SPIRAL SCHUBERT VARIETIES FOR AFFINE A_2

by

WENJING LI

(Under the Direction of William Graham)

ABSTRACT

The geometry of Schubert varieties $X(w)$ for a Kač-Moody group G is closely related to the corresponding affine Weyl group W. A great deal of geometric information is encoded in the Bruhat order on W. In particular, given a pair of elements $x \leq w$ in W, there are integers q_x^w defined using the Bruhat order which can be used to determine rational smoothness of $X(w)$. We prove general results relating the Bruhat order for W of type $\widetilde{A}_2$ to the action of W on $\mathbb{R}^2$, using the bijection of W with the center points of the alcoves on $\mathbb{R}^2$. We apply these results to an interesting family of elements $w(\ell) \in W$ $(\ell \in \mathbb{N})$ called spiral elements. We show that $x \leq w(\ell)$ if and only if the corresponding center point xq lies in a region $R(\ell)$ which is close to a triangle. Using this we determine all the $q_x^{w(\ell)}$ and determine the set of rationally smooth points of $X(w(\ell))$. This leads to the proof of the lookup conjecture for spiral Schubert varieties $X(w(\ell))$.

INDEX WORDS: Schubert variety; Rational smoothness; Affine Weyl group; Bruhat order

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WENJING LI

B.S., South China University of Technology, 1999

M.S., University of South Alabama, 2006

M.A., University of Georgia, 2011

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WENJING LI

Major Professor: William Graham

Committee: Brian D. Boe
Daniel K. Nakano
Edward A. Azoff

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
August 2012

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INTRODUCTION

This thesis concerns an interesting family of Schubert varieties in type $\widetilde{A}_2$, called spiral Schubert varieties. The study of these varieties is motivated by a conjecture called the Lookup Conjecture. We begin by explaining some of the background.

Let G be the Kač-Moody group associated to a Kač-Moody Lie algebra and let B denote a standard Borel subgroup of G. For example, G could be a finite dimensional semisimple algebraic group, or an affine Kač-Moody group constructed from a finite-dimensional algebraic group. The flag variety is the coset space G/B; this has the structure of a projective ind-variety [12, Chapter VII]. The Schubert varieties in G/B are the B-orbit closures $X(w) := \overline{B \cdot wB}$, where w is in the Weyl group W of G [12, Proposition 7.1.21]. If G is the "untwisted" affine Kač-Moody group constructed from a finite dimensional semisimple algebraic group as in [12, Chapter 13], then W is obtained from the Weyl group W_{finite} of the finite-dimensional group by adding one additional generator. The Bruhat order arises from inclusions among B orbit-closures. If $u, v \in W$, then $u < v$ in Bruhat order $\Leftrightarrow X(u) \subset X(v)$ (see [12], Definition 7.1.13).

Let X be a complex algebraic variety of dimension n and $x \in X$. If X is smooth in a neighborhood of x, then

$$H^i(X, X - \{x\}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 2n, \\ 0, & \text{else.} \end{cases}$$

This motivates the following definition. The variety X is rationally smooth at a point x if and only if

$$H^i(X, X - \{x\}; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 2n, \\ 0, & \text{else.} \end{cases}$$

Rational smoothness is a weaker condition than smoothness.

Kazhdan and Lusztig [11] discovered deep connections between the Bruhat order, representation theory, and singularities of Schubert varieties. For each $w \in W$, there are Kazhdan-Lusztig polynomials $P_{x,w}(q)$ for $x \leq w$ [10] which give a large amount information about the geometry of the Schubert variety $X(w)$. The polynomials $P_{x,w}(q)$ can be used to compute dimensions of the intersection homology groups for the Schubert variety $X(w)$ [11]. They are connected to representation theory because if G is a finite dimensional semisimple algebraic group, then the values $P_{x,w}(1)$ are equal to multiplicities in composition series of certain representations of the Lie algebra of G. Rational smoothness is significant because Kazhdan and Lusztig proved that $X(w)$ is rationally smooth at a point $xB \in G/B$ if and only if $P_{x,w}(q) = 1$. There is an algorithm to compute Kazhdan-Lusztig polynomials but it is very time consuming even on a computer to use this to test rational smoothness.

For each $w \in W$ there are integers q_x^w for $x \leq w$ which are connected to the singularities of the affine Schubert variety $X(w)$. Carrell and Peterson [5] discovered a simpler criterion for rational smoothness of Schubert varieties in terms of the q_x^w. Given $x \leq w$ in W, define

$$\Psi_x^w := \{r \in R \mid rx \leq w\}$$
$$q_x^w := |\Psi_x^w| - \ell(w).$$

Here R is the set of reflections in W, and $\ell(w) = \dim X(w)$ is the length of w. By [6], $|\Psi_x^w| \geq \ell(w)$. The integers q_x^w depend only on Bruhat order. Like the Bruhat order, they have geometric significance. There are $|\Psi_x^w|$ T-invariant curves in $X(w)$ through the point xB. The integer q_x^w is the difference between the number of such curves and the minimum number $\ell(w)$. The following is the Carrell-Peterson criterion.

THEOREM 1.1. [12, Theorem 12.2.14] Let $X(w)$ be an affine Schubert variety. xB is not rationally smooth in $X(w)$ if and only if $q_y^w > 0$ for some y with $x \leq y \leq w$.

The Carrell-Peterson test requires computing q_y^w for all $x \leq y \leq w$. Unfortunately in large examples there may be many $y's$ $(x \leq y \leq w)$ for which we need to compute q_y^w. Also, it is difficult to compare elements in the Bruhat order. Boe and Graham [4] make the following "Lookup Conjecture" which states that the Carrell-Peterson criterion can be simplified.

CONJECTURE 1.2. [4, Conjecture 1.1] xB is not rationally smooth in $X(w) \Leftrightarrow q_x^w > 0$ or $q_{rx}^w > 0$, some $x < rx \leq w$ with $r \in R$.

The Lookup Conjecture asserts that to test rational smoothness, it is only necessary to compute q_y^w for the bottom elements $y = x$ and $y = rx$ with r reflection in the Bruhat graph. This would greatly reduce the amount of computation.

Given a pair $x \leq w \in W$, the trivial case of the lookup conjecture is the case that $q_x^w \neq 0$. The nontrivial case of the lookup conjecture is the case that $q_x^w = 0$ and $q_{rx}^w \neq 0$, some $x \leq rx \leq w$ with $r \in R$. The lookup conjecture asserts that in this case $q_{rx}^w \neq 0$ for some reflection r, i.e., there is an element connected to x by a single edge in the Bruhat graph such that the corresponding $q_{rx}^w \neq 0$. In type A, only the trivial case of the lookup conjecture occurs, but in all other known cases, the nontrivial case of the lookup conjecture occurs.

We have written a computer program using GAP/Chevie4 [7] [13] to calculate q_x^w and test the lookup conjecture for rational smoothness of affine Schubert varieties. We have tested the lookup conjecture for pairs (x, w) with $x \leq w$ in the following cases: type $\widetilde{A}_2$: all w with $\ell(w) \leq 19$, $\widetilde{B}_2$: all w with $\ell(w) \leq 9$, $\widetilde{G}_2$: all w with $\ell(w) \leq 12$ and $\widetilde{A}_3$: all w with $\ell(w) \leq 10$. The lookup conjecture is true for all the above pairs we have calculated and tested. Computer data suggests that in type $\widetilde{A}_2$, the nontrivial case of the lookup conjecture occurs only for a family of elements we call spiral. More precisely:

CONJECTURE 1.3. In type $\widetilde{A}_2$, if w is not spiral, then xB is n.r.s. in $X(w) \Leftrightarrow q_x^w > 0$.

This conjecture is still open. It motivates us to look at spiral elements, where it is false that xB is not rationally smooth $\Leftrightarrow q_x^w > 0$. The term spiral has been used in [1], [2] in a related but slightly different context.

For the remainder of this introduction we work in type $\widetilde{A}_2$. In this case, $W = \langle s_1, s_2, s_3 \rangle$. Let $w(\ell) = s_1 s_2 s_3 s_1 s_2 s_3 \cdots$ (ℓ factors). We say $w(\ell)$ is spiral of length ℓ. We call these elements and the varieties associated to them spiral. By a similar argument we could study the elements of this form $s_i s_j s_k s_i s_j s_k \cdots$ ($\{i, j, k\} = \{1, 2, 3\}$), which are also called spiral.

In order to approach the Lookup Conjecture for spiral Schubert varieties, we need to overcome the obstacle that there are algorithms for comparing elements in the Bruhat order, but it is difficult to use these algorithms to prove theorems. Our strategy is to study the Bruhat order in affine A_2 by relating it to the action of W on $\mathbb{R}^2$. We use this action to better understand the combinatorics of rational smoothness. These techniques are also useful in determining the set of smooth points.

For affine Weyl group W of type $\widetilde{A}_2$, the plane $\mathbb{R}^2$ is spanned by elements α_1, α_2. Let $\widetilde{\alpha} = \alpha_1 + \alpha_2$. Let $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\widetilde{\alpha}\}$. If β is in Φ, let $H_{\beta,n}$ denote the hyperplane $\{p \in \mathbb{R}^2 \mid (p, \beta) = n\}$. The alcoves are the connected components of $\mathbb{R}^2 \setminus \{H_{\beta,n} \mid \beta = \alpha_1, \alpha_2, \widetilde{\alpha}, n \in \mathbb{Z}\}$. The alcove bounded by the hyperplanes $H_{\alpha_1,0}, H_{\alpha_2,0}, H_{\widetilde{\alpha},1}$ is called the fundamental alcove and denoted by $A_\circ$. Let q be the center of $A_\circ$. The Weyl group W acts on $\mathbb{R}^2$, and there is a bijection $w \mapsto wq$ between W and the set of center points of the alcoves. By examining the locations of the center points of alcoves, we obtain information about the Bruhat order on W.

We now describe the results of the thesis in more detail. Chapter 2 gives preliminary results. We define the window notation for elements of W and define the W-action on $\mathbb{R}^2$.

The main result of Section 2.4 is Proposition 2.13 which describes the set $Wq \cap L$, where L is a line parallel to β through some uq $(u \in W)$ where uq is center of some alcove. In Section 2.5 we give formulas relating the W-action on $\mathbb{R}^2$ to window notation.

Chapter 3 gives general results about the Bruhat order. Section 3.1 gives a characterization of the Bruhat order in type $\widetilde{A}_{n-1}$ analogous to the characterization in type A_n. The proof of this uses the characterization of Bruhat order in type $\widetilde{A}_{n-1}$ in [3]. In Section 3.2, we study the W action on the set of alcoves. Section 3.3 gives the endpoint theorem, which states that if yq lies between xq and zq on a line L as above, then $y \leq x$ or $y \leq z$ in the Bruhat order. The endpoint theorem follows from a more detailed result about the Bruhat order which we call the six region theorem (Theorem 3.19).

Chapter 4 studies the reflection sets Ψ_x^w and some properties of the reflection sets. In Section 4.1 we prove that in type $\widetilde{A}_2$, the sets Ψ_x^w look like unions of intervals (Theorem 4.2). Section 4.2 contains some general results about Ψ_x^w which are used later.

In Chapter 5, for a spiral element $w(\ell)$, we give a geometric description of the set $\{x \in W \mid x \leq w(\ell)\}$. We define a triangle $\overline{\triangle}(\ell) \subset \mathbb{R}^2$, and let $\triangle(\ell)$ consist of the points Wq inside $\overline{\triangle}(\ell)$. We define $R(\ell)$ as follows. If ℓ is even, then $R(\ell) = \triangle(\ell)$; if ℓ is even, then $R(\ell) = \triangle(\ell) \setminus \{A_1(\ell)q, A_2(\ell)q\}$ where $A_1(\ell), A_2(\ell)$ are elements of W (see Definition 5.9). We prove two "triangle theorems". The first is the following.

THEOREM 1.4. Let $\ell \geq 6$. Let $w(\ell)$ be a spiral element of length ℓ. Then

$$x \leq w(\ell) \Leftrightarrow xq \in R(\ell).$$

The proof that $x \leq w(\ell) \Rightarrow xq \in R(\ell)$ is naturally divided into many cases. The strategy is to use the identification $W = L(\Phi^\vee) \rtimes W_{finite}$. Suppose $x = t(A\alpha_1^\vee + B\alpha_2^\vee)u$, where $u \in W_{finite}$. We write x and $w(\ell)$ in window notation, then use $x^{\leqslant i} \leq w(\ell)^{\leqslant i}$ to get some inequalities involving A and B. These inequalities show that xq is in $\triangle(\ell)$. If ℓ is odd, we show $A_1(\ell), A_2(\ell)$ are not $\leq w(\ell)$, so xq is in $R(\ell)$. The proof that $xq \in R(\ell) \Rightarrow$

$x \leq w(\ell)$ is by induction on length ℓ, using the endpoint theorem. The second triangle theorem is the following result. The proof is similar to the proof of Theorem 1.4.

THEOREM 1.5. If ℓ is odd, then $xq \in \triangle(\ell) \Leftrightarrow x \leq w(\ell)$ or $x \leq A_1(\ell) := t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)$.

Chapter 6 contains the translation theorem and some consequences. Let $w = w(\ell)$. Given $x \leq w$ in W, for any non-negative integer ℓ define

$$\Lambda_x^w := \{r \in R \mid rxq \in \triangle(\ell)\}$$

$$p_x^w := |\Lambda_x^w| - \ell(w).$$

If ℓ is even, $p_x^w = q_x^w$. If ℓ is odd, then either $p_x^w = q_x^w$ or $p_x^w = q_x^w + 1$ (which occurs depends on x). The reason for defining the p_x^w is that they satisfy the following translation theorem.

THEOREM 1.6. Let w be a length ℓ spiral element. If $y = t(\gamma^\vee)x$ for $\gamma^\vee$ in the coroot lattice and both $xq, yq \in \triangle(\ell)$, then

$$p_x^w = p_y^w.$$

The idea of proof of the translation theorem is to compare the sets Λ_x^w and $\Lambda_{t(\beta^\vee)x}^w$ ($\beta \in \Phi$) using the inequalities defining the triangle.

THEOREM 1.7. If $w = w(\ell)$ is a spiral element, then $p_x^w = 0$ or 1. Hence $q_x^w = 0$ or 1.

To prove this, for any element x of W, we write it in terms of translation of an element $w(\ell - i)$ ($i \in \{0, 1, 2, 3, 4, 5\}$) in the finite Weyl group. We show that each $p_{w(\ell-i)}^w$ is 0 or 1 ($i \in \{0, 1, 2, 3, 4, 5\}$). The translation theorem then implies that for all $x \leq w$, $p_x^w = 0$ or 1. Since $0 \leq q_x^w \leq p_x^w$, we see that q_x^w is also 0 or 1. Note that if w is not spiral, then q_x^w need not be 0 or 1.

In Chapter 7, we identify the set of rationally smooth points in $X(w(\ell))$. This leads to the proof of the Lookup Conjecture for spiral Schubert varieties $X(w(\ell))$.

THEOREM 1.8. Let $w = w(\ell)$ be a spiral element of length ℓ with length $\ell \geq 6$. Then the point $xB \in X(w(\ell))$ is not rationally smooth $\Leftrightarrow xq \in \triangle(\ell - 3)$.

We describe the idea of the proof in case ℓ is odd (if ℓ is even, the argument is similar but slightly different). To show ($\Leftarrow$), observe that by the triangle theorem if $xq \in \triangle(\ell - 3)$, then $x \leq w(\ell - 3)$. Since $q_{w(\ell-3)}^{w(\ell)} = 1$ by calculation, x is not rationally smooth. For the other direction, if $xq \notin \triangle(\ell - 3)$, then for all $x \leq y \leq w(\ell)$, $q_y^w = 0$. This uses the Translation Theorem for p_x^w plus additional work. The Carrell-Peterson Theorem then implies that x is rationally smooth.

Using Theorem 1.8, we prove the following result, which is the Lookup Conjecture for spiral Schubert varieties.

THEOREM 1.9. Let $w(\ell)$ be spiral element. Then xB is not rationally smooth in $X(w(\ell))$ if and only if either $q_x^w \neq 0$ or $q_y^w \neq 0$ for $y = rx$, $r \in R, x < y \leq w$, where R is the set of reflections. In other words, the lookup conjecture holds for $X(w(\ell))$.

The implication ($\Leftarrow$) is an immediate consequence of the Carrell-Peterson criterion. For the other implication suppose the point xB is not rationally smooth. If $q_x^w > 0$, then we are done. Otherwise, we want to show that $q_y^w = 1, y = rx$ for some reflection r. By the previous theorem, $xq \in \triangle(\ell - 3)$ and we are able to explicitly identify a lookup direction.

2

PRELIMINARY RESULTS ABOUT AFFINE WEYL GROUP OF $\widetilde{A}_2$

This chapter contains basic results about the affine Weyl group W of type $\widetilde{A}_2$. In section 2.1, we define W. In Section 2.2, we recall that W acts on $\mathbb{Z}$ and introduce window notation, as in [3]. In Section 2.3 we recall the W-action on $\mathbb{R}^2$, and define alcoves, the fundamental alcove $A_\circ$, and the center point q of $A_\circ$. Let β be α_1, α_1 or $\widetilde{\alpha}$. The main result of Section 2.4 is Proposition 2.13 which describes the set $Wq \cap L$, where L is a line parallel to β through some uq $(u \in W)$. In Section 2.5, for $x \in W$, we give the window notation formulas for x and the formulas for $xq = u\alpha_1^\vee + v\alpha_2^\vee$ (Proposition 2.16).

2.1 Definition of the affine Weyl group W of type $\widetilde{A}_2$

The definition of W in terms of generators and relations is given as follows. In affine type $\widetilde{A}_2$, W has generators $\{s_1, s_2, s_3\}$ and relations $(s_i)^2 = 1, (s_is_j)^3 = 1, i \neq j$. The elements s_i are called simple reflections. Define reflections as ws_iw^{-1} where $w \in W$ and s_i is a simple reflection. If $w \in W$, we define the length $\ell(w)$ of w as $\ell(w) = r$ if $w = s_{i_1} \cdots s_{i_r}$ where s_{i_j} are simple reflection and r is as small as possible. In this case, $s_{i_1} \cdots s_{i_r}$ is called reduced.

2.2 The W action on $\mathbb{Z}$

In this section, we study the affine Weyl group of type $\widetilde{A}_2$ action on integers. The affine Weyl group of type $\widetilde{A}_2$ can be realized as infinite permutations. Let

$$W' = \{w \in Perm(\mathbb{Z}) \mid w(1) + w(2) + w(3) = 6, w(n+3) = w(n) + 3 \text{ for all } n \in \mathbb{Z}\}.$$

Any element $w \in W'$ is determined by the values $w(1), w(2)$, and $w(3)$. If $w \in W'$, we write $w = [w(1), w(2), w(3)]$ and refer to this as the window notation (see [3]). If $i \not\equiv j$ (mod 3), there is an element $t_{ij} \in W'$ satisfying $t_{ij}(i) = j, t_{ij}(j) = i, t_{ij}(k) = k$ if k is not congruent to i or j (mod 3) (see [3]). There is an isomorphism $W \to W'$ defined by

$$s_1 \mapsto t_{12} = [2, 1, 3],\ s_2 \mapsto t_{23} = [1, 3, 2],\ s_3 \mapsto t_{34} = [0, 2, 4].$$

We use this isomorphism to identify W with W', so we view W as a subgroup of $Perm(\mathbb{Z})$.

2.3 The W action on the plane

We relate Bruhat order to the geometry of a group acting by affine reflections. The affine Weyl group of type $\widetilde{A}_2$ acts on $\mathbb{R}^2 \cong \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} x_i = 0\}$. $\mathbb{R}^2$ is equipped with the inner product

$$(x, y) = \sum_{i=1}^{3} x_i y_i.$$

Define $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \widetilde{\alpha} = \alpha_1 + \alpha_2, \alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$. In type $\widetilde{A}_2$, $\alpha_i^\vee = \alpha_i$ but in other types, this is not always the case, so we often write $\alpha_i^\vee$ to be consistent with the notation from [9]. Given $\gamma \in \mathbb{R}^2$, let $t(\gamma) : \mathbb{R}^2 \to \mathbb{R}^2$ be translation by γ. Define

$$s_\beta = s_{\beta,0} \text{ and } s_{\beta,n} = t(n\beta^\vee)s_\beta.$$

Then $\mathbb{R}^2 =$ span of $\alpha_1^\vee, \alpha_2^\vee$. Let $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\widetilde{\alpha}\}$ and $L(\Phi^\vee)$ be the translation group corresponding to the coroot lattice (see [9, I, 4.2]). β is one of these, let $H_{\beta,n} = \{p \in \mathbb{R}^2 \mid (p, \beta) = n\}$. Observe that $s_{\beta,n}$ is reflection across the hyperplane $H_{\beta,n}$ (see [9, I, 4.1]).

Alcoves are the connected components of $\mathbb{R}^2 \setminus \{H_{\beta,n} \mid \beta = \alpha_1, \alpha_2, \widetilde{\alpha}, n \in \mathbb{Z}\}$. Informally, the hyperplanes $H_{\beta,n}$ divide the plane into triangles and the alcoves are the interiors. The alcove bounded by the hyperplanes $H_{\alpha_1,0}, H_{\alpha_2,0}, H_{\widetilde{\alpha},1}$ is called $A_\circ$. In other words, $A_\circ = \{p \mid (p, \alpha_1) > 0, (p, \alpha_2) > 0, (p, \widetilde{\alpha}) < 1\}$. Let $q = \frac{1}{3}\widetilde{\alpha}^\vee$ be the center of the alcove

$A_\circ$, so xq is the center of the alcove $x(A_\circ)$. There are bijections between W and alcoves:

$$W \to \{\text{alcoves}\} \to Wq$$

$$w \mapsto wA_\circ \mapsto wq.$$

Let Aff($\mathbb{R}^2$) be the semidirect product $GL(\mathbb{R}^2) \ltimes$(translations of $\mathbb{R}^2$). There is an injective homomorphism from W to the group Aff($\mathbb{R}^2$) of affine transformations of $\mathbb{R}^2$ given by

$$s_1 \mapsto s_{\alpha_1},\ s_2 \mapsto s_{\alpha_2},\ s_3 \mapsto s_{\widetilde{\alpha},1}$$

(see [9]). Using this homomorphism we identify W with a subgroup $L(\Phi^\vee) \rtimes W_{finite}$ of Aff($\mathbb{R}^2$). Here $W_{finite} = \langle s_{\alpha_1}, s_{\alpha_2} \rangle$ and $L(\Phi^\vee) = \{A\alpha_1^\vee + B\alpha_2^\vee\}_{A,B\in\mathbb{Z}}$ is the coroot lattice, viewed as acting on $\mathbb{R}^2$ by translations.

Recall that reflections are of the form ws_iw^{-1} where $w \in W$ and s_i is a simple reflection.

LEMMA 2.1. All reflections in W are of the form $s_{\delta,k}$ for $\delta \in \Phi$ and $k \in \mathbb{Z}$.

PROOF. We have $s_i = s_{\beta,\ell} = t(\ell\beta^\vee)s_\beta$ where $\beta \in \{\alpha_1, \alpha_2, \widetilde{\alpha}\}$ and ℓ is 0 or 1. Any $w \in W$ is of the form $w = t(\gamma^\vee)u$ for $u \in W_{finite}, \gamma^\vee \in L(\Phi^\vee)$. Then

$$\begin{aligned}
ws_iw^{-1} &= t(\gamma^\vee)ut(\ell\beta^\vee)s_\beta u^{-1}t(-\gamma^\vee)\\
&= t(\gamma^\vee)ut(\ell\beta^\vee)u^{-1}us_\beta u^{-1}t(-\gamma^\vee)\\
&= [t(\gamma^\vee + \ell u\beta^\vee)s_{u\beta}t(-\gamma^\vee)]s_{u\beta}s_{u\beta}\\
&= t(\gamma^\vee + \ell u\beta^\vee)t(s_{u\beta}(-\gamma^\vee))s_{u\beta}\\
&= t(\gamma^\vee - s_{u\beta}(\gamma^\vee) + \ell u\beta^\vee)s_{u\beta}\\
&= t(ku\beta^\vee)s_{u\beta} \text{ (some } k)\\
&= s_{u\beta,k}.
\end{aligned}$$

□

LEMMA 2.2. The following facts hold.

(a) $s_{\alpha_1,k}(u\alpha_1 + v\alpha_2) = (v - u + k)\alpha_1 + v\alpha_2$

(b) $s_{\alpha_2,k}(u\alpha_1 + v\alpha_2) = u\alpha_1 + (u - v + k)\alpha_2$

(c) $s_{\widetilde{\alpha},k}(u\alpha_1 + v\alpha_2) = (-v + k)\alpha_1 + (-u + k)\alpha_2$.

PROOF. (a)

$$\begin{aligned} s_{\alpha_1,k}(u\alpha_1 + v\alpha_2) &= t(k\alpha_1^\vee)s_{\alpha_1,0}(u\alpha_1 + v\alpha_2) \\ &= t(k\alpha_1^\vee)((v - u)\alpha_1 + v\alpha_2) \\ &= (v - u + k)\alpha_1 + v\alpha_2. \end{aligned}$$

(b)

$$\begin{aligned} s_{\alpha_2,k}(u\alpha_1 + v\alpha_2) &= t(k\alpha_2^\vee)s_{\alpha_2,0}(u\alpha_1 + v\alpha_2) \\ &= t(k\alpha_1^\vee)(u\alpha_1 + (u - v)\alpha_2) \\ &= u\alpha_1 + (u - v + k)\alpha_2. \end{aligned}$$

(c)

$$\begin{aligned} s_{\widetilde{\alpha},k}(u\alpha_1 + v\alpha_2) &= t(k\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}(u\alpha_1 + v\alpha_2) \\ &= t(k\widetilde{\alpha}^\vee)(-v\alpha_1 - u\alpha_2) \\ &= (-v + k)\alpha_1 + (-u + k)\alpha_2. \end{aligned}$$

□

LEMMA 2.3. The following holds.

(1) $s_1(\alpha_1^\vee) = -\alpha_1^\vee,\ s_1(\alpha_2^\vee) = \widetilde{\alpha}^\vee,\ s_1(\widetilde{\alpha}) = \alpha_2$.

(2) $s_2(\alpha_1) = \widetilde{\alpha},\ s_2(\alpha_2^\vee) = -\alpha_2,\ s_2(\widetilde{\alpha}) = \alpha_1$.

(3) $s_3(a\alpha_1^\vee + b\alpha_2^\vee) = (-b + 1)\alpha_1^\vee + (-a + 1)\alpha_2^\vee$.

In particular, $s_3(c\widetilde{\alpha}^\vee) = (-c + 1)\alpha_1^\vee + (-c + 1)\alpha_2^\vee = (-c + 1)\widetilde{\alpha}$.

PROOF. This follows from Lemma 2.2. □

LEMMA 2.4. Let $\beta = \alpha_1, \alpha_2$ or $\widetilde{\alpha}$. If $(\beta, \gamma^\vee) = k$, then $s_{\beta,k}t(\gamma^\vee) = t(\gamma^\vee)s_\beta$.

PROOF. We have

$$\begin{aligned} s_{\beta,k}t(\gamma^\vee) &= t(k\beta^\vee)s_\beta t(\gamma^\vee) \\ &= t(k\beta^\vee)s_\beta t(\gamma^\vee)s_\beta^{-1}s_\beta \\ &= t(k\beta^\vee)t(s_\beta\gamma^\vee)s_\beta \\ &= t(k\beta^\vee + \gamma^\vee - k\beta^\vee)s_\beta \\ &= t(\gamma^\vee)s_\beta. \end{aligned}$$

□

Expressions for translations $t(\alpha_1^\vee), t(\alpha_2^\vee)$ and $t(\widetilde{\alpha}^\vee)$ of Type $\widetilde{A}_2$ in terms of simple reflections are given as following.

LEMMA 2.5. The following holds

$$t(\alpha_1^\vee) = s_3s_2s_3s_1,\ t(\alpha_2^\vee) = s_3s_1s_3s_2,\ t(\widetilde{\alpha}^\vee) = s_3s_1s_2s_1.$$

PROOF. In the proof we use the facts that $ws_\beta w^{-1} = s_{w\beta}$ and $wt(x)w^{-1} = t(wx)$ for $\beta \in \Phi, x \in \mathbb{R}^2, w \in W_{finite}$ (see [9, I, Proposition 4.1] and [9, I.4.1]). In particular, $s_3 = t(\widetilde{\alpha}^\vee)s_{\widetilde{\alpha}} = s_{\widetilde{\alpha},0}t(-\widetilde{\alpha}^\vee)$.

$$\begin{aligned} s_3s_2s_3s_1 &= t(\widetilde{\alpha}^\vee) \circ s_{\widetilde{\alpha}}s_{\alpha_2}s_{\widetilde{\alpha}} \circ t(-\widetilde{\alpha}^\vee)s_{\alpha_1} = t(\widetilde{\alpha}^\vee) \circ s_{s_{\widetilde{\alpha}}(\alpha_2)} \circ t(-\widetilde{\alpha}^\vee)s_{\alpha_1} \\ &= t(\widetilde{\alpha}^\vee) \circ s_{-\alpha_1} \circ t(-\widetilde{\alpha}^\vee)s_{\alpha_1} = t(\widetilde{\alpha}^\vee)(s_{\alpha_1}t(-\widetilde{\alpha}^\vee)s_{\alpha_1}) \\ &= t(\widetilde{\alpha}^\vee)t(s_{\alpha_1}(-\widetilde{\alpha}^\vee)) = t(\widetilde{\alpha}^\vee - s_{\alpha_1}(\widetilde{\alpha}^\vee)) \\ &= t(\alpha_1^\vee). \end{aligned}$$

Since $s_1 s_2(\alpha_1^\vee) = \alpha_2^\vee$,

$$t(\alpha_2^\vee) = s_1 s_2 t(\alpha_1^\vee) s_2 s_1 = s_1 s_2 (s_3 s_2 s_3 s_1) s_2 s_1 = s_1 (s_3 s_2) s_2 s_1 s_2 = s_1 s_3 s_1 s_2 = s_3 s_1 s_3 s_2.$$

Since $s_2(\alpha_1^\vee) = \widetilde{\alpha}^\vee$,

$$t(\widetilde{\alpha}^\vee) = s_2 t(\alpha_1^\vee) s_2 = s_2 s_3 s_2 s_3 s_1 s_2 = s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1.$$

□

COROLLARY 2.6. If $x = t(\lambda^\vee) \in W$, for $\lambda^\vee \in \mathbb{Z}\Phi^\vee$, then $\ell(x)$ is even.

PROOF. If $\lambda = a\alpha_1^\vee + b\alpha_2^\vee$ then $x = y^a z^b$ where $y = t(\alpha_1^\vee), z = t(\alpha_2^\vee)$ for some $a, b \in \mathbb{Z}$. Since $\ell(x) = \ell(y) = 4$ by Lemma 2.5, $\ell(x)$ is even. □

LEMMA 2.7. The element $a_1\alpha_1^\vee + a_2\alpha_2^\vee$ is equal to wq for some $w \in W \Leftrightarrow$ one of the following holds.

(a) a_1 and $a_2 \in \mathbb{Z} + \frac{1}{3}$

(b) a_1 and $a_2 \in \mathbb{Z} + \frac{2}{3}$

(c) One of $a_1, a_2 \in \mathbb{Z}$ and the other is in $\mathbb{Z} + \frac{1}{3}$

(d) One of $a_1, a_2 \in \mathbb{Z}$ and the other is in $\mathbb{Z} + \frac{2}{3}$.

PROOF. ($\Rightarrow$): Observe that $\mathbb{Z} + \frac{2}{3} = \mathbb{Z} - \frac{1}{3}$. It suffices to check this for $w \in W_{finite}$ since any $w \in W$ is of the form $t(\beta^\vee)w_f$ where $\beta^\vee \in L(\Phi^\vee)$ and $w_f \in W_{finite}$. Since $q = \frac{1}{3}\widetilde{\alpha}^\vee = \frac{1}{3}\alpha_1^\vee + \frac{1}{3}\alpha_2^\vee$, we have $s_1 q = \frac{1}{3}\alpha_2^\vee, s_2 q = \frac{1}{3}\alpha_1^\vee, s_1 s_2 q = -\frac{1}{3}\alpha_1^\vee, s_2 s_1 q = -\frac{1}{3}\alpha_2^\vee, s_1 s_2 s_1 q = -\frac{1}{3}\alpha_1^\vee - \frac{1}{3}\alpha_2^\vee$. We see that for each $w \in W_{finite}$, wq is of the form stated.

($\Leftarrow$): The calculations above show that each of the possibilities listed in (a)-(d) occurs as $w_f q$ for some $w_f \in W_{finite}$. The result follows because $t(\beta^\vee)w_f$ is in W for any $\beta^\vee \in L(\Phi^\vee)$. □

COROLLARY 2.8. Let $\beta \in \Phi$. $(wq, \beta) \in \mathbb{Z} + \frac{1}{3}$ or $\mathbb{Z} + \frac{2}{3}$ (not in $\mathbb{Z}$).

PROOF. This follows from Lemma 2.7. □

2.4 The points Wq lying on a line

The main result of this section is Proposition 2.13 which describes the set $Wq \cap L$, where L is a line parallel to β through some uq $(u \in W)$ where uq is center of some alcove.

LEMMA 2.9. Suppose $p = wq, k < (p, \beta) < k + 1$. Then

(1) $s_{\beta,k+1}p = p + \frac{1}{3}\beta^\vee$ or $p + \frac{2}{3}\beta^\vee$.

(2) $s_{\beta,k}p = p - \frac{2}{3}\beta^\vee$ or $p - \frac{1}{3}\beta^\vee$.

PROOF. Let $a = (p, \beta)$. By Corollary 2.8, $a = k + \frac{1}{3}$ or $a = k + \frac{2}{3}$. Then

$$s_{\beta,k+1}p = p - ((p, \beta) - (k + 1))\beta^\vee = p + [(k + 1) - a]\beta^\vee = p + \frac{1}{3}\beta^\vee \text{ or } p + \frac{2}{3}\beta^\vee.$$

Similarly,

$$s_{\beta,k}p = p - ((p, \beta) - k)\beta^\vee = p + [k - a]\beta^\vee = p - \frac{2}{3}\beta^\vee \text{ or } p - \frac{1}{3}\beta^\vee.$$

□

LEMMA 2.10. Let $\beta \in \Phi$. Suppose $p = wq, k < (p, \beta) < k+1$, so $s_{\beta,k+1}p = p+c\beta^\vee, c = (k + 1) - (p, \beta)$ and $s_{\beta,k} = p + d\beta^\vee, d = k - (p, \beta)$.

(1) There is no t with $0 < t < c$ such that $p + t\beta^\vee = uq$ for some $u \in W$.

(2) There is no t with $d < t < 0$ such that $p + t\beta^\vee = uq$ for some $u \in W$.

PROOF. (1) By Corollary 2.8, $c = \frac{1}{3}$ or $c = \frac{2}{3}$. If $p + t\beta = uq$, then by Lemma 2.7, $3t \in \mathbb{Z}$, so if $0 < t$, then $1 \leq 3t$, so $\frac{1}{3} \leq t$. So if $c = \frac{1}{3}$, we are done. Suppose $c = \frac{2}{3}$. The only possible t with $0 < t < c$ and $p + t\beta^\vee = uq$ is $t = \frac{1}{3}$, and we must exclude this. In other words, we want to show that $p+\frac{1}{3}\beta^\vee$ is not of the form uq. Since $c = \frac{2}{3}$, $(k+1)-(p,\beta) = \frac{2}{3}$. Hence $(p,\beta) = k + \frac{1}{3}$. Then $(p + \frac{1}{3}\beta^\vee, \beta) = (k + \frac{1}{3}) + \frac{2}{3} = k + 1 \in \mathbb{Z}$, so by Corollary 2.8, $p + t\beta^\vee = uq$ is not of the form uq for $u \in W$.

(2) By Corollary 2.8, $d = -\frac{1}{3}$ or $-\frac{2}{3}$. If $p + t\beta = uq$, then by Lemma 2.7, $3t \in \mathbb{Z}$, so if $t < 0$, then $3t \leq -1$, so $t \leq -\frac{1}{3}$. So if $d = -\frac{1}{3}$, we are done. Suppose $d = -\frac{2}{3}$. The only possible t with $d < t < 0$ and $p + t\beta^\vee = uq$ is $t = -\frac{1}{3}$, and we must exclude this. In other words, we want to show that $p-\frac{1}{3}\beta^\vee$ is not of the form uq. Since $d = -\frac{2}{3}$, $k-(p,\beta) = -\frac{2}{3}$. Hence $(p,\beta) = k + \frac{2}{3}$. Then $(p - \frac{1}{3}\beta^\vee, \beta) = (k + \frac{2}{3}) - \frac{2}{3} = k \in \mathbb{Z}$, so by Corollary 2.8, $p + t\beta^\vee = uq$ is not of the form uq for $u \in W$. □

Definition 2.11. Suppose $p = uq$ for some $u \in W$ and suppose $k < (p,\beta) < k+1$. Define $\{p_j\}_{j\in\mathbb{Z}}$ inductively as follows. Let $p_0 = p$. For $i > 0$, let $p_{i+1} = s_{\beta,k+i+1}p_i$. For $i < 0$, let $p_i = s_{\beta,k+i+1}p_{i+1}$. Observe then that for all $i \in \mathbb{Z}$, $p_{i+1} = s_{\beta,k+i+1}p_i$.

LEMMA 2.12. Let p_i $(i \in \mathbb{Z})$ be as in Definition 2.11. Then

(1) $(p_i, \beta^\vee) = k + i + \frac{1}{3}$ or $(p_i, \beta^\vee) = k + i + \frac{2}{3}$.

(2) If $(p_i, \beta^\vee) = k + i + \frac{1}{3}$ then $p_{i+1} = p_i + \frac{2}{3}\beta^\vee$ and $(p_{i+1}, \beta^\vee) = k + (i+1) + \frac{2}{3}$.

(3) If $(p_i, \beta^\vee) = k + i + \frac{2}{3}$ then $p_{i+1} = p_i + \frac{1}{3}\beta^\vee$ and $(p_{i+1}, \beta^\vee) = k + (i+1) + \frac{1}{3}$.

(4) $p_{i+2} = p_i + \beta^\vee$ for all i.

PROOF. We first show (2) and (3). By Lemma 2.9, $p_{i+1} = s_{\beta,k+i+1}p_i = p_i + a\beta$ where $a = \frac{1}{3}$ or $a = \frac{2}{3}$. Then

$$(p_{i+1}, \beta^\vee) = (p_i + a\beta, \beta^\vee) = (p_i, \beta^\vee) + 2a.$$

By Corollary 2.8, $(p_{i+1}, \beta^\vee) \in \mathbb{Z} + \frac{1}{3}$ or $\mathbb{Z} + \frac{2}{3}$. Therefore, if $(p_i, \beta^\vee) = k + i + \frac{1}{3}$ then $a = \frac{2}{3}$ and $(p_{i+1}, \beta^\vee) = k + (i+1) + \frac{2}{3}$. If $(p_i, \beta^\vee) = k + i + \frac{2}{3}$ then $a = \frac{1}{3}$ and $(p_{i+1}, \beta^\vee) = k + (i+1) + \frac{1}{3}$.

Now we show (1). For $i \geq 0$, we use induction on i. Our hypothesis is $k < (p_0, \beta) < k + 1$. Corollary 2.8 implies that $(p_0, \beta) = k + \frac{1}{3}$ or $k + \frac{2}{3}$. Suppose the statement of (1) is true for i. Part (2) and (3) imply that the statement of (1) is true for all $i \geq 0$.

For $i \leq 0$, we use downward induction on i. The case of $i = 0$ was verified above. Suppose the statement of (1) is true for $i \leq 0$; we show it is true for $i - 1$. By hypothesis, $k + i < (p_i, \beta) < k + i + 1$. Since $p_{i-1} = s_{\beta,k+i} p_i$, we have $p_{i-1} = p_i + a\beta$ where $a = -\frac{1}{3}$ or $a = -\frac{2}{3}$. By Corollary 2.8, $(p_{i-1}, \beta^\vee) \in \mathbb{Z} + \frac{1}{3}$ or $\mathbb{Z} + \frac{2}{3}$. Therefore, if $(p_i, \beta) = k + i + \frac{1}{3}$ then $(p_{i-1}, \beta) = k + i + \frac{1}{3} + 2a$, so $a = -\frac{1}{3}$ and $(p_{i-1}, \beta) = k + i - \frac{1}{3} = k + (i-1) + \frac{2}{3}$. If $(p_i, \beta) = k + i + \frac{2}{3}$ then $(p_{i-1}, \beta) = k + i + \frac{2}{3} + 2a$, so $a = -\frac{2}{3}$ and $(p_{i-1}, \beta) = k + i - \frac{2}{3} = k + (i-1) + \frac{1}{3}$. Therefore the statement of (1) is true for $i - 1$. We conclude that (1) holds for all $i \leq 0$, so by the previous paragraph it holds for all $i \in \mathbb{Z}$.

We show (4) using part (2) and (3). If $(p_i, \beta^\vee) = k + i + \frac{1}{3}$ then $p_{i+1} = p_i + \frac{2}{3}\beta^\vee$ and $(p_{i+1}, \beta^\vee) = k + (i+1) + \frac{2}{3}$. Then

$$p_{i+2} = p_{i+1} + \frac{1}{3}\beta^\vee = p_i + \frac{2}{3}\beta^\vee + \frac{1}{3}\beta^\vee = p_i + \beta^\vee.$$

If $(p_i, \beta^\vee) = k + i + \frac{2}{3}$ then $p_{i+1} = p_i + \frac{1}{3}\beta^\vee$ and $(p_{i+1}, \beta^\vee) = k + (i+1) + \frac{1}{3}$. Then

$$p_{i+2} = p_{i+1} + \frac{2}{3}\beta^\vee = p_i + \frac{1}{3}\beta^\vee + \frac{2}{3}\beta^\vee = p_i + \beta^\vee.$$

□

We give the description of the points wq which lie on a line.

PROPOSITION 2.13. Let p_i be as in Definition 2.11. Define $t_i \in \mathbb{R}$ by the equation $p_i = p + t_i\beta^\vee, t_i \in \mathbb{R}$. Then

(1) $t_{-i} < t_{-i+1} < \cdots < t_0 = 0 < t_1 < \cdots < t_i$.

(2) The set $\{uq \mid uq = p + t\beta^\vee\} = \{p_i\}_{i\in\mathbb{Z}}$. Equivalently, $p + t\beta^\vee = uq$ for some $u \Leftrightarrow t = t_i$ for some i.

(3) $k + i < (p_i, \beta^\vee) < k + i + 1$.

(4) Suppose x, y are two elements of W such that xq, yq are on the line $xq + t\beta^\vee = L$ and $y = rx$, where r is a reflection. Then $r = s_{\beta,\ell}$ for some $\ell \in \mathbb{Z}$. Moreover, any $z \in W$ with $zq \in L$ is of the form $z = t(n\beta^\vee)x$ or $z = t(m\beta^\vee)y$.

(5) Let x, y be as in (4). Then any zq on the line $L = xq + t\beta^\vee$ is of the form $r_1 x$ and $x_2 y$, where $r_i = s_{\beta,k_i}$ for some $k_i \in \mathbb{Z}$ $(i = 1, 2)$.

PROOF. (1) By Lemma 2.12, $t_{i+1} - t_i = \frac{2}{3}$ or $\frac{1}{3}$. Hence $\cdots < t_{-1} < t_0 = 0 < t_1 < \cdots$.

(2) Suppose $p + t\beta^\vee = uq$. Then for some i, $t_i \le t \le t_{i+1}$. By Lemma 2.10, $t = t_i$ or $t = t_{i+1}$. Therefore if $p + t\beta = uq$, then $t = t_i$ for some i.

(3) This follows from Lemma 2.12.

(4) By Lemma 2.1, $r = s_{\delta,\ell}$ for $\delta \in \Phi$ and $\ell \in \mathbb{Z}$. Since $xq - rxq$ is parallel to δ and to β, we have $\delta = \pm\beta$, so (replacing ℓ by $-\ell$ if necessary), we see that $r = s_{\beta,\ell}$.

By (2), $xq = p_i$ and $yq = p_j$ for some i, j. By Lemma 2.12, $i \not\equiv j$ (mod 2) because if $i \equiv j$ (mod 2), then $y = t(\frac{j-i}{2}\beta^\vee)x$. Since any zq on L is of the form p_ℓ for some ℓ, we see that if $\ell \equiv i$ (mod 2), then $z = t(\frac{\ell-i}{2}\beta^\vee)x$ and if $\ell \equiv j$ (mod 2), then $z = t(\frac{\ell-j}{2}\beta^\vee)y$.

(5) This follows from (4). □

2.5 Facts relating window notation and the W action on $\mathbb{R}^2$

The affine Weyl group W is a semidirect product $L(\Phi^\vee) \rtimes W_{finite}$. The main result of this section is Proposition 2.16, which gives the window notation formula for $x = t(A\alpha_1^\vee + B\alpha_2^\vee)y$, for $A, B \in \mathbb{Z}$ and $y \in W_{finite}$, and also gives the formula for $xq = u\alpha_1^\vee + v\alpha_2^\vee$. We begin with two lemmas giving special cases of this proposition.

LEMMA 2.14. In window notation,

$$t(\alpha_1^\vee) = [4, -1, 3],\ t(\alpha_2^\vee) = [1, 5, 0],\ t(\widetilde{\alpha}^\vee) = [4, 2, 0].$$

PROOF. By Lemma 2.5, we have

$$\begin{aligned}
t(\alpha_1^\vee) &= s_3s_2s_3s_1 \\
&= [1,2,3] \circ s_3s_2s_3s_1 \\
&= [0,2,4] \circ s_2s_3s_1 \\
&= [0,4,2] \circ s_3s_1 \\
&= [-1,4,3] \circ s_1 \\
&= [4,-1,3]
\end{aligned}$$

$$\begin{aligned}
t(\alpha_2^\vee) &= s_3s_1s_3s_2 \\
&= [1,2,3] \circ s_3s_1s_3s_2 \\
&= [0,2,4] \circ s_1s_3s_2 \\
&= [2,0,4] \circ s_3s_2 \\
&= [1,0,5] \circ s_2 \\
&= [1,5,0]
\end{aligned}$$

$$\begin{aligned}
t(\widetilde{\alpha}^\vee) &= s_3s_1s_2s_1 \\
&= [1,2,3] \circ s_3s_1s_2s_1 \\
&= [0,2,4] \circ s_1s_2s_1 \\
&= [2,0,4] \circ s_2s_1 \\
&= [2,4,0] \circ s_1 \\
&= [4,2,0]
\end{aligned}$$

□

LEMMA 2.15. For all $k \in \mathbb{Z}$,

$$t(k\alpha_1^\vee) = [1+3k, 2-3k, 3],\ t(k\alpha_2^\vee) = [1, 2+3k, 3-3k],\ t(k\widetilde{\alpha}^\vee) = [1+3k, 2, 3-3k].$$

PROOF. We will first show the Lemma by induction on k for $k \geqslant 0$. For $k = 0$ the statement holds since translation by 0 is the identity $e = [1, 2, 3]$. Suppose the statement holds for k. By definition, Lemma 2.5 and Lemma 2.14 we have

$$\begin{aligned}
t((k+1)\alpha_1^\vee) &= t(k\alpha_1^\vee)t(\alpha_1^\vee) \\
&= t(k\alpha_1^\vee) \circ [4, -1, 3] \\
&= [t(k\alpha_1^\vee)(4), t(k\alpha_1^\vee)(-1), t(k\alpha_1^\vee)(3)] \\
&= [3 + t(k\alpha_1^\vee)(1), -3 + t(k\alpha_1^\vee)(2), 3] \\
&= [3 + 1 + 3k, -3 + 2 - 3k, 3] \\
&= [1 + 3(k+1), 2 - 3(k+1), 3]
\end{aligned}$$

$$\begin{aligned}
t((k+1)\alpha_2^\vee) &= t(k\alpha_2^\vee)t(\alpha_2^\vee) \\
&= t(k\alpha_2^\vee) \circ [1, 5, 0] \\
&= [t(k\alpha_2^\vee)(1), t(k\alpha_2^\vee)(5), t(k\alpha_2^\vee)(0)] \\
&= [1, 3 + t(k\alpha_2^\vee)(2), -3 + t(k\alpha_2^\vee)(3)] \\
&= [1, 3 + 2 + 3k, -3 + 3 - 3k] \\
&= [1, 2 + 3(k+1), 3 - 3(k+1)]
\end{aligned}$$

$$\begin{aligned}
t((k+1)\widetilde{\alpha}^\vee) &= t(k\widetilde{\alpha}^\vee)t(\widetilde{\alpha}^\vee) \\
&= t(k\widetilde{\alpha}^\vee) \circ [4, 2, 0] \\
&= [t(k\widetilde{\alpha}^\vee)(4), t(k\widetilde{\alpha}^\vee)(2), t(k\widetilde{\alpha}^\vee)(0)] \\
&= [3 + t(k\widetilde{\alpha}^\vee)(1), 2, -3 + t(k\widetilde{\alpha}^\vee)(3)] \\
&= [3 + 1 + 3k, 2, -3 + 3 - 3k] \\
&= [1 + 3(k+1), 2, 3 - 3(k+1)].
\end{aligned}$$

This proves the statement for $k \geqslant 0$.

