

CRITICALITY FOR THE RADIUS OF GYRATION OF CURVES

by

KENNETH L. LITTLE

(Under the direction of Joseph H.G. Fu)

ABSTRACT

The radius of gyration of a space curve is a measure of how tightly wound that curve is. In recent years, it has gained usage by molecular biologists investigating the geometric aspects of protein folding. Here, we explore the radius of gyration of sufficiently smooth space curves, primarily focusing on criticality. We use the calculus of variations to define criticality for the radius of gyration, then show that among curves of a fixed length, only a curve which is a straight line segment is critical.

INDEX WORDS: Radius of Gyration, Space Curve, Calculus of Variations

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DEDICATION

To my wife, whose support makes all things possible.

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CHAPTER 1

INTRODUCTION

1.1 MOTIVATION

Proteins control a variety of life processes from muscle movement to food digestion, and the behavior of a particular protein is dictated by the shape in which it is folded. There are three main factors that control protein folding: chemical composition, surrounding environment, and the geometry of the folding process. To explore how geometric constraints affect the way proteins fold, molecular biologists model proteins with space curves and utilize the tools of geometry. One geometric measurement that stands out in this type of investigation is the radius of gyration of a curve.

For a curve γ in \mathbb{R}^3 , the radius of gyration of γ is the root mean square distance to the center of mass of γ , where the center of mass $c(\gamma)$ is the average of all points on γ weighted by the length distribution function. For a smooth, unit length, arclength parameterized curve, we have simply

$$R(\gamma) := \sqrt{\int_0^1 |\gamma(s) - c(\gamma)|^2 ds}.$$

In essence, the radius of gyration is a measure of how densely compacted a curve is. It is used for a variety of reasons in the study of protein folding. For instance, in [1], the radius of gyration is used to compare the native state (properly folded state) of a given protein to a variety of its denatured states (improperly folded states). It is also used there to compare the compactedness of theoretical models of proteins to that of observed proteins as a means of verifying the validity of the model.

Alternatively, $R(\gamma)$ can be the constraint under which optimal curves are sought, and this is the role in which we are interested. In [2], the authors model protein chains with discretized space curves. They then borrow the notion of rope length from knot theory and seek to minimize the rope length of a curve under different constraints to see if there is a correlation between these types of optimal curves and the secondary structures observed in naturally occurring proteins.

As discussed in [3], single protein chains have three main levels of organization, which are their primary, secondary, and tertiary structures. The primary structure of a protein is the sequence of amino acids that constitute its chemical makeup. The initial geometric structure that a chain folds into is called the secondary structure and can be either a helix, known as an α *helix*, or a pleated sheet, known as a β *pleated sheet*. The secondary structure of a chain acts as a type of backbone which allows the chain to fold again, creating two depths of folding. The structure created by the outer folding is called the tertiary structure. We will be interested in the secondary structure of proteins, and in particular, α helices.

As defined in [2], the *thickness* of a curve γ is the maximum radius of a uniform tube with the curve as its axis which is smooth and has no self intersections, and the *rope length* of a curve is $\rho(\gamma) := \frac{\text{arclength}}{\text{thickness}}$. Also, the *local radius of curvature* $\omega(x)$ at a point x is defined as the minimum of the radii of the circles passing through x and all other points along the curve in a given neighborhood of x and the *global radius of curvature* is defined as the minimum of the radii of all circles passing through three points on the curve.

The authors of [2] impose a variety of constraints on the curves and use computer simulations to minimize rope length under each constraint. The most interesting results occur when the constraint is an upper bound on the local radius of gyration, which is defined as the radius of gyration of a fixed number of points of γ . Letting g_x be the minimum of the radii of circles passing through x and two points outside a given neighborhood of x , the optimal curve found under such a constraint is experimentally always a helix with the special

property that

$$f(\gamma) := \text{avg} \left\{ \frac{g_x}{\omega(x)} \text{ for all } x \text{ on } \gamma \right\}$$

$$= 1.$$

For helices such that $f(\gamma) < 1$, the thickness is determined by the pitch of the helix, which limits the thickness in a non-local manner. For helices such that $f(\gamma) > 1$, the thickness is determined by the radius of the helix, which limits the thickness in a local manner. Experimentally found optimal curves occur at the boundary of these two sets of curves. The authors then measure the value of f for naturally occurring α helices and note that on average, $f = 1.01 \pm 0.03$, which is very close to optimal.

This work is intended to be the first steps in a formal proof that given such constraints on space curves, the helices found in [2] are indeed optimal. We will work under the additional assumption that our curves are continuous and, in fact, C^∞ .

1.2 SUMMARY

In Chapter 2, we develop the necessary tools to discuss the criticality of $R(\gamma)$. As $R(\gamma)$ is a functional of γ , we can use the calculus of variations to find curves which are critical for it. We begin by restricting our attention to C^∞ , embedded curves and develop the notion of criticality for a general functional $I(\gamma)$. In doing so we develop the first variation of $I(\gamma)$, denoted $\delta_\gamma I(\xi)$, which is itself a functional of smooth vector fields. $\delta_\gamma I(\xi)$ is analogous to the first derivative in single variable Calculus, and a curve γ is said to be *critical* for $I(\gamma)$ if

$$\delta_\gamma I(\xi) = 0$$

for all ξ which we permit ourselves to consider.

We then produce an example of the first variation of a functional by considering the length functional of a curve, which for a curve

$$\gamma(t) : [a, b] \rightarrow \mathbb{R}^3$$

is

$$L(\gamma) = \int_a^b |\dot{\gamma}| dt$$

where $\dot{\gamma} = \frac{d}{dt}\gamma$. The first variation of $L(\gamma)$ is

$$\delta_\gamma L(\xi) = \int_a^b \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} dt$$

and plays an important role in the investigation of $R(\gamma)$. Specifically, if we are to have a perturbation which preserves length locally along γ , then it must be that

$$\int_a^x \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} dt = 0$$

for all x in $[a, b]$, which implies

$$\langle \dot{\gamma}(t), \dot{\xi}(t) \rangle = 0$$

for all t . These will be the vector fields we focus on.

In Chapter 3, we define the center of mass $c(\gamma)$ of a curve. Even though $c(\gamma)$ is not quite a functional as its range is \mathbb{R}^3 , we can still develop what is in essence its first variation $\delta_\gamma c(\xi)$ by considering it a triplet of functionals. We then formally define $R(\gamma)$ of a curve and note that as we are only interested in critical curves for $R(\gamma)$ we can consider instead

$$r(\gamma) := \frac{1}{2}R^2(\gamma),$$

which for unit length, arc length parametrized curves is

$$r(\gamma) = \frac{1}{2} \int_0^1 (\gamma - c(\gamma))^2 ds.$$

We then develop the first variation of $r(\gamma)$, denoted $\delta_\gamma r(\xi)$, which is cumbersome in its general form, and endeavor to simplify it. As a start, we establish that we can assume without loss of generality that γ is arclength parameterized and that $c(\gamma) = 0$.

In Chapter 4, we define criticality for $R(\gamma)$. We establish that we can assume without loss of generality that γ is unit length, thus further simplifying $\delta_\gamma r(\xi)$, and that we need only

consider smooth, locally length preserving vector fields ξ . This yields the final simplification of $\delta_\gamma r(\xi)$ and we have

$$\delta_\gamma r(\xi) = \int_0^1 \langle \gamma, \xi \rangle ds.$$

Thus, a curve γ is critical for $R(\gamma)$ if

$$\begin{aligned} \delta_\gamma r(\xi) &= \int_0^1 \langle \gamma, \xi \rangle ds \\ &= 0 \end{aligned}$$

for all C^∞ , locally length preserving vector fields ξ . We wrap up with our main theorem: the only critical curve for $R(\gamma)$ is a straight line segment.

Chapter 5 is dedicated to the proof of our main theorem. We begin by proving that a straight line segment is critical for $R(\gamma)$. We then break the remaining curves into disjoint sets based on their behavior at their endpoints and find an appropriate vector field ξ for each set such that

$$\delta_\gamma r(\xi) \neq 0,$$

thus demonstrating that such γ are not critical for $R(\gamma)$. Specifically, for curves which have an endpoint, say $\gamma(1)$, which does not point towards or away from $c(\gamma)$, i.e. $T(1) \neq \pm \frac{c(\gamma) - \gamma(1)}{|c(\gamma) - \gamma(1)|}$, we decrease $R(\gamma)$ by essentially pushing a small segment of γ which is near $\gamma(1)$ towards $c(\gamma)$. If instead $\gamma(1)$ points towards $c(\gamma)$, i.e. $T(1) = \frac{c(\gamma) - \gamma(1)}{|c(\gamma) - \gamma(1)|}$, we decrease $R(\gamma)$ by essentially perturbing an appropriate section of γ along its normal field. Lastly, if $\gamma(1)$ points away from $c(\gamma)$, i.e. $T(1) = -\frac{c(\gamma) - \gamma(1)}{|c(\gamma) - \gamma(1)|}$, we show that the same perturbation increases $R(\gamma)$, which completes the proof of our theorem.

CHAPTER 2

PRELIMINARIES

2.1 THE CALCULUS OF VARIATIONS

We follow the approach taken in [4] in our setup of the calculus of variations. Given a curve

$$\gamma(t) : [a, b] \rightarrow \mathbb{R}^3$$

and a C^∞ function

$$F(x, v, w) : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R},$$

the problems we wish to address involve functionals of the form

$$I(\gamma) := \int_a^b F(t, \gamma(t), \dot{\gamma}(t)) dt \tag{2.1.1}$$

where $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$. We wish to maximize or minimize $I(\gamma)$, and to do so we use the calculus of variations. For definiteness, we will minimize $I(\gamma)$, but maximizing is analogous.

Typically, we do not wish to minimize $I(\gamma)$ over all possible curves γ , but rather to minimize $I(\gamma)$ over a set of curves W which satisfy some constraints. We begin by constraining our curves to be C^∞ and embedded.

