Prewavelet Solutions

to

Poisson Equations

by

Haipeng Liu

(Under the direction of Ming-Jun Lai)

Abstract

Finite element method is one of powerful numerical methods to solve PDE. Usually, if a finite element solution to a Poisson equation based on a triangulation of the underlying domain is not accurate enough, one will discard the solution and then refine the triangulation uniformly and compute a new finite element solution over the refined triangulation. It is wasteful to discard the original finite element solution. We propose a Prewavelet method to save the original solution by adding a Prewavelet subsolution to obtain the refined level finite element solution. To increase the accuracy of numerical solution to Poisson equations, we can keep adding Prewavelet subsolutions.

Our Prewavelets are orthogonal in the H^1 norm and they are locally supported except for one globally supported basis function in a rectangular domain. We have implemented these Prewavelet basis functions in MATLAB and used them for numerical solution of Poisson equation with Dirichlet boundary conditions. Numerical simulation demonstrates that our Prewavelet solution is much more efficient than the standard finite element method. Prewavelets over other boundary domains, such as triangle, L-shape domain, are also constructed.

Index words: Prewavelet, Poisson, type-one, Triangulation, Multiresolution.

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Chapter 1

Introduction

Finite element method is one of powerful numerical methods to solve PDE. Usually, if a finite element solution to a Poisson equation based on one level triangulation of the underlying domain is not accurate enough, one will discard the solution and then refine the triangulation and compute a new finite element solution at the refined level. It is wasteful to throw the original finite element solution away. In order to save the original solution and get the more accurate new solution, we have to add H_0^1 orthogonal subsolution. That is, let V_h be a finite element space over a triangulation Δ_h and $V_{h/2}$ be the finite element space over the refined triangulation. Since $V_h \subset V_{h/2}$, let $W_h = V_{h/2} \ominus V_h$ under H_0^1 norm, if $\Phi_h \in V_h$ is a finite element solution of Poisson equation with Dirichlet boundary condition, we can find $\Psi_h \in W_h$ so that $\Phi_h + \Psi_h$ is the finite element solution in $V_{h/2}$. In addition, suppose that ϕ_h is the most accurate solution that a computer can compute in the sense that it would be out of memory when computing a finite element solution $\Phi_{h/2}$ in $V_{h/2}$ directly. Since the size of the linear system associated with Ψ_h is smaller than $\Phi_{h/2}$, if the computer can solve Ψ_h, we can add Ψ_h to Φ_h to get $\Phi_{h/2}$ achieving the next level of accuracy. In this dissertation, we discuss how to compute Ψ_h. We shall construct locally supported basis functions and a few global supported basis functions $\psi_{h,k}, k = 1, \cdots, N_h$ which span W_h. $\psi_{h,k}$'s are called prewavelets and Ψ_h is a linear combination of these $\psi_{h,k}$'s and hence is called a Prewavelet subsolution.

Prewavelets under L_2 norm instead of H_0^1 norm have been studied for more than 15 years pioneered by Jia and Micchelli [9], see also [5]. There are many methods available to construct locally supported prewavelets over 2D domains under the L_2 norm. That is, $W_h = V_{h/2} \ominus V_h$ under L_2 norm, e.g., in a series of papers [6], [7], [8], [11], and [4]. In 1997, Bastin and Laubin ([2]) explained how to construct compactly supported orthonormal wavelets in Sobolev space in the univariate setting. See also [1] for compactly supported biorthogonal wavelets in Sobolev space. People were also looking for prewavelets in Sobolev space in 2D domain for numerical solution of PDE, also in the univariate setting. However in [13], Lorentz and Oswald showed that there is no locally supported prewavelets in Sobolev space or under H_0^1 norm based on integer translations of a box spline over $\mathbf{R}^2$. Since continuous piecewise linear finite element can be expressed by using box spline B_{111}, the result in [13] ruins a hope to find locally supported prewavelets under H_0^1 norm over R^2. But this is not an end of story. It is possible to construct locally supported prewavelets in a semi-norm in the univariate setting in [10]. In [14], Jia and Liu used the Prewavelet to solve boundary valued ODE problem. It is also possible to construct compactly supported prewavelets in H^r norm over each nested subspace, but the union of these prewavelets over all levels fails to be a stable basis for a Sobolev space (cf. [12]). Our new question is if we can find a Prewavelet basis with as few as possible global supported Prewavelet functions. Our answer is affirmative. That is, there is a Prewavelet basis for W_h with only one global supported basis function under the H_0^1 norm over rectangular domains, and there is a locally supported Prewavelet basis for W_h over triangular domains, and there is a Prewavelet basis for W_h with three global supported basis function under the H_0^1 norm over L-shape domains. These are the main results in this dissertation

The dissertation is organized as follows: We first explain a construction method to convert the Dirichlet boundary value problem of Poisson equation into a Poisson equation with zero boundary condition. An explicit conversion will be given. Thus the H^1 norm is now equivalent to the H_0^1 semi-norm. Then we introduce some notation to explain the weak solution of Poisson equation and its approximation to the exact solution. These explanations are well-known and given in the Preliminary section §2. In §3, we explain how to construct locally supported prewavelets under H_0^1 semi-norm. In §4, we explain how to implement our Prewavelet method for numerical solution of Poisson equation. In §5 we present some numerical results. Our numerical experiment show that the time for computing a finite element solution by our Prewavelet method is about half of the time by the standard finite element method using the direct method for inverting the linear systems. If using the conjugate gradient method for the linear systems for the finite element method, the Prewavelet method is still faster than for sufficiently accurate iterative solutions. In §6, I describe how to construct locally supported Prewavelet under triangular domain, the numerical result shows the result works. In §7, I got the following result: there is no local supported Prewavelet basis under H_0^1 norm with rectangular boundary if we constructed the Prewavelet from linear box splines, while there is a compacted local supported Prewavelet basis when the boundary is triangle. That is, the existence of locally supported Prewavelet basis constructed from linear box splines under H_0^1 norm is dependent on the boundary shape. In §8, I will show the same kind of results on Prewavelet basis over L-shaped domain.

Chapter 2

Preliminary

2.1 Simplification of the Poisson Equation

Let us start with a square domain $\Omega = (0,1) \times (0,1) \in R^2$. Consider the Dirichlet boundary value problem for Poisson equation:

$$\begin{cases} -\Delta u(x,y) = g(x,y), \quad (x,y) \in \Omega \\ u(x,y) = f_1(x), \quad for \quad y = 0 \quad and \quad 0 \le x \le 1 \\ u(x,y) = f_2(x), \quad for \quad y = 1 \quad and \quad 0 \le x \le 1 \\ u(x,y) = f_3(y), \quad for \quad x = 0 \quad and \quad 0 \le y \le 1 \\ u(x,y) = f_4(y), \quad for \quad x = 1 \quad and \quad 0 \le y \le 1 \end{cases}$$

Without lose of generality, we may assume that $f_1(1) = f_2(1) = f_3(1) = f_4(1) = f_1(0) = f_2(0) = f_3(0) = f_4(0) = 0$. Otherwise, letting $f_1(0) = f_3(0) = a_1$, $f_3(1) = f_2(0) = a_2$, $f_2(1) = f_4(1) = a_3$, $f_4(0) = f_1(1) = a_4$, we define $h(x,y) = a_1 + (a_4 - a_1)x + (a_2 - a_1)y + (a_3 + a_1 - a_4 - a_2)xy$, and $v(x,y) = u(x,y) - h(x,y)$. Then the above Dirichlet problem becomes to:

$$\begin{cases} -\Delta v(x,y) = g(x,y), \quad (x,y) \in \Omega \\ v(x,y) = f_1(x) - h(x,0), \quad for \quad y = 0 \quad and \quad 0 \le x \le 1 \\ v(x,y) = f_2(x) - h(x,1), \quad for \quad y = 1 \quad and \quad 0 \le x \le 1 \\ v(x,y) = f_3(y) - h(0,y), \quad for \quad x = 0 \quad and \quad 0 \le y \le 1 \\ v(x,y) = f_4(y) - h(1,y), \quad for \quad x = 1 \quad and \quad 0 \le y \le 1 \end{cases}$$

which satisfy the above assumption.

Now let $w(x) = v(x,y) - x(f_4(y) - h(1,y)) - (1-x)(f_3(y) - h(0,y)) - y(f_2(x) - h(x,1)) - (1-y)(f_1(x) - h(x,0))$. Then $w(x)$ satisfies the equation

$$\begin{cases} -\Delta w(x,y) = g_1(x,y), & (x,y) \in \Omega \\ w(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}$$

with $g_1(x,y) = g(x,y) + \frac{\partial^2}{\partial y^2}[-x(f_4(y) - h(1,y)) - (1-x)(f_3(y) - h(0,y))] + \frac{\partial^2}{\partial x^2}[-y(f_2(x) - h(x,1)) - (1-y)(f_1(x) - h(x,0))]$.

If we can find solution for w, it is easy to get $u(x,y)$. In the remaining dissertation, we only consider the Poisson equation with zero boundary condition:

$$\begin{cases} -\Delta u(x,y) = g(x,y), & (x,y) \in \Omega \\ u(x,y) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{2.1}$$

2.2 Seminorm

Next we define

$$H_0^1(\Omega) = \{v \in L^2(\Omega) : \quad \langle v, v\rangle_s < \infty \quad and \quad v(x,y) = 0, (x,y) \in \partial\Omega\},$$

where the inner product $\langle u, v\rangle_s$ is defined by

$$\langle u, v\rangle_s = \int_0^1 \int_0^1 \left[\frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y}\right] dxdy.$$

By using Poincare's inequality, $\|u\|_s = \sqrt{\langle u, u\rangle_s}$ is a standard Sobolev norm for $H_0^1(\Omega)$. Suppose $u, v \in H_0^1(\Omega)$. Integration by parts yields

$$\begin{aligned} \langle g, v\rangle &= \int_0^1 \int_0^1 g(x,y)v(x,y)dxdy \\ &= \int_0^1 \int_0^1 -\Delta u(x,y)v(x,y)dxdy \\ &= \int_0^1 \int_0^1 \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y}dxdy \\ &= \langle u, v\rangle_s. \end{aligned}$$

Thus, a weak solution u to (2.1) is characterized by finding $u \in H_0^1(\Omega)$ such that

$$\langle u, v\rangle_s = \langle g, v\rangle, \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

The following result is well-known. For convenience, we present a short proof.

Theorem 2.2.1. *Suppose g and $u \in C^2(\Omega)$ is a weak solution satisfying (2.2). Then u is a classic solution satisfying (2.1).*

Proof. Let $v \in H_0^1(\Omega)$. Then integration by parts gives

$$\begin{aligned}\langle g, v\rangle &= \langle u, v\rangle_s \\ &= \int_0^1\int_0^1 \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} dxdy \\ &= \int_0^1\int_0^1 -\Delta u(x,y)v(x,y)dxdy \\ &= \langle -\Delta u(x,y), v\rangle.\end{aligned}$$

It follows that $\langle g - (-\Delta u(x,y)), v\rangle = 0$ for all $v \in H_0^1(\Omega)$. That is, $g \equiv -\Delta u$ and hence, u satisfies (2.1). □

2.3 Type-I Triangulation

Next we introduce continuous linear spline space on $\Omega = [0,1] \times [0,1]$. For convenience, let $N_j = (2^j - 1)^2$ and $j \geq 1$. Denote $x_{ji} = \frac{i}{2^j} = y_{ji}$ for $i = 1, .., 2^j - 1$. Clearly, the lines segment of $x = x_{ji}$ and $y = y_{jk}$ divide the square Ω into N_j sub-squares. The diagonal going from down-left to up-right of each sub-square divides the sub-square into two congruent triangle. We will refer to the set of all such triangles as a Type-1 triangulation of Ω (see Figure 1).

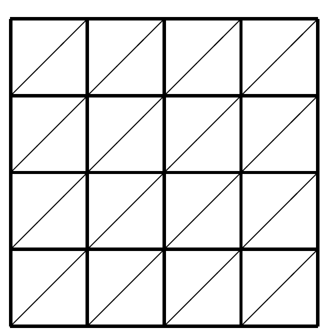

Figure 1. Type-I triangulation with j=2.

Define ϕ_{ik}^{j} to be linear spline with support on the hexagon with following vertices

$$\{(x_{j(i-1)}, y_{j(k-1)}), (x_{ji}, y_{j(k-1)}), (x_{j(i+1)}, y_{j(k)}), (x_{j(i+1)}, y_{j(k+1)}), (x_{j(i+1)}, y_{j(k)}), (x_{j(i-1)}, y_{j(k)})\}$$

and $\phi_{ik}(x_{ji'}, y_{jk'}) = \delta_{i,i'}\delta_{k,k'}$, where $\delta_{i,i'} = 0$ if $i' \neq i$ and 1 if $i' = i$.

Let $V_j = span\{\phi_{ik}^{j}, i = 1, .., 2^j - 1, k = 1, .., 2^j - 1\}$ be the subspace of $H_0^1(\Omega)$. By following lemma, there exists a unique $u_j \in V_j$ satisfying

$$\langle u_j, v\rangle_s = \langle g, v\rangle \ \ \forall v \in V_j. \tag{2.3}$$

u_j is the standard finite element solution in V_j.

2.4 Error Approximation

The following result is well-known. For completeness, we include a short proof.

Lemma 2.4.1. *Given $g \in L^2(\Omega)$, (2.3) has a unique solution.*

Proof. Reorder the basis functions $\phi_{ik}^{(j)}$ to ϕ_m, $m = 1, ..., N_j$ and let $u_j = \sum a_m\phi_m$. Denote $k_{mn} = \langle \phi_m, \phi_n\rangle_s$ and $F_m = \langle g, \phi_m\rangle$ for $m = 1, ..., N_j$. Set $A = (a_m)$ to be the coefficient vector, $K = [k_{mn}]_{1\le m,n\le N_j}$ to be the stiff matrix, and $F = (F_m)$ to be the right hand side vector. Then the solutions in (2.3) is written in the following matrix equation form

$$KA = F. \tag{2.4}$$

We claim that the solution for above equation always exists and is unique. Otherwise there is a nonzero vector $\mathbf{c}$ such that $K\mathbf{c} = 0$. Write $\mathbf{c} = (c_m, m = 1, ..., N_j)$ and let $v = \sum_{i=1}^{N_j} c_i\phi_i$ be the linear spline. Then $K\mathbf{c} = 0$ is equivalent to

$$\langle v, \phi_m\rangle_s = 0 \quad \forall m = 1, \cdots, N_j.$$

Multiplying $\langle v, \phi_m\rangle_s$ by c_m and summing over m yields $\langle v, v\rangle_s = 0$. Thus,

$$\int_0^1\int_0^1 \frac{\partial v(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x}dxdy = 0,$$

and

$$\int_0^1 \int_0^1 \frac{\partial v(x,y)}{\partial y} \frac{\partial v(x,y)}{\partial y} dxdy = 0,$$

it follows that, $\frac{\partial v(x,y)}{\partial x} = 0$ and $\frac{\partial v(x,y)}{\partial y} = 0$, it is a constant function. Boundary condition implies $v \equiv 0$. Since $\{\phi_m\}$ are linear independent, $\mathbf{c} \equiv 0$ and hence, the solution is unique. □

Let us discuss the error between u and u_j. It is standard in finite element analysis (cf. [3]). For completeness we present a simple derivation. Subtracting (2.3) from (2.2) implies

$$\langle u - u_j, w \rangle_s = 0 \ , \qquad \forall w \in V_j. \tag{2.5}$$

Then for any $v \in V_j$

$$\begin{aligned} \|u - u_j\|_s^2 &= \langle u - u_j, u - u_j \rangle_s \\ &= \langle u - u_j, u - v \rangle_s + \langle u - u_j, v - u_j \rangle_s \\ &= \langle u - u_j, u - v \rangle_s \\ &\leq \|u - u_j\|_s \|u - v\|_s. \end{aligned}$$

It follows that $\|u - u_j\|_s \leq \|u - v\|_s$ for any $v \in V_j$. Thus we have proved the following, (cf [3]).

Lemma 2.4.2. *(Céa's Lemma)* $\|u - u_j\|_s = \min\{\|u - v\|_s : v \in V_j\}$.

Given $u \in C^0(\Omega)$, let $u_I \in V_j$ be the interpolation of u:

$$u_I = \sum_{ik} u(x_{ji}, y_{jk}) \phi_{ik}^{(j)}.$$

The following error estimate is well-known.

Lemma 2.4.3. *Suppose* $u \in C^2(\Omega)$. *Then*

$$\|u - u_I\|_s \leq \frac{\sqrt{12}}{2^j} \sqrt{\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty}^2 + \left\|\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right\|_{L^\infty}^2 + \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}^2}.$$

Proof. Let $w = u - u_I$. Let us first give estimates of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in the region of triangle T with vertex $\{(0,0),(0,1/2^j),(1/2^j,1/2^j)\}$, by the definition, $w = 0$ on the vertex of the region T. For $(x,y) \in T$, the Taylor expansion yield the following equations,

$$\begin{aligned} 0 = \quad & w(x,y) + (\nabla w, (-x,-y)) \\ & + x^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x,(1-t)y)dt \\ & + 2xy \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial}((1-t)x,(1-t)y)dt \\ & + y^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x,(1-t)y)dt \end{aligned}$$

$$\begin{aligned} 0 = \quad & w(x,y) + (\nabla w, (1/2^j - x, -y)) \\ & + (1/2^j - x)^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x + t/2^j,(1-t)y)dt \\ & + 2(1/2^j - x)(-y) \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x + t/2^j,(1-t)y)dt \\ & + y^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x + t/2^j,(1-t)y)dt \\ 0 = \quad & w(x,y) + (\nabla w, (1/2^j - x, 1/2^j - y)) \\ & + (1/2^j - x)^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x + t/2^j,(1-t)y + t/2^j)dt \\ & + 2(1/2^j - x)(1/2^j - y) \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x + t/2^j,(1-t)y + t/2^j)dt \\ & + (1/2^j - y)^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x + t/2^j,(1-t)y + t/2^j)dt. \end{aligned}$$

Equivalently, the above equations can be written in the following matrix form,

$$\begin{pmatrix} 1 & -x & -y \\ 1 & \frac{1}{2^j} - x & -y \\ 1 & \frac{1}{2^j} - x & \frac{1}{2^j} - y \end{pmatrix} \begin{pmatrix} w \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

$$\begin{aligned}
r_1 = &- x^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x, (1-t)y)dt \\
&- 2xy \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x, (1-t)y)dt \\
&- y^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x, (1-t)y)dt \\
r_2 = &- (1/2^j - x)^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x + t/2^j, (1-t)y)dt \\
&- 2(1/2^j - x)(-y) \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x + t/2^j, (1-t)y)dt \\
&- y^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x + t/2^j, (1-t)y)dt
\end{aligned}$$

$$\begin{aligned}
r_3 = &- (1/2^j - x)^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x + t/2^j, (1-t)y + t/2^j)dt \\
&- 2(1/2^j - x)(1/2^j - y) \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x + t/2^j, (1-t)y + t/2^j)dt \\
&- (1/2^j - y)^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x + t/2^j, (1-t)y + t/2^j)dt
\end{aligned}$$

It is easy to solve $\frac{\partial w}{\partial y}(x,y)$ and $\frac{\partial w}{\partial x}(x,y)$ from the above system of linear equations.

$$\begin{aligned}
\frac{\partial w}{\partial x}(x,y) = \quad &- 2^j x^2 \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x, (1-t)y)dt \\
&- 2xy2^j \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x, (1-t)y)dt \\
&- y^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x, (1-t)y)dt \\
&+ (1/2^j - x)^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x + t/2^j, (1-t)y)dt \\
&+ 2(1/2^j - x)(-y)2^j \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x + t/2^j, (1-t)y)dt \\
&+ y^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x + t/2^j, (1-t)y)dt
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial y}(x,y) =&(1/2^j-x)^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x+t/2^j,(1-t)y+t/2^j)dt \\
&+2(1/2^j-x)(1/2^j-y)2^j \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x+t/2^j,(1-t)y+t/2^j)dt \\
&+(1/2^j-y)^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x+t/2^j,(1-t)y+t/2^j)dt \\
&-(1/2^j-x)^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial x^2}((1-t)x+t/2^j,(1-t)y)dt \\
&-2(1/2^j-x)(-y)2^j \int_0^1 (1-t)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}((1-t)x+t/2^j,(1-t)y)dt \\
&-y^2 2^j \int_0^1 (1-t)\frac{\partial^2 w}{\partial y^2}((1-t)x+t/2^j,(1-t)y)dt
\end{aligned}$$

Thus we can get the estimation for $\left(\frac{\partial w}{\partial x}(x,y)\right)^2$ and $\left(\frac{\partial w}{\partial y}(x,y)\right)^2$ with:

$$\begin{aligned}
\left(\frac{\partial w}{\partial x}(x,y)\right)^2 \leq&\ 6/2^{2j}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty}^2 + 6/2^{2j}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}^2 + 6/2^{2j}\left\|\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\|_{L^\infty}^2 \\
\left(\frac{\partial w}{\partial y}(x,y)\right)^2 \leq&\ 6/2^{2j}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty}^2 + 6/2^{2j}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}^2 + 6/2^{2j}\left\|\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\|_{L^\infty}^2.
\end{aligned}$$

it follows

$$\begin{aligned}
\|u-u_I\|_s^2 &\leq \int_0^1\int_0^1 \left(\frac{\partial w}{\partial x}(x,y)\right)^2 + \left(\frac{\partial w}{\partial y}(x,y)\right)^2 dxdy \\
&\leq \int_0^1\int_0^1 12/2^{2j}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty}^2 + 12/2^{2j}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}^2 + 12/2^{2j}\left\|\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\|_{L^\infty}^2 dxdy \\
&\leq \left(12/2^{2j}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty}^2 + 12/2^{2j}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}^2 + 12/2^{2j}\left\|\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\|_{L^\infty}^2\right).
\end{aligned}$$

Therefore

$$\|u-u_I\|_s \leq \sqrt{12}\frac{\sqrt{\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^\infty} + \left\|\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right\|_{L^\infty} + \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^\infty}}}{2^j}.$$

that means $\|u-u_I\|_s$ goes to zero as j goes to infinity. According Theorem(3.2), we will get $\|u-u_j\|_s \leq \|u-u_I\|_s$, and hence, we get the result of this lemma. □

2.5 Multiresolution

We start with the definition of multi-resolution approximation of $H_0^1(\Omega)$:

Definition 2.5.1. *A multiresolution approximation of $H_0^1(\Omega)$ is a sequence of finite dimensions subspaces V_j, $j \in Z_+$ of $H_0^1(\Omega)$ such that*

(1) $V_j \subset V_{j+1}, \quad j \in Z_+$;

(2) $\bigcup_{j=1}^{\infty} V_j$ *is dense in* $H_0^1(\Omega)$.

Let Γ^j be the type-1 triangulation with $2N_j$ triangles. Naturally, let Γ^{j+1} be the uniform refinement of Γ^j. Let V_j be the continuous piecewise linear spline space defined on the previous section. That is, $V_j = span\{\phi_{ik}^j, i = 1, .., 2^j - 1, k = 1, .., 2^j - 1\}$, where ϕ_{ik}^j are continuous piecewise linear functions which is 1 at (x_{ji}, y_{jk}) and zero at all other vertices. Let $V_{j+1} = span\{\phi_{ik}^{j+1}, i = 1, .., 2^{j+1}-1, k = 1, .., 2^{j+1}-1\}$, and $(x_{j+1,i}, y_{j+1,k})$ are the vertices on the j+1 level Type-1 triangulation. Then the refinement equation is easily seen to be

$$\phi_{ik}^j = \phi_{2i,2k}^{j+1} + \frac{1}{2}\phi_{2i-1,2k}^{j+1} + \frac{1}{2}\phi_{2i-1,2k-1}^{j+1} + \frac{1}{2}\phi_{2i,2k-1}^{j+1} + \frac{1}{2}\phi_{2i+1,2k}^{j+1} + \frac{1}{2}\phi_{2i+1,2k+1}^{j+1} + \frac{1}{2}\phi_{2i,2k+1}^{j+1}.$$

See the Figure 2.

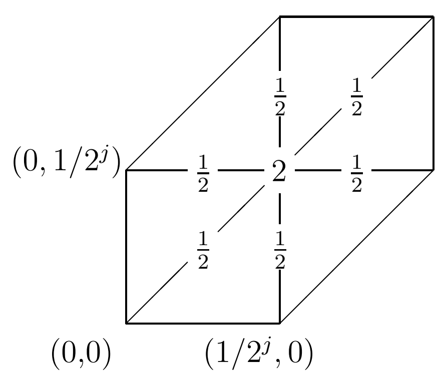

Figure 2. Dilation relations

The main purpose of this dissertation is to build a basis for the orthogonal complement W_j of V_j in V_{j+1} under the inner product $\langle\cdot,\cdot\rangle_s$. Suppose we have the W_j. Then $V_{j+1} = V_j + W_j$ under the $H_0^1(\Omega)$ inner product. For a solution u_j satisfying (3), we do not have to find out the solution for

$$u_{j+1} \in V_{j+1} \text{ such that } \langle u_{j+1}, v\rangle_s = \langle g, v\rangle, \ \forall v \in V_{j+1}.$$

Instead, we only need to find solutions for

$$w_j \in W_j \text{ such that } \langle w_j, v\rangle_s = \langle g, v\rangle, \;\; \forall v \in W_j.$$

Then we have $w_j + u_j = u_{j+1}$. Ideally, we hope the supports of basis functions for W_j are small, since the small supports can accelerate the calculations of $\langle g, v\rangle_s$. As explained in the Introduction, there is no locally supported prewavelets for W_j. Nevertheless, we shall construct basis functions with only a few globally supported basis function for W_j in the following chapters.

Clearly the Γ_j can be continuously refined and hence we will have a nested sequence of subspaces

$$V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5......$$

to span $H_0^1(\Omega)$ by Lemma 2.4.3, since $C^2(\Omega)$ is dense in $H_0^1(\Omega)$.

Let $W_j \subset V_{j+1}$ be the orthogonal complement of V_j in V_{j+1} for each refinement level j, i.e.,

$$V_{j+1} = V_j \bigoplus W_j.$$

Then we get the decomposition

$$V_{j+1} = V_1 \bigoplus W_1 \bigoplus W_2 \bigoplus W_3 \bigoplus ...... \bigoplus W_j$$

for any $j \geq 1$. The weak solution u_{j+1} to the Poisson equation (2.1) at V_{j+1} can be built by

$$u_{j+1} = u_1 + w_1 + w_2 + \cdots + w_j.$$

The following chapters focus on building basis functions for the orthogonal complement W_j.

Chapter 3

Prewavelets over Type-I Triangulations

Next by direct calculation, we obtain the following lemma immediately.

Lemma 3.0.1. *We have* $\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k}, \rangle_s = 2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k}, \rangle_s = 1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i,2k-1}, \rangle_s = 1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k}, \rangle_s = 1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k+1}, \rangle_s = 1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k-1}, \rangle_s = 1, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k+1}, \rangle_s = 1,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i,2k-2}, \rangle_s = -1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k+2}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k-2}, \rangle_s = 0, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k+2}, \rangle_s = 0,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k-1}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k+1}, \rangle_s = -1, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k+2}, \rangle_s = -1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k+1}, \rangle_s = -1/2, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k-1}, \rangle_s = -1, \quad \langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k-2}, \rangle_s = -1/2,$
$\langle \phi^j_{ik}, \phi^{j+1}_{i',k'}, \rangle_s = 0$, *for other* i', k' *which are not listed above.*

3.1 Prewavelets Construction

Let ψ^j be a function in W_j. Since $W_j \subset V_{j+1}$, let us write $\psi^j = \sum_{ik} \phi^{j+1}_{ik} b_{ik}$ for some unknown coefficients b_{ik}. We need to describe the concept of locally supported function in v^j more precisely.