If $k < 0$, then

$$t(k\alpha_1^\vee) = t(-k\alpha_1^\vee)^{-1} = [1-3k, 2+3k, 3]^{-1} = [1+3k, 2-3k, 3].$$

$$t(k\alpha_2^\vee) = t(-k\alpha_2^\vee)^{-1} = [1, 2-3k, 3+3k]^{-1} = [1, 2+3k, 3-3k].$$

$$t(k\widetilde{\alpha}^\vee) = t(-k\widetilde{\alpha}^\vee)^{-1} = [1-3k, 2, 3+3k]^{-1} = [1+3k, 2, 3-3k].$$

□

PROPOSITION 2.16. Let $x = zy$ where $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$ and $y \in W_{finite}$. Then the following table Table 2.1 gives x in window notation, the expression $xq = u\alpha_1^\vee + v\alpha_2^\vee$, and $v - u$.

x	x in window notation	$xq = u\alpha_1^\vee + v\alpha_2^\vee$	$v-u$
z	$[1+3A, 2-3A+3B, 3-3B]$	$(A+\frac{1}{3})\alpha_1^\vee + (B+\frac{1}{3})\alpha_2^\vee$	$B-A$
zs_1	$[2-3A+3B, 1+3A, 3-3B]$	$A\alpha_1^\vee + (B+\frac{1}{3})\alpha_2^\vee$	$B-A+\frac{1}{3}$
zs_2	$[1+3A, 3-3B, 2-3A+3B]$	$(A+\frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee$	$B-A-\frac{1}{3}$
zs_1s_2	$[2-3A+3B, 3-3B, 1+3A]$	$(A-\frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee$	$B-A+\frac{1}{3}$
zs_2s_1	$[3-3B, 1+3A, 2-3A+3B]$	$A\alpha_1^\vee + (B-\frac{1}{3})\alpha_2^\vee$	$B-A-\frac{1}{3}$
$zs_1s_2s_1$	$[3-3B, 2-3A+3B, 1+3A]$	$(A-\frac{1}{3})\alpha_1^\vee + (B-\frac{1}{3})\alpha_2^\vee$	$B-A$

Table 2.1: x in window notation and $xq = u\alpha_1^\vee + v\alpha_2^\vee$, $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$

PROOF. In the Table 2.1, let $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$.

Cases $y = 1$. We have $x = t(A\alpha_1^\vee + B\alpha_2^\vee)$ with $A, B \in \mathbb{Z}$. Then by Lemma 2.15,

$$\begin{aligned} x &= t(A\alpha_1^\vee + B\alpha_2^\vee) \\ &= t(A\alpha_1^\vee) \circ t(B\alpha_2^\vee) \\ &= t(A\alpha_1^\vee) \circ [1, 2+3B, 3-3B] \\ &= [1, 2+3B, 3-3B] \circ t(A\alpha_1^\vee) \\ &= [1, 2+3B, 3-3B] \circ [1+3A, 2-3A, 3] \\ &= [1+3A, 2-3A+3B, 3-3B]. \end{aligned}$$

$$x = [1 + 3A, 2 - 3A + 3B, 3 - 3B] \tag{2.17}$$

$$\begin{aligned} xq &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ (\frac{1}{3}\widetilde{\alpha}) \\ &= t(A\alpha_1^\vee + B\alpha_2^\vee)(\frac{1}{3}(\alpha_1 + \alpha_2)) \\ &= \frac{1}{3}(\alpha_1 + \alpha_2) + A\alpha_1^\vee + B\alpha_2^\vee \\ &= (A + \frac{1}{3})\alpha_1^\vee + (B + \frac{1}{3})\alpha_2^\vee. \end{aligned}$$

$$xq = (A + \frac{1}{3})\alpha_1^\vee + (B + \frac{1}{3})\alpha_2^\vee. \tag{2.18}$$

Case $y = s_1$.

$$\begin{aligned} x &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_1 \\ &= [1 + 3A, 2 - 3A + 3B, 3 - 3B] \circ s_1 \\ &= [2 - 3A + 3B, 1 + 3A, 3 - 3B]. \end{aligned}$$

$$x = [2 - 3A + 3B, 1 + 3A, 3 - 3B] \tag{2.19}$$

$$\begin{aligned} xq &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_1(\frac{1}{3}\widetilde{\alpha}) \\ &= t(A\alpha_1^\vee + B\alpha_2^\vee)(\frac{1}{3}\alpha_2) \\ &= \frac{1}{3}\alpha_2 + A\alpha_1^\vee + B\alpha_2^\vee \\ &= A\alpha_1^\vee + (B + \frac{1}{3})\alpha_2^\vee. \end{aligned}$$

$$xq = A\alpha_1^\vee + (B + \frac{1}{3})\alpha_2^\vee. \tag{2.20}$$

Case $y = s_2$.

$$\begin{aligned} x &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_2 \\ &= [1+3A, 2-3A+3B, 3-3B] \circ s_2 \\ &= [1+3A, 3-3B, 2-3A+3B]. \end{aligned}$$

$$x = [1+3A, 3-3B, 2-3A+3B] \tag{2.21}$$

$$\begin{aligned} xq &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_2(\frac{1}{3}\widetilde{\alpha}) \\ &= t(A\alpha_1^\vee + B\alpha_2^\vee)(\frac{1}{3}\alpha_1) \\ &= \frac{1}{3}\alpha_1 + A\alpha_1^\vee + B\alpha_2^\vee \\ &= (A+\frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee. \end{aligned}$$

$$xq = (A+\frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee. \tag{2.22}$$

Case $y = s_1 s_2$.

$$\begin{aligned} x &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_1 s_2 \\ &= [1+3A, 2-3A+3B, 3-3B] \circ s_1 s_2 \\ &= [2-3A+3B, 1+3A, 3-3B] \circ s_2 \\ &= [2-3A+3B, 3-3B, 1+3A]. \end{aligned}$$

$$x = [2-3A+3B, 3-3B, 1+3A] \tag{2.23}$$

$$\begin{aligned}
xq &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_1 s_2(\frac{1}{3}\widetilde{\alpha}) \\
&= t(A\alpha_1^\vee + B\alpha_2^\vee)(-\frac{1}{3}\alpha_1) \\
&= -\frac{1}{3}\alpha_1 + A\alpha_1^\vee + B\alpha_2^\vee \\
&= (A - \frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee.
\end{aligned}$$

$$xq = (A - \frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee \tag{2.24}$$

Case $y = s_2 s_1$.

$$\begin{aligned}
x &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_2 s_1 \\
&= [1 + 3A, 2 - 3A + 3B, 3 - 3B] \circ s_2 s_1 \\
&= [1 + 3A, 3 - 3B, 2 - 3A + 3B] \circ s_1 \\
&= [3 - 3B, 1 + 3A, 2 - 3A + 3B].
\end{aligned}$$

$$x = [3 - 3B, 1 + 3A, 2 - 3A + 3B] \tag{2.25}$$

$$\begin{aligned}
xq &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_2 s_1(\frac{1}{3}\widetilde{\alpha}) \\
&= t(A\alpha_1^\vee + B\alpha_2^\vee)(-\frac{1}{3}\alpha_2) \\
&= -\frac{1}{3}\alpha_2 + A\alpha_1^\vee + B\alpha_2^\vee \\
&= A\alpha_1^\vee + (B - \frac{1}{3})\alpha_2^\vee.
\end{aligned}$$

$$xq = A\alpha_1^\vee + (B - \frac{1}{3})\alpha_2^\vee. \tag{2.26}$$

Case $y = s_1 s_2 s_1$.

$$\begin{aligned} x &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_1 s_2 s_1 \\ &= [1+3A, 2-3A+3B, 3-3B] \circ s_1 s_2 s_1 \\ &= [2-3A+3B, 3-3B, 1+3A] \circ s_1 \\ &= [3-3B, 2-3A+3B, 1+3A]. \end{aligned}$$

$$x = [3-3B, 2-3A+3B, 1+3A] \tag{2.27}$$

$$\begin{aligned} xq &= t(A\alpha_1^\vee + B\alpha_2^\vee) \circ s_1 s_2 s_1 (\frac{1}{3}\widetilde{\alpha}) \\ &= t(A\alpha_1^\vee + B\alpha_2^\vee)(-\frac{1}{3}\widetilde{\alpha}) \\ &= -\frac{1}{3}\alpha_1 - \frac{1}{3}\alpha_2 + A\alpha_1^\vee + B\alpha_2^\vee \\ &= (A - \frac{1}{3})\alpha_1^\vee + (B - \frac{1}{3})\alpha_2^\vee. \end{aligned}$$

$$xq = (A - \frac{1}{3})\alpha_1^\vee + (B - \frac{1}{3})\alpha_2^\vee. \tag{2.28}$$

□

3

GENERAL RESULTS ABOUT THE BRUHAT ORDER

This chapter contains general results about the Bruhat order. In Section 3.1 we use the characterization of Bruhat order in type $\widetilde{A}_{n-1}$ in [3] to give a characterization analogous to the characterization in type A_{n-1}. Section 3.2 studies the W action on the set of alcoves. These results are used in Section 3.3 to prove the six region and endpoint theorems, which closely relate the Bruhat order to the W action on the plane.

3.1 Definition and relation to window notation

In this section, $w \in W$ where W is the affine Weyl group of type $\widetilde{A}_n$. Define Bruhat order as generated by $x < xr$ if r is a reflection and $\ell(x) < \ell(xr)$. Let $<$ denote the Bruhat order on W. We have $u \leq v$ in Bruhat order iff there is a chain $u \leq u_2 \leq \cdots \leq u_p = v$ where $u_q s = u_{q+1}$ for some reflection s(depending on q) and $\ell(u_{q+1}) > \ell(u_q)$. There is a lot of geometric information hidden in the Bruhat ordering. There are algorithms to tell if two elements are comparable in Bruhat order. We can compute particular elements using the algorithm, but it is hard to use to prove theorems. Our goals are understanding Bruhat order for spiral elements and using this better understanding of Bruhat order to prove lookup conjecture for spiral elements.

Define $u^{\leqslant i} := \{u(i), u(i-1), u(i-2), \cdots\}$. We show that the following Theorem is equivalent to the characterization of Bruhat order for $\widetilde{A}_n$ in [3].

Definition 3.1. Define $x^{\leqslant A} \leqslant y^{\leqslant A}$ if we reorder elements in

$$x^{\leqslant A} = \{x(A), x(A-1), x(A-2), \cdots\},\ y^{\leqslant A} = \{y(A), y(A-1), y(A-2), \cdots\}$$

in decreasing order and name them

$$a_A > a_{A-1} > a_{A-2} > \cdots , \; b_A > b_{A-1} > b_{A-2} > \cdots$$

then $b_A \geqslant a_A, b_{A-1} \geqslant a_{A-1}, b_{A-2} \geqslant a_{A-2}, \cdots$.

LEMMA 3.2. $x^{\leqslant A} \leqslant y^{\leqslant A}$ for all $A \in \mathbb{Z} \Leftrightarrow x^{\leqslant A} \leqslant y^{\leqslant A}$ for all $A = 1, 2, \cdots, n$.

PROOF. Let $x \in W$. Suppose the elements of $x^{\leqslant A}$, written in decreasing order, are $\{a_A, a_{A-1}, \cdots\}$. Since $i \in x^{\leqslant A} \Leftrightarrow i + n \in x^{\leqslant A+n}$, the elements of $x^{\leqslant A+n}$, written in decreasing order, are $\{a_A + n, a_{A-1} + n, \cdots\}$. Therefore, if $x, y \in W$, then $x^{\leqslant A} \leqslant y^{\leqslant A} \Leftrightarrow x^{\leqslant A+n} \leqslant y^{\leqslant A+n}$. □

Definition 3.3. Given $v \in W$, define $v[i, j] := |\{a \leq i : v(a) \geq j\}|$ for all $i, j \in \mathbb{Z}$.

Since $v^{\leqslant i}$ is bounded above (by $\max\{v(i), \cdots, v(i-n+1)\}$), the set $\{a \leq i : v(a) \geq j\}$ is finite. Since $v[i, j]$ is the number of elements in this set, it is a nonnegative integer.

The following theorem gives a characterization of the Bruhat order in type $\widetilde{A}_{n-1}$ similar to the well known characterization in type A_{n-1} (see [9, Section 5.9, p.119]). It is deduced from a characterization of the Bruhat order given in [3, Theorem II.8.3.7].

THEOREM 3.4. For affine type $\widetilde{A}_{n-1}$, we have

$$\begin{aligned} x \leqslant y \text{ in Bruhat order } &\Leftrightarrow x^{\leqslant A} \leqslant y^{\leqslant A} \text{ for all } A \in \mathbb{Z} \\ &\Leftrightarrow x^{\leqslant A} \leqslant y^{\leqslant A} \text{ for } A = 1, 2, \cdots, n. \end{aligned}$$

PROOF. By [3, Theorem II.8.3.7], for $x, y \in W, x \leq y \Leftrightarrow x[A, i] \leqslant y[A, i]$ for all $i \in \mathbb{Z}$. Hence it is enough to show that for any $A, i \in \mathbb{Z}$

$$x[A, i] \leqslant y[A, i] \text{ for all } i \in \mathbb{Z} \Leftrightarrow x^{\leqslant A} \leqslant y^{\leqslant A}. \tag{3.5}$$

($\Leftarrow$): We reorder elements in

$$x^{\leqslant A} = \{x(A), x(A-1), x(A-2), \cdots\},\ y^{\leqslant A} = \{y(A), y(A-1), y(A-2), \cdots\}$$

name them

$$a_A > a_{A-1} > a_{A-2} > \cdots,\ b_A > b_{A-1} > b_{A-2} > \cdots$$

Since $x^{\leqslant A} \leqslant y^{\leqslant A}$, we have $b_A \geqslant a_A, b_{A-1} \geqslant a_{A-1}, b_{A-2} \geqslant a_{A-2}, \cdots$.

Given $i \in \mathbb{Z}$, we consider three possibilities.

(1) Case $a_A \geqslant i$. There is a nonnegative integer k such that

$$a_A > \cdots > a_{A-k} \geqslant i > a_{A-k-1} > \cdots,$$

then

$$\{x(d) \geqslant i \mid d \leqslant A\} = \{a_A, a_{A-1}, \cdots, a_{A-k}\}.$$

If $a_j \geqslant i$, then $b_j \geqslant a_j \geqslant i$ for all j. So there is a nonnegative integer l such that

$$\{y(d) \geqslant i \mid d \leqslant A\} = \{b_A, b_{A-1}, \cdots, b_{A-k}, b_{A-k-1}, \cdots, b_{A-k-l}\} \supseteq \{b_A, \cdots, b_{A-k}\}.$$

Thus

$$|\{x(d) \geqslant i \mid d \leqslant A\}| \leqslant |\{y(d) \geqslant i \mid d \leqslant A\}|.$$

Thus $x[A, i] \leqslant y[A, i]$.

(2) Case $b_A \geqslant i > a_A$. There is a nonnegative integer k' such that

$$b_A > \cdots > b_{A-k'} \geqslant i > b_{A-k'-1} > \cdots,$$

then

$$\{x(d) \geqslant i \mid d \leqslant A\} = \emptyset,$$

$$\{y(d) \geqslant i \mid d \leqslant A\} = \{b_A, b_{A-1}, \cdots, b_{A-k'}\}.$$

Hence

$$|\{x(d) \geqslant i \mid d \leqslant A\}| = 0 < |\{y(d) \geqslant i \mid d \leqslant A\}|.$$

Thus $x[A,i] \leqslant y[A,i]$.

(3) Case $i > b_A \geqslant a_A$. Then

$$\{x(d) \geqslant i \mid d \leqslant A\} = \emptyset,\ \{y(d) \geqslant i \mid d \leqslant A\} = \emptyset.$$

Hence

$$|\{x(d) \geqslant i \mid d \leqslant A\}| = 0 = |\{y(d) \geqslant i \mid d \leqslant A\}|.$$

Thus $x[A,i] \leqslant y[A,i]$.

Therefore,

$$x[A,i] \leqslant y[A,i] \text{ for all } i \in \mathbb{Z}.$$

($\Rightarrow$): Given $A \in \mathbb{Z}$, since $x[A,i] \leqslant y[A,i]$ for all $i \in \mathbb{Z}$, we have

$$|\{d \leqslant A \mid x(d) \geqslant i\}| \leqslant |\{d \leqslant A \mid y(d) \geqslant i\}|, \text{ for all } i \in \mathbb{Z}.$$

We reorder elements in

$$x^{\leqslant A} = \{x(A), x(A-1), x(A-2), \cdots\},\ y^{\leqslant A} = \{y(A), y(A-1), y(A-2), \cdots\}$$

name them

$$a_A > a_{A-1} > a_{A-2} > \cdots,\ b_A > b_{A-1} > b_{A-2} > \cdots$$

We show $a_A > b_A, a_{A-1} > b_{A-1}, \cdots$. Let $i = a_A$. Since

$$1 = |\{a_A\}| = |\{x(d) \geqslant a_A \mid d \leqslant A\}| = |\{d \leqslant A \mid x(d) \geqslant a_A\}|$$
$$\leqslant |\{d \leqslant A \mid y(d) \geqslant a_A\}| = |\{y(d) \geqslant a_A \mid d \leqslant A\}|,$$

we see

$$\{y(d) \geqslant a_A \mid d \leqslant A\} = \{b_A, \cdots\} \supseteq \{b_A\}$$

has at least one element. Since one element of the set $\{b_A, b_{A-1}, \cdots\}$ is bigger than a_A and b_A is the biggest element of the set $\{b_A, b_{A-1}, \cdots\}$, so $b_A \geqslant a_A$. Similarly, let $i = a_{A-1}$. Since

$$\begin{aligned} 2 &= |\{a_A, a_{A-1}\}| = |\{x(d) \geqslant a_{A-1} \mid d \leqslant A\}| = |\{d \leqslant A \mid x(d) \geqslant a_{A-1}\}| \\ &\leqslant |\{d \leqslant A \mid y(d) \geqslant a_{A-1}\}| = |\{y(d) \geqslant a_{A-1} \mid d \leqslant A\}|, \end{aligned}$$

we see

$$\{y(d) \geqslant a_{A-1} \mid d \leqslant A\} = \{b_A, b_{A-1}, \cdots\} \supseteq \{b_A, b_{A-1}\}$$

has at least two elements. Since two elements of the set $\{b_A, b_{A-1}, \cdots\}$ is bigger than a_{A-1} and b_A, b_{A-1} are the biggest two elements of the set $\{b_A, b_{A-1}, \cdots\}$, so $b_A \geqslant a_{A-1}, b_{A-1} \geqslant a_{A-1}$. Iterating this process, we have $b_A \geqslant a_A, b_{A-1} \geqslant a_{A-1}, b_{A-2} \geqslant a_{A-2}, \cdots$. Therefore,

$$x^{\leqslant A} \leqslant y^{\leqslant A}.$$

□

3.2 The W action on alcoves

To relate the Bruhat order to the affine Weyl group action on $\mathbb{R}^2$, we need to study the affine Weyl group action on alcoves.

Definition 3.6. Let $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = -\widetilde{\alpha}$. Let

$$E(a_1, a_2, a_3) = \{p \mid (p, \beta_i) \geq A_i \text{ for } i = 1, 2, 3\},$$

$$O(a_1, a_2, a_3) = \{p \mid (p, \beta_i) \leq A_i \text{ for } i = 1, 2, 3\}$$

where $A_i = a_i + \varepsilon_i, \varepsilon_1 = \varepsilon_2 = 0, \varepsilon_3 = -1$. Denote either one of these sets by $X(a_1, a_2, a_3)$, where $X = E$ or O.

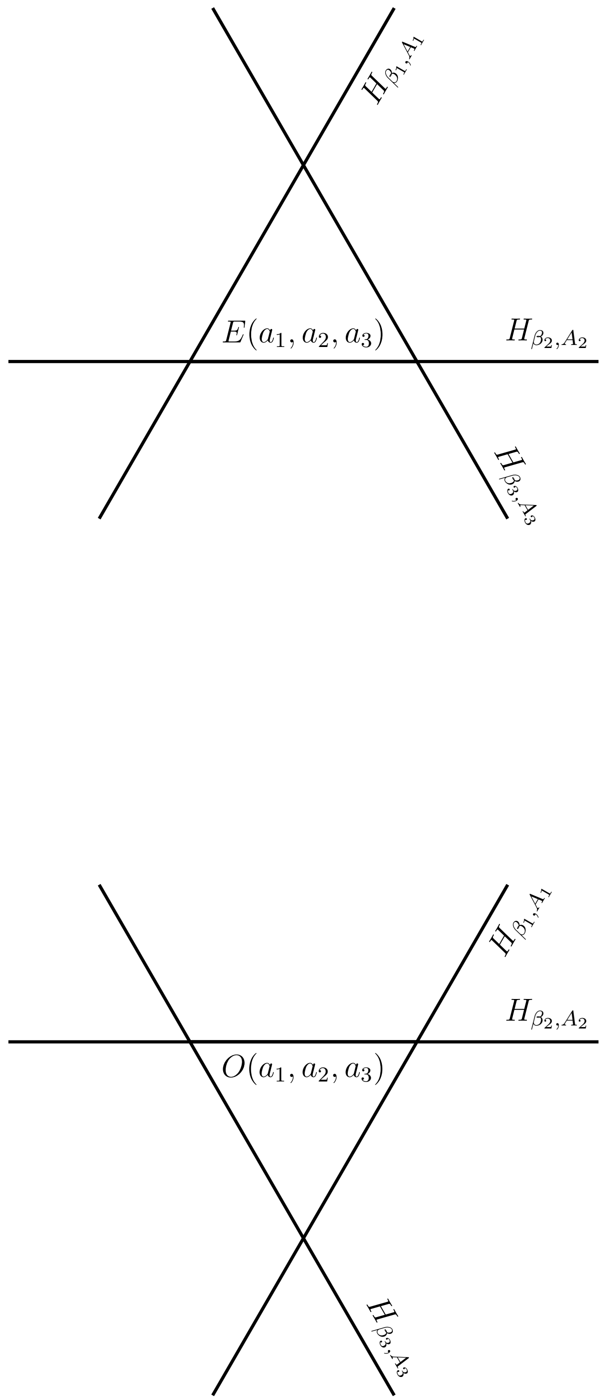

Figure 3.1: $E(a_1, a_2, a_3)$ and $O(a_1, a_2, a_3)$

Remark 3.7. In this section we use the convention that if an alcove is denoted $X(a_1, a_2, a_3)$ or $X(b_1, b_2, b_3)$, then A_i will denote $a_i + \varepsilon_i$ and B_i will denote $b_i + \varepsilon_i$. Also, the ε_i will be as defined in Definition 3.6.

Remark 3.8. $\beta_1 + \beta_2 + \beta_3 = 0$. Also for $\{i, j, k\} = \{1, 2, 3\}, (\beta_i, \beta_j) = -1, (\beta_i, \beta_i) = 2$ and $s_{\beta_i}\beta_j = -\beta_k$.

Let $A_\circ = E(0, 0, 0)$ (not to confused with the integers A_1, A_2, A_3). Note that $(p, \beta_3) \leq A_3 \Leftrightarrow (p, \widetilde{\alpha}) \geq -A_3 = -a_3 + 1$.

PROPOSITION 3.9. If $xA_\circ = X(a_1, a_2, a_3)$, then $A_1 + A_2 + A_3 = (-1)^{\ell(x)+1}$. In other words, we have

(a) If $\ell(x)$ is even, then $xA_\circ = E(a_1, a_2, a_3)$ where $a_1+a_2+a_3 = 0$, i.e. $A_1+A_2+A_3 = -1$.

(b) If $\ell(x)$ is odd, then $xA_\circ = O(a_1, a_2, a_3)$ where $a_1+a_2+a_3 = 2$, i.e. $A_1+A_2+A_3 = 1$.

PROOF. Let $xA_\circ = X(a_1, a_2, a_3), x \in W$. We show that if $X = E$, then $a_1 + a_2 + a_3 = 0$ and if $X = O$, then $a_1 + a_2 + a_3 = 2$.

Step 1. We show that it holds for $x \in W_{finite}$.

Claim. $p \in X(a_1, a_2, a_3) \Leftrightarrow s_{\beta_i}p \in X'(b_1, b_2, b_3)$ where X' is of opposite type to X, and $B_1 + B_2 + B_3 = -(A_1 + A_2 + A_3)$.

Proof of Claim: We give the proof assuming X is E. (If X is O, the proof is the same except with inequalities reversed.) We have

$$(s_{\beta_i}p, \beta_i) = (s_{\beta_i}p, \beta_i) = (s_{\beta_i}s_{\beta_i}p, s_{\beta_i}\beta_i) = -(p, \beta_i),$$

$$(s_{\beta_i}p, \beta_j) = (s_{\beta_i}p, \beta_j) = (s_{\beta_i}s_{\beta_i}p, s_{\beta_i}\beta_i) = -(p, \beta_k).$$

Switching the role of β_j and β_k, we have $(s_{\beta_i}p, \beta_k) = -(p, \beta_j)$.

$$\begin{aligned} p \in E(a_1, a_2, a_3) &\Leftrightarrow (p, \beta_\ell) \geq A_\ell \text{ for } \ell = 1, 2, 3 \\ &\Leftrightarrow (s_{\beta_i}p, \beta_i) \leq -A_i, (s_{\beta_i}p, \beta_j) \leq -A_k, (s_{\beta_i}p, \beta_k) \leq -A_j \\ &\Leftrightarrow s_{\beta_i}p \in O(b_1, b_2, b_3) \end{aligned}$$

where $B_i = -A_i, B_j = -A_k, B_k = -A_j$ $(B_i = b_i + \varepsilon_i)$. So $B_1 + B_2 + B_3 = -(A_1 + A_2 + A_3)$. This proves the claim.

These calculations imply that if $x \in W_{finite}$, and $xA_\circ = X(a_1, a_2, a_3)$ then $A_1 + A_2 + A_3 = (-1)^{\ell(x)+1}$. This follows by induction on $\ell(x)$. If $\ell(x) = 0$, then $xA_\circ = E(0, 0, 0)$ and $A_1 + A_2 + A_3 = -1$. Otherwise, $x = s_i y$ where $\ell(y) < \ell(x)$ and $i = 1$ or 2 (so $s_i = s_{\beta_i}$). By induction, $yA_\circ = X(a_1, a_2, a_3)$ with $A_1 + A_2 + A_3 = (-1)^{\ell(y)+1}$. Then the above claim implies that $xA_\circ = s_i y A_\circ = s_i X(a_1, a_2, a_3) = X'(b_1, b_2, b_3)$ with $B_1 + B_2 + B_3 = -(A_1 + A_2 + A_3) = -(-1)^{\ell(y)+1} = (-1)^{\ell(x)+1}$.

Step 2. For general x, $x = t(\lambda^\vee)y, y \in W_{finite}$. Since $\ell(t(\lambda^\vee))$ is even by Corollary 2.6, $\ell(x)$ and $\ell(y)$ have the same parity.

First suppose $\ell(y)$ is even. By Step 1, $yA_\circ = E(b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$. Then $xA_\circ = E(a_1, a_2, a_3)$ with $a_i = (\lambda^\vee, \beta_i) + b_i$. We have

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + (\lambda^\vee, \beta_1 + \beta_2 + \beta_3) = 0 + 0 = 0.$$

Now suppose $\ell(y)$ is odd. By Step 1, $yA_\circ = E(b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 2$. Then $xA_\circ = O(a_1, a_2, a_3)$ with $a_i = (\lambda^\vee, \beta_i) + b_i$. We have

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + (\lambda^\vee, \beta_1 + \beta_2 + \beta_3) = 2 + 0 = 2.$$

□

COROLLARY 3.10. For each $i = 1, 2, 3$, we have

(a) If $p \in E(a_1, a_2, a_3)$, then $A_i \le (p, \beta_i) \le A_i + 1$.

(b) If $p \in O(a_1, a_2, a_3)$, then $A_i - 1 \le (p, \beta_i) \le A_i$.

PROOF. (a) If $p \in E(a_1, a_2, a_3)$, then $(p, \beta_i) \ge A_i$. We want to show that $(p, \beta_i) \le A_i + 1$. Since $-(p, \beta_j) \le -A_j, -(p, \beta_k) \le -A_k$, so $(p, -(\beta_j + \beta_k)) \le -A_j - A_k$. By Proposition 3.9, $A_1 + A_2 + A_3 = -1$. Then $-A_j - A_k = A_i + 1$. Hence $(p, \beta_i) \le A_i + 1$.

(a) If $p \in O(a_1, a_2, a_3)$, then $(p, \beta_i) \le A_i$. We want to show that $(p, \beta_i) \ge A_i - 1$. Since $-(p, \beta_j) \ge -A_j, -(p, \beta_k) \ge -A_k$, so $(p, -(\beta_j + \beta_k)) \ge -A_j - A_k$. By Proposition 3.9, $A_1 + A_2 + A_3 = 1$. Then $-A_j - A_k = A_i - 1$. Hence $(p, \beta_i) \ge A_i - 1$. □

PROPOSITION 3.11. The center of $X(a_1, a_2, a_3)$ is $\frac{1}{3}(a_1\beta_1^\vee + a_2\beta_2^\vee + a_3\beta_3^\vee) - \frac{1}{3}\beta_3^\vee = \frac{1}{3}(A_1\beta_1^\vee + A_2\beta_2^\vee + A_3\beta_3^\vee)$.

PROOF. Let p_1 be the intersection point of hyperplanes H_{β_1,A_1} and H_{β_3,A_3}, p_2 be the intersection point of hyperplanes H_{β_2,A_2} and H_{β_3,A_3}, p_3 be the intersection point of hyperplanes H_{β_1,A_1} and H_{β_2,A_2}. Then the center of $X(a_1, a_2, a_3)$ is $\frac{1}{3}(p_1 + p_2 + p_3)$.

Let $p_1 = a\alpha_1^\vee + b\alpha_2^\vee$. Then $(p_1, \alpha_1) = 2a - b = A_1, (p_1, \beta_3) = -(p_1, \widetilde{\alpha}) = -a - b = A_3$. Hence $a = \frac{1}{3}A_1 - \frac{1}{3}A_3, b = -\frac{1}{3}A_1 - \frac{2}{3}A_3$. Therefore,

$$p_1 = \frac{1}{3}[(A_1 - A_3)\alpha_1^\vee + (-A_1 - 2A_3)\alpha_2^\vee].$$

Similarly,

$$p_2 = \frac{1}{3}[(-A_2 - 2A_3)\alpha_1^\vee + (A_2 - A_3)\alpha_2^\vee].$$

Let $p_3 = a\alpha_1^\vee + b\alpha_2^\vee$. Then $(p_3, \alpha_1) = 2a - b = A_1, (p_1, \alpha_2) = -a + 2b = A_2$. Hence $a = \frac{1}{3}(A_1 + 2A_3), b = \frac{1}{3}(2A_1 + A_2)$. Therefore,

$$p_1 = \frac{1}{3}[(2A_1 + A_2)\alpha_1^\vee + (A_1 + 2A_2)\alpha_2^\vee].$$

Hence

$$p = \frac{1}{9}[(3(A_1 - A_3)\alpha_1^\vee + 3(A_2 - A_3)\alpha_2^\vee]$$
$$= \frac{1}{3}[(A_1\alpha_1^\vee + A_2\alpha_2^\vee) - A_3(\alpha_1^\vee + \alpha_2^\vee)]$$
$$= \frac{1}{3}(A_1\beta_1^\vee + A_2\beta_2^\vee + A_3\beta_3^\vee).$$

□

PROPOSITION 3.12. Let p be the center of $xA_\circ = X(a_1, a_2, a_3)$. For all i,

$$(p, \beta_i) = A_i + \frac{(-1)^{\ell(x)}}{3}.$$

PROOF. Let $\{i, j, k\} = \{1, 2, 3\}$. By Proposition 3.9, if $xA_\circ = X(a_1, a_2, a_3)$ then $A_1 + A_2 + A_3 = (-1)^{\ell(x)+1}$. By Proposition 3.11, the center of $X(a_1, a_2, a_3)$ is $\frac{1}{3}(A_1\beta_1^\vee + A_2\beta_2^\vee + A_3\beta_3^\vee)$. So

$$(p, \beta_i) = (\frac{1}{3}(A_1\beta_1^\vee + A_2\beta_2^\vee + A_3\beta_3^\vee), \beta_i)$$
$$= \frac{1}{3}A_i(\beta_i^\vee, \beta_i) + \frac{1}{3}A_j(\beta_j^\vee, \beta_i) + \frac{1}{3}A_k(\beta_k^\vee, \beta_i)$$
$$= \frac{1}{3}A_i(2) - \frac{1}{3}A_j - \frac{1}{3}A_k$$
$$= \frac{2}{3}A_i - \frac{1}{3}(A_j + A_k)$$
$$= \frac{2}{3}A_i - \frac{1}{3}(-A_i + (-1)^{\ell(x)})$$
$$= A_i + \frac{(-1)^{\ell(x)}}{3}.$$

□

LEMMA 3.13. Let $p \in \mathbb{R}^2$. Then

(a) $(s_{\beta_i,n}p, \beta_i) = 2n - (p, \beta_i)$

(b) $(s_{\beta_i,n}p, \beta_j) = -n - (p, \beta_k)$

(c) $(s_{\beta_i,n}p, \beta_k) = -n - (p, \beta_j)$.

PROOF. (a) We have

$$\begin{aligned}(s_{\beta_i,n}p, \beta_i) &= (n\beta_i^\vee + s_{\beta_i}p, \beta_i) \\ &= n(\beta_i^\vee, \beta_i) + (s_{\beta_i}p, \beta_i) \\ &= 2n + (s_{\beta_i}p, \beta_i) \\ &= 2n + (s_{\beta_i}s_{\beta_i}p, s_{\beta_i}\beta_i) \\ &= 2n - (p, \beta_i).\end{aligned}$$

(b) Similarly,

$$\begin{aligned}(s_{\beta_i,n}p, \beta_j) &= (n\beta_i^\vee + s_{\beta_i}p, \beta_j) \\ &= n(\beta_i^\vee, \beta_j) + (s_{\beta_i}p, \beta_j) \\ &= -n + (s_{\beta_i}s_{\beta_i}p, s_{\beta_i}\beta_j) \\ &= -n - (p, \beta_k).\end{aligned}$$

(c) Switching the role of j and k in (b), proves (c). □

PROPOSITION 3.14. Let $\{i, j, k\} = \{1, 2, 3\}$. Then

(a) $s_{\beta_i,A_i}E(a_1, a_2, a_3) = O(b_1, b_2, b_3)$ where $b_i = a_i, b_j = a_j + 1, b_k = a_k + 1$.

(b) $s_{\beta_i,A_i+1}E(a_1, a_2, a_3) = O(b_1, b_2, b_3)$ where $b_i = a_i + 2, b_j = a_j, b_k = a_k$.

(c) $s_{\beta_i,A_i}O(a_1, a_2, a_3) = E(b_1, b_2, b_3)$ where $b_i = a_i, b_j = a_j - 1, b_k = a_k - 1$.

(d) $s_{\beta_i,A_i-1}O(a_1, a_2, a_3) = E(b_1, b_2, b_3)$ where $b_i = a_i - 2, b_j = a_j, b_k = a_k$.

PROOF. (a) We show that $p \in E(a_1,a_2,a_3) \Leftrightarrow s_{\beta_i,A_i}p \in O(b_1,b_2,b_3)$ where b_1,b_2,b_3 are as in the statement. Suppose $(p,\beta_\ell) \geq A_\ell$ for $\ell = 1,2,3$. We need to show that $(s_{\beta_i,A_i}p,\beta_\ell) \leq B_\ell$ for ℓ where $B_i = A_i, B_j = A_j+1, B_k = A_k+1$, as if $b_\ell = a_\ell+1$ then $B_\ell = A_\ell+1$. (Here $A_\ell = a_\ell+\varepsilon_\ell, B_\ell = b_\ell+\varepsilon_\ell$.) We will show the following.

(i) $(p,\beta_i) \geq A_i \Leftrightarrow (s_{\beta_i,A_i}p,\beta_i) \leq B_i = A_i$

(ii) $(p,\beta_k) \geq A_k \Leftrightarrow (s_{\beta_i,A_i}p,\beta_j) \leq B_j = A_j+1$

(iii) $(p,\beta_j) \geq A_j \Leftrightarrow (s_{\beta_i,A_i}p,\beta_k) \leq B_k = A_k+1$.

First, Lemma 3.13 implies $(s_{\beta_i,A_i}p,\beta_i) = 2A_i-(p,\beta_i)$. Therefore,

$$(p,\beta_i) \geq A_i \Leftrightarrow -(p,\beta_i) \leq -A_i \Leftrightarrow 2A_i-(p,\beta_i) \leq A_i \Leftrightarrow (s_{\beta_i,A_i}p,\beta_i) \leq A_i.$$

This proves (i). By Proposition 3.9, $A_1+A_2+A_3 = -1$. By Lemma 3.13, $(s_{\beta_i,A_i}p,\beta_j) = -A_i-(p,\beta_k)$. Therefore,

$$\begin{aligned}(p,\beta_k) \geq A_k &\Leftrightarrow -(p,\beta_k) \leq -A_k \\ &\Leftrightarrow -A_i-(p,\beta_k) \leq -A_i-A_k = A_j+1 \\ &\Leftrightarrow (s_{\beta_i,A_i}p,\beta_j) \leq A_j+1.\end{aligned}$$

This proves (ii). Switching the role of j and k, proves (iii).

(b) We show that $p \in E(a_1,a_2,a_3) \Leftrightarrow s_{\beta_i,A_i+1}p \in O(b_1,b_2,b_3)$ where b_1,b_2,b_3 are as in the statement. Suppose $(p,\beta_\ell) \geq A_\ell$ for $\ell = 1,2,3$. We need to show that $(s_{\beta_i,A_i+1}p,\beta_\ell) \leq B_\ell$ for ℓ where $B_i = A_i+2, B_j = A_j, B_k = A_k$, as if $b_\ell = a_\ell+1$ then $B_\ell = A_\ell+1$. (Here $A_\ell = a_\ell+\varepsilon_\ell, B_\ell = b_\ell+\varepsilon_\ell$.) We will show the following.

(i) $(p,\beta_i) \geq A_i \Leftrightarrow (s_{\beta_i,A_i+1}p,\beta_i) \leq B_i = A_i+2$

(ii) $(p,\beta_k) \geq A_k \Leftrightarrow (s_{\beta_i,A_i+1}p,\beta_j) \leq B_j = A_j$

(iii) $(p,\beta_j) \geq A_j \Leftrightarrow (s_{\beta_i,A_i+1}p,\beta_k) \leq B_k = A_k$.

First, Lemma 3.13 implies $(s_{\beta_i,A_i+1}p,\beta_i) = 2A_i+2-(p,\beta_i)$. Therefore,

$$(p,\beta_i) \geq A_i \Leftrightarrow -(p,\beta_i) \leq -A_i \Leftrightarrow 2A_i+2-(p,\beta_i) \leq A_i+2 \Leftrightarrow (s_{\beta_i,A_i+1}p,\beta_i) \leq A_i+2.$$

This proves (i). By Proposition 3.9, $A_1+A_2+A_3=-1$. By Lemma 3.13, $(s_{\beta_i,A_i+1}p,\beta_j)=-A_i-1-(p,\beta_k)$. Therefore,

$$\begin{aligned}(p,\beta_k)\geq A_k &\Leftrightarrow -(p,\beta_k)\leq -A_k\\ &\Leftrightarrow -A_i-1-(p,\beta_k)\leq -A_i-1-A_k=A_j\\ &\Leftrightarrow (s_{\beta_i,A_i+1}p,\beta_j)\leq A_j.\end{aligned}$$

This proves (ii). Switching the role of j and k, proves (iii).

(c) By (a), reversing the roles of a_i and b_i, $s_{\beta_i,B_i}E(b_1,b_2,b_3)=O(a_1,a_2,a_3)$ where $a_i=b_i, a_j=b_j+1, a_k=b_k+1$. Since $A_i=B_i$ and $s^2_{\beta_i,A_i}=id$, applying s_{β_i,A_i} to both sides gives $s_{\beta_i,A_i}O(a_1,a_2,a_3)=E(b_1,b_2,b_3)$ where $b_i=a_i, b_j=a_j-1, b_k=a_k-1$.

(d) By (b), reversing the roles of a_i and b_i, $s_{\beta_i,B_i+1}E(b_1,b_2,b_3)=O(a_1,a_2,a_3)$ where $a_i=b_i+2, a_j=b_j, a_k=b_k$. Since $A_i=B_i+2, A_i-1=B_i+1$, applying s_{β_i,A_i-1} to both sides gives $s_{\beta_i,A_i-1}O(a_1,a_2,a_3)=E(b_1,b_2,b_3)$ where $b_i=a_i-2, b_j=a_j, b_k=a_k$.
□

Definition 3.15. If $xA_\circ=X(a_1,a_2,a_3)$, we define $\ell(X(a_1,a_2,a_3))=\ell(x)$.

PROPOSITION 3.16. The following holds.

(a) $\ell(E(a_1,a_2,a_3))=|a_1|+|a_2|+|a_3|$.

(b) $\ell(O(a_1,a_2,a_3))=|a_1-1|+|a_2-1|+|a_3-1|$.

PROOF. (a) By [9, I, Theorem 4.5], the length $\ell(w)$ of an element w equals to the cardinality of the set

$$\mathcal{L}(w):=\{H\in\mathcal{H}\mid H \text{ separates } A_\circ \text{ and } wA_\circ\}.$$

Let $p_0\in A_\circ, p_1\in E(a_1,a_2,a_3)$. Hyperplanes of the form $H_{\beta_i,n}$ separating $A_\circ$ and $E(a_1,a_2,a_3)$ are the following.

Using Corollary 3.10, we have

$$A_i \leq (p_1, \beta_i) \leq A_i + 1$$

$$\varepsilon_i \leq (p_0, \beta_i) \leq \varepsilon_i + 1.$$

If $A_i \geq \varepsilon_i + 1$, then for any $j = \varepsilon_i + 1 \cdots, A_i, (p_0, \beta_i) < j < (p_1, \beta_i)$. H_{β_i, A_i} is a separating hyperplane and H_{β_i, A_i+1} is not a separating hyperplane. So $H_{\beta_i, \varepsilon_i+1}, H_{\beta_i, \varepsilon_i+2}, \cdots, H_{\beta_i, A_i}$ are separating hyperplanes. Hence the number of separating hyperplanes is $A_i - \varepsilon_i = a_i$.

If $A_i = \varepsilon_i$, there are no separating hyperplanes.

If $A_i \leq \varepsilon_i - 1$, the separating hyperplanes are $H_{\beta_i, \varepsilon_i}, H_{\beta_i, \varepsilon_i - 1}, \cdots, H_{\beta_i, A_i+1}$ are separating hyperplanes. Hence the number of separating hyperplanes is

$$\varepsilon_i - (A_i + 1) + 1 = \varepsilon_i - A_i = |A_i - \varepsilon_i| = |a_i|.$$

So there are $|a_i|$ hyperplanes of the form $H_{\beta_i, n}$ separating $A_\circ$ and $E(a_1, a_2, a_3)$. Similarly, there are $|a_j|$ hyperplanes of the form $H_{\beta_j, n}$ separating $A_\circ$ and $E(a_1, a_2, a_3)$. Similarly, there are $|a_k|$ hyperplanes of the form $H_{\beta_k, n}$ separating $A_\circ$ and $E(a_1, a_2, a_3)$. Therefore $\ell(E(a_1, a_2, a_3)) = |a_1| + |a_2| + |a_3|$.

(b) Let $p_0 \in A_\circ, p_1 \in O(a_1, a_2, a_3)$. Hyperplanes of the form $H_{\beta_i, n}$ separating $A_\circ$ and $O(a_1, a_2, a_3)$ are the following.

Using Corollary 3.10, we have

$$A_i - 1 \leq (p_1, \beta_i) \leq A_i$$

$$\varepsilon_i \leq (p_0, \beta_i) \leq \varepsilon_i + 1.$$

If $A_i \geq \varepsilon_i + 2$, then for any $j = \varepsilon_i + 1 \cdots, A_i - 1, (p_0, \beta_i) < j < (p_1, \beta_i)$. $H_{\beta_i, A_i - 1}$ is a separating hyperplane and H_{β_i, A_i} is not a separating hyperplane. So $H_{\beta_i, \varepsilon_i+1}, H_{\beta_i, \varepsilon_i+2}, \cdots,$ $H_{\beta_i, A_i - 1}$ are separating hyperplanes. Hence the number of separating hyperplanes is $A_i - 1 - \varepsilon_i = a_i - 1$.

If $A_i = \varepsilon_i + 1$, there are no separating hyperplanes.

If $A_i \leq \varepsilon_i$, the separating hyperplanes are $H_{\beta_i, A_i}, H_{\beta_i, A_i+1}, \cdots, H_{\beta_i, \varepsilon_i}$ are separating hyperplanes. Hence the number of separating hyperplanes is

$$\varepsilon_i - A_i + 1 = |A_i - \varepsilon_i - 1| = |a_i - 1|.$$

So there are $|a_i - 1|$ hyperplanes of the form $H_{\beta_i, n}$ separating $A_\circ$ and $O(a_1, a_2, a_3)$.

Similarly, there are $|a_j - 1|$ hyperplanes of the form $H_{\beta_j, n}$ separating $A_\circ$ and $O(a_1, a_2, a_3)$. Similarly, there are $|a_k - 1|$ hyperplanes of the form $H_{\beta_k, n}$ separating $A_\circ$ and $O(a_1, a_2, a_3)$. Therefore $\ell(O(a_1, a_2, a_3)) = |a_1 - 1| + |a_2 - 1| + |a_3 - 1|$. □

3.3 The six region and endpoint theorems

In this section we prove the six region theorem (Theorem 3.19), which describes how $\ell(rx)$ differs from x in terms of the location of the alcove $x(A_\circ)$, where r is a reflection along a hyperplane close to the alcove $x(A_\circ)$. As a consequence, we obtain the endpoint theorem (Theorem 3.23), which states that if $x, y, z \in W, xq, yq, zq$ all lie on a line parallel to $\beta^\vee$ and yq is between xq and zq, then $y \leq x$ or $y \leq z$ or both.

Let $L_{\alpha_1} := \{\lambda \mid (\lambda, \alpha_1) = \frac{1}{3}\}, L_{\alpha_2} := \{\lambda \mid (\lambda, \alpha_2) = \frac{1}{3}\}, L_{\widetilde{\alpha}} := \{\lambda \mid (\lambda, \widetilde{\alpha}) = \frac{2}{3}\}$. The three lines $L_{\alpha_1}, L_{\alpha_2}, L_{\widetilde{\alpha}}$ pass through the center point q of the fundamental alcove and separate $\mathbb{R}^2$ into 6 regions. These are Weyl chambers translated to $q = \frac{1}{3}\widetilde{\alpha}$.

(1) $Chamber(\widetilde{\alpha}) := \{\lambda \mid (\lambda, \alpha_1) \geq \frac{1}{3}, (\lambda, \alpha_2) \geq \frac{1}{3}\}$, contains $\widetilde{\alpha}$

(2) $Chamber(\alpha_2) := \{\lambda \mid (\lambda, \alpha_1) \leq \frac{1}{3}, (\lambda, \widetilde{\alpha}) \geq \frac{2}{3}\}$, contains α_2

(3) $Chamber(-\alpha_1) := \{\lambda \mid (\lambda, \alpha_2) \geq \frac{1}{3}, (\lambda, \widetilde{\alpha}) \leq \frac{2}{3}\}$, contains $-\alpha_1$

(4) $Chamber(-\widetilde{\alpha}) := \{\lambda \mid (\lambda, \alpha_1) \leq \frac{1}{3}, (\lambda, \alpha_2) \leq \frac{1}{3}\}$, contains $-\widetilde{\alpha}$

(5) $Chamber(-\alpha_2) := \{\lambda \mid (\lambda, \alpha_1) \geq \frac{1}{3}, (\lambda, \widetilde{\alpha}) \leq \frac{2}{3}\}$, contains $-\alpha_2$

(6) $Chamber(\alpha_1) := \{\lambda \mid (\lambda, \alpha_2) \leq \frac{1}{3}, (\lambda, \widetilde{\alpha}) \geq \frac{2}{3}\}$, contains α_1

PROPOSITION 3.17. Let $\{i, j, k\} = \{1, 2, 3\}$. Let p be the center of $x(A_\circ) = X(a_1, a_2, a_3)$. Then

(a) $p \in Chamber(\beta_i) \Leftrightarrow a_j \leq 0$ and $a_k \leq 0$.

(b) $p \in Chamber(-\beta_i) \Leftrightarrow$ either ℓ is even, $a_j \geq 0$ and $a_k \geq 0$ or ℓ is odd, $a_j \geq 1$ and $a_k \geq 1$.

Moreover, if $\ell(x)$ is odd, and either $p \in Chamber(\beta_i)$ or $p \in Chamber(-\beta_i)$ then p is in the interior of the chamber.

PROOF. Let $\ell = \ell(x)$. First we show that the statements hold for $\beta_i = \beta_1$.

