MATHEMATICAL MUSIC ANALYSIS: A HOLISTIC APPROACH

by

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(Under the Direction of Caner Kazanci)

ABSTRACT

Music theory undertakes the audacious goal of understanding why music is pleasing to the ear and why it is effective at expressing emotions. Applying mathematical thinking to music analysis is nothing new; the field of mathematical music theory is well-established and dates back to Pythagoras. This said, there is little literature on understanding music through a holistic lens. Ecologists and economists are more familiar with analyses which address indirect, nonlinear, obscure relationships between component parts of networks, often using mathematical tools. Network Environ Analysis, one of these methods, views networks as transactional systems and uses hard mathematics to describe more fully the effects which arise as systems grow in size.

In this document, we investigate how Network Environ Analysis can be applied to musical pieces to understand better their musical structures. We describe multiple ways of interpreting musical pieces as transactional networks and apply the corresponding mathematics in several cases. We attempt to draw links between mathematical indices which evaluate these networks' indirect structures and musical quality, and we use these measures to give rigorous meaning to subjective music theoretic judgements. Further, we develop statistical tools which use holistic information related to these transactional networks to define and demonstrate mathematical manifestations of the style and tonality of a piece of music.

INDEX WORDS: Mathematical music theory, Mathematical ecology, Network analysis

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DEDICATION

To the musicians I know, that we might better understand what makes our art so special.

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I would like to thank all of my mathematics teachers throughout the years, particularly Debbie Poss and Don Slater who retired from Lassiter High School this year, and Ted Shifrin who retired from UGA in 2015, for their incredible mentorship and inspiration. Your long careers of unparalleled dedication to students caused many a young nerd like me to pursue mathematics deeply and helped all of your pupils discover more of the beauty that there is in the structures all around.

And, of course, to my first mathematical teacher, my father, Chris.

Contents

1 Introduction

The search for why music "sounds good" has been on the minds of musicians and philosophers alike for centuries. The use of mathematical reasoning to aid in this process dates back to Pythagoras, who wondered extensively on why certain frequencies and harmonies are "pleasing to the ear." Mathematical music theory as a field addresses many of these questions about frequencies, harmony, and sound construction from an analytical point of view. It also seeks to understand music composition and symmetric or mathematically-developed musical creations. However, it does not often address some of the big questions about what *kinds* of mid- or large-scale musical structures are subjectively appealing.

Philosophically, like any art form, music is not just about what pleases the ear, but about expressing emotions to the observer. Bottom-up approaches to understanding unquantifiable events like feelings associated with the arts and music, while often insightful, give a limited scope of the big picture, and do not capture expressive qualities easily. The word "analysis" still means exactly what the Greeks meant—to split into parts—and to split loses holistic insights. Top-down, general thinking about musical pieces and *patterns within them* is an underdeveloped area within mathematical music theory, and one which could offer answers in this vein of subjectivity.

This document attempts to address the lack of this holistic thinking. In it, we explore several mathematical methods for big-picture music analysis, which help to mathematically identify characteristics of the music and to explain in more objective terms several general, subjective music theoretic definitions. We consider network decompositions of musical pieces on several scales of organization and discuss their musical relevance. Finally, we introduce new mathematical tools which can accurately recognize large-scale manifestations of musical style.

1.1 Network Environ Analysis

Network Environ Analysis (NEA) is a powerful, general tool that makes it possible to study objects as part of a connected system and to identify and quantify the direct and indirect effects in that system [2]. It is based on the conservative *transactions* of a consistent currency through the *compartments* in an interconnected network. In ecosystem models, an original context for NEA, this currency is usually energy or matter. This methodology can also be applied to any system which can be understood as a transactional compartmental network, as in economics or chemistry or, in our case, music.

A main advantage of network analysis is that it allows for investigation of direct and indirect interdependencies and relationships between components of a system without removing them from the system. This is the meaning of holism (components operating in systems), and understanding full-scale system behavior and its underlying structure requires such a perspective. The ideas of holistically evaluating ecosystem health using mathematics like NEA have parallels in ideas of quantifying "musical quality." Obviously, musical quality is a supremely subjective measure, and one cannot hope to fully understand this complex issue by using mathematics or models. This said, the entire point of using NEA to evaluate networks is to understand the behind-the-scenes mechanics, the interactions beneath the surface, from a full-view perspective.

One of the core hypotheses about transactional networks is that indirect effects account for a large amount of the behavior of a system, often times more significantly affecting the system in their sum than direct, easily understood relationships [11]. A good purely mathematical analogy is the divergence of the harmonic series $\sum \frac{1}{n}$: the tail end terms, while each insignificant, together account for a large majority of the complete sum. In the same way that neuroscientists can observe specific connections between parts of the brain and their function, but cannot explain consciousness, so too might indirect relationships within a piece of music or art capture the intangible qualities and expressiveness within them. Using NEA to understand musical structures could shed a logical light on these expressive aspects of the art itself and how they work.

Comprehensive introductions to the many facets of NEA exist in both mathematical and ecological literature [2, 6]. Software packages and Internet resources for computation of mathematical network properties abound as well [1, 5]. We have made available our own software packages for music analysis using NEA concepts in conjunction with these tools [8].

1.1.1 NEA mathematics

Before we demonstrate the application of NEA, we will give some background on the mathematics involved [6]. NEA primarily involves square matrix manipulations; a matrix can represent the whole network and its connections. To do this, each compartment is labeled with an integer.

For our purposes, we primarily consider *structure*- and *flow-based analyses*; *storage analysis* is more difficult to interpret meaningfully in music (see Section 6.1.1) [2].

- A primary object is the *flow matrix* F, whose (i, j) entry is the weight of the *direct* transactional "flow" (often, energy or matter) from compartment j to compartment i.
- Environmental inputs and outputs are stored in the *input* and *output vectors* z, y, resp.
- The *throughflow vector* T codifies the total input (equal to total output) for compartment j:

$$T_j \;=\; \sum_i F_{ij} + z_j \;=\; \sum_i F_{ji} + y_j$$

- The *total system throughflow* TST is the sum of throughflows across all compartments, $\sum_j T_j$
- Normalizing F by T gives G, with entries $G_{ij} = \frac{F_{ij}}{T_j}$

Normalizing by T ensures that the matrix series $N = I + G + G^2 + G^3 + \cdots = (I - G)^{-1}$ is convergent. N codifies the transactionary action of the entire system upon itself over all future time. *Indirect effects*, defined as multi-step transactional interactions and connections, correspond concretely to the high-order terms from this series.

Utility analysis considers net interactions between compartments and thus eliminates bidirectional relationships [2]. This is especially helpful in our musical network interpretations since the connections which arise often go both directions, complicating matters of interpreting cumulative indirect structure meaningfully.

- The net relation matrix D has entries $D_{ij} = \frac{F_{ij} - Fji}{T_i}$.
- Since D is normalized, the *utility matrix* $U = I + D + D^2 + \cdots = (I - D)^{-1}$ makes sense.

Most of the system measures we consider utilize this information to obtain different metrics on the indirect structures present in a network [6]:

- The *Indirect to Direct Effects ratio* (I/D ratio) is self-explanatory and is computed as

$$\text{I/D ratio} \;=\; \frac{\sum_j (G^2 + G^3 + G^4 + \dots)T}{\sum_j GT} \;=\; \frac{\sum_j (N - I - G)T}{\sum_j GT}$$

- The *indirect effects index* (IEI) is simply a normalized version of the I/D ratio which takes values between 0 and 1:

$$\text{IEI} \;=\; \frac{\text{I/D ratio}}{1+\text{I/D ratio}}$$

- *Finn's Cycling Index* (FCI) computes the proportion of the total system throughflow which arises due to indirect cycling effects:

$$\text{FCI} \;=\; \frac{1}{\text{TST}} \sum_j T_j \frac{N_{jj}-1}{N_{jj}}$$

- *Amplification* (Amp) is the number of nondiagonal entries in N which are larger than one. That these entries exist is surprising—one unit of input can be "amplified" via cycling.

- *Synergism* (Syn) is the ratio of the sum of the positive to negative entries of U:

$$\text{Syn} \;=\; \frac{\sum_{i,j} U_{ij} \text{ s.t. } U_{ij}>0}{-\sum_{i,j} U_{ij} \text{ s.t. } U_{ij}<0}$$

- *Mutualism* is the ratio of the number of positive to number of negative entries of U.

- *Homogenization* codifies how well-mixed the transactional material becomes due to indirect cycling. High values correspond to mixing due mostly to indirect effects. This is computed using the coefficients of variation of G and N:

$$\text{Homogenization} = \frac{\text{CV}(G)}{\text{CV}(N)} = \frac{\text{standard deviation}(G)/\text{avg}(G)}{\text{standard deviation}(N)/\text{avg}(N)}$$

As it pertains to music, we expect that these metrics, particularly homogenization and synergism, can help describe the unbalanced but synergetic importances given to certain musical elements via repetition in particular musical styles, as in the so-called system of *tonality* (see Section 4) [7]. This could help compare how stylistic systems in which musical elements are given biases sometimes actually end up creating pieces with more "stability" or organization than systems or styles which do not prioritize anything. Random or uncoordinated sounds do not always make for good or expressive music, in most aesthetic opinions.

2 Building Network Models out of Music

To apply network analysis to music, we first will need a transactional network corresponding to some musical structure. At its core, a piece of music is simply a series of frequencies, which make up *pitches*. Music is embedded in time, with pitches held for certain *rhythmic durations*, much as organisms might hold and then release a carbon atom or unit of energy.

Using recorded pitches and encoding them into a network would involve significant computer science problems which are well beyond the scope of this document. In particular, *polyphonic pitch recognition* in sound files is an as-yet intractable issue, since each pitch a voice or instrument makes is actually composed of several simultaneous frequencies. Further, capturing all the nuances of timing, pitch bending, volume, and timbre in an expressive performance seems frankly impossible.

Accordingly, we focus not upon the frequencies actually heard in a performance, but on the written score, the plan for which pitches will be played. Even improvised music like jazz often employs a large-scale plan for harmonic changes, melodic repetition, and the like. Considering this unchanging aspect of a piece of music removes us from the actual experience of listening to it, but this viewpoint enables us to analyze it and understand its structure in clear-cut ways.

For the purposes of this document, the fundamental network of a piece of music consists of a compartment for the start of every *note* and its *duration*, where the transactions are the end of one note and the start of the next. A note is a pitch, as it falls into the common 12-note-per-octave frequency division system in so-called Western music from the past few centuries (the frequencies in between these 12 logarithmic divisions are not considered) [7]. This model of note-duration pairs is a simple chain; it is not particularly interesting, nor possesses any of the relevant network properties that NEA seeks to quantify. As such, we will reduce it in a number of musically meaningful ways.

To create models based on notes, we need a sequence of notes for every piece we hope to analyze. Written music has frequently become digitized over the past few decades. An ubiquitous format for this is the MIDI file, which contains information about each note and its start and end time, regardless of instrument or frequency content. As such, these MIDI files are well-suited to our purposes. Fortuitously, tools for extracting the note information in MATLAB already exist [12].

2.1 Two types of network interpretations

2.1.1 Note transition networks

One simple way to describe a piece of music as a non-chained network is to view each note in the 12-note system as a compartment, regardless of its duration or specific place in the chained time-sequence. Then the flows between compartments correspond to direct transitions from one note to the next whenever one follows another. The weights on these flow connections are determined by how frequently their respective note-to-note transition occurs across the entire piece, so that TST is the total number of notes in the piece. Inputs and outputs correspond to the first and last notes.

This methodology requires special consideration for what to do with *tuple-stops*, or many notes happening simultaneously. We take the approach here of considering each note of an n-tuple to represent a fractional $1/n$ portion of a note, so that a transition to or away from it counts for a similarly scaled fractional amount.

It is also worth mentioning that the note compartments can be reduced modulo 12 under a typical octave equivalence relation. Keeping the octave information incorporates some aspect of whether notes transition upwards or downwards; however, this also increases the number of compartments in the model and exhibits different network properties. In addition, the way that a human ear processes sound naturally imposes some of this modulo 12 octave equivalence, so we generally take this 12-compartment perspective. See Figures 1 and 2 and Sections 2.3 and 3.1.

2.1.2 Formal networks

On an even larger scale, analysis of musical *form* gives a big-picture view as to how large sections of music repeat and follow each other. These sections can correspond to entire five-minute chunks of a large symphonic work, or even to the verses and choruses of a pop song.

Since there are usually fewer than ten large formal sections, the small networks and relationships they comprise are usually well-understood, at least in music theoretic terms. The ubiquity of a small variety of general formal templates points to effective musical expression due to their organization; if any random ordering of sections and repetition was effective at expressing musical beauty, there would be hundreds of common forms. We address several of these small networks in Section 3.3 in an attempt to give qualitative music theoretic judgments about form a mathematical backing.

