}, \theta_{\mathfrak{p}_2})$ is not Galois invariant;
- $\pi_p^+ = L(\omega_{\mu_p}\nu \times \omega_{\mu_p} \rtimes \chi_p\nu^{-1/2})$ (type Vd) for $\omega_{\mu_p} \neq 1$ and $\theta_{\mathfrak{p}} = \chi_p \circ \mathrm{Nm}_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ for χ_p a quadratic character of $\mathrm{GL}_1(\mathbb{Q}_p)$ (equivalently $\mu_p = \chi_p \times \omega_{\mu_p}\chi_p$);
- $\pi_p^+ = L(\nu \times 1_{\mathrm{GL}_1(\mathbb{Q}_p)} \rtimes \chi_p\nu^{-1/2})$ (type VId) for $\omega_{\mu_p} = 1$ and $\theta_{\mathfrak{p}} = (\chi_p, \chi_p)$ is Galois invariant (equivalently $\mu_p = \chi_p \times \chi_p$);
- $\pi_p^+ = L(\omega_{\mu_p}\nu \rtimes \mu_p\nu^{-1/2})$ (type IXb) for $\omega_{\mu_p} \neq 1$ and $\theta_{\mathfrak{p}}$ not Galois invariant (equivalently μ_p is supercuspidal).

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ the basepoint π_∞^+ is the Langlands quotient of the Klingen induced representation $\omega_{\mu_\infty}\nu \rtimes \mu_\infty\nu^{-1/2}$ where:

- $\pi_\infty^+ = \theta_{\mathfrak{w}_1}\theta_{\mathfrak{w}_2}^{-1} \rtimes \theta_{\mathfrak{w}_2} 1_{\mathrm{GL}_2(\mathbb{R})}$ (minimal K_∞-type $(k+1, 1)$) for $\omega_{\mu_\infty} = 1$ and $\theta_{\mathfrak{w}} = (\theta_{\mathfrak{w}_1}, \theta_{\mathfrak{w}_2})$ is not Galois invariant;
- $\pi_\infty^+ = L(\omega_{\mu_\infty}\nu \times \omega_{\mu_\infty} \rtimes \chi_\infty\nu^{-1/2})$ (minimal K_∞-type $(1,1)$) for $\omega_{\mu_\infty} \neq 1$ and $\theta_{\mathfrak{w}} = \chi_\infty \circ \mathrm{Nm}_{F_{\mathfrak{w}}/\mathbb{R}}$ for χ_∞ a quadratic character of $\mathrm{GL}_1(\mathbb{R})$;
- $\pi_\infty^+ = L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_\infty\nu^{-1/2})$ (minimal K_∞-type $(0,0)$) for $\omega_{\mu_\infty} = 1$ and $\theta_{\mathfrak{w}} = (\chi_{\mathfrak{w}_1}, \chi_{\mathfrak{w}_2})$ is Galois invariant;

- $\pi_\infty^+ = L(\omega_{\mu_\infty}\nu \rtimes \mathcal{D}(k)\nu^{-1/2})$ (minimal K_∞-type $(k+1,1)$) for $\omega_{\mu_\infty} \neq 1$ and $\theta_{\mathfrak{\infty}}$ not Galois invariant.

The global A-packet $\Pi(\psi) = \{\pi = \bigotimes_v \pi_v | \pi_v \in \Pi(\psi_v), \pi_v = \pi_v^+$ for almost all places v of $\mathbb{Q}\}$ contains the automorphic representation $\pi^+ = \bigotimes_v \pi_v^+$ an isobaric constituent of the Klingen induced representation $\omega_\mu\nu \rtimes \mu\nu^{-1/2}$. By Arthur's multiplicity formula the Soudry packets are always stable: for $\pi \in \Pi(\mu)$ we have $m(\pi) = 1$.

Theta Lifts We now explain how the above A-packets can be constructed by theta lifts. Consider the representations of $\mathrm{GO}_2(\mathbb{A}_F)$ parameterized as follows. For $\mathfrak{v}$ a place of F over a place v of $\mathbb{Q}$ and $\theta_\mathfrak{v}$ the quadratic character of $\mathrm{GL}_1(F_\mathfrak{v}^\times)$ corresponding to $F_\mathfrak{v}/\mathbb{Q}_v$ we have the local representation $\widetilde{\theta}_\mathfrak{v} = \mathrm{Ind}_{\mathrm{GSO}_2(F_\mathfrak{v})}^{\mathrm{GO}_2(F_\mathfrak{v})}\theta_\mathfrak{v}$ of $\mathrm{GO}_2(F_\mathfrak{v})$, where $\mathrm{GSO}_2(F_\mathfrak{v}) \simeq \mathrm{GL}_1(F_\mathfrak{v})$ and $\mathrm{GO}_2(F_\mathfrak{v}) \simeq \langle\tau_\mathfrak{v}\rangle \ltimes \mathrm{GSO}_2(F_\mathfrak{v})$ where $\tau_\mathfrak{v}$ is the nontrivial element of $\mathrm{Gal}(F_\mathfrak{v}/\mathbb{Q}_v)$; when $\theta_\mathfrak{v}$ is not Galois invariant we have that $\mathrm{Ind}_{\mathrm{GSO}_2(F_\mathfrak{v})}^{\mathrm{GO}_2(F_\mathfrak{v})}\theta_\mathfrak{v} = \widetilde{\theta}_\mathfrak{v}^+$ is an irreducible 2-dimensional representation of $\mathrm{GO}_2(F_\mathfrak{v})$, and when $\theta_\mathfrak{v}$ is Galois invariant we have that $\mathrm{Ind}_{\mathrm{GSO}_2(F_\mathfrak{v})}^{\mathrm{GO}_2(F_\mathfrak{v})}\theta_\mathfrak{v} = \widetilde{\theta}_\mathfrak{v}^+ \oplus \widetilde{\theta}_\mathfrak{v}^-$ splits as a direct sum of irreducible characters of $\mathrm{GO}_2(F_\mathfrak{v})$. Then for $\theta : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ the quadratic Hecke character corresponding to $F/\mathbb{Q}$ and for S a finite set of places $\mathfrak{v}$ of F for which $\theta_\mathfrak{v}$ is Galois invariant with $\#S$ even we have the global representation $\widetilde{\theta}^S = \bigotimes_{\mathfrak{v}\in S}\widetilde{\theta}_\mathfrak{v}^- \otimes \bigotimes_{v\notin S}\widetilde{\theta}_\mathfrak{v}^+$.

For $v = p$ a finite place of $\mathbb{Q}$ the theta lift produces an irreducible admissible representation π_p from a local representation $\widetilde{\theta}_\mathfrak{p}$ corresponding to μ_p. By conjugation of Soudry A-parameters to Howe-Piatetski-Shapiro A-parameters the local L-packets $\Pi(\psi_p)$ for the local A-parameter $\psi_p = \mu_p \boxtimes \nu_2$ are given by $\Pi(\psi_p) = \{\pi_p^+\}$ where π_p^+ is the theta lift of the local representation $\widetilde{\theta}_\mathfrak{p}^+$ when $\theta_\mathfrak{p}$ is not Galois invariant (necessarily the basepoint), and by $\Pi(\psi_p) = \{\pi_p^+, \pi_p^-\}$ where $\pi_p^\pm$ are the theta lifts of the local representations $\widetilde{\theta}_\mathfrak{p}^\pm$ when $\theta_\mathfrak{p}$ is Galois invariant. We obtain the following description for the members of the local L-packets $\Pi(\psi_p)$ of Soudry type, along with their corresponding L-parameters and epsilon-values (compare to [106, Table 3]):

Type		ω_{μ_p}	$\theta_\mathfrak{p}$	π_p	φ_p	$\epsilon(\pi_p)$
III	b	$\omega_{\mu_p} = 1$	$(\theta_{\mathfrak{p}_1}, \theta_{\mathfrak{p}_2}), \theta_{\mathfrak{p}_1} \neq \theta_{\mathfrak{p}_2}$	$\theta_{\mathfrak{p}_1}\theta_{\mathfrak{p}_2}^{-1} \rtimes \theta_{\mathfrak{p}_2}1_{\mathrm{GL}_2(\mathbb{Q}_p)}$	$(\theta_{\mathfrak{p}_1} \oplus \theta_{\mathfrak{p}_2}) \otimes \varphi_1$	1
V	a*	$\omega_{\mu_p} \neq 1$	$\chi_p \circ \mathrm{Nm}_{F_\mathfrak{p}/\mathbb{Q}_p}$	$\delta^*([\omega_{\mu_p}, \omega_{\mu_p}\nu] \rtimes \chi_p\nu^{-1/2})$	$(\chi_p \oplus \chi_p\omega_{\mu_p}) \otimes \varphi_{\mathrm{St}}$	-1
	d			$L(\omega_{\mu_p}\nu \times \omega_{\mu_p} \rtimes \chi_p\nu^{-1/2})$	$(\chi_p \oplus \chi_p\omega_{\mu_p}) \otimes \varphi_1$	1
VI	c	$\omega_{\mu_p} = 1$	(χ_p, χ_p)	$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \chi_p\nu^{-1/2})$	$\chi_p\varphi_1 \oplus \chi_p\varphi_{\mathrm{St}}$	-1
	d			$L(\nu \times 1_{\mathrm{GL}_1(\mathbb{Q}_p)} \rtimes \chi_p\nu^{-1/2})$	$(\chi_p \oplus \chi_p) \otimes \varphi_1$	1
IX	b	$\omega_{\mu_p} \neq 1$	Not Gal-invariant	$L(\omega_{\mu_p}\nu \rtimes \mu_p\nu^{-1/2})$	$\phi \otimes \varphi_1$	1

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ the theta lift produces an irreducible admissible representation π_∞ from a local representation $\widetilde{\theta}_{\boldsymbol{\infty}}$ corresponding to μ_∞. By conjugation of Soudry A-parameters to Howe-Piatetski-Shapiro A-parameters the local L-packets $\Pi(\psi_\infty)$ for the local A-parameter $\psi_\infty = \mu_\infty \boxtimes \nu_2$ are given by $\Pi(\psi_\infty) = \{\pi_\infty^+\}$ where π_∞^+ is the theta lift of the local representation $\widetilde{\theta}_{\boldsymbol{\infty}}^+$ when $\theta_{\boldsymbol{\infty}}$ is not Galois invariant (necessarily the basepoint), and by $\Pi(\psi_\infty) = \{\pi_\infty^+, \pi_\infty^-\}$ where $\pi_\infty^\pm$ are the theta lifts of the local representations $\widetilde{\theta}_{\boldsymbol{\infty}}^\pm$ when $\theta_{\boldsymbol{\infty}}$ is Galois invariant. We obtain the following description for the members of the local L-packets $\Pi(\psi_\infty)$ of Soudry type, along with their corresponding L-parameters and epsilon-values (compare to [106, Table 3]):

Type	ω_{μ_∞}	$\theta_{\boldsymbol{\infty}}$	π_∞	φ_∞	$\epsilon(\pi_\infty)$
	$\omega_{\mu_\infty} = 1$	$(\theta_{\boldsymbol{\infty}_1}, \theta_{\boldsymbol{\infty}_2}), \theta_{\boldsymbol{\infty}_1} \neq \theta_{\boldsymbol{\infty}_2}$	$\theta_{\boldsymbol{\infty}_1}\theta_{\boldsymbol{\infty}_2}^{-1} \rtimes \theta_{\boldsymbol{\infty}_2} 1_{\mathrm{GL}_2(\mathbb{R})}$	$(\theta_{\boldsymbol{\infty}_1} \oplus \theta_{\boldsymbol{\infty}_2}) \otimes \varphi_1$	1
$(2,2)$	$\omega_{\mu_\infty} \neq 1$	$\chi_\infty \circ \mathrm{Nm}_{F_{\boldsymbol{\infty}}/\mathbb{R}}$	$\mathcal{D}_{1,0}^{\mathrm{H}}$	$(\chi_\infty \oplus \chi_\infty\omega_{\mu_\infty}) \otimes \varphi_{\mathcal{D}_1}$	-1
$(1,1)$			$L(\omega_{\mu_\infty}\nu \times \omega_{\mu_\infty} \rtimes \chi_\infty\nu^{-1/2})$	$(\chi_\infty \oplus \chi_\infty\omega_{\mu_\infty}) \otimes \varphi_1$	1
$(0,0)$	$\omega_{\mu_\infty} = 1$	$(\chi_\infty, \chi_\infty)$	$L(\mathcal{D}_1\nu^{1/2} \rtimes \chi_\infty\nu^{-1/2})$	$\chi_\infty\varphi_1 \oplus \chi_\infty\varphi_{\mathrm{St}}$	-1
$(1,-1)$			$L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_\infty\nu^{-1/2})$	$(\chi_\infty \oplus \chi_\infty) \otimes \varphi_1$	1
$(k+1,1)$	$\omega_{\mu_\infty} \neq 1$	Not Gal-invariant	$L(\omega_{\mu_\infty}\nu \rtimes \mathcal{D}_k\nu^{-1/2})$	$\phi \otimes \varphi_1$	1

Klingen-CAP Cohomology Consider the Klingen-CAP part of the automorphic discrete spectrum which by Arthur's classification admits a spectral decomposition

$$L^2_{\mathrm{disc},[P_2]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc},[P_2]}(G,\omega) \\ (\psi = \mu \boxtimes \nu_2)}} \bigoplus_{\pi \in \Pi(\psi)} \pi$$

The Klingen-CAP part $H^*_{\mathrm{disc},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ of the intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\mathrm{disc},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc},[P_2]}(G,\omega) \\ (\psi = \mu \boxtimes \nu_2)}} \bigoplus_{\pi \in \Pi(\psi)} \pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

as a representation of $\mathcal{H}_{K_{\text{fin}}}$. Similarly, the Klingen-CAP part $H^*_{\text{disc},[P_2]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)$ of the ℓ-adic intersection cohomology $H^*_{\text{disc}}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\text{disc},[P_2]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\text{disc},[P_2]}(G,\omega)\\(\psi=\mu\boxtimes\nu_2)}} \bigoplus_{\pi\in\Pi(\psi)} \pi_{\text{fin}}^{K_{\text{fin}}} \boxtimes \rho_\pi$$

as a representation of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$.

To determine the structure of the representations $H^*_{\text{disc},[P_2]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)_{\{\pi_{\text{fin}}\}} = \pi^{K_{\text{fin}}} \boxtimes \rho_\pi$ we use the trace formula:

Theorem 4.4.10. Let $V_\lambda \in \text{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1+\lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\text{fin}}}$ for a compact open subgroup $K_{\text{fin}} \subseteq G(\mathbb{A}_\mathbb{Q}^\infty)$. Let $\psi = \mu \boxtimes \nu_2 \in \Psi_{\text{disc},[P_2]}(G,\omega)$ be an A-parameter of Soudry type, where μ is a unitary cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\mu = \omega$.

If $\pi \in \Pi(\psi)$ contributes nontrivially in the intersection cohomology $H^*_{\text{disc},[P_2]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)$, then the contribution occurs only for $\lambda_2 = 0$ and is given as a representation of $\mathcal{H}_{K_{\text{fin}}} \times \text{Gal}$ by

$$H^*_{\text{disc},[P_2]}(\mathcal{S}_{K_{\text{fin}}},\mathbb{V}_\lambda)_{\{\pi_{\text{fin}}\}} \simeq \rho_\mu \oplus \rho_\mu\mathbb{L}$$

Proof. For $\psi = \mu \boxtimes \nu_2 \in \Psi_{\text{disc},[P_2]}(G,\omega)$ an A-parameter of Soudry type and for $\pi \in \Pi(\psi)$ an automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ lifted from the automorphic representation $\pi^H = (\omega_\mu \circ \det) \boxtimes \mu$ of $H(\mathbb{A}_\mathbb{Q})$ with $\pi_{\text{fin}}^{K_{\text{fin}}} \neq 0$ and with π_∞ a cohomological $(\mathfrak{g}, K'_\infty)$-module with central and infinitesimal characters determined by those of V_λ, we have that $\lambda_2 = 0$ and $\pi_\infty = \pi_\infty^{(2)}$ is the Langlands quotient $\pi_\infty^{(2)} = L(\omega_{\mu_\infty}\nu \rtimes \mathcal{D}_{\lambda_1+2}\nu^{-1/2})$ which has nonzero $(\mathfrak{g}, K'_\infty)$ concentrated in degrees 2 and 4 with Hodge numbers (λ_1+2, λ_2) and (λ_1, λ_2+2), respectively $(\lambda_1+3, \lambda_2+1)$ and $(\lambda_1+1, \lambda_2+3)$. Such a representation π has multiplicity $m(\pi) = 1$: the packet $\Pi(\psi)$ is stable.

Choose a correspondence $f_G^{p,\infty} \in C_c^\infty(K^p \setminus G(\mathbb{A}_\mathbb{Q}^{p,\infty})/K^p)$ which is a projection onto the A-packet $\Pi(\psi_{\text{fin}})$, and choose a matching correspondence $f_H^{p,\infty}$ which is a projection onto π_{fin}^H. Recall that the test function f_∞^G satisfies $\text{tr}(\pi_\infty(f_\infty^G)) = 1$. Recall that the test function ϕ_j^G satisfies

$$\text{tr}(\Pi(\psi_p)(\phi_p^G)) = p^{\frac{3}{2}j}\Big(c_1(\mu_p)^j p^{\frac{1}{2}j} + c_2(\mu_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j p^{-\frac{1}{2}j} + c_2(\mu_p)^j p^{-\frac{1}{2}j}\Big)$$

Consider the test function $f^G = \phi_j^G f_G^{p,\infty} f_\infty^G$ and the matching test function $f^H = \phi_j^H f_H^{p,\infty} f_\infty^H$. Now the contribution of the A-packet $\Pi(\psi)$ to $\mathrm{STF}^G(f^G)$ is given by

$$\begin{aligned}
&\sum_{\pi\in\Pi(\psi)} m(\pi)\mathrm{tr}(\pi_\infty(f_\infty^G))\mathrm{tr}(\Pi(\psi_{\mathrm{fin}})(f_G^{p,\infty}))p^{\frac{3}{2}j}\left(c_1(\mu_p)^j p^{\frac{1}{2}j} + c_2(\mu_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j p^{-\frac{1}{2}j} + c_2(\mu_p)^j p^{-\frac{1}{2}j}\right) \\
&= p^{\frac{3}{2}j}\left(c_1(\mu_p)^j p^{\frac{1}{2}j} + c_2(\mu_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j p^{-\frac{1}{2}j} + c_2(\mu_p)^j p^{-\frac{1}{2}j}\right)
\end{aligned}$$

By stability of the packet $\Pi(\psi)$ and by matching, the contribution of π^H to $\mathrm{STF}^H(f^H)$ vanishes. It follows that we have the trace

$$\mathrm{tr}(\mathrm{Frob}_p^j | H^*_{\mathrm{disc},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}}) = p^{\frac{\lambda_1+\lambda_2}{2}}\mathrm{STF}^G(f^G) = p^{\frac{\lambda_1+\lambda_2+3}{2}j}\left(c_1(\mu_p)^j p^{\frac{1}{2}j} + c_2(\mu_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j p^{-\frac{1}{2}j} + c_2(\mu_p)^j p^{-\frac{1}{2}j}\right)$$

Since $\lambda_2 = 0$ we can write

$$\begin{aligned}
&p^{\frac{\lambda_1+\lambda_2+3}{2}j}(c_1(\mu_p)^j p^{\frac{1}{2}j} + c_2(\mu_p)^j p^{\frac{1}{2}j} + c_1(\mu_p)^j p^{-\frac{1}{2}j} + c_2(\mu_p)^j p^{-\frac{1}{2}j}) \\
&= p^{\frac{\lambda_1+2}{2}}(c_1(\mu_p) + c_2(\mu_p)) + p^{\frac{\lambda_1+2}{2}+1}(c_1(\mu_p) + c_2(\mu_p))
\end{aligned}$$

which is the trace of Frob_p^j on $\rho_\mu \oplus \rho_\mu\mathbb{L}$ whose contribution to $H^*_{\mathrm{disc},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees 2 and 4. It follows that

$$H^*_{\mathrm{disc},[P_2]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq \rho_\mu \oplus \rho_\mu\mathbb{L}$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$. The result follows. □

4.4.8 Howe-Piatetski-Shapiro Lifts

We review the Howe-Piatetski-Shapiro lift, following [106, Section 2] and [105, Section 1] specialized to the case $F = \mathbb{Q}$; while this can be used to construct (holomorphic) Siegel modular forms of weights 1 and 2, none of these contribute to the cohomology of local systems on Siegel modular threefolds. To that end we quickly review the construction of the Howe-Piatetski-Shapiro packets in terms of theta lifts from $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ to $\mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})$, where the theta lift from $\mathrm{Mp}_2(\mathbb{R})$ to $\mathrm{PGSp}_4(\mathbb{R})$ provides the information relevant for applying the Vogan-Zuckerman classification in order to rule out cohomological occurrence.

