FUTURE MIDDLE SCHOOL TEACHERS' PERFORMANCE ON PROPORTIONAL RELATIONSHIP TASKS AND THEIR USE OF MEANINGS FOR MULTIPLICATION AND DIVISION

by

MERVE NUR KURSAV

(Under the Direction of Andrew Izsák)

ABSTRACT

This study examines the solution methods that future middle school teachers chose when solving a problem from two perspectives on proportional relationships and the extent to which they explicitly utilized features from instruction including the use of equations, math drawings, and quantitative meanings for multiplication and division. The perspectives are called multiple batches and variable parts, and each supports multiple solution methods. The data were collected from a sample of 22 future middle-grade teachers' exams completed as part of a content course at a large university in the Southeastern United States. Findings revealed that (a) future middle grade teachers were able to use the two perspectives after completing a two-semester sequence of content courses emphasizing topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication, (b) future teachers who used strategies based in multiple batches and variable parts performed well on proportional relations tasks, and (c) when allowed to choose methods future teachers used one based on partitive division most often.

INDEX WORD: Proportional relationship, Proportional reasoning, Variable parts perspective,

Multiple batches perspective, Division and Multiplication, Partitive division, Quotitive division, Multiplicative situation

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MERVE NUR KURSAV

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MERVE NUR KURSAV

Major Professor: Andrew Izsák
Committee: Sybilla Beckmann-Kazez
Amy Ellis

Electronic Version Approved:
Suzanne Barbour
Dean of the Graduate School
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DEDICATION

I dedicate this work to my parents, Mustafa and Fahriye Kursav,

my sister Muberra, and my grandmother Fatma

For their endless love and support

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CHAPTER 1

INTRODUCTION

Middle school mathematics has long been a topic of interest for researchers and educators. This interest stems from challenges and difficulties in improving teaching and learning mathematics in the United States and from a general decline in mathematics achievement (National Mathematics Advisory Panel, 2008). In middle school mathematics, the concept of proportional relationships forms a crucial base for further concepts such as functions, graphing, algebraic equations and measurements (Karplus, Pulos, & Stage, 1983; Langrall & Swafford, 2000; Lobato & Ellis, 2010; Lobato, Orrill, Druken, & Jacobson, 2011; Thompson & Saldanha, 2003). According to Vergnaud (1983, 1988, 1994), ratios and proportional relationships are part of the multiplicative conceptual field, which is "a web of interrelated ideas that also include whole-number multiplication and division, fractions, linear functions, and more" (as cited in Beckmann & Izsák, 2015, p. 18). Therefore, to address underachievement in middle school mathematics in the United States, learning and teaching the foundational concept of proportional relationships is critical (National Mathematics Advisory Panel, 2008).

Although proportional relationships are a cornerstone of middle school mathematics, students in middle school face difficulties completing tasks that make use of proportional relationships. The psychological complexity can be underestimated because of its operational simplicity (Greer, 1992). While the operational aspect of proportional relationships requires procedural knowledge, the psychologically complex component of proportional relationship essentially requires conceptual knowledge. On one hand, procedural knowledge is defined with respect to two kinds of knowledge:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols (Hiebert & Lefevre, 1986, pp. 7-8).

On the other hand, conceptual knowledge is usually defined as:

> ... knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network (Hiebert & Lefevre, 1986, pp.3-4).

In tasks about proportional relationships, connecting aspects of procedural knowledge (i.e., formulations, definitions, and mathematical operations) to properties of conceptual knowledge (i.e., linking the all pieces of information and reasoning) is critical for academic achievement in middle school mathematics. Yet, because of the tendency to teach mathematical operations—such as cross multiplying for missing-value word problems—without conceptual learning, students' mathematics knowledge is more procedural than conceptual. Thus, understanding proportional relationships is more than performing mathematical operations and applying formulae. Understanding this concept requires proportional reasoning. Consequently, educators should be well aware that fostering students' procedural and conceptual knowledge is essential for proportional relationships and proportional reasoning.

Despite the growing body of research on proportional reasoning and proportional relationships, the studies that have explored future middle school teachers' understandings of ratios and proportional relationships in terms of quantities are rather limited. Lobato et. al. (2011) stated that:

> There are relatively few studies aimed at teachers' understanding of the same topic. These studies suggest that many elementary and middle grades teachers and future teachers lack a deep understanding of proportional reasoning and rely too heavily on rote procedures such as the cross-multiplication algorithm (p.3).

Thus, there is a need for research on how future middle-grade teachers reason about proportional relationships because "teachers are among the most, if not the most, significant factors in children's learning and the linchpins in educational reforms of all kinds" (Cochran-Smith & Zeichner, 2009, p. 1).

In particular, we need new approaches and perspectives to think about how future middle school teachers' reasoning about proportional relationships can be supported. With this objective in mind, Beckmann and Izsák (2015) developed a new approach comprising two perspectives and four methods that comprise a coherent understanding of proportion al relationships that includes multiplication and division. Their approach was distinctive because they connected multiplication, division, and proportional relationships into a single coherent framework that highlighted two complementary perspectives on ratios and proportional relationships. These perspectives are called variable parts and multiple batches. The multiple-batches perspective is well known (e.g., Abels, Wijers, Pligge, & Hedges, 2006; Lamon, 1995; Orrill, & Brown, 2012), but the variable-parts perspective has not been examined at any length in the literature (Beckmann & Izsák, 2015). In line with Beckmann and Izsák's (2015) approach, this study specifically focused on future middle school teachers' use of meanings for multiplication and division to make sense of proportional relationships in terms of quantities according to the two perspectives and four strategies that stem from the two perspectives.

Rationale

This study investigates the performances of future middle school teachers in understanding proportional relationships from the two perspectives and the role of multiplication and division in their reasoning. The data were collected from the final paper-and-pencil exam for a mathematics content course. To explore participants' performance on proportional relationship tasks, it is important to investigate if participants recognized 'the multiplicative relationship of a proportional situation in a table, graph, equation, diagram, or verbal descriptions' (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). While much has been written about proportional relationships, there is an absence of literature which explicitly addresses how future teachers use the meaning of multiplication and division in proportional relationships tasks by considering the two quantitative perspectives on ratios and proportional relationships (Beckman and Izsák, 2015). Therefore, the present study examines the following research questions:

1. Which solution methods do future middle school teachers choose when solving a problem at the end of a content course that introduce two perspectives on proportional relationships?
2. To what extent do future middle school teachers make explicit use of specific features from instruction including use of equations, math drawings, and quantitative meanings for multiplication and division in their solution methods?

CHAPTER 2

LITERATURE REVIEW

This chapter reviews the literature on proportional relationships. The chapter first discusses key terms used in the literature and then summarizes reports of proportional relationships and proportional reasoning and partitive division and quotitive division.

Proportional Relationships and Proportional Reasoning

Ratios are a multiplicative comparison between the measures of two quantities, and a proportion is the equivalence of two ratios. Thus, proportional relationships are relationships between two equal ratios (Touirniaire, 1985). Proportional reasoning is "a term that denotes reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus, Pulos, & Stage, 1983, p. 219) and the outcomes of perceiving rational numbers fundamentally (e.g., Hart, 1988; Lamon, 1996). Additionally, proportional reasoning explores "the holistic relationship between two rational expressions such as rates, ratios, quotients, and fractions" (Lesh, Post, & Behr, 1988, p. 93).

The concept of proportional relationships is fundamental in the U.S. middle school mathematics curricula. Correspondingly, proportional reasoning is the capstone of elementary school mathematics (Lesh et al., 1988, p.93), the heart of middle school mathematics (Pantziara & Pitta-Pantazi, 2005), and the cornerstone of high school mathematics (Lesh et al., 1988). The reason for the prominence of these ideas is the mathematical centrality of proportional relationships. Furthermore, this concept is also very important in numerous non-school settings. For example, Hoyles, Noss, and Pozzi (2001) reviewed "how expert nurses assume the calculation of drug dosages. This calculation is error-critical in nursing practice and maps onto the concepts of ratio and proportion" (p. 4). Similarly, Masingila (1994) investigated the use of proportional relationships during estimating situations of carpet layers—for instance, Gene is an estimator who was "measuring a blueprint in the process of preparing an estimate for a commercial job. The blueprints were drawn in a scale of 1/4 inch to 1 foot, and Gene was using a drafting ruler to measure the maximum length and width of each room" (p. 443). As seen in these examples, proportional relationships and proportional reasoning appear in real life circumstances.

Even though proportional relationships and proportional reasoning are milestones in school and non-school settings, and a person might have learned about them when he or she was young, many adults are not fluent (Cramer & Post, 1993; Newton et al. 1981; Pallrand, 1979). For example, in one study, when adult women shoppers were asked which of two sizes of a common item sold in a store was the better buy, only 1/3 could determine the ratio (2:3) by using proportional reasoning (Capon & Kuhn, 1979, p. 450). This study demonstrated that the development of proportional reasoning could be incomplete among adults and takes time (Van de Walle, 2006).

Given the importance of proportional relationships and reasoning in real life, it makes sense that many studies have been conducted at different grade levels in school settings, which has been well documented (e.g., Beckmann & Izsák, 2015; Greer, 1992; Karplus, 1981; Karplus et al., 1983; Kaput & West, 1994; Harel, Behr, Lesh, & Post, 1994; Hart, 1988; Lamon 1993, 1994, 1996, 1993, 2007; Noelting, 1980; Simon & Blume, 1994; Thompson, 1994; Tourniaire, 1986; Vergnaud, 1983, 1988, 2009). These studies have explored reasoning on three types of tasks: missing value, numerical comparison, and qualitative prediction and comparison (Lamon, 2007; The Rational Number Project, 1979). According to Cramer and Post (1993), a missing-value problem includes three given pieces of information with one unknown piece of information (e.g., "Lisa and Rachel drove equally fast along a country road. It took Lisa 6 minutes to drive 4 miles. How long did it take Rachel to drive six miles?"). Also, when two complete rates are given and compared, and a numerical answer is not required, it is a numerical comparison problem (e.g., "Anne and Linda are using different road maps of the city. On Anne's map, a road 3 inches long is 15 miles long. On Linda's map, a road 9 inches long is 45 miles long. Who is using the larger city map? a) Anne b) Linda c) Their maps are the same d) Not enough information to tell") (p. 405). The last one, qualitative prediction and comparison tasks, includes no numerical values with a "counterbalancing of variables in measure spaces" requirement (Cramer et al., 1993, p.10) (e.g., "If Nick mixed less lemonade mix with more water than he did yesterday, his lemonade drink would taste ______. a) Stronger b) Weaker c) Exactly the same d) Not enough information to tell") (p. 405).

A growing body of research includes well-known examples of proportional relationship tasks by considering missing-value, numerical comparison, and qualitative prediction and comparison task types (Cramer et al., 1993). For example, Karplus et. al. (1974) used a missing-value task regarding Mr. Tall and Mr. Short. In this problem, three pieces of information for the question are given, and one piece is unknown. Participants are given a chain of six paper clips and shown that this chain represents Mr. Short's height in paper clips. Also, Mr. Short measures four large buttons tall. Students in the study are told but not shown that Mr. Tall is six large buttons tall. Students are asked to find the height of Mr. Tall with the clips. Then they explain their answers. The information in missing-value problems can be shown as rates: 6 paper clips/ 4 buttons is a complete rate and x paper clips/ 6 buttons is an incomplete rate. This study indicated that students ages 12-14 found this problem to be very hard.

On the other hand, Noelting (1980) conducted a study with 321 participants ages 6-16 that used the orange juice comparison problem. In this study, instead of giving a numerical answer, students were required to compare the rates of the orange juice mix. The researcher told students that the shaded glasses represented orange juice, and the unshaded glasses represented water. Participants imagined that the orange juice mix was the pitcher and determined which pitcher had the strongest tasting orange juice, or if the taste was the same. The authors reported that almost 67% of students 12 years old or older understood the orange juice problem. With this study, Noelting (1980) supported the Piagetian view, which proposes proportional reasoning provides the skill to define the relationship between two quantities beyond just exploring the relationship (Baxter & Juker, 2001).

There is a general understanding that fluency with ratios and proportional relationships extends beyond solving missing-value and comparison problems with numerical operations (e.g., Lamon, 2007; Lobato & Ellis, 2010). This study correspondingly included a missing-value problem and extended the investigation well beyond the ability to just solve a missing-value problem through numerical computations.

Partitive Division and Quotitive Division

A considerable amount of research has shown that future and current teachers struggle to recognize mathematical operations, especially for multiplication and division situations (e.g., Graeber & Tirsoh, 1988; Harel & Behr, 1995; Harel et. al.1994; Izsák & Jacobson, 2015; Tirsoh & Graeber, 1990). Some of these studies have investigated mathematical operations regarding the decimal numbers, but in reality teachers should be competent in mathematical operations with different numbers such as whole numbers, fractions, and decimals. Also, they should be able to identify these operations as models for a range of contextualized situations.

Fischbein, Deri, Nello, and Marino (1985) proposed primitive psychological models that supported the four arithmetic operations. These authors gave multiplication and division word problems to students in grades 5, 7 and 9. These participants were asked to select the appropriate operation to solve various word problems. According to Fischbein et. al. (1985), mathematical operations are “attached to the primitive behavioral models which have an impact on the choice of an operation” (p. 3) because, according to Simon (1993), “even after learners have had solid formal-algorithmic training, they continue to be influenced by primitive intuitive models” (p. 235). Harel, Behr, Post, and Lesh (1989) also stated that primitive models for division affect the selection of operations (i.e., multiplication versus division).

Fischbein et. al. (1985) proposed that “the model of multiplication is repeated addition” (p. 3), and proposed two models for the partitive and quotitive meanings for division. Simon (1993) also stated that partitive division could be defined as follows:

> In the first model, which might also be termed sharing division, an object or collection of objects is divided into a number of equal fragments or sub collections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (operand) (p. 235).

On the other hand, quotitive division could be defined as follows:

> In the second model, which might also be termed measurement division, one seeks to determine how many times a given quantity is contained in a larger quantity. In this case, the only constraint is that the dividend must be larger than the divisor. If the quotient is a whole number, the model can be seen as a repeated subtraction (p. 235).

According to Ölmez (2014), when the question “v items divided into w groups” (p. 6) is asked, the operation would be division. However, a brainteaser is whether “we are looking for the number of groups (how many groups) or for the size of each group (how many in each group)" (p. 6). An appropriate response would be: it is quotitive (how many groups) division when v items are divided by w (i.e., v ÷ w) in each group to find the number of groups. In contrast, it is partitive (how many in each group) division when v items are shared by w groups equally to find the number of units in one group. For example, when 8 cookies are divided by 2 cookies on each plate (i.e., 8 ÷ 2), there exist 4 plates (the number of groups), which represents quotitive division, but then 8 cookies are distributed evenly into 2 plates (i.e., 8 ÷ 2), there exist 4 cookies in each plate, which represents partitive division (Ölmez, 2014).

According to Bell et. al. (1981), when students were given a series of word problems with the same structure, changing the numbers might cause students to change their opinions about the required operation. For example, students' performance on the question "How much do 5 gallons of petrol cost if one gallon costs £2" was better than their performance?" and on the question "How much does 0.22 gallons of petrol cost if one gallon costs £1.2?" For the second question, students preferred 1.20 ÷ 0.22 instead of 1.20 × 0.22 because of the decimal numbers. Because students believed that cost of 1 gallon should be higher than the cost of 0.22 gallon, they thought division was appropriate for this question. Moreover, Hart (1981, p. 91) stated that when students were given the question "A 15 cm eel has 9 cm of food; how much food should be given to a 25 cm eel?", they did not multiply 9 by $\frac{5}{3}$. Participants used more complicated solutions, one of which was "10 is the two-thirds of 15, two-thirds of 9 is 6, and 25 is 15 +10. Therefore, one has to add: 9+6=15". With respect to Bell et al.'s (1981) explanation, the answer for the eel problem must be greater than 9, and students could multiply 9 by $\frac{5}{3}$. However, no student multiplied 9 by $\frac{5}{3}$. Considering the findings of these studies, Fischbein et al. (1985) developed "tacit models of problem situations":

> Each fundamental operation of arithmetic generally remains linked to an implicit unconscious, and primitive intuitive model. Identification of the operation needed to solve a problem with two items of numerical data takes directly but as mediated by the model. The model imposes its own constraints on the search process (p.4)

For example, they proposed that as an intuitive model for multiplication, 3 times 5 means 5 + 5 + 5, in which the operator just must be a whole number. An operator cannot be 0.22 or $\frac{5}{3}$. These models may prevent a student from performing the proper operation. Similar to Fischbein et al. (1985), Bell et al. (1981) stated that a problem can be more challenging if it consists of decimals. Bell et al. (1981) also proposed that students' knowledge for "multiplication makes bigger" and "division makes smaller" can cause difficulties (as cited in Fischbein et al.,1985, p. 5).

Fischbein et al. (1985) supposed multiplication is repeated addition in which the "operator" is a whole number referring to "the number of equivalent collections" and the "operand" is any positive quantity and refers to "the magnitude of each collection." On the other hand, Izsák et al. (2011) highlighted the importance of identifying "a quantitative structure either involving A groups of size B or a multiplicative "times as many" comparison. A and B can be whole numbers, fractions, or decimals for multiplicative situations."

In other words, by considering tacit models, Fischbein supposed that a divisor is a whole number such that the dividend is greater than the divisor and the quotient for partitive division, and the dividend is greater than the divisor for quotitive division. On the other hand, Ölmez (2014) proposed that there is not any limitation for the definition of partitive and quotitive division.

Ölmez (2014) disagreed with Fishbein et. al.'s (1985) assumptions since the problem "4 pizzas are divided by 10 pizzas in each box. How many boxes do you need?" still requires quotitive division, and the problem “4 pizzas are shared evenly among 10 boxes. How many pizzas do you need in each box?” still requires partitive division (p. 6). Additionally, Vergnaud (1983) developed “a theory of epistemological obstacles for students who face difficulties while learning multiplicative structures (p. 5). Thinking about the multiplicative structures requires multiplicative reasoning. Multiplicative reasoning encompasses various topics such as “fractions, decimals, ratios, percent, proportions, linear functions and more advanced topics” (Izsák et al., 2011). These topics have been taught separately even though Vergnaud (1983, 1988) proposed unifying these topics—for instance, fraction division and proportionality have been explained independently, but in fact, they have the same joining themes within the same domain.

In this section, I reviewed the literature on proportional relationships, proportional reasoning and partitive and quotitive division. In light of the literature, the next section examines the the framework used in the present study.

CHAPTER 3

THEORITICAL FRAMEWORK

This study uses Beckmann and Izsák's (2015) perspective on multiplication, which integrates multiplication, division, and proportional relationships into a coherent whole. A key feature of their perspective is the identification of two distinct perspectives on proportional relationships. I used Beckmann and Izsák's work to examine which solution methods future middle school teachers chose when solving a problem at the end of a content course that introduce two perspectives on proportional relationships and to what extent future middle school teachers made explicit use of specific features from instruction including use of equations, math drawings, and quantitative meanings for multiplication and division in their solution methods.