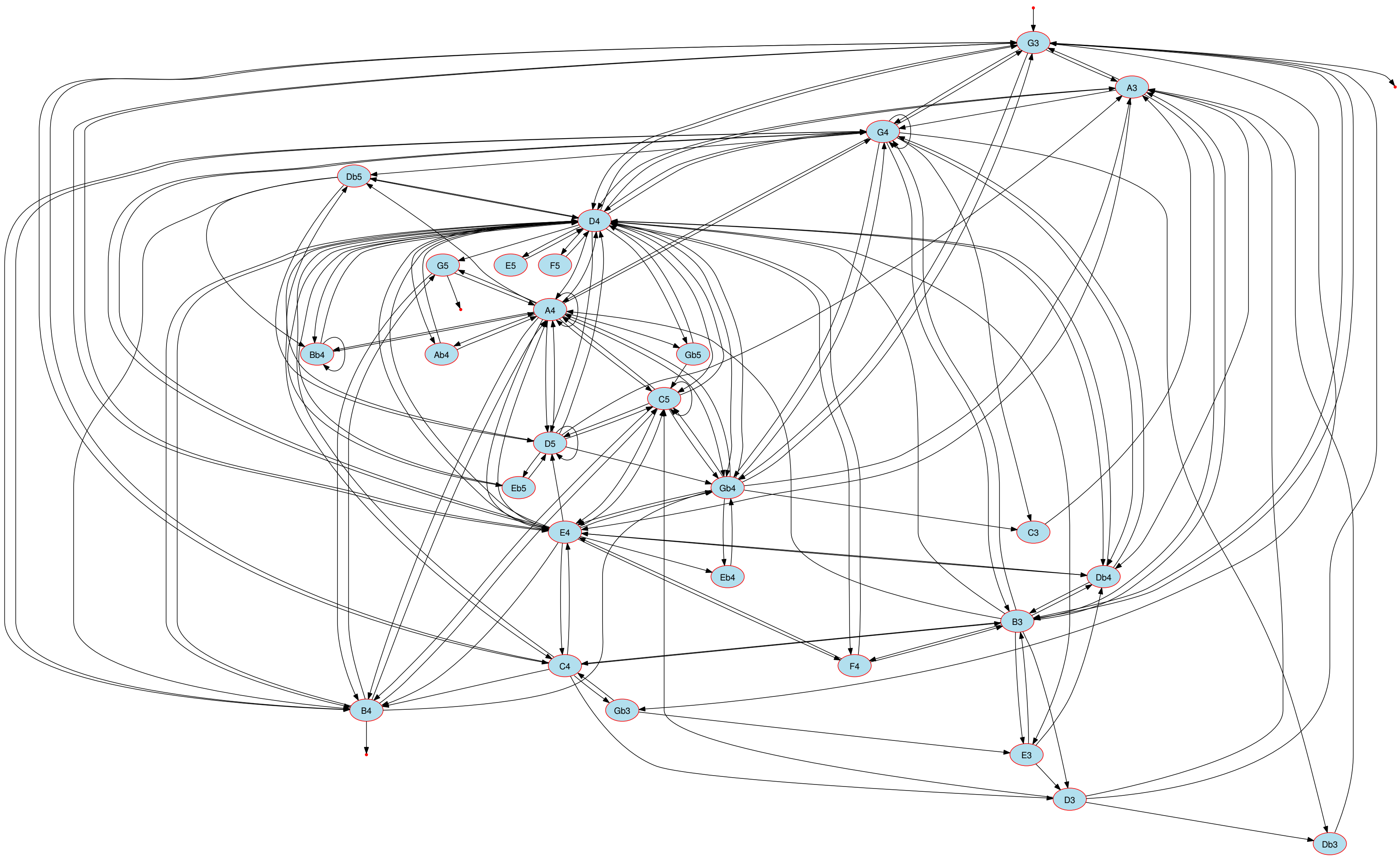


Figure 1: Note transition network for J.S. Bach's Prelude to his G major Cello Suite No.1, retaining octave information; each compartment has a note name and its octave number [5]. A single piece might span from two to six or seven octaves. Note how some notes, like D4, have more connections than others, and that there is only one input but three outputs (coming from a triple-stop on the last note). Here we use flats, as in Bb = B-flat, rather than sharps (each note has multiple *enharmonically equivalent* names).

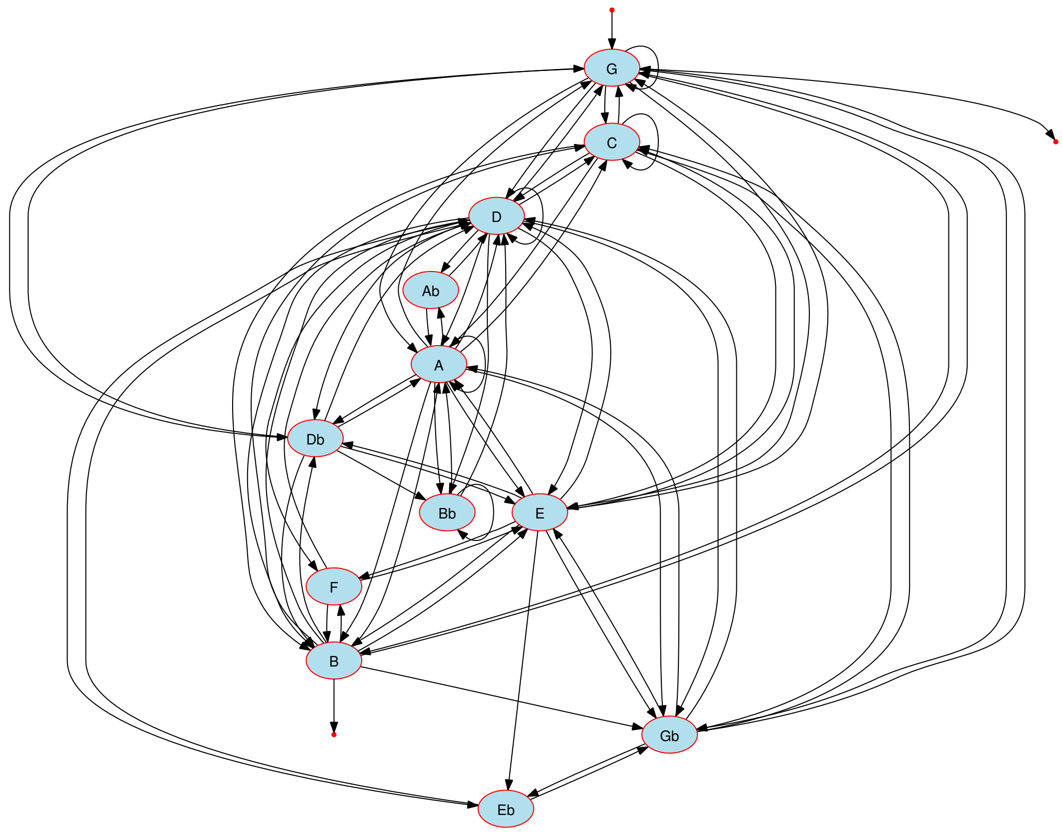


Figure 2: Note transition network modulo 12 for the G major Prelude [5]. This reduces any piece to just 12 compartments. Note that by reducing, many more notes have self-loops than in Figure 1, and many more share bidirectional connections; these tell us some notes are directly repeated, and most pairs happen in both orders at different times. Also, the three outputs have been reduced to two, since we have identified G3 and G5 together.

2.2 Hierarchical structure and aggregation

As seen in the (limited) variety of possible network interpretations above and in Figure 3, music has multiple levels of organization. Just as in ecology there are cellular, organismal, species, and communal scales of transactional activity, so in music there are small and large scale interactions, from notes to motives to melodies to sections. To fully judge a piece based upon one level of structure alone ignores a great deal of information.

Music theory and patterns throughout history point to what structural archetypes at various fixed scales seem to be particularly expressive, but give no hints as to which levels of detail contribute most to expression. A piece may have a well-knit and effectively expressive structure on one level, but poor organization on other levels, and still become famous; for instance, P.I. Tchaikovsky's beautiful melodies affect audiences deeply, despite poor formal structures (he was well aware of these compositional strengths and weaknesses) [16]. Applying network analysis to multiple levels of musical organization could give insight as to which levels are more expressively important to the listener, and to the tightness of structure and unity in a piece across scales.

The so-called *aggregation* process has been considered through multiple lenses in different fields [4, 15]. One may simplify a network into a smaller one by grouping compartments together and combining their relationships with other compartments, called aggregating. There are an enormous number of ways to aggregate compartments in a given network, since any subset can be grouped.

From the information theoretic point of view, it has been proven that information stored in a network model can only be lost when aggregating compartments in this manner [15]. Some aggregations preserve more information than others, and an intelligent aggregational strategy can be difficult to rigorously define. However, intuitive groupings of compartments which serve a similar function in the modeled system can end up correlating with minimal information loss [4].

Figure 3 shows a variety of possible intuitive aggregations of the fundamental musical network, namely, all the note-duration pairs chained in direct sequence. Traveling up in the hierarchy involves aggregations of musical elements while respecting their time-sequential embeddings, so that the network associated with each block is still a sequential chain, but of similarly-scaled musical elements and not just note-duration pairs. To apply NEA meaningfully, we leave the hierarchy by aggregating a second time, now identifying identical elements regardless of their sequential time-embedding. Section 2.3.2 contains more on the reasoning for this second aggregation.

Fortunately, the reductions we make are in line with music theoretic functional groupings, and with the musical interpretations of a human listener; people generally can identify repetitions of musical elements as the same entity and recognize them as such. A listener naturally aggregates identical sounds and even sequences of sounds across an entire piece. Musical motives or even key areas are frequently ingrained in a listener's memory, as evidenced by how frequently people complain of having a melody stuck in their heads. As such, we believe that these aggregations mirror the mathematically optimal choices and reflect the listener's point of view.

Further, despite all the losses described here, we show in Section 4 that there is still a significant amount encoded in the cumulative information stored in these aggregated networks.

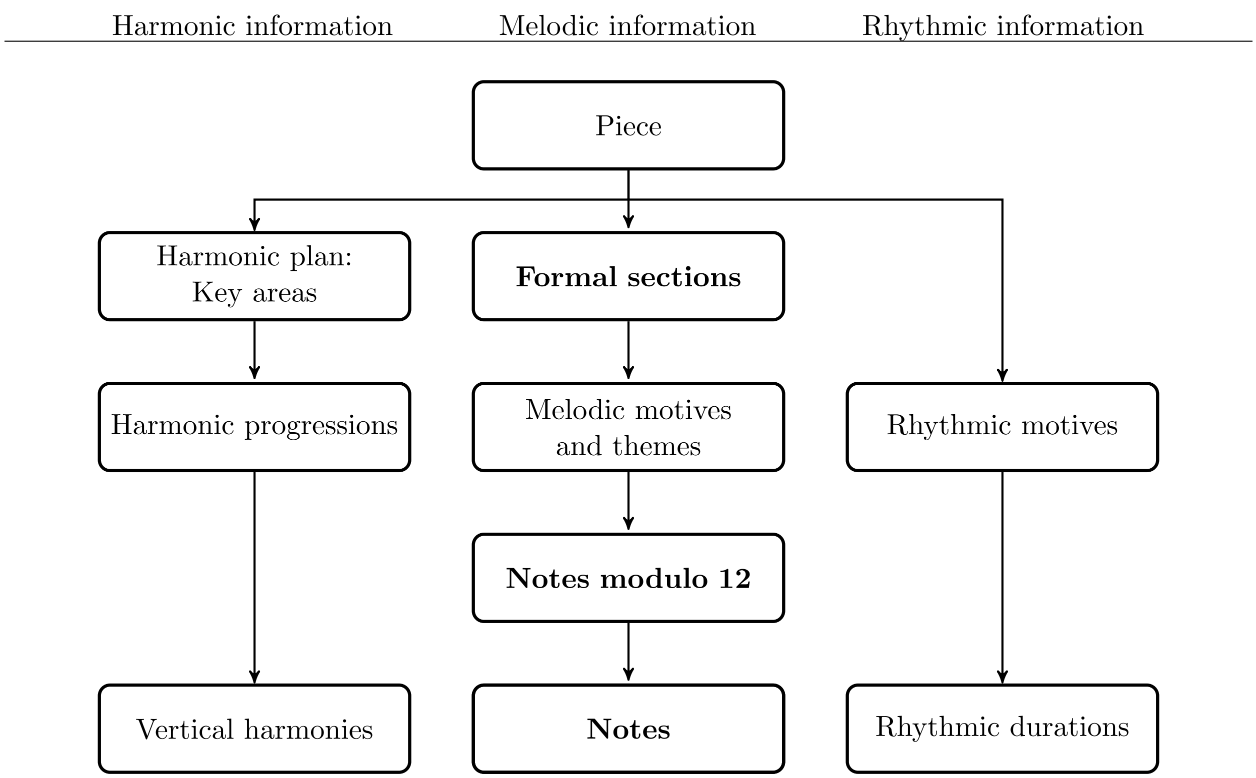


Figure 3: Aggregational hierarchy of a few possible information groupings within a piece of music, from less to more specific. Transactional network thinking can be applied to each block by aggregating identical elements within the block. Those addressed in this document are in boldface.

2.3 Implicit assumptions

Naturally, there are several assumptions made about these network interpretations, some of which have already been mentioned. Clearly, the conceptual model underlying the analysis has an effect on the network properties because they are properties of models and not of reality, per se [2]. The questions of scale and reduction are important, and rather than choose one as an ideal, we consider several possibilities. The reductions we have made, such as the modulo 12 equivalence, are in line with our musical intuition, and how listeners actually process music; thus we believe the network models we present represent musically relevant information. Below are several other types of assumptions about the networks we build and analyze.

