Classifying the Representation Type

of Infinitesimal Blocks

of Category $\mathcal{O}_S$

by

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(Under the Direction of Brian D. Boe)

Abstract

Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{C}$ of complex numbers, with root system Φ relative to a fixed maximal toral subalgebra $\mathfrak{h}$. Let S be a subset of the simple roots Δ of Φ, which determines a standard parabolic subalgebra of $\mathfrak{g}$. Fix an integral weight $\mu \in \mathfrak{h}^*$, with singular set $J \subseteq \Delta$. We determine when an infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J) := \mathcal{O}_S^\mu$ of parabolic category $\mathcal{O}_S$ is nonzero using the theory of nilpotent orbits. We extend work of Futorny-Nakano-Pollack, Brüstle-König-Mazorchuk, and Boe-Nakano toward classifying the representation type of the nonzero infinitesimal blocks of category $\mathcal{O}_S$ by considering arbitrary sets S and J, and observe a strong connection between the theory of nilpotent orbits and the representation type of the infinitesimal blocks. We classify certain infinitesimal blocks of category $\mathcal{O}_S$ including all the semisimple infinitesimal blocks in type A_n, and all of the infinitesimal blocks for F_4 and G_2.

Index words: Category $\mathcal{O}$; Representation type; Verma modules

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Dedication

For Emily, whose support and patience have meant more to me than I can with due justice express.

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Introduction

1.1 Historical Background

Representation theory is concerned with realizing a group or an algebra as a collection of matrices. In this way, one can understand the way the group or algebra acts linearly on a vector space, where the action respects the operations in the group or algebra. In the process, one is able to understand more completely the structure of the group or algebra. Representation theory has found uses in many areas, particularly where symmetry arises. These areas include physics, chemistry, and mathematics itself.

Suppose $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over the field $\mathbb{C}$ of complex numbers. If V is a vector space on which there is an action of $\mathfrak{g}$ defined which respects the bracket in $\mathfrak{g}$, then V is called a $\mathfrak{g}$-module. One often discusses representations of $\mathfrak{g}$ via the equivalent language of $\mathfrak{g}$-modules.

In 1976, the Russian mathematicians I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand [BGG] introduced a category of $\mathfrak{g}$-modules, called category $\mathcal{O}$. The letter O stands for a Russian word meaning "basic" or "principal". This is appropriate because this category contains many $\mathfrak{g}$-modules which are important in applications (including the finite dimensional modules), and in some sense is the smallest category containing the finite dimensional modules with the right properties to facilitate the study of these modules.

For any subset S of simple roots for $\mathfrak{g}$, one can construct a parabolic subalgebra $\mathfrak{p}_S$ of $\mathfrak{g}$. In the early 1980's, A. Rocha-Caridi [RC] introduced category $\mathcal{O}_S$, which generalized ordinary category $\mathcal{O}$. Since $\mathfrak{p}_S$-modules are generally more easily understood than $\mathfrak{g}$-modules, category $\mathcal{O}_S$ is useful for inductive arguments involving $\mathfrak{g}$-modules.

Category $\mathcal{O}_S$ is a highest weight category [CPS, Sec. 3], and it decomposes as a direct sum of certain subcategories, called infinitesimal blocks, which are defined in terms of the infinitesimal characters of the universal enveloping algebra. Under this decomposition, a $\mathfrak{g}$-module in category $\mathcal{O}_S$ decomposes as a direct sum of submodules with each summand belonging to one of the infinitesimal blocks. For each infinitesimal block, there is a finite dimensional quasi-hereditary algebra A such that the block is equivalent to the category of finitely generated A-modules [CPS, Thm. 3.6], [FM, Thm. 3, Ex. 5.2]. An infinitesimal block contains at most finitely many simple $\mathfrak{g}$-modules, and some contain only the zero $\mathfrak{g}$-module.

Suppose that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, so $\mathfrak{g}$ has a root system Φ of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2. If ρ is the half sum of positive roots and $\mu + \rho$ is an antidominant integral weight for S, then the infinitesimal block $\mathcal{O}_S^{\mu}$ contains all the simple modules with highest weight linked to μ via the dot action of the Weyl group $\mathcal{W}$ of Φ.

The integral infinitesimal blocks of category $\mathcal{O}_S$ are determined (up to equivalence of categories) by subsets J of the simple roots; we allow $J = \varnothing$, which corresponds to the regular infinitesimal blocks. Write $\mathcal{O}(\mathfrak{g}, S, J)$ for the infinitesimal block determined by the triple $(\mathfrak{g}, S, J)$.

Indecomposable modules of a finite dimensional algebra provide a complete description of all the modules of the algebra. Consequently, classifying the indecomposable modules for a fixed finite dimensional algebra A is a central theme in the representation theory of such algebras. One of the first questions one can ask is, "How classifiable are the indecomposable modules of A?" The algebra A will fall into one of three classes depending on the classifiability of its indecomposable modules. If there are only finitely many isomorphism classes of indecomposable A-modules, then we say that A has *finite representation type*, and if there are infinitely many such isomorphism classes, then A has *infinite representation type.* If A has infinite representation type, it can be further classified as having *tame representation* *type* if, roughly speaking, these indecomposable modules can be parameterized in some way, and *wild representation type* otherwise (see [Dro] and [CB, Sec. 6]).

For $S, J \subseteq \Delta$, let $A_{S,J}$ be a finite dimensional quasi-hereditary algebra for which the category $\mathrm{mod}_f(A_{S,J})$ of finitely generated $A_{S,J}$-modules is equivalent to the infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$. We will say that $\mathcal{O}(\mathfrak{g}, S, J)$ has finite (respectively, infinite, tame, wild) representation whenever $A_{S,J}$ has finite (respectively, infinite, tame, wild) representation type.

The classification of the representation type of the infinitesimal blocks of category $\mathcal{O}_S$ began with the classification of the representation type of the blocks of ordinary category $\mathcal{O}$ (where $S = \varnothing$). This was done by Futorny, Nakano, and Pollack [FNP] and, using different techniques, by Brüstle, König, and Mazorchuk [BKM]. Boe and Nakano [BN] later classified the representation type of all infinitesimal blocks of $\mathcal{O}_S$ with $S \cap J = \varnothing$.

1.2 Investigations

The simple modules are the building blocks of arbitrary modules of an algebra. However, an infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$ may not contain any simple modules. Consequently, one can ask, "under what conditions does $\mathcal{O}(\mathfrak{g}, S, J)$ contain at least one simple module?" Even though category $\mathcal{O}_S$ has been studied for over a quarter of a century, an easy description of the nonzero blocks has been elusive. Enright and Hunziker [EH, Sec. 2.5] gave a criterion for a block to be nonzero, but it is not easy to apply. In this dissertation, the answer will be given via nilpotent orbits of $\mathfrak{g}$. The description of the nonzero infinitesimal blocks will be given in terms of a partial ordering defined on the nilpotent orbits, and the Richardson orbits will provide the key.

An infinitesimal block is semisimple if and only if there are no extensions between its simple modules. An infinitesimal block with only one simple module is necessarily semisimple, but there are semisimple blocks with more than one simple module. In fact, we will show that in every type but A_n, there are semisimple blocks with more than one simple module.

In type A_n, a result of J. Brundan [Bru] implies that the semisimple infinitesimal blocks are precisely those blocks having one simple module. The question of when an infinitesimal block in type A_n contains exactly one simple module is answered in this thesis using partitions, and is related to the nilpotent and Richardson orbits. We will use this result to provide a sufficient condition for infinitesimal blocks in types B_n and C_n to be semisimple.

Collected in this work are results for the representation type of infinitesimal blocks $\mathcal{O}(\mathfrak{g}, S, J)$ when $\mathfrak{g}$ is of type A_n, B_n, or C_n and (Φ_S, Φ_J) is a Hermitian symmetric pair. This follows from the classification of non-empty and semisimple infinitesimal blocks in these types, and the work of Boe and Nakano [BN], Boe and Hunziker [BH], and Enright [E].

The representation type of all of the infinitesimal blocks for type F_4 and G_2 have been computed here. The complete results for these two cases required the use of a computer. Examples in the classical types are also given in this thesis. A strong link between representation type of infinitesimal blocks and nilpotent orbits is observed in these cases, and the classification of the representation type of the infinitesimal blocks in types F_4 and G_2 is given in terms of ideas from nilpotent orbits.

Chapter 2

Preliminaries

2.1 Notation

Write $\mathbb{Z}$ for the integers, and $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{<0}$ for the non-negative positive, and negative integers, respectively. Denote the real numbers by $\mathbb{R}$ and the field of complex numbers by $\mathbb{C}$. Denote the trace of a square matrix or an endomorphism x by $\mathrm{tr}(x)$.

In this thesis, we will work over the field $\mathbb{C}$ of complex numbers. Take $\mathfrak{g}$ to be a complex simple Lie algebra; for example,

$$\begin{aligned}
\mathfrak{sl}_{n+1}(\mathbb{C}) &= \{\, x \in \mathfrak{gl}_n(\mathbb{C}) \mid \mathrm{tr}(x) = 0 \,\} \\
\mathfrak{so}_{2n+1}(\mathbb{C}) &= \left\{ \begin{pmatrix} m & b & p \\ c & 0 & -b^t \\ q & -c^t & -m^t \end{pmatrix} \middle| \, m, p, q \in \mathfrak{gl}_n(\mathbb{C}),\ b^t, c \in \mathbb{C}^n,\ q = -q^t,\ p = -p^t \right\} \\
\mathfrak{sp}_{2n}(\mathbb{C}) &= \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} \middle| \, m, p, q \in \mathfrak{gl}_n(\mathbb{C}),\ q = q^t,\ p = p^t \right\} \qquad (2.1) \\
\mathfrak{so}_{2n}(\mathbb{C}) &= \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} \middle| \, m, p, q \in \mathfrak{gl}_n(\mathbb{C}),\ q = -q^t,\ p = -p^t \right\}
\end{aligned}$$

with bracket given by $[x, y] = xy - yx$. The material in this section can be found in [Hum].

If V is a complex vector space, then let $\mathfrak{gl}(V)$ denote the usual Lie algebra of endomorphisms of V. Let ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote the adjoint representation of $\mathfrak{g}$. If V is finite dimensional and $x \in \mathfrak{gl}(V)$, then x has a Jordan canonical form; i.e., there is an ordered basis of V such that the matrix of x with respect to this basis is the direct sum of **Jordan**

blocks:

$$x_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}$$

Therefore, x is the sum of a diagonal matrix and a nilpotent matrix, and these two matrices commute with each other. We say that $x \in \mathfrak{gl}(V)$ is **semisimple** if x is diagonalizable. We say that $x \in \mathfrak{gl}(V)$ is **nilpotent** if $x^k = 0$ for some $k > 0$.

Given $x \in \mathfrak{g}$, there exist unique $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$ with $[x_s, x_n] = 0$, ad $x_s \in \mathfrak{gl}(\mathfrak{g})$ is semisimple, and ad $x_n \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent. This is called the **Jordan-Chevalley decomposition** of x. If $x = x_s$, then we say that x is **semisimple**, and we say x is **nilpotent** if $x = x_n$.

Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. For example, take the set of diagonal matrices in each of the Lie algebras in (2.1). Let $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be the dual of $\mathfrak{h}$. Denote by Φ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and for each $\alpha \in \Phi$, let $\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}$ denote the α-root space of $\mathfrak{g}$. We have the **root space decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$.

Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denote the Killing form on $\mathfrak{g}$. For each $\lambda \in \mathfrak{h}^*$, there is a unique element $h_\lambda \in \mathfrak{h}$ such that $\kappa(h_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$, and so we have a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ defined by $(\lambda, \mu) = \kappa(h_\lambda, h_\mu)$.

We will take $n = \dim_{\mathbb{C}} \mathfrak{h}^*$ to be the rank of Φ, and denote the fixed set of simple roots by $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Denote the set of positive (respectively, negative) roots with respect to Δ by Φ^+ (respectively, Φ^-).

If

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$$

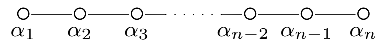
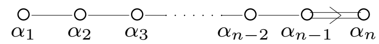
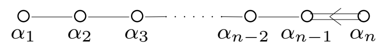
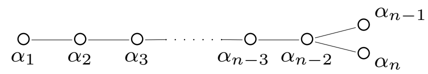
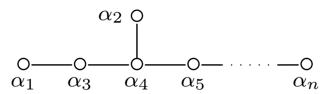
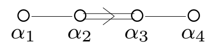
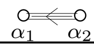
Φ	Dynkin Diagram
A_n $(n \geq 1)$	
B_n $(n \geq 2)$	
C_n $(n \geq 3)$	
D_n $(n \geq 4)$	
E_n $(n = 6, 7, 8)$	
F_4	
G_2	

Table 2.1: Dynkin Diagrams of Root Systems of Simple Lie Algebras

then the Lie subalgebras $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ of $\mathfrak{g}$ are called **Borel subalgebras**.

The **coroot** of $\alpha \in \Phi$ is defined to be:

$$\check{\alpha} = \frac{2\alpha}{(\alpha, \alpha)}$$

For each $\alpha \in \Phi$, the reflection in $\mathfrak{h}^*$ through the hyperplane $P_\alpha = \{\, \beta \in \mathfrak{h}^* \mid (\beta, \alpha) = 0 \,\}$ is given by $s_\alpha(\beta) = \beta - (\beta, \check{\alpha})\alpha$ for each $\beta \in \mathfrak{h}^*$. Let $\mathcal{W}$ denote the **Weyl group** of Φ, which is generated by the reflections s_α for $\alpha \in \Phi$. In fact, the **simple reflections**, $s_i := s_{\alpha_i}$, $i = 1, 2, \ldots, n$ generate the Weyl group $\mathcal{W}$. If $w \in \mathcal{W}$ is written as $w = s_{i_1} s_{i_2} \cdots s_{i_t}$ with $i_j \in \{1, 2, \ldots, n\}$ and t minimal, then we call this a **reduced expression** for w and we call t the **length** of w, and write $l(w) = t$. By definition, $l(1) = 0$. Furthermore, there exists a unique longest element w_0 of $\mathcal{W}$.

Define a partial ordering on $\mathcal{W}$ as follows. For $w, w' \in \mathcal{W}$, write $w \to w'$ if $l(w) < l(w')$ and $w' = ws_\alpha$ for some $\alpha \in \Phi$. Write $w < w'$ if there is a sequence $w = w_0 \to w_1 \to \cdots \to w_r = w'$. This partial ordering is compatible with the length function, and is called the **Bruhat ordering**.

The simple Lie algebras are classified by the Dynkin diagrams given in Table 2.1. The **classical Lie algebras** $\mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$, and $\mathfrak{so}_{2n}(\mathbb{C})$ have respective root systems of types A_n, B_n, C_n and D_n. The Lie algebras with root systems of type E_6, E_7, E_8, F_4, or G_2 are called **exceptional Lie algebras**.

2.2 Representation Theory of Lie Algebras

Let V be a vector space over $\mathbb{C}$. A Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a **representation** of $\mathfrak{g}$ in V and the **dimension** of φ is the dimension of V. We say that V is a **$\mathfrak{g}$-module** if $\mathfrak{g}$ acts linearly on V and this action respects the bracket in $\mathfrak{g}$. There is a one-to-one correspondence between representations of $\mathfrak{g}$ and $\mathfrak{g}$-modules. A **simple $\mathfrak{g}$-module** is a $\mathfrak{g}$-module $V \neq 0$ whose only $\mathfrak{g}$-submodules are V and 0.

Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We have that V is a $\mathfrak{g}$-module if and only if V is a $\mathcal{U}(\mathfrak{g})$-module.

Let V be a (not necessarily finite dimensional) $\mathfrak{g}$-module. For each $\mu \in \mathfrak{h}^*$, define the **μ-weight space** of V to be:

$$V_\mu = \{\, v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h} \,\}$$

If $V_\mu \neq 0$, then we call μ a **weight** of V, and its **multiplicity** is $\dim V_\mu$. Denote the set of weights of V by $\mathrm{wt}(V)$.

The set $X = \{\, \mu \in \mathfrak{h}^* \mid (\mu, \check{\alpha}) \in \mathbb{Z} \text{ for all } \alpha \in \Phi \,\}$ is the **integral weight lattice**, and its elements are called **integral weights**. The set $X^+ = \{\, \mu \in X \mid (\mu, \check{\alpha}) \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Phi^+ \,\}$ is the set of **dominant integral weights**, and we say that $\mu \in X^+$ is a **strongly dominant integral weight** if $(\mu, \check{\alpha}) \in \mathbb{Z}_{>0}$ for all $\alpha \in \Phi^+$.

Since Δ is a basis for $\mathfrak{h}^*$, any weight can be written as a linear combination of simple roots. Consequently, there is a natural action of $\mathcal{W}$ on $\mathfrak{h}^*$. In fact, this is an action on X. The strongly dominant integral weight

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

plays a useful role in the representation theory of Lie algebras. One nice property that ρ has is that

$$(\rho, \check{\alpha}) = 1 \quad \text{for all } \alpha \in \Delta \tag{2.2}$$

(see [Hum, Sec. 13.3]). The Weyl group acts on $\mathfrak{h}^*$ via the dot action:

$$w \cdot \mu = w(\mu + \rho) - \rho$$

for all $w \in \mathcal{W}$, $\mu \in \mathfrak{h}^*$.

There is a partial order on $\mathfrak{h}^*$ defined as follows. For $\mu, \lambda \in \mathfrak{h}^*$, say $\lambda \prec \mu$ if and only if $\mu - \lambda$ is a sum of positive roots. If V is a $\mathfrak{g}$-module such that $V = \mathcal{U}(\mathfrak{g}) \cdot v$ for some $v \in V_\mu$ with the property that $\mathfrak{g}_{\alpha_i} \cdot v = 0$ for all i, then we call V a **highest weight module** with **highest weight** μ. The weight μ has the property that $\lambda \in \mathrm{wt}(V)$ implies that either $\lambda = \mu$ or $\lambda \prec \mu$.

2.2.1 Category $\mathcal{O}_S$

In the 1970's, Bernstein-Gelfand-Gelfand [BGG] defined an important category of $\mathcal{U}(\mathfrak{g})$-modules, called category $\mathcal{O}$. The key objects in category $\mathcal{O}$ are the **Verma modules**, which are constructed as follows. Let $\mu \in \mathfrak{h}^*$, and define an action of $\mathfrak{h}$ on $\mathbb{C}$ by $hz = \mu(h)z$ for all $h \in \mathfrak{h}$, $z \in \mathbb{C}$. We can inflate this to an action of $\mathfrak{b}^+$ on $\mathbb{C}$ by letting $\mathfrak{n}^+$ act trivially. Denote this $\mathfrak{b}^+$-module by $\mathbb{C}_\mu$. In fact, any finite dimensional irreducible $\mathcal{U}(\mathfrak{b}^+)$-module is equivalent to some $\mathbb{C}_\mu$ ($\mu \in \mathfrak{h}^*$). The induced $\mathcal{U}(\mathfrak{g})$-module $M(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\mu$ with the natural left action of $\mathcal{U}(\mathfrak{g})$ is the Verma module associated to μ. In fact, it is a highest weight module with highest weight μ (see [Dix, Sec. 7.1]).

We will work in special subcategories of category $\mathcal{O}$ determined by certain parabolic subalgebras of $\mathfrak{g}$. These categories were defined by Rocha-Caridi in the early 1980's.

Fix $S \subseteq \Delta$, viewed where appropriate as a subset of $\{1, \dots, n\}$ via the fixed ordering on simple roots. For each $i \in S$, there exist $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$, and $h_i \in \mathfrak{h}$ such that $[x_i, y_i] = h_i$ and $\alpha_i(h_i) = 2$ [Dix, 1.10]. Set $\mathfrak{h}_S = \langle h_i \mid i \in S \rangle$ and $\mathfrak{h}^S = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \text{ for all } i \in S\}$

and let

$$\Phi_S = \Phi \cap \sum_{i \in S} \mathbb{Z}\alpha_i$$

and $\Phi_S^+ = \Phi^+ \cap \Phi_S$. Define:

$$\mathfrak{n}_S^+ = \bigoplus_{\alpha \in \Phi_S^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_S^- = \bigoplus_{\alpha \in \Phi_S^+} \mathfrak{g}_{-\alpha}$$

If we set $\mathfrak{g}_S = \mathfrak{n}_S^+ \oplus \mathfrak{h}_S \oplus \mathfrak{n}_S^-$, then $\mathfrak{g}_S$ is a semisimple Lie algebra with maximal toral subalgebra $\mathfrak{h}_S$. Furthermore, viewed as elements of $\mathfrak{h}_S^*$ (by restricting to $\mathfrak{h}_S$), Φ_S is a root system of $\mathfrak{g}_S$ with respect to $\mathfrak{h}_S$ having positive roots Φ_S^+ and simple roots S.

Define $\mathfrak{m}_S = \mathfrak{n}_S^+ \oplus \mathfrak{h}_S \oplus \mathfrak{h}^S \oplus \mathfrak{n}_S^-$ and set:

$$\mathfrak{u}_S^+ = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_S^+} \mathfrak{g}_\alpha$$

Since $[\mathfrak{m}_S, \mathfrak{u}_S^+] \subseteq \mathfrak{u}_S^+$, we have that $\mathfrak{p}_S = \mathfrak{m}_S \oplus \mathfrak{u}_S^+$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, called a **standard parabolic subalgebra** of $\mathfrak{g}$. We call $\mathfrak{m}_S$ the **Levi factor** and $\mathfrak{u}_S^+$ the **nilradical** of $\mathfrak{p}_S$ (see [RC]). The Levi factor $\mathfrak{m}_S$ of $\mathfrak{p}_S$ also has root system Φ_S. Denote the Weyl group of Φ_S by $\mathcal{W}_S$. It can be considered as a subgroup of $\mathcal{W}$. Let w_S be the longest element of $\mathcal{W}_S$.

Let $\mathfrak{p}_S$ be the standard parabolic subalgebra of $\mathfrak{g}$ determined by S. The category $\mathcal{O}_S$ is defined as follows.

Definition 2.2.1 *Let $\mathcal{O}_S$ be the full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules consisting of modules V which satisfy the following conditions:*

(i) V is a finitely generated $\mathcal{U}(\mathfrak{g})$-module.

(ii) As a $\mathcal{U}(\mathfrak{m}_S)$-module, V is the direct sum of finite dimensional $\mathcal{U}(\mathfrak{m}_S)$-modules.

(iii) For all $v \in V$, $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{u}_S^+)v < \infty$.

Define $X_S^+ = \{\, \mu \in \mathfrak{h}^* \mid (\mu, \check{\alpha}) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi_S^+ \,\}$. We have the following theorem [Hum, Sec. 21].

Theorem 2.2.2 *There is a one-to-one correspondence between the finite dimensional irreducible $\mathfrak{m}_S$-modules (up to equivalence) and the set X_S^+.*

Let $F(\mu)$ denote the finite dimensional irreducible $\mathfrak{m}_S$-module of highest weight $\mu \in X_S^+$. If $V \in \mathcal{O}_S$, then by Definition 2.2.1(ii), as an $\mathfrak{m}_S$-module

$$V = \bigoplus_{\mu \in X_S^+} m_\mu F(\mu)$$

for some $m_\mu \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Set $\mathrm{wt}_{\mathfrak{m}}(V) = \{\, \mu \in X_S^+ \mid m_\mu \neq 0 \,\}$.

When $S = \varnothing$, $\mathcal{O}_S$ is the category $\mathcal{O}$ defined in [BGG]. For each $\mu \in \mathfrak{h}^*$, define:

$$D(\mu) = \{\, \mu - (a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) \mid a_i \in \mathbb{Z}_{\geq 0} \,\}$$

Category $\mathcal{O}_S$ has the following properties (see [RC, Sec. 3] for details).

Theorem 2.2.3 *Let $V \in \mathcal{O}_S$.*

(i) If $U \subseteq V$ is a submodule, then $U \in \mathcal{O}_S$ and $V/U \in \mathcal{O}_S$.

(ii) If $V_1, \ldots, V_r \in \mathcal{O}_S$, then the finite direct sum $\bigoplus_{i=1}^r V_i$ is in $\mathcal{O}_S$.

(iii) If $V \in \mathcal{O}_S$, then

$$V = \bigoplus_{\mu \in X_S^+} m_\mu F(\mu)$$

for some $m_\mu \in \mathbb{Z}_{\geq 0}$.

(iv) There exist $\mu_1, \mu_2, \ldots, \mu_r \in X_S^+$ such that $\mathrm{wt}_{\mathfrak{m}}(V) \subseteq D(\mu_1) \cup D(\mu_2) \cup \cdots \cup D(\mu_r)$.

(v) V has a Jordan-Hölder series.

The key objects in category $\mathcal{O}_S$ are the parabolic Verma modules, which are constructed as follows. Start with a finite dimensional irreducible $\mathfrak{m}_S$-module $F(\mu)$ ($\mu \in X_S^+$). Extend $F(\mu)$ to a $\mathfrak{p}_S$-module by letting $\mathfrak{u}_S^+$ act trivially (this gives the most general finite dimensional irreducible $\mathfrak{p}_S$-module). The induced module

$$V(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_S)} F(\mu)$$

is a **parabolic Verma module** (or PVM for short).

PVMs have properties similar to Verma modules, and in fact, generalize the ordinary Verma modules as shown below [RC, Sec. 3].

Theorem 2.2.4 *Let $V(\mu)$ be the PVM associated to $\mu \in X_S^+$.*

(i) $V(\mu) \in \mathcal{O}_S$.

(ii) $V(\mu)$ is a highest weight module for $\mathfrak{g}$ with highest weight μ.

(iii) $V(\mu)$ has a unique maximal submodule, and hence a unique irreducible quotient module, denoted $L(\mu)$ (this is the same unique irreducible quotient module of $M(\mu)$).

(iv) If $S = \varnothing$, then $V(\mu) = M(\mu)$ and if $S = \Delta$, then $V(\mu) = L(\mu)$.

(v) If $V \in \mathcal{O}_S$ is irreducible, then V is isomorphic to $L(\mu)$ for some $\mu \in X_S^+$.

We will now discuss briefly projective modules in category $\mathcal{O}_S$. First, there are enough projectives in category $\mathcal{O}_S$. In fact, we have the following (see [RC, Sec. 4]).

Proposition 2.2.5 *(i) For each $V \in \mathcal{O}_S$, there is a projective module P in $\mathcal{O}_S$ such that V is a quotient of P.*

(ii) If $P \in \mathcal{O}_S$ is a projective indecomposable module, then P has a unique maximal submodule and hence a unique irreducible quotient.

(iii) There is a one-to-one correspondence between the irreducible modules in $\mathcal{O}_S$ and the projective indecomposable modules in $\mathcal{O}_S$ given by $L(\mu) \leftrightarrow P(\mu)$ for all $\mu \in X_S^+$. In fact, $P(\mu)$ is the projective cover of $L(\mu)$.