(a) By the definition of the Chambers and Proposition 3.12,

$$\begin{aligned} p \in Chamber(\beta_1) &\Leftrightarrow (\lambda, \alpha_2) \leq \frac{1}{3},\ (\lambda, \widetilde{\alpha}) \geq \frac{2}{3} \\ &\Leftrightarrow (\lambda, \beta_2) \leq \frac{1}{3},\ (\lambda, \beta_3) \leq -\frac{2}{3} \\ &\Leftrightarrow A_2 + \frac{(-1)^\ell}{3} \leq \frac{1}{3},\ A_3 + \frac{(-1)^\ell}{3} \leq -\frac{2}{3} \\ &\Leftrightarrow a_2 + \frac{(-1)^\ell}{3} \leq \frac{1}{3},\ a_3 - 1 + \frac{(-1)^\ell}{3} \leq -\frac{2}{3} \\ &\Leftrightarrow a_2 \leq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3},\ a_3 \leq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3}. \end{aligned}$$

If ℓ is even, then this is equivalent to $a_2 \leq 0$ and $a_3 \leq 0$. If ℓ is odd, then this is equivalent to $a_2 \leq \frac{2}{3}$ and $a_3 \leq \frac{2}{3}$. Hence $a_2 \leq 0$ and $a_3 \leq 0$ as $a_i \in \mathbb{Z}$. Moreover, if $\ell(x)$ is odd and if $p \in Chamber(\beta_1)$, then since $a_2 < \frac{2}{3}$ and $a_3 < \frac{2}{3}$. $p \in Interior(Chamber(\beta_1))$.

(b) By Proposition 3.12,

$$p \in Chamber(-\beta_1) \Leftrightarrow (\lambda, \alpha_2) \geq \frac{1}{3},\ (\lambda, \widetilde{\alpha}) \leq \frac{2}{3}.$$

By the reasoning of (a), with inequalities reversed, this is equivalent to

$$a_2 \geq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3},\ a_3 \geq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3}.$$

If ℓ is even, then this is equivalent to $a_2 \geq 0$ and $a_3 \geq 0$. If ℓ is odd, then this is equivalent to $a_2 \geq \frac{2}{3}$ and $a_3 \geq \frac{2}{3}$. Hence $a_2 \geq 1$ and $a_3 \geq 1$ as $a_i \in \mathbb{Z}$. Moreover, if $\ell(x)$ is odd and if $p \in Chamber(-\beta_1)$, then since $a_2 > \frac{2}{3}$ and $a_3 > \frac{2}{3}$. $p \in Interior(Chamber(-\beta_1))$.

Similar arguments show that the statements hold for $\beta_i = \beta_2$.

Now we show that the statements hold for $\beta_i = \beta_3$.

(a) By the definition of the Chambers and Proposition 3.12,

$$\begin{aligned} p \in Chamber(\beta_3) &\Leftrightarrow (\lambda, \alpha_1) \leq \frac{1}{3},\ (\lambda, \alpha_2) \leq \frac{1}{3} \\ &\Leftrightarrow (\lambda, \beta_1) \leq \frac{1}{3},\ (\lambda, \beta_2) \leq \frac{1}{3} \\ &\Leftrightarrow A_1 + \frac{(-1)^\ell}{3} \leq \frac{1}{3},\ A_2 + \frac{(-1)^\ell}{3} \leq \frac{1}{3} \\ &\Leftrightarrow a_1 + \frac{(-1)^\ell}{3} \leq \frac{1}{3},\ a_2 + \frac{(-1)^\ell}{3} \leq \frac{1}{3} \\ &\Leftrightarrow a_1 \leq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3},\ a_2 \leq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3}. \end{aligned}$$

If ℓ is even, then this is equivalent to $a_1 \leq 0$ and $a_2 \leq 0$. If ℓ is odd, then this is equivalent to $a_1 \leq \frac{2}{3}$ and $a_2 \leq \frac{2}{3}$. Hence $a_1 \leq 0$ and $a_2 \leq 0$ as $a_i \in \mathbb{Z}$. Moreover, if $\ell(x)$ is odd and if $p \in Chamber(\beta_3)$, then since $a_1 < \frac{2}{3}$ and $a_2 < \frac{2}{3}$. $p \in Interior(Chamber(\beta_3))$.

(b) By Proposition 3.12,

$$p \in Chamber(-\beta_3) \Leftrightarrow (\lambda, \alpha_1) \geq \frac{1}{3},\ (\lambda, \alpha_2) \geq \frac{1}{3}.$$

By the reasoning of (a), with inequalities reversed, this is equivalent to

$$a_1 \geq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3},\ a_2 \geq \frac{1}{3} + \frac{(-1)^{\ell+1}}{3}.$$

If ℓ is even, then this is equivalent to $a_1 \geq 0$ and $a_2 \geq 0$. If ℓ is odd, then this is equivalent to $a_1 \geq \frac{2}{3}$ and $a_2 \geq \frac{2}{3}$. Hence $a_1 \geq 1$ and $a_2 \geq 1$ as $a_i \in \mathbb{Z}$. Moreover, if $\ell(x)$ is odd and if $p \in Chamber(-\beta_3)$, then since $a_1 > \frac{2}{3}$ and $a_2 > \frac{2}{3}$. $p \in Interior(Chamber(-\beta_3))$. $\square$

We have the following six region theorem about length going up by 1 or 3. If $x(A_\circ) = X(a_1, a_2, a_3)$, then H_{β_i, A_i} is a wall of $x(A_\circ)$. Let p be the center of $x(A_\circ)$. If $\ell(x)$ is even, then $A_i < (p, \beta_i) \leq A_i + 1$. If $\ell(x)$ is odd, then $A_i - 1 < (p, \beta_i) \leq A_i$.

THEOREM 3.18. Let $x(A_\circ) = X(a_1, a_2, a_3)$. Then $|\ell(x) - \ell(s_{\beta_i,A_i}x)| = 1$. Suppose first that $\ell(x)$ is even. Then the following hold:

(1) Suppose $a_i \geq 1$. Then $s_{\beta_i,A_i}x < x$ and $\ell(x) - \ell(s_{\beta_i,A_i}x) = 1$.

(2) Suppose $a_i \leq 0$. Then $x < s_{\beta_i,A_i}x$ and $\ell(s_{\beta_i,A_i}x) - \ell(x) = 1$.

(3) Suppose $a_i \geq 0$. Then $x < s_{\beta_i,A_i+1}x$. Also if $xq \in Chamber(\beta_i)$, then $\ell(s_{\beta_i,A_i+1}x) - \ell(x) = 3$. Otherwise $\ell(s_{\beta_i,A_i+1}x) - \ell(x) = 1$.

(4) Suppose $a_i \leq -1$. Then $s_{\beta_i,A_i+1}x < x$. Also if $xq \in Interior(Chamber(-\beta_i))$, then $\ell(x) - \ell(s_{\beta_i,A_i+1}x) = 3$. Otherwise $\ell(x) - \ell(s_{\beta_i,A_i+1}x) = 1$.

Suppose next that $\ell(x)$ is odd. Then the following hold:

(5) Suppose $a_i \geq 1$. Then $x < s_{\beta_i,A_i}x$ and $\ell(s_{\beta_i,A_i}x) - \ell(x) = 1$.

(6) Suppose $a_i \leq 0$. Then $s_{\beta_i,A_i}x < x$ and $\ell(x) - \ell(s_{\beta_i,A_i}x) = 1$.

(7) Suppose $a_i \geq 2$. Then $s_{\beta_i,A_i-1}x < x$ and if $xq \in Interior(Chamber(\beta_i))$, then $\ell(x) - \ell(s_{\beta_i,A_i-1}x) = 3$. Otherwise $\ell(x) - \ell(s_{\beta_i,A_i-1}x) = 1$.

(8) Suppose $a_i \leq 1$. Then $x < s_{\beta_i,A_i-1}x$ and if $xq \in Chamber(-\beta_i)$, then $\ell(s_{\beta_i,A_i-1}x) - \ell(x) = 3$. Otherwise $\ell(s_{\beta_i,A_i-1}x) - \ell(x) = 1$.

PROOF. Let $p \in E(a_1, a_2, a_3)$. We have $A_i < (p_1, \beta_i) \leq A_i + 1$. In this case H_{β_i,A_i} is a wall and H_{β_i,A_i+1} is not a wall.

We first prove statements (1) and (2), which deal with $s_{\beta_i,A_i}x$. We have $s_{\beta_i,A_i}E(a_1, a_2, a_3) = O(b_1, b_2, b_3)$ where $b_i = a_i, b_j = a_j + 1, b_k = a_k + 1$, which has length $|b_1 - 1| + |b_2 - 1| + |b_3 - 1| = |a_i - 1| + |a_j| + |a_k|$. Also, the length of $E(a_1, a_2, a_3)$ is $|a_1| + |a_2| + |a_3|$. Hence

$$\ell(s_{\beta_i,A_i}E(a_1, a_2, a_3)) - \ell(E(a_1, a_2, a_3)) = |a_i - 1| - |a_i| = \begin{cases} -1, & a_i \geq 1 \text{ (gives (1))} \\ 1, & a_i \leq 0 \text{ (gives (2))} . \end{cases}$$

We now turn to statements (3) and (4), which deal with $s_{\beta_i,A_i+1}x$. We have $s_{\beta_i,A_i}E(a_1,a_2,a_3) = O(b_1,b_2,b_3)$ where $b_i = a_i+2, b_j = a_j, b_k = a_k$, which has length $|b_1-1|+|b_2-1|+|b_3-1| = |a_i+1|+|a_j-1|+|a_k-1|$. Hence

$$\begin{aligned}&\ell(s_{\beta_i,A_i+1}E(a_1,a_2,a_3)) - \ell(E(a_1,a_2,a_3))\\&= (|a_i+1|-|a_i|)+(|a_j-1|-|a_j|)+(|a_k-1|-|a_k|)\\&= \begin{cases}1, & \text{if } a_i \geq 0\\ -1, & \text{if } a_i \leq -1\end{cases} + \begin{cases}1, & \text{if } a_j \leq 0\\ -1, & \text{if } a_j \geq 1\end{cases} + \begin{cases}1, & \text{if } a_k \leq 0\\ -1, & \text{if } a_k \geq 1.\end{cases}\end{aligned}$$

Also by Proposition 3.9, $a_1+a_2+a_3 = 0$. So $a_i = -a_j - a_k$.

Proof of (3): If $a_i \geq 0$, then $-a_j - a_k \geq 0$. We cannot have both $a_j \geq 1$ and $a_k \geq 1$, since then $-a_j - a_k \leq -2$. Therefore either $a_j \leq 0$ or $a_k \leq 0$ or both. This implies that

$$\begin{aligned}&\ell(s_{\beta_i,A_i+1}E(a_1,a_2,a_3)) - \ell(E(a_1,a_2,a_3))\\&= \begin{cases}3, & \text{if both } a_j \leq 0 \text{ and } a_k \leq 0 (\Leftrightarrow xq \in Chamber(\beta_i))\\ 1, & \text{if one of } a_j \leq 0 \text{ and } a_k \leq 0 \text{ but not both } (\Leftrightarrow xq \notin Chamber(\beta_i)).\end{cases}\end{aligned}$$

by Proposition 3.17. This proves (3).

Proof of (4): If $a_i \leq -1$, then $-a_j - a_k \leq -1$. We can not have both $a_j \leq 0$ and $a_k \leq 0$, since then $-a_j - a_k \geq 0$. Therefore either $a_j \geq 1$ or $a_k \geq 1$ or both. Therefore

$$\begin{aligned}&\ell(s_{\beta_i,A_i+1}E(a_1,a_2,a_3)) - \ell(E(a_1,a_2,a_3))\\&= \begin{cases}-3, & \text{if both } a_j \geq 1 \text{ and } a_k \geq 1 (\Leftrightarrow xq \in Interior(Chamber(-\beta_i)))\\ -1, & \text{if one of } a_j \geq 1 \text{ and } a_k \geq 1 \text{ but not both.}\end{cases}\end{aligned}$$

This proves (4).

Let $p \in O(a_1,a_2,a_3)$. We have $A_i - 1 < (p_1, \beta_i) \leq A_i$. In this case H_{β_i,A_i} is a wall and H_{β_i,A_i-1} is not a wall.

We first prove statements (5) and (6), which deal with $s_{\beta_i,A_i}x$. We have $s_{\beta_i,A_i}O(a_1,a_2,a_3) = E(b_1,b_2,b_3)$ where $b_i = a_i, b_j = a_j - 1, b_k = a_k - 1$, which has length $|b_1| + |b_2| + |b_3| = |a_i| + |a_j - 1| + |a_k - 1|$. Also, the length of $O(a_1, a_2, a_3)$ is $|a_i - 1| + |a_j - 1| + |a_k - 1|$. Hence

$$\ell(s_{\beta_i,A_i} O(a_1,a_2,a_3)) - \ell(O(a_1,a_2,a_3)) = |a_i| - |a_i - 1| = \begin{cases} 1, & a_i \geq 1 \text{ (gives (5))} \\ -1, & a_i \leq 0 \text{ (gives (6))} . \end{cases}$$

We now turn to statements (7) and (8), which deal with $s_{\beta_i,A_i-1}x$. We have $s_{\beta_i,A_i-1}O(a_1,a_2,a_3) = E(b_1,b_2,b_3)$ where $b_i = a_i - 2, b_j = a_j, b_k = a_k$, which has length $|b_i| + |b_j| + |b_k| = |a_i - 2| + |a_j| + |a_k|$. Hence

$$\begin{aligned} &\ell(s_{\beta_i,A_i-1}O(a_1,a_2,a_3)) - \ell(O(a_1,a_2,a_3)) \\ &= (|a_i - 2| - |a_i - 1|) + (|a_j| - |a_j - 1|) + (|a_k| - |a_k - 1|) \\ &= \begin{cases} 1, & \text{if } a_i \leq 1 \\ -1, & \text{if } a_i \geq 2 \end{cases} + \begin{cases} 1, & \text{if } a_j \geq 1 \\ -1, & \text{if } a_j \leq 0 \end{cases} + \begin{cases} 1, & \text{if } a_k \geq 1 \\ -1, & \text{if } a_k \leq 0. \end{cases} \end{aligned}$$

Also by Proposition 3.9, $a_1 + a_2 + a_3 = 2$. So $a_i = 2 - a_j - a_k$.

Proof of (8): If $a_i \leq 1$, then $2 - a_j - a_k \leq 1$. We can not have both $a_j \leq 0$ and $a_k \leq 0$, since then $2 - a_j - a_k \geq 2$. Therefore either $a_j \geq 1$ or $a_k \geq 1$ or both. This implies that

$$\begin{aligned} &\ell(s_{\beta_i,A_i-1}O(a_1,a_2,a_3)) - \ell(O(a_1,a_2,a_3)) \\ &= \begin{cases} 3, & \text{if both } a_j \geq 1 \text{ and } a_k \geq 1 (\Leftrightarrow xq \in Chamber(-\beta_i)) \\ 1, & \text{if one of } a_j \geq 1 \text{ and } a_k \geq 1 \text{ but not both } (\Leftrightarrow xq \notin Chamber(-\beta_i)). \end{cases} \end{aligned}$$

by Proposition 3.17. This proves (8).

Proof of (7): If $a_i \geq 2$, then $2 - a_j - a_k \geq 2$. We can not have both $a_j \geq 1$ and $a_k \geq 1$, since then $2 - a_j - a_k \leq 0$. Therefore either $a_j \leq 0$ or $a_k \leq 0$ or both. Therefore

$$\ell(s_{\beta_i,A_i-1}O(a_1,a_2,a_3)) - \ell(O(a_1,a_2,a_3))$$
$$= \begin{cases} -3, & \text{if both } a_j \le 0 \text{ and } a_k \le 0 (\Leftrightarrow xq \in Chamber(\beta_i)) \\ -1, & \text{if one of } a_j \le 0 \text{ and } a_k \le 0 \text{ but not both.} \end{cases}$$

This proves (7). □

THEOREM 3.19. Let β be of the form α_1, α_2, or $\widetilde{\alpha}$. Suppose $k - 1 < (xq, \beta^\vee) < k$, so either $H_{\beta,k-1}$ is a wall or $H_{\beta,k}$ is a wall of the alcove $x(A_\circ)$.

(1) Suppose $k \ge 1$. Then $x < s_{\beta,k}x$. If $H_{\beta,k-1}$ is a wall of the alcove $x(A_\circ)$ and xq is in $Chamber(\beta)$, then $\ell(s_{\beta,k}x) - \ell(x) = 3$. Otherwise $\ell(s_{\beta,k}x) - \ell(x) = 1$.

(2) Suppose $k \le 0$. Then $s_{\beta,k}x < x$. If $H_{\beta,k-1}$ is a wall of $x(A_\circ)$ and xq is in $Interior(Chamber(-\beta))$, then $\ell(x) - \ell(s_{\beta,k}x) = 3$. Otherwise $\ell(x) - \ell(s_{\beta,k}x) = 1$.

(3) Suppose $k \ge 2$. Then $s_{\beta,k-1}x < x$. If $H_{\beta,k}$ is a wall of $x(A_\circ)$ and xq is in $Interior(Chamber(\beta))$, then $\ell(x) - \ell(s_{\beta,k-1}x) = 3$. Otherwise $\ell(x) - \ell(s_{\beta,k-1}x) = 1$.

(4) Suppose $k \le 1$. Then $x < s_{\beta,k-1}x$. If $H_{\beta,k}$ is a wall of the alcove $x(A_\circ)$ and xq is in $Chamber(-\beta)$, then $\ell(s_{\beta,k-1}x) - \ell(x) = 3$. Otherwise $\ell(s_{\beta,k-1}x) - \ell(x) = 1$.

PROOF. Let $\beta = \beta_1$ or β_1 or $-\beta_3$ $(i = 1, 2, 3)$. Let $x(A_\circ) = X(a_1, a_2, a_3)$.

Suppose $x(A_\circ) = E(a_1, a_2, a_3)$, so $\ell(x)$ is even. Then $A_i < (xq, \beta_i^\vee) \le A_i + 1$. By assumption, $k - 1 < (xq, \beta^\vee) < k$. If $i = 1, 2$, then $\beta = \beta_i$, so $k = A_i + 1$. So $A_i = a_i = k - 1$. Hence $H_{\beta,k-1} = H_{\beta_i,A_i}$ is a wall, $H_{\beta,k} = H_{\beta_i,A_i+1}$ is not a wall. Also, $s_{\beta,k-1} = s_{\beta_i,A_i}, s_{\beta,k} = s_{\beta_i,A_i+1}$.

If $i = 3$, then $\beta_3 = -\beta$,

$$A_3 < -(xq, \beta_3^\vee) \le A_3 + 1$$

$$-(A_3+1) < (xq, \beta_3^\vee) \leq -A_3,$$

so $k = -A_3, a_3 = A_3 + 1 = -k + 1$. Hence $H_{\beta,k} = H_{-\beta_3,k} = H_{-\beta_3,-A_3} = H_{\beta_3,A_3}$ is a wall, $H_{\beta,k-1} = H_{-\beta_3,k-1} = H_{-\beta_3,-A_3-1} = H_{\beta_3,A_3+1}$ is not a wall. Also, $s_{\beta,k-1} = s_{\beta_3,A_3+1}, s_{\beta,k} = s_{\beta_3,A_3}$.

Suppose next that $x(A_\circ) = O(a_1, a_2, a_3)$, so $\ell(x)$ is odd. Then $A_i - 1 < (xq, \beta_i^\vee) \leq A_i$. By assumption, $k-1 < (xq, \beta^\vee) < k$. If $i = 1, 2$, then $\beta = \beta_i$, so $k = A_i$. Hence $H_{\beta,k} = H_{\beta_i,A_i}$ is a wall, $H_{\beta,k-1} = H_{\beta_i,A_i-1}$ is not a wall. Also, $s_{\beta,k-1} = s_{\beta_i,A_i-1}, s_{\beta,k} = s_{\beta_i,A_i}$.

If $i = 3$, then $\beta = -\beta_3$,

$$A_3 - 1 < -(xq, \beta_3^\vee) \leq A_3$$

$$-A_3 < (xq, \beta_3^\vee) \leq -A_3 + 1,$$

so $k = -A_3 + 1, A_3 = -k + 1$. Also, $a_3 = A_3 + 1 = -k + 1 + 1 = -k + 2$. Hence $H_{\beta,k-1} = H_{\beta,-A_3} = H_{-\beta_3,-A_3} = H_{\beta_3,A_3}$ is a wall, $H_{\beta,k} = H_{\beta,-A_3+1} = H_{-\beta_3,-A_3+1} = H_{\beta_3,A_3-1}$ is not a wall. Also, $s_{\beta,k-1} = s_{\beta_3,A_3}, s_{\beta,k} = s_{\beta_3,A_3-1}$.

We now check the parts of the theorem.

(1) First suppose $\ell(x)$ is even.

If $i = 1, 2$, then $\beta_i = \beta, k = A_i + 1$, $k - 1 = a_i$. Since $k \geq 1, a_i \geq 0$. Theorem 3.18 (3) says $x < s_{\beta_i,A_i+1}x = s_{\beta,k}x$. If $i = 3$, then $\beta_3 = -\beta$, and $k = -A_3, a_3 = -k+1$. Since $k \geq 1, a_3 \leq 0$. Theorem 3.18 (2) says $x < s_{\beta_3,A_3}x = s_{\beta,k}x$.

Now suppose $H_{\beta,k-1}$ is a wall. Then $i = 1, 2, \beta = \beta_i$. Then Theorem 3.18 (3) says that if $xq \in Chamber(\beta_i) = Chamber(\beta)$, then $\ell(s_{\beta_i,A_i+1}x) - \ell(x) = \ell(s_{\beta,k}x) - \ell(x) = 3$, otherwise 1 which is what we want. Suppose $H_{\beta,k-1}$ is not a wall. Then $i = 3$ and Theorem 3.18 (2) says $\ell(s_{\beta_3,A_3}x) - \ell(x) = \ell(s_{\beta,k}x) - \ell(x) = 1$ which is what we want.

Next suppose $\ell(x)$ is odd. If $i = 1, 2$, then $\beta = \beta_i, k = A_i = a_i$. We assume $k = a_i \geq 1$. Theorem 3.18 (5) says $x < s_{\beta_i,A_i}x = s_{\beta,k}x$. If $i = 3$, then $\beta = -\beta_3$, $k = -A_3 + 1, a_3 = -k + 2$. We assume $k \geq 1$, so $a_3 \leq 1$. Theorem 3.18 (8) says $x < s_{\beta_3,A_3-1}x = s_{\beta,k}x$.

Now suppose $H_{\beta,k-1}$ is a wall. Then $i = 3, \beta = -\beta_3$. Then Theorem 3.18 (8) says that if $xq \in Chamber(-\beta_3) = Chamber(\beta)$, then $\ell(s_{\beta_3,A_3-1}x) - \ell(x) = \ell(s_{\beta,k}x) - \ell(x) = 3$, otherwise 1 which is what we want. Suppose $H_{\beta,k-1}$ is not a wall. Then $i = 1, 2$ and Theorem 3.18 (5) says $\ell(s_{\beta_i,A_i}x) - \ell(x) = \ell(s_{\beta,k}x) - \ell(x) = 1$ which is what we want.

(2) First suppose $\ell(x)$ is even.

If $i = 1, 2$, then $\beta_i = \beta, k = A_i + 1, k - 1 = a_i$. Since $k \leq 0, a_i \leq -1$. Theorem 3.18 (4) says $s_{\beta,k}x = s_{\beta_i,A_i+1}x < x$. If $i = 3$, then $\beta_3 = -\beta$, and $k = -A_3, a_3 = -k+1$. Since $k \leq 0, a_3 \geq 1$. Theorem 3.18 (1) says $s_{\beta,k}x = s_{\beta_3,A_3}x < x$.

Now suppose $H_{\beta,k-1}$ is a wall. Then $i = 1, 2, \beta = \beta_i$. Then Theorem 3.18 (4) says that if $xq \in Interior(Chamber(-\beta_i)) = Interior(Chamber(-\beta))$, then $\ell(x) - \ell(s_{\beta_i,A_i+1}x) = \ell(x) - \ell(s_{\beta,k}x) = 3$, otherwise 1 which is what we want. Suppose $H_{\beta,k-1}$ is not a wall. Then $i = 3$ and Theorem 3.18 (1) says $\ell(x) - \ell(s_{\beta_i,A_i}x) = \ell(x) - \ell(s_{\beta,k}x) = 1$ which is what we want.

Next suppose $\ell(x)$ is odd.

If $i = 1, 2$, then $\beta = \beta_i, k = A_i = a_i$. We assume $k = a_i \leq 0$. Theorem 3.18 (6) says $s_{\beta,k}x = s_{\beta_i,A_i}x < x$. If $i = 3$, then $\beta = -\beta_3$, $k = -A_3 + 1, a_3 = -k + 2$. We assume $k \leq 0$, so $a_3 \geq 2$. Theorem 3.18 (7) says $s_{\beta,k}x = s_{\beta_3,A_3-1}x < x$.

Now suppose $H_{\beta,k-1}$ is a wall. Then $i = 3, \beta = -\beta_3$. Then Theorem 3.18 (7) says that if $xq \in Interior(Chamber(\beta_3)) = Interior(Chamber(-\beta))$, then $\ell(x) - \ell(s_{\beta_3,A_3-1}x) = \ell(x) - \ell(s_{\beta,k}x) = 3$, otherwise 1 which is what we want. Suppose $H_{\beta,k-1}$ is not a wall. Then $i = 1, 2$ and Theorem 3.18(6) says $\ell(x) - \ell(s_{\beta_i,A_i}x) = \ell(x) - \ell(s_{\beta,k}x) = 1$ which is what we want.

(3) First suppose $\ell(x)$ is even.

If $i = 1, 2$, then $\beta_i = \beta, k = A_i + 1, k - 1 = a_i$. Since $k \geq 2, a_i \geq 1$. Theorem 3.18 (1) says $s_{\beta,k-1}x = s_{\beta_i,A_i}x < x$. If $i = 3$, then $\beta_3 = -\beta$, and $k = -A_3, a_3 = -k+1$. Since $k \geq 2, a_3 \leq -1$. Theorem 3.18 (4) says $s_{\beta,k-1}x = s_{\beta_3,A_3+1}x < x$.

Suppose $H_{\beta,k}$ is a wall. Then $i = 3, \beta = -\beta_3$. Then Theorem 3.18 (4) says that if $xq \in Interior(Chamber(-\beta_3)) = Interior(Chamber(\beta))$, then $\ell(x) - \ell(s_{\beta_3,A_3+1}x) = \ell(x) - \ell(s_{\beta,k-1}x) = 3$, otherwise 1. This is what we want. Suppose $H_{\beta,k}$ is not a wall. Then $i = 1, 2$ and Theorem 3.18 (1) says $\ell(x) - \ell(s_{\beta_i,A_i}x) = \ell(x) - \ell(s_{\beta,k-1}x) = 1$, which is what we want.

Next suppose $\ell(x)$ is odd.

If $i = 1, 2$, then $\beta = \beta_i, k = A_i = a_i$. We assume $k = a_i \geq 2$. Theorem 3.18 (7) says $s_{\beta,k-1}x = s_{\beta_i,A_i-1}x < x$. If $i = 3$, then $\beta = -\beta_3$, $k = -A_3 + 1, a_3 = -k + 2$. We assume $k \geq 2$, so $a_3 \leq 0$. Theorem 3.18 (6) says $s_{\beta,k-1}x = s_{\beta_3,A_3}x < x$.

Now suppose $H_{\beta,k}$ is a wall. Then $i = 1, 2, \beta = \beta_i$. Then Theorem 3.18 (7) says that if $xq \in Interior(Chamber(\beta_i)) = Interior(Chamber(\beta))$, then $\ell(x) - \ell(s_{\beta_i,A_i-1}x) = \ell(x) - \ell(s_{\beta,k-1}x) = 3$, otherwise 1 which is what we want. Suppose $H_{\beta,k}$ is not a wall. Then $i = 3$ and Theorem 3.18 (6) says $\ell(x) - \ell(s_{\beta_3,A_3}x) = \ell(x) - \ell(s_{\beta,k-1}x) = 1$ which is what we want.

(4) First suppose $\ell(x)$ is even.

If $i = 1, 2$, then $\beta_i = \beta, k = A_i + 1$, $k - 1 = a_i$. Since $k \leq 1, a_i \leq 0$. Theorem 3.18 (2) says $x < s_{\beta,k-1}x = s_{\beta_i,A_i}x$. If $i = 3$, then $\beta_3 = -\beta$, and $k = -A_3, a_3 = -k + 1$. Since $k \leq 1, a_3 \geq 0$. Theorem 3.18 (3) says $x < s_{\beta,k-1}x = s_{\beta_3,A_3+1}x < x$.

Suppose $H_{\beta,k}$ is a wall. Then $i = 3, \beta = -\beta_3$. Then Theorem 3.18 (3) says that if $xq \in Chamber(\beta_3) = Chamber(-\beta)$, then $\ell(s_{\beta_3,A_3+1}x) - \ell(x) = \ell(s_{\beta_3,k-1}x) - \ell(x) = 3$, otherwise 1. This is what we want. Suppose $H_{\beta,k}$ is not a wall. Then $i = 1, 2$ and Theorem 3.18 (2) says $\ell(s_{\beta_i,A_i}x) - \ell(x) = \ell(s_{\beta,k-1}x) - \ell(x) = 1$, which is what we want.

Next suppose $\ell(x)$ is odd.

If $i = 1, 2$, then $\beta = \beta_i, k = A_i = a_i$. We assume $k = a_i \leq 1$. Theorem 3.18 (8) says $x < s_{\beta_i,A_i-1}x = s_{\beta,k-1}x$. If $i = 3$, then $\beta = -\beta_3$, $k = -A_3 + 1, a_3 = -k + 2$. We assume $k \leq 1$, so $a_3 \geq 1$. Theorem 3.18 (5) says $x < s_{\beta_3,A_3}x = s_{\beta,k-1}x$.

Now suppose $H_{\beta,k}$ is a wall. Then $i = 1, 2, \beta = \beta_i$. Then Theorem 3.18 (8) says that if $xq \in Chamber(-\beta_i) = Chamber(-\beta)$, then $\ell(s_{\beta_i,A_i-1}x)-\ell(x) = \ell(s_{\beta,k-1}x)-\ell(x) = 3$, otherwise 1 which is what we want. Suppose $H_{\beta,k}$ is not a wall. Then $i = 3$ and Theorem 3.18 (5) says $\ell(s_{\beta_3,A_3}x) - \ell(x) = \ell(s_{\beta,k-1}x) - \ell(x) = 1$ which is what we want. □

COROLLARY 3.20. Let β be α_1, α_1 or $\widetilde{\alpha}$. Let L be a line parallel to β through some uq ($u \in W$).

(1) For each $i \in \mathbb{Z}$ there is a unique $p_i = w_i q$ on L such that $i < (w_i q, \beta) < i + 1$. Moreover, The set $Wq \cap L = \{\cdots, p_{-1},\ p_0,\ p_1,\ p_2, \cdots\}$.

(2) In Bruhat order, $w_0 < w_1 < w_2 < \cdots$ and $w_0 < w_{-1} < w_{-2} < \cdots$.

PROOF. (1) This follows by Proposition 2.13.

(2) If $i \geq 0$, then $p_{i+1} = s_{\beta,i+1}p$ (by Definition 2.11, since $0 < p_0 < 1$). By Theorem 3.19 (1), with $i + 1$ playing the role of k, $w_i < s_{\beta,i+1}w_i = w_{i+1}$. Therefore,

$$w_0 < w_1 < w_2 < \cdots .$$

If $i \leq -1$, then by Definition 2.11, $p_i = s_{\beta,i+1}p_{i+1}$. By Theorem 3.19 (4), with $x = w_{i+1}$, and $i + 2$ playing the role of k, so $k - 1 = i + 1$ and $i \leq -1 \Leftrightarrow k \leq 1$, we have $w_{i+1} < s_{\beta,i+1}w_{i+1} = w_i$. Therefore,

$$w_0 < w_{-1} < w_{-2} < \cdots .$$

□

COROLLARY 3.21. On the line $xq + t\beta^\vee$, if $0 < (xq, \beta^\vee) < 1$, then

$$x < s_{\beta,1}x < s_{\beta,2}s_{\beta,1}x < \cdots$$

$$x < s_{\beta,0}x < s_{\beta,-1}s_{\beta,0}x < \cdots .$$

PROOF. Let $w_0 = x$. From the proof of Corollary 3.20, we see that if $i \geq 0$, then $w_{i+1} = s_{\beta,i+1}s_{\beta,i} \cdots s_{\beta,1}x$ and if $i \leq -1$ then $w_{i+1} = s_{\beta,i+1}s_{\beta,i+2} \cdots s_{\beta,0}x$. Now the statement follows from Corollary 3.20. □

COROLLARY 3.22. For any ℓ, there are at most 2 elements y such that yq is on the line $xq + t\beta^\vee$ and $\ell(y) = \ell$.

PROOF. This follows from Corollary 3.20. □

The following endpoint theorem follows from Corollary 3.20.

THEOREM 3.23. Let β be one of $\alpha_1, \alpha_2, \widetilde{\alpha}$. Suppose $x, y, z \in W,\ xq, yq, zq$ all lie on a line parallel to $\beta^\vee$ and yq is between xq and zq. Then $y \leq x$ or $y \leq z$ or both.

PROOF. By Corollary 3.20, there are integers $a < b < c$ such that $x = w_a, y = w_b, z = w_c$ (with notation as in that corollary). By Corollary 3.20, if $b \geq 0$ then $y = w_b \leq z = w_c$. If $b \leq 0$ then $y = w_b \leq x = w_a$. □

4

REFLECTION SETS AND RATIONAL SMOOTHNESS

In this chapter we define the reflection sets Ψ_x^w. In Section 4.1, we use the endpoint theorem to show that in type $\widetilde{A}_2$, these sets look like unions of intervals (Theorem 4.2). In Section 4.2, we state the Carrell-Peterson criterion, the lookup conjecture, and prove some general results which will be applied to the case of spiral Schubert varieties in Chapter 7.

4.1 Reflection sets

In what follows, if $a \leq b$ are integers, we will call $\{a, a+1, \cdots, b\}$ an integral interval and denote it by $[a, b]$. If $\beta \in \Phi$, let $[a, b]_\beta := \{s_{\beta,a}, s_{\beta,a+1}, \ldots, s_{\beta,b}\}$. We will refer to $[a, b]_\beta$ as an interval.

We will adopt the convention that when we say $[a, b]_\beta$ this interval may be the empty set. It will be convenient for us to say that the empty set is represented by the interval $[1, 0]$. Then for example, if $[a, b] = [1, 0]$, then $[a, b+1] = [1, 1]$ which is the set $\{1\}$. This convention will allow us to make uniform statements. Similarly, $[1, 0]_\alpha$ is the empty set, but $[1, 1]_\alpha = \{s_{\alpha,1}\}$. The following theorem states that nothing in the interval can be missing. In other words, the interval is not broken.

THEOREM 4.1. Let w, x be any elements of W. Let β be one of $\alpha_1, \alpha_2, \widetilde{\alpha}$. Then the set $\{k \mid s_{\beta,k}x \leq w\}$ is an integral interval $[a, b]$ $(a, b \in \mathbb{Z})$ or the empty set.

PROOF. Let a, b be the smallest (respectively, largest) integers such that $s_{\beta,a}x \leq w$ and $s_{\beta,b}x \leq w$. We must show that $s_{\beta,c}x \leq w \Leftrightarrow a \leq c \leq b$.

(⇒): This is by definition of a, b.

(⇐): If $a \leq c \leq b$, then $s_{\beta,c}xq$ is between $s_{\beta,a}xq$ and $s_{\beta,b}xq$ on the line through them. By the endpoint theorem Theorem 3.23, either $s_{\beta,c}x < s_{\beta,a}x \leq w$ or $s_{\beta,c}x < s_{\beta,b}x \leq w$. □

THEOREM 4.2. $\Psi_x^w = [a_1, b_1]_{\alpha_1} \cup [a_2, b_2]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b}]_{\widetilde{\alpha}}$.

PROOF. By definition, $\Psi_x^w = \{r \mid rx \leq w\}$. Since all reflections are of the form $s_{\beta,k}$ for $\beta \in \Phi^+$ and $k \in \mathbb{Z}$ by Lemma 2.1,

$$\Psi_x^w = \{s_{\alpha_1,k} \mid s_{\alpha_1,k}x \leq w\} \cup \{s_{\alpha_2,k} \mid s_{\alpha_2,k}x \leq w\} \cup \{s_{\widetilde{\alpha},k} \mid s_{\widetilde{\alpha},k}x \leq w\}.$$

By Theorem 4.1, this is equivalent to $[a_1, b_1]_{\alpha_1} \cup [a_2, b_2]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b}]_{\widetilde{\alpha}}$. □

As discussed earlier, in the previous corollary, some of the "intervals" may be the empty set.

4.2 Rational smoothness and reflection sets

This section provides the statement of the lookup conjecture. This section also contains some general results about Ψ_x^w which are used later. Given $x \leq w$ in W, define

$$\Psi_x^w := \{r \in R \mid rx \leq w\}$$

$$q_x^w := |\Psi_x^w| - \ell(w).$$

By [6], $|\Psi_x^w| \geq \ell(w)$. The following is the Carrell-Peterson criterion.

THEOREM 4.3. [12, Theorem 12.2.14] Let $X(w)$ be an affine Schubert variety. xB is not rationally smooth in $X(w)$ if and only if $q_y^w > 0$ for some y with $x \leq y \leq w$.

Recall the statement of the lookup conjecture which states that the Carrell-Peterson criterion can be simplified.

CONJECTURE 4.4. [4, Conjecture 1.1] xB is not rationally smooth in $X(w) \Leftrightarrow q_x^w > 0$ or $q_{rx}^w > 0$, some $x < rx \leq w$ with $r \in R$.

Let W be any affine Weyl group. Now we prove some general results which will be applied to the case of spiral Schubert varieties in Chapter 7.

LEMMA 4.5. Let s_a be a simple reflection. Suppose $ws_a < w$. Then $x \leq w \Leftrightarrow xs_a \leq w$. Similarly, suppose $s_a w < w$, then $x \leq w \Leftrightarrow s_a x \leq w$.

PROOF. Suppose $ws_a < w$. ($\Rightarrow$) By [9, 5.9] $x \leq w \Rightarrow xs_a \leq w$ or $xs_a \leq ws_a$. As $ws_a < w$, in either case, $xs_a \leq w$. ($\Leftarrow$) This follows by the other implication, interchanging x and xs_a. The statement where $s_a w < w$ is proved similarly. □

By [4, Proposition 4.6] and [4, Lemma 2.1], we have the following.

PROPOSITION 4.6. Let s be a simple reflection.

(a) If $x < w, ws < w$, then $q_x^w = q_{xs}^w$.

(b) If $x < w, sw < w$, then $q_x^w = q_{sx}^w$.

PROOF. We prove (a). By Lemma 4.5, $\Psi_x^w = \{r \in R \mid rx \leq w\} = \{r \in R \mid rxs_a \leq w\} = \Psi_{xs_a}^w$. The proof of (b) is similar. □

PROPOSITION 4.7. Let s be a simple reflection. Assume $x \leq w$ and $ws < w$. Then xB is not rationally smooth in $X(w) \Leftrightarrow xs$ is not rationally smooth in $X(w)$.

PROOF. ($\Rightarrow$) x is not rationally smooth $\Rightarrow$ there exists $y, x \leq y \leq w$, such that $q_y^w > 0$. Both $y, ys \leq w$ by Lemma 4.5. Since $x < y$, either $xs < y$ or $xs < ys$ by [9, 5.9]. By Proposition 4.6, $q_{ys}^w = q_y^w > 0$. Therefore xs is not rationally smooth. ($\Leftarrow$) This follows by the other implication, interchanging x and xs. □

Let W be of type $\widetilde{A}_{n-1}$. Recall that if $i \not\equiv j \pmod{n}$, there is an element $t_{ij} \in W$ satisfying $t_{ij}(i) = j, t_{ij}(j) = i, t_{ij}(k) = k$ if k is not congruent to i or j (mod n).

LEMMA 4.8. Let W be of type $\widetilde{A}_{n-1}$. Let s be a simple reflection. Suppose $r \neq xsx^{-1}$. Then $x < rx \Leftrightarrow xs < rxs$.

PROOF. As in the proof of Proposition 4.7, it is enough to show ($\Rightarrow$). First, if $r = t_{i,j}$ in window notation, then

$$x < rx \Leftrightarrow x^{-1}(i) < x^{-1}(j). \tag{4.9}$$

The reason is that in window notation: $r = t_{i,j}, i < j$. By [3, Proposition 8.3.6], $x < t_{i,j}x = x(x^{-1})t_{i,j}x = xt_{x^{-1}(i),x^{-1}(j)} \Leftrightarrow x^{-1}(i) < x^{-1}(j)$.

Assume $x < rx$. We want to show $xs < rxs$. By (4.9), this is equivalent to $(xs)^{-1}(i) = sx^{-1}(i) < (xs)^{-1}(j) = sx^{-1}(j)$. Since s is a simple reflection, $s = t_{p,p+1}$ for some $p = 1, \cdots, n$. To show $sx^{-1}(i) < sx^{-1}(j)$, it is enough to show that we do not have $x^{-1}(i) = nk + p$ and $x^{-1}(j) = nk + p + 1$ for some $k \in \mathbb{Z}$. By hypothesis, $x^{-1}rx \neq s$. If $x^{-1}(i) = nk + p$ and $x^{-1}(j) = nk + p + 1$, then $x^{-1}rx = x^{-1}t_{i,j}x = t_{x^{-1}(i),x^{-1}(j)} = t_{nk+p,nk+p+1} = t_{p,p+1} = s$, a contradiction. □

PROPOSITION 4.10. Let W be of type $\widetilde{A}_{n-1}$ and let s be a simple reflection. Assume $x \leq w, ws < w$. Suppose xB is not rationally smooth in $X(w)$. Then the lookup conjecture holds for $x \Leftrightarrow$ the lookup conjecture holds for xs.

PROOF. As in the proof of Proposition 4.7, it is suffices to show ($\Rightarrow$). By Proposition 4.7, x is not rationally smooth $\Leftrightarrow$ xs is not rationally smooth. Assume the lookup conjecture holds for x. If $q_x^w > 0$, then by Proposition 4.6 $q_{xs}^w = q_x^w > 0$, done. Otherwise $q_x^w = 0$ (so $q_{xs}^w = 0$). There exists $r \in R$ such that $x < rx < w$ and $q_{rx}^w > 0$. We have $rx \neq xs$ because $q_{xs}^w = 0,\ q_{rx}^w > 0$. Therefore $x^{-1}rx \neq s$. So by Lemma 4.5 and Lemma 4.8, $xs < rxs \leq w$. By Proposition 4.6, $q_{rxs}^w = q_{rx}^w > 0$. So the lookup conjecture holds for xs. □

5

GEOMETRY AND THE BRUHAT ORDER FOR SPIRAL ELEMENTS

In this chapter we give a characterization of Bruhat ordering for spiral elements in Type $\widetilde{A}_2$ in terms of the geometry of the triangle region. More precisely, we define a triangle $\overline{\triangle}(\ell)$ in $\mathbb{R}^2$, and let $\triangle(\ell) = \overline{\triangle}(\ell) \cap Wq$. If ℓ is even, we let $R(\ell) = \triangle(\ell)$; if ℓ is odd, we let $R(\ell) = \triangle(\ell) \setminus \{A_1(\ell)q, A_2(\ell)q\}$ where $A_1(\ell), A_1(\ell)$ are particular elements of W. The main results of this chapter are two triangle theorems. Theorem 5.41 states that $x \leq w(\ell) \Leftrightarrow xq \in R(\ell)$. If ℓ is even, $R(\ell) = \triangle(\ell)$, but if ℓ is odd, $R(\ell) \neq \triangle(\ell)$, and we prove (Theorem 5.44) that for odd ℓ, $x \leq w(\ell)$ or $x \leq A_1(\ell) \Leftrightarrow xq \in \triangle(\ell)$. Theorem 5.41 will allow us to prove the translation theorems in the next chapter, which will give us a good understanding of the integers $q_x^{w(\ell)}$. To identify the set of rationally smooth points in Chapter 7, we need Theorem 5.44 in addition to Theorem 5.41.

5.1 Spiral elements and some basic properties

This section studies the spiral elements and some basic properties. We begin by defining spiral elements.

Definition 5.1. For $\ell \geq 1$, define $w(\ell) = s_1 s_2 s_3 s_1 s_2 s_3 s_1 \cdots$ (ℓ factors). We call $w(\ell)$ a spiral element. We call these elements and the varieties associated to them spiral.

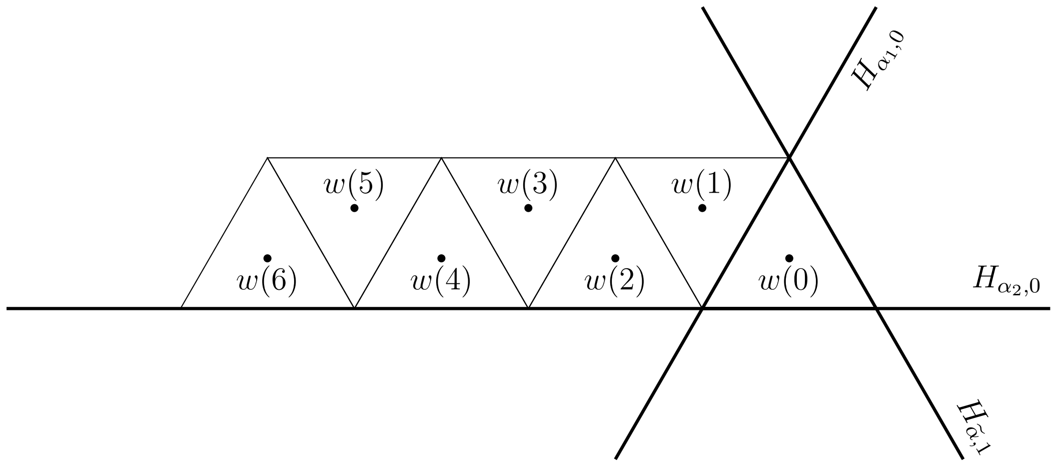

Figure 5.1: Spiral elements $w(\ell)$

We can also define spiral elements of the form $s_i s_j s_k s_i s_j s_k s_i s_j \cdots$ where $\{i, j, k\} = \{1, 2, 3\}$ as sets. All the results we prove for $w(\ell)$ will have analogues for these other spiral elements. We first need to prove that the length of $w(\ell)$ is ℓ. We begin with a lemma.

LEMMA 5.2. Given any permutation σ of $\{1, 2, 3\}$, there is a homomorphism $f : W \longrightarrow W$ such that $f(s_i) = s_{\sigma(i)}$. This satisfies $\ell(f(w)) = \ell(w)$.

PROOF. Given permutation σ of $\{1, 2, 3\}$, there is a map

$$f : W \longrightarrow W$$

$$f(s_i) = s_{\sigma(i)}.$$

Observe that W has generators $s_i \in S$ and relations $(s_i)^2 = 1, (s_i s_j)^3 = 1, i \neq j$. Then we have $f((s_i)^2) = (s_{\sigma(i)})^2 = 1$ and $f((s_i s_j)^3) = (s_{\sigma(i)} s_{\sigma(j)})^3 = 1$ for $i \neq j$. Since f preserves the generators and relations, $f : W \longrightarrow W$ is a homomorphism, then we can define $f : W \longrightarrow W$, $f(s_{i_1} s_{i_2} \cdots s_{i_n}) = s_{\sigma(i_1)} s_{\sigma(i_2)} \cdots s_{\sigma(i_n)}$. Similarly, we define a homomorphism

$$g : W \longrightarrow W$$
$$g(s_i) = s_{\sigma^{-1}(i)}.$$

$$g(f(s_{i_1}s_{i_2}\cdots s_{i_n})) = g(s_{\sigma(i_1)}s_{\sigma(i_2)}\cdots s_{\sigma(i_n)}) = s_{\sigma^{-1}(\sigma(i_1))}\cdots s_{\sigma^{-1}(\sigma(i_n))} = s_{i_1}s_{i_2}\cdots s_{i_n}.$$

Then $gf = id$, and $fg = id$, so f and g are isomorphisms.

Suppose $\ell(w) = m$. Let $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ be a reduced expression. If $f(w) = s_{\sigma(i_1)}s_{\sigma(i_2)}\cdots s_{\sigma(i_m)}$ is not a reduced expression, then by [9, Corollary 5.8], we can delete some of the $s_{\sigma(i_k)}$ to get a shorter expression

$$f(w) = s_{\sigma(i_1)}\cdots \hat{s}_{\sigma(i_t)}\cdots \hat{s}_{\sigma(i_r)}\cdots s_{\sigma(i_m)}.$$

Hence

$$\begin{aligned} w &= g(f(w)) \\ &= g(s_{\sigma(i_1)}\cdots \hat{s}_{\sigma(i_t)}\cdots \hat{s}_{\sigma(i_r)}\cdots s_{\sigma(i_m)}) \\ &= s_{\sigma^{-1}(\sigma(i_1))}\cdots \hat{s}_{\sigma^{-1}(\sigma(i_t))}\cdots \hat{s}_{\sigma^{-1}(\sigma(i_r))}\cdots s_{\sigma^{-1}(\sigma(i_m))} \\ &= s_{i_1}\cdots \hat{s}_{i_t}\cdots \hat{s}_{i_r}\cdots s_{i_n} \end{aligned}$$

which has length less than m. This is a contradiction. So $f(w) = s_{\sigma(i_1)}s_{\sigma(i_2)}\cdots s_{\sigma(i_m)}$ is a reduced expression. Therefore $\ell(f(w)) = \ell(w) = m$. □

PROPOSITION 5.3. Let $\{i, j, k\} = \{1, 2, 3\}$ (as sets). Let $u, v \in W$. The following facts hold.