2.3.1 Polyphony

When constructing transition matrices for polyphonic pieces (where several instruments or voices make different notes simultaneously), there is room for interpretation on how to classify note transitions. For this document, we have used the convention of separating each individual melody line and counting each note transition within it regularly, then adding up. Just as easily, we could have considered every simultaneous note event as a large tuple-stop and counted fractional transitions, as we typically do with tuple-stopping in the same melodic line. However, this is prone to some problems: if individual lines are moving at different speeds, there might be a transition from a 24-note unison to a single note in a quickly moving voice, which is not in line with how a human would listen to the note events. The intuitive ear is readily able to tell the difference between a trumpet and a cello, and processes note transitions within each voice independently, so separating "by voice" makes sense.

2.3.2 Model-induced information losses, preservations, and relevance

In the network interpretations described above, aggregating inside of one column from Figure 3 inherently loses information corresponding to other columns. For example, considering note transitions alone forgets about the durational, rhythmic dimension of the fundamental network.

Further, music is embedded in time, and the idea of modeling music as a transactional network in the schemes above disregards this time-dependent, directional aspect by identifying repetitions of musical elements together. The network properties with which NEA is concerned have to do with steady-state ecosystems which experience the same transactions at every time-step. By viewing music through this infinite time-stepping lens, then, the NEA indices which utilize infinite matrix series actually draw on network qualities that are not present in actual musical pieces' performance and experience. The second aggregation still preserves information about the direct *transitions* between notes, harmonies, motives, or sections but throws away information about their specific embeddings in the large timeline of a piece.

However, only by identifying repetitions of a musical element as the same, regardless of their timing, can we obtain meaningful understanding of the organization of a piece and how its component parts function together to form a well-knit whole. NEA identifies information about a network's cycling processes and holistic construction; seeing a piece as a linear progression without relation to its past is simplistic. For instance, the I/D ratio and Indirect Effects Index (IEI) can be computed with "particle tracking" [5, 10]. For the fundamental musical network, a single chain with n note-duration compartments, this complicated computation is unnecessary; there are simply $\binom{n}{2} - (n-1)$ indirect connections and $n-1$ direct ones. This I/D ratio is not a very discerning index, because it simply scales with n, i.e. piece length. Since we would like to interpret music in a way which gives indices like these musical and mathematical relevance, we have to change our viewpoint.

These reductions do not mean that the typical indices and computations from NEA say nothing about musical structure or aesthetic quality, the philosophical motivators for any kind of music theory. The subconscious understanding of a piece which a listener obtains in real time contains all the information about the transactional neighbor relations between compartments (notes, motives, sections, etc.). This information about direct connections is all that we use to construct our network models. However, one's subjective enjoyment of a piece may well incorporate a subconscious grasp of the indirect effects of the structure which is *implied* by these direct connections. Therefore, we let these measures stand for themselves as descriptors of a hidden kind of musical structure which music theory is not equipped to describe.

3 Applying NEA to Music

We will apply this musical network analysis to several musical pieces to illustrate the construction of networks and usage of the mathematical tools. We then attempt to relate the mathematical indices to a measure of "musical quality" in Section 3.2.

3.1 Note transition analysis: J.S. Bach - Cello Suites

First, we will delve into famous works for a single instrument, Johann Sebastian Bach's Cello Suites. The most famous, of course, is the first, in G major, whose Prelude tops all other works for cello in popularity terms. Below in Table 1 is the *transition matrix* (i.e. flow matrix) corresponding to a modulo-12 note network. The unreduced transition matrix can be found in Table 2.

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
C	1		**20**		8		18	3		4		14
C♯			**9**		2			3		3		1
D	**12**	**7**	**4**	**2**	**18**	**4**	**5**	**5**	**4**	**21**	**1**	**11**
D♯			**2**		1		3					
E	10	1	**13**			4	17	7		9		2
F			**3**		5							4
F♯	14		**7**	4	13			25		20		1
G	$3\frac{2}{3}$	2	**6**		5		20	4		33		20
G♯			**4**							1		
A	12	2	**16**		8		21	29	1	13	2	24
A♯		1	**1**							1	1	
B	$15\frac{1}{3}$	5	**9**		3	4		18		23		

Table 1: Flow matrix for 12-note transition network of Bach's G major Prelude [8]. Zero is omitted for simplicity.

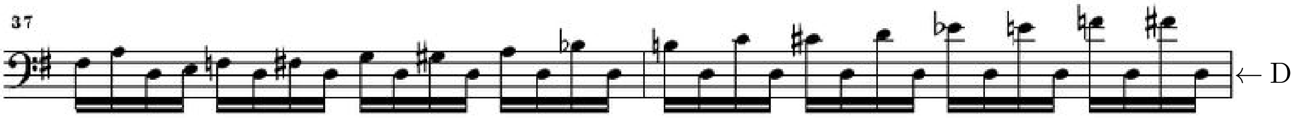
Entry ij may be interpreted as the number of times note i follows note j. As is typical, we enumerate the twelve notes upwards from C, so that C $\leftrightarrow$ 0, C♯ $\leftrightarrow$ 1, etc.

Note the prevalence of zero. For instance, the note C never transitions immediately to the note C♯, meaning the first entry in the second row is zero. Further, the only note which transitions to or from every other note, i.e. with a full row or column, is D, whose entries are in boldface.

	C3	C♯3	D3	E3	F♯3	G3	A3	B3	C4	C♯4	**D4**	D♯4	E4	F4	F♯4	G4	C♯4	A4	A♯4	B4	C5	C♯5	D5	D♯5	E5	F5	F♯5	G5
C3															2	1												
C♯3			1													1												
D3				2				1	1																			
E3					2			2			1																	
F♯3						3			1												$\frac{1}{3}$							
G3			1				3	3	3		2		2		5	1												
A3	3	2	1			1		6			1				1								1					
B3				2		3	3		5	4				4		5												
C4					2	2		2			14		4															
C♯4				1			1	1			3		1			1												
D4						3	5	2	7	1			14	3	4	2	4	13	1	8	3	1	1	1	1	1	1	
D♯4													1		3													
E4						3	2		6	1	11			4	15	4		7			4							
F4								4			2		5															
F♯4						4					5	4	13			18		19		1	8		1					
G4						2	2	4		2	2		3		15	1		23		7								
G♯4											4							1										
A4								1			9		8		20	22	1	13	2	17	9		4					6
A♯4											1							1	1			1						
B4									1		9		1			4		20			$9\frac{1}{3}$	1						6
C5			1								4		4		8			4		12	1		1				6	
C♯5											1					1		2					4					
D5											2		1					3			1	5	1	1				
D♯5											1												1					
E5											1																	
F5											1										5							
F♯5											1							1			$\frac{1}{3}$							
G5											1							5		6								

Table 2: Full transition matrix (not reduced modulo 12) for Bach's G major Prelude [8]. Octave information is not thrown away. Zero is omitted for simplicity. As seen in Figure 1, the note D4 has the most connections.

The importance of D is apparent in the following musical passage, right before the end:



This is an example of a *pedal note*, a technique where the same note is held or emphasized repeatedly for an extended period of time. This technique is often used to make one note, or harmony, important. The pedal on D and the prevalence of zero, i.e. the absence of some transition connections, are both consequences of this piece's compositional style: Bach wrote it with the tonality of G major in mind, and this tonality simply dictates that certain notes and transitions are more important than others, like D.

The different weights on transition connections codify cumulative information about note relationships in the entire piece, rather than single time events, as ecological models often represent. This shows how often certain transition patterns arise, which is directly related to musical style and a listener's perception across the entire piece. Tonal relationships and their mathematical manifestations in holistic patterns are explored further in Section 4.

Now, we may compute the network properties associated with this flow matrix using appropriate input, output, and storage vectors. An outline of this process for the modulo 12 network follows. Since the piece starts on the note G, it is only natural that the input vector has a 1 in its eighth entry. The final triple-stop contains B and two G's, meaning the output vector should have two nonzero entries, $\frac{2}{3}$ in the eighth column and $\frac{1}{3}$ in the twelfth. Indeed, this balances the inputs and outputs for every compartment. Conveniently, any such model satisfies this *steady-state assumption.*

	Index type	Modulo 12	Unreduced
1	# nodes, n	12	28
2	# links, L	76	158
3	connectance, L/n^2	.5278	.2015
4	link density, L/n	6.3333	5.6429
6	Total System Throughflow	654	654
7	Finn's Cycling Index	.9829	.9673
13	Amplification	132	754
15	I/D ratio	656.8596	661.3721
17	Homogenization	2.0785	2.7525
27	Synergism	8.5124	4.3957
28	Mutualism	1.1493	1.1538

Table 3: Outputs from NEA on the matrices from Tables 1 and 2 [1]. See Section 1.1.1 for the unlisted mathematical definitions [6].

Letting the storage of each compartment remain at one and computing using a publicly available MATLAB tool for NEA yields the selected network indices in Table 3 [1]. Notable in these indices are Finn's Cycling Index, I/D ratio, Homogenization, and Synergism. They encode some of the hidden structures in the piece, with respect to the model that they are attached to, and may correspond to intangible qualities that make music expressive.

Some of the others, like amplification, carry meaning in ecology but do not tell us much by nature of their definition, and simply correlate with network size or total system throughflow in well-connected systems like our musical models.

Note how much more connected the modulo 12 network is, despite (of course) having the same total system throughflow. A vast majority of pieces use nearly all or most of the 12 notes when octave information is discarded, though they rarely have all $2 \times \binom{12}{2}$ possible directed connections. Still, every compartment usually has a connection in one or two steps to any other in the modulo 12 network. This is different from most ecological models, and is part of the reason that utility analysis' *net* relationships could help decode important connections between musical elements.

Interestingly, homogenization is higher in the larger network, perhaps because it is less dense and directly connected. Synergism, which we would expect to increase in larger networks due to its summational nature, actually decreases in the less connected unreduced network, despite mutualism being similar; the notes' net relationships function more synergetically together when viewed under our equivalence relation, perhaps because directed, one-sided relations are more balanced out.

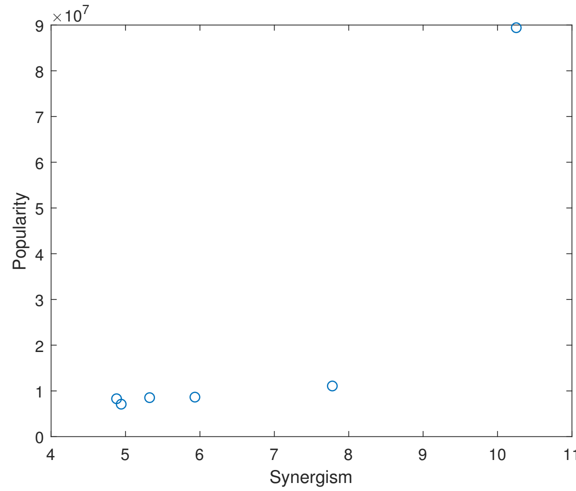
3.2 Correlating network properties with musical quality

In this section, we would like to see whether any of the network property indices given by NEA for note transition networks can correlate with some measure of musical quality, since good musical structure correlates with listener appreciation in practice. One measure of a piece's quality is its popularity. Of course, different genres and artists gain societal recognition and fame for a variety of external factors. To minimize the effects of these, we have chosen to compare pieces which share the same author, era and style of composition, and instrumentation: Bach's Cello Suites. As these pieces are well-known in the public sphere, their popularity data should reflect their quality better than more obscure songs', except perhaps the G major suite's Prelude, whose popularity has been inflated from its frequent inclusion in commercials and visual media.

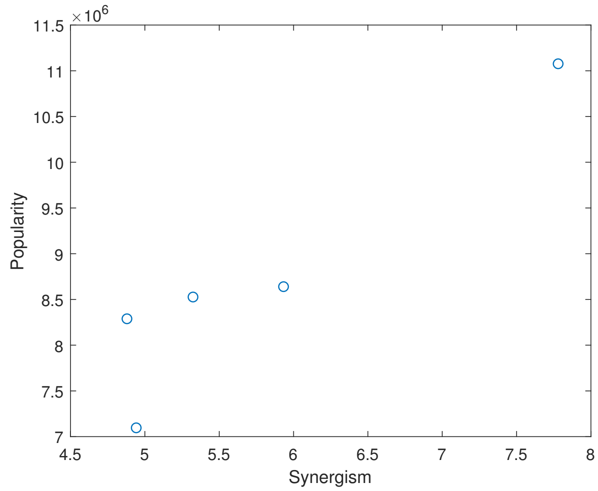
All the data here is extracted from the play counts of albums of the complete set of Cello Suites, recorded by the same performer, available to the public through the music database Spotify.

For example, as seen in Figure 4a, the values for synergism correlate extremely well with popularity in the first G major Cello Suite. On the other hand, in Figure 4c, synergism correlates negatively with popularity in the second Suite. Figure 4d shows popularity against synergism across all six Cello Suites. Indeed, this erratic behavior with wildly varying correlations shows up in most of the indices NEA provides.