We say that $V \in \mathcal{O}_S$ has a **parabolic Verma filtration** if there is a filtration

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{r-1} \supseteq V_r = 0$$

such that $V_{i-1}/V_i \cong V(\mu_i)$ for some $\mu_i \in X_S^+$ $(1 \le i \le r)$. If V has a parabolic Verma composition series, then let $[V : V(\mu)]$ denote the multiplicity of $V(\mu)$ as a subquotient of V in the series (this is well-defined; see [RC, Sec. 5]). If $[V(\lambda) : L(\mu)]$ is the multiplicity of $L(\mu)$ in a Jordan-Hölder series of $V(\lambda)$, then we have the following reciprocity law [RC, Sec. 6].

Theorem 2.2.6 (Reciprocity in $\mathcal{O}_S$) *Every projective module in $\mathcal{O}_S$ has a parabolic Verma filtration and if $\lambda, \mu \in X_S^+$, then:*

$$[P(\mu) : V(\lambda)] = [V(\lambda) : L(\mu)]$$

2.2.2 Infinitesimal Blocks of $\mathcal{O}_S$

Let Z be the center of $\mathcal{U}(\mathfrak{g})$ and denote the set of algebra homomorphisms $Z \to \mathbb{C}$ by $Z^\sharp$. If $V \in \mathcal{O}_S$ and there exists $\chi \in Z^\sharp$ such that $zv = \chi(z)v$ for all $z \in Z$ and all $v \in V$, then we say that V has **central character** (or **infinitesimal character**) χ.

For each $\chi \in Z^\sharp$, let $\mathcal{O}_S^\chi$ be the full subcategory of $\mathcal{O}_S$ consisting of modules $V \in \mathcal{O}_S$ such that for all $z \in Z$, each $v \in V$ is annihilated by some power of $z - \chi(z)$. If $V \in \mathcal{O}_S$, then for each $\chi \in Z^\sharp$, there exists $V^\chi \in \mathcal{O}_S^\chi$ such that:

$$V = \bigoplus_{\chi \in Z^\sharp} V^\chi$$

We thus have the decomposition

$$\mathcal{O}_S = \bigoplus_{\chi \in Z^\sharp} \mathcal{O}_S^\chi$$

of the category $\mathcal{O}_S$. We call $\mathcal{O}_S^\chi$ an **infinitesimal block** of category $\mathcal{O}_S$.

For each $\mu \in \mathfrak{h}^*$, the ordinary Verma module $M(\mu)$ has a central character which we will denote by $\chi_\mu \in Z^\sharp$. Since $V(\mu)$ for $\mu \in X_S^+$ is a highest weight module with highest weight μ, $V(\mu)$ is a quotient of $M(\mu)$. Thus, as quotients of $M(\mu)$, each of $V(\mu)$ and $L(\mu)$ also has central character χ_μ. Furthermore, if $\chi \in Z^\sharp$ is a central character, then there exists $\mu \in \mathfrak{h}^*$ such that $\chi = \chi_\mu$. If $\chi = \chi_\mu$, we can write $\mathcal{O}_S^\mu = \mathcal{O}_S^{\chi_\mu} = \mathcal{O}(\mathfrak{g}, S, \mu)$ for $\mathcal{O}_S^\chi$.

If $\mu \in X_S^+$, then we have $V(\mu), L(\mu) \in \mathcal{O}_S^\mu$. Furthermore, we have the following linkage principle.

Theorem 2.2.7 (Harish-Chandra Linkage Principle) *If $\lambda, \mu \in \mathfrak{h}^*$, then:*

$$\chi_\mu = \chi_\nu \quad \Longleftrightarrow \quad \nu \in \mathcal{W} \cdot \mu$$

Thus, $V(\nu), L(\nu) \in \mathcal{O}_S^\mu$ if and only if $\nu = w_S w \cdot \mu$ for some $w \in \mathcal{W}$. Since PVM's are constructed from the finite dimensional $\mathfrak{m}_S$-modules with highest weights in X_S^+, the set of PVM's in $\mathcal{O}_S^\mu$ is $\{ V(w_S w \cdot \mu) \mid w_S w \cdot \mu \in X_S^+ \}$. Consequently, the PVM's (as well as the simple modules and projective indecomposable modules) in $\mathcal{O}_S^\mu$ are parameterized by $\{ w \in \mathcal{W} \mid w_S w \cdot \mu \in X_S^+ \}$.

Assume from now on that μ is an integral weight and $\mu+\rho$ is antidominant, i.e., $(\mu+\rho, \check{\alpha}) \in \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Phi^+$; (if it is not antidominant, we can replace it by a $\mathcal{W}$-translate, so we are justified in making this assumption). Let

$$\Phi_\mu = \{ \alpha \in \Phi \mid (\mu + \rho, \check{\alpha}) = 0 \}.$$

If $\Phi_\mu = \varnothing$, then $\mu+\rho$ is called a **regular weight**. If $\mu+\rho$ and $\nu+\rho$ are both regular weights, then the category $\mathcal{O}_S^\mu$ is equivalent to $\mathcal{O}_S^\nu$ by the Jantzen-Zuckerman translation principle. If $\mu + \rho$ is a regular weight, then $\{ w \in \mathcal{W} \mid w_S w \cdot \mu \in X_S^+ \}$ is the set

$$\begin{aligned} {}^S\mathcal{W} &= \{ w \in \mathcal{W} \mid l(s_\alpha w) = l(w) + 1 \text{ for all } \alpha \in S \} \\ &= \{ w \in \mathcal{W} \mid w^{-1}(\Phi_S^+) \subseteq \Phi^+ \} \end{aligned}$$

which is the set of smallest length representatives for the right cosets of $\mathcal{W}_S$ in $\mathcal{W}$.

Now, if $\mu \in \mathfrak{h}^*$ is such that $\Phi_\mu \neq \varnothing$, then Φ_μ is a subroot system of Φ, and in this case $\mu + \rho$ is called a **singular weight**. Suppose $\alpha \in \Phi^+ \cap \Phi_\mu$. Then $(\mu + \rho, \check{\alpha}) = 0$ and also we can write $\alpha = \sum_{i=1}^n a_i \alpha_i$ for some $a_i \in \mathbb{Z}_{\geq 0}$. Since $\mu + \rho$ is antidominant, $(\mu + \rho, \check{\alpha}_i) \leq 0$ for each $1 \leq i \leq n$. Consequently, $0 = (\mu+\rho, \check{\alpha}) = \sum_{i=1}^n a_i(\mu+\rho, \check{\alpha}_i)$ implies that $(\mu+\rho, \check{\alpha}_i) = 0$ for any $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$. Setting

$$J = \{ \alpha \in \Delta \mid (\mu + \rho, \check{\alpha}) = 0 \}$$

we have that Φ_μ is the root system Φ_J which has simple roots J. Note that the Weyl group $\mathcal{W}_J$ of Φ_J is the stabilizer of $\mu + \rho$.

If $\mu + \rho$ is singular, then $w_S w \cdot \mu \in X_S^+$ if and only if $w\mathcal{W}_J \subseteq {}^S\mathcal{W}$. Since $\mathcal{W}_J$ stabilizes $\mu + \rho$, the set

$$ {}^S\mathcal{W}^J = \{\, w \in {}^S\mathcal{W} \mid w < ws_\alpha \text{ and } ws_\alpha \in {}^S\mathcal{W} \text{ for all } \alpha \in J \,\} $$

is the set of smallest length representatives for the left cosets $w\mathcal{W}_J$ contained in ${}^S\mathcal{W}$. Consequently, the set ${}^S\mathcal{W}^J$ parameterizes the set of inequivalent irreducible modules in the infinitesimal block $\mathcal{O}_S^\mu$. That is, the set of simple modules in $\mathcal{O}_S^\mu$ is the set $\{\, L(w_S w \cdot \mu) \mid w \in {}^S\mathcal{W}^J \,\}$ (see [BN, Prop. 2.2]).

As with S, we will frequently view the set J as a subset of $\{1, \ldots, n\}$. We will use the notation

$$ \mathcal{O}_S^\mu = \mathcal{O}(\Phi, S, J) = \mathcal{O}(\mathfrak{g}, S, J) $$

when $\Phi_\mu = \Phi_J$.

Chapter 3

Non-Zero Infinitesimal Blocks

It is possible that an infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$ contains only the zero module. Consequently, one of the first questions one can ask is: under what conditions is a given infinitesimal block guaranteed to contain at least one simple module? First we will discuss this question in the case when $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$. The answer generally will take us into the realm of nilpotent orbits of $\mathfrak{g}$, where combinatorial and geometric tools will provide the answer.

3.1 The Idea: A Look at Type A_n

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$. Recall that if $\{\varepsilon_1, \cdots, \varepsilon_{n+1}\}$ is the standard orthonormal basis of $\mathfrak{h}^* \cong \mathbb{R}^{n+1}$, then $\Phi = \{\, \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n+1,\ i \neq j \,\}$ with simple roots $\Delta = \{\, \alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n \,\}$ and positive roots $\Phi^+ = \{\, \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1 \,\}$. The Weyl group $\mathcal{W}$ is isomorphic to the symmetric group S_{n+1}, with the simple reflection s_i acting as a transposition which interchanges ε_i and ε_{i+1} and fixes every other basis element.

Fix a set $S \subseteq \Delta$ and let $\mu + \rho$ be an antidominant integral weight, where Φ_μ has simple roots $J \subseteq \Delta$. Suppose that $w \in {}^S\mathcal{W}^J$ and let $\nu = w_S w(\mu + \rho)$ so that $\nu - \rho = w_S w \cdot \mu \in X_S^+$. Write $\nu = (\nu_1, \nu_2, \ldots, \nu_n, \nu_{n+1})$ (in the ε-basis) and note that $\check{\alpha} = \alpha$ for all $\alpha \in \Phi$. If $\alpha_i \in S$, then

$$(\nu - \rho, \alpha_i) = (\nu, \alpha_i) - (\rho, \alpha_i) = \nu_i - \nu_{i+1} - 1$$

by (2.2). Now, $(\nu - \rho, \alpha_i) \in \mathbb{Z}_{\geq 0}$ and so $\nu_i - \nu_{i+1} \in \mathbb{Z}$ and $\nu_i > \nu_{i+1}$ whenever $\alpha_i \in S$.

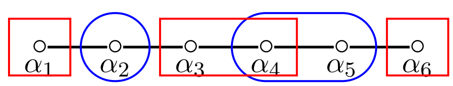
On the other hand, if $\alpha_i \in J$ and $\mu + \rho = (\mu_1, \ldots, \mu_n, \mu_{n+1})$, then:

$$0 = (\mu + \rho, \alpha_i) = \mu_i - \mu_{i+1} \quad \Longleftrightarrow \quad \mu_i = \mu_{i+1}$$

3.1.1 An Example

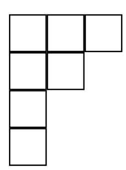
Consider $\mathfrak{g} = \mathfrak{sl}_7(\mathbb{C})$. Let $S = \{\alpha_2, \alpha_4, \alpha_5\}$ and $\mu + \rho = (0, 0, 1, 1, 1, 2, 2)$. Then $J = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$, and the simple roots in S require ν to be of the form $\nu = (\nu_1|\nu_2, \nu_3|\nu_4, \nu_5, \nu_6|\nu_7)$ where $\nu_2 > \nu_3$ and $\nu_4 > \nu_5 > \nu_6$. For convenience, we will say that the coordinates between two consecutive bars in ν are in the same 'corral'. In this example, there are four corrals. The elements in ${}^S\mathcal{W}^J$ are precisely those elements $w \in \mathcal{W} \cong S_{n+1}$ such that $w_S w(\mu + \rho) = \nu$ with the coordinates in each corral of ν arranged in decreasing order. For example, if $\nu = (1|2, 1|2, 1, 0|0)$, then $\nu - \rho$ is a weight in X_S^+.

Put ovals around the nodes in the Dynkin diagram of A_6 corresponding to the simple roots in S, with α_i, α_j in the same oval if and only if $\alpha_k \in S$ for each $i \leq k \leq j$. In the same manner, put boxes around the nodes corresponding to simple roots in J. Then we represent this example as follows:



We can construct a *Young diagram* π_S from ν in the following way. For each corral of ν, write a sequence of squares, one square for each coordinate in the corral. Arrange these sequences of squares from longest to shortest. Put the first sequence as the top row, the second sequence as the next row, and so on. In this example, we have the Young diagram:

π_S:



We can construct a *Young tableau* $\overset{*}{\pi}_J$ from the weight $\mu + \rho$ by using the coordinates which are equal for each row, arranged from longest to shortest. For this example, we have

$\overset{*}{\pi}_J$:

1	1	1
2	2	
0	0	

By the **content** of $\overset{*}{\pi}_J$, we mean the set of integers (including multiplicities) appearing in $\overset{*}{\pi}_J$. Notice that both π_S and $\overset{*}{\pi}_J$ have $n + 1$ boxes.

Now, $w \in {}^S\mathcal{W}^J$ if and only if $\nu_i > \nu_j$ whenever $i < j$ and ν_i, ν_j belong to the same 'corral'. Consequently, there is a one-to-one correspondence between the elements in ${}^S\mathcal{W}^J$ and the Young tableaux of shape π_S and content the same as that of $\overset{*}{\pi}_J$ such that each row is strictly decreasing. Such tableaux are in one-to-one correspondence with *tabloids* (tableaux with unordered rows) of shape π_S and content the same as that of $\overset{*}{\pi}_J$ with each row containing distinct elements. For our example, there are five elements in ${}^S\mathcal{W}^J$, associated to the following five tabloids:

$$\begin{array}{lll} \hline 2 & 1 & 0 \\ \hline 2 & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1} 0 & & \\ \cline{1-1} \end{array} \qquad \begin{array}{lll} \hline 2 & 1 & 0 \\ \hline 2 & 1 & \\ \cline{1-2} 0 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array} \qquad \begin{array}{lll} \hline 2 & 1 & 0 \\ \hline 2 & 0 & \\ \cline{1-2} 1 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array}$$

$$\begin{array}{lll} \hline 2 & 1 & 0 \\ \hline 1 & 0 & \\ \cline{1-2} 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array} \qquad \begin{array}{lll} \hline 2 & 1 & 0 \\ \hline 1 & 0 & \\ \cline{1-2} 1 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} \end{array}$$

Denote a tabloid of shape π_S and content the same as that of $\overset{*}{\pi}_J$ by $\{\overset{*}{\pi}_{S,J}\}$.

3.1.2 Partitions

The set of **partitions** of an integer N is the set

$$\mathcal{P}(N) := \{\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in \mathbb{Z}^N \mid \pi_1 \geq \pi_2 \geq \cdots \geq \pi_N \geq 0,\ \sum_{i=1}^{N} \pi_i = N\}$$

and we write $\pi \vdash N$ if $\pi \in \mathcal{P}(N)$. We call π_i the i**th part** of π. For any $\pi \in \mathcal{P}(N)$, the partition $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_N) \vdash N$ defined as

$$\tilde{\pi}_i := \#\{j \mid \pi_j \geq i\} \quad \text{for} \quad 1 \leq i \leq N$$

is called the **dual partition** to π. By convention, we will usually omit the trailing 0's when we write down a partition of n. Any partition π is described via a Young diagram, with part π_i represented as the ith row of length π_i. Furthermore, the Young diagram of $\tilde{\pi}$ is the reflection of the Young diagram of π along its main diagonal.

There is a useful partial ordering $\trianglelefteq$ on partitions of N, called the **dominance ordering**, defined as follows. If $\pi, \eta \vdash N$, then $\pi \trianglelefteq \eta$ if and only if

$$\pi_1 + \pi_2 + \cdots + \pi_i \leq \eta_1 + \eta_2 + \cdots + \eta_i$$

for every i.

Let's define the partitions π_S and π_J which represent the Young diagrams determined by S and J that we introduced in Section 3.1.1. Since $\mathfrak{g}$ is of type A_n, any subroot system Φ' is isomorphic to $A_{r_1} \times A_{r_2} \times \cdots \times A_{r_k}$ with $r_1 \geq r_2 \geq \cdots \geq r_k \geq 1$. For each such subroot system, define a partition

$$\pi = (r_1 + 1, r_2 + 1, \ldots, r_k + 1, 1^m) \tag{3.1}$$

of $n+1$. For $S, J \subseteq \Delta$, denote the partitions determined by Φ_S and Φ_J as, respectively, π_S and π_J. Note that each A_{r_i} in Φ_S corresponds to a unique corral of ν, and it has r_i+1 coordinates. Furthermore, there are exactly $m = k_0 := (n+1) - (|S|+k)$ corrals with only one coordinate because ν has $n+1$ coordinates and $(r_1+1)+(r_2+1)+\cdots+(r_k+1) = |S|+k$ of them are in corrals with at least two coordinates. On the other hand, each A_{r_i} in Φ_J corresponds to a unique coordinate of $\mu + \rho$ of multiplicity $r_i + 1$ and there are $m = l_0 := (n+1) - (|J| + l)$ coordinates of multiplicity 1. Thus, π_S determines the Young diagram and $\overset{*}{\pi}_J$ determines the Young tableau (filled with the coordinates of $\mu + \rho$) defined in Section 3.1.1.

3.2 Nilpotent Orbits

In this section, most of the material can be found in [CMcG]. We will assume here that $\mathfrak{g}$ is a simple Lie algebra and that G is its adjoint group, a connected complex Lie group. For $g \in G$, let $g \cdot x$ denote the adjoint action of G on $x \in \mathfrak{g}$. Let $\mathcal{N}(\mathfrak{g})$ be the variety of nilpotent elements of $\mathfrak{g}$, called the **nullcone** of $\mathfrak{g}$. The restriction of the adjoint action of G on $\mathfrak{g}$ to $\mathcal{N}(\mathfrak{g})$ is an action on $\mathcal{N}(\mathfrak{g})$ and furthermore, $\mathcal{N}(\mathfrak{g})$ has finitely many G-orbits under this action [L]. For each $x \in \mathcal{N}(\mathfrak{g})$, the orbit $G \cdot x$ is called the **nilpotent orbit** of x in $\mathfrak{g}$.

There is a one-to-one correspondence between the set of nilpotent orbits of $\mathfrak{g}$ and certain weighted Dynkin diagrams; i.e., the Dynkin diagram of $\mathfrak{g}$ together with some admissible combination of the integers 0,1, and 2 attached to the nodes [CMcG, Ch. 3].

A parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ is said to be **distinguished** if $\dim \mathfrak{m} = \dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}])$. For example, a Borel subalgebra is distinguished. The following theorem, due to Bala and Carter [BC1, BC2], provides a link between nilpotent orbits of $\mathfrak{g}$ and parabolic subalgebras of $\mathfrak{g}$ (via Levi subalgebras).

Theorem 3.2.1 (Bala-Carter) *There is a natural one-to-one correspondence between the nilpotent orbits of* $\mathfrak{g}$ *and the* G*-conjugacy classes of pairs* $(\mathfrak{m}, \mathfrak{p}_{\mathfrak{m}})$*, where* $\mathfrak{m}$ *is a Levi subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{p}_{\mathfrak{m}}$ *is a distinguished parabolic subalgebra of the semisimple Lie algebra* $[\mathfrak{m}, \mathfrak{m}]$*.*

Since any Borel subalgebra is distinguished, there is a nilpotent orbit O_S associated to the G-conjugacy class of the pair $(\mathfrak{m}_S, \mathfrak{b}_{\mathfrak{m}_S})$, where $\mathfrak{b}_{m_S}$ is a Borel subalgebra of $[\mathfrak{m}_S, \mathfrak{m}_S]$. In fact, one can choose orbit representatives naturally in this case as follows. For each $\alpha \in S$, let x_α be a fixed nonzero element of the root space $\mathfrak{g}_\alpha$ and define a **regular nilpotent element**

$$x_S = \sum_{\alpha \in S} x_\alpha$$

for $\mathfrak{m}_S$. Then $O_S = G \cdot x_S$.

Proposition 3.2.2 *If* $S, S' \subseteq \Delta$*, then* $O_S = O_{S'}$ *if and only if* Φ_S *and* $\Phi_{S'}$ *are* $\mathcal{W}$*-conjugate.*

Proof. $O_S = O_{S'}$ if and only if $(\mathfrak{m}_S, \mathfrak{b}_{\mathfrak{m}_S})$ and $(\mathfrak{m}_{S'}, \mathfrak{b}_{\mathfrak{m}_{S'}})$ are G-conjugate, which is true if and only if $\mathfrak{m}_S$ and $\mathfrak{m}_{S'}$ are G-conjugate. By [ColMcG, Lemma 3.8.1], this is true if and only if Φ_S and $\Phi_{S'}$ are $\mathcal{W}$-conjugate. $\square$

In consequence of Proposition 3.2.2, we will call an orbit O_S a **root system orbit of type** Φ_S. It turns out that in a few cases (mainly to be dealt with in types D_n and E_7) $O_S \neq O_{S'}$ even though Φ_S and $\Phi_{S'}$ are of the same type.

There is a partial order on the set of nilpotent orbits of $\mathfrak{g}$ defined by $O \leq O'$ if and only if $\overline{O} \subseteq \overline{O'}$.

3.2.1 Nilpotent Orbits for Classical Lie Algebras

Suppose $\mathfrak{g}$ is a classical Lie algebra so that $\mathfrak{g}$ is of type X_n, where X is one of A, B, C or D. Then the nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ are parameterized by certain partitions of some integer $N(X_n)$ [CMcG, Sec. 5.1]. This parametrization is as follows.

Type A_n: If $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, then any nilpotent element $x \in \mathcal{N}(\mathfrak{g})$ is conjugate under $G = GL_n(\mathbb{C})$ to a nilpotent Jordan-block matrix x_π with Jordan block sizes given by the parts of the partition $\pi \vdash n+1$. Consequently, the nilpotent orbits of $\mathfrak{sl}_{n+1}(\mathbb{C})$ are parameterized by the set $\mathcal{P}_A(n+1) := \mathcal{P}(n+1)$ of all partitions of $N(A_n) = n+1$.

Type B_n: If $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$, then the nilpotent orbits of $\mathfrak{g}$ are parameterized by the set $\mathcal{P}_B(2n+1)$ of partitions of $N(B_n) = 2n+1$ for which the even parts occur with even multiplicity.

Type C_n: If $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, then the nilpotent orbits are parameterized by the set $\mathcal{P}_C(2n)$ of partitions of $N(C_n) = 2n$ for which the odd parts occur with even multiplicity.

Type D_n: If $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$, then the nilpotent orbits are parameterized by the set $\mathcal{P}_D(2n)$ of partitions of $N(D_n) = 2n$ for which the even parts occur with even multiplicity, except that if $\pi \vdash 4n$ is a very even partition (i.e., π has only even parts and each even part occurs with even multiplicity), then there are exactly two orbits corresponding to π.

We will denote an orbit corresponding to the partition π by O_π. For type D_{2n}, if $\pi \vdash 4n$ is a very even partition, then denote by O_π^I and O_π^{II} the two orbits associated to π. If $\mathfrak{g}$ is of type A_n, B_n, or C_n and $O_\pi, O_{\pi'}$ correspond respectively to the partitions $\pi, \pi' \vdash N$, then $\overline{O_\pi} \subseteq \overline{O_{\pi'}}$ if and only if $\pi \trianglelefteq \pi'$. The same statement is true in type D_n unless $\pi = \pi'$ is very even and $O_\pi = O_\pi^I$ and $O_{\pi'} = O_\pi^{II}$ in which case O_π^I and O_π^{II} are incomparable. See [Ger, Hes].

If $S \subseteq \Delta$, then $O_S = O_{\pi_S}$ for some $\pi_S \in \mathcal{P}_X(N)$ and, in fact, π_S is constructed as follows [Spalt]. If $\mathfrak{g}$ is of type X_n, then $\mathfrak{m}_S$ is of type $X_m \times A_{n_1-1} \times A_{n_2-1} \times \cdots \times A_{n_r-1}$ for $m \geq 0$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 2$.

- If $X = A$, then $m = 0$ and there exists $k \in \mathbb{Z}_{\geq 0}$ such that $n_1 + n_2 + \cdots + n_r + k = n+1$ and $\pi_S = (n_1, n_2, \ldots, n_r, 1^k)$. In fact, $k = (n+1) - (|S| + r)$ and π_S is the partition defined in Section 3.1.2 with the same symbol.

- If $X = B$, then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $(2m+1)+2n_1+2n_2+\cdots+2n_r+k = 2n+1$ and π_S is obtained by arranging the set

  $$\{2m+1, n_1, n_1, n_2, n_2, \ldots, n_r, n_r, 1, \ldots, 1\}$$

  (k 1's) in decreasing order. Note that we distinguish a root system of type A_1 from that of B_1 (short roots).

- If $X = C$, then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $2m + 2n_1 + 2n_2 + \cdots + 2n_r + k = 2n$ and arranging the set

  $$\{2m, n_1, n_1, n_2, n_2, \ldots, n_r, n_r, 1, \ldots, 1\}$$

  (k 1's) in decreasing order gives π_S. Again we distinguish a root system of type A_1 from that of C_1 (long roots).

- If $X = D$, then $m \neq 1$ and there exists $k \in \mathbb{Z}_{\geq 0}$ such that $2m+2n_1+2n_2+\cdots+2n_r+k = 2n$. If $m = 0$, then π_S is obtained by arranging the set

  $$\{n_1, n_1, n_2, n_2, \ldots, n_r, n_r, 1 \ldots, 1\}$$

  (k 1's) in decreasing order; if $m \geq 2$, then π_S is obtained by arranging the set

  $$\{2m-1, n_1, n_1, n_2, n_2, \ldots, n_r, n_r, 1, 1, \ldots, 1\}$$

  ($k+1$ 1's) in decreasing order. Note that we are distinguishing the subroot system of type D_2 from that of type $A_1 \times A_1$ and the subroot system of type D_3 from that of A_3.

3.2.2 Nilpotent Orbits for Exceptional Lie Algebras

If $\mathfrak{g}$ is of type E_6, E_7, E_8, F_4, or G_2, then the correspondence of Theorem 3.2.1 is given in the tables in [CMcG, Sec. 8.4]. Furthermore, the partial orderings on nilpotent orbits of the exceptional Lie algebras are given by Hasse diagrams in [Cart, pp. 439–445]. Remark: there are some small corrections to the Hasse diagram for E_8 which are given in [UGA, Sec. 7].

3.3 Richardson Orbits

Let $\mathfrak{g}$ be any simple Lie algebra. Let $S \subseteq \Delta$ and let $\mathfrak{p}_S = \mathfrak{m}_S \oplus \mathfrak{u}_S$ be the corresponding parabolic subalgebra of $\mathfrak{g}$. Then $G \cdot \mathfrak{u}_S$ is a closed, irreducible subvariety of $\mathcal{N}(\mathfrak{g})$ and there exists a unique nilpotent orbit R_S such that $\overline{R_S} = G \cdot \mathfrak{u}_S$. The orbit R_S is called the **Richardson orbit** corresponding to S.