(a) If $w = us_is_js_i$ and $\ell(w) = \ell(u) + 3$, then $\ell(ws_k) = \ell(us_is_js_is_k) = \ell(w) + 1$.

(b) If $w = us_is_js_k$ and $\ell(w) = \ell(u) + 3$, then $\ell(ws_i) = \ell(us_is_js_ks_i) = \ell(w) + 1$.

(c) If $w = u's_js_is_js_k$ and $\ell(w) = \ell(u') + 4$, then $\ell(ws_j) = \ell(u's_js_is_js_ks_j) = \ell(w) + 1$.

Remark. If $w = u's_ks_is_js_k$ and $\ell(w) = \ell(u')+4$, then the length of $ws_j = u's_ks_is_js_ks_j$ can be either $\ell(w) + 1$ or $\ell(w) - 1$.

PROOF. We can assume $i = 1, j = 2, k = 3$. The reason is that given three different integers $i, j, k \in \{1, 2, 3\}$, there is a permutation σ of $\{1, 2, 3\}$ such that $\sigma(1) = i, \sigma(2) = j, \sigma(3) = k$. Then by Lemma 5.2, there is a length-preserving homomorphism $f : W \longrightarrow W$ such that $f(s_i) = s_{\sigma(i)}$.

Suppose $w = us_1s_2s_1$, $\ell(w) = n, \ell(u) = n-3$. Let $u = [a, b, c]$ be in window notation. Since $\ell(us_1) = n - 2$, we have $a < b$ and $us_1 = [b, a, c]$. Since $\ell(us_1s_2) = n - 1$, we have $a < c$ and $us_1s_2 = [b, c, a]$. Since $\ell(us_1s_2s_1) = n$, we have $b < c$ and $w = us_1s_2s_1 = [c, b, a]$. Then $a - 3 < c$, so $\ell(ws_3) = \ell(w) + 1$.

Suppose $w = us_1s_2s_3$, $\ell(w) = n, \ell(u) = n - 3$. Let $u = [a, b, c]$ be in window notation. Since $\ell(us_1) = n-2$, we have $a < b$ and $us_1 = [b, a, c]$. Since $\ell(us_1s_2) = n-1$, we have $a < c$ and $us_1s_2 = [b, c, a]$. Since $\ell(us_1s_2s_3) = n$, we have $a - 3 < b$ and $us_1s_2s_3 = [a - 3, c, b + 3]$. Then $a - 3 < c$, so $\ell(ws_1) = \ell(w) + 1$.

Suppose $w = u's_2s_1s_2s_3$, $\ell(w) = n, \ell(u') = n - 4$. Let $u' = [a, b, c]$ be in window notation. Since $\ell(u's_2) = n-3$, we have $b < c$ and $u's_2 = [a, c, b]$. Since $\ell(u's_2s_1) = n-2$, we have $a < c$ and $u's_2s_1 = [c, a, b]$. Since $\ell(u's_2s_1s_2) = n - 1$, we have $a < b$ and $u's_2s_1s_2 = [c, b, a]$. Since $\ell(u's_2s_1s_2s_3) = n$, we have $a - 3 < c$ and $u's_2s_1s_2s_3 = [a - 3, b, c + 3]$. Then $b < c + 3$, so $\ell(ws_2) = \ell(w) + 1$. □

COROLLARY 5.4. The length of $w(\ell)$ is ℓ. Moreover, if $\ell_1 < \ell_2$ then $w(\ell_1) < w(\ell_2)$.

PROOF. The statement that $\ell(w(\ell)) = \ell$ follows by induction on length from Proposition 5.3 (ii). This implies that $s_1s_2\cdots$ is a reduced expression for $w(\ell)$. If $\ell_1 < \ell_2$, then $w(\ell_1)$ is the product of elements in a subexpression of a reduced expression of $w(\ell)$. Therefore $w(\ell_1) < w(\ell_2)$. □

The following result states that the spiral element $w(6)$ is a translation.

LEMMA 5.5. $w(6) = t(-2\alpha_1^\vee - \alpha_2^\vee)$.

PROOF. Using Lemma 2.5,

$$\begin{aligned} t(-2\alpha_1^\vee - \alpha_2^\vee) &= t(\widetilde{\alpha}^\vee)^{-1} \circ t(\alpha_1^\vee)^{-1} \\ &= (s_1 s_2 s_1 s_3) \circ (s_1 s_3 s_2 s_3) \\ &= s_1 s_2 (s_1 s_3 s_1 s_3) s_2 s_3 \\ &= s_1 s_2 (s_3 s_1) s_2 s_3 = w(6). \end{aligned}$$

□

LEMMA 5.6. Let $w(\ell)$ be a length ℓ spiral element. Let $q = \frac{1}{3}\widetilde{\alpha}$. Then

$$w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + (\frac{2-\ell}{6} + \varepsilon)\alpha_2^\vee$$

where $\varepsilon = 0$ if ℓ is even, and $\varepsilon = \frac{1}{6}$ if ℓ is odd (cf. Definition 5.15). In other words:

(a) If ℓ is even, then $w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee$.

(b) If ℓ is odd, then $w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee$.

PROOF. (a) We proceed by induction. We first verify the result for $\ell = 0, 2, 4$. Then $w(0)q = q = q$. Also,

$$w(2)q = s_1 s_2 q = s_1 s_2 (\frac{1}{3}\widetilde{\alpha}) = \frac{1}{3}s_1(\alpha_1) = -\frac{1}{3}\alpha_1.$$

Also,

$$\begin{aligned} w(4)q &= s_1 s_2 s_3 s_1 (\frac{1}{3}\widetilde{\alpha}) \\ &= s_1 s_2 s_{\widetilde{\alpha},1}(\frac{1}{3}\alpha_2) \\ &= s_1 s_2 (\frac{2}{3}\alpha_1 + \alpha_2) \\ &= s_1(\frac{2}{3}\widetilde{\alpha} - \alpha_2) = -\alpha_1 - \frac{1}{3}\alpha_2. \end{aligned}$$

For the inductive step, it suffices to show that if the result holds for ℓ, then the result holds for $\ell+6$. By (5.5), $w(6) = t(-2\alpha_1^\vee - \alpha_2^\vee)$. Then

$$\begin{aligned} w(\ell+6)q &= t(-2\alpha_1^\vee - \alpha_2^\vee)w(\ell)q \\ &= (\frac{1-\ell}{3}\alpha_1 + \frac{2-\ell}{6}\alpha_2) - 2\alpha_1^\vee - \alpha_2^\vee \\ &= \frac{1-(\ell+6)}{3}\alpha_1 + \frac{2-(\ell+6)}{6}\alpha_2. \end{aligned}$$

(b) We first verify the result for $\ell = 1, 3, 5$. Then $w(1)q = s_1(\frac{1}{3}\widetilde{\alpha}) = \frac{1}{3}\alpha_2$. Also,

$$\begin{aligned} w(3)q &= s_1 s_2 s_3(\frac{1}{3}\widetilde{\alpha}) \\ &= s_1 s_2 s_{\widetilde{\alpha},1}(\frac{1}{3}\widetilde{\alpha}) \\ &= s_1(\frac{2}{3}\alpha_1) = -\frac{2}{3}\alpha_1. \end{aligned}$$

Also

$$\begin{aligned} w(5)q &= s_1 s_2 s_3 s_1 s_2(\frac{1}{3}\widetilde{\alpha}) \\ &= s_1 s_2 s_{\widetilde{\alpha},1} s_1(\frac{1}{3}\alpha_1) \\ &= s_1 s_2(\alpha_1 + \frac{4}{3}\alpha_2) \\ &= s_1(\widetilde{\alpha} - \frac{4}{3}\alpha_2) = -\frac{4}{3}\alpha_1 - \frac{1}{3}\alpha_2. \end{aligned}$$

For the inductive step, it suffices to show that if the result holds for ℓ, then the result holds for $\ell+6$. By (5.5), $w(6) = t(-2\alpha_1^\vee - \alpha_2^\vee)$. Then

$$\begin{aligned} w(\ell+6)q &= t(-2\alpha_1^\vee - \alpha_2^\vee)w(\ell)q \\ &= (\frac{1-\ell}{3}\alpha_1 + \frac{3-\ell}{6}\alpha_2) - 2\alpha_1^\vee - \alpha_2^\vee \\ &= \frac{1-(\ell+6)}{3}\alpha_1 + \frac{3-(\ell+6)}{6}\alpha_2. \end{aligned}$$

□

PROPOSITION 5.7. Let $\ell \in \mathbb{Z}, \ell \geq 5$. The elements $w(\ell - i)$ for $i = 0, \cdots, 5$ form a complete set of coset representatives for $L(\Phi^\vee)\backslash W$. Equivalently, any $x \in W$ can be written uniquely as

$$x = t(\gamma^\vee)u$$

where $u = w(\ell - i)$, $i = 0, \cdots, 5$.

PROOF. By [9, I, Proposition 4.2], the affine Weyl group W is the semidirect product of the finite Weyl group W_{finite} and the translation group corresponding to the coroot lattice $L = L(\Phi^\vee)$ ($W = L(\Phi^\vee) \rtimes W_{finite}$). Hence $x \in W$, $x = t(\nu^\vee)y$ for $t(\nu^\vee) \in L(\Phi^\vee), y \in W_{finite}$. We have in type $\widetilde{A}_2$, $W_{finite} = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$,

$$|L(\Phi^\vee)\backslash W| = 6.$$

We show that $w(\ell) = t(\gamma_1^\vee)u_1, w(\ell-1) = t(\gamma_2^\vee)u_2, w(\ell-2) = t(\gamma_3^\vee)u_3, w(\ell-3) = t(\gamma_4^\vee)u_4, w(\ell-4) = t(\gamma_5^\vee)u_5, w(\ell-5) = t(\gamma_6^\vee)u_6$, where $u_1, u_2, u_3, u_4, u_5, u_6 \in W_{finite}$ are all different. By Lemma 5.5, $w(6) = t(-2\alpha_1^\vee - \alpha_2^\vee)$. Hence

$$w(6k) = (s_1s_2s_3s_1s_2s_3)^k = t(k(-2\alpha_1^\vee - \alpha_2^\vee)) \in L(\Phi^\vee) \cdot 1 \in L(\Phi^\vee)\backslash W.$$

$$w(6k+1) = w(6k) \cdot s_1 \in L(\Phi^\vee) \cdot s_1$$

$$w(6k+2) = w(6k) \cdot s_1s_2 \in L(\Phi^\vee) \cdot s_1s_2$$

By Lemma 2.5, $t(\alpha_1^\vee) = s_3s_2s_3s_1$. Thus

$$w(6k+3) = w(6k)s_1s_2s_3 = w(6k)(s_1s_2s_3s_2)s_2 = w(6k)t(-\alpha_1^\vee)s_2 \in L(\Phi^\vee)s_2$$

$$w(6k+4) = w(6k)s_1s_2s_3s_1 = w(6k)(s_1s_2s_3s_2)s_2s_1 = w(6k)t(-\alpha_1^\vee)s_2s_1 \in L(\Phi^\vee)s_2s_1$$

$$\begin{aligned} w(6k+5) &= w(6k)s_1s_2s_3s_1s_2 = w(6k)(s_1s_2s_3s_2)s_2s_1s_2 \\ &= w(6k)t(-\alpha_1^\vee)s_2s_1s_2 \in L(\Phi^\vee)s_2s_1s_2. \end{aligned}$$

Therefore the elements $w(\ell - i)$ for $i = 0, \cdots, 5$ form a complete set of coset representatives for $L(\Phi^\vee)\backslash W$. $\square$

We give formulas for spiral elements in window notation.

LEMMA 5.8. Let k be a nonnegative integer.

$$w(6k) = [1-6k, 2+3k, 3+3k],\ w(6k+1) = [2+3k, 1-6k, 3+3k],$$

$$w(6k+2) = [2+3k, 3+3k, 1-6k],\ w(6k+3) = [-2-6k, 3+3k, 5+3k],$$

$$w(6k+4) = [3+3k, -2-6k, 5+3k],\ w(6k+5) = [3+3k, 5+3k, -2-6k].$$

PROOF. We prove the lemma by induction on k. For $k = 0$, since $s_1 = t_{1,2}, s_2 = t_{2,3}, s_3 = t_{1,0}$ are transpositions and multiplying on the right by $s_{ij} = t_{i,j}$ switches the ith and jth position, then we have

$$\begin{aligned}
w(0) &= e = [1,2,3] \\
w(1) &= s_1 = es_1 = [1,2,3]s_1 = [2,1,3] \\
w(2) &= s_1 s_2 = [2,1,3]s_2 = [2,3,1] \\
w(3) &= s_1 s_2 s_3 = [2,3,1]s_3 = [-2,3,5] \\
w(4) &= s_1 s_2 s_3 s_1 = [-2,3,5]s_1 = [3,-2,5] \\
w(5) &= s_1 s_2 s_3 s_1 s_2 = [3,-2,5]s_2 = [3,5,-2] \\
w(6) &= s_1 s_2 s_3 s_1 s_2 s_3 = [3,5,-2]s_3 = [-5,5,6].
\end{aligned}$$

Hence the statements hold for $k = 0$. Suppose all six formulas hold for k, then for $k+1$ we have

$$
\begin{aligned}
w(6(k+1)) &= w(6k+6) \\
&= w(6k+5)s_3 \\
&= [3+3k, 5+3k, -2-6k]s_3 \\
&= [-2-6k-3, 3k+5, 3+3k+3] \\
&= [1-6(k+1), 2+3(k+1), 3+3(k+1)]
\end{aligned}
$$

$$
\begin{aligned}
w(6(k+1)+2) &= w(6(k+1)+1)s_2 \\
&= [2+3(k+1), 1-6(k+1), 3+3(k+1)]s_2 \\
&= [2+3(k+1), 3+3(k+1), 1-6(k+1)]
\end{aligned}
$$

$$
\begin{aligned}
w(6(k+1)+3) &= w(6(k+1)+2)s_3 \\
&= [2+3(k+1), 3+3(k+1), 1-6(k+1)]s_3 \\
&= [1-6(k+1)-3, 3+3(k+1), 2+6(k+1)+3] \\
&= [-2-6(k+1), 3+3(k+1), 5+6(k+1)]
\end{aligned}
$$

$$
\begin{aligned}
w(6(k+1)+4) &= w(6(k+1)+3)s_1 \\
&= [-2-6(k+1), 3+3(k+1), 5+6(k+1)]s_1 \\
&= [3+3(k+1), -2-6(k+1), 5+6(k+1)]
\end{aligned}
$$

$$
\begin{aligned}
w(6(k+1)+5) &= w(6(k+1)+4)s_2 \\
&= [3+3(k+1), -2-6(k+1), 5+6(k+1)]s_2 \\
&= [3+3(k+1), 5+6(k+1), -2-6(k+1)].
\end{aligned}
$$

Hence all six formulas hold for $k+1$. $\square$

From Lemma 5.8 we can determine the sets $w(\ell)^{\leq i}$. We record some of this in Table 5.1 and 5.2 for future use.

	$\ell = 6k$	$\ell = 6k+2$	$\ell = 6k+4$
$w(\ell)^{\leq 1}$	$\{3k, -1+3k, \cdots\}$ $= \{\frac{\ell}{2}, \frac{\ell-2}{2}, \cdots\}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+2}{2}, \frac{\ell-2}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+2}{2}, \frac{\ell}{2}, \cdots\}$
$w(\ell)^{\leq 2}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+4}{2}, \frac{\ell}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+4}{2}, \frac{\ell+2}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+2}{2}, \frac{\ell}{2}, \cdots\}$
$w(\ell)^{\leq 3}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+6}{2}, \frac{\ell+4}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+4}{2}, \frac{\ell+2}{2}, \cdots\}$	$\{5+3k, 3+3k, \cdots\}$ $= \{\frac{\ell+6}{2}, \frac{\ell+2}{2}, \cdots\}$

Table 5.1: The sets $w(\ell)^{\leq i}$ for ℓ even

	$\ell = 6k+1$	$\ell = 6k+3$	$\ell = 6k+5$
$w(\ell)^{\leq 1}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+3}{2}, \frac{\ell-1}{2}, \cdots\}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+1}{2}, \frac{\ell-3}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+1}{2}, \frac{\ell-1}{2}, \cdots\}$
$w(\ell)^{\leq 2}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+3}{2}, \frac{\ell-1}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+3}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{5+3k, 3+3k, \cdots\}$ $= \{\frac{\ell+5}{2}, \frac{\ell+1}{2}, \cdots\}$
$w(\ell)^{\leq 3}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+5}{2}, \frac{\ell+3}{2}, \cdots\}$	$\{5+3k, 3+3k, \cdots\}$ $= \{\frac{\ell+7}{2}, \frac{\ell+3}{2}, \cdots\}$	$\{5+3k, 3+3k, \cdots\}$ $= \{\frac{\ell+5}{2}, \frac{\ell+1}{2}, \cdots\}$

Table 5.2: The sets $w(\ell)^{\leq i}$ for ℓ odd

5.2 The elements $A_1(\ell)$ and $A_2(\ell)$

If ℓ is odd, there are two special elements $A_1(\ell)$ and $A_2(\ell)$. In this section, we study these elements.

Definition 5.9. For ℓ odd, define $A_1(\ell), A_2(\ell)$ as the following.

(a) $A_1(\ell) := t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)$

(b) $A_2(\ell) := s_1 A_1(\ell)$.

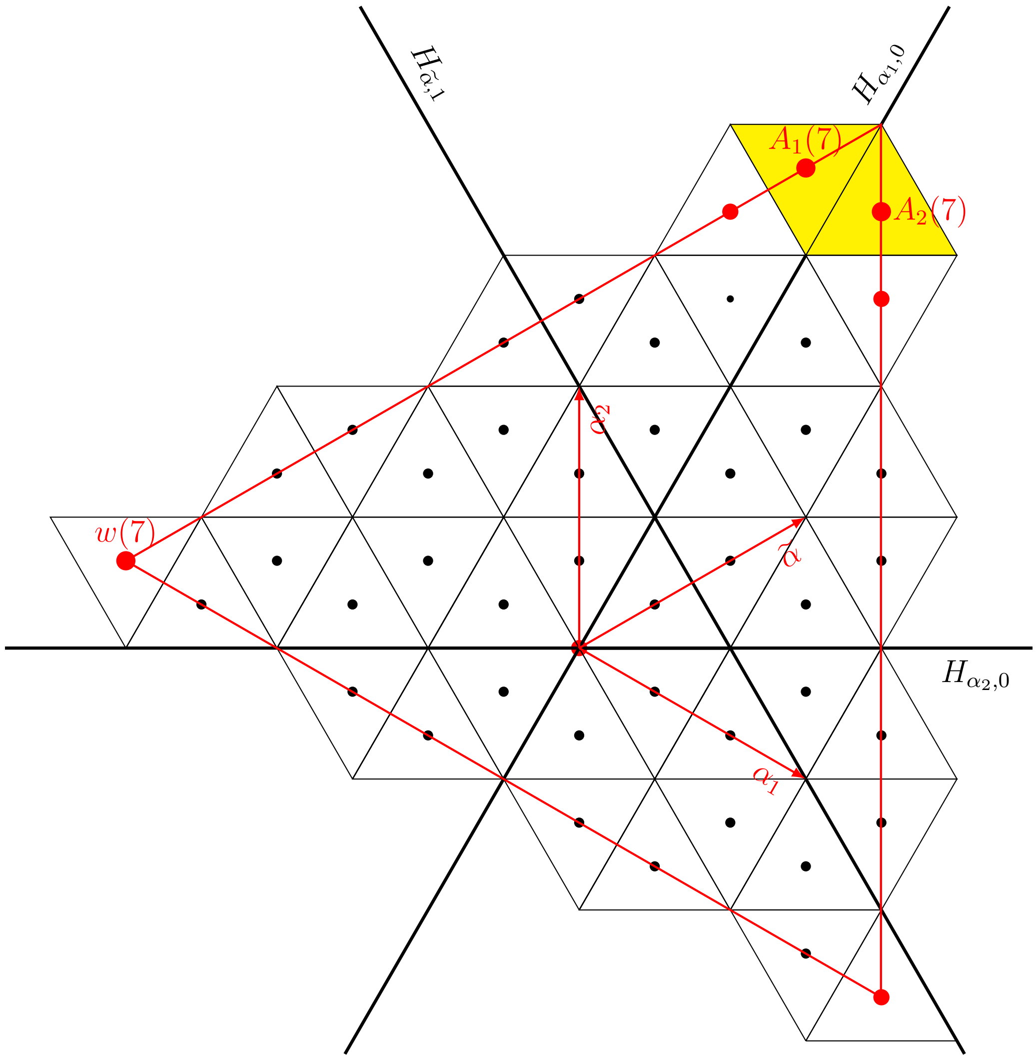

Figure 5.2: The elements $A_1(\ell)$ and $A_2(\ell)$ for ℓ odd $(\ell = 7)$

LEMMA 5.10. Let ℓ be odd.

(1) $A_1(\ell)q = \frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee$.

(2) $A_2(\ell)q = \frac{1+\ell}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee$.

(3) $A_1(\ell) > A_2(\ell)$.

PROOF. Since $w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee$ for ℓ odd, by Lemma 5.6

$$A_1(\ell)q = t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)q = (\frac{1-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee) + \frac{\ell-1}{2}\widetilde{\alpha}^\vee = \frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee$$

$$A_2(\ell)q = s_1A_1(\ell)q = s_1(\frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee) = \frac{1-\ell}{6}\alpha_1^\vee + \frac{\ell}{3}\widetilde{\alpha}^\vee = \frac{1+\ell}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee.$$

Then

$$(A_2(\ell)q, \alpha_1^\vee) = (\frac{1+\ell}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee, \alpha_1^\vee) = 2(\frac{1+\ell}{6}) - \frac{\ell}{3} = \frac{1}{3}.$$

Hence $0 < (A_2(\ell)q, \alpha_1^\vee) < 1$. Therefore $A_2(\ell)$ is the smallest along the line and $A_1(\ell) > A_2(\ell)$. $\square$

LEMMA 5.11. Let ℓ be odd.

(1) $A_1(\ell) = s_{\widetilde{\alpha},0}w(\ell+1)$.

(2) $A_2(\ell) = s_2s_1w(\ell+1)$.

PROOF. (1) By Lemma 5.6,

$$\begin{aligned} s_{\widetilde{\alpha},0}w(\ell+1)q &= s_{\widetilde{\alpha},0}(\frac{1-(\ell+1)}{3}\alpha_1^\vee + \frac{2-(\ell+1)}{6}\alpha_2^\vee) \\ &= -\frac{-\ell}{3}\alpha_2^\vee - \frac{1-\ell}{6}\alpha_1^\vee \\ &= \frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee. \end{aligned}$$

By Lemma 5.10, $s_{\widetilde{\alpha},0}w(\ell+1)q = A_1(\ell)q$. Hence $A_1(\ell) = s_{\widetilde{\alpha},0}w(\ell+1)$.

(2) By Lemma 5.6,

$$\begin{aligned} s_2 s_1 w(\ell+1)q &= s_2 s_1\left(\frac{1-(\ell+1)}{3}\alpha_1^\vee + \frac{2-(\ell+1)}{6}\alpha_2^\vee\right) \\ &= s_2\left(-\frac{-\ell}{3}\alpha_1^\vee + \frac{1-\ell}{6}\widetilde{\alpha}^\vee\right) \\ &= -\frac{-\ell}{3}\widetilde{\alpha}^\vee + \frac{1-\ell}{6}\alpha_1^\vee \\ &= \frac{\ell+1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee. \end{aligned}$$

By Lemma 5.10, $s_2 s_1 w(\ell+1)q = A_2(\ell)q$. Hence $A_2(\ell) = s_2 s_1 w(\ell+1)$. $\square$

LEMMA 5.12. Let $\ell \geq 6, \ell$ odd. The formulas for $A_1(\ell)$ in window notation, and tables for $A_1(\ell)^{\leq i}$ are given as follows.

(1) If $\ell = 6k+1$, then $A_1(\ell) = [2+3k, 1+3k, 3-6k]$.

(2) If $\ell = 6k+3$, then $A_1(\ell) = [1+3k, -6k, 5+3k]$.

(3) If $\ell = 6k+5$, then $A_1(\ell) = [-3-6k, 5+3k, 4+3k]$.

	$\ell = 6k+1$	$\ell = 6k+3$	$\ell = 6k+5$
$A_1(\ell)^{\leq 1}$	$\{2+3k, -1+3k, \cdots\} = \{\frac{\ell+3}{2}, \frac{\ell-3}{2}, \cdots\}$	$\{2+3k, 1+3k, \cdots\} = \{\frac{\ell+1}{2}, \frac{\ell-1}{2}, \cdots\}$	$\{2+3k, 1+3k, \cdots\} = \{\frac{\ell-1}{2}, \frac{\ell-3}{2}, \cdots\}$
$A_1(\ell)^{\leq 2}$	$\{2+3k, 1+3k, \cdots\} = \{\frac{\ell+3}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{2+3k, 1+3k, \cdots\} = \{\frac{\ell+1}{2}, \frac{\ell-1}{2}, \cdots\}$	$\{5+3k, 2+3k, \cdots\} = \{\frac{\ell+5}{2}, \frac{\ell-1}{2}, \cdots\}$
$A_1(\ell)^{\leq 3}$	$\{2+3k, 1+3k, \cdots\} = \{\frac{\ell+3}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{5+3k, 2+3k, \cdots\} = \{\frac{\ell+7}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{5+3k, 4+3k, \cdots\} = \{\frac{\ell+5}{2}, \frac{\ell+3}{2}, \cdots\}$

Table 5.3: The sets $A_1(\ell)^{\leq i}$

PROOF. Case $\ell = 6k+1$. By Lemma 5.8, $w(\ell) = [2+3k, 1-6k, 3+3k]$. By Lemma 2.15,

$$\begin{aligned}A_1(\ell) &= t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)\\ &= t(3k\widetilde{\alpha}^\vee)w(\ell)\\ &= [1+3(3k), 2, 3-3(3k)][2+3k, 1-6k, 3+3k]\\ &= [2+3k, 1+3k, 3-6k].\end{aligned}$$

Case $\ell = 6k+3$. By Lemma 5.8, $w(\ell) = [-2-6k, 3+3k, 5+3k]$. By Lemma 2.15,

$$\begin{aligned}A_1(\ell) &= t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)\\ &= t((3k+1)\widetilde{\alpha}^\vee)w(\ell)\\ &= [1+3(3k+1), 2, 3-3(3k+1)][-2-6k, 3+3k, 5+3k]\\ &= [1+3k, -6k, 5+3k].\end{aligned}$$

Case $\ell = 6k+5$. By Lemma 5.8, $w(\ell) = [3+3k, 5+3k, -2-6k]$. By Lemma 2.15,

$$\begin{aligned}A_1(\ell) &= t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)\\ &= t((3k+2)\widetilde{\alpha}^\vee)w(\ell)\\ &= [1+3(3k+2), 2, 3-3(3k+2)][3+3k, 5+3k, -2-6k]\\ &= [-3-6k, 5+3k, 4+3k].\end{aligned}$$

From this, we can determine the sets $A_1(\ell)^{\leq i}, i = 1, 2, 3.$ □

LEMMA 5.13. Let $\ell \geq 6, \ell$ odd. Then $A_1(\ell)$ and $A_2(\ell)$ are not $\leq w(\ell)$.

PROOF. By Lemma 5.10 $A_1(\ell) \geq A_2(\ell)$, it is enough to check that $A_2(\ell) \nleq w(\ell)$. Since $s_1 w(\ell) < w(\ell)$ and $s_1 A_1(\ell) = A_2(\ell)$, Lemma 4.5 implies that $A_1(\ell) < w(\ell) \Leftrightarrow A_2(\ell) < w(\ell)$. Therefore it is enough to check that $A_1(\ell) \nleq w(\ell)$.

Case $\ell = 6k+1$. We have $A_1(\ell) = [2+3k, 1+3k, 3-6k]$. Thus $(A_1(\ell))^{\leq 2} = \{2+3k, 1+3k, \cdots\}$, $w(\ell)^{\leq 2} = \{2+3k, 3k, \cdots\}$. Hence $A_1(\ell) \nleq w(\ell)$ as $1+3k \nleq 3k$.

Case $\ell = 6k+3$. We have $A_1(\ell) = [1+3k, -6k, 5+3k]$. Thus $(A_1(\ell))^{\leq 1} = \{2+3k, 1+3k, \cdots\}$, $w(\ell)^{\leq 1} = \{2+3k, 3k, \cdots\}$. Hence $A_1(\ell) \not\leq w(\ell)$ as $1+3k \not\leq 3k$.

Case $\ell = 6k+5$. We have $A_1(\ell) = [-3-6k, 5+3k, 4+3k]$. Thus $(A_1(\ell))^{\leq 3} = \{5+3k, 4+3k, \cdots\}$, $w(\ell)^{\leq 3} = \{5+3k, 3+3k, \cdots\}$. Hence $A_1(\ell) \not\leq w(\ell)$ as $4+3k \not\leq 3+3k$. $\square$

5.3 The region $R(\ell)$

In this section, we study the region $R(\ell)$, which is close to a triangle region. Let us first define the triangle region.

Definition 5.14. If $z = u\alpha_1^\vee + v\alpha_2^\vee$, define

$$\lambda_u(z) := u,\ \lambda_v(z) := v,\ \lambda_{vu}(z) := v-u.$$

Definition 5.15. Let $\overline{\triangle}(\ell)$ be the (triangle) region defined to be the set of all $z = u\alpha_1^\vee + v\alpha_2^\vee$ satisfying:

$$\begin{aligned}(I_1(\ell)):&\ v-u \leq \frac{\ell}{6}+\varepsilon\\ (I_2(\ell)):&\ u \leq \frac{\ell}{6}+\varepsilon\\ (I_3(\ell)):&\ v \geq -\frac{\ell}{6}+\frac{1}{3}+\varepsilon\end{aligned}$$

where $\varepsilon = 0$ if ℓ is even, $\varepsilon = \frac{1}{6}$ if ℓ is odd. In other words, $z = u\alpha_1^\vee + v\alpha_2^\vee$ is in $\overline{\triangle}(\ell)$ if and only if

$$\lambda_{vu}(z) \leq \frac{\ell}{6}+\varepsilon, \lambda_u(z) \leq \frac{\ell}{6}+\varepsilon, \lambda_v(z) \geq -\frac{\ell}{6}+\frac{1}{3}+\varepsilon.$$

Let $\triangle(\ell) = \{xq \mid w \in W\} \cap \overline{\triangle}(\ell)$. For each $k = 1, 2, 3$, define $L_k(\ell)$ to be the set of all $z = u\alpha_1^\vee + v\alpha_2^\vee$ such that z satisfies $I_k(\ell)$ with equality holding.

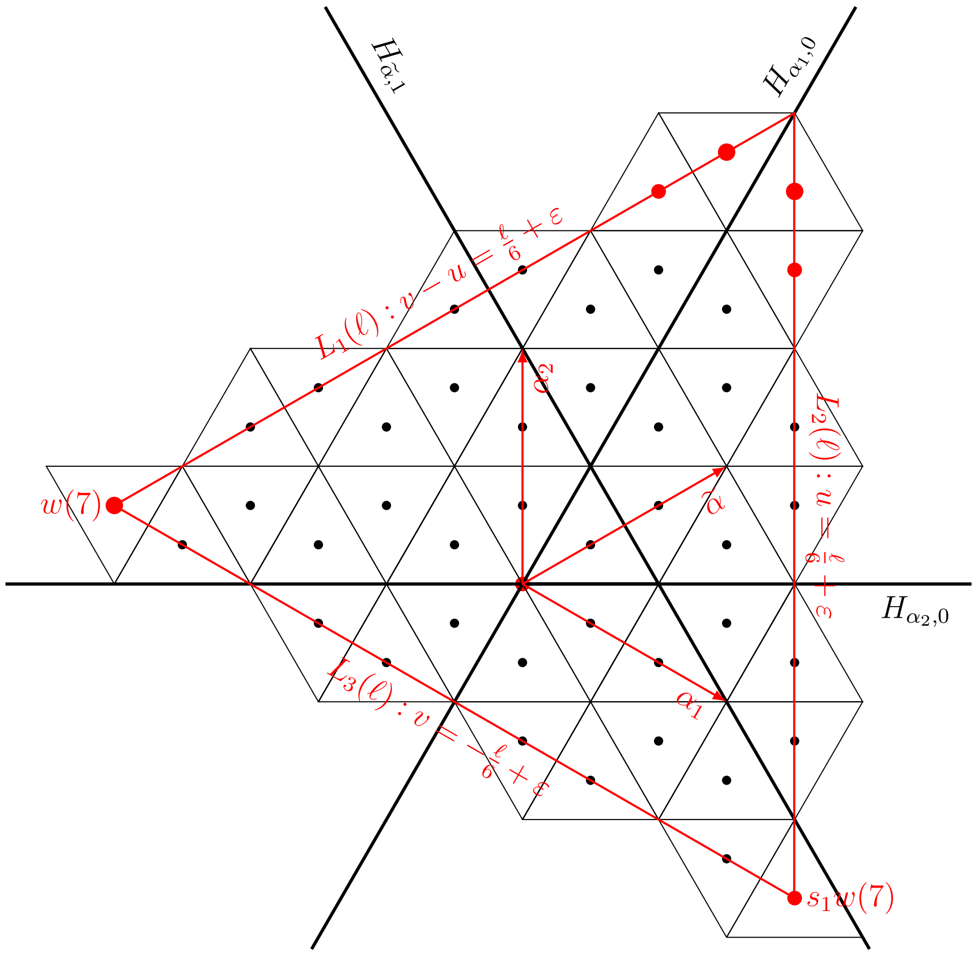

Figure 5.3: $\triangle(\ell)$ for ℓ odd ($\ell = 7$)

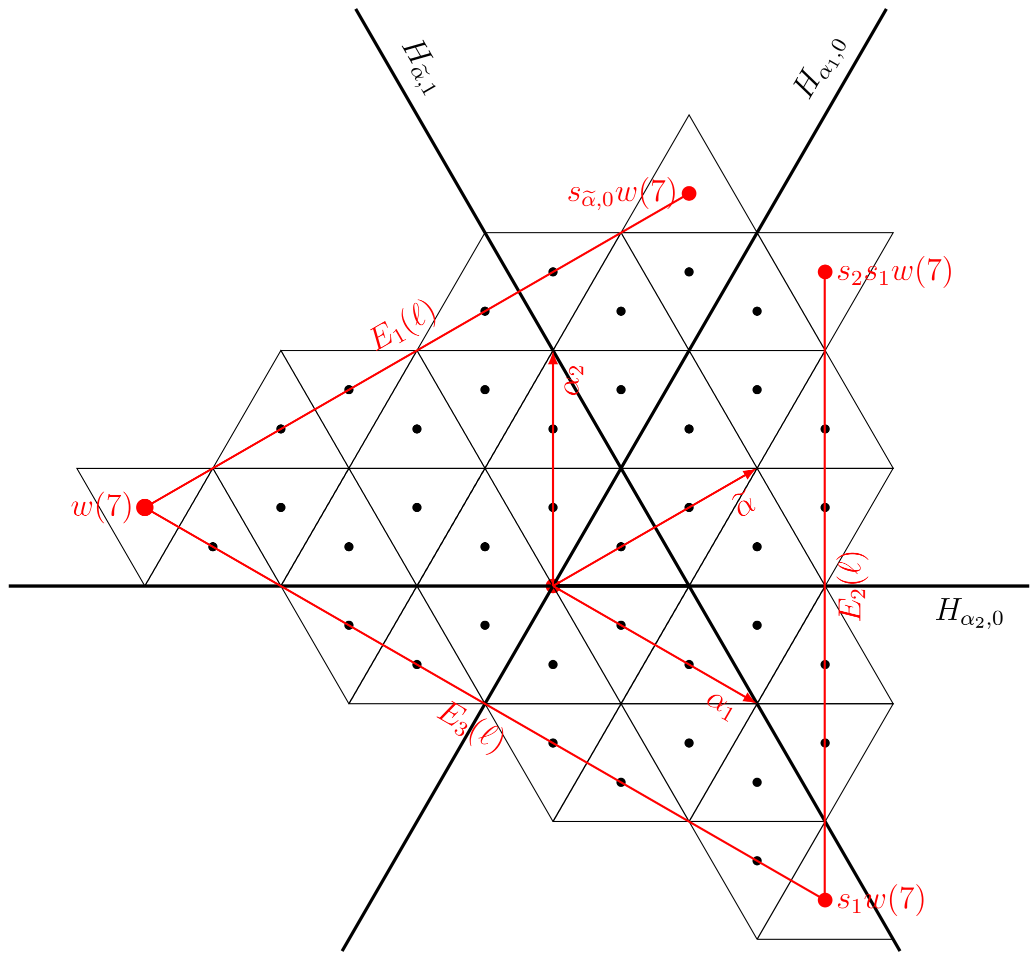

Figure 5.4: $R(\ell)$ for ℓ odd ($\ell = 7$)

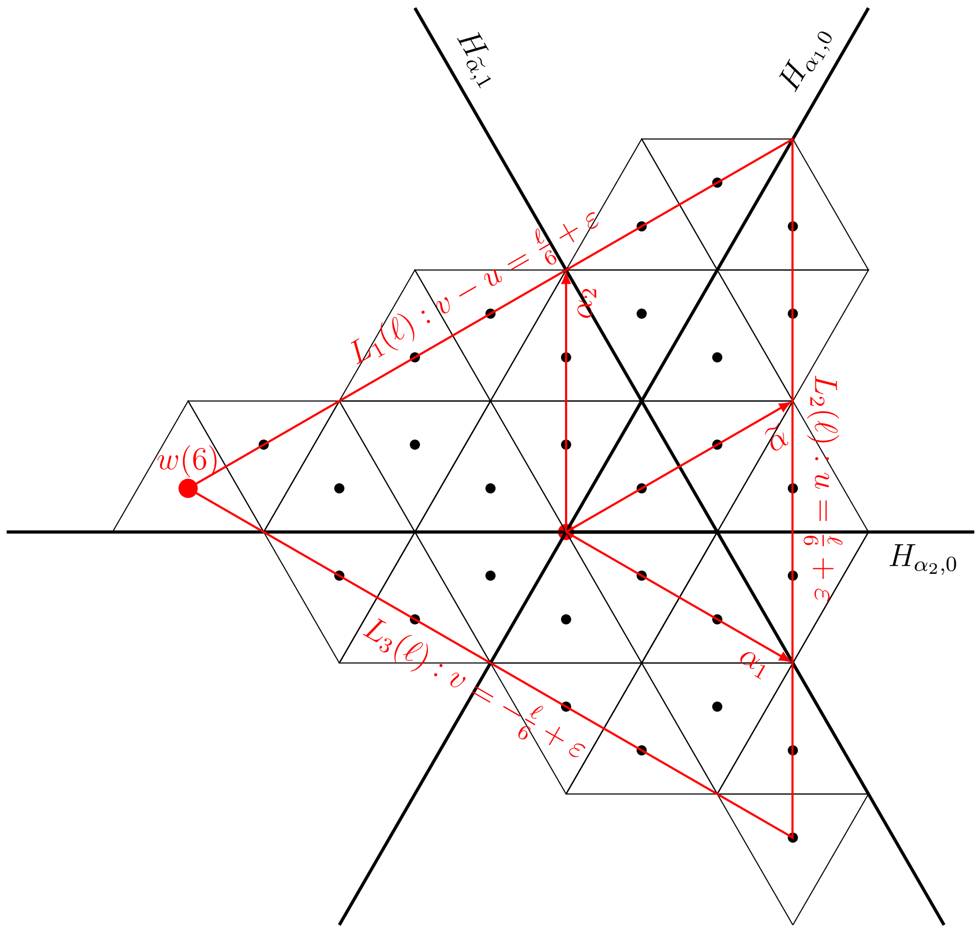

Figure 5.5: $\triangle(\ell)$ for ℓ even $(\ell = 6)$

Observe that the line $L_1(\ell)$ is parallel to $\widetilde{\alpha}$. The line $L_2(\ell)$ of the triangle $\overline{\triangle}(\ell)$ is parallel to α_2. The line $L_3(\ell)$ of the triangle $\overline{\triangle}(\ell)$ is parallel to α_1.

LEMMA 5.16. $\overline{\triangle}(\ell+1) \supseteq \overline{\triangle}(\ell)$ and $\triangle(\ell+1) \supseteq \triangle(\ell)$ for all ℓ.

PROOF. If $\ell = 2k$, then $I_1(2k) \Rightarrow I_1(2k+1), I_2(2k) \Rightarrow I_2(2k+1)$ and $I_3(2k)$ is the same as $I_3(2k+1)$, so $z \in \overline{\triangle}(2k) \Rightarrow z \in \overline{\triangle}(2k+1)$.

If $\ell = 2k-1$, then $I_1(2k-1)$ is the same as $I_1(2k)$, $I_2(2k-1)$ is the same as $I_2(2k)$, and $I_3(2k-1) \Rightarrow I_3(2k)$, so $z \in \overline{\triangle}(2k-1) \Rightarrow z \in \overline{\triangle}(2k)$. Hence $\triangle(\ell+1) \supseteq \triangle(\ell)$ for all ℓ. ☐

Definition 5.17. If ℓ is even, let $R(\ell) = \triangle(\ell)$.

If ℓ is odd, let $R(\ell) = \triangle(\ell) \setminus \{A_1(\ell)q, A_2(\ell)q\}$.

Definition 5.18. Let $E_i(\ell) = L_i(\ell) \cap R(\ell)$ for $i = 1, 2, 3$. If $L_i(\ell)$ is parallel to β, and $z \in L_i(\ell)$, the endpoints of $E_i(\ell)$ are the points on $E_i(\ell)$ of the form $z + t_1\beta$ and $z + t_2\beta$ where t_1 and t_2 are chosen as small (respectively, large) as possible.

Note that these definitions depend on ℓ. If ℓ is understood, we will sometimes omit this from the notation and write, for example, L_i for $L_i(\ell)$.

LEMMA 5.19. If $z \in \triangle(\ell)$, then

$$\lambda_u(z) \geq -\frac{\ell}{3} + \frac{1}{3},\ \lambda_v(z) \leq \frac{\ell}{3} + 2\varepsilon,\ \lambda_{vu}(z) \geq -\frac{\ell}{3} + \frac{1}{3}$$

where $\epsilon = 0$ if ℓ is even, $\epsilon = \frac{1}{6}$ if ℓ is odd.

PROOF. Since $z \in \triangle(\ell)$, $\lambda_{vu}(z) \leq \frac{\ell}{6} + \varepsilon$ and $-\lambda_v(z) \leq \frac{\ell}{6} - \frac{1}{3} - \varepsilon$. Therefore, $(\lambda_{vu} - \lambda_v)(z) = -\lambda_u(z) \leq \frac{\ell}{3} - \frac{1}{3}$ which gives the first inequality.

Since $z \in \triangle(\ell)$, we have $\lambda_{vu}(z) = \lambda_v(z) - \lambda_u(z) \leq \frac{\ell}{6} + \varepsilon$ and $\lambda_u(z) \leq \frac{\ell}{6} + \varepsilon$. Adding these inequalities gives $\lambda_v(z) \leq \frac{\ell}{3} + 2\varepsilon$ which is the second inequality.

Since $z \in \triangle(\ell)$, $\lambda_v(z) \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon$ and $-\lambda_u(z) \geq -\frac{\ell}{6} - \varepsilon$. Adding these inequalities gives $(\lambda_v - \lambda_u)(z) = \lambda_{vu}(z) \geq -\frac{\ell}{3} + \frac{1}{3}$. □

COROLLARY 5.20. $\overline{\triangle}(\ell)$ is a triangle and its interior.

PROOF. By Lemma 5.19, $\overline{\triangle}(\ell)$ is a bounded region defined by 3 linear inequalities, hence $\overline{\triangle}(\ell)$ is a triangle and its interior. □

LEMMA 5.21. $s_1(\triangle(\ell)) = \triangle(\ell)$ and $s_1(R(\ell)) = R(\ell)$.

$s_1(E_1(\ell)) = E_2(\ell)$, and $s_1(E_3(\ell)) = E_3(\ell)$.

PROOF. (a) We show $s_1(L_1(\ell)) = L_2(\ell)$. Consider

$$\{x = u\alpha_1 + v\alpha_2 \mid v - u \leq \frac{\ell}{6} + \varepsilon\}$$

where $\varepsilon = 0$ if ℓ is even, $\varepsilon = \frac{1}{6}$ if ℓ is odd. We have $s_1 x = s_1(u\alpha_1 + v\alpha_2) = (v-u)\alpha_1 + v\alpha_2 = u'\alpha_1 + v'\alpha_2$ where $u' = v - u, v' = v$. Since $u' = v - u, v' = v$, we have $u = v' - u'$. Then

$$v' - (v' - u') \leq \frac{\ell}{6} + \varepsilon$$

$$u' \leq \frac{\ell}{6} + \varepsilon$$

Therefore $s_1(L_1(\ell)) = L_2(\ell)$. Since $L_3(\ell) = \{x = u\alpha_1 + v\alpha_2 \mid v - u = -\frac{\ell}{6} + \frac{1}{3} + +\varepsilon\}$ is perpendicular to the hyperplane $H_{\alpha_1,0}$, $s_1(L_3(\ell)) = L_3(\ell)$. Therefore $s_1(\overline{\triangle}(\ell)) = \overline{\triangle}(\ell)$. Since $\triangle(\ell) = \{xq \mid x \in W\} \cap \overline{\triangle}(\ell)$, we have $s_1(\triangle(\ell)) = \triangle(\ell)$. Since $s_1(A_1(\ell)q) = A_2(\ell)q$, we have $s_1(R(\ell)) = R(\ell)$. We showed $s_1(L_1(\ell)) = L_2(\ell)$, so

$$s_1(E_1(\ell)) = s_1(L_1(\ell) \cap R(\ell)) = s_1 L_1(\ell) \cap s_1 R(\ell) = L_2(\ell) \cap R(\ell) = E_2(\ell).$$

Similarly,

$$s_1(E_3(\ell)) = s_1(L_3(\ell) \cap R(\ell)) = s_1 L_3(\ell) \cap s_1 R(\ell) = L_3(\ell) \cap R(\ell) = E_3(\ell).$$

$\square$

LEMMA 5.22. The following holds.

(1) $L_1(\ell) \cap L_3(\ell) = w(\ell)q$.

(2) $L_2(\ell) \cap L_3(\ell) = s_1 w(\ell)q$.

PROOF. $L_1(\ell) \cap L_3(\ell)$ is the solution $u\alpha_1^\vee + v\alpha_2^\vee$ to the linear equations

$$v - u = \frac{\ell}{6} + \varepsilon,\ v = -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

The formula for $w(\ell)q$ from Lemma 5.6 shows that $w(\ell)q$ is this solution. Also $L_2(\ell) \cap L_3(\ell)$ is the point $(\frac{\ell}{6} + \varepsilon)\alpha_1^\vee + (-\frac{\ell}{6} + \frac{1}{3} + \varepsilon)\alpha_2^\vee$. Using the formulas from Lemma 5.6 we see that this is $s_1 w(\ell)q$. $\square$

Remark. The points $L_1(\ell) \cap L_3(\ell)$ and $L_2(\ell) \cap L_3(\ell)$ each lie on some edge $E_i(\ell)$, so they are endpoints. But $L_1(\ell) \cap L_2(\ell)$ does not lie on $E_i(\ell)$ since it is not of the form xq. So we must do some work to figure out the other endpoints.

LEMMA 5.23. If $w(\ell)q + t\widetilde{\alpha}^\vee \in \triangle(\ell)$, then $0 \leq t \leq \frac{\ell}{2} - \frac{1}{3} + \varepsilon$.

PROOF. By Lemma 5.6,

$$w(\ell)q + t\widetilde{\alpha}^\vee = (\frac{1-\ell}{3} + t)\alpha_1^\vee + (\frac{2-\ell}{6} + \varepsilon + t)\alpha_2^\vee.$$

Since $w(\ell)q + t\widetilde{\alpha}^\vee \in \triangle(\ell)$, by Definition 5.15 and Lemma 5.19,

$$-\frac{\ell}{3} + \frac{1}{3} \leq \lambda_u(w(\ell)q + t\widetilde{\alpha}^\vee) = \frac{1-\ell}{3} + t \leq \frac{\ell}{6} + \varepsilon.$$

Hence

$$0 \le t \le \frac{\ell}{6} + \frac{\ell}{3} - \frac{1}{3} + \varepsilon = \frac{\ell}{2} - \frac{1}{3} + \varepsilon.$$

□

Remark 5.24. If ℓ is odd and $w(\ell)q + t\widetilde{\alpha}^\vee \in R(\ell)$, then $t < \frac{\ell-1}{2}$ since $t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)q = A_1(\ell)q \notin R(\ell)$.

PROPOSITION 5.25. The endpoints of the $E_i(\ell)$ are as follows.

(a) The endpoints of $E_1(\ell)$ are $w(\ell)q$ and $s_{\widetilde{\alpha},0}w(\ell)q$.

(b) The endpoints of $E_2(\ell)$ are $s_1w(\ell)q$ and $s_2s_1w(\ell)q$.