Definition 3.1.1. $\psi^j = \sum_{ik} \phi^{j+1}_{ik} b_{ik}$ *is said to be locally supported if there exists a positive integer I which is independent j, such that the number of the none zero coefficient* b_{ik} *is less than I.*

By orthogonal condition $\langle \phi^j_{i'k'}, \psi^j \rangle_s = 0$, we need to solve the following equations.

$$0 = \langle \phi^j_{i'k'}, \sum_{i,k} b_{ik} \phi^{j+1}_{ik} \rangle_s = \sum_{i,k} b_{ik} \langle \phi^j_{i'k'}, \phi^{j+1}_{ik} \rangle_s. \tag{3.1}$$

Each $(i^{'}, k^{'})$ determines one equation. Since there are N_j elements in the set V_j, they determine the N_j equations. These N_j equations with N_{j+1} coefficients, $b_{i,k}$. There are at least $N_{j+1} - N_j$ degrees of freedom. The solution space of these equation system should be the W_j. The linear independence of $\phi^{j}_{i',k'}$ implies that the coefficient matrix of the above linear system is of full rank. Hence, there are $N_{j+1} - N_j$ linear independent solutions which constitute a basis for W_j.

Definition 3.1.2. *Let $V^m_{j+1} = span\{\phi^{j+1}_{ik}, i = 1, .., 2m-1, k = 1, .., 2m-1\}$ be a subspace of V_{j+1}. Let W^m_j be subspace of W_j such that $W^m_j = W_j \bigcap V^m_{j+1}$.*

Obviously $\emptyset \subset V^1_{j+1} \subset V^2_{j+1} \subset \ldots \subset V^{2^j}_{j+1} = V_{j+1}$, and $\emptyset \subset W^1_j \subset W^2_j \subset \ldots \subset W^{2^j}_j = W_j$. There is no nonzero solution of (3.1) in space of V^1_{j+1}. However, there are five solution of (3.1) in space V^2_{j+1}. They are solutions of the following system of linear equations.

$$\sum_{1\le i,k\le 3} b_{ik} \left\langle \phi^{j+1}_{ik}, \phi^{j}_{1,1} \right\rangle_s = 0, \quad \sum_{1\le i,k\le 3} b_{ik} \left\langle \phi^{j+1}_{ik}, \phi^{j}_{2,1} \right\rangle_s = 0,$$
$$\sum_{1\le i,k\le 3} b_{ik} \left\langle \phi^{j+1}_{ik}, \phi^{j}_{1,2} \right\rangle_s = 0, \quad \sum_{1\le i,k\le 3} b_{ik} \left\langle \phi^{j+1}_{ik}, \phi^{j}_{2,2} \right\rangle_s = 0.$$

They are equivalent to the following equations.

$$\begin{pmatrix} \langle \phi^j_{1,1}, \phi^{j+1}_{1,1} \rangle_s & \langle \phi^j_{1,1}, \phi^{j+1}_{2,1} \rangle_s & \ldots. & \langle \phi^j_{1,1}, \phi^{j+1}_{3,3} \rangle_s \\ \langle \phi^j_{2,1}, \phi^{j+1}_{1,1} \rangle_s & \langle \phi^j_{2,1}, \phi^{j+1}_{2,1} \rangle_s & \ldots. & \langle \phi^j_{2,1}, \phi^{j+1}_{3,3} \rangle_s \\ \langle \phi^j_{1,2}, \phi^{j+1}_{1,1} \rangle_s & \langle \phi^j_{1,2}, \phi^{j+1}_{2,1} \rangle_s & \ldots. & \langle \phi^j_{1,2}, \phi^{j+1}_{3,3} \rangle_s \\ \langle \phi^j_{2,2}, \phi^{j+1}_{1,1} \rangle_s & \langle \phi^j_{2,2}, \phi^{j+1}_{2,1} \rangle_s & \ldots. & \langle \phi^j_{2,2}, \phi^{j+1}_{3,3} \rangle_s \end{pmatrix} \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{1,2} \\ b_{2,2} \\ b_{3,2} \\ b_{1,3} \\ b_{2,3} \\ b_{3,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using Lemma 3.0.1, we obtain the following equations.

$$\begin{pmatrix} 1 & 1/2 & -1 & 1/2 & 2 & 1/2 & -1 & 1/2 & 1 \\ 0 & -1/2 & 1 & 0 & -1/2 & 1/2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & -1/2 & -1/2 & 0 & 1 & 1/2 & -1 \end{pmatrix} \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{1,2} \\ b_{2,2} \\ b_{3,2} \\ b_{1,3} \\ b_{2,3} \\ b_{3,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The rank of the left matrix is four, because $\phi^j_{1,1}$, $\phi^j_{2,1}$, $\phi^j_{1,2}$, $\phi^j_{2,2}$, are linear independent. So there are five solutions which can be found to be.

$$\begin{pmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{1,2} \\ b_{2,2} \\ b_{3,2} \\ b_{1,3} \\ b_{2,3} \\ b_{3,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

More precisely,

$$\psi^{j,1}_{0,1} = 2\phi^{j+1}_{1,2} + \phi^{j+1}_{1,3} \quad \text{as shown in Figure 3;} \tag{3.2}$$

$$\psi^{j,2}_{1,0} = 2\phi^{j+1}_{2,1} + \phi^{j+1}_{3,1} \quad \text{as shown in Figure 4;} \tag{3.3}$$

$$\psi^{j,3}_{1,1} = -\phi^{j+1}_{2,2} + \phi^{j+1}_{3,2} + \phi^{j+1}_{2,3} + \phi^{j+1}_{3,3} \quad \text{as shown in Figure 5;} \tag{3.4}$$

$$\psi^{j,4}_{1,1} = \phi^{j+1}_{1,1} + \phi^{j+1}_{2,1} + \phi^{j+1}_{1,2} - \phi^{j+1}_{2,2} \quad \text{as shown in Figure 6;} \tag{3.5}$$

$$\psi^{j,5}_{1,1} = \phi^{j+1}_{1,2} + \phi^{j+1}_{2,3} - \phi^{j+1}_{2,1} - \phi^{j+1}_{3,2} \quad \text{as shown in Figure 7.} \tag{3.6}$$

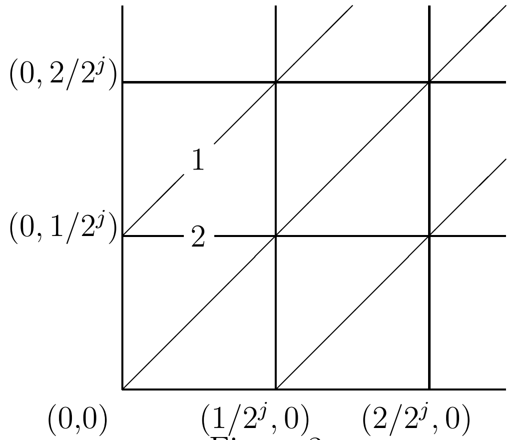

Figure 3.

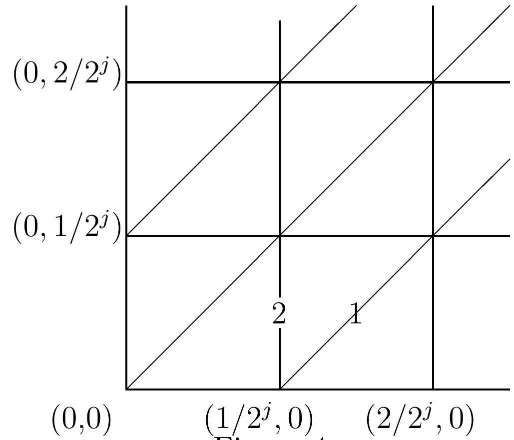

Figure 4.

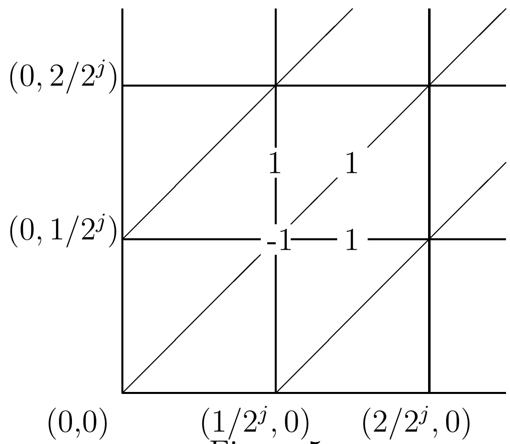

Figure 5.

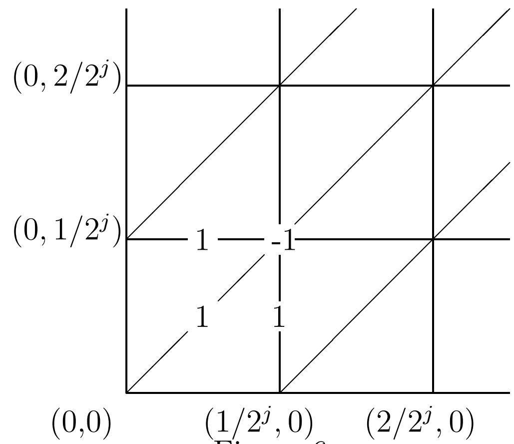

Figure 6.

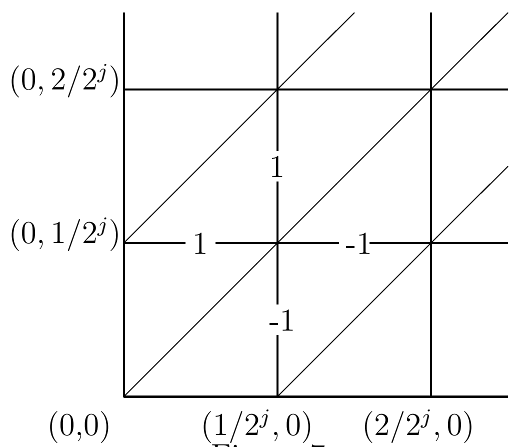

Figure 7.

Now we consider V_j^3. Similarly, there are 25 non-zero coefficient for linear system (3.1) and the coefficient matrix of rank 9. So the dimension of solution space of W_j^3 is $25-9=16$. The first five of them are the same to the wavelet functions in (3.2)–(3.6). The other 11 are given below.

$\psi_{0,2}^{j,1} = 2\phi_{1,4}^{j+1} + \phi_{1,5}^{j+1}$ as shown in Figure 8;

$\psi_{2,0}^{j,2} = 2\phi_{4,1}^{j+1} + \phi_{5,1}^{j+1}$ as shown in Figure 9;

$\psi_{1,2}^{j,3} = \phi_{3,5}^{j+1} + \phi_{3,4}^{j+1} + \phi_{2,5}^{j+1} - \phi_{2,4}^{j+1}$ as shown in Figure 10;

$\psi_{2,2}^{j,3} = \phi_{5,5}^{j+1} + \phi_{5,4}^{j+1} + \phi_{4,5}^{j+1} - \phi_{4,4}^{j+1}$ as shown in Figure 11;

$\psi_{2,1}^{j,3} = \phi_{5,3}^{j+1} + \phi_{5,2}^{j+1} + \phi_{4,3}^{j+1} - \phi_{4,2}^{j+1}$ as shown in Figure 12;

$\psi_{2,1}^{j,4} = \phi_{3,2}^{j+1} + \phi_{4,1}^{j+1} + \phi_{3,1}^{j+1} - \phi_{4,2}^{j+1}$ as shown in Figure 13;

$\psi_{2,2}^{j,4} = \phi_{3,3}^{j+1} + \phi_{4,3}^{j+1} + \phi_{3,4}^{j+1} - \phi_{4,4}^{j+1}$ as shown in Figure 14;

$\psi_{1,2}^{j,4} = \phi_{1,3}^{j+1} + \phi_{2,3}^{j+1} + \phi_{1,4}^{j+1} - \phi_{2,4}^{j+1}$ as shown in Figure 15;

$\psi_{1,2}^{j,5} = \phi_{1,4}^{j+1} + \phi_{2,5}^{j+1} - \phi_{2,3}^{j+1} - \phi_{3,4}^{j+1}$ as shown in Figure 16;

$\psi_{2,2}^{j,5} = \phi_{3,4}^{j+1} + \phi_{4,5}^{j+1} - \phi_{4,3}^{j+1} - \phi_{5,4}^{j+1}$ as shown in Figure 17;

$\psi_{2,1}^{j,5} = \phi_{3,2}^{j+1} + \phi_{4,3}^{j+1} - \phi_{4,1}^{j+1} - \phi_{5,2}^{j+1}$ as shown in Figure 18.

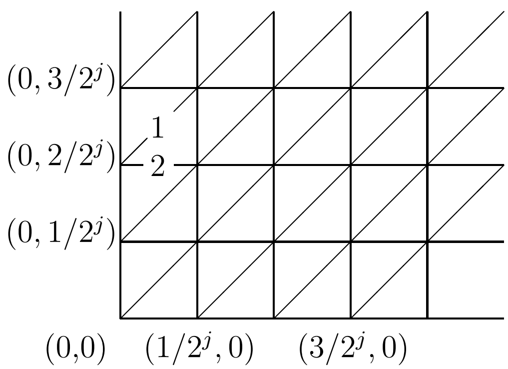

Figure 8.

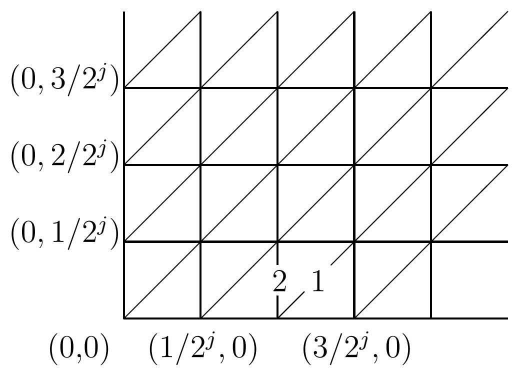

Figure 9.

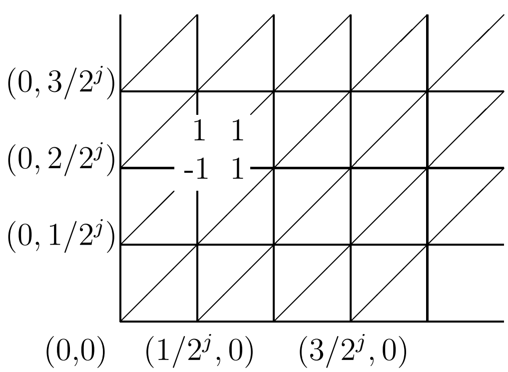

Figure 10.

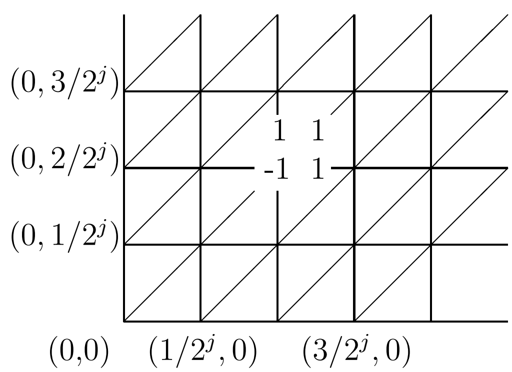

Figure 11.

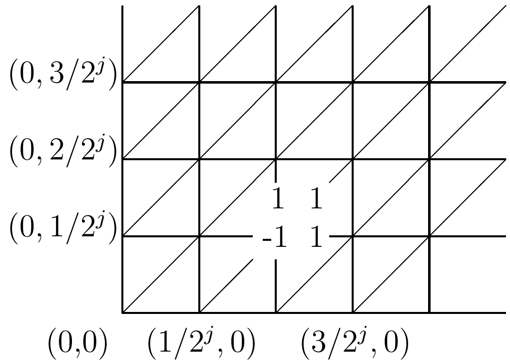

Figure 12.

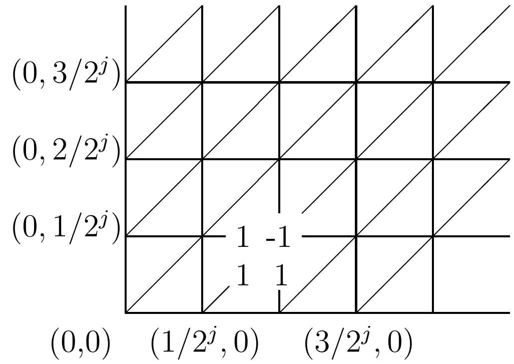

Figure 13.

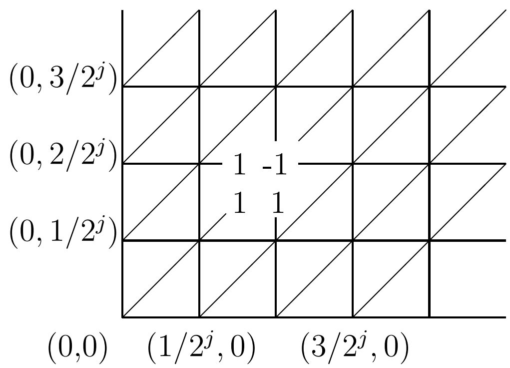

Figure 14.

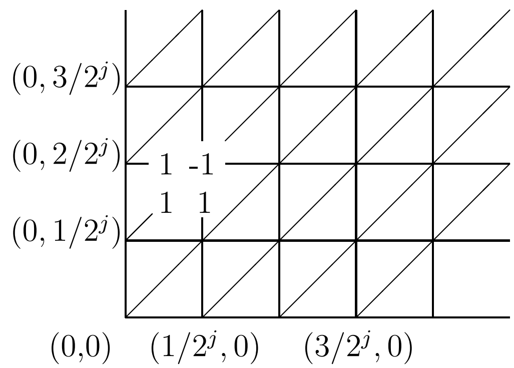

Figure 15.

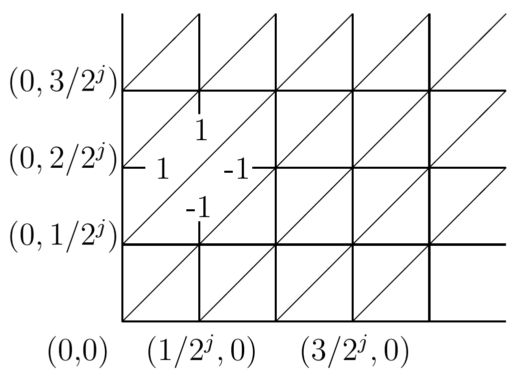

Figure 16.

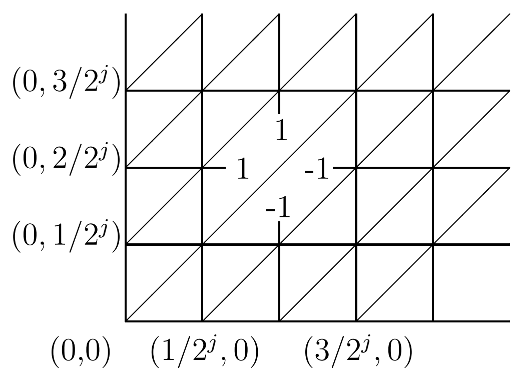

Figure 17.

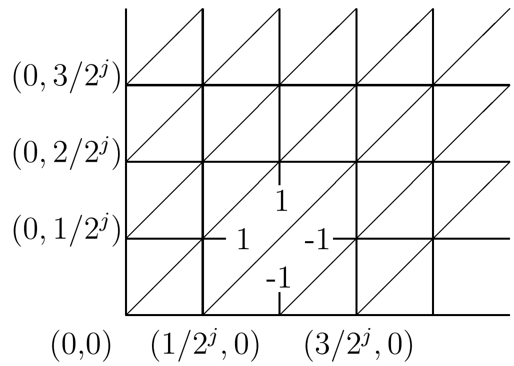

Figure 18.

The above computation can be carried out on V_j^n for $n = 3, ...2^j - 1$. We have thus obtained five types of wavelet functions:

$$\psi_{0,k}^{j,1} = 2\phi_{1,k+1}^{j+1} + \phi_{1,k+2}^{j+1}$$

is supported next to the vertical boundary and is called vertical boundary wavelet.

$$\psi_{k,0}^{j,2} = 2\phi_{k+1,1}^{j+1} + \phi_{k+2,1}^{j+1}$$

called horizontal boundary wavelet, is supported next to the horizontal boundary. The next three types are supported inside the domain. The following

$$\psi_{i,k}^{j,3} = -\phi_{i+1,k+1}^{j+1} + \phi_{i+2,k+1}^{j+1} + \phi_{i+1,k+2}^{j+1} + \phi_{i+2,k+2}^{j+1}$$

is called interior wavelet of first kind. We call

$$\psi_{i,k}^{j,4} = -\phi_{2i,2k}^{j+1} + \phi_{2i-1,2k}^{j+1} + \phi_{2i,2k-1}^{j+1} + \phi_{2i-1,2k-1}^{j+1}$$

interior wavelet of second kind. The last one

$$\psi_{i,k}^{j,5} = \phi_{2i-1,2k}^{j+1} + \phi_{2i,2k+1}^{j+1} - \phi_{2i,2k-1}^{j+1} - \phi_{2i+1,2k}^{j+1}$$

is called interior wavelet of third kind.

Theorem 3.1.1. *All the five types of wavelets in the V_{j+1}^n are linear independent for $1 \leq n \leq 2j-1$. That is, for each $1 \leq n \leq 2j-1$, the following functions*

$$\begin{aligned}
&\psi_{0,k}^{j,1}, \quad k = 1, .., n-1,\\
&\psi_{k,0}^{j,2}, \quad k = 1, .., n-1,\\
&\psi_{i,k}^{j,3}, \quad 1 \leq i, k \leq n-1,\\
&\psi_{i,k}^{j,4}, \quad 1 \leq i, k \leq n-1,\\
&\psi_{i,k}^{j,5}, \quad 1 \leq i, k \leq n-1
\end{aligned}$$

are linear independent.

Proof. Let us prove it by induction. It is true for $n = 2$ and for $n = 3$. Suppose it is true for $n = p$, that is,

$$\begin{aligned}
&\psi_{0,k}^{j,1}, \quad k = 1, .., p-1;\\
&\psi_{k,0}^{j,2}, \quad k = 1, .., p-1;\\
&\psi_{i,k}^{j,3}, \quad 1 \leq i, k \leq p-1;\\
&\psi_{i,k}^{j,4}, \quad 1 \leq i, k \leq p-1;\\
&\psi_{i,k}^{j,5}, \quad 1 \leq i, k \leq p-1;
\end{aligned}$$

are linear independent. For $n = p+1$, there are $6p-1$ new functions which are

$$\begin{aligned}
&\psi_{0,k}^{j,1}, \quad k = p;\\
&\psi_{k,0}^{j,2}, \quad k = p;\\
&\psi_{i,k}^{j,3}, \quad i \quad or \quad k = p;\\
&\psi_{i,k}^{j,4}, \quad i \quad or \quad k = p;\\
&\psi_{i,k}^{j,5}, \quad i \quad or \quad k = p.
\end{aligned}$$

Suppose they are not linear independent. That is, one can find

$$\begin{aligned}
&a^1,\\
&a^2,\\
&a_{i,k}^3, \quad i \quad or \quad k = p;\\
&a_{i,k}^4, \quad i \quad or \quad k = p;\\
&a_{i,k}^5, \quad i \quad or \quad k = p
\end{aligned}$$

such that

$$a^1\psi_{0,p}^{j,1} + a^2\psi_{p,0}^{j,2} + \sum_{i \text{ or } k=p} a_{i,k}^3\psi_{i,k}^{j,3} + \sum_{i \text{ or } k=p} a_{i,k}^4\psi_{i,k}^{j,4} + \sum_{i \text{ or } k=p} a_{i,k}^5\psi_{i,k}^{j,5} + \psi^{'} = 0, \tag{3.7}$$

where $\psi^{'}$ is linear combination of the following functions:

$$\begin{aligned}
&\psi_{0,k}^{j,1}, \quad k = 1, .., p-1;\\
&\psi_{k,0}^{j,2}, \quad k = 1, .., p-1;\\
&\psi_{i,k}^{j,3}, \quad 1 \leq i, k \leq p-1;\\
&\psi_{i,k}^{j,4}, \quad 1 \leq i, k \leq p-1;\\
&\psi_{i,k}^{j,5}, \quad 1 \leq i, k \leq p-1.
\end{aligned}$$

By the definition, $\phi_{2i+1,2k+1}^{j+1}$, with i=p or k=p appear only once in $\psi_{i,k}^{j,3}$, with i=p or $k = p$, $\psi_{0,p}^{j,1}$ and $\psi_{p,0}^{j,2}$. Since ϕ^{j+1} are linear independent, that is, $a_{i,k}^3 = 0$, when i or k =p, $a^1 = 0$, and $a^2 = 0$. Thus the equation (3.7) can be simplified to

$$\sum_{i\, or\, k=p} a_{i,k}^4\psi_{i,k}^{j,4} + \sum_{i\, or\, k=p} a_{i,k}^5\psi_{i,k}^{j,5} + \psi^{'} = 0. \tag{3.8}$$

By the similar reason, $\phi_{2i,2k}^{j+1}$, when i=p or $k = p$ appear only once in $\psi_{i,k}^{j,4}$, when i=p or $k = p$. Since ϕ_{ik}^{j+1} are linear independent, $a_{i,k}^4 = 0, i\, or\, k = p$. Thus the equation (3.8) can be further simplified to the following equation

$$\sum_{i\, or\, k=p} a_{i,k}^5\psi_{i,k}^{j,5} + \psi^{'} = 0.$$

Similarly, $a_{i,k}^5 = 0$, when i , or , k =p too. Thus the equation (3.7) is reduced to

$$\psi^{'} = 0.$$

By induction hypothesis, all the coefficient of $\psi^{'} = 0$ are zeros. Hence,

$$\begin{aligned}
&\psi_{0,k}^{j,1}, \quad k = 1, .., n-1,\\
&\psi_{k,0}^{j,2}, \quad k = 1, .., n-1,\\
&\psi_{i,k}^{j,3}, \quad 1 \leq i, k \leq n-1,\\
&\psi_{i,k}^{j,4}, \quad 1 \leq i, k \leq n-1,\\
&\psi_{i,k}^{j,5}, \quad 1 \leq i, k \leq n-1
\end{aligned}$$

are linear independent. □

Theorem 3.1.2. *All the five types of wavelets in the* W_j^n *form a basis of* W_j^n *for* $1 \leq n \leq 2j-1$. *That is,*

$$W_j^n = span\{\psi_{0,k}^{j,1}, \psi_{k,0}^{j,2}, \psi_{i,k}^{j,3}, \psi_{i,k}^{j,4}, \psi_{i,k}^{j,5}, 1 \leq i,k \leq n-1\}$$

for $1 \leq n \leq 2j-1$.

Proof. The dimension of W_j^n is $(2n-1)^2 - (n)^2 = 3n^2 - 4n + 1$. It is easy to count that there are $(2n-1)^2 - (n)^2 = 3n^2 - 4n + 1$ functions in the following set

$$\begin{aligned} &\psi_{0,k}^{j,1}, \quad k = 1, .., n; \\ &\psi_{k,0}^{j,2}, \quad k = 1, .., n; \\ &\psi_{i,k}^{j,3}, \quad 1 \leq i,k \leq n; \\ &\psi_{i,k}^{j,4}, \quad 1 \leq i,k \leq n; \\ &\psi_{i,k}^{j,5}, \quad 1 \leq i,k \leq n \end{aligned}$$

which all belong to the space W_j^n. Since they are linear independent, they form a basis for space W_j^n, where $1 \leq n \leq 2j-1$. □

Finally we need to find wavelets in $W_j^{2^j} \backslash W_j^{2^j-1}$. The computations are the same to the above except for that there is one globally supported basis function. In fact the following pictures show the basis functions located on the top boundary of the domain Ω. (We omit the pictures for the basis functions on the right vertical boundary which are symmetric with respect to the line y=x are those basic functions on the top horizontal boundary of Ω.)

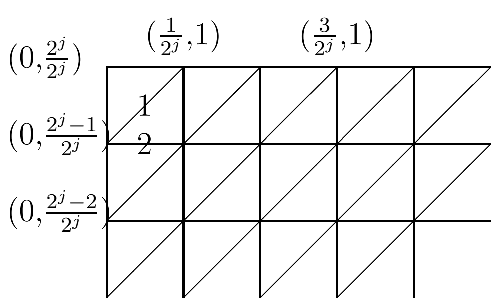

Figure 19.

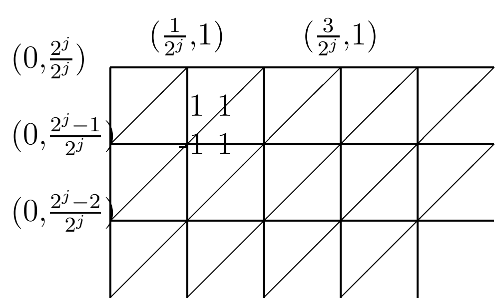

Figure 20.