Definition 2.1.1. *Let $\gamma(t)$ be a C^∞ , embedded curve. Then let*

$$\Gamma(t, u) : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}^3$$

be a C^∞ function such that for each u ,

$$\gamma_u(t) := \Gamma(t, u)$$

is a C^∞ , embedded curve and

$$\gamma_0(t) = \gamma(t).$$

Then $\gamma_u(t)$ is called a **comparison curve** to $\gamma(t)$ and Γ is said to be a **family of comparison curves** to $\gamma(t)$.

Alternatively, as $\Gamma(t, u)$ provides a smooth deformation of $\gamma(t)$, $\Gamma(t, u)$ is also called a **perturbation** of $\gamma(t)$.

The two interpretations of $\Gamma(t, u)$ serve their separate purposes and we will use both as needed. If we consider $\Gamma(t, u)$ as a perturbation of $\gamma(t)$, then $\left. \frac{\partial \Gamma}{\partial u} \right|_{u=0}$ is the *initial velocity vector field* of $\Gamma(t, u)$ in the u direction and we denote it by

$$\xi(t) := \left(\frac{\partial}{\partial u} \Gamma(t, u) \right) \Big|_{u=0}.$$

Definition 2.1.2. Let ξ be a C^∞ vector field and Γ be a perturbation such that

$$\left. \frac{\partial \Gamma}{\partial u} \right|_{u=0} = \xi.$$

then $\Gamma(t, u)$ is called an **associated perturbation** of ξ .

Notice that since $\Gamma(t, u)$ is C^∞ , ξ is C^∞ . Notice as well that if we are given a smooth vector field $\xi(t)$ along $[a, b]$, we can construct a related function $\Gamma(t, u)$ which is a perturbation of $\gamma(t)$ by defining

$$\Gamma(t, u) := \gamma(t) + u\xi(t).$$

Thus every perturbation $\Gamma(t, u)$ has a related C^∞ vector field $\xi(t)$ and every C^∞ vector field $\xi(t)$ has at least one related perturbation $\Gamma(t, u)$.

Alternatively, considering $\Gamma(t, u)$ as a family of comparison curves provides us with the tools needed to construct the notion of a neighborhood of γ .

Definition 2.1.3. Given a C^∞ , embedded curve $\gamma(t)$ and a comparison curve $\beta(t)$, let

$$d(\beta, \gamma) := \sup\{|\gamma(t) - \beta(t)|, |\dot{\gamma}(t) - \dot{\beta}(t)| \text{ for } t \text{ in } [a, b]\}$$

be the distance from γ to β .

Noice for any ϵ and any family of comparison curve $\Gamma(t, u)$, we can find a $\delta_\Gamma > 0$ such that if $-\delta_\Gamma < u < \delta_\Gamma$, then

$$d(\gamma_u, \gamma) < \epsilon.$$

This allows us the following definition.

Definition 2.1.4. A *neighborhood* N_ϵ of γ is defined as

$$N_\epsilon := \{\beta \mid d(\beta, \gamma) < \epsilon\}.$$

Definition 2.1.5. γ is a *local minimum* (resp. *maximum*) of $I(\gamma)$ if there exists a neighborhood N_ϵ of γ such that for any β in N_ϵ , $I(\gamma) \leq I(\beta)$ (resp. $I(\gamma) \geq I(\beta)$).

If $I(\gamma) \leq I(\beta)$ (resp. $I(\gamma) \geq I(\beta)$) for all other admissible curves β , then γ is a **global minimum** (resp. **maximum**).

Given a C^∞ , embedded curve γ and a perturbation $\Gamma(t, u)$, we limit our attention to a suitably small neighborhood of γ and write

$$i(u) := i_\Gamma(u) := \int_a^b F(t, \gamma_u(t), \dot{\gamma}_u(t)) dt.$$

If γ is a minimum of $I(\gamma)$, we have that $u = 0$ is a minimum of the single variable function $i(u)$. Thus, it must be that $i'(0) = 0$. Differentiating $i(u)$ yields

$$i'(u) = \frac{\partial}{\partial u} \int_a^b F(t, \gamma_u(t), \dot{\gamma}_u(t)) dt \quad (2.1.2)$$

$$= \int_a^b \frac{\partial}{\partial u} F(t, \Gamma(t, u), \dot{\Gamma}(t, u)) dt \quad (2.1.3)$$

$$= \int_a^b \left(\sum_{i=1}^3 \frac{\partial F}{\partial v_i} \frac{\partial v_i}{\partial u} + \sum_{i=1}^3 \frac{\partial F}{\partial w_i} \frac{\partial w_i}{\partial u} \right) \Big|_{\substack{v=\Gamma_i \\ w=\dot{\Gamma}_i}} dt \quad (2.1.4)$$

$$= \int_a^b \sum_{i=1}^3 \frac{\partial F}{\partial v_i} \Big|_{v_i=\Gamma_i} \frac{\partial \Gamma_i}{\partial u} dt + \int_a^b \sum_{i=1}^3 \frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\Gamma}_i} \left(\frac{\partial}{\partial u} \frac{\partial \Gamma_i}{\partial t} \right) dt \quad (2.1.5)$$

$$= \int_a^b \sum_{i=1}^3 \frac{\partial F}{\partial v_i} \Big|_{v_i=\Gamma_i} \frac{\partial \Gamma_i}{\partial u} dt + \int_a^b \sum_{i=1}^3 \frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\Gamma}_i} \left(\frac{\partial}{\partial t} \frac{\partial \Gamma_i}{\partial u} \right) dt \quad (2.1.6)$$

$$= \int_a^b \sum_{i=1}^3 \frac{\partial F}{\partial v_i} \Big|_{v_i=\Gamma_i} \frac{\partial \Gamma_i}{\partial u} dt + \left(\sum_{i=1}^3 \frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\Gamma}_i} \frac{\partial \Gamma_i}{\partial u} \right) \Big|_{t=a}^{t=b} \quad (2.1.7)$$

$$- \int_a^b \sum_{i=1}^3 \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\Gamma}_i} \right) \frac{\partial \Gamma_i}{\partial u} dt$$

$$= \left(\sum_{i=1}^3 \frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\Gamma}_i} \frac{\partial \Gamma_i}{\partial u} \right) \Big|_{t=a}^{t=b} \quad (2.1.8)$$

$$+ \int_a^b \sum_{i=1}^3 \left(\frac{\partial F}{\partial v_i} \Big|_{v_i=\Gamma_i} - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\Gamma}_i} \right) \right) \frac{\partial \Gamma_i}{\partial u} dt,$$

and thus,

$$i'(0) = \left(\sum_{i=1}^3 \frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \xi_i \right) \Big|_{t=a}^{t=b} + \int_a^b \sum_{i=1}^3 \left(\frac{\partial F}{\partial v_i} \Big|_{v_i=\gamma_i} - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \right) \right) \xi_i dt \quad (2.1.9)$$

$$= 0$$

for any family of comparison curves we choose to consider.

Definition 2.1.6. $i'(0)$ is the **first variation** of the functional $I(\gamma)$ and we denote it by

$$\delta_\gamma I(\xi) := i'(0).$$

It is worth noting that $\delta_\gamma I(\xi)$ is dependent only upon ξ . Thus, $\delta_\gamma I(\xi)$ is a functional of ξ and is in fact clearly linear in ξ .

Now, if γ is to be a minimum of $I(\gamma)$ then Equation (2.1.9) must be true for all C^∞ vector fields. In particular, Equation (2.1.9) must be true for all C^∞ vector fields $\xi(t)$ such that

$$\xi(a) = \xi(b) = 0.$$

Focusing our attention on such C^∞ vector fields we have

$$\begin{aligned} \delta_\gamma r(\xi) &= \int_a^b \sum_{i=1}^3 \left(\left. \frac{\partial F}{\partial v_i} \right|_{v_i=\dot{\gamma}_i} - \frac{\partial}{\partial t} \left(\left. \frac{\partial F}{\partial w_i} \right|_{w_i=\dot{\gamma}_i} \right) \right) \xi_i dt \\ &= 0. \end{aligned} \tag{2.1.10}$$

And, as Equation (2.1.10) must be true for all such ξ , we must have

$$\left. \frac{\partial F}{\partial v_i} \right|_{v_i=\dot{\gamma}_i} - \frac{\partial}{\partial t} \left(\left. \frac{\partial F}{\partial w_i} \right|_{w_i=\dot{\gamma}_i} \right) = 0$$

for $i = 1, 2$, and 3 . This equation is known as the *Euler-Lagrange differential equation*, and it provides a necessary condition for a curve to be a minimum of $I(\gamma)$.

Definition 2.1.7. *Let U be the set of C^∞ , embedded curves satisfying some set of constraints and let W be the set of initial velocity vector fields of perturbations $\Gamma(t, u)$ such that each γ_u is in U . If*

$$\delta_\gamma I(\xi) = 0$$

*for all vector fields in W then γ is **critical for $I(\gamma)$ over W** . If U is simply the set of C^∞ , embedded curves, it is equivalent to require γ to satisfy the Euler-Lagrange equations for $i = 1, 2$, and 3 .*

2.2 ISOPERIMETRIC PROBLEMS

We often wish to constrain our curves to satisfy some auxillary functional. Consider a function

$$G(x, v, w) : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

and the functional

$$J(\gamma) := \int_a^b G(t, \gamma, \dot{\gamma}) dt.$$

If we wish to minimize $I(\gamma)$ subject to the constraint

$$J(\gamma) = L \tag{2.2.1}$$

for some L in \mathbb{R} , then our problem is said to be an *isoperimetric problem* and Equation (2.2.1) is said to be an *isoperimetric constraint*. As discussed in [4], these types of problems take their name from the historic problem of finding a planar curve of fixed length which encloses the largest area.