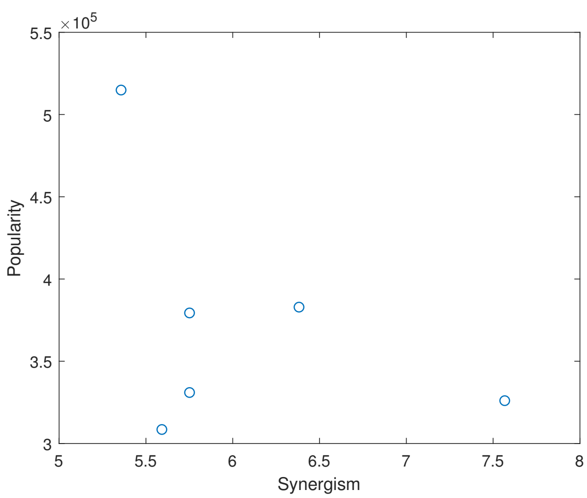
For both the unreduced and modulo 12 note transition networks, there is no statistically significant positive or negative correlation between *any* network property and the popularity of the 36 movements of the six Cello Suites. Interestingly, if we do not consider the G major Prelude, a bit of an outlier due to its high popularity, the correlation coefficients almost all double in magnitude, but are still insignificant. This could have a number of factors, from small sample size of popularity data to the small number of pieces compared. Alternatively, the lack of correlations here could mean that the note transition level of organization has little to do with popularity, but some other, larger scale could have more relevance.



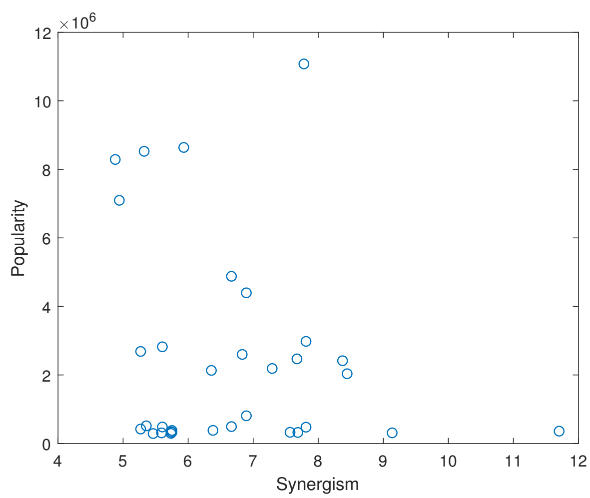
(a) Popularity (# of plays) to Synergism across six movements of Bach's Cello Suite No.1. This set has a Pearson correlation coefficient of 0.8808.



(b) Popularity (# of plays) to Synergism across five movements of Bach's Cello Suite No.1, excluding the inflated Prelude. This set has a Pearson correlation coefficient of 0.9402.



(c) Popularity (# of plays) to Synergism across six movements of Bach's Cello Suite No.2. This set has a Pearson correlation coefficient of -0.4135.



(d) Popularity (# of plays) to Synergism across all 34 movements of the six Cello Suites which have a synergism value defined, except the G major Prelude. This set has a Pearson correlation coefficient of -0.1713, though with the Prelude included, this is 0.3567.

Figure 4: Popularity to Synergism as it correlates with multiple subsets of the 36 movements of the six Cello Suites. Plotting popularity agains other NEA indices yields similarly scattered images. Note the clusters in (d), usually movements from the same Suite.

3.3 Formal analysis

The subject of musical form is the highest level of musical structure addressed here. There are relatively few archetypal large-scale formal patterns in music. In music theoretic terms, a well-structured piece has a musical form which involves the right balance of repetition and variation. In practice, recognizing the reappearance of mid- and large-scale musical elements adds much to an observer's appreciation. The hard question in understanding musical quality as it relates to form then becomes what exactly "the right balance" means.

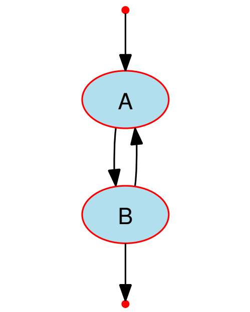
This is where applying NEA and holistic thinking can really help. Interesting indirect relationships only arise when a network has cycles; the network properties mentioned above are heavily influenced and dependent upon cycling. For instance, the formal sections in non-repeating musical forms (called *through-composed*) have transactional relationships with each other of only finite order; their indirect effects are minimal. Though expressive through-composed music certainly exists, the fact that indirect network effects and systemic unity depend upon cycling gives a mathematical backing to the music theorits' subjective concept that "repetition is good."

Some of the forms here exist on different mid- and large-scale hierarchical levels, and can even be aggregated into other forms on the list. This aggregation technically throws away information about the system, but it can still give meaningful insight into a different, larger level of hierarchical ogranization if done intelligently [4].

We dedicate the rest of this section to exploring the network properties of a handful of common musical forms. Detailed music theoretic explanations of formal analysis abound in textbooks [7].

3.3.1 Simple Binary form

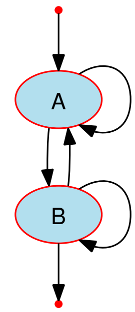
Simple binary form is the most basic musical form with nontrivial cycling and can be seen as a prototype for many others. It consists of two sections, which will be compartments A and B, which transition back and forth an arbitrary number of times. Often, each section is itself repeated once before transitioning to the other, corresponding to a self loop (or to storage, though this interpretation is not discussed here). Since the form begins with A and ends with B, we assign an input to A and an output to B, of weight one.



Flow matrix:

	A	B
A	0	$n-1$
B	n	0

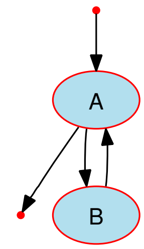
(a) Simple binary form n times without repeats ABAB...



Flow matrix:

	A	B
A	n	$n-1$
B	n	n

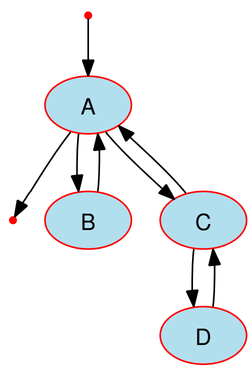
(b) Simple binary form n times with repeats AABBAABB...



Flow matrix:

	A	B
A	0	1
B	1	0

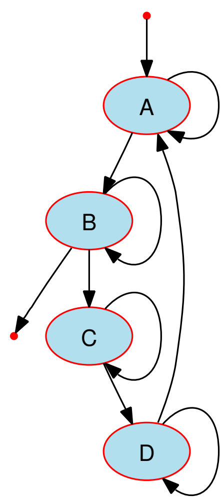
(c) Ternary form ABA



Flow matrix:

	A	B	C	D
A	0	2	1	0
B	2	0	0	0
C	1	0	0	1
D	0	0	1	0

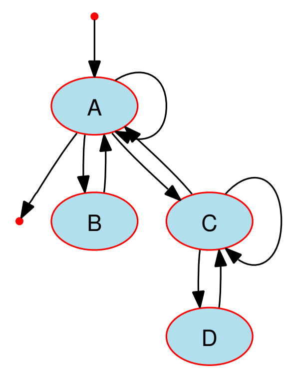
(d) Ternary form with nested ternary ABACDCABA



Flow matrix:

	A	B	C	D
A	1	0	0	1
B	2	1	0	0
C	0	1	1	0
D	0	0	1	1

(e) Ternary form with nested simple binary AABBCCDDAB



Flow matrix:

	A	B	C	D
A	1	3	1	0
B	3	0	0	0
C	1	0	1	2
D	0	0	2	0

(f) Ternary form with nested rounded binary AABABABACCDCDCABA

Figure 5: Network structures of simple binary and several ternary forms [5]. The topology of two-part verse-chorus forms is the same as that of 5b above.

3.3.2 Ternary form

Ternary form is related to rounded binary form (see below). ABA is the common overarching structure. This changes the network topology of simple binary form, by moving the output from B to A, and has a different balance of flow values.

3.3.3 Nested ternary form

In this form, each of the sections in the large-scale ABA form is itself composed of a smaller binary or ternary form. If nesting ternary forms, this leads to a structure ABA CDC ABA, which is reminiscent of rondo form (see Section 3.3.6). The repeats on the smaller forms are omitted on the return of the first sectional group; with simple binary, this gives AABB CCDD AB; with rounded binary (see below), AABABA CCDCDC ABA. Once again, A has the input and output.

3.3.4 Rounded binary form

Rounded binary form is a relative of both simple binary and ternary forms, in which some or all of the A section returns, sometimes in a different key, during the second half of the B section. When both sections are repeated, this yields a pattern A A BA BA. In network terms, this simply changes the weighting of the transitions between A and B; the network's topology remains the same as that of binary form. Most of the dance movements in Bach's Cello Suites follow this exact formula; the minuets follow a nested ternary form (see above).

3.3.5 Sonata form

A dramatic expansion of rounded binary form, *sonata form* eventually became the idiom in which a significant proportion of Western European "classical" music has been written. The large-scale structure consists of an *exposition*, A, which is repeated; a *development* section, B; and a *recapitulation*, a repeat of A, the latter half of which is in a new key. This leaves an overarching AABA, which is split still smaller by standalone sections which are characterized by their melodic *themes*. The A section usually contains four to five distinct themes; the B section often will contain one or more of these themes, but varied in a way to sound quite different and express a different feeling. The B section follows no particular pattern and has no set length.

As such, we will consider sonata form to have an ABCD ABCD E ABCD structure, which can be aggregated into the smaller rounded binary form, usually with only the first half repeated.

3.3.6 Rondo form

A *rondo* consists of a rondo section, A, and arbitrarily many other sections B, C, D, ... which reprise A in between each new one. So, one possible and common rondo form is ABACADA. Another variation is the hybrid *sonata-rondo* form, ABA C ABA, which is titled as such because it follows the harmonic plan of sonata form (see Section 3.3.5) and involves a recapitulation of the sectional group ABA.

3.3.7 Arch form

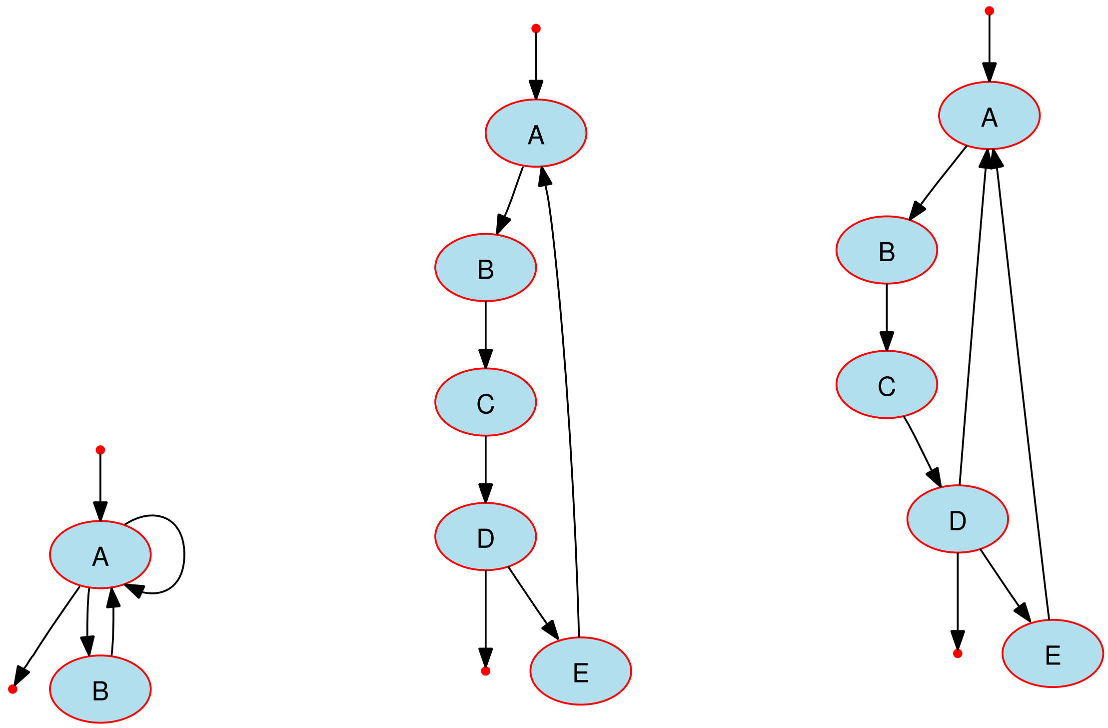
A favorite of Béla Bartók, *arch form* is not particularly common throughout music history, but offers a nice comparison to sonata-rondo form. It follows a palindromic, symmetrical pattern, as in ABCBA or ABCDCBA.

3.3.8 Popular song forms

To touch upon the most popular forms in music created today, and in songs dating back centuries, we turn to a couple of simplified formulas of *strophic* forms with reprised *choruses*:

- Three verse-chorus AABABB
- Four verse-chorus AABABABB
- Verse-chorus with bridge AABABCB

It is notable that the network topology for the forms above with only two distinct sections is exactly that of simple binary form, though the sequence of repeats (self-loops) is different, leading to noticeable differences in mathematical properties.



