# Special Values of Hurwitz Zeta Functions and Dirichlet $L$-functions

by

Milton H. Nash

(Under the direction of Robert S. Rumely)

## Abstract

We generalize a result of Ball and Rivoal. This result shows, among other things, that the Riemann zeta function is irrational at infinitely many positive odd integers. We will show that their techniques can be extended to prove a more general result for linear combinations of Hurwitz zeta functions.

Index words: $L$-functions, Special Values, Transcendence

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Milton H. Nash

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Milton H. Nash

Approved:

Major Professor: Robert S. Rumely

Committee: Malcolm Adams
Akos Magyar
Roy Smith
Robert Varley

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
August 2004

# Dedication

This work is dedicated to my mother Gloria Trawick. Without her, none of this would be possible. I dedicate it also to my nephew DJ and nieces Nyia and Nakiria who are the hope of the future.

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# Chapter 1

## Introduction

A significant line of research has been the investigation of the special values of transcendental functions. Given a function defined by a convergent series having algebraic coefficients with prescribed growth, what can be said about the arithmetic nature of the values that the function takes on at integer values of the argument? An example of such a function is the Riemann $\zeta$- function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s$ is a complex variable with $\mathrm{Re}(s) > 1$. Euler proved:

$$\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}$$

for any positive integer $m \geq 1$ where the $B_k$ are Bernoulli numbers. From the presence of $\pi$ in this formula, one sees that the $\zeta$-function takes on transcendental values at the positive even integers. It has long been conjectured that $\zeta(2m+1)$ is transcendental $\forall m \in \mathbb{N}$ and another, much stronger, conjecture is that these values are algebraically independent (see Cartier). In 1978, Apéry proved that $\zeta(3)$ is irrational but until recently very little else was known. Then in 2001, Ball and Rivoal proved that infinitely many of these values are irrational. If $\delta(a)$ is the dimension of the $\mathbb{Q}$-vector space generated by the set $\{1, \zeta(3), \zeta(5), ..., \zeta(a)\}$, they proved that for each $\epsilon > 0$ and $a > A(\epsilon)$:

$$\delta(a) \geq \frac{1-\epsilon}{1+\log(2)} \log(a)$$

More generally, one may ask if there is a comparable result for linear combinations of Hurwitz $\zeta$-functions or, in particular, for Dirichlet $L$-functions. This is the question pursued in the present work. We now give a brief description of the the general result.

Fix an integer $Q \geq 1$ and let $c = (c_1, c_2, ..., c_Q) \in \mathbb{C}^Q$ be such that $c \neq 0$. If we define $L(s,c)$ by:

$$L(s,c) = \sum_{d=1}^{Q} c_d \sum_{k=0}^{\infty} \frac{1}{(kQ+d)^s}$$

then we will establish the following:

**Theorem 1.1** *Let $\delta_c(a)$ denote the $\mathbb{Q}$-dimension of the space generated by the set $\{c_1, ..., c_Q, L(2,c), L(3,c), ..., L(a,c)\}$. Moreover, suppose that:*

$$\sum_{d=1}^{Q} c_d \neq 0$$

*Then for each $\epsilon > 0$, there exists an integer $A(\epsilon)$ such that for $a > A(\epsilon)$:*

$$\delta_c(a) \geq \frac{1-\epsilon}{Q+\log(2)} \log\left(\frac{a}{Q}\right)$$

As in Ball and Rivoal, the key lemma is a criterion of linear independence due to Nesterenko:

**Proposition 1.2** *Let $\theta = (\theta_1, ..., \theta_M) \in \mathbb{C}^M$ $(M > 2)$ and suppose that for each $n$ there is a linear form with integer coefficients:*

$$L_n(\vec{x}) = \sum_{l=1}^{M} p_{l,n} x_l$$

*such that*

*1. There are $\alpha_1$ and $\alpha_2$, $0 < \alpha_1 \leq \alpha_2 < 1$, such that:*

$$\alpha_1^{n+o_1(n)} \leq |L_n(\theta)| \leq \alpha_2^{n+o_2(n)}$$

*2. There is a $\beta > 1$ such that:*

$$\max_{1 \leq l \leq M} |p_{l,n}| \leq \beta^{n+o_3(n)}$$

*Then, if* $\delta_{\mathbb{Q}} = dim_{\mathbb{Q}}\{\mathbb{Q}\theta_1 + ... + \mathbb{Q}\theta_M\}$*:*

$$\delta_{\mathbb{Q}} \geq \frac{\ln(\beta) - \ln(\alpha_1)}{\ln(\beta) - \ln(\alpha_1) + \ln(\alpha_2)}$$

Following Ball and Rivoal, we introduce an auxiliary function from which we derive the requisite sequence of linear forms. Recall that we fix an integer $Q \geq 1$ and $c = (c_1, c_2, ..., c_Q) \in \mathbb{C}^Q$ is such that $c \neq 0$. Extend this sequence periodically, requiring $c_d = c_{d'}$ if $d \equiv d' (\bmod Q)$. Of particular interest is the case when $\chi$ is a Dirichlet character with conductor $Q$ and $c_d = \chi(d)$. Let $n$, $r$, and $a$ be integers such that $1 \leq r < \frac{a}{2}$, $n \in \mathbb{N}$. Furthermore, in the special case $c_d = \chi(d)$, the parity of $a$ and $n$ will depend on $\chi$ in a manner to be described below. We define $m = m(c) = 0$ if:

$$\sum_{d=1}^{Q} dc_d \neq 0$$

Otherwise, let $m$ be the least positive integer such that $c_m \neq 0$. Since $c \neq 0$, $m \leq Q$. Then, a fact which will be useful later, it is not difficult to see that:

$$\sum_{d=1+m}^{Q+m} dc_d \neq 0$$

since if $m = 0$ the sum is nonzero by definition and if $m \neq 0$ it equals $Qc_m$. We will analyze the function:

$$S_{n,c}(z) = \sum_{k=0}^{\infty} \left( \sum_{d=1+m}^{Q+m} c_d R_{n,d,Q}(k) \right) z^{-kQ}$$

where:

$$R_{n,Q}(t) = Q^{2rn} n!^{a-2r} \frac{\prod_{l=0}^{rn-1} \left(t - \frac{l}{Q}\right) \prod_{l=0}^{rn-1} \left(t + n + 2 + \frac{l}{Q}\right)}{\prod_{l=0}^{n} (t+1+l)^a}$$

and we define:

$$R_{n,d,Q}(t) = R_{n,Q}\left(t + \frac{d}{Q}\right)$$

For $k = 1, 2, 3, ...$, the Pochammer symbol is defined to be:

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \prod_{l=0}^{k-1} (\alpha + l)$$

Thus:

$$R_{n,d,Q}(k) = n!^{a-2r} \frac{(Qk+d-rn+1)_{rn}\,(Qk+Qn+2Q+d)_{rn}}{\left(k+1+\frac{d}{Q}\right)^{a}_{n+1}}$$

The series defining $S_{n,c}(z)$ converges for all $z \in \mathbb{C}$ with $|z| > 1$ because, as a rational function in $k$, $R_{n,d,Q}(k)$ has total degree $\leq -2$ due to our assumption that $1 \leq r < \frac{a}{2}$.

We begin the proof of Theorem (1.1) by establishing in chapter two that:

$$S_{n,c}(1) = \sum_{l=2}^{a} P_{l,n}(1) L(l,c) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1) c_d \tag{1.1}$$

where $P_{l,n}(z), \widehat{P}_{d,n}(z) \in \mathbb{Q}[z]$. We here briefly sketch the argument. First expand the $R_{n,d,Q}(k)$ into partial fractions:

$$R_{n,d,Q}(k) = \sum_{l=1}^{a}\sum_{j=0}^{n} \frac{c^{(d)}_{l,j,n}}{\left(k+1+j+\frac{d}{Q}\right)^{l}}$$

where $c^{(d)}_{l,j,n} \in \mathbb{Q}$ and by the uniqueness of the partial fraction expansion:

$$c^{(d)}_{l,j,n} = c^{(d')}_{l,j,n} \text{ for all } d, d'$$

The poles are all of order $a$ and, when summed over $k$ and character values, survive as $L(l,c)$ up to $L(a,c)$. Substituting this expansion into the series defining $S_{n,c}$, rearranging the sums, and evaluating at $z = 1$ gives the result.

Furthermore, the coefficients $P_{l,n}(1)$ have additional properties which allow us to specialize equation (1.1) when we have more information about $c = (c_1, c_2, ..., c_Q)$. The symmetry of the zeroes of $R_{n,Q}(k)$ about its poles yields the functional equation:

$$R_{n,Q}(k) = (-1)^{(n+1)a} R_{n,Q}(-n-k)$$

This translates via the partial fraction expansion to:

$$P_{l,n}(1) = (-1)^{(n+1)a+l} P_{l,n}(1)$$

Thus, for example, if $c_d = \chi(d)$ where $\chi$ is an primitive even Dirichlet character for which it is known that:

$$L(2m, \chi) = (-1)^{m+1} \left(\frac{2\pi}{Q}\right)^{2m} A_m(\chi)$$

where $A_m(\chi) \in \mathbb{Q}(e^{2\pi i/Q})$, we take $a$ odd and $n$ even. Hence $P_{l,n}(1) = 0$ for $l$ even and we have:

$$S_{n,\chi}(1) = \sum_{l=1}^{\frac{a-1}{2}} P_{2l+1,n}(1) L(2l+1, \chi) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1)\chi(d)$$

Defining $d_{n,Q} = lcm\{1, 2, ..., (n+1)Q + m\}$, we prove in chapter eight that:

$$L_n = d_{2n,Q}^a S_{2n,c}(1)$$

is a sequence of linear forms with *integer* coefficients and $d_{n,Q} = e^{nQ+o(n)}$ by the Prime Number Theorem.

With $R = \frac{r}{Q}$ and $A = \frac{a}{Q}$ we easily compute, using the residue theorem and standard estimates on multinomial coefficients, that:

$$\limsup_{n\to\infty} |P_{l,n}(1)|^{1/Qn} \leq Q^{2R+1} 2^{A-2R} (2R+1)^{2R+1}$$

The same estimate is proved for $\widehat{P}_{d,n}(1)$. So:

$$\beta = \left[e^a Q^{2R+1} 2^{A-2R} (2R+1)^{2R+1}\right]^{2Q}$$

suffices to satisfy the second part of Nesterenko's criterion.

The technical heart of the proof is in satisfying the first part of the criterion. In fact, we explicitly compute the limit:

$$\lim_{n\to\infty} |S_{n,c}(1)|^{1/Qn}$$

This requires a detailed investigation of the asymptotics of $S_{n,c}(1)$ which is the task undertaken in chapters three through seven. The main term comes from the function:

$$F(x) = \left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A} \frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}$$

If we define:

$$\phi_{Q,r,a} = \max_{R \leq x < \infty} Q^{2R} F(x)$$

we prove in chapter four that there exists a unique $x_0 \in (R, \infty)$ such that $\phi_{Q,r,a} = Q^{2R} F(x_0)$. Moreover:

$$0 < \phi_{Q,r,a} \leq \frac{Q^{2R} 2^{R+1}}{R^{A-2R}}$$

In chapters five through seven we establish:

$$\lim_{n \to \infty} |S_{n,c}(1)|^{1/nQ} = \phi_{Q,r,a} \tag{1.2}$$

In chapter three, we prove using Stirling's formula that for sufficiently small $\delta$ and sufficiently large $n$ depending on $\delta$, if $n(R+\delta) \leq k$ then:

$$R_{n,d,Q}(k) = Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x)^{nQ} H(x) G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right)$$

where $x = \frac{k}{n}$. Moreover $R_{n,d,Q}(k)$ is nondecreasing for $k \leq n(R+\delta)$ for sufficiently small $\delta$ and $n$ large.

In chapter six, we use these facts to prove (1.2) under the hypothesis that

$$\sum_{d=1}^{Q} c_d \neq 0$$

Establishing the limit is technically most difficult when $\sum_{d=1}^{Q} c_d = 0$. Indeed, under this assumption, we use the asymptotic product formula above to prove in chapter seven the following:

There is a constant $C > 0$ such that if:

$$I_n = \left[x_0 - C\sqrt{\frac{\log n}{n}}, x_0 + C\sqrt{\frac{\log n}{n}}\right]$$

then:

$$\sum_{\frac{k}{n} \notin I_n} \left( \sum_{d=1+m}^{Q+m} c_d R_{n,d,Q}(k) \right) = Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x_0)^{nQ} O\left(\frac{1}{n}\right)$$

and:

$$\sum_{\frac{k}{n}\in I_n}\left(\sum_{d=1+m}^{Q+m} c_d R_{n,d,Q}(k)\right) = Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}\frac{F(x_0)^{nQ}}{\sqrt{n}}\left(\frac{\tau' H_0 A_0}{2\tau Q}\sum_{d=1+m}^{Q+m} dc_d + o(1)\right)$$

where $H_0$ and $A_0$ are nonzero constants. Furthermore:

$$\tau = -\frac{1}{2}\left(\frac{d}{dx}\right)^2 \log\left(F(x)\right)|_{x=x_0}$$

and:

$$\tau' = \frac{1}{2}\left(\frac{d}{dx}\right)^3 \log\left(F(x)\right)|_{x=x_0}$$

In particular, neither is zero as is proved in chapter four.

However, the reader is strongly cautioned that there is a neglected term in the main computation which begins on page 57. This term exactly cancels the main term found in the asymptotic above. In fact, with a bit more care, what we have in fact established is that:

$$S_{n,c}(1) = Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}F(x_0)^{nQ}O\left(\frac{1}{n}\right)$$

It is hoped that the computational method given in this chapter will eventually show that the next term in the asymptotic is nonvanishing. This would be sufficient to prove the following:

**Conjecture 1.3** *Let $\chi$ be a Dirichlet character with conductor $Q$. Suppose $\chi(-1) = -1$. Take $a$ even and define $\delta_\chi(a)$ to be the $\mathbb{Q}$-dimension of the space generated by the set $\{\chi(1), ..., \chi(Q), L(2,\chi), L(4,\chi), ..., L(a,\chi)\}$. If $\chi(-1) = 1$, take $a$ odd and define $\delta_\chi(a)$ to be the $\mathbb{Q}$-dimension of the space generated by the set $\{\chi(1), ..., \chi(Q), L(3,\chi), L(5,\chi), ..., L(a,\chi)\}$. Then for each $\epsilon > 0$, there exists an integer $A(\epsilon)$ such that for $a > A(\epsilon)$:*

$$\delta_\chi(a) \geq \frac{1-\epsilon}{Q+\log(2)}\log\left(\frac{a}{Q}\right)$$

Having established the limit:

$$\lim_{n\to\infty} |S_{n,c}(1)|^{1/nQ} = \phi_{Q,r,a}$$

we take:

$$\alpha_1 = \alpha_2 = (e^a \phi_{Q,r,a})^{2Q}$$

Theorem (1.1) follows by applying Nesterenko's criterion.

# Chapter 2

## Linear Forms In Special Values

Recall the definition of the rational function $R_{n,d,Q}(t)$:

$$R_{n,d,Q}(t) = Q^{2rn} n!^{a-2r} \frac{\prod_{l=0}^{rn-1}\left(t+\frac{d}{Q}-\frac{l}{Q}\right)\prod_{l=0}^{rn-1}\left(t+\frac{d}{Q}+n+2+\frac{l}{Q}\right)}{\prod_{l=0}^{n}\left(t+1+l+\frac{d}{Q}\right)^{a}}$$

We first investigate the coefficients in the partial fraction expansion of $R_{n,d,Q}(t)$:

$$R_{n,d,Q}(t) = \sum_{l=1}^{a}\sum_{j=0}^{n} \frac{c_{l,j,n}^{(d)}}{\left(t+1+j+\frac{d}{Q}\right)^{l}} \tag{2.1}$$

where letting:

$$D_\lambda = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}$$

we have:

$$c_{l,j,n}^{(d)} = D_{a-l}\left(R_{n,d,Q}(t)\left(t+j+1+\frac{d}{Q}\right)^{a}\right)|_{t=-j-1-\frac{d}{Q}} \in \mathbb{Q}$$

which is the $l$th coefficient in the principal part of the Laurent expansion of $R_{n,d,Q}(t)$ about the pole $t=-j-1-\frac{d}{Q}$. These coefficients are independent of the congruence class of $d$ since, by Cauchy's Theorem:

$$\begin{aligned} c_{l,j,n}^{(d)} &= \frac{1}{2\pi i}\int_{|z+j+1+\frac{d}{Q}|=\frac{1}{2}} R_{n,d,Q}(z)\left(z+j+1+\frac{d}{Q}\right)^{l-1} dz \\ &= \frac{1}{2\pi i}\int_{|w+j+1|=\frac{1}{2}} R_{n,Q}(w)(w+j+1)^{l-1} dw \end{aligned}$$

under the linear transformation $w=T(z)=z+\frac{d}{Q}$ where $|w+j+1|=\frac{1}{2}$ denotes the circle of radius $1/2$ and center $w=-j-1$. Henceforth, we will simply write $c_{l,j,n}$ for $c_{l,j,n}^{(d)}$ .

We now insert (2.1) into the function $S_{n,c}(z)$. By suitably rearranging sums and evaluating this function at $z = 1$ with a limit argument, we'll derive a sequence of linear forms with rational coefficients in special values of $L(s, c)$. Recall:

$$S_{n,c}(z) \;=\; \sum_{k=0}^{\infty}\left(\sum_{d=1+m}^{Q+m} c_d R_{n,d,Q}(k)\right) z^{-kQ}$$

So we have:

$$\begin{aligned}
S_{n,c}(z) &= \sum_{k=0}^{\infty}\left(\sum_{d=1+m}^{Q+m} c_d \sum_{l=1}^{a}\sum_{j=0}^{n}\frac{c_{l,j,n}}{(k+1+j+\frac{d}{Q})^l}\right) z^{-kQ} \\
&= \sum_{l=1}^{a}\sum_{j=0}^{n} c_{l,j,n}Q^l \sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty}\frac{1}{((k+j+1)Q+d)^l} z^{-kQ} \\
&= \sum_{l=1}^{a}\sum_{j=0}^{n} Q^l c_{l,j,n} z^{(j+1)Q} \sum_{d=1+m}^{Q+m} c_d \sum_{k=j+1}^{\infty}\frac{1}{(kQ+d)^l} z^{-kQ} \\
&= \sum_{l=1}^{a}\sum_{j=0}^{n} Q^l c_{l,j,n} z^{(j+1)Q} \sum_{d=1+m}^{Q+m} c_d \left(\sum_{k=0}^{\infty}\frac{1}{z^{kQ}}\frac{1}{(kQ+d)^l}\right. \\
&\qquad \left. - \sum_{k=0}^{j}\frac{1}{z^{kQ}}\frac{1}{(kQ+d)^l}\right) \\
&= \sum_{l=1}^{a}\sum_{j=0}^{n} Q^l c_{l,j,n} z^{(j+1)Q} \sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty}\frac{1}{z^{kQ}}\frac{1}{(kQ+d)^l} \\
&\qquad - \sum_{l=1}^{a}\sum_{j=0}^{n} Q^l c_{l,j,n} \sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{j}\frac{1}{(kQ+d)^l} z^{(j-k+1)Q}
\end{aligned}$$

Define:

$$\widetilde{P}_{d,n}(z) = -\sum_{l=1}^{a}\sum_{j=0}^{n} Q^l c_{l,j,n} \sum_{k=0}^{j}\frac{1}{(kQ+d)^l} z^{(j-k+1)Q}$$

and:

$$P_{l,n}(z) = \sum_{j=0}^{n} Q^l c_{l,j,n} z^{(j+1)Q}$$

Thus:

$$S_{n,c}(z) \;=\; \sum_{l=1}^{a} P_{l,n}(z) \sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty}\frac{1}{z^{kQ}}\frac{1}{(kQ+d)^l} + \sum_{d=1+m}^{Q+m}\widetilde{P}_{d,n}(z) c_d$$

There are identities which further simplify this expression. First we have:

$$P_{1,n}(1) = Q\sum_{j=0}^{n} c_{1,j,n} = Q\sum_{j=0}^{n} Res_{t=-j-1-\frac{d}{Q}} R_{n,d,Q}(t) = 0$$

since, from the assumption that $r < \frac{a}{2}$, $R_{n,d,Q}(t)$ has total degree $\leq -2$ as a rational function in $t$ and thus has a zero at infinity. Also, the function $\sum_{k=0}^{\infty} \frac{1}{z^{kQ}} \frac{1}{(kQ+d)}$ has a logarithmic singularity at $z = 1$. So:

$$\lim_{\substack{z\to 1\\ |z|>1}} \sum_{j=0}^{n} Qc_{1,j,n} z^{(j+1)Q} \sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty} \frac{1}{z^{kQ}} \frac{1}{(kQ+d)} = 0$$

Furthermore, for $l \geq 2$ and $m = 0$:

$$\sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty} \frac{1}{(kQ+d)^l} = L(l,c)$$

If $m \neq 0$:

$$\sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty} \frac{1}{(kQ+d)^l} = L(l,c) - \frac{c_m}{m^l}$$

Hence for $m = 0$:

$$S_{n,c}(1) = \sum_{l=2}^{a} P_{l,n}(1)L(l,c) + \sum_{d=1+m}^{Q+m} \widetilde{P}_{d,n}(1)c_d$$

while if $m \neq 0$:

$$\begin{aligned} S_{n,c}(1) &= \sum_{l=2}^{a} P_{l,n}(1)L(l,c) - c_m \sum_{l=2}^{a} \frac{1}{m^l} P_{l,n}(1) + \sum_{d=1+m}^{Q+m} \widetilde{P}_{d,n}(1)c_d \\ &= \sum_{l=2}^{a} P_{l,n}(1)L(l,c) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1)c_d \end{aligned}$$

where if $d \neq Q+m$, $\widehat{P}_{d,n}(1) = \widetilde{P}_{d,n}(1)$ and the coefficient of $c_m = c_{Q+m}$ is:

$$\widehat{P}_{Q+m,n}(1) = \widetilde{P}_{Q+m,n}(1) - \sum_{l=2}^{a} \frac{1}{m^l} P_{l,n}(1)$$

For the sake of consistency of notation, if $m = 0$ we shall also write $\widehat{P}_{d,n}(1)$ for $\widetilde{P}_{d,n}(1)$ $(d = 1+m, ..., Q+m)$. In summary, we've proven the following:

**Proposition 2.1** *There is a finite expansion:*

$$S_{n,c}(1) = \sum_{l=2}^{a} P_{l,n}(1) L(l,c) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1) c_d$$

*where* $P_{l,n}(z), \widehat{P}_{d,n}(z) \in \mathbb{Q}[z]$.

Now, it is possible to say more about the coefficients $P_{l,n}(1)$. More specifically, they satisfy a symmetry relation expressed in the following proposition.