Since it is generally not the case that, given a one parameter family of comparison curves $\Gamma(t, u)$,

$$J(\gamma_u) = L,$$

we must consider instead a two parameter family

$$\gamma_{u,q}(t) := \Gamma(t, u, q)$$

where $\gamma_{0,0} = \gamma$ and where the second parameter q is not independent of u , but rather determined by the equation

$$J(\gamma_{u,q}) = L.$$

Then we wish to minimize the two variable function

$$i(u, q) := I(\gamma_{u,q})$$

subject to the constraint

$$\begin{aligned} j(u, q) &:= J(\gamma_{u,q}) \\ &= L, \end{aligned}$$

and a necessary condition for a curve to be a solution of such a problem (if a solution exists) is given by the method of Lagrange multipliers.

Defining

$$F^*(t, \gamma_{u,q}, \dot{\gamma}_{u,q}, \lambda) := F(t, \gamma_{u,q}, \dot{\gamma}_{u,q}) + \lambda G(t, \gamma_{u,q}, \dot{\gamma}_{u,q}),$$

$$I^*(t, \gamma_{u,q}, \dot{\gamma}_{u,q}, \lambda) := \int_a^b F^*(t, \gamma_{u,q}, \dot{\gamma}_{u,q}, \lambda) dt,$$

and

$$\begin{aligned} i^*(u, q, \lambda) &:= I^*(t, \gamma_{u,q}, \dot{\gamma}_{u,q}) \\ &= i(u, q) + \lambda j(u, q), \end{aligned}$$

the method of Lagrange multipliers tells us that a necessary condition for γ to be critical for $I(\gamma)$ is

$$\left(\frac{\partial}{\partial u} i^*(u, q, \lambda) \right) \Big|_{u,q=0} = 0,$$

$$\left(\frac{\partial}{\partial q} i^*(u, q, \lambda) \right) \Big|_{u,q=0} = 0,$$

and

$$\left(\frac{\partial}{\partial \lambda} i^*(u, q, \lambda) \right) \Big|_{u,q=0} = 0$$

for all families of comparison curves. Thus, allowing

$$\xi(t) := \frac{\partial}{\partial u} F^*(t, \gamma_{u,q}, \dot{\gamma}_{u,q}, \lambda) \Big|_{u,q=0}$$

and

$$\eta(t) := \frac{\partial}{\partial q} F^*(t, \gamma_{u,q}, \dot{\gamma}_{u,q}, \lambda) \Big|_{u,q=0},$$

the same calculation as in Equation (2.1.2) yields

$$\begin{aligned} \left(\frac{\partial}{\partial u} i^*(u, q, \lambda) \right) \Big|_{u, q=0} &= \left(\sum_{i=1}^3 \frac{\partial F^*}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \xi_i \right) \Big|_{t=a}^{t=b} \\ &+ \int_a^b \sum_{i=1}^3 \left(\frac{\partial F^*}{\partial v_i} \Big|_{v_i=\dot{\gamma}_i} - \frac{\partial}{\partial t} \left(\frac{\partial F^*}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \right) \right) \xi_i dt \\ &= 0 \end{aligned} \quad (2.2.2)$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial q} i^*(u, q, \lambda) \right) \Big|_{u, q=0} &= \left(\sum_{i=1}^3 \frac{\partial F^*}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \eta_i \right) \Big|_{t=a}^{t=b} \\ &+ \int_a^b \sum_{i=1}^3 \left(\frac{\partial F^*}{\partial v_i} \Big|_{v_i=\dot{\gamma}_i} - \frac{\partial}{\partial t} \left(\frac{\partial F^*}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \right) \right) \eta_i dt \\ &= 0 \end{aligned} \quad (2.2.3)$$

As Equations (2.2.2) and (2.2.3) must be true for an arbitrary choice of ξ and η , it must be that any constrained critical curve γ satisfies the Euler-Lagrange equation

$$\frac{\partial F^*}{\partial v_i} \Big|_{v_i=\dot{\gamma}_i} - \frac{\partial}{\partial t} \left(\frac{\partial F^*}{\partial w_i} \Big|_{w_i=\dot{\gamma}_i} \right) = 0 \quad (2.2.4)$$

for $i = 1, 2$, and 3 . However, solving Equation (2.2.4) generates two constants of integration in addition to the unknown λ . Thus, to find an explicit γ which is critical for $I(\gamma)$ under the given constraint requires auxillary functions, usually fixed endpoint conditions, and the isoparametric constraint of Equation (2.2.1).

Alternatively, we can take a less general approach to an isoperimetric problem by using the isoperimetric constraint to constrain the set of vector fields we consider. This will hopefully simplify the functional $I(\gamma)$ that we wish to minimize. Notice that the constraint of Equation (2.2.1) implies that, letting $j(u) := J(\gamma_u)$,

$$j'(u) = 0$$

for all u . Thus, we must have that

$$\delta_\gamma J(\xi) = 0$$

if ξ is to be a vector field we wish to consider. This condition can sometimes simplify the functional to be minimized, and it will be the method that we use.

2.3 LENGTH AND ITS FIRST VARIATION

As an example that will be useful later on, let us find the first variation of the length functional. Let γ be a C^∞ , embedded curve and $L(\gamma)$ be the length of γ . Then we have

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| dt.$$

As in Section 2.1, we let $\Gamma(t, u)$ be a perturbation of $\gamma(t)$ with $\gamma_u(t) = \Gamma(t, u)$ and $l(u) = L(\gamma_u)$. We wish to understand the first variation of the length of γ , $\delta_\gamma L(\xi) = l'(0)$. Differentiating $l(u)$ yields

$$l'(u) = \frac{\partial}{\partial u} L(\gamma_u) \tag{2.3.1}$$

$$= \frac{\partial}{\partial u} \int_a^b |\dot{\gamma}_u| dt \tag{2.3.2}$$

$$= \int_a^b \frac{\partial}{\partial u} |\dot{\gamma}_u| dt \tag{2.3.3}$$

$$= \int_a^b \frac{\partial}{\partial u} \left| \frac{\partial}{\partial t} \Gamma(t, u) \right| dt \tag{2.3.4}$$

$$= \int_a^b \frac{\partial}{\partial u} \sqrt{\left\langle \frac{\partial}{\partial t} \Gamma(t, u), \frac{\partial}{\partial t} \Gamma(t, u) \right\rangle} dt \tag{2.3.5}$$

$$= \int_a^b \frac{\left\langle \frac{\partial}{\partial t} \Gamma(t, u), \frac{\partial}{\partial u} \frac{\partial}{\partial t} \Gamma(t, u) \right\rangle}{\sqrt{\left\langle \frac{\partial}{\partial t} \Gamma(t, u), \frac{\partial}{\partial t} \Gamma(t, u) \right\rangle}} dt \tag{2.3.6}$$

$$= \int_a^b \frac{\left\langle \frac{\partial}{\partial t} \Gamma(t, u), \frac{\partial}{\partial t} \frac{\partial}{\partial u} \Gamma(t, u) \right\rangle}{\sqrt{\left\langle \frac{\partial}{\partial t} \Gamma(t, u), \frac{\partial}{\partial t} \Gamma(t, u) \right\rangle}} dt, \tag{2.3.7}$$

and thus, setting $u = 0$,

$$\delta_\gamma L(\xi) = \int_a^b \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} dt. \tag{2.3.8}$$

Definition 2.3.1. *If for a given γ we have*

$$\delta_\gamma L(\xi) = 0,$$

we call ξ globally length preserving.

If we have a family of comparison curves $\Gamma(t, u)$ such that each comparison curve has the same length and parameterization as γ , i.e. the speed of $\gamma_u(t)$ is equal to the speed of $\gamma(t)$ for each t in $[a, b]$, then as the speed of $\gamma_u(t)$ is defined as $|\dot{\gamma}_u(t)|$, we have

$$|\dot{\gamma}_u(t)| = |\dot{\gamma}(t)|$$

for each t in $[a, b]$ and for all u . Thus, it follows that

$$\frac{\partial}{\partial u} |\dot{\gamma}_u(t)| = 0 \tag{2.3.9}$$

for all u and for all t in $[a, b]$. In particular, Equation (2.3.9) must hold for $u = 0$ and we have

$$\begin{aligned} \left(\frac{\partial}{\partial u} |\dot{\gamma}_u(t)| \right) \Big|_{u=0} &= \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} \\ &= 0, \end{aligned}$$

or, simply,

$$\langle \dot{\gamma}, \dot{\xi} \rangle = 0$$

for all t in $[a, b]$. Notice that given such a vector field,

$$\int_a^x \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} dt = 0$$

for all x in $[a, b]$ trivially, and we have the following definition.

Definition 2.3.2. *If for a given γ we have*

$$\langle \dot{\xi}(t), \dot{\gamma}(t) \rangle = 0 \text{ for all } t \text{ in } [a, b],$$

we call ξ locally length preserving.

We have the following relationship between globally and locally length preserving vector fields.

Proposition 2.3.3. *Let γ be an C^∞ , embedded curve which is arclength parametrized. Let ξ be a globally length preserving, C^∞ vector field and let $\Gamma(t, u)$ be an associated perturbation. Then*

$$\bar{\xi} := \xi - T \int_0^t \langle \dot{\xi}, T \rangle dx \quad (2.3.10)$$

is a C^∞ vector field which is locally length preserving and has an associated perturbation $\bar{\Gamma}(t, u)$ which is a reparametrization of $\Gamma(t, u)$ such that $\bar{\gamma}_u$ is arclength parametrized for each u .