Flow matrix:

	A	B	C	D
A	1	3	1	0
B	3	0	0	0
C	1	0	1	2
D	0	0	2	0

(a) Rounded binary form AABABA

Flow matrix:

	A	B	C	D	E
A	0	0	0	0	1
B	2	0	0	0	0
C	0	2	0	0	0
D	0	0	2	0	0
E	0	0	0	1	0

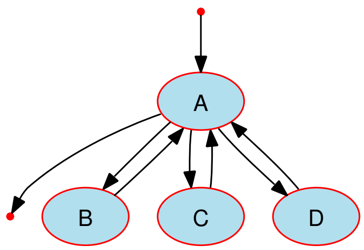
(b) Sonata form with one exposition ABCDEABCD

Flow matrix:

	A	B	C	D	E
A	0	0	0	1	1
B	3	0	0	0	0
C	0	3	0	0	0
D	0	0	3	0	0
E	0	0	0	1	0

(c) Sonata form with repeated exposition ABCDABCDEABCD

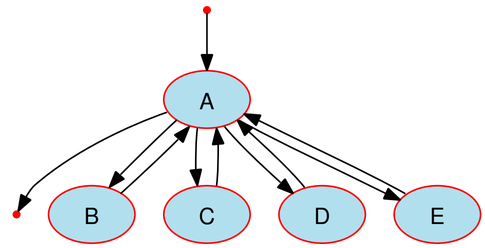
Figure 6: Network structures of rounded binary and sonata forms [5]. Note that in 6b and 6c, aggregating ABCD into one compartment A yields at two-compartment structure: the repeated exposition in 6c corresponds to the D→A connection, becoming the self-loop in 6a, while 6b takes on the topology of ternary form, 5c.



Flow matrix:

	A	B	C	D
A	0	1	1	1
B	1	0	0	0
C	1	0	0	0
D	1	0	0	0

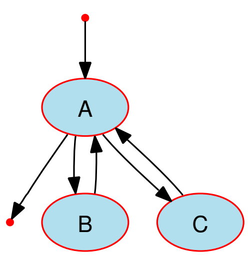
(a) 4-part rondo form ABACADA



Flow matrix:

	A	B	C	D	E
A	0	1	1	1	1
B	1	0	0	0	0
C	1	0	0	0	0
D	1	0	0	0	0
E	1	0	0	0	0

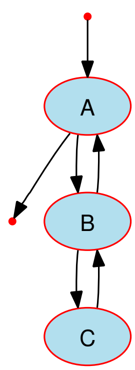
(b) 5-part rondo form ABACADAEA



Flow matrix:

	A	B	C
A	0	2	1
B	2	0	0
C	1	0	0

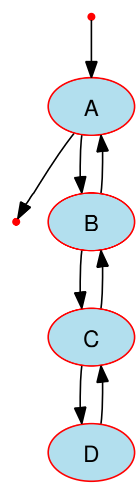
(c) Sonata-rondo form ABACABA



Flow matrix:

	A	B	C
A	0	1	0
B	1	0	1
C	0	1	0

(d) Arch form ABCBA



Flow matrix:

	A	B	C	D
A	0	1	0	0
B	1	0	1	0
C	0	1	0	1
D	0	0	1	0

(e) Arch form ABCDCBA

Figure 7: Network structures of several rondo and arch forms [5]. Note the lack of self-loops and balanced connections between every compartment, i.e. symmetric flow matrices.

Form	Letter sequence	Cycling index	IEI	Synergism	Homogenization
Simple Binary, no repeats	ABAB	.5	.7	5	4.46799
Simple Binary, no repeats	ABABAB	.66667	.814815	7	6.6
Simple Binary, twice	AABBAABB	.125	.34375	9	2.34038
Simple Binary, thrice	AABBAABBAABB	.16667	.404762	13	2.47386
Simple Binary, four times	AABBAABBAABBAABB	.1875	.43125	17	2.57558
Ternary	ABA	.5	.714286	undefined[1]	4.46799
Nested Ternary-Ternary	ABACDCABA	.72222	.904762	undefined	3.74306
Nested Ternary-Binary	AABBCCDDAB	.05555	.476923	4.89189	2.42239
Nested Ternary-Rounded	AABABACCDCDCABA	.570707	.817073	undefined	2.91068
Rounded Binary	AABABA	.5	.714286	undefined	4.46799
Rounded Binary, twice	AABABAAABABA	.5	.714286	undefined	4.46799
Sonata	ABCDABCDEABCD	.653846	.913669	5.18056	6.29035
Sonata, single exposition	ABCDEABCD	.5	.868852	5.70588	6.73056
4-part Rondo	ABACADA	.642857	.866667	undefined	2.51523
Sonata-Rondo	ABACABA	.690476	.866667	undefined	2.90273
Arch, 3-part	ABCBA	.633333	.866667	undefined	3.15303
Arch, 4-part	ABCDCBA	.702381	.922078	undefined	2.99597
Three verse-chorus	AABABB	.222222	.461538	7	3
Four verse-chorus	AABABABB	.375	.609375	9	4.07342
Verse-chorus w/bridge	AABABCBB	.329167	.642857	10	2.50359
Verse-chorus w/bridge	AABABCB	.442177	.74359	10.3333	2.78328

Table 4: NEA index value comparison for several different forms, grouped by similarity.

3.3.9 Interpreting data on formal networks

The four particular indices in Table 4 capture network information which can be used to give mathematical reasoning to music theoretic judgements and heuristics, explained here.[1]

As it pertains to the simple binary forms, these indices reflect the heuristic that more repetition of the entire short form can enhance the expressiveness of the piece, but with decreasing marginal returns. When self-looping repeats are omitted, there is more direct variation and mixing of musical material (homogenization), which keeps the listener interested while still cycling often. This lines up with modern performance practice: performers of Bach's Cello Suites often skip these repeats, to shorten the six-movement works for a concert audience (historically, music for dance gatherings was often repeated, for maximal utility from minimal musical material).

Ternary form shares cycling and homogenization values with both simple binary form and rounded binary form, which points to their topological relationships. Interestingly, repeating

[1]Interestingly, a number of the forms considered here do not have synergism defined. This is because their flow matrices are perfectly symmetric, meaning $D = 0$, so that $U = (I - D)^{-1} = I$ has no negative entries, and the ratio of sums which defines synergism is not defined [2, 6].

rounded binary form twice directly adds nothing to these network properties, which may be relevant to why binary form dances (like those in the Cello Suites) are written with just one iteration. Nesting other forms complicates by adding more compartments, though these indices and the less-connected network topology (see Figure 5e) accurately mirror the relative scarcity of pieces with simple binary form as the nested component.

Sonata form, which represents a much longer form than the others listed (sometimes up to 20 minute long pieces), carries with it high values of homogenization and cycling. Notably, not repeating the long exposition leads to higher values of synergism and homogenization, which point to the monotony of the four theme groups ABCD happening three separate times. Indeed, music performers today frequently omit the repeated exposition when playing strictly written sonata forms, as they find it boring or too long for the listener.

Sonata-rondo form shares similar index values with sonata form, except for its homogenization, which makes sense; it has fewer thematic groups to add musical variation. However, it carries higher index values than typical rondo form, perhaps due to its more tightly-knit network topology (Figures 7a,7b,7c) and meaningful repetition of B. Arch form, with its symmetry, cycles effectively and has a well-connected topology but does not mix variation in as well as some other forms.

The pop song forms reflect the heuristic notion that adding a bridge or instrumental break between choruses at the end adds to the indirect effects and cycling between sections as they interact synergistically. Interestingly, though sharing network topology with simple binary forms, the different order and repetition patterns of basic verse-chorus forms lead to different flow weights, which yield much higher index values for the same (or less) total system throughflow. It is also noteworthy that removing one repetition of the chorus at the end of a verse-chorus with bridge form actually increases every index value, including synergism. Indeed, pop songs which fade out on an extra repetition of the chorus often get a bad rap for "having a boring ending."

Aggregationally, there are several relationships worth mentioning, and many more irrelevant ones (cf. Section 2.2) [4]. For instance, sonata-rondo form can be aggregated into ternary form in different symmetric ways: (ABA) C (ABA), which follows the intuition about recapitulating, or A (BACAB) A, which groups strangely in functional terms. Similarly, sonata form can be aggregated into rounded binary form, which is its historical predecessor: (ABCD) (ABCD) E (ABCD). Interestingly, after aggregation, one cannot tell the difference between sonata-rondo and sonata form from the indices in Table 4. This aligns with the loss of information from aggregation.

All in all, the network properties discussed here shed light on why exactly certain formal structures gain popularity and contain expressive value beyond simply random sectional orderings or chains. They give mathematical justification for subjective comparisons between forms by music theorists and performers about repetition and variation.

4 Tonality

4.1 Background

In the history of Western musical development, a system of *tonality* arose, in which some notes are more important than others. In particular, there is a *tonal center* around which other important pitches are organized. Systems of harmony and harmonic progression also came into play, adding horizontal and vertical dimensions of pitch organization which frequently followed common tropes.

Music theory attempts to understand tonal relationships in terms of "harmonic tension and release." A tonal center is easily recognized by a listener from the absence of tension around it; other notes and harmonies are characterized by their feeling of direction back towards this tonal center. Musicians write pieces by exploiting these fundamental qualities of melody and harmony for dramatic effect. While these are subjective qualities not easy to distinguish for a computer, analogous mathematical patterns arise, which reflect the style and tonality of a piece.

In the next sections, we explore the mathematical manifestations and meanings of music theoretic concepts. We utilize music theoretic knowledge about prototypical tonal relationships to apply and develop several indices which extract meaningful information using only simple holistic data. These indices can both distinguish between a variety of styles of musical composition and help determine tonal centers in tonal music.

While we describe all the necessary musical concepts and terminology for the purposes of this mathematical context, in-depth treatment can be found in music theory textbooks [7].

4.2 Detecting atonality

Before we define tonal relationships in more depth, we will consider mathematical ways to determine if a piece lacks the defining features of tonality, specifically, a hierarchical organization of notes. Some styles of composition, usually termed *atonal*, give every note equal importance. Serialism is one stark example, in which all twelve notes are used before repeating any one of them, and the entire piece is built (rather mathematically) out of variations on the initial 12-note sequence. Free atonality is a more general style, containing no specific small-scale melodic tropes or sequences.

The importance of a note should manifest itself in the number of times it is played across the entire piece. As far as networks are concerned, this note frequency is itself equal to the total number of note-to-note inputs into a single compartment j, simply corresponding to its entry in the throughflow vector, T_j.

Going back to the G major Prelude, these totals are shown in Figure 8. As is apparent, the seven notes of the G major scale (G, A, B, C, D, E, F♯) are frequently visited, and the other five notes are not. Totals for the first movement of Beethoven's Fifth Symphony, in C minor, are shown in Figure 9. Here again, notes of the C *minor* scale are far more frequent.[2] Even more dramatically, a tonal pop song like Michael Jackson's "Bad" emphasizes yet fewer notes.

On the other hand, an atonal piece such as Anton Webern's symphonic Passacaglia or a serial piece like his Klavierstücke have much more even note distributions, as in Figures 11 and 12, respectively. Though it is reductionist to say so, these pieces can sound more chaotic and disorderly, despite their equally-weighted note-hierarchical structure.

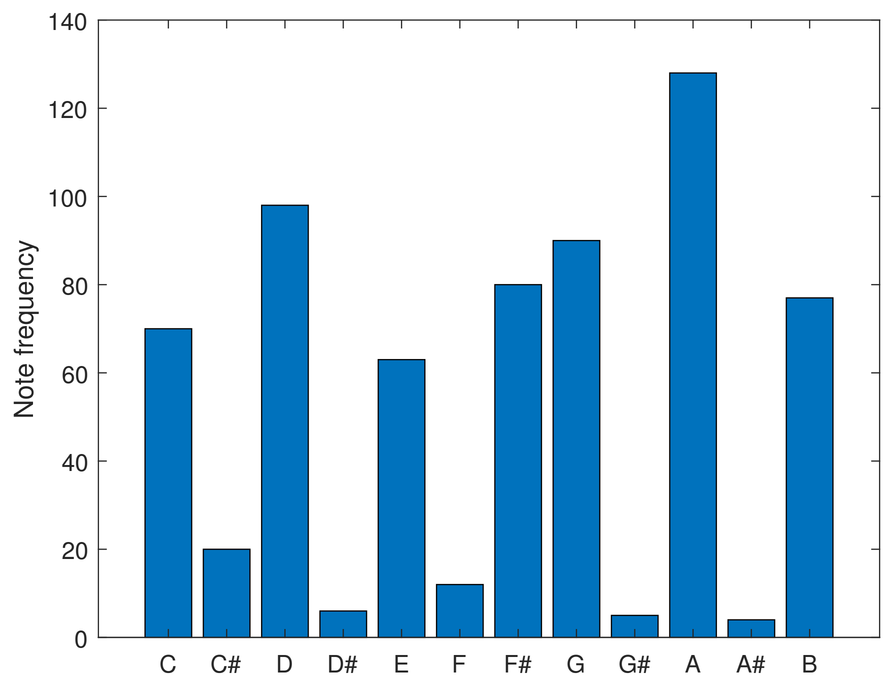


Figure 8: Frequency of each note in Bach's G major Prelude

[2]There are multiple "minor" scales. The *natural minor* is simply the major diatonic scale built on the note three notes above; here, this is D♯ = E♭, so the notes are (C, D, E♭, F, G, A♭, B♭).