Given $S, S' \subseteq \Delta$, it is possible to have $R_S = R_{S'}$ even if $S \neq S'$. For $S, S' \subseteq \Delta$, write $S \sim S'$ if and only if $R_S = R_{S'}$. This is an equivalence relation on Δ and Hirai classified the equivalence classes in [H] as follows.

Theorem 3.3.1 (Hirai) *Let $S, S' \subseteq \Delta$.*

(i) If S and S' are orthogonal to each other, and $T_1, T_2 \subseteq S$ and $T_1', T_2' \subseteq S'$ with $T_1 \sim T_2$ in Φ_S and $T_1' \sim T_2'$ in $\Phi_{S'}$, then $T_1 \cup T_1' \sim T_2 \cup T_2'$ in Φ.

(ii) We have the following relations.

(a) In A_n, if there exists $w \in \mathcal{W}$ (the Weyl group of A_n) such that $wS = S'$, then $S \sim S'$.

(b) In B_n or C_n, if $n = 3k-1$ for $k \geq 1$, then $(\Delta - \{\alpha_{2k-1}\}) \sim (\Delta - \{\alpha_{2k}\})$.

(c) In D_4, $\{\alpha_1, \alpha_2\} \sim \{\alpha_1, \alpha_3, \alpha_4\}$.

(d) In D_n, if $n = 2k+1$ or $n = 3k+1$ for $k \geq 2$, then $(\Delta - \{\alpha_{2k}\}) \sim (\Delta - \{\alpha_{2k+1}\})$.

(e) In E_6, $(\Delta - \{\alpha_1\}) \sim (\Delta - \{\alpha_6\})$, $(\Delta - \{\alpha_3\}) \sim (\Delta - \{\alpha_5\})$, and $(\Delta - \{\alpha_4\}) \sim (\Delta - \{\alpha_2, \alpha_5\})$.

(f) *In* E_8, $(\Delta - \{\alpha_5\}) \sim (\Delta - \{\alpha_2, \alpha_3\})$.

(g) *In* F_4, $\{\alpha_1, \alpha_2, \alpha_4\} \sim \{\alpha_1, \alpha_3, \alpha_4\} \sim \{\alpha_2, \alpha_3\}$.

(h) *In* G_2, $\{\alpha_1\} \sim \{\alpha_2\}$.

(iii) *The relations in (ii) generate the whole equivalence relation* $\sim$ *under the property given in (i).*

Recall that $\mathcal{W} \cong N_G(T)/T$, where T is a maximal toral subgroup of G. For $w \in \mathcal{W}$, let $w \mapsto g_w T \in N_G(T)/T$ under this isomorphism.

Proposition 3.3.2 *If* $S, S' \subseteq \Delta$ *and* $O_S = O_{S'}$*, then* $R_S = R_{S'}$*.*

Proof. If $O_S = O_{S'}$, then by Proposition 3.2.2, there exists $w \in \mathcal{W}$ such that $\Phi_{S'} = w\Phi_S$. Hence, if $g_w \in G$ is a representative of w, then $\mathfrak{u}_{S'} = g_w \cdot \mathfrak{u}_S$ because if $\alpha \in \Phi \backslash \Phi_S$ and $x_\alpha \in \mathfrak{g}_\alpha$, then $g_w \cdot x_\alpha \in \mathfrak{g}_{w\alpha}$ and $w\alpha \in \Phi \backslash \Phi_{S'}$ since $\Phi_{S'} = w\Phi_S$. Consequently,

$$\overline{R}_S = G \cdot \mathfrak{u}_S = (Gg_w) \cdot \mathfrak{u}_S = G \cdot (g_w \cdot \mathfrak{u}_S) = G \cdot \mathfrak{u}_{S'} = \overline{R}_{S'}$$

which implies that $R_S = R_{S'}$. □

We may have $R_S = R_{S'}$ even though $O_S \neq O_{S'}$. For example, suppose $\mathfrak{g}$ is of type D_4 and $S = \{\alpha_1, \alpha_2\}$ and $S' = \{\alpha_1, \alpha_3, \alpha_4\}$. Then Theorem 3.3.1(ii)(d) implies that $R_S = R_{S'}$. But Φ_S is of type A_2 and $\Phi_{S'}$ is of type A_1^3, and so they are not $\mathcal{W}$-conjugate and therefore $O_S \neq O_{S'}$ by Proposition 3.2.2.

Let X be one of the letters B, C, or D. Given any partition $\pi \vdash N(X_n)$, there is a unique partition $\pi_X \in \mathcal{P}_X(N(X_n))$, called the X**-collapse** of π, with the property that $\pi_X \trianglelefteq \pi$ and $\nu \trianglelefteq \pi_X$ for any partition $\nu \in \mathcal{P}_X(N(X_n))$ with $\nu \trianglelefteq \pi$. For completeness, for any $\pi \in \mathcal{P}_A(n+1)$ set $\pi_A = \pi$.

Let $\mathfrak{g}$ be a simple Lie algebra of type X_n, where X is one of the letters A, B, C, or D. Define a map $d_X : \mathcal{P}_X(N(X_n)) \to \mathcal{P}_X(N(X_n))$ by $d_X(\pi) = (\widetilde{\pi})_X$. Then d_X induces a map

d'_X on the set of nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ via $d'_X(O_\pi) = O_{d_X(\pi)} = O_{(\tilde{\pi})_X}$ if π is not very even whenever $X_n = D_{2m}$, and if $X_n = D_{2m}$ with $\pi \vdash 4m$ very even, then:

$$d'_D(O^I_\pi) = \begin{cases} O^I_{\tilde{\pi}} & \text{if } m \text{ is even} \\ O^{II}_{\tilde{\pi}} & \text{if } m \text{ is odd} \end{cases} \quad \text{and} \quad d'_D(O^{II}_\pi) = \begin{cases} O^{II}_{\tilde{\pi}} & \text{if } m \text{ is even} \\ O^I_{\tilde{\pi}} & \text{if } m \text{ is odd} \end{cases}$$

The following theorem is due to Kraft [Kr] for type A_n and Spaltenstein [Spalt] for types B_n, C_n, and D_n.

Theorem 3.3.3 *If $\mathfrak{g}$ is a simple Lie algebra of type X_n, where X is one of the letters A, B, C, or D, then d_X is an order-reversing map with the property that $d^2_X(\pi) \trianglerighteq \pi$ for any $\pi \in \mathcal{P}_X(N(X_n))$. Furthermore, the induced map d'_X is an order reversing map on the set of nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ such that for all $S \subseteq \Delta$, $d'_X(O_S) = R_S$.*

To get an idea of how the transpose partitions come up, consider $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$. If $\pi \vdash n+1$, then:

$$O_\pi = \{\, x \in \mathfrak{sl}_{n+1}(\mathbb{C}) \mid \dim \operatorname{Ker} x^i = \sum_{j=1}^{i} \tilde{\pi}_j,\ i = 1, 2, \ldots, n \,\}$$

For example, if $\pi = (4, 2, 1)$, then there are three columns of zeros in the nilpotent Jordan block matrix x_π corresponding to π (one for each Jordan block). Since x^2_π increases the number of columns of zeros by one in each block of x_π of size at least 2 and does nothing to blocks of size 1, x^2_π has $3 + 2 = 5$ zero columns. In general, x^i_π has one more column of zeros in each block of x_π of size at least i than x^{i-1}_π and the same number of columns of zeros in blocks of size less than i. Thus, we get the transpose partition $\tilde{\pi} = (3, 2, 1, 1)$.

3.4 Nonzero Blocks of Category $\mathcal{O}_S$

Let $\mathfrak{g}$ be any simple Lie algebra and fix $S, J \subseteq \Delta$. We are now ready to use the machinery we have developed to determine if the block $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero. We start with some lemmas.

Lemma 3.4.1 *If $w \in {}^S\mathcal{W}^J$, then $w_S w(\Phi_J) \cap \Phi_S = \varnothing$. Conversely, if $w_S w(\Phi_J) \cap \Phi_S = \varnothing$ for some $w \in \mathcal{W}$, then $w_1 w w_2 \in {}^S\mathcal{W}^J$ for some $w_1 \in \mathcal{W}_S$ and $w_2 \in \mathcal{W}_J$.*

Proof. Suppose $w \in {}^S\mathcal{W}^J$ and let $\mu + \rho \in \mathfrak{h}^*$ be an antidominant integral weight with $\Phi_\mu = \Phi_J$. Then $w_S w \cdot \mu \in X_S^+$ so that $(w_S w \cdot \mu, \check{\alpha}) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi_S^+$. If $\alpha \in \Phi_S^+$, then

$$0 \leq (w_S w \cdot \mu, \check{\alpha}) = (w_S w(\mu + \rho) - \rho, \check{\alpha}) = (w_S w(\mu + \rho), \check{\alpha}) - (\rho, \check{\alpha})$$

so that $(w_S w(\mu + \rho), \check{\alpha}) \geq (\rho, \check{\alpha}) \in \mathbb{Z}_{>0}$ for all $\alpha \in \Phi_S^+$. Since $w_S = w_S^{-1}$, we thus have that $(\mu + \rho, w^{-1} w_S \check{\alpha}) > 0$ for all $\alpha \in \Phi_S^+$. This means that $w^{-1} w_S \alpha \notin \Phi_J$ for any $\alpha \in \Phi_S^+$. That is, $w^{-1} w_S(\Phi_S) \cap \Phi_J = \varnothing$ which implies that $\Phi_S \cap w_S w(\Phi_J) = \varnothing$ as required.

For the converse, suppose $w_S w(\Phi_J) \cap \Phi_S = \varnothing$ and again let $\mu + \rho \in \mathfrak{h}^*$ be an antidominant integral weight with $\Phi_\mu = \Phi_J$. Since $w_S \in \mathcal{W}_S$, $w_S(\Phi_S) = \Phi_S$ and so $\Phi_J \cap w^{-1}(\Phi_S) = \varnothing$ and therefore $w^{-1}\alpha \notin \Phi_J$ for any $\alpha \in \Phi_S$. Consequently, $(\mu + \rho, w^{-1}\check{\alpha}) \neq 0$ for any $\alpha \in \Phi_S$, which implies that $(w(\mu + \rho), \check{\alpha}) \neq 0$ for any $\alpha \in \Phi_S$. Thus, $w(\mu + \rho)$ is a regular integral weight with respect to Φ_S and so there exists $w_1 \in \mathcal{W}_S$ such that $(w_S w_1 w(\mu + \rho), \check{\alpha}) > 0$ for all $\alpha \in S$. Hence, $(w_S w_1 w(\mu + \rho) - \rho, \check{\alpha}) \geq 0$ for all $\alpha \in S$, since $(\rho, \check{\alpha}) = 1$. Therefore, $w_S w_1 w \cdot \mu \in X_S^+$ so that $w_1 w \mathcal{W}_J \subseteq {}^S\mathcal{W}$. Let $w_2 \in \mathcal{W}_J$ be such that the length of $w_1 w w_2$ is minimal among all elements in the coset $w_1 w \mathcal{W}_J$. Then $w_1 w w_2 \in {}^S\mathcal{W}^J$. □

Let x_J be a regular nilpotent element for $\mathfrak{m}_J$ as defined in Section 3.2. The proofs of the next two lemmas are adaptations of [Jan2, Sec. 2.5].

Lemma 3.4.2 *If $w \in {}^S\mathcal{W}^J$, then $g_{w_S w} \cdot x_J \in \mathfrak{u}_S$. Conversely, if $g_{w_S w} \cdot x_J \in \mathfrak{u}_S$, then $w_1 w w_2 \in {}^S\mathcal{W}^J$ for some $w_1 \in \mathcal{W}_S$ and $w_2 \in \mathcal{W}_J$.*

Proof. Suppose $w \in {}^S\mathcal{W}^J = {}^S\mathcal{W} \cap w_S \mathcal{W}^J$, where $\mathcal{W}^J = \{\, w \in \mathcal{W} \mid w(\Phi_J^+) \subseteq \Phi^+ \,\}$ [BN, Cor. 2.2]. Then $w = w_S y$ for some $y \in \mathcal{W}^J$ so that $y = w_S w$. Now,

$$g_{w_S w} \cdot x_J = g_y \cdot x_J = \sum_{\alpha \in J} g_y \cdot x_\alpha$$

and $g_y \cdot x_\alpha$ has weight $y(\alpha)$ for each $\alpha \in J$. Since $y \in \mathcal{W}^J$, $y(\alpha) \in \Phi^+$ for each $\alpha \in J$. Furthermore, by Lemma 3.4.1, $y(\Phi_J) \cap \Phi_S = \varnothing$ and so $y(\alpha) \notin \Phi_S^+$ for any $\alpha \in J$. Therefore, $g_y \cdot x_\alpha \in \mathfrak{u}_S$ for each $\alpha \in J$ and so $g_y \cdot x_J \in \mathfrak{u}_S$.

Conversely, suppose $g_{w_S w} \cdot x_J \in \mathfrak{u}_S$. Then $x_J \in g_{w_S w}^{-1} \cdot \mathfrak{u}_S = g_{w^{-1} w_S} \cdot \mathfrak{u}_S$. Now, $g_{w^{-1} w_S} \cdot \mathfrak{u}_S$ is the direct sum of all root spaces $\mathfrak{g}_\alpha$ with $\alpha \in w^{-1} w_S(\Phi^+ \backslash \Phi_S)$. Since $x_J \in g_{w^{-1} w_S} \cdot \mathfrak{u}_S$, each $\alpha \in J$ must lie in $w^{-1} w_S(\Phi^+ \backslash \Phi_S^+)$. Thus $\Phi_J^+ \subseteq w^{-1} w_S(\Phi^+ \backslash \Phi_S)$ so that $\Phi_J \subseteq w^{-1} w_S(\Phi \backslash \Phi_S) = w^{-1} w_S(\Phi) \backslash w^{-1} w_S(\Phi_S)$. That is $\Phi_J \cap w^{-1} w_S(\Phi_S) = \varnothing$ which implies that $w_S w(\Phi_J) \cap \Phi_S = \varnothing$. By Lemma 3.4.1, there exists $w_1 \in \mathcal{W}_S$ and $w_2 \in \mathcal{W}_J$ such that $w_1 w w_2 \in {}^S\mathcal{W}^J$. □

Recall the **Bruhat decomposition** of G as the disjoint union

$$G = \bigsqcup_{w \in \mathcal{W}} U g_w B$$

where U is a unipotent subgroup and B is a Borel subgroup of G (see [Jan1, Sec. 1.9]).

Lemma 3.4.3 *If $x_J \in G \cdot \mathfrak{u}_S$, then ${}^S\mathcal{W}^J$ is not empty.*

Proof. Suppose $x_J \in G \cdot \mathfrak{u}_S$. Then $x_J \in g \cdot \mathfrak{u}_S$ for some $g \in G$. Using the Bruhat decomposition of G, we can write $g = u g_w b$ for some $w \in \mathcal{W}$, $u \in U$, and $b \in B$. Since B normalizes $\mathfrak{u}_S$, we have $x_J \in u g_w \cdot \mathfrak{u}_S$, and so $u^{-1} \cdot x_J \in g_w \cdot \mathfrak{u}_S$. On the one hand, $u^{-1} \cdot x_J = x_J + x$, where $x \in \bigoplus_{\beta \in (\Phi^+ \backslash J)} \mathfrak{g}_\beta$. On the other hand, $g_w \cdot \mathfrak{u}_S = \bigoplus_{\alpha \in w(\Phi^+ \backslash \Phi_S^+)} \mathfrak{g}_\alpha$. Consequently, $J \subseteq w(\Phi^+) \backslash w(\Phi_S^+)$, so $\Phi_J \subseteq w(\Phi) \backslash w(\Phi_S)$ which implies that $\Phi_J \cap w(\Phi_S) = \varnothing$. Write $w = (w')^{-1} w_S$ for some $w' \in \mathcal{W}$ so that $w^{-1} = w_S w'$. Then $w_S w'(\Phi_J) \cap \Phi_S = \varnothing$ implies that $w_1 w' w_2 \in {}^S\mathcal{W}^J$ for some $w_1 \in \mathcal{W}_S$ and $w_2 \in \mathcal{W}_J$. □

We are now ready for the main theorem of this section.

Theorem 3.4.4 *Suppose $S, J \subseteq \Delta$. The following are equivalent.*

(i) $\mathcal{O}(\mathfrak{g}, S, J)$ contains at least one simple module.

(ii) $O_J \leq R_S$.

(iii) $O_S \leq R_J$.

Proof. First, $\mathcal{O}(\mathfrak{g}, S, J)$ contains a simple module if and only if ${}^S\mathcal{W}^J$ is not empty, since the simple modules in $\mathcal{O}(\mathfrak{g}, S, J)$ are parameterized by ${}^S\mathcal{W}^J$. If $w \in {}^S\mathcal{W}^J$, then $g_{w_S w} \cdot x_J \in \mathfrak{u}_S$

by Lemma 3.4.2, which means that:

$$x_J \in g_{w^{-1}w_S} \cdot \mathfrak{u}_S \subseteq G \cdot \mathfrak{u}_S$$

Therefore, $G \cdot x_J \subseteq G \cdot \mathfrak{u}_S = \overline{R_S}$ and so $\overline{G \cdot x_J} \subseteq \overline{R_S}$. Since $O_J = G \cdot x_J$, we have $(i) \Rightarrow (ii)$. To show $(ii) \Rightarrow (i)$, if $\overline{O_J} = \overline{G \cdot x_J} \subseteq \overline{R_S} = G \cdot \mathfrak{u}_S$, then $x_J \in G \cdot \mathfrak{u}_S$. Therefore, by Lemma 3.4.3, ${}^S\mathcal{W}^J$ is not empty.

We will now show $(ii) \Leftrightarrow (iii)$. First, we have that ${}^S\mathcal{W}^J = ({}^J\mathcal{W}^S)^{-1} = \{ w^{-1} \mid w \in {}^J\mathcal{W}^S \}$ [BN, Cor. 2.4.1]. Furthermore, $O_J \leq R_S$ if and only if ${}^S\mathcal{W}^J$ is not empty. But ${}^S\mathcal{W}^J = ({}^J\mathcal{W}^S)^{-1}$ and so if one of these sets is nonempty, then so is the other one. Switching the roles of the subsets $S, J \subseteq \Delta$, we have $O_S \leq R_J$ if and only if ${}^J\mathcal{W}^S$ is nonempty. Consequently, $(ii) \Leftrightarrow (iii)$ and the the theorem follows. □

When the orbits are labeled by partitions, then we have an easy criterion for determining whether or not $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero.

Corollary 3.4.5 *Suppose $\mathfrak{g}$ is a Lie algebra of type $X = A, B, C$, or D. If $S, J \subseteq \Delta$, then the following are equivalent.*

(i) $\mathcal{O}(\mathfrak{g}, S, J)$ *contains at least one simple module.*

(ii) $\pi_J \trianglelefteq (\widetilde{\pi}_S)_X$.

(iii) $\pi_S \trianglelefteq (\widetilde{\pi}_J)_X$.

Proof. This follows from Theorem 3.4.4 because $O_J = O_{\pi_J}$ and $R_S = O_{(\widetilde{\pi}_S)_X}$ by Theorem 3.3.3. □

Another consequence of Theorem 3.4.4 comes from the fact that $R_S = R_{S'}$ if and only $S \sim S'$ (Hirai equivalence).

Corollary 3.4.6 *If $S \sim S'$ and $J \sim J'$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero if and only if $\mathcal{O}(\mathfrak{g}, S', J')$ is nonzero.*

Proof. First, $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero if and only if $O_J \leq R_S = R_{S'}$ if and only if $\mathcal{O}(\mathfrak{g}, S', J)$ is nonzero. On the other hand, $\mathcal{O}(\mathfrak{g}, S', J)$ is nonzero if and only if $O_{S'} \leq R_J = R_{J'}$ if and only if $\mathcal{O}(\mathfrak{g}, S', J')$ is nonzero. □

The final consequence below provides insight into the partial orderings on the nilpotent orbits for $\mathfrak{so}_{2n+1}(\mathbb{C})$ and for $\mathfrak{sp}_{2n}(\mathbb{C})$. If $\mathfrak{so}_{2n+1}(\mathbb{C})$ has simple roots Δ and $\mathfrak{sp}_{2n}(\mathbb{C})$ has simple roots Δ', then using the labeling of simple roots given in Figure 2.1 for B_n and C_n, we have a one-to-one correspondence $\alpha_i \leftrightarrow \alpha'_i$ between simple roots in Δ and those in Δ'. Hence, if $S \subseteq \Delta$, then $S \leftrightarrow S'$ for some $S' \subseteq \Delta'$.

Corollary 3.4.7 *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ and let $\mathfrak{g}' = \mathfrak{sp}_{2n}(\mathbb{C})$ with respective bases Δ and Δ'. If $S, J \subseteq \Delta$ and $S', J' \subseteq \Delta'$ with $S \leftrightarrow S'$ and $J \leftrightarrow J'$, then $\pi_J \trianglelefteq (\widetilde{\pi}_S)_B$ if and only if $\pi_{J'} \trianglelefteq (\widetilde{\pi}_{S'})_C$.*

Proof. First, the respective Weyl groups $\mathcal{W}$, $\mathcal{W}'$ of B_n and C_n are isomorphic, and they have the same Bruhat ordering. Consequently, ${}^S\mathcal{W}^J$ is nonempty if and only if ${}^{S'}(\mathcal{W}')^{J'}$ is nonempty. Therefore, $\pi_J \trianglelefteq (\widetilde{\pi}_S)_B$ if and only if $\mathcal{O}(\mathfrak{so}_{2n+1}(\mathbb{C}), S, J)$ is nonzero if and only if $\mathcal{O}(\mathfrak{sp}_{2n}(\mathbb{C}), S', J')$ is nonzero if and only if $\pi_{J'} \trianglelefteq (\widetilde{\pi}_{S'})_C$. □

3.5 Nonzero Blocks for the Classical Types

If $\mathfrak{g}$ is a classical Lie algebra, Corollary 3.4.5 provides us with just the right tool for determining when an infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero. We will consider here four examples, one for each classical type.

Example 1: A_5. The first case of a non-linear ordering on nilpotent orbits for $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ occurs for $n = 5$. For type A_n, every nilpotent orbit is both a Richardson orbit and a root system orbit. The Hirai equivalence is just the conjugacy classes of subroot systems in this case, which are labeled by their Cartan type. The Hasse diagram in Figure 3.1 exhibits the partial ordering.

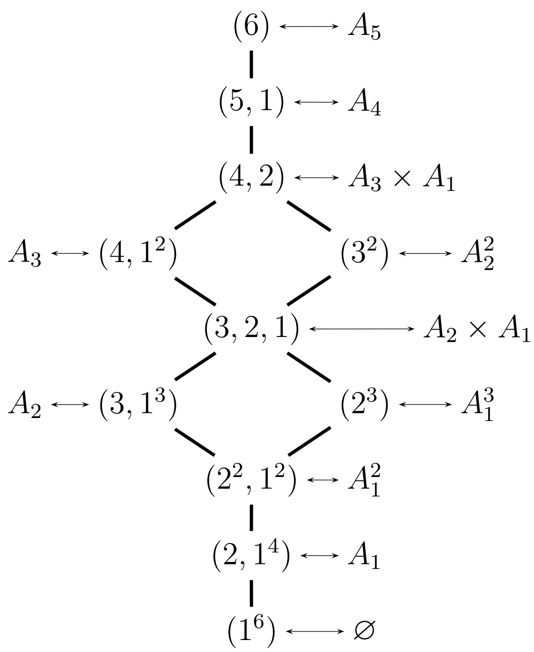


Figure 3.1: Nilpotent Orbits for $\mathfrak{g} = \mathfrak{sl}_6(\mathbb{C})$

Consider, for example, any $S \subseteq \Delta = \{\alpha_1, \ldots, \alpha_5\}$ for which Φ_S is of type A_3. Then $\pi_S = (4, 1^2)$ and $\tilde{\pi}_S = (3, 1^3)$. Thus, if $J \subseteq \Delta$ is such that Φ_J is of type A_2, A_1^2, A_1 or $\varnothing$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero, and for any other J, $\mathcal{O}(\mathfrak{g}, S, J)$ will be zero.

Example 2: B_4. There are 13 nilpotent orbits for $\mathfrak{so}_9(\mathbb{C})$, labeled by the partitions of 9 with even parts having even multiplicity. The partial ordering on these nilpotent orbits is shown in the Hasse diagram on the left in Figure 3.2. If Φ is of type B_n, let A_1 denote any root subsystem generated by a single long root, and let B_1 denote the root subsystem generated by the short root α_n. If a nilpotent orbit is a root system orbit, the corresponding root system is given in Figure 3.2 as well.