Howe-Piatetski-Shapiro Lifts We consider the trivial central character Howe-Piatetski-Shapiro lift corresponding to the A-parameters for $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of the form $\psi = (\chi_1 \boxtimes \nu_2) \boxplus (\chi_2 \boxtimes \nu_2)$ for distinct quadratic Hecke characters $\chi_1, \chi_2 : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ coming from the embedding of dual groups

$$\{\pm 1\} \times \{\pm 1\} \to \mathrm{Sp}_4(\mathbb{C}) \qquad (t_1, t_2) \mapsto \mathrm{diag}(t_1, t_2, t_2, t_1)$$

where for distinct quadratic Hecke characters $\chi_1, \chi_2 : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ we have an global A-packet $\Pi(\psi)$ of automorphic representations π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with an equality of L-functions

$$L(s, \pi) = L(s + \tfrac{1}{2}, \chi_1) L(s - \tfrac{1}{2}, \chi_1) L(s + \tfrac{1}{2}, \chi_2) L(s - \tfrac{1}{2}, \chi_2)$$

For $\chi_1 = \bigotimes_v \chi_{1,v}$ and $\chi_2 = \bigotimes_v \chi_{2,v}$ where $\chi_{1,v}, \chi_{2,v} : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ are quadratic characters we have $\Pi(\pi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets $\Pi(\psi_v)$ corresponding to the local A-parameter $\psi_v : L_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$ given by

$$(w, 1) \mapsto \mathrm{diag}(\chi_{1,v}(w), \chi_{2,v}(w), \chi_{2,v}(w), \chi_{1,v}(w)) \qquad (1, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \mapsto \begin{pmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d \end{pmatrix}$$

which are fibered over characters of the centralizer group $\mathcal{S}_{\psi_v} = S_{\psi_v}/S_{\psi_v}^0 Z = \mathbb{Z}/2\mathbb{Z}$. In other words, we have $\Pi(\psi) = \bigotimes_v \Pi(\psi_v)$ for local L-packets $\Pi(\psi_v) = \{\pi_v^+, \pi_v^-\}$.

The local L-packet $\Pi(\psi_v)$ contains the unique unitary non-tempered non-generic irreducible admissible representation π_v^+ (the basepoint) with local L-parameter $\varphi_v : L_{\mathbb{Q}_v} \to \mathrm{Sp}_4(\mathbb{C})$ given by $w \mapsto \psi_v(w, \mathrm{diag}(|w|_v^{1/2}, |w|_v^{-1/2}))$. For $v = p$ a finite place of $\mathbb{Q}$ the basepoint π_p^+ is the Langlands quotient of the Borel induced representation $\chi_{1,p}\chi_{2,p}\nu \times \chi_{1,p}\chi_{2,p} \rtimes \chi_{2,p}\nu^{-1/2}$ where:

- $\pi_p^+ = L(\chi_{1,p}\chi_{2,p}\nu \times \chi_{1,p}\chi_{2,p} \rtimes \chi_{2,p}\nu^{-1/2})$ (type Vd) for $\chi_{1,p} \neq \chi_{2,p}$;
- $\pi_p^+ = L(\nu \times 1_{\mathrm{GL}_1(\mathbb{Q}_p)} \rtimes \chi_{2,p}\nu^{-1/2})$ (type VId) for $\chi_{1,p} = \chi_{2,p}$.

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ the basepoint π_∞^+ is the Langlands quotient of the Borel induced representation $\chi_{1,\infty}\chi_{2,\infty}\nu \times \chi_{1,\infty}\chi_{2,\infty} \rtimes \chi_{2,\infty}\nu^{-1/2}$ where:

- $\pi_\infty^+ = L(\mathrm{sign}\nu \times \mathrm{sign} \rtimes \nu^{1/2})$ (minimal K_∞-type $(1,1)$) for $\chi_{1,\infty} \neq \chi_{2,\infty}$;
- $\pi_\infty^+ = L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_{2,\infty}\nu^{-1/2})$ (minimal K_∞-type $(0,0)$) for $\chi_{1,\infty} = \chi_{2,\infty}$.

The global A-packet $\Pi(\psi) = \{\pi = \bigotimes_v \pi_v | \pi_v \in \Pi(\psi_v), \pi_v = \pi_v^+$ for almost all places v of $\mathbb{Q}\}$ contains the automorphic representation $\pi^+ = \bigotimes_v \pi_v^+$ an isobaric constituent of the Borel induced representation $\chi_1\chi_2\nu \times \chi_1\chi_2 \rtimes \chi_2\nu^{-1/2}$. By Arthur's multiplicity formula the Howe-Piatetski-Shapiro packets are unstable: for $\pi \in \Pi(\psi)$ we have $m(\pi) = 1$ precisely if $\epsilon(\pi) = 1$. That is, for $\pi = \bigotimes_v \pi_v \in \Pi(\psi)$ and for Σ the set of places v of $\mathbb{Q}$ such that π_v is non-generic we have $m(\pi) = 1$ precisely if $\#\Sigma$ is even. In particular $\Pi(\chi_1, \chi_2)$ contains 1 discrete element if $\#\Sigma = 0$ and contains $2^{\#\Sigma-1}$ discrete elements otherwise.

Theta Lifts We now explain how the above A-packets can be constructed by theta lifts. Consider the Weil representations of $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ parameterized as follows. For v a place of $\mathbb{Q}$ and $\chi_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ a quadratic character we have the local Weil representation $\widetilde{\pi}_{\chi_v} = \widetilde{\pi}_{\chi_v}^+ \oplus \widetilde{\pi}_{\chi_v}^-$ of $\mathrm{Mp}_2(\mathbb{Q}_v)$ where $\widetilde{\pi}_{\chi_v}^+$ is the even local Weil representation and $\widetilde{\pi}_{\chi_v}^-$ is the supercuspidal odd local Weil representation. Then for $\chi = \bigotimes_v \chi_v : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a nontrivial quadratic Hecke character and for S a finite set of places of $\mathbb{Q}$ with $\#S$ even we have the global Weil representation $\widetilde{\pi}_\chi^S = \bigotimes_{v\in S} \widetilde{\pi}_{\chi_v}^- \otimes \bigotimes_{v\notin S} \widetilde{\pi}_{\chi_v}^+$. Now by [106, Lemma 2.1], as S runs through the set of places of $\mathbb{Q}$ with $\#S$ even, the theta lifts of the global Weil representations $\widetilde{\pi}_\chi^S$ from $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ to $\mathrm{SO}_5(\mathbb{A}_\mathbb{Q}) \simeq \mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})$ run through the discrete automorphic representations in the global A-packet $\Pi(\psi)$ with parameter $\psi = (\chi \boxtimes \nu_2) \boxplus (1 \boxtimes \nu_2)$. The general case is obtained by twisting: for $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ a unitary Hecke character and for $\chi_1, \chi_2 : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ unitary Hecke characters with $\chi_1^2 = \chi_2^2 = \omega$, as S runs through the set of places of $\mathbb{Q}$ with $\#S$ even, the twists by χ_2 of the theta lifts of the global Weil repersentations $\widetilde{\pi}_{\chi_1/\chi_2}^S$ from $\mathrm{Mp}_2(\mathbb{A}_\mathbb{Q})$ to $\mathrm{SO}_5(\mathbb{A}_\mathbb{Q}) \simeq \mathrm{PGSp}_4(\mathbb{A}_\mathbb{Q})$ from run through the discrete automorphic representations in the global A-packet $\Pi(\psi)$ with parameter $\psi = (\chi_1 \boxtimes \nu_2) \boxplus (\chi_2 \boxtimes \nu_2)$.

For $v = p$ a finite place of $\mathbb{Q}$ and for $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ a quadratic character the theta lift produces irreducible admissible representations $\pi_p^\pm$ of $\mathrm{SO}_5(\mathbb{Q}_v) \simeq \mathrm{PGSp}_4(\mathbb{Q}_p)$ from the local Weil representations $\widetilde{\pi}_{\chi_p}^\pm$. We obtain the following theta lifts (compare to [106, 17]):

Type		$\widetilde{\pi}_{\chi_p}$	π_p	φ_p
V	a*	$\widetilde{\pi}_{\chi_p}^-$	$\delta^*([\chi_p, \chi_p\nu] \rtimes \nu^{1/2})$	$\chi_p\varphi_{\mathrm{St}} \oplus \varphi_{\mathrm{St}}$
	d	$\widetilde{\pi}_{\chi_p}^+$	$L(\chi_p\nu \times \chi_p \rtimes \nu^{-1/2})$	$\chi_p\varphi_1 \oplus \varphi_1$
VI	c	$\widetilde{\pi}_{1_p}^-$	$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \nu^{-1/2})$	$\varphi_1 \oplus \varphi_{\mathrm{St}}$
	d	$\widetilde{\pi}_{1_p}^+$	$L(\nu \times 1_{\mathrm{GL}_1(\mathbb{Q}_p)} \rtimes \nu^{-1/2})$	$\varphi_1 \oplus \varphi_1$

where φ_1 is the L-parameter of the trivial representation $1_{\mathrm{GL}_2(\mathbb{Q}_p)}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ and where φ_{St} is the L-parameter of the Steinberg representation $\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. By [106, Lemma 2.1] it follows that for a quadratic character $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ the local L-packets $\Pi(\psi_p)$ for the local A-parameter $\psi_p = (\chi_p \boxtimes \nu_2) \boxplus (1 \boxtimes \nu_2)$ are given by $\Pi(\psi_p) = \{\pi_p^+, \pi_p^-\}$ where $\pi_p^\pm$ are the theta lifts of the local Weil representations $\widetilde{\pi}^\pm_{\chi_p}$. The general case is obtained by twisting: for characters $\chi_{1,p}, \chi_{2,p} : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ the local L-packets $\Pi(\psi_p)$ for the local A-parameter $\psi_p = (\chi_{1,p} \otimes \nu_2) \boxplus (\chi_{2,p} \boxtimes \nu_2)$ are given by $\Pi(\psi_p) = \{\pi_p^+, \pi_p^-\}$ where $\pi_p^\pm$ are twists by $\chi_{2,p}$ of the theta lifts of the local Weil representations $\widetilde{\pi}^\pm_{\chi_{1,p}/\chi_{2,p}}$. We obtain the following description for the members of the local L-packets $\Pi(\psi_p)$ of Howe-Piatetski-Shapiro type, along with their corresponding L-parameters and epsilon-values (compare to [106, Table 1]):

Type		$(\chi_{1,p}, \chi_{2,p})$	π_p	φ_p	$\epsilon(\pi_p)$
V	a*	$\chi_{1,p} \neq \chi_{2,p}$	$\delta^*([\chi_{1,p}\chi_{2,p}, \chi_{1,p}\chi_{2,p}\nu] \rtimes \chi_{2,p}\nu^{-1/2})$	$\chi_{1,p}\varphi_{\mathrm{St}} \oplus \chi_{2,p}\varphi_{\mathrm{St}}$	-1
	d		$L(\chi_{1,p}\chi_{2,p}\nu \times \chi_{1,p}\chi_{2,p} \rtimes \chi_{2,p}\nu^{-1/2})$	$\chi_{1,p}\varphi_1 \oplus \chi_{2,p}\varphi_1$	1
VI	c	$\chi_{1,p} = \chi_{2,p}$	$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_p)}\nu^{1/2} \rtimes \chi_{2,p}\nu^{-1/2})$	$\chi_{1,p}\varphi_1 \oplus \chi_{2,p}\varphi_{\mathrm{St}}$	-1
	d		$L(\nu \times 1_{\mathrm{GL}_1(\mathbb{Q}_p)} \rtimes \chi_{2,p}\nu^{-1/2})$	$\chi_{1,p}\varphi_1 \oplus \chi_{2,p}\varphi_1$	1

For $v = \infty$ the Archimedean place of $\mathbb{Q}$ and for $\chi_\infty : \mathbb{R}^\times \to \mathbb{C}^\times$ a quadratic charcter the theta lift produces irreducible admissible representations $\pi_\infty^\pm$ of $\mathrm{SO}_5(\mathbb{R}) \simeq \mathrm{PGSp}_4(\mathbb{R})$ from the local Weil representations $\widetilde{\pi}^\pm_{\chi_\infty}$. We obtain the following theta lifts:

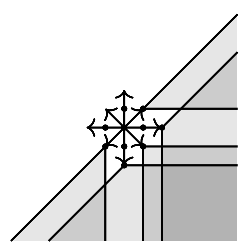
Type	$\widetilde{\pi}_{\chi_\infty}$	π_∞	φ_∞
$(2,2)$	$\widetilde{\pi}^-_{\chi_\infty}$	$\mathcal{D}^{\mathrm{H}}_{1,0}$	$\varphi_{\mathcal{D}_1} \oplus \varphi_{\mathcal{D}_1}$
$(1,1)$	$\widetilde{\pi}^+_{\chi_\infty}$	$L(\mathrm{sign}\nu \times \mathrm{sign} \rtimes \nu^{-1/2})$	$\mathrm{sign}\varphi_1 \oplus \varphi_1$
$(1,-1)$	$\widetilde{\pi}^-_{1_\infty}$	$L(\mathcal{D}_1\nu^{1/2} \rtimes \nu^{-1/2})$	$\varphi_1 \oplus \varphi_{\mathcal{D}_1}$
$(0,0)$	$\widetilde{\pi}^+_{1_\infty}$	$L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \nu^{-1/2})$	$\varphi_1 \oplus \varphi_1$

where φ_1 is the L-parameter of the trivial representation $1_{\mathrm{GL}_2(\mathbb{R})}$ of $\mathrm{GL}_2(\mathbb{R})$ and $\varphi_{\mathcal{D}_1}$ is the L-parameter of the discrete series representation $\mathcal{D}_1$ of $\mathrm{GL}_2(\mathbb{R})$ with minimal $\mathrm{O}(2)$-type 2 and trivial central character. By [106, Lemma 2.1] it follows that for a quadratic character $\chi_\infty : \mathbb{R}^\times \to \mathbb{C}^\times$ the local L-packets $\Pi(\psi_\infty)$ for the local A-parameter $\psi_\infty = (\chi_\infty \boxtimes \nu_2) \boxplus (1 \boxtimes \nu_2)$ are given by $\Pi(\psi_\infty) = \{\pi_\infty^+, \pi_\infty^-\}$ where $\pi_\infty^\pm$ are the theta lifts of the local Weil representations $\widetilde{\pi}^\pm_{\chi_\infty}$. The general case is obtained by twisting: for characters $\chi_{1,\infty}, \chi_{2,\infty} : \mathbb{R}^\times \to \mathbb{C}^\times$ the local L-packets Π_{ψ_∞} for the local A-parameter $\psi_\infty = (\chi_{1,\infty} \otimes \nu_2) \boxplus (\chi_{2,\infty} \boxtimes \nu_2)$ are given by $\Pi(\psi_\infty) =$

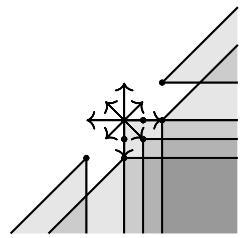
$\{\pi_\infty^+, \pi_\infty^-\}$ where $\pi_\infty^\pm$ are twists by $\chi_{2,\infty}$ of the theta lifts of the local Weil representations $\widetilde{\pi}^\pm_{\chi_{1,\infty}/\chi_{2,\infty}}$. We obtain the following description for the members of the local L-packets $\Pi(\psi_\infty)$ of Howe-Piatetski-Shapiro type, along with their corresponding L-parameters and epsilon-values (compare to [106, Table 1]):

Type	$(\chi_{1,\infty}, \chi_{2,\infty})$	π_∞	φ_∞	$\epsilon(\pi_\infty)$
$(2,2)$	$\chi_{1,\infty} \neq \chi_{2,\infty}$	$\mathcal{D}^{\mathrm{H}}_{1,0}$	$\varphi_{\mathcal{D}_1} \oplus \varphi_{\mathcal{D}_1}$	-1
$(1,1)$		$L(\mathrm{sign}\nu \times \mathrm{sign} \rtimes \nu^{-1/2})$	$\mathrm{sign}\varphi_1 \oplus \varphi_1$	1
$(1,-1)$	$\chi_{1,\infty} = \chi_{2,\infty}$	$L(\mathcal{D}_1\nu^{1/2} \rtimes \chi_{2,\infty}\nu^{-1/2})$	$\chi_{1,\infty}\varphi_1 \oplus \varphi_{\mathcal{D}_1}$	-1
$(0,0)$		$L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_{2,\infty}\nu^{-1/2})$	$\chi_{1,\infty}\varphi_1 \oplus \chi_{2,\infty}\varphi_1$	1

The K_∞-types in the table are obtained as follows. For a quadratic character $\chi_{2,\infty} : \mathbb{R}^\times \to \mathbb{C}^\times$ the representation $\mathrm{sign}\nu \times \mathrm{sign} \rtimes \chi_{2,\infty}\nu^{-1/2}$ has four irreducible constituents: $\mathcal{D}^{\mathrm{W}}_{1,0}$ the large generic limit discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with minimal K_∞-type $(1,-1)$ occurring with multiplicity 1, $L(\mathcal{D}_1\nu^{1/2} \rtimes \chi_{2,\infty}\nu^{-1/2})$ the non-tempered Langlands quotient of $\mathcal{D}_1\nu^{1/2} \rtimes \chi_{2,\infty}\nu^{-1/2}$ with minimal K_∞-type $(1,-1)$ occurring with multiplicity 2, and $L(\mathrm{sign}\nu \times \mathrm{sign} \rtimes \chi_{2,\infty}\nu^{-1/2})$ the Langlands quotient of $\mathrm{sign}\nu \times \mathrm{sign} \rtimes \chi_{2,\infty}\nu^{-1/2}$ with minimal K_∞-type $(1,1)$ occurring with multiplicity 1. For this last constituent the K_∞-type $(1,1)$ is obtained by subtracting the K_∞-types of the other constituents from the K_∞-types of the induced representation. We obtain the following K_∞-type regions (compare to [106, 20]):



For a quadratic character $\chi_{2,\infty} : \mathbb{R}^\times \to \mathbb{C}^\times$ the representation $\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_{2,\infty}\nu^{-1/2}$ has four irreducible constituents: $\mathcal{D}^{\mathrm{H}}_{1,0}$ the holomorphic limit discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with minimal K_∞-type $(2,2)$ occurring with multiplicity 1, $\mathcal{D}^{\mathrm{W}}_{1,0}$ the large generic limit discrete series representation of $\mathrm{PGSp}_4(\mathbb{R})$ with minimal K_∞-type $(1,-1)$ occurring with multiplicity 1, $L(\mathcal{D}_1\nu^{1/2} \rtimes \chi_{2,\infty}\nu^{-1/2})$ the non-tempered Langlands quotient of $\mathcal{D}_1\nu^{1/2} \rtimes \chi_{2,\infty}\nu^{-1/2}$ with minimal K_∞-type $(1,-1)$ occurring with multiplicity 1, and $L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_{2,\infty}\nu^{-1/2})$ the Langlands quotient of $\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_{2,\infty}\nu^{-1/2}$ with minimal $\mathrm{U}(2)$-type $(0,0)$ occurring with multiplicity 1. For this last constituent the K_∞-type $(0,0)$ is obtained by subtracting the K_∞-types of the other constituents from the K_∞-types of the induced representation. Note that $\mathcal{D}^{\mathrm{H}}_{1,0}$ is the representation underlying holomorphic Siegel modular forms of weight 2. We obtain the following K_∞-type regions (compare to [106, 21]):



Borel-CAP Cohomology We finally explain how the above description of the (Archimedean) local L-packets implies that Howe-Piatetski-Shapiro lifts do not contribute to the cohomology of local systems. Consider the Borel-CAP part of the automorphic discrete spectrum which by Arthur's classification admits a spectral decomposition

$$L^2_{\mathrm{disc},[P_0]}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) = \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_0]}(G,\omega)\\ (\psi=(\chi_1\boxtimes\nu_2)\boxplus(\chi_2\boxtimes\nu_2))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} \pi$$

We will consider the Borel-CAP part $IH^*_{[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ of the intersection cohomology $IH^*(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \mathbb{H}^*(\overline{\mathcal{S}}_{K_{\mathrm{fin}}}, j_{!*}\mathbb{V}_\lambda)$ which by the above admits a spectral decomposition

$$H^*_{\mathrm{disc},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) = \bigoplus_{\omega} \bigoplus_{\substack{\psi\in\Psi_{\mathrm{disc},[P_0]}(G,\omega)\\ (\psi=(\chi_1\boxtimes\nu_2)\boxplus(\chi_2\boxtimes\nu_2))}} \bigoplus_{\substack{\pi\in\Pi(\psi)\\ \langle\cdot,\pi\rangle=1}} m(\pi)\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \pi_\infty \otimes V_\lambda)$$

Theorem 4.4.11. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1+\lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$ for a compact open subgroup $K_{\mathrm{fin}} \subseteq G(\mathbb{A}^\infty_\mathbb{Q})$. Then the Borel-CAP cohomology $H^*_{\mathrm{disc},[P_0]}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is trivial.