(c) The endpoints of $E_3(\ell)$ are $w(\ell)q$ and $s_1w(\ell)q$.

PROOF. (a) All points on $E_1(\ell)$ are of the form $w(\ell)q + t\widetilde{\alpha}^\vee, t \in \mathbb{R}$. Lemma 5.22 and Lemma 5.23 imply that one endpoint of $E_1(\ell)$ is $w(\ell)q$. The other endpoint of $E_1(\ell)$ is the point

$$xq = w(\ell)q + t\widetilde{\alpha}^\vee$$

on $L_1(\ell)$ where $x \in W$ is chosen so that t is as large as possible. Any point xq on $L_1(\ell)$ is of the form

$$t(a\widetilde{\alpha}^\vee)w(\ell)q = w(\ell)q + a\widetilde{\alpha} \text{ or } t(b\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}w(\ell)q = w(\ell)q + (\frac{\ell}{2} - \frac{2}{3} - \varepsilon)\widetilde{\alpha},\ (a, b \in \mathbb{Z}).$$

because $w(\ell)$ and $s_{\widetilde{\alpha},0}w(\ell)$ differ by reflection (by Proposition 2.13). Here we have used the fact that by Lemma 5.6, $w(\ell)q = (\frac{1-\ell}{3})\alpha_1^\vee + (\frac{2-\ell}{6} + \varepsilon)\alpha_2^\vee$, so

$$\begin{aligned} s_{\widetilde{\alpha},0}w(\ell)q &= w(\ell)q - \frac{2(\widetilde{\alpha}, w(\ell)q)}{(\widetilde{\alpha},\widetilde{\alpha})}\widetilde{\alpha} \\ &= w(\ell)q - (\frac{2-2\ell}{6} + \frac{2-\ell}{6} + \varepsilon)\widetilde{\alpha} \\ &= w(\ell)q - (-\frac{\ell}{2} + \frac{2}{3} + \varepsilon)\widetilde{\alpha} \\ &= w(\ell)q + (\frac{\ell}{2} - \frac{2}{3} - \varepsilon)\widetilde{\alpha}. \end{aligned}$$

By Lemma 5.23, if $w(\ell)q + t\widetilde{\alpha}^\vee \in \triangle(\ell)$ $(t \in \mathbb{R})$, then $t \leq \frac{\ell}{2} - \frac{1}{3} + \varepsilon$. Therefore, the other endpoint of $E_1(\ell)$ is the point $w(\ell)q + c\widetilde{\alpha}^\vee$ where c is the largest value of a or $\frac{\ell}{2} - \frac{2}{3} - \varepsilon + b$ $(a, b \in \mathbb{Z})$ satisfying $c \leq \frac{\ell}{2} - \frac{1}{3} + \varepsilon$, and such that if ℓ is odd, $c < \frac{\ell-1}{2}$ (By Remark 5.24).

We claim: If $t(a\widetilde{\alpha}^\vee)w(\ell)q \in \triangle(\ell)$, then

$$a \leq \begin{cases} \frac{\ell-2}{2}, & \text{if } \ell \text{ is even} \\ \frac{\ell-1}{2}, & \text{if } \ell \text{ is odd.} \end{cases}$$

The reason is that the largest value of $a \in \mathbb{Z}$ such that $a \leq \frac{\ell}{2} - \frac{1}{3} + \varepsilon$ is $\lfloor \frac{\ell}{2} - \frac{1}{3} + \varepsilon \rfloor$. If ℓ is even, $\ell = 2m, \varepsilon = 0$, then $\lfloor m - \frac{1}{3} \rfloor = m - 1 = \frac{\ell}{2} - 1 = \frac{\ell-2}{2}$. If ℓ is odd, $\ell = 2m+1, \varepsilon = \frac{1}{6}$, then $\lfloor m + \frac{1}{2} - \frac{1}{3} + \frac{1}{6} \rfloor = m = \frac{\ell-1}{2}$. This proves the claim. As noted in the Remark 5.24, if ℓ is odd, then $t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell) = A_1(\ell) \notin R(\ell)$, so if $t(a\widetilde{\alpha}^\vee)w(\ell)q \in R(\ell)$, then

$$a \leq \begin{cases} \frac{\ell-2}{2}, & \text{if } \ell \text{ is even} \\ \frac{\ell-3}{2}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Observe that $s_{\widetilde{\alpha},0}w(\ell)q$ is on $E_1(\ell)$. The reason is that $s_{\widetilde{\alpha},0}w(\ell)q$ is a point of the form xq $(x \in W)$, it is in $\triangle(\ell)$, since

$$s_{\widetilde{\alpha},0}w(\ell)q = s_{\widetilde{\alpha},0}(\frac{1-\ell}{3}\alpha_1^\vee + (\frac{2-\ell}{6} + \varepsilon)\alpha_2^\vee) = (\frac{\ell-2}{6} - \varepsilon)\alpha_1^\vee + \frac{\ell-1}{3}\alpha_2^\vee,$$

which satisfies the inequalities from Definition 5.15, and if ℓ is odd, $s_{\widetilde{\alpha},0}w(\ell)q$ is not equal to $A_1(\ell)$ or $A_2(\ell)$ (by Lemma 5.10).

Claim. If $t(b\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}w(\ell)q \in R(\ell), (b \in \mathbb{Z})$, then $b \leq 0$.

We have $t(b\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}w(\ell)q = w(\ell)q + (\frac{\ell}{2} - \frac{2}{3} - \varepsilon + b)\widetilde{\alpha}^\vee$. If this is in $\triangle(\ell)$, then

$$\frac{\ell}{2} - \frac{2}{3} - \varepsilon + b \leq \frac{\ell}{2} - \frac{1}{3} + \varepsilon.$$

Then

$$b \leq \frac{1}{3} + 2\varepsilon \Rightarrow b \leq 0.$$

This proves the claim.

We have $\frac{\ell}{2} - \frac{2}{3} + \varepsilon = \frac{\ell - \frac{4}{3}}{2} + \varepsilon$. This is bigger than either $\frac{\ell-2}{2}$ or $\frac{\ell-3}{2}$. So the value of c is $\frac{\ell}{2} - \frac{2}{3} + \varepsilon$ and the point xq on $E_1(\ell)$ with $xq = w(\ell)q + t\widetilde{\alpha}^\vee$, t as large as possible, is $xq = s_{\widetilde{\alpha},0}w(\ell)q$.

(b) By (a), The endpoints of $E_1(\ell)$ are $w(\ell)q$ and $s_{\widetilde{\alpha},0}w(\ell)q$. By Lemma 5.21, $s_1(E_1(\ell)) = E_2(\ell)$. Therefore the endpoints of $E_2(\ell)$ are $s_1w(\ell)q$ and

$$s_1 s_{\widetilde{\alpha},0}w(\ell)q = s_1s_1s_2s_1w(\ell)q = s_2s_1w(\ell)q.$$

(c) This follows by Lemma 5.22. ☐

Now we show that the endpoints of $E_1(\ell), E_2(\ell)$, and $E_3(\ell)$ are $\leq w(\ell)$.

LEMMA 5.26. Let $w = w(\ell)$. Then $s_1w < w$, $s_{\widetilde{\alpha},0}w < w$ and $s_2s_1w < w$.

PROOF. Since $s_1w = s_2s_3s_1s_2s_3\cdots$ is spiral starting with s_2, and there are $\ell - 1$ factors, $\ell(s_1w) = \ell - 1 < \ell(w) = \ell$. Hence $s_1w < w$. Similarly, $s_2s_1w = s_3s_1s_2s_3\cdots$ is spiral starting with s_3, and there are $\ell - 2$ factors, so $\ell(s_2s_1w) = \ell - 2 < \ell(s_1w) = \ell - 1$. Hence $s_2s_1w < s_1w < w$. By Lemma 2.5, $s_{\widetilde{\alpha},0} = t(-\widetilde{\alpha}^\vee)s_3 = s_1s_2s_1s_3s_3 = s_1s_2s_1$. So

$$s_{\widetilde{\alpha},0}w = (s_1s_2s_1)(s_1s_2s_3)w(\ell - 3) = s_1s_3w(\ell - 3).$$

Since $s_3w(\ell - 3) = s_3s_1s_2s_3\cdots$ is spiral starting with s_3, and there are $\ell - 2$ factors, so $s_3w(\ell - 3)$ has length $\ell - 2$. This implies $\ell(s_1s_3w(\ell - 3)) \leq \ell - 1$. Then $\ell(s_{\widetilde{\alpha},0}w) \leq \ell - 1 < \ell(w(\ell))$. Hence $s_{\widetilde{\alpha},0}w < w$. ☐

PROPOSITION 5.27. If xq is on $E_i(\ell)$ with $i = 1, 2, 3$, then $x \leq w(\ell)$.

PROOF. We show this by reducing to the endpoints. By Theorem 3.23, for any point xq on $E_1(\ell)$, $x \leq w(\ell)$ or $x \leq s_{\tilde{\alpha},0}w(\ell)$. For any point xq on $E_2(\ell)$, $x \leq s_1w(\ell)$ or $x \leq s_2s_1w(\ell)$. For any point xq on $E_3(\ell)$, $x \leq w(\ell)$ or $x \leq s_1w(\ell)$. By Lemma 5.26, $s_1w(\ell) \leq w(\ell)$, $s_{\tilde{\alpha},0}w(\ell) \leq w(\ell)$ and $s_2s_1w(\ell) \leq w(\ell)$. □

PROPOSITION 5.28. The following holds.

(a) $R(2k+1) = R(2k) \cup E_1(2k+1) \cup E_2(2k+1)$

(b) $R(2k) = R(2k-1) \cup E_3(2k) \cup \{A_1(2k-1), A_2(2k-1)\}$.

PROOF. ($\supseteq$): Observe that $R(\ell) \supset R(\ell-1)$ for any ℓ. Indeed, if ℓ is even, then $\ell-1$ is odd, and $R(\ell) = \triangle(\ell) \supset \triangle(\ell-1) \supset R(\ell-1)$ (using Lemma 5.16). If ℓ is odd, then $R(\ell) = \triangle(\ell) \setminus \{A_1(\ell)q, A_2(\ell)q\}$. But $A_1(\ell)q, A_2(\ell)q$ are not in $\triangle(\ell-1)$, since the formula from Lemma 5.10 show that $A_1(\ell)q$ fails $I_1(\ell-1)$ and $A_2(\ell)q$ fails $I_2(\ell-1)$. Hence $R(\ell) = \triangle(\ell) \setminus \{A_1(\ell)q, A_2(\ell)q\} \supset \triangle(\ell-1) = R(\ell-1)$ (using Lemma 5.16).

This proves that in statements (a) and (b) of the Proposition, the inclusion ($\supseteq$) holds, since $R(\ell+1) \supseteq R(\ell)$ and also, by definition, $R(\ell+1) \supset E_i(\ell+1)$.

($\subseteq$): We must prove the reverse inclusion for (a) and (b).

(a) Let $xq \in R(2k+1)$. If the inequality $I_1(2k+1)$ is not strict, then xq lies on $L_1(2k+1) \cap \triangle(2k+1)$, so $xq \in E_1(2k+1)$. If the inequality $I_2(2k+1)$ is not strict, then xq lies on $L_2(2k+1) \cap \triangle(2k+1)$, so $xq \in E_2(2k+1)$.

If the inequalities $I_1(2k+1)$ and $I_2(2k+1)$ are both strict, we claim that $xq \in R(2k)$. Since $x \in W$, by Lemma 2.7, $xq = u\alpha_1^\vee + v\alpha_2^\vee \Rightarrow 3u, 3v \in \mathbb{Z}$. If $I_1(2k+1)$ is strict, then

$$v - u < \frac{2k+1}{6} + \frac{1}{6} = \frac{k}{3} + \frac{1}{3}.$$

So

$$3(v-u) < k+1.$$

As $3(v-u) \in \mathbb{Z}, 3(v-u) \leq k$, so $v-u \leq \frac{k}{3}$. Similarly, if $I_2(2k+1)$ is strict, then

$$u < \frac{2k+1}{6} + \frac{1}{3} = \frac{k}{3} + \frac{1}{3}.$$

So

$$3u < k+1.$$

As $3u \in \mathbb{Z}, 3u \leq k$, so $u \leq \frac{k}{3}$. Therefore xq satisfies $I_1(2k)$ and $I_2(2k)$, so $xq \in R(2k)$, proving the claim. Since the inequalities $I_3(2k+1)$and $I_3(2k)$ are the same, xq also satisfies $I_3(2k)$ and therefore $xq \in R(2k)$.

(b) Let $xq \in R(2k)$. If the inequality $I_3(2k)$ is not strict, then xq lies on $L_3(2k) \cap \triangle(2k)$, so $xq \in E_3(2k)$.

If the inequality $I_3(2k)$ is strict, we show that

$$xq \in \triangle(2k-1) = R(2k-1) \cup \{A_1(2k-1), A_2(2k-1)\}.$$

Since $x \in W$, by Lemma 2.7, $xq = u\alpha_1^\vee + v\alpha_2^\vee \Rightarrow 3u, 3v \in \mathbb{Z}$. If $I_3(2k)$ is strict, then

$$v > -\frac{2k}{6} + \frac{1}{3} = -\frac{k}{3} + \frac{1}{3},$$

so

$$3v > -k+1.$$

Since $3v \in \mathbb{Z}, 3v \geq -k+2$, so $v \geq -\frac{k}{3} + \frac{2}{3}$ which is $I_3(2k-1)$. Since the inequalities $I_1(2k)$ and $I_1(2k-1)$ are the same, and $I_2(2k)$ and $I_2(2k-1)$ are the same, $xq \in \triangle(2k-1)$.
□

5.4 Triangle Theorems

In this section we prove that $x \leq w(\ell)$ if and only if $xq \in R(\ell)$. If ℓ is even, then $R(\ell) = \triangle(\ell)$. If ℓ is odd, then $R(\ell) \neq \triangle(\ell)$. We prove that $x \leq w(\ell)$ or $x \leq A_1(\ell) \Leftrightarrow xq \in \triangle(\ell)$. We call these theorems triangle theorem even though $R(\ell)$ is not quite a triangle. We will write these inequalities in a different form. Consider the following sets of three inequalities.

Definition 5.29. Let ℓ be even. Define

$(BA_1(\ell)) : B - A \leq \frac{\ell}{6} - \frac{1}{3}$

$(BA_2(\ell)) : B - A \leq \frac{\ell}{6}$

$(BA_3(\ell)) : B - A \leq \frac{\ell}{6} + \frac{1}{3}$

$(A_1(\ell)) : A \leq \frac{\ell}{6} - \frac{1}{3}$

$(A_2(\ell)) : A \leq \frac{\ell}{6}$

$(A_3(\ell)) : A \leq \frac{\ell}{6} + \frac{1}{3}$

$(B_1(\ell)) : B \geq -\frac{\ell}{6} + \frac{2}{3}$

$(B_2(\ell)) : B \geq -\frac{\ell}{6} + \frac{1}{3}$

$(B_3(\ell)) : B \geq -\frac{\ell}{6}$.

Let ℓ be odd. Define

$(BA_1(\ell)) : B - A \leq \frac{\ell}{6} - \frac{1}{6}$

$(BA_2(\ell)) : B - A \leq \frac{\ell}{6} + \frac{1}{6}$

$(BA_3(\ell)) : B - A \leq \frac{\ell}{6} + \frac{1}{2}$

$(A_1(\ell)) : A \leq \frac{\ell}{6} - \frac{1}{6}$

$(A_2(\ell)) : A \leq \frac{\ell}{6} + \frac{1}{6}$

$(A_3(\ell)) : A \leq \frac{\ell}{6} + \frac{1}{2}$

$(B_1(\ell)) : B \geq -\frac{\ell}{6} + \frac{5}{6}$

$(B_2(\ell)) : B \geq -\frac{\ell}{6} + \frac{1}{2}$

$(B_3(\ell)) : B \geq -\frac{\ell}{6} + \frac{1}{6}$.

Note that we could combine the definitions for ℓ odd and ℓ even, but it is convenient to separate them for future use.

Definition 5.30. We say A, B satisfy $T(i,j,k)(\ell)$ if A, B satisfy the inequalities $BA_i(\ell)$, $A_j(\ell)$, $B_k(\ell)$. If ℓ is understood, we may omit it from the notation.

Remark. Since $BA_1 \Rightarrow BA_2 \Rightarrow BA_3$, if $i < i'$, then $T(i,j,k) \Rightarrow T(i',j,k)$. Since $A_1 \Rightarrow A_2 \Rightarrow A_3$, if $j < j'$, then $T(i,j,k) \Rightarrow T(i,j',k)$. Since $B_1 \Rightarrow B_2 \Rightarrow B_3$, if $k < k'$, then $T(i,j,k) \Rightarrow T(i,j,k')$.

LEMMA 5.31. Let $x = zy$, where $y \in W_{finite}$ and $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$. Then $x \in \triangle(\ell) \Leftrightarrow$ x satisfies $T(i,j,k)(\ell)$ where $T(i,j,k) = T(i,j,k)(\ell)$ is as in the following table.

x	$x \in \triangle(\ell)$
z	$T(2,1,3)$
zs_1	$T(1,2,3)$
zs_2	$T(3,1,2)$
zs_1s_2	$T(1,3,2)$
zs_2s_1	$T(3,2,1)$
$zs_1s_2s_1$	$T(2,3,1)$

PROOF. Table 2.1 is reproduced here for convenience. It records some information which is needed in the proof. In Table 2.1, let $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$.

x	x in window notation	$xq = u\alpha_1^\vee + v\alpha_2^\vee$	$v - u$
z	$[1+3A, 2-3A+3B, 3-3B]$	$(A+\frac{1}{3})\alpha_1^\vee + (B+\frac{1}{3})\alpha_2^\vee$	$B-A$
zs_1	$[2-3A+3B, 1+3A, 3-3B]$	$A\alpha_1^\vee + (B+\frac{1}{3})\alpha_2^\vee$	$B-A+\frac{1}{3}$
zs_2	$[1+3A, 3-3B, 2-3A+3B]$	$(A+\frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee$	$B-A-\frac{1}{3}$
zs_1s_2	$[2-3A+3B, 3-3B, 1+3A]$	$(A-\frac{1}{3})\alpha_1^\vee + B\alpha_2^\vee$	$B-A+\frac{1}{3}$
zs_2s_1	$[3-3B, 1+3A, 2-3A+3B]$	$A\alpha_1^\vee + (B-\frac{1}{3})\alpha_2^\vee$	$B-A-\frac{1}{3}$
$zs_1s_2s_1$	$[3-3B, 2-3A+3B, 1+3A]$	$(A-\frac{1}{3})\alpha_1^\vee + (B-\frac{1}{3})\alpha_2^\vee$	$B-A$

We have $xq \in \triangle(\ell) \Leftrightarrow$ if $xq = u\alpha_1^\vee + v\alpha_2^\vee$, then (Definition 5.15)

$$v - u \leq \frac{\ell}{6} + \varepsilon,\ u \leq \frac{\ell}{6} + \varepsilon, v \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon \tag{5.32}$$

where $\varepsilon = 0$ if ℓ is even, $\varepsilon = \frac{1}{6}$ if ℓ is odd.

Case 1. $x = z = t(A\alpha_1^\vee + B\alpha_2^\vee)$ Then the inequalities from (5.32) can be written as

$$B - A \leq \frac{\ell}{6} + \varepsilon,\ A + \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon,\ B + \frac{1}{3} \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

These simplify to

$$B - A \leq \frac{\ell}{6} + \varepsilon,\ A \leq \frac{\ell}{6} - \frac{1}{3} + \varepsilon,\ B \geq -\frac{\ell}{6} + \varepsilon. \tag{5.33}$$

which are $(BA_2(\ell)), (A_1(\ell)), (B_3(\ell))$. Therefore $xq \in \triangle(\ell) \Leftrightarrow A, B$ satisfies $T(2,1,3)$.

Case 2. $x = zs_1 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_1$ Then the inequalities from (5.32) can be written as

$$B - A + \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon,\ A \leq \frac{\ell}{6} + \varepsilon,\ B + \frac{1}{3} \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

These simplify to

$$B - A \leq \frac{\ell}{6} - \frac{1}{3} + \varepsilon,\ A \leq \frac{\ell}{6} + \varepsilon,\ B \geq -\frac{\ell}{6} + \varepsilon. \tag{5.34}$$

which are $(BA_1(\ell)), (A_2(\ell)), (B_3(\ell))$. Therefore $xq \in \triangle(\ell) \Leftrightarrow A, B$ satisfies $T(1,2,3)$.

Case 3. $x = zs_2 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_2$ Then the inequalities from (5.32) can be written as

$$B - A - \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon,\ A + \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon, B \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

These simplify to

$$B - A \leq \frac{\ell}{6} + \frac{1}{3} + \varepsilon,\ A \leq \frac{\ell}{6} - \frac{1}{3} + \varepsilon, B \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon. \tag{5.35}$$

which are $(BA_3(\ell)), (A_1(\ell)), (B_2(\ell))$. Therefore $xq \in \triangle(\ell) \Leftrightarrow A, B$ satisfies $T(3,1,2)$.

Case 4. $x = zs_1s_2 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_1s_2$ Then the inequalities from (5.32) can be written as

$$B - A + \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon,\ A - \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon, B \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

These simplify to

$$(5.36) \qquad B - A \leq \frac{\ell}{6} - \frac{1}{3} + \varepsilon,\ A \leq \frac{\ell}{6} + \frac{1}{3} + \varepsilon, B \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

which are $(BA_1(\ell)), (A_3(\ell)), (B_2(\ell))$. Therefore $xq \in \triangle(\ell) \Leftrightarrow A, B$ satisfies $T(1,3,2)$.

Case 5. $x = zs_2s_1 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_2s_1$. Then the inequalities from (5.32) can be written as

$$B - A - \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon,\ A \leq \frac{\ell}{6} + \varepsilon,\ B - \frac{1}{3} \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

These simplify to

$$(5.37) \qquad B - A \leq \frac{\ell}{6} + \frac{1}{3} + \varepsilon,\ A \leq \frac{\ell}{6} + \varepsilon,\ B \geq -\frac{\ell}{6} + \frac{2}{3} + \varepsilon.$$

which are $(BA_3(\ell)), (A_2(\ell)), (B_1(\ell))$. Therefore $xq \in \triangle(\ell) \Leftrightarrow A, B$ satisfies $T(3,2,1)$.

Case 6. $x = zs_1s_2s_1 = t(A\alpha_1^\vee + B\alpha_2^\vee)_1s_2s_1$. Then the inequalities from (5.32) can be written as

$$B - A \leq \frac{\ell}{6} + \varepsilon,\ A - \frac{1}{3} \leq \frac{\ell}{6} + \varepsilon,\ B - \frac{1}{3} \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

These simplify to

$$(5.38) \qquad B - A \leq \frac{\ell}{6} + \varepsilon,\ A \leq \frac{\ell}{6} + \frac{1}{3} + \varepsilon,\ B \geq -\frac{\ell}{6} + \frac{2}{3} + \varepsilon.$$

which are $(BA_2(\ell)), (A_3(\ell)), (B_1(\ell))$. Therefore $xq \in \triangle(\ell) \Leftrightarrow A, B$ satisfies $T(2,3,1)$. □

LEMMA 5.39. Let $x = zy$, where $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$ and $y \in W_{finite}$. Suppose $x \leq w(\ell)$. The integer $2 - 3A + 3B \in x^{\leq i}$ for some $i \in \{1, 2, 3\}$ and this implies that one of the inequalities $BA_k(\ell)$ holds (where k depends on i and ℓ). Similarly, $1 + 3A \in x^{\leq i}$, and this implies that one of the inequalities $A_k(\ell)$, and $3 - 3B \in x^{\leq i}$, and this implies that one of the inequalities $B_k(\ell)$. The following tables list the particular inequality which holds in each case.

ℓ	i with $2 - 3A + 3B \in x^{\leq i}$	Inequalities that holds
$6k+1$	1	$BA_1(\ell)$
	2	$BA_1(\ell)$
	3	$BA_2(\ell)$
$6k+3$	1	$BA_1(\ell)$
	2	$BA_1(\ell)$
	3	$BA_3(\ell)$
$6k+5$	1	$BA_1(\ell)$
	2	$BA_2(\ell)$
	3	$BA_2(\ell)$

ℓ	i with $1 + 3A \in x^{\leq i}$	Inequalities that holds
$6k+1$	1	$A_1(\ell)$
	2	$A_1(\ell)$
	3	$A_3(\ell)$
$6k+3$	1	$A_1(\ell)$
	2	$A_2(\ell)$
	3	$A_3(\ell)$
$6k+5$	1	$A_1(\ell)$
	2	$A_2(\ell)$
	3	$A_2(\ell)$

ℓ	i with $3 - 3B \in x^{\leq i}$	Inequalities that holds
$6k+1$	1	$B_1(\ell)$
	2	$B_1(\ell)$
	3	$B_3(\ell)$
$6k+3$	1	$B_1(\ell)$
	2	$B_2(\ell)$
	3	$B_2(\ell)$
$6k+5$	1	$B_1(\ell)$
	2	$B_1(\ell)$
	3	$B_1(\ell)$

ℓ	i with $2-3A+3B \in x^{\leq i}$	Inequalities that holds
$6k$	1	$BA_1(\ell)$
	2	$BA_2(\ell)$
	3	$BA_3(\ell)$
$6k+2$	1	$BA_1(\ell)$
	2	$BA_2(\ell)$
	3	$BA_2(\ell)$
$6k+4$	1	$BA_1(\ell)$
	2	$BA_1(\ell)$
	3	$BA_3(\ell)$

ℓ	i with $1+3A \in x^{\leq i}$	Inequalities that holds
$6k$	1	$A_1(\ell)$
	2	$A_2(\ell)$
	3	$A_2(\ell)$
$6k+2$	1	$A_1(\ell)$
	2	$A_1(\ell)$
	3	$A_1(\ell)$
$6k+4$	1	$A_1(\ell)$
	2	$A_1(\ell)$
	3	$A_3(\ell)$

ℓ	i with $3-3B \in x^{\leq i}$	Inequalities that holds
$6k$	1	$B_1(\ell)$
	2	$B_2(\ell)$
	3	$B_3(\ell)$
$6k+2$	1	$B_1(\ell)$
	2	$B_2(\ell)$
	3	$B_2(\ell)$
$6k+4$	1	$B_1(\ell)$
	2	$B_1(\ell)$
	3	$B_3(\ell)$

PROOF. Table 5.2 and Table 5.1 are reproduced here for convenience. They record some information which is needed in the proof.

	$\ell=6k+1$	$\ell=6k+3$	$\ell=6k+5$
$w(\ell)^{\leq 1}$	$\{2+3k,3k,\cdots\}$ $=\{\frac{\ell+3}{2},\frac{\ell-1}{2},\cdots\}$	$\{2+3k,3k,\cdots\}$ $=\{\frac{\ell+1}{2},\frac{\ell-3}{2},\cdots\}$	$\{3+3k,2+3k,\cdots\}$ $=\{\frac{\ell+1}{2},\frac{\ell-1}{2},\cdots\}$
$w(\ell)^{\leq 2}$	$\{2+3k,3k,\cdots\}$ $=\{\frac{\ell+3}{2},\frac{\ell-1}{2},\cdots\}$	$\{3+3k,2+3k,\cdots\}$ $=\{\frac{\ell+3}{2},\frac{\ell+1}{2},\cdots\}$	$\{5+3k,3+3k,\cdots\}$ $=\{\frac{\ell+5}{2},\frac{\ell+1}{2},\cdots\}$
$w(\ell)^{\leq 3}$	$\{3+3k,2+3k,\cdots\}$ $=\{\frac{\ell+5}{2},\frac{\ell+3}{2},\cdots\}$	$\{5+3k,3+3k,\cdots\}$ $=\{\frac{\ell+7}{2},\frac{\ell+3}{2},\cdots\}$	$\{5+3k,3+3k,\cdots\}$ $=\{\frac{\ell+5}{2},\frac{\ell+1}{2},\cdots\}$

	$\ell = 6k$	$\ell = 6k+2$	$\ell = 6k+4$
$w(\ell)^{\leq 1}$	$\{3k, -1+3k, \cdots\}$ $= \{\frac{\ell}{2}, \frac{\ell-2}{2}, \cdots\}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+2}{2}, \frac{\ell-2}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+2}{2}, \frac{\ell}{2}, \cdots\}$
$w(\ell)^{\leq 2}$	$\{2+3k, 3k, \cdots\}$ $= \{\frac{\ell+4}{2}, \frac{\ell}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+4}{2}, \frac{\ell+2}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+2}{2}, \frac{\ell}{2}, \cdots\}$
$w(\ell)^{\leq 3}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+6}{2}, \frac{\ell+4}{2}, \cdots\}$	$\{3+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+4}{2}, \frac{\ell+2}{2}, \cdots\}$	$\{5+3k, 3+3k, \cdots\}$ $= \{\frac{\ell+6}{2}, \frac{\ell+2}{2}, \cdots\}$

Table: $\mathbf{2 - 3A + 3B \in x^{\leq i}}$, ℓ odd. Case $\ell = 6k+1$. The largest element of $w(\ell)^{\leq 1}$ or $w(\ell)^{\leq 2}$ is $\frac{\ell+3}{2}$. So if $2-3A+3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$2 - 3A + 3B \leq \frac{\ell+3}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{6}$$

which is $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $2-3A+3B \in x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell+5}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} + \frac{1}{6},$$

which is $BA_2(\ell)$.

Case $\ell = 6k+3$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+1}{2}$. So if $2-3A+3B \in x^{\leq 1}$, then

$$2 - 3A + 3B \leq \frac{\ell+1}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{2}.$$

which implies $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ is $\frac{\ell+3}{2}$. So if $2-3A+3B \in x^{\leq 2}$, then

$$2 - 3A + 3B \leq \frac{\ell+3}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{6},$$

which is $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+7}{2}$. So if $2-3A+3B \in x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell+7}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} + \frac{1}{2}.$$

which is $BA_3(\ell)$.

Case $\ell = 6k+5$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+1}{2}$. So if $2-3A+3B \in x^{\leq 1}$, then

$$2 - 3A + 3B \leq \frac{\ell+1}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{2}.$$

which implies $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ and $w(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $2-3A+3B$

is in $x^{\leq 2}$ or $x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell + 5}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} + \frac{1}{6}.$$

which is $BA_2(\ell)$.

Table: $\mathbf{1 + 3A \in x^{\leq i}}$, ℓ odd. Case $\ell = 6k + 1$. The largest element of $w(\ell)^{\leq 1}$ or $w(\ell)^{\leq 2}$ is $\frac{\ell+3}{2}$. So if $1 + 3A$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$1 + 3A \leq \frac{\ell + 3}{2} \Leftrightarrow A \leq \frac{\ell}{6} + \frac{1}{6} = k + \frac{1}{3}.$$

Since $A \in \mathbb{Z}, A \leq k = \frac{\ell-1}{6}$, which is $A_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $1 + 3A \in x^{\leq 3}$, then

$$1 + 3A \leq \frac{\ell + 5}{2} \Leftrightarrow A \leq \frac{\ell}{6} + \frac{1}{2}.$$

which is $A_3(\ell)$.

Case $\ell = 6k + 3$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+1}{2}$. So if $1 + 3A \in x^{\leq 1}$, then

$$1 + 3A \leq \frac{\ell + 1}{2} \Leftrightarrow A \leq \frac{\ell}{6} - \frac{1}{6}.$$

which is $A_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ is $\frac{\ell+3}{2}$. So if $1 + 3A \in x^{\leq 2}$, then

$$1 + 3A \leq \frac{\ell + 3}{2} \Leftrightarrow A \leq \frac{\ell}{6} + \frac{1}{6}.$$

which is $A_2(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+7}{2}$. So if $1 + 3A \in x^{\leq 3}$, then

$$1 + 3A \leq \frac{\ell + 7}{2} \Leftrightarrow A \leq \frac{\ell}{6} + \frac{5}{6} = k + \frac{4}{3}.$$

As $A \in \mathbb{Z}, A \leq k + 1 = \frac{\ell}{6} + \frac{1}{2}$, which is $A_3(\ell)$.

Case $\ell = 6k + 5$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+1}{2}$. So if $1 + 3A \in x^{\leq 1}$, then

$$1 + 3A \leq \frac{\ell + 1}{2} \Leftrightarrow A \leq \frac{\ell}{6} - \frac{1}{6}.$$

which is $A_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ and $w(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $1 + 3A$ is in $x^{\leq 2}$ or

$x^{\leq 3}$, then

$$1 + 3A \leq \frac{\ell+5}{2} \Leftrightarrow A \leq \frac{\ell}{6} + \frac{1}{2} = k + \frac{4}{3}.$$

As $A \in \mathbb{Z}, A \leq k + 1 = \frac{\ell}{6} + \frac{1}{6}$, which is $A_2(\ell)$.

Table: $\mathbf{3 - 3B \in x^{\leq i}}$, ℓ odd. Case $\ell = 6k + 1$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $3 - 3B \in x^{\leq 3}$, then

$$3 - 3B \leq \frac{\ell+5}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{1}{6}$$

which is $B_3(\ell)$. The largest element of $w(\ell)^{\leq 1}$ or $w(\ell)^{\leq 2}$ is $\frac{\ell+3}{2}$. So if $3 - 3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$3 - 3B \leq \frac{\ell+3}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{1}{2} = k + \frac{1}{3}.$$

Since $B \in \mathbb{Z}, B \geq -k + 1 = -\frac{\ell}{6} + \frac{7}{6}$, which implies $B_1(\ell)$

Case $\ell = 6k + 3$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+7}{2}$. So if $3 - 3B \in x^{\leq 3}$, then

$$3 - 3B \leq \frac{\ell+7}{2} \Leftrightarrow B \geq -\frac{\ell}{6} - \frac{1}{6} = -k - \frac{2}{3}.$$

Since $B \in \mathbb{Z}, B \geq -k = -\frac{\ell}{6} + \frac{1}{2}$, which is $B_2(\ell)$. The largest element of $w(\ell)^{\leq 2}$ is $\frac{\ell+3}{2}$. So if $3 - 3B \in x^{\leq 2}$, then

$$3 - 3B \leq \frac{\ell+3}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{1}{2}$$

which is $B_2(\ell)$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+1}{2}$. So if $3 - 3B \in x^{\leq 1}$, then

$$3 - 3B \leq \frac{\ell+1}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{5}{6}$$

which is $B_1(\ell)$.

Case $\ell = 6k + 5$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+1}{2}$. So if $3 - 3B \in x^{\leq 1}$, then

$$3 - 3B \leq \frac{\ell+1}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{5}{6}.$$

which is $B_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ and $w(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $3 - 3B$ is in $x^{\leq 2}$ or

$x^{\leq 3}$, then

$$3 - 3B \leq \frac{\ell+5}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{1}{6} = -k - \frac{2}{3}.$$

So $B \geq -k = -\frac{\ell}{6} + \frac{5}{6}$, which is $B_1(\ell)$.

Table: $\mathbf{2 - 3A + 3B \in x^{\leq i}}$, ℓ even. Case 1. $\ell = 6k$. The largest element of $w(\ell)^{\leq 2}$ is $\frac{\ell+4}{2}$. So if $2 - 3A + 3B \in x^{\leq 2}$, then

$$2 - 3A + 3B \leq \frac{\ell+4}{2} \Leftrightarrow B - A \leq \frac{\ell}{6}.$$

which is $BA_2(\ell)$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell}{2}$. So if $2 - 3A + 3B \in x^{\leq 1}$, then

$$2 - 3A + 3B \leq \frac{\ell}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{2}{3}.$$

which implies $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+6}{2}$. So if $2 - 3A + 3B \in x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell+6}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} + \frac{1}{3}.$$

which is $BA_3(\ell)$.

Case $\ell = 6k + 2$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+2}{2}$. So if $2 - 3A + 3B \in x^{\leq 1}$, then

$$2 - 3A + 3B \leq \frac{\ell+2}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{3}.$$

which is $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ and $w(\ell)^{\leq 3}$ is $\frac{\ell+4}{2}$. So if $2 - 3A + 3B$ is in $x^{\leq 2}$ or $x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell+4}{2} \Leftrightarrow B - A \leq \frac{\ell}{6}.$$

which is $BA_2(\ell)$.

Case $\ell = 6k + 4$. The largest element of $w(\ell)^{\leq 1}$ and $w(\ell)^{\leq 2}$ is $\frac{\ell+2}{2}$. So if $2 - 3A + 3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$2 - 3A + 3B \leq \frac{\ell+2}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{3},$$

which is $BA_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+6}{2}$. So if $2-3A+3B \in x^{\leq 3}$, then

$$2-3A+3B \leq \frac{\ell+6}{2} \Leftrightarrow B-A \leq \frac{\ell}{6}+\frac{1}{3},$$

which is $BA_3(\ell)$.

Table: $\mathbf{1+3A \in x^{\leq i}}$, ℓ even. Case $\ell = 6k$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell}{2}$. So if $1+3A \in x^{\leq 1}$, then

$$1+3A \leq \frac{\ell}{2} \Leftrightarrow A \leq \frac{\ell}{6}-\frac{1}{3}.$$

which is $A_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ is $\frac{\ell+4}{2}$. So if $1+3A \in x^{\leq 2}$, then

$$1+3A \leq \frac{\ell+4}{2} \Leftrightarrow A \leq \frac{\ell}{6}+\frac{1}{3} = k+\frac{1}{3}.$$

Since $A \in \mathbb{Z}, A \leq k = \frac{\ell}{6}$, which is $A_2(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+6}{2}$. So if $1+3A \in x^{\leq 3}$, then

$$1+3A \leq \frac{\ell+6}{2} \Leftrightarrow A \leq \frac{\ell}{6}+\frac{2}{3} = k+\frac{2}{3}.$$

As $A \in \mathbb{Z}, A \leq k = \frac{\ell}{6}$, which is $A_2(\ell)$.

Case $\ell = 6k+2$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+2}{2}$. So if $1+3A \in x^{\leq 1}$, then

$$1+3A \leq \frac{\ell+2}{2} \Leftrightarrow A \leq \frac{\ell}{6} = k+\frac{1}{3}.$$

Since $A \in \mathbb{Z}, A \leq k = \frac{\ell}{6}-\frac{1}{3}$,which is $A_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ and $w(\ell)^{\leq 3}$ is $\frac{\ell+4}{2}$. So if $1+3A$ is in $x^{\leq 2}$ or $x^{\leq 3}$, then

$$1+3A \leq \frac{\ell+4}{2} \Leftrightarrow A \leq \frac{\ell}{6}+\frac{1}{3} = k+\frac{2}{3}.$$

As $A \in \mathbb{Z}, A \leq k = \frac{\ell}{6}-\frac{1}{3}$, which is $A_1(\ell)$.

Case $\ell = 6k+4$. The largest element of $w(\ell)^{\leq 1}$ and $w(\ell)^{\leq 2}$ is $\frac{\ell+2}{2}$. So if $1+3A$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$1+3A \leq \frac{\ell+2}{2} \Leftrightarrow A \leq \frac{\ell}{6} = k+\frac{2}{3}.$$

Since $A \in \mathbb{Z}, A \leq k = \frac{\ell-4}{6}$, which implies $A_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+6}{2}$. So if $1+3A \in x^{\leq 3}$, then

$$1+3A \leq \frac{\ell+6}{2} \Leftrightarrow A \leq \frac{\ell}{6}+\frac{2}{3}=k+\frac{4}{3}.$$

As $A \in \mathbb{Z}, A \leq k+1=\frac{\ell}{6}+\frac{1}{3}$, which is $A_3(\ell)$.

Table: $\mathbf{3-3B \in x^{\leq i}}$, ℓ even. Case $\ell=6k$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+6}{2}$. So if $3-3B \in x^{\leq 3}$, then

$$3-3B \leq \frac{\ell+6}{2} \Leftrightarrow B \geq -\frac{\ell}{6}$$

which is $B_3(\ell)$. The largest element of $w(\ell)^{\leq 2}$ is $\frac{\ell+4}{2}$. So if $3-3B \in x^{\leq 2}$, then

$$3-3B \leq \frac{\ell+4}{2} \Leftrightarrow B \geq -\frac{\ell}{6}+\frac{1}{3}$$

which is $B_2(\ell)$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell}{2}$. So if $3-3B \in x^{\leq 1}$, then

$$3-3B \leq \frac{\ell}{2} \Leftrightarrow B \geq -\frac{\ell}{6}+1$$

which implies $B_1(\ell)$.

Case $\ell=6k+2$. The largest element of $w(\ell)^{\leq 1}$ is $\frac{\ell+2}{2}$. So if $3-3B \in x^{\leq 1}$, then

$$3-3B \leq \frac{\ell+2}{2} \Leftrightarrow B \geq -\frac{\ell}{6}+\frac{2}{3}.$$

which is $B_1(\ell)$. The largest element of $w(\ell)^{\leq 2}$ and $w(\ell)^{\leq 3}$ is $\frac{\ell+4}{2}$. So if $3-3B$ is in $x^{\leq 2}$ or $x^{\leq 3}$, then

$$3-3B \leq \frac{\ell+4}{2} \Leftrightarrow B \geq -\frac{\ell}{6}+\frac{1}{3}.$$

which is $B_2(\ell)$.

Case $\ell = 6k+4$. The largest element of $w(\ell)^{\leq 1}$ and $w(\ell)^{\leq 2}$ is $\frac{\ell+2}{2}$. So if $3-3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$3-3B \leq \frac{\ell}{2} + \frac{2}{2} \Leftrightarrow B \geq -\frac{\ell}{6} + \frac{2}{3}.$$

which is $B_1(\ell)$. The largest element of $w(\ell)^{\leq 3}$ is $\frac{\ell+6}{2}$. So if $3-3B \in x^{\leq 3}$, then

$$3-3B \leq \frac{\ell+6}{2} \Leftrightarrow B \geq -\frac{\ell}{6}.$$

which is $B_3(\ell)$. □

COROLLARY 5.40. Let $x \leq w(\ell)$. Write $x = zy$, where $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$ and $y \in W_{finite}$. Then for $i \in \{1,2,3\}$ we have:

(1) $2-3A+3B \in x^{\leq i} \Rightarrow BA_i(\ell)$.

(2) $1+3A \in x^{\leq i} \Rightarrow A_i(\ell)$.

(3) $3-3B \in x^{\leq i} \Rightarrow B_i(\ell)$.

PROOF. This follows from Lemma 5.39, using $BA_1 \Rightarrow BA_2 \Rightarrow BA_3$, $A_1 \Rightarrow A_2 \Rightarrow A_3$ and $B_1 \Rightarrow B_2 \Rightarrow B_3$. □

For spiral elements, we give a close connection between the Bruhat order and the geometry of affine Weyl group W acting on the plane $\mathbb{R}^2$.

THEOREM 5.41. Let $\ell \geq 6$. Let $w(\ell)$ be a spiral element of length ℓ. Then

$$x \leq w(\ell) \Leftrightarrow xq \in R(\ell).$$

PROOF. ($\Leftarrow$) We want: If $xq \in R(\ell)$, then $x \leq w$. We show this by induction on ℓ.

(a) If $\ell = 2k+1$, then $xq \in R(2k)$ or $xq \in E_1(\ell) \cup E_2(\ell)$ by Proposition 5.28. If $xq \in R(2k)$, then $x \leq w(2k) < w(\ell)$ by induction hypothesis. If xq is on $E_1(\ell) \cup E_2(\ell)$, then $x \leq w(\ell)$ by Proposition 5.27.

(b) If $\ell = 2k$, then $xq \in R(2k-1)$ or $xq \in E_3(\ell)$ by Proposition 5.28. If $xq \in R(2k-1)$, then $x \leq w(2k-1) < w(\ell)$ by induction hypothesis. If xq is on $E_3(\ell)$, then $x \leq w(\ell)$ by Proposition 5.27.

($\Rightarrow$) Table 2.1 is reproduced here for convenience. It records some information which is needed in the proof. In Table 2.1, let $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$.

x	x in window notation
z	$[1+3A, 2-3A+3B, 3-3B]$
zs_1	$[2-3A+3B, 1+3A, 3-3B]$
zs_2	$[1+3A, 3-3B, 2-3A+3B]$
zs_1s_2	$[2-3A+3B, 3-3B, 1+3A]$
zs_2s_1	$[3-3B, 1+3A, 2-3A+3B]$
$zs_1s_2s_1$	$[3-3B, 2-3A+3B, 1+3A]$

Suppose $x \leq w(\ell)$. We must show that $x \in R(\ell)$. To verify this, we will show that $x \in \triangle(\ell)$. If ℓ is even, then $\triangle(\ell) = R(\ell)$. If ℓ is odd, this suffices since Lemma 5.13 below implies that $A_1(\ell), A_2(\ell)$ are not $\leq w(\ell)$, so $x \in \triangle(\ell) \setminus \{A_1(\ell), A_2(\ell)\} \subset R(\ell)$ as desired. The proof will be divided into 6 cases, corresponding to $x = t(A\alpha_1^\vee + B\alpha_2^\vee)y$, as y runs over the 6 elements of the finite Weyl group.

Case 1. $x = z = t(A\alpha_1^\vee + B\alpha_2^\vee)$. By Lemma 5.31, we must check that A, B satisfy $T(2,1,3)$. Since $2-3A+3B \in x^{\leq 2}$, Corollary 5.40 implies that BA_2 holds. Since $1+3A \in x^{\leq 1}$, Corollary 5.40 implies that A_1 holds. Since $3-3B \in x^{\leq 3}$, Corollary 5.40 implies that B_3 holds. Hence A, B satisfy $T(2,1,3)$.

Case 2. $x = zs_1 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_1$. By Lemma 5.31, we must check that A, B satisfy $T(1,2,3)$. Since $2-3A+3B \in x^{\leq 1}$, Corollary 5.40 implies that BA_1 holds. Since $1+3A \in x^{\leq 2}$, Corollary 5.40 implies that A_2 holds. Since $3-3B \in x^{\leq 3}$, Corollary 5.40 implies that B_3 holds. Hence A, B satisfy $T(1,2,3)$.

Case 3. $x = zs_2 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_2$. By Lemma 5.31, we must check that A, B satisfy $T(3,1,2)$. Since $2-3A+3B \in x^{\leq 3}$, Corollary 5.40 implies that BA_3 holds. Since $1+3A \in x^{\leq 1}$, Corollary 5.40 implies that A_1 holds. Since $3-3B \in x^{\leq 2}$, Corollary 5.40 implies that B_2 holds. Hence A, B satisfy $T(3,1,2)$.

Case 4. $x = zs_1s_2 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_1s_2$. By Lemma 5.31, we must check that A, B satisfy $T(1, 3, 2)$. Since $2 - 3A + 3B \in x^{\le 1}$, Corollary 5.40 implies that BA_1 holds. Since $1 + 3A \in x^{\le 3}$, Corollary 5.40 implies that A_3 holds. Since $3 - 3B \in x^{\le 2}$, Corollary 5.40 implies that B_2 holds. Hence A, B satisfy $T(1, 3, 2)$.

Case 5. $x = zs_2s_1 = t(A\alpha_1^\vee + B\alpha_2^\vee)s_2s_1$. By Lemma 5.31, we must check that A, B satisfy $T(3, 2, 1)$. Since $2 - 3A + 3B \in x^{\le 3}$, Corollary 5.40 implies that BA_3 holds. Since $1 + 3A \in x^{\le 2}$, Corollary 5.40 implies that A_2 holds. Since $3 - 3B \in x^{\le 1}$, Corollary 5.40 implies that B_1 holds. Hence A, B satisfy $T(3, 2, 1)$.

Case 6. $x = zs_1s_2s_1 = t(A\alpha_1^\vee + B\alpha_2^\vee)_1s_2s_1$. By Lemma 5.31, we must check that A, B satisfy $T(2, 3, 1)$. Since $2 - 3A + 3B \in x^{\le 2}$, Corollary 5.40 implies that BA_2 holds. Since $1 + 3A \in x^{\le 3}$, Corollary 5.40 implies that A_3 holds. Since $3 - 3B \in x^{\le 1}$, Corollary 5.40 implies that B_1 holds. Hence A, B satisfy $T(2, 3, 1)$. □

LEMMA 5.42. Let ℓ be odd. Let $x = zy$, where $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$ and $y \in W_{finite}$. Suppose $x \le A_1(\ell)$. The integer $2 - 3A + 3B \in x^{\le i}$ for some $i \in \{1, 2, 3\}$ and this implies that one of the inequalities $BA_k(\ell)$ holds (where k depends on i and ℓ). Similarly, $1 + 3A \in x^{\le i}$, and this implies that one of the inequalities $A_k(\ell)$, and $3 - 3B \in x^{\le i}$, and this implies that one of the inequalities $B_k(\ell)$. The following tables list the particular inequality which holds in each case.