**Proposition 2.2** *For* $l = 1, ...a$, *the coefficients in Proposition (2.1) satisfy the relation:*

$$P_{l,n}(1) = (-1)^{(n+1)a+l} P_{l,n}(1)$$

*In particular,* $P_{l,n}(1) = 0$ *if* $a$ *is odd and* $n$ *and* $l$ *are even* or $P_{l,n}(1) = 0$ *if* $a$ *is even and* $l$ *is odd.*

*proof*

To derive this relation, recall that:

$$P_{l,n}(1) = \sum_{j=0}^{n} Q^l c_{l,j,n}$$

where:

$$c_{l,j,n} = D_{a-l} \left( R_{n,d,Q}(t) \left( t + j + 1 + \frac{d}{Q} \right)^a \right) |_{t=-j-1-\frac{d}{Q}}$$

With $t = -x - 1 - \frac{d}{Q}$, rewrite:

$$c_{l,j,n} = (-1)^{a-l} D_{a-l} \left( \Phi_{n,j}(x) \right) |_{x=j}$$

where:

$$\begin{aligned} \Phi_{n,j}(x) &= R_{n,d,Q} \left( -x - 1 - \frac{d}{Q} \right) (j-x)^a \\ &= Q^{2rn} n!^{a-2r} (j-x)^a (-x)_{n+1}^{-a} \\ &\quad \times \prod_{l=0}^{rn-1} \left( -x - 1 - \frac{l}{Q} \right) \\ &\quad \times \prod_{l=0}^{rn-1} \left( -x + n + 1 + \frac{l}{Q} \right) \end{aligned}$$

So we have:

$$\begin{aligned}
\Phi_{n,n-j}(n-x) &= Q^{2rn}n!^{a-2r}(n-j-(n-x))^a(-(n-x))_{n+1}^{-a} \\
&\quad \times \prod_{l=0}^{rn-1}\left(-(n-x)-1-\frac{l}{Q}\right) \\
&\quad \times \prod_{l=0}^{rn-1}\left(-(n-x)+n+1+\frac{l}{Q}\right) \\
&= Q^{2rn}n!^{a-2r}(-1)^a(j-x)^a(-1)^{a(n+1)}(-x)_{n+1}^{-a} \\
&\quad \times(-1)^{rn}\prod_{l=0}^{rn-1}\left(-x-1-\frac{l}{Q}\right) \\
&\quad \times(-1)^{rn}\prod_{l=0}^{rn-1}\left(-x+n+1+\frac{l}{Q}\right) \\
&= (-1)^{an}\Phi_{n,j}(x)
\end{aligned}$$

Hence $\forall k \geq 0$:

$$\Phi^{(k)}_{n,n-j}(n-x) = (-1)^k(-1)^{na}\Phi^{(k)}_{n,j}(x)$$

Letting $k = a - l$ and $x = j$:

$$c_{l,n-j,n} = (-1)^{a-l}(-1)^{an}c_{l,j,n}$$

so that:

$$P_{l,n}(1) = (-1)^{(n+1)a+l}P_{l,n}(1)$$

It is well known that the even (resp. odd) positive values of $L(s,\chi)$ are given by a formula resembling Euler's formula for $\zeta(2m)$ (see, for example, Iwasawa): Define $\epsilon_\chi = 0$ if $\chi(-1) = 1$ and $\epsilon_\chi = 1$ if $\chi(-1) = -1$. Then for $\chi$ primitive and $l \equiv \epsilon_\chi \bmod 2$ $(l \geq 2)$:

$$L(l,\chi) = (-1)^{1+\frac{l-\epsilon_\chi}{2}}\frac{\tau(\chi)}{2i^{\epsilon_\chi}}\left(\frac{2\pi}{Q}\right)^l\frac{B_{l,\overline{\chi}}}{l!}$$

where the generalized Bernoulli numbers and Gauss sum $B_{l,\overline{\chi}}, \tau(\chi) \in \mathbb{Q}\left(e^{2\pi i/Q}\right)$. This formula, under the conditions given, shows that $L(l,\chi)$ is, in fact, transcendental because of the presence of $\pi$.

So, if $c_d = \chi(d)$, to restrict our attention to the case where values of $L(l,\chi)$ are unknown, we take $a$ odd and $n$ even if $\chi(-1) = 1$. Then $(n+1)a + l$ is odd for even $l$ and $P_{l,n}(1) = 0$ by Proposition (2.2). If, on the other hand, $\chi(-1) = -1$, we take $a$ even. Then $P_{l,n}(1) = 0$ for odd $l$. If $\chi(-1) = 1$, we have:

$$S_{n,c}(1) = \sum_{l=1}^{\frac{a-1}{2}} P_{2l+1,n}(1)L(2l+1,c) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1)c_d$$

If $\chi(-1) = -1$:

$$S_{n,c}(1) = \sum_{l=1}^{\frac{a}{2}} P_{2l,n}(1)L(2l,c) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1)c_d$$

# Chapter 3

## Analysis of $R_{n,d,Q}(t)$

In this chapter we will assemble some facts about the growth of $R_{n,d,Q}(k)$ for non-negative integers $k$. We begin with a simple observation:

**Proposition 3.1** $R_{n,d,Q}(k) = 0$ *for* $k = 0, 1, ..., \lfloor \frac{rn-d-1}{Q} \rfloor$

*proof*

By definition, $R_{n,d,Q}(k) = 0$ for any integer $k$ with $0 \le k = \frac{l-d}{Q}$ for $l = 0, ..., rn-1$. However, as we shall show in the next proposition, $R_{n,d,Q}(k)$ is in fact nondecreasing on a larger range.

**Proposition 3.2** *For each sufficiently small* $\delta > 0$ *(where small depends only upon* $a$*,* $r$*, and* $Q$*) there is an* $N_\delta$ *such that for each* $d = 1+m, ..., Q+m$ *,* $R_{n,d,Q}(k)$ *is monotonically increasing for each* $n > N_\delta$ *and for* $\frac{rn-d}{Q} \le k \le \left(\frac{r}{Q} + \delta\right) n$*.*

*proof*

In this chapter and those that follow, we shall frequently use the notation $R = \frac{r}{Q}$, $A = \frac{a}{Q}$, and $D = \frac{d}{Q}$. We shall proceed with the proof of the proposition by using the inequality:

$$\int_{u+1}^{u+s+1} \frac{1}{x}\, dx \le \sum_{k=1}^{s} \frac{1}{u+k} \le \int_{u}^{u+s} \frac{1}{x}\, dx$$

valid for $u > 0$ and $s \in \mathbb{N}$, to show that $\frac{d}{dt} \log\left(R_{n,d,Q}(t)\right)$ is positive on an interval for large $n$.

We have:

$$
\begin{aligned}
\frac{d}{dt}\log\left(R_{n,d,Q}(t)\right) &= \sum_{l=0}^{rn-1}\frac{Q}{Qt-l+d}+\sum_{l=0}^{rn-1}\frac{Q}{Qt+Q(n+2)+l+d} \\
&\quad -a\sum_{l=0}^{n}\frac{1}{t+1+l+\frac{d}{Q}} \\
&= Q\sum_{l=1}^{rn}\frac{1}{Qt-rn+d+l}+Q\sum_{l=1}^{rn}\frac{1}{Qt+Q(n+2)+d-1+l} \\
&\quad -a\sum_{l=1}^{n+1}\frac{1}{t+\frac{d}{Q}+l} \\
&\geq Q\int_{Qt-rn+d+1}^{Qt+d+1}\frac{1}{x}\,dx+Q\int_{Qt+Q(n+2)+d}^{Qt+Q(n+2)+d+rn}\frac{1}{x}\,dx \\
&\quad -a\int_{t+\frac{d}{Q}}^{t+\frac{d}{Q}+n+1}\frac{1}{x}\,dx \\
&= Q\left(\ln(Qt+d+1)-\ln(Qt-rn+d+1)\right) \\
&\quad +Q\left(\ln(Qt+Q(n+2)+d+rn)-\ln(Qt+Q(n+2)+d)\right) \\
&\quad -a\left(\ln\left(t+\frac{d}{Q}+n+1\right)-\ln\left(t+\frac{d}{Q}\right)\right)
\end{aligned}
$$

We continue the analysis by dividing through by $Qn$ in all terms, noting that this leaves the expression unchanged, and then with $x=\frac{t}{n}$:

$$
\begin{aligned}
\frac{d}{dt}\log\left(R_{n,d,Q}(t)\right) &\geq Q\left(\ln\left(x+\frac{d+1}{Qn}\right)-\ln\left(x-R+\frac{d+1}{Qn}\right)\right) \\
&\quad +Q\left(\ln\left(x+1+R+\frac{2Q+d}{Qn}\right)-\ln\left(x+1+\frac{2Q+d}{Qn}\right)\right) \\
&\quad -a\left(\ln\left(x+1+\frac{d+Q}{Qn}\right)-\ln\left(x+\frac{d}{Qn}\right)\right) \\
&= Q\left[\ln(x)-\ln(x-R)+\ln(x+1+R)-\ln(x+1)\right. \\
&\quad \left.-A\ln(x+1)+A\ln(x)\right] \\
&\quad +Q\ln\left(\frac{x+\frac{d+1}{Qn}}{x}\right)-Q\ln\left(\frac{x-R+\frac{d+1}{Qn}}{x-R}\right) \\
&\quad +Q\ln\left(\frac{x+1+R+\frac{2Q+d}{Qn}}{x+1+R}\right)-Q\ln\left(\frac{x+1+\frac{2Q+d}{Qn}}{x+1}\right)
\end{aligned}
$$

$$-a\ln\left(\frac{x+1+\frac{d+Q}{Qn}}{x+1}\right)+a\ln\left(\frac{x+\frac{d}{Qn}}{x}\right)$$

$$\begin{aligned} = & \; Q\left[(A+1)\ln\left(\frac{x}{x+1}\right)+\ln\left(\frac{x+1+R}{x-R}\right)\right] \\ & +Q\ln\left(1+\frac{d+1}{Qnx}\right)-Q\ln\left(1+\frac{d+1}{Qn(x-R)}\right) \\ & +Q\ln\left(1+\frac{2Q+d}{Qn(x+1+R)}\right)-Q\ln\left(1+\frac{2Q+d}{Qn(x+1)}\right) \\ & -a\ln\left(1+\frac{d+Q}{Qn(x+1)}\right)+a\ln\left(1+\frac{d}{Qnx}\right) \end{aligned}$$

Since $\frac{1}{2}\cdot t \leq \ln(1+t) \leq t$ if $0 \leq t \leq 1$, then:

$$\begin{aligned} \frac{d}{dt}\log\left(R_{n,d,Q}(t)\right) \;\geq\; & Q\left[(A+1)\ln\left(\frac{x}{x+1}\right)+\ln\left(\frac{x+1+R}{x-R}\right)\right] \\ & +\frac{Q}{2}\cdot\frac{d+1}{Qnx}-Q\cdot\frac{d+1}{Qn(x-R)}+\frac{Q}{2}\cdot\frac{2Q+d}{Qn(x+1+R)} \\ & -Q\cdot\frac{2Q+d}{Qn(x+1)}-a\cdot\frac{d+Q}{Qn(x+1)}+\frac{a}{2}\cdot\frac{d}{Qnx} \end{aligned}$$

if $n$ is large enough that $\frac{d+1}{Qnx} < 1$, $\frac{2Q+d}{Qn(x+1+R)} < 1$, $\frac{2Q+d}{Qn(x+1)} < 1$, $\frac{d+Q}{Qn(x+1)} < 1$, and $\frac{d}{Qnx} < 1$ for $x \geq R$ and $d = 1+m, ..., Q+m$. Furthermore, $x$ must be chosen so that $\frac{d+1}{Qn(x-R)} \leq 1$. Since $Qn(x-R) = Qt - rn$, this last is the case when $t \geq \frac{rn+d+1}{Q}$. Thus if $\delta > 0$ and $\left(R+\frac{d+1}{Qn}\right)n \leq t \leq (R+\delta)n$ or, equivalently $R+\frac{d+1}{Qn} \leq x \leq R+\delta$, then:

$$\begin{aligned} \frac{d}{dt}\log\left(R_{n,d,Q}(t)\right)|_{t=nx} \;\geq\; & Q\left[(A+1)\ln\left(\frac{x}{x+1}\right)+\ln\left(\frac{x+1+R}{x-R}\right)\right] \\ & -Q-\frac{2Q+d}{n(R+1)}-a\cdot\frac{d+Q}{Qn(R+1)} \end{aligned}$$

where we've dropped the positive terms, used the fact that $\frac{d+1}{Qn(x-R)} \leq 1$, and increased the remaining terms using $x \geq R$. Since $R < x \leq R+\delta$, we have $|x-R| \leq \delta$ and hence the first term can be made as large as we like for all $x$ in the range under consideration simply by taking $\delta$ sufficiently small. In particular, we can make it positive and greater than $Q$. Then the remaining terms can be made negligible by taking $n$ large. Thus for $\delta$ sufficiently small (depending only on $A$ and $R$), there is an $N$ such that for all $n \geq N$, all $d = 1+m, ..., Q+m$, and all $x \in \left[R + \frac{d+1}{Qn}, R+\delta\right]$, $\frac{d}{dt}\log\left(R_{n,d,Q}(t)\right)|_{t=nx} > 0$.

To conclude the proof of the proposition, observe that if we restrict $t$ to integer values there are only finitely many possible values between $\lfloor \frac{rn-d-1}{Q} \rfloor$ and $\frac{rn}{Q} + \frac{d+1}{Q}$. These are at most 4 in number since $d \leq 2Q$ and they are all contained in the set:

$$\left\{\frac{rn-d}{Q}, \frac{rn-d+1}{Q}, ..., \frac{rn}{Q}, \frac{rn+1}{Q}, ..., \frac{rn+d+1}{Q}\right\}$$

Hence we are done if we can show that $R_{n,d,Q}(t)$ increases monotonically on this set for large $n$. Indeed for $-d \leq \gamma \leq d$:

$$\begin{aligned}\frac{R_{n,d,Q}\left(\frac{rn+\gamma+1}{Q}\right)}{R_{n,d,Q}\left(\frac{rn+\gamma}{Q}\right)} &= \frac{rn+\gamma+d+1}{\gamma+d+1} \cdot \frac{2rn+\gamma+d+Q(n+2)}{rn+\gamma+d+Q(n+2)} \\ &\cdot \prod_{l=1}^{n+1}\left(\frac{rn+\gamma+d+Ql}{rn+\gamma+d+1+Ql}\right)^a\end{aligned}$$

As $n$ approaches infinity, the first factor goes to infinity while the second converges to $\frac{2r+Q}{r+Q} > 1$. To complete the argument, it suffices to show that the third factor is bounded from below away from 0. Indeed, we have:

$$\begin{aligned}\prod_{l=1}^{n+1}\left(\frac{rn+\gamma+d+Ql}{rn+\gamma+d+1+Ql}\right)^a &\geq \prod_{l=1}^{n+1}\left(\frac{rn+\gamma+d+Ql}{rn+\gamma+d+Q+Ql}\right)^a \\ &= \left(\frac{rn+\gamma+d+Q}{rn+\gamma+d+Q+Q(n+1)}\right)^a \\ &= \left(\frac{r+\frac{\gamma+d+Q}{n}}{r+Q+\frac{\gamma+d+2Q}{n})}\right)^a\end{aligned}$$

and as $n$ tends to infinity, this converges to $\left(\frac{r}{r+Q}\right)^a$. Thus we see that there is an $N'$ such that for $n > N'$:

$$\frac{R_{n,d,Q}\left(\frac{rn+\gamma+1}{Q}\right)}{R_{n,d,Q}\left(\frac{rn+\gamma}{Q}\right)} > 1$$

for all $d = 1+m, ..., Q+m$ and $-d \leq \gamma \leq d$. The proposition follows with $N_\delta = \max\{N, N'\}$.

Finally, we establish an asymptotic product formula for $R_{n,d,Q}(t)$ when $t > (R+\delta)n$, for any $\delta > 0$. First, we define:

$$\begin{aligned}
f(x) &= \log\left(\left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A}\frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}\right)\\
F(x) &= \exp(f(x)) = \left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A}\frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}\\
H(x) &= \sqrt{\frac{x}{x-R}\cdot\frac{x+1}{x+1+R}}\cdot\left(\frac{x+1+R}{x+1}\right)^{2Q}\cdot\left(\sqrt{\frac{x}{(x+1)^3}}\right)^a\\
g(x) &= (A+1)\log\left(\frac{x}{x+1}\right)+\log\left(\frac{x+1+R}{x-R}\right)\\
G(x) &= \exp(g(x)) = \left(\frac{x}{x+1}\right)^{A+1}\frac{x+1+R}{x-R}\\
g_1(x) &= \frac{1}{2Q}\left(\frac{A+1}{x}-\frac{A+1}{x+1}+\frac{1}{x+1+R}-\frac{1}{x-R}\right)\\
h_1(x) &= \frac{a-2r}{12}+\frac{a}{12x}-\frac{13a}{12(x+1)}+\frac{2Q}{x+1+R}-\frac{2Q}{x+1}\\
&\quad \frac{1}{12Qx}-\frac{1}{12Q(x-R)}-\frac{1}{12Q(x+1)}+\frac{1}{12Q(x+1+R)}\\
&\quad +\frac{1}{x+1}-\frac{1}{x+1+R}\\
h_2(x) &= \frac{A}{2x}-\frac{3A}{2(x+1)}+\frac{2}{x+1+R}-\frac{2}{x+1}+\frac{1}{2Qx}\\
&\quad -\frac{1}{2Q(x+1+R)}+\frac{1}{2Q(x+1)}-\frac{1}{2Q(x-R)}
\end{aligned}$$

**Proposition 3.3** *For each $\delta > 0$ and $n \geq N_{asym}$ (depending on $\delta$), if $x = \frac{k}{n}$ then with the notation above:*

$$R_{n,d,Q}(k) = Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}F(x)^{nQ}H(x)G(x)^d\exp\left(O_\delta\left(\frac{1}{n}\right)\right)$$

*uniformly for $k \geq n(R+\delta)$ and furthermore:*

$$\exp\left(O_\delta\left(\frac{1}{n}\right)\right) = \exp\left(\frac{d^2}{n}g_1(x)+\frac{1}{n}h_1(x)+\frac{d}{n}h_2(x)+O_\delta\left(\frac{1}{n^2}\right)\right)$$

*proof*

Recall the definition of the Pochammer symbol:

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \prod_{l=0}^{k-1}(\alpha+l)$$

and apply Stirling's formula in the form:

$$\Gamma(s) = s^{s-\frac{1}{2}} e^{-s} \sqrt{2\pi} e^{\frac{1}{12s} + O\left(\frac{1}{|s|^2}\right)}$$

to:

$$R_{n,d,Q}(k) = n!^{a-2r} \frac{(Qk+d-rn+1)_{rn}\,(Qk+Qn+2Q+d)_{rn}}{\left(k+1+\frac{d}{Q}\right)^a_{n+1}}$$

We have:

$$\begin{aligned}
R_{n,d,Q}(k) &= n!^{a-2r} \frac{\Gamma(Qk+d+1)}{\Gamma(Qk+d-rn+1)} \cdot \frac{\Gamma(Qk+Qn+2Q+d+rn)}{\Gamma(Qk+Qn+2Q+d)} \\
&\quad \cdot \left[ \frac{\Gamma\left(k+2+\frac{d}{Q}+n\right)}{\Gamma\left(k+1+\frac{d}{Q}\right)} \right]^{-a} \\
&= \left((n+1)^{n+1/2} e^{-(n+1)} \sqrt{2\pi}\right)^{a-2r} \\
&\quad \cdot \frac{(Qk+d+1)^{Qk+d+1/2} e^{-(Qk+d+1)}}{(Qk+d-rn+1)^{Qk+d-rn+1/2} e^{-(Qk+d-rn+1)}} \\
&\quad \cdot \frac{(Qk+Qn+2Q+d+rn)^{Qk+Qn+2Q+d+rn-1/2}}{(Qk+Qn+2Q+d)^{Qk+Qn+2Q+d-1/2}} \\
&\quad \cdot \frac{e^{-(Qk+Qn+2Q+d+rn)}}{e^{-(Qk+Qn+2Q+d)}} \cdot \frac{\left(k+2+\frac{d}{Q}+n\right)^{-a\left(k+2+\frac{d}{Q}+n-1/2\right)}}{\left(k+1+\frac{d}{Q}\right)^{-a\left(k+1+\frac{d}{Q}-1/2\right)}} \\
&\quad \frac{e^{a\left(k+2+\frac{d}{Q}+n\right)}}{e^{a\left(k+1+\frac{d}{Q}\right)}} \cdot \exp\left(O\left(\frac{1}{n}\right)\right) \\
&= ((n+1)^{n+1/2} e^{-(n+1)} \sqrt{2\pi})^{a-2r} Q^{2rn} n^{-(a-2r)n-a} e^{(a-2r)n+a} \\
&\quad \cdot \frac{\left(\frac{k}{n}+\frac{d+1}{Qn}\right)^{Qk+d+1/2}}{\left(\frac{k}{n}-\frac{r}{Q}+\frac{d+1}{Qn}\right)^{Qk+d-rn+1/2}} \\
&\quad \cdot \frac{\left(\frac{k}{n}+1+\frac{r}{Q}+\frac{2Q+d}{Qn}\right)^{Qk+Qn+2Q+d+rn-1/2}}{\left(\frac{k}{n}+1+\frac{2Q+d}{Qn}\right)^{Qk+Qn+2Q+d-1/2}} \\
&\quad \cdot \frac{\left(\frac{k}{n}+1+\frac{2Q+d}{Qn}\right)^{-a\left(k+2+\frac{d}{Q}+n-1/2\right)}}{\left(\frac{k}{n}+\frac{Q+d}{Qn}\right)^{-a\left(k+1+\frac{d}{Q}-1/2\right)}} \cdot \exp\left(O\left(\frac{1}{n}\right)\right)
\end{aligned}$$

It should be noted at this point that for arbitrary $\delta > 0$, if $\frac{k}{n} = x > R + \delta$ then all of the arguments in the gamma functions are positive and tend to infinity with $n$. Thus this formula is valid for all $x > R + \delta$ and the $O\left(\frac{1}{n}\right) = O_\delta\left(\frac{1}{n}\right)$ is uniform in $x$. For brevity we have written $O_\delta\left(\frac{1}{n}\right)$ in place of the following expanded expression which uses the $e^{\frac{1}{12s}}$ term in Stirling's formula:

$$\begin{aligned}
O_\delta\left(\frac{1}{n}\right) &= \frac{a-2r}{12(n+1)} + \frac{1}{12(Qk+d+1)} - \frac{1}{12(Qk+d-rn+1)} \\
&+\frac{1}{12(Qk+Qn+2Q+d+rn)} - \frac{1}{12(Qk+Qn+2Q+d)} \\
&-\frac{a}{12\left(k+2+\frac{d}{Q}+n\right)} + \frac{a}{12\left(k+1+\frac{d}{Q}\right)} + O_\delta\left(\frac{1}{n^2}\right) \\
&= \frac{a-2r}{12n}\cdot\frac{1}{1+\frac{1}{n}} + \frac{1}{12nQx}\cdot\frac{1}{1+\frac{d+1}{nQx}} \\
&-\frac{1}{12nQ(x-R)}\cdot\frac{1}{1+\frac{d+1}{nQ(x-R)}} \\
&+\frac{1}{12nQ(x+1+R)}\cdot\frac{1}{1+\frac{2Q+d}{nQ(x+1+R)}} \\
&-\frac{1}{12nQ(x+1)}\cdot\frac{1}{1+\frac{2Q+d}{nQ(x+1)}} \\
&-\frac{a}{12n(x+1)}\cdot\frac{1}{1+\frac{2Q+d}{nQ(x+1)}} + \frac{a}{12nx}\cdot\frac{1}{1+\frac{Q+d}{nQx}} + O_\delta\left(\frac{1}{n^2}\right) \\
&= \frac{a-2r}{12n} + \frac{1}{12nQx} - \frac{1}{12nQ(x-R)} + \frac{1}{12nQ(x+1+R)} \\
&-\frac{1}{12nQ(x+1)} - \frac{a}{12n(x+1)} + \frac{a}{12nx} + O_\delta\left(\frac{1}{n^2}\right)
\end{aligned}$$

Here, the $O_\delta\left(\frac{1}{n^2}\right)$ is uniform in $x > R + \delta$ for $n$ sufficiently large. Letting $\frac{k}{n} = x$, combining these results and writing them in logarithmic form we have:

$$\frac{1}{nQ}\log R_{n,d,Q}(k) =$$

$$\begin{aligned}
&\frac{A-2R}{n}\left(\log(\sqrt{2\pi}) - 1 + \left(n+\frac{1}{2}\right)\log(n+1) - n\log(n)\right) \\
&+2R\log(Q) - \frac{A\log(n)}{n} + \frac{A}{n} + \left(x+\frac{D}{n}+\frac{1}{2nQ}\right)\log\left(x+\frac{d+1}{Qn}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(x-R+\frac{D}{n}+\frac{1}{2nQ}\right)\log\left(x-R+\frac{d+1}{Qn}\right)\\
&+\left(x+1+R+\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\log\left(x+1+R+\frac{2Q+d}{Qn}\right)\\
&-\left(x+1+\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\log\left(x+1+\frac{2Q+d}{Qn}\right)\\
&-A\left(x+1+\frac{2}{n}+\frac{D}{n}-\frac{1}{2n}\right)\log\left(x+1+\frac{2Q+d}{Qn}\right)\\
&+A\left(x+\frac{1}{n}+\frac{D}{n}-\frac{1}{2n}\right)\log\left(x+\frac{Q+d}{Qn}\right)\\
&+\frac{A-2R}{12n^2}+\frac{1}{12n^2Q^2x}-\frac{1}{12n^2Q^2(x-R)}+\frac{1}{12n^2Q^2(x+1+R)}\\
&-\frac{1}{12n^2Q^2(x+1)}-\frac{A}{12n^2(x+1)}+\frac{A}{12n^2x}+O_\delta\left(\frac{1}{n^3}\right)
\end{aligned}
$$