Proof. Let $\psi \in \Psi_{\mathrm{disc},[P_0]}(G,\omega)$ be an A-parameter of Howe-Piatetski-Shapiro type and let $\pi = \bigotimes_v \pi_v \in \Pi(\psi)$ be a member of the corresponding L-packet. Then π_∞ can only be of the form $L(\mathrm{sign}\nu \times \mathrm{sign} \rtimes \nu^{-1/2})$, $\mathcal{D}^{\mathrm{H}}_{1,0}$, $L(\nu \times 1_{\mathrm{GL}_1(\mathbb{R})} \rtimes \chi_\infty\nu^{-1/2})$, or $L(\mathcal{D}_1\nu^{1/2} \rtimes \chi_1\nu^{-1/2})$ with minimal K_∞-types $(1,1)$, $(2,2)$, $(0,0)$, and $(1,-1)$ respectively, none of which are cohomological by the Vogan-Zuckerman classification for GSp_4. The result follows. $\square$

Despite not contributing to cohomology, the Howe-Piatetski-Shapiro packets are still of some use to us: by conjugation of Soudry A-parameters to Howe-Piatetski-Shapiro A-parameters, one can reduce the description of Soudry packets to the description of Howe-Piatetski-Shapiro packets, at least in the case of trivial central character.

4.4.9 1-Dimensional Cohomology

There is one last case to consider, namely the A-parameters of 1-dimensional type, which contribute to intersection cohomology, but never to cuspidal cohomology. There is little to review in this case; we jump straight to the proof.

Consider the 1-dimensional part of the automorphic discrete spectrum $L^2_{\mathrm{disc,1dim}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q}))$ which by Arthur's classification admits a spectral decomposition

$$L^2_{\mathrm{disc,1dim}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}_\mathbb{Q})) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc,1dim}}(G,\omega) \\ (\psi = \chi \boxtimes \nu_4)}} \chi \circ \mathrm{sim}$$

The 1-dimensional part $H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ of the intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}(\mathbb{C}), \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc,1dim}}(G,\omega) \\ (\psi = \chi \boxtimes \nu_4)}} (\chi_{\mathrm{fin}} \circ \mathrm{sim})^{K_{\mathrm{fin}}} \otimes H^*(\mathfrak{g}, K'_\infty; \chi_\infty \circ \mathrm{sim} \otimes V_\lambda)$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}}$. Similarly, the 1-dimensional part $H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ of the ℓ-adic intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ admits a spectral decomposition

$$H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda) \simeq \bigoplus_{\omega} \bigoplus_{\substack{\psi \in \Psi_{\mathrm{disc,1dim}}(G,\omega) \\ (\psi = \chi \boxtimes \nu_4)}} (\chi_{\mathrm{fin}} \circ \mathrm{sim})^{K_{\mathrm{fin}}} \boxtimes \rho_{\chi \circ \mathrm{sim}}$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$.

To determine the structure of the representations $H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} = (\chi_{\mathrm{fin}} \circ \mathrm{sim})^{K_{\mathrm{fin}}} \boxtimes \rho_{\chi \circ \mathrm{sim}}$ we use the trace formula:

Theorem 4.4.12. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{S}_{K_{\mathrm{fin}}}$ for a compact open subgroup

$K_{\mathrm{fin}} \subseteq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Let $\psi = \chi \otimes \nu_4 \in \Psi_{\mathrm{disc,1dim}}(G,\omega)$ be an A-parameter of 1-dimensional type, where $\chi : \mathbb{Q}^{\times} \setminus \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times}$ is a unitary Hecke character with $\chi^4 = \omega$.

If $\pi \in \Pi(\psi)$ contributes nontrivially in the intersection cohomology $H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_{\lambda})$, then the contribution occurs only for $\lambda_1 = \lambda_2 = 0$ and is given as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$ by

$$H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_{\lambda})_{\{\pi_{\mathrm{fin}}\}} \simeq (\chi_{\mathrm{fin}} \circ \mathrm{sim})_{\mathrm{fin}}^{K_{\mathrm{fin}}} \boxtimes (\mathbb{L}_{\chi}^{0} \oplus \mathbb{L}_{\chi}^{1} \oplus \mathbb{L}_{\chi}^{2} \oplus \mathbb{L}_{\chi}^{3})$$

Proof. For $\psi = \chi \otimes \nu_4 \in \Psi_{\mathrm{disc,1dim}}(G,\omega)$ an A-parameter of 1-dimensional type and for $\pi \in \Pi(\psi)$ an automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ with $\pi_{\mathrm{fin}}^{K_{\mathrm{fin}}} \neq 0$ and with π_{∞} a cohomological $(\mathfrak{g}, K'_{\infty})$-module with central and infinitesimal characters determined by those of V_{λ}, we have that $\pi_{\infty} = \chi_{\infty} \circ \mathrm{sim}$ is a 1-dimensional representation which has nonzero $(\mathfrak{g}, K'_{\infty})$-cohomology concentrated in degrees $0, 2, 4, 6$ with Hodge numbers $(0,0)$, $(1,1)$, $(2,2)$, and $(3,3)$ respectively. Such a representation π has multiplicity $m(\pi) = 1$: the packet $\Pi(\psi)$ is stable.

Choose a correspondence $f_G^{p,\infty} \in C_c^{\infty}(K^p \setminus G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})/K^p)$ which is a projection onto $\chi_{\mathrm{fin}} \circ \mathrm{sim}$. Recall that the test function f_{∞}^G satisfies $\mathrm{tr}(\chi_{\infty} \circ \mathrm{sim}(f_{\infty}^G)) = 1$. Recall that the test function ϕ_j^G satisfies

$$\mathrm{tr}(\Pi(\psi_p)(\phi_j^G)) = p^{\frac{3}{2}j} \left(c(\chi_p)^j p^{\frac{3}{2}j} + c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{3}{2}j} \right)$$

Consider the test function $f^G = \phi_j^G f_G^{p,\infty} f_{\infty}^G$. Now the contribution of the A-packet $\Pi(\psi)$ to $\mathrm{STF}^G(f^G)$ is given by

$$\begin{aligned}
&\sum_{\pi \in \Pi(\psi)} m(\pi) \mathrm{tr}(\pi_{\infty}(f_{\infty}^G)) \mathrm{tr}(\Pi(\psi_{\mathrm{fin}})(f_G^{p,\infty})) p^{\frac{3}{2}j} \left(c(\chi_p)^j p^{\frac{3}{2}j} + c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{3}{2}j} \right) \\
&= p^{\frac{3}{2}j} \left(c(\chi_p)^j p^{\frac{3}{2}j} + c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{3}{2}j} \right)
\end{aligned}$$

By stability of the packet $\Pi(\psi)$ and by matching, only $\mathrm{STF}^G(f^G)$ contributes to $\mathrm{tr}(\mathrm{Frob}_p^j | H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_{\lambda})_{\{\pi_{\mathrm{fin}}\}})$. It follows that we have the trace

$$\begin{aligned}
&\mathrm{tr}(\mathrm{Frob}_p^j | H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_{\lambda})_{\{\pi_{\mathrm{fin}}\}}) = \mathrm{STF}^G(f^G) \\
&= p^{\frac{3}{2}j} \left(c(\chi_p)^j p^{\frac{3}{2}j} + c(\chi_p)^j p^{\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{1}{2}j} + c(\chi_p)^j p^{-\frac{3}{2}j} \right)
\end{aligned}$$

which is the trace of Frob_p^j on $\mathbb{L}_\chi^0 \oplus \mathbb{L}_\chi^1 \oplus \mathbb{L}_\chi^2 \oplus \mathbb{L}_\chi^3$ whose contribution to $H^*_{\mathrm{disc,gen}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)$ is concentrated in degrees $0, 2, 4, 6$. It follows that

$$H^*_{\mathrm{disc,1dim}}(\mathcal{S}_{K_{\mathrm{fin}}}, \mathbb{V}_\lambda)_{\{\pi_{\mathrm{fin}}\}} \simeq (\chi_{\mathrm{fin}} \circ \mathrm{sim})^{K_{\mathrm{fin}}} \boxtimes (\mathbb{L}_\chi^0 \oplus \mathbb{L}_\chi^1 \oplus \mathbb{L}_\chi^2 \oplus \mathbb{L}_\chi^3)$$

as a representation of $\mathcal{H}_{K_{\mathrm{fin}}} \times \mathrm{Gal}$. The result follows. $\square$

4.5 Example: Cohomology of Local Systems on $\mathcal{A}_2$

Let $\mathcal{A}_2$ be the moduli stack of principally polarized Abelian surfaces. We revisit earlier results and compute the cohomology $H^*(\mathcal{A}_2, \mathbb{V}_\lambda)$ as a Gal-module, reproving theorems of [55] and [96].

For the intersection cohomology and cuspidal cohomology we have the following:

Proposition 4.5.1. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2$. The intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{A}_2, \mathbb{V}_\lambda)$ is concentrated in degrees 2, 3, 4 and given by

$$H^0_{\mathrm{disc}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^0 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^2_{\mathrm{disc}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{L}^1 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} H^3_{\mathrm{disc}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] + &\begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \\ &+ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1} \end{aligned}$$

$$H^4_{\mathrm{disc}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{L}^2 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^6_{\mathrm{disc}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^3 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the cuspidal cohomology $H^*_{\mathrm{cusp}}(\mathcal{A}_2, \mathbb{V}_\lambda)$ (equivalently the inner cohomology $H^*_!(\mathcal{A}_2, \mathbb{V}_\lambda)$) is concentrated in degrees 2, 3, 4 and given by

$$H^2_{\mathrm{cusp}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{cusp}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] \oplus \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\oplus\, s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1}$$

$$H^4_{\mathrm{cusp}}(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the intersection cohomology, we note the following. The Yoshida lifts with $\pi_\infty = \pi_\infty^{\mathrm{H}}$ cannot occur for reasons of parity in the multiplicity formula $m(\pi) = \frac{1}{2}(1 + \langle \mu_\infty, \pi_\infty \rangle)$ (so only the Yoshida lifts with $\pi_\infty = \pi_\infty^{\mathrm{W}}$ contribute, yielding the contribution $s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1}$ in degree 3). The Saito-Kurokawa lifts with $\pi_\infty = \pi_\infty^{\mathrm{H}}$ can only occur for $\lambda_1 = \lambda_2$ odd (and contribute $\mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]$ in degree 3 in this case) and the Saito-Kurokawa lifts with $\pi_\infty = \pi_\infty^{(1)+}$ can only occur for $\lambda_1 = \lambda_2$ even (and contribute $s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1}$ and $s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+2}$ in degrees 2 and 4 in this case), in both cases this is for reasons of parity in the multiplicity formula $m(\pi) = \frac{1}{2}(1 + (-1)^{\lambda_1}\langle \mu_\infty, \pi_\infty \rangle)$. The Soudry lifts cannot occur since there are no CM cusp forms in level 1.

For the cuspidal cohomology we note the following. The general type contributions are always cuspidal. The contributions from Yoshida lifts are cuspidal as soon as $\mu_1 \neq \mu_2$ (which is automatic since $\lambda_1+\lambda_2+4 > \lambda_1-\lambda_2+2$ as soon as $\lambda_1 \geq \lambda_2 \geq 0$). The contributions from Saito-Kurokawa lifts are cuspidal as soon as $L(\frac{1}{2}, \mu) = 0$ (which is automatic by the functional equation in the case $\lambda_1 = \lambda_2$ odd, and which explains the modification in degrees 2 and 4 in the case $\lambda_1 = \lambda_2$ even). The contributions from 1-dimensional representations are never cuspidal. □

For the Eisenstein cohomology, we have the following:

Proposition 4.5.2. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2 > 0$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2$. The compactly supported Eisenstein cohomology is concentrated in degrees 2, 3, 4 and given by

$$
\begin{aligned}
H^2_{\mathrm{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) &= \mathbb{S}_{\Gamma(1)}[\lambda_2+2] \\
&\oplus s_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^0 \\
&\oplus \begin{cases} \mathbb{L}^0 & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} \\
H^3_{\mathrm{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) &= \mathbb{S}_{\Gamma(1)}[\lambda_1+3] \\
&\oplus \begin{cases} s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases} \\
&\oplus \begin{cases} \mathbb{L}^0 & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{L}^1 & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases} \\
H^4_{\mathrm{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) &= \begin{cases} s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Proof. We collect the results from 4.3.13, 4.3.22, 4.3.35. It follows that the Eisenstein cohomology is concentrated in degrees 2, 3, 4 and given by

$$
\begin{aligned}
H^2_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) &= \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_1+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} \\
H^3_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) &= \mathbb{S}_{\Gamma(1)}[\lambda_1 + 3]\mathbb{L}^{\lambda_2+1} \\
&\oplus \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+2} & \text{otherwise} \end{cases} \\
&\oplus \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+2} & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases} \\
H^4_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) &= \mathbb{S}_{\Gamma(1)}[\lambda_2 + 2]\mathbb{L}^{\lambda_1+2} \\
&\oplus s_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_1+\lambda_2+3} \\
&\oplus \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

The result follows by Poincare duality. □

Collecting the above contributions yields the following:

Theorem 4.5.3. [96, Theorem 2.1] Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2 > 0$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2$. The compactly supported cohomology $H^*_c(\mathcal{A}_2, \mathbb{V}_\lambda)$ is concentrated in degrees 2, 3, 4, and given up to semisimplification by

$$
\begin{aligned}
H^2_c(\mathcal{A}_2, \mathbb{V}_\lambda) &= \mathbb{S}_{\Gamma(1)}[\lambda_2 + 2] + s_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^0 \\
&\quad + \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^0 & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} \\
H^3_c(\mathcal{A}_2, \mathbb{V}_\lambda) &= \mathbb{S}^{\mathrm{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] + \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \\
&\quad + s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1} + \mathbb{S}_{\Gamma(1)}[\lambda_1 + 3] \\
&\quad + \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^0 & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^1 & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases} \\
H^4_c(\mathcal{A}_2, \mathbb{V}_\lambda) &= \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Proof. Recall that we have $H^i_c(\mathcal{A}_2, \mathbb{V}_\lambda) \simeq H^i_{\mathrm{cusp}}(\mathcal{A}_2, \mathbb{V}_\lambda) + H^i_{\mathrm{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda)$ up to semisimplification, and Poincare dually we have $H^i(\mathcal{A}_2, \mathbb{V}_\lambda) \simeq H^i_{\mathrm{cusp}}(\mathcal{A}_2, \mathbb{V}_\lambda) \oplus H^i_{\mathrm{Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda)$ up to semisimplification. Combining 4.5.1 and 4.5.2 it follows that the cohomology $H^*(\mathcal{A}_2, \mathbb{V}_\lambda)$ is concentrated in degrees 2, 3, 4, and given up to semisimplification by

$$H^2(\mathcal{A}_2, \mathbb{V}_\lambda) = \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+1} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$H^3(\mathcal{A}_2, \mathbb{V}_\lambda) = \mathbb{S}^{\text{gen}}_{\Gamma(1)}[\lambda_1 - \lambda_2, \lambda_2 + 3] + \begin{cases} \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$+ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_2+1} + \mathbb{S}_{\Gamma(1)}[\lambda_1 + 3]\mathbb{L}^{\lambda_2+1}$$

$$+ \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \text{ even} \\ s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_1+2} & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 = \lambda_2 \text{ odd} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+2} & \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^4(\mathcal{A}_2, \mathbb{V}_\lambda) = \mathbb{S}_{\Gamma(1)}[\lambda_2 + 2]\mathbb{L}^{\lambda_1+2} + s_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2]\mathbb{L}^{\lambda_1+\lambda_2+3}$$

$$+ \begin{cases} s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

The result follows by Poincare duality. □

Mixed Motives The above results for $H^*(\mathcal{A}_2, \mathbb{V}_\lambda)$ only hold up to semisimplification, and it is expected that there are nontrivial extensions between contributions to cuspidal cohomology and Eisenstein cohomology. Recall that we have short exact sequences

$$0 \to H^i_!(\mathcal{A}_2, \mathbb{V}_\lambda) \to H^i(\mathcal{A}_2, \mathbb{V}_\lambda) \to H^i_{\text{Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) \to 0$$

$$0 \to H^i_{\text{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda) \to H^i_{\text{c}}(\mathcal{A}_2, \mathbb{V}_\lambda) \to H^i_!(\mathcal{A}_2, \mathbb{V}_\lambda) \to 0$$

Suppose that $\lambda_1 = \lambda_2 > 0$ is odd. Then the Saito-Kurokawa lift yields a summand

$$\mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] = H^3_{\text{cusp},[P_1]} \subseteq H^3_!(\mathcal{A}_2, \mathbb{V}_\lambda)$$

while the (compactly supported) Siegel Eisenstein cohomology yields summands

$$s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_1+2} = H^3_{\mathrm{Eis},[P_1]} \subseteq H^3_{\mathrm{Eis}}(\mathcal{A}_2,\mathbb{V}_\lambda)$$
$$s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} = H^3_{\mathrm{c,Eis},[P_1]} \subseteq H^3_{\mathrm{c,Eis}}(\mathcal{A}_2,\mathbb{V}_\lambda)$$

which give rise to extensions

$$0 \to H^3_{\mathrm{cusp},[P_1]} \to H^3_{[P_1]} \to H^3_{\mathrm{Eis},[P_1]} \to 0$$
$$0 \to H^3_{\mathrm{c,Eis},[P_1]} \to H^3_{\mathrm{c},[P_1]} \to H^3_{\mathrm{cusp},[P_1]} \to 0$$

regarded as extensions in the (conjectural) category $\mathcal{MM}_{\mathbb{Q}}$ of mixed motives over $\mathbb{Q}$, with coefficients in $\mathbb{Q}$.