Beckmann and Izsák's (2015) approach is grounded in Vergnaud's (1983, 1988, 1994) multiplicative conceptual field that places "ratios and proportional relationships in a web of interrelated ideas including whole number multiplication and division, fractions, ratios and proportions, linear functions, and more" (as cited in Beckmann & Izsák, 2014, p. 18). This study is framed by considering future middle-grades teachers use of the two perspectives, equations, meanings for multiplication and division, and math drawings relevant to selected methods.

Equation: M•N= P

Beckmann and Izsák (2015) formulized an equation as "M • N= P", where M is the number of the groups, N is the number of the units in each whole group, and P is the product amount. A key feature of the perspective is consistently writing the multiplier and multiplicand in the same order. Following Beckmann and Izsák, in this thesis I will write multiplication expressions as multiplier • multiplicand.

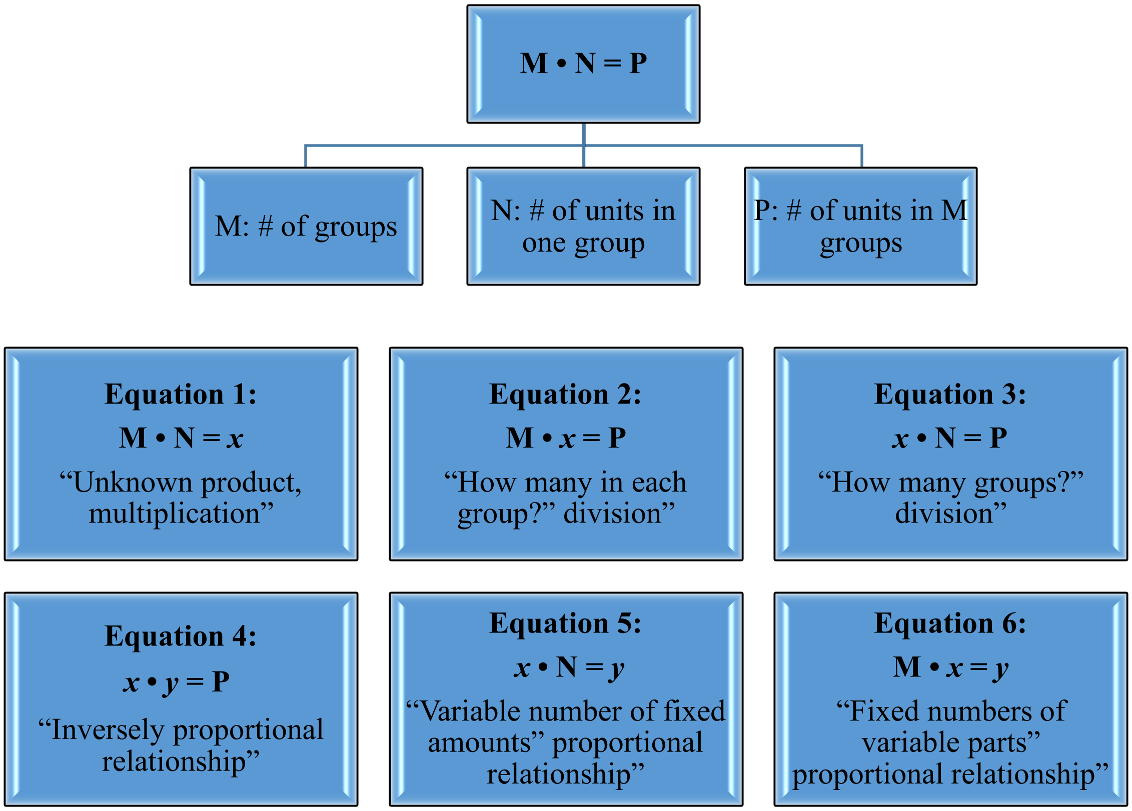


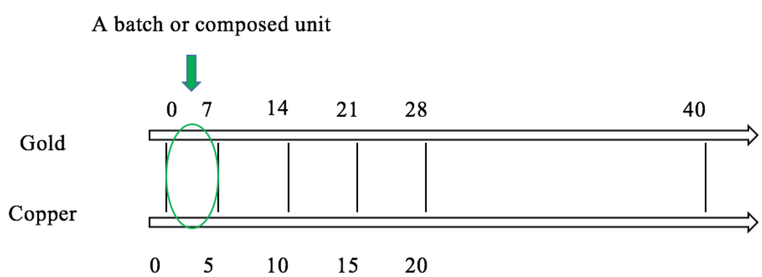
Figure 1. Multiplicative relationships (Adapted from Beckmann & Izsak, 2015, p. 19)

With respect to Figure 1, the equation M • N = P can be used to organize multiplication and division situations, including proportions. While M, N, and P are constants, x and y are values that can vary. M is "the number of the groups", that is the "multiplier," N is "the number of units in one group", that is the "multiplicand," and P is the "product." In the Figure 1, the first row represents situations in which two of M, N, and P are known, and the purpose is to find the third value, whereas the second row represents situations where one of M, N, and P is known, and x and y are "to-be-determined or are co-varying values" (p. 19).

Perspectives: Multiple Batches and Variable Parts

Beckmann and Izsák (2015) proposed two perspectives, multiple batches and variable parts, by considering the roles of the multiplier and multiplicand in proportional relationships. In this study, I will demonstrate two perspectives by using the following Gold and Copper problem: To make jewelry, jewelers often mix gold and copper in a 7 to 5 ratio. How much copper should a jeweler mix with 40 grams of gold?

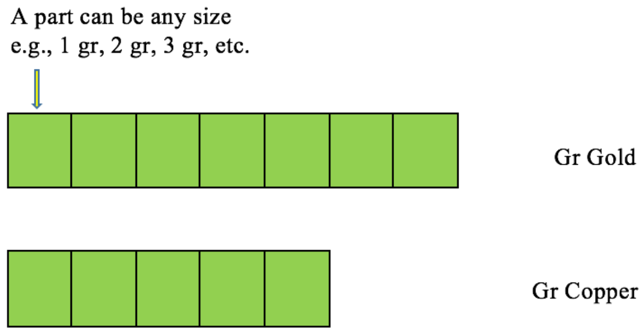
For the multiple batches perspective, they stated that “the original batch (A units of the first quantity and B units of the second quantity) are fixed multiplicands, and the multiplier varies; therefore, the proportional relationships can include “all of pairs (rA, rB)”, where r > 0 (Beckmann, Izsák, & Olmez., 2015, p. 519). Figure 2 shows one way to represent multiple batches in the Gold and Copper problem.



Number of Batches	Gr Gold	Gr Copper
1	**1 •** 7	**1 •** 5
2	**2 •** 7	**2 •** 5
3	**3 •** 7	**3 •** 5
4	**4 •** 7	**4 •** 5

Figure 2: Multiple Batches Perspective (Beckmann et al., 2015)

Another perspective is variable parts. Beckmann et al. (2015) considered the two quantities as consisting of A parts and B parts, respectively, where each part contains the same number of units. This time the multipliers are fixed by the numbers of parts, whereas the multiplicand varies with “the number of the measurement units” in every part (see Figure 3). Similar to the multiple-batches perspective, variable-parts proportional relationships include “all of pairs (Ar, Br)” for r > 0 (Beckmann et al., 2015, p. 520). Figure 3 shows one way to represent variable parts in the Gold and Copper problem.



Gr per part	Gr Gold	Gr Copper
1	7 • 1	5 • 1
2	7 • 2	5 • 2
3	7 • 3	5 • 3
4	7 • 4	5 • 4

Figure 3: Variable Parts Perspective (Beckmann et al., 2015)

The multiple-batches perspective supports at least two solution strategies: the how many batches method (i.e., multiply one batch method (number of the groups)) and the how much of a measurement unit method (i.e., multiply unit-rate batch method (unit rate or mini batch)). Similarly, the variable-parts perspective supports at least two solution strategies: the how much in one-part method (i.e., multiply one-part method (units per part)) and the how many total amounts method (i.e., multiply total amount method (whole groups)). I use the Gold and Copper problem to illustrate and contrast the four solution methods.

First, the multiply one batch method requires taking 7 grams of gold and 3 grams of copper as 1 batch. The question asks how many batches of 7 grams are in 40 grams.

$$? \bullet 7 = 40$$

$$(\text{\# batches}) \bullet (\text{\# grams gold per batch}) = (\text{\# grams gold})$$

Finding the number of batches is the same as finding the number of groups. There are 40 ÷ 7 batches of gold, so there are also 40 ÷ 7 batches of copper. Thus, $\frac{40}{7} \times 5 = \frac{200}{7}$ grams of copper are needed.

When the multiply unit-batch method is used, one considers the unit rate of $\frac{5}{7}$ grams of copper per 1 gram of gold. The unit rate can be viewed as a new batch. Since there are 40 groups of 1 gram in the gold, there are also 40 groups of $\frac{5}{7}$ grams in the copper.

$$40 \bullet \frac{5}{7} = \frac{200}{7}$$

(# batches) • (# grams copper per batch) = (# grams copper)

When the multiply one-part method is used with the variable-parts perspective, the jewelry is viewed as 7 parts gold and 5 parts copper. One can use information about the gold to find how many grams are in each part of gold and of copper.

$$7 \bullet ? = 40$$

(# parts) • (# grams gold per part) = (# grams gold)

Thus, since there are 40 ÷ 7 grams per part of gold, there are also this many grams per part of copper. Therefore, $5 \times \frac{40}{7} = \frac{200}{7}$ grams of copper are needed.

When the multiply total amount method with the variable-parts perspective is used, the total amount of gold can be taken as 1 group consisting of 40 grams. Then if 1 group is 40 grams of gold, the copper is $\frac{5}{7}$ of a group. Thus, $\frac{5}{7}$ groups • 40 grams per group is equal to $\frac{200}{7}$ grams copper.

Meanings for Multiplication and Division

In this study, the meaning of multiplication and division is framed with Beckmann and Izsák (2015) and Izsák et al. (2001) by considering M, N and P with respect to Figure 1.

Vergnaud (1983) organized multiplicative structures into three subgroups that he called isomorphism of measures, product of measures, and multiple proportion other than the product. The isomorphism of measures group includes situations in which there is a direct relationship between two measure spaces (denoted as M_1 and M_2 in Figure 4). The quantities within each measure space can be any “integers, fractions, or decimals” (Greer, 1992, p. 282).

M_1	M_2
1	a
b	c

Figure 4. Vergnaud’ schematic diagram for the isomorphism of measures (1983, 1988)

M_1	M_2
1	?
3	6

Figure 5. Vergnaud’s schematic diagram for the isomorphism of measures (1983, 1988)

A second type of division compares M_1 and M_2 directly. Consider the following problem: *Pizza problem 2*: 6 pizzas are distributed equally into 2 pizzas in each box. How many boxes do we need? The second type of division is different than the first one and requires connections between the two measure spaces, one for boxes and one for pizzas. By using the function operator, it can be reasoned there should be 3 boxes when 2 pizzas in each box. This is similar to the quotitive division (see Figure 6).

M_1	M_2
1	2
?	6

Figure 6. Vergnaud’s schematic diagram for the isomorphism of measures (1983, 1988)

In this study, multiplication and division were investigated as multiplicative situations (MS), quotitive division situations (QDS), and partitive division situations (PDS) (see Figure 7).

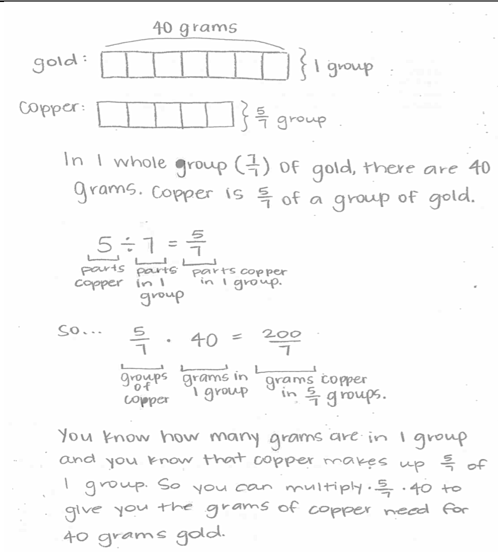
Multiplication/Division	**M • N = P** **(#of groups) • (# of units in each/one whole group) = (# of units in M group)**
Multiplication Situations (MS)	M• N= ☐ where P is unknown
Quotitive Division Situations (QDS)	☐ • N= P where M is unknown
Partitive Division Situations (PDS)	M • ☐= P where N is unknown

Figure 7. Multiplication and Division

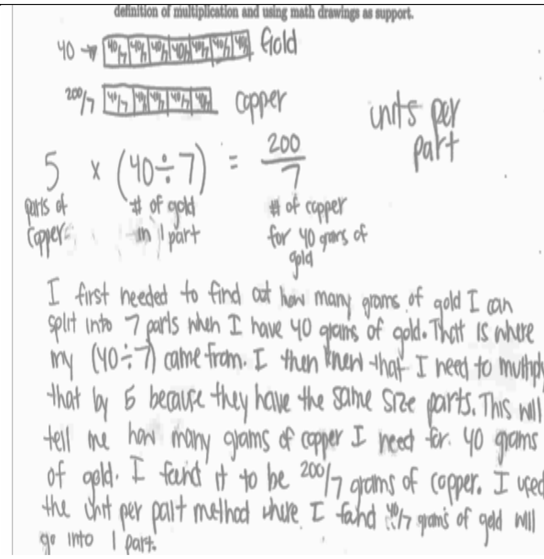
In an equation M • N = P, by considering the Figure 7,

- when M and N are known and P is unknown, it is a multiplication situation.
- when N and P are known, the division is quotitive, measurement, or how many groups
- when M and P are known, the division is partitive, sharing, or how many units in 1 group

To illustrate how meanings for multiplication and division appear in solutions to problems about proportional relationships, I show the solutions of two future teachers to the Gold and Copper problem (Figure 8). In Figure 8, on one hand, LF included in her equation a known N, an unknown M, and a division operation (i.e., 5 ÷ 7), so there is evidence for the identification of QDS. Also, after determining M, LF found P using the equation 5/7 • 40, which is MS. On the other hand, KA's solution includes a known M, an unknown N, and a division operation (i.e., 40 ÷ 7), so there is evidence for the identification of PDS. After determining N, KA found P using the equation 5 • 40/7, which is MS, similar to LF.



LF's solution
Identification of QDS
Identification of MS



KA's solution
Identification of PDS
Identification of MS

Figure 8. QDS, PDS, and MS samples

Math drawing

The present study also investigates how future middle school teachers expressed their reasoning about proportional relationship through math drawings. In this study, I mainly discuss two types of math drawings related to the two perspectives, double number lines (DNLs) and strip diagrams. Beckmann and Izsák (2015) indicated that DNLs fit well with the multiple-batches perspective and that the strip diagrams fit well with the variable-parts perspective. DNLs represent quantities visually as lengths and afford such operations as iterating, partitioning, or addition. Strip diagrams represent quantities in terms of variable parts. Various studies have been conducted to emphasize the importance of math drawings in the problems (e.g., Aprea & Ebner, 1999; Hall, Bailey, & Tillman, 1997; Reed, 1999)

Supporting a solution with a math drawing is critical since "a diagram is (sometimes) worth ten thousand words" (Larkin & Simon, 1987). These authors mentioned that math drawings can include more information than a written statement to support the solution of a problem. According to De Corte (1996), instead of being given a drawing, allowing future teachers to create their own math drawing provides them with deep knowledge about a task (Aprea & Ebner, 1999; Dirkes, 1991), so it represents a way to learn a concept effectively (De Corte, 1996). Additionally, math drawings support conceptualizing the question correctly and help to identify any incorrect understanding, so a solution can be generated for any issue (Van Essen & Hamaker, 1990).

In the resent study, future teachers' math drawings have been investigated by following the requirements of the two perspectives and four methods. Figure 9 illustrates the four strategies with a different version of the Gold and Copper problem. Double number lines (DNLs) are often used to support reasoning about proportional relationships from the multiple-batches perspective "by supporting the coordination of two values in ways similar to Cartesian graphs, but relying on the more familiar ideas of linear measurement" (Orrill & Brown, 2012, p. 382). Orrill and Brown also stated that DNLs organize the relationship between the values and are very beneficial for future teachers' development of proportional reasoning. They can be accepted "as a tool for mathematical communication and reasoning" (Corina et al., 2004, p. 142). Additionally, strip diagrams have been used to support students' understanding proportions (Cohen, 2013).

Figure 9 shows solutions for a version of the Gold and Copper problem that illustrate the two perspectives and four methods and how those methods are coordinated with equations following the multiplier • multiplicand convention. The figure shows how the two perspectives, four methods, and two different type of math drawings (i.e., DNLs and strip diagrams) are coordinated in four distinct ways.

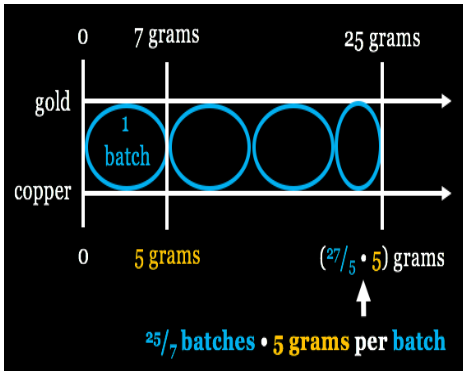
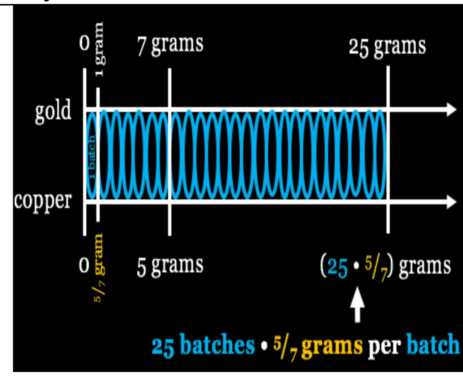
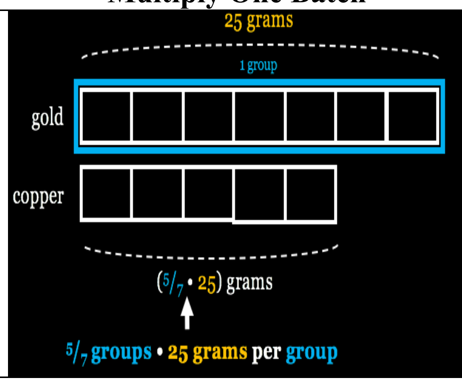
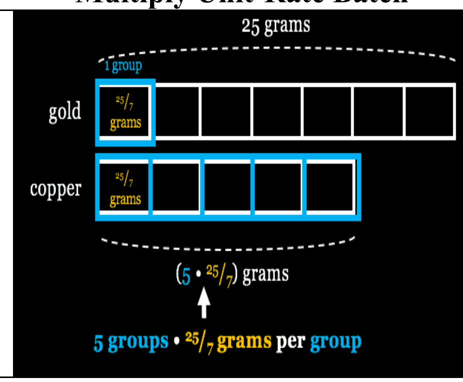
<table>
<tr><td>Problem
Perspective</td><td colspan="2">Gold and copper problem: “A company makes jewelry gold using gold and copper. The company uses different weights of gold and copper on different days, but always in the same ratio of 7 to 5. If the company uses 25 grams of gold on one day, how much copper will they use?”</td></tr>
<tr><td>Multiple Batches</td><td>
</td><td>
</td></tr>
<tr><td>Strategy</td><td>Multiply One Batch</td><td>Multiply Unit-Rate Batch</td></tr>
<tr><td>Variable Parts</td><td>
</td><td>
</td></tr>
<tr><td>Strategy</td><td>Multiply Total Amount</td><td>Multiply One Part</td></tr>
</table>

Figure 9. Solutions to the Gold and Copper Problem using two perspectives on proportional relationships and four strategies (Reproduced Kulow, 2016)

In the next section, methodology of this present study is represented.