Proof. We assume γ is unit length without loss of generality. Let

$$\bar{\xi}(t) := \xi(t) + f(t)T(t), \quad (2.3.11)$$

then using the Frenet formulas as defined in [5], we have

$$\dot{\bar{\xi}}(t) = \dot{\xi}(t) + f'(t)T(t) + f(t)\kappa(t)N(t)$$

where $N(t)$ is the principal normal and $\kappa(t)$ is the curvature of $\gamma(t)$. It follows that

$$\begin{aligned} \langle \dot{\gamma}, \dot{\bar{\xi}} \rangle &= \langle \dot{\gamma}, \dot{\xi} \rangle + f' \langle \dot{\gamma}, T \rangle + f \kappa \langle \dot{\gamma}, N \rangle \\ &= \langle T, \dot{\xi} \rangle + f' \langle T, T \rangle + f \kappa \langle T, N \rangle \\ &= \langle T, \dot{\xi} \rangle + f'. \end{aligned}$$

Therefore $\langle \dot{\gamma}, \dot{\bar{\xi}} \rangle = 0$ if and only if $f' = -\langle \dot{\xi}, T \rangle$. Thus,

$$f(t) := - \int_0^t \langle \dot{\xi}(x), T(x) \rangle dx + C.$$

Since we would like for $\bar{\Gamma}(t, u)$ to be merely a reparametrization of $\Gamma(t, u)$, we would like for both $\bar{\xi}$ and ξ to have the same effect on the endpoints 0 and 1, i.e. $\bar{\xi}(0) = \xi(0)$ and $\bar{\xi}(1) = \xi(1)$, which is true if and only if $f(1) = f(0) = 0$. Hence we have

$$\bar{\xi} := \xi - T \int_0^t \langle \dot{\xi}, T \rangle dx,$$

and $\bar{\xi}$ is clearly C^∞ .

Now we wish to see that $\bar{\xi}$ is the initial velocity vector field of a reparametrization of $\Gamma(t, u)$. Define

$$\tau(s, u) : [0, 1] \times \mathbb{R} \rightarrow [0, 1]$$

and

$$\Phi(s, u) = (\tau(s, u), u)$$

such that

$$\bar{\Gamma}(s, u) = \Gamma \circ \Phi(s, u)$$

is arclength parametrized for each u . Then let

$$\Phi^{-1}(t, u) = (S(t, u), u)$$

where for fixed u we have

$$t = \tau(S(t, u), u).$$

As $\bar{\Gamma}(t, u)$ is arclength parametrized, we must have

$$S(t, u) = \int_0^t |\dot{\gamma}_u| dr$$

for each u . It follows that

$$\frac{\partial}{\partial t} S(t, u) = |\dot{\gamma}_u(t)|$$

and

$$\frac{\partial}{\partial s} \tau(s, u) = \frac{1}{|\dot{\gamma}_u(\tau(s, u))|}.$$

By the same calculation as Equation (2.3.1), we have

$$\begin{aligned} \frac{\partial}{\partial u} S(t, u) \Big|_{u=0} &= \frac{\partial}{\partial u} \left(\int_0^t |\dot{\gamma}_u| dr \right) \Big|_{u=0} \\ &= \int_0^t \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} dr \\ &= \int_0^t \langle \dot{\gamma}, \dot{\xi} \rangle dr. \end{aligned}$$

Now letting I be the \mathbb{R}^2 identity map, we have

$$\begin{aligned} I(t, u) &= \Phi \circ \Phi^{-1}(t, u) \\ &= (\tau(S(t, u), u), u) \end{aligned}$$

and

$$\frac{\partial}{\partial u} I(t, u) = (0, 1).$$

Thus

$$\begin{aligned} 0 &= \frac{\partial}{\partial u} \tau(S(t, u), u) \\ &= \left(\frac{\partial \tau}{\partial s} \Big|_{s=S(t, u)} \right) \frac{\partial S}{\partial u} + \frac{\partial \tau}{\partial u} \Big|_{s=S(t, u)}, \end{aligned}$$

which implies

$$\frac{\partial \tau}{\partial u} \Big|_{s=S(t, u)} = - \left(\frac{\partial \tau}{\partial s} \Big|_{s=S(t, u)} \right) \frac{\partial S}{\partial u},$$

or equivalently,

$$\frac{\partial \tau}{\partial u} = - \frac{\partial \tau}{\partial s} \left(\frac{\partial S}{\partial u} \Big|_{t=\tau(s, u)} \right).$$

And, since γ is arclength parametrized,

$$\begin{aligned} \tau(s, 0) &= s \\ &= S(t, 0) \\ &= t. \end{aligned}$$

Hence,

$$\begin{aligned} \left(\frac{\partial}{\partial u} \tau(s, u) \right) \Big|_{u=0} &= \frac{1}{|\dot{\gamma}(s)|} \int_0^s \langle \dot{\gamma}, \dot{\xi} \rangle dr \\ &= \int_0^s \langle \dot{\gamma}, \dot{\xi} \rangle dr. \end{aligned}$$

Finally, we consider $\bar{\xi}(s)$.

$$\begin{aligned}
\bar{\xi}(s) &= \left(\frac{\partial}{\partial u} \bar{\Gamma}(s, u) \right) \Big|_{u=0} \\
&= \left(\frac{\partial}{\partial u} \Gamma(\tau(s, u), u) \right) \Big|_{u=0} \\
&= \left(\frac{\partial \Gamma}{\partial t} \frac{\partial t}{\partial u} \right) \Big|_{\substack{u=0, \\ t=\tau(s,0)}} + \left(\frac{\partial \Gamma}{\partial u} \right) \Big|_{\substack{u=0, \\ t=\tau(s,0)}} \\
&= -T(s) \int_0^s \langle \dot{\gamma}, \dot{\xi} \rangle dr + \xi(s) \\
&= \xi(s) - T(s) \int_0^s \langle \dot{\gamma}, \dot{\xi} \rangle dr
\end{aligned}$$

as desired. □

Corollary 2.3.4. *Let ξ be any C^∞ vector field along γ . The adjustment of Equation (2.3.10) gives a related smooth vector field $\bar{\xi}$ which is locally length preserving.*

CHAPTER 3

THE RADIUS OF GYRATION

3.1 THE CENTER OF MASS

Geometrically, the center of mass of γ is the average of the points along $\gamma(t)$. Analytically, this translates to the following.

Definition 3.1.1. *The **center of mass** $c(\gamma)$ of a curve $\gamma(t)$ is the closest constant function in the L^2 sense to $\gamma(t)$. To find $c(\gamma)$ we minimize*

$$f(x) := \sqrt{\int_a^b |\gamma(t) - x|^2 |\dot{\gamma}(t)| dt}.$$

Equivalently, we can minimize $f^2(x)$.

$$\begin{aligned} \frac{\partial}{\partial x_i} f^2(x) &= -2 \int_a^b (\gamma_i(t) - x_i) |\dot{\gamma}(t)| dt \\ &= 0, \end{aligned}$$

which implies

$$\begin{aligned} \int_a^b x_i |\dot{\gamma}(t)| dt &= \int_a^b \gamma_i(t) |\dot{\gamma}(t)| dt \\ x_i &= \frac{1}{\int_a^b |\dot{\gamma}(t)| dt} \int_a^b \gamma_i(t) |\dot{\gamma}(t)| dt. \end{aligned}$$

for $i = 1, 2$, and 3 . It follows that

$$x = \frac{1}{\int_a^b |\dot{\gamma}(t)| dt} \int_a^b \gamma(t) |\dot{\gamma}(t)| dt$$

is the minimum of $f(x)$ as it is the only critical point of an upward-opening quadratic function. Therefore

$$c(\gamma) := \frac{1}{L(\gamma)} \int_a^b \gamma(t) |\dot{\gamma}(t)| dt.$$

3.2 THE FIRST VARIATION OF $c(\gamma)$

Although $c(\gamma)$ is not a functional in the strictest sense since its range is \mathbb{R}^3 , we can approach the manner in which it changes under perturbations as we have our other functionals up to this point by considering it instead as a triplet of functionals.

As in Section 2.1, we let $\Gamma(t, u)$ be a perturbation of $\gamma(t)$ with $\gamma_u(t) = \Gamma(t, u)$ and $\sigma(u) = c(\gamma_u)$. Then we are interested in $\delta_\gamma c(\xi) = \sigma'(0)$. Differentiating $\sigma(u)$ yields

$$\begin{aligned}\sigma'(u) &= \frac{\partial}{\partial u} \left(\frac{1}{l(u)} \int_a^b \gamma_u |\dot{\gamma}_u| dt \right) \\ &= \frac{-l'(u)}{l^2(u)} \int_a^b \gamma_u |\dot{\gamma}_u| dt + \frac{1}{l(u)} \left(\frac{\partial}{\partial u} \int_a^b \gamma_u |\dot{\gamma}_u| dt \right) \\ &= \frac{-l'(u)}{l^2(u)} \int_a^b \gamma_u |\dot{\gamma}_u| dt + \frac{1}{l(u)} \int_a^b \left(\frac{\partial}{\partial u} \gamma_u \right) |\dot{\gamma}_u| + \gamma_u \left(\frac{\partial}{\partial u} |\dot{\gamma}_u| \right) dt\end{aligned}$$

and thus

$$\delta_\gamma c(\xi) = \frac{-\delta_\gamma L(\xi)c(\gamma)}{L^2(\gamma)} + \frac{1}{L(\gamma)} \int_a^b \xi |\dot{\gamma}| + \gamma \frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} dt.$$

3.3 THE RADIUS OF GYRATION

Definition 3.3.1. *The **radius of gyration** $R(\gamma)$ of a curve γ is the root mean square of the distances from points $\gamma(t)$ to the center of mass of γ weighted by the speed of γ at t :*

$$R(\gamma) := \sqrt{\int_a^b |\gamma - c(\gamma)|^2 |\dot{\gamma}| dt}.$$

Notice the weighting term $|\dot{\gamma}|$ ensures that $R(\gamma)$ is independent of the parametrization of γ .