Let $f \in S_{\lambda_1+\lambda_2+4}(\Gamma(1))$ be a normalized cuspidal Hecke eigenform, and let $\mathbb{Q}(f)$ be the number field generated by the Hecke eigenvalues of f. Let $H^3_{\mathrm{cusp},[P_1]}(f) \subseteq H^3_{\mathrm{cusp},[P_1]}$ be the corresponding 2-dimensional summand and let $H^3_{\mathrm{Eis},[P_1]}(f) \subseteq H^3_{\mathrm{Eis},[P_1]}$ and $H^3_{\mathrm{c,Eis},[P_1]}(f) \subseteq H^3_{\mathrm{c,Eis},[P_1]}$ be the corresponding 1-dimensional summands. These should give rise to extensions

$$0 \to H^3_{\mathrm{cusp},[P_1]}(f) \to H^3_{[P_1]}(f) \to H^3_{\mathrm{Eis},[P_1]}(f) \to 0$$
$$0 \to H^3_{\mathrm{c,Eis},[P_1]}(f) \to H^3_{\mathrm{c},[P_1]}(f) \to H^3_{\mathrm{cusp},[P_1]}(f) \to 0$$

regarded as extensions in the (conjectural) category $\mathcal{MM}_{\mathbb{Q}} \otimes \mathbb{Q}(f)$ of mixed motives over $\mathbb{Q}$, with coefficients in $\mathbb{Q}(f)$. As motives, $H^3_{\mathrm{cusp},[P_1]}(f) = M(f)$ is the pure motive of weight $\lambda_1+\lambda_2+3$ attached to f, while $H^3_{\mathrm{Eis},[P_1]}(f) = \mathbb{Q}(-\lambda_1-2)$ and $H^3_{\mathrm{c,Eis},[P_1]}(f) = \mathbb{Q}(-\lambda_2-1)$ are Tate motives of weights λ_1+2 and λ_2+1 respectively. We consider the extension classes

$$[H^3_{[P_1]}(f)] \in \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(-\lambda_1-2), M(f))$$
$$[H^3_{\mathrm{c},[P_1]}(f)] \in \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(M(f), \mathbb{Q}(-\lambda_2-1))$$

In the first case we have $\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(-\lambda_1-2), M(f)) \simeq \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(0), M(f)(\lambda_1+2))$ where $M(f)(\lambda_1+2)$ is pure of weight -1. Then Beilinson's conjectures predict that

$$\dim_{\mathbb{Q}}\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(0), M(f)(\lambda_1+2)) = \mathrm{ord}_{s=\lambda_1+2}L(f,s)$$

In the second case we have $\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(M(f), \mathbb{Q}(-\lambda_2-1)) \simeq \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(0), M(f)^\vee(-\lambda_2-1))$ and since $M(f)^\vee(-\lambda_2-1) \simeq M(f)(\lambda_1+2)$ we obtain the same prediction, dual to the first case.

If $L(f,\lambda_1+2)=0$ then the expectation is that these extensions split as Beilinson's conjectures predict that $\dim_{\mathbb{Q}}\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(0), M(f)(\lambda_1+2)) = 0$. The Manin-Drinfeld theorem holds in this case.

If $L(f,\lambda_1+2)\neq 0$ then the situation is more interesting. The expectation is that these extensions do not split. Beilinson's conjectures predict that $\dim_{\mathbb{Q}}\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}\otimes\mathbb{Q}(f)}(\mathbb{Q}(0), M(f)(\lambda_1+2)) > 0$.

Congruences As we have seen, the 4-dimensional ℓ-adic Galois representations attached to automorphic representations for $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ are in general not irreducible: for Yoshida lifts, Saito-Kurokawa lifts, and Soudry lifts, these decompose further into 1-dimensional and 2-dimensional ℓ-adic Galois representations, whose contributions to cohomology behave irregularly and may be concentrated outside of middle degree.

The 4-dimensional ℓ-adic Galois representations attached to automorphic representations for $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ of general type are irreducible. On the other hand, they need not remain irreducible after reduction modulo a prime $\mathfrak{l}$. When this happens, we obtain congruences for the Hecke eigenvalues of such automorphic representations. Such congruences, and the divisivility of L-values which controls them, have been conjectured by Harder, and are closely related to Eisenstein series. We recall these conjectures now, in the case of level 1.

Let $f \in S_k(\Gamma(1))$ be a cuspidal Hecke eigenform of weight k for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. let p be a prime, and let $(\alpha_{0,p}(f), \alpha_{p,1}(f)) \in \mathbb{C}^2$ be the Satake parameters of f at p. For $q = p^n$ a power of p consider the Hecke eigenvalues

$$\lambda_q(f) = \alpha_{p,0}(f)^n + (\alpha_{p,0}(f)\alpha_{p,1}(f))^n$$

Let $F \in S_{k_1,k_2}(\Gamma(1))$ be a cuspidal Hecke eigenform of weight (k_1,k_2) for $\Gamma(1) = \mathrm{Sp}_4(\mathbb{Z})$, let p be a prime, and let $(\alpha_{0,p}(F), \alpha_{p,1}(F), \alpha_{p,2}(F)) \in \mathbb{C}^3$ be the Satake parameters of f at p. For $q = p^n$ a power of p consider the

Hecke eigenvalues

$$\lambda_q(F) = \alpha_{p,0}(f)^n + (\alpha_{p,0}(f)\alpha_{p,1}(f))^n + (\alpha_{p,0}(f)\alpha_{p,2}(f))^n + (\alpha_{p,0}(f)\alpha_{p,1}(f)\alpha_{p,2}(f))^n$$

Harder predicts certain congruences between the Hecke eigenvalues $\lambda_p(F)$ and $\lambda_p(f)$ modulo certain "large" congruence primes dividing certain expressions involving L-values which are related to the constant terms of Siegel and Klingen Eisenstein series, so that these Eisenstein series behave as cusp forms modulo such congruence primes.

Harder conjectures the existence of certain mod ℓ congruences between cuspidal Hecke eigenforms coming from the Siegel Eisenstein contribution

$$s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} \subseteq H^3_{\mathrm{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda)$$

Let $f \in S_k(\Gamma(1))$ be a cuspidal Hecke eigenform of weight $k \geq 2$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ with Satake parameters $\{(\alpha_{0,p}(f), \alpha_{p,1}(f))\}_p$. We consider the completed L-function $\Lambda(f,s) = L_\infty(f,s)L(f,s)$ where

$$L(f,s) = \prod_p \frac{1}{(1-\alpha_{p,0}(f)p^{-s})(1-\alpha_{p,0}(f)\alpha_{p,1}(f)p^{-s})}$$

and where $L_\infty(f,s) = \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. The completed L-function $\Lambda(f,s)$ admits an analytic continuation in $s \in \mathbb{C}$ and satisfies the functional equation $\Lambda(f,s) = (-1)^{\frac{k}{2}}\Lambda(f,k-s)$. We consider the critical values $\Lambda(f,s_0)$ at integers $\frac{k}{2} \leq s_0 \leq k-1$ (which determine the remaining critical values at integers $0 \leq s_0 \leq \frac{k}{2}-1$ by the functional equation). By Manin-Vishnik there exist periods $\omega^\pm(f) \in \mathbb{C}$ such that $\Lambda(f,s_0)/\omega^+(f) \in \mathbb{Q}(f)$ for $\frac{k}{2} \leq s_0 \leq k-1$ even and $\Lambda(f,s_0)/\omega^-(f) \in \mathbb{Q}(f)$ for $\frac{k}{2} \leq s_0 \leq k-1$ odd.

The constant terms of Siegel Eisenstein series attached to f are related to the expression

$$\frac{\Lambda(f,\lambda_1+2)\zeta(\lambda_1-\lambda_2+1)}{\Lambda(f,\lambda_1+3)\zeta(\lambda_1-\lambda_2+2)}$$

and we are interested in the denominators of $\frac{\Lambda(f,\lambda_1+3)}{\omega^\pm(f)}$ where we have divided by Deligne period $\omega^\pm(f)$ in order to obtain an element of $\mathbb{Q}(f)$. Harder explains how to choose these periods $\omega^\pm(f)$ so as to be well-defined up to multiplication by a unit in $\mathcal{O}^\times_{\mathbb{Q}(f)}$, rather than only up to multiplication by a unit in $\mathbb{Q}(f)^\times$, so that this question makes sense. Let $\mathfrak{Den}(\frac{\omega^{\pm}(f)}{\Lambda(f,\lambda_1+3)})$ be the corresponding fractional ideal of $\mathcal{O}_{\mathbb{Q}(f)}$. Then we have the following conjecture:

Conjecture. (Harder) Let $\lambda_1 \geq \lambda_2 \geq 0$ be integers and let $f \in S_{\lambda_1+\lambda_2+4}(\Gamma(1))$ be a normalized cuspidal Hecke eigenform of weight $\lambda_1 + \lambda_2 + 4$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Suppose that $\mathfrak{l}$ is a "large" prime of $\mathbb{Q}(f)$ such that

$$\mathfrak{l}^n \mid \mathfrak{Den}\Big(\frac{\omega^{\pm}(f)}{\Lambda(f,\lambda_1+3)}\Big)$$

Then there exists a normalized cuspidal Hecke eigenform $F \in S^{\mathrm{gen}}_{\lambda_1-\lambda_2,\lambda_2+3}(\Gamma(1))$ of weight $(\lambda_1 - \lambda_2, \lambda_2 + 3)$ for $\Gamma(1) = \mathrm{Sp}_4(\mathbb{Z})$ of general type such that

$$\lambda_p(F) \equiv \lambda_p(f) + p^{\lambda_1+2} + p^{\lambda_2+1} \mod \mathfrak{l}^n$$

for every prime p.

The first example of such a congruence was discovered by Harder:

Example 4.5.4. (Harder) Let $(\lambda_1, \lambda_2) = (11, 7)$. Let $f \in S_{22}(\Gamma(1))$ be the normalized cuspidal Hecke eigenform of weight 22 for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. In this case the "large" prime $\ell = 41$ divides $\frac{\Lambda(f,14)}{\omega_f^+}$. Then there exists a normalized cuspidal Hecke eigenform $F \in S_{4,10}(\Gamma(1))$ of weight $(4, 10)$ for $\Gamma(1) = \mathrm{Sp}_4(\mathbb{Z})$ such that

$$\lambda_p(F) \equiv \lambda_p(f) + p^{13} + p^8 \mod 41$$

for every prime p.

We should also have conjectural Eisenstein congruences coming from the Klingen Eisenstein contribution

$$\mathbb{S}_{\Gamma(1)}[\lambda_1 + 3] \subseteq H^3_{\mathrm{c,Eis}}(\mathcal{A}_2, \mathbb{V}_\lambda)$$

Let $f \in S_k(\Gamma(1))$ be a normalized cuspidal Hecke eigenform of weight $k \geq 2$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ with Satake parameters $\{(\alpha_{0,p}(f), \alpha_{p,1}(f))\}_p$, and let $\mathbb{Q}(f)$ be the number field generated by the $\lambda_p(f)$ for primes p. We consider the completed L-function $\Lambda(\mathrm{Sym}^2(f), s) = L_\infty(\mathrm{Sym}^2(f), s) L(\mathrm{Sym}^2(f), s)$ where

$$L(\mathrm{Sym}^2(f), s) = \prod_p \frac{1}{(1-\alpha_{p,0}(f)p^{-s})(1-\alpha_{p,0}(f)\alpha_{p,1}(f)p^{-s})(1-\alpha_{p,0}(f)\alpha_{p,1}(f)^2 p^{-s})}$$

and where $L_\infty(\mathrm{Sym}^2(f), s) = \pi^{-\frac{3}{2}(s+k-1)}\Gamma(\frac{s+k}{2})\Gamma(\frac{s+2k-2}{2})\Gamma(\frac{s+2k-1}{2})$.

The completed L-function $\Lambda(\mathrm{Sym}^2(f), s)$ admits an analytic continuation in $s \in \mathbb{C}$ which is entire since f is not CM, and satisfies the functional equation $L(\mathrm{Sym}^2(f), s) = L(\mathrm{Sym}^2(f), 2k-1-s)$. We consider the critical values $\Lambda(\mathrm{Sym}^2(f), s_0)$ for even integers $k \leq s_0 \leq 2k-2$ and for odd integers $1 \leq s_0 \leq k-1$. By Manin-Vishnik there exist periods $\omega^\pm(\mathrm{Sym}^2(f)) \in \mathbb{C}$ such that $\Lambda(\mathrm{Sym}^2(f), s_0)/\omega^+(\mathrm{Sym}^2(f)) \in \mathbb{Q}(f)$ for $k \leq s_0 \leq 2k-2$ even and $\Lambda(\mathrm{Sym}^2(f), s_0)/\omega^-(f) \in \mathbb{Q}(f)$ for $1 \leq s_0 \leq k-1$ odd.

The constant terms of Klingen Eisenstein series attached to f are related to the expression

$$\frac{\Lambda(\mathrm{Sym}^2(f), \lambda_1+\lambda_2+3)}{\Lambda(\mathrm{Sym}^2(f), \lambda_1+\lambda_2+4)}$$

and we are interested in the denominators of $\frac{\Lambda(\mathrm{Sym}^2(f), \lambda_1+\lambda_2+4)}{\omega^\pm(\mathrm{Sym}^2(f))}$ where we have divided by Deligne period $\omega^\pm(\mathrm{Sym}^2(f))$ in order to obtain an element of $\mathbb{Q}(f)$. Harder explains how to choose these periods $\omega^\pm(\mathrm{Sym}^2(f))$ so as to be well-defined up to multiplication by a unit in $\mathcal{O}_{\mathbb{Q}(f)}^\times$, rather than only up to multiplication by a unit in $\mathbb{Q}(f)^\times$, so that this question makes sense. Let $\mathfrak{Den}(\frac{\omega^\pm(\mathrm{Sym}^2(f))}{\Lambda(\mathrm{Sym}^2(f), \lambda_1+\lambda_2+4)})$ be the corresponding fractional ideal of $\mathcal{O}_{\mathbb{Q}(f)}$. Then we have the following conjecture:

Conjecture. (Harder) Let $\lambda_1 \geq \lambda_2 \geq 0$ be integers and let $f \in S_{\lambda_1+3}(\Gamma(1))$ be a normalized cuspidal Hecke eigenform of weight λ_1+3 for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Suppose that $\mathfrak{l}$ is a "large" prime in $\mathbb{Q}(f)$ such that

$$\mathfrak{l}^n \mid \mathfrak{Den}\Big(\frac{\omega^\pm(\mathrm{Sym}^2(f))}{\Lambda(\mathrm{Sym}^2(f), \lambda_1+\lambda_2+4)}\Big)$$

Then there exists a normalized cuspidal Hecke eigenform $F \in S^{\mathrm{gen}}_{\lambda_1-\lambda_2, \lambda_2+3}(\Gamma(1))$ of weight $(\lambda_1-\lambda_2, \lambda_2+3)$ for $\Gamma(1) = \mathrm{Sp}_4(\mathbb{Z})$ of general type such that

$$\lambda_p(F) \equiv \lambda_p(f)(p^{\lambda_2+1}+1) \mod \mathfrak{l}^n$$

for every prime p.

The first example of such a congruence was discovered by Kurokawa:

Example 4.5.5. (Kurokawa) Let $(\lambda_1, \lambda_2) = (17, 17)$. Let $f \in S_{20}(\Gamma(1))$ be the normalized cuspidal Hecke eigenform of weight 20 for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. In this case the "large" prime power $\ell = 71^2$ divides $\frac{\Lambda(\mathrm{Sym}^2(f),38)}{\omega^+(\mathrm{Sym}^2(f))}$. Then there exists a normalized cuspidal Hecke eigenform $F \in S_{0,20}(\Gamma(1))$ of weight $(0, 20)$ for $\Gamma(1) = \mathrm{Sp}_4(\mathbb{Z})$ of general type such that

$$\lambda_p(F) \equiv \lambda_p(f)(p^{18} + 1) \mod 71^2$$

for every prime p.

Such congruences, which are controlled by the divisibility of L-values, correspond to the reducibility of Galois representations modulo ℓ (up to semisimplification). The (conjectural) situation can be summarized by the following diagrams:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Denominator}\\ \mathrm{ord}_{\mathfrak{l}}\left(\frac{\Lambda(f,\lambda_1+3)}{\omega^{\pm}(f)}\right) > 0\end{array}} \arrow[d, bend right, "\text{Eisenstein series}"'] \\
\boxed{\begin{array}{c}\text{Congruence}\\ \lambda_p(F) \equiv \lambda_p(f) + p^{\lambda_1+2} + p^{\lambda_2+1} \mod \mathfrak{l}\end{array}} \arrow[u, bend right, "\text{Bloch-Kato}"'] \arrow[d, bend right, "\substack{\text{Chebotarev density}\\ \text{Brauer-Nesbitt}}"'] \\
\boxed{\begin{array}{c}\text{Reducibility}\\ \overline{\rho}_F^{\mathrm{ss}} = \overline{\rho}_f \oplus \overline{\chi}^{\lambda_1+2} \oplus \overline{\chi}^{\lambda_2+1}\end{array}} \arrow[u, bend right, "\substack{\text{Deligne}\\ \text{Fontaine}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Denominator}\\ \mathrm{ord}_{\mathfrak{l}}\left(\frac{\Lambda(\mathrm{Sym}^2(f),\lambda_1+\lambda_2+4)}{\omega^{\pm}(\mathrm{Sym}^2(f))}\right) > 0\end{array}} \arrow[d, bend right, "\text{Eisenstein series}"'] \\
\boxed{\begin{array}{c}\text{Congruence}\\ \lambda_p(F) \equiv \lambda_p(f)(p^{\lambda_2+1} + 1) \mod \mathfrak{l}\end{array}} \arrow[u, bend right, "\text{Bloch-Kato}"'] \arrow[d, bend right, "\substack{\text{Chebotarev density}\\ \text{Brauer-Nesbitt}}"'] \\
\boxed{\begin{array}{c}\text{Reducibility}\\ \overline{\rho}_F^{\mathrm{ss}} = \overline{\rho}_f \otimes (\overline{1} \oplus \overline{\chi}^{\lambda_2+1})\end{array}} \arrow[u, bend right, "\substack{\text{Deligne}\\ \text{Fontaine}}"']
\end{tikzcd}$$

4.6 Example: Cohomology of Local Systems on $\mathcal{A}_2[2]$

Let $\mathcal{A}_2[2]$ be the moduli stack of principally polarized Abelian surfaces with full level 2 structure. The group $\mathrm{GSp}_4(\mathbb{F}_2)$ acts on $\mathcal{A}_2[2]$. We compute the cohomology $H^*(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ as a representation of $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$, resolving conjectures of Bergstrom-Faber-van der Geer.

We have an isomorphism $\mathrm{GSp}_4(\mathbb{F}_2) \simeq S_6$ so that the irreducible representations of $\mathrm{GSp}_4(\mathbb{F}_2)$ can be labeled by partitions of 6. We have

Representation of $\mathrm{GSp}_4(\mathbb{F}_2)$	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	$X_4(1)$	$\chi_5(1)$	$\chi_6(1)$	$\chi_7(1)$	$\chi_8(1)$
Representation of S_6	V_6	$V_{4,2}$	V_{1^6}	V_{2^3}	$V_{5,1}$	$V_{3,2,1}$	$V_{2^2,1^1}$	V_{3^2}	$V_{4,1^2}$	$V_{2,1^4}$	$V_{3,1^3}$
Dimension	1	9	1	5	5	16	9	5	10	5	10

In particular $\theta_0 = 1_{\mathrm{GSp}_4(\mathbb{F}_2)}$ is the trivial representation and $\theta_5 = \mathrm{St}_{\mathrm{GSp}_4(\mathbb{F}_2)}$ is the Steinberg representation.