CHAPTER 4

METHODOLOGY

The purpose of this qualitative study was to explore future middle grade teachers' performance on paper-and-pencil test items about proportional relationships. I concentrated on their use of the two perspectives and four strategies discussed in the previous chapter. In particular, I examined how the future teachers reasoned in terms of quantities with an equation, various math drawings and notations, and specific meanings for multiplication and division.

In this chapter, I will discuss qualitative research methodology and give my rationale for selecting the research design. Then I will describe the data collection and analysis procedures I used.

Research Design

The aim of this study was to explore which solution methods future middle school teachers choose when solving a problem at the end of a content course that introduced two perspectives on proportional relationships and to what extent future middle school teachers made explicit use of specific features from instruction including use of equations, math drawings, and quantitative meanings for multiplication and division in their solution methods. "Qualitative research methodologies have become increasingly important modes of inquiry for social sciences and applied fields" (Marshall & Rossman, 2014, p. 1), and they are used to discover the meanings created by the participants in an activity or context (Wolcott, 2009). More specifically, Creswell (2008) defined qualitative research as "an inquiry process of understanding based on distinct methodological traditions of inquiry that explore a social or human problem. The researcher builds a complex, holistic picture, analyzes words, reports detailed views of informants, and conducts the study in a natural setting" (p. 15).

In addition, when a researcher tries to understand, rather than explain, by assuming a personal role, qualitative research becomes appropriate research methodology (Denzin & Lincoln, 2005; Marshall & Rossman, 2014; Stake, 1995). It is suggested that qualitative research methodologies, which allow the researcher to collect data from multiple sources, are preferable for studies that seek to discover experiences of people. Patton (1990) further stated that "qualitative methods permit [the researcher] to study selected issues in depth and detail, and approach fieldwork without being constrained by predetermined categories of analysis that contribute to the depth, opened and detail of the qualitative inquiry" (p. 13). In this qualitative study, the researcher aimed to understand 22 future middle school teachers' performance and use of a quantitative meaning for multiplication and division when solving a problem at the end of a content course that introduced two perspectives on proportional relationships. The cohort of 22 future teachers was enrolled in a sequence of two content courses. The second course was designed to deepen and strengthen future middle school teachers' knowledge of topics related to multiplication including fraction division, ratio, proportional relationships, inversely proportional relationships, and deriving and explaining equations and solution methods in terms of two perspectives. The future teachers were instructed with respect to the variable-parts and multiple-batches perspectives, the four methods, and the use of double number lines and strip diagrams explained in Chapter 3.

Participants and Setting

Data for the present study were collected from 23 future middle school teachers at a large, public university in the Southeastern United States during the Spring 2016 semester. However, one future teacher was removed from the dataset because the future teacher did not provide consent, so the number of participants in the present study is 22. The setting where this study was conducted was the second semester of a two-semester sequence of mathematics content courses. The first semester focused on numbers and operations including multiplication, division, and fractions; the second semester focused on topics related to fraction division, ratio, proportional relationships (including the two perspectives discussed in Chapter 3), and algebra. Both courses emphasized the meaning of multiplication introduced in Chapter 3. Both courses were also intended to help future teachers develop practices outlined in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The textbook for both courses was *Mathematics for Elementary Teachers with Activities* (Beckmann, 2014). In both courses, future middle grades teachers studied through individual and group work during class, homework assignments, and examinations at a large, public university in the Southeastern United States.

The timelines below summarize the topics covered in the first and second semester courses. As shown in Figure 10, number operations were included mainly in the first course and algebra was included in the second course.

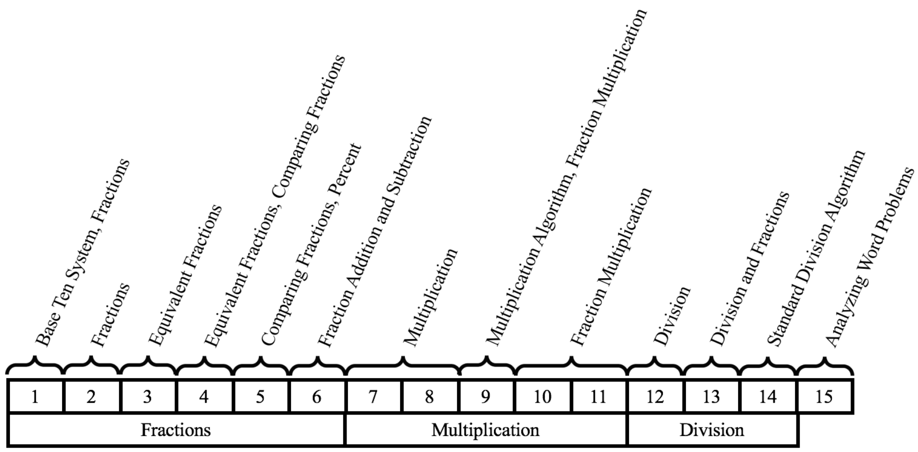


Figure 10. Timeline of the First Course in Fall 2015

As shown in Figure 11, the second course addressed fraction division, ratio and proportional relationships, statistics, probability, and number theory.

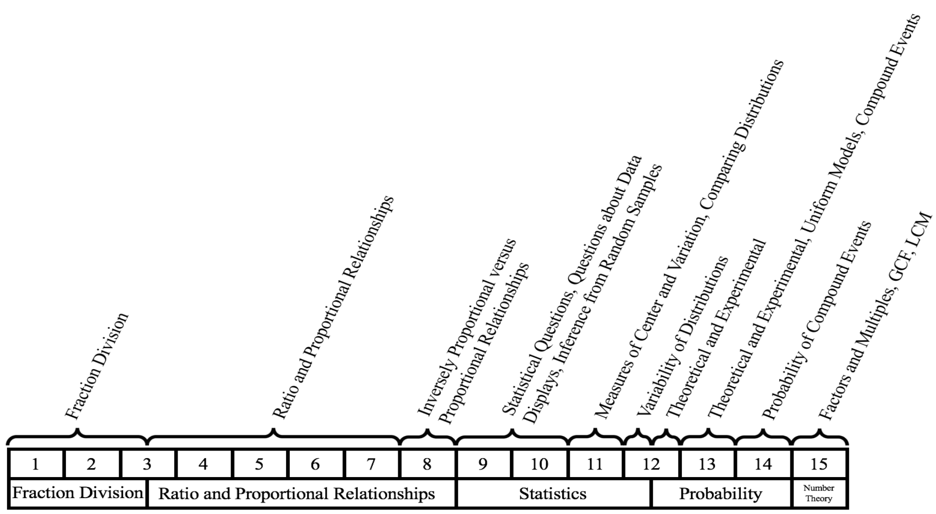


Figure 11. Timeline of the Second Course in Spring 2016

Data Collection

The future teachers in the present study were given different assignments and exams regarding the two perspectives during the spring semester in 2016. Figure 12 provides sample tasks about proportional relationships used in the course.

Task 1	Peacock Purple Paint Problem: PaintPals paint store mixed 9 pints blue and 4 pints pink and called the result Peacock Purple Paint. Now PaintPals need to make Peocock Purple Paint using 50 pints blue paint. How many pints of pink paint will they need? Explain in detail how to reason about multiplication and division with quantities to solve the Peacock Purple Paint Problem in 4 different ways: (1) from the multiple batches perspective using (a) the "unit rate" (mini batch) method and (b) the "number of the groups" method, and (2) from the variable parts perspective using (a) the "units per part" method and (b) the "whole groups" method. In each case, Use a suitable math drawing to develop your explanation; Express the answer as a product A•B, where A and B are derived from quantities of paint in the problem and explain how our definition of multiplication applies; When you use division, explain what kind of division it is (how-many-groups or how-many-units-in-1-group). Put your four explanations in order from most accessible to most difficult (in your view).
Task 2	A paint store mixed 3 quarts yellow paint with 2 quarts blue paint to make 5 quarts Garden Green paint. Make math drawings showing two different ways to organize 12 quarts yellow paint and 8 quarts blue paint so that: You can tell from the way the quarts of are organized that when they are mixed, they will make the same shade of green as Garden Green paint; Your two ways of organizing the quarts illustrate the two perspectives on ratio we discussed (multiple batches and variable parts).
Task 3	There are 600 sheep on a farm. A vet takes a random sample of 15 sheep and finds that 4 of the sheep have an infection. Based on this sample, what is the best estimate you can give for the number of infected sheep on the farm? Explain how to solve the problem in two very distinctly different ways, none of which involve cross-multiplying. Use math drawings to support your reasoning.

Figure 12. Task Item

After reviewing tasks on the midterm and final exams of the course during the Spring 2016 semester, I chose the following version of the Golden and Copper task from the final exam (see Figure 13) because, unlike other exam tasks, the Golden and Copper task invited teachers to chose from among the four methods summarized in Chapter 3.

Task	To make jewelry, jewelers often mix gold and copper in a 7 to 5 ratio. How much copper should a jeweler mix with 40 grams of gold? Write two different products A•B for the amount of the copper, where A and B are numbers derived from 7, 5, and 40. Explain each product A•B in detail in terms of the situation using our definition of multiplication and using math drawings as support.

Figure 13. Task Item

Analysis

The data were collected during the Spring 2016 semester in a content course at a large public university in the Southeastern United States. I identified all tasks on the midterm and final exams that addressed proportional relationships. Then I looked for items that allowed future middle school teachers to choose their own methods as opposed to items that directed future teachers to use a particular method. I selected one task from the final exam of the second semester course. This task, the Gold and Copper task shown in Figure 13 above, allowed me to explore future middle grade teachers' use of equations, math drawings, explanations for the four methods discussed in Chapter 3, and the meaning of multiplication and division also discussed in Chapter 3. By the time of the final exam, the future teachers had studied the distinction between PDS (how-many-units-in-one-group division) and QDS (how-many-groups division), and the DNL and strip diagram math drawings. First, I sorted the future teachers' solutions based on the perspective they chose (multiple batches or variable parts) and then based on methods that fit with those two perspectives. I analyzed the future teachers' solutions according their drawings, equations, and explanations. In the following part, I discuss the appropriate values of M, N and P for each of the four methods for solving the Gold and Copper problem.

- one part (variable parts; going through one part): M • N is 5 • 40/7
  - When M is known (M = 7) and N is unknown in the equation M • N = P, that is 7 • ? = 40. In this case, ? = (40 ÷ 7) and N = ? = 40/7. So the division is PDS.
  - When M is known (M = 5) and N is known (N = 40/7) and P is unknown in the equation M • N = P that is 5 • 40/7 = ?, so P = ? = 200/7, it is MS.
- total amount (variable parts; whole group): M • N is 5/7 • 40

- When M is unknown and N is known (N = 7) in the equation M • N = P that is ? • 7 = 5, so ? = (5 ÷ 7) and M = ? = 5/7. So the division QDS.
- When M is known (M = 5/7) and N is known (N = 40) and P is unknown in the equation M • N = P that is 5/7 • 40 = ? so P = ? = 200/7, it is MS.

▪ one batch (multiple batches): M • N is 40/7 • 5

- When M is unknown and N is known (N = 7) in the equation M • N = P, that is ? • 7 = 40, so ? = (40 ÷ 7) and M =? = 40/7, it is QDS.
- When M is known (M = 40/7) and N is known (N = 5) and P is unknown in the equation M • N = P that is 40/7 • 5 = ? so P = ? = 200/7, it is MS.

▪ unit rate batch (multiple batches): M • N is 40 • 5/7

- When M is known (M = 7) and N is unknown in the equation M • N = P, that is 7 • ? = 5, so ? = (5 ÷ 7) and N = ? = 5/7, it is PDS.
- When M is known (M = 40) and N is known (N = 5/7) and P is unknown in the equation M • N = P that is 40 • 5/7 = ? so P = ? = 200/7, it is MS.

I analyzed future teachers' use of the class meaning for multiplication and division in their solutions to the Gold and Copper task and how they applied the two perspectives in their solution. I focused on every word, number, and drawing to gather evidence for the future teachers' thinking and analyzed items according to the coding schema.

Using the coding schema shown in Figure 14, I placed all solutions for the Gold and Copper task into category 1, 2, or 3. I took category 2 as a standard level. Solutions in category 2 were complete and accurate in terms of the two perspectives and the given method. Solutions in this category included an equation with M and N where M and N had appropriate values for the given method. If M and N were switched in position but the values were still appropriate for the method, solutions were still placed in category 2 if they met all remaining criteria for that category. I placed solutions that did not mention the amount of gold and copper into category 1.

Although some future teachers misread the Gold and Copper problem, I rated their work against their reading of the problem and placed their work in category 2 when all other criteria were met. Additionally, in some cases, future teachers provided merely an equation and math drawing without any explanation. I evaluated their work and, if they provided adequate information to meet the standard category, I placed their solution in category 2 (see the Appendix 1 and AA's solution). In addition to meeting all criteria in category 2, if future teachers also included division with any explicit indicator (i.e., use of the ÷ symbol, the definition of division, or use of multiplication with a missing factor), I placed their solution in category 3. If all the requirements for category 2 were not met, I placed the solution into category 1.

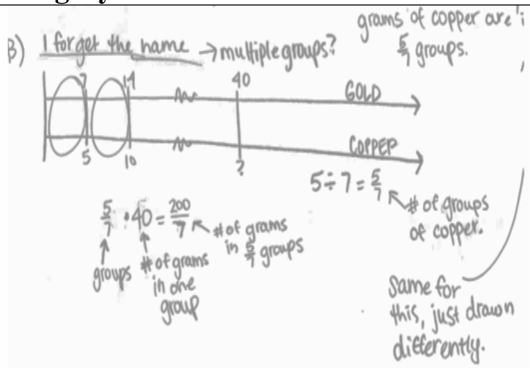
<table>
<tr><th colspan="3">Criteria for Category 2</th></tr>
<tr><th></th><th>Variable Parts</th><th>Multiple Batches</th></tr>
<tr><td rowspan="2">Equation</td><td colspan="2">▪ the M and N have appropriate values given the method- M and N might be switched in position but values are appropriate</td></tr>
<tr><td>▪ going through one part: M is 5, N is 40/7
▪ whole group: M is 5/7 and N is 40</td><td>▪ unit rate batch: M is 40 and N is 5/7
▪ one batch: M is 40/7 and N is 5

Notes:
▪ unit rate batch: M is 40 and N is 5/7 (when strip diagrams are used with multiply unit rate batch method)
▪ one batch: M is 40/7 and N is 5 (when strip diagrams are used with multiply one batch method)</td></tr>
<tr><td rowspan="2">Math Drawing</td><td colspan="2">▪ show total amount of gold and copper: DNL or strip diagram
▪ indicate target amount (e.g., tick mark for 40 grams gold and ? grams copper or 40 grams copper and ? grams gold)</td></tr>
<tr><td>▪ strip diagrams with correct number of parts with respect</td><td>▪ one batch: DNL indicate initial batch (e.g., tick mark for 7 grams</td></tr>
</table>

	to the problem (i.e., copper is 5 parts and gold is 7 parts)	gold and 5 grams copper) ▪ **unit rate batch:** DNL indicate unit-rate batch (e.g., tick mark for 1 gram gold and 5/7 grams copper OR 1 gram copper and 7/5 grams gold)
Any part of solution (equation, explanation or math drawing):	▪ indicate 1 group and base units in 1 group (i.e. they need to identify the units as grams)	
Notes	▪ some future teachers misread problem so they interpreted the problem as 40 grams copper instead of 40 grams gold and solved accordingly. So, if their work is appropriate for these numbers, then use same criteria ▪ some future teachers misread problem so they interpreted the problem as 7 parts copper instead of 7 parts gold and 5 parts gold instead of 5 parts copper and solved accordingly. So, if their work is appropriate for these numbers, then use same criteria ▪ some future teachers provide only equation and math drawing (without explanation), which is considered to be acceptable ▪ sometimes infer information based on other solutions provided (e.g., infer groups and base unit in 1 group for Candace's third and fourth solutions given annotation for first two solutions)	
Criteria for Category 1		
▪ Not meet the criteria for category 2		
Criteria for Category 3		
▪ Meets all criteria Category 2 ▪ Division in the equation and/or written explanation ▪ Use of division operation sign ÷ ▪ Mention division (e.g., distributing 40 grams evenly among 7 parts or finding the amount of one part ▪ Use the multiplication with a missing factor (e.g., 7 • ? = 40)		

Figure 14. Coding Schema

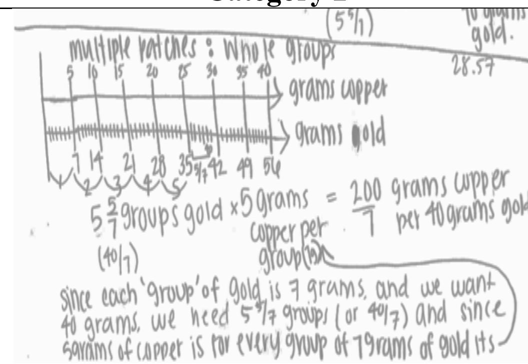
In order to see how I rated a solution with respect to my coding schema, I present Figure 15 that shows a sample work for the Gold and Cooper task and the multiply one batch method. In the example, I placed AH's work in category 2 because she included an equation that included appropriate values and units for M and N and her drawing showed the total amount of copper and gold. MJ included the same steps as AH and also identified QDS. So I placed her work is category 3. On the other hand, I placed JP's work in category 1 because JP used the one batch method, but he did not provide an equation with respect to his method. Since for the one batch method, the equation should be 40/7• 5 by considering M and N. The solution and method are not consistent.