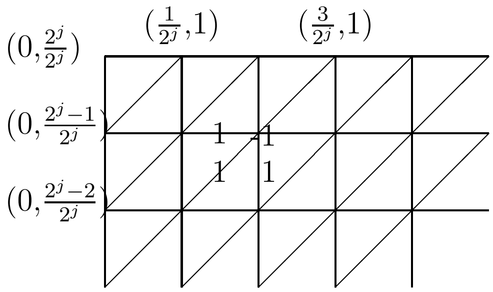

Figure 21.

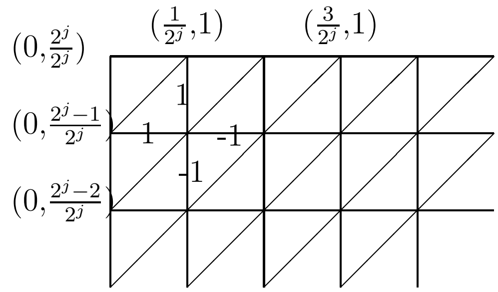

Figure 22.

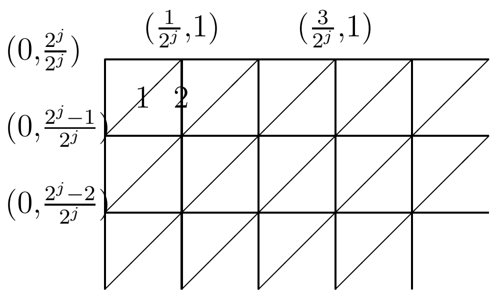

Figure 23.

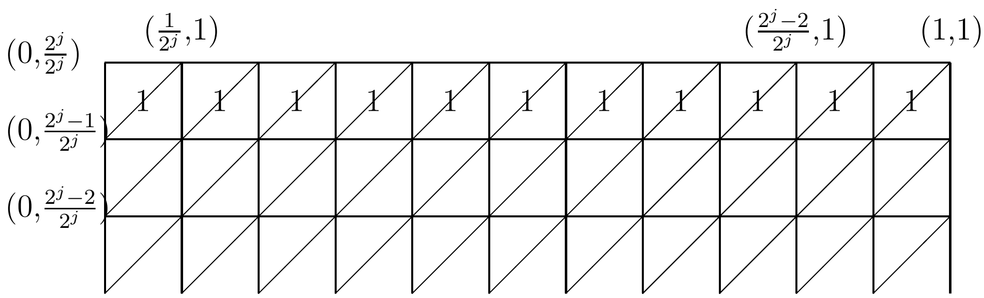

Figure 24.

The last one (cf. Figure 24) is the only special basis function since it is not local supported. The numbers of all these wavelets in $W_j^{2^j} \backslash W_j^{2^j-1}$ amount to $2^{j+3} - 8$ which is equal to the number of dimension of $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$.

Theorem 3.1.3. *All the wavelets in the $W_j^{2^j} \backslash W_j^{2^j-1}$ are linear independent and form a basis for $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$ which is spanned by the functions in $\{\phi_{i,k}^{j+1}, 2^{j+1} - 2 \leq i, k \leq 2^{j+1} - 1\}$.*

Proof. Let us just concentrate on the basis functions in $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$ and in $W_j^{2^j} \backslash W_j^{2^j-1}$. Then the scaling matrix between two sets of basis functions is the following matrix up to a constant

$$A = \begin{pmatrix} D & & & & & & & & & & \\ B1 & B2 & & & & & & & & & \\ & B1 & B2 & & & & & & & & \\ & & B1 & B2 & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & B1 & B2 & & & & & \\ & & & & & C1 & C2 & & & & \\ C3 & C3 & C3 & \dots & C3 & C3 & C3 & & & & \\ & & & & & & C4 & & & & \\ & & & & & & B2' & B1' & & & \\ & & & & & & & B2' & B1' & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & B2' & B1' \\ & & & & & & & & & & D' \end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}, \quad B1 = \begin{pmatrix} 1 & 0 & 2 & \\ & 1 & 0 & -1 \\ & 1 & 1 & 0 \\ & & 1 & -1 \end{pmatrix}, \quad B2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$D' = \begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix}, \quad B1' = \begin{pmatrix} -1 & 1 & & \\ 0 & 1 & 1 & \\ -1 & 0 & 1 & \\ & 2 & 0 & 1 \end{pmatrix}, \quad B2' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C1 = \begin{pmatrix} 1 & 0 & 2 & \\ & 1 & 0 & -1 \\ & 1 & 1 & 0 \\ & & 1 & -1 \end{pmatrix}, \quad C2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad C4 = \begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix},$$

$$C3 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $E = (m \; n \; 0 \; 0)$. By the row operations we have

$$\begin{pmatrix} E & & \\ B1 & B2 & \\ & B1 & B2 \end{pmatrix} = \begin{pmatrix} m & n & 0 & & & & & & & \\ 1 & 0 & 2 & & & & & & & \\ & 1 & 0 & -1 & & & & & & \\ & 1 & 1 & 0 & 0 & -1 & & & & \\ & & 1 & -1 & 1 & 1 & & & & \\ & & & & 1 & 0 & 2 & & & \\ & & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 1 & 0 & 0 & -1 \\ & & & & & & 1 & -1 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} m & n & & & & & & & & \\ & -n & 2m & & & & & & & \\ & & 2m & -n & & & & & & \\ & & & n & m & & & & & \\ & & & & 2m+n & 2n & 0 & 0 & & \\ & & & & 1 & 0 & 2 & & & \\ & & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 1 & 0 & 0 & -1 \\ & & & & & & 1 & -1 & 1 & 1 \end{pmatrix}.$$

Similar for B'. Thus by row operations,

$$A \to \begin{pmatrix} A_1 & G_1 & & & & & & & & & \\ & A_2 & G_2 & & & & & & & & \\ & & A_3 & G_3 & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & A_{2^j-2} & G_{2^j-2} & & & & & \\ & & & & & C'_1 & C'_2 & & & & \\ & & & & & & G'_{2^j-2} & A'_{2^j-2} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & G'_2 & A'_2 & \\ & & & & & & & & & G'_1 & A'_1 \end{pmatrix},$$

where A_n is an upper triangular matrix of size 4×4 while A'_n is a lower triangular matrix of size 4×4 which are given below.

$$A_1 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ & -1 & 1 & 0 \\ & & 1 & -1 \\ & & & 2 \end{pmatrix}, G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ & -1 & 2 & 0 \\ & & 2 & -1 \\ & & & 1 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A_n = \begin{pmatrix} n & 2 & 0 & 0 \\ & -1 & n & 0 \\ & & n & -1 \\ & & & 2 \end{pmatrix}, G_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 \end{pmatrix},$$

$$A'_n = \begin{pmatrix} 2 & & & \\ -1 & n & & \\ 0 & n & -1 & \\ 0 & 0 & 2 & n \end{pmatrix}, G'_n = \begin{pmatrix} 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix $(C_1' \quad C_2')$ is the following matrix

$$(C_1' \quad C_2') = \begin{pmatrix} 2^{j+1}-5 & 2 & & & & & & \\ 1 & 0 & 2 & & & & & \\ & 1 & 0 & -1 & 0 & 0 & 1 & \\ & 1 & 1 & 0 & 0 & -1 & -1 & \\ & & 1 & -1 & 1 & 1 & & \\ 2^{j+1}-5 & 0 & 0 & 0 & 1 & 0 & & \\ & & & & & 2 & 0 & 1 \\ & & & & & & 2 & 2^{j+1}-5 \end{pmatrix}.$$

It is easy to see the rank of $(C_1' \quad C_2')$ is 8. Thus the rank of A is $8(2^j) - 8$. Thus, all the Prewavelet functions constructed above in the $W_j^{2^j} \backslash W_j^{2^j-1}$ are linear independent and hence form a basis of $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$. □

It is easy to see that the coefficients of the Prewavelet functions in $W_j^{2^j-1}$ in terms of the basis functions of $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$ are all zeros. Thus the Prewavelet functions in $W_j^{2^j-1}$ together with the Prewavelet functions in $V_{j+1}^{2^j} \backslash V_{j+1}^{2^j-1}$ are linear independent. It follows the main result in this dissertation.

3.2 the Main Result

Theorem 3.2.1. *All the locally supported Prewavelet functions in the $W_j^{2^j} \backslash W_j^{2^j-1}$ and the locally supported and one globally supported Prewavelet functions in $W_j^{2^j-1}$ form a basis for W_j.*

The above construction find a Prewavelet basis for W_j. We shall use them for numerical solution of Poisson equation with zero boundary condition in the next chapter. We shall also show that it is necessary to have a globally supported Prewavelet function in the basis for W_j.

Chapter 4

The Prewavelet Method For Poisson Equation

Let us use the basis functions of V_j and W_j to solve Poisson equation (2.1). Mainly we explain how to compute $h_j \in W_j$. Let $g_j \in V_j$ and $g_{j+1} \in V_{j+1}$ be two FEM solutions. We aim to show that $h_j + g_j = g_{j+1}$.

By a reordering the indices $(i, k), 1 \leq i, k \leq 2^j$ in a linear fashion, let $V_j = span\{\phi_1^j, ..\phi_{N_j}^j\}$. Also, we reorder all five type wavelet functions as well as the globally supported wavelet to denote $W_j = span\{\psi_1^j, ..., \psi_{N_{j+1}-N_j}^j\}$. Let Φ^j, Ψ^j be following vectors,

$$\Phi^j = \begin{pmatrix} \phi_1^j \\ \phi_2^j \\ \vdots \\ \phi_{N_j}^j \end{pmatrix}, \qquad \Psi^j = \begin{pmatrix} \psi_1^j \\ \psi_2^j \\ \vdots \\ \psi_{N_{j+1}-N_j}^j \end{pmatrix}.$$

Then we have the following equations

$$\Phi^j = B_j \Phi^{j+1}, \qquad \Psi^j = C_j \Phi^{j+1},$$

where B_j is $N_j \times N_{j+1}$ scaling matrix, and C_j is a wavelet matrix of size $(N_{j+1} - N_j) \times N_{j+1}$. Let D_j and E_j be the following matrices:

$$D_j = \begin{pmatrix} \langle \phi_1^j, \phi_1^j \rangle_s & \langle \phi_1^j, \phi_2^j \rangle_s & \cdots\cdots & \langle \phi_1^j, \phi_{N_j}^j \rangle_s \\ \langle \phi_2^j, \phi_1^j \rangle_s & \langle \phi_2^j, \phi_2^j \rangle_s & \cdots\cdots & \langle \phi_2^j, \phi_{N_j}^j \rangle_s \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_{N_j}^j, \phi_1^j \rangle_s & \langle \phi_{N_j}^j, \phi_2^j \rangle_s & \cdots\cdots & \langle \phi_{N_j}^j, \phi_{N_j}^j \rangle_s \end{pmatrix}$$

$$E_j = \begin{pmatrix} \langle\psi_1^j, \psi_1^j\rangle_s & \langle\psi_1^j, \psi_2^j\rangle_s & \cdots\cdots & \langle\psi_1^j, \psi_{N_{j+1}-N_j}^j\rangle_s \\ \langle\psi_2^j, \psi_1^j\rangle_s & \langle\psi_2^j, \psi_2^j\rangle_s & \cdots\cdots & \langle\psi_2^j, \psi_{N_{j+1}-N_j}^j\rangle_s \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{N_{j+1}-N_j}^j, \psi_1^j\rangle_s & \langle\psi_{N_{j+1}-N_j}^j, \psi_2^j\rangle_s & \cdots\cdots & \langle\psi_{N_{j+1}-N_j}^j, \psi_{N_{j+1}-N_j}^j\rangle_s. \end{pmatrix}.$$

It is easy to see that $B_j D_{j+1} C_j^T = 0$ is equivalent to $V_j \perp W_j$. Clearly, we have $D_j = B_j D_{j+1} B_j^T$ and $E_j = C_j D_{j+1} C_j^T$.

Let g_j be the projection of g in V_j, and h_j be the projection of g in W_j. Since $V_j \bigoplus W_j = V_{j+1}$, $g_j + h_j$ will be equal to g_{j+1}. Let us write $g_j = \sum_{j=1}^{N_j} a_i \phi_i^j = (a_1, a_2, ..., a_{N_j})\Phi^j$. Similarly, $h_j = (b_1, b_2, ..., b_{N_{j+1}-N_j})\Psi^j$, and $g_{j+1} = (c_1, c_2, ..., c_{N_{j+1}})\Phi^{j+1}$. By computing the weak solutions h_j, g_j, and g_{j+1} in W_j, V_j, and V_{j+1}, respectively, we have

$$D_j \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_j} \end{pmatrix} = \begin{pmatrix} \langle\phi_1^j, g\rangle \\ \langle\phi_2^j, g\rangle \\ \vdots \\ \langle\phi_{N_j}^j, g\rangle \end{pmatrix},$$

$$E_j \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N_{j+1}-N_j} \end{pmatrix} = \begin{pmatrix} \langle\psi_1^j, g\rangle \\ \langle\psi_2^j, g\rangle \\ \vdots \\ \langle\psi_{N_{j+1}-N_j}^j, g\rangle \end{pmatrix},$$

$$D_{j+1} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N_{j+1}} \end{pmatrix} = \begin{pmatrix} \langle\phi_1^{j+1}, g\rangle \\ \langle\phi_2^{j+1}, g\rangle \\ \vdots \\ \langle\phi_{N_{j+1}}^{j+1}, g\rangle \end{pmatrix}.$$

It follows

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_j} \end{pmatrix} = (D_j)^{-1} \begin{pmatrix} \langle\phi_1^j, g\rangle \\ \langle\phi_2^j, g\rangle \\ \vdots \\ \langle\phi_{N_j}^j, g\rangle \end{pmatrix},$$

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N_{j+1}-N_j} \end{pmatrix} = (E_j)^{-1} \begin{pmatrix} \langle \psi_1^j, g\rangle \\ \langle \psi_2^j, g\rangle \\ \vdots \\ \langle \psi_{N_{j+1}-N_j}^j, g\rangle \end{pmatrix},$$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N_{j+1}} \end{pmatrix} = (D_{j+1})^{-1} \begin{pmatrix} \langle \phi_1^{j+1}, g\rangle \\ \langle \phi_2^{j+1}, g\rangle \\ \vdots \\ \langle \phi_{N_{j+1}}^{j+1}, g\rangle \end{pmatrix}.$$

The above linear systems provide a computational method to find g_j, h_j.

We now show $h_j + g_j = g_{j+1}$. That is, g_{j+1} can be computed by using h_j and g_j only. Indeed, we have

$$\begin{aligned} g_j &= (a_1, a_2, ..., a_{N_j})\Phi^j = (\Phi^j)^T (a_1, a_2, ..., a_{n_j})^T \\ &= (\Phi^{j+1})^T B_j^T (a_1, a_2, ..., a_{N_j})^T \\ &= (\Phi^{j+1})^T B_j^T D_j^{-1} (\langle \phi_1^j, g\rangle, \langle \phi_2^j, g\rangle, \cdots \langle \phi_{N_j}^j, g\rangle)^T \\ &= ((\Phi^{j+1}))^T B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j (\langle \phi_1^{j+1}, g\rangle, \langle \phi_2^{j+1}, g\rangle, \cdots \langle \phi_{N_{j+1}}^{j+1}, g\rangle)^T. \end{aligned}$$

Similarly,

$$h_j \;=\; ((\Phi^{j+1}))^T C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j (\langle \phi_1^{j+1}, g\rangle, \langle \phi_2^{j+1}, g\rangle, \cdots \langle \phi_{N_{j+1}}^{j+1}, g\rangle)^T.$$

and

$$g_{j+1} \;=\; ((\Phi^{j+1}))^T D_{j+1}^{-1} (\langle \phi_1^{j+1}, g\rangle, \langle \phi_2^{j+1}, g\rangle, \cdots \langle \phi_{N_{j+1}}^{j+1}, g\rangle)^T.$$

In order to show $h_j + g_j = g_{j+1}$, we only need to prove

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j = D_{j+1}^{-1}. \tag{4.1}$$

Notice that B_j and C_j are not square matrices. That is we can not invert B_j and C_j. Consider

$$\begin{pmatrix} B_j \\ C_j \end{pmatrix} D_{j+1} \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} = \begin{pmatrix} B_j D_{j+1} B_j^T & B_j D_{j+1} C_j^T \\ C_j D_{j+1} B_j^T & C_j D_{j+1} C_j^T \end{pmatrix}$$

$$= \begin{pmatrix} B_j D_{j+1} B_j^T & 0 \\ 0 & C_j D_{j+1} C_j^T \end{pmatrix}$$

by using the orthogonal conditions of V_j and W_j. Then we have the following equation

$$\begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} \begin{pmatrix} (B_j D_{j+1} B_j^T)^{-1} & 0 \\ 0 & (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} = I,$$

where I stands for the identity matrix. In other words, we have

$$\begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} \begin{pmatrix} B_j^T (B_j D_{j+1} B_j^T)^{-1} & C_j^T (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} = I$$

which can be rewritten in the following form

$$\begin{pmatrix} B_j^T (B_j D_{j+1} B_j^T)^{-1} & C_j^T (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} \begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} = I.$$

Hence we have

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j D_{j+1} + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j D_{j+1} = I$$

or

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j = D_{j+1}^{-1}.$$

which is (4.1) and hence $h_j + g_j = g_{j+1}$. The above computational procedure have been implemented in MATLAB and numerical experiments will be reported in the next chapter.

Chapter 5

Numerical Experiments

We have implemented the Prewavelet method for numerical solution of Poisson equations over rectangular domains in MATLAB. We would like to demonstrate that our prewavelet method is more efficient than the standard FEM method.

In the following we provide three tables of CPU times for numerical solutions based on our prewavelet method and the standard finite element method for various levels of refinement of an initial triangulation (Γ_0 which consists of two triangles) of the standard domain $[0,1] \times [0,1]$.

Let V_j be the continuous linear finite element space over triangulation Γ_j which is the jth refinement of Γ_0. For a test function u which is the exact solution of Poisson equation (2.1), the finite element method is to compute $u_j \in V_j$ directly while our prewavelet method computes u_j by computing $w_k, k = 1, \cdots, j$, i.e., $u_j = u_1 + w_1 + \cdots + w_{j-1}$.

In the following we present three tables of CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ for three test solutions by using these two methods. Note that we use the direct method coded in MATLAB to solve the associated linear equations. We shall present tables of CPU times based on Conjugate Gradient Method for the systems of equations next.

For an exact solution $u(x,y) = \sin(2\pi x)\sin(2\pi y)$ which clearly satisfies the zero boundary conditions, we list CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ by using these two methods in Table 1. (The Figure is shown in Figure 5.1 to 5.6.)

Table 1. CPU times to compute u_j by the two methods

	FEM method	Prewavelet Method
j=4	0.164531 seconds	0.204067 seconds
j=5	0.593587 seconds	0.519293 seconds
j=6	13.960323 seconds	6.222679 seconds

For an exact solution $u(x, y) = xy(1 - x)(1 - y)$, the CPU times for numerical solutions by these two methods are given in Table 2. (The Figure is shown in Figure 5.7 to 5.12.)

Table 2. CPU times for computing u_j by the two methods

CPU time	FEM method	Prewavelet Method
j=4	0.150836 seconds	0.218282 seconds
j=5	0.574085 seconds	0.558071 seconds
j=6	13.896825 seconds	6.202557 seconds

We list the CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ of $u(x, y) = xy(1 - x)(1 - y)e^{8xy}$ by using these two methods in Table 3. (The Figure is shown in Figure 5.13 to 5.18.)

Table 3. CPU times for computing u_j by the two methods

CPU time	FEM method	Prewavelet Method
j=4	0.144159 seconds	0.186389 seconds
j=5	0.584828 seconds	0.459181 seconds
j=6	13.877403 seconds	6.139101 seconds

It is clear from these three tables that the prewavelet method is much more efficient.

Next we use the Conjugate Gradient Method to solve the linear systems associated with FEM. Let us consider iterative solutions to u_j for $j = 6$ with various accuracy ϵ. First let us consider the exact solution $u(x, y) = \sin(2\pi x) \sin(2\pi y)$.

Table 4. CPU times for approximating the FEM solution u_6 by Conjugate Gradient Method

ϵ	CPU times
10^{-8}	5.411852 seconds
10^{-9}	5.783497 seconds
10^{-10}	6.221683 seconds
10^{-11}	6.616816 seconds
10^{-12}	6.917468 seconds
10^{-13}	7.836775 seconds

To approximate the FEM solution u_6 of the exact solution $u(x,y) = xy(1-x)(1-y)$ by the Conjugate Gradient Method, we list the CPU times in Table 5 for various accuracy ϵ.

Table 5. CPU times for approximating the FEM solution u_j by Conjugate Gradient Method

ϵ	CPU times
10^{-8}	4.476794 seconds
10^{-9}	4.878259 seconds
10^{-10}	5.306747 seconds
10^{-11}	5.887849 seconds
10^{-12}	6.811317 seconds
10^{-13}	6.754465 seconds

Finally let us consider the CPU times to approximate the FEM solution u_6 of $u(x,y) = xy(1-x)(1-y)e^{8xy}$ by the Conjugate Gradient Method.

Table 6. CPU times for approximating the FEM solution by Conjugate Gradient Method

ϵ	CPU times
10^{-8}	10.110517 seconds
10^{-9}	10.740035 seconds
10^{-10}	11.319618 seconds
10^{-11}	11.810142 seconds
10^{-12}	12.320903 seconds
10^{-13}	13.103407 seconds

It is clear from all six tables, if we want an accurate iterative solution of u_6 within 10^{-12}, the prewavelet method appears better.