Recall that $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2) = S_3$ has three isomorphism classes of irreducible representations: the trivial representation $V_3 = 1_{\mathrm{GL}_2(\mathbb{F}_2)}$, the Steinberg representation $V_{2,1} = \mathrm{St}_{\mathrm{GL}_2(\mathbb{F}_2)}$, and the sign representation V_{1^3} which is the unique cuspidal representation. We have the parabolically induced representations

$$\mathrm{Ind}_{P_1(\mathbb{F}_2)}^{\mathrm{GSp}_4(\mathbb{F}_2)}(V_{1^3} \otimes 1) = V_{3^2} \oplus V_{4,1^2} \qquad \mathrm{Ind}_{P_2(\mathbb{F}_2)}^{\mathrm{GSp}_4(\mathbb{F}_2)}(V_2 \otimes V_{1^4}) = V_{2,1^4} \oplus V_{3,1^3}$$

$$\mathrm{Ind}_{P_1(\mathbb{F}_2)}^{\mathrm{GSp}_4(\mathbb{F}_2)}(V_{2,1} \otimes 1) = V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1} \qquad \mathrm{Ind}_{P_2(\mathbb{F}_2)}^{\mathrm{GSp}_4(\mathbb{F}_2)}(V_2 \otimes V_{2^2}) = V_{4,2} \oplus V_{2^3} \oplus V_{3,2,1}$$

$$\mathrm{Ind}_{P_1(\mathbb{F}_2)}^{\mathrm{GSp}_4(\mathbb{F}_2)}(V_3 \otimes 1) = V_6 \oplus V_{4,2} \oplus V_{2^3} \qquad \mathrm{Ind}_{P_2(\mathbb{F}_2)}^{\mathrm{GSp}_4(\mathbb{F}_2)}(V_2 \otimes V_4) = V_6 \oplus V_{4,2} \oplus V_{5,1}$$

We include a table of the dimensions $s_{\Gamma(1)}[k]$, $s_{\Gamma_0(2)}^{\mathrm{new},+}[k]$, $s_{\Gamma_0(2)}^{\mathrm{new},-}[k]$, and $s_{\Gamma_0(4)}^{\mathrm{new}}[k]$; the entries in bold are those with vanishing central L-value.

k	6	8	10	12	14	16	18	20	22	24
$s_{\Gamma(1)}[k]$	0	0	0	1	0	1	**1**	1	**1**	2
$s_{\Gamma_0(2)}^{\mathrm{new},+}[k]$	0	1	0	0	**1**	1	0	1	**1**	1
$s_{\Gamma_0(2)}^{\mathrm{new},-}[k]$	0	0	1	0	1	0	1	**1**	1	0
$s_{\Gamma_0(4)}^{\mathrm{new}}[k]$	1	0	1	**1**	1	**1**	2	**1**	2	**2**

Theorem 4.6.1. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The Siegel inner cohomology $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4$ and given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} V_{3^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \\ \quad \oplus V_{1^6} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \\ \quad \oplus V_{5,1} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} & \lambda_1 = \lambda_2 \text{ odd} \\ V_{4,2} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \\ \quad \oplus V_{2^3} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \\ \quad \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} & \lambda_1 = \lambda_2 \text{ even} \end{cases}$$

$$H^{3}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} V_{4,2} \boxtimes \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\ \quad \oplus V_{2^3} \boxtimes \mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\ \quad \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ odd} \\ V_{3^2} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] \\ \quad \oplus V_{1^6} \boxtimes \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\ \quad \oplus V_{5,1} \boxtimes \mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] & \lambda_1 = \lambda_2 \text{ even} \end{cases}$$

In particular the Siegel inner cohomology $H^3_{!,[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given as a Gal-module by

$$H^{2i}_{!,[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)=\begin{cases}5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \\ \quad\oplus s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \\ \quad\oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} & \lambda_1=\lambda_2 \text{ odd} \\ 9s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \\ \quad\oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \\ \quad\oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} & \lambda_1=\lambda_2 \text{ even}\end{cases}$$

$$H^{3}_{!,[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)=\begin{cases}9\mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \quad\oplus 5\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \quad\oplus 15\mathbb{S}_{\Gamma(1)}[\lambda_1+\lambda_2+4] & \lambda_1=\lambda_2 \text{ odd} \\ 5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \\ \quad\oplus \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \quad\oplus 5\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] & \lambda_1=\lambda_2 \text{ even}\end{cases}$$

Proof. Let $\lambda_1=\lambda_2$. Then by [100, Example 5.5] we have:

$$IH^{2i}_{[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)=\bigoplus_\mu\begin{cases}r_{\mathscr{K}_2}(\Pi(\mu_2,\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))\boxtimes\mathbb{L}^{\lambda_2+i} & \epsilon(\frac{1}{2},\mu_2)=-(-1)^{\lambda_1} \\ r_{\mathscr{K}_2}(\Pi(\mu_2,1_{\mathrm{GL}_2(\mathbb{Q}_2)}))\boxtimes\mathbb{L}^{\lambda_2+i} & \epsilon(\frac{1}{2},\mu_2)=(-1)^{\lambda_1}\end{cases}$$

$$IH^{3}_{[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)=\bigoplus_\mu\begin{cases}r_{\mathscr{K}_2}(\Pi(\mu_2,\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))\boxtimes\rho_\mu & \epsilon(\frac{1}{2},\mu_2)=(-1)^{\lambda_1} \\ r_{\mathscr{K}_2}(\Pi(\mu_2,1_{\mathrm{GL}_2(\mathbb{Q}_2)}))\boxtimes\rho_\mu & \epsilon(\frac{1}{2},\mu_2)=-(-1)^{\lambda_1}\end{cases}$$

where the direct sums are taken over cuspidal automorphic representations μ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character, with $\mu_\infty=\mathcal{D}_{\lambda_1+\lambda_2+3}$, spherical outside $p=2$, and depth 0 at $p=2$, corresponding to a cusp form of weight $\lambda_1+\lambda_2+4$ for $\Gamma(2)$. Now we have the following cases:

Type		μ_2	μ_2^Σ	$\Pi(\mu_2, \mu_2^\Sigma)$	$r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma))$	Dimension
II	b	$\chi_2 \times \chi_2^{-1}$	$1_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$\chi_2 1_{\mathrm{GL}_2(\mathbb{Q}_2)} \rtimes \chi_2^{-1}$	$V_6 \oplus V_{4,2} \oplus V_{2^3}$	15
V	a*	$\xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$\delta^*([\xi, \xi\nu], \nu^{-1/2})$	V_{1^6}	1
	b		$1_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$L(\xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \nu^{1/2} \rtimes \nu^{-1/2})$	$V_{4,2}$	9
VI	b	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$\tau(T, \nu^{-1/2})$	V_{2^3}	5
	c		$1_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \nu^{1/2} \rtimes \nu^{-1/2})$	$V_{5,1}$	5
XI	a*	Supercuspidal	$\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$\delta^*(\mu_2 \nu^{1/2} \rtimes \nu^{-1/2})$	0	0
	b		$1_{\mathrm{GL}_2(\mathbb{Q}_2)}$	$L(\mu_2 \nu^{1/2} \rtimes \nu^{-1/2})$	V_{3^2}	5

(i) $((\mu_2, \mu_2^\Sigma) = (\chi_2 \times \chi_2^{-1}, 1_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f of weight $\lambda_1 + \lambda_2 + 4$ for $\Gamma(1)$ yields the representation $\Pi(\mu_2, \mu_2^\Sigma) = \chi_2 1_{\mathrm{GL}_2(\mathbb{Q}_2)} \rtimes \chi_2^{-1}$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma)) = V_6 + V_{4,2} + V_{2^3}$ of dimension 15, so the contribution to $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$(V_6 + V_{4,2} + V_{2^3}) \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] \qquad (\lambda_1 = \lambda_2 \text{ odd})$$

and the contribution to $H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$(V_6 + V_{4,2} + V_{2^3}) \boxtimes s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2 + i} \qquad (\lambda_1 = \lambda_2 \text{ even})$$

(ii) $((\mu_2, \mu_2^\Sigma) = (\xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f of weight $\lambda_1 + \lambda_2 + 4$ for $\Gamma_0(2)$ with Atkin-Lehner eigenvalue $\epsilon(\frac{1}{2}, \mu_2) = 1$ yields the supercuspidal representation $\Pi(\mu_2, \mu_2^\Sigma) = \delta^*([\xi, \xi\nu], \nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma)) = V_{1^6}$ of dimension 1, so the contribution to $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{1^6} \boxtimes \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \qquad (\lambda_1 = \lambda_2 \text{ even})$$

and the contribution to $H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{1^6} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2 + i} \qquad (\lambda_1 = \lambda_2 \text{ odd})$$

(iii) $((\mu_2, \mu_2^\Sigma) = (\xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, 1_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ with Atkin-Lehner eigenvalue $\epsilon(\frac{1}{2}, \mu_2) = 1$ yields the Langlands quotient $\Pi(\mu_2, \mu_2^\Sigma) = L(\xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \nu^{1/2} \rtimes \nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma)) = V_{4,2}$ of dimension 9, so the contribution to $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{4,2} \boxtimes \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \qquad (\lambda_1 = \lambda_2 \text{ odd})$$

and the contribution to $H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{4,2} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \qquad (\lambda_1 = \lambda_2 \text{ even})$$

(iv) $((\mu_2, \mu_2^\Sigma) = (\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ with Atkin-Lehner eigenvalue $\epsilon(\frac{1}{2}, \mu_2) = -1$ yields the representation $\Pi(\mu_2, \mu_2^\Sigma) = \tau(T, \nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma)) = V_{2^3}$ of dimension 5, so the contribution to $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{2^3} \boxtimes \mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \qquad (\lambda_1 = \lambda_2 \text{ odd})$$

and the contribution to $H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{2^3} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \qquad (\lambda_1 = \lambda_2 \text{ even})$$

(v) $((\mu_2, \mu_2^\Sigma) = (\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, 1_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ with Atkin-Lehner eigenvalue $\epsilon(\frac{1}{2}, \mu_2) = -1$ yields the Langlands quotient $\Pi(\mu_2, \mu_2^\Sigma) = L(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \nu^{1/2} \rtimes \nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma)) = V_{5,1}$ of dimension 5, so the contribution to $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{5,1} \boxtimes \mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \qquad (\lambda_1 = \lambda_2 \text{ even})$$

and the contribution to $H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{5,1} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \qquad (\lambda_1=\lambda_2 \text{ odd})$$

(vi) $((\mu_2, \mu_2^\Sigma) = (\theta_2, 1_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(4)$ yields the Langlands quotient $\Pi(\mu_2, \mu_2^\Sigma) = L(\mu_2\nu^{1/2} \rtimes \nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi(\mu_2, \mu_2^\Sigma)) = V_{3^2}$ of dimension 5, so the contribution to $H^3_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{3^2} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \qquad (\lambda_1=\lambda_2 \text{ even})$$

and the contribution to $H^{2i}_{!,[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given:

$$V_{3^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \qquad (\lambda_1=\lambda_2 \text{ odd})$$

The result follows by collecting all these contributions. $\square$

Proposition 4.6.2. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1+\lambda_2)$ with $\lambda_1+\lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) \oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$\begin{gathered}
\begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]V_{3,1^3} \\ \oplus s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]V_{4,1^2} \\ \oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{3^2}\oplus V_{4,1^2}) \end{pmatrix} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} \\
\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]V_{4,1^2} \\ \oplus s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{3,2,1}) \\ \oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4(V_{5,1}\oplus V_{3,2,1}) \\ \oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1}) \end{pmatrix} \boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} \\
\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4](V_{3^2}\oplus V_{4,1^2}) \\ \oplus s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1}) \\ \oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1}\oplus V_{3^2}\oplus V_{4,1^2}) \end{pmatrix} \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}
\end{gathered}$$

In particular the contribution to $H^{2,1}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) \oplus H^{1,2}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given as Gal-module by

$$\begin{pmatrix} 10s^{\text{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \oplus 10s^{\text{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4] \end{pmatrix} \mathbb{S}^{\text{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$
$$\oplus \begin{pmatrix} 10s^{\text{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \oplus 25s^{\text{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 21s^{\text{new},\mp}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \oplus 30s_{\Gamma(1)}[\lambda_1+\lambda_2+4] \end{pmatrix} \mathbb{S}^{\text{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$
$$\oplus \begin{pmatrix} 15s^{\text{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \oplus 30s^{\text{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 45s_{\Gamma(1)}[\lambda_1+\lambda_2+4] \end{pmatrix} \mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$

Proof. The contribution to $H^{2,1}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) \oplus H^{1,2}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given

$$H^{2,1}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) \oplus H^{1,2}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) \simeq \bigoplus_{\pi=(\mu_1,\mu_2)} r_{\mathscr{K}_2}(\Pi^+(\mu_1,\mu_2)) \boxtimes \rho_{\mu_2}\mathbb{L}^{\lambda_2+1}$$

taken over cuspidal automorphic representations $\pi = (\mu_1, \mu_2)$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $\pi_\infty = (\mathcal{D}_{\lambda_1+\lambda_2+3}, \mathcal{D}_{\lambda_1-\lambda_2+1})$, with π_p spherical for $p \neq 2$, and with π_2 of depth 0, corresponding to modular forms f_1 and f_2 of weights $\lambda_1+\lambda_2+4$ and $\lambda_1-\lambda_2+2$ for $\Gamma(2)$. Now we have the following cases:

$(\mu_{1,2}, \mu_{2,2})$	$\Pi^+(\mu_{1,2}, \mu_{2,2})$	$r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2}, \mu_{2,2}))$	dim
$(\chi_{1,2} \times \chi_{2,2}, \chi'_{1,2} \times \chi'_{2,2})$	$\chi'_{1,2}\chi_{1,2}^{-1} \times \chi'_{2,2}\chi_{1,2}^{-1} \rtimes \chi_{1,2}$	$V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1} \oplus V_{3^2} \oplus V_{4,1^2}$	45
$(\chi_{1,2} \times \chi_{2,2}, \chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)})$	$L(\chi_2\chi_{1,2}^{-1}\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \rtimes \chi_{1,2})$	$V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}$	30
$(\chi_{1,2} \times \chi_{2,2}, \chi_2\pi_2)$	$\chi_2^{-1}\pi_2 \rtimes \chi_2$	$V_{3^2} \oplus V_{4,1^2}$	15
$(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)})$	$\tau(S, \xi\nu^{-1/2})$	$V_{4,2} \oplus V_{3,2,1}$	25
$(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \chi_2\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)})$	$\delta(\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}\nu^{1/2} \rtimes \chi_2\nu^{-1/2})$	$V_{5,1} \oplus V_{3,2,1}$	21
$(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \chi_2\pi_2)$	$\delta(\chi_2^{-1}\pi_2\nu^{1/2} \rtimes \chi_2\nu^{-1/2})$	$V_{4,1^2}$	10
(π_2, π_2)	$\tau(S, \pi_2)$	$V_{3,1^3}$	10

(i) $((\mu_{1,2}, \mu_{2,2}) = (\chi_{1,2} \times \chi_{2,2}, \chi'_{1,2} \times \chi'_{2,2}))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma(1)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma(1)$ yields the unramified representation $\Pi^+(\mu_{1,2}, \mu_{2,2}) = \chi'_{1,2}\chi_{1,2}^{-1} \times \chi'_{2,2}\chi_{1,2}^{-1} \rtimes \chi_{1,2}$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2}, \mu_{2,2})) = V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1} \oplus V_{3^2} \oplus V_{4,1^2}$ with dimension 45, so the contribution to $H^{2,1}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) \oplus H^{1,2}_{!,\text{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$

is given:

$$s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1}\oplus V_{3^2}\oplus V_{4,1^2})\boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$

(ii) $((\mu_{1,2},\mu_{2,2})=(\chi_{1,2}\times\chi_{2,2},\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma(1)$, or a cuspidal newform f_1 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(2)$ and a cuspidal newform f_2 of weight $\lambda_1+\lambda_2+4$ for $\Gamma(1)$ yields the Langlands quotient representation $\Pi^+(\mu_{1,2},\mu_{2,2}) = L(\chi_2\chi_{1,2}^{-1}\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}\rtimes\chi_{1,2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2},\mu_{2,2}))=V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1}$ with dimension 30, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$\begin{gathered}s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1})\boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}\\ \oplus\, s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{5,1}\oplus V_{3,2,1})\boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}\end{gathered}$$

(iii) $((\mu_{1,2},\mu_{2,2})=(\chi_{1,2}\times\chi_{2,2},\chi_2\pi_2))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(4)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma(1)$, or a cuspidal newform f_1 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(4)$ and a cuspidal newform f_2 of weight $\lambda_1+\lambda_2+4$ for $\Gamma(1)$ yields the representation $\Pi^+(\mu_{1,2},\mu_{2,2})=\chi_2^{-1}\pi_2\rtimes\chi_2$ where π_2 is supercuspidal of depth 0 with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2},\mu_{2,2}))=V_{3^2}\oplus V_{4,1^2}$ with dimension 15, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$\begin{gathered}s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{3^2}\oplus V_{4,1^2})\boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}\\ \oplus\, s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4](V_{3^2}\oplus V_{4,1^2})\boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}\end{gathered}$$

(iv) $((\mu_{1,2},\mu_{2,2})=(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(2)$ with equal Atkin-Lehner eigenvalues yields the representation $\Pi^+(\mu_{1,2},\mu_{2,2})=\tau(S,\xi\nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2},\mu_{2,2}))=V_{4,2}\oplus V_{3,2,1}$ with dimension 25, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{4,2}\oplus V_{3,2,1})\boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$

(v) $((\mu_{1,2},\mu_{2,2})=(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(2)$ with opposite Atkin-Lehner eigenvalues yields the representation $\Pi^+(\mu_{1,2},\mu_{2,2})=\delta(\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}\nu^{1/2}\rtimes\chi_2\nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2},\mu_{2,2}))=V_{5,1}\oplus V_{3,2,1}$ with dimension 21, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{5,1}\oplus V_{3,2,1})\boxtimes\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$

(vi) $((\mu_{1,2},\mu_{2,2})=(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\pi_2))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(4)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(2)$, or a cuspidal newform f_1 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(4)$ and a cuspidal newform f_2 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ yields the representation $\Pi^+(\mu_{1,2},\mu_{2,2})=\delta(\chi_2^{-1}\pi_2\nu^{1/2}\rtimes\chi_2\nu^{-1/2})$ where π_2 is supercuspidal of depth 0 with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2},\mu_{2,2}))=V_{4,1^2}$ with dimension 10, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$\begin{gathered}s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]V_{4,1^2}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}\\ \oplus\, s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]V_{4,1^2}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}\end{gathered}$$

(vii) $((\mu_{1,2},\mu_{2,2})=(\pi_2,\pi_2))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(4)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(4)$ yields the representation $\Pi^+(\mu_{1,2},\mu_{2,2})=\tau(S,\pi_2)$ where π_2 is supercuspidal of depth 0 with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^+(\mu_{1,2},\mu_{2,2}))=V_{3,1^3}$ with dimension 10, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]V_{3,1^3}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1}$$

The result follows by collecting all these contributions. □

Proposition 4.6.3. Let $V_\lambda\in\mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda=(\lambda_1,\lambda_2;\lambda_1+\lambda_2)$ with $\lambda_1+\lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The contribution to $H^{3,0}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{0,3}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given as a $\mathrm{GSp}_4(\mathbb{F}_2)\times\mathrm{Gal}$-module by

$$\begin{aligned}&s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]V_{2,1^4}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\\&\oplus(s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]V_{2^3}\oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]V_{1^6})\boxtimes\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\end{aligned}$$

In particular the contribution to $H^{3,0}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{0,3}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given as a Gal-module by

$$\begin{aligned}&5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\\&\oplus(5s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2])\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\end{aligned}$$

Proof. The contribution to $H^{3,0}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{0,3}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given

$$H^{3,0}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{0,3}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\simeq\bigoplus_{\pi=(\mu_1,\mu_2)}r_{\mathscr{K}_2}(\Pi^-(\mu_1,\mu_2))\boxtimes\rho_{\mu_1}$$

taken over cuspidal automorphic representations $\pi=(\mu_1,\mu_2)$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})\times\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $\pi_\infty=(\mathcal{D}_{\lambda_1+\lambda_2+3},\mathcal{D}_{\lambda_1-\lambda_2+1})$, with π_p spherical for $p\neq 2$, and with π_2 of depth 0, corresponding to modular forms f_1 and f_2 of weights $\lambda_1+\lambda_2+4$ and $\lambda_1-\lambda_2+2$ for $\Gamma(2)$. Now we have the following cases:

$(\mu_{1,2},\mu_{2,2})$	$\Pi^-(\mu_{1,2},\mu_{2,2})$	$r_{\mathscr{K}_2}(\Pi^-(\mu_{1,2},\mu_{2,2}))$	dim
$(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)})$	$\tau(T,\chi_2\nu^{-1/2})$	V_{2^3}	5
$(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)})$	supercuspidal	V_{1^6}	1
$(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\pi_2)$	supercuspidal	0	0
(π_2,π_2)	$\tau(T,\pi_2)$	$V_{2,1^4}$	5
$(\pi_{1,2},\pi_{2,2})\quad\pi_{1,2}\neq\pi_{2,2}$	supercuspidal	0	0

(i) ($(\mu_{1,2},\mu_{2,2})=(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)})$) A cuspidal newform f_1 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(2)$ and a cuspidal newform f_2 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ with equal Atkin-Lehner eigenvalues yields the representation $\Pi^-(\mu_{1,2},\mu_{2,2})=\tau(T,\chi_2\nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^-(\mu_{1,2},\mu_{2,2}))=V_{2^3}$ with dimension 5, so the contribution to $H^{3,0}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{0,3}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]V_{2^3}\boxtimes\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]$$