<table>
<tr><th>Category 1</th><th>Category 2</th><th>Category 3</th></tr>
<tr>
<td>


JP's solution

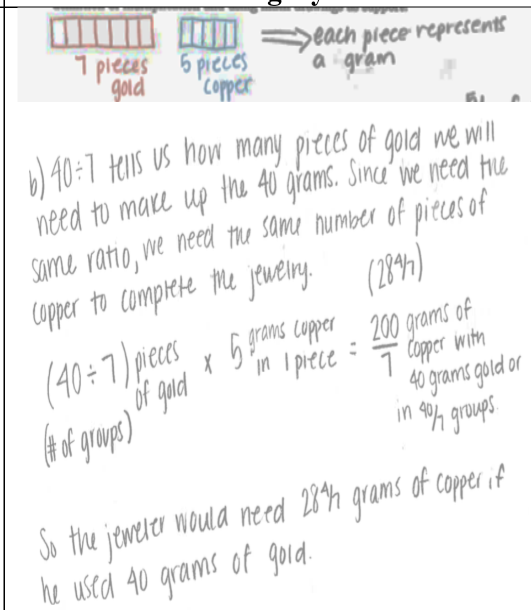
Include an equation by considering the M and N which have appropriate values given the method where M = 5/7 ("groups"), N = 40 ("# grams in one group"), and P = 200/7 ("# grams in 5/7 groups").
Use of division (i.e. 5÷7)
Identification of QDS
Indicator: division
Identification of MS with the equation
Use the mathematical drawing

show total amount of gold and copper
DNL indicated target amount (e.g., tick mark for 40 gold)
DNL indicated initial batch (e.g., tick mark for 7 gold and 5 copper)

Mention "gram" in the equation part
JP used the one batch method, but he did not provide an equation with respect to his method. Since for the one batch method, the
</td>
<td>


AH's solution
Include an equation by considering the M and N which have appropriate values given the method where M = 40/7 ("groups gold"), N = 5 ("copper per group"), and P = 200/7("grams copper per 40 grams gold")
No use of the division (i.e. 40÷7)
No evidence for identification of QDS
Identification of MS with the equation
Use the mathematical drawing

show total amount of gold and copper
DNL indicated target amount (e.g., tick mark for 40 grams of gold)

DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper)
</td>
<td>


MJ's solution

Include an equation by considering the M and N which are M = 40/7 ("pieces of gold-# of groups"), N = 5 ("grams copper in 1 piece"), and P = 200/7 ("grams of copper with 40 grams gold or in 40/7 groups").
M and N were switched in position
Use of division (i.e. 40÷7)
Identification of QDS
Identification of MS with the equation
Not use the appropriate mathematical drawing

show total amount of gold and

</td>
</tr>
</table>

equation should be 40/7• 5 by considering M and N. The solution and method are not consistent.		copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in the math drawing part ▪ mention "…we will need to make up the 40 grams…" in the explanation part.

Figure 15. Samples for variable parts perspective with multiply one batch method

CHAPTER 5

RESULTS

This study presents three main results. First, future middle grade teachers who completed the two-semester sequence of content courses emphasizing topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication were able to use appropriately the multiple-batches and variable-parts perspectives; and, they tended to use the variable-parts perspective instead of the multiple-batches perspective when allowed to chose their own method. Second, the future teachers were most successful coordinating the class meaning of multiplication and division with their math drawings, equations, and explanations when using the multiply one-part method with the variable-parts perspective. Third, the future teachers were better at identifying PDS than identifying QDS, whereas all future teachers identified MS. The following sections present descriptive results, results in terms of perspectives, results for labeling units, results for division, interesting solutions, and solutions that were challenging to classify.

Descriptive Results

Table 1 presents counts for classification of solutions to the Gold and Copper task. Recall that the task asked for two solutions. The counts in Table 1 show that 44 solutions were provided by 22 future teachers: 19 future teachers used two different methods, two future teachers used one method, and one future teacher used four methods, as shown in Table 1. According to these results, the future teachers used the variable-parts perspective in 30 solutions and the multiple batches perspective in 14 solutions. I placed 27 of the solutions for the Gold and Copper problem in category 2, 16 solutions in category 3, and three solutions in category 1.

Table 1

Frequency of each method by category

Perspective	Total	Method	Cat. 1	Cat. 2	Cat. 3	Total
Variable-Parts Perspective	**29**	Multiply Total Amount	0	11	1	**12**
		Multiply One Part	1	5	11	**17**
Multiple-Batches Perspective	**15**	Multiply One Batch	2	4	2	**8**
		Multiply Unit-Rate Batch	0	6	1	**7**
	Total		3	26	15	**44**

The total number of solutions in which future teachers used the variable-parts perspective with the multiply-total-amount method was 12, whereas the total number of solutions in which future teachers used the variable-parts perspective with multiply-one-part method was 17. Additionally, the total number of solutions in which the future teachers used the multiple-batches perspective with multiply one batch method was 8, and the total number of the solutions in which future teachers used the multiple-batches perspective with multiply unit rate batch method was 7. Some future teachers used the multiple batches perspective with multiply one batch method logically in combination with a strip diagram instead of a DNL and some future teachers used the multiple batches perspective with multiply unit-rate batch method logically in combination with a strip diagram instead of a DNL. Fifteen of he 44 solutions included explicit use of division. I identified PDS in 13 solutions and, QDS in two solutions.

Results in terms of Perspectives

In this study, a solution was rated with respect to the equation, math drawing, and explanation. I used three categories, and category 2 was the standard level. In order to be rated in category 2, the equation should include the multiplier, M, and multiplicand, N, with the appropriate values according to the given method, and a math drawing should show total amount of gold and copper. Additionally, the solution should indicate one group and base units in one group in any part of the solution (e.g., equation, math drawing, or explanation). When some future teachers misread the Gold and Copper problem but solved appropriately according to their interpretation, I placed the solutions in category 2. If solutions did not meet criteria for category 2 (see the coding scheme), I placed them in category 1. As well as meeting the category 2 criteria, if the future teachers made explicit use of division in their solutions, I placed their solutions in category 3. In the following sections, I present some examples from each category according to the selected perspective and method.

Category 2 for Variable-Parts Perspective with the Multiply Total Amount Method

Future teachers who used the variable parts perspective with the multiply total amount method included an equation which mainly included appropriate values for M and N (i.e., M is 5/7 and N is 40). I mainly checked how they described the M, N, and P in their equations. For instance, the future teacher LM defined M = 5/7 as "# of groups", N = 40 as "# of grams in one whole group", and P = 200/7 is "# grams in 5/7 group". In this solution, future teacher did not use division, so there is no evidence for QDS. On the other hand, LM showed the total amount of gold and copper in the math drawing (see Figure 16).

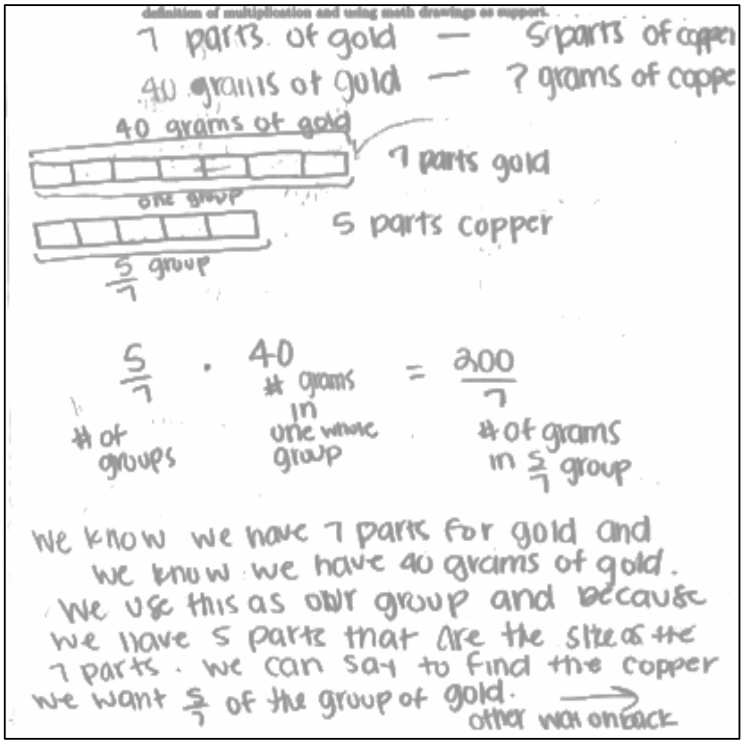


Figure 16. LM's solution

Category 3 for Variable-Parts Perspective with the Multiply Total Amount Method

In some cases, as well as providing very careful drawings, equations, and explanations, future teachers gave explicit indications of division. As it is seen in Figure 17, LF used the division symbol to indicate division, identified QDS, and supported her answer with a drawing that fit with the selected perspective and strategy.

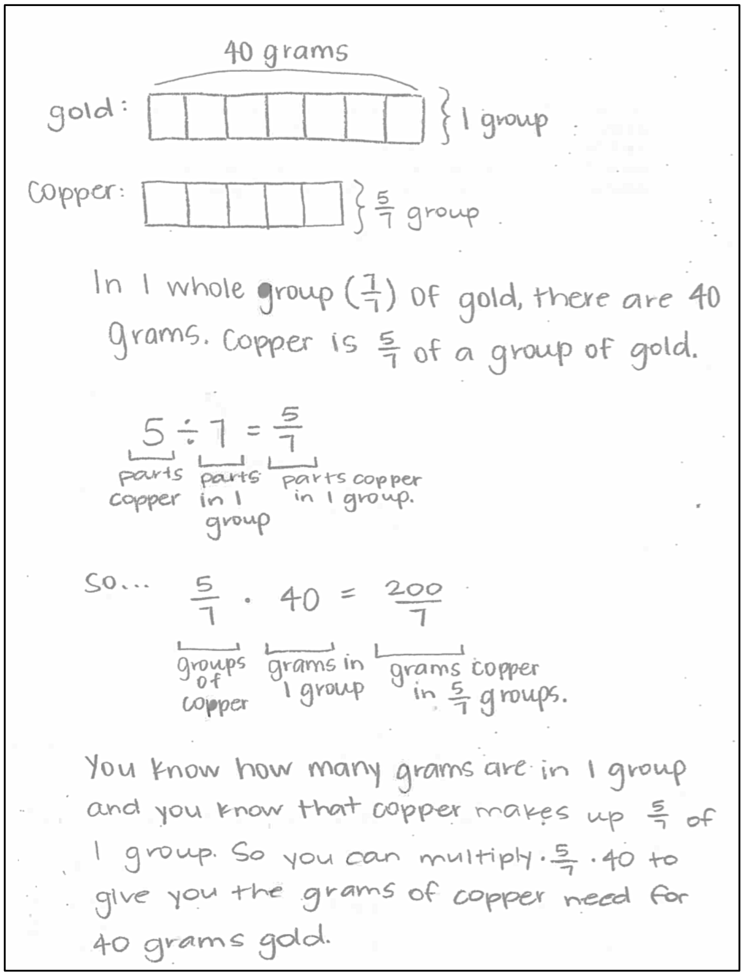


Figure 17. LF's solution

Category 2 for Variable-Parts Perspective with the Multiply One Part Method

Future teachers who used the variable parts perspective with the multiply one-part method included an equation with appropriate values for M and N (i.e., M = 5, N = 40/7, and P = 200/7). The future teacher BM stated M is "# of groups", N is "units per group", and P is "amount of copper needed" (Figure 18).

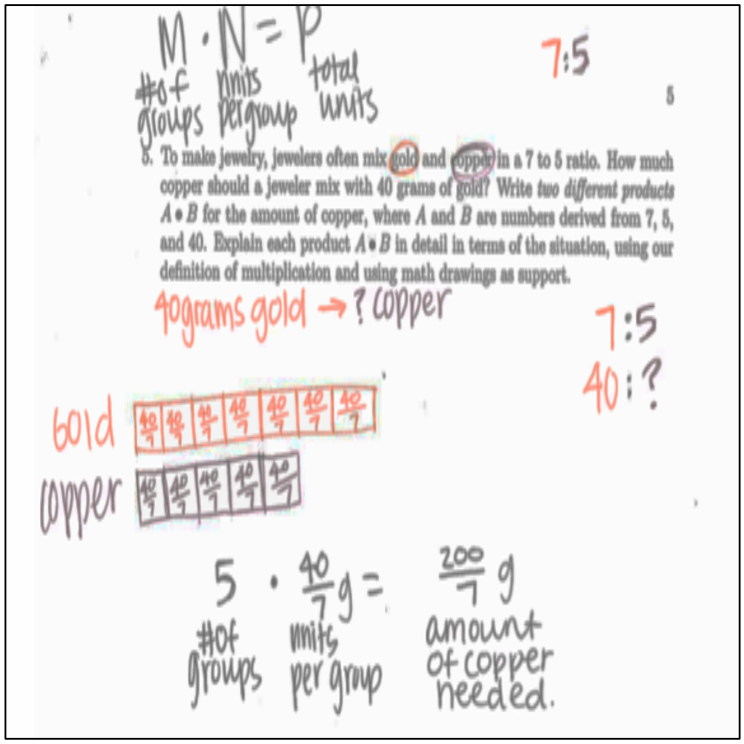


Figure 18. BM's solution

Category 1 for Variable-Parts Perspective with the Multiply One Part Method

Although AA misread problem, interpreting the problem as stating there are 7 parts copper instead of 7 parts gold and 5 parts gold instead of 5 parts copper, she solved the problem accordingly in the first equation. However, in her second equation (i.e. 7 • 8), there is problem since the units are not appropriate. She stated that 7 is both grams of copper and the number of group, and it is very ambiguous. Thus, there is no evidence that this future teacher could state units appropriately. Therefore, it was rated in category 1 (see Figure 19).

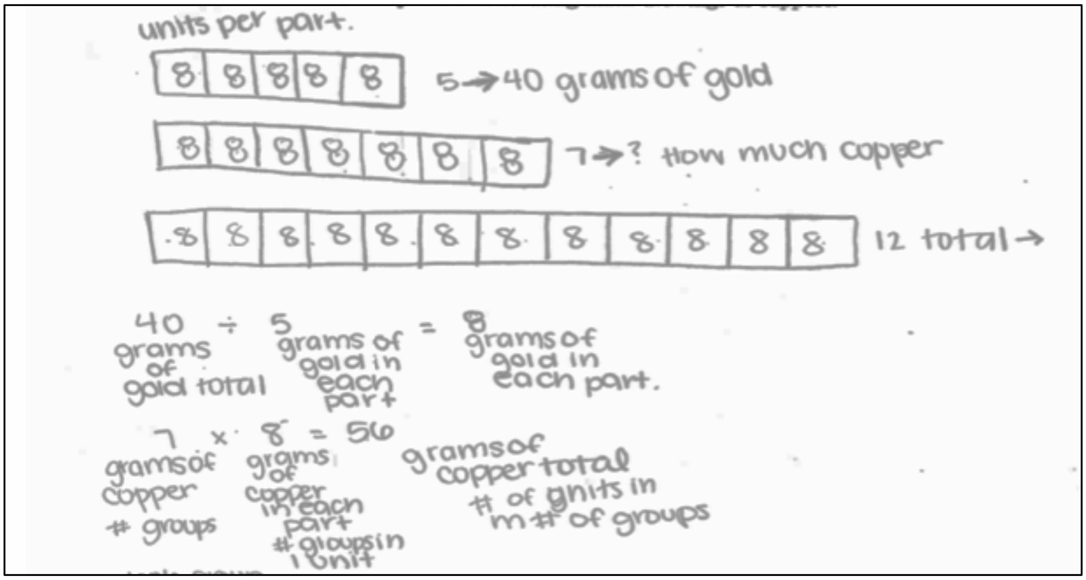


Figure 19. AA's solution

Category 3 for Variable-Parts Perspective with the Multiply One Part Method

I placed solutions that included not only equations with appropriate M and N values and consistent drawings but also explicit discussion of division in category 3. KA is one of the future teachers who provided explicit indicators of division with the division sign and the statement "how many grams of gold I can split into 7 parts when I have 40 grams of gold," and thus identified PDS (see Figure 20).

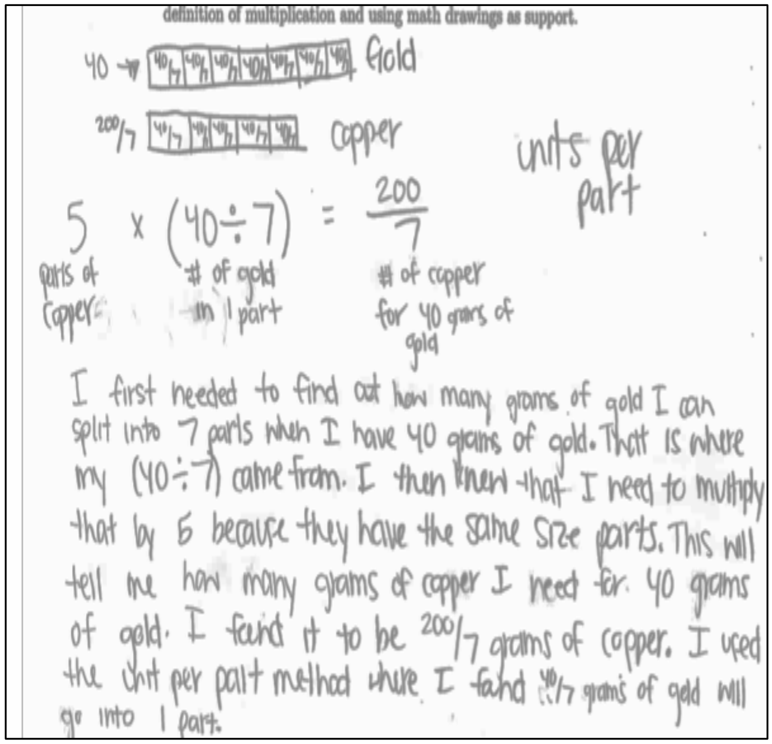


Figure 20. KA's solution

Category 2 for Multiple-Batches Perspective with the Multiply One Batch Method

Future teachers who used the multiple-batches perspective with the multiply one batch method included an equation which mainly included appropriate values for M and N (i.e., M = 40/7, N = 5, and P = 200/7). Figure 21 includes AH's solution using the one batch method that included explicit descriptions for M, N, and P such as M is "groups gold", N is "grams copper per group", and P is "grams copper per 40 grams gold."

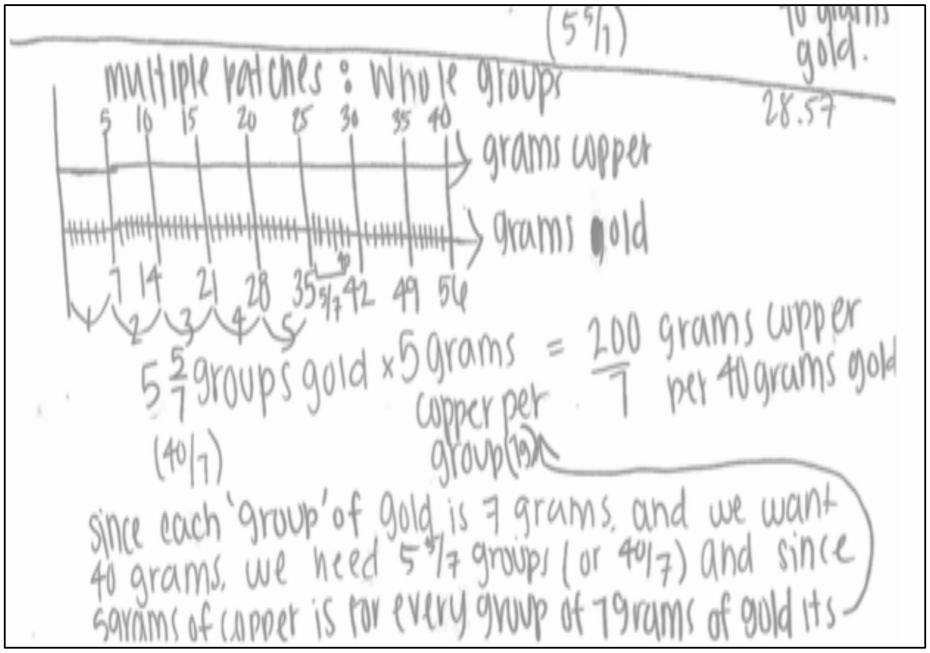


Figure 21. AH's solution

In Figure 22, JP used strip diagrams with the multiply one batch method because of the statement 7 gold: 40 gold. JP's solution met all criteria for category 2 (see Figure 22). This future teacher appeared to use the multiple-batches perspective with strip diagrams that in instruction were only discussed in combination with variable parts.