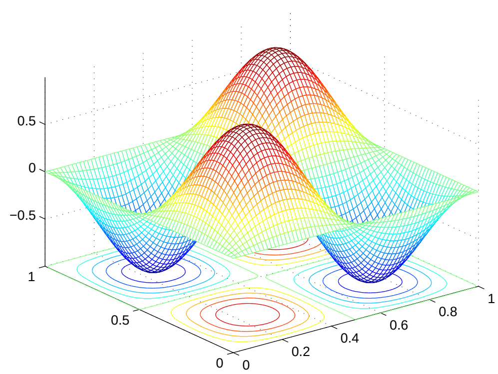

Figure 5.1: Scaling level:j=6 by prewavelets

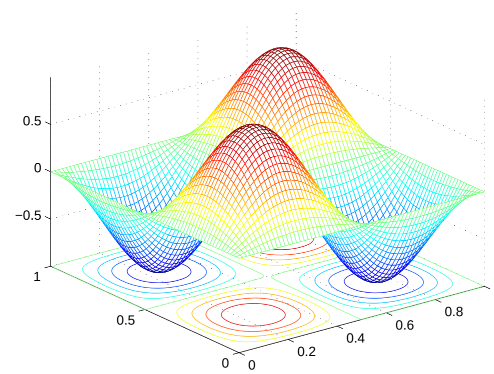

Figure 5.2: Scaling level:j=6 by finite element

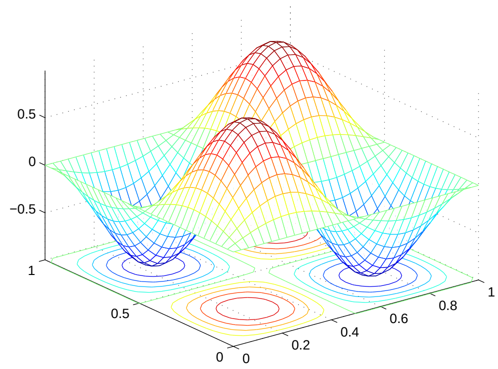

Figure 5.3: Scaling level:j=5 by pre-wavelets

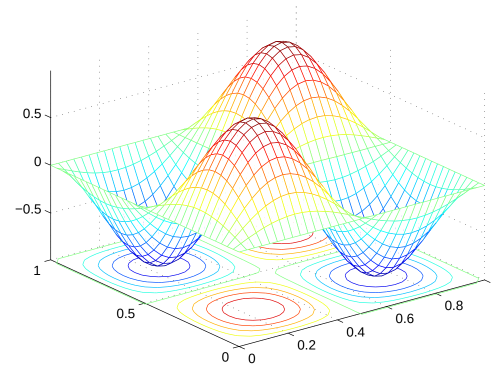

Figure 5.4: Scaling level:j=5 by finite element

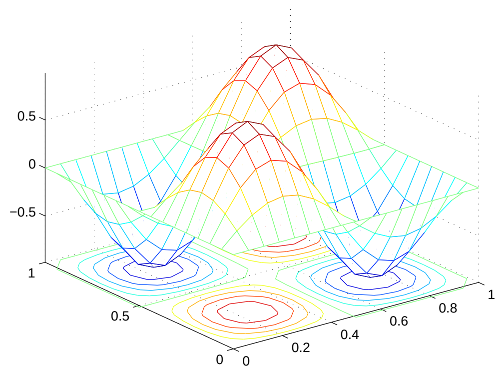

Figure 5.5: Scaling level:j=4 by pre-wavelets

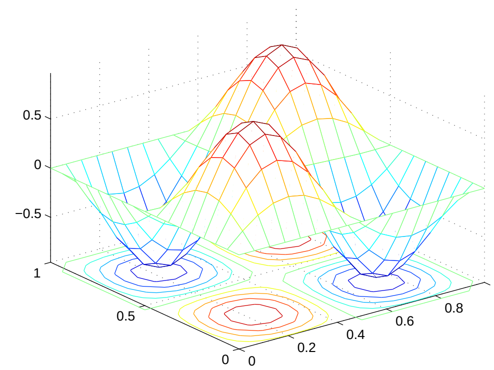

Figure 5.6: Scaling level:j=4 by finite element

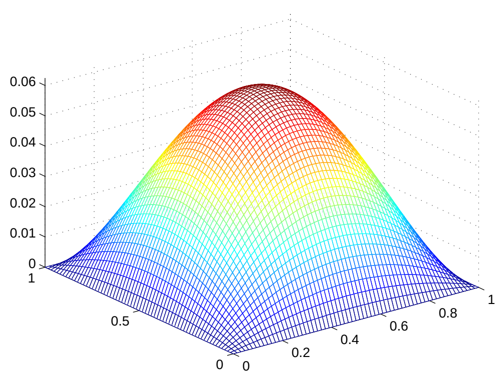

Figure 5.7: Scaling level:j=6, by pre-wavelets

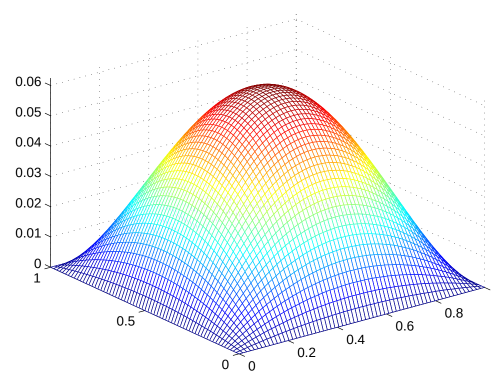

Figure 5.8: Scaling level:j=6, by finite element

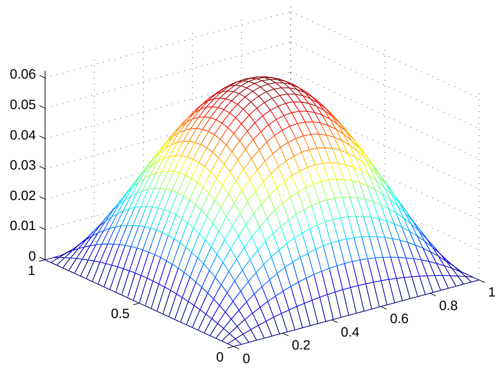

Figure 5.9: Scaling level:j=5 by pre-wavelets

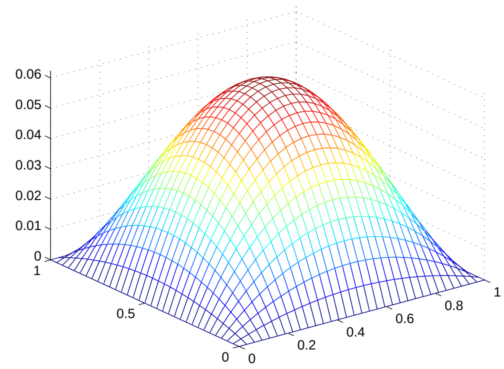

Figure 5.10: Scaling level:j=5 by finite element

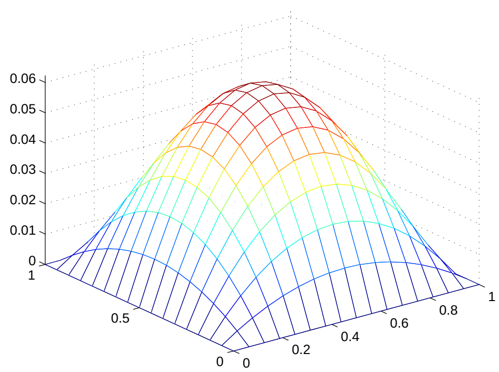

Figure 5.11: Scaling level:j=4 by pre-wavelets

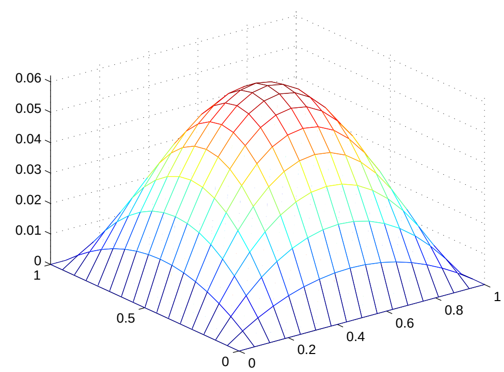

Figure 5.12: Scaling level:j=4 by finite element

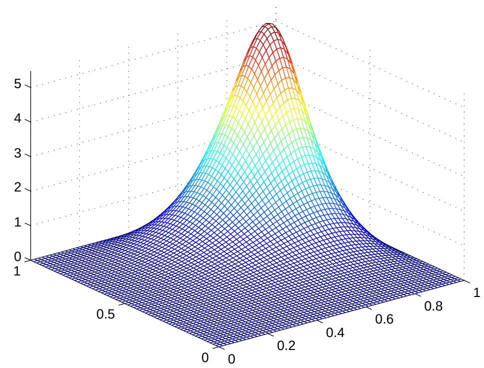

Figure 5.13: Scaling level:j=6, by pre-wavelets

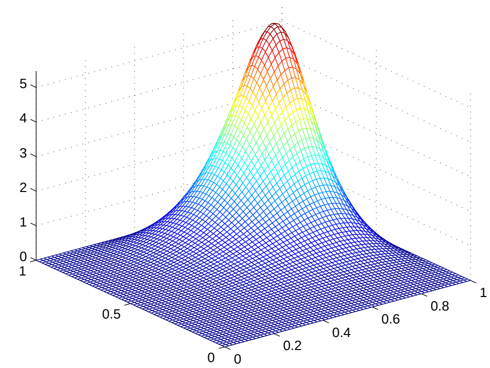

Figure 5.14: Scaling level:j=6, by finite element

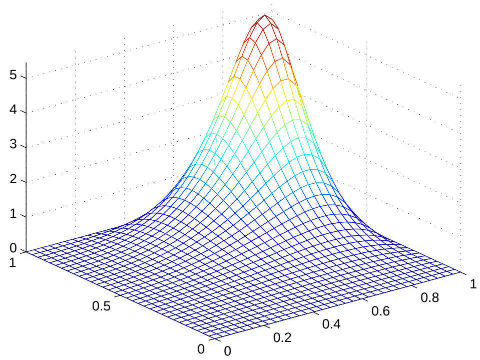

Figure 5.15: Scaling level:j=5 by pre-wavelets

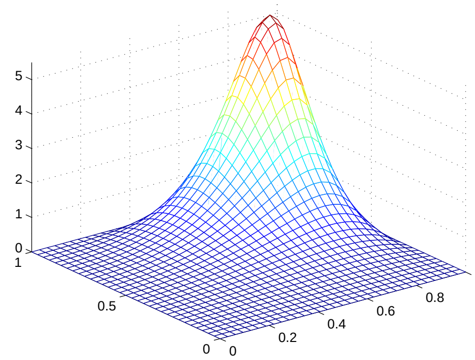

Figure 5.16: Scaling level:j=5 by finite element

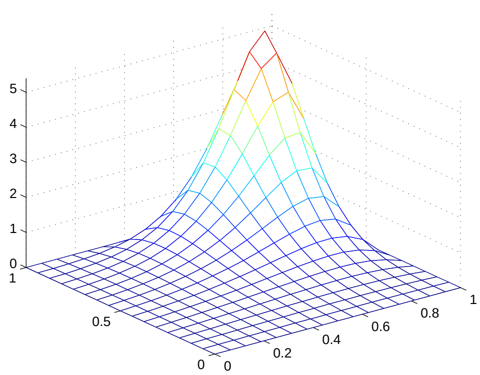

Figure 5.17: Scaling level:j=4 by pre-wavelets

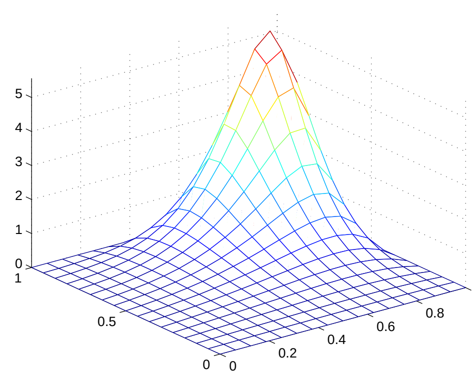

Figure 5.18: Scaling level:j=4 by finite element

Chapter 6

Prewavelets Solution to Poisson Equation over Triangular Domain

In this chapter, I deal with the Dirichlet boundary value problem for Poisson equation with triangular boundary. In reality, this boundary shape is almost as important as the rectangle boundary.

6.1 Simplification of the Poisson Equation

Let us start with a triangular domain $\Omega \in R^2$, which is determined by three vertices (0,0),(0,1),(1,0), consider the Dirichlet boundary value problem for Poisson equation:

$$\begin{cases} -\Delta u(x,y) = g(x,y), & (x,y) \in \Omega \\ u(x,y) = f_1(x), & for \quad y = 0 \quad and \quad 0 \le x \le 1 \\ u(x,y) = f_2(y), & for \quad x = 0 \quad and \quad 0 \le y \le 1 \\ u(x,y) = f_3(x), & for \quad x + y = 1 \quad and \quad 0 \le y \le 1. \end{cases}$$

Without lose of generality, we may assume that $f_1(1) = f_2(1) = f_3(1) = f_1(0) = f_2(0) = f_3(0)$. Otherwise, letting $f_1(0) = f_2(0) = a_1$, $f_1(1) = f_3(1) = a_2$, $f_2(1) = f_3(0) = a_3$, we define $h(x,y) = a_1 + a_2 x + a_3 y$, and $v(x,y) = u(x,y) - h(x,y)$. Then the above Dirichlet problem becomes to:

$$\begin{cases} -\Delta v(x,y) = g(x,y), & (x,y) \in \Omega \\ v(x,y) = f_1(x) - h(x,0), & for \quad y = 0 \quad and \quad 0 \le x \le 1 \\ v(x,y) = f_2(y) - h(0,y), & for \quad x = 0 \quad and \quad 0 \le y \le 1 \\ v(x,y) = f_3(x) - h(x,1-x), & for \quad x + y = 1 \quad and \quad 0 \le y \le 1 \end{cases}$$

Now let $g_2(x,y) = (1-x)^2(f_2(\frac{y}{1-x}) - h(0,\frac{y}{1-x})) - (1-y)^2(f_1(\frac{x}{1-y}) - h(\frac{x}{1-y},0)) - (x+y)^2(f_3(\frac{x}{x+y}) - h(\frac{x}{x+y},\frac{y}{x+y}))$, and $w(x,y) = v(x,y) - g_2(x+y)$. Then $w(x,y)$ is well defined and has the second order smoothness, and $w(x,y)$ satisfies the equation

$$\begin{cases} -\Delta w(x,y) = g_1(x,y), & (x,y) \in \Omega \\ w(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}.$$

with $g_1(x,y) = g(x,y) + \frac{\partial^2}{\partial y^2} g_2(x,y) + \frac{\partial^2}{\partial x^2} g_2(x,y)$. If we can find solution for w, it is easy to get $u(x,y)$. In the remaining part of this chapter, we only consider the Poisson equation with zero boundary condition:

$$\begin{cases} -\Delta u(x,y) = g(x,y), & (x,y) \in \Omega \\ u(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}. \tag{6.1}$$

6.2 Seminorm

Let us define

$$H_0^1(\Omega) = \{v \in L^2(\Omega) : \quad \langle v, v\rangle_s < \infty \quad and \quad v(x,y) = 0, (x,y) \in \partial\Omega\}$$

where seminorm $\langle v, v\rangle_s$ is defined by

$$\langle f, f\rangle_s = \|f\|_s^2 = \int_0^1 \int_0^y \left[\frac{\partial f(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{\partial f(x,y)}{\partial y}\right] dxdy,$$

Thus $H_0^1(\Omega)$ is a standard Sobolev space. Suppose $u \in H_0^1(\Omega)$, for any function $v \in H_0^1(\Omega)$ such that $v(x,y) = 0$, for $(x,y) \in \partial\Omega$ then integration by parts of (6.1) yields

$$\begin{aligned} \langle g, v\rangle &= \int_0^1 \int_0^y g(x,y)v(x,y)dxdy \\ &= \int_0^1 \int_0^y -\Delta u(x,y)v(x,y)dxdy \\ &= \int_0^1 \int_0^y \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y}dxdy \\ &= \langle u, v\rangle_s, \end{aligned}$$

We can say that the weak solution u to (6.1) is in $V \in H_0^1(\Omega)$ if

$$u \in V \quad such \quad that \ \langle u, v\rangle_s = \langle g, v\rangle, \quad \forall v \in V. \tag{6.2}$$

Theorem 6.2.1. *Suppose $g \in C(\Omega)$. If $u \in C^2(\Omega)$ is a weak solution satisfying (6.2), then u is a classic solution satisfying (6.1), .*

Proof. Let $v \in H_0^1(\Omega)$. Then integration by parts gives

$$\begin{aligned}\langle g, v\rangle &= \langle u, v\rangle_s \\ &= \int_0^1 \int_0^y \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{\partial v(x,y)}{\partial y} dxdy \\ &= \int_0^1 \int_0^y -\Delta u(x,y) v(x,y) dxdy \\ &= \langle -\Delta u(x,y), v\rangle.\end{aligned}$$

Thus, $\langle g - (-\Delta u(x,y)), v\rangle = 0$ for all $v \in H_0^1(\Omega)$. Claim $\omega = g + \Delta u(x,y) \in C^0(\Omega)$ is identically zero. If $\omega \neq 0$ then ω is of one sign in some square $\{((x_1, y_1) \times (x_2, y_2)) \in (\Omega)\}$. Choose $v(x,y) = (x-x_1)^2(x-x_2)^2(y-y_1)^2(y-y_2)^2$ in the square $((x_1, y_1) \times (x_2, y_2))$ and $v = 0$ outside the square. Then $\langle w, v\rangle \neq 0$, which is a contradiction. Thus $g \equiv -\Delta u(x,y)$.

□

6.3 Linear Spline Space

For convenient, let $N_j = (2^j - 1)(2^{(j-1)} - 1)$. Let $x_{ji} = \frac{i}{2^j}$ for $i = 1, .., 2^j - 1$ and $y_{ji} = \frac{i}{2^j}$ for $i = 1, .., 2^j - 1$ then the segment of $x = x_{ji}$, $y = y_{ji}$ and $y + x = y_{ji}$ divide the Ω into 4^j small subtriangle.

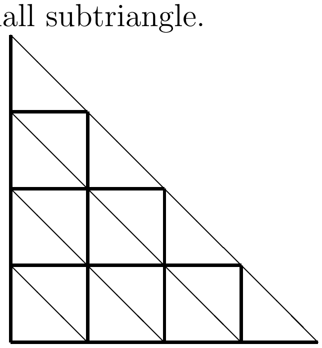

Figure 25. Type-I triangulation,j=2.

Define ϕ^j_{ik} to be linear spline with support on the hexagon with vertices

$$\{(x_{j(i-1)}, y_{j(k+1)}), (x_{ji}, y_{j(k-1)}), (x_{j(i+1)}, y_{j(k)}), (x_{j(i+1)}, y_{j(k-1)}), (x_{j(i)}, y_{j(k+1)}), (x_{j(i-1)}, y_{j(k)})\}$$

and $\phi_{ik}(x_{ji'}, y_{jk'}) = \delta_{ii'}\delta_{kk'}$. Let $V_j = span\{\phi^j_{ik}, i > 0, j > 0, j + i < 2^j\}$ be the subspace of $H^1_0(\Omega)$. Let

$$u_j \in V_j \ \ such \ \ that \ \ \langle u_j, v\rangle_s = \langle f, v\rangle \ \ \forall v \in V_j \tag{6.3}$$

Theorem 6.3.1. *Given* $f \in L^2(\Omega)$, *(6.3) has a unique weak solution.*

Proof. By the same proof as theorem 2.4.1, (6.3) has a unique weak solution. □

Let us observe relationship between u and u_j. Subtracting (6.3) from (6.2) implies

$$\langle u - u_j, w\rangle_s = 0 \qquad\qquad \forall w \in V_j. \tag{6.4}$$

Then for any $v \in V_j$

$$\begin{aligned} \|u - u_j\|^2_s &= \langle u - u_j, u - u_j\rangle_s \\ &= \langle u - u_j, u - v\rangle_s + \langle u - u_j, v - u_j\rangle_s \\ &= \langle u - u_j, u - v\rangle_s \\ &\leq \|u - u_j\|_s \|u - v\|_s \end{aligned}$$

It follows that $\|u - u_j\|_s \leq \|u - v\|_s$ for any $v \in V_j$. Thus we have proved the following theorem.

Theorem 6.3.2. $\|u - u_j\|_s = min\ \{\|u - v\|_s : v \in V_j\}$.

6.4 Error Approximation

Given $u \in C^0(\Omega)$, let $u_j \in V_j$ be the interpolation of u:

$$u_I = \sum_{ik} u(x_{ji}, y_{jk})\phi^{(j)}_{ik}.$$

The following error estimate is well-known.

Lemma 6.4.1. *Suppose $u \in C^2(\Omega)$. Then*

$$\|u-u_j\|_s \le \frac{\sqrt{12}}{2^j}\sqrt{\left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L^\infty}+\left\|\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right\|^2_{L^\infty}+\left\|\frac{\partial^2 u}{\partial y^2}\right\|^2_{L^\infty}},$$

.

Proof. the proof is similar to the lemma 2.4.3. □

6.5 Prewavelets Construction over Triangular Domain

By direct calculation, we obtain the following result immediately.

Lemma 6.5.1. *We have* $\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k}, \rangle_s = 2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k}, \rangle_s = 1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k-1}, \rangle_s = 1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k}, \rangle_s = 1/2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k+1}, \rangle_s = 1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k-1}, \rangle_s = 1$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k+1}, \rangle_s = 1$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k}, \rangle_s = -1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k}, \rangle_s = -1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k-2}, \rangle_s = -1/2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i,2k+2}, \rangle_s = -1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k-2}, \rangle_s = 0$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k+2}, \rangle_s = 0$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i-2,2k-1}, \rangle_s = -1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k+1}, \rangle_s = -1$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k+2}, \rangle_s = -1/2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{2i+2,2k+1}, \rangle_s = -1/2$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i+1,2k-1}, \rangle_s = -1$, $\langle \phi^j_{ik}, \phi^{j+1}_{2i-1,2k-2}, \rangle_s = -1/2$,
$\langle \phi^j_{ik}, \phi^{j+1}_{i',k'}, \rangle_s = 0$, *for other i', k', which are not listed above.*

In the following, I will give one method to find the locally supported basis for W_j. Suppose $\psi^j = \sum_{ik} \phi^{j+1}_{ik} b_{ik} \in W_j$. Then by orthogonal condition, we need to solve the following equations.

$$\begin{aligned} \langle \phi^j_{i'k'}, \psi^j \rangle_s &= 0 \\ \langle \phi^j_{i'k'}, \sum_{ik} \phi^{j+1}_{ik} b_{ik} \rangle_s &= 0 \\ \sum_{ik} \left\langle \phi^j_{i'k'}, \phi^{j+1}_{ik} \right\rangle_s b_{ik} &= 0. \end{aligned} \tag{6.5}$$

Each (i', k') determines one equation. Thus, there are N_j elements in the set V_j and hence they determined the N_j equations. These N_j equations with N_{j+1} coefficients, b_{ik}, implies that there are $N_{j+1} - N_j$ linear independent solutions of these equation system which compose a basis for W_j.

Definition 6.5.1. *Let V_{j+1}^m be a subspace of V_{j+1} such that $V_{j+1}^m = span\{\phi_{ik}^{j+1}, i > 0, k > 0, i + k < 2m\}$. Let W_j^m be subspace of W_j such that $W_j^m = W_j \bigcap V_{j+1}^m$.*

Obviously $\emptyset = V_{j+1}^1 \subset V_{j+1}^2 \subset \ldots \subset V_{j+1}^{2^j} = V_j$, and $\emptyset = W_j^1 \subset W_j^2 \subset \ldots \subset W_j^{2^j} = W_j$. There is no nonzero solution of (6.5) in space of V_{j+1}^1, and there are two solution of (6.5) in space V_{j+1}^2, they are solutions of the following equation for $i > 0, k > 0$.

$$\left\langle \sum_{1 \le i+k < 4} \phi_{ik}^{j+1} b_{ik}, \phi_{1,1}^j \right\rangle_s = 0$$

it is equivalent to the following equation

$$\begin{pmatrix} \langle \phi_{1,1}^j, \phi_{1,1}^{j+1} \rangle_s & \langle \phi_{1,1}^j, \phi_{2,1}^{j+1} \rangle_s & \langle \phi_{1,1}^j, \phi_{1,2}^{j+1} \rangle_s \end{pmatrix} \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ b_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

By Lemma 6.5.1, we obtain the following equation.

$$\begin{pmatrix} -1 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ b_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

The rank of the left matrix is 1, so there are two solutions shown as the follows.

$$\begin{pmatrix} b_{1,1} \\ b_{2,1} \\ b_{1,2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} or \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\psi_{0,1}^{j,1} = 2\phi_{1,2}^{j+1} + \phi_{1,1}^{j+1} \quad \text{as shown in Figure 26;} \tag{6.6}$$

$$\psi_{1,0}^{j,1} = 2\phi_{2,1}^{j+1} + \phi_{1,1}^{j+1} \quad \text{as shown in Figure 27;} \tag{6.7}$$

Definition 6.5.2. $\psi^{j,1}_{0,k} = 2\phi^{j+1}_{1,2k} + \phi^{j+1}_{1,2k-1}$ *is the first kind of wavelets on the vertical edge.* $\psi^{j,1}_{k,0} = 2\phi^{j+1}_{2k,1} + \phi^{j+1}_{2k-1,1}$ *is the second kind of wavelet on the horizontal edge.*

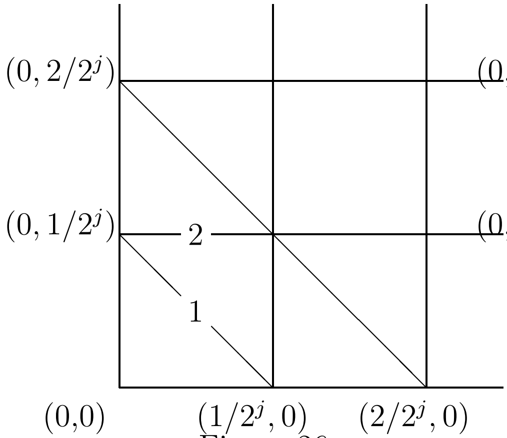

Figure 26.

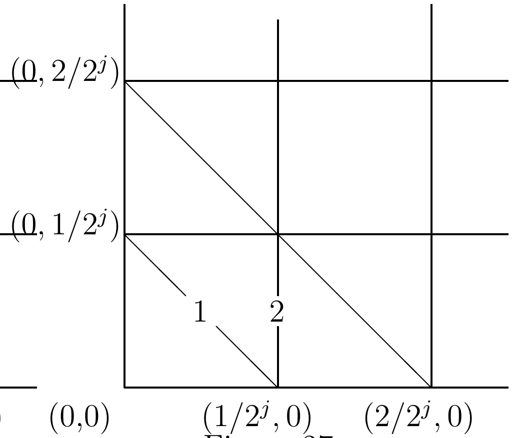

Figure 27.

Now we consider V^3_j. Similarly, there are 10 non-zero coefficients for linear system (6.5), and there are 3 linear independent equations. So the dimension of solution space of W^3_j is 10-3=7. The first two of them are same to the wavelet functions in (6.6) and (6.7), let me show the other 5 in the following figures.

$$\psi^{j,1}_{0,2} = 2\phi^{j+1}_{1,4} + \phi^{j+1}_{1,3} \quad \text{as shown in Figure 28;} \tag{6.8}$$

$$\psi^{j,1}_{2,0} = 2\phi^{j+1}_{4,1} + \phi^{j+1}_{3,1} \quad \text{as shown in Figure 29;} \tag{6.9}$$

$$\psi^{j,2}_{1,1} = \phi^{j+1}_{1,3} + \phi^{j+1}_{1,2} + \phi^{j+1}_{2,3} - \phi^{j+1}_{2,2} \quad \text{as shown in Figure 30;} \tag{6.10}$$

$$\psi^{j,3}_{1,1} = \phi^{j+1}_{3,1} + \phi^{j+1}_{3,2} + \phi^{j+1}_{2,1} - \phi^{j+1}_{2,2} \quad \text{as shown in Figure 31;} \tag{6.11}$$

$$\psi^{j,4}_{1,1} = \phi^{j+1}_{3,2} + \phi^{j+1}_{2,3} - \phi^{j+1}_{1,2} - \phi^{j+1}_{2,1} \quad \text{as shown in Figure 32;} \tag{6.12}$$

Definition 6.5.3. $\psi^{j,2}_{i,k} = -\phi^{j+1}_{2i,2k} + \phi^{j+1}_{2i,2k+1} + \phi^{j+1}_{2i-1,2k} + \phi^{j+1}_{2i-1,2k+1}$ *is the second kind of wavelet.* $\psi^{j,3}_{i,k} = -\phi^{j+1}_{2i,2k} + \phi^{j+1}_{2i+1,2k} + \phi^{j+1}_{2i,2k-1} + \phi^{j+1}_{2i+1,2k-1}$ *is the third kind of wavelets,* $\psi^{j,4}_{i,k} = \phi^{j+1}_{2i+1,2k} + \phi^{j+1}_{2i,2k+1} - \phi^{j+1}_{2i,2k-1} - \phi^{j+1}_{2i-1,2k}$ *is the forth kind of wavelets.*

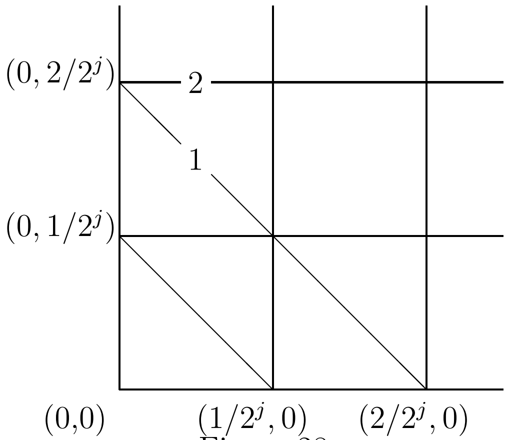

Figure 28.

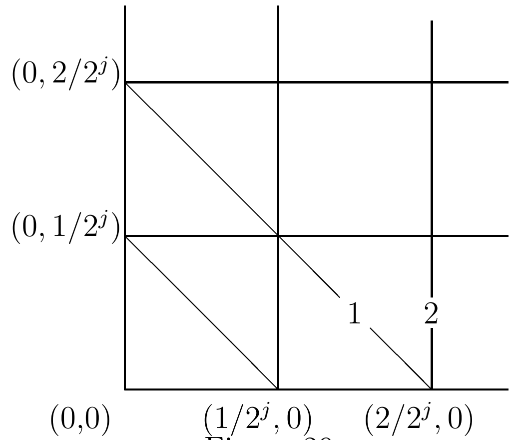

Figure 29.

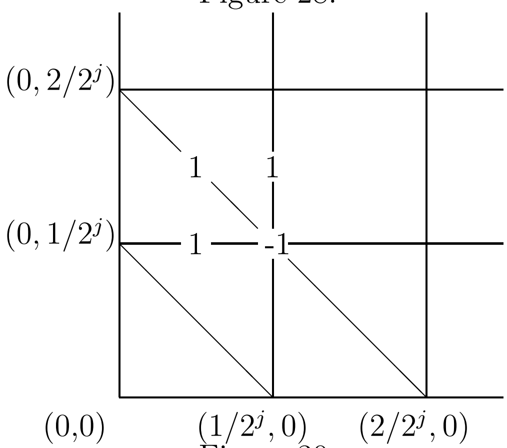

Figure 30.

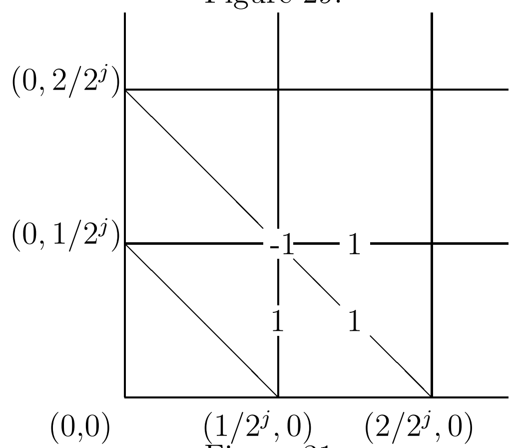

Figure 31.

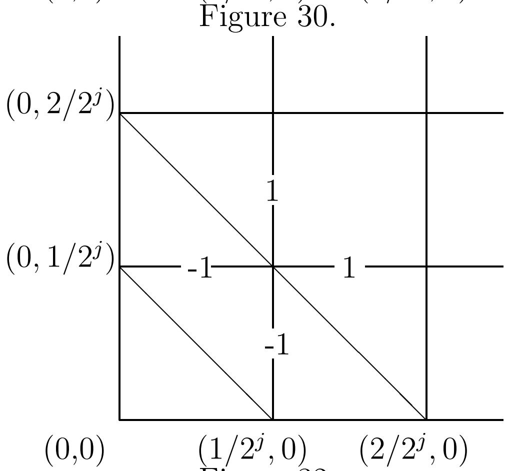

Figure 32.

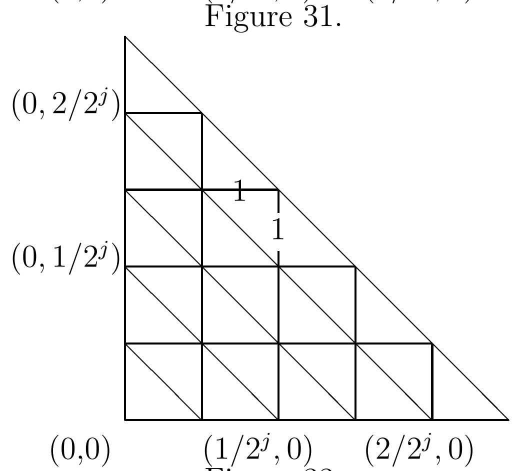

Figure 33.

The above computation can be carried out until V_j^m, $m = 2, 3, ...2^j - 1$. We obtain four types of wavelet functions in $W_j^{2^j-1}$

Theorem 6.5.1. *All the five types of wavelet in the V_j^n are linear independent for $1 \leq n \leq 2^j - 1$. That means for $i > 0$ and $k > 0$*

$$\psi_{0,k}^{j,1}, \quad k = 1,..,n-1;$$
$$\psi_{k,0}^{j,1}, \quad k = 1,..,n-1;$$
$$\psi_{i,k}^{j,2}, \quad 1 \leq i+k \leq n-1;$$
$$\psi_{i,k}^{j,3}, \quad 1 \leq i+k \leq n-1;$$
$$\psi_{i,k}^{j,4}, \quad 1 \leq i+k \leq n-1;$$

are linear independent for $1 \leq n \leq 2^j - 1$.