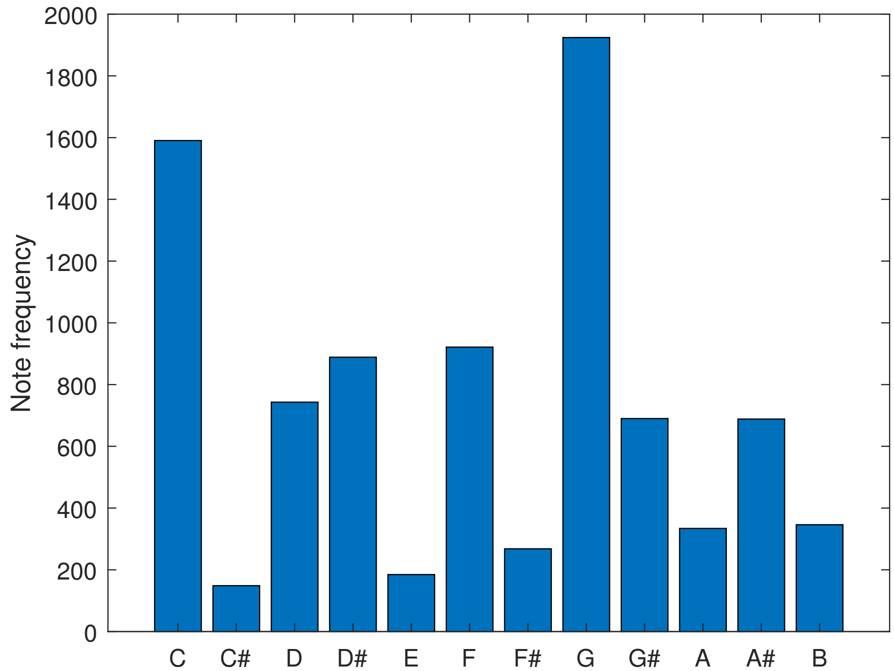


Figure 9: Frequency of each note in Beethoven's Symphony No.5, first movement

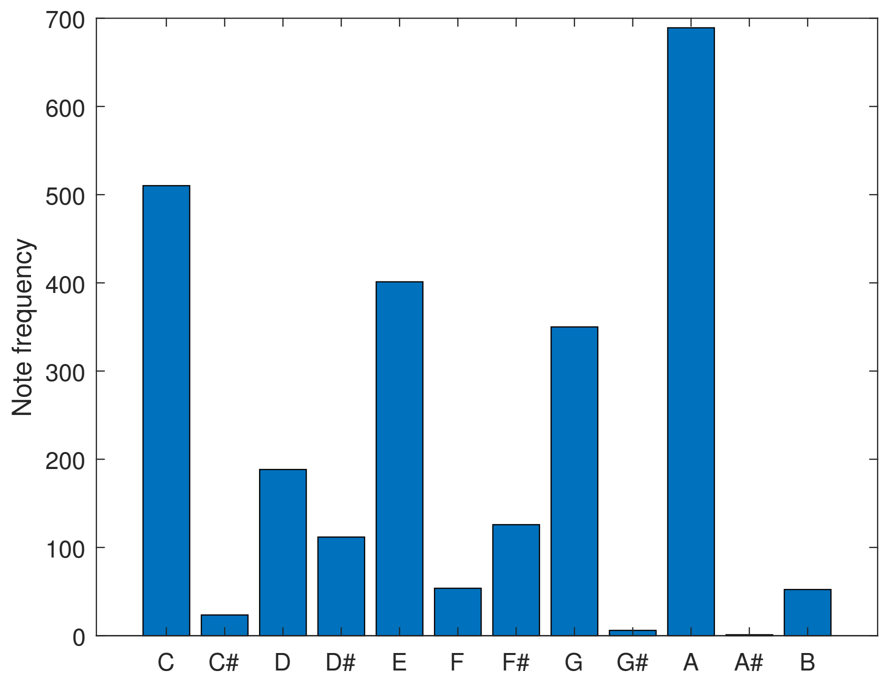


Figure 10: Frequency of each note in Michael Jackson's "Bad"

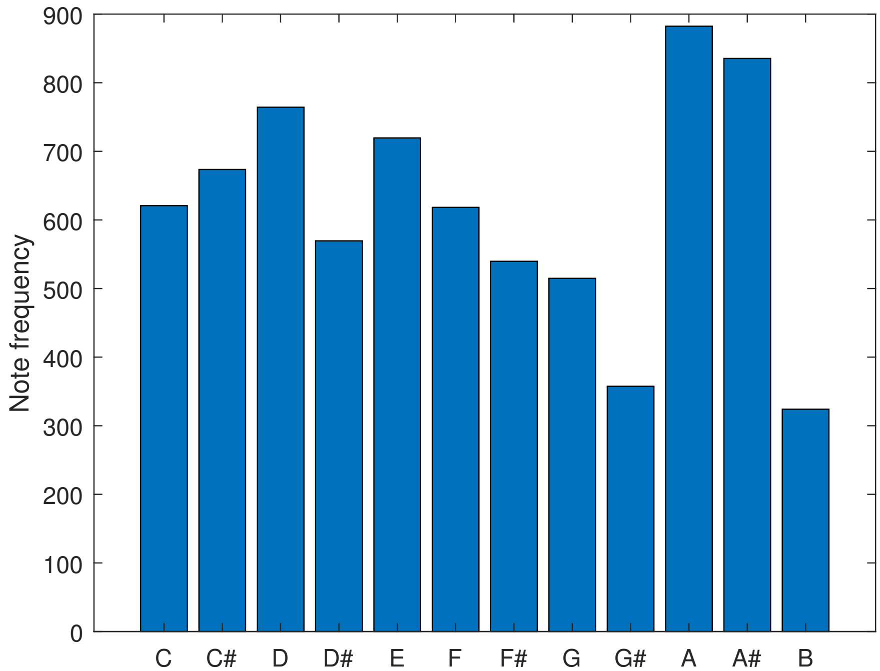


Figure 11: Frequency of each note in Webern's Passacaglia Op.1

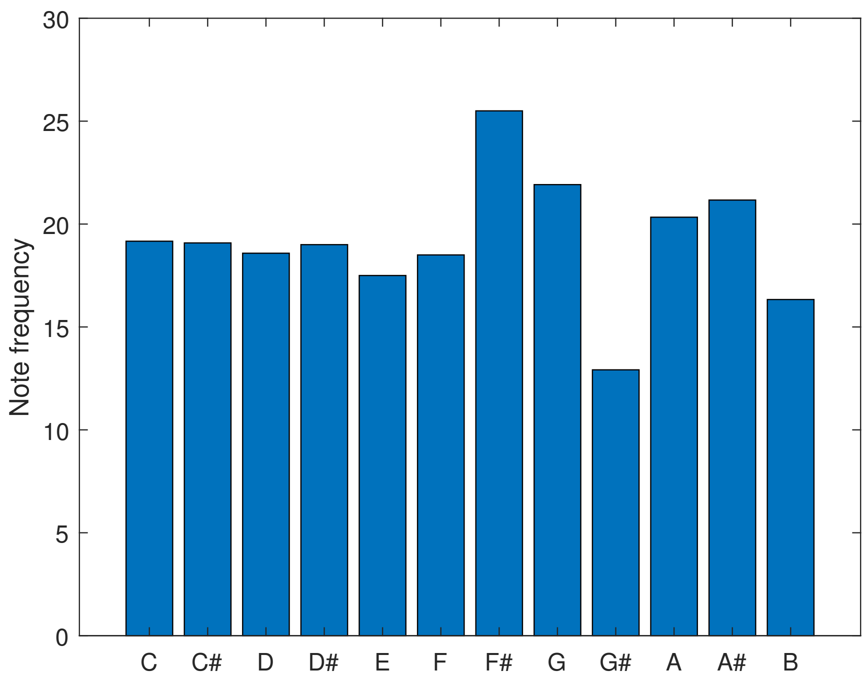


Figure 12: Frequency of each note in Webern's Klavierstücke Op.posth. (WoO 18)

4.2.1 Diversity measures

There exist mathematical tools used in ecology to evaluate "species evenness" or diversity in a community [13, 14]. Some of these include the *Shannon index*, H' and the *Simpson index*, λ:

$$H' = -\sum_{j=1}^{R} P_j \log(P_j) \tag{1}$$

$$\lambda = \sum_{j=1}^{R} {P_j}^2 \tag{2}$$

Here, P_j is the proportion of the total community population belonging to species j (so that $\sum P_j = 1$), and R is the *richness* or total number of species in the community.

The Shannon index is based upon the weighted geometric mean of the P_j's. In NEA contexts, it has been called *throughflow diversity* [6]. As every $P_j \to \frac{1}{R}$, i.e. as the species distribution approaches evennness, $H' \to \log(R)$. As the distribution approaches a highly uneven weighting, such as with the domination of one single species, $H' \to 0$.

The Simpson index simply measures the probability that two objects chosen randomly from the community are the same. As the distribution approaches evenness, $\lambda \to \frac{1}{R}$. As it approaches unevenness, $\lambda \to 1$. This is the opposite behavior of what one might intuitively want from such an index, so we consider here $1 - \lambda$ instead.

These two measures are each related to different formulations of *true diversity*, which is simply computed as the reciprocal of a generalized mean of the proportions P_j [3]. As it happens, these are the weighted geometric mean and weighted arithmetic means, respectively.

$$H' = \log\left(\frac{1}{\prod_{j=1}^{R} P_j^{P_j}}\right) = \log(\text{true diversity of order 1}) \tag{3}$$

$$\lambda = \frac{1}{\text{true diversity of order 2}} \tag{4}$$

Now, as seen in Table 5, when applied to the diversity of note frequencies in musical pieces, these ecological indices can actually differentiate between tonal and atonal note distributions. Notably, they do so in nonlinear fashions due to their calculation. Rescaling is necessary to see much differentiation. Also, these values are somewhat consistent within the large classifications of tonal and atonal musics; if we wish to distinguish more, we need to delve deeper.

4.2.2 Differentiating between tonal styles

Tonal pieces from different eras have different levels of note unevenness, for a variety of musical reasons. Briefly put, popular music from the past few decades emphasizes a small number of notes in short songs; centuries-old Baroque-era pieces (like Bach's Cello Suites) emphasize a few more notes, mostly from the same key; later Romantic-era music (as with Beethoven's Fifth Symphony and later tonal music) develops towards more note equality, with the addition of many *chromatic* notes between diatonic scales and *modulation* to distantly related key areas. This historical evolution is sometimes termed the *dissolution of tonality* and ends with complete atonality.

Mathematically, we should simply see a range of diversities, even within tonal music. The indicators above did not show much distinction within tonal music. As such, we propose our own Note Evenness Index NEI to quantify how even a note frequency vector D is (i.e. D_j is the frequency of note j), where $\bar{D}$ and $\|D\|_1$ are the arithmetic mean and linear sum of the elements of D, respectively:

$$\text{NEI}\,(D) \;=\; \frac{12}{11}\sum_{j=1}^{12}\frac{|D_j-\bar{D}|}{\|D\|_1} \;=\; \frac{12}{11}\sum_{j=1}^{12}P_j-\bar{P} \tag{5}$$

After rescaling by $\frac{12}{11}$, this index is 0 for D perfectly even and 2 for D with only one note. Rather than just computing a type of weighted geometric or arithmetic mean of proportional frequencies, as with (1) and (2), this metric incorporates each note's frequency difference from the average. This helps to clarify yet further how even the distribution D is, since it factors in individual evenness.

Note that D in our network scenario can be computed in the same way as the throughflow vector T. However, this evenness index could also be interesting in other ecological situations to measure diversity in a new way.

4.2.3 Differentiating between atonal styles

We would also like to distinguish further between different atonal styles. In freely atonal pieces, the frequencies of the *transitions between* notes are often similarly weighted, as well as those of notes. In serial music, however, each note appears with similar frequency, but usually appears as part of one of a few 12-note sequences. This means that certain transitions will necessarily occur more frequently than others.

We could simply apply the Shannon or Simpson diversity measures, but again, we would like to capture component-wise distances from a perfectly even distribution. To differentiate between these two types of atonality, we propose an analogous Transition Evenness Index TEI to quantify how even a note transition matrix F is.

For these purposes, we remove the diagonal of F to obtain F'; keeping track of repeated notes does not point to particular styles.[3] $\mathbb{M}$ is the 12×12 matrix of ones less its diagonal, and $f,\ m$ are the sums of $F',\ \mathbb{M}$, respectively, to normalize each to 1.

$$\text{TEI}\ (F) \ = \ \sum_{i,j} \left| \frac{1}{m}\ \mathbb{M}_{ij} \ - \ \frac{1}{f}\ F'_{ij} \right|, \qquad f = \sum_{i \neq j} F'_{ij}, \quad m = \sum_{i \neq j} \mathbb{M} = 12^2 - 12 = 132 \qquad (6)$$

For a perfectly even transition matrix (disregarding the diagonal), $\text{TEI}(F) = 0$. For a perfectly uneven F with only one nondiagonal transition type, $\text{TEI}(F) = 2 - \frac{2}{132} = 1.9848$.