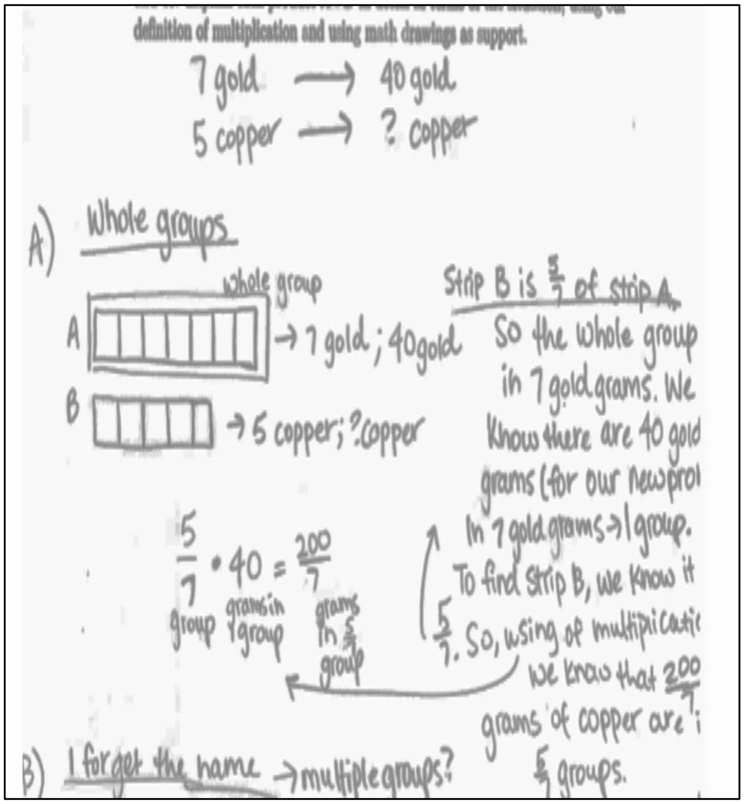


Figure 22. JP's solution

Category 1 for Multiple-Batches Perspective with Multiply One Batch Method

Whereas AH provided appropriate values for M, N, and P, JP's equation did not coordinate his drawing with his method appropriately. Because for the one batch method, the equation should be 40/7 • 5, the solution and method were not consistent (see Figure 23).

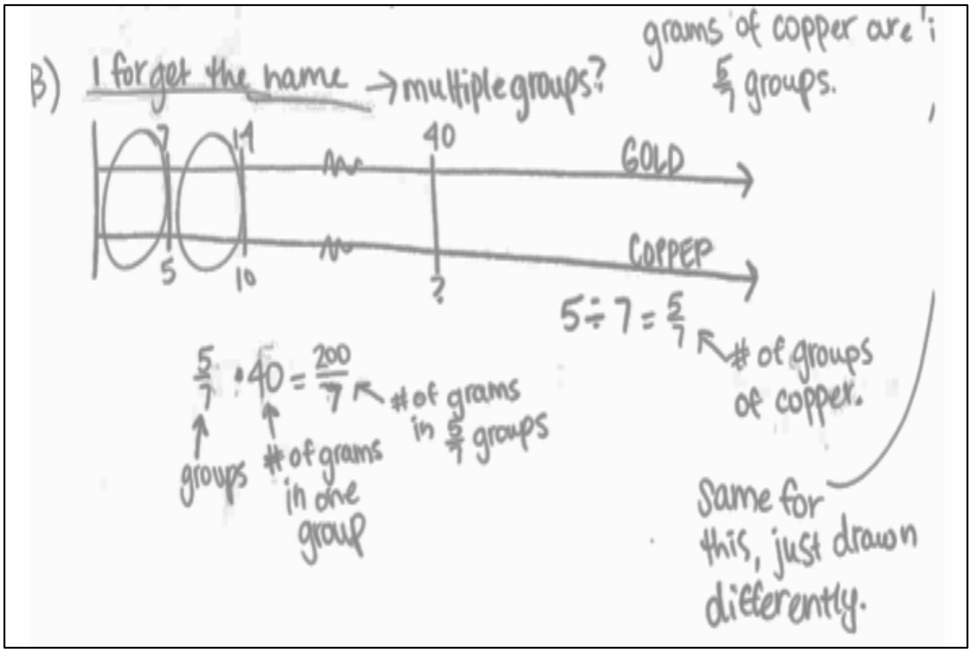


Figure 23. JP's solution

Category 3 for Multiple Batches Perspective with Multiply One Batch Method

In this part, I present CP's solution since CP solved the problem in two ways and used division. It seems that CP understood the mixture of gold and copper in a 7 to 5 ratio and that there was 40 grams of gold but then switched the ratio 5 to 7 for gold and copper. One explanation might be that she understood she needed to switch the ratios when asked to provide two different products, A • B, where A and B are numbers derived from 7, 5, and 40 (see Figure 24). CP used multiple batches perspective with multiply one batch method. Even though there was an inconsistency between the answers, CP's solution met the criteria for the category 2. Also, CP used division in the both solutions accordingly; therefore, I placed this solution in category 3.

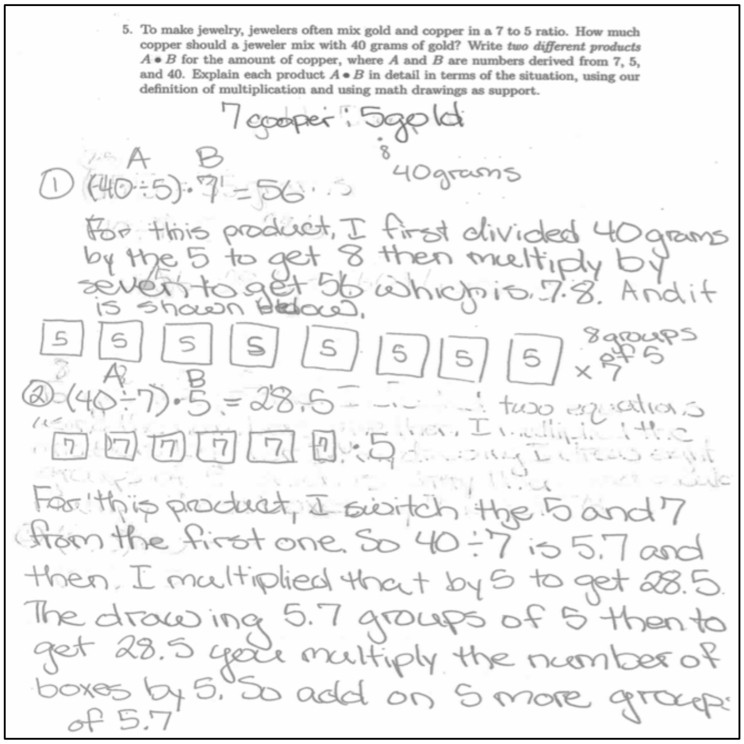


Figure 24. CP's solution

Category 2 for Multiple-Batches Perspective with the Multiply Unit-Rate Batch Method

Future teachers who used the multiple-batches perspective with the multiply unit-rate batch method included an equation which mainly included appropriate values for M and N (i.e., M is 40, N is 5/7, and P is 200/7). In Figure 25, KC used the mathematical drawing, showed total amount of gold and copper. More specifically, in KC's solution, DNL indicated target amount (e.g., tick mark for 40 grams of gold) and DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper). I placed KC's solution in category 2 because this solution met all criteria for category 2, and there was no explicit indicator for division (see Figure 25).

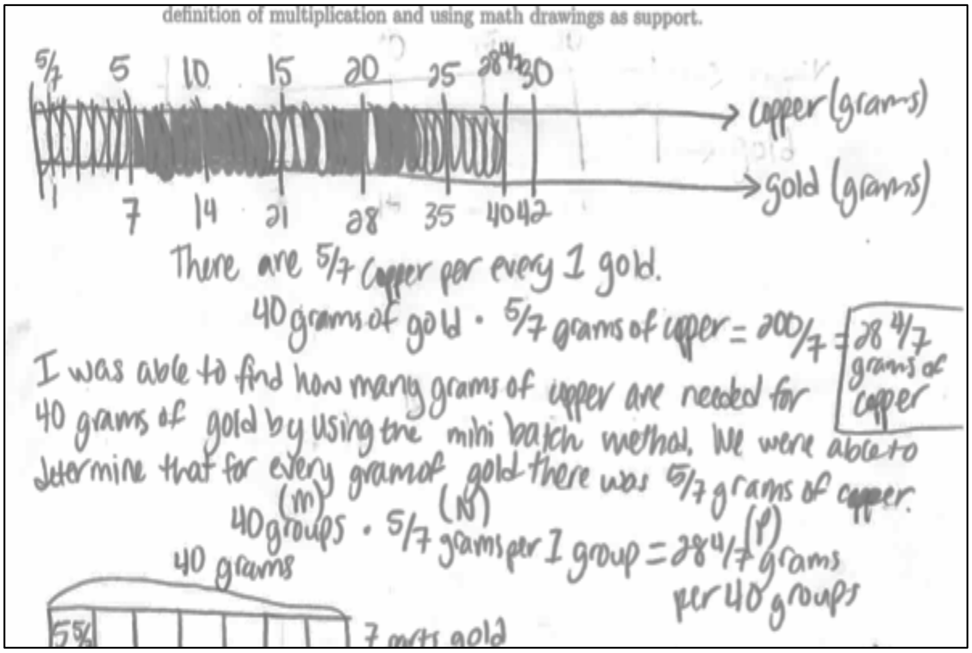


Figure 25. KC's solution

Two future teachers appeared to use the multiple-batches perspective with strip diagrams that in instruction were only discussed in combination with variable parts. In these solutions, the future teachers determined the number of groups and then iterated 40 times consistent with the multiple-batches perspective and the multiply unit-rate batch method. Thus, these future teachers mixed aspects of different methods. In these solutions, the future teachers included an equation by considering the M = 40 and N = 5/7. In both solutions there was no explicit indication of division through notation, such as 5 ÷ 7, or in the explanation part, or through a multiplication equation with a missing factor (see Figures 26 and Figure 27)

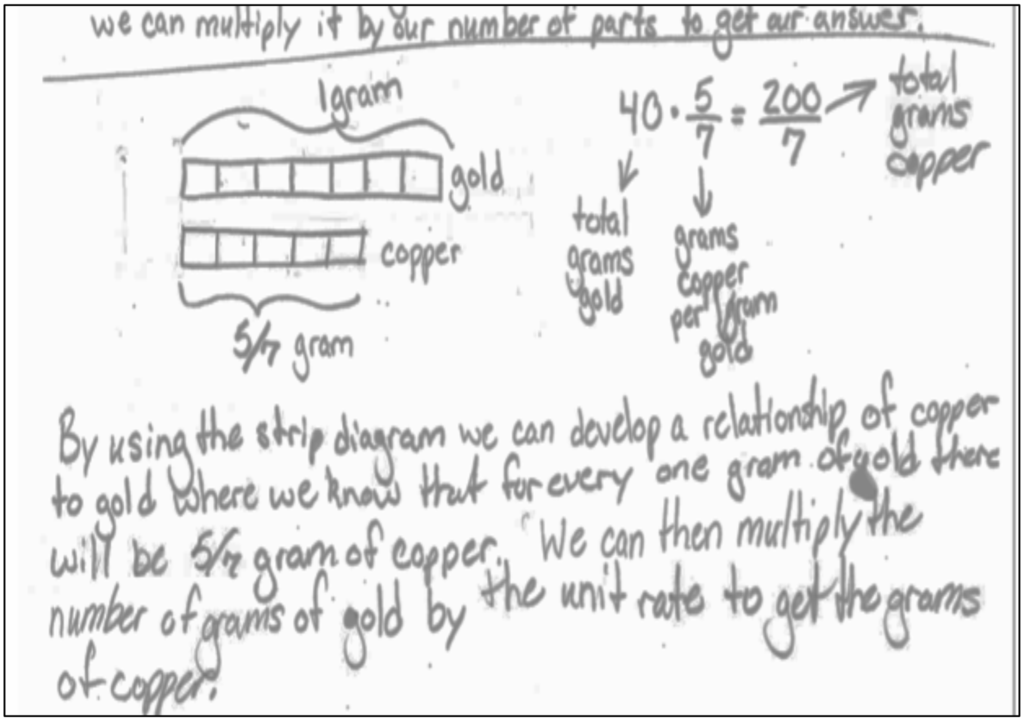


Figure 26. CB's solution

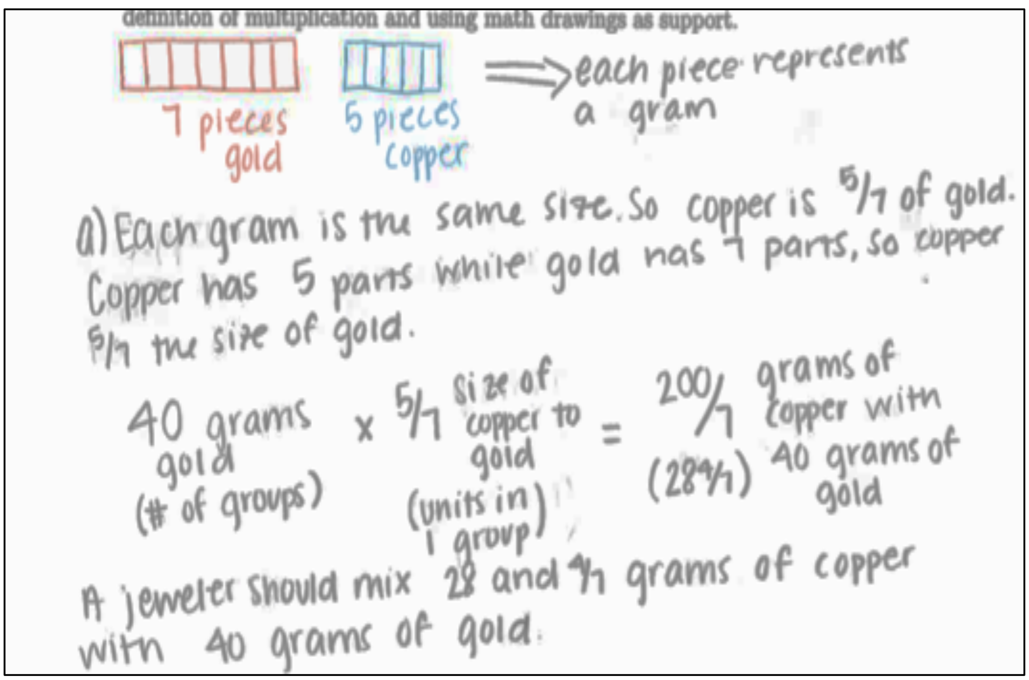


Figure 27. MJ's solution

Category 3 for Multiple-Batches Perspective with the Multiply Unit-Rate Batch Method

In Figure 28, MU provided the division indicator 5 ÷ 7 and identified PDS in the solution, so I placed the solution in the category 3. For labeling 40, MU mentioned the word gram in the explanation part. No solutions were placed in category 1 for the unit-rate batch method. This shows that at least the future teachers who used the unit-rate batch method used M and N with appropriate values and units and they showed the total amount of gold and copper.

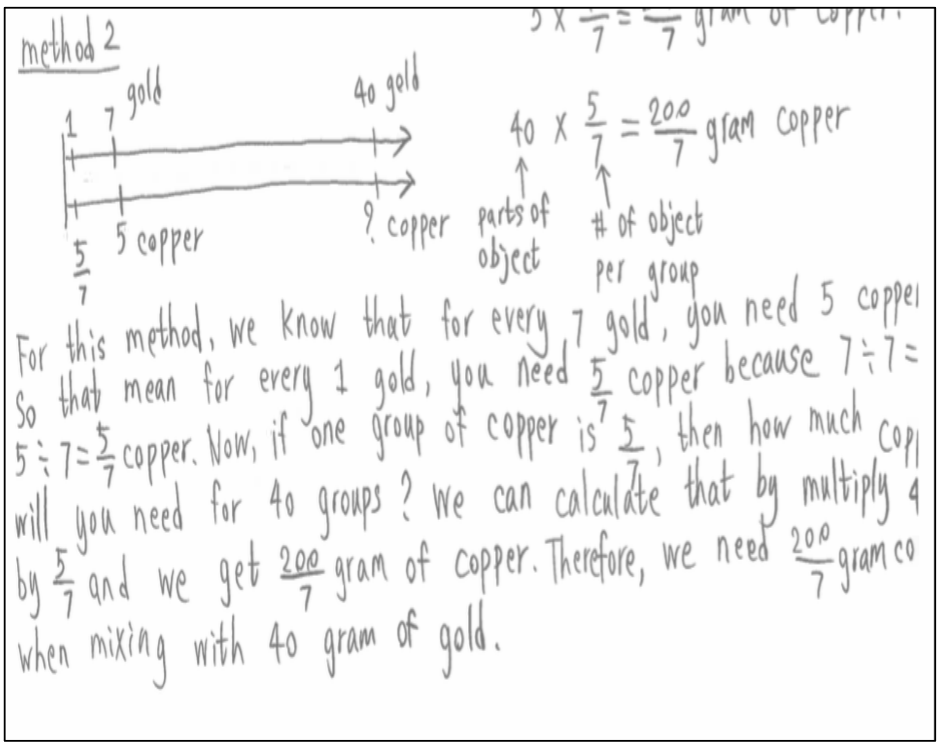
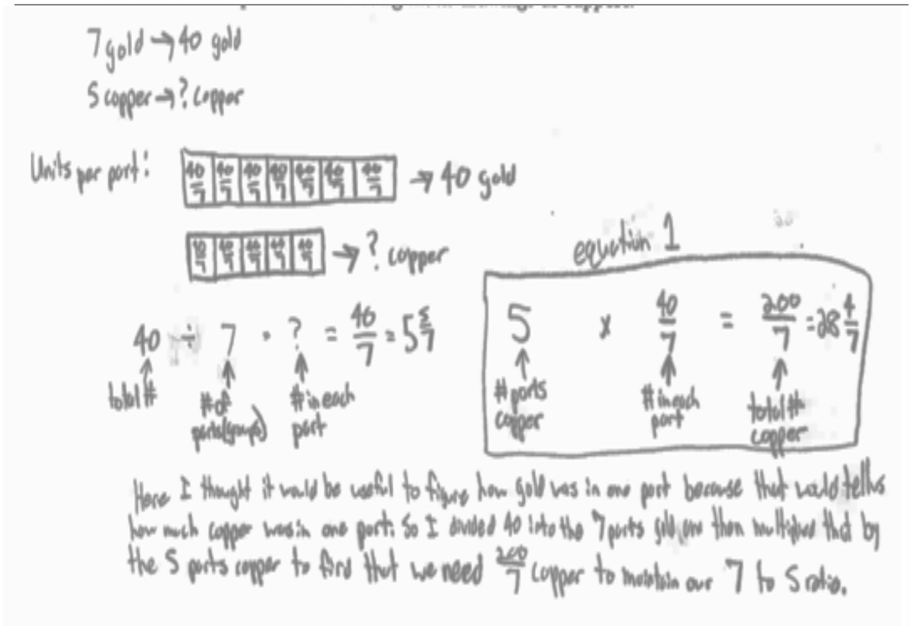