Since we are only concerned with the critical curves of $R(\gamma)$, it is logically equivalent and computationally simpler to consider instead

$$\begin{aligned}r(\gamma) &:= \frac{1}{2} R^2(\gamma) \\ &= \frac{1}{2} \int_a^b |\gamma - c(\gamma)|^2 |\dot{\gamma}| dt.\end{aligned}$$

3.4 THE FIRST VARIATION OF $r(\gamma)$

Again as in Section 2.1, we let $\Gamma(t, u)$ be a perturbation of γ with $\gamma_u(t) = \Gamma(t, u)$ and $\rho(u) = r(\gamma_u)$. We wish to understand the first variation of $r(\gamma)$, $\delta_\gamma r(\xi) = \rho'(0)$. Differentiating $\rho(u)$ yields

$$\begin{aligned} \rho'(u) &= \frac{\partial}{\partial u} \left(\frac{1}{2} \int_a^b |\gamma_u - \sigma(u)|^2 |\dot{\gamma}_u| dt \right) \\ &= \frac{1}{2} \int_a^b 2 \left\langle \gamma_u - \sigma(u), \frac{\partial}{\partial u} \gamma_u + \frac{d}{du} \sigma(u) \right\rangle |\dot{\gamma}_u| + |\gamma_u - \sigma(u)|^2 \left(\frac{\partial}{\partial u} |\dot{\gamma}_u| \right) dt \end{aligned}$$

and thus

$$\delta_\gamma r(\xi) = \frac{1}{2} \int_a^b 2 \langle \gamma - c(\gamma), \xi - \delta_\gamma c(\xi) \rangle |\dot{\gamma}| + |\gamma - c(\gamma)|^2 \left(\frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} \right) dt.$$

3.5 $\delta_\gamma r(\xi)$ SIMPLIFIED

Intuitively, $R(\gamma)$ is translationally invariant. We would like to see this explicitly.

Proposition 3.5.1. *Let γ be a C^∞ , embedded curve and let*

$$\bar{\gamma}(s) := \gamma(s) + p$$

for some p in \mathbb{R}^3 . Then

$$R(\gamma) = R(\bar{\gamma}),$$

and given any C^∞ vector field ξ ,

$$\delta_\gamma r(\xi) = \delta_{\bar{\gamma}} r(\xi).$$

Proof. First, we examine $R(\bar{\gamma})$:

$$\begin{aligned}
R(\bar{\gamma}) &= \sqrt{\int_a^b |\bar{\gamma} - c(\bar{\gamma})|^2 |\dot{\bar{\gamma}}| dt} \\
&= \sqrt{\int_a^b \left| \gamma + p - \frac{\int_a^b (\gamma + p) |\dot{\gamma}| dt}{L(\bar{\gamma})} \right|^2 |\dot{\gamma}| dt} \\
&= \sqrt{\int_a^b \left| \gamma + p - \frac{\int_a^b \gamma |\dot{\gamma}| dt}{L(\gamma)} - \frac{p \int_a^b |\dot{\gamma}| dt}{L(\gamma)} \right|^2 |\dot{\gamma}| dt} \\
&= \sqrt{\int_a^b |\gamma + p - c(\gamma) - p|^2 |\dot{\gamma}| dt} \\
&= R(\gamma).
\end{aligned}$$

This implies that $r(\bar{\gamma}) = r(\gamma)$ as well.

Now, given a family of comparison functions $\Gamma(t, u)$ to γ , we create a family of comparison functions $\bar{\Gamma}(t, u)$ to $\bar{\gamma}$ by defining

$$\bar{\Gamma}(t, u) := \Gamma(t, u) + p.$$

Then clearly $\bar{\xi} = \xi$ and as $r(\bar{\gamma}_u) = r(\gamma_u)$ for all u , it must be that

$$\begin{aligned}
\delta_{\bar{\gamma}} r(\xi) &= \delta_{\bar{\gamma}} r(\bar{\xi}) \\
&= \left(\frac{\partial}{\partial u} r(\bar{\gamma}_u) \right) \Big|_{u=0} \\
&= \left(\frac{\partial}{\partial u} r(\gamma_u) \right) \Big|_{u=0} \\
&= \delta_{\gamma} r(\xi)
\end{aligned}$$

as desired. □

Hence, we can assume without loss of generality that $c(\gamma) = 0$. Also, as $R(\gamma)$ is not dependent upon the parametrization of γ , we can assume γ is arclength parametrized. We therefore constrain our curves γ to be arclength parametrized, C^∞ , embedded curves such that $c(\gamma) = 0$.

These assumptions allow us an initial simplification of $\delta_\gamma r(\xi)$, and we have

$$\begin{aligned} \delta_\gamma r(\xi) &= \frac{1}{2} \int_a^b 2 \langle \gamma - c(\gamma), \xi - \delta_\gamma c(\xi) \rangle |\dot{\gamma}| + |\gamma - c(\gamma)|^2 \left(\frac{\langle \dot{\gamma}, \dot{\xi} \rangle}{|\dot{\gamma}|} \right) dt \\ &= \frac{1}{2} \int_0^{L(\gamma)} 2 \left\langle \gamma, \left(\xi - \frac{1}{L(\gamma)} \int_0^{L(\gamma)} (\xi + \gamma \langle \dot{\gamma}, \dot{\xi} \rangle) ds \right) \right\rangle + |\gamma|^2 \langle \dot{\gamma}, \dot{\xi} \rangle ds. \end{aligned}$$

We will work under these simplifying assumptions going forward.

CHAPTER 4

CRITICAL CURVES FOR THE RADIUS OF GYRATION

4.1 CRITICALITY FOR $R(\gamma)$

Notice that $R(\gamma)$ is not homothety invariant. If

$$\bar{\gamma}(t) := a\gamma(t)$$

for some a in \mathbb{R} ,

$$\begin{aligned} R(\bar{\gamma}) &= \sqrt{\int_a^b |\bar{\gamma}|^2 |\dot{\bar{\gamma}}| dt} \\ &= \sqrt{\int_a^b |a\gamma|^2 |a\dot{\gamma}| dt} \\ &= \sqrt{a^2 |a| \int_a^b |\gamma|^2 |\dot{\gamma}| dt} \\ &= |a|^{\frac{3}{2}} R(\gamma). \end{aligned}$$

As we are interested in what $R(\gamma)$ tells us about how densely a curve is wound, we do not wish to compare curves of differing lengths. Thus, we constrain our curves γ to curves such that $L(\gamma) = 1$ and our vector fields to globally length preserving, C^∞ vector fields.

In fact, we can limit our attention even further by considering only vector fields that preserve length *locally* without affecting our notion of criticality.

Proposition 4.1.1. *Given a C^∞ , embedded curve γ , let W be the set of all C^∞ , globally length preserving vector fields and let W' be the set of all C^∞ , locally length preserving vector fields. Then γ is critical for $R(\gamma)$ over W if and only if it is critical for $R(\gamma)$ over W' .*

Proof. By Section 3.5 and the above, without loss of generality we can restrict our curves to C^∞ , embedded, arclength parametrized, unit length curves with $c(\gamma) = 0$. If γ is critical for $R(\gamma)$ over globally length preserving perturbations, then by definition it is critical for $R(\gamma)$ for all ξ such that $\int_0^1 \langle \dot{\gamma}, \dot{\xi} \rangle ds = 0$. If we have a η such that $\langle \dot{\gamma}, \dot{\eta} \rangle = 0$ for all s in $[0, 1]$, then $\int_0^1 \langle \dot{\gamma}, \dot{\eta} \rangle ds = 0$ trivially. Thus γ is critical for $R(\gamma)$ over locally length preserving perturbations as well.

If instead γ is critical for $R(\gamma)$ over locally length preserving, C^∞ vector fields, then for any ξ such that $\int_0^1 \langle \dot{\gamma}, \dot{\xi} \rangle ds = 0$, we consider instead

$$\bar{\xi}(s) = \xi(s) - f(s)T(s)$$

where $f(s) = \int_0^s \langle \dot{\xi}, T \rangle dt$. Note that $f(0) = f(1) = 0$. By Proposition 2.3.3, we have then that $\bar{\xi}$ is locally length preserving. Thus, $\delta_\gamma r(\bar{\xi}) = 0$ by assumption and since $\delta_\gamma r(\xi)$ is linear, we know that

$$\delta_\gamma r(\bar{\xi}) = \delta_\gamma r(\xi) - \delta_\gamma r(f(s)T).$$

We investigate $\delta_\gamma r(f(s)T)$.

$$\begin{aligned}
\delta_\gamma r(f(s)T) &= \frac{1}{2} \int_0^1 2 \left\langle \gamma, \left(f(s)T - \int_0^1 (f(s)T + \gamma \langle \dot{\gamma}, f'(s)T + f(s)\kappa N \rangle) ds \right) \right\rangle ds \\
&\quad + \frac{1}{2} \int_0^1 |\gamma|^2 \langle \dot{\gamma}, f'(s)T + f(s)\kappa N \rangle ds \\
&= \frac{1}{2} \int_0^1 2 \left\langle \gamma, \left(f(s)T - \int_0^1 (f(s)T + \gamma \langle T, f'(s)T \rangle) ds \right) \right\rangle ds \\
&\quad + \frac{1}{2} \int_0^1 |\gamma|^2 \langle T, f'(s)T \rangle ds \\
&= \frac{1}{2} \int_0^1 2 \left\langle \gamma, \left(f(s)T - \int_0^1 f(s)T ds - \int_0^1 f'(s)\gamma ds \right) \right\rangle ds \\
&\quad + \frac{1}{2} \int_0^1 f'(s)|\gamma|^2 ds \\
&= \frac{1}{2} \int_0^1 2 \left\langle \gamma, \left(f(s)T - \int_0^1 f(s)T ds - (f(s)\gamma)|_0^1 + \int_0^1 f(s)T ds \right) \right\rangle ds \\
&\quad + \left(\frac{f(s)}{2} |\gamma|^2 \right) \Big|_0^1 - \frac{1}{2} \int_0^1 2f(s) \langle \gamma, T \rangle ds \\
&= \frac{1}{2} \int_0^1 2 \langle \gamma, f(s)T \rangle ds - \int_0^1 f(s) \langle \gamma, T \rangle ds \\
&= \int_0^1 f(s) \langle \gamma, T \rangle ds - \int_0^1 f(s) \langle \gamma, T \rangle ds \\
&= 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\delta_\gamma r(\xi) &= \delta_\gamma r(\bar{\xi}) \\
&= 0
\end{aligned}$$

and that γ is critical for $R(\gamma)$ over globally length preserving perturbations. \square

It follows that we can constrain our vector fields to be C^∞ , locally length preserving vector fields, and this will be the set of vector fields we work over.