These new indices are easy to compute and can consistently distinguish within tonal and atonal styles of composition. See a comparison with other diversity measures below in Table 5. Every column except for TEI computes the diversity of the note frequency distribution D.

Piece	**Style**	H'	$\exp(H')/12$	λ	$\frac{12}{11}(1-\lambda)$	NEI	TEI
Michael Jackson, Bad	Pop Tonal	1.9524	0.5871	0.1724	0.9028	0.9658	1.4098
Bach, Prelude	Baroque Tonal	2.1386	0.7073	0.1317	0.9472	0.7587	1.1625
Beethoven, Fifth Symphony	Romantic Tonal	2.2350	0.7788	0.1278	0.9515	0.6079	0.9517
Webern, Passacaglia	Freely atonal	2.4476	0.9634	0.0892	0.9936	0.2311	0.4773
Webern, Klavierstücke	Serial atonal	2.4730	0.9882	0.0853	0.9979	0.1162	1.3039

Table 5: Comparison of diversity index values between five example pieces in different styles. We have computed the Shannon index with base e. Note that the Shannon and Simpson indices increase with respect to diversity, while NEI and TEI do the opposite. The two atonal pieces have significantly different note diversity index values from the tonal pieces. NEI shows even more differentiation within different tonal styles in the ways we would expect, and TEI distinguishes strongly between the two atonal styles.

[3] For pieces in any of our tonal, serial, and freely atonal styles, the same single note might be repeated constantly for extended periods for musical effect. For instance, Webern's Passacaglia contains many repeated notes, changing the values of TEI and distinguishing between styles less effectively. $\text{TEI}(F') = 0.4773$, while leaving the diagonal in yields a value of 0.5634, looking more like a serial piece.

4.3 Determining the tonal center in tonal music

Determining the tonal center of a piece strictly from holistic data is more challenging than determining whether it simply fits into the style of tonal music. A simple look at the note frequency distributions in Figures 8 and 9 does not yield immediate hints as to the tonal centers of these pieces; the question is not simply answered by determining the most common note. Still, the relationships dictated by tonality give us enough information to uncover this. Detectable mathematical patterns arise which reflect the rules of tonal music theory. We examine some of these below and exploit them to create tools for predicting a tonal center or *key area*.

4.3.1 Simple defining features of tonality and key areas

A set of inter-note relationships and *resolution tendencies* is a major characteristic of tonality. For instance, the note relationship of a *perfect fifth* carries strong importance in the tonal system. It has particular harmonic meaning; as Pythagoras noted, it corresponds to a frequency ratio of 3:2, the simplest integer ratio possible after the octave, 2:1.

In the time-based context of a piece of music, transitioning down a fifth sounds like a release of harmonic tension, particularly when other simulatneous notes are added in vertical chords. For instance, the *dominant triad* is a specific set of three notes which, when played together, all sound like they are leading towards the tonal center. In particular, the note directly below the tonal center, the *leading tone*, has a strong tendency to resolve up to the tonal center. This release of tension towards the tonal center is a defining feature of being in a particular key.

Confusingly, the typical terminology *fifth* comes not from the actual seven-note interval between the two notes in the twelve-note system, but from the five-note interval within a seven-note subset, a *diatonic scale*. A tonal key area is characterized by a tonal center and a choice of diatonic scale. A *major* key's diatonic scale, the basic prototype for tonal music, can actually be built out of successive perfect fifth intervals, by starting seven notes below the tonal center and choosing every seventh note modulo 12. The most important notes that define this major key are those which are generated from the first steps in this process, which we try to capture mathematically. As an example, the sequence of generating a G major scale is shown in Table 6.

Consider the note frequency distribution in Figure 9. As is readily seen, the notes G and C are

1		3		5		7	2		4		6
C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B

Table 6: Order of the generation process for the G major diatonic scale (G, A, B, C, D, E, F♯).

extremely frequent, almost twice that of any other note in this C minor piece. The first row of the transition matrix also gives information about how often the note G transitions to C; see Table 7. Beethoven firmly establishes the key of C minor by emphasizing the G→C fifth relationship, and by repeating C and G more than any other notes in the piece. It is clear from Table 8 that he uses the B→C leading tone relationship to solidify the key as well.

C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
850	18	**118**	99	30	58	7	**218**	19	9	63	**110**

Table 7: First row of the transition matrix for the first movement of Beethoven's Fifth Symphony. Each value is the number of times a note is followed by C, the tonal center of the piece, rounded to the nearest integer. In bold are the notes of the dominant triad, which are each expected to freqeuntly resolve to C in the tonal system. Note that G transitions to C more than any other note (besides C itself).

C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
0.534	0.118	0.159	0.112	0.161	0.063	0.026	0.113	0.028	0.026	0.092	**0.317**

Table 8: First row of Table 12a, which is the transition matrix for Beethoven's Fifth, normalized by the frequency of each respective note. This can be interpreted as the probability that note j transitions immediately to C. In bold, the leading tone B transitions to C almost a third of the time.

4.3.2 Tonality indicator using note frequencies

Seeing that notes with particularly high frequencies are often important notes in a piece, we define an indicator $F(j)$ to tell if note j and the note a perfect fifth below it are both frequently visited, since this could point in the direction of a relevant tonal relationship. For these purposes, it is important that all note values are modulo 12.

$$F(j) = \begin{cases} 1, & \text{if } j \text{ and } j-7 \text{ appear with above average frequency} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Recognizing that the diatonic scale is built out of fifths, we define another simple indicator to capture whether the fifth above j also shows up frequently:

$$M(j) = F(j) + F(j+7) \tag{8}$$

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
F	0	0	1	0	1	0	1	1	0	1	0	1
M	1	0	2	0	2	0	1	2	0	2	0	2

Table 9: Behavior of F and M for each note in Bach's G major Prelude.

A powerful extension of this indicator can be obtained by using the AND operator & on $F(j)$ rather than addition, and by iterating recursively. Define

$$M'_i(j) \;=\; \begin{cases} F(j)\ , \quad j = 0 \\ M'_{i-1}(j) \;\;\&\;\; M'_{i-1}(j+7)\ , \quad j > 0 \end{cases} \tag{9}$$

As i increases, this recursion considers notes around the so-called *circle of fifths* from note $j - 7$. Thus the i^{th} iteration of this modified indicator effectively determines if the all the notes up to i steps around the circle of fifths appear often. Up to $i = 6$, this keeps track of how many notes in j's major diatonic scale (constructed by starting at $j - 7$) occur frequently. Its behavior is listed in Table 10.

i	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
0	0	0	1	0	1	0	1	**1**	0	1	0	1
1	0	0	1	0	1	0	0	**1**	0	1	0	1
2	0	0	1	0	1	0	0	**1**	0	1	0	0
3	0	0	1	0	0	0	0	**1**	0	1	0	0
4	0	0	1	0	0	0	0	**1**	0	0	0	0
5	0	0	0	0	0	0	0	**1**	0	0	0	0
6	0	0	0	0	0	0	0	**0**	0	0	0	0

Table 10: Value of $M'_i(j)$ for successive iterations i in Bach's G major Prelude. Note that the row $i = 1$ of this table corresponds to those in the M row in Table 9 which have value 2.

Indeed, $M'_i(j)$ can consistently identify the diatonic scale used in a piece (or, if a piece changes keys often, sections of a piece), sometimes requiring a less stringent definition of F to distinguish between the notes of the scale and the five other less frequent notes. To search for a possible minor tonal center at note j, we may simply apply this indicator to identify the major diatonic scale of

i	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
0	**1**	0	1	1	1	0	1	1	0	1	0	1
1	**1**	0	1	0	1	0	0	1	0	1	0	0
2	**1**	0	1	0	0	0	0	1	0	1	0	0
3	**1**	0	1	0	0	0	0	1	0	0	0	0
4	**1**	0	0	0	0	0	0	1	0	0	0	0
5	**1**	0	0	0	0	0	0	0	0	0	0	0
6	**0**	0	0	0	0	0	0	0	0	0	0	0

Table 11: Value of $M'_i(j+3)$ for successive iterations i in Beethoven's Fifth Symphony, in C minor. Here we have lowered the frequency cutoff in the definition of F. This properly identifies the natural minor scale for C, which is actually the diatonic scale of the relative major, E♭.

the *relative major*, $j+3$, since this is exactly the natural minor scale for note j.

Unfortunately, using this recursive indicator alone does not determine the key of a piece, merely the diatonic scale used. To determine the tonal center, we may use the choice of scale to inform our usage of other mathematical hints, like the leading tone relationship exemplified in Table 8 and those explained in the following sections and tables.

4.3.3 Tonality information in the note transition matrix

Hints towards particular tonalities exist in note transition information as well. Transitions between notes frequently become more important than others in particular tonal keys, as we saw in the development of our Transition Evenness Index. The tendency of a particular note to resolve to one or two particular other notes shows up in the transition matrix. The leading tone relationship mentioned earlier is just an important one of these tendencies which can be wisely looked for once the diatonic scale of a piece is known. For instance, knowing that Beethoven's Fifth uses the diatonic scale for E♭, we might compare the relationship D→E♭ to the stronger B→C correlation to correctly guess that the piece is in C minor, not E♭ major.

Other specific tendencies which require in-depth music theoretic explanation can be seen in Table 12. We highlight just a few of these without going into detail.

4.3.4 Tonality information in individual voices

Also worth mentioning is a ranking of sound types and ranges in polyphonic tonal music. The lowest sounding notes in the *bass voice* are of the utmost importance in defining a harmony and the attached subjective harmonic feeling. For instance, in virtually all forms of tonal music, the fifth transition from dominant down to tonic in the bass voice holds huge priority in the tension-releasing feeling of reaching the tonal center. By investigating the transition matrix for just the bass voice in a piece like Beethoven's Fifth, clues for C minor are even more apparent than in any other individual voice, or all the voices together. We will not delve into too much detail here, but provide Table 13 for comparison and contrast with Table 12.

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
C	.5344	.1178	.1589	.1115	.1612	.0632	.0261	**.1133**	.0275	.0262	.0915	**.3173**
C♯	.0130	.7020	.0020	.0094	0	.0005	.0019	.0012	.0076	.0037	.0019	.0072
D	.0759	.0758	.3062	.0968	.0718	.1501	.0506	.0297	.0052	.0524	.0247	.1048
D♯	.0413	.0589	.1333	.3024	0	**.1501**	.0765	.0867	.0388	.0112	.1084	.0485
E	.0060	0	.0276	.0037	.4228	.0244	0	.0184	.0109	.0045	.0011	.0159
F	.0316	.0034	.0841	.2161	.1301	.2429	.1409	.0746	.2022	.0279	.0341	.0426
F♯	.0050	.0017	.0054	.0183	.0217	.0586	.3545	.0269	.0022	.0801	.0073	.0043
G	**.1310**	.0034	.1181	.0746	.0921	.1796	**.2085**	.5211	.1930	.1287	.1318	.1316
G♯	.0418	.0135	.0204	.0274	.0136	.0817	.0009	.0661	.4475	.0120	.0683	.0496
A	.0244	.0168	.0188	.0084	.0014	.0068	.0961	.0256	.0112	.3885	.0628	.0260
A♯	.0454	.0034	.0279	.1080	.0298	.0298	.0345	.0236	.0475	.1876	.4557	.0058
B	.0502	.0034	.0972	.0234	.0556	.0122	.0095	.0128	.0066	.0771	.0123	.2464

(a) Transition matrix of Beethoven's Fifth but where each entry is normalized by the frequency of the note in the corresponding column, so that each column sums to 1. This can be interpreted as the probability that note i follows note j, after already reaching note j. Important harmonic tendencies such as the leading tone B→C, secondary leading tone F♯→G, and dominant seventh F→E♭ are in boldface.

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
C		.3955	.2290	.1598	.2793	.0835	.0405	**.2367**	.0498	.0428	.1681	**.4210**
C♯	.0280		.0029	.0134	0	.0007	.0029	.0025	.0138	.0061	.0036	.0096
D	.1631	.2542		.1388	.1244	.1982	.0784	.0619	.0093	.0857	.0454	.1390
D♯	.0888	.1977	.1921		0	**.1983**	.1185	.1811	.0702	.0184	.1991	.0643
E	.0128	0	.0398	.0052		.0323	0	.0385	.0197	.0073	.0020	.0211
F	.0679	.0113	.1212	.3097	.2254		.2182	.1557	.3659	.0457	.0627	.0566
F♯	.0108	.0056	.0078	.0263	.0376	.0774		.0561	.0039	.1310	.0133	.0058
G	**.2813**	.0113	.1703	.1070	.1596	.2372	**.3231**		.3492	.2105	.2422	.1746
G♯	.0898	.0452	.0294	.0392	.0235	.1079	.0014	.1380		.0196	.1254	.0658
A	.0523	.0565	.0272	.0121	.0023	.0090	.1488	.0534	.0203		.1154	.0345
A♯	.0976	.0113	.0403	.1549	.0516	.0394	.0535	.0494	.0859	.3068		.0077
B	.1077	.0113	.1402	.0336	.0962	.0161	.0148	.0266	.0119	.1261	.0227	

(b) Transition matrix of Beethoven's Fifth, normalized as in (a), but without diagonal elements, to clarify resolution tendencies. This can be interpreted as the probability that note i follows note $j \neq i$, after already reaching note j. Here we see that G and C are strongly related—they transition to each other more than to any other note, since each is the maximal value in the other's column.