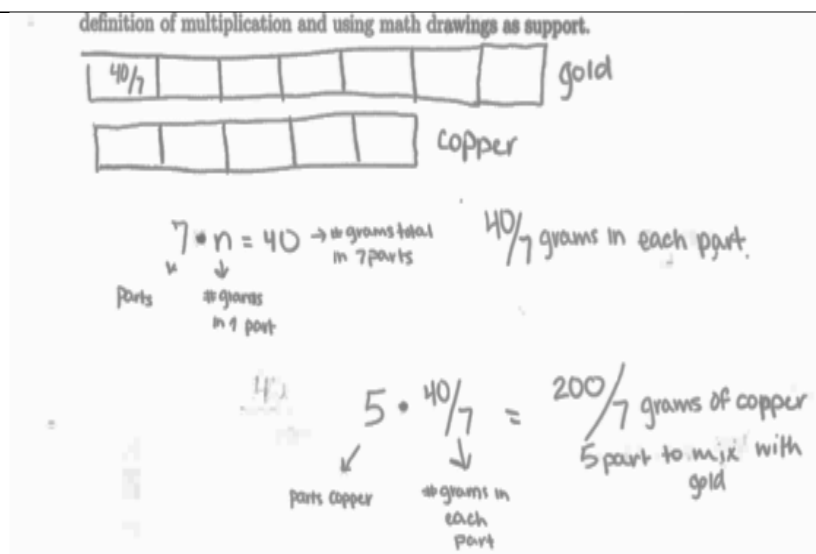
Figure 28. MU's solution

Results for labeling units (e.g., use of "gram")

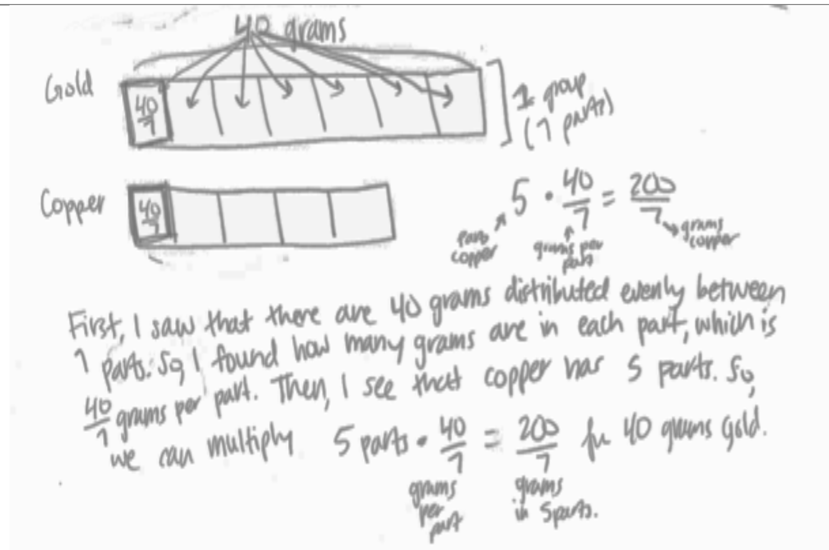
I carefully evaluated every detail of the future teachers' written work since each was important for categorizing the solution; for instance, identifying units appropriately was important for my analysis. When all other criteria were met for category 2, I placed solutions in category 1 when future teachers did not use appropriate labels for units. In particular, I looked to see if future teachers mentioned "grams" in any place of their solution (e.g., 40 grams). Figure 28 shows three samples that illustrate differences in terms of labeling units with the word grams. PM did not indicate grams in any place of her solution, whereas BB mentioned 40 as grams in the equation and LB showed "40 grams" explicitly in her drawing (see Figure 29).



(a) PM's solution



(b) BB's solution



(c) LB's solution

Figure 29. Categorization sample in terms of the word "gram" (a) PM's solution, (b) BB's solution, (c) LB's solution.

Results for Division

For category 3, I used three indicators for division. In three solutions, future teachers used a division sign, in four solutions future teachers used a division statement, and in one solution a future teacher used multiplication with a missing factor. The last of these was one way that division was indicated in class instruction. Also, in seven solutions future teachers used division in a statement and multiplication with a missing factor. Figure 30 shows each indicator separately.

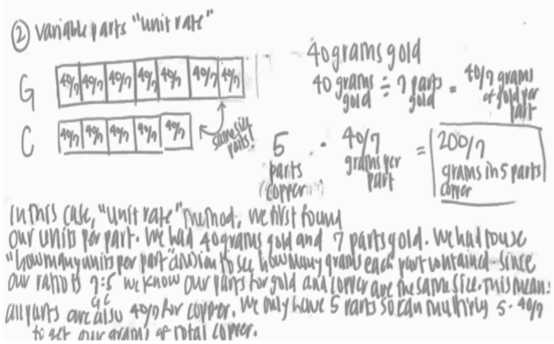
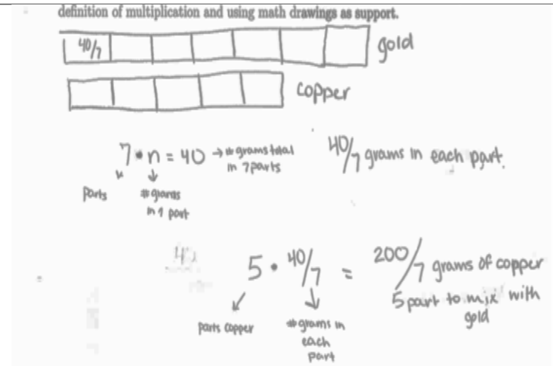
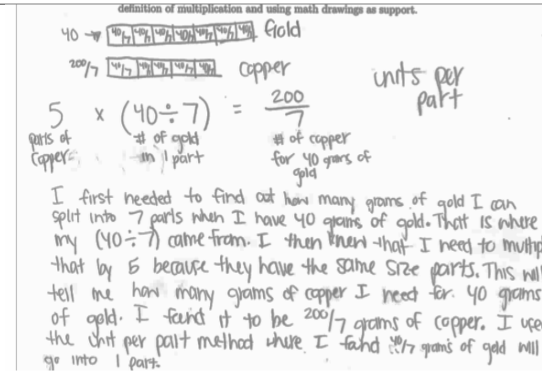
 BK's solution Indicator: Use of division operation sign ÷	 LF's solution Indicator: Use the multiplication with a missing factor (e.g., 7 • n = 40)	 KA's solution Indicator: Mention division (e.g., distributing 40 grams evenly among 7 parts or finding the amount of one part and used of division operation sign ÷)

Figure 30. Indicators for division

Interesting Results

In this section, I present some results which are just employed by only one student and some of them were not included in the instruction during the semester.

JG was the only future teacher who provided “1 group gold • 40 g in 1 group = 40 g gold” in her equation (Figure 31). This result was interesting because the future teacher developed her own equation (1 group gold • 40 g in 1 group = 40 g) without any instruction. That was crucial to see to understand how JG coordinated 40 grams of gold and 1 group with her drawing.

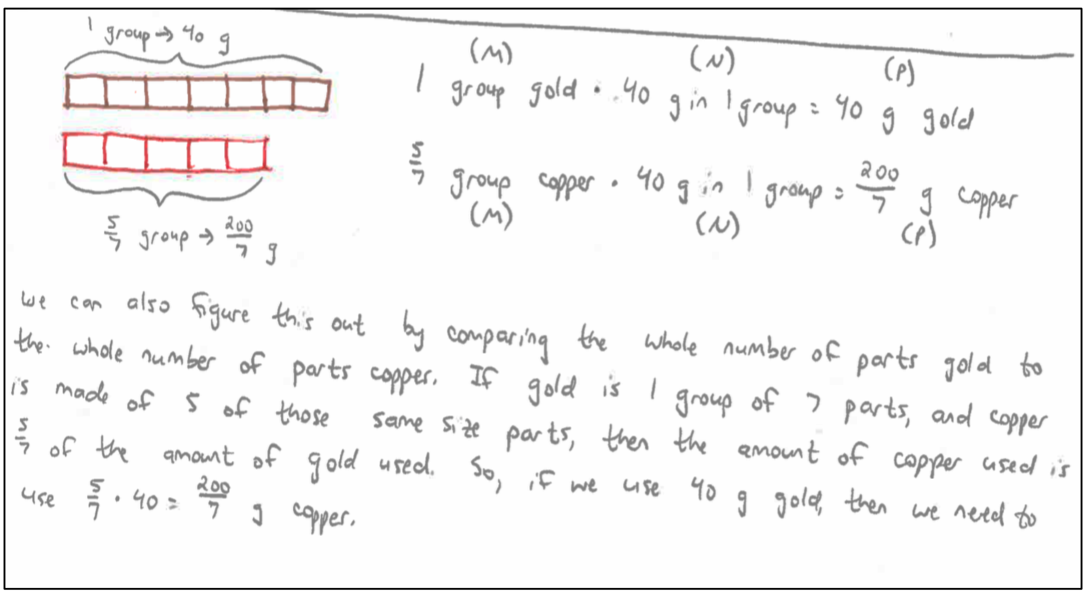


Figure 31. JG’ s solution

CS was the only future teacher who solved the problem using all four methods (Figure 32). Even though CS did not include substantial explanation for each method respectively, this solution indicated that CS was able to differentiate and apply the four methods appropriately.

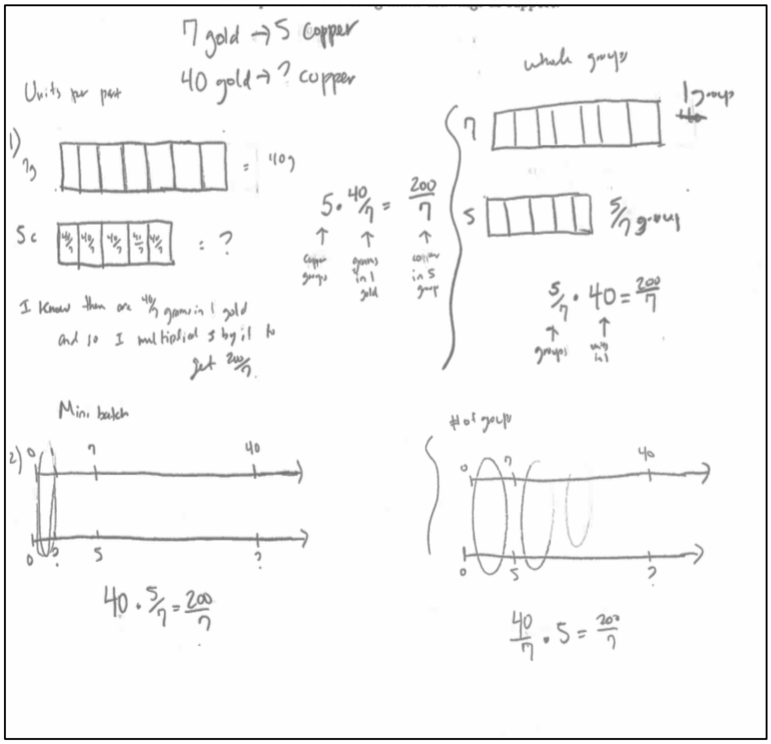


Figure 32. CS's solution

I placed AA' s solution in category 2 since M and N had appropriate values given the method even though AA reversed the gold and copper (Figure 33). AA's reasoning was appropriate for her method (see the coding scheme).

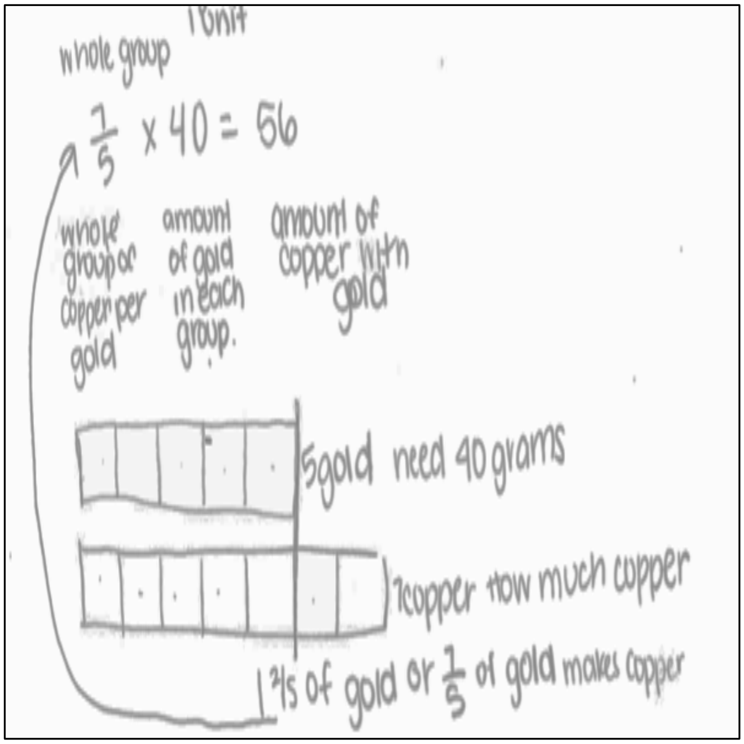


Figure 33. AA's solution

It is also important to note that LF is the only future teacher who used the meaning of division and identified QDS by using multiply total amount method with variable parts perspective (see Figure 34). While most of future teachers who used the meaning of division by using multiply one-part method, just LF used the meaning of division by using multiply total amount method.

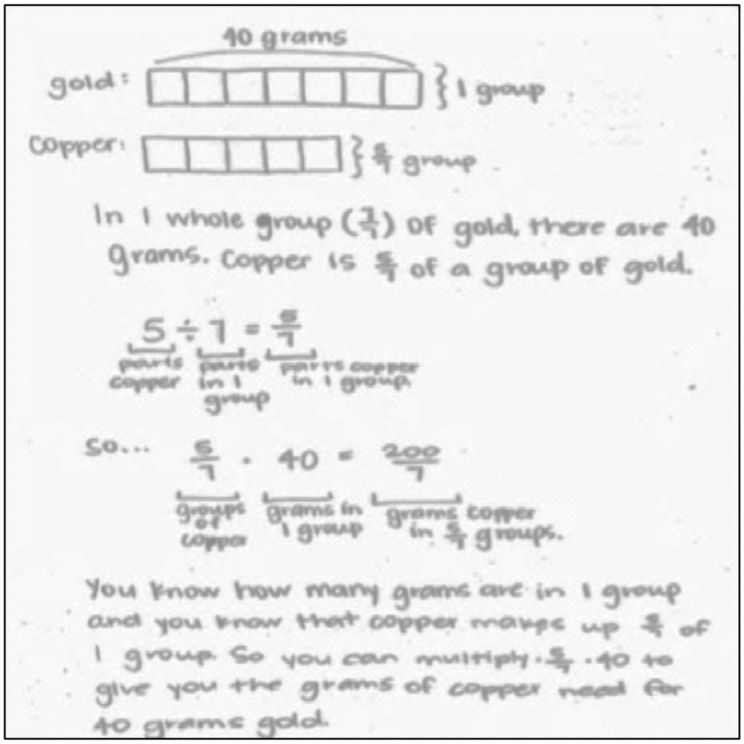


Figure 34. LF's solution

MW used the multiple-batches perspective with multiply one batch method (Figure 35). I placed this solution in category 2. Even though she did not provide a standard drawing such as a DNL for the multiple batches perspective, she represented it with two-column table by considering the ratio 7 to 5 between gold and copper. In category 2, a DNL should indicate initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper) as one of criteria for the drawing. In her table, MW indicated the 7 as gold and 5 as copper.

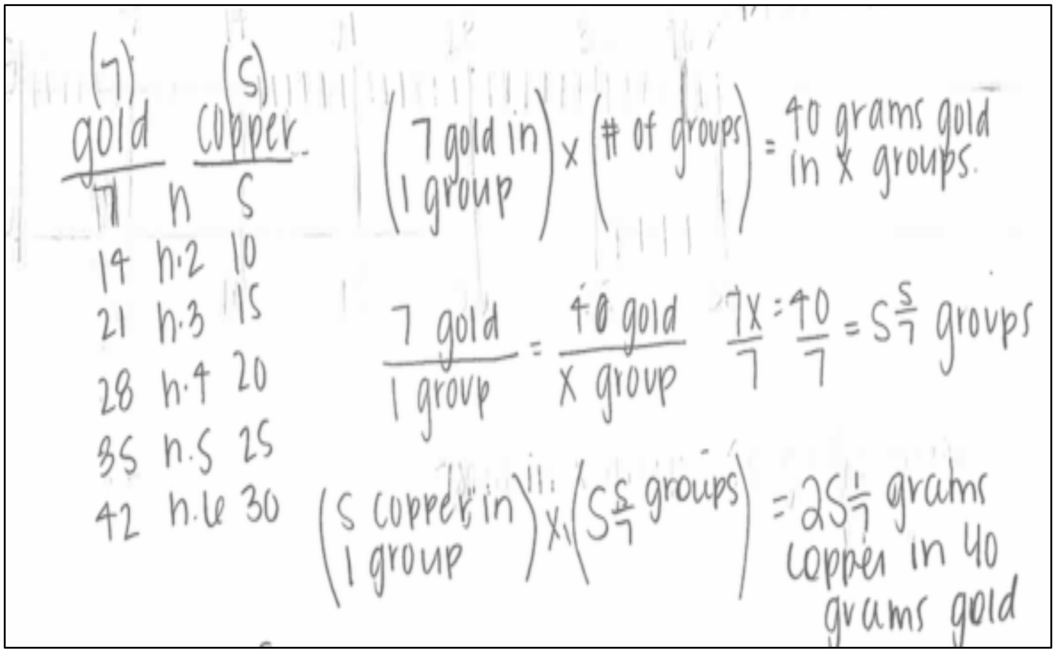


Figure 35. MW's solution

Solutions that were challenging to classify

The appendix shows the categories into which I placed all 44 solutions. Applying my coding scheme was not straight forward and there were examples that did not fit well into the three categories. In these cases, I compared the solution to other relevant solutions and discussed the solutions with three experts until we agreed on the category for every solution. I present some of these difficult cases in this section.

For some solutions, it was challenging to decide which category was the best fit. For example, CS included all four methods on piece of paper (Figure 36). She provided units for M, N and P for some equations but not others. Furthermore, CS used an equation for the multiply whole group method (top right method in Figure 36) with appropriate values and units for M and N. She wrote M = 5/7 ("groups"), N = 40 ("units in 1"), and P = 200/7; identified MS; and drew an appropriate strip diagram. Although CS did not show total amount of gold and copper for this method, she showed it for other methods, so I granted that CS indicated total amounts of gold and copper and placed her solution in category 2. CS also showed four methods and identified them well without explaining. Even though I would have liked to see the total amount of the gold and copper identified separately for each method, I finally decided it was enough to mention the total amount at least once.