Proof. Let us prove it by induction, it is true for n=2 and n=3, Suppose it is true for n=p, that means

$$\psi_{0,k}^{j,1}, \quad k = 1,..,p-1,$$
$$\psi_{k,0}^{j,1}, \quad k = 1,..,p-1,$$
$$\psi_{i,k}^{j,2}, \quad 1 \leq i+k \leq p-1,$$
$$\psi_{i,k}^{j,3}, \quad 1 \leq i+k \leq p-1,$$
$$\psi_{i,k}^{j,4}, \quad 1 \leq i+k \leq p-1,$$

are linear independent. For $n = p+1$, there are $3p-1$ new elements, which are

$$\psi_{0,k}^{j,1}, \quad k = p,$$
$$\psi_{k,0}^{j,1}, \quad k = p,$$
$$\psi_{i,k}^{j,2}, \quad i+k = p,$$
$$\psi_{i,k}^{j,3}, \quad i+k = p,$$
$$\psi_{i,k}^{j,4}, \quad i+k = p.$$

Suppose they are not linear independent. Then I can find

$$a_{0,p}^1,$$
$$a_{p,0}^1,$$
$$a_{i,k}^2, \quad i+k = p,$$
$$a_{i,k}^3, \quad i+k = p,$$
$$a_{i,k}^4, \quad i+k = p,$$

such that

$$a^1_{0,p}\psi^{j,1}_{0,p} + a^1_{p,0}\psi^{j,2}_{p,0} + \sum_{i+k=p} a^2_{i,k}\psi^{j,2}_{i,k} + \sum_{i+k=p} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{i+k=p} a^4_{i,k}\psi^{j,4}_{i,k} + \psi^{'} = 0, \tag{6.13}$$

where $\psi^{'}$ is a linear combination of the following elements:

$$\begin{aligned} &\psi^{j,1}_{0,k}, \quad k = 1, .., p-1; \\ &\psi^{j,1}_{k,0}, \quad k = 1, .., p-1; \\ &\psi^{j,2}_{i,k}, \quad 1 \le i+k \le p-1; \\ &\psi^{j,3}_{i,k}, \quad 1 \le i+k \le p-1; \\ &\psi^{j,4}_{i,k}, \quad 1 \le i+k \le p-1. \end{aligned}$$

By the definition, $\phi^{j+1}_{1,2p}$ appears only once in $\psi^{j,1}_{0,p}$, $\phi^{j+1}_{2p,1}$ appears only once in $\psi^{j,1}_{p,0}$, in equation (6.13), that means $a^1_{0,p} = a^1_{p,0} = 0$. By the same reason, $a^2_{i,k} = a^3_{i,k} = a^4_{i,k} = 0$, for $i+k=p$. Therefore the equation (6.13) is simplified to following expression:

$$\psi^{'} = 0$$

By the induction, all the coefficients of $\psi^{'} = 0$ are zeroes. That means

$$\begin{aligned} &\psi^{j,1}_{0,k}, \quad k = 1, .., n-1, \\ &\psi^{j,2}_{k,0}, \quad k = 1, .., n-1, \\ &\psi^{j,3}_{i,k}, \quad i+k \le n-1, \\ &\psi^{j,4}_{i,k}, \quad i+k \le n-1, \text{ and} \\ &\psi^{j,5}_{i,k}, \quad i+k \le n-1 \end{aligned}$$

are linear independent for $1 \le n \le 2^j - 1$. □

Theorem 6.5.2. *All the five types of wavelet in the V^n_j compose a linear independent basis for W^n_j for $1 \le n \le 2^j - 1$. That means*

$$W^n_j = span \left\{ \begin{array}{ll} \psi^{j,1}_{0,k}, & k = 1, .., n-1; \\ \psi^{j,1}_{k,0}, & k = 1, .., n-1; \\ \psi^{j,2}_{i,k}, & i+k \le n-1; \\ \psi^{j,3}_{i,k}, & i+k \le n-1; \\ \psi^{j,4}_{i,k}, & i+k \le n-1; \end{array} \right\}$$

for $1 \leq n \leq 2^j - 1$.

Proof. The dimension of W_j^n is $(2n-1)(n-1)-(n)(n-1)/2 = (4n-2-n)(n-1)/2$. After counting, there are the same amount of elements in the following set:

$$\left\{ \begin{array}{ll} \psi_{0,k}^{j,1}, & k = 1,..,n-1; \\ \psi_{k,0}^{j,1}, & k = 1,..,n-1; \\ \psi_{i,k}^{j,2}, & i+k \leq n-1; \\ \psi_{i,k}^{j,3}, & i+k \leq n-1; \\ \psi_{i,k}^{j,4}, & i+k \leq n-1; \end{array} \right\}$$

which are in the space W_j^n. Since they are linear independent, they form a basis for space W_j^n, for $1 \leq n \leq 2^j - 1$. □

Now only work left is to find prewavelets in $W_j^{2^j} \backslash W_j^{2^j-1}$. Let me define one more kind of prewavelets function $\psi_{i,k}^{j,5}$ in this space, see Figure33.

$$\psi_{i,k}^{j,5} = \phi_{2i-1,2k}^{j+1} + \phi_{2i,2k-1}^{j+1}, \quad i+k = 2^j.$$

Thus we know $W_j^{2^j} \backslash W_j^{2^j-1}$ span by the following wavelets.

$$W_j^{2^j} \backslash W_j^{2^j-1} = span \left\{ \begin{array}{ll} \psi_{0,k}^{j,1}, & k = 2^j - 1; \\ \psi_{k,0}^{j,1}, & k = 2^j - 1; \\ \psi_{i,k}^{j,2}, & i+k = 2^j - 1; \\ \psi_{i,k}^{j,3}, & i+k = 2^j - 1; \\ \psi_{i,k}^{j,4}, & i+k = 2^j - 1; \\ \psi_{i,k}^{j,5}, & i+k = 2^j; \end{array} \right\}$$

By counting, all wavelets in $W_j^{2^j} \backslash W_j^{2^j-1}$ amount to $2^{j+2} - 5$, it is right the number of dimension of $V_j^{2^j} \backslash V_j^{2^j-1}$.

Theorem 6.5.3. *All the five types of wavelet in the* $W_j^{2^j} \backslash W_j^{2^j-1}$ *are linear independent for scaling functions of* $V_j^{2^j} \backslash V_j^{2^j-1}$, *which is set* $\{\phi_{i,k}^{j+1}, 2^{j+1} - 2 \leq i+k \leq 2^{j+1} - 1\}$.

Proof. Let us just concentrate on the basis functions in $V_j^{2^j} \backslash V_j^{2^j-1}$ and in $W_j^{2^j} \backslash W_j^{2^j-1}$. Then the scaling matrix between two sets of basis functions is the following matrix up to a constant.

Let A be a $(2^{j+2} - 5) \times (2^{j+2} - 5)$ matrix below. If matrix A is invertible, then the prewavelets basis I choose for $W_j^{2^j} \backslash W_j^{2^j-1}$ are linear independent.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 & 1 \\ & 1 & 1 & -1 \\ & & 1 & 0 & 1 \\ & & & -1 & 1 & 1 \\ & & & & & \ddots \\ & & & & & 1 & 0 & 1 \\ & & & & & & 1 & 1 & -1 \\ & & & & & & & 1 & 0 & 1 \\ & & & & & & & & -1 & 1 & 1 \\ & & & & & & & & & & \ddots \\ & & & & & & & & & & 1 & 0 & 1 \\ & & & & & & & & & & & 1 & 1 & -1 \\ & & & & & & & & & & & & 1 & 0 & 1 \\ & & & & & & & & & & & & & -1 & 1 & 1 \\ & & & & & & & & & & & & & & 1 & 0 & 1 \\ & & & & & & & & & & & & & & & 1 & 2 \end{pmatrix}.$$

If we denote

$$E = \begin{pmatrix} 2 & 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 & \\ & 1 & 1 & -1 \\ & & 1 & 0 \\ & & & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$C1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

we can rewrite matrix A in the following expression.

$$A = \begin{pmatrix} E & & & & & & & & \\ B & C & & & & & & & \\ & B & C & & & & & & \\ & & B & C & & & & & \\ & & & & \ddots & & & & \\ & & & & & B & C & & \\ & & & & & & B & C & \\ & & & & & & & B & C1 \\ & & & & & & & & F \end{pmatrix}.$$

let $E_n = \begin{pmatrix} 2n & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, by the row operation we have

$$\begin{pmatrix} E_n \\ B \end{pmatrix} = \begin{pmatrix} 2n & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 1 & -1 & & \\ & & 1 & 0 & 1 & \\ & & & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 2n & 1 & & & & \\ & -1 & 2n & & & \\ & & 2n+1 & -1 & & \\ & & & 1 & 2n+1 & \\ & & & & 2n+2 & 1 \end{pmatrix}$$

$$\text{let } B_n = \begin{pmatrix} 2n & 1 & & \\ & -1 & 2n & \\ & & 2n+1 & -1 \\ & & & 1 \end{pmatrix}, G_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2n+1 & 0 & 0 & 0 \end{pmatrix},$$

$$G' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2^{j+1}-3 & 0 & 0 \end{pmatrix}, \; F' = \begin{pmatrix} 2^{j+1}-2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$ Thus by induction, we can get,

$$A \to \begin{pmatrix} B_1 & G_1 & & & & \\ & B_2 & G_2 & & & \\ & & \ddots & & & \\ & & & B_{2^j-3} & G_{2^j-3} & \\ & & & & B_{2^j-2} & G' \\ & & & & & F' \end{pmatrix}.$$

here B_i are invertible for $i = 1, .., 2^j - 2$, and F' is invertible too, so is A. That is, all the five types of prewavelet in the $W_j^{2^j} \backslash W_j^{2^j-1}$ are linear independent for basis of $V_j^{2^j} \backslash V_j^{2^j-1}$. □

All the five types of wavelet in the $W_j^{2^j} \backslash W_j^{2^j-1}$ are linear independent for the basis of $V_j^{2^j} \backslash V_j^{2^j-1}$, we know the coefficient of wavelet in $W_j^{2^j-1}$ for the basis of $V_j^{2^j} \backslash V_j^{2^j-1}$ are all zeros, that means all the wavelets in $W_j^{2^j} = W_j$ are linear independent. We have thus established the following theorem.

Theorem 6.5.4. *All the five types of wavelet in the $W_j^{2^j} \backslash W_j^{2^j-1}$ and the wavelets in $W_j^{2^j-1}$ compose a basis of W_j. That is, under H_0^1 norm, there exists a locally supported box spline prewavelet basis over triangular domain.*

Chapter 7

The Existence of Locally Supported Prewavelet Using Linear Box Splines

In the previous Chapters, prewavelets under H_0^1 norm over two different domains were discussed. There exists a basis of locally supported prewavelets if the domain is a triangle. If the domain is a rectangle, there exists a basis of prewavelets, which are locally supported except for one basis function. Now here comes the question, does there exist a locally supported prewavelet basis when the domain is a rectangle? The following result will answer this question.

Lemma 7.0.2. *There is no locally supported prewavelet basis under H_0^1 norm over square domain if the prewavelets were constructed from linear box splines.*

Proof. Recall the triangulation and definition of linear box spline in Chapter 2. By uniform refinement 2 times from the initial triangulation of $\Omega = [0,1]^2$, we obtain the triangulation in the Figure XX1. This refinement is corresponding to level j=2 in chapter 2. It follows that the space V_2 includes 9 box splines and V_3 includes 49 box splines, and the prewavelets in W_2 will be constructed using the 49 box splines in V_3.

In Figure XX1, 25 box splines were marked, each of them represents one linear box spline. Each of the splines is corresponding to one circle with value 1 at the circle and 0 at the others, for example, the spline $\phi_{1,1}^3$ is corresponding to the circle on the low left corner, the spline $\phi_{5,5}^3$ is corresponding to the circle on the upper right corner, and so on for each of $\{\phi_{i,k}^3, i = 1,..,5, k = 1,..,5\}$. To construct prewavelet from these 25 spline, we require the prewavelet ψ satisfy the following two conditions:

$$\psi = \sum_{1 \le i,k < 5} \phi_{i,k}^3 b_{i,k}$$

ψ is orthogonal to $\phi^2_{1,1}$, $\phi^2_{1,2}$, $\phi^2_{1,3}$, $\phi^2_{2,1}$, $\phi^2_{2,2}$, $\phi^2_{2,3}$, $\phi^2_{3,1}$, $\phi^2_{3,2}$, $\phi^2_{3,3}$.

It follows that there are exactly 25-9=16 degree of freedom for ψ. Therefore there are exactly 16 prewavelets over this region. According to the construction method in Chapter 3, there are 4 boundary prewavelets, 4 type-I prewavelets, 4 type-II prewavelets and 4 type-III prewavelets. Thus there are 16 prewavelets, which means there are no other prewavelets with support in the region marked with circle in Figure XX1. By the same reason, we can construct 16 locally supported prewavelet in the region marked with circle in figure XX2 or XX3 or XX4. There are no others prewavelets in these region either.

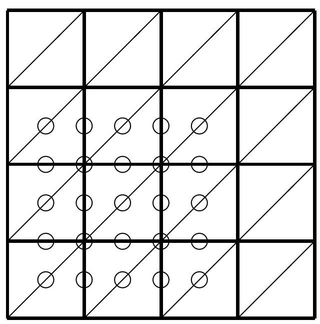

Figure XX1.

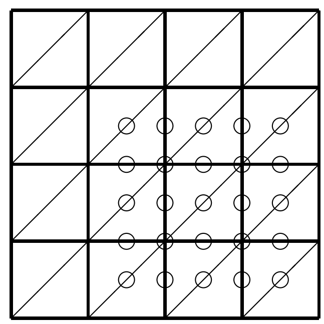

Figure XX2.

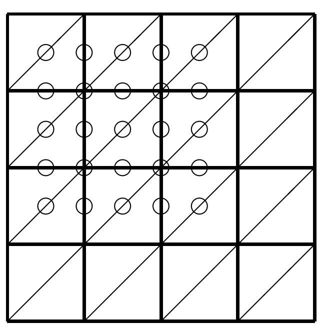

Figure XX3.

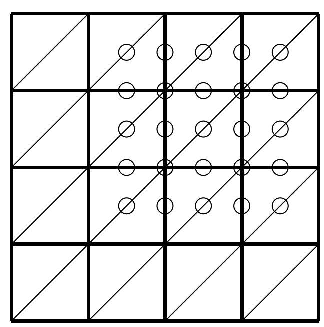

Figure XX4.

Now, recall the result in chapter 3, the dimension of the $V_3 = 7 \times 7 = 49$, and dimension of $V_2 = 3 \times 3 = 9$, and the dimension of the prewavelet space W_2 should be 49-9=40. Counting all the locally supported prewavelet functions constructed in above four Figures, there are 12 edge prewavelets, 9 type-I prewavelets, 9 type-II prewavelets and 9 type-III prewavelets. Thus there are totally 39 locally supported prewavelets. In other words, one more prewavelet basis function is needed, but this prewavelet can not be constructed in the region marked by the circle in Figures XX1, or XX2, or XX3, or even XX4 as explained above.

By the above statement, the last prewavelet must have one none zero coefficient corresponding the circle in figure XX5. Otherwise all the coefficients of the last prewavelet comes from the region marked by the circle in figure XX6, which is the subset of the region marked by the circle in figure XX1. Then the last prewavelet in the region marked by the circle in figure XX1, that is a contradiction to the result we already had.

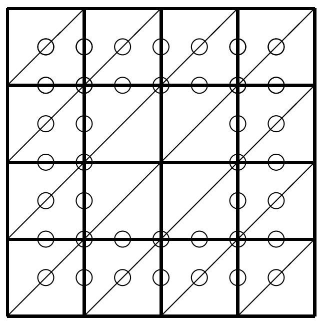

Figure XX5.

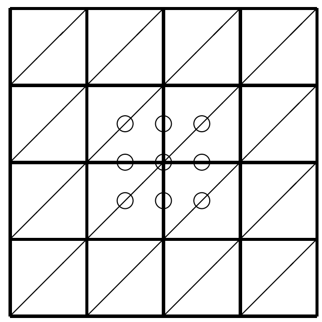

Figure XX6.

Now assume the last prewavelet has one coefficient coming from the region in the Figure XX5. We divide the study into three cases, case 1: the coefficient comes from the region in The Figure XX7; case 2: the coefficient comes from the region in The Figure XX9; case 3: the coefficient comes from the region in The Figure XX11.

Case 1: If the last prewavelet must have one coefficient coming from the region marked by the circle in Figure XX7, then there must be one coefficient coming from the region marked by the circle in figure XX8. Otherwise, this prewavelet will stay in the region shown in Figure XX3. According the above statement, it is impossible. Now one coefficient of this prewavelet comes from the region in Figure XX7, another coefficient comes from the region in Figure XX8, and the support of this prewavelet has to connect these two regions. Therefore this prewavelet is not locally supported.

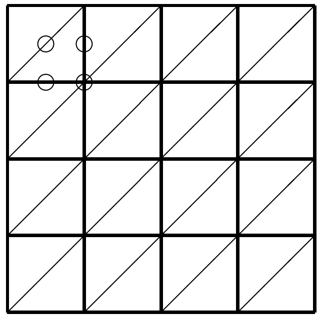

Figure XX7.

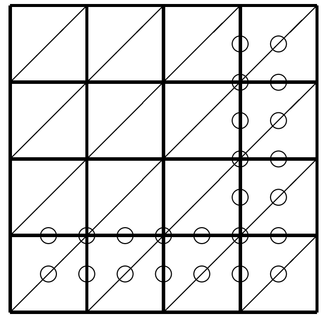

Figure XX8.

Case 2: If the last prewavelet must have one coefficient coming from the region marked by the circle in Figure XX9, then there must be one coefficient coming from the region marked by the circle in figure XX10. Otherwise, this prewavelet will stay in the region shown in Figure XX3. According the above statement, it is impossible. Now one coefficient of this prewavelet comes from the region in Figure XX9, another coefficient comes from the region in Figure XX10, and the support of this prewavelet has to connect these two regions. Therefore this prewavelet is not locally supported.

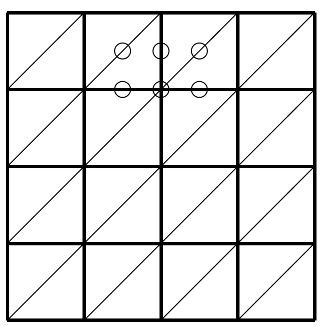

Figure XX9.

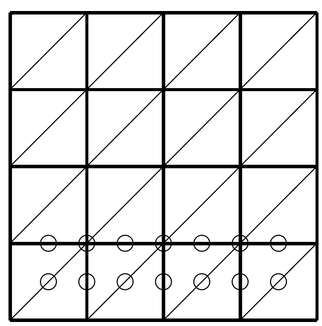

Figure XX10.

Case 3: If the last prewavelet must have one coefficient coming from the region marked by the circle in Figure XX11, then there must be one coefficient coming from the region marked by the circle in figure XX12. Otherwise, this prewavelet will stay in the region shown in Figure XX4. According the above statement, it is impossible. Now one coefficient of this prewavelet comes from the region in Figure XX11, another coefficient comes from the region in Figure XX12, and the support of this prewavelet has to connect these two regions. Therefore this prewavelet is not locally supported.

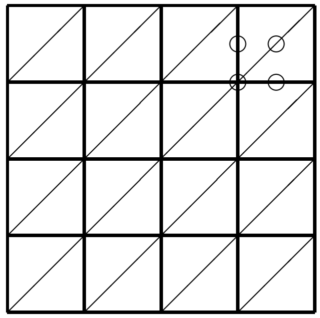

Figure XX11.

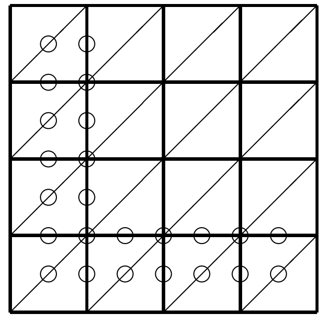

Figure XX12.

□

Combine the above three cases, it follows that the lemma is true.

Theorem 7.0.5. *There is at least one global supported prewavelet basis under H_0^1 norm over rectangle domain, if the prewavelets are constructed from linear box splines.*

On the other hand, I have proved that there exists localed supported prewavelet basis when the domain is triangle. Then the existence of locally supported prewavelet basis constructed from linear box spline under H_0^1 norm is dependent on the boundary shape of the domain.

Chapter 8

Prewavelets Solution to Poisson Equation over L-shape Domain

In this chapter, I will discuss the existence of a locally supported prewavelet basis over L-shaped domain , and explain how to construct it.

8.1 Triangulation

For L-shaped domain, there are at least two kinds of triangulations, e.g. the Figures YY1 and YY2.

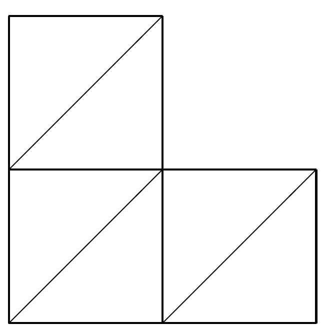

Figure YY1.

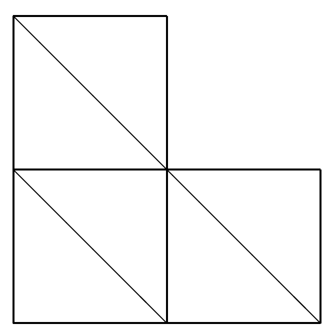

Figure YY2.

Which one should we choose? In fact, these two kinds of triangulations have no big difference for prewavelet construction. Therefore, in this chapter, I will use the first kind of triangulation.

8.2 Simplification of the Poisson Equation

Let us start with a L-shape domain $\Omega = [(0,2)\times(0,2)]\backslash[(1,2)\times(1,2)] \in R^2$. Consider the Dirichlet boundary value problem for Poisson equation:

$$\begin{cases} -\Delta u(x,y) = g(x,y), \quad (x,y)\in\Omega \\ u(x,y) = f_1(x), \quad for \quad y=0 \quad and \quad 0\le x\le 2 \\ u(x,y) = f_2(x), \quad for \quad y=1 \quad and \quad 1\le x\le 2 \\ u(x,y) = f_3(x), \quad for \quad y=2 \quad and \quad 0\le x\le 1 \\ u(x,y) = f_4(y), \quad for \quad x=0 \quad and \quad 0\le y\le 2 \\ u(x,y) = f_5(y), \quad for \quad x=1 \quad and \quad 1\le y\le 2 \\ u(x,y) = f_6(y), \quad for \quad x=2 \quad and \quad 0\le y\le 1 \end{cases}$$

Without lose of generality, we may assume that $u(x,y)$ is equal to zero at each of vertices, that means $f_1(0) = f_1(2) = f_2(1) = f_2(2) = f_3(0) = f_3(1) = f_4(0) = f_4(2) = f_5(1) = f_5(2) = f_6(0) = f_6(1) = 0$. Otherwise, letting $f_1(0) = f_4(0) = a_1$, $f_1(2) = f_6(0) = a_2$, $f_6(0) = f_2(2) = a_3$, $f_2(1) = f_5(1) = a_4$, $f_5(2) = f_3(1) = a_5$, $f_3(0) = f_4(2) = a_6$, let $b_1 = a_1$, $b_2 = (a_2 - a_1)/4$, $b_6 = (a_6 - a_1)/4$, $b_3 = a_3/2 - a_4 + a_1/2 + a_6/4 - a_2/4$, $b_5 = a_5/2 - a_4 + a_1/2 + a_2/4 - a_6/4$, $b_4 = a_4 - a_1 + (a_2 - a_1)/4 - (a_6 - a_1)/4 + b_3 + b_5$, we define $h(x,y) = b_1 + b_2x^2 + b_6y^2 + b_3x^2y + b_4xy + b_5xy^2$, and $v(x,y) = u(x,y) - h(x,y)$. Then the above Dirichlet problem becomes to:

$$\begin{cases} -\Delta u(x,y) = g(x,y), \quad (x,y)\in\Omega \\ u(x,y) = f_1(x) - h(x,0), \quad for \quad y=0 \quad and \quad 0\le x\le 2 \\ u(x,y) = f_2(x) - h(x,1), \quad for \quad y=1 \quad and \quad 1\le x\le 2 \\ u(x,y) = f_3(x) - h(x,2), \quad for \quad y=2 \quad and \quad 0\le x\le 1 \\ u(x,y) = f_4(y) - h(0,y), \quad for \quad x=0 \quad and \quad 0\le y\le 2 \\ u(x,y) = f_5(y) - h(1,y), \quad for \quad x=1 \quad and \quad 1\le y\le 2 \\ u(x,y) = f_6(y) - h(2,y), \quad for \quad x=2 \quad and \quad 0\le y\le 1 \end{cases}$$

which satisfy the above assumption.

Now let $w(x) = v(x,y) - (1-3y/2+y^2/2)(f_1(x) - h(x,0)) - (1-3x/2+x^2/2)(f_4(y) - h(0,y)) - (y^2/2 - y/2)(f_3(x) - h(x,2)) - (x^2/2 - x/2)(f_6(y) - h(2,y)) - (2y - y^2)(f_2(x) - h(x,1)) - (2x - x^2)(f_5(x) - h(x,1))$. Then $w(x)$ satisfies the equation

$$\begin{cases} -\Delta w(x,y) = g_1(x,y), & (x,y) \in \Omega \\ w(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}$$

with $g_1(x,y) = g(x,y) + \frac{\partial^2}{\partial y^2}[-x(f_4(y) - h(1,y)) - (1-x)(f_3(y) - h(0,y))] + \frac{\partial^2}{\partial x^2}[-y(f_2(x) - h(x,1)) - (1-y)(f_1(x) - h(x,0))]$.

If we can find solution for w, it is easy to get $u(x,y)$. In the remaining dissertation, we only consider the Poisson equation with zero boundary condition:

$$\begin{cases} -\Delta u(x,y) = g(x,y), & (x,y) \in \Omega \\ u(x,y) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{8.1}$$

8.3 the Non-Existence of Locally Supported Prewavelet Base

Lemma 8.3.1. *There is no locally supported prewavelet basis under H_0^1 norm over L-shape domain if the prewavelets were constructed from linear box spline.*

Proof. Here the Figure YY1 was zoomed in to refinement level j=3, we get the Figure YY3. In this region, there should be 161 elements in space V_4, and the prewavelets should be construct from these 161 linear box spline which compose the space V_4, and the prewavelets should be orthogonal to the 33 linear box splines in space V_3, it follows there should be 161-33=128 prewavelets in this region.

In Figure YY4, 65 box splines were marked, each of them represents one linear box spline. Each of the splines is corresponding to one circle with value 1 at the circle and 0 at the others, for example, $\phi^4_{1,1}$ is corresponding to the circle on the low left corner. The prewavelet constructed from this 65 element should be orthonormal to 24 upper level splines, so there should be 65-24=41 prewavelet in this region. Using the method shown in Chapter 3, there are 36 inner locally supported prewavelets and 5 edge prewavelets, it follows there are exactly 36+5=41 prewavelets. Then there is no other prewavelets in this region. By the same reason, after locally supported prewavelet construction using the method in Chapter 3, there is no other prewavelets in region shown in Figures YY5, YY6, YY7, YY8 or YY9.

Counting all prewavelets shown in YY4 to YY9, there are exactly 125 prewavelets, there must be three more prewavelets to match the dimension of W_3, 128, for L-shape and the last three prewavelet can not construct only in the region shown in figure YY4, YY5, YY6, YY7, YY8 or YY9.

Since the last three prewavelet can not constructed in the region shown in YY6, the last three prewavelets must have one none zero coefficient coming from the complement of the region shown in Figure YY6, so the last three prewavelets must have one none zero coefficient corresponding the circles in Figure YY10, which is the complement of the region shown in Figure YY6. In order to show one of the last three prewavelet can not be locally supported, the study will be divided into 13 cases.