We use Theorem 3.3.1 to determine the Hirai equivalence classes on subsets of Δ. Because every subroot system of B_4 has irreducible components of type A_k or B_k, if $S, S' \subseteq \Delta$ with $S \sim S'$, then $|S| = |S'|$. Theorem 3.3.1 (ii)(a) yields $\{\alpha_1\} \sim \{\alpha_2\} \sim \{\alpha_3\}$ since

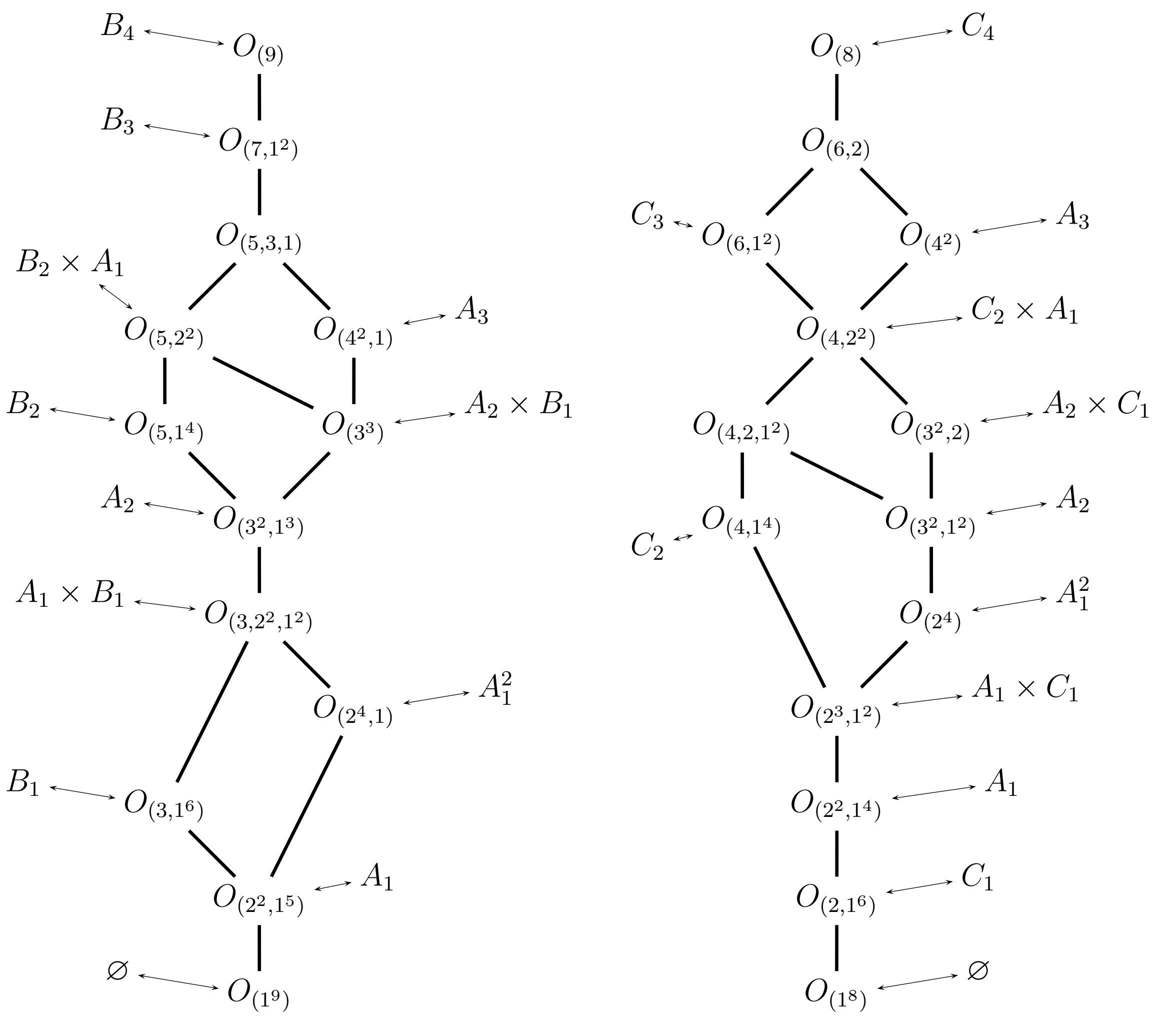


Figure 3.2: Ordering on nilpotent orbits for $\mathfrak{so}_9(\mathbb{C})$ (left) and $\mathfrak{sp}_8(\mathbb{C})$ (right)

Richardson Orbit B_4	Richardson Orbit C_4	Corresponding Hirai Equivalence Class
$O_{(1^9)}$	$O_{(1^8)}$	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
$O_{(3,1^6)}$	$O_{(2^2,1^4)}$	$\{\alpha_2, \alpha_3, \alpha_4\}$
$O_{(3,2^2,1^2)}$	$O_{(2^4)}$	$\{\alpha_1, \alpha_2, \alpha_3\}$
$O_{(3^2,1^3)}$	$O_{(3^2,1^2)}$	$\{\alpha_1, \alpha_3, \alpha_4\}$
$O_{(3^3)}$	$O_{(3^2,2)}$	$\{\alpha_1, \alpha_2, \alpha_4\}$
$O_{(5,1^4)}$	$O_{(4,2,1^2)}$	$\{\alpha_3, \alpha_4\}$
$O_{(5,2^2)}$	$O_{(4,2^2)}$	$\{\alpha_1, \alpha_2\} \sim \{\alpha_2, \alpha_3\}$
$O_{(5,3,1)}$	$O_{(4^2)}$	$\{\alpha_1, \alpha_3\} \sim \{\alpha_1, \alpha_4\} \sim \{\alpha_2, \alpha_4\}$
$O_{(7,1^2)}$	$O_{(6,2)}$	$\{\alpha_1\} \sim \{\alpha_2\} \sim \{\alpha_3\} \sim \{\alpha_4\}$
$O_{(9)}$	$O_{(8)}$	$\varnothing$

Table 3.1: Richardson Orbits and Hirai Equivalence Classes for B_4 and C_4

$s_i s_{i+1}\alpha_i = \alpha_{i+1}$ for $i = 1, 2$. Furthermore, $\{\alpha_3, \alpha_4\}$ generates a root subsystem of type B_2 and so $\{\alpha_3\} \sim \{\alpha_4\}$ in B_4 by Theorem 3.3.1 (ii)(b) and (i) (taking $S' = \varnothing$). Take $S = \{\alpha_1, \alpha_2\}$ and $S' = \{\alpha_4\}$. Applying Theorem 3.3.1 (i) with $T_1 = \{\alpha_1\}$, $T_2 = \{\alpha_2\}$, and $T_1' = T_2' = S'$, we have $\{\alpha_1, \alpha_4\} \sim \{\alpha_2, \alpha_4\}$. Setting $S = \{\alpha_1\}$ and $S' = \{\alpha_3, \alpha_4\}$, Theorem 3.3.1 (i) yields $\{\alpha_1, \alpha_4\} \sim \{\alpha_1, \alpha_3\}$ by taking $T_1 = \{\alpha_1\} = T_2$, $T_1' = \{\alpha_3\}$, and $T_2' = \{\alpha_4\}$. It is apparent that $\{\alpha_1, \alpha_4\}$ is not in the same equivalence class as $\{\alpha_1, \alpha_2\}$ nor $\{\alpha_3, \alpha_4\}$ (note that S and S' have to be orthogonal to apply Theorem 3.3.1 (i)). On the other hand, $s_1 s_2 s_3\{\alpha_1, \alpha_2\} = \{\alpha_2, \alpha_3\}$ and so $\{\alpha_1, \alpha_2\} \sim \{\alpha_2, \alpha_3\}$. However, $\{\alpha_1, \alpha_2\}$ is not in the same equivalence class as $\{\alpha_3, \alpha_4\}$. No two subsets of Δ with three elements can be in the same equivalence class because there is not enough room in a rank 4 root system to satisfy orthogonality required to apply Theorem 3.3.1 (i). This argument yields the set of ten distinct equivalence classes given in Table 3.1. Notice that there are Richardson orbits that are not root system orbits, and some root system orbits that are not Richardson orbits.

As an example of how Corollary 3.4.5 is applied in this case, suppose $S \subseteq \Delta$ is such that Φ_S is of type A_1^2 so that $\pi_S = (2^4, 1)$. Now, $\widetilde{\pi}_S = (5, 4)$ with collapse $(\widetilde{\pi}_S)_B = (5, 3, 1)$. Hence, the Richardson orbit corresponding to S is $O_{(5,3,1)}$, which agrees with what we have in the Table 3.1. Using Figure 3.2, we see that $\mathcal{O}(\mathfrak{g}, S, J)$ is non-zero if we take J so that Φ_J is of type $B_2 \times A_1$ or type A_3 (as well as of any type below these in the diagram).

Example 3: C_4. There are 14 nilpotent orbits for $\mathfrak{sp}_8(\mathbb{C})$, labeled by the partitions of 8 having odd parts of even multiplicity. The partial ordering on these nilpotent orbits is shown in the Hasse diagram on the right in Figure 3.2. Similar to what we do for B_n, if Φ is of type C_n, let A_1 denote any root subsystem generated by a single short root, and let C_1 denote the root subsystem generated by the long root α_n. Again we show the root system orbits with their corresponding root system in Figure 3.2. As with B_4, there are Richardson orbits that are not root system orbits, and root system orbits that are not Richardson orbits. Theorem 3.3.1 yields the same ten distinct equivalence classes as B_4, given in Table 3.1.

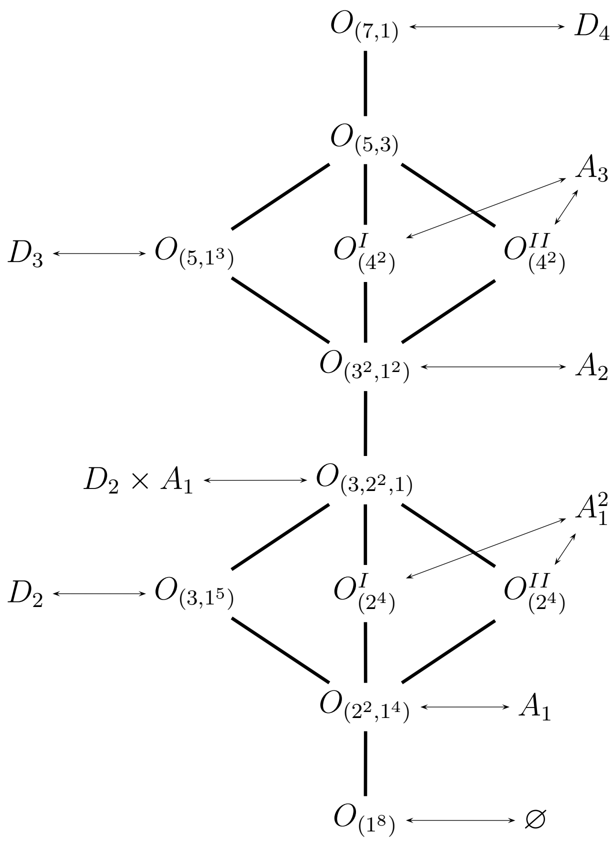


Figure 3.3: Ordering on nilpotent orbits for $\mathfrak{so}_8(\mathbb{C})$

We apply Corollary 3.4.5 the same way in this case as we did for B_4. For example, take $S \subseteq \Delta$ so that Φ_S is of type A_1^2. Then the Richardson orbit corresponding to S is $O_{(4^2)}$, and we can see from Figure 3.2 that $\mathcal{O}(\mathfrak{g}, S, J)$ is non-zero if we take J so that Φ_J is of type A_3 as well as of any type below this in the diagram. Notice that $C_2 \times A_1$ lies below A_3 in Figure 3.2. This agrees with what we found in our B_4 example, even though the diagram for nilpotent orbits in type B_4 is different than that for C_4. We expected the agreement, based on Corollary 3.4.7, but this example shows that the corollary is certainly not trivial!

Example 4: D_4. There are 11 nilpotent orbits for $\mathfrak{so}_8(\mathbb{C})$, labeled by the partitions of 8 with even parts having even multiplicity, with two orbits corresponding to each of the two very

Richardson Orbit	Corresponding Hirai Equivalence Class
$O_{(1^8)}$	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
$O_{(3,1^5)}$	$\{\alpha_2, \alpha_3, \alpha_4\}$
$O^I_{(2^4)}$	$\{\alpha_1, \alpha_2, \alpha_3\}$
$O^{II}_{(2^4)}$	$\{\alpha_1, \alpha_2, \alpha_4\}$
$O_{(3^2,1^2)}$	$\{\alpha_1, \alpha_3, \alpha_4\} \sim \{\alpha_1, \alpha_2\} \sim \{\alpha_2, \alpha_3\} \sim \{\alpha_2, \alpha_4\}$
$O_{(5,1^3)}$	$\{\alpha_3, \alpha_4\}$
$O^I_{(4^2)}$	$\{\alpha_1, \alpha_3\}$
$O^{II}_{(4^2)}$	$\{\alpha_1, \alpha_4\}$
$O_{(5,3)}$	$\{\alpha_1\} \sim \{\alpha_2\} \sim \{\alpha_3\} \sim \{\alpha_4\}$
$O_{(7,1)}$	$\varnothing$

Table 3.2: Richardson Orbits and Hirai Equivalence Classes for D_4

even partitions (2^4) and (4^2) of 8. The partial ordering on these nilpotent orbits is shown in the Hasse diagram in Figure 3.3. If Φ is of type D_n, let D_2 denote the root subsystem generated by the simple roots $\{\alpha_{n-1}, \alpha_n\}$ and let D_3 denote the root subsystem generated by the simple roots $\{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$. The root system orbits are shown in Figure 3.2 with their corresponding root system, noting that the root systems A_1^2 and A_3 each have two root system orbits (since there are two orbits corresponding to a very even partition). Notice that the nilpotent orbit corresponding to $(5, 3)$ is a Richardson orbit that is not a root system orbit, and $(2^2, 1^4)$ and $(3, 2^2, 1)$ correspond to root system orbits that are not Richardson orbits. Note further that the only time collapsing is not necessary to get the Richardson orbit R_S from $\widetilde{\pi}_S$ is when π_S (and therefore $\widetilde{\pi}_S$) is very even.

Using Theorem 3.3.1, we have that there are ten Hirai equivalence classes of subsets of simple roots in D_4, given in Table 3.2. If we use the conventions that $O^I_{(2^4)}$ corresponds to $S_1 = \{\alpha_1, \alpha_3\}$, $O^{II}_{(2^4)}$ corresponds to $S_1' = \{\alpha_1, \alpha_4\}$, $O^I_{(4^2)}$ corresponds to $S_2 = \{\alpha_1, \alpha_2, \alpha_3\}$, and $O^{II}_{(4^2)}$ corresponds to $S_2' = \{\alpha_1, \alpha_2, \alpha_4\}$, then by Theorem 3.3.3, since $m = 2$, we have

$R_{S_1} = O^{I}_{(4^2)}$, $R_{S_1'} = O^{II}_{(4^2)}$, $R_{S_2} = O^{I}_{(2^4)}$, and $R_{S_2'} = O^{II}_{(2^4)}$. For example, $\mathcal{O}(\mathfrak{so}_8(\mathbb{C}), S_1, J)$ is nonzero if $J = S_2$ but it is zero if $J = S_2'$.

3.6 Nonzero Blocks for the Exceptional Types

We will now classify the nonzero infinitesimal blocks in the cases where $\mathfrak{g}$ is an exceptional Lie algebra. We will start with the smallest rank (type G_2) and work up to the largest rank (type E_8).

First, the nilpotent orbits in $\mathfrak{g}$ are parameterized by certain weighted Dynkin diagrams, where each node of such a Dynkin diagram is labeled by one of the integers 0, 1, or 2 [CMcG, Ch. 3]. On the other hand, the nilpotent orbits of $\mathfrak{g}$ can be labeled by the G-conjugacy classes of the pairs $(\mathfrak{m}, \mathfrak{p}_{\mathfrak{m}})$ as in Theorem 3.2.1. Using the notation introduced by Bala and Carter, we will label the nilpotent orbit corresponding to G-conjugacy classes of the pair $(\mathfrak{m}, \mathfrak{p}_{\mathfrak{m}})$ by $X_N(a_i)$, where X_N is the type of the semisimple Lie algebra $[\mathfrak{m}, \mathfrak{m}]$ and i is the number of simple roots in any Levi subalgebra of the distinguished parabolic subalgebra $\mathfrak{p}_{\mathfrak{m}}$ of $[\mathfrak{m}, \mathfrak{m}]$. Write X_N rather than $X_N(a_0)$; these are exactly the root system orbits, with corresponding root system the same as the label. If there are two nilpotent orbits of the same type X_N with the same value for i, we will choose one and label it $X_N(a_i)$ and label the other one $X_N(b_i)$.

A nilpotent orbit is **even** if and only if its weighted Dynkin diagram involves only the weights 0 and 2. Every even nilpotent orbit is Richardson, and its corresponding Hirai equivalence class includes the subset

$$S = \{\, \alpha \in \Delta \mid \text{the } \alpha\text{-node on the Dynkin diagram has weight } 0 \,\}$$

of simple roots. In [H] there is a complete list of Richardson orbits that are not even, along with one subset of simple roots in the corresponding Hirai equivalence class of each such Richardson orbit.

G_2
|
$G_2(a_1)$
|
$\widetilde{A}_1$
|
A_1
|
$\varnothing$

Figure 3.4: Ordering on nilpotent orbits for G_2

Richardson Orbit	Corresponding Hirai Equivalence Class
$\varnothing$	$\{\alpha_1, \alpha_2\}$
$G_2(a_1)$	$\{\alpha_1\} \sim \{\alpha_2\}$
G_2	$\varnothing$

Table 3.3: Richardson Orbits and Hirai Equivalence Classes for G_2

3.6.1 Type G_2

Since there are two root lengths in G_2, denote by $\widetilde{A}_1$ the root subsystem generated by the short root α_1 and by A_1 the root subsystem generated by the long root α_2.

There are five nilpotent orbits in type G_2, with the linear ordering given in Figure 3.4. The three Richardson orbits are described in Table 3.3.

3.6.2 Type F_4

As for G_2 there are two root lengths in type F_4, so denote by A_1 a root subsystem generated by one of the long roots α_1 or α_2, and by $\widetilde{A}_1$ a root subsystem generated by one of the short roots α_3 or α_4. Similarly, denote by A_2 the root subsystem generated by $\{\alpha_1, \alpha_2\}$ and by $\widetilde{A}_2$ the root subsystem generated by $\{\alpha_3, \alpha_4\}$.

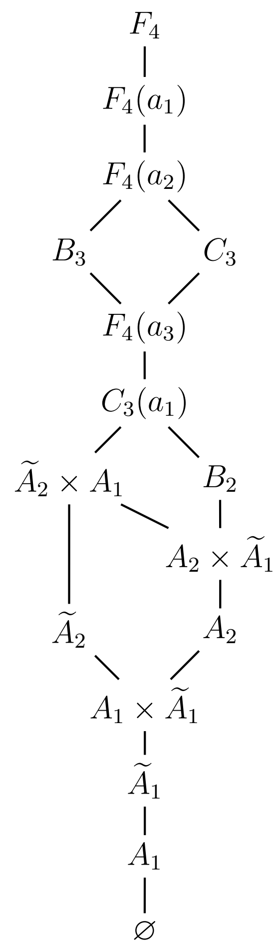


Figure 3.5: Ordering on nilpotent orbits for F_4

Richardson Orbit	Corresponding Hirai Equivalence Class
$\varnothing$	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
A_2	$\{\alpha_2, \alpha_3, \alpha_4\}$
$\widetilde{A}_2$	$\{\alpha_1, \alpha_2, \alpha_3\}$
$F_4(a_3)$	$\{\alpha_1, \alpha_3, \alpha_4\} \sim \{\alpha_1, \alpha_2, \alpha_4\} \sim \{\alpha_2, \alpha_3\}$
B_3	$\{\alpha_3, \alpha_4\}$
C_3	$\{\alpha_1, \alpha_2\}$
$F_4(a_2)$	$\{\alpha_1, \alpha_3\} \sim \{\alpha_1, \alpha_4\} \sim \{\alpha_2, \alpha_4\}$
$F_4(a_1)$	$\{\alpha_1\} \sim \{\alpha_2\} \sim \{\alpha_3\} \sim \{\alpha_4\}$
F_4	$\varnothing$

Table 3.4: Richardson Orbits and Hirai Equivalence Classes for F_4

Richardson Orbit	Corresponding Hirai Equivalence Class
$\varnothing$	E_6
A_1^2	$D_5(2)$
A_2	A_5
A_2^2	D_4
$A_2 \times A_1^2$	$A_4 \times A_1(2)$
A_3	$A_4(4)$
$D_4(a_1)$	$A_2^2 \times A_1$, $A_3 \times A_1(4)$
A_4	$A_3(5)$
D_4	A_2^2
$A_4 \times A_1$	$A_2 \times A_1^2(5)$
$D_5(a_1)$	$A_2 \times A_1(10)$
$E_6(a_3)$	$A_1^3(5)$, $A_2(5)$
D_5	$A_1^2(10)$
$E_6(a_1)$	$A_1(6)$
E_6	$\varnothing$

Table 3.5: Richardson Orbits and Hirai Equivalence Classes for E_6

Figure 3.5 gives the partial ordering on the 16 nilpotent orbits in F_4, and Table 3.4 lists the nine Richardson orbits in F_4. Notice that the nilpotent orbit labeled $C_3(a_1)$ is neither a root system orbit nor a Richardson orbit.

For an example of how Theorem 3.4.4 can be used to determine the non-zero blocks in type F_4, take $S = \{\alpha_3, \alpha_4\}$. From Table 3.4, one has that the Richardson orbit R_S is B_3. Hence, if $J = \{\alpha_1, \alpha_2, \alpha_3\}$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is nonzero. In fact, using Figure 3.5, we can take any J corresponding to a root system orbit that is less than the nilpotent orbit labeled B_3 in the Hasse diagram to obtain a nonzero block $\mathcal{O}(\mathfrak{g}, S, J)$. However, note that if we take $J = \{\alpha_2, \alpha_3, \alpha_4\}$, then $\Phi_J = C_3$, and the associated nilpotent orbit is incomparable to the nilpotent orbit labeled B_3 and therefore, for this J, $\mathcal{O}(\mathfrak{g}, S, J)$ is zero.

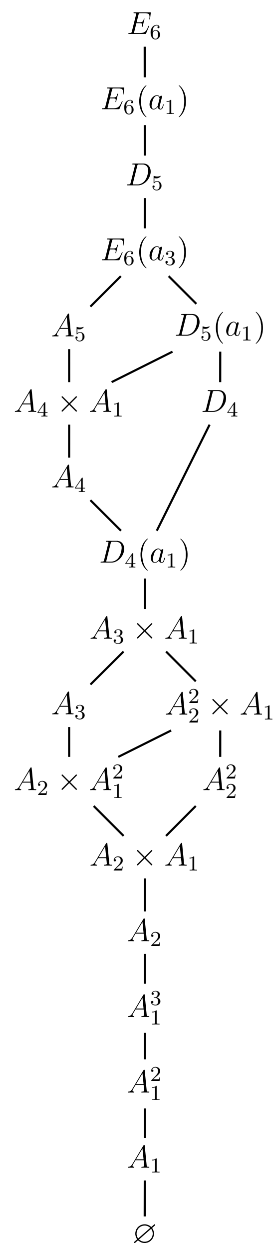


Figure 3.6: Ordering on nilpotent orbits for E_6

Root System Label	Corresponding Subsets of Simple Roots
$(A_5)'$	$\{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} \sim \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$
$(A_5)''$	$\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$
$(A_3 \times A_1)'$	9 out of 11 subsets of Δ generating root systems of type $A_3 \times A_1$
$(A_3 \times A_1)''$	$\{\alpha_2, \alpha_4, \alpha_5, \alpha_7\} \sim \{\alpha_1, \alpha_3, \alpha_4, \alpha_7\}$
$(A_1^3)'$	10 out of 11 subsets of Δ generating root systems of type A_1^3
$(A_1^3)''$	$\{\alpha_2, \alpha_5, \alpha_7\}$

Table 3.6: Root Subsystem Types of E_7 Split Under Hirai Equivalence

3.6.3 Type E_6

There are 21 nilpotent orbits in type E_6, given with their partial ordering in Figure 3.6. Table 3.5 lists the 15 Richardson orbits in E_6. Since there are 64 subsets of the simple roots in E_6, we will introduce some notation to write the Hirai equivalence classes more compactly. If $S, S' \subseteq \Delta$ are such that Φ_S and $\Phi_{S'}$ are of the same type, then $S \sim S'$ in type E_6 and we will write $\Phi_S(k)$ in Table 3.5 for the k subsets of Δ generating root subsystems of the same type as Φ_S. If $k = 1$, then we will just write Φ_S. For example, there are six simple roots in E_6, each generating a root subsystem of type A_1, and so $A_1(6)$ represents these six (singleton) subsets of Δ.

3.6.4 Type E_7

The partial ordering on the 45 nilpotent orbits in type E_7 is given in Figure 3.7, 29 of which are Richardson orbits. The Richardson orbits are listed in Table 3.7. In E_7, there are 128 subsets of the simple roots, so we will again use the notation introduced for type E_6 to write the Hirai equivalence classes more compactly. However, in type E_7, there is a small difficulty: there are subsets $S, S' \subseteq \Delta$ such that Φ_S and $\Phi_{S'}$ are of the same type, but $R_S \neq R_{S'}$. However, this only happens in three cases: when Φ_S is of type A_5, $A_3 \times A_1$, or A_1^3.