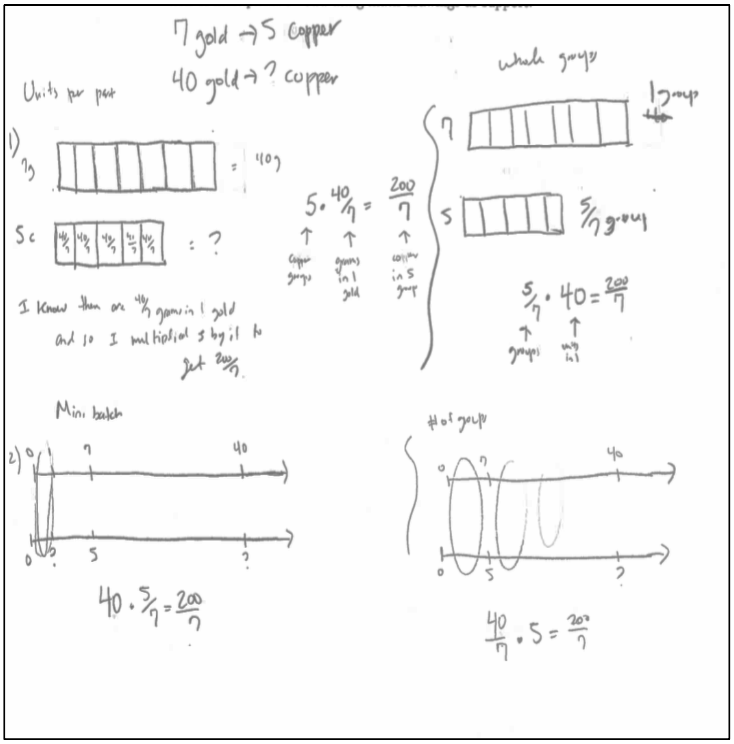


Figure 36. CS's solution

Furthermore, AA included an equation by considering M and N with appropriate values given the method where M = 7/5 ("whole group or copper per gold"), N = 40 ("amount of gold in each group"), and P = 200/7 (amount of copper with gold). According to criteria for category 2, some future teachers misinterpreted the problem as 7 parts copper instead of 7 parts gold and 5 parts gold instead of 5 parts copper. Because AA's solution was consistent with this interpretation of the jewelry gold situation, I placed the solution in category 2 (see Figure 37).

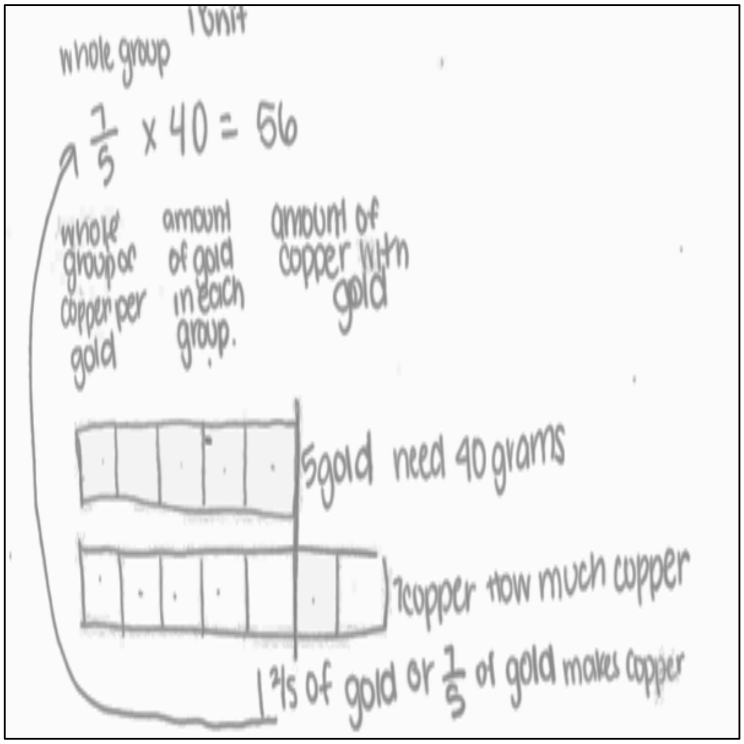


Figure 37. AA' solution

AG included an equation by considering the M and N which have appropriate values given the method where M = 5/7 ("strip"), N = 40 ("grams"), and P = 200/7 ("grams copper"). Even though she used unusual annotation for the equation, there was enough explanation "…we can say that copper strip is 5/7 of the gold strip. And because we know that one gold strip carrying 40 grams, we can set up a multiplication problem." This shows that she took one strip to be one whole group and used the word "strip" instead of the word "group" (see Figure 38).

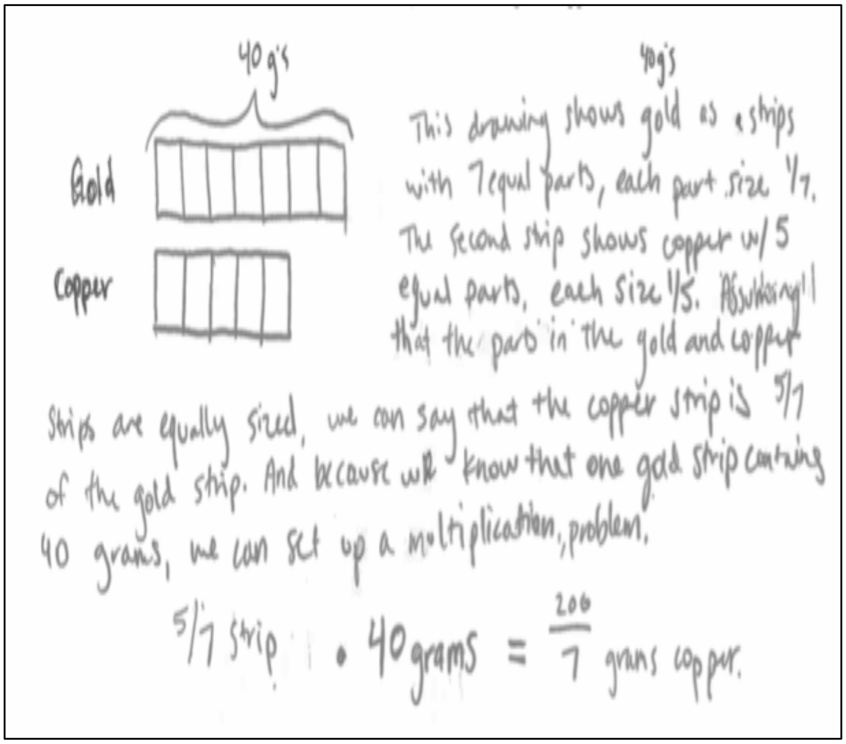


Figure 38 AG's solution

JP included an equation by considering the M and N with appropriate values given the method where M = 5/7 ("group"), N = 40 ("grams in 1 group"), and P = 200/7 ("grams in 5/7 group"), identified MS, and used a strip diagram. Although this solution seemed to be in category 2, she mentioned "So the whole group in 7 gold grams" instead of mentioning 7 parts stand for 40 grams of gold. Therefore, I placed this solution in category 1 (see Figure 39).

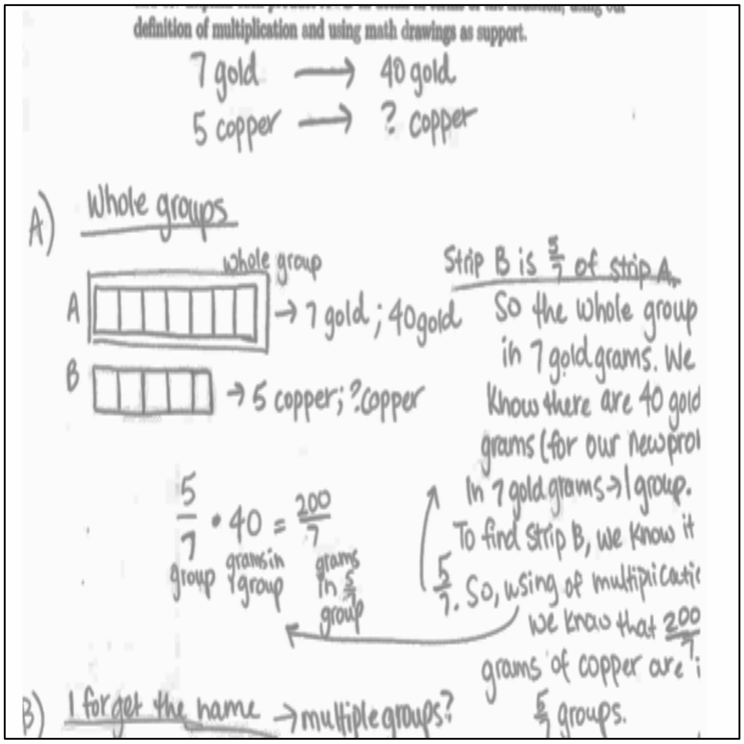


Figure 39. JP's solution

I placed JP's other solution in category 1 because she used the one batch method but did not provide an appropriate equation. Because for the one batch method, the equation should be 40/7 • 5, the solution and method are not consistent (see Figure 40).

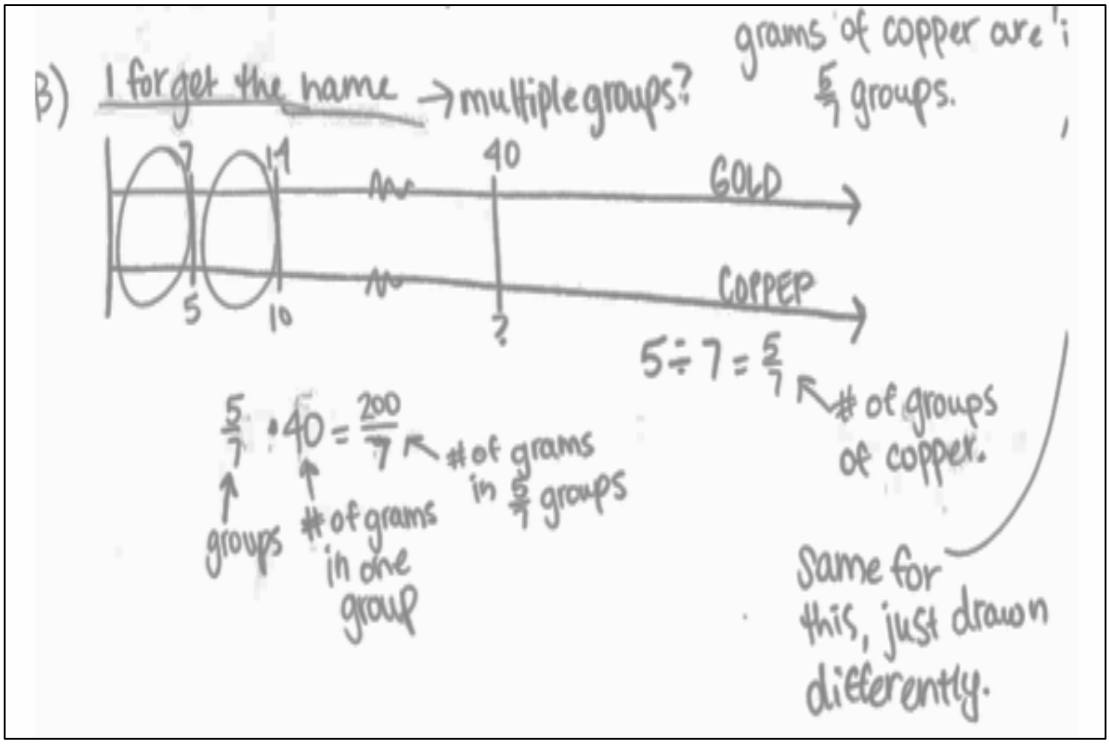
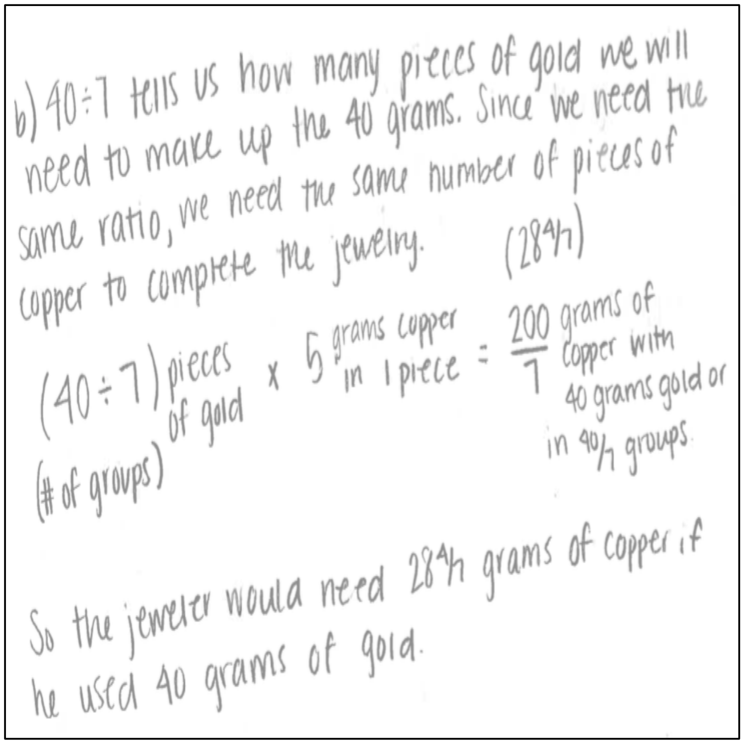


Figure 40. JP's solution

It was hard to decide if future teachers defined appropriate units for M, N, and P. I showed how future teachers' solutions were different with respect to precision about the units. Although MJ' s work met most of the criteria in category 2, similar to BM's work, MJ mentioned that 40 ÷ 7 is the "pieces of gold". That seemed unreasonable at first, but then I decided that it would fit with a "piece" being like a piece of jewelry: Each piece of jewelry contains 7 grams of gold and 5 grams of copper (see Figure 41).


b) 40÷7 tells us how many pieces of gold we will need to make up the 40 grams. Since we need the same ratio, we need the same number of pieces of copper to complete the jewelry. $(28\frac{4}{7})$

$(40 \div 7)$ pieces of gold (# of groups) x 5 grams copper in 1 piece = $\frac{200}{7}$ grams of copper with 40 grams gold or in $\frac{40}{7}$ groups.

So the jeweler would need $28\frac{4}{7}$ grams of copper if he used 40 grams of gold.

Figure 41. MJ's solution

In Figure 42, MU provided the division indicator 5 ÷ 7 and identified PDS in the solution, so I placed the solution in the category 3. For labeling 40, MU mentioned the word gram in the explanation part. No solutions were placed in category 1 for the unit-rate batch method. This shows that at least the future teachers who used the unit-rate batch method used M and N with appropriate values and units and they showed the total amount of gold and copper. Although there is a possibility not to see evidence for partitive division, because the future teachers might not explain the division in terms of the quantities in the situation. For example, the future teachers didn't work with the idea of distributing 5 units equally among 7 groups. To claim the future teacher was reasoning with partitive division, I would expect some idea like that to be expressed. Although the future teacher used the division, it seems like the future teacher could just be thinking of that division purely as a numerical process, and not in terms of reasoning about the quantities in the situation. In this case, I abided by my coding schema which indicated the criteria for each category and included using division operation is enough to decide if the future teacher identified PDS or QDS.

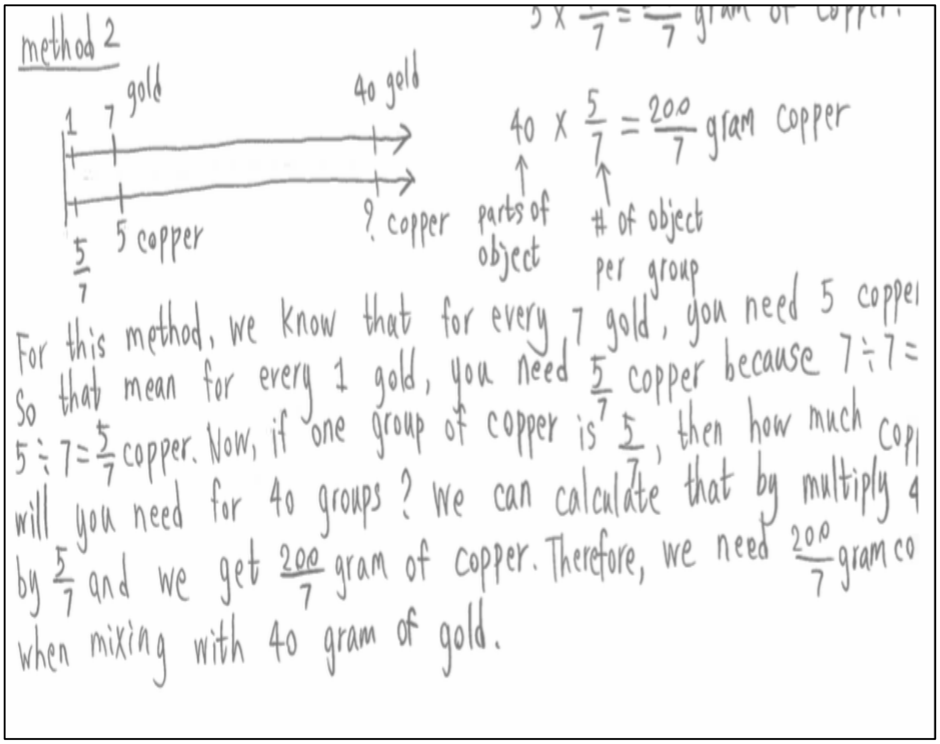


Figure 42. MU's solution

CHAPTER 6

DISCUSSION AND CONCLUSION

This study investigated the perspectives future middle school teachers used to solve a problem about proportional relationships and the extent to which explicit attention to multiplication and division played a role in their reasoning. The findings of this study indicated that when topics related to ratio, proportional relationships, fraction division, algebra, and the meaning of multiplication were emphasized in a two-sequence content course, future middle school teachers were able to use the multiple-batches and variable-parts perspectives and the associated methods in an appropriate way on an exam problem. Thus, the instructional approach to topics in the multiplicative conceptual field appeared to support future middle school teachers' development of their understanding of proportional relationships as well as their understanding of the meaning of multiplication and division and the use of features of each perspective.

According to Beckmann and Izsák (2015), the variable-parts perspective offers students an approach to thinking about variations of quantities in proportional relationship problems. In this study, students used the variable-parts perspective ($n = 29$) more often than the multiple-batches perspective ($n = 15$). This result represents the first determination regarding students' tendency when choosing with which perspective to work.

This study also found that students used the multiply total amount method ($n = 13$) less than the multiply one-part method ($n = 17$) when using the variable-parts perspective. Beckmann et al. (2015) investigated 26 students' use of the two perspectives and the four methods in two proportional relationship problems. According to their findings, 19 students used the multiply one-part method, 5 students used the multiply total amount method, and 1 student employed both methods using the variable parts perspective. Although the multiply total amount method was the most difficult method to use (Beckmann et al., 2015), almost 30% of the solutions in the present study incorporated this method.

In Beckmann et al. (2015), 15 of 26 students used the multiply one batch method, 12 used the multiply unit-rate batch method (i.e., multiple batches methods), and 1 used both. Similarly, in the present study, 8 future middle school teachers preferred the multiply one batch method, and 7 preferred the multiply unit-rate batch method.