4.2 $\delta_\gamma r(\xi)$ FURTHER SIMPLIFIED

Let us summarize the final conditions under which we will investigate $R(\gamma)$.

Definition 4.2.1. Let U be the set of C^∞ , embedded, arc length parametrized, unit length curves with center of mass at the origin. If γ is in U , it is said to be an **admissible curve**.

Definition 4.2.2. If $\Gamma(t, u)$ is a family of comparison functions such that γ_u is an admissible curve for each u then $\Gamma(t, u)$ is called an **admissible family** or **perturbation**.

Definition 4.2.3. Let $\Gamma(t, u)$ be an admissible perturbation. Then we call its initial velocity vector field ξ an **admissible vector field**. Notice this is precisely the set of locally length preserving, C^∞ vector fields.

These constraints allow us to greatly simplify $\delta_\gamma r(\xi)$. Given an admissible curve γ and an admissible vector field ξ ,

$$\begin{aligned} \delta_\gamma r(\xi) &= \frac{1}{2} \int_0^{L(\gamma)} 2 \left\langle \gamma, \left(\xi - \frac{1}{L(\gamma)} \int_0^{L(\gamma)} (\xi + \gamma \langle \dot{\gamma}, \dot{\xi} \rangle) ds \right) \right\rangle + \gamma^2 \langle \dot{\gamma}, \dot{\xi} \rangle ds \\ &= \int_0^1 \langle \gamma, \xi - \int_0^1 \xi ds \rangle ds. \end{aligned}$$

Recalling that $c(\gamma) = \int_0^1 \gamma ds$ yields

$$\begin{aligned} \delta_\gamma r(\xi) &= \int_0^1 \langle \gamma, \xi \rangle ds - \int_0^1 \langle \gamma, \int_0^1 \xi ds \rangle ds \\ &= \int_0^1 \langle \gamma, \xi \rangle ds - \left\langle \int_0^1 \gamma ds, \int_0^1 \xi ds \right\rangle \\ &= \int_0^1 \langle \gamma, \xi \rangle ds - \langle c(\gamma), \int_0^1 \xi ds \rangle \\ &= \int_0^1 \langle \gamma, \xi \rangle ds - \langle 0, \int_0^1 \xi ds \rangle \\ &= \int_0^1 \langle \gamma, \xi \rangle ds. \end{aligned}$$

This is the version of $\delta_\gamma r(\xi)$ we will work with.

4.3 CRITICAL CURVES FOR $R(\gamma)$

Now that $\delta_\gamma r(\xi)$ is adequately simplified, we are ready to search for critical curves for $R(\gamma)$.

Theorem 4.3.1 (Criticality Theorem). *Let U_L be the set of C^∞ , embedded curves of length L and W be the set of globally length preserving, C^∞ vector fields. γ is critical for $R(\gamma)$ over W if and only if γ is a straight line segment.*

The proof of Theorem 4.3.1 is the entirety of Chapter 5.

CHAPTER 5

PROOF OF THE CRITICALITY THEOREM

5.1 STRAIGHT LINE CRITICALITY FOR $R(\gamma)$

Proposition 5.1.1. *Let γ be a straight line segment in \mathbb{R}^3 with length L . Then γ is critical for $R(\gamma)$ over globally length preserving, C^∞ vector fields.*

Proof. We first restrict our attention to admissible curves and vector fields as defined in Definitions 4.2.1 and 4.2.3 without loss of generality. Then γ is of the form

$$\gamma(s) := \left(s - \frac{1}{2}\right) v$$

for s in $[0, 1]$ and for some v in \mathbb{R}^3 with $|v| = 1$.

Since ξ is an admissible vector field, we have

$$\begin{aligned} \langle \dot{\gamma}, \dot{\xi} \rangle &= \langle v, \dot{\xi} \rangle \\ &= 0 \end{aligned}$$

for all s in $[0, 1]$. Thus, letting w_1 and w_2 be an orthonormal basis for v^\perp , we have that

$$\dot{\xi}(s) = f_1(s)w_1 + f_2(s)w_2$$

where

$$f_i(s) : \mathbb{R} \rightarrow \mathbb{R}$$

is a C^∞ function for $i = 1, 2$. It follows that, letting

$$F_1(s) := \int_a^b f_1(s) + C_1$$

and

$$F_2(s) := \int_a^b f_2(s) + C_2,$$

we have

$$\xi(s) = F_1(s)w_1 + F_2(s)w_2 + Cv$$

for some constants C_1 , C_2 , and C in \mathbb{R} . Hence,

$$\begin{aligned} \delta_\gamma r(\xi) &= \int_0^1 \langle \gamma, \xi \rangle ds \\ &= \int_0^1 \left(s - \frac{1}{2} \right) \langle v, F_1(s)w_1 + F_2(s)w_2 + Cv \rangle ds \\ &= C \int_0^1 \left(s - \frac{1}{2} \right) \langle v, v \rangle ds \\ &= C \int_0^1 \left(s - \frac{1}{2} \right) (1) ds \\ &= 0. \end{aligned}$$

Therefore, a straight line segment is critical for radius of gyration. □

5.2 THE CASES FOR ALL OTHER CURVES

Now we would like to see that, given any admissible curve γ that is not a straight line segment, γ is not critical for $R(\gamma)$. To do so, given such a γ , we will find an admissible vector field ξ such that $\delta_\gamma r(\xi) \neq 0$. Thus, by Definition 2.1.7, γ will not be critical for $R(\gamma)$. We consider two cases based on behavior of a curve at its endpoints. In particular, we are interested in how the endpoints of a given curve relate to its center of mass. We will then concentrate an appropriate perturbation, and consequently its related vector field, on an appropriate segment of γ .

Definition 5.2.1. *Let U be the set of C^∞ , embedded curves which are not straight line segments, and consider the equation*

$$T(x) = \pm \frac{\gamma(x) - c(\gamma)}{|\gamma(x) - c(\gamma)|}. \tag{5.2.1}$$

Allowing γ to be parametrized over $[a, b]$, define V to be the set of curves in U such that $\gamma(a) \neq c(\gamma)$ and Equation (5.2.1) holds for $x = a$, or $\gamma(b) \neq c(\gamma)$ and Equation (5.2.1) holds for $x = b$, or both. Lastly, define

$$\bar{V} = U - V.$$

Notice that for γ in \bar{V} , Equation (5.2.1) fails for at least one endpoint which is not equal to the center of mass.

5.3 MOVING TOWARDS $c(\gamma)$ DECREASES $R(\gamma)$

First we consider the set \bar{V} , and we assume γ is admissible without loss of generality. We will wish to avoid an endpoint e if

$$e = c(\gamma).$$

Since γ is embedded, it cannot be that

$$\gamma(0) = \gamma(1) = c(\gamma) = 0.$$

Thus, we assume without loss of generality that $\gamma(1) \neq c(\gamma)$ and focus exclusively on $\gamma(1)$.

Our general approach will be to perturb a small segment of γ near $\gamma(1)$ towards $c(\gamma)$ as in Figure 5.1, adjusting our perturbation to be locally length preserving. Loosely speaking, however, we will perturb $\gamma(s)$ along the vector field $(c(\gamma) - \gamma(s)) = -\gamma(s)$.

Proposition 5.3.1. *Let U be the set of C^∞ , embedded curves with length L and let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a curve in U such that $\gamma(b) \neq c(\gamma)$ and $T(b) \neq \frac{|\gamma(b)|}{|\gamma(b)|}$. Then γ is not critical for $R(\gamma)$ over globally length preserving, C^∞ vector fields.*

Proof. Without loss of generality, we restrict our attention to admissible curves and vector fields. Then by assumption, $T(1) \neq \pm \frac{\gamma(1)}{|\gamma(1)|}$. By Cauchy-Schwarz, this implies

$$|\langle T(1), \gamma(1) \rangle| < |\gamma(1)|,$$

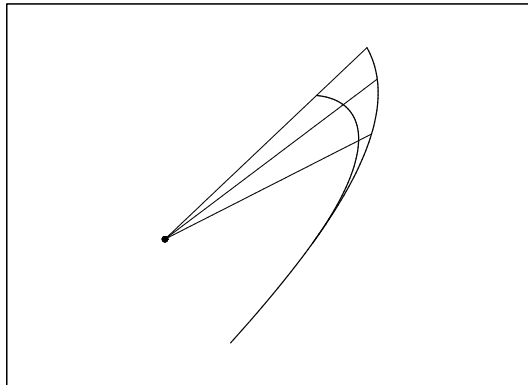


Figure 5.1: Decreasing $R(\gamma)$ for γ in \overline{V}

and by continuity, there exists a δ such that

$$|\langle T, \gamma \rangle| < (1 - \epsilon)|\gamma(s)|$$

for all s in $[\delta, 1]$.