(ii) $((\mu_{1,2},\mu_{2,2})=(\chi_2\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)},\chi_2\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}))$ A cuspidal newform f_1 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(2)$ and a cuspidal newform f_2 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(2)$ with opposite Atkin-Lehner eigenvalues yields the a supercuspidal representation $\Pi^-(\mu_{1,2},\mu_{2,2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^-(\mu_{1,2},\mu_{2,2}))=V_{1^6}$ with dimension 1, so the contribution to $H^{3,0}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{0,3}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]V_{1^6}\boxtimes\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]$$

(iii) $((\mu_{1,2},\mu_{2,2})=(\pi_2,\pi_2))$ A cuspidal newform f_1 of weight $\lambda_1+\lambda_2+4$ for $\Gamma_0(4)$ and a cuspidal newform f_2 of weight $\lambda_1-\lambda_2+2$ for $\Gamma_0(4)$ yields the representation $\Pi^-(\mu_{1,2},\mu_{2,2})=\tau(T,\pi_2)$ where π_2 is supercuspidal of depth 0 with hyperspecial parahoric restriction $r_{\mathscr{K}_2}(\Pi^-(\mu_{1,2},\mu_{2,2}))=V_{2,1^4}$ with dimension 5, so the contribution to $H^{2,1}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)\oplus H^{1,2}_{!,\mathrm{endo}}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given:

$$s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]V_{2,1^4}\boxtimes\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]$$

The result follows by collecting all these contributions. $\square$

Theorem 4.6.4. Let $V_\lambda \in \operatorname{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The endoscopic inner cohomology $H^*_{!,\mathrm{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is concentrated in degree 3 and given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$
\begin{aligned}
H^3_{!,\mathrm{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] V_{2,1^4} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] \\
&\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] V_{3,1^3} \\ \oplus s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] V_{4,1^2} \\ \oplus s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4](V_{3^2} \oplus V_{4,1^2}) \end{pmatrix} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2 + 1} \\
&\oplus (s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] V_{2^3} \oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] V_{1^6}) \boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\
&\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] V_{4,1^2} \\ \oplus s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4](V_{4,2} \oplus V_{3,2,1}) \\ \oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4(V_{5,1} \oplus V_{3,2,1}) \\ \oplus s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4](V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \end{pmatrix} \boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2 + 1} \\
&\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4](V_{3^2} \oplus V_{4,1^2}) \\ \oplus s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4](V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \\ \oplus s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4](V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1} \oplus V_{3^2} \oplus V_{4,1^2}) \end{pmatrix} \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2 + 1}
\end{aligned}
$$

In particular the endoscopic inner cohomology $H^3_{!,\mathrm{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given as a Gal-module by

$$
\begin{aligned}
H^3_{!,\mathrm{endo}}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= 5 s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] \\
&\oplus \begin{pmatrix} 10 s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] \oplus 10 s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\ \oplus 15 s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] \end{pmatrix} \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2 + 1} \\
&\oplus (5 s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] \oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2]) \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\
&\oplus \begin{pmatrix} 10 s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] \oplus 25 s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\ \oplus 21 s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \oplus 30 s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] \end{pmatrix} \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2 + 1} \\
&\oplus \begin{pmatrix} 15 s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4] \oplus 30 s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4] \\ \oplus 45 s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4] \end{pmatrix} \mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2 + 1}
\end{aligned}
$$

We summarize these contributions to inner cohomology as follows:

Theorem 4.6.5. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The inner cohomology $H_!^*(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4$ and given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$H_!^{2i}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} \begin{array}{c} V_{3^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \\ \oplus V_{1^6} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \oplus V_{5,1} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \end{array} & \lambda_1 = \lambda_2 \text{ odd} \\ \begin{array}{c} V_{4,2} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \oplus V_{2^3} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \\ \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0} \mathbb{L}^{\lambda_2+i} \end{array} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$H_!^{3}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[\lambda_1 - \lambda_2, \lambda_2 + 3] \oplus \begin{cases} \begin{array}{c} V_{4,2} \boxtimes \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \oplus V_{2^3} \boxtimes \mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1+\lambda_2+4] \end{array} & \lambda_1 = \lambda_2 \text{ odd} \\ \begin{array}{c} V_{3^2} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \\ \oplus V_{1^6} \boxtimes \mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \oplus V_{5,1} \boxtimes \mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \end{array} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$\oplus\, s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] V_{2,1^4} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4]$$

$$\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] V_{3,1^3} \\ \oplus s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] V_{4,1^2} \\ \oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{3^2} \oplus V_{4,1^2}) \end{pmatrix} \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2+1}$$

$$\oplus\, (s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] V_{2^3} \oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] V_{1^6}) \boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]$$

$$\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] V_{4,1^2} \\ \oplus s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{4,2} \oplus V_{3,2,1}) \\ \oplus s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{5,1} \oplus V_{3,2,1}) \\ \oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \end{pmatrix} \boxtimes \mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2+1}$$

$$\oplus \begin{pmatrix} s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4](V_{3^2} \oplus V_{4,1^2}) \\ \oplus s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4](V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \\ \oplus s_{\Gamma(1)}[\lambda_1+\lambda_2+4](V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1} \oplus V_{3^2} \oplus V_{4,1^2}) \end{pmatrix} \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2] \mathbb{L}^{\lambda_2+1}$$

In particular the inner cohomology $H^*_!(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given as a Gal-module by

$$H^{2i}_!(\mathcal{A}_2[2],\mathbb{V}_\lambda)=\begin{cases} \substack{5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i} \\ \oplus s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i}\oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+i}} & \lambda_1=\lambda_2 \text{ odd} \\ \substack{9s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1}\oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} \\ \oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1}} & \lambda_1=\lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_!(\mathcal{A}_2[2],\mathbb{V}_\lambda)=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[\lambda_1-\lambda_2,\lambda_2+3]\oplus\begin{cases} \substack{9\mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\oplus 5\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 15\mathbb{S}_{\Gamma(1)}[\lambda_1+\lambda_2+4]} & \lambda_1=\lambda_2 \text{ odd} \\ \substack{5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\oplus\mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 5\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]} & \lambda_1=\lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &\oplus 5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4] \\ &\oplus \left(\substack{10s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\oplus 10s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4]}\right)\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} \\ &\oplus (5s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]+s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2])\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ &\oplus \left(\substack{10s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\oplus 25s^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 21s^{\mathrm{new},\mp}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\oplus 30s_{\Gamma(1)}[\lambda_1+\lambda_2+4]}\right)\mathbb{S}^{\mathrm{new},\pm}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} \\ &\oplus \left(\substack{15s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\oplus 30s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4] \\ \oplus 45s_{\Gamma(1)}[\lambda_1+\lambda_2+4]}\right)\mathbb{S}_{\Gamma(1)}[\lambda_1-\lambda_2+2]\mathbb{L}^{\lambda_2+1} \end{aligned}$$

Proof. The result follows by combining 4.6.1 and 4.6.4. □

We have the following contributions to $H^*_!(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ in the range $0\le\lambda_1+\lambda_2\le 10$:

(λ_1,λ_2)	$H^2_!$	$H^3_!$	$H^4_!$
$(0,0)$	0	0	0
$(2,0)$	0	0	0
$(1,1)$	0	0	0
$(4,0)$	0	$10\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}$	0
$(3,1)$	0	0	0
$(2,2)$	0	$\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]$	0
$(6,0)$	0	$\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10]+31\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L}$	0
$(5,1)$	0	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10]+20\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^2$	0
$(4,2)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,5]$	0
$(3,3)$	0	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10]$	0
$(8,0)$	0	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12]+25\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10]\mathbb{L}+40\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10]\mathbb{L}$	0
$(7,1)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,4]+40\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L}^2$	0
$(6,2)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,5]+5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12]+25\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^3$	0
$(5,3)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,6]$	0
$(4,4)$	0	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12]$	0
$(10,0)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[8,4]+5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[14]+30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12]\mathbb{L}+75\mathbb{S}_{\Gamma(1)}[12]\mathbb{L}$	0
$(9,1)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[8,4]+5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[14]+30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10]\mathbb{L}^2+\mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[14]+5\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[14]+56\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10]\mathbb{L}^2$	0
$(8,2)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]+5\mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[14]+\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[14]+56\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L}^3$	0
$(7,3)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]+5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[14]+30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^4$	0
$(6,4)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,7]$	0
$(5,5)$	$\mathbb{L}^6$	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[0,8]+9\mathbb{S}^{\mathrm{new},+}_{\Gamma_0(2)}[14]+5\mathbb{S}^{\mathrm{new},-}_{\Gamma_0(2)}[14]$	$\mathbb{L}^7$

The decomposition of the terms $\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[\lambda_1-\lambda_2,\lambda_2+3]$ is as follows:

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,4]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,4]_{2^2,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,4]_{3,1^3}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,5]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,5]_{2^2,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,5]_{3,1^3}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,6]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,6]_{3,2,1}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,6]_{3,1^3}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[8,4]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[8,4]_{3,2,1}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[8,4]_{2^2,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[8,4]_{3,1^3}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]_{3,2,1}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]_{2^2,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]_{3^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]_{4,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,5]_{3,1^3}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]_{4,2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]_{2^3}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]_{3,2,1}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]_{2^2,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,6]_{3,1^3}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,7]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,7]_{3,2,1}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,7]_{2^2,1^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,7]_{3^2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,7]_{4,1^2}$$

$$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[0,8]=\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[0,8]_{4,2}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[0,8]_{2^3}\oplus\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[0,8]_{3,1^3}$$

Theorem 4.6.6. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The compactly supported Siegel Eisenstein cohomology $H^*_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4$ and given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$
\begin{aligned}
H^2_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= (V_{3^2} \oplus V_{4,1^2}) \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 - \lambda_2 + 2] \\
&\oplus (V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \boxtimes s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1 - \lambda_2 + 2] \\
&\oplus (V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1}) \boxtimes s_{\Gamma(1)}[\lambda_1 - \lambda_2 + 2] \\
H^3_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= V_{4,1^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} \\
&\oplus V_{3,2,1} \boxtimes s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} \\
&\oplus (V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \boxtimes s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} \\
&\oplus \begin{cases} V_{3^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus V_{5,1} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ odd} \\ V_{3^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus V_{5,1} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases} \\
&\oplus \begin{cases} V_{4,2} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \text{ even} \\ V_{4,2} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases} \\
H^4_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= \begin{cases} V_{3^2} \boxtimes s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \\ \quad \oplus V_{5,1} \boxtimes s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ odd} \\ V_{4,2} \boxtimes s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \\ \quad \oplus (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes s_{\Gamma(1)}[\lambda_1 + \lambda_2 + 4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \lambda_1 = \lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

In particular the compactly supported Siegel Eisenstein cohomology $H^*_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given as a Gal-module by

$$H^2_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) = 15s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1-\lambda_2+2] \oplus 30s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1-\lambda_2+2] \oplus 45s_{\Gamma(1)}[\lambda_1-\lambda_2+2]$$

$$\begin{aligned}
H^3_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) &= 10s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} \oplus 16s^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} \oplus 30s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} \\
&\oplus \begin{cases} 5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1=\lambda_2 \text{ odd} \\ 5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases} \\
&\oplus \begin{cases} 9s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})=0}\mathbb{L}^{\lambda_2+1} & \lambda_1=\lambda_2 \text{ even} \\ 9s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \\ \quad \oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4]\mathbb{L}^{\lambda_2+1} & \text{otherwise} \end{cases}
\end{aligned}$$

$$H^4_{\mathrm{c,Eis},[P_1]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) = \begin{cases} 5s^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \\ \quad \oplus 5s^{\mathrm{new},-}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \lambda_1=\lambda_2 \text{ odd} \\ 9s^{\mathrm{new},+}_{\Gamma_0(2)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \\ \quad \oplus 15s_{\Gamma(1)}[\lambda_1+\lambda_2+4]_{L(\frac{1}{2})\neq 0}\mathbb{L}^{\lambda_2+2} & \lambda_1=\lambda_2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Taking $\chi = 1$ everywhere in 4.3.12 we have

$$H^2_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} \displaystyle\bigoplus_{\substack{\pi=\mu\otimes 1 \\ \mu_\infty=\mathcal{D}_{\lambda_1+\lambda_2+3} \\ L(\mu,\frac{1}{2})\neq 0}} r_{\mathscr{K}_2}(\mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})) \boxtimes \mathbb{L}^{\lambda_2+1} & \lambda_1 = \lambda_2 \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} \displaystyle\bigoplus_{\substack{\pi=\mu\otimes 1 \\ \mu_\infty=\mathcal{D}_{\lambda_1+\lambda_2+3}}} r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)\Big) \boxtimes \mathbb{L}^{\lambda_1+2} & \lambda_1 = \lambda_2 \\ \displaystyle\bigoplus_{\substack{\pi=\mu\otimes 1 \\ \mu_\infty=\mathcal{D}_{\lambda_1+\lambda_2+3}}} r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{\lambda_1-\lambda_2+1}{2})) \boxtimes \mathbb{L}^{\lambda_1+2} & \text{otherwise} \end{cases}$$

$$H^4_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \bigoplus_{\substack{\pi=\mu\otimes 1 \\ \mu_\infty=\mathcal{D}_{\lambda_1-\lambda_2+1}}} r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{\lambda_1+\lambda_2+3}{2})) \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3}$$

It remains to compute parahoric restriction. Recalling that the local component of a newform for $\Gamma(1)$ is the unramified principal series representation $\mu_2 = \chi_2 \times \chi_2^{-1}$ with parahoric restriction $r_{\mathscr{K}_2}(\chi_2 \times \chi_2^{-1}) = V_3 \oplus V_{2,1}$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}})) \simeq \mathrm{Ind}^{G(\mathbb{F}_2)}_{P_1(\mathbb{F}_2)}((V_3 \oplus V_{2,1}) \otimes 1) = V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1}$$

and yields the kernel (comparing to 4.6.1)

$$\begin{aligned} & r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)\Big) \\ &= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})) \to r_{\mathscr{K}_2}(\mathcal{J}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}))\Big) \\ &= \ker(V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1} \to V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \\ &= V_6 \oplus V_{4,2} \oplus V_{2^3} \end{aligned}$$

Recalling that the local component of a newform for $\Gamma_0(2)$ is the (twisted) Steinberg representation $\mu_2 \in \{\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}\}$ with parahoric restriction $r_{\mathscr{K}_2}(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}) = r_{\mathscr{K}_2}(\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}) = V_{2,1}$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_2}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}})) \simeq \mathrm{Ind}_{P_1(\mathbb{F}_2)}^{G(\mathbb{F}_2)}(V_{2,1} \otimes 1) = V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}$$

and yields the kernel (comparing to 4.6.1)

$$\begin{aligned}
&r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)\Big) \\
&= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})) \to r_{\mathscr{K}_2}(\mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}))\Big) \\
&= \ker(V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1} \to V_{3,2,1}) \\
&= \begin{cases} V_{4,2} & \mu_2 = \xi \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \\ V_{5,1} & \mu_2 = \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)} \end{cases}
\end{aligned}$$

Recalling that the local component of a newform for $\Gamma_0(4)$ is the unique depth 0 supercuspidal representation $\mu_2 = \theta_2$ with parahoric restriction $r_{\mathscr{K}_2}(\theta_2) = V_{1^3}$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_2}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}})) \simeq \mathrm{Ind}_{P_1(\mathbb{F}_2)}^{G(\mathbb{F}_2)}(V_{1^3} \otimes 1) = V_{3^2} \oplus V_{4,1^2}$$

and yields the kernel (comparing to 4.6.1)

$$\begin{aligned}
&r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)\Big) \\
&= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})) \to r_{\mathscr{K}_2}(\mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}))\Big) \\
&= \ker(V_{3^2} \oplus V_{4,1^2} \to V_{4,1^2}) = V_{3^2}
\end{aligned}$$

The result follows by Poincare duality. □

Theorem 4.6.7. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The compactly supported Klingen Eisenstein cohomology $H^*_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4$ and given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times$ Gal-module by

$$\begin{aligned} H^2_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= (V_{2,1^4} \oplus V_{3,1^3}) \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_2+2] \\ &\oplus (V_{4,2} \oplus V_{2^3} \oplus V_{3,2,1}) \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_2+2] \\ &\oplus (V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} + V_{3,2,1}) \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_2+2] \\ H^3_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= (V_{2,1^4} \oplus V_{3,1^3}) \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+3] \\ &\oplus (V_{4,2} \oplus V_{2^3} \oplus V_{3,2,1}) \boxtimes \mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+3] \\ &\oplus (V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} + V_{3,2,1}) \boxtimes \mathbb{S}_{\Gamma(1)}[\lambda_1+3] \end{aligned}$$

In particular the compactly supported Klingen Eisenstein cohomology $H^*_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is given as a Gal-module by

$$\begin{aligned} H^2_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= 15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_2+2] \oplus 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_2+2] \oplus 45\mathbb{S}_{\Gamma(1)}[\lambda_2+2] \\ H^3_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= 15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[\lambda_1+3] \oplus 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[\lambda_1+3] \oplus 45\mathbb{S}_{\Gamma(1)}[\lambda_1+3] \end{aligned}$$

Proof. Taking $\chi = 1$ everywhere in 4.3.21 and recalling that there are no CM cusp forms with trivial central character for $\Gamma(2)$ we have

$$\begin{aligned} H^2_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= \bigoplus_{\substack{\pi=1\otimes\mu \\ \mu_\infty=\mathcal{D}_{\lambda_2+1}}} r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \lambda_1+2)) \boxtimes \rho_\mu \\ H^3_{\mathrm{c,Eis},[P_2]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) &= \bigoplus_{\substack{\pi=1\otimes\mu \\ \mu_\infty=\mathcal{D}_{\lambda_1+2}}} r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}}, \lambda_2+1)) \boxtimes \rho_\mu \end{aligned}$$

It remains to compute parahoric restriction. Recalling that the local component of a newform for $\Gamma(1)$ is the unramified principal series representation $\mu_2 = \chi_2 \times \chi_2^{-1}$ with parahoric restriction $r_{\mathscr{K}_2}(\chi_2 \times \chi_2^{-1}) = V_3 \oplus V_{2,1}$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(\pi_{\mathrm{fin}})) \simeq \mathrm{Ind}^{G(\mathbb{F}_2)}_{P_2(\mathbb{F}_2)}(V_2 \otimes (V_4 \oplus V_{2^2})) = V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1}$$

Recalling that the local component of a newform for $\Gamma_0(2)$ is the (twisted) Steinberg representation $\mu_2 \in \{\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}, \xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}\}$ with parahoric restriction $r_{\mathscr{K}_2}(\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}) = r_{\mathscr{K}_2}(\xi\mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_2)}) = V_{2,1}$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_2}(\mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}})) \simeq \mathrm{Ind}_{P_2(\mathbb{F}_2)}^{G(\mathbb{F}_2)}(V_2 \otimes V_{2^2}) = V_{4,2} \oplus V_{2^3} \oplus V_{3,2,1}$$

Recalling that the local component of a newform for $\Gamma_0(4)$ is unique depth 0 supercuspidal representation $\mu_2 = \theta_2$ with parahoric restriction $r_{\mathscr{K}_2}(\theta_2) = V_{1^3}$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_2}(\mathcal{I}_{P_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}})) \simeq \mathrm{Ind}_{P_2(\mathbb{F}_2)}^{G(\mathbb{F}_2)}(V_2 \otimes V_{1^4}) = V_{2,1^4} \oplus V_{3,1^3}$$