C	C♯	D	**D♯**	E	F	F♯	**G**	G♯	A	A♯	B
2.1060	0.0835	1.2984	1.3284	0.1787	1.6403	0.3756	2.2663	0.6852	0.5320	0.8983	0.6072

(c) Row sums of (b). This can be seen as an indicator of how often note i follows *any* other note. As is evident, the notes of the C minor scale and especially those of the C minor tonic triad (in bold) act as gravity wells for other notes to resolve to; all roads lead home to C minor.

Table 12: Note transition information modulo 12 for Beethoven's Fifth Symphony, viewed under different tonal contexts.

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
C	**.6842**	.1250	.1186	.0455	.0714	.0946		.2074	.1034	.1875	.0260	.1081
C♯	.0048	.8125	.0169	.0227								
D	.0287		.4576	.0455	.2143	.0811		.0370			.0260	.2162
D♯	.0239	.0625	.0169	.1818		.1486		.0296	.0345		.1558	
E			.0169		.5000	.0541	.0303	.0074				
F	.0239		.2034	.1364	.1429	.4054	.0606	.0667	.1207	.0313		
F♯	.0048		.0339	.0227			.8182	.0074			.0130	
G	.1579		.0339	.1364		.0811		**.5259**	.1897	.1250	.0260	.0270
G♯	.0096			.0455	.0714	.1351		.0593	.4828	.0625	.0519	.0270
A	.0048		.0169				.0909	.0074		.5938	.0390	.1081
A♯	.0096		.0339	.3636				.0148	.0345		.6623	.0541
B	.0478		.0508					.0370	.0345			.4595

(a) Transition matrix of the string bass line of Beethoven's Fifth, normalized as in Table 12a with zero omitted. Notably, C and G are repeated sequentially exceedingly often, as seen in boldface. Having the bass voice hang around dominant and tonic notes is typical in music from bluegrass to polkas to Beethoven (ask any tuba or bass player) and solidifies chords and key areas.

	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
C		.6667	.2188	.0556	.1429	.1591		**.4375**	.2	.4615	.0769	.2
C♯	.0152		.0313	.0278								
D	.0909			.0556	.4286	.1364		.0781			.0769	.4
D♯	.0758	.3333	.0313			.2500		.0625	.0667		**.4615**	
E			.0313			.0909	.1667	.0156				
F	.0758		.3750	.1667	.2857		.3333	.1406	.2333	.0769		
F♯	.0152		.0625	.0278				.0156			.0385	
G	**.5**		.0625	.1667		.1364			.3667	.3077	.0769	.05
G♯	.0303			.0556	.1429	.2273		.1250		.1538	.1538	.05
A	.0152		.0313				.5	.0156			.1154	.2
A♯	.0303		.0625	**.4444**				.0313	.0667			.1
B	.1515		.0938					.0781	.0667			

(b) Transition matrix of the string bass line of Beethoven's Fifth, normalized as in (a), but without diagonal elements, to clarify resolution tendencies. Here we see an even stronger correlation between G and C than in Table 12b, and also strong ties between A♯ and D♯ (enharmonically, B♭ and E♭), the dominant and tonic notes in E flat major, the relative major key to C minor. Beethoven actually modulates to E♭ for the second theme (the C compartment in our sonata form structure in Section 3.3.5), as is the standard operating procedure in sonata form. This strong hint for C minor does not appear in Table 12b.

C	C♯	D	**D♯**	E	F	F♯	**G**	G♯	A	A♯	B
2.6189	0.0742	1.2664	1.2810	0.3045	1.6874	0.1595	1.6668	0.9387	0.8774	0.7352	0.3901

(c) Row sums of (b). This can be seen as an indicator of how often note i follows *any* other note in the bass voice. As in Table 12c, notes in C minor are especially frequent; indeed, the tonal center, C, massively outweighs every other note.

Table 13: Note transition information modulo 12 for just the bass line in Beethoven's Fifth Symphony, viewed under different tonal contexts. Since the bass voice frequently contains far fewer notes in total than a quickly moving melodic line, and due to its importance in forming harmony, tonal note relationships can appear with more contrast than in other voices or in total.

5 Conclusion

In conclusion, big-picture mathematical analysis of music can yield a surprising amount of information, in both practically evident aspects and more obscure structures. NEA shows promise as a mathematical tool in understanding what makes for well-knit music. Its applicability to the various scales of hierarchical organization allows for an intricate mapping of a musical network and evaluation of its qualities. Despite the lack of an apparent correlation with popularity or subjective quality, the concepts and thought processes involved are still quite valuable in music analysis. Aided in their quest by mathematics and concepts like NEA, music theorists can understand how musical elements and structures of many types function together from a holistic point of view and with rigorous mathematical language.

In terms of stylistic systems like tonality and serialism, we have seen how constraints on note and inter-note transition use manifest themselves as patterns in holistic data. The mathematical tools presented and proposed in Section 4 hold potential for all sorts of variations and extensions, and are easily implemented with a computer, especially with publicly available software packages [8, 9, 12]. They are successful at distinguishing between a variety of musical styles using general statistical information, which supports the hypothesis that the fundamentally defining feature of a musical style is simply the set of priorities it gives to certain notes, resolutions, motives, harmonies, or other musical elements. This view and the compiled tools could help to rigorously describe the historical process sometimes called the "dissolution of tonality" in music theory and history studies.

Strategies for tonal key recognition based upon a score do exist, but often involve specialized tricks and a human intellect. A computer, however, can observe holistic patterns in a piece and interpret them meaningfully, pending the existence of information to be gleaned; we have shown how neat mathematical interpretations of diatonic scale construction can inform a computer search, and that there is indeed information in piece-wide statistics. The more esoteric music theoretic ideas explained in Figures 12 and 13 do not make for clean-cut tools like the recursive scale-recognizer M_i but still correspond to mathematical patterns apparent to the musically informed reader.

Mathematical music analysis only has upwards to go by using interdisciplinary methods and the holistic viewpoint. We look forward to seeing these ideas develop further.

6 Future directions

6.1 NEA applications

This document is just the beginning of applying NEA to music and viewing music as a transactional network. As Section 2.2 and Figure 3 show, there are numerous other scales of musical organization which we did not address here where NEA could come in handy.

6.1.1 Storage analysis

It is difficult to interpret what *storage analysis* means in the context of music [2]. Most of the mathematical properties which come from storage analysis are redundant in value with their throughflow-based definitions in our note transition models. All of them are rather irrelevant to our interpretations, in which we chose to have equal storage in every compartment; a storage value $x_j \neq 1$ would translate to the note j receiving x_j repetitions every time it appeared in the piece, which is certainly not representative of reality.

There may be value in pursuing storage analysis on differently-constructed networks. Adapting the fundamental sequential network to associate a storage value with the duration of each note might lend insight, but identifying notes, sections, or different elements together regardless of time-embedding becomes impossible when each has a distinct duration. Meaningfully applying storage analysis will require more consideration, but may yield a more accurate model of music, as it could incorporate both pitch and durational information, the two dimensions of the fundamental musical network.

6.1.2 Motivic analysis

One of the levels in between the note and formal viewpoints is that of small groups of notes. A *motive*, alternately called "motif" or "figure," is simply a short sequence (incorporating notes and/or rhythmic durations). Then, as elsewhere, transitions from one motive to the one immediately following become the flows. The defining pitch information in a motive is its *intervallic* content, i.e. the spaces between the notes, not the particular notes themselves. One difficulty in motive analysis is that important motives are usually *developed*, or manipulated into various forms, by starting on different notes, transforming upside-down or backwards, faster or slower, etc.

Unfortunately, although important musical motives (those which occur frequently, or which are emphasized by context) are often easy for a human ear to recognize, a computer has little "intuition" about what to look for without incorporating advanced machine learning. Still, it would be easily possible to create a tool for a comprehensive motivic analysis which simply considers *all* note- or interval-sequences of a fixed length, perhaps only considering those which appear more often than some fixed frequency tolerance, since musically important motives appear multiple times.

6.1.3 Frequency tolerance

In the vein of this "frequency tolerance," the actual mathematics involved in NEA could be adapted to glean more musically relevant information, or to more accurately model music. The matrix G^n for $n > 1$ in a note transition aggregation does not only reflect n-length steps in the fundamental network; because we have identified notes regardless of time embedding, G^n encodes some n-step indirect connections that may not actually happen in the timeline of the piece, especially for large n. However, these entries should have lower values than important (i.e. frequent) n-step connections that actually happen in the piece. If we impose a frequency tolerance in each power of G, or to D in utility analysis, we could distill down to more musically important indirect connections.

This viewpoint could also apply to ecological networks, where some modeled relationships are, in reality, quite negligible.

An investigation or survey of meaningful tolerances is necessary to develop these ideas further and make them viable. A dynamic tolerance, changing with n, would probably be insightful.

6.1.4 Harmonic analysis

Harmonic networks are slightly tougher to construct, since they involve recognition of multiple simultaneous notes for each compartment. Automated chord recognition today is actually quite poor, with both raw recordings and digitized MIDI files. This is probably due to the frequent use of dissonances and melodic elaborations around a longer-term harmonic structure which does not substantially change, from the listener's perspective. Tackling network construction of this type thus would involve a hands-on human harmonic analysis of any piece in question, which is not a cheap task in long, polyphonic works. While easy in short pop songs which repeat the same chords, the simple cyclical networks created probably do not provide much mathematical insight.

6.1.5 Popularity

A more detailed study of correlations with popularity or other versions of musical quality is in order. Obtaining robust popularity data may prove to be the most challenging aspect of this task. Once multiple levels of the network sturctures of a fixed set of pieces have been encoded and analyzed, a comparison of correlations with formal, harmonic, rhythmic, or motivic data could shed light on the importance of the various levels of musical organization in effective expression.

6.2 Mathematical reflections of musical style

The diversity measures and tonality indicators we introduced in Section 4 form a whole battery of mathematical tools for classifying musical styles and keys. Applying them to a large data set would illustrate their power and effectiveness and uncover any glitches, perhaps leading to refined formulations. A comprehensive strategy for stylistic determination could be formulated, using all music theoretic knowledge of tonal tendencies, which could ideally give probabilities for a musical piece fitting into a given style or key and assign the highest such value to the correct option.

References

[1] Stuart Borrett. "NEA.m". mathworks.com/matlabcentral/fileexchange/5261-nea-m, MATLAB Central File Exchange, 2004.

[2] Brian D. Fath and Bernard C. Patten. "Review of the Foundations of Network Environ Analysis". *Ecosystems* (1999) 2: 167-179.

[3] Mark O. Hill. "Diversity and evenness: a unifying notation and its consequences." *Ecology* (1973) 54: 427-432.

[4] Hironori Hirata and Robert E. Ulanowicz. "Information Theoretical Analysis of the Aggregation and Hierarchical Structure of Ecological Networks". *Journal of Theoretical Biology* (1985) 116: 321-341.

[5] Caner Kazancı. "EcoNet: A new software for ecological modeling, simulation and network analysis". *Ecological Modeling* (2007) 208/1: 3-8.

[6] Caner Kazanci and Qianqian Ma. "System-wide measures in ecological network analysis". *Advanced Modelling Techniques Studying Global Changes in Environmental Sciences* (2015) 27: 45-68

[7] Stefan M. Kostka and Dorothy Payne. *Tonal Harmony, with an Introduction to Twentieth-Century Music*. Boston: McGraw-Hill, 2004.

[8] Nicholas J. Lindell. "Analyzing musical compartmental networks using MIDI files". mathworks.com/matlabcentral/fileexchange/72168-analyzing-musical-compartmental-networks-using-midi-files, MATLAB Central File Exchange, 2019.

[9] Nicholas J. Lindell. "Determining a musical piece's style and key using maths". mathworks.com/matlabcentral/fileexchange/72167-determining-a-musical-piece-s-style-and-key-using-maths, MATLAB Central File Exchange, 2019.

[10] Qianqian Ma and Caner Kazanci. "Analysis of indirect effects within ecosystem models using pathway-based methodology". *Ecological Modeling* (2013) 252: 238-245.

[11] Bernard C. Patten. "Energy cycling in the ecosystem". *Ecological Modeling* (1985) 28: 1-71.

[12] Ken Schutte. "MATLAB and MIDI". kenschutte.com/midi, 2012.

[13] Claude E. Shannon. "A mathematical theory of communication". *The Bell System Technical Journal* (1948) 27: 379–423 and 623–656.

[14] Edward H. Simpson. "Measurement of diversity". *Nature* (1949) 163: 688.

[15] Henri Theil. *Economics and Information Theory.* Amsterdam: North-Holland, 1967.

[16] Henry Zajaczkowski. *Tchaikovsky's Musical Style.* Ann Arbor: UMI Research Press, 1987.