Next we expand the logarithms to isolate the main terms:

$$
\frac{1}{nQ}\log R_{n,d,Q}(k)=
$$

$$
\begin{aligned}
&(A-2R)\left(\frac{\log(\sqrt{2\pi})-1}{n}+\left(1+\frac{1}{2n}\right)\left(\log(n)+\log\left(1+\frac{1}{n}\right)\right)-\log(n)\right)\\
&-\frac{A(\log(n)-1)}{n}+2R\log(Q)\\
&+\left(x+\frac{D}{n}+\frac{1}{2nQ}\right)\left[\log(x)+\log\left(1+\frac{d+1}{Qnx}\right)\right]\\
&-\left(x-R+\frac{D}{n}+\frac{1}{2nQ}\right)\left[\log(x-R)+\log\left(1+\frac{d+1}{Qn(x-R)}\right)\right]\\
&+\left(x+1+R+\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\Big[\log(x+1+R)\\
&\quad+\log\left(1+\frac{2Q+d}{Qn(x+1+R)}\right)\Big]\\
&-\left(x+1+\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\left[\log(x+1)+\log\left(1+\frac{2Q+d}{Qn(x+1)}\right)\right]\\
&-A\left(x+1+\frac{2}{n}+\frac{D}{n}-\frac{1}{2n}\right)\left[\log(x+1)+\log\left(1+\frac{2Q+d}{Qn(x+1)}\right)\right]
\end{aligned}
$$

$$+A\left(x+\frac{1}{n}+\frac{D}{n}-\frac{1}{2n}\right)\left[\log(x)+\log\left(1+\frac{Q+d}{Qnx}\right)\right]$$
$$+\frac{A-2R}{12n^2}+\frac{1}{12n^2Q^2x}-\frac{1}{12n^2Q^2(x-R)}+\frac{1}{12n^2Q^2(x+1+R)}$$
$$-\frac{1}{12n^2Q^2(x+1)}-\frac{A}{12n^2(x+1)}+\frac{A}{12n^2x}+O_\delta\left(\frac{1}{n^3}\right)$$

We next expand the smaller order terms into Taylor series to third order error:

$$\frac{1}{nQ}\log R_{n,d,Q}(k)=$$

$$(A-2R)\left[\frac{\log(\sqrt{2\pi})-1}{n}+\frac{1}{2n}\log(n)+\left(1+\frac{1}{2n}\right)\left(\frac{1}{n}-\frac{1}{2n^2}\right.\right.$$
$$\left.\left.+O\left(\frac{1}{n^3}\right)\right)\right]-A\cdot\frac{\log(n)-1}{n}+2R\log(Q)$$
$$+x\log(x)-(x-R)\log(x-R)+(x+1+R)\log(x+1+R)$$
$$-(x+1)\log(x+1)-A\cdot(x+1)\log(x+1)+A\cdot(x)\log(x)$$
$$+\left(\frac{D}{n}+\frac{1}{2nQ}\right)\log(x)-\left(\frac{D}{n}+\frac{1}{2nQ}\right)\log(x-R)$$
$$+\left(\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\log(x+1+R)-\left(\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\log(x+1)$$
$$-A\left(\frac{2}{n}+\frac{D}{n}-\frac{1}{2n}\right)\log(x+1)+A\left(\frac{1}{n}+\frac{D}{n}-\frac{1}{2n}\right)\log(x)$$
$$+x\left[\frac{d+1}{Qnx}-\frac{1}{2}\left(\frac{d+1}{Qnx}\right)^2+O\left(\frac{1}{n^3}\right)\right]$$
$$-(x-R)\left[\frac{d+1}{Qn(x-R)}-\frac{1}{2}\left(\frac{d+1}{Qn(x-R)}\right)^2+O\left(\frac{1}{n^3}\right)\right]$$
$$+(x+1+R)\left[\frac{2Q+d}{Qn(x+1+R)}-\frac{1}{2}\left(\frac{2Q+d}{Qn(x+1+R)}\right)^2+O\left(\frac{1}{n^3}\right)\right]$$
$$-(x+1)\left[\frac{2Q+d}{Qn(x+1)}-\frac{1}{2}\left(\frac{2Q+d}{Qn(x+1)}\right)^2+O\left(\frac{1}{n^3}\right)\right]$$
$$-A(x+1)\left[\frac{2Q+d}{Qn(x+1)}-\frac{1}{2}\left(\frac{2Q+d}{Qn(x+1)}\right)^2+O\left(\frac{1}{n^3}\right)\right]$$
$$+Ax\left[\frac{Q+d}{Qnx}-\frac{1}{2}\left(\frac{Q+d}{Qnx}\right)^2+O\left(\frac{1}{n^3}\right)\right]$$

$$+\left(\frac{D}{n}+\frac{1}{2nQ}\right)\left(\frac{d+1}{Qnx}\right)-\left(\frac{D}{n}+\frac{1}{2nQ}\right)\left(\frac{d+1}{Qn(x-R)}\right)$$
$$+\left(\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\left(\frac{2Q+d}{Qn(x+1+R)}\right)$$
$$-\left(\frac{2}{n}+\frac{D}{n}-\frac{1}{2nQ}\right)\left(\frac{2Q+d}{Qn(x+1)}\right)-A\left(\frac{2}{n}+\frac{D}{n}-\frac{1}{2n}\right)\left(\frac{2Q+d}{Qn(x+1)}\right)$$
$$+A\left(\frac{1}{n}+\frac{D}{n}-\frac{1}{2n}\right)\left(\frac{Q+d}{Qnx}\right)$$
$$+\frac{A-2R}{12n^2}+\frac{1}{12n^2Q^2x}-\frac{1}{12n^2Q^2(x-R)}+\frac{1}{12n^2Q^2(x+1+R)}$$
$$-\frac{1}{12n^2Q^2(x+1)}-\frac{A}{12n^2(x+1)}+\frac{A}{12n^2x}+O_\delta\left(\frac{1}{n^3}\right)$$

Now we expand the products and collect terms so that finally:

$$\frac{1}{nQ}\log R_{n,d,Q}(k)=$$

$$(A-2R)\left[\frac{\log(\sqrt{2\pi})}{n}+\frac{1}{2n}\log(n)+O\left(\frac{1}{n^3}\right)\right]-\frac{A\log(n)}{n}$$
$$+\log\left(Q^{2R}\left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A}\frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}\right)$$
$$+\frac{1}{2Qn}\log\left(\frac{x}{x-R}\cdot\frac{x+1}{x+1+R}\right)+\frac{2}{n}\log\left(\frac{x+1+R}{x+1}\right)$$
$$+\frac{A}{2n}\log\left(\frac{x}{(x+1)^3}\right)$$
$$+\frac{D}{n}\left[(A+1)\log\left(\frac{x}{x+1}\right)+\log\left(\frac{x+1+R}{x-R}\right)\right]$$
$$+\frac{D^2}{n^2}\left[\frac{1}{2}\left(\frac{A+1}{x}-\frac{A+1}{x+1}+\frac{1}{x+1+R}-\frac{1}{x-R}\right)\right]$$
$$+\frac{1}{n^2}\left[\frac{A-2R}{12}+\frac{A}{12x}-\frac{13A}{12(x+1)}+\frac{2}{x+1+R}-\frac{2}{x+1}\right]$$
$$+\frac{1}{n^2}\left[\frac{1}{12Q^2x}-\frac{1}{12Q^2(x-R)}-\frac{1}{12Q^2(x+1)}+\frac{1}{12Q^2(x+1+R)}\right]$$
$$+\frac{1}{n^2}\left[\frac{1}{Q(x+1)}-\frac{1}{Q(x+1+R)}\right]$$
$$+\frac{D}{n^2}\left[\frac{A}{2x}-\frac{3A}{2(x+1)}+\frac{2}{x+1+R}-\frac{2}{x+1}+\frac{1}{2Qx}\right.$$
$$\left.\frac{1}{2Q(x+1+R)}+\frac{1}{2Q(x+1)}-\frac{1}{2Q(x-R)}\right]+O_\delta\left(\frac{1}{n^3}\right)$$

Recall the notation:

$$
\begin{aligned}
f(x) &= \log\left(\left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A}\frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}\right)\\
F(x) &= \exp(f(x)) = \left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A}\frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}\\
H(x) &= \sqrt{\frac{x}{x-R}\cdot\frac{x+1}{x+1+R}}\cdot\left(\frac{x+1+R}{x+1}\right)^{2Q}\cdot\left(\sqrt{\frac{x}{(x+1)^3}}\right)^{a}\\
g(x) &= (A+1)\log\left(\frac{x}{x+1}\right)+\log\left(\frac{x+1+R}{x-R}\right)\\
G(x) &= \exp(g(x)) = \left(\frac{x}{x+1}\right)^{A+1}\frac{x+1+R}{x-R}\\
g_1(x) &= \frac{1}{2Q}\left(\frac{A+1}{x}-\frac{A+1}{x+1}+\frac{1}{x+1+R}-\frac{1}{x-R}\right)\\
h_1(x) &= \frac{a-2r}{12}+\frac{a}{12x}-\frac{13a}{12(x+1)}+\frac{2Q}{x+1+R}-\frac{2Q}{x+1}\\
&\quad \frac{1}{12Qx}-\frac{1}{12Q(x-R)}-\frac{1}{12Q(x+1)}+\frac{1}{12Q(x+1+R)}\\
&\quad +\frac{1}{x+1}-\frac{1}{x+1+R}\\
h_2(x) &= \frac{A}{2x}-\frac{3A}{2(x+1)}+\frac{2}{x+1+R}-\frac{2}{x+1}+\frac{1}{2Qx}\\
&\quad -\frac{1}{2Q(x+1+R)}+\frac{1}{2Q(x+1)}-\frac{1}{2Q(x-R)}
\end{aligned}
$$

Then note that:

$$f'(x)=g(x)\,,\ \ f''(x)=g'(x)=2Qg_1(x)\ \text{and}\ \frac{1}{Q}H'(x)=H(x)h_2(x)$$

and that for $k \geq n(R+\delta)$ and $n \geq N_{asym}$:

$$R_{n,d,Q}(k)=Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}F(x)^{nQ}H(x)G(x)^d\exp\left(O_\delta\left(\frac{1}{n}\right)\right)$$

where:

$$\exp\left(O_\delta\left(\frac{1}{n}\right)\right)=\exp\left(\frac{d^2}{n}g_1(x)+\frac{1}{n}h_1(x)+\frac{d}{n}h_2(x)+O_\delta\left(\frac{1}{n^2}\right)\right)$$

as claimed in the proposition.

# Chapter 4

## Maximum of the Main Term

Let $R = \frac{r}{Q}$, $A = \frac{a}{Q}$, and recall:

$$F(x) = \exp\left(f(x)\right) = \left(\frac{x^x}{(x+1)^{x+1}}\right)^{1+A} \frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}$$

We define:

$$\phi_{Q,r,a} = \max_{R \leq x < \infty} Q^{2R} F(x)$$

**Proposition 4.1** *There exists a unique $x_0 \in (R, \infty)$ such that:*

$$\phi_{Q,r,a} = Q^{2R} F(x_0)$$

*Moreover:*

$$0 < \phi_{Q,r,a} \leq \frac{Q^{2R} 2^{R+1}}{R^{A-2R}}$$

*proof*

To show this, we follow the argument given in Ball and Rivoal ([1], pp. 199-201). We see that:

$$f'(x) = \frac{F'(x)}{F(x)} = \log\left(\frac{x^{1+A}(x+1+R)}{(x+1)^{1+A}(x-R)}\right)$$

Thus, a critical point of $F$ must satisfy:

$$x^{1+A}(x+1+R) = (x+1)^{1+A}(x-R)$$

The transformation $x = \frac{s}{1-s}$ or, equivalently, $s = \frac{x}{x+1}$ maps the interval $[R, \infty)$ bijectively to $\left[\frac{R}{R+1}, 1\right)$ and is order preserving. Under this transformation, we see that any critical point of $F$ must also satisfy the relation:

$$s^{1+A} = \frac{(R+1)s - R}{R+1-Rs} \tag{4.1}$$

and be a root of the function:

$$P_{R,A}(s) = s^{1+A}(-Rs + (R+1)) - ((R+1)s - R)$$

So it is necessary to further understand this critical point as a root of $P_{R,A}(s)$. We have $P_{R,A}(0) = R > 0$, $P_{R,A}(1) = 0$, and $P_{R,A}(s) > 0$ on $\left[0, \frac{R}{R+1}\right]$ since, on this interval $s^{1+A} > 0$, $-Rs + R + 1 > 0$, and $-(R+1)s + R \geq 0$. Furthermore:

$$P'_{R,A}(s) = (1+A)s^{A}(-Rs + R + 1) - Rs^{1+A} - (R+1)$$

So $P'_{R,A}(0) = -(R+1) < 0$ and $P'_{R,A}(1) = A - 2R > 0$. Also:

$$\begin{aligned} P''_{R,A}(s) &= A(1+A)s^{A-1}(-Rs + R + 1) - 2R(1+A)s^{A} \\ &= -(1+A)s^{A-1}(R(A+2)s - (R+1)A) \end{aligned}$$

Thus $P''_{R,A}(s) > 0$ on $[0, 1]$. Hence, $P_{R,A}(s)$ has a single root in $s_0 \in [0, 1)$ and $\frac{R}{R+1} < s_0 < 1$. Equivalently, on the interval $[R, \infty)$, $F(x)$ takes on its maximum value uniquely at $x_0 = \frac{s_0}{1-s_0}$. Writing $s = s_0$, it follows from equation (4.1) that:

$$\begin{aligned} \phi_{Q,r,a} &= Q^{2R} F\left(\frac{s}{1-s}\right) \\ &= Q^{2R}\left(\frac{\left(\frac{s}{1-s}\right)^{\frac{s}{1-s}}}{\left(\frac{s}{1-s}+1\right)^{\frac{s}{1-s}+1}}\right)^{1+A} \frac{\left(\frac{s}{1-s}+1+R\right)^{\frac{s}{1-s}+1+R}}{\left(\frac{s}{1-s}-R\right)^{\frac{s}{1-s}-R}} \\ &= Q^{2R}(1-s)^{A-2R} \cdot s^{\frac{s(1+A)}{1-s}} \cdot \frac{(s + (1+R)(1-s))^{\frac{s+(1+R)(1-s)}{1-s}}}{(s - R(1-s))^{\frac{s-R(1-s)}{1-s}}} \\ &= Q^{2R}(1-s)^{A-2R} \cdot s^{\frac{s(1+A)}{1-s}} \cdot \frac{(R+1-Rs)^{\frac{s+(1+R)(1-s)}{1-s}}}{((R+1)s - R)^{\frac{s-R(1-s)}{1-s}}} \end{aligned}$$

So we have that:

$$
\begin{aligned}
\phi_{Q,r,a} &= Q^{2R}(1-s)^{A-2R}(R+1-Rs)^{R+1}((R+1)s-R)^R \\
&\quad \cdot \left(s^{1+A}\cdot\frac{R+1-Rs}{(R+1)s-R}\right)^{\frac{s}{1-s}} \\
&= Q^{2R}(1-s)^{A-2R}(R+1-Rs)^{R+1}((R+1)s-R)^R
\end{aligned}
$$

and since $\frac{R}{R+1} < s_0 < 1$:

$$
\begin{aligned}
0 < \phi_{Q,r,a} &\leq Q^{2R}\left(1-\frac{R}{R+1}\right)^{A-2R}\left(R+1-R\cdot\frac{R}{R+1}\right)^{R+1}(R+1-R)^R \\
&= Q^{2R}\left(\frac{1}{R+1}\right)^{A-2R}\left(\frac{(R+1)^2-R^2}{R+1}\right)^{R+1} \\
&= Q^{2R}\left(\frac{1}{R+1}\right)^{A-2R}\left(\frac{2R+1}{R+1}\right)^{R+1} \\
&\leq \frac{Q^{2R}2^{R+1}}{R^{A-2R}}
\end{aligned}
$$

completing the proof of the proposition.

In subsequent work we will also require some information about $f''(x_0)$ and $f'''(x_0)$. In the remainder of this chapter we endeavor to establish the nonvanishing of these higher derivatives.

**Lemma 4.2** *For fixed $x$ and $0 \leq R < x$ the functions:*

$$
K_x(R) = \frac{1}{\ln(x+1+R)-\ln(x-R)}\left(\frac{1}{x-R}-\frac{1}{x+1+R}\right)
$$

*and:*

$$
H_x(R) = \frac{1}{\ln(x+1+R)-\ln(x-R)}\left(\frac{1}{(x-R)^2}-\frac{1}{(x+1+R)^2}\right)
$$

*are monotonically increasing in $R$.*

*proof*

Let $t = \frac{R+\frac{1}{2}}{x+\frac{1}{2}}$. Then $0 \leq t < 1$ and:

$$
K_x(R) = \frac{1}{x+\frac{1}{2}}\cdot\frac{1}{\ln(1+t)-\ln(1-t)}\left(\frac{1}{1-t}-\frac{1}{1+t}\right) =: \frac{K(t)}{x+\frac{1}{2}}
$$

and:

$$\begin{aligned}
H_x(R) &= \frac{1}{\left(x+\frac{1}{2}\right)^2} \cdot \frac{1}{\ln(1+t)-\ln(1-t)} \left(\frac{1}{(1-t)^2} - \frac{1}{(1+t)^2}\right) \\
&= \frac{H(t)}{\left(x+\frac{1}{2}\right)^2} \\
&= \frac{1}{\left(x+\frac{1}{2}\right)^2} \cdot \frac{2}{1-t^2} \cdot K(t)
\end{aligned}$$

Since $\frac{2}{1-t^2}$ is monotonically increasing for $0 \le t < 1$ it suffices to show that $K(t)$ is monotonically increasing for $0 \le t < 1$. This is equivalent to showing that:

$$K(t)^{-1} = \frac{\ln(1+t)-\ln(1-t)}{2t} \cdot (1-t^2)$$

is decreasing. We look at the Taylor expansion:

$$\begin{aligned}
K(t)^{-1} &= \frac{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} ... - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} ...\right)}{2t} \cdot (1-t^2) \\
&= \left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} ...\right) \cdot (1-t^2) \\
&= 1 + \left(\frac{1}{3} - 1\right) t^2 + \left(\frac{1}{5} - \frac{1}{3}\right) t^4 + ... \\
&= 1 - \sum_{k=1}^{\infty} \frac{2}{4k^2-1} \cdot t^{2k}
\end{aligned}$$

Therefore:

$$\frac{d}{dt} K(t)^{-1} = -\sum_{k=1}^{\infty} \frac{4k}{4k^2-1} \cdot t^{2k-1} < 0$$

for $0 < t < 1$ thus proving the assertion.

As we have seen, for $F_{A,R}(x) = F(x)$ and a given $A$ and $R$, $x_0$ is defined by the equation:

$$\frac{d}{dx} \log\left(F_{A,R}(x)\right) = 0$$

In the following, we shall fix $R$ and view this equation as defining a correspondence between $A$ and $x$.

Since we have that:

$$\begin{aligned}\frac{d}{dx}\log\left(F_{A,R}(x)\right) &= g(x) \\ &= (1+A)\left(\log(x)-\log(x+1)\right) \\ &\quad +\log(x+1+R)-\log(x-R)\end{aligned}$$

then:

$$\frac{d}{dx}\log\left(F_{A,R}(x)\right) = 0$$

if and only if:

$$A+1=\frac{\log(x+1+R)-\log(x-R)}{\log(x+1)-\log(x)}$$

which defines $A$ as a function of $x$.

**Proposition 4.3** *For fixed $R$ the correspondence between $A$ and $x_0$ is biholomorphic and monotonically decreasing between $A \in (2R, \infty)$ and $x_0 \in (R, \infty)$.*

*proof*

We consider the derivative $\frac{dA}{dx}$ and the asymptotic behavior of $A(x)$ as $x \to R^+$ and as $x \to \infty$. We compute:

$$\begin{aligned}\frac{dA}{dx} &= -\frac{\log(x+1+R)-\log(x-R)}{\log(x+1)-\log(x)} \\ &\quad \cdot \left[\frac{1}{\log(x+1+R)-\log(x-R)}\left(\frac{1}{x-R}-\frac{1}{x+1+R}\right)\right. \\ &\quad \left. -\frac{1}{\log(x+1)-\log(x)}\left(\frac{1}{x}-\frac{1}{x+1}\right)\right] \\ &= -\frac{\log(x+1+R)-\log(x-R)}{\log(x+1)-\log(x)}\left(K_x(R)-K_x(0)\right)\end{aligned}$$

The first factor is clearly positive and the second is positive by the lemma. Hence $\frac{dA}{dx} < 0$ for all $x > R$.

We next examine the asymptotics. We have:

$$\begin{aligned}\lim_{x\to\infty} A(x) &= -1+\lim_{x\to\infty}\frac{\log(x+1+R)-\log(x-R)}{\log(x+1)-\log(x)}\\ &= -1+\lim_{x\to\infty}\frac{\log\left(\frac{x+1+R}{x-R}\right)}{\log\left(\frac{x+1}{x}\right))}\\ &= -1+\lim_{x\to\infty}\frac{\log\left(1+\frac{2R+1}{x-R}\right)/\left(\frac{1}{x}\right)}{\log\left(1+\frac{1}{x}\right)/\left(\frac{1}{x}\right)}\\ &= -1+\frac{2R+1}{1}\\ &= 2R\end{aligned}$$

On the other hand:

$$\begin{aligned}\lim_{x\to R^+} A(x) &= -1+\lim_{x\to R^+}\frac{\log(x+1+R)-\log(x-R)}{\log(x+1)-\log(x)}\\ &= +\infty\end{aligned}$$

We have thus established that $A(x)$ is a one to one direction reversing holomorphic correspondence between $x \in (R,\infty)$ and $A \in (2R,\infty)$. The holomorphic version of the inverse function theorem implies that its inverse function is holomorphic as well. Hence, the map $A \to x_0(A)$ is a one to one holomorphic correspondence between $A \in (2R,\infty)$ and $x \in (R,\infty)$ which, moreover, decreases monotonically as $A$ increases. This proves the proposition. We obtain as a corollary of the proof the following:

**Proposition 4.4** *If* $f(x) = \log F_{A,R}(x)$ *and* $x_0 \in (R,\infty)$ *is the unique point where* $f'(x_0) = 0$ *then* $f''(x_0) < 0$.

*proof*

We have:

$$\frac{f''(x)}{2Q} = g_1(x) = (A+1)\left(\frac{1}{x}-\frac{1}{x+1}\right)+\frac{1}{x+1+R}-\frac{1}{x-R}$$

As before, if $x = x_0$ then:

$$A + 1 = \frac{\log(x+1+R) - \log(x-R)}{\log(x+1) - \log(x)}$$

and:

$$\begin{aligned} \frac{f''(x)}{2Q} &= \frac{\log(x+1+R) - \log(x-R)}{\log(x+1) - \log(x)} \left( \frac{1}{x} - \frac{1}{x+1} \right) \\ &\quad + \frac{1}{x+1+R} - \frac{1}{x-R} \\ &= -\left(\log(x+1+R) - \log(x-R)\right)\left(K_x(R) - K_x(0)\right) \\ &< 0 \end{aligned}$$

Lastly, we establish the nonvanishing of the third derivative of $f(x)$.