ℓ	i with $2 - 3A + 3B \in x^{\le i}$	Inequalities that holds
$6k + 1$	1	$BA_1(\ell)$
	2	$BA_1(\ell)$
	3	$BA_1(\ell)$
$6k + 3$	1	$BA_1(\ell)$
	2	$BA_1(\ell)$
	3	$BA_3(\ell)$
$6k + 5$	1	$BA_1(\ell)$
	2	$BA_2(\ell)$
	3	$BA_2(\ell)$

ℓ	i with $1+3A \in x^{\leq i}$	Inequalities that holds
$6k+1$	1	$A_1(\ell)$
	2	$A_1(\ell)$
	3	$A_1(\ell)$
$6k+3$	1	$A_1(\ell)$
	2	$A_1(\ell)$
	3	$A_3(\ell)$
$6k+5$	1	$A_1(\ell)$
	2	$A_2(\ell)$
	3	$A_2(\ell)$

ℓ	i with $3-3B \in x^{\leq i}$	Inequalities that holds
$6k+1$	1	$B_1(\ell)$
	2	$B_1(\ell)$
	3	$B_1(\ell)$
$6k+3$	1	$B_1(\ell)$
	2	$B_1(\ell)$
	3	$B_2(\ell)$
$6k+5$	1	$B_1(\ell)$
	2	$B_1(\ell)$
	3	$B_1(\ell)$

PROOF. Let ℓ be odd. By Lemma 5.12, we have

Case $\ell = 6k+1$. $A_1(\ell) = [2+3k, 1+3k, 3-6k]$.

Case $\ell = 6k+3$. $A_1(\ell) = [1+3k, -6k, 5+3k]$.

Case $\ell = 6k+5$. $A_1(\ell) = [-3-6k, 5+3k, 4+3k]$.

	$\ell = 6k+1$	$\ell = 6k+3$	$\ell = 6k+5$
$A_1(\ell)^{\leq 1}$	$\{2+3k, -1+3k, \cdots\}$ $= \{\frac{\ell+3}{2}, \frac{\ell-3}{2}, \cdots\}$	$\{2+3k, 1+3k, \cdots\}$ $= \{\frac{\ell+1}{2}, \frac{\ell-1}{2}, \cdots\}$	$\{2+3k, 1+3k, \cdots\}$ $= \{\frac{\ell-1}{2}, \frac{\ell-3}{2}, \cdots\}$
$A_1(\ell)^{\leq 2}$	$\{2+3k, 1+3k, \cdots\}$ $= \{\frac{\ell+3}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{2+3k, 1+3k, \cdots\}$ $= \{\frac{\ell+1}{2}, \frac{\ell-1}{2}, \cdots\}$	$\{5+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+5}{2}, \frac{\ell-1}{2}, \cdots\}$
$A_1(\ell)^{\leq 3}$	$\{2+3k, 1+3k, \cdots\}$ $= \{\frac{\ell+3}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{5+3k, 2+3k, \cdots\}$ $= \{\frac{\ell+7}{2}, \frac{\ell+1}{2}, \cdots\}$	$\{5+3k, 4+3k, \cdots\}$ $= \{\frac{\ell+5}{2}, \frac{\ell+3}{2}, \cdots\}$

Table: $\mathbf{2-3A+3B \in x^{\leq i}}$. Case $\ell = 6k+1$. The largest element of $A_1(\ell)^{\leq 1}$ or $A_1(\ell)^{\leq 2}$or $A_1(\ell)^{\leq 3}$ is $\frac{\ell+3}{2}$. So if $2-3A+3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$ or $x^{\leq 3}$, then

$$2-3A+3B \leq \frac{\ell+3}{2} \Leftrightarrow B-A \leq \frac{\ell}{6} - \frac{1}{6}$$

which is $BA_1(\ell)$.

Case $\ell = 6k+3$. The largest element of $A_1(\ell)^{\leq 1}$ or $A_1(\ell)^{\leq 2}$ is $\frac{\ell+1}{2}$. So if $2-3A+3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$2 - 3A + 3B \leq \frac{\ell+1}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{1}{2}$$

which implies $BA_1(\ell)$. The largest element of $A_1(\ell)^{\leq 3}$ is $\frac{\ell+7}{2}$. So if $2-3A+3B \in x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell}{2} + \frac{7}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} + \frac{1}{2}$$

which is $BA_3(\ell)$.

Case $\ell = 6k+5$. The largest element of $A_1(\ell)^{\leq 1}$ is $\frac{\ell-1}{2}$. So if $2-3A+3B \in x^{\leq 1}$, then

$$2 - 3A + 3B \leq \frac{\ell-1}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} - \frac{5}{6}$$

which implies $BA_1(\ell)$. The largest element of $A_1(\ell)^{\leq 2}$ and $A_1(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $2-3A+3B$ is in $x^{\leq 2}$ or $x^{\leq 3}$, then

$$2 - 3A + 3B \leq \frac{\ell+5}{2} \Leftrightarrow B - A \leq \frac{\ell}{6} + \frac{1}{6}$$

which is $BA_2(\ell)$.

Table: $\mathbf{1+3A \in x^{\leq i}}$. Case $\ell = 6k+1$. The largest element of $A_1(\ell)^{\leq 1}$ or $A_1(\ell)^{\leq 2}$or $A_1(\ell)^{\leq 3}$ is $\frac{\ell+3}{2}$. So if $1+3A$ is in $x^{\leq 1}$ or $x^{\leq 2}$ or $x^{\leq 3}$, then

$$1 + 3A \leq \frac{\ell+3}{2} \Leftrightarrow A \leq \frac{\ell+1}{6} = k + \frac{1}{3}.$$

Since $A \in \mathbb{Z}$, $A \leq k = \frac{\ell-1}{6}$, which is $A_1(\ell)$.

Case $\ell = 6k+3$. The largest element of $A_1(\ell)^{\leq 1}$ or $A_1(\ell)^{\leq 2}$ is $\frac{\ell+1}{2}$. So if $1+3A$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$1+3A \leq \frac{\ell}{2}+\frac{1}{2} \Leftrightarrow A \leq \frac{\ell}{6}-\frac{1}{6}$$

which implies $A_1(\ell)$. The largest element of $A_1(\ell)^{\leq 3}$ is $\frac{\ell+7}{2}$. So if $1+3A \in x^{\leq 3}$, then

$$1+3A \leq \frac{\ell}{2}+\frac{7}{2} \Leftrightarrow A \leq \frac{\ell}{6}+\frac{5}{6} = k+\frac{4}{3}.$$

As $A \in \mathbb{Z}, A \leq k+1 = \frac{\ell}{6}+\frac{1}{2}$, which is $A_3(\ell)$.

Case $\ell = 6k+5$. The largest element of $A_1(\ell)^{\leq 1}$ is $\frac{\ell-1}{2}$. So if $1+3A \in x^{\leq 1}$, then

$$1+3A \leq \frac{\ell-1}{2} \Leftrightarrow A \leq \frac{\ell}{6}-\frac{1}{2}$$

which implies $A_1(\ell)$. The largest element of $A_1(\ell)^{\leq 2}$ and $A_1(\ell)^{\leq 3}$ is $\frac{\ell+5}{2}$. So if $1+3A$ is in $x^{\leq 2}$ or $x^{\leq 3}$, then

$$1+3A \leq \frac{\ell+5}{2} \Leftrightarrow A \leq \frac{\ell}{6}+\frac{1}{2} = k+\frac{4}{3}.$$

As $A \in \mathbb{Z}, A \leq k+1 = \frac{\ell}{6}+\frac{1}{6}$, which is $A_2(\ell)$.

Table: $\mathbf{3-3B \in x^{\leq i}}$. Case $\ell = 6k+1$. The largest element of $A_1(\ell)^{\leq 1}$ or $A_1(\ell)^{\leq 2}$or $A_1(\ell)^{\leq 3}$ is $\frac{\ell+3}{2}$. So if $3-3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$ or $x^{\leq 3}$, then

$$3-3B \leq \frac{\ell+3}{2} \Leftrightarrow B \geq -\frac{\ell}{6}+\frac{1}{2} = -k+\frac{1}{3}.$$

Since $B \in \mathbb{Z}, B \geq -k+1 = -\frac{\ell}{6}+\frac{7}{6}$, which implies $B_1(\ell)$.

Case $\ell = 6k+3$. The largest element of $A_1(\ell)^{\leq 1}$ or $A_1(\ell)^{\leq 2}$ is $\frac{\ell+1}{2}$. So if $3-3B$ is in $x^{\leq 1}$ or $x^{\leq 2}$, then

$$3-3B \leq \frac{\ell}{2}+\frac{1}{2} \Leftrightarrow B \geq -\frac{\ell}{6}+\frac{5}{6}$$

which is $B_1(\ell)$. The largest element of $A_1(\ell)^{\leq 3}$ is $\frac{\ell+7}{2}$. So if $3-3B \in x^{\leq 3}$, then

$$3-3B \leq \frac{\ell}{2}+\frac{7}{2} \Leftrightarrow B \geq -\frac{\ell}{6}-\frac{1}{6} = -k-\frac{2}{3}.$$

Since $B \in \mathbb{Z}, B \geq -k = -\frac{\ell}{6}+\frac{1}{2}$, which is $B_2(\ell)$.

Case $\ell = 6k+5$. The largest element of $A_1(\ell)^{\le 1}$ is $\frac{\ell-1}{2}$. So if $3-3B \in x^{\le 1}$, then

$$3 - 3B \le \frac{\ell-1}{2} \Leftrightarrow B \ge -\frac{\ell}{6} + \frac{7}{6}$$

which implies $B_1(\ell)$. The largest element of $A_1(\ell)^{\le 2}$ and $A_1(\ell)^{\le 3}$ is $\frac{\ell+5}{2}$. So if $3-3B$ is in $x^{\le 2}$ or $x^{\le 3}$, then

$$3 - 3B \le \frac{\ell+5}{2} \Leftrightarrow B \ge -\frac{\ell}{6} + \frac{1}{6} = -k - \frac{5}{6} + \frac{1}{6} = -k - \frac{2}{3}.$$

So $B \ge -k = -\frac{\ell-5}{6} = -\frac{\ell}{6} + \frac{5}{6}$, which is $B_1(\ell)$. □

COROLLARY 5.43. Let $x \le A_1(\ell)$. Write $x = zy$, where $z = t(A\alpha_1^\vee + B\alpha_2^\vee)$ and $y \in W_{finite}$. Then for $i \in \{1,2,3\}$ we have:

(1) $2 - 3A + 3B \in x^{\le i} \Rightarrow BA_i(\ell)$.

(2) $1 + 3A \in x^{\le i} \Rightarrow A_i(\ell)$.

(3) $3 - 3B \in x^{\le i} \Rightarrow B_i(\ell)$.

PROOF. This follows from Lemma 5.42, using $BA_1 \Rightarrow BA_2 \Rightarrow BA_3$, $A_1 \Rightarrow A_2 \Rightarrow A_3$ and $B_1 \Rightarrow B_2 \Rightarrow B_3$. □

Remark: The Corollary is not as strong as can be shown from Lemma 5.42, but it is all we need.

Theorem 5.41 shows that if ℓ is even, then since $\triangle(\ell) = R(\ell)$, we have $x \in \triangle(\ell) \Leftrightarrow x \le w(\ell)$. The next result gives a criterion for $x \in \triangle(\ell)$ in case ℓ is odd. This result will be used when we determine the set of rationally smooth points.

THEOREM 5.44. If ℓ is odd, then

$$x \le w(\ell) \text{ or } x \le A_1(\ell) \Leftrightarrow xq \in \triangle(\ell).$$

PROOF. ($\Leftarrow$) If ℓ is odd, $\triangle(\ell) = R(\ell) \cup \{A_1(\ell), A_2(\ell)\}$ and $A_1(\ell) > A_2(\ell)$ by Lemma 5.10. $xq \in \triangle(\ell) \Rightarrow xq \in R(\ell)$ or $xq \in \{A_1(\ell), A_2(\ell)\} \Rightarrow x \leq w(\ell)$ or $x \leq A_1(\ell)$.

($\Rightarrow$) By Theorem 5.41, if $x \leq w(\ell)$ then $xq \in R(\ell) \subseteq \triangle(\ell)$. So it is enough to show that if $x \leq A_1(\ell)$ then $xq \in \triangle(\ell)$. The proof is identical to the proof that $x \leq w(\ell) \Rightarrow xq \in \triangle(\ell)$, using Corollary 5.43 in place of Corollary 5.40. $\square$

6

THE TRANSLATION THEOREM AND SOME CONSEQUENCES

If ℓ is even, then $R(\ell) = \triangle(\ell)$, which implies that q_x^w satisfies a certain translation theorem (Theorem 6.6). If ℓ is odd, then $R(\ell) \neq \triangle(\ell)$ and the translation theorem fails for q_x^w. To fix this, we define integers p_x^w which are equal to q_x^w if ℓ is even, but may differ if ℓ is odd. We prove that p_x^w satisfies the translation theorem (Theorem 6.6). Using this we show that q_x^w is always 0 or 1 (Theorem 6.16). In the next chapter these results will be used to prove the lookup conjecture for spiral elements.

6.1 The sets Λ_x^w and the integers p_x^w

If ℓ is odd, it is not always true that $q_x^w = q_{t(\gamma^\vee)x}^w$ (the translation theorem for q_x^w fails). To deal with this, we define the sets Λ_x^w and the integers p_x^w.

Definition 6.1. Given $x \leq w$ in W, for any non-negative integer ℓ define

$$\Lambda_x^w := \{r \in R \mid rxq \in \triangle(\ell)\}$$

$$p_x^w := |\Lambda_x^w| - \ell(w).$$

Recall that $\Psi_x^w := \{r \in R \mid rx \leq w\}$. If ℓ is even then $x \leq w(\ell) \Leftrightarrow xq \in \triangle(\ell)$ by Theorem 5.41, so $\Lambda_x^w = \Psi_x^w$ and hence $p_x^w = q_x^w$. If ℓ is odd, then $\Lambda_x^w \neq \Psi_x^w$. We study Λ_x^w because it has nice properties. In the case where ℓ is odd, we will use the p_x^w to study the q_x^w.

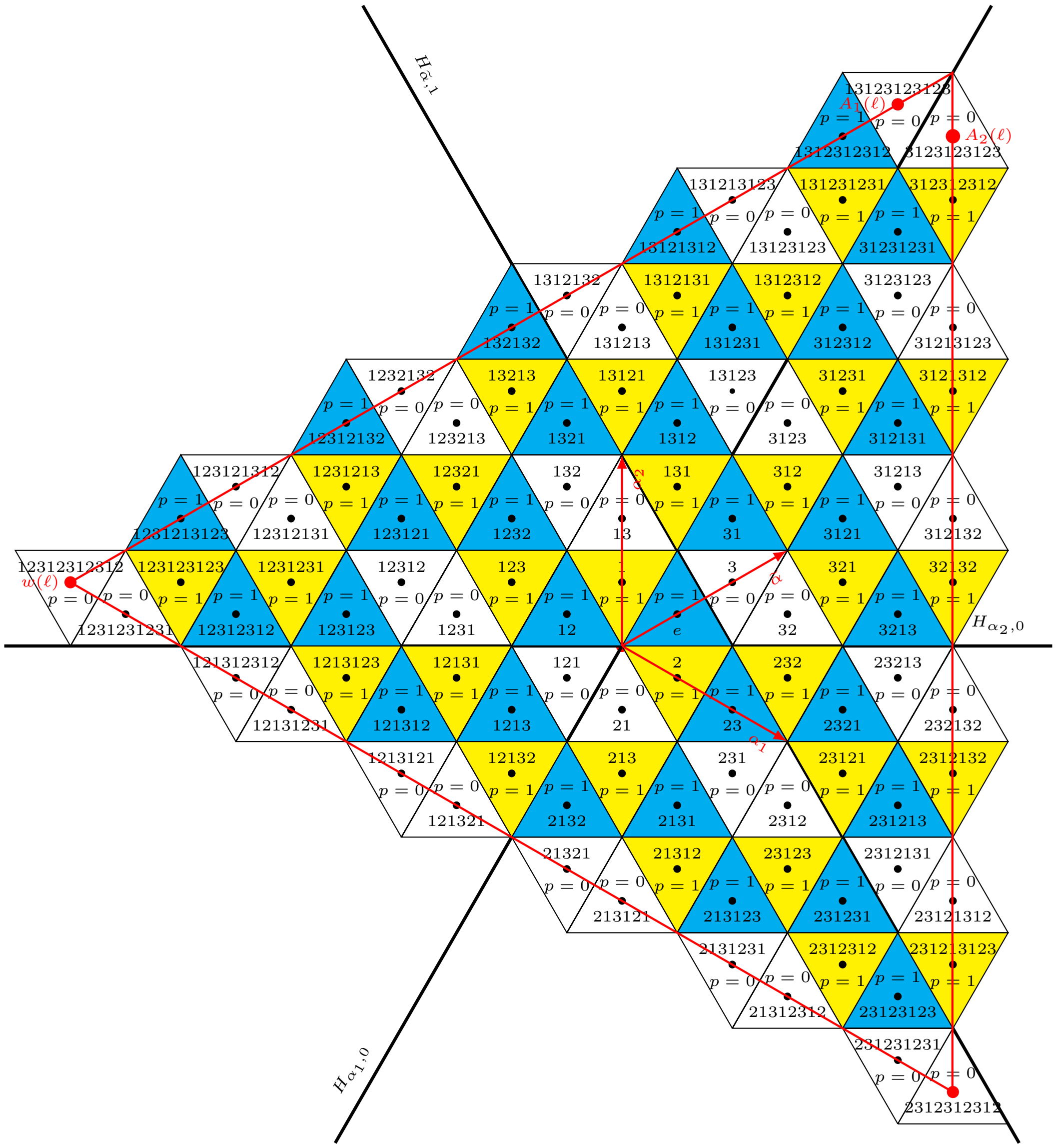

Figure 6.1: The integers $p_x^{w(\ell)}$ for ℓ odd ($\ell = 11$).
The colored alcoves correspond to x for which $p_x^{w(\ell)} = 1$, and the white alcoves to x for which $p_x^{w(\ell)} = 0$.

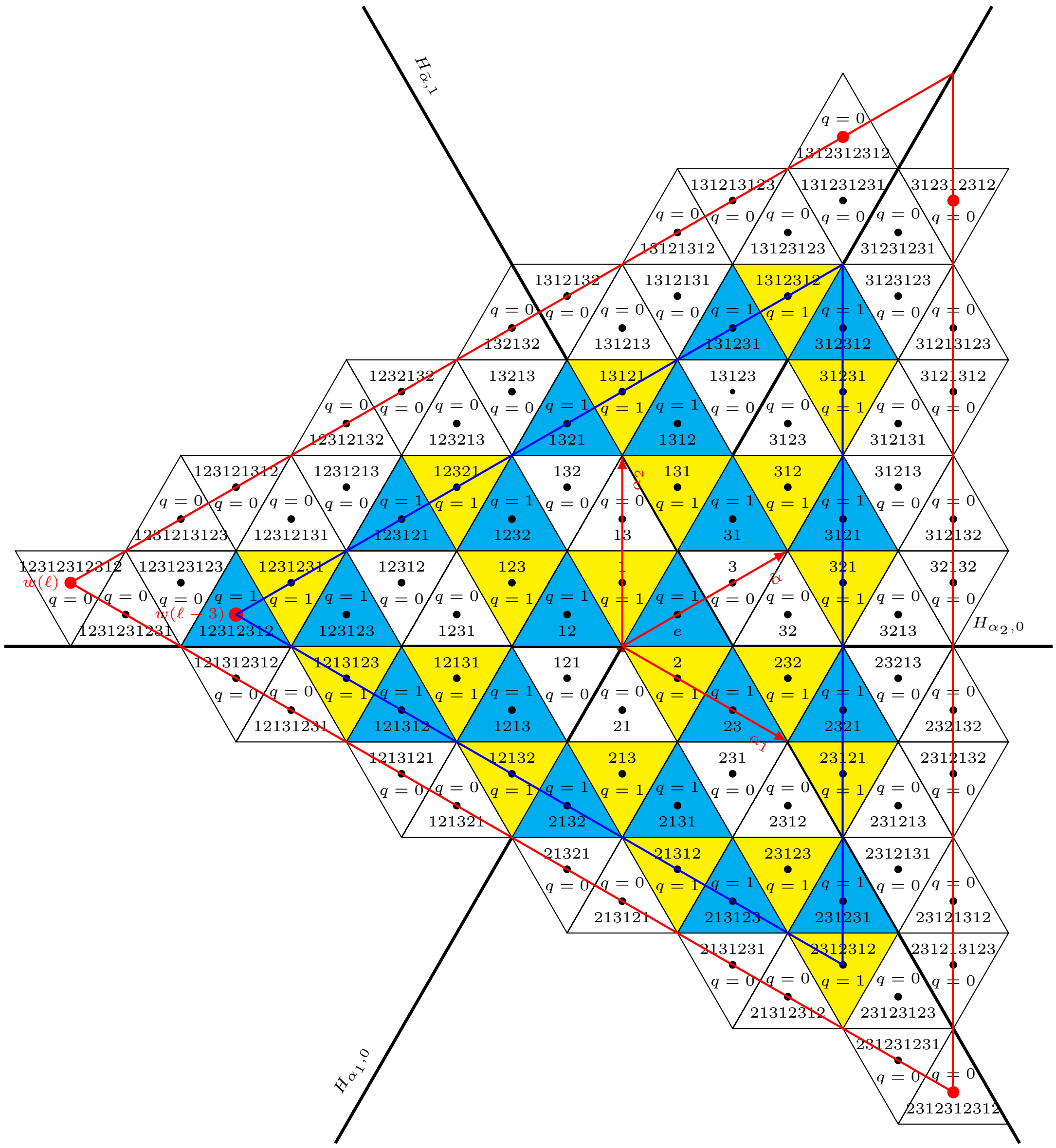

Figure 6.2: The integers $q_x^{w(\ell)}$ for ℓ odd ($\ell = 11$).
The colored alcoves correspond to x for which $q_x^{w(\ell)} = 1$, and the white alcoves to x for which $q_x^{w(\ell)} = 0$.

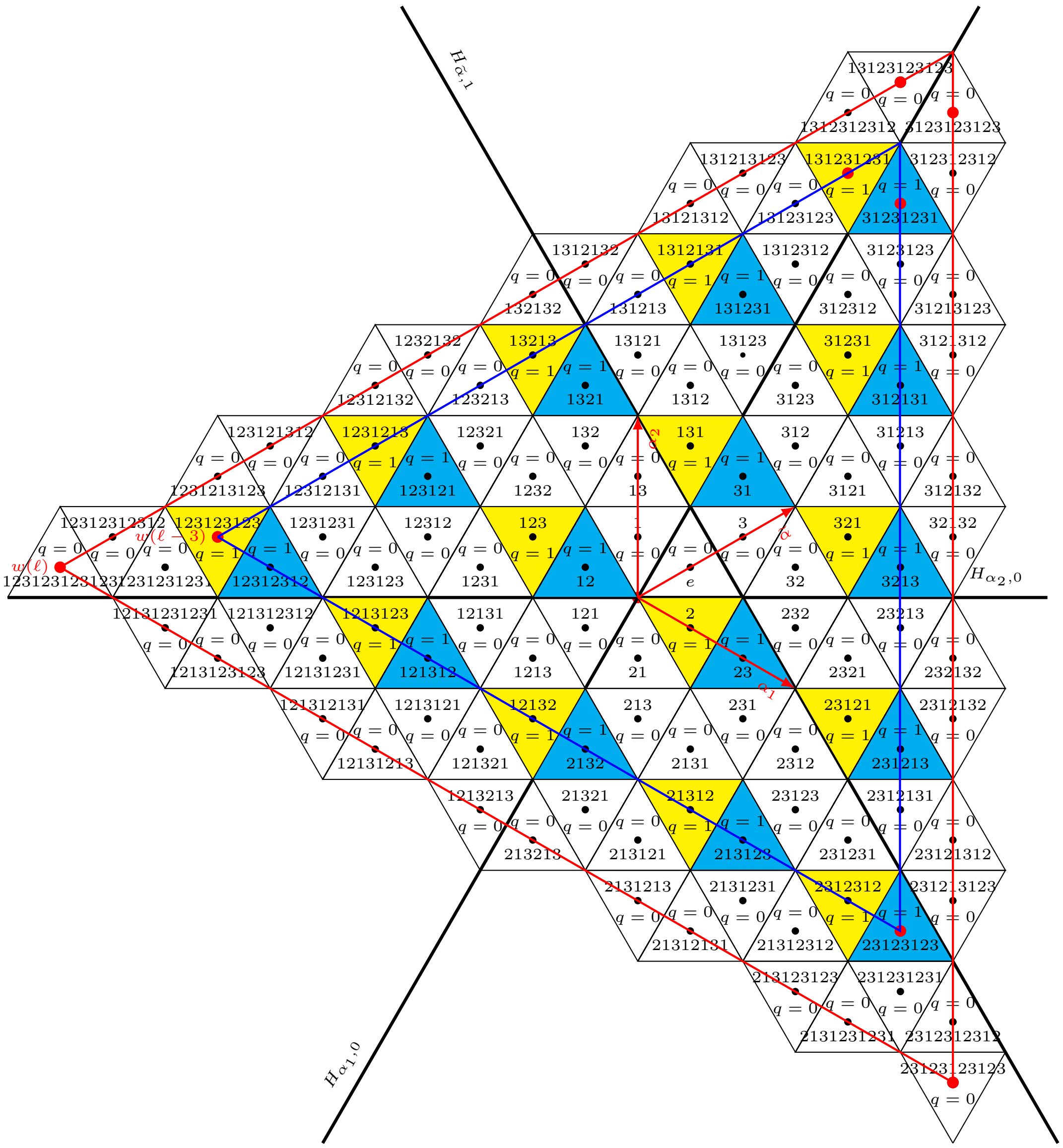

Figure 6.3: The integers $p_x^{w(\ell)} = q_x^{w(\ell)}$ for ℓ even ($\ell = 12$).
The colored alcoves correspond to x for which $q_x^{w(\ell)} = 1$, and the white alcoves to x for which $q_x^{w(\ell)} = 0$.

Remark: If $x, s_1x \in \triangle(\ell)$, then $p^w_{s_1x} = p^w_x$. The reason is the following. Let $B^w_x = \{r \mid xrq \in \triangle(\ell)\}$. Since $s_1(\triangle(\ell)) = \triangle(\ell)$, $xr \in \triangle(\ell) \Leftrightarrow s_1xr \in \triangle(\ell)$, we have $B^w_x = B^w_{s_1x}$. Then $p^w_x = p^w_{s_1x}$ as $p^w_x = |B^w_x| - \ell(w)$. (Note that $|B^w_x| = |\Lambda^w_x|$ since $B^w_x = x^{-1}\Lambda^w_x x$.)

6.2 The Translation Theorem for p^w_x

In this section, we prove the translation theorem for p^w_x and give some consequences. If ℓ is even, $p^w_x = q^w_x$. If ℓ is odd, then either $p^w_x = q^w_x$ or $p^w_x = q^w_x + 1$ (which occurs depends on x).

Definition 6.2. Let β be one of $\alpha_1, \alpha_1, \widetilde{\alpha}$, and let $a \leq b$ be integers. Define

$$[a, b]_\beta := \{s_{\beta,k} \mid a \leq k \leq b\}.$$

By convention, if $a \not\leq b$, then $[a, b]_\beta$ is the empty set.

THEOREM 6.3. Let $w = w(\ell)$ be a length ℓ spiral element and let $xq \in \triangle(\ell)$. Then $\Lambda^w_x = [a_1, b_1]_{\alpha_1} \cup [a_2, b_2]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b}]_{\widetilde{\alpha}}$ where

(1) $a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_{\alpha_1,0}xq)\rceil,\ b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\alpha_1,0}xq)\rfloor$

(2) $a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\alpha_2,0}xq)\rceil,\ b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_{\alpha_2,0}xq)\rfloor$

(3) $\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0}xq)\rceil,\ \widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0}xq)\rfloor.$

PROOF. We have $s_{\alpha_1,k}xq \in \triangle(\ell) \Leftrightarrow$

$$\lambda_{vu}(s_{\alpha_1,k}xq) = \lambda_{vu}(s_{\alpha_1,0}xq + k\alpha_1^\vee) = \lambda_{vu}(s_{\alpha_1,0}xq) - k \leq \frac{\ell}{6} + \varepsilon$$

$$\lambda_u(s_{\alpha_1,k}xq) = \lambda_u(s_{\alpha_1,0}xq + k\alpha_1^\vee) = \lambda_u(s_{\alpha_1,0}xq) + k \leq \frac{\ell}{6} + \varepsilon$$

$$\lambda_v(s_{\alpha_1,k}xq) = \lambda_v(xq) \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

The third inequality is satisfied since $xq \in \triangle(\ell)$ by hypothesis. So $s_{\alpha_1,k} \in \Lambda_x^w \Leftrightarrow s_{\alpha_1,k}xq \in \triangle(\ell) \Leftrightarrow$

$$-\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_{\alpha_1,0}xq) \leq k \leq \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\alpha_1,0}xq).$$

Therefore, $\{s_{\alpha_1,k} \mid s_{\alpha_1,k}xq \in \triangle(\ell)\} = [a_1, b_1]_{\alpha_1}$, where

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_{\alpha_1,0}xq) \rceil,\ b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\alpha_1,0}xq) \rfloor.$$

We have $s_{\alpha_2,k}xq \in \triangle(\ell) \Leftrightarrow$

$$\lambda_{vu}(s_{\alpha_2,k}xq) = \lambda_{vu}(s_{\alpha_2,0}xq + k\alpha_2^\vee) = \lambda_{vu}(s_{\alpha_2,0}xq) + k \leq \frac{\ell}{6} + \varepsilon$$

$$\lambda_u(s_{\alpha_1,k}xq) = \lambda_u(xq) \leq \frac{\ell}{6} + \varepsilon$$

$$\lambda_v(s_{\alpha_2,k}xq) = \lambda_v(s_{\alpha_2,0}xq + k\alpha_2^\vee) = \lambda_v(s_{\alpha_2,0}xq) + k \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

The second inequality is satisfied since $xq \in \triangle(\ell)$ by hypothesis. So $s_{\alpha_2,k} \in \Lambda_x^w \Leftrightarrow s_{\alpha_2,k}xq \in \triangle(\ell) \Leftrightarrow$

$$-\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\alpha_2,0}xq) \leq k \leq \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_{\alpha_2,0}xq).$$

Therefore, $\{s_{\alpha_2,k} \mid s_{\alpha_2,k}xq \in \triangle(\ell)\} = [a_2, b_2]_{\alpha_2}$, where

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\alpha_2,0}xq) \rceil,\ b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_{\alpha_2,0}xq) \rfloor.$$

We have $s_{\widetilde{\alpha},k}xq \in \triangle(\ell) \Leftrightarrow$

$$\lambda_{vu}(s_{\widetilde{\alpha},k}xq) = \lambda_{vu}(xq) \leq \frac{\ell}{6} + \varepsilon$$

$$\lambda_u(s_{\widetilde{\alpha},k}xq) = \lambda_u(s_{\widetilde{\alpha},0}xq + k\widetilde{\alpha}^\vee) = \lambda_u(s_{\widetilde{\alpha},0}xq) + k \leq \frac{\ell}{6} + \varepsilon$$

$$\lambda_v(s_{\widetilde{\alpha},k}xq) = \lambda_v(s_{\widetilde{\alpha},0}xq + k\widetilde{\alpha}^\vee) = \lambda_v(s_{\widetilde{\alpha},0}xq) + k \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon.$$

The first inequality is satisfied since $xq \in \triangle(\ell)$ by hypothesis. So $s_{\widetilde{\alpha},k} \in \Lambda_x^w \Leftrightarrow s_{\widetilde{\alpha},k}xq \in$

$\triangle(\ell) \Leftrightarrow$

$$-\frac{\ell}{6}+\frac{1}{3}+\varepsilon-\lambda_v(s_{\widetilde{\alpha},0}xq) \leq k \leq \frac{\ell}{6}+\varepsilon-\lambda_u(s_{\widetilde{\alpha},0}xq).$$

Therefore, $\{s_{\widetilde{\alpha},k} \mid s_{\widetilde{\alpha},k}xq \in \triangle(\ell)\} = [\widetilde{a},\widetilde{b}]_{\widetilde{\alpha}}$, where

$$\widetilde{a} = \lceil -\frac{\ell}{6}+\frac{1}{3}+\varepsilon-\lambda_v(s_{\widetilde{\alpha},0}xq)\rceil,\ \widetilde{b} = \lfloor \frac{\ell}{6}+\varepsilon-\lambda_u(s_{\widetilde{\alpha},0}xq)\rfloor.$$

□

COROLLARY 6.4. Let $w = w(\ell), xq \in \triangle(\ell)$. Then

$$|\Lambda_x^w| = b_1+b_2+\widetilde{b}-(a_1+a_2+\widetilde{a})+3$$

where $a_1, a_2, \widetilde{a}, b_1, b_2, \widetilde{b}$ are as in Theorem 6.3.

PROOF. It suffices to show that

$$|[a_1,b_1]_{\alpha_1}| = b_1-a_1+1,\ |[a_2,b_2]_{\alpha_2}| = b_2-a_2+1,\ |[\widetilde{a},\widetilde{b}]_{\widetilde{\alpha}}| = \widetilde{b}-\widetilde{a}+1.$$

Observe that if $a_1 \leq b_1$, then the first equality holds, because if $a_1 \leq b_1$, then

$$|[a_1,b_1]_{\alpha_1}| = |\{a_1, a_1+1, \cdots, b_1\}| = b_1-a_1+1.$$

Similarly, if $a_2 \leq b_2$, then the second equality holds, and if $\widetilde{a} \leq \widetilde{b}$, then the third equality holds.

If $a_1 \not\leq b_1$, then $[a_1,b_1]_{\alpha_1}$ is the empty set, so we need to show $b_1-a_1+1=0$, in other words, $a_1 = b_1+1$. Similarly, we must show that if $a_2 \not\leq b_2$, then $a_2 = b_2+1$, and if $\widetilde{a} \not\leq \widetilde{b}$, then $\widetilde{a} = \widetilde{b}+1$.

We know that $xq \in \triangle(\ell)$. So

$$\lambda_{vu}(xq) \leq \frac{\ell}{6}+\varepsilon,\ \lambda_u(xq) \leq \frac{\ell}{6}+\varepsilon,\ \lambda_v(xq) \geq -\frac{\ell}{6}+\frac{1}{3}+\varepsilon.$$

First we show that $a_1 \leq b_1$ (that is, $a_1 \not\leq b_1$ does not occur). By definition,

$$a_1 = \lceil A_1\rceil,\ b_1 = \lfloor B_1\rfloor$$

where A_1, B_1 are given by Theorem 6.3. Since $s_{\alpha_1,0}xq = xq + n\alpha_1^\vee$ for some n, $\lambda_{vu}(s_{\alpha_1,0}xq) = \lambda_{vu}(xq) - n$, $\lambda_u(s_{\alpha_1,0}xq) = \lambda_u(xq) + n$. Since $xq \in \triangle(\ell)$, we have $\lambda_v(xq) \geq -\frac{\ell}{6} + \frac{1}{3} + \varepsilon$. Therefore,

$$\begin{aligned} B_1 - A_1 &= (\frac{\ell}{6} + \varepsilon - \lambda_u(s_{\alpha_1,0}xq)) - (-\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_{\alpha_1,0}xq)) \\ &= \frac{\ell}{3} + 2\varepsilon - \lambda_u(xq) - n - \lambda_{vu}(xq) + n \\ &= \frac{\ell}{3} + 2\varepsilon - (\lambda_u - \lambda_v + \lambda_u)(xq) = \frac{\ell}{3} + 2\varepsilon + \lambda_v(xq) \\ &\geq \frac{\ell}{3} + 2\varepsilon - \frac{\ell}{6} + \frac{1}{3} + \varepsilon \\ &= \frac{\ell}{6} + 3\varepsilon + \frac{1}{3} \geq 1 \end{aligned}$$

since $\ell \geq 6$. The inequality $B_1 - A_1 \geq 1$ implies $b_1 - a_1 \geq 0$.

We next consider a_2, b_2. In this case we will see that $a_2 \nleq b_2$ can occur. If this happens, then $[a_2, b_2]_{\alpha_2}$ is the empty set, so we must show $b_2 - a_2 + 1 = 0$. By definition,

$$a_2 = \lceil A_2 \rceil,\ b_2 = \lfloor B_2 \rfloor$$

where A_2, B_2 are given by Theorem 6.3. Since $s_{\alpha_2,0}xq = xq + n\alpha_2^\vee$ for some n,

$$A_2 = \frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\alpha_2,0}xq) = \frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(xq) - n$$

$$B_2 = \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_{\alpha_2,0}xq) = \frac{\ell}{6} + \varepsilon - \lambda_{vu}(xq) - n.$$

By Lemma 5.19, $\lambda_v(xq) \geq -\frac{\ell}{3} + \frac{1}{3}$, so

$$\begin{aligned} B_2 - A_2 &= \frac{\ell}{6} + \varepsilon - \lambda_{vu}(xq) - n + \frac{\ell}{6} - \varepsilon + \lambda_v(xq) + n - \frac{1}{3} \\ &= \frac{\ell}{3} + \lambda_u(xq) - \frac{1}{3} \\ &\geq \frac{\ell}{3} - \frac{\ell}{3} + \frac{1}{3} - \frac{1}{3} = 0. \end{aligned}$$

So $B_2 - A_2 \geq 0 \Rightarrow b_2 - a_2 \geq -1$. If $a_2 \nleq b_2$, then $b_2 = a_2 - 1$.

Now we consider $\widetilde{a}, \widetilde{b}$. In this case we will see that $\widetilde{a} \not\leq \widetilde{b}$ can occur. If this happens, then $[\widetilde{a}, \widetilde{b}]_{\widetilde{\alpha}}$ is the empty set, so we must show $\widetilde{b} - \widetilde{a} + 1 = 0$. By definition,

$$\widetilde{a} = \lceil \widetilde{A} \rceil, \; \widetilde{b} = \lfloor \widetilde{B} \rfloor$$

where $\widetilde{A}, \widetilde{B}$ are given by Theorem 6.3. Since $s_{\widetilde{\alpha},0}xq = xq + n\widetilde{\alpha}^\vee$ for some n,

$$\widetilde{A} = -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0}xq) = -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(xq) - n$$

$$\widetilde{B} = \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0}xq) = \frac{\ell}{6} + \varepsilon - \lambda_u(xq) - n.$$

By Lemma 5.19, $\lambda_{vu}(xq) \geq -\frac{\ell}{3} + \frac{1}{3}$, so

$$\begin{aligned} \widetilde{B} - \widetilde{A} &= \frac{\ell}{6} + \varepsilon - \lambda_u(xq) - n + \frac{\ell}{6} - \varepsilon + \lambda_v(xq) + n - \frac{1}{3} \\ &= \frac{\ell}{3} + \lambda_{vu}(xq) - \frac{1}{3} \\ &\geq \frac{\ell}{3} - \frac{\ell}{3} + \frac{1}{3} - \frac{1}{3} = 0. \end{aligned}$$

So $\widetilde{B} - \widetilde{A} \geq 0 \Rightarrow \widetilde{b} - \widetilde{a} \geq -1$. If $\widetilde{a} \not\leq \widetilde{b}$, then $\widetilde{b} = \widetilde{a} - 1$. □

PROPOSITION 6.5. Let $w(\ell)$ be a length ℓ spiral element. Let $xq \in \triangle(\ell)$. Suppose $\Lambda_x^w = [a_1, b_1]_{\alpha_1} \cup [a_2, b_2]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b}]_{\widetilde{\alpha}}$.

(1) If $t(\alpha_1^\vee)xq \in \triangle(\ell)$, then $\Lambda^w_{t(\alpha_1^\vee)x} = [a_1 + 1, b_1 + 1]_{\alpha_1} \cup [a_2 - 1, b_2]_{\alpha_2} \cup [\widetilde{a} + 1, \widetilde{b}]_{\widetilde{\alpha}}$.

(2) If $t(\alpha_2^\vee)xq \in \triangle(\ell)$, then $\Lambda^w_{t(\alpha_2^\vee)x} = [a_1, b_1 - 1]_{\alpha_1} \cup [a_2 + 1, b_2 + 1]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b} + 1]_{\widetilde{\alpha}}$.

(3) If $t(\widetilde{\alpha}^\vee)xq \in \triangle(\ell)$, then $\Lambda^w_{t(\widetilde{\alpha}^\vee)x} = [a_1 + 1, b_1]_{\alpha_1} \cup [a_2, b_2 + 1]_{\alpha_2} \cup [\widetilde{a} + 1, \widetilde{b} + 1]_{\widetilde{\alpha}}$.

PROOF. Let β be one of α_1, α_1, or $\widetilde{\alpha}$. Let $y = t(\beta^\vee)x$, and let $\Lambda_y^w = [a_1', b_1']_{\alpha_1} \cup [a_2', b_2']_{\alpha_2} \cup [\widetilde{a}', \widetilde{b}']_{\widetilde{\alpha}}$.

Step 1: We calculate a_1', b_1'. We have $s_{\alpha_1,k}(z) = z - ((z, \alpha_1) - k)\alpha_1^\vee$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_{\alpha_1,0}xq) \rceil, \; b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\alpha_1,0}xq) \rfloor$$

and

$$a_1' = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_{\alpha_1,0}yq) \rceil,\ b_1' = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\alpha_1,0}yq) \rfloor.$$

Now, $s_{\alpha_1,0}yq = s_{\alpha_1,0}xq + s_{\alpha_1,0}\beta^\vee$. So

$$a_1' = a_1 + \lambda_{vu}(s_{\alpha_1,0}\beta^\vee),\ b_1' = b_1 - \lambda_u(s_{\alpha_1,0}\beta^\vee).$$

If $\beta = \alpha_1$, then $s_{\alpha_1,0}\beta^\vee = -\alpha_1^\vee$, so $\lambda_u(s_{\alpha_1,0}\beta^\vee) = -1$ and $\lambda_{vu}(s_{\alpha_1,0}\beta^\vee) = 1$. Hence

$$a_1' = a_1 + 1,\ b_1' = b_1 + 1.$$

If $\beta = \alpha_2$, then $s_{\alpha_1,0}\beta^\vee = \widetilde{\alpha}$, so $\lambda_u(s_{\alpha_1,0}\beta^\vee) = 1$ and $\lambda_{vu}(s_{\alpha_1,0}\beta^\vee) = 0$. Hence

$$a_1' = a_1,\ b_1' = b_1 - 1.$$

If $\beta = \widetilde{\alpha}$, then $s_{\alpha_1,0}\beta^\vee = \alpha_2$, so $\lambda_u(s_{\alpha_1,0}\beta^\vee) = 0$ and $\lambda_{vu}(s_{\alpha_1,0}\beta^\vee) = 1$. Hence

$$a_1' = a_1 + 1,\ b_1' = b_1.$$

Step 2: We calculate a_2', b_2'. We have $s_{\alpha_2,k}(z) = z - ((z,\alpha_2) - k)\alpha_2^\vee$. By Theorem 6.3,

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\alpha_2,0}xq) \rceil,\ b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_{\alpha_2,0}xq) \rfloor$$

and

$$a_2' = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\alpha_2,0}yq) \rceil,\ b_2' = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_{\alpha_2,0}yq) \rfloor.$$

Now, $s_{\alpha_2,0}yq = s_{\alpha_2,0}xq + s_{\alpha_2,0}\beta^\vee$. So

$$a_2' = a_2 - \lambda_v(s_{\alpha_2,0}\beta^\vee),\ b_2' = b_2 - \lambda_{vu}(s_{\alpha_2,0}\beta^\vee).$$

If $\beta = \alpha_1$, then $s_{\alpha_2,0}\beta^\vee = \widetilde{\alpha}^\vee$, so $\lambda_v(s_{\alpha_2,0}\beta^\vee) = 1$ and $\lambda_{vu}(s_{\alpha_2,0}\beta^\vee) = 0$. Hence

$$a_2' = a_2 - 1,\ b_2' = b_2.$$

If $\beta = \alpha_2$, then $s_{\alpha_2,0}\beta^\vee = -\alpha_2^\vee$, so $\lambda_v(s_{\alpha_2,0}\beta^\vee) = -1$ and $\lambda_{vu}(s_{\alpha_2,0}\beta^\vee) = -1$. Hence

$$a_2' = a_2 + 1,\ b_2' = b_2 + 1.$$

If $\beta = \widetilde{\alpha}$, then $s_{\alpha_2,0}\beta^\vee = \alpha_1^\vee$, so $\lambda_v(s_{\alpha_2,0}\beta^\vee) = 0$ and $\lambda_{vu}(s_{\alpha_2,0}\beta^\vee) = -1$. Hence

$$a_2' = a_2,\ b_2' = b_2 + 1.$$

Step 3: We calculate $\widetilde{a}', \widetilde{b}'$. We have $s_{\widetilde{\alpha},k}(z) = z - ((z, \widetilde{\alpha}) - k)\widetilde{\alpha}^\vee$. By Theorem 6.3,

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0}xq) \rceil,\ \widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0}xq) \rfloor$$

and

$$\widetilde{a}' = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0}yq) \rceil,\ \widetilde{b}' = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0}yq) \rfloor.$$

Now, $s_{\widetilde{\alpha},0}yq = s_{\widetilde{\alpha},0}xq + s_{\widetilde{\alpha},0}\beta^\vee$. So

$$\widetilde{a}' = \widetilde{a} - \lambda_v(s_{\widetilde{\alpha},0}\beta^\vee),\ \widetilde{b}' = \widetilde{b} - \lambda_u(s_{\widetilde{\alpha},0}\beta^\vee).$$

If $\beta = \alpha_1$, then $s_{\widetilde{\alpha},0}\beta^\vee = -\alpha_2^\vee$, so $\lambda_u(s_{\widetilde{\alpha},0}\beta^\vee) = 0$ and $\lambda_v(s_{\widetilde{\alpha},0}\beta^\vee) = -1$. Hence

$$\widetilde{a}' = \widetilde{a} + 1,\ \widetilde{b}' = \widetilde{b}.$$

If $\beta = \alpha_2$, then $s_{\widetilde{\alpha},0}\beta^\vee = -\alpha_1^\vee$, so $\lambda_u(s_{\widetilde{\alpha},0}\beta^\vee) = -1$ and $\lambda_v(s_{\widetilde{\alpha},0}\beta^\vee) = 0$. Hence

$$\widetilde{a}' = \widetilde{a},\ \widetilde{b}' = \widetilde{b} + 1.$$

If $\beta = \widetilde{\alpha}$, then $s_{\widetilde{\alpha},0}\beta^\vee = -\widetilde{\alpha}^\vee$, so $\lambda_u(s_{\widetilde{\alpha},0}\beta^\vee) = \lambda_v(s_{\widetilde{\alpha},0}\beta^\vee) = -1$. Hence

$$\widetilde{a}' = \widetilde{a} + 1,\ \widetilde{b}' = \widetilde{b} + 1.$$

□

The following theorem is the main result of this chapter.

THEOREM 6.6. Let w be a length ℓ spiral element. If $y = t(\gamma^\vee)x$ for $\gamma^\vee$ in the coroot lattice and both $xq, yq \in \triangle(\ell)$, then

$$p_x^w = p_y^w.$$

PROOF. It suffices to prove this in case γ is one of α_1, α_2, or $\widetilde{\alpha}$. Corollary 6.4 implies

$$p_x^w = \ell(w) - |\Lambda_x^w| = \ell(w) - (b_1 + b_2 + \widetilde{b}) + (a_1 + a_2 + \widetilde{a}) - 3$$

$$p_y^w = \ell(w) - |\Lambda_y^w| = \ell(w) - (b_1' + b_2' + \widetilde{b}') + (a_1' + a_2' + \widetilde{a}') - 3.$$

Here $a_1', a_2', \widetilde{a}', b_1', b_2', \widetilde{b}'$ are given in terms of $a_1, a_2, \widetilde{a}, b_1, b_2, \widetilde{b}$ by Proposition 6.5. These formulas imply that $|\Lambda_x^w| = |\Lambda_y^w|$, hence $p_x^w = p_y^w$. □

6.3 Calculations on coset representatives

To apply the translation theorem we must calculate p_x^w, where x runs over a collection of coset representatives for $L(\Phi^\vee)$ in W. By Theorem 5.7, the elements $\{w(\ell - i) \mid 0 \leq i \leq 5\}$ form such a collection. In this section we calculate $p^w_{w(\ell-i)}$.

LEMMA 6.7. Let $z = a\alpha_1^\vee + b\alpha_2^\vee$. Then

(1) $\lambda_u(s_1 z) = b - a,\ \lambda_v(s_1 z) = b,\ \lambda_{vu}(s_1 z) = a.$

(2) $\lambda_u(s_2 z) = a,\ \lambda_v(s_2 z) = a - b,\ \lambda_{vu}(s_2 z) = -b.$

(3) $\lambda_u(s_{\widetilde{\alpha},0} z) = -b,\ \lambda_v(s_{\widetilde{\alpha},0} z) = -a,\ \lambda_{vu}(s_{\widetilde{\alpha},0} z) = b - a.$

PROOF. This follows from $s_1(\alpha_1^\vee) = -\alpha_1^\vee, s_1(\alpha_2^\vee) = \widetilde{\alpha}^\vee, s_2(\alpha_1^\vee) = \widetilde{\alpha}^\vee, s_2(\alpha_2^\vee) = -\alpha_2^\vee, s_{\widetilde{\alpha},0}(\alpha_1^\vee) = -\alpha_2^\vee, s_{\widetilde{\alpha},0}(\alpha_2^\vee) = -\alpha_1^\vee$. □

The next proposition describes q_x^w when w is an even length spiral element and $x \in \{w(\ell), w(\ell-1), w(\ell-2), w(\ell-3), w(\ell-4), w(\ell-5)\}$. This gives q_x^w when w is an even length spiral element and $x \in W_{finite}$.