Given a δ_1 in $[\delta, 1]$, let $\Phi(s) = \Phi_{\delta_1}(s)$ be a nonnegative, nondecreasing ramp function with support $(\delta_1, 1]$ such that

$$\int_{\delta_1}^1 \Phi(s) ds = 1.$$

Let

$$M(\delta_1) := \max \{|\gamma(s)| \text{ for } s \text{ in } [\delta_1, 1]\},$$

and let

$$m(\delta_1) := \min \{|\gamma(s)| \text{ for } s \text{ in } [\delta_1, 1]\}.$$

Finally, let

$$\xi(s) := -\Phi(s)\gamma(s).$$

We would like to have a locally length preserving vector field, so we adjust ξ by the method set forth in Proposition 2.3.3 and let

$$\bar{\xi}(s) = \xi(s) - T(s) \int_0^s \langle \dot{\xi}(t), T(t) \rangle dt.$$

Thus we have

$$\begin{aligned} \bar{\xi} &= -\Phi\gamma - T \int_0^s \langle -\Phi'\gamma - \Phi T, T \rangle dt \\ &= -\Phi\gamma + T \int_0^s \Phi' \langle \gamma, T \rangle dt + T \int_0^s \Phi dt. \end{aligned}$$

Now we examine $\delta_\gamma r(\bar{\xi})$.

$$\begin{aligned} \delta_\gamma r(\bar{\xi}) &= \int_0^1 \langle \gamma, \bar{\xi} \rangle ds \\ &= \int_0^1 \langle \gamma, -\Phi\gamma + T \int_0^s \Phi' \langle \gamma, T \rangle dt + T \int_0^s \Phi dt \rangle ds \\ &= \int_0^1 -\Phi |\gamma|^2 ds + \int_0^1 \left(\int_0^s \Phi' \langle \gamma, T \rangle dt \right) \langle \gamma, T \rangle ds + \int_0^1 \left(\int_0^s \Phi dt \right) \langle \gamma, T \rangle ds. \end{aligned}$$

At this point, we will break the right hand side of this equation into three pieces for clarity as we proceed. Let

$$\begin{aligned} A &:= - \int_0^1 \Phi |\gamma|^2 ds, \\ B &:= \int_0^1 \left(\int_0^s \Phi' \langle \gamma, T \rangle dt \right) \langle \gamma, T \rangle ds, \end{aligned}$$

and

$$C := \int_0^1 \left(\int_0^s \Phi dt \right) \langle \gamma, T \rangle ds.$$

Thus we have

$$\begin{aligned} A &= - \int_{\delta_1}^1 \Phi |\gamma|^2 ds \\ &\leq - \int_{\delta_1}^1 \Phi m^2(\delta_1) ds \\ &= -m^2(\delta_1) \int_{\delta_1}^1 \Phi ds \\ &= -m^2(\delta_1), \end{aligned}$$

$$\begin{aligned}
B &= \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi' \langle \gamma, T \rangle dt \right) \langle \gamma, T \rangle ds \\
&< (1 - \epsilon)^2 \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi' |\gamma| dt \right) |\gamma| ds \\
&\leq (1 - \epsilon)^2 \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi' M(\delta_1) dt \right) M(\delta_1) ds \\
&= (1 - \epsilon)^2 M^2(\delta_1) \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi' dt \right) ds \\
&= (1 - \epsilon)^2 M^2(\delta_1) \int_{\delta_1}^1 \Phi ds \\
&= (1 - \epsilon)^2 M^2(\delta_1),
\end{aligned}$$

and

$$\begin{aligned}
C &= \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi dt \right) \langle \gamma, T \rangle ds \\
&< (1 - \epsilon) \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi dt \right) |\gamma| ds \\
&\leq (1 - \epsilon) \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi dt \right) M(\delta_1) ds \\
&< (1 - \epsilon) M(\delta_1) \int_{\delta_1}^1 1 ds \\
&= (1 - \delta_1)(1 - \epsilon) M(\delta_1).
\end{aligned}$$

Putting it all back together we have

$$\delta_\gamma r(\bar{\xi}) < -m^2(\delta_1) + (1 - \epsilon)^2 M^2(\delta_1) + (1 - \delta_1)(1 - \epsilon) M(\delta_1),$$

and since

$$\begin{aligned}
&\lim_{\delta_1 \rightarrow 1} -m^2(\delta_1) + (1 - \epsilon)^2 M^2(\delta_1) + (1 - \delta_1)(1 - \epsilon) M(\delta_1) \\
&= -|\gamma(1)|^2 + (1 - \epsilon)|\gamma(1)|^2 \\
&< 0,
\end{aligned}$$

we can choose δ_1 in $(\delta, 1)$ such that

$$\begin{aligned}
\delta_\gamma r(\bar{\xi}) &< -m^2(\delta_1) + (1 - \epsilon)^2 M^2(\delta_1) + (1 - \delta_1) \\
&< 0.
\end{aligned}$$

Thus, $\delta_\gamma r(\bar{\xi}) < 0$ and γ is not critical. \square

5.4 THE CASES FOR CURVES IN V

We now wish to consider curves such that Equation (5.2.1) holds for an endpoint which is not equal to the center of mass. Geometrically, this constraint requires that the tangent vector at x be collinear with the line joining $\gamma(x)$ and $c(\gamma)$. Without loss of generality, assume $x = 1$ and let

$$\delta := \min \left\{ x \text{ such that } T(s) = \pm \frac{\gamma(s)}{|\gamma(s)|} \text{ for all } s \text{ in } [x, 1] \right\}. \quad (5.4.1)$$

We call $e_a = \gamma(1)$ a *desirable endpoint* if

$$|\gamma(s) - c(\gamma)| = |\gamma(s)| \neq 0$$

for all s in $[\delta, 1]$. This assures us that $\frac{\gamma(s)}{|\gamma(s)|}$ is defined on $[\delta, 1]$ and that the sign of $\langle T(s), \gamma(s) \rangle$ remains constant on $[\delta, 1]$.

If e_a is not a desirable endpoint, then there exists a p in $[\delta, 1]$ such that $\gamma(p) = c(\gamma)$ and we must consider $e_b = \gamma(0)$ instead. If Equation (5.2.1) holds for $x = 0$ as well, then e_b must be a desirable endpoint (once we reparametrize γ such that $\gamma(1) = e_b$). If e_b were not a desirable endpoint, γ would clearly be a straight line segment, which by assumption it is not.

If Equation (5.2.1) does not hold for $x = 0$, then since γ is embedded,

$$e_b \neq \gamma(p) = c(\gamma)$$

and by Proposition 5.3.1, γ is not critical.

Thus we are left only with curves such that $\gamma(1)$ is a desirable endpoint and

$$T(1) = \frac{\gamma(1)}{|\gamma(1)|} \text{ or } T(1) = -\frac{\gamma(1)}{|\gamma(1)|}.$$

In general, our approach will be to find an appropriate segment of γ and perturb this segment along the vector field $N(s)$. While our approach for each case will be similar, we will consider them separately. In either case, we need some preliminary results.

Lemma 5.4.1. *Suppose $g(x)$ and $f(x)$ are C^2 functions on $[a, b]$ with $g(a) = f(a)$ and $f'(a) = g'(a)$. If $g''(x) \leq f''(x)$ for all x in $[a, b]$, then $g(x) \leq f(x)$ for all x in $[a, b]$. In particular, $g(b) \leq f(b)$.*

Proof. Given an x in $[a, b]$,

$$\begin{aligned} g(x) &= \int_a^x g'(t) dt + g(a) \\ &= \int_a^x \left(\int_a^t g''(z) dz + g'(a) \right) dt + g(a) \\ &\leq \int_a^x \left(\int_a^t f''(z) dz + f'(a) \right) dt + f(a) \\ &= \int_a^x f'(t) dt + f(a) \\ &= f(x) \end{aligned} \quad \square$$

Lemma 5.4.2. *Suppose $g(x)$ and $f(x)$ are C^2 functions on $[a, b]$ with $g(a) = f(a)$, $g'(a) = f'(a)$, and $g(b) > f(b)$. Then there exists a y in $[a, b]$ such that $g''(y) > f''(y)$.*

Proof. If not, by Lemma 5.4.1 we would have $g(b) < f(b)$. □

5.5 MOVING NORMALLY INCREASES $R(\gamma)$

We first consider curves such that

$$T(1) = \frac{\gamma(1)}{|\gamma(1)|}.$$

Lemma 5.5.1. *Let γ be an admissible curve and let δ be defined as in Equation (5.4.1). Assume also that $\gamma(1)$ a desirable endpoint as described in Section 5.4 and*

$$T(1) = \frac{\gamma(1)}{|\gamma(1)|}.$$

Then there exist δ_1, δ_2 in $[0, \delta)$ such that

$$\langle \gamma, T \rangle > 0$$

for all s in $[\delta_1, 1]$ and

$$\langle \gamma, N \rangle > 0$$

for all s in $[\delta_1, \delta_2]$.

Proof. By the definition of δ , we have

$$\begin{aligned} \langle \gamma(s), T(s) \rangle &= |\gamma(s)| \\ &> 0, \end{aligned}$$

for all s in $[\delta, 1]$. Since $\gamma(s)$ is C^∞ , there must exist an ϵ such that

$$\langle \gamma(s), T(s) \rangle > 0$$

for s in $[\delta - \epsilon, 1]$.

Turning our attention to $\langle \gamma, N \rangle$, we have

$$\begin{aligned} \langle \gamma, N \rangle &= \frac{1}{\kappa} (\langle \gamma, T \rangle' - \langle T, T \rangle) \\ &= \frac{1}{\kappa} (\langle \gamma, T \rangle' - 1) \\ &= \frac{1}{\kappa} \left(\frac{1}{2} \langle \gamma, \gamma \rangle'' - 1 \right) \end{aligned}$$

Thus,

$$\langle \gamma, N \rangle > 0$$

if and only if

$$\langle \gamma, \gamma \rangle'' > 2.$$

Now consider

$$\bar{\gamma}(s) = \gamma(\delta - s)$$

on $[0, \epsilon]$, which is the reverse parametrization of $\gamma(s)$ on $[\delta - \epsilon, \delta]$. Then we have

$$\begin{aligned} |\bar{\gamma}(s)|' &= \frac{-\langle \gamma(\delta - s), T(\delta - s) \rangle}{|\gamma(\delta - s)|} \\ &\geq -1 \end{aligned}$$

on $[0, \epsilon]$, and by the definition of δ and the fact that γ is C^∞ , we must have strict inequality on an open set in $[0, \epsilon]$. It follows that

$$\begin{aligned} |\bar{\gamma}(s)| &= \int_0^s |\bar{\gamma}(t)|' dt + |\bar{\gamma}(0)| \\ &\geq \int_0^s -1 ds + |\bar{\gamma}(0)| \\ &= |\bar{\gamma}(0)| - s \end{aligned}$$

on $(0, \epsilon]$ with

$$|\bar{\gamma}(\epsilon)| > |\bar{\gamma}(0)| - \epsilon. \quad (5.5.1)$$

Now let $L := |\bar{\gamma}(0)|$, $g(s) := |\bar{\gamma}(s)|^2$, and $f(s) := (L - s)^2$. Then we have that

$$\begin{aligned} g(0) &= f(0) = L^2, \\ g'(0) &= f'(0) = -2L, \end{aligned}$$

and by Equation (5.5.1),

$$\begin{aligned} g(\epsilon) &= |\bar{\gamma}(\epsilon)|^2 \\ &> (|\bar{\gamma}(0)| - \epsilon)^2 \\ &= f(\epsilon). \end{aligned}$$