**Proposition 4.5** *With:*

$$f'''(x_0) = \left(\frac{d}{dx}\right)^3 \log\left(F_{A,R}(x)\right)|_{x=x_0(A)}$$

*then* $f'''(x_0) > 0$.

*proof*

We begin by computing:

$$\left(\frac{d}{dx}\right)^3 \log\left(F_{A,R}(x)\right) = (1+A)\left(\frac{1}{(x+1)^2} - \frac{1}{x^2}\right) + \left(\frac{1}{(x-R)^2} - \frac{1}{(x+1+R)^2}\right)$$

Fix $R$. Using the one to one correspondence between $x = x_0(A)$ and $A = A(x_0)$, it suffices to show that the expression, resulting from substituting for $A$ its representation in terms of $x$, is nonzero for $R < x < \infty$. Proceeding thus:

$$\begin{aligned} \left(\frac{d}{dx}\right)^3 \log\left(F_{A,R}(x)\right) &= \frac{\log(x+1+R) - \log(x-R)}{\log(x+1) - \log(x)} \cdot \left(\frac{1}{(x+1)^2} - \frac{1}{x^2}\right) \\ &\quad + \left(\frac{1}{(x-R)^2} - \frac{1}{(x+1+R)^2}\right) \end{aligned}$$

Rewriting the righthand side of this equation, we see:

$$\begin{aligned}
\left(\frac{d}{dx}\right)^3 \log\left(F_{A,R}(x)\right) &= (\log(x+1+R) - \log(x-R)) \\
&\quad \cdot \left[\frac{1}{\log(x+1+R) - \log(x-R)}\right. \\
&\quad \cdot \left(\frac{1}{(x-R)^2} - \frac{1}{(x+1+R)^2}\right) \\
&\quad \left. - \frac{1}{\log(x+1) - \log(x)} \cdot \left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right)\right] \\
&= (\log(x+1+R) - \log(x-R)) \cdot [H_x(R) - H_x(0)]
\end{aligned}$$

The first factor is positive and, by Lemma (4.2), so is the second. Thus $f'''(x_0) > 0$ as was to be shown.

Henceforth, we shall fix $\delta_0 > 0$ satisfying Proposition (3.2) such that $R + \delta_0 < x_0$. Furthermore, we shall fix $0 < \delta < \delta_0$ and $n$ shall always be assumed large enough that both Proposition (3.2) and Proposition (3.3) are valid with $\delta_0$ and $\delta$ respectively. That is we will suppose $n \geq \max\{N_{\delta_0}, N_{asym}\}$ and make use of the asymptotic product formula valid on $[R + \delta, \infty]$.

## Chapter 5

## The Upper Bound

Now we are able to prove the following proposition which provides an upper bound for the linear forms constructed in the second chapter.

**Proposition 5.1**

$$\limsup_{n\to\infty} |S_{n,c}(1)|^{1/nQ} \leq \phi_{Q,r,a}$$

*proof*

We will require a lemma. First we define:

$$R_n(x) = Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}H(x_0)F(x)^{nQ}$$

where $x_0$ is the unique critical point of $F(x)$ in $[R,\infty)$.

**Lemma 5.2** *(Tail Estimate) There are an* $x_2 > 2x_0$ *and* $N_1$ *such that for each* $x_1 > x_2$ *and each* $n > N_1$*:*

$$R_n\left(\frac{1}{2}x_1\right) \geq \sum_{\frac{k}{n}>x_1} R_{n,d,Q}\left(k\right)$$

Since $R_n(x)$ is monotone decreasing for $x > x_0$, if $\eta > 0$ and $x_1$ is chosen such that $\frac{x_1}{2} > x_0 + \eta$, then:

$$\sum_{\frac{k}{n}>x_1} R_{n,d,Q}(k) \leq R_n\left(x_0+\eta\right) < R_n(x_0)$$

In the sequel, it will be assumed that $x_1$ has been chosen such that this estimate is valid for all large $n$.

Assuming the lemma for the moment, we prove the proposition. If we choose $n > N_1$, note that $R_n(x_0) > R_n\left(\frac{1}{2}x_1\right)$, and let $c_{\max} = \max\{|c_1|, ..., |c_Q|\}$:

$$\begin{aligned}
|S_{n,c}(1)| &= \left|\sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty} R_{n,d,Q}(k)\right| \\
&\leq c_{\max} \sum_{d=1+m}^{Q+m} \left|\sum_{k=0}^{\infty} R_{n,d,Q}(k)\right| \\
&= c_{\max} \sum_{d=1+m}^{Q+m} \left(\sum_{k=0}^{nx_1} R_{n,d,Q}(k) + \sum_{\frac{k}{n}>x_1} R_{n,d,Q}(k)\right) \\
&\leq c_{\max} \sum_{d=1+m}^{Q+m} \left(\sum_{k=0}^{nx_1} R_{n,d,Q}(k) + R_n(x_0)\right) \\
&\leq c_{\max} \sum_{d=1+m}^{Q+m} \left(nx_1 \max_{R+\delta<x<x_1} \left\{Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x)^{nQ} H(x) G(x)^d \right.\right. \\
&\quad \left.\left. \cdot \exp\left(O_\delta\left(\frac{1}{n}\right)\right)\right\}\right) + c_{\max} Q R_n(x_0) \\
&= c_{\max} Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x_0)^{nQ} \left(\sum_{d=1+m}^{Q+m} nx_1 \max_{R+\delta<x<x_1} \left\{\left(\frac{F(x)}{F(x_0)}\right)^{nQ}\right.\right. \\
&\quad \left.\left. \cdot H(x) G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right)\right\} + Q H(x_0)\right) \\
&\leq c_{\max} Q^{2rn+1} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x_0)^{nQ} \left(nx_1 C_\delta + H(x_0)\right)
\end{aligned}$$

Here we've used Lemma (5.2) and Proposition (3.2), the fact that for $d = 1+m, ..., Q+m$ the functions $H(x)G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right)$ are bounded on $[R+\delta, x_1]$ by some constant $C_\delta > 0$, and lastly that:

$$\max_{R+\delta<x<x_1} \left(\frac{F(x)}{F(x_0)}\right)^{nQ} \leq 1$$

Therefore:

$$\begin{aligned}
\limsup_{n\to\infty} |S_{n,c}(1)|^{1/nQ} &\leq Q^{2R} \lim_{n\to\infty} \left[ c_{\max} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x_0)^{nQ} \left(n x_1 C_\delta + H(x_0)\right)\right]^{1/nQ} \\
&= Q^{2R} F(x_0) \\
&= \phi_{Q,r,a}
\end{aligned}$$

We shall now proceed with the proof of the lemma. We must first examine the asymptotic behavior of $F(x)^{nQ}$, $H(x)$, and $G(x)$. We have:

$$H(x) = \sqrt{\frac{x}{x-R}\frac{x+1}{x+1+R}} \left(\sqrt{\frac{x}{x+1}}\right)^{a} \left(\frac{x+1+R}{x+1}\right)^{2Q} \frac{1}{(x+1)^a} \leq \frac{C_1}{x^a}$$

for some constant $C_1$ and $x$ large. Secondly:

$$G(x) = \left(\frac{x}{x+1}\right)^{1+A} \frac{x+1+R}{x-R} \leq C_2$$

for some $C_2 > 1$ and $x$ large, and finally:

$$\begin{aligned}
F(x)^{nQ} &= \left(\left(\frac{x^x}{(x+1)^{x+1}}\right)^{A+1} \frac{(x+1+R)^{x+1+R}}{(x-R)^{x-R}}\right)^{nQ} \\
&= \left[\left(\left(\frac{x}{x+1}\right)^{x} \frac{1}{x+1}\right)^{A+1} \left(\frac{x+1+R}{x-R}\right)^{x-R} (x+1+R)^{1+2R}\right]^{nQ} \\
&= \left[\left(\frac{1}{(1+\frac{1}{x})^x}\right)^{A+1} \left(\frac{x}{x+1}\right)^{A+1} \left(1+\frac{1+2R}{x-R}\right)^{x-R}\right. \\
&\qquad \left. \cdot \left(\frac{x+1+R}{x}\right)^{1+2R} \frac{x^{1+2R}}{x^{A+1}}\right]^{nQ} \\
&= \left[\left(\frac{1+o(1)}{e}\right)^{A+1} (1+o(1))^{A+1} e^{1+2R} (1+o(1))^{1+2R} \frac{1}{x^{A-2R}}\right]^{nQ} \\
&= \left[\frac{1+o(1)}{e^{A-2R}} \frac{1}{x^{A-2R}}\right]^{nQ} \\
&= \left(\frac{1+o(1)}{e^{a-2r}}\right)^{n} \frac{1}{x^{(a-2r)n}}
\end{aligned}$$

Thus we see there is an $x_2$ (depending on $A$ and $R$) such that for all $x \geq x_2$:

$$\frac{1}{2^n} \cdot \frac{1}{e^{(a-2r)n}} \cdot \frac{1}{x^{(a-2r)n}} \leq F(x)^{nQ} \leq 2^n \cdot \frac{1}{e^{(a-2r)n}} \cdot \frac{1}{x^{(a-2r)n}}$$

Now combining all the estimates above and using $d \leq Q + m$, we see:

$$\begin{aligned} R_{n,d,Q}(k) &= Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x)^{nQ} H(x) G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right) \\ &\leq Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x)^{nQ} \frac{C_1}{x^a} (C_2)^{Q+m} C_3 \\ &\leq \frac{C_1 (C_2)^{Q+m} C_3}{H(x_0)} Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} H(x_0) \frac{2^n}{x^a} \frac{1}{e^{(a-2r)n}} \frac{1}{x^{(a-2r)n}} \end{aligned}$$

for $x \geq x_2$. Let $C_4 = \frac{C_1 (C_2)^{Q+m} C_3}{H(x_0)}$. Then for any $x_1 > x_2$:

$$\sum_{\frac{k}{n} > x_1} R_{n,d,Q}(k) \leq C_4 Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} H(x_0) 2^n \frac{1}{e^{(a-2r)n}} \sum_{\frac{k}{n} > x_1} \frac{1}{\left(\frac{k}{n}\right)^{(a-2r)n+a}} \tag{5.1}$$

It remains to estimate the sum on the righthand side:

$$\begin{aligned} \sum_{\frac{k}{n} > x_1} \frac{1}{\left(\frac{k}{n}\right)^{(a-2r)n+a}} &= n^{(a-2r)n+a} \sum_{k > nx_1} \frac{1}{k^{(a-2r)n+a}} \\ &\leq n^{(a-2r)n+a} \frac{1}{(a-2r)n + a - 1} \cdot \frac{1}{(nx_1 - 1)^{(a-2r)n+a-1}} \\ &= \frac{1}{(a-2r) + \frac{a-1}{n}} \cdot \frac{1}{\left(x_1 - \frac{1}{n}\right)^{(a-2r)n+a-1}} \\ &\leq \frac{1}{a-2r} \cdot \frac{1}{\left(1 - \frac{1}{nx_1}\right)^{(a-2r)n}} \cdot \frac{1}{\left(1 - \frac{1}{nx_1}\right)^{a-1}} \cdot \frac{1}{x_1^{a-1}} \cdot \frac{1}{x_1^{(a-2r)n}} \end{aligned}$$

where the first inequality is obtained by observing that for any integer $s > 1$ and any $u > 1$:

$$\sum_{k > u} \frac{1}{k^s} < \int_{u-1}^{\infty} \frac{1}{t^s} dt = \frac{1}{s-1} \cdot \frac{1}{(u-1)^{s-1}}$$

which is true since $\int_{k-1}^{k} \frac{1}{t^s} dt \geq \frac{1}{k^s}$ for any integer $k$. Hence for $n > N_1$ (which depends on $x_1$, $A$, and $R$):

$$\sum_{\frac{k}{n} > x_1} \frac{1}{\left(\frac{k}{n}\right)^{(a-2r)n+a}} \leq \frac{1}{a-2r} \cdot 2e^{\frac{1}{x_1}(a-2r)} \cdot \frac{1}{x_1^{a-1}} \cdot \frac{1}{x_1^{(a-2r)n}}$$

Inserting this into equation (5.1) yields:

$$\begin{aligned}
\sum_{\frac{k}{n}>x_1} R_{n,d,Q}(k) &\leq C_4 Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}H(x_0)2^n\frac{1}{e^{(a-2r)n}} \\
&\quad \cdot\frac{1}{a-2r}\cdot 2e^{\frac{1}{x_1}(a-2r)}\cdot\frac{1}{x_1^{a-1}}\cdot\frac{1}{x_1^{(a-2r)n}} \\
&= \left(C_4\cdot\frac{1}{a-2r}\cdot 2e^{\frac{1}{x_1}(a-2r)}\cdot\frac{1}{x_1^{a-1}}\right)\cdot\left(\frac{2^n\cdot 2^n}{2^{(a-2r)n}}\right) \\
&\quad \cdot\left(Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}H(x_0)\right) \\
&\quad \cdot\left(\frac{1}{2^n}\cdot\frac{1}{e^{(a-2r)n}}\cdot\frac{1}{\left(\frac{x_1}{2}\right)^{(a-2r)n}}\right)
\end{aligned}$$

Since $a > 1$ by hypothesis, then for sufficiently large $x_3$ and $x_1 > x_3$:

$$C_4\cdot\frac{1}{a-2r}\cdot 2e^{\frac{1}{x_1}(a-2r)}\cdot\frac{1}{x_1^{a-1}} < 1$$

If $a - 2r \geq 2$ then:

$$\frac{2^n\cdot 2^n}{2^{(a-2r)n}} \leq 1$$

Lastly, if $\frac{x_1}{2} > x_2 > x_3 > 2x_0$, then:

$$\frac{1}{2^n}\cdot\frac{1}{e^{(a-2r)n}}\cdot\frac{1}{\left(\frac{x_1}{2}\right)^{(a-2r)n}} \leq F\left(\frac{x_1}{2}\right)^{nQ}$$

Combining all these estimates with $x_1$ chosen appropriately large, we have:

$$\begin{aligned}
\sum_{\frac{k}{n}>x_1} R_{n,d,Q}(k) &\leq Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}H(x_0)F\left(\frac{x_1}{2}\right)^{nQ} \\
&= R_n\left(\frac{x_1}{2}\right)
\end{aligned}$$

which completes the proof of the lemma.

## Chapter 6

## First Case of the Lower Bound

We will now complement the upper bound established in the previous chapter with a lower bound to prove:

**Proposition 6.1** *For any* $c = (c_1, ..., c_Q) \in \mathbb{C}^Q$*:*

$$\lim_{n\to\infty} |S_{n,c}(1)|^{1/nQ} = \phi_{Q,r,a}$$

*proof*

In this chapter, we will prove the result under the assumption that:

$$\sum_{d=1}^{Q} c_d = \sum_{d=1+m}^{Q+m} c_d \neq 0$$

The case where $\sum_{d=1}^{Q} c_d = 0$ is much more complicated and will be established in the next chapter. Henceforth, in this chapter, we shall suppose $\sum_{d=1}^{Q} c_d \neq 0$.

Recall that we defined:

$$R_n(x) = Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} H(x_0) F(x)^{nQ}$$

**Lemma 6.2** *For each* $\epsilon > 0$*, there exist an* $\eta > 0$ *and an* $N_0$ *such that for all* $n > N_0$ *and all* $1 + m \leq d \leq Q + m$*:*

$$(1-\epsilon) \sum_{\frac{k}{n}\in[x_0-\eta, x_0+\eta]} R_n\left(\frac{k}{n}\right) \leq \sum_{k=0}^{\infty} R_{n,d,Q}(k) \leq (1+\epsilon) \sum_{\frac{k}{n}\in[x_0-\eta, x_0+\eta]} R_n\left(\frac{k}{n}\right)$$

*proof of lemma*

Because $G(x_0) = 1$ and $H(x_0) > 0$, we can choose $\eta$ small enough so that for each $1 + m \leq d \leq Q + m$:

$$|G(x)^d H(x) - H(x_0)| < \frac{\epsilon}{10} H(x_0)$$

for $x \in [x_0 - \eta, x_0 + \eta]$. In addition $\eta$ is also chosen so that $R + \delta_0 < x_0 - \eta$. Choose $N_2$ such that for all $x = \frac{k}{n} \in [R + \delta, \infty)$ and all $n > N_2$:

$$\left| \exp\left( O_\delta \left( \frac{1}{n} \right) \right) - 1 \right| < \frac{\epsilon}{10}$$

In the manner of the tail estimate already proved in Lemma (5.2), we will proceed by estimating the contribution for the sum over $\frac{k}{n}$ in subintervals of $[R, \infty)$ in terms of $R_n\left(\frac{k}{n}\right)$. We begin on the central interval $[x_0 - \eta, x_0 + \eta]$. If $n > N_2$, then for $x \in [x_0 - \eta, x_0 + \eta]$:

$$\begin{aligned} &\left| G(x)^d H(x) \exp\left( O_\delta \left( \frac{1}{n} \right) \right) - H(x_0) \right| \\ &\quad = \left| \left( \exp\left( O_\delta \left( \frac{1}{n} \right) \right) - 1 \right) G(x)^d H(x) + G(x)^d H(x) - H(x_0) \right| \\ &\quad \leq \left| \exp\left( O_\delta \left( \frac{1}{n} \right) \right) - 1 \right| |G(x)^d H(x)| + |G(x)^d H(x) - H(x_0)| \\ &\quad \leq \frac{\epsilon}{10} \left( 1 + \frac{\epsilon}{10} \right) H(x_0) + \frac{\epsilon}{10} H(x_0) \\ &\quad \leq \frac{3\epsilon}{10} H(x_0) \end{aligned}$$

Thus, for $\frac{k}{n} \in [x_0 - \eta, x_0 + \eta]$, one has:

$$\begin{aligned} &\left| R_{n,d,Q}(k) - R_n\left( \frac{k}{n} \right) \right| \\ &\quad = \left| Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x)^{nQ} \left[ G(x)^d H(x) \exp\left( O_\delta \left( \frac{1}{n} \right) \right) - H(x_0) \right] \right| \\ &\quad \leq Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} F(x)^{nQ} \cdot \frac{3\epsilon}{10} \cdot H(x_0) \\ &\quad = \frac{3\epsilon}{10} R_n\left( \frac{k}{n} \right) \end{aligned}$$

and so:

$$\left| \sum_{\frac{k}{n} \in [x_0-\eta, x_0+\eta]} R_{n,d,Q}\left(k\right) - \sum_{\frac{k}{n} \in [x_0-\eta, x_0+\eta]} R_n\left(\frac{k}{n}\right) \right| \leq \frac{3\epsilon}{10} \sum_{\frac{k}{n} \in [x_0-\eta, x_0+\eta]} R_n\left(\frac{k}{n}\right)$$

We will now estimate the remaining terms, breaking the sum into four intervals: lower $[0, R+\delta]$; lower middle $[R+\delta, x_0-\eta]$; upper middle $[x_0+\eta, x_1]$; and the tail $[x_1, \infty)$. We will consider these intervals in turn estimating the sum in terms of $R_n(x)$ on each. Indeed, we are already in possession of such an estimate for the tail.

For the tail, using Lemma (5.2) we choose $x_1 > \max\{2x_0, x_0+\eta\}$ and $N_1$ such that the estimate:

$$R_n\left(x_0+\eta\right) \geq \sum_{\frac{k}{n} > x_1} R_{n,d,Q}\left(k\right)$$

holds for $n > N_1$.

Now we turn to estimating the contribution on the upper middle range. As noted in the previous chapter there is a $C_\delta > 0$ such that for $d = 1+m, ..., Q+m$ and $x \in [R+\delta, x_1]$:

$$\frac{H(x)G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right)}{H(x_0)} \leq C_\delta$$

where we have adjusted the constant to absorb the denominator. In particular this bound holds on the interval $[x_0+\eta, x_1]$. So for $x = \frac{k}{n} \in [x_0+\eta, x_1]$:

$$\begin{aligned} R_{n,d,Q}(k) &= Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}F(x)^{nQ}G(x)^d H(x)\exp\left(O_\delta\left(\frac{1}{n}\right)\right) \\ &\leq C_\delta \cdot Q^{2rn}\frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}H(x_0)F(x)^{nQ} \\ &= C_\delta \cdot R_n(x) \end{aligned}$$

and hence:

$$R_{n,d,Q}(k) \leq C_\delta \cdot \left(\frac{F(x_0+\eta)}{F\left(x_0+\frac{\eta}{2}\right)}\right)^{nQ} \cdot R_n\left(x_0+\frac{\eta}{2}\right)$$

Since the number of integers $k$ with $\frac{k}{n} \in [x_0+\eta, x_1]$ is at most $(x_1-(x_0+\eta))n+1 \leq C_5 n$ for some constant $C_5$, we have:

$$\sum_{\frac{k}{n}\in[x_0+\eta,x_1]} R_{n,d,Q}(k) \leq C_\delta \cdot R_n\left(x_0+\frac{\eta}{2}\right) \cdot C_5 \cdot n \left(\frac{F(x_0+\eta)}{F\left(x_0+\frac{\eta}{2}\right)}\right)^{nQ}$$

We have:

$$\lim_{n\to\infty} n \left(\frac{F(x_0+\eta)}{F\left(x_0+\frac{\eta}{2}\right)}\right)^{nQ} = 0$$

Thus there is an integer $N_3$ such that:

$$C_\delta \cdot C_5 \cdot n \left(\frac{F(x_0+\eta)}{F\left(x_0+\frac{\eta}{2}\right)}\right)^{nQ} \leq 1$$

for $n > N_3$ and hence, for these $n$:

$$\sum_{\frac{k}{n}\in[x_0+\eta,x_1]} R_{n,d,Q}(k) \leq R_n\left(x_0+\frac{\eta}{2}\right)$$

We now deal with lower middle range. As in the previous case, on the interval $[R+\delta, x_0-\eta]$, the functions:

$$\frac{G(x)^d H(x) \exp\left(O_\delta\left(\frac{1}{n}\right)\right)}{H(x_0)}$$

are uniformly bounded for $1+m \leq d \leq Q+m$ by $C_\delta$. So for $\frac{k}{n} \in [R+\delta, x_0-\eta]$:

$$\begin{aligned} R_{n,d,Q}(k) &\leq C_\delta \cdot R_n(x) \\ &= C_\delta \cdot Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} H(x_0) F(x)^{nQ} \end{aligned}$$

Again, because $F(x)$ is monotone increasing on $[R+\delta, x_0-\eta]$:

$$R_{n,d,Q}(k) \leq C_\delta \cdot \left(\frac{F(x_0-\eta)}{F\left(x_0-\frac{\eta}{2}\right)}\right)^{nQ} \cdot R_n\left(x_0-\frac{\eta}{2}\right)$$

As before:

$$\lim_{n\to\infty} n \left(\frac{F(x_0-\eta)}{F\left(x_0-\frac{\eta}{2}\right)}\right)^{nQ} = 0$$

and the number of integers $k$ such that $\frac{k}{n} \in [R+\delta, x_0-\eta]$ is at most $C_6 n$ for some constant $C_6$. Therefore, there is an $N_4$ such that:

$$\sum_{\frac{k}{n}\in[R+\delta,x_0-\eta]} R_{n,d,Q}(k) \leq R_n\left(x_0-\frac{\eta}{2}\right)$$

for $n > N_4$.