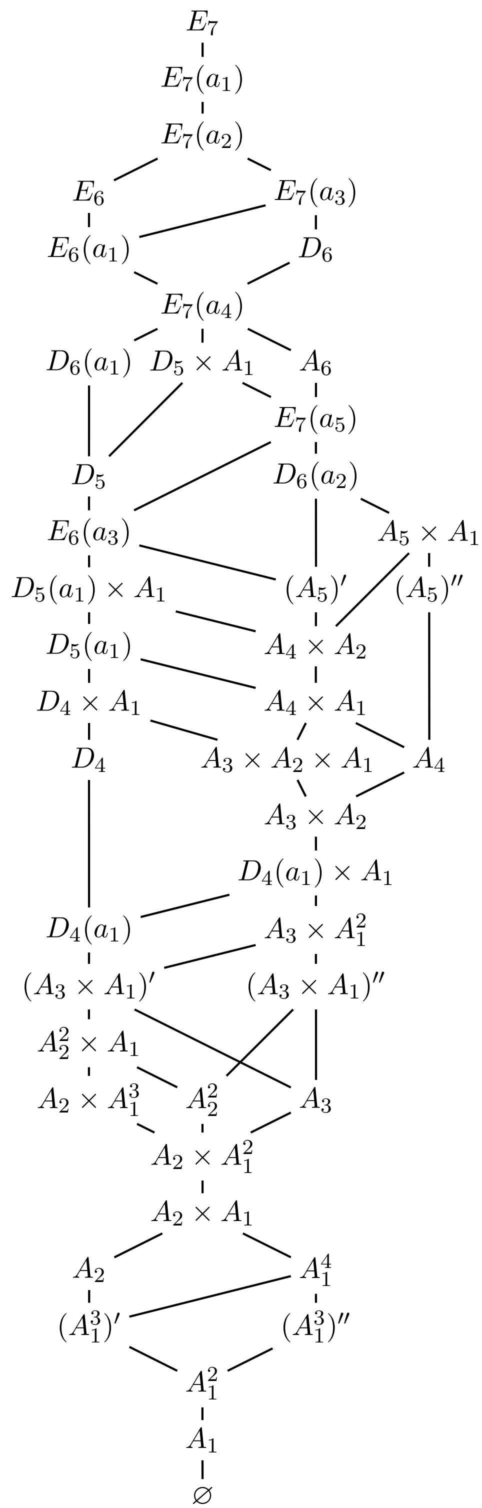


Figure 3.7: Ordering on nilpotent orbits for E_7

Richardson Orbit	Corresponding Hirai Equivalence Class
$\varnothing$	E_7
$(A_1^3)''$	E_6
A_2	D_6
A_2^2	$D_5 \times A_1$
$A_2 \times A_1^3$	A_6
$(A_3 \times A_1)''$	$D_5(2)$
$D_4(a_1)$	$A_5 \times A_1$
D_4	$(A_5)''$
$D_4(a_1) \times A_1$	$(A_5)'(2)$
A_4	$D_4 \times A_1$
$A_3 \times A_2 \times A_1$	$A_4 \times A_2$
$(A_5)''$	D_4
$A_4 \times A_1$	$A_4 \times A_1(5)$
$D_5(a_1)$	$A_4(5)$
$A_4 \times A_2$	$A_3 \times A_2 \times A_1$
$D_5(a_1) \times A_1$	$A_3 \times A_2(3)$
$E_6(a_3)$	$A_3 \times A_1^2(3)$
D_5	$(A_3 \times A_1)''(2)$
$E_7(a_5)$	$A_2^2 \times A_1(3)$, $(A_3 \times A_1)'(9)$
A_6	$A_2 \times A_1^3$
$D_5 \times A_1$	$A_2^2(4)$
$D_6(a_1)$	$A_3(6)$
$E_7(a_4)$	$A_2 \times A_1^2(12)$
$E_6(a_1)$	$A_1^4(2)$, $A_2 \times A_1(18)$
E_6	$(A_1^3)''$
$E_7(a_3)$	$(A_1^3)'(10)$, $A_2(6)$
$E_7(a_2)$	$A_1^2(15)$
$E_7(a_1)$	$A_1(7)$
E_7	$\varnothing$

Table 3.7: Richardson Orbits and Hirai Equivalence Classes for E_7

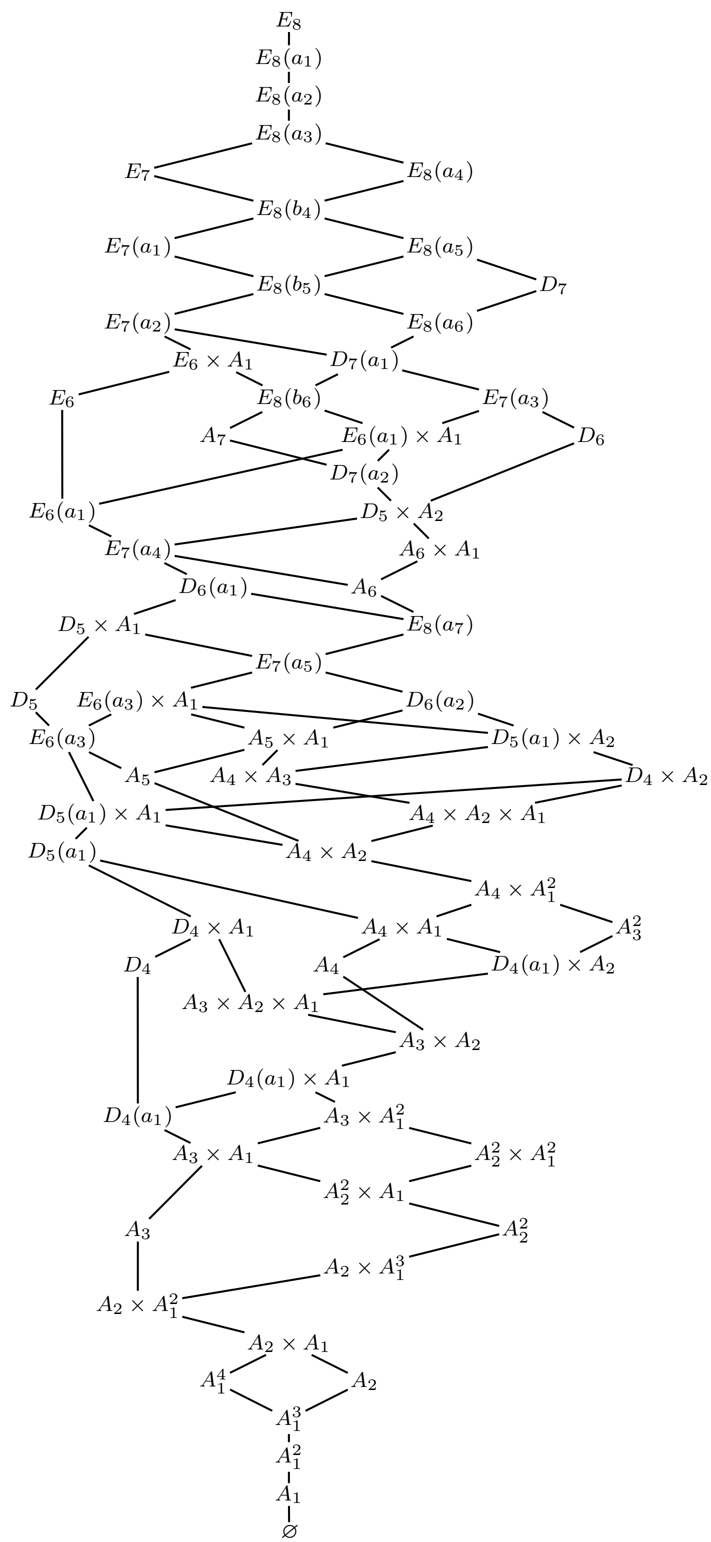


Figure 3.8: Ordering on nilpotent orbits for E_8

Richardson Orbit	Corresponding Hirai Equivalence Class
$\varnothing$	E_8
A_2	E_7
A_2^2	D_7
$D_4(a_1)$	$E_6 \times A_1$
D_4	E_6
A_4	D_6
$D_4(a_1) \times A_2$	A_7
$A_4 \times A_2$	$D_5 \times A_2$
$A_4 \times A_2 \times A_1$	$A_6 \times A_1$
$D_4 \times A_2$	$A_6(3)$
$E_6(a_3)$	$D_5 \times A_1(3)$
D_5	$D_5(2)$
$E_8(a_7)$	$A_4 \times A_3$, $A_5 \times A_1(3)$
A_6	$D_4 \times A_2$
$D_4(a_1)$	$A_5(4)$
$A_6 \times A_1$	$A_4 \times A_2 \times A_1$
$E_6(a_1)$	$D_4 \times A_1(2)$
$D_5 \times A_2$	$A_4 \times A_2(4)$
E_6	D_4
$D_7(a_2)$	$A_4 \times A_1^2(4)$, $A_3^2(2)$
$E_6(a_1) \times A_1$	$A_4 \times A_1(12)$
$E_7(a_3)$	$A_4(6)$
$E_8(b_6)$	$A_3 \times A_2 \times A_1(4)$
$D_7(a_1)$	$A_3 \times A_2(10)$
$E_8(a_6)$	$A_2^2 \times A_1^2(2)$, $A_3 \times A_1^2(10)$
$E_8(b_5)$	$A_2^2 \times A_1(8)$, $A_3 \times A_1(20)$
$E_7(a_1)$	$A_3(7)$
$E_8(a_5)$	$A_2 \times A_1^3(8)$, $A_2^2(8)$
$E_8(b_4)$	$A_2 \times A_1^2(28)$
$E_8(a_4)$	$A_1^4(7)$, $A_2 \times A_1(28)$
$E_8(a_3)$	$A_1^3(21)$, $A_2(7)$
$E_8(a_2)$	$A_1^2(21)$
$E_8(a_1)$	$A_1(8)$
E_8	$\varnothing$

Table 3.8: Richardson Orbits and Hirai Equivalence Classes for E_8

In each of these cases, there are two Hirai equivalence classes represented; we denote one by Φ'_S and the other by Φ''_S. We describe the corresponding subsets of simples in Table 3.6.

3.6.5 Type E_8

There are 70 nilpotent orbits in type E_8 arranged in the partial ordering given in Figure 3.8. The Hasse diagram for E_8 given in [Cart, pp. 439–445] has some errors, but the Hasse diagram for E_8 in Figure 3.8 corrects those errors. There are 34 Richardson orbits in E_8, and they are listed in Table 3.8. We will again use the notation introduced for type E_6 to write the Hirai equivalence classes of the 256 subsets of simple roots in E_8 more compactly. Note that the splitting of a single root system type into distinct Hirai equivalence classes which occurs in type E_7 does not occur in type E_8 (or any of the other exceptional Lie algebras).

Chapter 4

Representation Type of Infinitesimal Blocks

4.1 Representation Type

Given a simple Lie algebra $\mathfrak{g}$ and subsets S, J of simple roots, the infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$ is equivalent to the module category for some quasi-hereditary algebra A [CPS, Sec. 3]. Projective modules of A admit filtrations by certain standard modules (i.e., the parabolic Verma modules). Consequently, it is possible at times to deduce the structures of the projective modules directly. Using this information, one can express the algebra as a quiver with relations from which one can potentially determine the representation type of the algebra.

For the remainder of the paper the statement "representation type of $\mathcal{O}(\Phi, S, J)$" will mean one of the five mutually exclusive conditions for the block: zero, semisimple (see Section 4.2 below), finite representation type (but not semisimple), tame representation type, or wild representation type.

4.2 Radical Filtrations and Extensions

The **radical** of a $\mathfrak{g}$-module V, denoted rad V, is the smallest submodule of V such that $V/\text{rad}\, V$ is semisimple. If V is a $\mathfrak{g}$-module, set $\text{rad}^0\, V = V$ and for each $i \geq 1$, set $\text{rad}^i\, V = \text{rad}(\text{rad}^{i-1}\, V)$. We thus have the **radical filtration** of V:

$$V = \text{rad}^0 V \supseteq \text{rad}^1 V \supseteq \text{rad}^2 V \supseteq \cdots$$

If V is a finite length module (i.e., all chains of submodules in V have finite length), then for each $i \geq 0$, define $\text{rad}_i\, V = \text{rad}^i\, V/\text{rad}^{i+1}\, V$, which is called the i**th radical layer** of

V. Each PVM has a finite radical filtration. We will frequently write the radical layers of a module V with finite radical filtration as:

$$\begin{array}{c} \mathrm{rad}_0 V \\ \mathrm{rad}_1 V \\ \vdots \\ \mathrm{rad}_r V \end{array}$$

Recall that two rings R, S are said to be **Morita equivalent** if the category of R-modules is equivalent to the category of S-modules [Erd, Sec. I.2]. If Λ is a finite dimensional algebra over $\mathbb{C}$, then we say that Λ is a **basic algebra** if all simple Λ-modules are 1-dimensional. Fix a finite dimensional algebra A over $\mathbb{C}$. Then A is Morita equivalent to some basic algebra Λ [Erd, Cor. I.2.7]. Let $L_1, \dots, L_r$ be a complete set of non-isomorphic simple Λ-modules with corresponding projective covers $P_1, \dots, P_r$. The Ext^1_Λ**-quiver** $Q(\Lambda)$ of Λ is a directed graph with vertices in one-to-one correspondence with the simple modules $\{L_i\}$ and the number of arrows from vertex i to vertex j equal to $\dim_{\mathbb{C}} \mathrm{Ext}^1_\Lambda(L_i, L_j) = \dim_{\mathbb{C}} \mathrm{Hom}_\Lambda(P_j, \mathrm{rad}(P_i))/\mathrm{Hom}_\Lambda(P_j, \mathrm{rad}^2(P_i))$.

The **path algebra** $\mathbb{C}Q(\Lambda)$ of $Q(\Lambda)$ is the complex vector space whose basis is the set of all **paths** $\bullet \to \bullet \to \cdots \to \bullet$ in $Q(\Lambda)$, with the product of two paths defined to be the composition if it exists and zero otherwise [Erd, Sec. I.5]. From a theorem of Gabriel [Gab2], the basic finite dimensional algebra Λ is isomorphic to $\mathbb{C}Q(\Lambda)/I$ for some ideal I of the path algebra $\mathbb{C}Q(\Lambda)$. Therefore, the category of A-modules is equivalent to the category of representations of some path algebra of a quiver with relations.

Every extension between two simple modules in category $\mathcal{O}_S$ arises from an extension between them in layers 0 and 1 of the radical filtration of some PVM. An infinitesimal block $\mathcal{O}_S^\mu$ is **semisimple** if and only if there are no extensions between its simple modules. Because there are no self-extensions between simple modules in a highest weight category, an infinitesimal block with only one PVM (and hence only one simple module) is necessarily semisimple.

Let $\mathcal{S}$ be a set of simple modules in $\mathcal{O}_S^\mu$ corresponding to all the vertices in a single graph component of the Ext^1-quiver associated to $\mathcal{O}_S^\mu$. The full subcategory of $\mathcal{O}_S^\mu$ consisting of those modules whose composition factors are all contained in $\mathcal{S}$ is called a **linkage class** of $\mathcal{O}_S^\mu$. It is apparent that $\mathcal{O}_S^\mu$ is semisimple if and only if $\mathrm{rad}_1 V = 0$ for all PVM's V in $\mathcal{O}_S^\mu$ if and only if each linkage class of $\mathcal{O}_S^\mu$ consists of a single simple module.

4.3 The U_α-Algorithm

The U_α algorithm is a tool used to compute radical filtrations of PVM's in an infinitesimal block $\mathcal{O}_S^\mu$ (see [Irv, Sec. 6.3–7.1], [Vog, Sec. 3]).

First, let λ be a regular antidominant integral weight. Fix a simple reflection s_α for some $\alpha \in \Delta$. If one composes the translation functors 'onto' and 'out of' the α-wall, one gets an exact covariant functor θ_α on $\mathcal{O}_S$ called **translation through the α-wall**. For $w \in {}^S\mathcal{W}$, $\theta_\alpha L(w_S w \cdot \lambda) = 0$ unless $w < w s_\alpha \in {}^S\mathcal{W}$; in this case, $\theta_\alpha L(w_S w \cdot \lambda)$ has radical filtration layers:

$$L(w_S w \cdot \lambda)$$
$$U_\alpha L(w_S w \cdot \lambda)$$
$$L(w_S w \cdot \lambda)$$

where $U_\alpha L(w_S w \cdot \lambda)$ is a semisimple module defined as follows. Let ${}_S\mathcal{W} = \{\, x \in {}^S\mathcal{W} \mid x > x s_\alpha \text{ or } x s_\alpha \notin {}^S\mathcal{W} \,\}$. For $x, y \in {}^S\mathcal{W}$ with $x < y$, let $\mu_S(x,y)$ be the coefficient of $q^{(l(y)-l(x)-1)/2}$ in the relative Kazhdan-Lusztig polynomial $P^S_{x,y}(q)$, called the relative Kazhdan-Lusztig μ-function (see [Deo, Sec. 3], [CC, Sec. 3.26]). In fact, $\mu_S(x,y) = [\mathrm{rad}_1 V(w_S y \cdot \lambda) : L(w_S x \cdot \lambda)]$ (see [BN, Sec. 2.3]). Now:

$$U_\alpha L(w_S w \cdot \lambda) = L(w_S w s_\alpha \cdot \lambda) \oplus \bigoplus_{x \in {}_S\mathcal{W}} \mu_S(x,w) L(w_S x \cdot \lambda)$$

One can start with $V(w_S e \cdot \lambda) = L(w_S e \cdot \lambda)$ and use the fact that if $w \in {}^S\mathcal{W}$ with $w < w s_\alpha \in {}^S\mathcal{W}$, then $\theta_\alpha V(w_S w \cdot \lambda)$ is a non-split extension of $V(w_S w s_\alpha \cdot \lambda)$ by $V(w_S w \cdot \lambda)$ to compute inductively the composition factors of each $V(w_S w \cdot \lambda)$.

In fact, using a "graded" version of the U_α-algorithm, one can compute not just the composition factors but also the radical filtrations of the PVM's (see [Bac], [BGS], [BN], [Irv], [Str]). Given a module M with filtration $\{M^i\}$, define σM to be the same module with filtration $(\sigma M)^i = M^{i-1}$. Suppose $w, ws_\alpha \in {}^S\mathcal{W}$ with $w < ws_\alpha$ and that the radical filtration of $V(w_S w \cdot \lambda)$ is known. Compute the radical filtration $V = \mathrm{rad}^0 V \supseteq \mathrm{rad}^1 V \supseteq \mathrm{rad}^2 V \supseteq \cdots$ of $V := V(w_S w s_\alpha \cdot \lambda)$ as follows. First, the module $\theta_\alpha V(w_S w \cdot \lambda)$ has the following filtration. For each $i \geq 0$, let $L(w_S y \cdot \lambda)$ be a composition factor of $\mathrm{rad}_i V(w_S w \cdot \lambda)$ with $y, ys_\alpha \in {}^S\mathcal{W}$ and $y < ys_\alpha$ (so that $\theta_\alpha L(w_S y \cdot \lambda) \neq 0$). If $j = 0, 1, 2$, then $\mathrm{rad}_j \theta_\alpha L(w_S y \cdot \lambda)$ occurs in the $(i+j)$th layer of $\theta_\alpha V(w_S w \cdot \lambda)$. There is a short exact sequence

$$0 \to \sigma V(w_S w s_\alpha \cdot \lambda) \to \theta_\alpha V(w_S w \cdot \lambda) \to V(w_S w \cdot \lambda) \to 0$$

of filtered modules. Hence, deleting the known radical filtration of $V(w_S w \cdot \lambda)$ from $\theta_\alpha V(w_S w \cdot \lambda)$ leaves the radical filtration of $V(w_S w s_\alpha \cdot \lambda)$ (with all layers shifted up 1 in index).

Now suppose that μ is any antidominant integral weight and let $J \subseteq \Delta$ be the set of simple roots on which $\mu + \rho$ is singular. If $x, w \in {}^S\mathcal{W}^J$, then

$$[\mathrm{rad}_i V(w_S w \cdot \mu) : L(w_S x \cdot \mu)] = [\mathrm{rad}_i V(w_S w \cdot \lambda) : L(w_S x \cdot \lambda)]$$

[BN, Sec. 2.3], and consequently, the radical filtration of $V(w_S w \cdot \mu)$ is obtained from that of $V(w_S w \cdot \lambda)$ by ignoring all simple composition factors $L(w_S y \cdot \lambda)$ with $y \notin {}^S\mathcal{W}^J$.

4.4 Representation Type of Infinitesimal Blocks of Category $\mathcal{O}_S$

In this section, we compile some criteria to determine the representation type of a given infinitesimal block of category $\mathcal{O}_S$. These criteria can be used whenever the structure of the PVM's in the infinitesimal block $\mathcal{O}_S^\mu$ is known; however, one criterion for wild representation type depends only on knowing something about the Bruhat order on ${}^S\mathcal{W}^J$ (Proposition 4.4.4).

4.4.1 Triangular Infinitesimal Blocks

Suppose a linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ has m simple modules, labeled $L_1, \ldots, L_m$. If V_i is the PVM with simple quotient L_i, and $V_1, \ldots, V_m$ have radical filtration layers

$$\begin{array}{cccccc} V_1 & V_2 & V_3 & \cdots & V_{m-1} & V_m \\ \hline L_1 & L_2 & L_3 & \cdots & L_{m-1} & L_m \\ & L_1 & L_2 & \cdots & L_{m-2} & L_{m-1} \\ & & L_1 & \cdots & L_{m-3} & L_{m-2} \\ & & & \ddots & \vdots & \vdots \\ & & & & L_1 & L_2 \\ & & & & & L_1 \end{array} \tag{4.1}$$

then we say that the linkage class is *triangular of length* m. If $\mathcal{O}(\mathfrak{g}, S, J)$ has only one linkage class and it is triangular of length m, then we say that $\mathcal{O}(\mathfrak{g}, S, J)$ is a *triangular block of length* m.

The following theorem classifies the representation type of all triangular infinitesimal blocks. For its proof, see [FNP, Props. 5.3, 6.2, 7.1, 7.2].

Theorem 4.4.1 *Suppose $\mathcal{O}(\mathfrak{g}, S, J)$ is triangular of length m.*

(i) If $m = 1$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple.

(ii) If $m = 2$ or $m = 3$, then $\mathcal{O}(\mathfrak{g}, S, J)$ has finite representation type.

(iii) If $m = 4$, then $\mathcal{O}(\mathfrak{g}, S, J)$ has tame representation type.

(iv) If $m \geq 5$, then $\mathcal{O}(\mathfrak{g}, S, J)$ has wild representation type.

4.4.2 Finite Representation Type

Suppose a linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ has $m \geq 2$ simple modules. If these simple modules are labeled $L_1, \ldots, L_m$ and if V_i is the PVM with simple quotient module L_i and the PVM's have radical filtration layers

$$\begin{array}{cccccc} V_1 & V_2 & V_3 & \cdots & V_{m-1} & V_m \\ \hline L_1 & L_2 & L_3 & \cdots & L_{m-1} & L_m \\ & L_1 & L_2 & \cdots & L_{m-2} & L_{m-1} \end{array} \tag{4.2}$$

then we say that the linkage class is *uniserial of length 2*.

The following theorem says that there are very strict conditions placed on the structures of PVM's in a block having finite representation type. For details, see [DoRe, Sec. 1] and [BN, Sec. 3.1].

Theorem 4.4.2 *$\mathcal{O}(\mathfrak{g}, S, J)$ has finite representation type if and only if all the linkage classes of $\mathcal{O}(\mathfrak{g}, S, J)$ having more than one simple module are uniserial of length 2 or triangular of length 3.*

4.4.3 Wild Representation Type

Kite in the Ext^1-quiver

Let Λ be a finite dimensional 2-nilpotent algebra. Gabriel's Theorem [Gab1] asserts that the Ext^1-quiver of Λ separates into a union of quivers whose underlying graphs are Dynkin diagrams if and only if Λ has finite representation type. Furthermore, Dlab and Ringel [DlRi] proved that Λ has tame representation type if and only if the Ext^1-quiver of Λ separates into a union of quivers whose underlying graphs are Dynkin or extended Dynkin diagrams with at least one extended Dynkin diagram.

Suppose the infinitesimal block $\mathcal{O}_S^\mu$ is equivalent to the module category of a quasi-hereditary algebra A. Consider the finite dimensional 2-nilpotent algebra $\Lambda = A/\mathrm{rad}^2 A$. Since $A \twoheadrightarrow \Lambda$, if Λ has wild representation type then so does A. Furthermore, Λ and A have the same Ext^1-quivers since each extension between simple modules arises as an extension between layers 0 and 1 of the radical filtration of some PVM in $\mathcal{O}_S^\mu$.

Now suppose the Ext^1-quiver of Λ contains a 'kite' with any orientation on the arrows, such as the kite shown in Figure 4.1. Since the underlying graph is not a Dynkin diagram

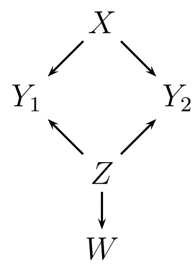


Figure 4.1: A Kite

nor an extended Dynkin diagram, Λ must have wild representation type. This proves the following proposition.

Proposition 4.4.3 *If the* Ext^1*-quiver associated to* $\mathcal{O}_S^\mu$ *contains a kite, then* $\mathcal{O}_S^\mu$ *has wild representation type.*

Diamond Linkage Classes

The following argument is an adaptation of the argument in [FNP, Sec. 4.2] proving that $\mathcal{O}(A_1 \times A_1, \varnothing, \varnothing)$ has wild representation type. Suppose $\mathfrak{g}$ is any simple Lie algebra and $\mathcal{L}$ is a linkage class of the infinitesimal block $\mathcal{O}_S^\mu$ such that $\mathcal{L}$ contains exactly four simple modules L_1, L_2, L_3, L_4 and the corresponding PVM's have the following radical filtration layers:

V_1	V_2	V_3	V_4
			L_4
	L_2	L_3	
L_1			$L_2 \quad L_3$
	L_1	L_1	
			L_1

(4.3)

Based on the structure of the Ext^1-quiver of $\mathcal{L}$, which is shown in Figure 4.2, we will call this a **diamond linkage class**. Using Theorem 2.2.6, we can compute the structures of the projective indecomposable modules. These structures are shown in Table 4.1.

For any idempotent e in an algebra A, A has wild representation type whenever eAe has wild representation type [Erd, I.4.7]. Let $P = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and set $A = \text{End}_{\mathcal{O}_S^\mu}(P)^{op}$.

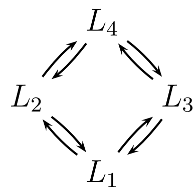


Figure 4.2: Ext^1-Quiver of a Diamond Linkage Class

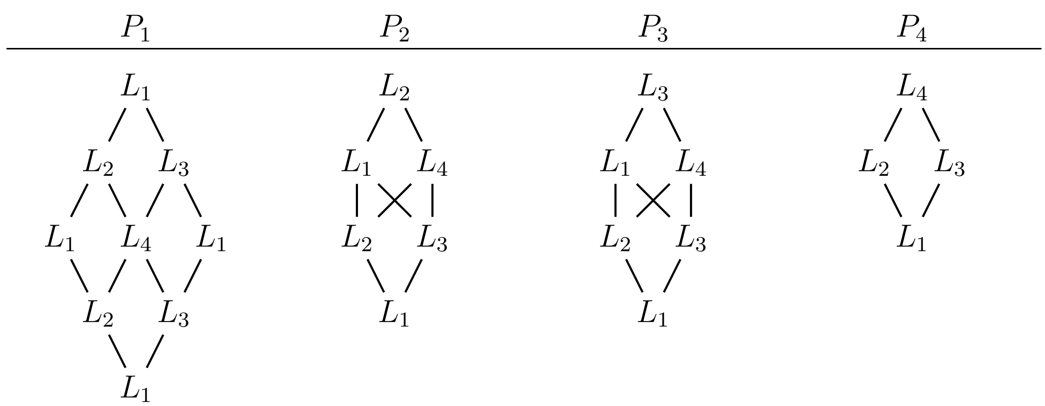


Table 4.1: Projective Indecomposable Modules in a Diamond Linkage Class

Then the diamond linkage class is Morita equivalent to the category of finitely-generated A-modules. Consider the idempotent $e = 1_{L_1} + 1_{L_4}$. Localizing at e and using the structure of P_1 and P_4, we conclude that the quiver of eAe has a subquiver shown in Figure 4.3. Consequently, by [Erd, I.10.8(i)] we have that eAe has wild representation type and therefore $\mathcal{O}_S^\mu$ has wild representation type.

One application of diamonds and kites is the following proposition due to [BN, Sec. 2.7]. We say that four distinct elements $w, x_1, x_2, y \in \mathcal{W}$ form a **diamond** if $y < x_i < w$ for $i = 1, 2$ and $l(w) = l(y) + 2$.

Proposition 4.4.4 (Boe-Nakano) *If ${}^S\mathcal{W}^J$ contains a diamond, then $\mathcal{O}(\mathfrak{g}, S, J)$ has wild representation type.*

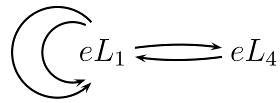


Figure 4.3: Subquiver of eAe

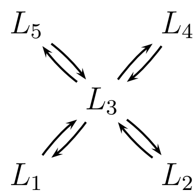


Figure 4.4: The X Ext^1-quiver

This follows because a diamond in ${}^S\mathcal{W}^J$ gives rise to either a kite in the Ext^1-quiver or else a diamond linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$.

X-Linkage Classes

Suppose $\mathcal{O}_S^\mu$ has a linkage class with exactly five PVM's and they have the following radical filtrations:

V_1	V_2	V_3	V_4	V_5
			L_4	L_5
		L_3		
L_1	L_2		L_3	L_3
		L_1 L_2		
			L_1	L_2

(4.4)

where L_i is the simple quotient module of V_i. In this case, we will call this linkage class of $\mathcal{O}_S^J$ an *X-linkage class*. Its Ext^1-quiver is shown in Figure 4.4.