Case 1: One coefficient comes from the region shown in Figure YY11. Since the prewavelet can not be constructed in the region shown in Figure YY4, so there must be one coefficient coming from the region shown in Figure YY12, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 2: One coefficient comes from the region shown in Figure YY13. Since the prewavelet can not be constructed in the region shown in Figure YY7, so there must be one coefficient coming from the region shown in Figure YY14, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 3: One coefficient comes from the region shown in Figure YY15. Since the prewavelet can not be constructed in the region shown in Figure YY9, so there must be one coefficient coming from the region shown in Figure YY16, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 4: One coefficient comes from the region shown in Figure YY17. Since the prewavelet can not be constructed in the region shown in Figure YY8, so there must be one coefficient coming from the region shown in Figure YY18, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 5: One coefficient comes from the region shown in Figure YY19. Since the prewavelet can not be constructed in the region shown in Figure YY5, so there must be one coefficient coming from the region shown in Figure YY20, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 6: One coefficient comes from the region shown in Figure YY21. Since the prewavelet can not be constructed in the region shown in Figure YY4, so there must be one coefficient coming from the region shown in Figure YY12, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 7: One coefficient comes from the region shown in Figure YY22. Since the prewavelet can not be constructed in the region shown in Figure YY7, so there must be one coefficient coming from the region shown in Figure YY14, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 8: One coefficient comes from the region shown in Figure YY23. Since the prewavelet can not be constructed in the region shown in Figure YY7, so there must be one coefficient coming from the region shown in Figure YY14, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 9: One coefficient comes from the region shown in Figure YY24. Since the prewavelet can not be constructed in the region shown in Figure YY9, so there must be one coefficient coming from the region shown in Figure YY16, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 10: One coefficient comes from the region shown in Figure YY25. Since the prewavelet can not be constructed in the region shown in Figure YY8, so there must be one coefficient coming from the region shown in Figure YY18, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 11: One coefficient comes from the region shown in Figure YY26. Since the prewavelet can not be constructed in the region shown in Figure YY9, so there must be one coefficient coming from the region shown in Figure YY16, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 12: One coefficient comes from the region shown in Figure YY27. Since the prewavelet can not be constructed in the region shown in Figure YY5, so there must be one coefficient coming from the region shown in Figure YY20, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

Case 13: One coefficient comes from the region shown in Figure YY28. Since the prewavelet can not be constructed in the region shown in Figure YY5, so there must be one coefficient coming from the region shown in Figure YY20, and the support of this prewavelet has to connected these two region. Therefore this prewavelet is not locally supported.

combine all above cases, it follows that the lemma is true.

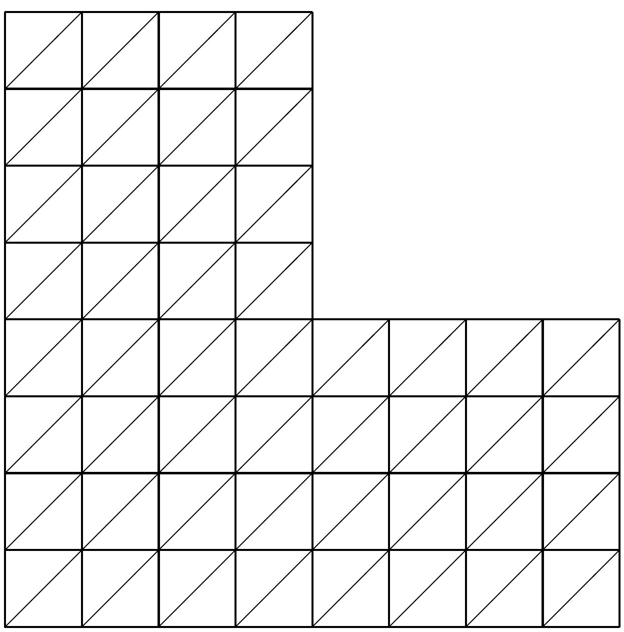

Figure YY3.

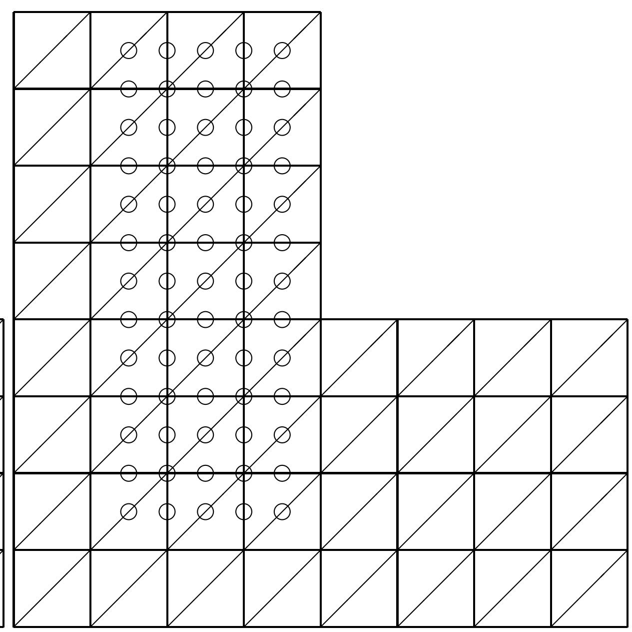

Figure YY4.

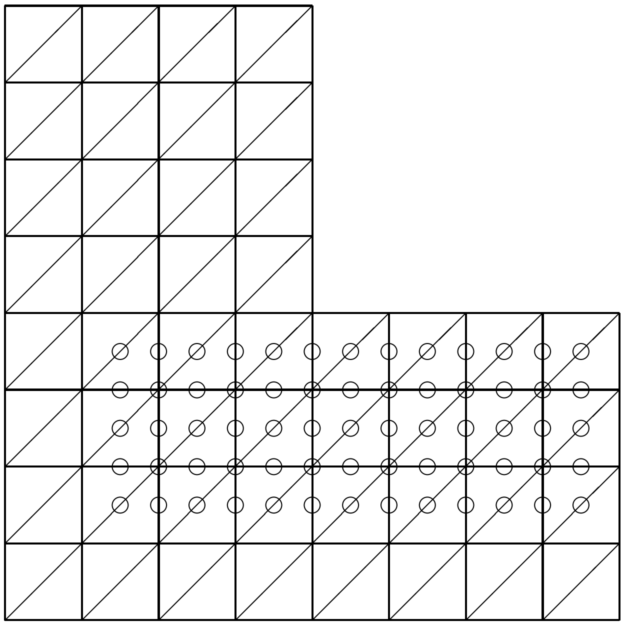

Figure YY5.

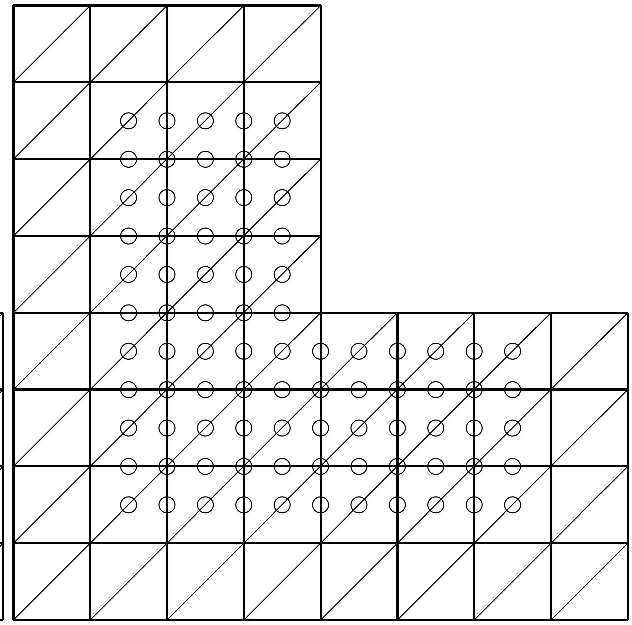

Figure YY6.

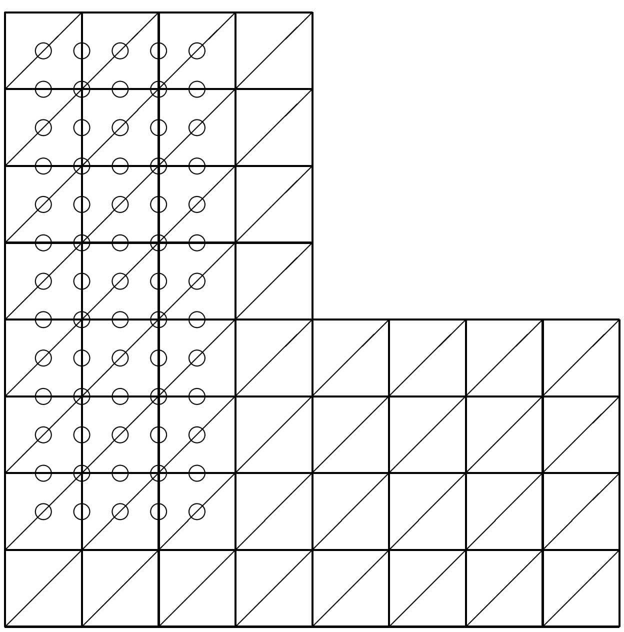

Figure YY7.

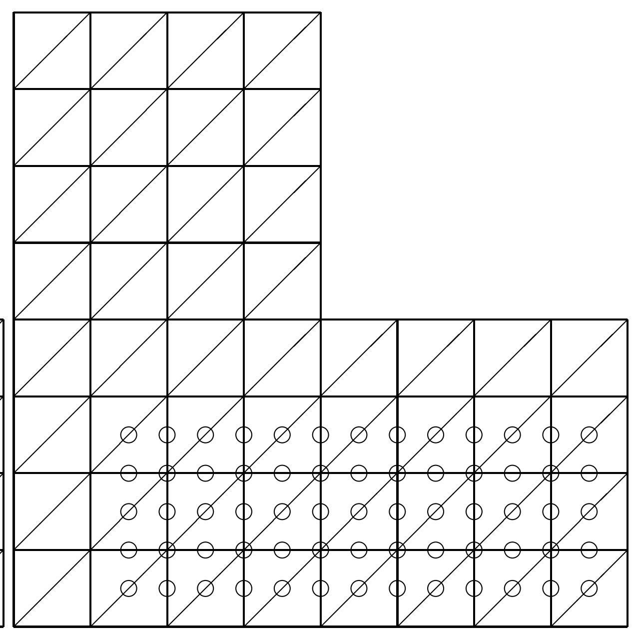

Figure YY8.

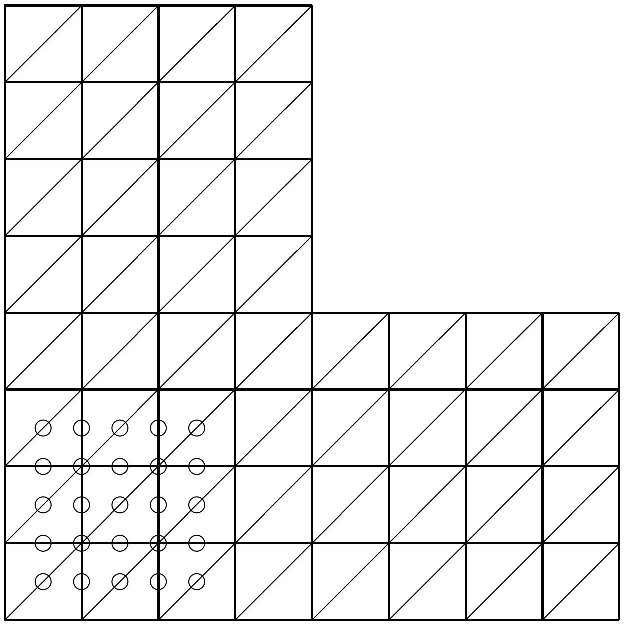

Figure YY9.

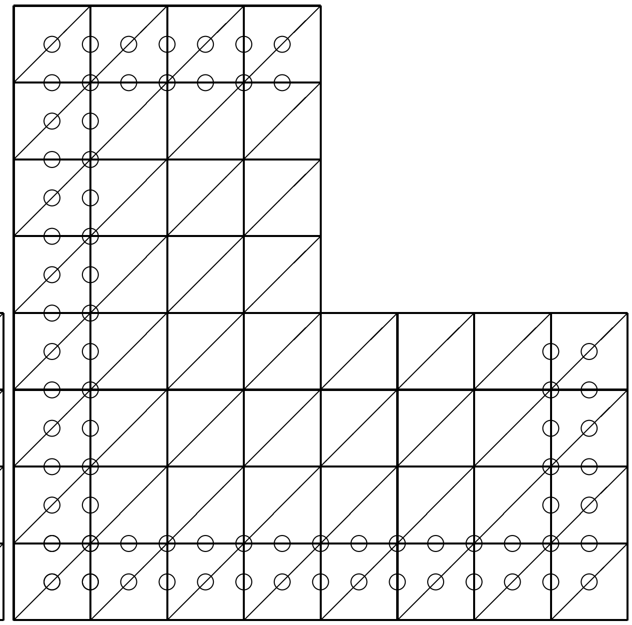

Figure YY10.

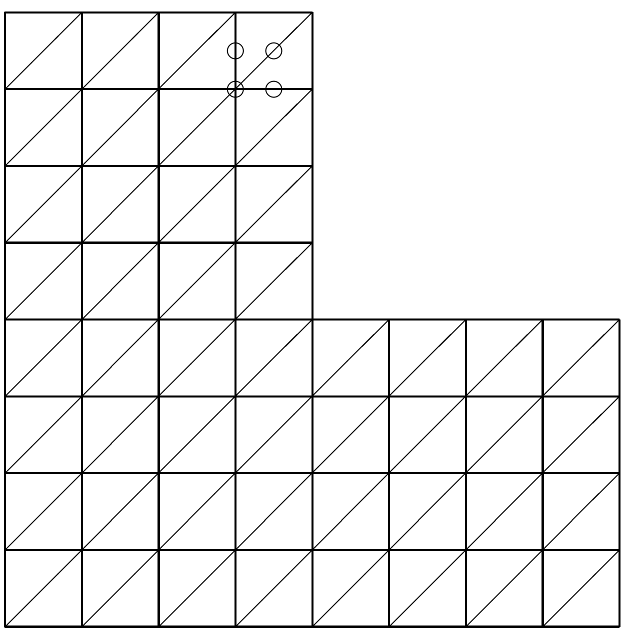

Figure YY11.

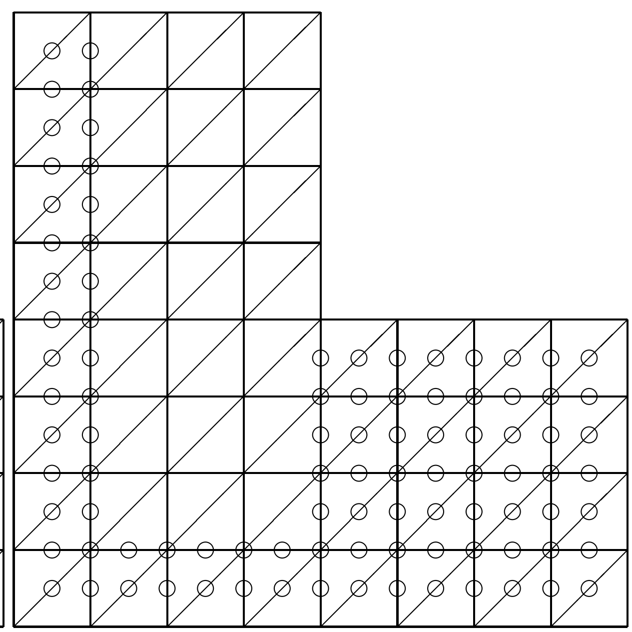

Figure YY12.

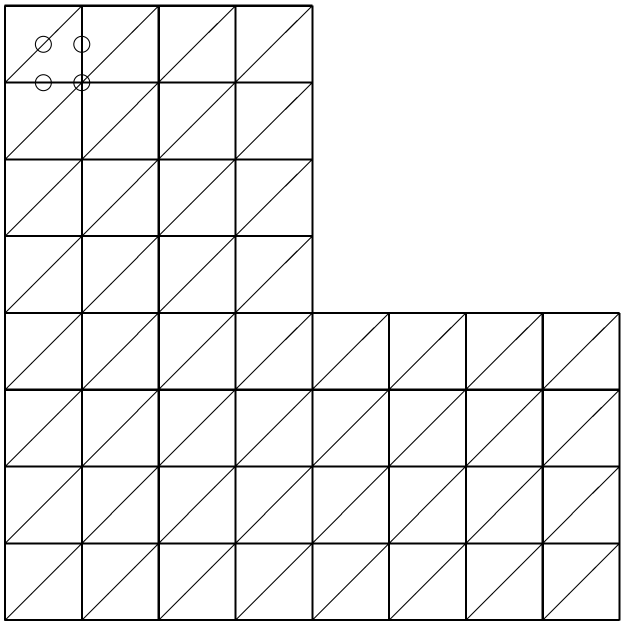

Figure YY13.

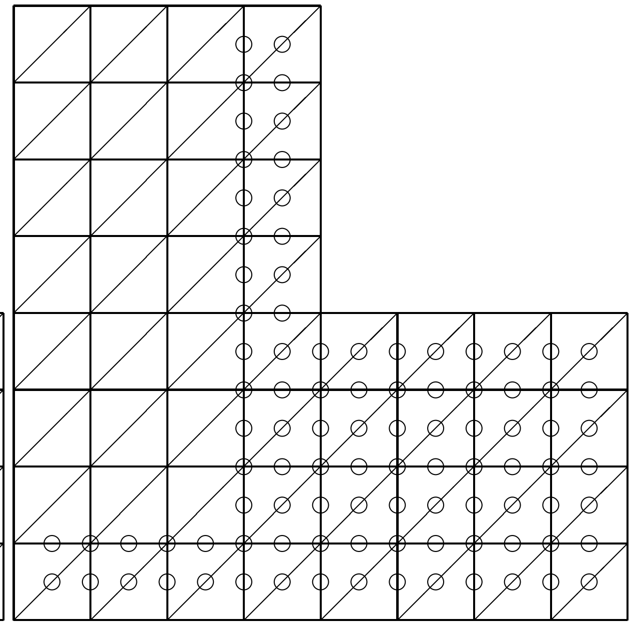

Figure YY14.

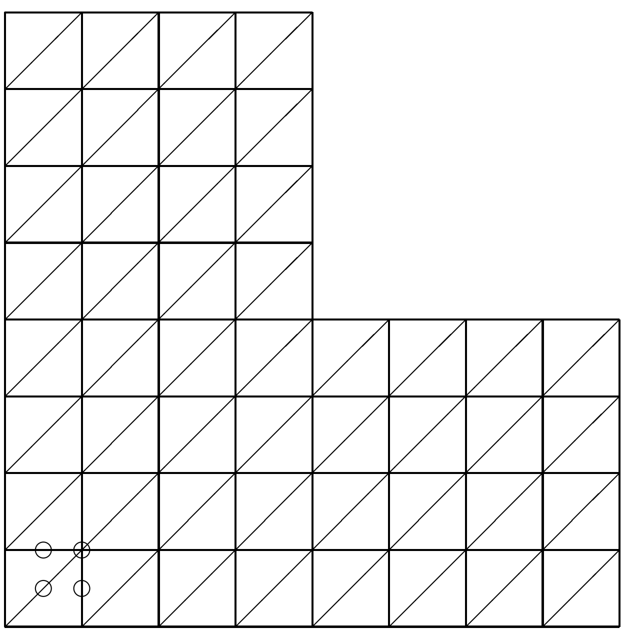

Figure YY15.

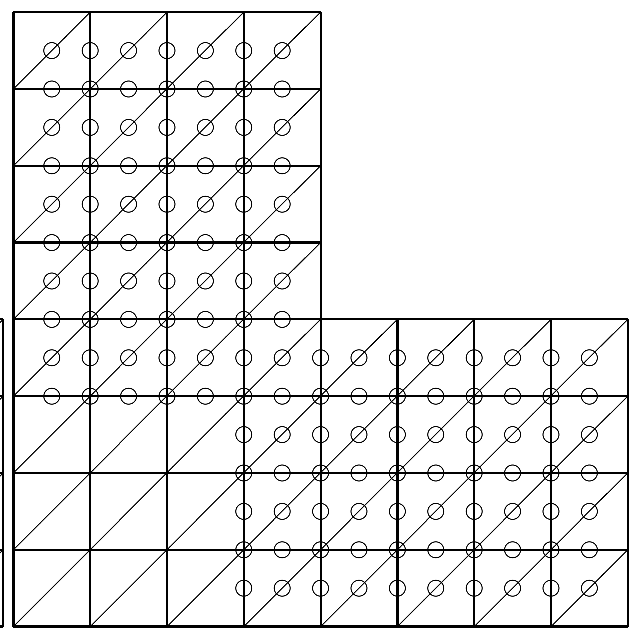

Figure YY16.

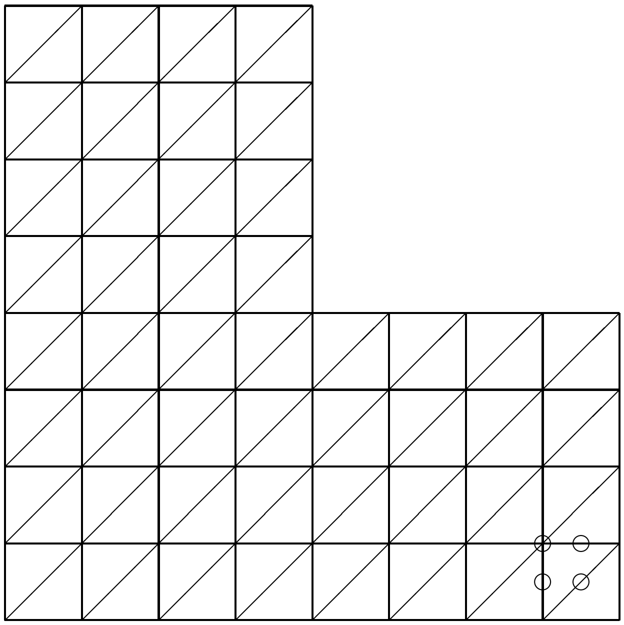

Figure YY17.

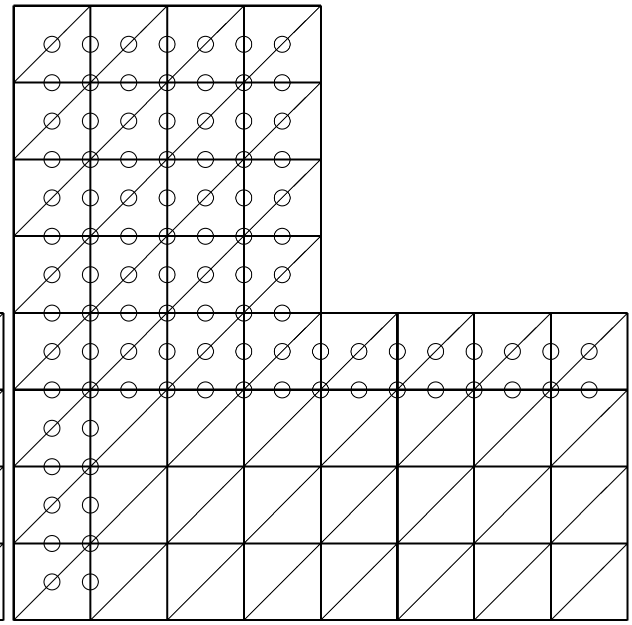

Figure YY18.

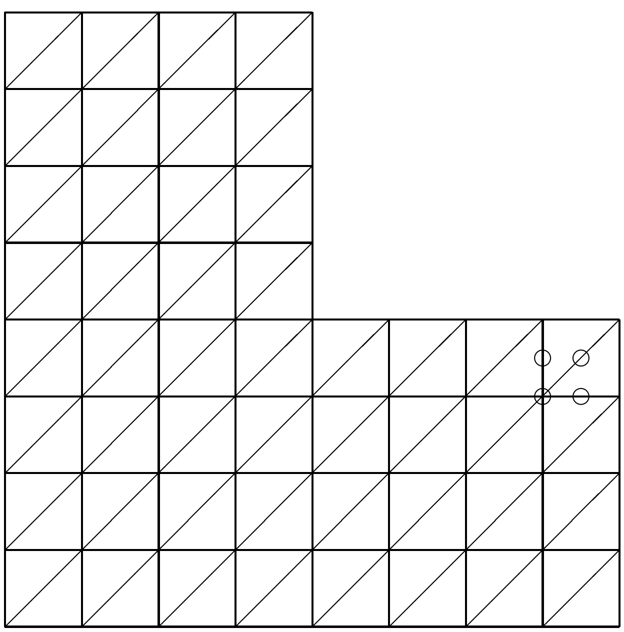

Figure YY19.

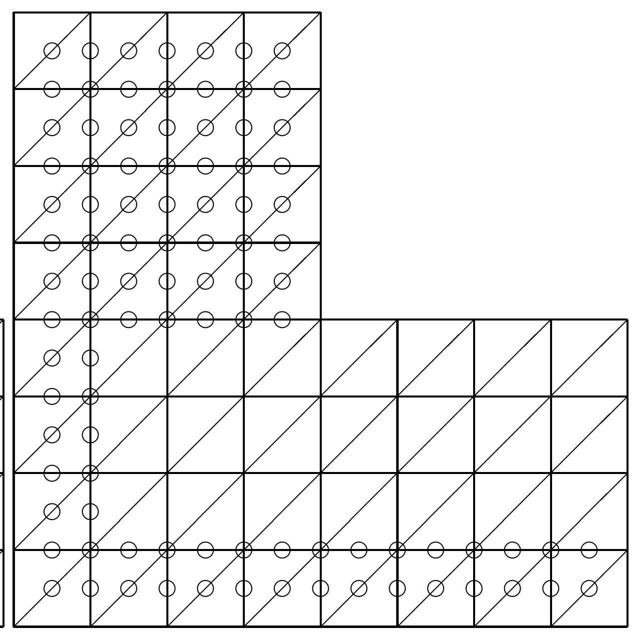

Figure YY20.

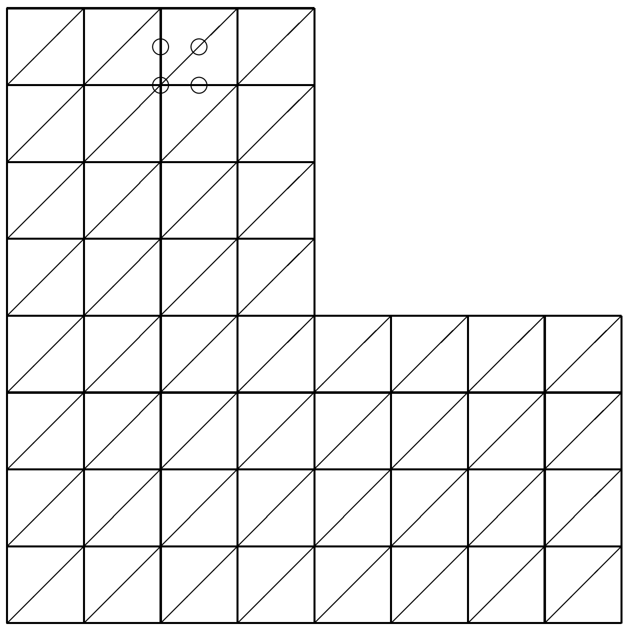

Figure YY21.

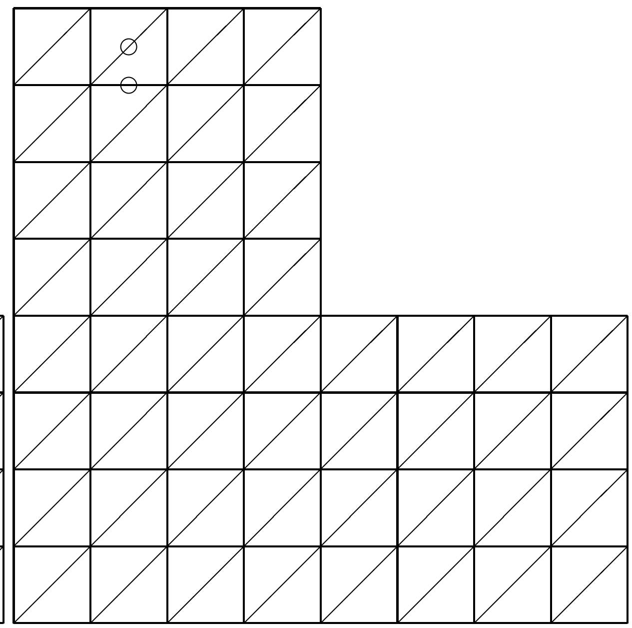

Figure YY22.