Finally, on the lower interval $0 \leq \frac{k}{n} < R+\delta$, we can be assured by the choice of $\delta$ that $R_{n,d,Q}(k)$ is nondecreasing by Proposition (3.2). If $k_1$ is the least integer with the property that $R+\delta_0 > \frac{k_1}{n} > R+\delta$ then for $k < k_1$:

$$\begin{aligned} R_{n,d,Q}(k) &\leq R_{n,d,Q}(k_1) \\ &\leq C_\delta \cdot R_n(x) \\ &\leq C_\delta \cdot \left(\frac{F(R+\delta_0)}{F(x_0-\eta)}\right)^{nQ} \cdot R_n(x_0-\eta) \end{aligned}$$

by the estimates on $[R+\delta, x_0-\eta]$ and the fact that $R_n(x)$ is increasing on this range. Again, there are at most $C_7 n$ integers $k$ with $0 \leq \frac{k}{n} < R+\delta$ for some constant $C_7$. Since:

$$\frac{F(R+\delta_0)}{F(x_0-\eta)} < 1$$

there is an $N_5$ such that for $n > N_5$ we have the estimate:

$$\sum_{0\leq\frac{k}{n}\leq R+\delta} R_{n,d,Q}(k) \leq R_n(x_0-\eta)$$

Now we are able to complete the proof of the lemma. Combining all of the estimates above:

$$\begin{aligned} \sum_{\frac{k}{n}\notin[x_0-\eta,x_0+\eta]} R_{n,d,Q}(k) &\leq R_n(x_0-\eta)+R_n\left(x_0-\frac{\eta}{2}\right) \\ &\quad +R_n\left(x_0+\frac{\eta}{2}\right)+R_n(x_0+\eta) \end{aligned}$$

if $n \geq \max\{N_1, N_2, N_3, N_4, N_5\}$. Now let $N_6$ be large enough so that for $n > N_6$:

$$\frac{4}{n\eta-1} < \frac{\epsilon}{2}$$

Since $F(x)$ and, consequently, $R_n(x)$ are monotone increasing for $R+\delta \le x < x_0$ and monotone decreasing for $x > x_0$, it follows that for $n > N_0 = \max\{N_1, ..., N_6\}$:

$$\begin{aligned}
\sum_{\frac{k}{n}\notin[x_0-\eta,x_0+\eta]} R_{n,d,Q}\left(k\right) &\le 4\max\left\{R_n\left(x_0-\frac{\eta}{2}\right), R_n\left(x_0+\frac{\eta}{2}\right)\right\} \\
&\le 4\frac{1}{n\eta-1}\sum_{\frac{k}{n}\in\left[x_0-\frac{\eta}{2},x_0+\frac{\eta}{2}\right]} R_n\left(\frac{k}{n}\right) \\
&\le \frac{\epsilon}{2}\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right)
\end{aligned}$$

Using our estimate on $[x_0-\eta, x_0+\eta]$ and the fact that $\frac{3}{10}\epsilon + \frac{\epsilon}{2} < \epsilon$, we have:

$$\left|\sum_{k=0}^{\infty} R_{n,d,Q}\left(k\right) - \sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right)\right|$$

$$\begin{aligned}
&\le \left|\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_{n,d,Q}\left(k\right) - \sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right)\right| + \left|\sum_{\frac{k}{n}\notin[x_0-\eta,x_0+\eta]} R_{n,d,Q}\left(k\right)\right| \\
&\le \frac{3}{10}\epsilon\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right) + \frac{\epsilon}{2}\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right) \\
&< \epsilon\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right)
\end{aligned}$$

Hence:

$$(1-\epsilon)\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right) \le \sum_{k=0}^{\infty} R_{n,d,Q}(k) \le (1+\epsilon)\sum_{\frac{k}{n}\in[x_0-\eta,x_0+\eta]} R_n\left(\frac{k}{n}\right)$$

completing the proof of the lemma.

Now we are in a position to prove the lower bound that will establish the main result of this chapter. First we observe that the function:

$$B(\vec{x}) = \sum_{d=1+m}^{Q+m} c_d(1-x_d)$$

is a nonconstant continuous function of $\vec{x} = (x_{1+m}, ..., x_{Q+m})$ such that:

$$B(0) = \sum_{d=1+m}^{Q+m} c_d = \tau(c) \ne 0$$

by hypothesis. By continuity, there is an $\epsilon$ such that for all $\vec{x} = (x_{1+m}, ..., x_{Q+m})$ with $|x_d| < \epsilon$ $(d = 1+m, ..., Q+m)$:

$$|B(\vec{x})| > \left|\frac{\tau(c)}{2}\right|$$

With this $\epsilon$, by Lemma (6.2) there is an $\eta$ such that for $d = 1+m, ...Q+m$ and sufficiently large $n$:

$$1 - \epsilon \leq \frac{\sum_{k=0}^{\infty} R_{n,d,Q}(k)}{\sum_{\frac{k}{n} \in [x_0 - \eta, x_0 + \eta]} R_n\left(\frac{k}{n}\right)} \leq 1 + \epsilon$$

Hence, for all sufficiently large $n$:

$$|S_{n,c}(1)| = \left|\sum_{d=1+m}^{Q+m} c_d \sum_{k=0}^{\infty} R_{n,d,Q}(k)\right| \geq \left|\frac{\tau(c)}{2}\right| \sum_{\frac{k}{n} \in [x_0 - \eta, x_0 + \eta]} R_n\left(\frac{k}{n}\right)$$

Now define:

$$c_\eta = \max_{x \in [x_0 - \eta, x_0 + \eta]} |F(x) - F(x_0)|$$

Then for $x \in [x_0 - \eta, x_0 + \eta]$:

$$\left(1 - \frac{c_\eta}{F(x_0)}\right) F(x_0) \leq F(x)$$

Therefore:

$$\begin{aligned}
&\lim_{n\to\infty} \left( \sum_{\frac{k}{n} \in [x_0 - \eta, x_0 + \eta]} R_n\left(\frac{k}{n}\right) \right)^{1/nQ} \\
&\geq \lim_{n\to\infty} \left[ 2n\eta Q^{2rn} \left( \left(1 - \frac{c_\eta}{F(x_0)}\right) F(x_0) \right)^{nQ} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}} H(x_0) \right]^{1/nQ} \\
&= Q^{2R} F(x_0) \left(1 - \frac{c_\eta}{F(x_0)}\right) \\
&= \phi_{Q,r,a} \left(1 - \frac{c_\eta}{F(x_0)}\right)
\end{aligned}$$

The lower bound we seek follows once we observe that, subject only to the requirement that it be sufficiently small, $\eta$ was arbitrary and $c_\eta$ tends to 0 with $\eta$. Combined with the upper bound of Proposition (5.1) we have:

$$\lim_{n\to\infty} |S_{n,c}(1)|^{1/nQ} = \phi_{Q,r,a}$$

# Chapter 7

## Second Case of the Lower Bound

In this chapter, we shall establish the lower bound under the assumption that:

$$\sum_{d=1}^{Q} c_d = \sum_{d=1+m}^{Q+m} c_d = 0$$

We develop $f(x)$ in a Taylor series about its critical point $x_0$:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + O(x-x_0)^3 \\ &= f(x_0) - \tau(x-x_0)^2 + O(x-x_0)^3 \end{aligned}$$

since $f'(x_0) = 0$ and $\tau = -\frac{f''(x_0)}{2!} = -Qg_1(x_0) > 0$ by Proposition (4.4). Hence:

$$\begin{aligned} F(x)^{nQ} &= \exp\left(f(x)\right)^{nQ} \\ &= F(x_0)^{nQ}\left(\exp\left(-\tau(x-x_0)^2 + O(x-x_0)^3\right)\right)^{nQ} \end{aligned}$$

We also have:

$$\begin{aligned} g(x) &= g(x_0) + g'(x_0)(x-x_0) + \frac{g''(x_0)}{2}(x-x_0)^2 + O(x-x_0)^3 \\ &= -2\tau(x-x_0) + \tau'(x-x_0)^2 + O(x-x_0)^3 \end{aligned}$$

since $g(x_0) = 0$, $g'(x_0) = 2Qg_1(x_0) = -2\tau$, and $\tau' = \frac{g''(x_0)}{2} = \frac{f'''(x_0)}{2}$. So:

$$\begin{aligned} G(x)^d &= \exp\left(dg(x)\right) \\ &= \exp\left(-2d\tau(x-x_0)\right)\exp\left(d\tau'(x-x_0)^2\right)\exp\left(O(x-x_0)^3\right) \\ &= \left(1 - 2d\tau(x-x_0) + \frac{4d^2\tau^2}{2}(x-x_0)^2 + O(x-x_0)^3\right) \\ &\quad \cdot \left(1 + d\tau'(x-x_0)^2 + O(x-x_0)^4\right)\cdot\left(1 + O(x-x_0)^3\right) \\ &= 1 - 2d\tau(x-x_0) + (2d^2\tau^2 + d\tau')(x-x_0)^2 + O(x-x_0)^3 \end{aligned}$$

Finally we write:

$$H(x) = H_0 + H_1(x - x_0) + H_2(x - x_0)^2 + O(x - x_0)^3$$

and we'll use the same notational convention for $h_1(x)$, $h_2(x)$, and $g_1(x)$. For example:

$$h_1(x) = h_{1,0} + h_{1,1}(x - x_0) + h_{1,2}(x - x_0)^2 + O(x - x_0)^3$$

Note also that $g_{1,0} = g_1(x_0) = \frac{f''(x_0)}{2Q} = -\frac{\tau}{Q}$. Lastly, we abbreviate:

$$C_n = Q^{2rn} \frac{(\sqrt{2\pi}\,)^{a-2r}}{(\sqrt{n}\,)^{a+2r}}$$

We begin by expanding the factor $\exp\left(O_\delta\left(\frac{1}{n}\right)\right)$ in the asymptotic product of $R_{n,d,Q}(k)$. We have:

$$\begin{aligned}
R_{n,d,Q}(k) &= C_n F(x)^{nQ} H(x) G(x)^d \exp\left(\frac{d^2}{n} g_1(x)\right) \exp\left(\frac{1}{n} h_1(x)\right) \\
&\quad \cdot \exp\left(\frac{d}{n} h_2(x)\right) \exp\left(O_\delta\left(\frac{1}{n^2}\right)\right) \\
&= C_n F(x)^{nQ} H(x) G(x)^d \\
&\quad \cdot \left(1 + \frac{d^2}{n} g_1(x) + \frac{1}{n} h_1(x) + \frac{d}{n} h_2(x) + O_\delta\left(\frac{1}{n^2}\right)\right) \\
&= C_n F(x)^{nQ} H(x) G(x)^d + \frac{d^2}{n} C_n F(x)^{nQ} H(x) G(x)^d g_1(x) \\
&\quad + \frac{1}{n} C_n F(x)^{nQ} H(x) G(x)^d h_1(x) + \frac{d}{n} C_n F(x)^{nQ} H(x) G(x)^d h_2(x) \\
&\quad + O_\delta\left(\frac{1}{n^2}\right) C_n F(x)^{nQ} H(x) G(x)^d
\end{aligned}$$

We shall expand further by substituting the Taylor series computed above for $F(x)$, $G(x)^d$, and $H(x)$. We then use the resulting formula to estimate $|S_{n,c}(1)|$ under the hypothesis that the components of the vector $c \in \mathbb{C}^Q$ sum to zero. Eventually we shall show that the main term is nonzero and is of exact order $C_n F(x_0)^{nQ} O\left(\frac{1}{\sqrt{n}}\right)$. So accordingly, before we proceed, we shall prove several lemmas showing that in the infinite series defining $S_{n,c}(1)$ the terms indexed outside of a small interval (specified below) contribute a negligible error. The first such result follows immediately from a formula proved in the previous chapter:

**Lemma 7.1** *If $\eta > 0$ is sufficiently small then there is a $0 < \lambda < 1$ such that for $n$ sufficiently large:*

$$\sum_{d=1+m}^{Q+m} c_d \sum_{\frac{k}{n} \notin [x_0-\eta, x_0+\eta]} R_{n,d,Q}(k) = C_n F(x_0)^{nQ} \cdot O\left(\lambda^{nQ}\right)$$

*proof*

From chapter six, for $\eta$ sufficiently small there is an $N_0$ such that for $n > N_0$:

$$\sum_{\frac{k}{n} \notin [x_0-\eta, x_0+\eta]} R_{n,d,Q}(k) \leq 4 \max \left\{ R_n\left(x_0 - \frac{\eta}{2}\right), R_n\left(x_0 + \frac{\eta}{2}\right) \right\}$$

But:

$$\begin{aligned} R_n\left(x_0 - \frac{\eta}{2}\right) &= C_n F(x_0)^{nQ} \left( \frac{F\left(x_0 - \frac{\eta}{2}\right)}{F(x_0)} \right)^{nQ} H(x_0) \\ &= C_n F(x_0)^{nQ} \lambda_1^{nQ} \cdot O(1) \end{aligned}$$

with $0 < \lambda_1 < 0$ and, similarly:

$$R_n\left(x_0 + \frac{\eta}{2}\right) = C_n F(x_0)^{nQ} \lambda_2^{nQ} \cdot O(1)$$

where $0 < \lambda_2 < 0$. Letting $\lambda = \max\{\lambda_1, \lambda_2\}$ then we have for all large $n$:

$$\sum_{\frac{k}{n} \notin [x_0-\eta, x_0+\eta]} R_{n,d,Q}(k) = C_n F(x_0)^{nQ} \lambda^{nQ} \cdot O_d(1)$$

Multiplying by $c_d$ and summing over $d$ from $1+m$ to $Q+m$, the lemma follows.

However, we can cut down to an even smaller interval. The typical term in the expansion will have the form:

$$C_n F(x_0)^{nQ} \exp\left(-nQ\tau \left(x - x_0\right)^2\right) \left(x - x_0\right)^m$$

The next two lemmas assure us that not only can we restrict the sum to a smaller interval with negligible error but that we need only make use of at most the first four terms of any Taylor series we use.

**Lemma 7.2** *There exists a constant $C > 0$ such that for $\eta > 0$ chosen sufficiently small:*

$$\sum_{\substack{\left|\frac{k}{n}-x_0\right|>C\sqrt{\frac{\log n}{n}} \\ \left|\frac{k}{n}-x_0\right|\leq\eta}} R_{n,d,Q}\left(k\right) = C_n F(x_0)^{nQ} \cdot O\left(\frac{1}{n}\right)$$

*proof*

Recall that:

$$R_{n,d,Q}(k) = C_n F(x)^{nQ} H(x) G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right)$$

and:

$$\begin{aligned} F(x)^{nQ} &= F(x_0)^{nQ}\left(\exp\left(-\tau(x-x_0)^2 + O(x-x_0)^3\right)\right)^{nQ} \\ &= F(x_0)^{nQ}\left(\exp\left(-\tau(x-x_0)^2\left(1-\frac{1}{\tau}O(x-x_0)\right)\right)\right)^{nQ} \\ &= F(x_0)^{nQ}\left(\exp\left(-nQ\tau(x-x_0)^2\right)\right)^{\left(1-\frac{1}{\tau}O(x-x_0)\right)} \end{aligned}$$

If $|x-x_0| \geq \sqrt{\frac{3}{Q\sigma}}\sqrt{\frac{\log n}{n}}$ for arbitrary $\sigma > 0$ , then:

$$-nQ\sigma(x-x_0)^2 \leq -3\log n \quad \text{or} \quad \exp\left(-nQ\sigma(x-x_0)^2\right) \leq \frac{1}{n^3}$$

and so we take $C = \sqrt{\frac{3}{Q\tau}}$ and $\sigma = \tau$. Then we choose $\eta > 0$ small enough so that for $\eta \geq |x-x_0| \geq C\sqrt{\frac{\log n}{n}}$:

$$1-\frac{1}{\tau}O(x-x_0) > 0 \quad \text{and} \quad 3\left|\frac{1}{\tau}O(x-x_0)\right| \leq 1$$

Then if $\eta \geq |x-x_0| \geq C\sqrt{\frac{\log n}{n}}$:

$$\begin{aligned} \left(\exp\left(-nQ\tau(x-x_0)^2\right)\right)^{\left(1-\frac{1}{\tau}O(x-x_0)\right)} &\leq \left(\frac{1}{n^3}\right)^{\left(1-\frac{1}{\tau}O(x-x_0)\right)} \\ &= \frac{1}{n^3}\cdot\left(\frac{1}{n^3}\right)^{-\frac{1}{\tau}O(x-x_0)} \\ &\leq \frac{1}{n^3}\cdot n^{3\left|\frac{1}{\tau}O(x-x_0)\right|} \\ &\leq \frac{1}{n^2} \end{aligned}$$

As we have seen $H(x)G(x)^d \exp\left(O_\delta\left(\frac{1}{n}\right)\right)$ is uniformly bounded by a constant $C_\delta > 0$ on $[x_0 - \eta, x_0 + \eta]$. Thus:

$$\begin{aligned}\sum_{\substack{\left|\frac{k}{n}-x_0\right|>C\sqrt{\frac{\log n}{n}}\\ \left|\frac{k}{n}-x_0\right|\le\eta}} R_{n,d,Q}\left(k\right) &\le C_nF(x_0)^{nQ}\cdot C_\delta\cdot(2n\eta)\cdot\frac{1}{n^2}\\ &= C_nF(x_0)^{nQ}\cdot O\left(\frac{1}{n}\right)\end{aligned}$$

**Lemma 7.3** *If $m \ge 0$ then with $C$ as in the previous lemma:*

$$\sum_{\left|\frac{k}{n}-x_0\right|\le C\sqrt{\frac{\log n}{n}}} C_nF(x_0)^{nQ}\exp\left(-nQ\tau\left(x-x_0\right)^2\right)\left(x-x_0\right)^m$$

$$\le C_nF(x_0)^{nQ}\left(3C^{m+1}\frac{(\log n)^{\frac{m+1}{2}}}{n^{\frac{m-1}{2}}}\right)$$

**Corollary 7.4** *If $m \ge 4$:*

$$\sum_{\left|\frac{k}{n}-x_0\right|\le C\sqrt{\frac{\log n}{n}}} C_nF(x_0)^{nQ}\exp\left(-nQ\tau\left(x-x_0\right)^2\right)\left(x-x_0\right)^m = C_nF(x_0)^{nQ}\cdot O\left(\frac{1}{n}\right)$$

*proof*

We have that:

$$\sum_{\left|\frac{k}{n}-x_0\right|\le C\sqrt{\frac{\log n}{n}}} C_nF(x_0)^{nQ}\exp\left(-nQ\tau(x-x_0)^2\right)\left(x-x_0\right)^m$$

$$\begin{aligned}&\le \sum_{\left|\frac{k}{n}-x_0\right|\le C\sqrt{\frac{\log n}{n}}} C_nF(x_0)^{nQ}\left(C\sqrt{\frac{\log n}{n}}\right)^m\\ &\le \left(2C\sqrt{\frac{\log n}{n}}\cdot n+1\right)C_nF(x_0)^{nQ}\left(C\sqrt{\frac{\log n}{n}}\right)^m\\ &\le C_nF(x_0)^{nQ}\left(3C^{m+1}\frac{(\log n)^{\frac{m+1}{2}}}{n^{\frac{m-1}{2}}}\right)\end{aligned}$$

as asserted in the lemma. The corollary follows immediately since if $m \geq 4$:

$$\begin{aligned} 3C^{m+1}\frac{(\log n)^{\frac{m+1}{2}}}{n^{\frac{m-1}{2}}} &= 3\left(\frac{C^2\log n}{n}\right)^{\frac{m-4}{2}}\sqrt{\frac{(C^2\log n)^5}{n}}\cdot\frac{1}{n} \\ &= O\left(\frac{1}{n}\right) \end{aligned}$$

Next we turn to evaluating the terms:

$$\sum_{\left|\frac{k}{n}-x_0\right|\leq C\sqrt{\frac{\log n}{n}}} C_n F(x_0)^{nQ}\exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)^m$$

for $m = 0, 1, 2,$ and 3. Define:

$$A_m = \int_{-\infty}^{\infty} |t|^m \exp\left(-Q\tau t^2\right) dt$$

In particular, we have the standard integrals:

**Lemma 7.5** *With $A_m$ defined as above:*

$$\begin{aligned} A_0 &= \sqrt{\frac{\pi}{Q\tau}} \\ A_2 &= \frac{1}{2Q\tau}\sqrt{\frac{\pi}{Q\tau}} = \frac{1}{2Q\tau}A_0 \end{aligned}$$

Also, with $C$ as in Lemma (7.2), we see:

$$\lim_{n\to\infty}\frac{1}{n^{\frac{m+1}{2}}}\sum_{|l|\leq C\sqrt{n\log n}}|l|^m\exp\left(-Q\tau\left(\frac{l^2}{n}\right)\right)$$

$$\begin{aligned} &= \lim_{n\to\infty}\sum_{\left|\frac{l}{\sqrt{n}}\right|\leq C\sqrt{\log n}}\left|\frac{l}{\sqrt{n}}\right|^m\exp\left(-Q\tau\left(\frac{l}{\sqrt{n}}\right)^2\right)\cdot\frac{1}{\sqrt{n}} \\ &= \int_{-\infty}^{\infty}|t|^m\exp\left(-Q\tau t^2\right)dt \\ &= A_m \end{aligned}$$

So let us now consider the sum:

$$\sum_{\left|\frac{k}{n}-x_0\right|\leq C\sqrt{\frac{\log n}{n}}}\exp\left(-nQ\tau\left(x-x_0\right)^2\right)$$

For each $n$, Let $k_n$ be an integer such that $\left|\frac{k_n}{n} - x_0\right|$ is minimal. Let $\frac{k_n}{n} - x_0 = \frac{\phi_n}{n}$ or, equivalently, $k_n - nx_0 = \phi_n$. Also, write $k = k_n + l$ so that:

$$\frac{k}{n} - x_0 = \frac{k_n}{n} + \frac{l}{n} - x_0 = \frac{l}{n} + \left(\frac{k_n}{n} - x_0\right) = \frac{l}{n} + \frac{\phi_n}{n}$$

Then:

$$\sum_{\left|\frac{k}{n} - x_0\right| \leq C\sqrt{\frac{\log n}{n}}} \exp\left(-nQ\tau\left(x - x_0\right)^2\right) \approx \sum_{|l| \leq C\sqrt{n \log n}} \exp\left(-nQ\tau\left(\frac{l}{n} + \frac{\phi_n}{n}\right)^2\right)$$

where the error results from the loss of at most two terms (since $|\phi_n| \leq 1$) which, because of the choice of C, are of order $O\left(\frac{1}{n^3}\right)$. In the sequel, we will suppress this error and treat the above as a strict equality. Continuing our analysis:

$$\begin{aligned}
&\exp\left(-nQ\tau\left(\frac{l}{n} + \frac{\phi_n}{n}\right)^2\right) \\
&= \exp\left(-Q\tau\frac{l^2}{n} - 2Q\tau\frac{l}{n}\phi_n - Q\tau\frac{\phi_n^2}{n}\right) \\
&= \exp\left(-Q\tau\frac{l^2}{n}\right)\left(1 - 2Q\tau\frac{l}{n}\phi_n + O\left(\frac{l^2}{n^2}\right)\right)\left(1 + O\left(\frac{1}{n}\right)\right) \\
&= \exp\left(-Q\tau\frac{l^2}{n}\right)\left(1 - 2Q\tau\frac{l}{n}\phi_n + O\left(\frac{l^2}{n^2}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{l}{n^2}\right) + O\left(\frac{l^2}{n^3}\right)\right)
\end{aligned}$$

So in the case where $m = 0$:

$$\begin{aligned}
&\sum_{|l| \leq C\sqrt{n \log n}} \exp\left(-nQ\tau\left(\frac{l}{n} + \frac{\phi_n}{n}\right)^2\right) \\
&= \sum_{|l| \leq C\sqrt{n \log n}} \exp\left(-Q\tau\frac{l^2}{n}\right) - 2Q\tau\frac{\phi_n}{n} \sum_{|l| \leq C\sqrt{n \log n}} l \exp\left(-Q\tau\frac{l^2}{n}\right) \\
&\quad + O\left(\frac{1}{n^2}\right) \sum_{|l| \leq C\sqrt{n \log n}} |l|^2 \exp\left(-Q\tau\frac{l^2}{n}\right) \\
&\quad + O\left(\frac{1}{n}\right) \sum_{|l| \leq C\sqrt{n \log n}} \exp\left(-Q\tau\frac{l^2}{n}\right)
\end{aligned}$$

$$
\begin{aligned}
& +O\left(\frac{1}{n^2}\right) \sum_{|l|\le C\sqrt{n\log n}} |l| \exp\left(-Q\tau\frac{l^2}{n}\right) \\
& +O\left(\frac{1}{n^3}\right) \sum_{|l|\le C\sqrt{n\log n}} |l|^2 \exp\left(-Q\tau\frac{l^2}{n}\right) \\
= \; & \sqrt{n}\cdot(A_0+\epsilon_n^{(0)}) + O\left(\frac{1}{\sqrt{n}}\right)\cdot(A_2+\epsilon_n^{(2)}) + O\left(\frac{1}{\sqrt{n}}\right)\cdot(A_0+\epsilon_n^{(0)}) \\
& +O\left(\frac{1}{n}\right)\cdot(A_1+\epsilon_n^{(1)}) + O\left(\frac{1}{n^{3/2}}\right)\cdot(A_2+\epsilon_n^{(2)}) \\
= \; & \sqrt{n}\cdot(A_0+\epsilon_n^{(0)}) + O\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

where $\epsilon_n^{(i)} = o(1)$ for $i = 0, 1, 2,$ and 3. Next we consider the case $m = 1$:

$$
\sum_{\left|\frac{k}{n}-x_0\right|\le C\sqrt{\frac{\log n}{n}}} \exp\left(-nQ\tau\left(x-x_0\right)^2\right)(x-x_0)
$$

$$
= \sum_{|l|\le C\sqrt{n\log n}} \exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)
$$