The result follows by Poincare duality. $\square$

Theorem 4.6.8. Let $V_\lambda \in \mathrm{Rep}(G)$ be an absolutely irreducible rational representation with highest weight $\lambda = (\lambda_1, \lambda_2; \lambda_1 + \lambda_2)$ with $\lambda_1 + \lambda_2$ even, and let $\mathbb{V}_\lambda$ be the corresponding ℓ-adic local system on $\mathcal{A}_2[2]$. The compactly supported Borel Eisenstein cohomology $H^*_{\mathrm{c,Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ is concentrated in degrees $2, 3, 4, 6$ and given as a $\mathrm{GSp}_4(\mathbb{F}_2) \times \mathrm{Gal}$-module by

$$H^2_{\mathrm{c,Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} V_{3,2,1} \boxtimes \mathbb{L}^0 & \lambda_1 = \lambda_2 = 0 \\ (V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}) \boxtimes \mathbb{L}^0 & \lambda_1 = \lambda_2 > 0; \lambda_1 \text{ even} \\ (V_{4,2} \oplus V_{2^3} \oplus V_{3,2,1}) \boxtimes \mathbb{L}^0 & \lambda_1 > \lambda_2 = 0 \\ (V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1}) \boxtimes \mathbb{L}^0 & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \end{cases}$$

$$H^3_{\mathrm{c,Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} (V_{4,2} \oplus V_{5,1}) \boxtimes \mathbb{L}^1 & \lambda_1 = \lambda_2 = 0 \\ (V_6 \oplus V_{4,2} \oplus V_{2^3}) \boxtimes \mathbb{L}^0 & \lambda_1 = \lambda_2 > 0; \lambda_1 \text{ odd} \\ (V_6 \oplus V_{4,2} \oplus V_{5,1}) \boxtimes \mathbb{L}^1 & \lambda_1 > \lambda_2 = 0 \\ 0 & \lambda_1 > \lambda_2 > 0 \end{cases}$$

$$H^4_{\mathrm{c,Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} V_6 \boxtimes \mathbb{L}^2 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^6_{\mathrm{c,Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} V_6 \boxtimes \mathbb{L}^3 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

In particular the compactly supported Borel Eisenstein cohomology $H^*_{c,\mathrm{Eis},[P_0]}(\mathcal{A}_2[2],\mathbb{V}_\lambda)$ is given as a Gal-module by

$$H^2_{c,\mathrm{Eis},[P_0]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) = \begin{cases} 16\mathbb{L}^0 & \lambda_1=\lambda_2=0 \\ 30\mathbb{L}^0 & \lambda_1=\lambda_2>0; \lambda_1 \text{ even} \\ 30\mathbb{L}^0 & \lambda_1>\lambda_2=0 \\ 45\mathbb{L}^0 & \lambda_1>\lambda_2>0; \lambda_1,\lambda_2 \text{ even} \end{cases}$$

$$H^3_{c,\mathrm{Eis},[P_0]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) = \begin{cases} 14\mathbb{L}^1 & \lambda_1=\lambda_2=0 \\ 15\mathbb{L}^0 & \lambda_1=\lambda_2>0; \lambda_1 \text{ odd} \\ 15\mathbb{L}^1 & \lambda_1>\lambda_2=0 \\ 0 & \lambda_1>\lambda_2>0 \end{cases}$$

$$H^4_{c,\mathrm{Eis},[P_0]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^2 & \lambda_1=\lambda_2=0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^6_{c,\mathrm{Eis},[P_0]}(\mathcal{A}_2[2],\mathbb{V}_\lambda) = \begin{cases} \mathbb{L}^3 & \lambda_1=\lambda_2=0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Taking $\chi_1 = \chi_2 = \chi = 1$ everywhere in 4.3.34 we have

$$H^0_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} r_{\mathscr{K}_2}(\mathrm{sim}) \boxtimes \mathbb{L}^0 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^2_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} r_{\mathscr{K}_2}(\mathrm{sim}) \boxtimes \mathbb{L}^1 & \lambda_1 = \lambda_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^3_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} r_{\mathscr{K}_2}(\mathcal{K}^3_0(1)) \boxtimes \mathbb{L}^2 & \lambda_1 = \lambda_2 = 0 \\ r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\det \otimes 1, \frac{\lambda_1+\lambda_2+3}{2})) \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 = \lambda_2 > 0; \lambda_1 \text{ odd} \\ r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, \lambda_1 + 2)) \boxtimes \mathbb{L}^{\lambda_1+2} & \lambda_1 > \lambda_2 = 0 \\ 0 & \lambda_1 > \lambda_2 > 0 \end{cases}$$

$$H^4_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[2], \mathbb{V}_\lambda) = \begin{cases} r_{\mathscr{K}_2}(\mathcal{K}^4_0(1)) \boxtimes \mathbb{L}^3 & \lambda_1 = \lambda_2 = 0 \\ r_{\mathscr{K}_2}(\mathcal{K}^4_1(1,1)) \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 = \lambda_2 > 0; \lambda_1 \text{ even} \\ r_{\mathscr{K}_2}(\mathcal{K}^4_2(1,1)) \boxtimes \mathbb{L}^{\lambda_1+3} & \lambda_1 > \lambda_2 = 0 \\ r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes 1 \otimes 1, \lambda + \rho_{P_0})) \boxtimes \mathbb{L}^{\lambda_1+\lambda_2+3} & \lambda_1 > \lambda_2 > 0; \lambda_1, \lambda_2 \text{ even} \end{cases}$$

It remains to compute parahoric restriction. We have $r_{\mathscr{K}_2}(\mathrm{sim}) = V_6$. For the induced representations we have

$$\begin{aligned} r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes 1 \otimes 1, \lambda + \rho_{P_0})) &= \mathrm{Ind}^{G(\mathbb{F}_2)}_{P_0(\mathbb{F}_2)}(1 \otimes 1 \otimes 1) = V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1} \\ r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\det \otimes 1, \tfrac{\lambda_1+\lambda_2+3}{2})) &= \mathrm{Ind}^{G(\mathbb{F}_2)}_{P_1(\mathbb{F}_2)}(\det \otimes 1) = V_6 \oplus V_{4,2} \oplus V_{2^3} \\ r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, \lambda_1 + 2)) &= \mathrm{Ind}^{G(\mathbb{F}_2)}_{P_1(\mathbb{F}_2)}(1 \otimes \det) = V_6 \oplus V_{4,2} \oplus V_{5,1} \end{aligned}$$

For $\mathcal{K}^3_0(1)$ we have

$$\begin{aligned} r_{\mathscr{K}_2}(\mathcal{K}^3_0(1)) &= r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2) \to \mathrm{sim}\Big)\Big) \\ &= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2)) \to r_{\mathscr{K}_2}(\mathrm{sim})\Big) \\ &= \ker(V_6 \oplus V_{4,2} \oplus V_{5,1} \to V_6) = V_{4,2} \oplus V_{5,1} \end{aligned}$$

For $\mathcal{K}_0^4(1)$ we have

$$\begin{aligned}
r_{\mathscr{K}_2}(\mathcal{K}_0^4(1)) &= r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes 1, \rho_{P_0}) \to \ker\Big(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\det\otimes 1, \tfrac{3}{2}) \oplus \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes \det, 2) \to \mathrm{sim}\Big)\Big)\Big)\\
&= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes 1, \rho_{P_0})) \to \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\det\otimes 1, \tfrac{3}{2})) \oplus r_{\mathscr{K}_2}(\mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes \det, 2)) \to r_{\mathscr{K}_2}(\mathrm{sim})\Big)\Big)\\
&= \ker\Big(V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1} \to \ker\Big((V_6 \oplus V_{4,2} \oplus V_{2^3}) \oplus (V_6 \oplus V_{4,2} \oplus V_{5,1}) \to V_6\Big)\Big)\\
&= \ker(V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1} \to V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1}) = V_{3,2,1}
\end{aligned}$$

For $\mathcal{K}_1^4(1,1)$ we have

$$\begin{aligned}
r_{\mathscr{K}_2}(\mathcal{K}_1^4(1,1)) &= r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes 1, \lambda+\rho_{P_0}) \to \mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\det\otimes 1, \tfrac{\lambda_1+\lambda_2+3}{2})\Big)\Big)\\
&= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes 1, \lambda+\rho_{P_0})) \to r_{\mathscr{K}_2}(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\det\otimes 1, \tfrac{\lambda_1+\lambda_2+3}{2}))\Big)\\
&= \ker(V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1} \to V_6 \oplus V_{4,2} \oplus V_{2^3}) = V_{4,2} \oplus V_{5,1} \oplus V_{3,2,1}
\end{aligned}$$

For $\mathcal{K}_2^4(1,1)$ we have

$$\begin{aligned}
r_{\mathscr{K}_2}(\mathcal{K}_2^4(1,1)) &= r_{\mathscr{K}_2}\Big(\ker\Big(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes 1, \lambda+\rho_{P_0}) \to \mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes \det, \lambda_1+2)\Big)\Big)\\
&= \ker\Big(r_{\mathscr{K}_2}(\mathcal{I}_{P_0(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes 1\otimes 1, \lambda+\rho_{P_0})) \to r_{\mathscr{K}_2}\mathcal{I}_{P_2(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(1\otimes \det, \lambda_1+2))\Big)\\
&= \ker(V_6 \oplus 2V_{4,2} \oplus V_{2^3} \oplus V_{5,1} \oplus V_{3,2,1} \to V_6 \oplus V_{4,2} \oplus V_{5,1}) = V_{4,2} \oplus V_{2^3} \oplus V_{3,2,1}
\end{aligned}$$

The result follows by Poincare duality. $\square$

We have the following contributions to $H_c^*(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ in the range $0 \leq \lambda_1 + \lambda_2 \leq 8$:

(λ_1,λ_2)	H_c^2	H_c^3	H_c^4	H_c^6
$(0,0)$	16	$14\mathbb{L}$	$\mathbb{L}^2$	$\mathbb{L}^3$

(λ_1, λ_2)	H^2_c	H^3_c	H^4_c
$(2,0)$	30	$30\mathbb{L}$	0
$(1,1)$	0	$10\mathbb{L}^2 + 15$	$5\mathbb{L}^3$
$(4,0)$	45	$10\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L} + 45\mathbb{L}$	0
$(3,1)$	0	$30\mathbb{L}^2 + 15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]$	0
$(2,2)$	30	$\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8] + 21\mathbb{L}^3$	$9\mathbb{L}^4$
$(6,0)$	60	$\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10] + 31\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L} + 60\mathbb{L}$	0
$(5,1)$	15	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10] + 20\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^2 + 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8] + 45\mathbb{L}^2$	0
$(4,2)$	45	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,5] + 45\mathbb{L}^3$	0
$(3,3)$	0	$15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6] + 5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10] + 35\mathbb{L}^4 + 15$	$10\mathbb{L}^5$
$(8,0)$	75	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12] + 25\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10]\mathbb{L} + 40\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10]\mathbb{L} + 75\mathbb{L}$	0
$(7,1)$	30	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,4] + 15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10] + 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10] + 40\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L}^2 + 60\mathbb{L}^2$	0
$(6,2)$	60	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,5] + 5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12] + 25\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^3 + 60\mathbb{L}^3$	0
$(5,3)$	0	$\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,6] + 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8] + 60\mathbb{L}^4$	0
$(4,4)$	$15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6] + 30$	$5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12] + 45\mathbb{L}^5$	$15\mathbb{L}^6$

We have the following contributions to $e_c(\mathcal{A}_2[2], \mathbb{V}_\lambda)$ in the range $0 \le \lambda_1 + \lambda_2 \le 8$:

(λ_1, λ_2)	e_c
$(0,0)$	$\mathbb{L}^3 + \mathbb{L}^2 - 14\mathbb{L} + 16$
$(2,0)$	$-30\mathbb{L} + 30$
$(1,1)$	$5\mathbb{L}^3 - 10\mathbb{L}^2 - 15$
$(4,0)$	$-10\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L} - 45\mathbb{L} + 45$
$(3,1)$	$-15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6] - 30\mathbb{L}^2$
$(2,2)$	$-\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8] + 9\mathbb{L}^4 - 21\mathbb{L}^3 + 30$
$(6,0)$	$-\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10] - 31\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L} - 60\mathbb{L} + 60$
$(5,1)$	$-5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10] - 20\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^2 - 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8] - 45\mathbb{L}^2 + 15$
$(4,2)$	$-\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,5] - 45\mathbb{L}^3 + 45$
$(3,3)$	$-15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6] - 5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10] + 10\mathbb{L}^5 - 35\mathbb{L}^4 - 15$
$(8,0)$	$-5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12] - 25\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10]\mathbb{L} - 40\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10]\mathbb{L} - 75\mathbb{L} + 75$
$(7,1)$	$-\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[6,4] - 15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[10] - 40\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8]\mathbb{L}^2 - 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[10] - 60\mathbb{L}^2 + 30$
$(6,2)$	$-\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[4,5] - 5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12] - 25\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6]\mathbb{L}^3 - 60\mathbb{L}^3 + 60$
$(5,3)$	$-\mathbb{S}^{\mathrm{gen}}_{\Gamma(2)}[2,6] - 30\mathbb{S}^{\mathrm{new}}_{\Gamma_0(2)}[8] - 60\mathbb{L}^4$
$(4,4)$	$15\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[6] - 5\mathbb{S}^{\mathrm{new}}_{\Gamma_0(4)}[12] + 15\mathbb{L}^6 - 45\mathbb{L}^5 + 30$

We should remark on the discrepancies between the above table and the table appearing in [14, Section 10]: some of the discrepancies in latter contradict known results of Harder [55] and Petersen [96] on the cohomology of local systems on $\mathcal{A}_2$, and some contradict results of Rösner [100] on the cuspidal cohomology of local systems on $\mathcal{A}_2[2]$. On the other hand the entries in the above table perfectly match the outputs of the computer program (kindly shared by Bergström) which relies on direct computations of point counts of genus 2 curves over finite fields. This provides strong computational evidence for the correctness of the expression for the Euler characteristics of local systems on $\mathcal{A}_2[2]$.

4.7 Example: Cohomology of $\mathcal{A}_2[3]$

Let $\mathcal{A}_2[3]$ be the moduli stack of principally polarized Abelian surfaces with full level 3 structure. The group $\mathrm{GSp}_4(\mathbb{F}_3)$ acts on $\mathcal{A}_2[3]$. We will compute the cohomology $H^*(\mathcal{A}_2[2], \mathbb{Q}_\ell)$ as a $\mathrm{GSp}_4(\mathbb{F}_3) \times \mathrm{Gal}$-module, recovering results of Hoffman-Weintraub.

Let us first recall the strategy employed by Lee-Weintraub. Let $\mathcal{A}_2[N]$ be the moduli stack of principally polarized Abelian surfaces with full level N structure, which is the Shimura variety $\mathcal{S}_{K'(N)}$ for the modified full level N congruence subgroup $K'(N) \subseteq \mathrm{GSp}_4(\mathbb{A}_\mathbb{Q}^\infty)$; in particular, it is connected. The same results can be deduced for the Shimura variety $\mathcal{S}_{K(N)}$ for the full level N congruence subgroup $K(N) \subseteq \mathrm{GSp}_4(\mathbb{A}_\mathbb{Q}^\infty)$ by taking character twists.

Let $\overline{\mathcal{A}}_2[N]$ be a toroidal compactification of $\mathcal{A}_2[N]$. We consider the Leray spectral sequence for the canonical inclusion $j : \mathcal{A}_2[N] \hookrightarrow \overline{\mathcal{A}}_2[N]$

$$E_2^{p,q} = H^p(\overline{\mathcal{A}}_2[N], \mathbb{R}^q j_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\mathcal{A}_2[N], \mathbb{Q}_\ell)$$

which by Deligne degenerates at E_3 with $E_3^{p,q} = \mathrm{Gr}^W_{p+2q} H^{p+q}(\mathcal{A}_2[N], \mathbb{Q}_\ell)$. We have $\mathrm{Gr}^W_i H^j(\mathcal{A}_2[N], \mathbb{Q}_\ell) = 0$ unless $j \le i \le 2j$, and $\mathrm{Gr}^W_i H^i(\mathcal{A}_2[N], \mathbb{Q}_\ell) = \mathrm{im}(H^i(\overline{\mathcal{A}}_2[N], \mathbb{Q}_\ell) \to H^i(\mathcal{A}_2[N], \mathbb{Q}_\ell))$. We have

$$H^p(\overline{\mathcal{A}}_2[N], \mathbb{R}^q j_* \mathbb{Q}_\ell) = \bigoplus_{\ell_1 < \ldots < \ell_q} H^p(D(\ell_1, \ldots, \ell_q), \epsilon)$$

where $\epsilon = \wedge^q \mathbb{Q}^{E_q}$ is a 1-dimensional ℓ-adic local system where E_q is the sheaf of germs of the q hypersurfaces intersecting $D(\ell_1, \ldots, \ell_q)$. The E_2-page differentials $d_2^{p,q} : E_2^{p,q} \to E^{p+2,q-1}$ are given by

$$d_2^{p,q} = \sum_i (-1)^i d_{i*} : \bigoplus_{\ell_1 < \ldots < \ell_q} H^p(D(\ell_1, \ldots, \ell_q), \epsilon) \to \bigoplus_{\ell_1 < \ldots < \ell_{q-1}} H^{p+2}(D(\ell_1, \ldots, \ell_{q-1}), \epsilon)$$

where $d_{i*} : H^p(D(\ell_1, \dots, \ell_q), \epsilon) \to H^{p+2}(D(\ell_1, \dots, \widehat{\ell_i}, \dots, \ell_q), \epsilon)$ is the Gysin morphism for the inclusion $d_i : D(\ell_1, \dots, \ell_q) \hookrightarrow D(\ell_1, \dots, \widehat{\ell_i}, \dots, \ell_q)$. The E_2-page has the following terms:

$$
\begin{array}{c|ccccccc}
3 & E_2^{0,3} & & & & & & \\
2 & E_2^{0,2} & E_2^{1,2} & E_2^{2,2} & & & & \\
1 & E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & & \\
0 & E_2^{0,0} & & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} & & E_2^{6,0} \\
\hline
E_2 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

By [57, Lemma 5.2] the differential $d_2^{0,1} : E_2^{0,1} = \bigoplus_\ell H^0(D(\ell), \epsilon) \to H^2(\overline{\mathcal{A}}_2[N], \mathbb{Q}) = E_2^{2,0}$ is injective, hence we have

$$H^2(\mathcal{A}_2[N], \mathbb{Q}) = H^2(\overline{\mathcal{A}}_2[N], \mathbb{Q}) / \bigoplus_\ell H^0(D(\ell), \mathbb{Q})$$

The differential $d_2^{1,1} : E_2^{1,1} = \bigoplus_\ell H^1(D(\ell), \epsilon) \to H^3(\overline{\mathcal{A}}_2[N], \mathbb{Q}) = E_2^{3,0}$ is injective since $\mathrm{Gr}_3^W H^2(\mathcal{A}_2[N], \mathbb{Q}_\ell) = 0$, hence we have

$$\mathrm{Gr}_3^W H^3(\mathcal{A}_2[N], \mathbb{Q}) = H^3(\overline{\mathcal{A}}_2[N], \mathbb{Q}) / \bigoplus_\ell H^1(D(\ell), \mathbb{Q})$$

The differential $d_2^{1,2} : E_2^{1,2} = \bigoplus_{\ell_1 < \ell_2} H^1(D(\ell_1, \ell_2), \epsilon) \to \bigoplus_\ell H^2(D(\ell), \epsilon) = E_2^{2,1}$ is an isomorphism since $\mathrm{Gr}_5^W H^3(\mathcal{A}_2[N], \mathbb{Q}_\ell) = 0$ and $\mathrm{Gr}_5^W H^4(\mathcal{A}_2[N], \mathbb{Q}_\ell) = 0$. Define the complex

$$
\begin{aligned}
S^\bullet &= \left(E_2^{0,2} \to E_2^{2,1} \to E_2^{4,0}\right) \\
&= \Big(\bigoplus_{\ell_1 < \ell_2} H^0(D(\ell_1, \ell_2), \epsilon) \to \bigoplus_\ell H^2(D(\ell), \epsilon) \to H^4(\overline{\mathcal{A}}_2[N], \mathbb{Q}) \Big)
\end{aligned}
$$

which is exact in $S^0 - E_2^{0,2}$ since $\mathrm{Gr}_4^W H^2(\mathcal{A}_2[N], \mathbb{Q}_\ell) = 0$. Define the complex

$$
\begin{aligned}
T^\bullet &= \left(E_2^{0,3} \to E_2^{2,2} \to E_2^{4,1} \to E_2^{6,0}\right) \\
&= \Big(\bigoplus_{\ell_1 < \ell_2 < \ell_3} H^0(D(\ell_1, \ell_2, \ell_3), \epsilon) \to \bigoplus_{\ell_1 < \ell_2} H^2(D(\ell_1, \ell_2), \epsilon) \to \bigoplus_\ell H^4(D(\ell), \epsilon) \to H^6(\overline{\mathcal{A}}_2[N], \mathbb{Q}) \Big)
\end{aligned}
$$

which is exact in $T^2 = E_2^{4,1}$ and $T^3 = E_2^{6,0}$ since $H^5(\mathcal{A}_2[N], \mathbb{Q}_\ell) = 0$ and $H^5(\mathcal{A}_6[N], \mathbb{Q}_\ell) = 0$. Now the E_3-page has the following terms:

$$
\begin{array}{c|ccccc}
3 & E_3^{0,3} & & & & \\
2 & & & E_3^{2,2} & & \\
1 & & & E_3^{2,1} & & \\
0 & E_3^{0,0} & & E_3^{2,0} & E_3^{3,0} & E_3^{4,0} \\
\hline
E_3 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

In particular, one obtains the following:

Proposition 4.7.1. [57, Theorem 5.3]

$$
\mathrm{Gr}_i^W H^0(\mathcal{A}_2[N], \mathbb{Q}_\ell) = \begin{cases} H^0(\overline{\mathcal{A}}_2[N], \mathbb{Q}_\ell) = \mathbb{Q}_\ell & i = 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\mathrm{Gr}_i^W H^2(\mathcal{A}_2[N], \mathbb{Q}_\ell) = \begin{cases} H^2(\overline{\mathcal{A}}_2[N], \mathbb{Q}_\ell) / \bigoplus_\ell H^0(D(\ell), \mathbb{Q}_\ell) & i = 3 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\mathrm{Gr}_i^W H^3(\mathcal{A}_2[N], \mathbb{Q}_\ell) = \begin{cases} H^3(\overline{\mathcal{A}}_2[N], \mathbb{Q}_\ell) / \bigoplus_\ell H^1(D(\ell), \mathbb{Q}_\ell) & i = 3 \\ H^1(S^\bullet) & i = 4 \\ H^0(T^\bullet) & i = 6 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\mathrm{Gr}_i^W H^4(\mathcal{A}_2[N], \mathbb{Q}_\ell) = \begin{cases} H^2(S^\bullet) & i = 4 \\ H^1(T^\bullet) & i = 6 \\ 0 & \text{otherwise} \end{cases}
$$

At this point the main problem is to analyze the complexes $S^\bullet$ and $T^\bullet$, and how their cohomology decomposes as $\mathrm{GSp}_4(\mathbb{F}_q)$-modules.