Using Theorem 2.2.6, we find that the structures of the projective indecomposable modules are as shown in Table 4.2. Let $P = P_1 \oplus P_2 \oplus P_4 \oplus P_5$ and set $A = \mathrm{End}_{\mathcal{O}_S^\mu}(P)^{op}$. Take the

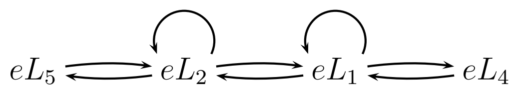


Figure 4.5: Subquiver of eAe for X-block

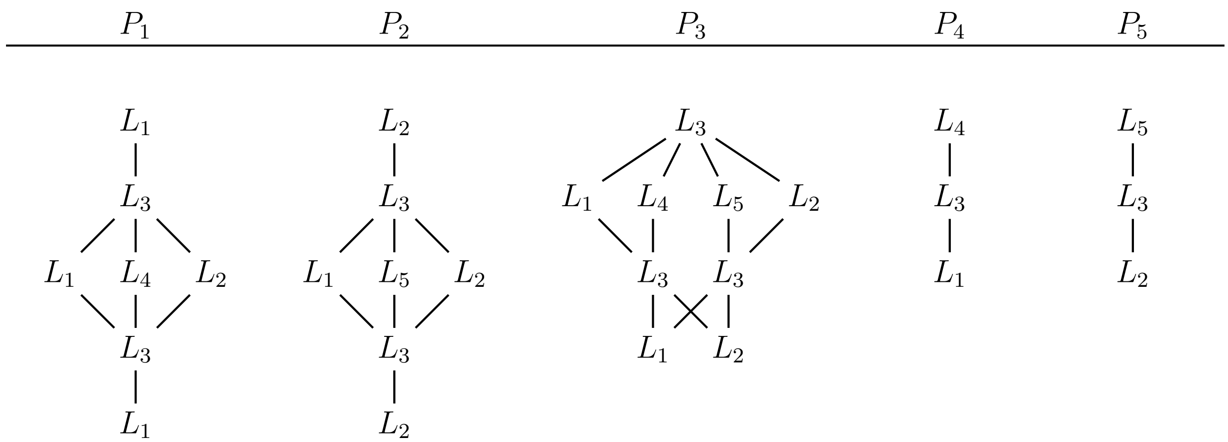


Table 4.2: Projective Indecomposable Modules for X-block

idempotent $e = 1_{L_1} + 1_{L_2} + 1_{L_4} + 1_{L_5}$. Localizing at e and using the structures of P_1, P_2, P_4, and P_5 we conclude that the quiver of eAe has a subquiver of the form shown in Figure 4.5. Now, [Erd, I.10.8(iv)] implies that eAe has wild representation type and therefore so does $\mathcal{O}_S^\mu$.

4.5 Conjectures

We close this chapter by stating two conjectures about the representation type of blocks that relate to nilpotent orbits and Hirai equivalence $\sim$ on subsets of Δ.

Conjecture 4.5.1 *If $\mathfrak{g}$ is any simple Lie algebra and $S, S', J, J' \subseteq \Delta$ are such that $S \sim S'$ and $J \sim J'$, then the infinitesimal blocks $\mathcal{O}(\mathfrak{g}, S, J)$ and $\mathcal{O}(\mathfrak{g}, S', J')$ have the same representation type (in the sense of the convention given in Section 4.1).*

A large number of computations support this conjecture. In the following two chapters, we will see this property manifest itself in various examples. In type A_n, Hirai equivalence of subsets of simple roots amounts to considering the $\mathcal{W}$-conjugacy classes of the roots systems that they generate (Theorem 3.3.1), and so we have the following corollary of the conjecture in this case.

Corollary 4.5.2 *If $\mathfrak{g}$ is of type A_n and $S, S', J, J' \subseteq \Delta$ are such that Φ_S and $\Phi_{S'}$ are $\mathcal{W}$-conjugate, and Φ_J and $\Phi_{J'}$ are $\mathcal{W}$-conjugate, then the infinitesimal blocks $\mathcal{O}(A_n, S, J)$ and $\mathcal{O}(A_n, S', J')$ have the same representation type.*

For $S \subseteq \Delta$, recall the notation O_S for a root system nilpotent orbit, and R_S for a Richardson orbit as defined in Sections 3.2 and 3.3.

Conjecture 4.5.3 *Let $\mathfrak{g}$ be any simple Lie algebra, and let $S, J \subseteq \Delta$. If $O_J = R_S$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple.*

We will prove this conjecture for type A_n in Section 5.1.1. In fact, in we will prove that the converse also holds for type A_n. We will also show that Conjecture 4.5.3 holds in types F_4 and G_2.

The converse of Conjecture 4.5.3 does not hold generally. For example, if $\mathfrak{g}$ is of type F_4, $S = \{\alpha_2, \alpha_3\} = J$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple, but $O_J = B_2 \neq R_S = F_4(a_3)$ (see Sections 6.2 and 6.3). Counterexamples also exist in types B_n and C_n. For example, if $\mathfrak{g}$ is of type C_4, $S = \{\alpha_4\}$ and $J = \{\alpha_2, \alpha_3, \alpha_4\}$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple even though $O_J = O_{(6,1^2)} \neq R_S = O_{(6,2)}$.

Chapter 5

Representation Type of Blocks for the Classical Lie Algebras

In this chapter, take $\mathfrak{g}$ to be a classical Lie algebra. We will look at the representation type of infinitesimal blocks $\mathcal{O}(\mathfrak{g}, S, J)$. We will classify the semisimple blocks in type A_n. Furthermore, we will classify the representation type of $\mathcal{O}(\mathfrak{g}, S, J)$ when $\mathfrak{g}$ is of type A_n or BC_n and $(\mathfrak{g}, S)$ is a Hermitian symmetric pair. By considering examples in each of the classical types, we will shed light on the classification of the representation type of the blocks of category $\mathcal{O}_S$. In fact, we will exhibit a strong connection between the theory of nilpotent orbits and the representation type of the infinitesimal blocks of category $\mathcal{O}_S$.

5.1 Type A_n

In this section, take $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$. A result of Brundan [Bru, Thm. 2] implies that each nonempty block $\mathcal{O}(\mathfrak{g}, S, J)$ contains exactly one linkage class. This property of infinitesimal blocks in type A_n will allow us to classify completely the semisimple infinitesimal blocks in this case. We will show later in this chapter and the one following that this property is unique to type A_n by exhibiting infinitesimal blocks containing more than one linkage class in every other type.

For type A_n, the Hermitian symmetric pairs $(\mathfrak{g}, S)$ are given exactly by the subsets $S \subseteq \Delta$ for which $\mathfrak{p}_S$ is a maximal parabolic subalgebra; i.e., $S = \Delta - \{\alpha\}$ for some $\alpha \in \Delta$. Given any S of this form, we will classify the representation type of the infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$ for any $J \subseteq \Delta$.

5.1.1 Semisimple Blocks in Type A_n

Fix $S, J \subseteq \Delta$, and recall the notation $\overset{*}{\pi}_J$, and $\{\overset{*}{\pi}_{S,J}\}$ as defined in Section 3.1.1. We will use the ideas from Section 3.1.1 to prove the following classification theorem for semisimple infinitesimal blocks in type A_n.

Theorem 5.1.1 *If $\mathfrak{g}$ is of type A_n, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple if and only if $\pi_S = \widetilde{\pi}_J$ if and only if $\pi_J = \widetilde{\pi}_S$.*

Proof. First, $\pi_S = \widetilde{\pi}_J \iff \widetilde{\pi}_S = \widetilde{\widetilde{\pi}}_J \iff \widetilde{\pi}_S = \pi_J$. Let $\pi_S = (r_1, r_2, \ldots, r_k)$ and let $\pi_J = (t_1, t_2, \ldots, t_l)$.

An infinitesimal block $\mathcal{O}(\mathfrak{sl}_{n+1}(\mathbb{C}), S, J)$ is semisimple if and only if it contains exactly one simple module [Bru, Thm. 2]. Hence, $\mathcal{O}(\mathfrak{sl}_{n+1}(\mathbb{C}), S, J)$ is semisimple if and only if ${}^S\mathcal{W}^J$ contains exactly one element. From the discussion in Section 3.1, it suffices to show that $\pi_S = \widetilde{\pi}_J$ if and only if there is exactly one tabloid $\{\overset{*}{\pi}_{S,J}\}$ of shape π_S and content from the tableau $\overset{*}{\pi}_J$ such that each row contains distinct elements.

Suppose that $\pi_S = \widetilde{\pi}_J$. Then there is a tabloid $\{\overset{*}{\pi}_{S,J}\}$ of shape π_S and content $\overset{*}{\pi}_J$ such that each row of the tabloid contains distinct elements, constructed by transposing the tableau $\overset{*}{\pi}_J$. Since each column of $\overset{*}{\pi}_J$ contains distinct elements, the rows of $\{\overset{*}{\pi}_{S,J}\}$ will have distinct elements. Suppose that $\{\overset{*}{\pi}_{S,J}\}'$ is another tabloid of shape π_S and content $\overset{*}{\pi}_J$ with each row containing distinct elements. We will show $\{\overset{*}{\pi}_{S,J}\} = \{\overset{*}{\pi}_{S,J}\}'$ by induction on the number of rows in π_S. If π_S has one row, then $\overset{*}{\pi}_J$ consists of one column and all of its elements are distinct. Hence, the content of $\overset{*}{\pi}_J$ must go in the one row of $\{\overset{*}{\pi}_{S,J}\}$ and we have $\{\overset{*}{\pi}_{S,J}\} = \{\overset{*}{\pi}_{S,J}\}'$. Notice that this argument does not depend on the integer $n+1$. Let Φ' be a root system of type A_m with simple roots Δ', and let $S', J' \subseteq \Delta'$. Suppose that $k \geq 2$ is such that if $\pi_{S'}$ is a partition whose Young diagram has at most $k-1$ rows, and if $\pi_{S'} = \widetilde{\pi}_{J'}$, then the tabloid $\{\overset{*}{\pi}_{S',J'}\}$ constructed by transposing $\overset{*}{\pi}_{J'}$ is the only tabloid of shape $\pi_{S'}$ and content $\pi_{J'}$ with each row containing distinct elements. Now, π_S has k rows and no row of $\{\overset{*}{\pi}_{S,J}\}'$ requires more distinct elements than the first. Furthermore, since all distinct elements of $\overset{*}{\pi}_J$ are in its first column and $\pi_S = \widetilde{\pi}_J$, there are exactly enough distinct elements in $\overset{*}{\pi}_J$ to fill the first row of $\{\overset{*}{\pi}_{S,J}\}'$ (having length r_1), and so it must be the same as row one of $\{\overset{*}{\pi}_{S,J}\}$. Set $m = n + 1 - r_1$ and take $\pi_{S'} = (r_2, \ldots, r_k)$ and $\pi_{J'} = (t_1 - 1, t_2 - 1, \ldots, t_l - 1)$ (note that $l = r_1$). Then $\pi_{S'}$ consists of the last $k - 1$ rows of π_S and $\pi_{J'}$ consists of the last $k - 1$ columns of π_J. Hence, $\pi_{S'} = \widetilde{\pi}_{J'}$. By the inductive hypothesis, the tabloid $\{\overset{*}{\pi}_{S',J'}\}$ of shape $\pi_{S'}$ and content that of $\overset{*}{\pi}_{J'}$ defined by transposing $\overset{*}{\pi}_{J'}$ is unique. Consequently, $\{\overset{*}{\pi}_{S,J}\} = \{\overset{*}{\pi}_{S',J'}\}$ as claimed. Hence, ${}^S\mathcal{W}^J$ contains only one element, and therefore there is only one simple module in $\mathcal{O}(\mathfrak{sl}_{n+1}(\mathbb{C}), S, J)$.

On the other hand, suppose that $\mathcal{O}(\mathfrak{sl}_{n+1}(\mathbb{C}), S, J)$ has only one simple module so that ${}^S\mathcal{W}^J$ contains exactly one element. This element corresponds to a tabloid $\{\overset{*}{\pi}_{S,J}\}$ of shape π_S and content from $\overset{*}{\pi}_J$ such that each row contains distinct elements. Write $\widetilde{\pi}_J = (\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{l'})$. Set $r_0 = 0 = \tilde{t}_0$ and suppose that for some $1 \leq p \leq k$, we have $r_j = \tilde{t}_j$ and the content from column j of $\overset{*}{\pi}_J$ was forced into row j of $\{\overset{*}{\pi}_{S,J}\}$ for all $0 \leq j \leq p - 1$ (which is vacuously true if $j = 0$). By Corollary 3.4.5, $\pi_S \trianglelefteq \widetilde{\pi}_J$ and so we must have $r_p \leq \tilde{t}_p$. Suppose $r_p < \tilde{t}_p$. We will construct a second tabloid $\{\overset{*}{\pi}_{S,J}\}'$ of shape π_S with content from $\overset{*}{\pi}_J$ with distinct elements on each row. Note that there are exactly $\tilde{t}_1$ distinct elements in the content of $\overset{*}{\pi}_J$. Furthermore, there are precisely $\tilde{t}_p$ distinct elements available, since (if $p \geq 2$) the other distinct elements were forced into the longer rows of $\{\overset{*}{\pi}_{S,J}\}$ preceding p. Only r_p of them are in row p of $\{\overset{*}{\pi}_{S,J}\}$. Hence, there is an element a_i appearing in some row $i > p$ which does not appear in row p of $\{\overset{*}{\pi}_{S,J}\}$. But $r_p \geq r_i$, and so row i of $\{\overset{*}{\pi}_{S,J}\}$ has no more elements than the number of elements in row p. Consequently, since a_i is in row i but not in row p, there is an element b_p in row p which is not in row i of $\{\overset{*}{\pi}_{S,J}\}$. Let $\{\overset{*}{\pi}_{S,J}\}'$ be the tabloid constructed from $\{\overset{*}{\pi}_{S,J}\}$ by interchanging element a_i in row i with b_p in row p. Then $\{\overset{*}{\pi}_{S,J}\}'$ is a tabloid of shape π_S with content from $\overset{*}{\pi}_J$ such that elements in each row are distinct, and it is different than the tabloid $\{\overset{*}{\pi}_{S,J}\}$. This contradicts that there is only one element in ${}^S\mathcal{W}^J$. Therefore we must have $l' = k$ and $r_i = \tilde{t}_i$ for each $1 \leq i \leq k$. That is, $\pi_S = \widetilde{\pi}_J$. $\square$

5.1.2 Maximal Parabolic Cases in A_n

Let $\mathfrak{p}_S$ be a maximal parabolic subalgebra of $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ so that $\Delta - S = \{\alpha_k\}$ for some $k \in \{1, \ldots, n\}$ and $\Phi_S = A_{n-k} \times A_{k-1}$. Set S_k to be the subset of Δ such that $\Delta - S_k = \{\alpha_k\}$. Since there is a Dynkin diagram automorphism sending α_i to α_{n-i+1} for each $i \in \{1, \ldots, n\}$, we can assume that $1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$.

Theorem 5.1.2 *If $1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, then*

(i) $\mathcal{O}(A_n, S_k, J)$ is nonzero if and only if $\Phi_J = A_1^l$ for $0 \leq l \leq k$.

(ii) $\mathcal{O}(A_n, S_k, J)$ is semisimple if and only if $\Phi_J = A_1^k$.

(iii) $\mathcal{O}(A_n, S_k, J)$ has finite representation type (but is not semisimple) if and only if $\Phi_J = A_1^{k-1}$.

(iv) $\mathcal{O}(A_n, S_k, J)$ has wild representation type if and only if $\Phi_J = A_1^l$ for $0 \leq l \leq k-2$.

(v) $\mathcal{O}(A_n, S_k, J)$ never has tame representation type.

Proof. For each $1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, the partition corresponding to S_k is $\pi_S = (n-k+1, k)$ and so $\widetilde{\pi}_S = (2^k, 1^{n-2k+1})$. Hence, using Corollary 3.4.5 we have that $\mathcal{O}(A_n, S_k, J)$ is nonzero if and only if $\pi_J \trianglelefteq \widetilde{\pi}_S$ if and only if $\pi_J = (2^l, 1^{n-2l+1})$ for $0 \leq l \leq k$ if and only if $\Phi_J = A_1^l$ for $0 \leq l \leq k$. Furthermore, by Theorem 5.1.1, $\mathcal{O}(A_n, S_k, J)$ is semisimple if and only if $\pi_J = \widetilde{\pi}_S$ if and only if $\Phi_J = A_1^k$.

Suppose $J \subseteq \Delta$ is such that $\Phi_J = A_1^{k-1}$. Since (A_n, Φ_{S_k}) corresponds to a Hermitian symmetric pair, $\mathcal{O}(A_n, S_k, J)$ is equivalent to $\mathcal{O}(A_{n-2k+2}, S, \varnothing)$, where $\Phi_S = A_{n-2k+1}$ [E, Sec. 3], [BH, Thm. 7.3], and this last infinitesimal block has finite representation type by [BN, Thm. 1.4].

Now suppose that J is such that $\Phi_J = A_1^l$ for $0 \leq l \leq k-2$. In this case, $\mathcal{O}(A_n, S_k, J)$ is equivalent to $\mathcal{O}(A_{n-2l}, S, \varnothing)$, where $\Phi_S = A_{k-l-1} \times A_{n-k-l}$, and this infinitesimal block has wild representation type by [BN, Thm. 1.4].

Since we have exhausted all possible subroot systems Φ_J of A_n, the theorem follows. □

5.1.3 An Example in Type A_n

We will use the machinery we have developed to compute the representation type of all infinitesimal blocks $\mathcal{O}(\mathfrak{sl}_6(\mathbb{C}), S, J)$. There are 32 subsets of simple roots in A_5, and so one could exhibit the representation type of the infinitesimal blocks for type A_5 as a 32×32 array, with choices for S given by the rows and choices for J given by the columns. However, Conjecture 4.5.1 is true (verified by direct computation) for type A_5. We will therefore express the representation type of the blocks in type A_5 via an 11×11 array, with rows and columns labeled by the 11 possible root subsystem types in A_5 (see Corollary 4.5.2), arranged according to the partial ordering on nilpotent orbits as given in Figure 3.1; this array is given in Table 5.1. The dashes ($-$) in the table represent zero infinitesimal blocks. They are computed using Corollary 3.4.5, and lie roughly in the upper left corner of the array. The semisimple blocks, labeled with 'SS', are determined by using Theorem 5.1.1, and they lie roughly along the diagonal of the array. The 'F', 'T', and 'W' entries correspond respectively to infinitesimal blocks having finite, tame, or wild representation type. There are three maximal parabolic cases, and they correspond to the rows labeled A_4, $A_3 \times A_1$, and A_2^2. All of the other entries in Table 5.1 were determined using a computer and the results in Chapter 4.

Of particular interest are the triangular infinitesimal blocks; Theorem 4.4.1 was used to determine their representation type. The infinitesimal blocks associated to $\Phi_S = A_2$ or A_1^3, $\Phi_J = A_2 \times A_1$ are triangular of length 3 and therefore have finite representation type. Those associated to $\Phi_S = A_1$, $\Phi_J = A_3 \times A_1$ are triangular of length 4 and therefore have tame representation type. Finally, those associated to $\Phi_S = \varnothing$, $\Phi_J = A_4$ are triangular of length 6 and therefore have wild representation type.

As can be observed in Table 5.1, there appears to be a strong connection between the ordering on nilpotent orbits for $\mathfrak{sl}_{n+1}(\mathbb{C})$ and the representation type of infinitesimal blocks for $\mathfrak{sl}_{n+1}(\mathbb{C})$. This connection is first made in Theorem 3.4.4 for nonzero blocks of any simple Lie algebra, then extended to include semisimple blocks for $\mathfrak{sl}_{n+1}(\mathbb{C})$ via Theorem 5.1.1,

Φ_J / Φ_S	A_5	A_4	$A_3 \times A_1$	A_3	A_2^2	$A_2 \times A_1$	A_2	A_1^3	A_1^2	A_1	$\varnothing$
A_5	–	–	–	–	–	–	–	–	–	–	SS
A_4	–	–	–	–	–	–	–	–	–	SS	F
$A_3 \times A_1$	–	–	–	–	–	–	–	–	SS	F	W
A_3	–	–	–	–	–	–	SS	–	F	W	W
A_2^2	–	–	–	–	–	–	–	SS	F	W	W
$A_2 \times A_1$	–	–	–	–	–	SS	F	F	W	W	W
A_2	–	–	–	SS	–	F	W	W	W	W	W
A_1^3	–	–	–	–	SS	F	W	W	W	W	W
A_1^2	–	–	SS	F	F	W	W	W	W	W	W
A_1	–	SS	T	W	W	W	W	W	W	W	W
$\varnothing$	SS	W	W	W	W	W	W	W	W	W	W

Table 5.1: Representation Type of Infinitesimal Blocks in Type A_5

and then extended further to maximal parabolic subalgebras of $\mathfrak{sl}_{n+1}(\mathbb{C})$. Table 5.1 supports the idea that this connection extends generally for $\mathfrak{sl}_{n+1}(\mathbb{C})$. In fact, the computations done indicate that the structure of an infinitesimal block $\mathcal{O}(\mathfrak{g}, S, J)$ is more complex (in terms of number of simple modules in any linkage class and how the simple modules fit together in the radical filtrations of the parabolic Verma modules) than the structure of any nonzero infinitesimal block $\mathcal{O}(\mathfrak{g}, S', J')$ whenever $O_S \leq O'_S$ and $O_J \leq O'_J$.

5.2 Type BC_n

The Weyl groups of the root systems of type B_n and C_n are isomorphic, and the Bruhat orders on each are the same. Consequently, the relative Kazhdan-Lusztig polynomials are the same in either case, and using the one-to-one correspondence $\alpha_i \leftrightarrow \alpha'_i$ for simple roots

α_i in B_n and α'_i in C_n, there is a one-to-one correspondence between the simple modules in the infinitesimal block $\mathcal{O}(B_n, S, J)$ and the infinitesimal block $\mathcal{O}(C_n, S', J')$, where $S \leftrightarrow S'$ and $J \leftrightarrow J'$, such that if the simple $L_i \in \mathcal{O}(B_n, S, J)$ corresponds to $L'_i \in \mathcal{O}(C_n, S', J')$, then the PVM's $V_i \in \mathcal{O}(B_n, S, J)$ and $V'_i \in \mathcal{O}(C_n, S', J')$ have the same structure. Hence, the projective indecomposable modules $P_i \in \mathcal{O}(B_n, S, J)$ and $P'_i \in \mathcal{O}(C_n, S', J')$ also have the same structure. Therefore, $\mathcal{O}(B_n, S, J)$ and $\mathcal{O}(C_n, S', J')$ have the same representation type. We will consider the representation type of infinitesimal blocks in types B_n and C_n together in most cases, and when doing so, will label them as BC_n.

5.2.1 Semisimple Blocks in Type BC_n

The question of when an infinitesimal block is semisimple in type BC_n is harder than it was in type A_n because there may be more than one linkage class in a semisimple block in this case. However, we will obtain a sufficient condition for an infinitesimal block to be semisimple.

First, let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$. If $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is the standard orthonormal basis of $\mathfrak{h}^* \cong \mathbb{R}^n$, then

$$\Phi = \{\, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i, j \leq n,\ i \neq j \,\} \cup \{\, \pm\varepsilon_i \mid 1 \leq i \leq n \,\}$$

with simple roots $\Delta = \{\, \alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1 \,\} \cup \{\, \alpha_n := \varepsilon_n \,\}$ and positive roots $\Phi^+ = \{\, \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \,\} \cup \{\, \varepsilon_i \mid 1 \leq i \leq n \,\}$. Now, the Weyl group $\mathcal{W}$ of Φ acts on $\mathfrak{h}^*$ as follows. For $1 \leq i \leq n-1$, the simple reflection s_i acts as a transposition which interchanges ε_i and ε_{i+1} and fixes every other basis element, while s_n acts by sending ε_n to $-\varepsilon_n$ and fixing every other basis element.

Suppose $\xi + \rho' = (\xi_1, \xi_2, \ldots, \xi_{2n+1})$ is an antidominant integral weight for $\mathfrak{sl}_{2n+1}(\mathbb{C})$ (written in the standard orthonormal basis of $\mathbb{R}^{2n+1}$). Let $\mathcal{W}' = \langle s'_i \mid 1 \leq i \leq 2n \rangle$ be the Weyl group of A_{2n}. For $1 \leq i \leq n$, set

$$t_i = \begin{cases} s'_i s'_{2n-i+1} & \text{if } 1 \leq i \leq n-1 \\ s'_n s'_{n+1} s'_n & \text{if } i = n \end{cases}$$

and consider the subgroup $\mathcal{W}'' = \langle t_i \mid 1 \le i \le n\rangle$ of $\mathcal{W}'$.