But:

$$
\exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)
$$

$$
\begin{aligned}
= \; & \exp\left(-Q\tau\frac{l^2}{n} - 2Q\tau\frac{l}{n}\phi_n - Q\tau\frac{\phi_n^2}{n}\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right) \\
= \; & \exp\left(-Q\tau\frac{l^2}{n}\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)\left(1-2Q\tau\frac{l}{n}\phi_n + O\left(\frac{l^2}{n^2}\right)\right)\left(1+O\left(\frac{1}{n}\right)\right) \\
= \; & \exp\left(-Q\tau\frac{l^2}{n}\right)\left(\frac{l}{n}+\frac{\phi_n}{n} - 2Q\tau\frac{l^2}{n^2}\phi_n + O\left(\frac{l^3}{n^3}\right) + O\left(\frac{l}{n^2}\right) + O\left(\frac{1}{n^2}\right)\right)
\end{aligned}
$$

So:

$$
\sum_{|l|\le C\sqrt{n\log n}} \exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)
$$

$$
\begin{aligned}
= \; & \frac{1}{n}\sum_{|l|\le C\sqrt{n\log n}} l\exp\left(-Q\tau\frac{l^2}{n}\right) + \frac{\phi_n}{n}\sum_{|l|\le C\sqrt{n\log n}} \exp\left(-Q\tau\frac{l^2}{n}\right) \\
& -2Q\tau\phi_n\frac{1}{n^2}\sum_{|l|\le C\sqrt{n\log n}} l^2\exp\left(-Q\tau\frac{l^2}{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +O\left(\frac{1}{n^3}\right) \sum_{|l|\leq C\sqrt{n\log n}} |l|^3 \exp\left(-Q\tau\frac{l^2}{n}\right) \\
& +O\left(\frac{1}{n^2}\right) \sum_{|l|\leq C\sqrt{n\log n}} |l| \exp\left(-Q\tau\frac{l^2}{n}\right) \\
& +O\left(\frac{1}{n^2}\right) \sum_{|l|\leq C\sqrt{n\log n}} \exp\left(-Q\tau\frac{l^2}{n}\right) \\
= \; & \phi_n \cdot \frac{1}{\sqrt{n}} \cdot (A_0 + \epsilon_n^{(0)}) - 2Q\tau\phi_n \cdot \frac{1}{\sqrt{n}} \cdot (A_2 + \epsilon_n^{(2)}) + A_3 O\left(\frac{1}{n}\right) \\
& +A_1 O\left(\frac{1}{n}\right) + A_0 O\left(\frac{1}{n^{3/2}}\right) \\
= \; & \phi_n \left(A_0 + \epsilon_n^{(0)} - 2Q\tau(A_2 + \epsilon_n^{(2)})\right) \frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right) \\
= \; & \frac{o(1)}{\sqrt{n}} + O\left(\frac{1}{n}\right)
\end{aligned}
$$

since $A_0 - 2Q\tau A_2 = 0$ and $|\phi_n| \leq 1$. Similarly in the case $m = 2$:

$$
\sum_{\left|\frac{k}{n}-x_0\right|\leq C\sqrt{\frac{\log n}{n}}} \exp\left(-nQ\tau\left(x-x_0\right)^2\right)\left(x-x_0\right)^2
$$

$$
= \sum_{|l|\leq C\sqrt{n\log n}} \exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2
$$

with:

$$
\exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2
$$

$$
\begin{aligned}
= \; & \exp\left(-Q\tau\frac{l^2}{n} - 2Q\tau\frac{l}{n}\phi_n - Q\tau\frac{\phi_n^2}{n}\right)\left(\frac{l^2}{n^2} + \frac{2l}{n^2}\cdot\phi_n + \frac{\phi_n^2}{n^2}\right) \\
= \; & \exp\left(-Q\tau\frac{l^2}{n}\right)\left(1 + O\left(\frac{l}{n}\right)\right)\left(1 + O\left(\frac{1}{n}\right)\right) \\
& \cdot\left(\frac{l^2}{n^2} + \frac{2l}{n^2}\cdot\phi_n + \frac{\phi_n^2}{n^2}\right) \\
= \; & \exp\left(-Q\tau\frac{l^2}{n}\right)\left(\frac{l^2}{n^2} + O\left(\frac{l}{n^2}\right) + O\left(\frac{l^3}{n^3}\right) + O\left(\frac{l^2}{n^3}\right)\right)
\end{aligned}
$$

So:

$$\sum_{|l|\leq C\sqrt{n\log n}} \exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2$$

$$\begin{aligned}
&= \frac{1}{n^2}\sum_{|l|\leq C\sqrt{n\log n}} l^2\exp\left(-Q\tau\frac{l^2}{n}\right) \\
&\quad +O\left(\frac{1}{n^2}\right)\sum_{|l|\leq C\sqrt{n\log n}} |l|\exp\left(-Q\tau\frac{l^2}{n}\right) \\
&\quad +O\left(\frac{1}{n^3}\right)\sum_{|l|\leq C\sqrt{n\log n}} |l|^3\exp\left(-Q\tau\frac{l^2}{n}\right) \\
&\quad +O\left(\frac{1}{n^3}\right)\sum_{|l|\leq C\sqrt{n\log n}} l^2\exp\left(-Q\tau\frac{l^2}{n}\right) \\
&= \frac{1}{\sqrt{n}}(A_2+\epsilon_n^{(2)})+A(1)O\left(\frac{1}{n}\right)+A(3)O\left(\frac{1}{n}\right)+A(2)O\left(\frac{1}{n^{3/2}}\right) \\
&= \frac{1}{\sqrt{n}}(A_2+\epsilon_n^{(2)})+O\left(\frac{1}{n}\right)
\end{aligned}$$

Lastly if $m=3$:

$$\sum_{\left|\frac{k}{n}-x_0\right|\leq C\sqrt{\frac{\log n}{n}}} \exp\left(-nQ\tau\left(x-x_0\right)^2\right)\left(x-x_0\right)^3$$

$$= \sum_{|l|\leq C\sqrt{n\log n}} \exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^3$$

with:

$$\exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^3$$

$$\begin{aligned}
&= \exp\left(-Q\tau\frac{l^2}{n}-2Q\tau\frac{l}{n}\phi_n-Q\tau\frac{\phi_n^2}{n}\right)\left(\frac{l^3}{n^3}+\frac{3l^2}{n^3}\cdot\phi_n+\frac{3l}{n^3}\cdot\phi_n^2+\frac{\phi_n^3}{n^3}\right) \\
&= \exp\left(-Q\tau\frac{l^2}{n}\right)\left(1+O\left(\frac{l}{n}\right)\right)\left(1+O\left(\frac{1}{n}\right)\right) \\
&\quad \cdot\left(\frac{l^3}{n^3}+\frac{3l^2}{n^3}\cdot\phi_n+\frac{3l}{n^3}\cdot\phi_n^2+\frac{\phi_n^3}{n^3}\right) \\
&= \exp\left(-Q\tau\frac{l^2}{n}\right)\left(O\left(\frac{l^3}{n^3}\right)+O\left(\frac{l^4}{n^4}\right)\right)
\end{aligned}$$

So:

$$\sum_{|l|\le C\sqrt{n\log n}} \exp\left(-nQ\tau\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^2\right)\left(\frac{l}{n}+\frac{\phi_n}{n}\right)^3 = O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^{3/2}}\right) = O\left(\frac{1}{n}\right)$$

We are finally able to embark upon the main computation of this chapter. First we observe:

$$\begin{aligned} F(x)^{nQ} &= F(x_0)^{nQ}\cdot\exp\left(-nQ\tau(x-x_0)^2\right)\cdot\exp\left(nO(x-x_0)^3\right) \\ &= F(x_0)^{nQ}\cdot\exp\left(-nQ\tau(x-x_0)^2\right)\cdot\left(1+O\left(\frac{(\log n)^{3/2}}{\sqrt{n}}\right)\right) \\ &= F(x_0)^{nQ}\cdot\exp\left(-nQ\tau(x-x_0)^2\right)\cdot(1+o(1)) \end{aligned}$$

if $|x-x_0| = O\left(\sqrt{\frac{\log n}{n}}\right)$ since in this case:

$$\exp\left(nO(x-x_0)^3\right) = \exp\left(O\left(\frac{(\log n)^{3/2}}{\sqrt{n}}\right)\right) = 1+O\left(\frac{(\log n)^{3/2}}{\sqrt{n}}\right)$$

Moreover, it should be noted that the implied constant in the big $O$ does not depend upon $d$. So returning to $R_{n,d,Q}(k)$ and substituting in the Taylor series computed in the beginning of the chapter we obtain:

$$\begin{aligned} R_{n,d,Q}(k) &= (1+o(1))\left[C_nF(x_0)^{nQ}\cdot\exp\left(-nQ\tau(x-x_0)^2\right)\right. \\ &\cdot\left(H_0+H_1(x-x_0)+H_2(x-x_0)^2+O(x-x_0)^3\right) \\ &\cdot\left(1-2d\tau(x-x_0)+(2d^2\tau^2+d\tau')(x-x_0)^2+O(x-x_0)^3\right) \\ &+\frac{d^2}{n}C_nF(x_0)^{nQ}\cdot\exp\left(-nQ\tau(x-x_0)^2\right) \\ &\cdot\left(H_0+H_1(x-x_0)+H_2(x-x_0)^2+O(x-x_0)^3\right) \\ &\cdot\left(1-2d\tau(x-x_0)+(2d^2\tau^2+d\tau')(x-x_0)^2+O(x-x_0)^3\right) \\ &\cdot\left(g_{1,0}+g_{1,1}(x-x_0)+O(x-x_0)^2\right) \\ &+\frac{1}{n}C_nF(x_0)^{nQ}\cdot\exp\left(-nQ\tau(x-x_0)^2\right) \\ &\cdot\left(H_0+H_1(x-x_0)+H_2(x-x_0)^2+O(x-x_0)^3\right) \end{aligned}$$

$$
\begin{aligned}
&\cdot \left(1 - 2d\tau(x-x_0) + (2d^2\tau^2 + d\tau')(x-x_0)^2 + O(x-x_0)^3\right) \\
&\cdot \left(h_{1,0} + h_{1,1}(x-x_0) + O(x-x_0)^2\right) \\
&+\frac{d}{n} C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) \\
&\cdot \left(H_0 + H_1(x-x_0) + H_2(x-x_0)^2 + O(x-x_0)^3\right) \\
&\cdot \left(1 - 2d\tau(x-x_0) + (2d^2\tau^2 + d\tau')(x-x_0)^2 + O(x-x_0)^3\right) \\
&\cdot \left(h_{2,0} + h_{2,1}(x-x_0) + O(x-x_0)^2\right) \\
&+O_\delta\left(\frac{1}{n^2}\right) C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) H(x)G(x)^d \Big]
\end{aligned}
$$

By multiplying the series, using $g_{1,0} = -\frac{\tau}{Q}$, and collecting terms we next obtain:

$$
\begin{aligned}
R_{n,d,Q}(k) \quad = \quad & \Big[C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) \cdot (H_0 + (H_1 - 2d\tau H_0)(x-x_0) \\
& +(H_2 - 2d\tau H_1 + 2d^2\tau^2 H_0 + d\tau' H_0)(x-x_0)^2 + O(x-x_0)^3\big) \\
& +\frac{d^2}{n} C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) \\
& \cdot \left(-\frac{\tau}{Q} H_0 + \left(-\frac{\tau}{Q} H_1 + H_0 g_{1,1} + \frac{2d\tau^2}{Q} H_0\right)(x-x_0) + O(x-x_0)^2\right) \\
& +\frac{1}{n} C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) \\
& \cdot \left(H_0 h_{1,0} + (H_1 h_{1,0} + H_0 h_{1,1} - 2d\tau H_0 h_{1,0})(x-x_0) + O(x-x_0)^2\right) \\
& +\frac{d}{n} C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) \\
& \cdot \left(H_0 h_{2,0} + (H_1 h_{2,0} + H_0 h_{2,1} - 2d\tau H_0 h_{2,0})(x-x_0) + O(x-x_0)^2\right) \\
& +O_\delta\left(\frac{1}{n^2}\right) C_n F(x_0)^{nQ} \cdot \exp\left(-nQ\tau(x-x_0)^2\right) H(x)G(x)^d \Big] \\
& \cdot(1 + o(1))
\end{aligned}
$$

We next multiply by $c_d$, sum $d$ from $1+m$ to $Q+m$, and then sum $k$ over the range $\left|\frac{k}{n} - x_0\right| \leq C\sqrt{\frac{\log n}{n}}$. Note that in the following we have suppressed the range of summation throughout. We thus obtain:

$$
\sum_{\left|\frac{k}{n} - x_0\right| \leq C\sqrt{\frac{\log n}{n}}} \left(\sum_{d=1+m}^{Q+m} c_d R_{n,d,Q}(k)\right) =
$$

$$C_nF(x_0)^{nQ}\left(\sum_{d=1+m}^{Q+m} c_d\right)\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)H_0$$
$$+C_nF(x_0)^{nQ}\left(H_1\sum_{d=1+m}^{Q+m} c_d - 2\tau H_0\sum_{d=1+m}^{Q+m} dc_d\right)$$
$$\cdot\sum_k \exp(-nQ\tau(x-x_0)^2)(x-x_0)$$
$$+C_nF(x_0)^{nQ}\left(H_2\sum_{d=1+m}^{Q+m} c_d - 2\tau H_1\sum_{d=1+m}^{Q+m} dc_d + 2\tau^2 H_0\sum_{d=1+m}^{Q+m} d^2c_d\right.$$
$$\left.+\tau' H_0\sum_{d=1+m}^{Q+m} dc_d\right)\cdot\left(\sum_k \exp(-nQ\tau(x-x_0)^2)(x-x_0)^2\right)$$
$$+C_nF(x_0)^{nQ}O\left(\sum_k \exp(-nQ\tau(x-x_0)^2)(x-x_0)^3\right)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}\left(H_0h_{1,0}\sum_{d=1+m}^{Q+m} c_d\right)\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}\left((H_1h_{1,0}+H_0h_{1,1})\sum_{d=1+m}^{Q+m} c_d - 2\tau H_0h_{1,0}\sum_{d=1+m}^{Q+m} dc_d\right)$$
$$\cdot\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}O\left(\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)^2\right)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}\left(\frac{-\tau}{Q}H_0\sum_{d=1+m}^{Q+m} d^2c_d\right)\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}\left(\frac{-\tau}{Q}H_1\sum_{d=1+m}^{Q+m} d^2c_d + H_0g_{1,1}\sum_{d=1+m}^{Q+m} d^2c_d\right.$$
$$\left.+\frac{2\tau^2}{Q}H_0\sum_{d=1+m}^{Q+m} d^3c_d\right)\cdot\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}O\left(\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)^2\right)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}\left(H_0h_{2,0}\sum_{d=1+m}^{Q+m} dc_d\right)\sum_k \exp\left(-nQ\tau(x-x_0)^2\right)$$

$$+\frac{1}{n}C_nF(x_0)^{nQ}\left((H_1h_{2,0}+H_0h_{2,1})\sum_{d=1+m}^{Q+m}dc_d-2\tau H_0h_{2,0}\sum_{d=1+m}^{Q+m}d^2c_d\right)$$
$$\cdot\sum_k\exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)$$
$$+\frac{1}{n}C_nF(x_0)^{nQ}O\left(\sum_k\exp\left(-nQ\tau(x-x_0)^2\right)(x-x_0)^2\right)$$
$$+O_\delta\left(\frac{1}{n^2}\right)C_nF(x_0)^{nQ}\left(\sum_{d=1+m}^{Q+m}c_d\sum_k\exp\left(-nQ\tau(x-x_0)^2\right)H(x)G(x)^d\right)$$
$$+o(1)\cdot M_n$$

Where we have denoted by $M_n$ the entirety of the sum preceding this last term. So using the formulas in Lemma (7.5), recalling that $\sum c_d=0$, and listing only terms that are not $O\left(\frac{1}{n}\right)$ we now have:

$$\sum_{\left|\frac{k}{n}-x_0\right|\le C\sqrt{\frac{\log n}{n}}}\left(\sum_{d=1+m}^{Q+m}c_dR_{n,d,Q}(k)\right)$$

$$\begin{aligned}
&= -2\tau H_0\left[\frac{o(1)}{\sqrt{n}}+O\left(\frac{1}{n}\right)\right]C_nF(x_0)^{nQ}\sum_{d=1+m}^{Q+m}dc_d\\
&\quad+(-2\tau H_1+\tau' H_0)\left[\frac{1}{\sqrt{n}}\cdot(A_2+\epsilon_n^{(2)})+O\left(\frac{1}{n}\right)\right]C_nF(x_0)^{nQ}\sum_{d=1+m}^{Q+m}dc_d\\
&\quad+2\tau^2H_0\left[\frac{1}{\sqrt{n}}\cdot(A_2+\epsilon_n^{(2)})+O\left(\frac{1}{n}\right)\right]C_nF(x_0)^{nQ}\sum_{d=1+m}^{Q+m}d^2c_d\\
&\quad+\frac{1}{\sqrt{n}}\left[-\frac{\tau}{Q}H_0(A_0+\epsilon_n^{(0)})+O\left(\frac{1}{n}\right)\right]C_nF(x_0)^{nQ}\sum_{d=1+m}^{Q+m}d^2c_d\\
&\quad+\frac{1}{\sqrt{n}}\left[H_0h_{2,0}(A_0+\epsilon_n^{(0)})+O\left(\frac{1}{n}\right)\right]C_nF(x_0)^{nQ}\sum_{d=1+m}^{Q+m}dc_d\\
&\quad+C_nF(x_0)O\left(\frac{1}{n}\right)+o(1)\cdot M_n\\
&= \frac{1}{\sqrt{n}}\left[H_0h_{2,0}A_0-2\tau H_1A_2+\tau' H_0A_2+o(1)\right]C_nF(x_0)^{nQ}\sum_{d=1+m}^{Q+m}dc_d
\end{aligned}$$

$$+\frac{1}{\sqrt{n}}\left[2\tau^2 H_0 A_2 - \frac{\tau}{Q}H_0 A_0 + o(1)\right] C_n F(x_0)^{nQ} \sum_{d=1+m}^{Q+m} d^2 c_d$$
$$+C_n F(x_0)^{nQ} O\left(\frac{1}{n}\right)$$

But $2\tau^2 H_0 A_2 - \frac{\tau}{Q}H_0 A_0 = H_0\left(2\tau^2 \frac{1}{2\tau Q}A_0 - \frac{\tau}{Q}A_0\right) = 0$. Furthermore:

$$\begin{aligned} H_0 h_{2,0} A_0 - 2\tau H_1 A_2 + \tau' H_0 A_2 &= H_0 h_{2,0} A_0 + (-2\tau H_1 + \tau' H_0) A_2 \\ &= H_0 h_{2,0} A_0 + (-2\tau H_1 + \tau' H_0)\frac{1}{2\tau Q} A_0 \\ &= \left(H_0 h_{2,0} - \frac{H_1}{Q} + \frac{\tau' H_0}{2\tau Q}\right) A_0 \\ &= \frac{\tau' H_0}{2\tau Q} \cdot A_0 \end{aligned}$$

since $\frac{H'(x)}{Q} = H(x)h_2(x)$ and, in particular, $\frac{H_1}{Q} = \frac{H'(x_0)}{Q} = H(x_0)h_2(x_0) = H_0 h_{2,0}$. So:

$$\sum_{\left|\frac{k}{n}-x_0\right| \le C\sqrt{\frac{\log n}{n}}} \left(\sum_{d=1+m}^{Q+m} c_d R_{n,d,Q}(k)\right)$$

$$\begin{aligned} &= C_n F(x_0)^{nQ}\left(\frac{1}{\sqrt{n}}\left[\frac{\tau' H_0 A_0}{2\tau Q} + o(1)\right]\sum_{d=1+m}^{Q+m} d c_d + O\left(\frac{1}{n}\right)\right) \\ &= \frac{C_n F(x_0)^{nQ}}{\sqrt{n}}\left(\frac{\tau' H_0 A_0}{2\tau Q}\sum_{d=1+m}^{Q+m} d c_d + o(1) + O\left(\frac{1}{\sqrt{n}}\right)\right) \end{aligned}$$

Here we have that:

$$\frac{\tau' H_0 A_0}{2\tau Q}\sum_{d=1+m}^{Q+m} d c_d \neq 0$$

since $\sum_{d=1+m}^{Q+m} d c_d \neq 0$ by the choice of $m$ and $\tau' = \frac{f'''(x_0)}{2} \neq 0$ by Proposition (4.5). Hence under the assumption that $\sum_{d=1+m}^{Q+m} c_d = 0$:

$$\lim_{n\to\infty} |S_{n,c}(1)|^{1/nQ} = \phi_{Q,r,a}$$

By the results of chapter six, this holds unconditionally thus proving Proposition (6.1).

# Chapter 8

## Estimates for Coefficients

Recall from chapter two that:

$$S_{n,c}(1) = \sum_{l=2}^{a} P_{l,n}(1)L(l,c) + \sum_{d=1+m}^{Q+m} \widehat{P}_{d,n}(1)c_d$$

In this chapter, we shall first establish an estimate for the coefficients of these linear forms. The concluding proposition and its corollary will show that when these forms are multiplied by an explicit factor (for which there is a standard estimate) we arrive at forms with integer coefficients.