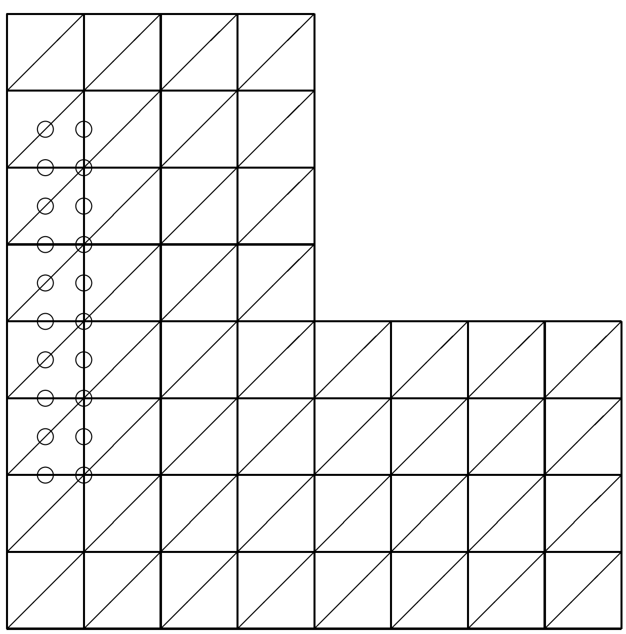

Figure YY23.

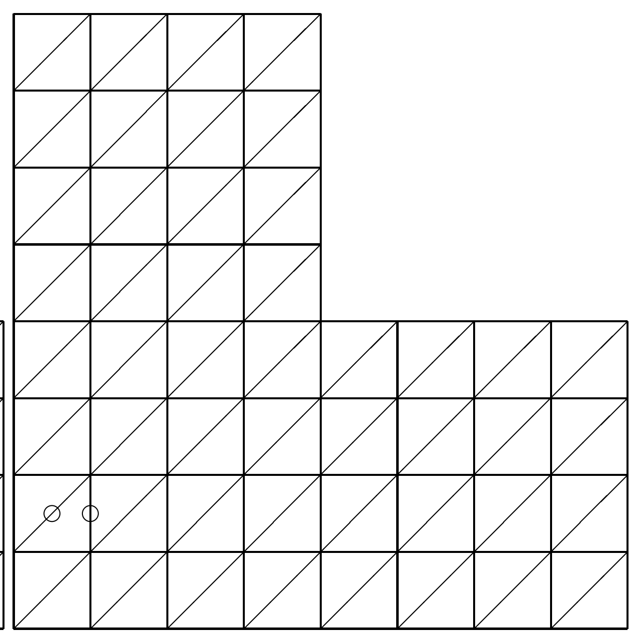

Figure YY24.

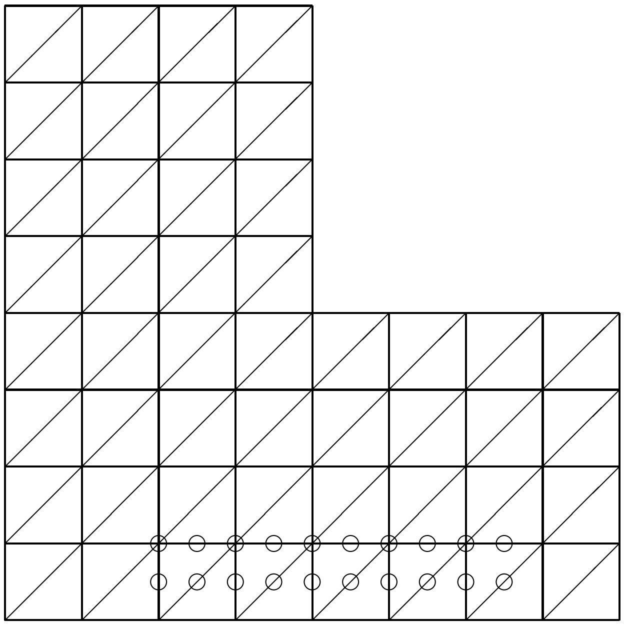

Figure YY25.

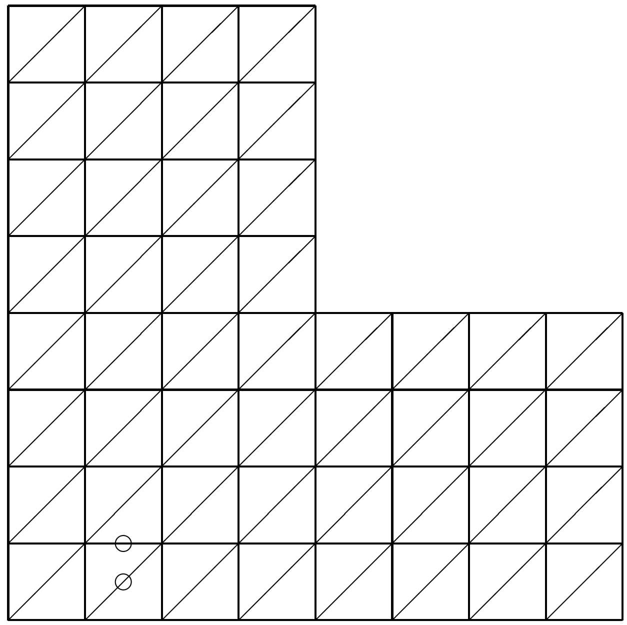

Figure YY26.

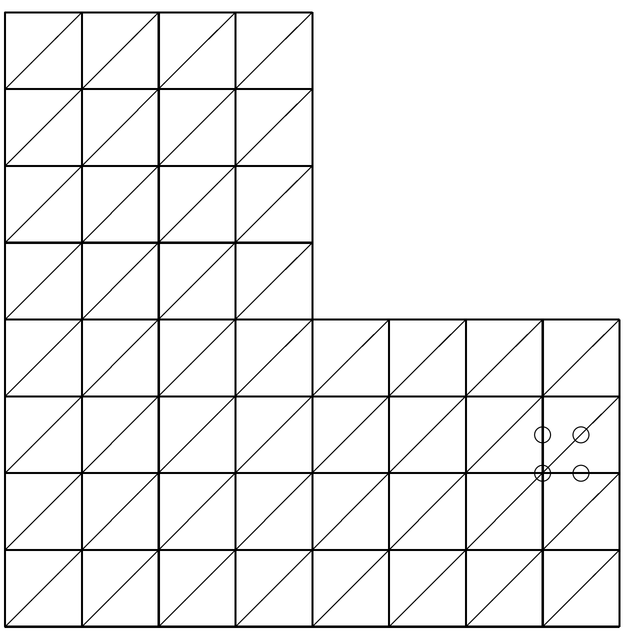

Figure YY27.

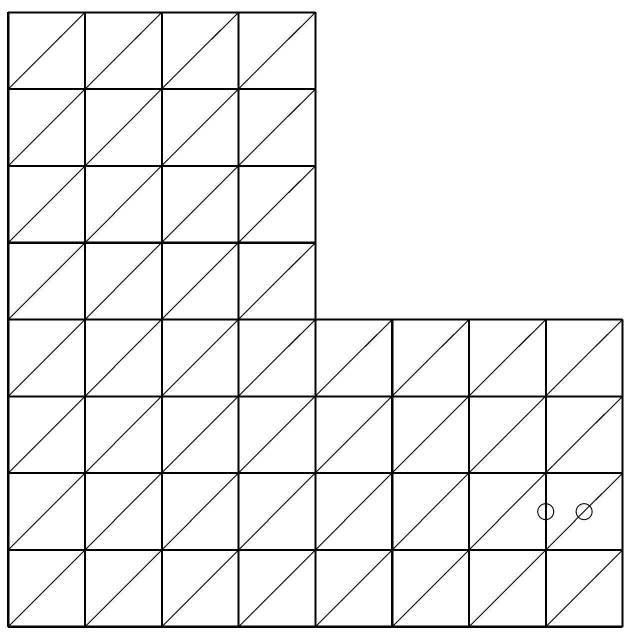

Figure YY28.

□

8.4 A Construction Method

Like the chapter 3, there 3 kind of locally supported inner prewavelet. But there are four kinds of locally supported edge prewavelets. The following are example of inner prewavelets

$$\psi_{1,1}^{j,3} = -\phi_{2,2}^{j+1} + \phi_{3,2}^{j+1} + \phi_{2,3}^{j+1} + \phi_{3,3}^{j+1} \quad \text{as shown in Figure YY29;}$$

$$\psi_{1,1}^{j,4} = \phi_{1,1}^{j+1} + \phi_{2,1}^{j+1} + \phi_{1,2}^{j+1} - \phi_{2,2}^{j+1} \quad \text{as shown in Figure YY30;}$$

$$\psi_{1,1}^{j,5} = \phi_{1,2}^{j+1} + \phi_{2,3}^{j+1} - \phi_{2,1}^{j+1} - \phi_{3,2}^{j+1} \quad \text{as shown in Figure YY31.}$$

The following are four example of edge prewavelets

$$\psi_{1,2}^{j,1} = 2\phi_{1,4}^{j+1} + \phi_{1,5}^{j+1} \quad \text{as shown in Figure YY32;}$$

$$\psi_{2,1}^{j,2} = 2\phi_{4,1}^{j+1} + \phi_{5,1}^{j+1} \quad \text{as shown in Figure YY33;}$$

$$\psi_{3,6}^{j,6} = 2\phi_{7,12}^{j+1} + \phi_{7,11}^{j+1} \quad \text{as shown in Figure YY34.}$$

$$\psi_{6,3}^{j,7} = 2\phi_{12,7}^{j+1} + \phi_{11,7}^{j+1} \quad \text{as shown in Figure YY35.}$$

The following are three global supported prewavelets when the refinement level is 3.

$$\psi^{j,8} = \phi_{7,15}^{j+1} + \phi_{7,13}^{j+1} + \phi_{7,11}^{j+1} + \phi_{7,9}^{j+1} + \phi_{7,7}^{j+1} + 2\phi_{8,7}^{j+1}$$

as shown in Figure YY36;

$$\psi^{j,9} = \phi_{15,7}^{j+1} + \phi_{13,7}^{j+1} + \phi_{11,7}^{j+1} + \phi_{9,7}^{j+1} + \phi_{7,7}^{j+1} + 2\phi_{7,8}^{j+1}$$

as shown in Figure YY37;

$$\psi^{j,10} = \phi_{1,1}^{j+1} + \phi_{1,3}^{j+1} + \phi_{1,5}^{j+1} + \phi_{1,7}^{j+1} + \phi_{1,9}^{j+1} + \phi_{1,11}^{j+1} + \phi_{1,13}^{j+1} + \phi_{1,15}^{j+1}$$

as shown in Figure YY38.

Recall the definition of V_{j+1}^m and W_j^m in chapter 3, for the L-shape, I will give the definition again in a different way.

Definition 8.4.1. *Let* $V_{j+1}^m = A_{j+1}^m \cup B_{j+1}^m$*, with* $A_{j+1}^m = span\{\phi_{ik}^{j+1}, i = 2^j - 1 - 2m, ..., 2^j - 1, k = 2^1 - 1 - 2m, ..., 2^{j+1} - 1\}$*,* $B_{j+1}^m = span\{\phi_{ik}^{j+1}, k = 2^j - 1 - 2m, ..., 2^j - 1, i = 2^1 - 1 - 2m, ..., 2^{j+1} - 1\}$*, Let* W_j^m *be subspace of* W_j *such that* $W_j^m = W_j \bigcap V_{j+1}^m$*.*

By the above definition, V_{j+1}^m is a subspace of V_{j+1}, W_j^m is a subspace of W_j.

Lemma 8.4.1. *All the three types of locally supported inner wavelets and four types of locally supported edge wavelets and the two global supported prewavelets in the* V_{j+1}^1 *are linear independent. That is, the following functions*

$$
\begin{aligned}
&\psi_{2^{j-1}-1,k}^{j,3}, && k = 2^{j-1} - 1, .., 2^j - 1,\\
&\psi_{k,2^{j-1}-1}^{j,3}, && k = 2^{j-1} - 1, .., 2^j - 1,\\
&\psi_{k,2^{j-1}-1}^{j,4}, && k = 2^{j-1} - 1, .., 2^j - 1,\\
&\psi_{2^{j-1}-1,k}^{j,4}, && k = 2^{j-1} - 1, .., 2^j - 1,\\
&\psi_{2^{j-1}-1,k}^{j,5}, && k = 2^{j-1} - 1, .., 2^j - 1,\\
&\psi_{k,2^{j-1}-1}^{j,5}, && k = 2^{j-1} - 1, .., 2^j - 1,\\
&\psi_{k,2^j-1}^{j,1}, && k = 2^{j-1} - 1\\
&\psi_{2^j-1,k}^{j,2}, && k = 2^{j-1} - 1\\
&\psi_{k,2^{j-1}-1}^{j,6}, && k = 2^{j-1} + 1, .., 2^j - 1,\\
&\psi_{2^{j-1}-1,k}^{j,7}, && k = 2^{j-1} + 1, .., 2^j - 1,\\
&\psi^{j,8},\\
&\psi^{j,9}
\end{aligned}
$$

are linear independent.

Proof. Suppose they are not linear independent. That is, one can find

$$
\begin{aligned}
&a^1, \\
&a^2, \\
&a^3_{i,2^{j-1}-1}, \quad 2^{j-1}-1 < i \le 2^j - 1; \\
&a^3_{2^{j-1}-1,i}, \quad 2^{j-1}-1 \le i \le 2^j - 1; \\
&a^4_{i,2^{j-1}-1}, \quad 2^{j-1}-1 < i \le 2^j - 1; \\
&a^4_{2^{j-1}-1,i}, \quad 2^{j-1}-1 \le i \le 2^j - 1; \\
&a^5_{i,2^{j-1}-1}, \quad 2^{j-1}-1 < i \le 2^j - 1; \\
&a^5_{2^{j-1}-1,i}, \quad 2^{j-1}-1 \le i \le 2^j - 1; \\
&a^6_{k,2^{j-1}-1}, \quad 2^{j-1}+1 \le k \le 2^j - 1, \\
&a^7_{2^{j-1}-1,k}, \quad 2^{j-1}+1 \le k \le 2^j - 1, \\
&a^8 \\
&a^9
\end{aligned}
$$

such that $a^1\psi^{j,1}_{2^{j-1}-1,2^j-1} + a^2\psi^{j,2}_{2^j-1,2^{j-1}-1} + \sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{ik} a^4_{i,k}\psi^{j,4}_{i,k} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} + \sum_{ik} a^6_{i,k}\psi^{j,6}_{i,k} + \sum_{ik} a^7_{i,k}\psi^{j,7}_{i,k} + a^8\psi^{j,8}_{i,k} + a^9\psi^{j,9}_{i,k} = 0$,

By the definition, $\phi^{j+1}_{2^j-3,2k-1}$ and $\phi^{j+1}_{2k-1,2^j-3}$ appear only once in $\psi^{j,4}_{i,k}$ and $\psi^{j,1}_{i,k}$ and $\psi^{j,2}_{i,k}$. Since ϕ^{j+1} are linear independent, that is, $a^4_{i,k}$, $a^1 = 0$, and $a^2 = 0$. Thus the above equation can be simplified to

$$\sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} + \sum_{ik} a^6_{i,k}\psi^{j,6}_{i,k} + \sum_{ik} a^7_{i,k}\psi^{j,7}_{i,k} + a^8\psi^{j,8}_{i,k} + a^9\psi^{j,9}_{i,k} = 0, \tag{8.2}$$

By the similar reason, $\phi^{j+1}_{2^j-3,2k}$ and $\phi^{j+1}_{2k,2^j-3}$ appear only once in $\psi^{j,5}_{i,k}$. Since ϕ^{j+1}_{ik} are linear independent, $a^5_{i,k} = 0$. Thus the equation (3.8) can be further simplified to the following equation

$$\sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{ik} a^6_{i,k}\psi^{j,6}_{i,k} + \sum_{ik} a^7_{i,k}\psi^{j,7}_{i,k} + a^8\psi^{j,8}_{i,k} + a^9\psi^{j,9}_{i,k} = 0, \tag{8.3}$$

Keep going this way, all the coefficient should be zeros. □

Theorem 8.4.1. *All the five types of wavelet in the V_j^n are linear independent for $1 \le n \le 2^j - 2$. That means for $i > 0$ and $k > 0$ $\psi^{j,1}, \psi^{j,2}, \psi^{j,3}, \psi^{j,4}, \psi^{j,5}, \psi^{j,6}, \psi^{j,7}, \psi^{j,8}$ in V_j^n are linear independent.*

Proof. Let us prove it by induction, it is true for n=1 ,Suppose it is true for n=p, that means

$$\begin{aligned}
&\psi^{j,3}_{i,k}, && 2^{j-1} - p < i, k \le 2^j - 1,\\
&\psi^{j,3}_{k,i}, && 2^{j-1} - p < i, k \le 2^j - 1,\\
&\psi^{j,4}_{k,i}, && 2^{j-1} - p < i, k \le 2^j - 1,\\
&\psi^{j,4}_{i,k}, && 2^{j-1} - p < i, k \le 2^j - 1,\\
&\psi^{j,5}_{i,k}, && 2^{j-1} - p < i, k \le 2^j - 1,\\
&\psi^{j,5}_{k,i}, && 2^{j-1} - p < i, k \le 2^j - 1,\\
&\psi^{j,1}_{k,2^j-1}, && 2^{j-1} - p < k \le 2^j - 1,\\
&\psi^{j,2}_{2^j-1,k}, && 2^{j-1} - p < k \le 2^j - 1,\\
&\psi^{j,6}_{k,2^{j-1}-1}, && k = 2^{j-1} + 1, .., 2^j - 1,\\
&\psi^{j,7}_{2^{j-1}-1,k}, && k = 2^{j-1} + 1, .., 2^j - 1,\\
&\psi^{j,8},\\
&\psi^{j,9}
\end{aligned}$$

are linear independent. For n=p+1, there $3(2^j + 2n + 1) + 2$ new element, they are

$$\begin{aligned}
&\psi^{j,3}_{2^{j-1}-n,k}, && k = 2^{j-1} - n, .., 2^j - 1,\\
&\psi^{j,3}_{k,2^{j-n}-1}, && k = 2^{j-1} - n, .., 2^j - 1,\\
&\psi^{j,4}_{k,2^{j-n}-1}, && k = 2^{j-1} - n, .., 2^j - 1,\\
&\psi^{j,4}_{2^{j-1}-n,k}, && k = 2^{j-1} - n, .., 2^j - 1,\\
&\psi^{j,5}_{2^{j-1}-n,k}, && k = 2^{j-1} - n, .., 2^j - 1,\\
&\psi^{j,5}_{k,2^{j-1}-n}, && k = 2^{j-1} - n, .., 2^j - 1,\\
&\psi^{j,1}_{k,2^j-1}, && k = 2^{j-1} - 1\\
&\psi^{j,2}_{2^j-1,k}, && k = 2^{j-1} - 1
\end{aligned}$$

Suppose they are not linear independent, so I can find

$$
\begin{aligned}
&a^3_{2^{j-1}-n,k}, \quad k = 2^{j-1}-n, .., 2^j-1,\\
&a^3_{k,2^{j-n}-1}, \quad k = 2^{j-1}-n, .., 2^j-1,\\
&a^4_{k,2^{j-n}-1}, \quad k = 2^{j-1}-n, .., 2^j-1,\\
&a^4_{2^{j-1}-n,k}, \quad k = 2^{j-1}-n, .., 2^j-1,\\
&a^5_{2^{j-1}-n,k}, \quad k = 2^{j-1}-n, .., 2^j-1,\\
&a^5_{k,2^{j-1}-n}, \quad k = 2^{j-1}-n, .., 2^j-1,\\
&a^1\\
&a^2
\end{aligned}
$$

such that $aa^1\psi^{j,1}_{2^{j-1}-1,2^j-1} + a^2\psi^{j,2}_{2^j-1,2^{j-1}-1} + \sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{ik} a^4_{i,k}\psi^{j,4}_{i,k} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} + \sum_{ik} a^6_{i,k}\psi^{j,6}_{i,k} + \sum_{ik} a^7_{i,k}\psi^{j,7}_{i,k} + a^8\psi^{j,8}_{i,k} + a^9\psi^{j,9}_{i,k} + \psi' = 0$ Where ψ' is linear combination of the following elements.

$$
\begin{aligned}
&\psi^{j,3}_{i,k}, \quad 2^{j-1}-n < i,k \leq 2^j-1,\\
&\psi^{j,3}_{k,i}, \quad 2^{j-1}-n < i,k \leq 2^j-1,\\
&\psi^{j,4}_{k,i}, \quad 2^{j-1}-n < i,k \leq 2^j-1,\\
&\psi^{j,4}_{i,k}, \quad 2^{j-1}-n < i,k \leq 2^j-1,\\
&\psi^{j,5}_{i,k}, \quad 2^{j-1}-n < i,k \leq 2^j-1,\\
&\psi^{j,5}_{k,i}, \quad 2^{j-1}-n < i,k \leq 2^j-1,\\
&\psi^{j,1}_{k,2^j-1}, \quad 2^{j-1}-n < k \leq 2^j-1,\\
&\psi^{j,2}_{2^j-1,k}, \quad 2^{j-1}-n < k \leq 2^j-1,\\
&\psi^{j,6}_{k,2^{j-1}-1}, \quad k = 2^{j-1}+1, .., 2^j-1,\\
&\psi^{j,7}_{2^{j-1}-1,k}, \quad k = 2^{j-1}+1, .., 2^j-1,\\
&\psi^{j,8},\\
&\psi^{j,9}
\end{aligned}
$$

By the definition, $\phi^{j+1}_{2^j-n-2,2k-1}$ and $\phi^{j+1}_{2k-1,2^j-n-2}$ appear only once in $\psi^{j,4}_{i,k}$ and $\psi^{j,1}_{i,k}$ and $\psi^{j,2}_{i,k}$. Since ϕ^{j+1} are linear independent, that is, $a^4_{i,k}$, $a^1 = 0$, and $a^2 = 0$. Therefor the above equation can be simplified to following expression:

Thus the above equation can be simplified to

$$\sum_{ik} a_{i,k}^3 \psi_{i,k}^{j,3} + \sum_{ik} a_{i,k}^5 \psi_{i,k}^{j,5} + \psi^{'} = 0, \tag{8.4}$$

By the same reason, all the coefficient should be zeros. Therefor the above equation can be simplified to following expression:

$$\psi^{'} = 0$$

By the induction, all the coefficient of $\psi^{'} = 0$ are zeroes. That means

$$\begin{array}{ll}
\psi_{i,k}^{j,3}, & 2^{j-1} - n < i, k \leq 2^j - 1, \\
\psi_{k,i}^{j,3}, & 2^{j-1} - n < i, k \leq 2^j - 1, \\
\psi_{k,i}^{j,4}, & 2^{j-1} - n < i, k \leq 2^j - 1, \\
\psi_{i,k}^{j,4}, & 2^{j-1} - n < i, k \leq 2^j - 1, \\
\psi_{i,k}^{j,5}, & 2^{j-1} - n < i, k \leq 2^j - 1, \\
\psi_{k,i}^{j,5}, & 2^{j-1} - n < i, k \leq 2^j - 1, \\
\psi_{k,2^j-1}^{j,1}, & 2^{j-1} - n < k \leq 2^j - 1, \\
\psi_{2^j-1,k}^{j,2}, & 2^{j-1} - n < k \leq 2^j - 1, \\
\psi_{k,2^{j-1}-1}^{j,6}, & k = 2^{j-1} + 1, .., 2^j - 1, \\
\psi_{2^{j-1}-1,k}^{j,7}, & k = 2^{j-1} + 1, .., 2^j - 1, \\
\psi^{j,8}, & \\
\psi^{j,9} &
\end{array}$$

are linear independent for V_j^n. □

Theorem 8.4.2. *All the five types of wavelets in the W_j^n form a basis of W_j^n for $1 \leq n \leq 2^j - 2$.*

Proof. By counting, the dimension all the prewavelets in W_j^n for $1 \leq n \leq 2^j - 2$ is right the dimension of W_j^n. □

Theorem 8.4.3. *All the wavelets in the $W_j^{2^j-1} \backslash W_j^{2^j-2}$ are linear independent and form a basis for $V_{j+1}^{2^j-1} \backslash V_{j+1}^{2^j-2}$ which is spanned by the functions in $\{\phi_{i,k}^{j+1}, 1 \leq i \leq 2 \text{ or } 1 \leq k \leq 2\}$.*

Proof. the proof is same ideal appeared in Chapter 3. □

Theorem 8.4.4. *All the prewavelet functions in the $W_j^{2^j-1}$ form a basis for W_j.*

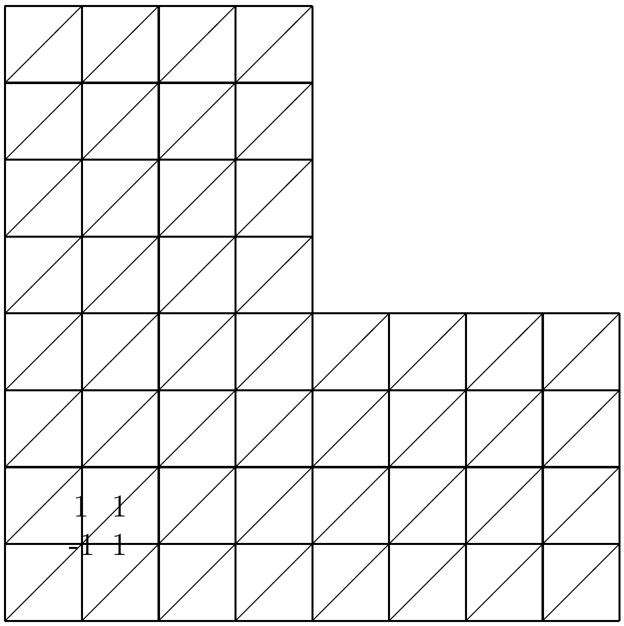

Figure YY29.

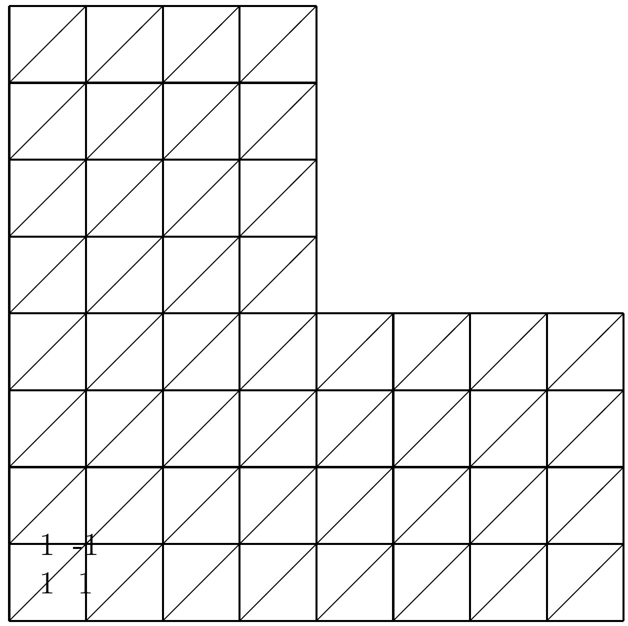

Figure YY30.

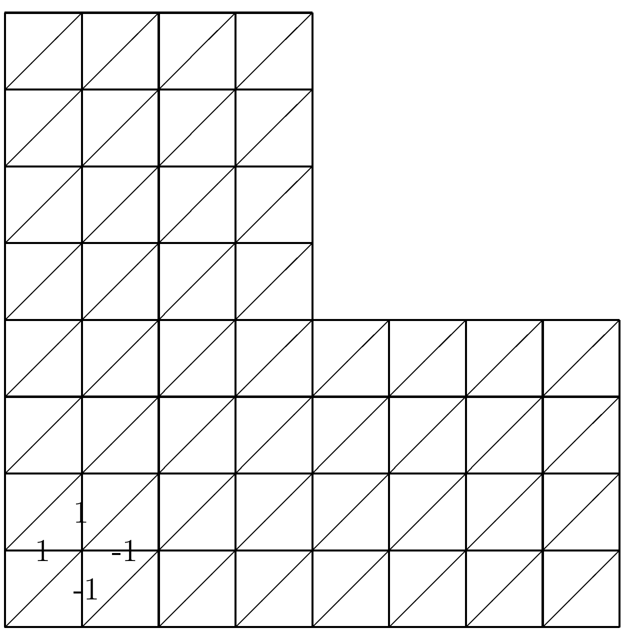

Figure YY31.