THEOREM 6.8. Let $w = w(\ell)$ be an even length spiral with length ℓ. Then

$$q_{w(\ell)}^{w(\ell)} = 0,\ q_{w(\ell-1)}^w = 0,\ q_{w(\ell-2)}^w = 0,\ q_{w(\ell-3)}^w = 1,\ q_{w(\ell-4)}^w = 1,\ q_{w(\ell-5)}^w = 0.$$

PROOF. Let $w(\ell) = s_1s_2s_3s_1\cdots s_is_js_k, ws_k < w$. By Proposition 4.6, $q_x^w = q_{xs_k}^w$, so $q_{w(\ell)}^{w(\ell)} = q_{w(\ell-1)}^{w(\ell)}, q_{w(\ell-3)}^{w(\ell)} = q_{w(\ell-4)}^{w(\ell)}$.

(1) We have $q_{w(\ell)}^{w(\ell)} = 0$. This is always true as

$$q_w^w = |\{r \in R \mid w < wr \leq w\}| - (\ell(w) - \ell(w)) = |\varnothing| = 0.$$

(2) We have $q_{w(\ell-1)}^w = 0$. This follows from (1) and Proposition 4.6.

(3) We show $q_{w(\ell-2)}^w = 0$. Since ℓ is even and $\ell - 2$ is even, by Lemma 5.6,

$$w(\ell-2)q = \frac{1-(\ell-2)}{3}\alpha_1^\vee + \frac{2-(\ell-2)}{6}\alpha_2^\vee = \frac{3-\ell}{3}\alpha_1^\vee + \frac{4-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda_{w(\ell-2)}^w$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1w(\ell-2)q)\rceil = \lceil -\frac{\ell}{6} + \frac{3}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + 1\rceil = -\frac{\ell}{2} + 1$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1w(\ell-2)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{2}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{2}{6}\rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2w(\ell-2)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} - \frac{2}{6} + \frac{\ell}{6}\rceil = \lceil 0\rceil = 0$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2w(\ell-2)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{4}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{4}{6}\rfloor = 0$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0}w(\ell-2)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{3}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{4}{3}\rceil = -\frac{\ell}{2} + 2$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0}w(\ell-2)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{4}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{4}{6}\rfloor = 0.$$

Therefore

$$\Lambda_{w(\ell-2)}^w = [-\frac{\ell}{2} + 1, 0]_{\alpha_1} \cup \{0\}_{\alpha_2} \cup [-\frac{\ell}{2} + 2, 0]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} q_{w(\ell-2)}^w &= |\Lambda_{w(\ell-2)}^w| - \ell(w) \\ &= (0 - (-\frac{\ell}{2} + 1) + 1) + 1 + (0 - (-\frac{\ell}{2} + 2) + 1) - \ell \\ &= 0. \end{aligned}$$

(4) We show $q^w_{w(\ell-3)} = 1$. Since ℓ is even and $\ell-3$ is odd, by Lemma 5.6,

$$w(\ell-3)q = \frac{1-(\ell-3)}{3}\alpha_1^\vee + \frac{3-(\ell-3)}{6}\alpha_2^\vee = \frac{4-\ell}{3}\alpha_1^\vee + \frac{6-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-3)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-3)q)\rceil = \lceil -\frac{\ell}{6} + \frac{4}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{4}{3}\rceil = -\frac{\ell}{2} + 2$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-3)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{2}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{2}{6}\rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-3)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} - \frac{2}{6} + \frac{\ell}{6}\rceil = \lceil 0\rceil = 0$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-3)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{6}{6} - \frac{\ell}{6}\rfloor = \lfloor 1\rfloor = 1$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell-3)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{4}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{5}{3}\rceil = -\frac{\ell}{2} + 2$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-3)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{6}{6} - \frac{\ell}{6}\rfloor = \lfloor 1\rfloor = 1.$$

Therefore

$$\Lambda^w_{w(\ell-3)} = [-\frac{\ell}{2} + 2, 0]_{\alpha_1} \cup [0, 1]_{\alpha_2} \cup [-\frac{\ell}{2} + 2, 1]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} q^w_{w(\ell-3)} &= |\Lambda^w_{w(\ell-3)}| - \ell(w) \\ &= (0 - (-\frac{\ell}{2} + 2) + 1) + 2 + (1 - (-\frac{\ell}{2} + 2) + 1) - \ell \\ &= 1. \end{aligned}$$

(5) $q^w_{w(\ell-4)} = 1$. This follows from (4) and Proposition 4.6.

(6) We show $q^w_{w(\ell-5)} = 0$. Since ℓ is even and $\ell-5$ is odd, by Lemma 5.6,

$$w(\ell-5)q = \frac{1-(\ell-5)}{3}\alpha_1^\vee + \frac{3-(\ell-5)}{6}\alpha_2^\vee = \frac{6-\ell}{3}\alpha_1^\vee + \frac{8-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-5)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-5)q)\rceil = \lceil -\frac{\ell}{6} + \frac{6}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{6}{3}\rceil = -\frac{\ell}{2} + 2$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-5)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{4}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{4}{6}\rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-5)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} - \frac{4}{6} + \frac{\ell}{6}\rceil = \lceil -\frac{1}{3}\rceil = 0$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-5)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{8}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{8}{6}\rfloor = 1$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell-5)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{6}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{7}{3}\rceil = -\frac{\ell}{2} + 3$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-5)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{8}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{8}{6}\rfloor = 1.$$

Therefore

$$\Lambda^w_{w(\ell-5)} = [-\frac{\ell}{2} + 2, 0]_{\alpha_1} \cup [0,1]_{\alpha_2} \cup [-\frac{\ell}{2} + 3, 1]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} q^w_{w(\ell-5)} &= |\Lambda^w_{w(\ell-5)}| - \ell(w) \\ &= (0 - (-\frac{\ell}{2} + 2) + 1) + (1 - 0 + 1) + (1 - (-\frac{\ell}{2} + 3) + 1) - \ell \\ &= 0. \end{aligned}$$

□

Remark: If w is not spiral, then we can have $q^w_x \geq 1$. For example, in type $\widetilde{A}_2$, $q^w_x = 2$ for $w = s_1 s_2 s_3 s_1 s_2 s_1 s_3, x = s_2$.

The next proposition describes p^w_x when w is an odd length spiral and $x \in \{w(\ell), w(\ell - 1), w(\ell-2), w(\ell-3), w(\ell-4), w(\ell-5)\}$. This gives p^w_x when w is an odd length spiral and $x \in W_{finite}$.

THEOREM 6.9. Let $w = w(\ell)$ be an odd length spiral element of length ℓ. Then

$$p^{w(\ell)}_{w(\ell)} = 0,\ p^w_{w(\ell-1)} = 0,\ p^w_{w(\ell-2)} = 1,\ p^w_{w(\ell-3)} = 1,\ p^w_{w(\ell-4)} = 1,\ p^w_{w(\ell-5)} = 1.$$

PROOF. (1) We show $p^w_{w(\ell)} = 0$. Since ℓ is odd, by Lemma 5.6, $w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee$. Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell)q)\rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{1}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{1}{6}\rceil = -\frac{\ell+1}{2} + 1$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} - \frac{1}{6} - \frac{\ell}{6}\rfloor = \lfloor 0\rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{\ell}{6}\rceil = \lceil \frac{2}{3}\rceil = 1$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{3}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{2}{3}\rfloor = 0$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{5}{6}\rceil = -\frac{\ell-1}{2} + 1$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{3}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{2}{3}\rfloor = 0.$$

Therefore

$$\Lambda^w_{w(\ell)} = [-\frac{\ell+1}{2} + 1, 0]_{\alpha_1} \cup \varnothing_{\alpha_2} \cup [-\frac{\ell-1}{2} + 1, 0]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} p^w_{w(\ell-1)} &= |\Lambda^w_{w(\ell-1)}| - \ell(w) \\ &= (0 - (-\frac{\ell+1}{2} + 1) + 1) + 0 + (0 - (-\frac{\ell-1}{2} + 1) + 1) - \ell \\ &= 0. \end{aligned}$$

(2) We show $p^w_{w(\ell-1)} = 0$. Since ℓ is odd and $\ell - 1$ is even, by Lemma 5.6,

$$w(\ell-1)q = \frac{1-(\ell-1)}{3}\alpha_1^\vee + \frac{2-(\ell-1)}{6}\alpha_2^\vee = \frac{2-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-1)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-1)q) \rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{2}{3} - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{1}{2} \rceil = -\frac{\ell-1}{2}$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-1)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{1}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{1}{3} \rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-1)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} - \frac{\ell}{6} \rceil = \lceil \frac{2}{3} \rceil = 1$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-1)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{3}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{2}{3} \rfloor = 0$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell-1)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{4}{6} - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{7}{6} \rceil = -\frac{\ell-1}{2} + 1$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-1)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{3}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{2}{3} \rfloor = 0.$$

Therefore

$$\Lambda^w_{w(\ell-1)} = [-\frac{\ell-1}{2}, 0]_{\alpha_1} \cup \varnothing_{\alpha_2} \cup [-\frac{\ell-1}{2} + 1, 0]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} p^w_{w(\ell-1)} &= |\Lambda^w_{w(\ell-1)}| - \ell(w) \\ &= (0 - (-\frac{\ell-1}{2}) + 1) + 0 + (0 - (-\frac{\ell-1}{2} + 1) + 1) - \ell \\ &= 0. \end{aligned}$$

(3) We show $p^w_{w(\ell-2)} = 1$. Since ℓ is odd and $\ell - 2$ is odd, by Lemma 5.6,

$$w(\ell-2)q = \frac{1-(\ell-2)}{3}\alpha_1^\vee + \frac{3-(\ell-2)}{6}\alpha_2^\vee = (1 - \frac{\ell}{3})\alpha_1^\vee + (\frac{5}{6} - \frac{\ell}{6})\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-2)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-2)q) \rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + 1 - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{5}{6} \rceil = -\frac{\ell-1}{2} + 1$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-2)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{1}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{1}{3} \rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-2)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{\ell}{6} \rceil = \lceil \frac{1}{3} \rceil = 1$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-2)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{5}{6} - \frac{\ell}{6} \rfloor = \lfloor 1 \rfloor = 1$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell-2)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + 1 - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{3}{2} \rceil = -\frac{\ell-1}{2} + 1$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-2)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{5}{6} - \frac{\ell}{6} \rfloor = \lfloor 1 \rfloor = 1.$$

Therefore

$$\Lambda^w_{w(\ell-2)} = [-\frac{\ell-1}{2} + 1, 0]_{\alpha_1} \cup \{1\}_{\alpha_2} \cup [-\frac{\ell-1}{2} + 1, 1]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} p^w_{w(\ell-2)} &= |\Lambda^w_{w(\ell-2)}| - \ell(w) \\ &= (0 - (-\frac{\ell-1}{2} + 1) + 1) + 1 + (1 - (-\frac{\ell-1}{2} + 1) + 1) - \ell \\ &= 1. \end{aligned}$$

(4) We show $p^w_{w(\ell-3)} = 1$. Since ℓ is odd and $\ell - 3$ is even, by Lemma 5.6,

$$w(\ell-3)q = \frac{1-(\ell-3)}{3}\alpha_1^\vee + \frac{2-(\ell-3)}{6}\alpha_2^\vee = \frac{4-\ell}{3}\alpha_1^\vee + \frac{5-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-3)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-3)q) \rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{4}{3} - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{7}{6} \rceil = -\frac{\ell-1}{2} + 1$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-3)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{3}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{2}{3} \rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-3)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} - \frac{3}{6} + \frac{\ell}{6} \rceil = \lceil 0 \rceil = 0$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-3)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{5}{6} - \frac{\ell}{6} \rfloor = \lfloor 1 \rfloor = 1$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell-3)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{4}{3} - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{11}{6} \rceil = -\frac{\ell-1}{2} + 2$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-3)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{5}{6} - \frac{\ell}{6} \rfloor = \lfloor 1 \rfloor = 1.$$

Therefore

$$\Lambda^w_{w(\ell-3)} = [-\frac{\ell-1}{2}+1, 0]_{\alpha_1} \cup [0,1]_{\alpha_2} \cup [-\frac{\ell-1}{2}+2, 1]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned} p^w_{w(\ell-3)} &= |\Lambda^w_{w(\ell-3)}| - \ell(w) \\ &= (0-(-\frac{\ell-1}{2}+1)+1) + (1-0+1) + (1-(-\frac{\ell-1}{2}+2)+1) - \ell \\ &= 1. \end{aligned}$$

(5) We show $p^w_{w(\ell-4)} = 1$. Since ℓ is odd and $\ell-4$ is odd, by Lemma 5.6,

$$w(\ell-4)q = \frac{1-(\ell-4)}{3}\alpha_1^\vee + \frac{3-(\ell-4)}{6}\alpha_2^\vee = \frac{5-\ell}{3}\alpha_1^\vee + \frac{7-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-4)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-4)q) \rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{5}{3} - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{3}{2} \rceil = -\frac{\ell-1}{2} + 1$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-4)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{3}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{2}{3} \rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-4)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} - \frac{3}{6} + \frac{\ell}{6} \rceil = \lceil 0 \rceil = 0$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-4)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{7}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{4}{3} \rfloor = 1$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} w(\ell-4)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{5}{3} - \frac{\ell}{3} \rceil = \lceil -\frac{\ell}{2} + \frac{13}{6} \rceil = -\frac{\ell-1}{2} + 2$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-4)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{7}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{4}{3} \rfloor = 1.$$

Therefore

$$\Lambda^w_{w(\ell-4)} = [-\frac{\ell-1}{2}+1, 0]_{\alpha_1} \cup [0,1]_{\alpha_2} \cup [-\frac{\ell-1}{2}+2, 1]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned}p^w_{w(\ell-4)} &= |\Lambda^w_{w(\ell-3)}| - \ell(w)\\ &= (0-(-\frac{\ell-1}{2}+1)+1)+(1-0+1)+(1-(-\frac{\ell-1}{2}+2)+1)-\ell\\ &= 1.\end{aligned}$$

(6) We show $p^w_{w(\ell-5)} = 1$. Since ℓ is odd and $\ell-5$ is even, by Lemma 5.6,

$$w(\ell-5)q = \frac{1-(\ell-5)}{3}\alpha_1^\vee + \frac{2-(\ell-5)}{6}\alpha_2^\vee = \frac{6-\ell}{3}\alpha_1^\vee + \frac{7-\ell}{6}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{w(\ell-5)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 w(\ell-5)q)\rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{6}{3} - \frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2} + \frac{11}{6}\rceil = -\frac{\ell-1}{2}+2$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 w(\ell-5)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{5}{6} - \frac{\ell}{6}\rfloor = \lfloor 1\rfloor = 1$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 w(\ell-5)q)\rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} - \frac{5}{6} + \frac{\ell}{6}\rceil = \lceil -\frac{1}{3}\rceil = 0$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 w(\ell-5)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{7}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{4}{3}\rfloor = 1$$

$$\widetilde{a} = \lceil -\frac{\ell}{6}+\frac{1}{3}+\varepsilon-\lambda_v(s_{\widetilde{\alpha},0} w(\ell-5)q)\rceil = \lceil -\frac{\ell}{6}+\frac{1}{3}+\frac{1}{6}+\frac{6}{3}-\frac{\ell}{3}\rceil = \lceil -\frac{\ell}{2}+\frac{15}{6}\rceil = -\frac{\ell-1}{2}+2$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} w(\ell-5)q)\rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{7}{6} - \frac{\ell}{6}\rfloor = \lfloor \frac{4}{3}\rfloor = 1.$$

Therefore

$$\Lambda^w_{w(\ell-5)} = [-\frac{\ell-1}{2}+2, 1]_{\alpha_1} \cup [0,1]_{\alpha_2} \cup [-\frac{\ell-1}{2}+2, 1]_{\widetilde{\alpha}}.$$

By Corollary 6.4,

$$\begin{aligned}p^w_{w(\ell-5)} &= |\Lambda^w_{w(\ell-3)}| - \ell(w)\\ &= (1-(-\frac{\ell-1}{2}+2)+1)+(1-0+1)+(1-(-\frac{\ell-1}{2}+2)+1)-\ell = 1.\end{aligned}$$

□

COROLLARY 6.10. Let $w = w(\ell)$. Then p_x^w is always 0 or 1.

PROOF. By Theorem 6.8 and Theorem 6.9 p_x^w is 0 or 1 if $x = w(\ell - i)$, $1 \leq i \leq 5$. (Note that if ℓ is even then $p_x^w = q_x^w$).

By Proposition 5.7, any $x \in W$ is of the form $x = t(\gamma^\vee)u$, where $u = w(\ell - i)$, some $i \in \{0, 1, 2, 3, 4, 5\}$. Therefore, $p_x^w = p_u^w$ is 0 or 1 by translation Theorem 6.6. □

6.4 The integers $q_x^{w(\ell)}$ for ℓ odd

If ℓ is even, then $p_x^w = q_x^w$. If ℓ is odd, then we need some further results to better understand q_x^w.

Definition 6.11. Define

$R_{2\triangle} := \{xq \mid x \in W, x \in R(\ell), rx = A_1(\ell)$ or $rx = A_2(\ell)$ for some reflection $r\}$ and set

$$r_x^w := \begin{cases} 1, & xq \in R_{2\triangle} \\ 0, & xq \notin R_{2\triangle}. \end{cases}$$

Observe that $r_x^w = |R_x^w|$, where

$$R_x^w = \{r \mid r \text{ is a reflection with } rx = A_i(\ell), i = 1 \text{ or } 2\}.$$

Since $A_2(\ell) = s_1 A_1(\ell)$, it is not possible for there to exist two reflections r_1, r_2 with $r_1 x = A_1(\ell)$ and $r_2 x = A_2(\ell)$.

$R_{2\triangle}$ can also be described as follows.

$$\begin{aligned} R_{2\triangle} &= \{rA_1(\ell)q \mid rA_1(\ell)q \in \triangle(\ell)\} \cup \{rA_2(\ell)q \mid rA_2(\ell)q \in \triangle(\ell)\} \\ &= \{rA_1(\ell)q \mid r \in \Lambda^{w(\ell)}_{A_1(\ell)}\} \cup \{rA_2(\ell)q \mid r \in \Lambda^{w(\ell)}_{A_2(\ell)}\}. \end{aligned}$$

We have $x \leq w(\ell) \Leftrightarrow xq \in R(\ell)$ (Theorem 5.41). This implies the following lemma.

LEMMA 6.12. Let $w(\ell)$ be an odd length spiral element. $q_x^w = p_x^w - r_x^w$.

PROOF. For an odd length spiral element $w(\ell)$, $\triangle(\ell) = R(\ell) \cup \{A_1(\ell), A_2(\ell)\}$. By Theorem 5.41, $rx \leq w(\ell) \Leftrightarrow rxq \in R(\ell)$. Hence $\Lambda_x^w = \Psi_x^w \sqcup R_x^w$. Then

$$|\Lambda_x^w| - \ell(w) = |\Psi_x^w| + |R_x^w| - \ell(w).$$

So $p_x^w = q_x^w + r_x^w$. $\square$

Our goal is to show that if $y \in R_{2\triangle}$, then $p_y^w = 1$. We first calculate $\Lambda^w_{A_1(\ell)}, \Lambda^w_{A_2(\ell)}$. Then we find four coset representatives.

PROPOSITION 6.13. Let ℓ be odd. Then

(1) $\Lambda^w_{A_1(\ell)} = \{0\}_{\alpha_1} \cup [1, \frac{\ell-1}{2}]_{\alpha_2} \cup [1, \frac{\ell-1}{2}]_{\widetilde{\alpha}}$.

(2) $\Lambda^w_{A_2(\ell)} = \{0\}_{\alpha_1} \cup [1, \frac{\ell-1}{2}]_{\alpha_2} \cup [1, \frac{\ell-1}{2}]_{\widetilde{\alpha}}$.

PROOF. Since ℓ is odd, by Lemma 5.10,

$$A_1(\ell)q = \frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee$$

$$A_2(\ell)q = \frac{\ell+1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee.$$

Using Lemma 6.7 we calculate. First we calculate $\Lambda^w_{A_1(\ell)}$. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 A_1(\ell)q) \rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{\ell}{6} - \frac{1}{6} \rceil = \lceil -\frac{1}{3} \rceil = 0$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 A_1(\ell)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} - \frac{1}{6} - \frac{\ell}{6} \rfloor = \lfloor 0 \rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 A_1(\ell)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{\ell}{6} \rceil = \lceil \frac{2}{3} \rceil = 1$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 A_1(\ell)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{\ell}{3} \rfloor = \lfloor \frac{\ell}{2} + \frac{1}{6} \rfloor = \lfloor \frac{\ell-1}{2} + \frac{4}{6} \rfloor = \frac{\ell-1}{2}$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} A_1(\ell)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{\ell}{6} - \frac{1}{6} \rceil = \lceil \frac{1}{3} \rceil = 1$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} A_1(\ell)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{\ell}{3} \rfloor = \lfloor \frac{\ell}{2} + \frac{1}{6} \rfloor = \frac{\ell-1}{2}.$$

Therefore

$$\Lambda^w_{A_1(\ell)} = \{0\}_{\alpha_1} \cup [1, \frac{\ell-1}{2}]_{\alpha_2} \cup [1, \frac{\ell-1}{2}]_{\widetilde{\alpha}}.$$

Now we calculate $\Lambda^w_{A_2(\ell)}$ using Lemma 6.7. By Theorem 6.3,

$$a_1 = \lceil -\frac{\ell}{6} - \varepsilon + \lambda_{vu}(s_1 A_2(\ell)q) \rceil = \lceil -\frac{\ell}{6} - \frac{1}{6} + \frac{\ell}{6} + \frac{1}{6} \rceil = \lceil 0 \rceil = 0$$

$$b_1 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_1 A_2(\ell)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{1}{6} - \frac{\ell}{6} \rfloor = \lfloor \frac{1}{3} \rfloor = 0$$

$$a_2 = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_2 A_2(\ell)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} - \frac{1}{6} + \frac{\ell}{6} \rceil = \lceil \frac{1}{3} \rceil = 1$$

$$b_2 = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_{vu}(s_2 A_2(\ell)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{\ell}{3} \rfloor = \lfloor \frac{\ell}{2} + \frac{1}{6} \rfloor = \lfloor \frac{\ell-1}{2} + \frac{4}{6} \rfloor = \frac{\ell-1}{2}$$

$$\widetilde{a} = \lceil -\frac{\ell}{6} + \frac{1}{3} + \varepsilon - \lambda_v(s_{\widetilde{\alpha},0} A_2(\ell)q) \rceil = \lceil -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} + \frac{\ell}{6} + \frac{1}{6} \rceil = \lceil \frac{2}{3} \rceil = 1$$

$$\widetilde{b} = \lfloor \frac{\ell}{6} + \varepsilon - \lambda_u(s_{\widetilde{\alpha},0} A_2(\ell)q) \rfloor = \lfloor \frac{\ell}{6} + \frac{1}{6} + \frac{\ell}{3} \rfloor = \lfloor \frac{\ell}{2} + \frac{1}{6} \rfloor = \frac{\ell-1}{2}.$$

Therefore

$$\Lambda^w_{A_2(\ell)} = \{0\}_{\alpha_1} \cup [1, \frac{\ell-1}{2}]_{\alpha_2} \cup [1, \frac{\ell-1}{2}]_{\widetilde{\alpha}}.$$

□

PROPOSITION 6.14. Let ℓ be odd. Let $m = \frac{\ell-1}{2}$.

(1) If $yq \in R_{2\triangle}$, then $y = t(\gamma^\vee)z$, where

$z \in \{s_{\alpha_2,m}A_1(\ell), s_{\widetilde{\alpha},m}A_1(\ell), s_{\alpha_2,m}A_2(\ell), s_{\widetilde{\alpha},m}A_2(\ell)\}$.

(2) $s_{\alpha_2,m}A_1(\ell) = t((m-1)\widetilde{\alpha}^\vee)w(\ell-3)$,

$s_{\alpha_2,m}A_2(\ell) = t((m-1)\widetilde{\alpha}^\vee)w(\ell-4)$,

$s_{\widetilde{\alpha},m}A_2(\ell) = t((m-1)\widetilde{\alpha}^\vee)w(\ell-2)$,

$s_{\widetilde{\alpha},m}A_1(\ell) = t((m-1)\widetilde{\alpha}^\vee - \alpha_1^\vee)w(\ell-5)$.

PROOF. (1) By Proposition 6.13, if $yq \in R_{2\triangle}$, then y is one of the following:

$$y = s_{\alpha_2,i}A_1(\ell) = t((i-m)\alpha_2^\vee)s_{\alpha_2,m}A_1(\ell)$$

$$y = s_{\widetilde{\alpha},i}A_1(\ell) = t((i-m)\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},m}A_1(\ell)$$

$$y = s_{\alpha_2,i}A_2(\ell) = t((i-m)\alpha_2^\vee)s_{\alpha_2,m}A_2(\ell)$$

$$y = s_{\widetilde{\alpha},i}A_2(\ell) = t((i-m)\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},m}A_2(\ell).$$

Note that $s_{\beta,i} = t((i-m)\beta^\vee)s_{\beta,m}$. Note also $s_{\alpha_1,0}A_1(\ell) = A_2(\ell)$ and $s_{\alpha_1,0}A_2(\ell) = A_1(\ell)$, not in $R_{2\triangle}$.

(2) Since ℓ is odd and $\ell - 3$ is even, by Lemma 5.10 and Lemma 5.6, we have

$$\begin{aligned} s_{\alpha_2,m}A_1(\ell)q &= t(m\alpha_2^\vee)s_2(\frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee) \\ &= t(\frac{\ell-1}{2}\alpha_2^\vee)(\frac{\ell-1}{6}\widetilde{\alpha}^\vee - \frac{\ell}{3}\alpha_2^\vee) \\ &= (\frac{\ell}{6} - \frac{1}{6})\alpha_1^\vee + (\frac{\ell}{3} - \frac{2}{3})\alpha_2^\vee \end{aligned}$$

$$\begin{aligned} t((m-1)\widetilde{\alpha}^\vee)w(\ell-3)q &= t((m-1)\widetilde{\alpha}^\vee)(\frac{1-(\ell-3)}{3}\alpha_1^\vee + \frac{2-(\ell-3)}{6}\alpha_2^\vee) \\ &= \frac{4-\ell}{3}\alpha_1^\vee + \frac{5-\ell}{6}\alpha_2^\vee + (\frac{\ell-1}{2} - 1)\widetilde{\alpha}^\vee \\ &= (\frac{\ell}{6} - \frac{1}{6})\alpha_1^\vee + (\frac{\ell}{3} - \frac{2}{3})\alpha_2^\vee. \end{aligned}$$

Therefore $s_{\alpha_2,m}A_1(\ell) = t((m-1)\widetilde{\alpha}^\vee)w(\ell-3)$. Similarly,

$$\begin{aligned} s_{\widetilde{\alpha},m}A_1(\ell)q &= t(m\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}(\frac{\ell-1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee) \\ &= t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)(-\frac{\ell-1}{6}\alpha_2^\vee - \frac{\ell}{3}\alpha_1^\vee) \\ &= (\frac{\ell}{6} - \frac{1}{2})\alpha_1^\vee + (\frac{\ell}{3} - \frac{1}{3})\alpha_2^\vee \end{aligned}$$

$$
\begin{aligned}
t((m-1)\widetilde{\alpha}^\vee - \alpha_1^\vee)w(\ell-5)q &= t((m-1)\widetilde{\alpha}^\vee - \alpha_1^\vee)(\frac{1-(\ell-5)}{3}\alpha_1^\vee + \frac{2-(\ell-5)}{6}\alpha_2^\vee) \\
&= \frac{6-\ell}{3}\alpha_1^\vee + \frac{7-\ell}{6}\alpha_2^\vee + (\frac{\ell-1}{2}-1)\widetilde{\alpha}^\vee - \alpha_1^\vee \\
&= (\frac{\ell}{6}-\frac{1}{2})\alpha_1^\vee + (\frac{\ell}{3}-\frac{1}{3})\alpha_2^\vee.
\end{aligned}
$$

Therefore $s_{\widetilde{\alpha},m}A_1(\ell) = t((m-1)\widetilde{\alpha}^\vee - \alpha_1^\vee)w(\ell-5)$. Similarly,

$$
\begin{aligned}
s_{\widetilde{\alpha},m}A_2(\ell)q &= t(m\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}(\frac{\ell+1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee) \\
&= t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)(-\frac{\ell+1}{6}\alpha_2^\vee - \frac{\ell}{3}\alpha_1^\vee) \\
&= (\frac{\ell}{6}-\frac{1}{2})\alpha_1^\vee + (\frac{\ell}{3}-\frac{2}{3})\alpha_2^\vee
\end{aligned}
$$

$$
\begin{aligned}
t((m-1)\widetilde{\alpha}^\vee)w(\ell-2)q &= t((m-1)\widetilde{\alpha}^\vee)(\frac{1-(\ell-2)}{3}\alpha_1^\vee + \frac{3-(\ell-2)}{6}\alpha_2^\vee) \\
&= \frac{3-\ell}{3}\alpha_1^\vee + \frac{5-\ell}{6}\alpha_2^\vee + (\frac{\ell-1}{2}-1)\widetilde{\alpha}^\vee \\
&= (\frac{\ell}{6}-\frac{1}{2})\alpha_1^\vee + (\frac{\ell}{3}-\frac{2}{3})\alpha_2^\vee.
\end{aligned}
$$

Therefore $s_{\widetilde{\alpha},m}A_2(\ell) = t((m-1)\widetilde{\alpha}^\vee)w(\ell-2)$. Finally,

$$
\begin{aligned}
s_{\alpha_2,m}A_2(\ell)q &= t(m\alpha_2^\vee)s_2(\frac{\ell+1}{6}\alpha_1^\vee + \frac{\ell}{3}\alpha_2^\vee) \\
&= t(\frac{\ell-1}{2}\alpha_2^\vee)(\frac{\ell+1}{6}\widetilde{\alpha}^\vee - \frac{\ell}{3}\alpha_2^\vee) \\
&= (\frac{\ell}{6}+\frac{1}{6})\alpha_1^\vee + (\frac{\ell}{3}-\frac{1}{3})\alpha_2^\vee
\end{aligned}
$$

$$
\begin{aligned}
t((m-1)\widetilde{\alpha}^\vee)w(\ell-4)q &= t((m-1)\widetilde{\alpha}^\vee)(\frac{1-(\ell-4)}{3}\alpha_1^\vee + \frac{3-(\ell-4)}{6}\alpha_2^\vee) \\
&= \frac{5-\ell}{3}\alpha_1^\vee + \frac{7-\ell}{6}\alpha_2^\vee + (\frac{\ell-1}{2}-1)\widetilde{\alpha}^\vee \\
&= (\frac{\ell}{6}+\frac{1}{6})\alpha_1^\vee + (\frac{\ell}{3}-\frac{1}{3})\alpha_2^\vee.
\end{aligned}
$$

Therefore $s_{\alpha_2,m}A_2(\ell) = t((m-1)\widetilde{\alpha}^\vee)w(\ell-4)$. □

COROLLARY 6.15. If $yq \in R_{2\triangle}$, then $p_y^w = 1$.

PROOF. This follows from Proposition 6.14 and Theorem 6.9 and the translation theorem Theorem 6.6. ☐

THEOREM 6.16. Let $w = w(\ell)$ be a spiral element in type $\widetilde{A}_2$. Then q_x^w is always 0 or 1.

PROOF. If ℓ is even, then $q_x^w = p_x^w$ and this is Corollary 6.10.

Suppose ℓ is odd. If $x \notin R_{2\triangle}$, then $q_x^w = p_x^w - r_x^w = p_x^w - 0 = p_x^w$ which is 0 or 1 by Corollary 6.10. If $x \in R_{2\triangle}$, then $p_x^w = 1$ by Proposition 6.15. Therefore $q_x^w = p_x^w - r_x^w = 1 - 1 = 0$ by Lemma 6.12. ☐

The following corollary is used in Chapter 7.

COROLLARY 6.17. Let ℓ be odd, $w = w(\ell), x \leq w$. If either $p_x^w = 0$ or $r_x^w = 1$, then $q_x^w = 0$.

PROOF. We have $p_x^w \geq q_x^w = p_x^w - r_x^w \geq 0$. By the coset theorem Theorem 6.9 and translation theorem Theorem 6.6, p_x^w is either 0 or 1. The result follows. ☐

7

THE RATIONALLY SMOOTH POINTS AND THE LOOKUP CONJECTURE

In this chapter we apply our results about the Bruhat order and the integers q_x^w to the study of rational smoothness. In Section 7.1, we describe the set of rationally smooth points of $X(w(\ell))$ in terms of the geometry of the triangle region. Precisely, we prove (Theorem 7.1) that if $x \leq w(\ell)$, then the point $xB \in X(w(\ell))$ is not rationally smooth if and only if $xq \in \triangle(\ell-3)$. Using this we identify the maximal non rationally smooth points of $X(w(\ell))$ (Theorem 7.2). In Section 7.2, we apply these results to prove the lookup conjecture for spiral Schubert varieties.

7.1 Identification of rationally smooth points

In this section we describe the set of rational smooth points of $X(w(\ell))$. Using this, we identify the maximal non rationally smooth points in $X(w(\ell))$. Since a point xB is not rationally smooth if and only if x is less than or equal to a maximal non rationally smooth points, this gives an alternative criterion for rational smoothness.

THEOREM 7.1. Let $w = w(\ell)$ be a spiral element of length ℓ with length $\ell \geq 6$. Then the point $xB \in X(w(\ell))$ is not rationally smooth $\Leftrightarrow$ $xq \in \triangle(\ell - 3)$.

From Theorem 7.1 we can deduce the following theorem, which identifies the maximal non rationally smooth points in $X(w(\ell))$. Recall that $A_1(\ell) := t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)$ for ℓ odd.

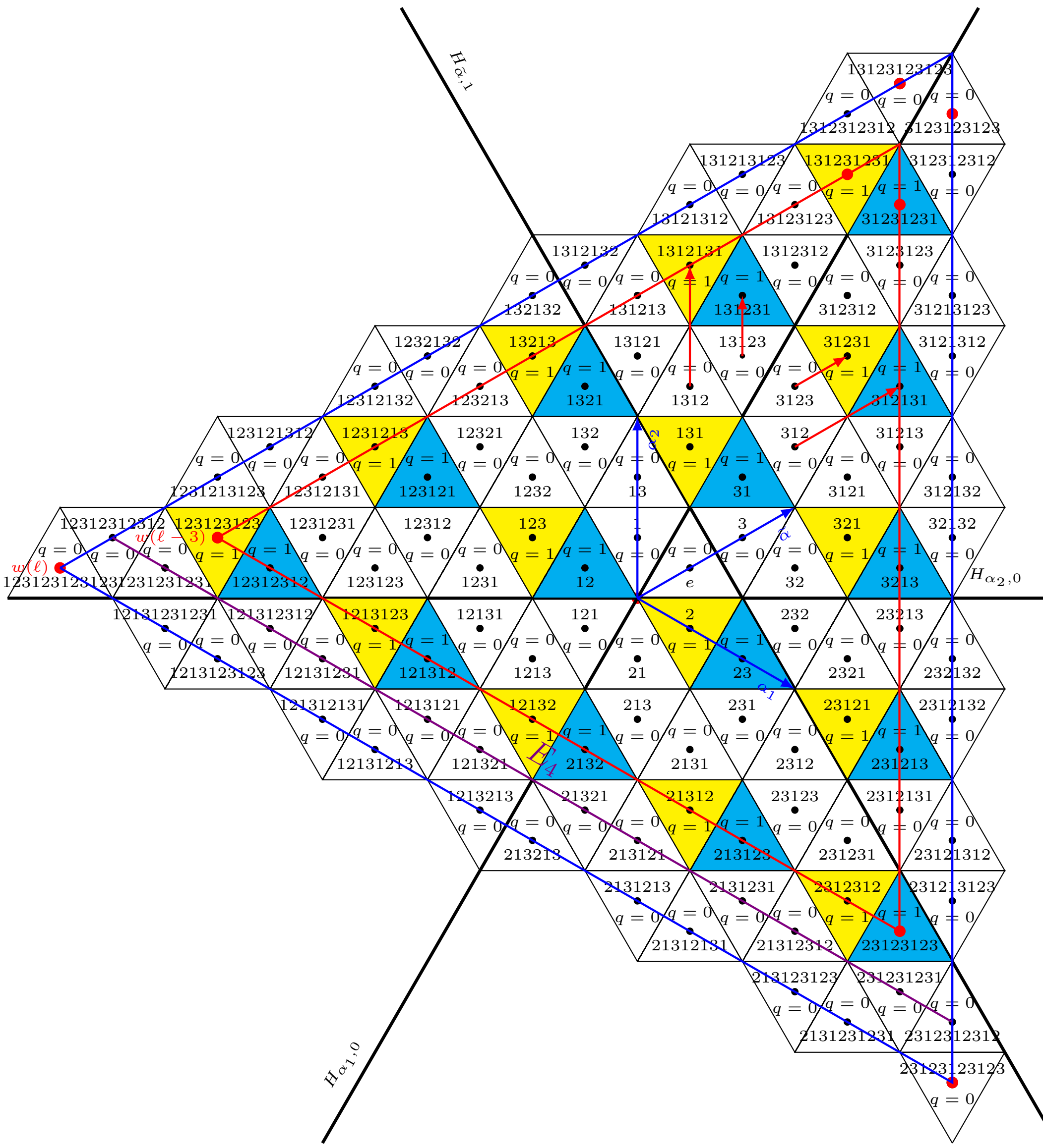

Figure 7.1: $x \leq w(\ell)$ n.r.s. $\Leftrightarrow x \in \triangle(\ell-3)$ and the lookup direction for ℓ even ($\ell = 12$). The lookup direction is pictured for $x = s_1s_3s_1s_2$, $x = s_1s_3s_1s_2s_3$, $x = s_3s_1s_2s_3$ and for $x = s_3s_1s_2$.

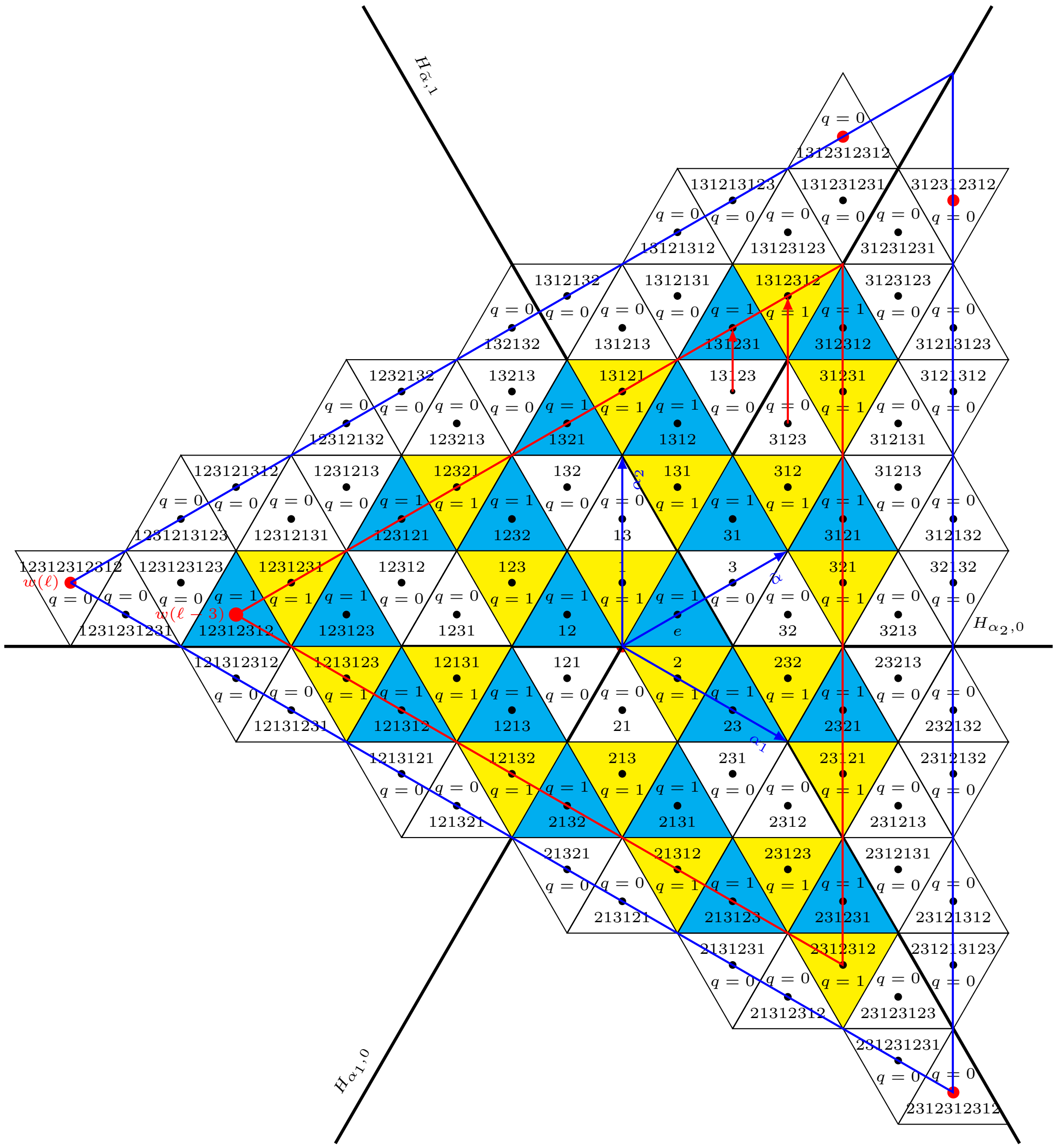

Figure 7.2: $x \leq w(\ell)$ n.r.s. $\Leftrightarrow x \in \triangle(\ell - 3)$ and the lookup direction for ℓ odd ($\ell = 11$). The lookup direction is pictured for $x = s_1 s_3 s_1 s_2 s_3$ and for $x = s_3 s_1 s_2 s_3$.

THEOREM 7.2. Suppose $\ell \geq 6$.

(a) If ℓ is odd, xB is not rationally smooth in $X(w(\ell))$ if and only if $x \leq w(\ell-3)$.

(b) If ℓ is even, xB is not rationally smooth in $X(w(\ell))$ if and only if $x \leq w(\ell-3)$ or $x \leq A_1(\ell-3)$.

PROOF. By Theorem 7.1, xB is not rationally smooth in $X(w(\ell)) \Leftrightarrow xq \in \triangle(\ell-3)$. By Theorem 5.41 and Theorem 5.44, if ℓ is odd, $xq \in \triangle(\ell-3) \Leftrightarrow x \leq w(\ell-3)$. If ℓ is even, $xq \in \triangle(\ell-3) \Leftrightarrow x \leq w(\ell-3)$ or $x \leq A_1(\ell-3)$. □

7.1.1 PROOF OF THEOREM 7.1

We first prove ($\Leftarrow$): First suppose ℓ is even. If $xq \in \triangle(\ell-3)$, then $xq \in R(\ell-3)$, so $x \leq w(\ell-3)$ or $x \in \{A_1(\ell-3), A_2(\ell-3)\}$. Since $A_1(\ell-3) = t(\frac{\ell-4}{2}\widetilde{\alpha}^\vee)w(\ell-3)$, Theorem 6.8 and the translation Theorem 6.6 imply

$$q^w_{w(\ell-3)} = q^w_{A_1(\ell-3)} = 1.$$

Since $A_2(\ell-3) < A_1(\ell-3)$ by Lemma 5.10, the Carrell-Peterson criterion [5, Theorem F] shows that xB is not rationally smooth.

Now suppose ℓ is odd. Since $xq \in \triangle(\ell-3)$ and $\ell-3$ is even, by Theorem 5.41 $x \leq w(\ell-3)$. By Proposition 6.9, $q^w_{w(\ell-3)} = p^w_{w(\ell-3)} = 1$, the Carrell-Peterson criterion [5, Theorem F] implies that xB is not rationally smooth.

($\Rightarrow$): To prove this implication we need a lemma.

LEMMA 7.3. If $xq \notin \triangle(\ell-3)$, then $q^w_x = 0$.

Using this Lemma, we complete the proof of ($\Rightarrow$) of Theorem 7.1. We prove the contrapositive: if $xq \notin \triangle(\ell-3)$ then xB is rationally smooth. Suppose that $xq \notin \triangle(\ell-3)$ and $y \geq x$. We claim that $yq \notin \triangle(\ell-3)$. First suppose ℓ is even. By Theorem 5.44, $xq \notin \triangle(\ell-3) \Leftrightarrow x \not\leq w(\ell-3)$ or $x \not\leq A_1(\ell-3)$. Since $x \leq y, y \not\leq w(\ell-3)$

or $y \not\leq A_1(\ell-3)$. So by Theorem 5.44, $yq \notin \triangle(\ell-3)$. Next suppose ℓ is odd. If $yq \in \triangle(\ell-3)$, then by Theorem 5.41, $y \leq w(\ell-3)$ as $\ell-3$ is even. By Theorem 5.41, $xq \notin \triangle(\ell-3)$ implies $x \not\leq w(\ell-3)$ as $\ell-3$ is even. Hence $x \not\leq y$. The claim is proved.

By Lemma 7.3, $q_y^w = 0$. So by the Carrell-Peterson criterion [5, Theorem F], xB is rationally smooth in $X(w(\ell))$, as desired.

7.1.2 Proof of Lemma 7.3: The Even Case.

Consider the four segments $E_1(\ell), E_2(\ell), E_3(\ell), E_3(\ell-1)$ from Definition 5.18. For simplicity we write $E_i = E_i(\ell)(i = 1, 2, 3)$ and $E_4 = E_3(\ell-1)$. By Definition 5.18, the point $xq = u\alpha_1^\vee + v\alpha_2^\vee \in \triangle(\ell)$ is on $E_i \Leftrightarrow xq$ is on the corresponding line L_i.

$$L_1(\ell) : v - u = \frac{\ell}{6},\ L_2(\ell) : u = \frac{\ell}{6},\ L_3(\ell) : v = -\frac{\ell}{6} + \frac{1}{3},\ L_4 = L_3(\ell-1) : v = -\frac{\ell}{6} + \frac{2}{3}.$$

First we will show that if $xq \notin \triangle(\ell-3)$, then $q_x^w = 0$. To do this we will show that if xq is not in $\triangle(\ell-3)$, then xq is on one of the 4 line segments E_1, E_2, E_3, E_4 and anything on E_1, E_2, E_3, E_4 is a translation of one of the 4 elements $w = w(\ell)$, $w(\ell-1), w(\ell-2), s_2w(\ell-2)$. We will show that if x is one of these elements, then $q_x^w = 0$. By the translation theorem Theorem 6.6, if xq is not in $\triangle(\ell-3)$, then $q_x^w = 0$.

Suppose xq is not in $\triangle(\ell-3)$. Then at least one of inequalities $I_1(\ell-3), I_2(\ell-3), I_3(\ell-3)$ (Definition 5.15) does not hold. So $v - u \not\leq \frac{\ell-2}{6}$ or $u \not\leq \frac{\ell-2}{6}$ or $v \not\geq -\frac{\ell}{6} + 1$. Since $x \leq w$, we have $xq = u\alpha_1^\vee + v\alpha_2^\vee$ lies in the triangular region $\triangle(\ell)$ defined by $I_1(\ell), I_2(\ell)$, and $I_3(\ell)$, so

$$v - u \leq \frac{\ell}{6},\ u \leq \frac{\ell}{6},\ v \geq -\frac{\ell}{6} + \frac{1}{3}.$$

If $v - u \not\leq \frac{\ell-2}{6}$, then $\frac{\ell-2}{6} < v - u \leq \frac{\ell}{6}$. Since ℓ is even and $v - u$ is in $\mathbb{Z}$ or $\mathbb{Z} \pm \frac{1}{3}$ by Lemma 2.7, we see that $v - u = \frac{\ell}{6}$. (This can be proved by considering the cases $\ell = 6k, \ell = 6k+2$, or $\ell = 6k+4$ separately.) Hence xq lies on $E_1 : v - u = \frac{\ell}{6}$.

If $u \not\leq \frac{\ell-2}{6}$, then $\frac{\ell-2}{6} < u \leq \frac{\ell}{6}$. As u is in $\mathbb{Z}$ or $\mathbb{Z} \pm \frac{1}{3}$, the same argument shows $u = \frac{\ell}{6}$. Hence xq lies on $E_2 : u = \frac{\ell}{6}$.

If $v \not\geq -\frac{\ell}{6}+1$, then $-\frac{\ell}{6}+\frac{1}{3} \leq v < -\frac{\ell}{6}+1$. As v is in $\mathbb{Z}$ or $\mathbb{Z} \pm \frac{1}{3}$, a similar argument shows $v = -\frac{\ell}{6}+\frac{1}{3}$ or $v = -\frac{\ell}{6}+\frac{2}{3}$. Hence xq lies on $E_3 : v = -\frac{\ell}{6}+\frac{1}{3}$ or $E_4 : v = -\frac{\ell}{6}+\frac{2}{3}$. This shows that if $xq \notin \triangle(\ell) \setminus \triangle(\ell-3)$, then xq lies on E_1, E_2, E_3 or E_4.