Lemma 5.2.1 *If $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathcal{W}'' = \langle t_i \mid 1 \le i \le n\rangle$, then $\mathcal{W}'' \cong \mathcal{W}$.*

Proof. First, we will show that $\mathcal{W}''$ is isomorphic to a subgroup of $\mathcal{W}$. We do this by showing that the generators of $\mathcal{W}''$ satisfy the relations defining $\mathcal{W}$ as a Coxeter group. First observe that if $1 \le i \le n-1$, then $2n - i + 1 \ge n + 2$. Consequently, since $(s_i')^2 = 1$ for $1 \le i \le 2n$ and $(s_i's_j')^2 = 1$ whenever $|i - j| \ge 2$, it is clear that $t_i^2 = 1$ and $(t_it_j)^2 = 1$ whenever $1 \le i, j \le n$ and $|i - j| \ge 2$. Furthermore, since $(s_i's_{i+1}')^3 = 1$ for all i, it is not hard to see that $(t_it_{i+1})^3 = 1$ for $1 \le i \le n-2$. We are left to verify that $(t_{n-1}t_n)^4 = 1$. We compute:

$$
\begin{aligned}
(s_{n-1}'s_{n+2}'s_n's_{n+1}'s_n')^4 &= (s_{n-1}'s_n's_{n+2}'s_{n+1}'s_n's_{n-1}'s_n's_{n+2}'s_{n+1}'s_n')^2\\
&= (s_{n-1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n's_{n-1}'s_{n+2}'s_{n+1}'s_n')^2\\
&= (s_{n-1}'s_n's_{n-1}'s_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n')^2\\
&= (s_n's_{n-1}'s_n's_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n')^2\\
&= s_n's_{n-1}'s_n's_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n'\\
&= s_n's_{n-1}'s_n's_{n+2}'s_{n+1}'s_{n+2}'s_n's_{n+1}'s_{n+2}'s_n's_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n'\\
&= s_n's_{n-1}'s_n's_{n+1}'s_{n+2}'s_{n+1}'s_n's_{n+1}'s_n's_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n'\\
&= s_n's_{n-1}'s_n's_{n+1}'s_{n+2}'s_n's_{n+1}'s_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n'\\
&= s_n's_{n-1}'s_n's_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n+2}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n'\\
&= s_n's_{n-1}'s_{n+1}'s_n's_{n+1}'s_{n+1}'s_{n+2}'s_{n+1}'s_{n+1}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n'\\
&= s_n's_{n-1}'s_{n+1}'s_n's_{n+2}'s_n's_{n+2}'s_{n+1}'s_{n-1}'s_n' = 1
\end{aligned}
$$

Hence, $\mathcal{W}''$ is isomorphic to a subgroup of $\mathcal{W}$. To show that $\mathcal{W}'' \cong \mathcal{W}$, we will show that $|\mathcal{W}''| \ge |\mathcal{W}|$. To see this, without loss of generality, assume that $\xi + \rho' = (-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n)$, and consider the antidominant weight $\mu + \rho = (-n, -(n-1), \ldots, -1)$ for $\mathfrak{g}$. Comparing the action of $\mathcal{W}''$ on $\xi + \rho'$ and the action of $\mathcal{W}$ on $\mu + \rho$, we see that if $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in \mathcal{W}$, $w' = t_{i_1}t_{i_2}\cdots t_{i_r} \in \mathcal{W}''$ (same indices as for w) and $w(\mu+\rho) = (\mu_1, \mu_2, \ldots \mu_n)$, then $w'(\xi+\rho') = (\mu_1, \mu_2, \ldots, \mu_n, 0, -\mu_n, \ldots, -\mu_2, -\mu_1)$. Notice that t_n essentially acts by changing the signs of the n and $n+2$ coordinates of $\xi+\rho'$ by exchanging those two coordinates. Considering just the first n coordinates of $w'(\xi+\rho')$, we thus have that the number of weights in the $\mathcal{W}$-orbit of $\mu+\rho$ is no more than the number of weights in the $\mathcal{W}''$-obit of $\xi+\rho'$. Since the $\mathcal{W}$-action on $\mu+\rho$ is faithful, we must have $|\mathcal{W}| \leq |\mathcal{W}''|$. Therefore, the lemma follows. $\square$

Fix a set $S \subseteq \Delta$ and let $\mu+\rho$ be an antidominant integral weight, where Φ_μ has simple roots $J \subseteq \Delta$. Suppose that $w \in {}^S\mathcal{W}^J$ and let $\nu = w_S w(\mu+\rho)$ so that $\nu - \rho = w_S w \cdot \mu \in X_S^+$. Write $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ (in the ε-basis). Note that $\check{\alpha}_i = \alpha_i$ if $1 \leq i \leq n-1$ and $\check{\alpha}_n = 2\alpha_n$. If $1 \leq i \leq n-1$, then

$$(\nu - \rho, \check{\alpha}_i) = (\nu, \alpha_i) - (\rho, \alpha_i) = \nu_i - \nu_{i+1} - 1$$

and $(\nu-\rho, \check{\alpha}_n) = (\nu, 2\alpha_n) - (\rho, 2\alpha_n) = 2\nu_n - 1$. Thus, if $\alpha_i \in S$, then $(\nu-\rho, \check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$; so for $1 \leq i \leq n-1$ we have $\nu_i - \nu_{i+1} \in \mathbb{Z}$ with $\nu_i > \nu_{i+1}$, and if $\alpha_n \in S$, then $\nu_n \in \frac{1}{2}\mathbb{Z}$ with $\nu_n \geq \frac{1}{2}$.

On the other hand, if $\alpha_i \in J$ and $\mu+\rho = (\lambda_1, \ldots, \lambda_n)$, then

$$0 = (\mu+\rho, \check{\alpha}_i) = \lambda_i - \lambda_{i+1} \quad \Longleftrightarrow \quad \lambda_i = \lambda_{i+1}$$

whenever $1 \leq i \leq n-1$, and $0 = (\mu+\rho, \check{\alpha}_n) = 2\lambda_n$ which implies $\lambda_n = 0$ whenever $\alpha_n \in J$.

For any $S \subseteq \Delta$, form a subset S' of the simple roots Δ' of A_{2n} by setting:

$$S' = \{\, \alpha_i' \in \Delta' \mid \alpha_i \in \Delta \,\} \cup \{\, \alpha_{2n-i+1}' \mid \alpha_i \in \Delta \,\}$$

Note that the partition $\pi_S \in \mathcal{P}_B(2n+1)$ is the same as the partition $\pi_{S'} \in \mathcal{P}_A(2n+1)$ defined in Section 3.2.1.

We claim that $w := s_{i_1}s_{i_2}\cdots s_{i_r} \in {}^S\mathcal{W}^J$ implies that $w' := t_{i_1}t_{i_2}\cdots t_{i_r} \in {}^{S'}(\mathcal{W}')^{J'}$. Let $\mu+\rho = (\mu_1, \ldots, \mu_n)$ be an antidominant integral weight for $\mathfrak{g}$, where Φ_μ has simple roots J. Without loss of generality, we will take $\mu_n \in \mathbb{Z}_{\leq 0}$. If $\nu = w_S w(\mu+\rho) = (\nu_1, \ldots, \nu_n)$, then taking the antidominant weight $\zeta + \rho' = (\mu_1, \ldots, \mu_n, 0, -\mu_n, \ldots, -\mu_1)$, the proof of Lemma 5.2.1 implies that $w_{S'}w'(\zeta + \rho') = (\nu_1, \ldots, \nu_n, 0, -\nu_n, \ldots, -\nu_1)$. If $w \in {}^S\mathcal{W}^J$, then $\nu_i > \nu_{i+1}$ whenever $\alpha_i \in S$ for $1 \leq i \leq n-1$. If $\alpha_n \in S$, then $\nu_n \geq 1 > 0 > -\nu_n$. Therefore, $w_{S'}w'(\zeta + \rho') \in X^+_{S'}$ and so $w' \in {}^{S'}(\mathcal{W}')^{J'}$. Thus we have $|{}^S\mathcal{W}^J| \leq |{}^{S'}(\mathcal{W}')^{J'}|$.

Now set $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$. Again taking the standard orthonormal basis $\{\varepsilon_1, \cdots, \varepsilon_n\}$ of $\mathfrak{h}^* \cong \mathbb{R}^n$, we have

$$\Phi = \{\, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i, j \leq n,\ i \neq j \,\} \cup \{\, \pm 2\varepsilon_i \mid 1 \leq i \leq n \,\}$$

with simple roots $\Delta = \{\, \alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1 \,\} \cup \{\, \alpha_n := 2\varepsilon_n \,\}$ and positive roots $\Phi^+ = \{\, \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \,\} \cup \{\, 2\varepsilon_i \mid 1 \leq i \leq n \,\}$. Now, the Weyl group $\mathcal{W}$ of Φ is isomorphic to the Weyl group for B_n, and acts on $\mathfrak{h}^*$ in the same way.

Take an antidominant integral weight $\xi + \rho' = (\xi_1, \xi_2, \ldots, \xi_{2n})$ for $\mathfrak{sl}_{2n}(\mathbb{C})$ and now let $\mathcal{W}' = \langle s'_i \mid 1 \leq i \leq 2n \rangle$ be the Weyl group of A_{2n-1}. This time, set

$$t_i = \begin{cases} s'_i s'_{2n-i} & \text{if } 1 \leq i \leq n-1 \\ s'_n & \text{if } i = n \end{cases}$$

for $1 \leq i \leq n$.

Lemma 5.2.2 *If $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ and $\mathcal{W}'' = \langle t_i \mid 1 \leq i \leq n \rangle$, then $\mathcal{W}'' \cong \mathcal{W}$.*

Proof. The proof is similar to that of Lemma 5.2.1. As before, the only relation we need to check to show that $\mathcal{W}''$ is isomorphic to a subgroup of $\mathcal{W}$ is $(t_{n-1}t_n)^4 = 1$. We compute:

$$\begin{aligned}
(s'_{n-1}s'_{n+1}s'_n)^4 &= (s'_{n-1}s'_{n+1}s'_n s'_{n-1}s'_{n+1}s'_n)^2 \\
&= (s'_{n-1}s'_n s'_{n+1}s'_n s'_{n-1}s'_n)^2 \\
&= (s'_{n-1}s'_n s'_{n+1}s'_{n-1}s'_n s'_{n-1})^2 \\
&= (s'_{n-1}s'_n s'_{n+1}s'_{n-1}s'_{n+1}s'_{n-1}s'_n s'_{n-1}) = 1
\end{aligned}$$

To show that $|\mathcal{W}''| \geq |\mathcal{W}|$, without loss of generality, assume that $\xi + \rho' = (-n, -(n-1), \ldots, -1, 1, \ldots, n-1, n)$, and consider the antidominant weight $\mu + \rho = (-n, -(n -$

$1), \ldots, -1)$ for $\mathfrak{g}$. If $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in \mathcal{W}$ and we set $w' = t_{i_1}t_{i_2}\cdots t_{i_r} \in \mathcal{W}''$, then $w(\mu+\rho) = (\mu_1, \mu_2, \ldots \mu_n)$ implies $w'(\xi+\rho') = (\mu_1, \mu_2, \ldots, \mu_n, -\mu_n, \ldots, -\mu_2, -\mu_1)$. Considering just the first n coordinates of $w'(\xi+\rho')$, we see that the number of weights in the $\mathcal{W}$-orbit of $\mu+\rho$ is no more than the number of weights in the $\mathcal{W}''$-obit of $\xi+\rho'$. Hence, $|\mathcal{W}| \leq |\mathcal{W}''|$ and so $\mathcal{W} \cong \mathcal{W}''$. □

Fix a set $S \subseteq \Delta$ and let $\mu+\rho$ be an antidominant integral weight, where Φ_μ has simple roots $J \subseteq \Delta$. Suppose that $w \in {}^S\mathcal{W}^J$ and let $\nu = w_S w(\mu+\rho)$ so that $\nu - \rho = w_S w \cdot \mu \in X_S^+$. Write $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ (in the ε-basis). Note that $\check{\alpha}_i = \alpha_i$ if $1 \leq i \leq n-1$ and $\check{\alpha}_n = \frac{1}{2}\alpha_n$. If $1 \leq i \leq n-1$, then

$$(\nu-\rho, \check{\alpha}_i) = (\nu, \alpha_i) - (\rho, \alpha_i) = \nu_i - \nu_{i+1} - 1$$

and $(\nu-\rho, \check{\alpha}_n) = (\nu, \check{\alpha}_n) - (\rho, \check{\alpha}_n) = \frac{1}{2}\nu_n - 1$. Thus, if $\alpha_i \in S$, then $(\nu-\rho, \check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$; so for $1 \leq i \leq n-1$ we have $\nu_i - \nu_{i+1} \in \mathbb{Z}$ with $\nu_i > \nu_{i+1}$, and if $\alpha_n \in S$, then $\nu_n \in \mathbb{Z}$ with $\nu_n \geq 2$.

On the other hand, if $\alpha_i \in J$ and $\mu+\rho = (\lambda_1, \ldots, \lambda_n)$, then

$$0 = (\mu+\rho, \check{\alpha}_i) = \lambda_i - \lambda_{i+1} \quad \Longleftrightarrow \quad \lambda_i = \lambda_{i+1}$$

whenever $1 \leq i \leq n-1$, and $0 = (\mu+\rho, \check{\alpha}_n) = \frac{1}{2}\lambda_n$ which implies $\lambda_n = 0$ whenever $\alpha_n \in J$.

For any $S \subseteq \Delta$, form a subset S' of the simple roots Δ' of A_{2n-1} by setting:

$$S' = \{\, \alpha_i' \in \Delta' \mid \alpha_i \in \Delta \,\} \cup \{\, \alpha_{2n-i}' \mid \alpha_i \in \Delta \,\}$$

Again the partition $\pi_S \in \mathcal{P}_B(2n+1)$ is the same as the partition $\pi_{S'} \in \mathcal{P}_A(2n)$ defined in Section 3.2.1.

An argument similar to what we did for $\mathfrak{so}_{2n+1}(\mathbb{C})$ shows that $w := s_{i_1}s_{i_2}\cdots s_{i_r} \in {}^S\mathcal{W}^J$ implies $w' := t_{i_1}t_{i_2}\cdots t_{i_r} \in {}^{S'}(\mathcal{W}')^{J'}$. Hence, $|{}^S\mathcal{W}^J| \leq |{}^{S'}(\mathcal{W}')^{J'}|$.

Theorem 5.2.3 *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$, and let $S, J \subseteq \Delta$. If $\pi_J = \widetilde{\pi}_S$, then $\mathcal{O}(BC_n, S, J)$ is semisimple with exactly one simple module.*

Proof. In either case, if $\pi_J = \widetilde{\pi}_S$, then $\pi_{J'} = \widetilde{\pi}_{S'}$. Consequently, using Theorems 3.4.5 and 5.1.1, we have $0 < |{}^S\mathcal{W}^J| \leq |{}^{S'}(\mathcal{W}')^{J'}| = 1$. Therefore, $|{}^S\mathcal{W}^J| = 1$ and the theorem follows. □

5.2.2 Hermitian Symmetric Cases

Suppose $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ so that $\Phi = B_n$ and let $S = \{\alpha_2, \ldots, \alpha_n\}$. Then $\Phi_S = B_{n-1}$ and (B_n, B_{n-1}) is a Hermitian symmetric pair.

Theorem 5.2.4 *Let Φ be of type B_n or C_n, and suppose $S, J \subseteq \Delta$ with $S = \{\alpha_2, \ldots, \alpha_n\}$.*

(i) $\mathcal{O}(BC_n, S, J)$ is nonzero if and only if $\Phi_J = A_1$, B_1, or $\varnothing$.

(ii) $\mathcal{O}(BC_n, S, J)$ is semisimple if and only if $\Phi_J = A_1$ or $\Phi_J = B_1$; in the first case, the infinitesimal block has two simple modules and in the second it has one.

(iii) $\mathcal{O}(BC_n, S, J)$ has finite representation type if and only if $\Phi_J = \varnothing$.

(iv) $\mathcal{O}(BC_n, S, J)$ never has wild or tame representation type.

Proof. First, if $S \leftrightarrow S'$ and $J \leftrightarrow J'$, then $\mathcal{O}(B_n, S, J)$ and $\mathcal{O}(C_n, S', J')$ have the same representation type, so we need only prove the theorem for $\mathcal{O}(B_n, S, J)$. The B-partition corresponding to Φ_S is $\pi_S = (2n-1, 1^2)$ and so $\widetilde{\pi}_S = (3, 1^{2n-2})$. Hence, $\mathcal{O}(B_n, S, J)$ is nonzero if and only if $\pi_J \trianglelefteq \widetilde{\pi}_S$ if and only if $\pi_J = \widetilde{\pi}_S$ or $\pi_J = (2^2, 1^{2n-3})$ or $\pi_J = (1^{2n+1})$ if and only if $\Phi_J = B_1$ or $\Phi_J = A_1$ or $\Phi_J = \varnothing$.

Now, (B_n, B_{n-1}) corresponds to a Hermitian symmetric pair so we will use [E, Sec. 3], [BH, Thm. 7.3] to prove the rest of the theorem. First, if $\Phi_J = A_1$, then $\mathcal{O}(B_n, S, J)$ is equivalent to two copies of the trivial infinitesimal block $\mathcal{O}(\varnothing, \varnothing, \varnothing)$ each of which has one simple module. Consequently, this infinitesimal block has two linkage classes, each with only one simple module, and hence it is semisimple with two simple modules. On the other hand, if $\Phi_J = B_1$, then $\mathcal{O}(B_n, S, J)$ is equivalent to one copy of $\mathcal{O}(\varnothing, \varnothing, \varnothing)$ and so the block is semisimple with one simple module. Finally, $\mathcal{O}(B_n, S, \varnothing)$ has finite representation type by [BN, Theorem 1.4]. We have exhausted all possible subroot systems Φ_J of B_n and so the theorem follows. $\square$

Now set $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ with root system $\Phi = C_n$. Let $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$ so that $\Phi_S = A_{n-1}$. Now we have Hermitian symmetric pair (C_n, A_{n-1}) that we will use to obtain additional results in type BC_n.

Theorem 5.2.5 *Let Φ be of type B_n or C_n, and suppose $S, J \subseteq \Delta$ with $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$. Set $m = \lfloor \frac{n}{2} \rfloor$.*

(i) $\mathcal{O}(BC_n, S, J)$ is nonzero if and only if $\Phi_J = A_1^l$ or $\Phi_J = A_1^l \times C_1$ $(0 \leq l \leq m)$.

(ii) $\mathcal{O}(BC_n, S, J)$ is semisimple if and only if one of the following is true.

(a) n is even and $\Phi_J = A_1^m$ or $\Phi_J = A_1^{m-1} \times C_1$; in the first case, the block has one simple module and in the second it has two.

(b) n is odd and $\Phi_J = A_1^m \times C_1$; the block has exactly one simple module in this case.

(iii) $\mathcal{O}(BC_n, S, J)$ has finite representation type if and only if one of the following is true.

(a) n is even and $\Phi_J = A_1^{m-1}$ or $\Phi_J = A_1^{m-2} \times C_1$.

(b) n is odd and $\Phi_J = A_1^m$ or $\Phi_J = A_1^{m-1} \times C_1$.

(iv) $\mathcal{O}(BC_n, S, J)$ has wild representation type if and only if one of the following is true.

(a) n is even and $\Phi_J = A_1^i$ for $0 \leq i \leq m-2$ or $\Phi_J = A_1^j \times C_1$ for $0 \leq j \leq m-3$.

(b) n is odd and $\Phi_J = A_1^i$ for $0 \leq i \leq m-1$ or $\Phi_J = A_1^j \times C_1$ for $0 \leq j \leq m-2$.

(v) $\mathcal{O}(BC_n, S, J)$ never has tame representation type.

Proof. We will prove the theorem for $\mathcal{O}(C_n, S, J)$ and the case for the corresponding block in type B_n will follow. Set $m = \lfloor \frac{n}{2} \rfloor$. Note that if n is even, then $n = 2m$ and if n is odd, then $n = 2m + 1$. The C-partition corresponding to Φ_S is $\pi_S = (n^2)$ and so $\widetilde{\pi}_S = (2^n)$. Hence, $\mathcal{O}(C_n, S, J)$ is nonzero if and only if $\pi_J \trianglelefteq \widetilde{\pi}_S$ if and only if $\pi_J = (2^i, 1^{2(n-i)})$ for $0 \leq i \leq n$. If $i = 2l$ for some $0 \leq l \leq m$, then $\Phi_J = A_1^l$. If $i = 2l + 1$ for some $0 \leq l \leq m$, then $\Phi_J = A_1^l \times C_1$.

If $\pi_J = (2^n) = \widetilde{\pi}_S$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple with one simple module by Theorem 5.2.3. If $n = 2m$, then $\Phi_J = A_1^m$, and if $n = 2m+1$, then $\Phi_J = A_1^m \times C_1$.

Suppose $\pi_J = (2^i, 1^{2(n-i)})$ for $0 \leq i \leq n-1$. Since (C_n, A_{n-1}) corresponds to a Hermitian symmetric pair, we will use [E, Sec. 3], [BH, Thm. 7.3]. First, we will consider the case $\Phi_J = A_1^l$. If n is even, then $0 \leq l \leq m-1$, and if n is odd, then $0 \leq l \leq m$. By [E, Sec. 3], we have

$$\mathcal{O}(C_n, A_{n-1}, A_1^l) \simeq \mathcal{O}(C_{n-2j}, A_{n-2j-1}, A_1^{l-j})$$

provided $0 \leq j \leq m$ if n is odd and $0 \leq j \leq m-1$ if n is even. Taking $j = l$, we have the following cases:

$$\mathcal{O}(C_n, A_{n-1}, A_1^l) \simeq \begin{cases} \mathcal{O}(C_{2(m-l)+1}, A_{2(m-l)}, \varnothing) & \text{if } n = 2m+1 \\ \mathcal{O}(C_{2(m-l)}, A_{2(m-l)-1}, \varnothing) & \text{if } n = 2m \text{ and } l \leq m-1 \end{cases}$$

If n is odd and $l = m$, then we have that $\mathcal{O}(C_n, A_{n-1}, A_1^m) \simeq \mathcal{O}(C_1, \varnothing, \varnothing)$ which has finite representation type by [BN, Thm. 1.3]; if $l < m$, then $\mathcal{O}(C_n, A_{n-1}, A_1^l)$ has wild representation type by [BN, Thm. 1.4], since $2(m-l)+1 \geq 3$. If n is even and $l = m-1$, then $\mathcal{O}(C_n, A_{n-1}, A_1^{m-1}) \simeq \mathcal{O}(C_2, A_1, \varnothing)$ which has finite representation type by [BN, Thm. 1.4]; if $l < m-1$, then $\mathcal{O}(C_n, A_{n-1}, A_1^l)$ has wild representation type by [BN, Thm. 1.4], since $2(m-l) \geq 4$.

Now suppose that $\Phi_{J'} = A_1^l \times C_1$ for $0 \leq l \leq m-1$. By [E, Thm. 3.2(b)], we have

$$\mathcal{O}(C_n, A_{n-1}, A_1^l \times C_1) \simeq \mathcal{O}(C_{n-2j}, A_{n-2j-1}, A_1^{l-j} \times C_1)$$

for any $0 \leq j \leq l$. Setting $j = l$, we have two cases:

$$\mathcal{O}(C_n, A_{n-1}, A_1^l \times C_1) \simeq \begin{cases} \mathcal{O}(C_{2(m-l)}, A_{2(m-l)-1}, C_1) & \text{if } n = 2m \\ \mathcal{O}(C_{2(m-l)+1}, A_{2(m-l)}, C_1) & \text{if } n = 2m+1 \end{cases}$$

If n is even and $l = m-1$, then $\mathcal{O}(C_n, A_{n-1}, A_1^{m-1} \times C_1)$ is equivalent to two copies of $\mathcal{O}(C_2, A_1, C_1)$ and so it is semisimple with two simple modules by [BN, Thm. 1.5]. If n is even and $l = m-2$, then $\mathcal{O}(C_n, A_{n-1}, A_1^{m-2} \times C_1)$ is equivalent to two copies of $\mathcal{O}(C_4, A_3, C_1)$,

Label	Subsets of Simple Roots
BC_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
BC_3	$\{\alpha_2, \alpha_3, \alpha_4\}$
A_3	$\{\alpha_1, \alpha_2, \alpha_3\}$
$A_1 \times BC_2$	$\{\alpha_1, \alpha_3, \alpha_4\}$
$A_2 \times BC_1$	$\{\alpha_1, \alpha_2, \alpha_4\}$
BC_2	$\{\alpha_3, \alpha_4\}$
A_2	$\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}$
A_1^2	$\{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_4\}, \{\alpha_2, \alpha_4\}$
A_1	$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}$
$\varnothing$	$\{\}$

Table 5.2: Equivalence Classes of Subsets of Simple Roots in BC_4

which has finite representation type by [BN, Thm. 1.5]. If n is odd and $l = m - 1$, then $\mathcal{O}(C_n, A_{n-1}, A_1^{m-1} \times C_1)$ is equivalent to two copies of $\mathcal{O}(C_3, A_2, C_1)$, which has finite representation type by [BN, Thm. 1.5]. Finally, if n is even and $l < m - 2$ then $2(m - l) \geq 6$; if n is odd $l < m - 1$ and $2(m - l) + 1 \geq 5$. In either case, $\mathcal{O}(C_n, A_{n-1}, A_1^l \times C_1)$ has wild representation type by [BN, Thm. 1.5].

We have exhausted all possible subroot systems Φ_J of C_n and so the theorem follows. □

5.2.3 An Example in Type BC_n

Let $\mathfrak{g} = \mathfrak{so}_9(\mathbb{C})$ or $\mathfrak{sp}_8(\mathbb{C})$, so we are considering type BC_4. We will compute the representation type of all infinitesimal blocks $\mathcal{O}(\mathfrak{g}, S, J)$. We will be able to compact the 16×16 matrix giving the representation type of $\mathcal{O}(\mathfrak{g}, S, J)$ in a similar fashion to how it was done for A_5 in Section 5.1.3 using Hirai equivalence. We can do this because Conjecture 4.5.1 is true (verified by direct computation) in type BC_4. Table 5.2 gives the 10 Hirai equivalence classes of subsets of simple roots in type BC_4 (see Section 3.5). Recall that if $A_1 = \langle \alpha_i \rangle$ for $1 \leq i \leq 3$, then $A_1 \sim BC_1 = \langle \alpha_4 \rangle$.

Φ_S \ Φ_J	BC_4	BC_3	A_3	$A_1 \times BC_2$	$A_2 \times BC_1$	BC_2	A_2	A_1^2	A_1	$\varnothing$
BC_4	–	–	–	–	–	–	–	–	–	SS
BC_3	–	–	–	–	–	–	–	–	SS	F
A_3	–	–	–	–	–	–	–	SS	F	W
$A_1 \times BC_2$	–	–	–	–	–	–	SS	F	W	W
$A_2 \times BC_1$	–	–	–	–	SS	–	F	W	W	W
BC_2	–	–	–	–	–	SS	F	W	W	W
A_2	–	–	–	SS	F	F	W	W	W	W
A_1^2	–	–	SS	F	W	W	W	W	W	W
A_1	–	SS	T	W	W	W	W	W	W	W
$\varnothing$	SS	W	W	W	W	W	W	W	W	W

Table 5.3: Representation Type of Infinitesimal Blocks in Type BC_4

In Table 5.3 we give the representation type of any infinitesimal block $\mathcal{O}(BC_4, S, J)$. Note that the equivalence classes are arranged according to the partial ordering on nilpotent orbits as given in Figure 3.2. We use the Hasse diagrams in Figure 3.2 along with Corollary 3.4.5 to determine the nonzero infinitesimal blocks, which lie in the upper left triangle of Table 5.3.

There are two Hermitian symmetric cases, given in the rows labeled BC_3 and A_3. The entries in Table 5.1 which could not be deduced from the theorems in this section were determined using a computer and the results in Chapter 4. There are some interesting points to make here. First, there are some triangular infinitesimal blocks. If $\Phi_S = A_1$ (or BC_1) and $\Phi_J = A_3$, then the corresponding blocks are triangular of length 4 and therefore have tame representation type. Those associated to $\Phi_S = \varnothing$ and $\Phi_J = BC_3$ are triangular of length 8 and therefore have wild representation type. Another point of interest is that if $S = \{\alpha_1, \alpha_4\}$

and $J = \{\alpha_1, \alpha_2, \alpha_4\}$, then the blocks $\mathcal{O}(\mathfrak{g}, S, J)$ and $\mathcal{O}(\mathfrak{g}, J, S)$ have one X-linkage class, and therefore have wild representation type.

As with A_5, we observe in Table 5.1 a strong connection between the ordering on nilpotent orbits for $\mathfrak{g}$ and the representation type of infinitesimal blocks for $\mathfrak{g}$.