**Proposition 8.1** *For $l = 1, ..., a$:*

$$\limsup_{n\to\infty} |P_{l,n}(1)|^{1/Qn} \leq Q^{2R+1}2^{A-2R}(2R+1)^{2R+1}$$

*Likewise, for $d = 1+m, ..., Q+m$:*

$$\limsup_{n\to\infty} |\widehat{P}_{d,n}(1)|^{1/Qn} \leq Q^{2R+1}2^{A-2R}(2R+1)^{2R+1}$$

*proof*

For $l = 1, 2, ..., a$ it suffices to bound the $c_{l,j,n}$ since $P_{l,n}(1) = \sum_{j=0}^{n} Q^l c_{l,j,n}$. Because the $c_{l,j,n}$ are independent of the congruence class of $d$, we can obtain estimates of these coefficients from restricting our attention to $R_{n,Q}(z)$. Recall:

$$R_{n,Q}(z) = Q^{2rn}n!^{a-2r}\frac{\prod_{l=0}^{rn-1}\left(z-\frac{l}{Q}\right)\prod_{l=0}^{rn-1}\left(z+n+2+\frac{l}{Q}\right)}{\prod_{l=0}^{n}(z+1+l)^a}$$

By definition:

$$c_{l,j,n} = \frac{1}{2\pi i} \int_{|z+j+1|=\frac{1}{2}} R_{n,Q}(z)(z+j+1)^{l-1} dz$$

If $z$ is restricted to the path of integration, then for $k = 0, 1, ..., n$:

$$\begin{aligned} |z+1+k| &= |(j-k) - (z+j+1)| \\ &\geq |j-k| - |z+j+1| \end{aligned}$$

From this inequality we see that for $0 \leq k \leq j-2$:

$$\begin{aligned} |z+1+k| &\geq j-k-1/2 \\ &\geq j-k-1 \end{aligned}$$

and for $j+2 \leq k \leq n$:

$$\begin{aligned} |z+1+k| &\geq k-j-1/2 \\ &\geq k-j-1 \end{aligned}$$

while for $k = j-1, j, j+1$:

$$|z+1+k| \geq \frac{1}{2}$$

Hence:

$$|(z+1)_{n+1}| = \left| \prod_{k=0}^{n} (z+1+k) \right| \geq 2^{-3}(j-1)!(n-j-1)!$$

From:

$$\begin{aligned} |z+n+2| &= |z+j+1+n-j+1| \\ &\leq |z+j+1| + n-j+1 \\ &= 1/2 + n - j + 1 \\ &\leq n-j+2 \end{aligned}$$

we have:

$$\begin{aligned}\left|Q^{rn}\prod_{l=0}^{rn-1}\left(z+n+2+\frac{l}{Q}\right)\right| &= |(Q(z+n+2))_{rn}| \\ &\leq (Q(n-j+2))_{rn} \\ &= \frac{(rn+Q(n-j+2)-1)!}{(Q(n-j+2)-1)!}\end{aligned}$$

Finally, on the path of integration, using $|z|-|-j-1| \leq |z-(-j-1)| = |z+j+1| = 1/2$, we have $|z| \leq |j+1| + 1/2 \leq j+2$. Hence:

$$|(z-rn+1)_{rn}| = |(-1)^{rn}(-z)_{rn}| \leq (|z|)_{rn} \leq (j+2)_{rn}$$

and:

$$\begin{aligned}\left|Q^{rn}\prod_{l=0}^{rn-1}\left(z-\frac{l}{Q}\right)\right| &= |(Qz-rn+1)_{rn}| \\ &\leq (|Qz|)_{rn} \\ &\leq (Q(j+2))_{rn} \\ &= \frac{(rn+Q(j+2)-1)!}{(Q(j+2)-1)!}\end{aligned}$$

Therefore:

$$\begin{aligned}|c_{l,j,n}| &\leq \frac{(rn+Q(j+2)-1)!}{(Q(j+2)-1)!(j!(n-j)!)^r} \cdot \frac{(rn+Q(n-j+2)-1)!}{(Q(n-j+2)-1)!(j!(n-j)!)^r} \\ &\quad \cdot \left(\frac{n!}{j!(n-j)!}\right)^{a-2r} \cdot (j(n-j))^a 8^a \\ &\leq (2r+1)^{2rn+Qn-2+4Q} 2^{(a-2r)n}(2n^2)^a \\ &\leq Q^{2rn+Qn-2+4Q}(2R+1)^{2rn+Qn-2+4Q} 2^{(a-2r)n}(2n^2)^a\end{aligned}$$

since $(j(n-j))^a 8^a \leq (2n^2)^a$ and from the following bounds:

$$\frac{(rn+Q(j+2)-1)!}{(Q(j+2)-1)!(j!(n-j)!)^r} \leq (2r+1)^{rn+Q(j+2)-1}$$

$$\frac{(rn+Q(n-j+2)-1)!}{(Q(n-j+2)-1)!(j!(n-j)!)^r} \leq (2r+1)^{rn+Q(n-j+2)-1}$$

These follow directly from a standard bound on multinomial coefficients:

$$\begin{pmatrix} n \\ n_1, n_2, ..., n_p \end{pmatrix} = \frac{n!}{n_1!n_2!...n_p!} \leq p^n$$

where $n_1, ..., n_p \geq 0$ and $n_1 + ... + n_p = n$. Thus we have:

$$\limsup_{n\to\infty} |P_{l,n}(1)|^{1/Qn} \leq Q^{2R+1} 2^{A-2R} (2R+1)^{2R+1}$$

Finally:

$$\widehat{P}_{d,n}(1) = -\sum_{l=1}^{a} \sum_{j=0}^{n} Q^l c_{l,j,n} \sum_{k=0}^{j} \frac{1}{(kQ+d)^l}$$

Since:

$$\sum_{k=0}^{j} \frac{1}{(kQ+d)^l} \leq \sum_{k=0}^{j} \frac{1}{(kQ+d)} \leq j+1 \leq n+1$$

it is also the case that for $d = 1+m, ..., Q+m$:

$$\limsup_{n\to\infty} |\widehat{P}_{d,n}(1)|^{1/Qn} \leq Q^{2R+1} 2^{A-2R} (2R+1)^{2R+1}$$

Since eventually we will require linear forms with integer coefficients, it will also be necessary to have estimates on the size of the denominators of the $c_{l,j,n}$:

**Proposition 8.2** *If $d_n = lcm\{1, ..., n\}$ then $d_n^{a-l} P_{l,n}(1) \in \mathbb{Z}$ for $l = 1, ..., a$.*

**Corollary 8.3** *If $d_{n,Q} = lcm\{1, ..., (n+1)Q+m\}$ then $d_{n,Q}^{a-l} P_{l,n}(1) \in \mathbb{Z}$ for $l = 1, ..., a$ and $d_{n,Q}^{a} \widehat{P}_{d,n}(1) \in \mathbb{Z}$ for $d = 1+m, ..., Q+m$.*

*proof*

To estimate the size of the denominators in the $c_{l,j,n}$ we write:

$$R_{n,Q}(t)(t+j+1)^a = \prod_{l=1}^{r} F_l(t) \times \prod_{l=1}^{r} G_l(t) \times H(t)^{a-2r}$$

where:

$$F_l(t) = \frac{(Qt-nl+1)_n}{(t+1)_{n+1}}(t+j+1)$$

$$G_l(t) = \frac{(Q(t+n+2)+(l-1)n)_n}{(t+1)_{n+1}}(t+j+1)$$

$$H(t) = \frac{n!}{(t+1)_{n+1}}(t+j+1)$$

We will decompose these into partial fractions and give the details of the calculation for $F_l(t)$. Since the numerator and denominator of $F_l(t)$ have the same degree, the constant in the partial fraction decomposition is $Q^n$. We write:

$$F_l(t) = \frac{(Qt-nl+1)_n}{\prod_{\substack{h=0\\h\neq j}}^{n}(t+h+1)} = Q^n + \sum_{\substack{k=0\\k\neq j}}^{n} \frac{c_k}{t+k+1}$$

for some constants $c_k$. To solve for these constants, we clear the denominators to obtain:

$$(Qt-nl+1)_n = Q^n \prod_{\substack{h=0\\h\neq j}}^{n}(t+h+1) + \sum_{\substack{k=0\\k\neq j}}^{n} c_k \prod_{\substack{h=0\\h\neq k,j}}^{n}(t+h+1)$$

If we evaluate at $t = -p-1$ ($p \neq j$), all terms but one drop out:

$$(Q(-p-1)-nl+1)_n = c_p \prod_{\substack{h=0\\h\neq p,j}}^{n}(-p-1+h+1)$$

Thus:

$$c_p = \frac{(-Qp-nl-Q+1)_n}{\prod_{\substack{h=0\\h\neq p,j}}^{n}(-p+h)} = (j-p)\cdot\frac{(-Qp-nl-Q+1)_n}{\prod_{\substack{h=0\\h\neq p}}^{n}(-p+h)}$$

So decomposing $F_l(t)$, $G_l(t)$, and $H(t)$ into partial fractions:

$$F_l(t) = Q^n + \sum_{\substack{p=0\\p\neq j}}^{n} \frac{(j-p)f_{p,l}}{t+p+1}$$

$$G_l(t) = Q^n + \sum_{\substack{p=0\\p\neq j}}^{n} \frac{(j-p)g_{p,l}}{t+p+1}$$

$$H(t) = \sum_{\substack{p=0\\p\neq j}}^{n} \frac{(j-p)h_p}{t+p+1}$$

where:

$$\begin{aligned}
f_{p,l} &= (-Qp - nl - Q + 1)_n \prod_{\substack{h=0 \\ h \neq p}}^{n} (-p+h)^{-1} \\
&= \frac{(-1)^n (Qp + nl + Q - n)_n}{(-1)^p p!(n-p)!} \\
&= (-1)^{n-p} \frac{(nl + Qp + Q - 1)!}{(nl + Qp + Q - n - 1)! n!} \frac{n!}{p!(n-p)!} \\
&= (-1)^{n-p} \begin{pmatrix} nl + Qp + Q - 1 \\ n \end{pmatrix} \begin{pmatrix} n \\ p \end{pmatrix} \in \mathbb{Z}
\end{aligned}$$

Similarly:

$$\begin{aligned}
g_{p,l} &= (Q(-p+n+1) + (l-1)n)_n \prod_{\substack{h=0 \\ h \neq p}}^{n} (-p+h)^{-1} \\
&= (-1)^p \frac{((Q+l)n - Qp + Q - 1)!}{((Q+l)n - Qp + Q - 1 - n)! n!} \frac{n!}{p!(n-p)!} \\
&= (-1)^p \begin{pmatrix} (Q+l)n - Qp + Q - 1 \\ n \end{pmatrix} \begin{pmatrix} n \\ p \end{pmatrix} \in \mathbb{Z}
\end{aligned}$$

and:

$$\begin{aligned}
h_p &= n! \prod_{\substack{h=0 \\ h \neq p}}^{n} (-p+h)^{-1} \\
&= \frac{(-1)^p n!}{p!(n-p)!} \\
&= (-1)^p \begin{pmatrix} n \\ p \end{pmatrix} \in \mathbb{Z}
\end{aligned}$$

Recall that $D_\lambda = \frac{1}{\lambda!} \left(\frac{d}{dt}\right)^\lambda$. Then we have:

$$(D_\lambda F_l(t))_{|t=-j-1} = Q^n \cdot \delta_{0,\lambda} + \sum_{\substack{p=0 \\ p \neq j}}^{n} (-1)^\lambda \frac{(j-p) f_{p,l}}{(p-j)^{\lambda+1}}$$

$$(D_\lambda G_l(t))_{|t=-j-1} = Q^n \cdot \delta_{0,\lambda} + \sum_{\substack{p=0 \\ p \neq j}}^{n} (-1)^\lambda \frac{(j-p) g_{p,l}}{(p-j)^{\lambda+1}}$$

$$(D_\lambda H(t))_{|t=-j-1} = \sum_{\substack{p=0\\p\neq j}}^{n} (-1)^\lambda \frac{(j-p)h_p}{(p-j)^{\lambda+1}}$$

where $\delta_{0,\lambda} = 1$ if $\lambda = 0$, $\delta_{0,\lambda} = 0$ if $\lambda \geq 1$. On the basis of these formulae and recalling that $d_n = \mathrm{lcm}\{1, ...n\}$, one deduces that:

$$d_n^\lambda \cdot (D_\lambda F_l(t))_{|t=-j-1},\ d_n^\lambda \cdot (D_\lambda G_l(t))_{|t=-j-1},\ d_n^\lambda \cdot (D_\lambda H(t))_{|t=-j-1}$$

are integers for all $\lambda \in \mathbb{N}$. From Leibniz's formula, for each $l = 1, ..., a$:

$$D_{a-l}(R_{n,Q}(t)(t+j+1)^a) =$$

$$\sum_\nu (D_{\nu_1}F_1)\cdots(D_{\nu_r}F_r)(D_{\nu_{r+1}}G_1)\cdots(D_{\nu_{2r}}G_r)(D_{\nu_{2r+1}}H)\cdots(D_{\nu_a}H)$$

where the sum is over all multi-indices $\nu \in \mathbb{N}^a$ such that $\nu_1 + ... + \nu_a = a - l$, we conclude that $d_n^{a-l} c_{l,j,n} \in \mathbb{Z}$ and therefore $d_n^{a-l} P_{l,n}(1) \in \mathbb{Z}$.

To prove the corollary, we observe that since $(n+1)Q + m > n$ then $d_n | d_{n,Q}$ and it follows immediately that $d_{n,Q}^{a-l} P_{l,n}(1) \in \mathbb{Z}$ for $l = 1, ..., a$. Lastly:

$$d_{n,Q}^a \widehat{P}_{d,n}(1) = -\sum_{l=1}^{a}\sum_{j=0}^{n} Q^l d_{n,Q}^{a-l} c_{l,j,n} \sum_{k=0}^{j} \frac{d_{n,Q}^l}{(kQ+d)^l} \in \mathbb{Z}$$

for $d = 1 + m, ..., Q + m$ since the base of the denominator in the inner sum is no greater than $nQ + Q + m = (n+1)Q + m$ and a similar computation shows that, in the case $m \neq 0$, $d_{n,Q}^a \widehat{P}_{Q+m,n}(1) \in \mathbb{Z}$ since:

$$d_{n,Q}^a \widehat{P}_{Q+m,n}(1) = d_{n,Q}^a \widetilde{P}_{Q+m,n}(1) - \sum_{l=2}^{a} \left(\frac{d_{n,Q}}{m}\right)^l \cdot d_{n,Q}^{a-l} \cdot P_{l,n}(1)$$

and these are integers by what we have shown above.

# Chapter 9

## Nesterenko's Criterion

Before we state the main result, we'll first introduce some notation and terminology.

For $u, v \in \mathbb{C}^M$, $\mathcal{L} \subset \mathbb{C}^M$ a linear subspace, $(u, v) = \overline{u} \cdot v$, $||u|| = (u, u)^{1/2}$, $\mathrm{pr}_{\mathcal{L}}(u)$ will denote the projection of $u$ on $\mathcal{L}$, and $\rho(u, \mathcal{L}) = ||\mathrm{pr}_{\mathcal{L}^\perp}(u)||$.

If $u_1, ..., u_s \in \mathbb{C}^M$ then $\det ||(u_k, u_j)|| > 0$ if and only if $u_1, ..., u_s$ are linearly independent. We define the volume of the parallelepiped constructed on these vectors by $V(u_1, ...., u_s) = (\det ||(u_k, u_j)||)^{1/2}$.

If $u_1, ..., u_s$ is the basis of a linear subspace $\mathcal{L} \subset \mathbb{C}^M$ then for every $v \in \mathbb{C}^M$:

$$V(v, u_1, ...., u_s) = \rho(v, \mathcal{L}) V(u_1, ...., u_s)$$

We will say that a linear subspace $\mathcal{L} \subset \mathbb{C}^M$ is a rational subspace if it can be specified by linear equations with $\mathbb{Q}$-rational coefficients. If $\mathcal{L}$ is a rational subspace and $\dim \mathcal{L} = M - s$ then the set of linear forms with rational coefficients which vanish on $\mathcal{L}$ forms an $s$-dimensional linear space over $\mathbb{Q}$. The subset of forms having integer coefficients form a lattice. The volume of the base parallelipiped of this lattice will be denoted by $V(\mathcal{L})$. Then it is clearly the case that $V(\mathcal{L}) \geq 1$.

If $L(x) = (a, x)$ is a linear form then we define $||L|| = ||a||$. If the coefficients are integers such that $(a_1, ..., a_M) = 1$, $\vec{a} \neq 0$, and if $\mathcal{L}$ is the hyperplane of $\mathbb{C}^M$ defined by the equation $L(x) = 0$ then $V(\mathcal{L}) = ||a||$ and for any $u \in \mathbb{C}^M$, $\rho(u, \mathcal{L}) = |L(u)| / V(\mathcal{L})$.

The main result of this chapter is the following version of a theorem of Nesterenko, originally stated and proved over a real vector space:

**Proposition 9.1** *Let* $\theta = (\theta_1, \theta_2, ..., \theta_M) \in \mathbb{C}^M$ *with* $M > 2$ *and* $\theta \neq 0$. *Furthermore, suppose that there exist* $M$ *sequences* $\{p_{l,n}\}_{n\geq 0}$ *such that:*

*(i)* $\forall l \in \{1, ..., M\}$, $p_{l,n} \in \mathbb{Z}$

*(ii) letting* $L_n(\vec{x}) = \sum_{l=1}^{M} p_{l,n} x_l$, *there are* $\alpha_1, \alpha_2$, $0 < \alpha_1 \leq \alpha_2 < 1$ *with:*

$$\alpha_1^{n+o_1(n)} \leq |L_n(\theta)| \leq \alpha_2^{n+o_2(n)}$$

*(iii)* $\exists \beta > 1$ *such that* $\max_{1\leq l\leq M} |p_{l,n}| \leq \beta^{n+o_3(n)}$

*Then:*

$$dim_{\mathbb{Q}}\{\mathbb{Q}\theta_1 + \mathbb{Q}\theta_2 + .... + \mathbb{Q}\theta_M\} \geq \frac{\ln(\beta) - \ln(\alpha_1)}{\ln(\beta) - \ln(\alpha_1) + \ln(\alpha_2)}$$

*proof*

Fix $\delta > \ln\left(\frac{\alpha_2}{\alpha_1}\right)$. Under the hypotheses we shall prove that for any integer $r$ with $0 \leq r < \frac{\ln(\beta)-\ln(\alpha_1)}{\ln(\beta)+\delta}$, there exists a constant $\gamma_r > 0$ (depending on $\theta$) such that for every rational subspace $\mathcal{L} \subset \mathbb{C}^M$, $\dim \mathcal{L} = r$:

$$\rho(\theta, \mathcal{L}) \geq \gamma_r V(\mathcal{L})^{-\frac{\ln(\beta)-\ln(\alpha_1)}{\ln(\beta)-\ln(\alpha_1)-r(\ln(\beta)+\delta)}} \tag{9.1}$$

The theorem follows immediately since if $r$ is the maximum number of linearly independent numbers over $\mathbb{Q}$ from among $\theta_1, ..., \theta_M$ , then there are $M - r$ linearly independent forms $L_1(x), ..., L_{M-r}(x)$ having rational coefficients with $L_j(\theta) = 0$ for $j = 1, ..., M - r$. If $\mathcal{L}$ is the rational subspace determined by the equations $L_1(x) = ... = L_{M-r}(x) = 0$ then $\dim \mathcal{L} = r$ and $\theta \in \mathcal{L}$. Therefore $\rho(\theta, \mathcal{L}) = 0$. So the assumption that $r < \frac{\ln(\beta)-\ln(\alpha_1)}{\ln(\beta)+\delta}$ leads to a contradiction. Hence we must have $r \geq \frac{\ln(\beta)-\ln(\alpha_1)}{\ln(\beta)+\delta}$ for any $\delta > \ln\left(\frac{\alpha_2}{\alpha_1}\right)$.

Before we proceed with the proof, we first require two lemmas.

**Lemma 9.2** *If* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are linear subspaces of* $\mathbb{C}^M$ *such that* $\mathcal{L}_2 \subset \mathcal{L}_1$ *and* $\theta \in \mathbb{C}^M$ *then:*

$$\rho(\theta, \mathcal{L}_2) \geq \rho(\theta, \mathcal{L}_1)$$

*proof*

The assertion is elementary but we prove it nevertheless. We have:

$$||\theta||^2 = ||\mathrm{pr}_{\mathcal{L}_1}(\theta)||^2 + ||\mathrm{pr}_{\mathcal{L}_1^\perp}(\theta)||^2$$

$$||\theta||^2 = \left|\left|\mathrm{pr}_{\mathcal{L}_2}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right)\right|\right|^2 + \left|\left|\mathrm{pr}_{\mathcal{L}_2^\perp}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right)\right|\right|^2 + ||\mathrm{pr}_{\mathcal{L}_1^\perp}(\theta)||^2 \tag{9.2}$$

But also:

$$||\theta||^2 = ||\mathrm{pr}_{\mathcal{L}_2}(\theta)||^2 + ||\mathrm{pr}_{\mathcal{L}_2^\perp}(\theta)||^2$$

Since $\mathcal{L}_2 \subset \mathcal{L}_1$, it must be that $\mathcal{L}_1^\perp \subset \mathcal{L}_2^\perp$. So we also have:

$$\begin{aligned} \theta &= \mathrm{pr}_{\mathcal{L}_1}(\theta) + \mathrm{pr}_{\mathcal{L}_1^\perp}(\theta) \\ &= \mathrm{pr}_{\mathcal{L}_2}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right) + \mathrm{pr}_{\mathcal{L}_2^\perp}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right) + \mathrm{pr}_{\mathcal{L}_1^\perp}(\theta) \end{aligned}$$

with $\mathrm{pr}_{\mathcal{L}_2^\perp}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right) + \mathrm{pr}_{\mathcal{L}_1^\perp}(\theta) \in \mathcal{L}_2^\perp$. By the uniqueness of projection $\mathrm{pr}_{\mathcal{L}_2}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right) = \mathrm{pr}_{\mathcal{L}_2}(\theta)$. Hence:

$$||\theta||^2 = \left|\left|\mathrm{pr}_{\mathcal{L}_2}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right)\right|\right|^2 + ||\mathrm{pr}_{\mathcal{L}_2^\perp}(\theta)||^2$$

Subtracting equation (9.2) from this we have:

$$0 = ||\mathrm{pr}_{\mathcal{L}_2^\perp}(\theta)||^2 - ||\mathrm{pr}_{\mathcal{L}_1^\perp}(\theta)||^2 - \left|\left|\mathrm{pr}_{\mathcal{L}_2^\perp}\left(\mathrm{pr}_{\mathcal{L}_1}(\theta)\right)\right|\right|^2$$

from which we easily deduce:

$$\rho(\theta, \mathcal{L}_2) = ||\mathrm{pr}_{\mathcal{L}_2^\perp}(\theta)|| \geq ||\mathrm{pr}_{\mathcal{L}_1^\perp}(\theta)|| = \rho(\theta, \mathcal{L}_1)$$

and the lemma is proved.

**Lemma 9.3** *Suppose $\mathcal{L}$, $\mathcal{L}_1$ are rational subspaces of $\mathbb{C}^M$, $\dim \mathcal{L}_1 = M-1$, $\mathcal{L} \not\subset \mathcal{L}_1$, and $\mathcal{M} = \mathcal{L} \cap \mathcal{L}_1$. Then for $\theta \in \mathbb{C}^M$*

*1) $V(\mathcal{M}) \leq V(\mathcal{L})V(\mathcal{L}_1)$*

*2) $V(\mathcal{M})\rho(\theta, \mathcal{M}) \leq V(\mathcal{L})V(\mathcal{L}_1)(\rho(\theta, \mathcal{L}) + \rho(\theta, \mathcal{L}_1))$*

*proof*

We first observe that when $\theta = 0$, $\rho(\theta, \mathcal{M}) = \rho(\theta, \mathcal{L}) = \rho(\theta, \mathcal{L}_1) = 0$ and the second inequality is trivially true. Hence, in the sequel, we assume $\theta \neq 0$. Now suppose that $(a_1, x), ..., (a_r, x)$ and $(b, x)$ are bases of lattices of integer forms that vanish on $\mathcal{L}$ and $\mathcal{L}_1$ respectively. Then:

$$\begin{aligned} V(\mathcal{M}) &\leq V(b, a_1, ..., a_r) \\ &= ||\text{pr}_{\mathcal{L}}(b)||V(a_1, ..., a_r) \\ &\leq ||b||V(a_1, ..., a_r) \\ &= V(\mathcal{L}_1)V(\mathcal{L}) \end{aligned}$$

thus proving the first inequality. To prove the second, we will show that it suffices to prove:

$$||\text{pr}_{\mathcal{L}}(b)|| \leq \rho(\theta, \mathcal{L}) + \rho(\theta, \mathcal{L}_1) \tag{9.3}$$

when $||\theta|| = 1$. First, since we have $||b|| = V(\mathcal{L}_1)$ and $V(\mathcal{M}) \leq ||\text{pr}_{\mathcal{L}}(b)||V(\mathcal{L})$, as seen above, it is sufficient to prove:

$$\rho(\theta, \mathcal{M})||\text{pr}_{\mathcal{L}}(b)|| \leq ||b||(\rho(\theta, \mathcal{L}) + \rho(\theta, \mathcal{L}_1))$$

We can make a further reduction by observing that we may assume $\theta \in \mathcal{M}^\perp$ since if we set $\tau = \text{pr}_{\mathcal{M}^\perp}(\theta) \in \mathcal{M}^\perp$ then $\theta - \tau = \text{pr}_{\mathcal{M}}(\theta) \in \mathcal{M} = \mathcal{L} \cap \mathcal{L}_1$. Hence:

$$\text{pr}_{\mathcal{M}^\perp}(\theta - \tau) = \text{pr}_{\mathcal{L}^\perp}(\theta - \tau) = \text{pr}_{\mathcal{L}_1^\perp}(\theta - \tau) = 0$$

and therefore:

$$\rho(\theta, \mathcal{M}) = ||\text{pr}_{\mathcal{M}^\perp}(\theta)|| = ||\text{pr}_{\mathcal{M}^\perp}(\tau)|| = \rho(\tau, \mathcal{M})$$

In similar fashion $\rho(\theta, \mathcal{L}) = \rho(\tau, \mathcal{L})$ and $\rho(\theta, \mathcal{L}_1) = \rho(\tau, \mathcal{L}_1)$. Under the assumption $\theta \in \mathcal{M}^\perp$, $\rho(\theta, \mathcal{M}) = ||\theta||$. So, lastly, we may assume $||\theta|| = 1$ since projection is linear and the general result follows by application to $\frac{\theta}{||\theta||}$.