We claim that any xq on E_1, E_2, E_3, E_4 is of the form $x = t(\nu^\vee)u$, where $u \in S = \{w(\ell), w(\ell-1), w(\ell-2), s_2 w(\ell-2)\}$. First observe that $w(\ell)$ and $w(\ell-1)$ lie on E_1, $w(\ell)$ and $s_2 w(\ell-2)$ lie on E_3, $w(\ell-1)$ and $w(\ell-2)$ lie on E_4, $s_1 w(\ell)$ and $s_1 w(\ell-1)$ lie on E_2. The reason is that

$$w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee$$

$$s_1 w(\ell)q = \frac{\ell}{6}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee$$

$$w(\ell-1)q = \frac{1-(\ell-1)}{3}\alpha_1^\vee + \frac{3-(\ell-1)}{6}\alpha_2^\vee = \frac{2-\ell}{3}\alpha_1^\vee + \frac{4-\ell}{6}\alpha_2^\vee$$

$$s_1 w(\ell-1)q = \frac{\ell}{6}\alpha_1^\vee + \frac{4-\ell}{6}\alpha_2^\vee$$

$$w(\ell-2)q = \frac{1-(\ell-2)}{3}\alpha_1^\vee + \frac{2-(\ell-2)}{6}\alpha_2^\vee = \frac{3-\ell}{3}\alpha_1^\vee + \frac{4-\ell}{6}\alpha_2^\vee$$

$$s_2 w(\ell-2)q = \frac{3-\ell}{3}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee.$$

Hence $w(\ell)$ lies on E_1 as $\frac{2-\ell}{6} - \frac{1-\ell}{3} = \frac{\ell}{6}$, $w(\ell-1)$ lies on E_1 as $\frac{4-\ell}{6} - \frac{2-\ell}{3} = \frac{\ell}{6}$. Similarly $w(\ell)$ and $s_2 w(\ell-2)$ lie on E_3, $w(\ell-1)$ and $w(\ell-2)$ lie on E_4, $s_1 w(\ell)$ and $s_1 w(\ell-1)$ lie on E_2.

Proposition 2.13 shows that if x and y lie on E_i and differ by a reflection, then any element is a translation of x or y. Therefore any element on $E_1 \cup E_2 \cup E_3 \cup E_4$ is a translation of one of the elements

$$\{w(\ell), w(\ell-1), w(\ell-2), s_2 w(\ell-2), s_1 w(\ell), s_1 w(\ell-1)\}.$$

Observe that $s_1 w(\ell-1)$ lies on E_3 and $s_1 w(\ell-1)$ lies on E_4, so these two elements are translations of the other 4 elements in the set. The claim is proved.

We claim that if u is in the set $S = \{w(\ell), w(\ell-1), w(\ell-2), s_2w(\ell-2)\}$, then $q^w_u = 0$. By Proposition 6.8, $q^{w(\ell)}_{w(\ell)} = q^w_{w(\ell-1)} = q^w_{w(\ell-2)} = 0$. Also we have $t(\widetilde{\alpha}^\vee)s_2w(\ell-2) = w(\ell-5)$ as $t(\widetilde{\alpha}^\vee)s_2w(\ell-2) = s_2s_2t(\widetilde{\alpha}^\vee)s_2w(\ell-2) = s_2t(\alpha_1^\vee)w(\ell-2) = s_2s_3s_2s_3s_1w(\ell-2) = s_3s_2s_1w(\ell-2) = w(\ell-5)$. So $q^w_{s_2w(\ell-2)} = q^w_{w(\ell-5)} = 0$ by Theorem 6.6 and Proposition 6.8. The claim is proved. Therefore if xq is not in $\triangle(\ell-3)$, then $q^w_x = 0$ by the translation theorem Theorem 6.6. This proves Lemma 7.3.

7.1.3 Proof of Lemma 7.3: The Odd Case.

Let $\ell = 2k+1$. First we claim that if xq is not in $\triangle(\ell-3)$, then xq is on one of $E_1(\ell), E_1(\ell-1), E_2(\ell), E_2(\ell-1), E_3(\ell)$. By Definition 5.18, to show this it is enough to show that xq is on one of $L_1(\ell), L_1(\ell-1), L_2(\ell), L_2(\ell-1), L_3(\ell)$. These lines are defined by the equations

$$L_1(\ell): v-u = \frac{\ell}{6}+\frac{1}{6},\ L_2(\ell): u = \frac{\ell}{6}+\frac{1}{6},\ L_3(\ell): v = -\frac{\ell}{6}+\frac{1}{2},$$

$$L_1(\ell-1): v-u = \frac{\ell-1}{6},\ L_2(\ell-1): u = \frac{\ell-1}{6}.$$

If xq is not in $\triangle(\ell-3)$, then at least one inequality $I_k(\ell)$ (Definition 5.15) does not hold. So

$$v-u \not\leq \frac{\ell-3}{6} \text{ or } u \not\leq \frac{\ell-3}{6} \text{ or } v \not\geq -\frac{\ell-3}{6}+\frac{1}{3}.$$

Since $x \leq w$, we have $xq = u\alpha_1^\vee + v\alpha_2^\vee$ lies in the triangular region $\triangle(\ell)$ defined by

$$v-u \leq \frac{\ell}{6}+\frac{1}{6},\ u \leq \frac{\ell}{6}+\frac{1}{6},\ v \geq -\frac{\ell}{6}+\frac{1}{3}+\frac{1}{6}.$$

If $v-u \not\leq \frac{\ell-3}{6}$, then $\frac{\ell-3}{6} < v-u \leq \frac{\ell}{6}+\frac{1}{6}$. Since $\ell = 2k+1$, $k = \frac{\ell-1}{2}$,

$$k-1 < 3(v-u) \leq k+1.$$

By Lemma 2.7, $3(v-u) \in \mathbb{Z}$. So either

$$3(v-u) = k \Rightarrow v-u = \frac{k}{3} = \frac{\ell-1}{6} = \frac{\ell}{6}-\frac{1}{6}$$

or

$$3(v-u) = k+1 \Rightarrow v-u = \frac{k}{3} + \frac{1}{3} = \frac{\ell-1}{6} + \frac{1}{3} = \frac{\ell}{6} + \frac{1}{6}.$$

Hence xq lies on $L_1(\ell) : v-u = \frac{\ell}{6} + \frac{1}{6}$ or $L_1(\ell-1) : v-u = \frac{\ell-1}{6}$. If $u \not\leq \frac{\ell-3}{6}$, then $\frac{\ell-3}{6} < u \leq \frac{\ell}{6} + \frac{1}{6}$. So

$$k-1 < 3u \leq k+1.$$

By Lemma 2.7, $3u \in \mathbb{Z}$. So either

$$3u = k \Rightarrow u = \frac{k}{3} = \frac{\ell-1}{6} = \frac{\ell}{6} - \frac{1}{6}$$

or

$$3u = k+1 \Rightarrow u = \frac{k}{3} + \frac{1}{3} = \frac{\ell-1}{6} + \frac{1}{3} = \frac{\ell}{6} + \frac{1}{6}.$$

Hence xq lies on $L_2(\ell) : u = \frac{\ell}{6} + \frac{1}{6}$ or $L_2(\ell-1) : u = \frac{\ell-1}{6}$. If $v \not\geq -\frac{\ell-3}{6} + \frac{1}{3}$, then $-\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6} \leq v < -\frac{\ell-3}{6} + \frac{1}{3}$. Then $\frac{-\ell+3}{2} \leq 3v < \frac{-\ell+5}{6}$. By Lemma 2.7, $3v \in \mathbb{Z}$, so $3v = \frac{-\ell+3}{2}, v = \frac{-\ell+3}{6}$. Hence xq lies on $L_3(\ell) : v = -\frac{\ell}{6} + \frac{1}{3} + \frac{1}{6}$. The claim is proved.

We show that if $xq \in E_1(\ell)$, then $q_x^w = 0$. By Proposition 2.13, every element x with $xq \in E_1(\ell)$ is of the form $t(i\widetilde{\alpha}^\vee)A_1(\ell)$ or $t(j\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}A_1(\ell)$ for some i, j. If $x = t(i\widetilde{\alpha}^\vee)A_1(\ell)$, then $x = t(i\widetilde{\alpha}^\vee)t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell) = t(i\widetilde{\alpha}^\vee + \frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell)$. By translation theorem Theorem 6.6 and coset theorem Theorem 6.9, $p_x^w = 0$, so by Corollary 6.17, $q_x^w = 0$. If $x = t(j\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}A_1(\ell)$, then $x \in R_{2\triangle}$. So $r_x^w = 1$. So by Corollary 6.17, $q_x^w = 0$.

We show that if $xq \in E_1(\ell-1)$, then $q_x^w = 0$. Since $w(\ell-1)q$ lies on $E_1(\ell-1)$, by Proposition 2.13, every element x with $xq \in E_1(\ell-1)$ is of the form $t(i\widetilde{\alpha}^\vee)w(\ell-1)$ or $t(j\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}w(\ell-1)$ for some i, j. If $x = t(i\widetilde{\alpha}^\vee)w(\ell-1)$, then $p_x^w = 0$ by translation theorem Theorem 6.6 and coset theorem Theorem 6.9,. So by Corollary 6.17, $q_x^w = 0$. If $x = t(j\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}w(\ell-1)$, then $x \in R_{2\triangle}$. So $r_x^w = 1$. So by Corollary 6.17, $q_x^w = 0$.

Since $s_1(E_1(\ell)) = E_2(\ell)$ and $s_1(E_1(\ell-1)) = E_2(\ell-1)$ and $q_x^w = q_{s_1x}^w$, we conclude $q_x^w = 0$ for xq on $E_2(\ell)$ and $q_x^w = 0$ for xq on $E_2(\ell-1)$.

We show that $q_x^w = 0$ for xq on $E_3(\ell)$. Since for xq on $E_3(\ell)$, $x = t(i\alpha_1)w(\ell)$ or $x = t(j\alpha_1)w(\ell-1)$ for some i, j, and $p^w_{w(\ell)} = p^w_{w(\ell-1)} = 0$ by Proposition 6.9, we have $p_x^w = 0$ for xq on $E_3(\ell)$. If xq is on $E_3(\ell)$, then $p_x^w = q_x^w$. So $q_x^w = 0$. This proves the Lemma.

7.2 Lookup conjecture for spiral Schubert varieties

The main purpose of this section is to prove the following theorem, which states that the lookup conjecture is true for spiral Schubert varieties $X(w(\ell))$.

THEOREM 7.4. Let $w(\ell)$ be spiral element. Then xB is not rationally smooth in $X(w(\ell))$ if and only if either $q_x^w \neq 0$ or $q_y^w \neq 0$ for $y = rx, r \in R, x < y \leq w$, where R is the set of reflections. In other words, the lookup conjecture holds for $X(w(\ell))$.

The implication ($\Leftarrow$) follows immediately by the Carrell-Peterson criterion [5, Theorem F]. So we need to show ($\Rightarrow$).

7.2.1 PROOF OF THEOREM 7.4: THE EVEN CASE

By Theorem 7.1, $x \leq w(\ell)$ is n.r.s. $\Leftrightarrow$ x is in $\triangle(\ell-3)$. So we must show: if $x \in \triangle(\ell-3)$, then either $q_x^w \neq 0$ or $q_y^w \neq 0$ for $y = xr$, $x < y \leq w$.

Suppose xq is in $\triangle(\ell-3)$. By Proposition 5.7, any $xq \in \triangle(\ell-3)$ is of the form $x = t(\gamma^\vee)u$ where $u = w(\ell-i)$, some $i \in \{3,4,5,6,7,8\}$. By Proposition 6.8,

$$q^w_{w(\ell-3)} = q^w_{w(\ell-4)} = 1,\ q^w_{w(\ell-5)} = q^w_{w(\ell-6)} = q^w_{w(\ell-7)} = q^w_{w(\ell-8)} = 0.$$

If $x = t(\gamma^\vee)w(\ell-3)$ or $x = t(\gamma^\vee)w(\ell-4)$, then $q_x^w = q^w_{w(\ell-3)} = 1$ or $q_x^w = q^w_{w(\ell-4)} = 1$ by translation Theorem 6.6. This shows that the lookup conjecture is true for $x = t(\gamma^\vee)w(\ell-3)$ and $x = t(\gamma^\vee)w(\ell-4)$.

Let $x = t(\gamma^\vee)w(\ell-5)$. Suppose $k < (\widetilde{\alpha}^\vee, xq) < k+1$. Since $xq = t(\gamma^\vee)w(\ell-5)q = w(\ell-5)q + \gamma^\vee$,

$$(\widetilde{\alpha}, xq) = (\widetilde{\alpha}, \gamma^\vee) + (\widetilde{\alpha}, w(\ell-5)q).$$

By Lemma 5.6,

$$w(\ell-5)q = \frac{1-(\ell-5)}{3}\alpha_1 + \frac{3-(\ell-5)}{6}\alpha_2 = (2-\frac{\ell}{3})\alpha_1^\vee + (\frac{4}{3}-\frac{\ell}{6})\alpha_2^\vee.$$

Then since $(\widetilde{\alpha}, \alpha_1^\vee) = (\widetilde{\alpha}, \alpha_2^\vee) = 1$, we have

$$(\widetilde{\alpha}, w(\ell-5)q) = (2-\frac{\ell}{3})(1) + (\frac{4}{3}-\frac{\ell}{6})(1) = 3\frac{1}{3} - \frac{\ell}{2} \tag{7.5}$$

At least one of $s_{\widetilde{\alpha},k}x$ and $s_{\widetilde{\alpha},k+1}x$ is greater than x. We will show that at least one of these is a lookup direction from x. We show that both $s_{\widetilde{\alpha},k}x$ and $s_{\widetilde{\alpha},k+1}x$ are $\leq w$. Let $\gamma^\vee = A\alpha_1^\vee + B\alpha_2^\vee$. Then

$$xq = t(\gamma^\vee)w(\ell-5)q = (A+2-\frac{\ell}{3})\alpha_1^\vee + (B+\frac{4}{3}-\frac{\ell}{6})\alpha_2^\vee.$$

By Lemma 2.9, and using $\widetilde{\alpha}^\vee = \alpha_1^\vee + \alpha_2^\vee$, we have

$$s_{\widetilde{\alpha},k}xq = xq - \frac{1}{3}\widetilde{\alpha} = (A+\frac{5}{3}-\frac{\ell}{3})\alpha_1^\vee + (B+1-\frac{\ell}{6})\alpha_2^\vee. \tag{7.6}$$

Again by Lemma 2.9,

$$s_{\widetilde{\alpha},k+1}xq = xq + \frac{2}{3}\widetilde{\alpha} = (A+\frac{8}{3}-\frac{\ell}{3})\alpha_1^\vee + (B+2-\frac{\ell}{6})\alpha_2^\vee. \tag{7.7}$$

Let $\ell = 2m$. By hypothesis, $xq \in \triangle(\ell-3)$, the inequalities which say that xq is in $\triangle(\ell-3)$ are

$$(B+\frac{4}{3}-\frac{\ell}{6}) - (A+2-\frac{\ell}{3}) \leq \frac{\ell-3}{6} + \frac{1}{6}$$

$$(A+2-\frac{\ell}{3}) \leq \frac{\ell-3}{6} + \frac{1}{6}$$

$$(B+\frac{4}{3}-\frac{\ell}{6}) \geq -\frac{\ell-3}{6} + \frac{1}{3} + \frac{1}{6}.$$

These simplify to

$$B - A \leq \frac{1}{3},\ A \leq \frac{\ell}{2} - \frac{7}{3},\ B \geq -\frac{1}{3}. \tag{7.8}$$

Then $A \leq m - \frac{7}{3}$. Since $A, m \in \mathbb{Z}$, $A \leq m - 3$. Then $A \leq m - \frac{8}{3}$. To show $s_{\widetilde{\alpha},k}x \leq w$, we need

$$(B + 1 - \frac{\ell}{6}) - (A + \frac{5}{3} - \frac{\ell}{3}) \leq \frac{\ell}{6}$$

$$A + \frac{5}{3} - \frac{\ell}{3} \leq \frac{\ell}{6}$$

$$B + 1 - \frac{\ell}{6} \geq -\frac{\ell}{6} + \frac{1}{3}.$$

These inequalities simplify to

$$B - A \leq \frac{2}{3},\ A \leq \frac{\ell}{2} - \frac{5}{3},\ B \geq -\frac{2}{3}$$

which follow from (7.8). Hence $s_{\widetilde{\alpha},k}x \leq w$ by Theorem 5.41. Also to show $s_{\widetilde{\alpha},k+1}x \leq w$, we need

$$(B + 2 - \frac{\ell}{6}) - (A + \frac{8}{3} - \frac{\ell}{3}) \leq \frac{\ell}{6}$$

$$A + \frac{8}{3} - \frac{\ell}{3} \leq \frac{\ell}{6}$$

$$B + 2 - \frac{\ell}{6} \geq -\frac{\ell}{6} + \frac{1}{3}.$$

These inequalities simplify to

$$B - A \leq \frac{2}{3},\ A \leq \frac{\ell}{2} - \frac{8}{3},\ B \geq -\frac{5}{3}$$

which follow from (7.8). Hence $s_{\widetilde{\alpha},k+1}x \leq w$ by Theorem 5.41.

Because $s_{\widetilde{\alpha},k+1}x = t(\widetilde{\alpha})s_{\widetilde{\alpha},k}x$, by the translation Theorem 6.6, $q^w_{s_{\widetilde{\alpha},k}x} = q^w_{s_{\widetilde{\alpha},k+1}x}$. To show that one is a lookup direction, enough to show $q^w_{s_{\widetilde{\alpha},k}x} = 1$. Let $a = (\widetilde{\alpha}, \gamma^\vee)$. We will show that $s_{\widetilde{\alpha},k}x = t(s_{\widetilde{\alpha},0}(\gamma^\vee) + a\widetilde{\alpha})w(\ell - 4)$. Then the translation theorem will imply that $q^w_{s_{\widetilde{\alpha},k}x} = q^w_{w(\ell-4)} = 1$ as desired.

First consider the case $\gamma^\vee = 0$, then $x = w(\ell-5)$. Then $(\widetilde{\alpha}, xq) = 3\frac{1}{3} - \frac{\ell}{2}$ by (7.5) and $k = 3 - \frac{\ell}{2}$. By Lemma 2.9, we have

$$s_{\widetilde{\alpha},3-\frac{\ell}{2}} w(\ell-5)q = w(\ell-5)q - \frac{1}{3}\widetilde{\alpha}.$$

On the other hand by Lemma 5.6,

$$\begin{aligned} w(\ell-4)q &= \frac{1-(\ell-4)}{3}\alpha_1 + \frac{2-(\ell-4)}{6}\alpha_2 \\ &= (\frac{5}{3} - \frac{\ell}{3})\alpha_1^\vee + (\frac{4}{3} - \frac{\ell}{6})\alpha_2^\vee \\ &= w(\ell-5)q - \frac{1}{3}\widetilde{\alpha}. \end{aligned}$$

Therefore,

$$s_{\widetilde{\alpha},3-\frac{\ell}{2}} w(\ell-5) = w(\ell-4).$$

Hence $q^w_{s_{\widetilde{\alpha},3-\frac{\ell}{2}}x} = q^w_{w(\ell-4)} = 1$ by the translation Theorem 6.6.

In general, $k = (\widetilde{\alpha}, \gamma^\vee) + 3 - \frac{\ell}{2} = a + 3 - \frac{\ell}{2}$. We have

$$s_{\widetilde{\alpha},k} = t(k\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0} = s_{\widetilde{\alpha},0}s_{\widetilde{\alpha},0}t(k\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}^{-1} = s_{\widetilde{\alpha},0}t(-k\widetilde{\alpha}^\vee)$$

$$\begin{aligned} s_{\widetilde{\alpha},k}t(\gamma^\vee) &= s_{\widetilde{\alpha},k}t(\gamma^\vee)s_{\widetilde{\alpha},k}^{-1}s_{\widetilde{\alpha},k} \\ &= s_{\widetilde{\alpha},0}t(-k\widetilde{\alpha}^\vee)t(\gamma^\vee)t(k\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},0}s_{\widetilde{\alpha},k} \\ &= s_{\widetilde{\alpha},0}t(\gamma^\vee)s_{\widetilde{\alpha},0}^{-1}s_{\widetilde{\alpha},k} \\ &= t(s_{\widetilde{\alpha},0}(\gamma^\vee))s_{\widetilde{\alpha},k} \\ &= t(s_{\widetilde{\alpha},0}(\gamma^\vee))s_{\widetilde{\alpha},a+3-\frac{\ell}{2}} \\ &= t(s_{\widetilde{\alpha},0}(\gamma^\vee))t(a\widetilde{\alpha})s_{\widetilde{\alpha},3-\frac{\ell}{2}} \\ &= t(s_{\widetilde{\alpha},0}(\gamma^\vee) + a\widetilde{\alpha})s_{\widetilde{\alpha},3-\frac{\ell}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} s_{\widetilde{\alpha},k}x &= s_{\widetilde{\alpha},k}t(\gamma^\vee)w(\ell-5) \\ &= t(s_{\widetilde{\alpha},0}(\gamma^\vee)+a\widetilde{\alpha})s_{\widetilde{\alpha},3-\frac{\ell}{2}}w(\ell-5) \\ &= t(s_{\widetilde{\alpha},0}(\gamma^\vee)+a\widetilde{\alpha})w(\ell-4). \end{aligned}$$

Hence $q^w_{s_{\widetilde{\alpha},k}x} = q^w_{w(\ell-4)} = 1$ by the translation Theorem 6.6. This shows that the lookup conjecture is true for $t(\gamma^\vee)w(\ell-5)$.

Suppose $w(\ell)$ ends with s_a, in other words, $w(\ell)s_a < w(\ell)$. To show the lookup conjecture holds for $t(\gamma^\vee)w(\ell-8)$, it suffices to show that $t(\alpha_1^\vee)w(\ell-5)s_a = w(\ell-8)$. The reason is that then $t(\gamma^\vee)w(\ell-8) = t(\gamma^\vee+\alpha_1^\vee)w(\ell-5)s_a$, and since the lookup conjecture holds for $t(\gamma^\vee+\alpha_1^\vee)w(\ell-5)$, by Proposition 4.10 it holds for $t(\gamma^\vee+\alpha_1^\vee)w(\ell-5)s_a$. So we must prove the following claim.

Claim: $t(\alpha_1^\vee)w(\ell-5)s_a = w(\ell-8)$.

In the proof, we use that $w(6n) = t(n(-2\alpha_1^\vee-\alpha_2^\vee))$ to calculate.

Case $\ell = 6n$. Observe $w(\ell-5)s_3 = w(\ell-6)s_1s_3$. By Lemma 2.3, $s_1s_3(\frac{1}{3}\widetilde{\alpha}) = s_1(\frac{1}{3}\alpha_1+\frac{1}{3}\alpha_2) = s_1((-\frac{1}{3}+1)\alpha_1+(-\frac{1}{3}+1)\alpha_2) = s_1(\frac{2}{3}\widetilde{\alpha}) = \frac{2}{3}\alpha_2$.

$$\begin{aligned} t(\alpha_1)w(\ell-5)s_3q &= t(\alpha_1)w(\ell-6)s_1s_3(\frac{1}{3}\widetilde{\alpha}) \\ &= t(\alpha_1)t((n-1)(-2\alpha_1^\vee-\alpha_2^\vee))(\frac{2}{3}\alpha_2^\vee) \\ &= t(\alpha_1+(n-1)(-2\alpha_1^\vee-\alpha_2^\vee))(\frac{2}{3}\alpha_2^\vee) \\ &= (-2n+3)\alpha_1^\vee+(-n+\frac{5}{3})\alpha_2^\vee \end{aligned}$$

Also, $w(\ell-8) = w(\ell-6)s_3s_2$, so

$$\begin{aligned}
w(\ell-8)q &= w(\ell-6)s_3s_2(\frac{1}{3}\widetilde{\alpha}) \\
&= t((n-1)(-2\alpha_1^\vee - \alpha_2^\vee))s_3(\frac{1}{3}\alpha_1^\vee) \\
&= t((n-1)(-2\alpha_1^\vee - \alpha_2^\vee))(\alpha_1 + \frac{2}{3}\alpha_2) \\
&= (-2n+2+1)\alpha_1^\vee + (-n+1+\frac{2}{3})\alpha_2^\vee \\
&= (-2n+3)\alpha_1^\vee + (-n+\frac{5}{3})\alpha_2^\vee.
\end{aligned}$$

Hence $t(\alpha_1)w(\ell-5)s_3 = w(\ell-8)$.

Case $\ell = 6n+2$. We show that $t(\alpha_1)w(\ell-5)s_2 = w(\ell-8)$. First observe that by Lemma 2.3, $s_1s_2s_3s_2(\frac{1}{3}\widetilde{\alpha}) = s_1s_2s_3(\frac{1}{3}\alpha_1) = s_1s_2(\alpha_1 + \frac{2}{3}\alpha_2) = s_1(\alpha_1 + \alpha_2 - \frac{2}{3}\alpha_2) = s_1(\alpha_1 + \frac{1}{3}\alpha_2) = -\alpha_1 + \frac{1}{3}(\alpha_1+\alpha_2) = -\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$. Therefore,

$$\begin{aligned}
t(\alpha_1)w(\ell-5)s_2q &= t(\alpha_1)w(\ell-8)s_1s_2s_3s_2(\frac{1}{3}\widetilde{\alpha}) \\
&= t(\alpha_1)t((n-1)(-2\alpha_1^\vee - \alpha_2^\vee))(-\frac{2}{3}\alpha_1^\vee + \frac{1}{3}\alpha_2^\vee) \\
&= [1-2n+2-\frac{2}{3}]\alpha_1^\vee + [-2n+1+\frac{1}{3}]\alpha_2^\vee \\
&= [-2n+\frac{7}{3}]\alpha_1^\vee + [-2n+\frac{4}{3}]\alpha_2^\vee
\end{aligned}$$

$$\begin{aligned}
w(\ell-8)q &= w(6(n-1))(\frac{1}{3}\widetilde{\alpha}) \\
&= t((n-1)(-2\alpha_1^\vee - \alpha_2^\vee))(\frac{1}{3}\widetilde{\alpha}) \\
&= \frac{1}{3}\alpha_1 + \frac{1}{3}\alpha_2 + (n-1)(-2\alpha_1^\vee - \alpha_2^\vee) \\
&= [-2n+\frac{7}{3}]\alpha_1^\vee + [-2n+\frac{4}{3}]\alpha_2^\vee.
\end{aligned}$$

Hence $t(\alpha_1)w(\ell-5)s_2 = w(\ell-8)$.

Case $\ell = 6n+4$. We show that $t(\alpha_1)w(\ell-5)s_1 = w(\ell-8)$. Observe that by Lemma 2.3, $s_1s_2s_3s_1s_2s_1(\frac{1}{3}\widetilde{\alpha}) = s_1s_2s_3(-\frac{1}{3}\widetilde{\alpha}) = s_1s_2(\frac{4}{3}\widetilde{\alpha}) = s_1(\frac{4}{3}\alpha_1) = -\frac{4}{3}\alpha_1$. Therefore,

$$\begin{aligned} t(\alpha_1)w(\ell-5)s_1q &= t(\alpha_1)w(\ell-10)s_1s_2s_3s_1s_2s_1(\frac{1}{3}\widetilde{\alpha}) \\ &= t(\alpha_1)w(\ell-10)(-\frac{4}{3}\alpha_1) \\ &= t(\alpha_1+(n-1)(-2\alpha_1^\vee-\alpha_2^\vee))(-\frac{4}{3}\alpha_1^\vee) \\ &= [1-2n+2-\frac{4}{3}]\alpha_1^\vee+[-n+1]\alpha_2^\vee \\ &= [-2n+\frac{5}{3}]\alpha_1^\vee+[-n+1]\alpha_2^\vee \end{aligned}$$

$$\begin{aligned} w(\ell-8)q &= w(\ell-10)s_1s_2(\frac{1}{3}\widetilde{\alpha}) \\ &= w(6(n-1))(-\frac{1}{3}\alpha_1) \\ &= t((n-1)(-2\alpha_1^\vee-\alpha_2^\vee))(-\frac{1}{3}\alpha_1) \\ &= [-2n+\frac{5}{3}]\alpha_1^\vee+[-n+1]\alpha_2^\vee. \end{aligned}$$

Hence $t(\alpha_1)w(\ell-5)s_1 = w(\ell-8)$. The claim is proved. This shows that the lookup conjecture is true for $t(\gamma^\vee)w(\ell-8)$.

Now we will prove the conjecture for $x = t(\gamma^\vee)w(\ell-6)$. Let $x = t(\gamma^\vee)w(\ell-6)$. Suppose $k < (\alpha_2, xq) < k+1$. Since $xq = t(\gamma^\vee)w(\ell-6)q = w(\ell-6)q+\gamma^\vee$, and $\ell-6$ is even, by Lemma 2.9,

$$(\alpha_2, xq) = (\alpha_2, \gamma^\vee) + (\alpha_2, w(\ell-6)q) = (\alpha_2, \gamma^\vee) + \frac{1}{3}.$$

Hence $k = (\alpha_2, \gamma^\vee)$. At least one of $s_{\alpha_2,k}x$ and $s_{\alpha_2,k+1}x$ is greater than x. We will show that at least one of these is a lookup direction from x. First, we show that both $s_{\alpha_2,k}x$ and $s_{\alpha_2,k+1}x$ are $\leq w$. Let $\gamma^\vee = A\alpha_1^\vee + B\alpha_2^\vee$. Then

$$xq = t(\gamma^\vee)w(\ell-6)q = (A+\frac{7}{3}-\frac{\ell}{3})\alpha_1^\vee + (B+\frac{4}{3}-\frac{\ell}{6})\alpha_2^\vee.$$

By Lemma 2.9,

$$s_{\alpha_2,k}xq = xq - \frac{1}{3}\alpha_2^\vee = (A+\frac{7}{3}-\frac{\ell}{3})\alpha_1^\vee + (B+1-\frac{\ell}{6})\alpha_2^\vee. \tag{7.9}$$

By Lemma 2.9,

$$s_{\alpha_2,k+1}xq = xq + \frac{2}{3}\alpha_2^\vee = (A + \frac{7}{3} - \frac{\ell}{3})\alpha_1^\vee + (B + 2 - \frac{\ell}{6})\alpha_2^\vee. \tag{7.10}$$

Since xq is in $\triangle(\ell - 3)$,

$$(B + \frac{4}{3} - \frac{\ell}{6}) - (A + \frac{7}{3} - \frac{\ell}{3}) \leq \frac{\ell - 3}{6} + \frac{1}{6}$$

$$(A + \frac{7}{3} - \frac{\ell}{3}) \leq \frac{\ell - 3}{6} + \frac{1}{6}$$

$$(B + \frac{4}{3} - \frac{\ell}{6}) \geq -\frac{\ell - 3}{6} + \frac{1}{3} + \frac{1}{6}.$$

These inequalities simplify to

$$B - A \leq \frac{2}{3}, A \leq \frac{\ell}{2} - \frac{8}{3}, B \geq -\frac{1}{3}. \tag{7.11}$$

Since $A, B \in \mathbb{Z}$, $B - A \leq 0$. To show $s_{\alpha_2,k}x \leq w$, we need

$$(B + 1 - \frac{\ell}{6}) - (A + \frac{7}{3} - \frac{\ell}{3}) \leq \frac{\ell}{6}$$

$$(A + \frac{7}{3} - \frac{\ell}{3}) \leq \frac{\ell}{6}$$

$$B + 1 - \frac{\ell}{6} \geq -\frac{\ell}{6} + \frac{1}{3}.$$

These simplify to

$$B - A \leq \frac{4}{3}, A \leq \frac{\ell}{2} - \frac{7}{3}, B \geq -\frac{2}{3}$$

which follow from (7.11). Hence $s_{\alpha_2,k}x \leq w$ by Theorem 5.41. Also to show $s_{\alpha_2,k+1}x \leq w$, we need

$$(B + 2 - \frac{\ell}{6}) - (A + \frac{7}{3} - \frac{\ell}{3}) \leq \frac{\ell}{6}$$

$$A + \frac{7}{3} - \frac{\ell}{3} \leq \frac{\ell}{6}$$

$$B + 2 - \frac{\ell}{6} \geq -\frac{\ell}{6} + \frac{1}{3}.$$

These simplify to

$$B - A \leq \frac{1}{3}, A \leq \frac{\ell}{2} - \frac{7}{3}, B \geq -\frac{5}{3}$$

which follow from (7.11). Hence $s_{\alpha_2,k+1}x \leq w$ by Theorem 5.41.

Next, observe that by the translation Theorem 6.6, $q^w_{s_{\alpha_2,k}x} = q^w_{s_{\alpha_2,k+1}x}$, because $s_{\alpha_2,k+1}x = t(\alpha_2)s_{\alpha_2,k}x$. To show that one is a lookup direction, it is enough to show $q^w_{s_{\alpha_2,k}x} = 1$. First consider the case $\gamma^\vee = 0$, then $x = w(\ell - 6)$, $k = 0$, so $s_{\alpha_2,k} = s_2$.

Then we claim: $s_2 x = t(\alpha_1^\vee)w(\ell - 3)$. By Lemma 5.6,

$$\begin{aligned} s_2 x q &= s_2 w(\ell - 6)q \\ &= s_2((\frac{7}{3} - \frac{\ell}{3})\alpha_1^\vee + (\frac{4}{3} - \frac{\ell}{6})\alpha_2^\vee) \\ &= (\frac{7}{3} - \frac{\ell}{3})\widetilde{\alpha}^\vee - (\frac{4}{3} - \frac{\ell}{6})\alpha_2^\vee \\ &= (\frac{7}{3} - \frac{\ell}{3})\alpha_1^\vee + (1 - \frac{\ell}{6})\alpha_2^\vee. \end{aligned}$$

By Lemma 5.6,

$$\begin{aligned} t(\alpha_1^\vee)w(\ell - 3)q &= t(\alpha_1^\vee)(\frac{1 - (\ell - 3)}{3}\alpha_1^\vee + \frac{3 - (\ell - 3)}{6}\alpha_2^\vee) \\ &= (\frac{4}{3} - \frac{\ell}{3})\alpha_1^\vee + (1 - \frac{\ell}{6})\alpha_2^\vee + \alpha_1^\vee \\ &= (\frac{7}{3} - \frac{\ell}{3})\alpha_1^\vee + (1 - \frac{\ell}{6})\alpha_2^\vee. \end{aligned}$$

Thus $s_2 x = t(\alpha_1^\vee)w(\ell - 3)$. The claim is proved. Therefore by the translation Theorem 6.6, $q^w_{s_2 w(\ell-6)} = q^w_{w(\ell-3)} = 1$.

In general, $x = t(\gamma^\vee)w(\ell - 6)$. Since $(\alpha_2, \gamma^\vee) = k$, by Lemma 2.4, $s_{\alpha_2,k}t(\gamma^\vee) = t(\gamma^\vee)s_2$. Hence

$$s_{\alpha_2,k}x = s_{\alpha_2,k}t(\gamma^\vee)w(\ell - 6) = t(\gamma^\vee)s_2 w(\ell - 6).$$

Therefore,

$$q^w_{s_{\alpha_2,k}x} = q^w_{t(\gamma^\vee)s_2 w(\ell-6)} = q^w_{s_2 w(\ell-6)} = 1.$$

This shows that the lookup conjecture is true for $x = t(\gamma^\vee)w(\ell - 6)$.

Because if $w(\ell)$ ends with s_a, $ws_a < w$, then $w(\ell-7) = w(\ell-6)s_a$. By Proposition 4.10, lookup conjecture holds for $x = t(\gamma^\vee)w(\ell-7)$.

7.2.2 PROOF OF THEOREM 7.4: THE ODD CASE

Suppose xB is not rationally smooth, we must show the lookup conjecture holds for $X(w(\ell))$. Because xB is not rationally smooth, $xq \in \triangle(\ell-3)$ by Theorem 7.1. We assume this for the rest of the proof.

By Proposition 5.7, x is of the form $x = t(\gamma^\vee)u$ where $u = w(\ell-i)$, some $i \in \{0,1,2,3,4,5\}$. Since p_x^w satisfies the translation property, by Proposition 6.9,

$$p^w_{w(\ell-3)} = p^w_{w(\ell-4)} = p^w_{w(\ell-5)} = 1$$

$$p^w_{w(\ell-6)} = p^{w(\ell)}_{w(\ell)} = 0,\ p^w_{w(\ell-7)} = p^w_{w(\ell-1)} = 0,\ p^w_{w(\ell-8)} = p^w_{w(\ell-2)} = 1.$$

We have $q_x^w = p_x^w$ for $xq \in \triangle(\ell-3)$. If $x = t(\gamma^\vee)w(\ell-i)$ for $i \in \{3,4,5,8\}$, then since $q_x^w = p_x^w$ for $xq \in \triangle(\ell-3)$, by the translation Theorem 6.6, $q_x^w = p_x^w = p^w_{w(\ell-i)} = 1$. This shows that the lookup conjecture is true for $x = t(\gamma^\vee)w(\ell-i)$ for $i \in \{3,4,5,8\}$.

(1) We show that if x is on the outside edges of small triangle $\triangle(\ell-3)$, $q_x^w = 1$. Since $x \in \triangle(\ell-3)$, $q_x^w = p_x^w$, so we will show $p_x^w = 1$. First we show $p_x^w = 1$ for xq on $E_1(\ell-3)$. We have $w(\ell-3)q$ is on $E_1(\ell-3) : v-u = \frac{\ell-3}{6}$ because

$$w(\ell-3)q = \frac{1-(\ell-3)}{3}\alpha_1 + \frac{2-(\ell-3)}{6}\alpha_2 = (\frac{4}{3}-\frac{\ell}{3})\alpha_1^\vee + (\frac{5}{6}-\frac{\ell}{6})\alpha_2^\vee$$

with $u = \frac{4}{3}-\frac{\ell}{3}, v = \frac{5}{6}-\frac{\ell}{6}$. Also, $w(\ell-4)q$ is on $E_1(\ell-3) : v-u = \frac{\ell-3}{6}$ because

$$w(\ell-4)q = \frac{1-(\ell-4)}{3}\alpha_1 + \frac{3-(\ell-4)}{6}\alpha_2 = (\frac{5}{3}-\frac{\ell}{3})\alpha_1^\vee + (\frac{7}{6}-\frac{\ell}{6})\alpha_2^\vee$$

with $u = \frac{5}{3}-\frac{\ell}{3}, v = \frac{7}{6}-\frac{\ell}{6}$. Since $w(\ell-3)q$ and $w(\ell-4)q$ are on $E_1(\ell-3)$ and differ by a reflection, so any xq on $E_1(\ell-3)$ is either $x = t(n\widetilde{\alpha}^\vee)w(\ell-3)$ or $x = t(m\widetilde{\alpha}^\vee)w(\ell-4)$ by Proposition 2.13. Since $p^w_{w(\ell-3)} = p^w_{w(\ell-4)} = 1$, by translation theorem $q_x^w = p_x^w = 1$ for any x on $E_1(\ell-3)$.

Next show that $p_x^w = 1$ for xq on $E_2(\ell-3)$. By Lemma 5.21, $s_1(\triangle(\ell-3)) = \triangle(\ell-3), s_1(E_1(\ell-3)) = E_2(\ell-3)$. Then by Lemma 5.21, $p_x^w = p_{s_1 x}^w$. Therefore $p_x^w = 1$ for xq on $E_2(\ell-3)$.

Finally we show $p_x^w = 1$ for xq on $E_3(\ell-3)$. To apply Proposition 2.13 for $E_3(\ell-3)$, we show that $w(\ell-2)q, w(\ell-3)q$ are on $E_3(\ell-3) : v = -\frac{\ell-3}{6} + \frac{1}{3}$. By Lemma 5.6,

$$w(\ell-2)q = \frac{1-(\ell-2)}{3}\alpha_1 + \frac{3-(\ell-2)}{6}\alpha_2 = (1-\frac{\ell}{3})\alpha_1^\vee + (\frac{5}{6}-\frac{\ell}{6})\alpha_2^\vee$$

$$w(\ell-3)q = \frac{1-(\ell-3)}{3}\alpha_1 + \frac{2-(\ell-3)}{6}\alpha_2 = (\frac{4}{3}-\frac{\ell}{3})\alpha_1^\vee + (\frac{5}{6}-\frac{\ell}{6})\alpha_2^\vee.$$

Then $v = \frac{5}{6} - \frac{\ell}{6}$. Thus $w(\ell-2)q, w(\ell-3)q$ are on $E_3(\ell-3) : v = -\frac{\ell-3}{6} + \frac{1}{3}$. Since $w(\ell-2)q$ and $w(\ell-3)q$ are on $E_1(\ell-3)$ and differ by a reflection, so any xq on $E_1(\ell-3)$ is either $x = t(n\widetilde{\alpha}^\vee)w(\ell-2)$ or $x = t(m\widetilde{\alpha}^\vee)w(\ell-3)$ by Proposition 2.13. Since $q^w_{t(\alpha_1^\vee)w(\ell-2)} = q^w_{w(\ell-3)} = 1$, by translation theorem $q_x^w = p_x^w = 1$ for any x on $E_1(\ell-3)$.

(2) Suppose x is in the small triangle $\triangle(\ell-3)$, but not on the outside edges of small triangle $\triangle(\ell-3)$: If $q_x^w \neq 0$, then we are done. If $q_x^w = 0$, then $x = t(\gamma^\vee)w(\ell-6)$ or $x = t(\gamma^\vee)w(\ell-7)$. We will find y so that $q_y^w = 1$, $x < y = xs \leq w$. The y we find will be in $\triangle(\ell-3)$, so $p_y^w = q_y^w$. So we show $p_y^w = 1$.

Step 1. We locate a direction where $p_y^w = 1$. Suppose $x = t(\gamma^\vee)w(\ell-6)$. Suppose $k < (\alpha_2, xq) < k+1$. Since $xq = t(\gamma^\vee)w(\ell-6)q = w(\ell-6)q + \gamma^\vee$, and $\ell-6$ is odd, by Lemma 2.9,

$$(\alpha_2, xq) = (\alpha_2, \gamma^\vee) + (\alpha_2, w(\ell-6)q) = (\alpha_2, \gamma^\vee) + \frac{1}{3}.$$

Hence $k = (\alpha_2, \gamma^\vee)$. At least one of $s_{\alpha_2,k}x$ and $s_{\alpha_2,k+1}x$ is greater than x. We will show that at least one of these is a lookup direction from x.

We are assuming xq is not on the edge of the small triangle $\triangle(\ell-3)$, so $xq \notin E_1(\ell-3), E_2(\ell-3), E_3(\ell-3)$. We show that both $s_{\alpha_2,k}x$ and $s_{\alpha_2,k+1}x$ stay in the small triangle $\triangle(\ell-3)$. So $s_{\alpha_2,k+1}xq = xq + \frac{1}{3}\alpha_2^\vee$, $s_{\alpha_2,k}xq = xq - \frac{2}{3}\alpha_2^\vee$ by Lemma 2.9.

(a) We show $s_{\alpha_2,k+1}xq = xq + \frac{1}{3}\alpha_2^\vee = u\alpha_1^\vee + (v+\frac{1}{3})\alpha_2^\vee \in \triangle(\ell-3)$. Since $xq = u\alpha_1^\vee + v\alpha_2^\vee$ is not on $E_1(\ell-3)$, so there is a strict inequality $v-u < \frac{\ell-3}{6}$. Then $3v-3u < \frac{\ell-3}{2} \in \mathbb{Z}$, so

$$3v - 3u \leq \frac{\ell-3}{2} - 1$$

$$v - u \leq \frac{\ell-3}{6} - \frac{1}{3}$$

so $(v+\frac{1}{3}) - u \leq \frac{\ell-3}{6}$. Hence $s_{\alpha_2,k+1}x$ satisfies $I_1(\ell-3)$. The u value is same for xq and $s_{\alpha_2,k+1}xq$. Since xq satisfies $I_3(\ell-3)$, so $u \leq \frac{\ell-3}{6}$, so does $s_{\alpha_2,k+1}x$. Also $v+\frac{1}{3} \geq -\frac{\ell-3}{6} + \frac{1}{3}$, since $v \geq \frac{\ell-3}{6}$ (as $xq \in \triangle(\ell-3)$), so $s_{\alpha_2,k+1}x$ satisfies $I_3(\ell-3)$. Therefore $s_{\alpha_2,k+1}xq \in \triangle(\ell-3)$.

(b) We show $s_{\alpha_2,k}xq \in \triangle(\ell-3)$. Suppose $x = t(\gamma^\vee)w(\ell-6)$, $xq = u\alpha_2^\vee + v\alpha_2^\vee \in \triangle(\ell-3)$. We have $s_{\alpha_2,k}xq = xq - \frac{2}{3}\alpha_2^\vee = u\alpha_1^\vee + (v-\frac{2}{3})\alpha_2^\vee = u'\alpha_1^\vee + v'\alpha_2^\vee$ where $u' = u, v' = v - \frac{2}{3}$. Our assumption implies that $xq \notin L_3(\ell-3)$. Because $w(\ell-4), w(\ell-5)$ are reflection of each other, any x on the edge $E_3(\ell-4) = E_3(\ell-5)$ is a translation of either $w(\ell-4)$ or $w(\ell-5)$, so $x = t(\gamma^\vee)w(\ell-6)$ can not be on the edge $E_3(\ell-4)$. Therefore, xq satisfies $I_3(\ell-6)$. Since ℓ is odd, $v \geq -\frac{\ell-6}{6} + \frac{1}{3} + \frac{1}{6} = -\frac{\ell}{6} + \frac{3}{2}$, so $v - \frac{2}{3} \geq -\frac{\ell}{6} + \frac{5}{6}$. Thus $s_{\alpha_2,k}xq = u\alpha_2^\vee + (v-\frac{2}{3})\alpha_2^\vee$ satisfies $I_3(\ell-3) : v \geq -\frac{\ell-3}{6} + \frac{1}{3} = -\frac{\ell}{6} + \frac{5}{6}$, as $v' = v - \frac{2}{3} \geq -\frac{\ell-3}{6} + \frac{1}{3} = \frac{-\ell+3+2}{6} = \frac{-\ell+5}{6}$, (note that $\ell-3$ is even). Since $v - u \leq -\frac{\ell-3}{6}$, thus $(v-\frac{2}{3}) - u \leq -\frac{\ell-3}{6}$. So $s_{\alpha_2,k}x = u\alpha_2^\vee + (v-\frac{2}{3})\alpha_2^\vee$ satisfies $I_1(\ell-3)$. The u values for xq and $s_{\alpha_2,k}xq$ is the same. Since xq satisfies $I_2(\ell-3)$, so does $s_{\alpha_2,k}xq$. Therefore $s_{\alpha_2,k}xq \in \triangle(\ell-3)$.

Now we show $p^w_{s_{\alpha_2,k}(x)} = 1$ and $p^w_{s_{\alpha_2,k+1}(x)} = 1$ by the translation theorem. Observe that $p^w_{s_{\alpha_2,k}(x)} = p^w_{s_{\alpha_2,k+1}(x)}$ because $s_{\alpha_2,k+1}x = t(\alpha_2)s_{\alpha_2,k}x$ and by the translation theorem. To show that one is a lookup direction, enough to show $p^w_{s_{\alpha_2,k}(x)} = 1$. First consider the case $\gamma^\vee = 0$, then $x = w(\ell-6)$, $k = 0$, so $s_{\alpha_2,k} = s_2$.

Then we claim: $s_2 x = t(\alpha_1^\vee)w(\ell-3)$. By Lemma 5.6,

$$\begin{aligned} s_2 x q &= s_2 w(\ell-6)q \\ &= s_2((\frac{7}{3}-\frac{\ell}{3})\alpha_1^\vee + (\frac{3}{2}-\frac{\ell}{6})\alpha_2^\vee) \\ &= (\frac{7}{3}-\frac{\ell}{3})\widetilde{\alpha} - (\frac{3}{2}-\frac{\ell}{6})\alpha_2^\vee \\ &= (\frac{7}{3}-\frac{\ell}{3})\alpha_1^\vee + \frac{5-\ell}{6}\alpha_2^\vee. \end{aligned}$$

By Lemma 5.6, $t(\alpha_1)w(\ell-3)q = t(\alpha_1)(\frac{1-(\ell-3)}{3}\alpha_1 + \frac{2-(\ell-3)}{6}\alpha_2) = (\frac{7}{3}-\frac{\ell}{3})\alpha_1^\vee + \frac{5-\ell}{6}\alpha_2^\vee$. Thus $s_2 x = t(\alpha_1^\vee)w(\ell-3)$. The claim is proved. Therefore $p^w_{s_2 w(\ell-6)} = p^w_{w(\ell-3)} = 1$.

In general, $x = t(\gamma^\vee)w(\ell-6)$. Since $k = (\alpha_2, \gamma^\vee)$, by Lemma 2.4, $s_{\alpha_2,k}t(\gamma^\vee) = t(\gamma^\vee)s_2$. Hence

$$s_{\alpha_2,k}x = s_{\alpha_2,k}t(\gamma^\vee)w(\ell-6) = t(\gamma^\vee)s_2 w(\ell-6).$$

Therefore,

$$p^w_{s_{\alpha_2,k}x} = p^w_{t(\gamma^\vee)s_2 w(\ell-6)} = p^w_{s_2 w(\ell-6)} = 1.$$

This shows that the lookup conjecture is true for $x = t(\gamma^\vee)w(\ell-6)$.

If $w(\ell)$ ends with s_a, so does $w(\ell-6)$, so $w(\ell-7) = w(\ell-6)s_a$. As $w(\ell)s_a < w(\ell)$, by Proposition 4.10, the lookup conjecture is true for $t(\gamma^\vee)w(\ell-7)$.

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