5.2.4 Multiple Linkage Classes in an Infinitesimal Block

We have already observed infinitesimal blocks that have two linkage classes in type BC_n. This observation goes back to Enright-Shelton [ES], where these multiple linkage classes were observed for the semiregular blocks in the Hermitian symmetric cases. Theorems 5.2.4 and 5.2.5 show that infinitesimal blocks with two linkage classes occur for any $n \geq 2$. One may wonder if there are infinitesimal blocks that have more than two linkage classes. The answer is yes. We will exhibit some that have four linkage classes.

Let $\mathfrak{g} = \mathfrak{so}_{13}(\mathbb{C})$, which has a root system of type B_6. Take S to be either of the two subsets of simple roots in B_6 which generate a subroot system of type $A_3 \times A_1$, and take J to be either of the two subsets of simple roots which generate a subroot system of type $A_2 \times B_2$. Then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple with four simple modules, and hence it has four linkage classes. One can see this by observing that the lengths of any two of the four elements in ${}^S\mathcal{W}^J$ differ by an even integer, and so there cannot be an extension between the two corresponding simple modules.

It may seem plausible that for some n and some subsets S and J of simple roots in BC_n, the infinitesimal block $\mathcal{O}(BC_n, S, J)$ contains more than four linkage classes. However, the question of the existence of such an infinitesimal block is still unknown.

5.3 Type D_n

5.3.1 An Example in Type D_n

Let $\mathfrak{g} = \mathfrak{so}_8(\mathbb{C})$, which is of type D_4. We will compute the representation type of all infinitesimal blocks $\mathcal{O}(\mathfrak{g}, S, J)$. Other than Corollary 3.4.5 for determining the nonzero blocks, we will

Label	Subsets of Simple Roots
D_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
D_3	$\{\alpha_2, \alpha_3, \alpha_4\}$
A_3^I	$\{\alpha_1, \alpha_2, \alpha_3\}$
A_3^{II}	$\{\alpha_1, \alpha_2, \alpha_4\}$
A_2	$\{\alpha_1, \alpha_3, \alpha_4\} \sim \{\alpha_1, \alpha_2\} \sim \{\alpha_2, \alpha_3\} \sim \{\alpha_2, \alpha_4\}$
D_2	$\{\alpha_3, \alpha_4\}$
$(A_1^2)^I$	$\{\alpha_1, \alpha_3\}$
$(A_1^2)^{II}$	$\{\alpha_1, \alpha_4\}$
A_1	$\{\alpha_1\} \sim \{\alpha_2\} \sim \{\alpha_3\} \sim \{\alpha_4\}$
$\varnothing$	$\varnothing$

Table 5.4: Equivalence Classes of Subsets of Simple Roots in D_4

have to rely on computer calculations for most of the blocks in this case. Since Conjecture 4.5.1 holds in type D_4 (verified through computation), we will label the infinitesimal blocks for D_4 with the ten Hirai equivalence classes of subsets of simple roots. These are given in Table 5.4, together with their labels. The partial ordering on nilpotent orbits of $\mathfrak{so}_8(\mathbb{C})$ is given in Figure 3.3, from which we can find the nonzero infinitesimal blocks for D_4.

In Table 5.5, we give the representation type of any infinitesimal block $\mathcal{O}(D_4, S, J)$ in the row corresponding to Φ_S and the column corresponding to Φ_J. Note that again the equivalence classes are arranged according to the partial ordering on nilpotent orbits as given in Figure 3.3.

If $S = \{\alpha_1, \alpha_3, \alpha_4\} = J$, then $\mathcal{O}(D_4, S, J)$ is semisimple with two simple modules. This is an example of a simply laced case for which an infinitesimal block splits into two linkage classes. The same holds for any block for which $\Phi_S = A_2 = \Phi_J$. However, if $\Phi_S = A_2$ and $\Phi_J = A_1 \times D_2$ or vice versa, the block $\mathcal{O}(D_4, S, J)$ is semisimple but with only one simple module.

Φ_J / Φ_S	D_4	D_3	A_3^I	A_3^{II}	A_2	D_2	$(A_1^2)^I$	$(A_1^2)^{II}$	A_1	$\varnothing$
D_4	–	–	–	–	–	–	–	–	–	SS
D_3	–	–	–	–	–	SS	–	–	F	W
A_3^I	–	–	–	–	–	–	SS	–	F	W
A_3^{II}	–	–	–	–	–	–	–	SS	F	W
A_2	–	–	–	–	SS	F	F	F	W	W
D_2	–	SS	–	–	T	W	W	W	W	W
$(A_1^2)^I$	–	–	SS	–	T	W	W	W	W	W
$(A_1^2)^{II}$	–	–	–	SS	T	W	W	W	W	W
A_1	–	F	F	F	W	W	W	W	W	W
$\varnothing$	SS	W	W	W	W	W	W	W	W	W

Table 5.5: Representation Type of Infinitesimal Blocks in Type D_4

If $\Phi_S = D_2$ or A_1^2 and $\Phi_J = A_2$ or $A_1 \times D_2$, then $\mathcal{O}(D_4, S, J)$ is a triangular infinitesimal block of length 4 and therefore it has tame representation type.

As in the other classical cases, Table 5.5 demonstrates a close relationship between the representation type of infinitesimal blocks in type D_4 and the partial ordering of nilpotent orbits.

Chapter 6

Representation Type of Blocks for the Exceptional Lie Algebras

We will address the representation type of infinitesimal blocks for the exceptional Lie algebras in this chapter. In fact, we will compute the representation type of all the blocks in types F_4 and G_2 with brute computational force. We will also collect a few results and observations in types E_6, E_7, and E_8, even though their sizes makes brute computation much more difficult.

6.1 An Order-Reversing Map

For this section only, let $\mathfrak{g}$ be any finite dimensional complex simple Lie algebra. Denote by $\mathcal{N}$ the set of nilpotent orbits of $\mathfrak{g}$. There is a map $d : \mathcal{N} \to \mathcal{N}$ with the following properties:

(i) $d(O_S) = R_S$ for all $S \subseteq \Delta$;

(ii) if $O, O' \in \mathcal{N}$ with $O \leq O'$, then $d(O') \leq d(O)$;

(iii) $d^2(O) \geq O$ for all $O \in \mathcal{N}$;

(iv) if $\mathcal{R} := \text{Im } d$, then $d^2(O) = O$ for all $O \in \mathcal{R}$.

See [Spalt, Ch. 3]. The set $\mathcal{R}$ is called the set of **special nilpotent orbits** of $\mathfrak{g}$. If $\mathfrak{g}$ is a classical Lie algebra, the map d is defined in Theorem 3.3.3.

By property (i) of the map d, if $S \subseteq \Delta$, then $d^{-1}(R_S)$ contains any root system orbit $O_{S'}$ with $S \sim S'$.

Φ_S \ Φ_J	G_2	A_1	$\widetilde{A}_1$	$\varnothing$
G_2	–	–	–	SS (1)
A_1	–	SS (2)	SS (3)	F (6)
$\widetilde{A}_1$	–	SS (3)	SS (2)	F (6)
$\varnothing$	SS (1)	W^T (6)	W^T (6)	W^D (12)

Table 6.1: Representation Type of $\mathcal{O}(G_2, \Phi_S, \Phi_J)$

6.2 Representation Type of Infinitesimal Blocks in Type G_2

We will now classify the representation type of each infinitesimal block $\mathcal{O}(G_2, S, J)$. Denote the root subsystem of G_2 generated by the short root α_1 as $\widetilde{A}_1$, and the root subsystem generated by the long root α_2 as A_1. Then there is a one-to-one correspondence between the four subsets of Δ and the four root systems $\varnothing$, $\widetilde{A}_1$, A_1, and G_2.

We first use Section 3.6.1 to determine the nonzero infinitesimal blocks. To classify the nonzero infinitesimal blocks, we generate the Hasse diagram of ${}^S\mathcal{W}^J$ and, when necessary, the radical filtrations of the PVM's in the infinitesimal block using the U_α-algorithm. Where convenient, these calculations were done using a computer.

The representation type of the infinitesimal block $\mathcal{O}(G_2, S, J)$ is given on the row labeled with the root system Φ_S and the column labeled by the root system Φ_J in Table 6.1. As in Chapter 5, a dash (–) means the block is zero, SS means the block is semisimple, F means it has finite representation type, and W means it has wild representation type; there are no tame blocks for G_2. The number below each SS in the table represents the number of simple modules in the semisimple block; this is also the number of linkage classes in the block. The number below each F or W indicates the number of simple modules in the corresponding

$$\begin{array}{c} G_2 \\ | \\ G_2(a_1) \\ | \\ \varnothing \end{array}$$

Figure 6.1: Special Nilpotent Orbits for G_2

block. If an infinitesimal block is not semisimple, then the block has only one linkage class. The infinitesimal blocks having finite representation type are uniserial of length two as in (4.2). The superscript above each W indicates what condition was used to determine that the infinitesimal block has wild representation type. If there is a diamond in the poset of ${}^S\mathcal{W}^J$, then it is marked W^D in the table. If the poset contains no diamonds but there is a kite in the Ext^1-quiver, then it is marked W^K in the table. If the infinitesimal block is triangular of length at least five, then it is marked with W^T. Note that there are no infinitesimal blocks for G_2 labeled W^K. However, we will need this label when we discuss the infinitesimal blocks in type F_4.

The set $\mathcal{R}$ of special nilpotent orbits for G_2 contains three orbits, given in Figure 6.1. The involution $d|_{\mathcal{R}}$ is given by reflecting about the horizontal line of symmetry in Figure 6.1. The pre-images of these three special orbits are listed in Table 6.2. For each orbit $O \in \mathcal{R}$, the set of all Φ_S such that O_S is in $d^{-1}(O)$ is enclosed between horizontal and vertical lines in Table 6.1. Notice that these correspond exactly to the Hirai equivalence classes of subsets of Δ, and that Conjecture 4.5.1 holds in type G_2.

We classify the representation type of the infinitesimal blocks for G_2 in the following way.

Theorem 6.2.1 *(i)* $\mathcal{O}(G_2, S, J)$ *is nonzero if and only if* $O_J \le d(O_S)$ *if and only if* $O_S \le d(O_J)$.

Special Orbit	Pre-Image
G_2	$\varnothing$
$G_2(a_1)$	$G_2(a_1)$, $\widetilde{A}_1$, A_1
$\varnothing$	G_2

Table 6.2: Pre-Images of Orbits in G_2

(ii) *$\mathcal{O}(G_2, S, J)$ is semisimple if and only if $d^2(O_J) = d(O_S)$ if and only if $d(O_J) = d^2(O_S)$. If $O_J = d(O_S)$ or $O_S = d(O_J)$, then $\mathcal{O}(G_2, S, J)$ has one simple module; if $|S| = 1 = |J|$, then $\mathcal{O}(G_2, S, J)$ is semisimple with two linkage classes if $S = J$ and three linkage classes if $S \neq J$.*

(iii) *$\mathcal{O}(G_2, S, J)$ has finite representation type if and only if $|S| = 1$ and $J = \varnothing$.*

(iv) *$\mathcal{O}(G_2, S, J)$ has wild representation type if and only if $S = \varnothing$ and $|J| \leq 1$.*

6.3 Representation Type of Infinitesimal Blocks in Type F_4

Now we will classify all of the infinitesimal blocks $\mathcal{O}(F_4, S, J)$. First, we recall the following notation for the subroot systems of F_4. Suppose $T \subseteq \Delta$. If $T = \{\alpha_1, \alpha_2\}$ (long roots), then Φ_T is denoted by A_2, whereas if $T = \{\alpha_3, \alpha_4\}$ (short roots), then Φ_T is denoted by $\widetilde{A}_2$. If $|T| = 1$, then write A_1 if T contains a long root and $\widetilde{A}_1$ if it contains a short root. Since there are two long simple roots and two short simple roots, write A_1' if $T = \{\alpha_2\}$ and $\widetilde{A}_1'$ if $T = \{\alpha_4\}$. Using this notation, we can write the root system generated by one the 16 subsets of simple roots in F_4 in a unique way (see Table 6.3).

The representation type of the infinitesimal block $\mathcal{O}(F_4, S, J)$ is given on the row labeled with the root system Φ_S and the column labeled by the root system Φ_J in Table 6.3, using the notation we set up in Section 6.2. There are no tame infinitesimal blocks in type F_4. A pair of numbers below an F or W in the table indicates that the corresponding infinitesimal block

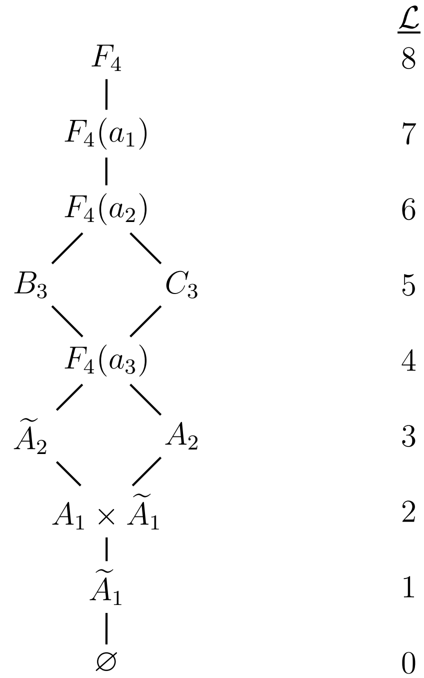


Figure 6.2: Special Orbits for F_4

splits into two linkage classes having the specified numbers of simple modules. For example, note that $\mathcal{O}(F_4, A_2, \widetilde{A}_2)$ has two linkage classes, and they have, respectively, 20 and 12 simple modules for a total of 32 simple modules in the infinitesimal block. If an infinitesimal block is not semisimple, then it does not split into more than two linkage classes in type F_4.

All of the infinitesimal blocks $\mathcal{O}(F_4, S, J)$ having finite representation type are composed of linkage classes which are uniserial length two as in (4.2). Notice that as one moves right in a row or down in a column, one expects to eventually find diamonds in the poset of ${}^S\mathcal{W}^J$.

There are 11 special nilpotent orbits in $\mathcal{R}$ for F_4. They are given in Figure 6.2 (see [Cart, Sec. 13.4]). The involution $d|_{\mathcal{R}}$ is given by reflecting the Hasse diagram in Figure 6.1 about the horizontal line of symmetry. The pre-images of these special orbits are listed in Table 6.4. For each orbit $O \in \mathcal{R}$, the set of all Φ_S such that O_S is in $d^{-1}(O)$ is enclosed between horizontal and vertical lines in Table 6.1. As with G_2 these correspond exactly to the Hirai

Φ_S \ Φ_J	F_4	B_3	C_3	$A_2 \times \widetilde{A}_1'$	$\widetilde{A}_2 \times A_1$	B_2	A_2	$\widetilde{A}_2$	$A_1 \times \widetilde{A}_1$	$A_1' \times \widetilde{A}_1'$	$A_1 \times \widetilde{A}_1'$	A_1	A_1'	$\widetilde{A}_1$	$\widetilde{A}_1'$	$\varnothing$
F_4	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	SS (1)
B_3	–	–	–	–	–	–	–	SS (1)	F (2)	F (2)	F (2)	W^K (6)	W^K (6)	W^K (9)	W^K (9)	W^D (24)
C_3	–	–	–	–	–	–	SS (1)	–	F (2)	F (2)	F (2)	W^K (9)	W^K (9)	W^K (6)	W^K (6)	W^D (24)
$A_2 \times \widetilde{A}_1'$	–	–	–	SS (3)	SS (5)	SS (4)	F (6)	F (6,6)	W^K (17)	W^K (17)	W^K (17)	W^D (36)	W^D (36)	W^D (44)	W^D (44)	W^D (96)
$\widetilde{A}_2 \times A_1$	–	–	–	SS (5)	SS (3)	SS (4)	F (6,6)	F (6)	W^K (17)	W^K (17)	W^K (17)	W^D (44)	W^D (44)	W^D (36)	W^D (36)	W^D (96)
B_2	–	–	–	SS (4)	SS (4)	SS (9)	F (6,6)	F (6,6)	W^K (24)	W^K (24)	W^K (24)	W^D (60)	W^D (60)	W^D (60)	W^D (60)	W^D (144)
A_2	–	–	SS (1)	W^T (6)	W^T (6,6)	W^T (6,6)	W^D (12)	W^D (20,12)	W^D (36)	W^D (36)	W^D (36)	W^D (72)	W^D (72)	W^D (48,48)	W^D (96)	W^D (192)
$\widetilde{A}_2$	–	SS (1)	–	W^T (6,6)	W^T (6)	W^T (6,6)	W^D (20,12)	W^D (12)	W^D (36)	W^D (36)	W^D (36)	W^D (96)	W^D (48,48)	W^D (72)	W^D (72)	W^D (192)
$A_1 \times \widetilde{A}_1$	–	F (2)	F (2)	W^K (17)	W^K (17)	W^K (24)	W^D (36)	W^D (36)	W^D (61)	W^D (61)	W^D (61)	W^D (132)	W^D (132)	W^D (132)	W^D (132)	W^D (288)
$A_1' \times \widetilde{A}_1'$	–	F (2)	F (2)	W^K (17)	W^K (17)	W^K (24)	W^D (36)	W^D (36)	W^D (61)	W^D (61)	W^D (61)	W^D (132)	W^D (132)	W^D (132)	W^D (132)	W^D (288)
$A_1 \times \widetilde{A}_1'$	–	F (2)	F (2)	W^K (17)	W^K (17)	W^K (24)	W^D (36)	W^D (36)	W^D (61)	W^D (61)	W^D (61)	W^D (132)	W^D (132)	W^D (132)	W^D (132)	W^D (288)
A_1	–	W^K (6)	W^K (9)	W^D (36)	W^D (44)	W^D (60)	W^D (72)	W^D (96)	W^D (132)	W^D (132)	W^D (132)	W^D (264)	W^D (264)	W^D (288)	W^D (288)	W^D (576)
A_1'	–	W^K (6)	W^K (9)	W^D (36)	W^D (44)	W^D (60)	W^D (72)	W^D (96)	W^D (132)	W^D (132)	W^D (132)	W^D (264)	W^D (264)	W^D (288)	W^D (288)	W^D (576)
$\widetilde{A}_1$	–	W^K (9)	W^K (6)	W^D (44)	W^D (36)	W^D (60)	W^D (96)	W^D (72)	W^D (132)	W^D (132)	W^D (132)	W^D (288)	W^D (288)	W^D (264)	W^D (264)	W^D (576)
$\widetilde{A}_1'$	–	W^K (9)	W^K (6)	W^D (44)	W^D (36)	W^D (60)	W^D (96)	W^D (72)	W^D (132)	W^D (132)	W^D (132)	W^D (288)	W^D (288)	W^D (264)	W^D (264)	W^D (576)
$\varnothing$	SS (1)	W^D (24)	W^D (24)	W^D (96)	W^D (96)	W^D (144)	W^D (192)	W^D (192)	W^D (288)	W^D (288)	W^D (288)	W^D (576)	W^D (576)	W^D (576)	W^D (576)	W^D (1152)

Table 6.3: Representation Type of $\mathcal{O}(F_4, \Phi_S, \Phi_J)$

Special Orbit O	Pre-Image of O	$\mathcal{H}(O)$
F_4	$\varnothing$	$\varnothing$
$F_4(a_1)$	A_1, $\widetilde{A}_1$	$\{\alpha_1\} \sim \{\alpha_2\} \sim \{\alpha_3\} \sim \{\alpha_4\}$
$F_4(a_2)$	$A_1 \times \widetilde{A}_1$	$\{\alpha_1, \alpha_3\} \sim \{\alpha_1, \alpha_4\} \sim \{\alpha_2, \alpha_4\}$
B_3	$\widetilde{A}_2$	$\{\alpha_3, \alpha_4\}$
C_3	A_2	$\{\alpha_1, \alpha_2\}$
$F_4(a_3)$	$F_4(a_3)$, $C_3(a_1)$, B_2, $\widetilde{A}_2 \times A_1$, $A_2 \times \widetilde{A}_1$	$\{\alpha_1, \alpha_3, \alpha_4\} \sim \{\alpha_1, \alpha_2, \alpha_4\} \sim \{\alpha_2, \alpha_3\}$
A_2	C_3	$\{\alpha_2, \alpha_3, \alpha_4\}$
$\widetilde{A}_2$	B_3	$\{\alpha_1, \alpha_2, \alpha_3\}$
$A_1 \times \widetilde{A}_1$	$F_4(a_2)$	
$\widetilde{A}_1$	$F_4(a_1)$	
$\varnothing$	F_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Table 6.4: Pre-Images of Special Orbits in F_4

equivalence classes of subsets of Δ, and we see that Conjecture 4.5.1 holds in type F_4. For any special nilpotent orbit $O \in \mathcal{R}$, let $\mathcal{H}(O)$ denote the Hirai equivalence class associated to O; these are listed in the third column of Table 6.4. Since Conjecture 4.5.1 holds for F_4, it makes sense to talk about the representation type of the collection of infinitesimal blocks $\mathcal{O}(F_4, \mathcal{H}(O_1), \mathcal{H}(O_2))$.

Define $\mathcal{L} : \mathcal{R} \to \mathbb{Z}_{\geq 0}$, where $\mathcal{L}(O)$ is the level of O in the poset given in Figure 6.2. We classify the infinitesimal blocks for F_4 in the following way.

Theorem 6.3.1 *Let $\mathfrak{g}$ be of type F_4.*

(i) $\mathcal{O}(F_4, S, J)$ is nonzero if and only if $O_J \leq d(O_S)$ if and only if $O_S \leq d(O_J)$.

(ii) $\mathcal{O}(F_4, S, J)$ is semisimple if and only if $d^2(O_J) = d(O_S)$ if and only if $d(O_J) = d^2(O_S)$. $\mathcal{O}(F_4, S, J)$ is semisimple with more than one linkage class if and only if $d(O_S) = F_4(a_3) = d(O_J)$.

(iii) If $\mathcal{O}(F_4, S, J)$ has finite representation type, then $\mathcal{L}(d^2(O_J)) = \mathcal{L}(d(O_S)) - 1$ and $\mathcal{L}(d^2(O_S)) = \mathcal{L}(d(O_J)) - 1$. On the other hand, if either $\mathcal{L}(d^2(O_J)) = \mathcal{L}(d(O_S)) - 1$

or $\mathcal{L}(d^2(O_S)) = \mathcal{L}(d(O_J)) - 1$ *and* $\mathcal{O}(F_4, S, J)$ *does not have finite representation type, then* $O_S \in \{A_2, \widetilde{A}_2\}$, $d(O_J) = F_4(a_3)$, *and* $\mathcal{O}(F_4, S, J)$ *consists of exactly one or two linkage classes which are triangular of length six; therefore it has wild representation type.*

(iv) If $\mathcal{L}(d^2(O_J)) \leq \mathcal{L}(d(O_S)) - 2$ *or* $\mathcal{L}(d^2(O_S)) \leq \mathcal{L}(d(O_J)) - 2$, *then* $\mathcal{O}(F_4, S, J)$ *has wild representation type.*

Proof. For (i), we use Theorem 3.4.4 and the fact that $R_S = d(O_S)$ for any $S \subseteq \Delta$. Inspecting Table 6.3 and using Figure 6.2 with Table 6.4, we obtain (ii)–(iv). Note that $\mathcal{O}(F_4, \mathcal{H}(C_3), \mathcal{H}(F_4(a_3)))$ and $\mathcal{O}(F_4, \mathcal{H}(B_3), \mathcal{H}(F_4(a_3)))$ are the only collections of infinitesimal blocks satisfying $\mathcal{L}(d^2(O_J)) = \mathcal{L}(d(O_S)) - 1$ and $\mathcal{L}(d^2(O_S)) = \mathcal{L}(d(O_J)) - 1$ but do not have finite representation type. In these collections, the infinitesimal blocks $\mathcal{O}(F_4, A_2, A_2 \times \widetilde{A}_1')$ and $\mathcal{O}(F_4, \widetilde{A}_2, \widetilde{A}_2 \times A_1)$ have one triangular linkage class of length 6, while the other four infinitesimal blocks in the collection have two triangular linkage classes of length 6. As a remark, these are the only triangular linkage classes of length greater than 2 appearing in any of the infinitesimal blocks for F_4. □

We observe that Table 6.3 is "almost" symmetric across the main diagonal; the only non-symmetric entries occur between the collections $\mathcal{O}(F_4, \mathcal{H}(C_3), \mathcal{H}(F_4(a_3)))$ and $\mathcal{O}(F_4, \mathcal{H}(F_4(a_3)), \mathcal{H}(C_3)))$, and between $\mathcal{O}(F_4, \mathcal{H}(B_3), \mathcal{H}(F_4(a_3)))$ and $\mathcal{O}(F_4, \mathcal{H}(F_4(a_3), \mathcal{H}(B_3)))$. This proves the following.

Proposition 6.3.2 *If* Φ *is of type* F_4, *then* $\mathcal{O}(F_4, S, J)$ *has the same representation type as* $\mathcal{O}(F_4, J, S)$ *except in the following cases: each of the infinitesimal blocks in the collections*

$$\mathcal{O}(F_4, \mathcal{H}(F_4(a_3)), \mathcal{H}(B_3)) \quad \textit{and} \quad \mathcal{O}(F_4, \mathcal{H}(F_4(a_3)), \mathcal{H}(C_3))$$

has finite representation type, but each of the infinitesimal blocks in the collections

$$\mathcal{O}(F_4, \mathcal{H}(B_3), \mathcal{H}(F_4(a_3))) \quad \textit{and} \quad \mathcal{O}(F_4, \mathcal{H}(C_3), \mathcal{H}(F_4(a_3)))$$

has wild representation type.

Before leaving our discussion of type F_4, note that $\mathcal{O}(F_4, B_2, B_2)$ is semisimple with 9 simple modules. This is the largest semisimple block discovered in our investigations!

6.4 Remarks About Types E_6, E_7, and E_8

In principle, one can use a computer to determine the representation types of the blocks in types E_6, E_7, and E_8. However, their sizes make this a much more difficult task; therefore, a better approach would be desirable in these cases. Preliminary calculations suggest that there is likely an analogue of Theorem 6.3.1 for all the exceptional Lie algebras.

We noted multiple linkage classes in every type we have discussed so far, except type A_n for which it has been proven that multiple linkage classes in a single infinitesimal block do not exist. We will exhibit multiple linkage classes in infinitesimal blocks for types E_6, E_7, and E_8. These were found using a computer.

- In type E_6, take $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\} = J$. Then $\mathcal{O}(E_6, S, J)$ is semisimple with 3 simple modules.

- In type E_7, take $S = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ and $J = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$. Then $\mathcal{O}(E_7, S, J)$ is semisimple with 3 simple modules.

- In type E_8, take $S = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ and $J = \{\alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_8\}$. Then $\mathcal{O}(E_8, S, J)$ is semisimple with 2 simple modules.

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