By Lemma 5.4.2, therefore, there exists an x in $[0, \epsilon]$ such that

$$g''(x) > f''(x) = 2.$$

As we have that

$$\begin{aligned}\bar{g}''(s) &= \frac{d^2}{ds^2} |\bar{\gamma}(s)| \\ &= \frac{d^2}{ds^2} |\gamma(\delta - s)| \\ &= \frac{d^2}{ds^2} \langle \gamma(\delta - s), \gamma(\delta - s) \rangle,\end{aligned}$$

we let $y = \delta - x$ and obtain

$$\langle \gamma(s), \gamma(s) \rangle''|_{s=y} > 2.$$

Finally, since γ is C^∞ , there is an interval $[\delta_1, \delta_2]$ containing y such that

$$\langle \gamma, \gamma \rangle'' > 2$$

on $[\delta_1, \delta_2]$ as desired. □

We are now prepared to see that such a curve is not critical. Again, our general approach will be to perturb a segment of γ along the vector field $N(s)$ as pictured in Figure 5.2. We will, however, have to adjust our perturbation to be locally length preserving.

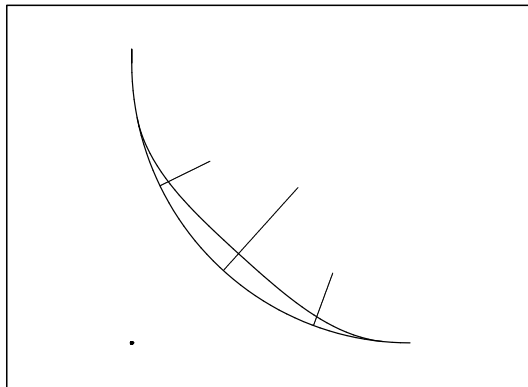


Figure 5.2: Increasing $R(\gamma)$ for Appropriate γ in V

Proposition 5.5.2. *Let U be the set of C^∞ , embedded curves with length L , and let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a curve in U such that*

$$T(b) = \frac{\gamma(b)}{|\gamma(b)|}$$

and

$$|\gamma(t) - c(\gamma)| = |\gamma(t)| \neq 0$$

for all t in $[\delta, b]$ where

$$\delta := \min \left\{ x \text{ such that } T(s) = \frac{\gamma(s)}{|\gamma(s)|} \text{ for all } s \text{ in } [x, 1] \right\}.$$

Then γ is not critical for $R(\gamma)$ over globally length preserving, C^∞ vector fields.

Proof. Without loss of generality, we restrict ourselves to admissible curves and vector fields as summarized in Section 4.2. By Lemma 5.5.1, there exists δ_1, δ_2 such that $\langle \gamma, T \rangle > 0$ on $[\delta_1, 1]$ and $\langle \gamma, N \rangle > 0$ on $[\delta_1, \delta_2]$. Let $\Phi(s)$ be a nonnegative bump function with support on $[\delta_1, \delta_2]$ and

$$\Phi(\delta_1) = \Phi(\delta_2) = 0,$$

and let

$$\xi(s) := \Phi(s)N(s).$$

We would like to have a locally length preserving perturbation of γ , so we adjust ξ by the method described in Proposition 2.3.3 and let

$$\bar{\xi}(s) := \xi(s) - T(s) \int_0^s \langle \dot{\xi}(t), T(t) \rangle dt.$$

Then by the Frenet formulas as defined in [5], we have

$$\begin{aligned} \bar{\xi}(s) &= \Phi N - T \int_0^s \langle \Phi' N + \Phi(\tau B - \kappa T), T \rangle dt \\ &= \Phi N + T \int_0^s \Phi \kappa dt. \end{aligned}$$

Now we examine $\delta_\gamma r(\bar{\xi})$.

$$\begin{aligned}\delta_\gamma r(\bar{\xi}) &= \int_0^1 \langle \gamma, \bar{\xi} \rangle ds \\ &= \int_0^1 \langle \gamma, \Phi N + T \int_0^s \Phi \kappa dt \rangle ds \\ &= \int_0^1 \Phi \langle \gamma, N \rangle ds + \int_0^1 \left(\int_0^s \Phi \kappa dt \right) \langle \gamma, T \rangle ds \\ &= \int_{\delta_1}^{\delta_2} \Phi \langle \gamma, N \rangle ds + \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi \kappa dt \right) \langle \gamma, T \rangle ds,\end{aligned}$$

and as $\langle \gamma, N \rangle$ and $\langle \gamma, T \rangle$ are both positive on their respective intervals of integration by Lemma 5.5.1, $\delta_\gamma r(\bar{\xi}) > 0$. Thus, γ is not critical. \square

5.6 MOVING NORMALLY DECREASES $R(\gamma)$

Finally we consider curves such that

$$T(1) = -\frac{\gamma(1)}{|\gamma(1)|}.$$

Our approach will be similar to that of Section 5.5 and we will omit repetitive details.

Lemma 5.6.1. *Let γ be an admissible curve and let δ be defined as in Equation (5.4.1)*

Assume also that $\gamma(1)$ a desirable endpoint as described in Section 5.4 and

$$T(1) = -\frac{\gamma(1)}{|\gamma(1)|}.$$

Then there exist δ_1, δ_2 in $[0, \delta)$ such that

$$\langle \gamma, T \rangle < 0$$

for all s in $[\delta_1, 1]$ and

$$\langle \gamma, N \rangle < 0$$

for all s in $[\delta_1, \delta_2]$.

Proof. Clearly there exists an ϵ such that $\langle \gamma, T \rangle < 0$ on $[\delta - \epsilon, 1]$.

Turning to $\langle \gamma, N \rangle$, we follow the proof of Lemma 5.5.1 closely. We now have

$$\langle \gamma, N \rangle < 0$$

if and only if

$$\langle \gamma, \gamma \rangle'' < 2.$$

Again allowing $L := |\gamma(\delta)|$ and $g(s) := |\gamma(\delta - s)|^2$ but defining $f(s) := (L + s)^2$, we have

$$f(0) = g(0) = L^2,$$

$$f'(0) = g'(0) = 2,$$

and

$$f(\epsilon) > g(\epsilon).$$

Again by Lemma 5.4.2 and since $\gamma(s)$ is C^∞ , we have that there exist δ_1 and δ_2 such that

$$\langle \gamma, \gamma \rangle'' < 2$$

on $[\delta_1, \delta_2]$ as desired. □

To see such curves are not critical, our general approach will be to again perturb γ along the vector field $N(s)$ as in Figure 5.3, adjusting our perturbation to be locally length preserving.

Proposition 5.6.2. *Let U be the set of C^∞ , embedded curves with length L , and let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a curve in U such that*

$$T(b) = -\frac{\gamma(b)}{|\gamma(b)|}$$

and

$$|\gamma(t) - c(\gamma)| = |\gamma(t)| \neq 0$$

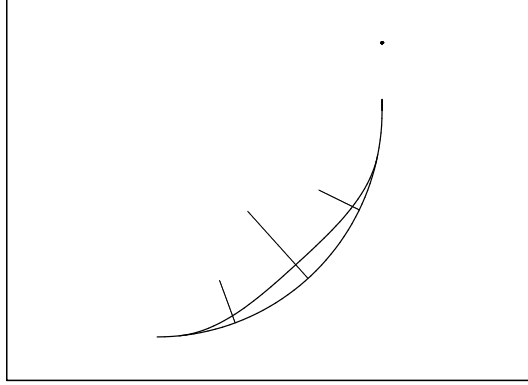


Figure 5.3: Decreasing $R(\gamma)$ for Appropriate γ in V

for all t in $[\delta, b]$ where

$$\delta := \min \left\{ x \text{ such that } T(s) = -\frac{\gamma(s)}{|\gamma(s)|} \text{ for all } s \text{ in } [x, 1] \right\}.$$

Then γ is not critical for $R(\gamma)$ over globally length preserving, C^∞ vector fields.

Proof. Without loss of generality, we restrict our attention to admissible curves and admissible vector fields as summarized in Section 4.2. By Lemma 5.6.1, there exists δ_1, δ_2 such that $\langle \gamma, T \rangle < 0$ on $[\delta_1, 1]$ and $\langle \gamma, N \rangle < 0$ on $[\delta_1, \delta_2]$. Again we let $\Phi(s)$ be a nonnegative bump function with support on $[\delta_1, \delta_2]$ with

$$\Phi(\delta_1) = \Phi(\delta_2) = 0,$$

$$\xi(s) := \Phi(s)N(s),$$

and

$$\begin{aligned} \bar{\xi}(s) &:= \xi(s) - T(s) \int_0^s \langle \dot{\xi}(t), T(t) \rangle dt \\ &= \Phi N + T \int_0^s \Phi \kappa dt. \end{aligned}$$

Now we examine $\delta_\gamma r(\bar{\xi})$:

$$\begin{aligned}\delta_\gamma r(\bar{\xi}) &= \int_0^1 \langle \gamma, \bar{\xi} \rangle ds \\ &= \int_{\delta_1}^{\delta_2} \Phi \langle \gamma, N \rangle ds + \int_{\delta_1}^1 \left(\int_{\delta_1}^s \Phi \kappa dt \right) \langle \gamma, T \rangle ds.\end{aligned}$$

As $\langle \gamma, N \rangle < 0$ and $\langle \gamma, T \rangle < 0$ on their respective intervals of integration by Lemma 5.6.1, $\delta_\gamma r(\bar{\xi}) < 0$. Thus, γ is not critical. \square

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