We now specialize to the case $N = 3$. The following representations of $\mathrm{GSp}_4(\mathbb{F}_3)$ play a role:

Representation of $\mathrm{GSp}_4(\mathbb{F}_3)$	θ_0	θ_1	θ_4	θ_5	τ_1	τ_2	τ_3	χ_5	χ_6
Dimension	1	24	15	81	10	30	90	20	60

In particular $\theta_0 = 1_{\mathrm{GSp}_4(\mathbb{F}_3)}$ is the trivial representatioon and $\theta_5 = \mathrm{St}_{\mathrm{GSp}_4(\mathbb{F}_3)}$ is the Steinberg representation.

By [57, Proposition 6.2] which is obtained by analyzing the action of $\mathrm{GSp}_4(\mathbb{F}_3)$ on boundary components of the toroidal compactification, the E_2-page has the following terms:

$$
\begin{array}{c|cccc}
3 & \tau_1 + \tau_2 + \chi_0\tau_2 + \tau_3 & & & \\
2 & \theta_1 + \theta_4 + \theta_5 + \tau_2 + \tau_3 & \theta_1 + \theta_4 + \theta_5 + \tau_2 + \tau_3 & & \\
1 & \theta_0 + \theta_1 + \theta_4 & 2\theta_0 + 3\theta_1 + 3\theta_4 + \theta_5 + \tau_2 + \tau_3 + \chi_5 + \chi_6 & \theta_0 + \theta_1 + \theta_4 & \\
0 & \theta_0 & 2\theta_0 + \theta_1 + \theta_4 + \chi_5 & 2\theta_0 + \theta_1 + \theta_4 + \chi_5 & \theta_0 \\
\hline
E_2 & 0 & 2 & 4 & 6
\end{array}
$$

By computing the cohomology of the complexes $S^\bullet$ and $T^\bullet$, the E_3 page has the following terms:

$$
\begin{array}{c|cc}
3 & \tau_1 + \chi_0\tau_2 & \\
2 & & \theta_5 \\
1 & & \theta_1 + \theta_4 + \chi_6 \\
0 & \theta_0 & \theta_0 + \chi_5 \\
\hline
E_3 & 0 & 2
\end{array}
$$

In particular, one obtains the following:

Theorem 4.7.2. (Hoffman-Weintraub, [57, Theorem 5.7, Theorem 6.1]) The cohomology $H^*(\mathcal{A}_2[3], \mathbb{Q}_\ell)$ is concentrated in degrees 0, 2, 3, 4, and given as a $\mathrm{GSp}_4(\mathbb{F}_3) \times \mathrm{Gal}$-module by

$$
\begin{aligned}
H^0(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \theta_0 \boxtimes \mathbb{L}^0 \\
H^2(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= (\theta_0 \oplus \chi_5) \boxtimes \mathbb{L}^1 \\
H^3(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= (\theta_1 \oplus \theta_4 \oplus \chi_6) \boxtimes \mathbb{L}^2 + (\tau_1 \oplus \chi_0\tau_2) \boxtimes \mathbb{L}^3 \\
H^4(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \theta_5 \boxtimes \mathbb{L}^3
\end{aligned}
$$

In particular the cohomology $H^*(\mathcal{A}_2[3], \mathbb{Q}_\ell)$ is given as a Gal-module by

$$H^0(\mathcal{A}_2[3], \mathbb{Q}_\ell) = \mathbb{L}^0$$
$$H^2(\mathcal{A}_2[3], \mathbb{Q}_\ell) = 21\mathbb{L}^1$$
$$H^3(\mathcal{A}_2[3], \mathbb{Q}_\ell) = 99\mathbb{L}^2 + 40\mathbb{L}^3$$
$$H^4(\mathcal{A}_2[3], \mathbb{Q}_\ell) = 81\mathbb{L}^3$$

We now explain how the above can be quickly deduced from earlier results.

Theorem 4.7.3. The intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{A}_2[3], \mathbb{Q}_\ell)$ is concentrated in degrees 0, 2, 4, 6 and given as a $\mathrm{GSp}_4(\mathbb{F}_3) \times \mathrm{Gal}$-module by

$$H^0_{\mathrm{disc}}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = \theta_0 \boxtimes \mathbb{L}^0$$
$$H^2_{\mathrm{disc}}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = (\theta_0 + \chi_5) \boxtimes \mathbb{L}^1$$
$$H^4_{\mathrm{disc}}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = (\theta_0 + \chi_5) \boxtimes \mathbb{L}^2$$
$$H^6_{\mathrm{disc}}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = \theta_0 \boxtimes \mathbb{L}^3$$

Proof. To compute intersection cohomology we only need to consider contributions from Siegel-CAP representations and 1-dimensional representations in this case. Recall that $S_4^{\mathrm{new}}[\Gamma_0(9)]$ is 1-dimensional, generated by the cusp form $f = \eta(3\tau)^8 = q\prod_{n\geq 1}(1-q^{3n})^8 = q - 8q^4 + 20q^7 + \ldots$ which has CM by $\mathbb{Q}(\sqrt{-3})$ and hence has depth 0 supercuspidal local component $\mu_3 = \mathrm{Ind}_{Z(\mathbb{Q}_3)\mathrm{GL}_2(\mathbb{Z}_3)}^{\mathrm{GL}_2(\mathbb{Q}_3)}\theta$ corresponding to the admissible pair $(\mathbb{Q}_9^\times, \theta)$ where $\mathbb{Q}_9 = \mathbb{Q}[s]/(s^2+2s+2)$ is an unramified quadratic extension and $\theta : \mathbb{Q}_9^\times \to \mathbb{Q}(i)$ is either of the characters of $\mathbb{Q}_9^\times$ trivial on $\mathbb{Q}_3^\times$ given by $s \mapsto \pm i$ and $3 \mapsto 9$. Since $\epsilon(\mu_3, \frac{1}{2}) = 1$, a Saito-Kurokawa lift contributing to IH^3 has local component $\Pi(\mu_3, \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_3)}) = \theta_-(\mu_3, \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_3)})$ an anisotropic theta lift with hyperspecial parahoric restriction $r_{\mathscr{K}_3}(\Pi(\mu_3, \mathrm{St}_{\mathrm{GL}_2(\mathbb{Q}_3)})) = 0$, and a Saito-Kurokawa lift contributing to IH^2, IH^4 has local component $\Pi(\mu_3, 1_{\mathrm{GL}_2(\mathbb{Q}_3)}) = L(\mu_3\nu^{1/2} \rtimes \nu^{-1/2})$ with hyperspecial parahoric restriction $r_{\mathscr{K}_3}(\Pi(\mu_3, 1_{\mathrm{GL}_2(\mathbb{Q}_3)})) = \chi_5$ an irreducible representation of dimension 20. $\square$

The cohomology $H^*(\overline{\mathcal{A}}_2[3], \mathbb{Q}_\ell)$ then differs from the intersection cohomology $H^*_{\mathrm{disc}}(\mathcal{A}_2[3], \mathbb{Q}_\ell)$ in degrees 2 and 4 by the $\mathrm{GSp}_4(\mathbb{F}_3)$-module $\bigoplus_\ell H^0(D(\ell), \mathbb{Q}_\ell) \simeq 1_{\mathbb{F}_3^\times} \rtimes 1_{\mathrm{GL}_2(\mathbb{F}_3)} = \theta_0 + \theta_1 + \theta_4$. It follows that the cohomology $H^*(\overline{\mathcal{A}}_2[3],\mathbb{Q}_\ell)$ is concentrated in degrees $0,2,4,6$, and given as a $\mathrm{GSp}_4(\mathbb{F}_3)\times\mathrm{Gal}$-module by

$$\begin{aligned}
H^0(\overline{\mathcal{A}}_2[3],\mathbb{Q}_\ell) &= H^0_{\mathrm{disc}}(\mathcal{A}_2[3],\mathbb{Q}_\ell) = \theta_0\boxtimes\mathbb{L}^0\\
H^2(\overline{\mathcal{A}}_2[3],\mathbb{Q}_\ell) &= H^2_{\mathrm{disc}}(\mathcal{A}_2[3],\mathbb{Q}_\ell) + 1_{\mathbb{F}_3^\times}\rtimes 1_{\mathrm{GL}_2(\mathbb{F}_3)} = (2\theta_0+\theta_1+\theta_4+\chi_5)\boxtimes\mathbb{L}^1\\
H^4(\overline{\mathcal{A}}_2[3],\mathbb{Q}_\ell) &= H^4_{\mathrm{disc}}(\mathcal{A}_2[3],\mathbb{Q}_\ell) + 1_{\mathbb{F}_3^\times}\rtimes 1_{\mathrm{GL}_2(\mathbb{F}_3)} = (2\theta_0+\theta_1+\theta_4+\chi_5)\boxtimes\mathbb{L}^2\\
H^6(\overline{\mathcal{A}}_2[3],\mathbb{Q}_\ell) &= H^6_{\mathrm{disc}}(\mathcal{A}_2[3],\mathbb{Q}_\ell) = \theta_0\boxtimes\mathbb{L}^3
\end{aligned}$$

which explains the first row of the spectral sequence.

Rather than analyzing the rest of the spectral sequence as done by Hoffman-Weintraub, we compute the cohomology directly. In this case the only contributions come from Siegel Eisenstein cohomology and Borel Eisenstein cohomology.

Theorem 4.7.4. The Siegel Eisenstein cohomology $H^*_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{Q}_\ell)$ is concentrated in degrees $2,3$ and given as a $\mathrm{GSp}_4(\mathbb{F}_3)\times\mathrm{Gal}$-module by

$$\begin{aligned}
H^2_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{Q}_\ell) &= \chi_5\boxtimes\mathbb{L}^1\\
H^3_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{Q}_\ell) &= \chi_6\boxtimes\mathbb{L}^2
\end{aligned}$$

In particular the Borel Eisenstein cohomology $H^*_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3],\mathbb{Q}_\ell)$ is given as a Gal-module by

$$\begin{aligned}
H^2_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{Q}_\ell) &= 20\mathbb{L}^1\\
H^3_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{Q}_\ell) &= 60\mathbb{L}^2
\end{aligned}$$

Proof. Taking $\chi=1$ everywhere in 4.3.12 and noting that $L(\mu,\frac{1}{2})\neq 0$ for the cuspidal automorphic representation μ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by the newform $f\in S_4^{\mathrm{new}}[\Gamma_0(9)]$ considered previously, we have

$$\begin{aligned}
H^2_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{Q}_\ell) &= r_{\mathscr{K}_3}(\mathcal{J}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}},\tfrac{1}{2}))\boxtimes\mathbb{L}^1\\
H^3_{\mathrm{Eis},[P_1]}(\mathcal{A}_2[3],\mathbb{V}_\lambda) &= r_{\mathscr{K}_3}\Big(\ker\Big(\mathcal{I}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}},\tfrac{1}{2})\to\mathcal{J}_{P_1(\mathbb{A}_\mathbb{Q}^\infty)}^{G(\mathbb{A}_\mathbb{Q}^\infty)}(\pi_{\mathrm{fin}},\tfrac{1}{2})\Big)\Big)\boxtimes\mathbb{L}^2
\end{aligned}$$

It remains to compute parahoric restriction. Recalling that the local component of the newform $f \in S_4^{\mathrm{new}}[\Gamma_0(9)]$ is a depth 0 supercuspidal representation $\mu_3 = \mathrm{Ind}_{Z(\mathbb{Q}_3)\mathrm{GL}_2(\mathbb{Z}_3)}^{\mathrm{GL}_2(\mathbb{Q}_3)}\theta$, such a newform yields the parahoric restriction

$$r_{\mathscr{K}_3}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})) = \chi_5 + \chi_6$$

and yields the kernel

$$\begin{aligned}
&r_{\mathscr{K}_3}\Big(\ker\Big(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}) \to \mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})\Big)\Big) \\
&= \ker\Big(r_{\mathscr{K}_3}(\mathcal{I}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2})) \to r_{\mathscr{K}_3}(\mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \tfrac{1}{2}))\Big) \\
&= \ker(\chi_5 + \chi_6 \to \chi_5) = \chi_6
\end{aligned}$$

Here we note that $r_{\mathscr{K}_3}(\mathcal{J}_{P_1(\mathbb{A}_{\mathbb{Q}}^{\infty})}^{G(\mathbb{A}_{\mathbb{Q}}^{\infty})}(\pi_{\mathrm{fin}}, \frac{1}{2})) \simeq \chi_5$ is the same parahoric restriction appearing in the description of the Siegel CAP part of intersection cohomology, and we are using $\pi_\Lambda \rtimes 1 \simeq \chi_5(\Lambda, 1) + \chi_6(\Lambda, 1)$ in the case $\Lambda = \omega_\Lambda \circ \mathrm{Nm}_2$ by the tables in the appendix. The result follows. $\square$

Theorem 4.7.5. The Borel Eisenstein cohomology $H^*_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell)$ is concentrated in degrees 0, 2, 3, 4 and given as a $\mathrm{GSp}_4(\mathbb{F}_3) \times \mathrm{Gal}$-module by

$$\begin{aligned}
H^0_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \theta_0 \boxtimes \mathbb{L}^0 \\
H^2_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \theta_0 \boxtimes \mathbb{L}^1 \\
H^3_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= (\theta_1 + \theta_4) \boxtimes \mathbb{L}^2 + (\tau_1 + \chi_0\tau_2) \boxtimes \mathbb{L}^3 \\
H^4_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \theta_5 \boxtimes \mathbb{L}^3
\end{aligned}$$

In particular the Borel Eisenstein cohomology $H^*_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell)$ is given as a Gal-module by

$$\begin{aligned}
H^0_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \mathbb{L}^0 \\
H^2_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= \mathbb{L}^1 \\
H^3_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= 39\mathbb{L}^2 + 40\mathbb{L}^3 \\
H^4_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) &= 81\mathbb{L}^3
\end{aligned}$$

Proof. Taking $\chi_1 \in \{1, \chi_0\}$ and $\chi_2 = \chi = 1$ everywhere in 4.3.34 we have

$$H^0_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = r_{\mathscr{K}_3}(\mathrm{sim}) \boxtimes \mathbb{L}^0$$
$$H^2_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = r_{\mathscr{K}_3}(\mathrm{sim}) \boxtimes \mathbb{L}^1$$
$$H^3_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = r_{\mathscr{K}_3}(\mathcal{K}^3_0(1)) \boxtimes \mathbb{L}^2 \oplus r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{0,\mathrm{fin}} \circ \det \otimes 1, \tfrac{3}{2})) \boxtimes \mathbb{L}^3$$
$$H^4_{\mathrm{Eis},[P_0]}(\mathcal{A}_2[3], \mathbb{Q}_\ell) = r_{\mathscr{K}_3}(\mathcal{K}^4_0(1)) \boxtimes \mathbb{L}^3$$

It remains to compute parahoric restriction. We have $r_{\mathscr{K}_3}(\mathrm{sim}) = \theta_0$. For the induced representations we have

$$r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes 1 \otimes 1, \rho_{P_0})) = \mathrm{Ind}^{G(\mathbb{F}_3)}_{P_0(\mathbb{F}_3)}(1 \otimes 1 \otimes 1) = \theta_0 + 2\theta_1 + \theta_3 + \theta_4 + \theta_5$$
$$r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\det \otimes 1, \tfrac{3}{2})) = \mathrm{Ind}^{G(\mathbb{F}_3)}_{P_0(\mathbb{F}_3)}(\det \otimes 1) = \theta_0 + \theta_1 + \theta_3$$
$$r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2)) = \mathrm{Ind}^{G(\mathbb{F}_3)}_{P_0(\mathbb{F}_3)}(1 \otimes \det) = \theta_0 + \theta_1 + \theta_4$$
$$r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\chi_{0,\mathrm{fin}} \circ \det \otimes 1, \tfrac{3}{2})) = \mathrm{Ind}^{G(\mathbb{F}_3)}_{P_0(\mathbb{F}_3)}(\chi_0 \circ \det \otimes 1) = \tau_1 + \chi_0\tau_2$$

For $\mathcal{K}^3_0(1)$ we have

$$\begin{aligned} r_{\mathscr{K}_3}(\mathcal{K}^3_0(1)) &= r_{\mathscr{K}_3}\Big(\ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2) \to \mathrm{sim}\Big)\Big) \\ &= \ker\Big(r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2)) \to r_{\mathscr{K}_3}(\mathrm{sim})\Big) \\ &= \ker(\theta_0 + \theta_1 + \theta_4 \to \theta_0) = \theta_1 + \theta_4 \end{aligned}$$

For $\mathcal{K}^4_0(1)$ we have

$$\begin{aligned} r_{\mathscr{K}_3}(\mathcal{K}^4_0(1)) &= r_{\mathscr{K}_3}\Big(\ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes 1 \otimes 1, \rho_{P_0}) \to \ker\Big(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\det \otimes 1, \tfrac{3}{2}) \oplus \mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2) \to \mathrm{sim}\Big)\Big)\Big) \\ &= \ker\Big(r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_0(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes 1 \otimes 1, \rho_{P_0})) \to \ker\Big(r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_1(\mathbb{A}^\infty_\mathbb{Q})}(\det \otimes 1, \tfrac{3}{2})) \oplus r_{\mathscr{K}_3}(\mathcal{I}^{G(\mathbb{A}^\infty_\mathbb{Q})}_{P_2(\mathbb{A}^\infty_\mathbb{Q})}(1 \otimes \det, 2)) \to r_{\mathscr{K}_3}(\mathrm{sim})\Big)\Big) \\ &= \ker\Big(\theta_0 + 2\theta_1 + \theta_3 + \theta_4 + \theta_5 \to \ker\Big((\theta_0 + \theta_1 + \theta_3) \oplus (\theta_0 + \theta_1 + \theta_4) \to \theta_0\Big)\Big) \\ &= \ker(\theta_0 + 2\theta_1 + \theta_3 + \theta_4 + \theta_5 \to \theta_0 + 2\theta_1 + \theta_3 + \theta_4) = \theta_5 \end{aligned}$$

The result follows. $\square$

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