Since $||b|| \geq 1$ then we may suppose $||b|| = 1$. The remainder of the argument will not make use of integrality. Now to prove (9.3), we have:

$$1 = ||\theta||^2 = ||\mathrm{pr}_{\mathcal{L}}(\theta)||^2 + ||\mathrm{pr}_{\mathcal{L}^\perp}(\theta)||^2$$

and so:

$$||\mathrm{pr}_{\mathcal{L}}(b)||^2 = (||\mathrm{pr}_{\mathcal{L}}(\theta)|| \cdot ||\mathrm{pr}_{\mathcal{L}}(b)||)^2 + (||\mathrm{pr}_{\mathcal{L}^\perp}(\theta)|| \cdot ||\mathrm{pr}_{\mathcal{L}}(b)||)^2 \tag{9.4}$$

Furthermore:

$$\dim(\mathcal{L} \cap \mathcal{M}^\perp) \geq \dim \mathcal{L} + \dim \mathcal{M}^\perp - M = \dim \mathcal{L} - \dim \mathcal{M} \geq 1$$

However, we also have:

$$0 = \dim(\mathcal{L}_1 \cap \mathcal{L} \cap \mathcal{M}^\perp) \geq \dim \mathcal{L}_1 + \dim(\mathcal{L} \cap \mathcal{M}^\perp) - M = \dim(\mathcal{L} \cap \mathcal{M}^\perp) - 1$$

Hence $\dim(\mathcal{L} \cap \mathcal{M}^\perp) = 1$. Now, since $\mathcal{L} \supset \mathcal{M}$, $\mathcal{L}^\perp \subset \mathcal{M}^\perp$. Thus $\mathrm{pr}_{\mathcal{L}}(\theta) = \theta - \mathrm{pr}_{\mathcal{L}^\perp}(\theta) \in \mathcal{L} \cap \mathcal{M}^\perp$. We also have $\mathrm{pr}_{\mathcal{L}}(b) = b - \mathrm{pr}_{\mathcal{L}^\perp}(b) \in \mathcal{L} \cap \mathcal{M}^\perp$ since $b \in \mathcal{L}_1^\perp \subset \mathcal{M}^\perp$. If $\mathrm{pr}_{\mathcal{L}}(\theta) \neq 0$ and $\mathrm{pr}_{\mathcal{L}}(b) \neq 0$ then:

$$\frac{\mathrm{pr}_{\mathcal{L}}(\theta)}{||\mathrm{pr}_{\mathcal{L}}(\theta)||} = c \cdot \frac{\mathrm{pr}_{\mathcal{L}}(b)}{||\mathrm{pr}_{\mathcal{L}}(b)||}$$

for some $c \in \mathbb{C}$ with $|c| = 1$. Therefore:

$$\begin{aligned} ||\mathrm{pr}_{\mathcal{L}}(\theta)|| \cdot ||\mathrm{pr}_{\mathcal{L}}(b)|| &= |(\mathrm{pr}_{\mathcal{L}}(\theta), \mathrm{pr}_{\mathcal{L}}(b))| \\ &= |(\mathrm{pr}_{\mathcal{L}}(\theta), b)| \\ &= |(\theta, b) - (\mathrm{pr}_{\mathcal{L}^\perp}(\theta), b)| \\ &\leq |(\theta, b)| + |(\mathrm{pr}_{\mathcal{L}^\perp}(\theta), b)| \end{aligned}$$

This holds trivially if $\mathrm{pr}_{\mathcal{L}}(\theta) = 0$ or $\mathrm{pr}_{\mathcal{L}}(b) = 0$. We have by Schwarz's inequality:

$$|(\mathrm{pr}_{\mathcal{L}^\perp}(\theta), b)| = |(\mathrm{pr}_{\mathcal{L}^\perp}(\theta), \mathrm{pr}_{\mathcal{L}^\perp}(b))| \leq ||\mathrm{pr}_{\mathcal{L}^\perp}(\theta)|| \cdot ||\mathrm{pr}_{\mathcal{L}^\perp}(b)||$$

and, since $||b|| = 1$, $\text{pr}_{\mathcal{L}_1^\perp}(\theta) = (\theta, b)b$ and:

$$|(\theta, b)| = ||\text{pr}_{\mathcal{L}_1^\perp}(\theta)||$$

Combining these, we obtain:

$$||\text{pr}_{\mathcal{L}}(\theta)|| \cdot ||\text{pr}_{\mathcal{L}}(b)|| \leq ||\text{pr}_{\mathcal{L}_1^\perp}(\theta)|| + ||\text{pr}_{\mathcal{L}^\perp}(\theta)|| \cdot ||\text{pr}_{\mathcal{L}^\perp}(b)||$$

Substituting the righthand side of this inequality into equation (9.4) and using the fact that $1 = ||b||^2 = ||\text{pr}_{\mathcal{L}^\perp}(b)||^2 + ||\text{pr}_{\mathcal{L}}(b)||^2$ and $||\text{pr}_{\mathcal{L}^\perp}(b)|| \leq 1$, we have:

$$\begin{aligned} ||\text{pr}_{\mathcal{L}}(b)||^2 &\leq ||\text{pr}_{\mathcal{L}_1^\perp}(\theta)||^2 + 2||\text{pr}_{\mathcal{L}_1^\perp}(\theta)|| \cdot ||\text{pr}_{\mathcal{L}^\perp}(\theta)|| \cdot ||\text{pr}_{\mathcal{L}^\perp}(b)|| + ||\text{pr}_{\mathcal{L}^\perp}(\theta)||^2 \\ &\leq \Big(||\text{pr}_{\mathcal{L}_1^\perp}(\theta)|| + ||\text{pr}_{\mathcal{L}^\perp}(\theta)||\Big)^2 \end{aligned}$$

This concludes the proof of the lemma.

The proof of Proposition (9.1) is by induction on $r$. For $r = 0$ there is a unique zero dimensional rational subspace of $\mathbb{C}^M$. Moreover, we have $V(\mathcal{L}) = 1$ and $\rho(\theta, \mathcal{L}) = ||\theta||$. Hence, formula (9.1) holds with $\gamma_0 = ||\theta||$.

Now suppose that $1 \leq r < \frac{\ln(\beta) - \ln(\alpha_1)}{\ln(\beta) + \delta}$ and that formula (9.1) holds for every rational subspace of dimension $r - 1$. Fix $\epsilon$ satisfying the inequalities:

$$0 < \epsilon < \frac{\ln(\alpha_2) - \ln(\alpha_1) - \delta}{\ln(\alpha_1)}$$

Also there is an $N_1$ such that for $n \geq N_1$ and $i = 1, 2,$ and $3$, $n + 1 + o_i(n+1) \leq (1 + \epsilon)(n + o_i(n))$. For $k \leq r$, define:

$$\lambda_k = \frac{\ln(\beta) - \ln(\alpha_1)}{\ln(\beta) - \ln(\alpha_1) - k(\ln(\beta) + \delta)}$$

Note that $\lambda_k > 1$. Now let $\tau_i = -\ln(\alpha_i)$, $\tau_n^{(i)} = -\left(1 + \frac{o_i(n)}{n}\right)\ln(\alpha_i)$ for $i = 1, 2$ and $\beta_n = \beta^{1 + \frac{o_3(n)}{n}}$. We see that $\lim_{n\to\infty} \tau_n^{(i)} = \tau_i$ for $i = 1, 2$ and $\lim_{n\to\infty} \beta_n = \beta$. Then we have:

**Lemma 9.4** *There is an $N$ such that for all $n \geq N$:*

$$\frac{\tau_n^{(1)} + \ln(\beta_n)}{\lambda_r}(1+\epsilon) - \frac{\tau_n^{(2)} + \ln(\beta_n)}{\lambda_{r-1}} < -\ln(\beta_n)$$

*proof*

Observe that $\ln(\beta) + \tau_1 - r(\ln(\beta) + \delta) \leq \tau_1 - \delta$ since $r \geq 1$. We compute:

$$\begin{aligned}
&\frac{\tau_1 + \ln(\beta)}{\lambda_r}(1+\epsilon) - \frac{\tau_2 + \ln(\beta)}{\lambda_{r-1}} \\
&= \frac{(\ln(\beta) + \tau_1 - r(\ln(\beta)+\delta))(1+\epsilon)(\ln(\beta)+\tau_1)}{\ln(\beta)+\tau_1} \\
&\quad - \frac{(\ln(\beta) + \tau_1 - (r-1)(\ln(\beta)+\delta))(\ln(\beta)+\tau_2)}{\ln(\beta)+\tau_1} \\
&= \frac{(\ln(\beta) + \tau_1 - r(\ln(\beta)+\delta))\left[(1+\epsilon)(\ln(\beta)+\tau_1) - (\ln(\beta)+\tau_2)\right]}{\ln(\beta)+\tau_1} \\
&\quad - \frac{(\ln(\beta)+\tau_2)(\ln(\beta)+\delta)}{\ln(\beta)+\tau_1} \\
&\leq \frac{(\tau_1 - \delta)\left[(1+\epsilon)(\ln(\beta)+\tau_1) - (\ln(\beta)+\tau_2)\right]}{\ln(\beta)+\tau_1} \\
&\quad - \frac{(\ln(\beta)+\tau_2)(\ln(\beta)+\delta)}{\ln(\beta)+\tau_1} \\
&= \frac{(\tau_1-\delta)(1+\epsilon)(\ln(\beta)+\tau_1) - (\ln(\beta)+\tau_2)(\ln(\beta)+\tau_1)}{\ln(\beta)+\tau_1} \\
&= \left(\frac{\ln(\beta)+\tau_1}{\ln(\beta)+\tau_1}\right)(\tau_1-\delta)(1+\epsilon) - (\ln(\beta)+\tau_2)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{\tau_1 + \ln(\beta)}{\lambda_r}(1+\epsilon) - \frac{\tau_2 + \ln(\beta)}{\lambda_{r-1}} &\leq (\tau_1 - \tau_2 - \delta) + \epsilon(\tau_1 - \delta) - \ln(\beta) \\
&= (\tau_1 - \tau_2 - \delta) + \epsilon\tau_1 - \ln(\beta) - \epsilon\delta \\
&\leq -\ln(\beta) - \epsilon\delta
\end{aligned}$$

since we chose:

$$\begin{aligned}
\epsilon &< \frac{\ln(\alpha_2) - \ln(\alpha_1) - \delta}{\ln(\alpha_1)} \\
&= \frac{-(\tau_1 - \tau_2 - \delta)}{\tau_1}
\end{aligned}$$

Now, since $\lim_{n\to\infty} \ln(\beta_n) = \ln(\beta)$ and:

$$\begin{aligned}\lim_{n\to\infty} \frac{\tau_n^{(1)} + \ln(\beta_n)}{\lambda_r} &= \frac{\tau_1 + \ln(\beta)}{\lambda_r} \\ \lim_{n\to\infty} \frac{\tau_n^{(2)} + \ln(\beta_n)}{\lambda_{r-1}} &= \frac{\tau_2 + \ln(\beta)}{\lambda_{r-1}}\end{aligned}$$

it is possible to choose $N$ such that for $n \geq N$:

$$\frac{\tau_n^{(1)} + \ln(\beta_n)}{\lambda_r}(1+\epsilon) - \frac{\tau_n^{(2)} + \ln(\beta_n)}{\lambda_{r-1}} < -\ln(\beta_n)$$

which completes the proof of the lemma. Henceforth, we will take $N_1$ large enough so that the lemma holds for $n \geq N_1$. Now we choose $\mu > 0$ such that the following inequalities hold:

$$\mu\alpha_1^{-(N_1+o_1(N_1))}||L_{N_1}|| < 1$$

and:

$$2\mu^{\frac{\lambda_{r-1}}{\lambda_r}}(\sqrt{M})^{\frac{\lambda_{r-1}}{\lambda_r}+\lambda_{r-1}-1} < \gamma_{r-1}$$

Then choose $0 < \gamma_r < \mu$. Now let's assume that $\mathcal{L}$ is a rational subspace of dimension $r$ with:

$$\rho(\theta, \mathcal{L}) < \gamma_r V(\mathcal{L})^{-\lambda_r}$$

and let $N$ be the largest integer with:

$$V(\mathcal{L})^{\lambda_r} \geq \mu\alpha_1^{-(N+o_1(N))}||L_N||$$

Such an $N$ exists since the set of such numbers includes $N_1$ (and is hence nonempty) and is bounded since $||L_n|| \geq 1$ for all $n$ and $\lim_{n\to\infty} n+o_i(n) = \infty$ for $i = 1, 2,$ and 3. In particular, this is true when $i = 1$.

We let $\mathcal{L}_1$ be the rational subspace defined by $L_N(x) = 0$. Then:

$$\rho(\theta, \mathcal{L}_1) = \frac{|L_N(\theta)|}{||L_N||} \geq \alpha_1^{N+o_1(N)}||L_N||^{-1} \geq \mu V(\mathcal{L})^{-\lambda_r} \geq \gamma_r V(\mathcal{L})^{-\lambda_r} > \rho(\theta, \mathcal{L})$$

By Lemma (9.2), $\mathcal{L} \not\subset \mathcal{L}_1$. Let $\mathcal{M} = \mathcal{L} \cap \mathcal{L}_1$. Then by Lemma (9.3):

$$\begin{aligned} V(\mathcal{M})\rho(\theta,\mathcal{M}) &\leq V(\mathcal{L})V(\mathcal{L}_1)(\rho(\theta,\mathcal{L}) + \rho(\theta,\mathcal{L}_1)) \\ &\leq V(\mathcal{L})||L_N||2\rho(\theta,\mathcal{L}_1) \\ &= 2V(\mathcal{L})|L_N(\theta)| \end{aligned}$$

and:

$$V(\mathcal{M}) \leq V(\mathcal{L})V(\mathcal{L}_1) \leq V(\mathcal{L})||L_N||$$

Using these two inequalities, the fact that $\lambda_{r-1} \geq 1$, and applying the induction hypothesis to $\mathcal{M}$ we have:

$$\gamma_{r-1}V(\mathcal{M})^{-\lambda_{r-1}} \leq \rho(\theta,\mathcal{M}) \leq 2V(\mathcal{L})|L_N(\theta)|V(\mathcal{M})^{-1}$$

Hence, using $V(\mathcal{M}) \leq V(\mathcal{L})||L_N||$:

$$\begin{aligned} \gamma_{r-1} &\leq 2V(\mathcal{L})|L_N(\theta)|V(\mathcal{M})^{\lambda_{r-1}-1} \\ &\leq 2V(\mathcal{L})^{\lambda_{r-1}}||L_N||^{\lambda_{r-1}-1}|L_N(\theta)| \end{aligned}$$

By hypothesis, $||L_n|| \leq \sqrt{M}\beta^{n+o_3(n)}$ and from the choice of $N$:

$$V(\mathcal{L})^{\lambda_r} < \mu\alpha_1^{-(N+1+o_1(N+1))}||L_{N+1}|| \leq \mu\alpha_1^{-(N+1+o_1(N+1))}\sqrt{M}\beta^{N+1+o_3(N+1)}$$

So we have:

$$\begin{aligned} \gamma_{r-1} &\leq 2\left(\mu\alpha_1^{-(N+1+o_1(N+1))}\sqrt{M}\beta^{N+1+o_3(N+1)}\right)^{\frac{\lambda_{r-1}}{\lambda_r}}||L_N||^{\lambda_{r-1}-1}|L_N(\theta)| \\ &\leq 2\left(\mu\alpha_1^{-(N+1+o_1(N+1))}\sqrt{M}\beta^{N+1+o_3(N+1)}\right)^{\frac{\lambda_{r-1}}{\lambda_r}} \\ &\quad\cdot\left(\sqrt{M}\beta^{N+o_3(N)}\right)^{\lambda_{r-1}-1}\alpha_2^{N+o_2(N)} \\ &\leq 2\mu^{\frac{\lambda_{r-1}}{\lambda_r}}(\sqrt{M})^{\frac{\lambda_{r-1}}{\lambda_r}+\lambda_{r-1}-1}\alpha_1^{-(N+o_1(N))(1+\epsilon)\frac{\lambda_{r-1}}{\lambda_r}}\beta^{(N+o_3(N))(1+\epsilon)\frac{\lambda_{r-1}}{\lambda_r}} \\ &\quad\cdot\beta^{(N+o_3(N))(\lambda_{r-1}-1)}\alpha_2^{N+o_2(N)} \\ &= 2\mu^{\frac{\lambda_{r-1}}{\lambda_r}}(\sqrt{M})^{\frac{\lambda_{r-1}}{\lambda_r}+\lambda_{r-1}-1} \\ &\quad\cdot\left(\frac{\alpha_2^{1+\frac{o_2(N)}{N}}}{\alpha_1^{\left(1+\frac{o_1(N)}{N}\right)\frac{\lambda_{r-1}}{\lambda_r}(1+\epsilon)}}\cdot\beta^{\left(1+\frac{o_3(N)}{N}\right)\left(\frac{\lambda_{r-1}}{\lambda_r}(1+\epsilon)+\lambda_{r-1}-1\right)}\right)^N \end{aligned}$$

So we have:

$$\gamma_{r-1} \leq 2\mu^{\frac{\lambda_{r-1}}{\lambda_r}}(\sqrt{M})^{\frac{\lambda_{r-1}}{\lambda_r}+\lambda_{r-1}-1}e^{\left[\lambda_{r-1}\left(\frac{\tau_N^{(1)}+\ln(\beta_N)}{\lambda_r}(1+\epsilon)-\frac{\tau_N^{(2)}+\ln(\beta_N)}{\lambda_{r-1}}+\ln(\beta_N)\right)\right]N}$$

Since $2\mu^{\frac{\lambda_{r-1}}{\lambda_r}}(\sqrt{M})^{\frac{\lambda_{r-1}}{\lambda_r}+\lambda_{r-1}-1} < \gamma_{r-1}$, this is a contradiction if:

$$\frac{\tau_N^{(1)}+\ln(\beta_N)}{\lambda_r}(1+\epsilon)-\frac{\tau_N^{(2)}+\ln(\beta_N)}{\lambda_{r-1}} \leq -\ln(\beta_N)$$

which is, in fact, the case by Lemma (9.4) since $N \geq N_1$. This completes the proof of Nesterenko's criterion.

# Chapter 10

## Conclusion

Now we are able to establish Theorem (1.1). Recall from chapter eight that $d_{n,Q} = lcm\{1, 2, ..., (n+1)Q + m\}$. We let $p_{l,n} = d_{n,Q}^a P_{l,n}(1)$ and $\widehat{p}_{d,n} = d_{n,Q}^a \widehat{P}_{d,n}(1)$. Then:

$$L_n = \sum_{l=2}^{a} p_{l,n} L(l, c) + \sum_{d=1+m}^{Q+m} \widehat{p}_{d,n} c_d$$

is a sequence of linear forms with integer coefficients by the corollary to Proposition (8.2). Likewise, in the case where $c_d = \chi(d)$, if $\chi(-1) = -1$ we let $p_{l,n} = d_{n,Q}^a P_{2l,n}(1)$ and $\widehat{p}_{d,n} = d_{n,Q}^a \widehat{P}_{d,n}(1)$. Then we have:

$$L_n = \sum_{l=1}^{\frac{a}{2}} p_{l,n} L(2l, \chi) + \sum_{d=1+m}^{Q+m} \widehat{p}_{d,n} \chi(d)$$

Lastly, if $\chi(-1) = 1$ we let $p_{l,n} = d_{2n,Q}^a P_{2l+1,2n}(1)$ and $\widehat{p}_{d,n} = d_{2n,Q}^a \widehat{P}_{d,2n}(1)$. Then we have:

$$L_n = \sum_{l=1}^{\frac{a-1}{2}} p_{l,n} L(2l+1, \chi) + \sum_{d=1+m}^{Q+m} \widehat{p}_{d,n} \chi(d)$$

By the prime number theorem we estimate $d_{n,Q} = e^{nQ+o(n)}$ and so we see that if we're not in the case where $c_d = \chi(d)$ and $\chi(-1) = 1$:

$$\log |L_n| = n \log \left(\phi_{Q,r,a} e^a\right)^Q + o(n)$$

by Proposition (6.1). We also have by Proposition (8.1):

$$\log |p_{l,n}| \leq n \log \left(e^a Q^{2R+1} 2^{A-2R} (2R+1)^{2R+1}\right)^Q + o(n)$$

and so we take:

$$\beta = \left[Q^{2R+1} e^a 2^{A-2R} (2R+1)^{2R+1}\right]^Q$$

With this $\beta$, the $L_n$ satisfy (iii) in Nesterenko's criterion where $M = a + Q - 1$ for general $c$ and $M = \frac{a+2Q}{2}$ if $c_d = \chi(d)$ and $\chi(-1) = -1$. We then let:

$$\alpha_1 = \alpha_2 = (e^a \phi_{Q,r,a})^Q$$

Otherwise, if $c_d = \chi(d)$ and $\chi(-1) = 1$, with $M = \frac{a+2Q-1}{2}$ we take:

$$\beta = \left[Q^{2R+1} e^a 2^{A-2R} (2R+1)^{2R+1}\right]^{2Q}$$

and:

$$\alpha_1 = \alpha_2 = (e^a \phi_{Q,r,a})^{2Q}$$

Then by Proposition (9.1), in all three of these cases, we have:

$$\begin{aligned} \delta_c(a) &\geq \frac{\log(\beta) - \log(\alpha_1)}{\log(\beta)} \\ &= \frac{(2R+1)\log(Q) + (A-2R)\log(2) + (2R+1)\log(2R+1) - \log(\phi_{Q,r,a})}{a + (2R+1)\log(Q) + (A-2R)\log(2) + (2R+1)\log(2R+1)} \end{aligned}$$

Using the inequalities $2R < 2R+1 < 2(R+1)$ and $\phi_{Q,r,a} \leq \frac{Q^{2R} 2^{R+1}}{R^{A-2R}}$ we obtain:

$$\delta_c(a) \geq \frac{\log(R) + \frac{A-R}{A+1}\log(2) + \frac{\log(Q)}{A+1}}{Q + \log(2) + \frac{2R+1}{A+1}\log(R+1) + \frac{2R+1}{A+1}\log(Q)}$$

Finally, we take $R \approx A/(\log(A))^2$ with $r$ the integer part of $a/(\log(A))^2$. Since:

$$\log(R) + \frac{A-R}{A+1}\log(2) + \frac{\log(Q)}{A+1} = (1+o(1))\log(A)$$

and:

$$Q + \log(2) + \frac{2R+1}{A+1}\left(\log(R+1) + \log(Q)\right) = Q + \log(2) + o(1)$$

As $A \to \infty$ we have:

$$\delta_c(a) \geq \frac{(1+o(1))\log\left(\frac{a}{Q}\right)}{Q + \log(2) + o(1)}$$

which proves Theorem (1.1). This also suffices to prove Conjecture (1.3) assuming that Proposition (6.1) is true unconditionally.

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