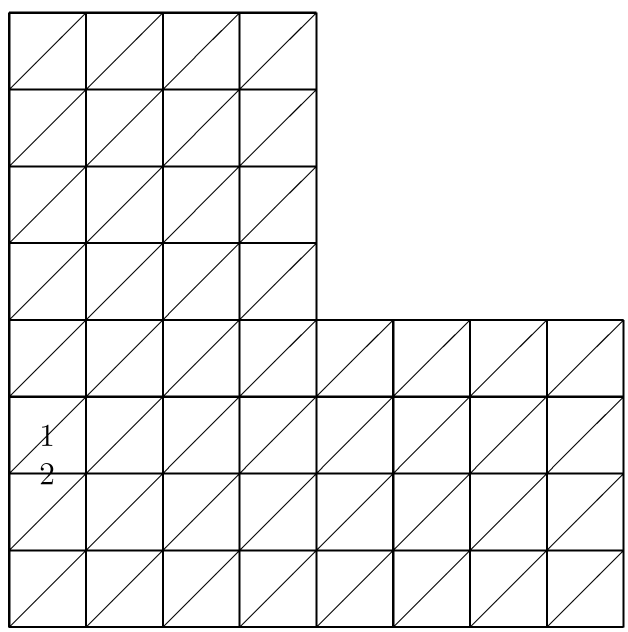

Figure YY32.

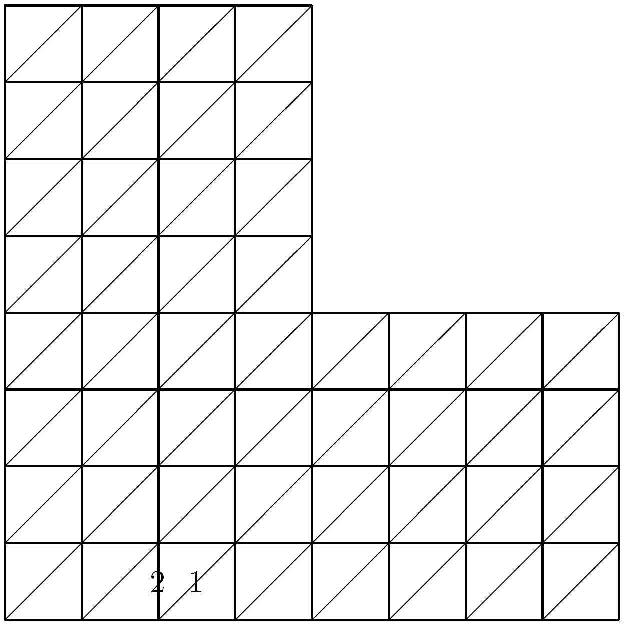

Figure YY33.

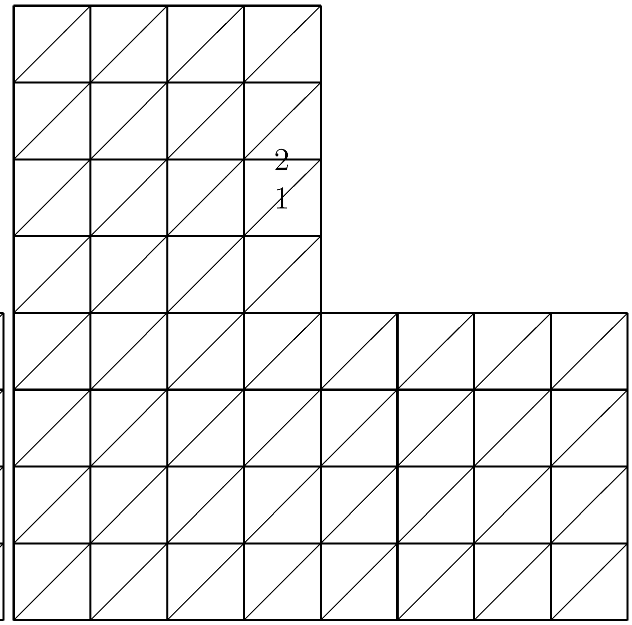

Figure YY34.

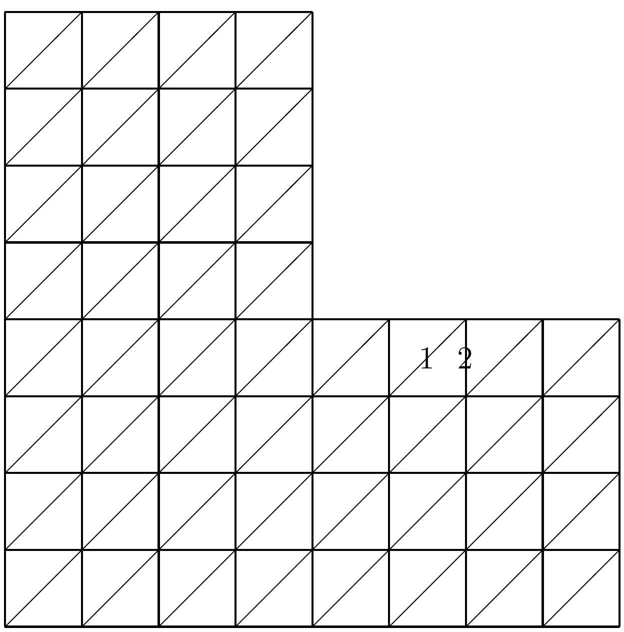

Figure YY35.

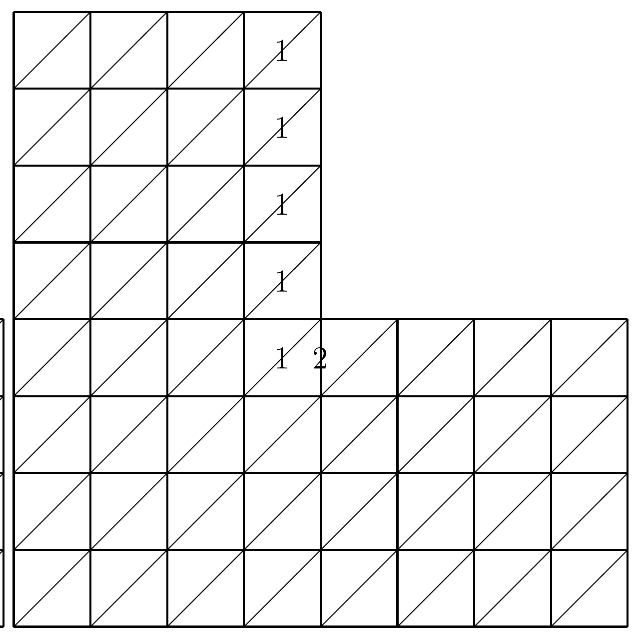

Figure YY36.

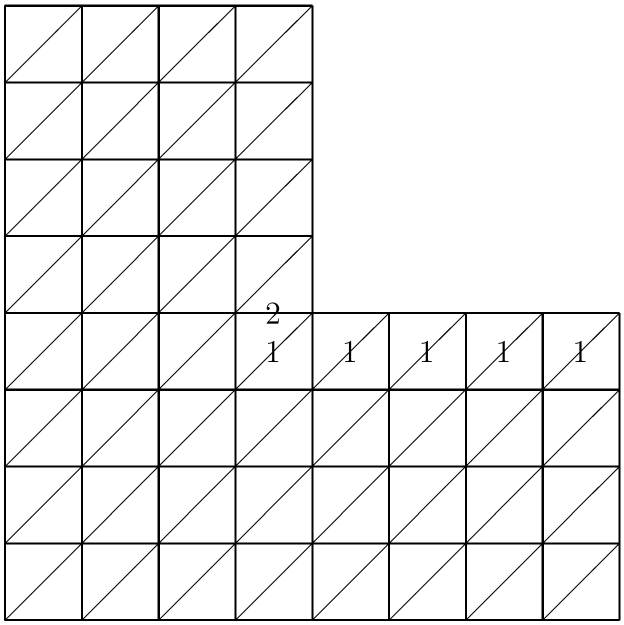

Figure YY37.

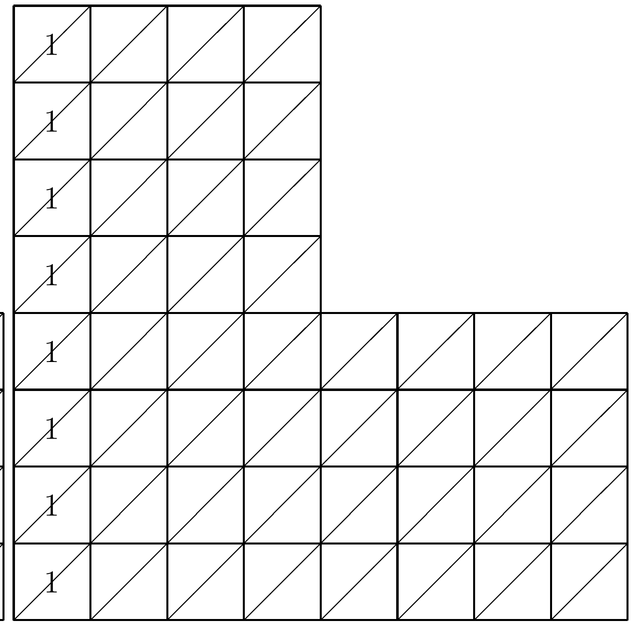

Figure YY38.

8.5 Another Construction Method

There is another method to construct prewavelet basis in L-shape domain. The only different from the previous section is the last global supported prewavelet $\psi^{j,10}$, all the others, $\psi^{j,3}$, $\psi^{j,4}$, $\psi^{j,5}$, $\psi^{j,1}$, $\psi^{j,2}$, $\psi^{j,6}$, $\psi^{j,7}$, $\psi^{j,8}$, $\psi^{j,9}$, is same to the one in the previous section.

The $\psi^{j,10}$ are constructed in the following way.

$$\psi^{j,10} = \phi_{15,1}^{j+1} + \phi_{15,3}^{j+1} + \phi_{15,5}^{j+1} + \phi_{15,7}^{j+1}$$

as shown in Figure YY39.

In order to show this basis is right for space W_j, the definition of V_{j+1}^m and W_j^m should be given in a different way.

Definition 8.5.1. *Let* $V_{j+1}^m = span\{\phi_{ik}^{j+1}, i = 1, ..., 2m+1, k = 1, ..., 2^j - 1\}$ *for* $m = 1, ..., 2^j - 1$, $V_{j+1}^m = span\{\phi_{ik}^{j+1}, i = 1, ..., 2^j - 1, k = 2^j, ..., 2m - 2^j + 1\} \cup V_{j+1}^{2^j - 1}$, *for* $m = 2^j, ..., 2^j + 2^{j-1} - 1$, *Let* W_j^m *be subspace of* W_j *such that* $W_j^m = W_j \bigcap V_{j+1}^m$.

By above definition, the space of V_{j+1}^{1} is the space shown as in Figure YY40, and the space of V_{j+1}^{9} is the space shown as in Figure YY41.

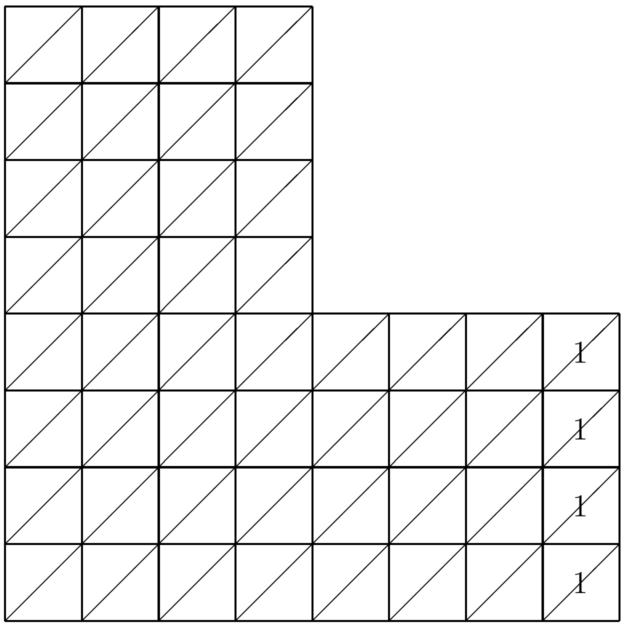

Figure YY39.

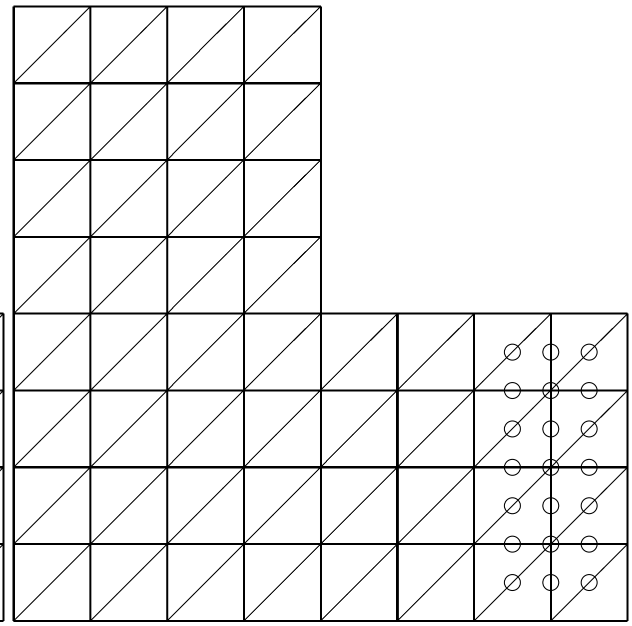

Figure YY40.

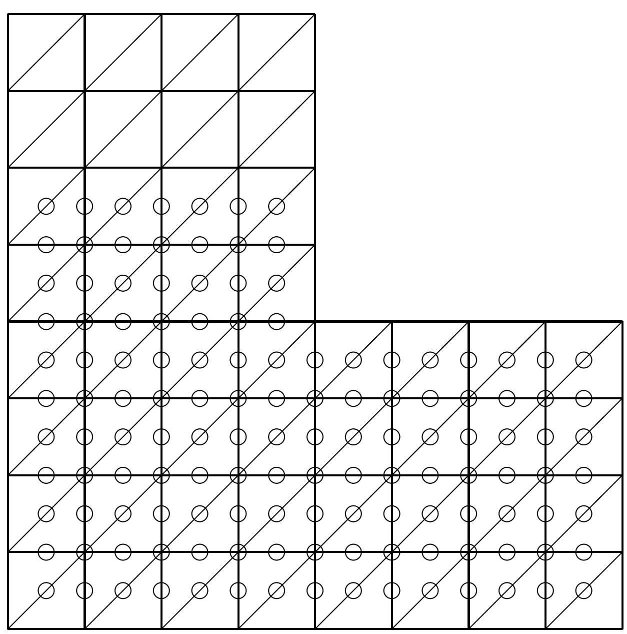

Figure YY41.

By the above definition, V_{j+1}^m is a subspace of V_{j+1}, W_j^m is a subspace of W_j.

Lemma 8.5.1. *All the three types of locally supported inner wavelets and three types of locally supported edge wavelets and the one global supported prewavelet in the* V_{j+1}^1 *are linear independent. That is, the following functions*

$$\begin{array}{ll} \psi^{j,3}_{2^j-1,k}, & k = 2^{j-1}-1, .., 2^j-1, \\ \psi^{j,4}_{2^j-1,k}, & k = 2^{j-1}-1, .., 2^j-1, \\ \psi^{j,5}_{2^j-1,k}, & k = 2^{j-1}-1, .., 2^j-1, \\ \psi^{j,1}_{2^j-1,k}, & k = 1, .., 2^{j-1}-1 \\ \psi^{j,2}_{2^j-1,k}, & k = 2^{j-1}-1 \\ \psi^{j,7}_{2^j-1,k}, & k = 2^{j-1}-1 \end{array}$$

are linear independent.

Proof. Suppose they are not linear independent. That is, one can find

$$\begin{array}{ll} a^3_{2^j-1,k}, & k = 2^{j-1}-1, .., 2^j-1, \\ a^4_{2^j-1,k}, & k = 2^{j-1}-1, .., 2^j-1, \\ a^5_{2^j-1,k}, & k = 2^{j-1}-1, .., 2^j-1, \\ a^1_{2^j-1,k}, & k = 1, .., 2^{j-1}-1 \\ a^2_{2^j-1,k}, & k = 2^{j-1}-1 \\ a^7_{2^j-1,k}, & k = 2^{j-1}-1 \end{array}$$

such that $\sum_{ik} a^1_{2^j-1,k}\psi^{j,1}_{2^j-1,k} + \sum_{ik} a^2_{2^j-1,k}\psi^{j,2}_{2^j-1,k} + \sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{ik} a^4_{i,k}\psi^{j,4}_{i,k} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} + \sum_{ik} a^7_{i,k}\psi^{j,7}_{i,k} = 0$,

By the definition, $\phi^{j+1}_{2^{j+1}-3,2k-1}$ appear only once in $\psi^{j,4}_{i,k}$ and $\psi^{j,7}_{2^j-1,2^{j-1}-1}$. Since ϕ^{j+1} are linear independent, that is, $a^4_{i,k}$, $a^7 = 0$. Thus the above equation can be simplified to

$$\sum_{ik} a^1_{2^j-1,k}\psi^{j,1}_{2^j-1,k} + \sum_{ik} a^2_{2^j-1,k}\psi^{j,2}_{2^j-1,k} + \sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} = 0, \tag{8.5}$$

By the similar reason, $\phi^{j+1}_{2^{j+1}-3,2k}$ appear only once in $\psi^{j,5}_{i,k}$. Since ϕ^{j+1}_{ik} are linear independent, $a^5_{i,k} = 0$. Thus the equation (8.5) can be further simplified to the following equation

$$\sum_{ik} a^1_{2^j-1,k}\psi^{j,1}_{2^j-1,k} + \sum_{ik} a^2_{2^j-1,k}\psi^{j,2}_{2^j-1,k} + \sum_{ik} a^3_{i,k}\psi^{j,3}_{i,k} = 0, \tag{8.6}$$

Keep going this way, all the coefficient should be zeros. □

Theorem 8.5.1. *All types of wavelet in the* V_j^n *are linear independent for* $1 \leq n \leq 2^j - 1$*. That means for* $i > 0$ *and* $k > 0$

Proof. the proof is similar to the previous section.

□

Theorem 8.5.2. *All types of wavelet in the* $V_j^{2^j}$ *are linear independent.*

Proof. By previous theorem, All types of wavelet in the $V_j^{2^j-1}$ are linear independent. Let ψ' be linear combination of the spline functions in $V_j^{2^j-1}$, in order to prove the theorem, the following box splines

$$
\begin{array}{ll}
\psi^{j,3}_{i,2^{j-1}}, & i = 1, .., 2^{j-1} - 1, \\
\psi^{j,4}_{i,2^{j-1}}, & i = 1, .., 2^{j-1} - 1, \\
\psi^{j,5}_{i,2^{j-1}}, & i = 1, .., 2^{j-1} - 1, \\
\psi^{j,1}_{1,2^{j-1}}, & \\
\psi^{j,9} & \\
\psi' &
\end{array}
$$

should be linear independent. Suppose they are not linear independent, so I can find

$$
\begin{array}{ll}
a^3_{i,2^{j-1}}, & i = 1, .., 2^{j-1} - 1, \\
a^4_{i,2^{j-1}}, & i = 1, .., 2^{j-1} - 1, \\
a^5_{i,2^{j-1}}, & i = 1, .., 2^{j-1} - 1, \\
a^1_{1,2^{j-1}}, & \\
a^9 &
\end{array}
$$

such that $a^1_{1,2^{j-1}}\psi^{j,1}_{1,2^{j-1}} + \sum_{ik} a^3\psi^{j,3} + \sum_{ik} a^4_{i,k}\psi^{j,4}_{i,k} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} + a^9\psi^{j,9} + \psi' = 0$, By the definition, $\phi^{j+1}_{2k-1,2^j+1}$ and appear only once in $\psi^{j,4}_{i,k}$ and $\psi^{j,1}_{1,2^j}$. Since ϕ^{j+1} are linear independent, that is, $a^4_{i,k}$, $a^1 = 0$. Therefor the above equation can be simplified to following expression:

$$
\sum_{ik} a^3\psi^{j,3} + \sum_{ik} a^5_{i,k}\psi^{j,5}_{i,k} + a^9\psi^{j,9} + \psi' = 0, \tag{8.7}
$$

By the definition, $\phi^{j+1}_{2k,2^j+1}$ and appear only once in $\psi^{j,5}_{i,k}$. Since ϕ^{j+1} are linear independent, that is, $a^5_{i,k} = 0$. Therefor the above equation can be simplified to following expression:

$$\sum_{ik} a^3\psi^{j,3} + a^9\psi^{j,9} + \psi^{'} = 0, \tag{8.8}$$

By the definition, $\phi^{j+1}_{2k,2^j}$ and appear only once in $\psi^{j,3}_{i,k}$. Since ϕ^{j+1} are linear independent, that is, $a^3_{i,k} = 0$. Therefor the above equation can be simplified to following expression:

$$a^9\psi^{j,9} + \psi^{'} = 0, \tag{8.9}$$

By the definition, $\phi^{j+1}_{2^j-1,2^j}$ and appear only once in $\psi^{j,9}_{i,k}$. Since ϕ^{j+1} are linear independent, that is, $a^9 = 0$. Therefor the above equation can be simplified to following expression:

$$\psi^{'} = 0, \tag{8.10}$$

By previous theorem, all the coefficient of $\psi^{'} = 0$ are zeroes. That means

$$\begin{aligned}
&\psi^{j,3}_{i,2^{j-1}}, \quad i = 1, .., 2^{j-1} - 1,\\
&\psi^{j,4}_{i,2^{j-1}}, \quad i = 1, .., 2^{j-1} - 1,\\
&\psi^{j,5}_{i,2^{j-1}}, \quad i = 1, .., 2^{j-1} - 1,\\
&\psi^{j,1}_{1,2^{j-1}},\\
&\psi^{j,9}\\
&\psi^{'}
\end{aligned}$$

are linear independent for V^n_j. □

Theorem 8.5.3. *All types of wavelet in the V^n_j are linear independent for $1 \leq n \leq 2^j + 2^{j-1} - 2$.*

Proof. the proof is similar to the previous section. □

Theorem 8.5.4. *All the five types of wavelet in the V^n_j compose linear independent basis of W^n_j for $1 \leq n \leq 2^j + 2^{j-1} - 2$.*

Proof. By counting, the number of prewavelet in space V_j^n is exactly the dimension of W_j^n for $1 \leq n \leq 2^j + 2^{j-1} - 2$. □

Theorem 8.5.5. *All types of prewavelet in the* $W_j^{2^j+2^{j-1}-1} \backslash W_j^{2^j+2^{j-1}-2}$ *are linear independent for scaling functions of* $V_j^{2^j+2^{j-1}-1} \backslash V_j^{2^j+2^{j-1}-2}$*, which is set* $\{\phi_{i,k}^{j+1}, 2^{j+1} - 2 \leq k \leq 2^{j+1} - 1\}$.

Proof. Let us just concentrate on the basis functions in $V_j^{2^j+2^{j-1}-1} \backslash V_j^{2^j+2^{j-1}-2}$ and in $W_j^{2^j+2^{j-1}-1} \backslash W_j^{2^j+2^{j-1}-2}$. Then the scaling matrix between two sets of basis functions is the following matrix up to a constant.

Where A is $(2^{j+1} - 2) \times (2^{j+1} - 2)$ matrix, so if matrix A is invertible, then then the wavelets basis I choose for $W_j^{2^j+2^{j-1}-1} \backslash W_j^{2^j+2^{j-1}-2}$ are linear independent for the scaling functions of $V_j^{2^j+2^{j-1}-1} \backslash V_j^{2^j+2^{j-1}-2}$.

Claim the matrix has rank $2^{j+1} - 2$, so it is invertible. By the calculation

$$A = \begin{pmatrix}
1 & 2 & & & & & & & & & & & & & & \\
1 & 0 & 2 & & & & & & & & & & & & & \\
 & 1 & 0 & -1 & & & & & & & & & & & & \\
 & 1 & 1 & 0 & 0 & -1 & & & & & & & & & & \\
 & & 1 & -1 & 1 & 1 & & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & & & \\
 & & & & & 1 & 0 & 2 & & & & & & & & \\
 & & & & & & 1 & 0 & -1 & & & & & & & \\
 & & & & & & 1 & 1 & 0 & 0 & -1 & & & & & \\
 & & & & & & & 1 & -1 & 1 & 1 & & & & & \\
 & & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & 1 & 0 & 2 & & & \\
 & & & & & & & & & & & 1 & 0 & -1 & & \\
 & & & & & & & & & & & 1 & 1 & 0 & 0 & -1 \\
 & & & & & & & & & & & & 1 & -1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & 0 & 1 & 0
\end{pmatrix}.$$

$$A = \begin{pmatrix}
D & & & & & & & \\
B1 & B2 & & & & & & \\
 & B1 & B2 & & & & & \\
 & & \ddots & & & & & \\
 & & & B1 & B2 & & & \\
 & & & & \ddots & & & \\
 & & & & & B1 & B2 & \\
 & & & & & & B1 & C1 \\
 & & & & & & & C2
\end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}, \quad B1 = \begin{pmatrix} 1 & 0 & 2 & \\ & 1 & 0 & -1 \\ & 1 & 1 & 0 \\ & & 1 & -1 \end{pmatrix}, \quad B2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$C1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C2 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

Let $E = (m\ n\ 0\ 0)$. By the row operations we have

$$\begin{pmatrix} E & & \\ B1 & B2 & \\ & B1 & B2 \end{pmatrix} = \begin{pmatrix} m & n & 0 & & & & & & & \\ 1 & 0 & 2 & & & & & & & \\ & 1 & 0 & -1 & & & & & & \\ & 1 & 1 & 0 & 0 & -1 & & & & \\ & & 1 & -1 & 1 & 1 & & & & \\ & & & & 1 & 0 & 2 & & & \\ & & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 1 & 0 & 0 & -1 \\ & & & & & & 1 & -1 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} m & n & & & & & & & & \\ & -n & 2m & & & & & & & \\ & & 2m & -n & & & & & & \\ & & & n & m & & & & & \\ & & & & 2m+n & 2n & 0 & 0 & & \\ & & & & 1 & 0 & 2 & & & \\ & & & & & 1 & 0 & -1 & & \\ & & & & & 1 & 1 & 0 & 0 & -1 \\ & & & & & & 1 & -1 & 1 & 1 \end{pmatrix}.$$

Thus by row operations,

$$A \rightarrow \begin{pmatrix} A_1 & G_1 & & & & \\ & A_2 & G_2 & & & \\ & & A_3 & G_3 & & \\ & & & \ddots & & \\ & & & & A_{2^j-2} & G_{2^j-2} \\ & & & & & C_1' \end{pmatrix},$$

where A_n is an upper triangular matrix of size 4×4 while A_n' is a lower triangular matrix of size 4×4 which are given below.

$$A_1 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ & -1 & 1 & 0 \\ & & 1 & -1 \\ & & & 2 \end{pmatrix},\ G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ & -1 & 2 & 0 \\ & & 2 & -1 \\ & & & 1 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ A_n = \begin{pmatrix} n & 2 & 0 & 0 \\ & -1 & n & 0 \\ & & n & -1 \\ & & & 2 \end{pmatrix},\ G_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 \end{pmatrix},$$

and the matrix C_1' is the following matrix

$$C_1' = \begin{pmatrix} 2^{j+1}-5 & 2 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see the rank of C_1' is 2. Thus the rank of A is $2^{j+1}-2$. Thus, all the prewavelet functions constructed above in the $W_j^{2^j+2^{j-1}-1} \backslash W_j^{2^j+2^{j-1}-2}$ are linear independent and hence form a basis of $V_j^{2^j+2^{j-1}-1} \backslash V_j^{2^j+2^{j-1}-2}$. □

All types of prewavelet in the $W_j^{2^j+2^{j-1}-1} \backslash W_j^{2^j+2^{j-1}-2}$ are linear independent for the basis of $V_j^{2^j+2^{j-1}-1} \backslash V_j^{2^j+2^{j-1}-2}$, we know the coefficient of wavelet in $W_j^{2^j+2^{j-1}-2}$ for the basis of $V_j^{2^j+2^{j-1}-1} \backslash V_j^{2^j+2^{j-1}-2}$ are all zeros, that means all the wavelets in $W_j^{2^j+2^{j-1}-1}$ are linear independent. then we come to the following theorem.

Theorem 8.5.6. *All types of prewavelets constructed in this section compose the basis of* $W_j^{2^j+2^{j-1}-1}$.

Proof. By counting, the number of the prewavelet is exactly the dimension of the space of $W_j^{2^j+2^{j-1}-1}$, combine with the independent property of these prewavelets, they compose the basis. □

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