Beckmann et al. (2005) also stated that some students used division for one of the task items even though the item did not specify the use of division. Correspondingly, in the present study, the use of division in the Gold and Copper task was not specified since the aim was to investigate whether or not the future middle school teachers could use the meaning of division without any direction. Meanings for division were incorporated into 15 of the 44 solutions. The indicators for the use of division were the use of the division operation sign ÷, the mention of division (e.g., distributing 40 grams evenly among 7 parts or finding the amount of one part), and/or the use multiplication with a missing factor (e.g., 7 • ? = 40). I identified future middle school teachers' uses of division as PDS or QDS. I detected QDS in only three solution but PDS in 12 solutions. This result is consistent with the idea QDS could be more challenging than PDS (Greer 1992).

In summary, the total number of solutions in this study was 44. Every future teacher used two methods except one student, who used four solutions, and two students who used only one solution. The total number of solutions in which students used variable-parts perspective with multiply total amount method was 12, whereas the total number of solutions in which students used variable-parts perspective with multiply one-part method was 17. Thus, 29 solutions made use of the variable-parts perspective, while 15 solutions made use of the multiple-batches perspective. In addition, the total number of solutions in which students used the multiple-batches perspective with multiply one batch method was 8, and the total number of solutions in which students used the multiple-batches perspective with multiply unit rate batch method was 7. In this study, 26 solutions of the Gold and Copper problem were rated in the category 2, whereas 15 solutions were rated in category 3 and 3 were rated in category 1. More importantly, 15 of 44 solutions included division and identified PDS or QDS. However, it is interesting that students' performances to recognize PDS (n = 12) was better than QDS (n = 3).

Implications

Proportional relationships are at the heart of middle school mathematics, so learning and teaching this concept is crucial. In order to improve achievement learning the concept, we need to educate future teachers. Thus, there is a need for research on the mathematical training of future middle-grade teachers for better teaching and learning of proportional relationships between co-varying quantities. In order to reach this goal, the education program for future middle school teachers should be designed to support proportional reasoning. More specifically, the course to teach proportional relationships should be designed by the considering multiplicative conceptual field and by connecting multiplication, division, ratios, and proportional relationships. Hence, mathematics courses should include both meanings for division in terms of PDS and QDS and both perspectives and methods on proportional relationships.

Future Research

In this study, I found that two perspectives are important since both have been designed by combining multiplication, division, and proportional relationships. In the future I would like to conduct a similar study with a larger sample. I also would like to support my research with interviews in order to make generalizations on my data more efficiently. I would like to develop a fully developed case study by considering the research process in which one gives detailed analysis to the development of a group or person or whatever constitutes the case.

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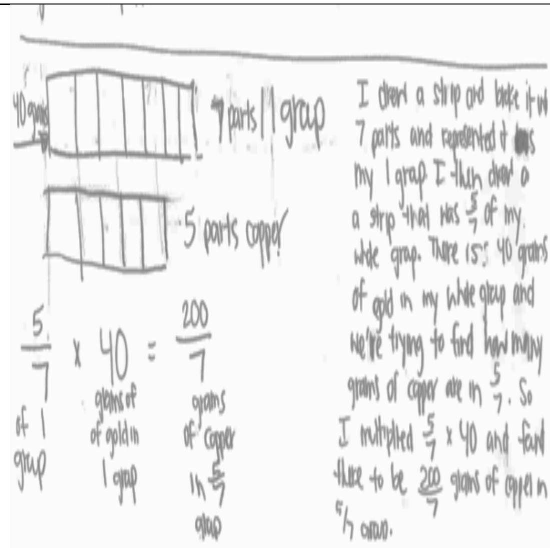
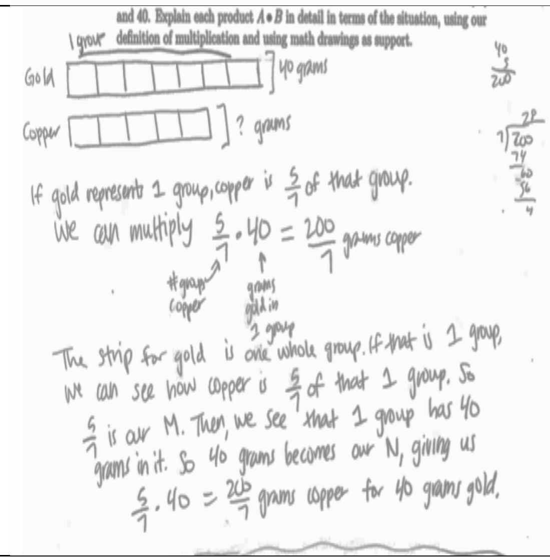
APPENDICES

Appendix 1

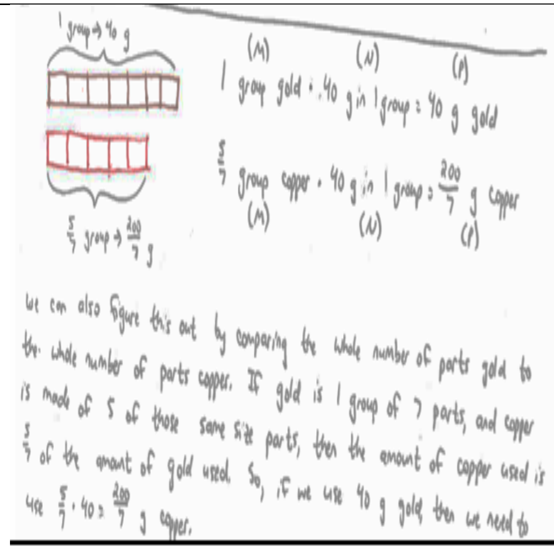
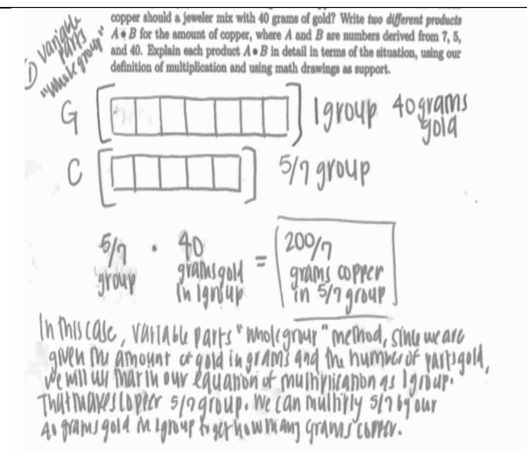
1. Variable Parts with Multiply Total Amount Method

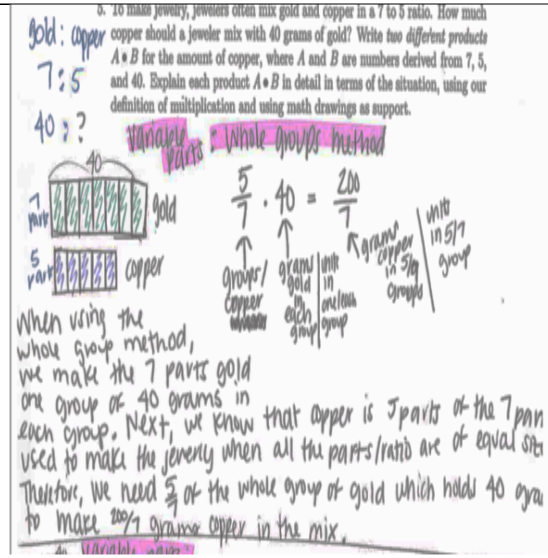
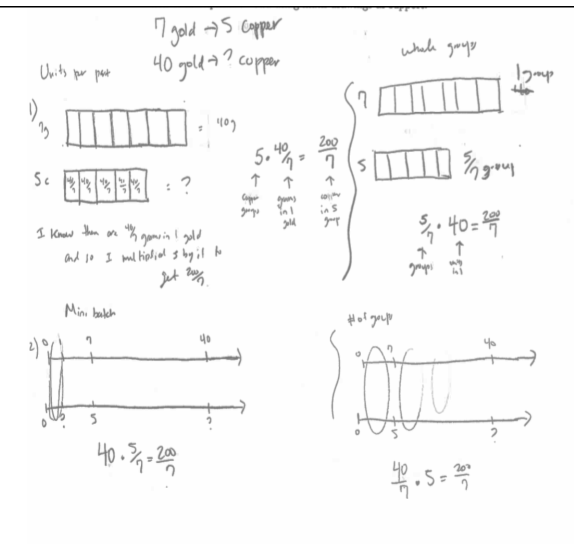
1.1. Category 2

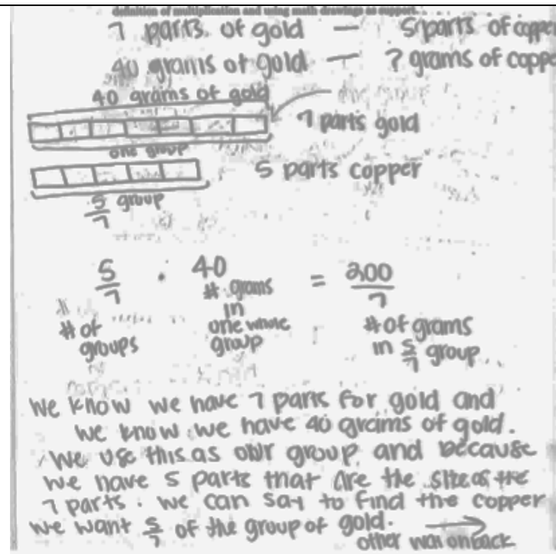
Perspective: Variable Parts
Method: Multiply total amount method
Category: Category 2
Criteria:
<u>Equation:</u>
the M and N have appropriate values given the method- M and N might be switched in position but values are appropriate
going through one part: M is 5, N is 40/7
whole group: M is 5/7 and N is 40
<u>Math Drawing:</u>
show total amount of gold and copper
Variable parts perspective (going through one part, whole group): strip diagrams with correct number of parts according to the problem (i.e., copper is 5 parts and gold is 7 parts)
<u>Any part of solution (equation, explanation or math drawing)</u>: indicate 1 group and base units in 1 group (i.e. they need to identify the units as grams)

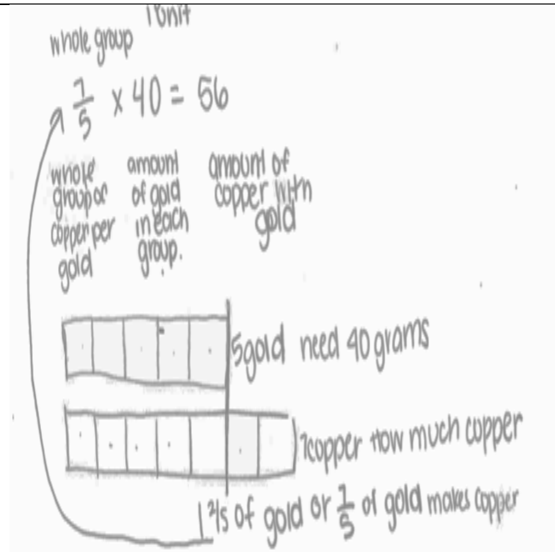
KA		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 ("of 1 group"), N= 40 ("grams of gold in 1 group"), and P= 200/7 ("grams of copper in 5/7 group"). No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gr in the math drawing part
LB		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 ("# of group copper"), N= 40 ("grams gold in 1 group"), and P= 200/7 ("grams copper"). No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gr in the math drawing part

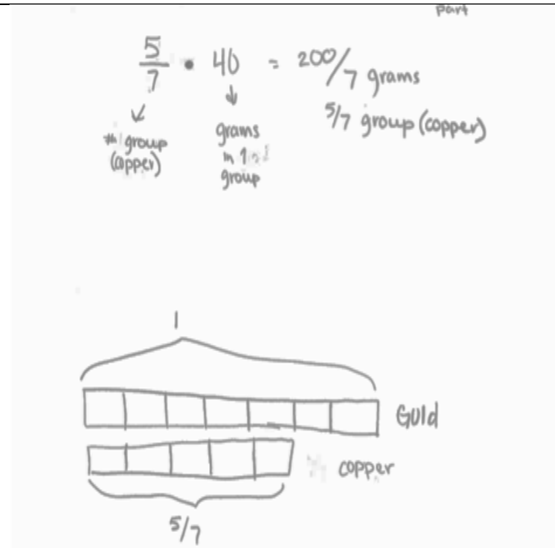
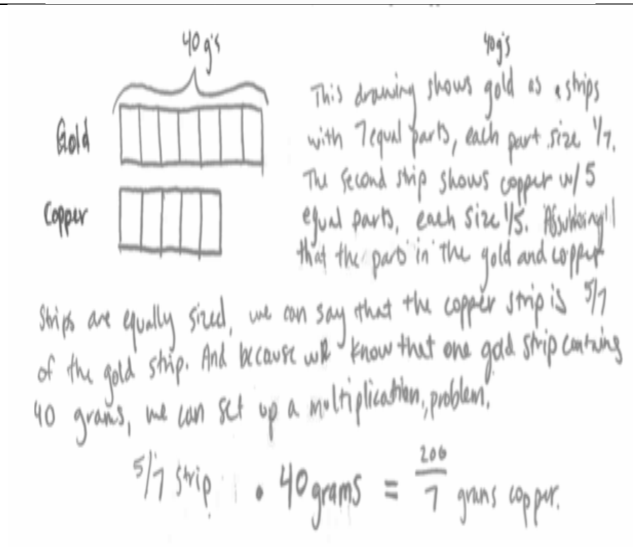
<table>
<tr>
<td>BG</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“# equal groups”), N= 40 (“# units (grams) in one group”), and P= 200/7 (“to the mixed with 40-gram gold (grams copper in 5/7 group)”).
No use of division (i.e. 5÷7)
No evidence for identification of QDS
Identification of MS with the equation
Use the mathematical drawing
▪ show total amount of gold and copper
▪ know 7 parts gold and 5 parts copper where 7 parts gold is one whole which also 40 with respect to the solution in the math drawing part
▪ know the 40 is “# units (grams) in one group” in the equation part
▪ mention “…we should mix up 40 grams of gold” in the explanation part</td>
</tr>
</table>

<table>
<tr>
<td>JG</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“# of groups of copper”), N= 40 (“# gold in 1 group”), and P= 200/7 (“grams copper”).
No use of division (i.e. 5÷7)
No evidence for identification of QDS
Identification of MS with the equation
Use the mathematical drawing
▪ show total amount of gold and copper:
▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gr in the math drawing part</td>
</tr>
<tr>
<td>BK</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“group”), N= 40 (“grams gold in 1 group”), and P= 200/7 (“grams copper in 5/7 group”).
No use of division (i.e. 5÷7)
No evidence for identification of QDS
Identification of MS with the equation
Use the mathematical drawing
▪ show total amount of gold and copper
▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gr in the math drawing part</td>
</tr>
</table>

MS		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“groups/copper”), N= 40 (“grams gold in each group”), and P= 200/7 (“grams copper in 5/7 groups”). No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing show total amount of gold and copper know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gold in the math drawing part ▪ know 40 is “grams gold in each group” in the equation part ▪ mention “…we make the 7 parts gold one group of 40 grams in each group…” in the explanation part.
CS		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“groups”), N= 40 (“units in 1”), and P= 200/7. No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing show total amount of gold and copper even the future teacher crossed out the 40 next to the 7 parts, this work has been rated in category 2 since CS’ other work with the multiply one-part method also included 40 gr gold in 7 parts. The crossed out 40 does not include any indicator for gram, his multiply one-part method showed that

		CS has already known that 40 is related to gram P.S. Although CS did not show total amount of gold and copper for this method, she showed it for other methods, so CS has been accepted that she showed total amount of gold and copper.
LM		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“# of groups”), N= 40 (“# of grams in one whole group”), and P= 200/7 (“# grams in 5/7 group”). No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 grams of gold in the math drawing part

AA		Include an equation by considering the M and N which have appropriate values given the method where M= 7/5 ("whole group or copper per gold"), N= 40 ("amount of gold in each group"), and P= 200/7 (amount of copper with gold). No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 5 parts gold and 7 parts copper where 5 parts gold is 1 group which also 40 grams of gold in the math drawing part ▪ some future teachers misread problem so interpret problem as 7 parts copper instead of 7 parts gold and 5 parts gold instead of 5 parts copper and solved accordingly so if their work is appropriate for these numbers then use same criteria P.S. AA was rated in category 2 since M and N have appropriate values given the method even though she reversed the gold and copper. Her reasoning is appropriate for her method. The only issue about her solution is she switched the gold and copper.

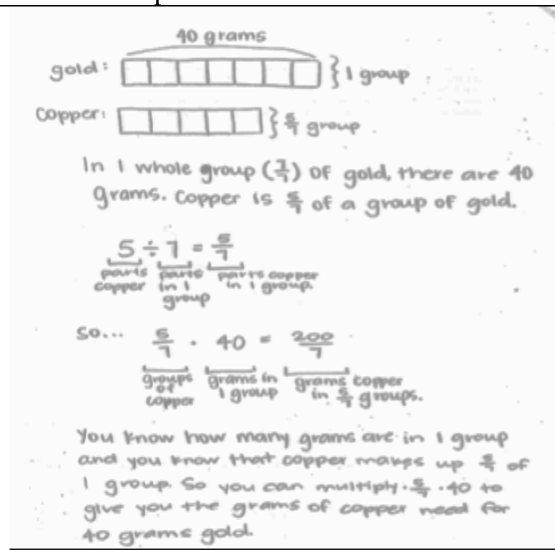
<table>
<tr><td>BB</td><td>
</td><td>Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“# group (copper)”), N= 40 (“grams in 1 group”), and P= 200/7 (“5/7 group (copper)”).
No use of division (i.e. 5÷7)
No evidence for identification of QDS
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts gold is 1 group
know 40 is “grams in 1 group” in the equation part</td></tr>
<tr><td>AG</td><td>
</td><td>Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“strip”), N= 40 (“grams”), and P= 200/7 (“grams copper”).
No use of division (i.e. 5÷7)
No evidence for identification of QDS
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 g gold in the math drawing part.
Even though she is using unusual annotation for the equation, there is enough explanation “…we can say that copper strip is 5/7 of the gold strip. Because we know that one gold strip carrying 40 grams, we can set up a multiplication problem.” This shows that she accepted 1 strip is 1 whole</td></tr>
</table>

		group and instead of using the word "group" she used the word "strip".

1.2. Category 1

Perspective: Variable parts **Method:** Multiply total amount method **Category:** Category 1 **Criteria:** Not meet the criteria for category 2
NO SOLUTION

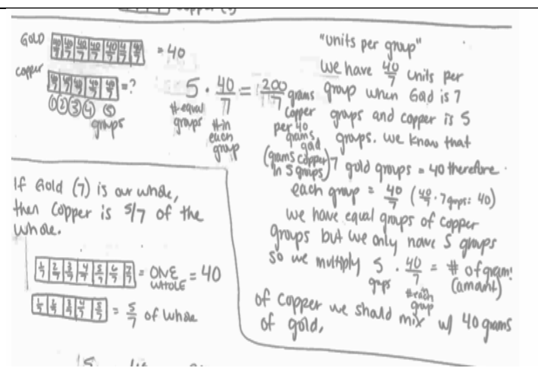
1.3. Category 3

Perspective: Variable Parts **Method:** Multiply total amount method **Category:** Category 3 **Indicators:** Meets all criteria Category 2 Division in the equation and/or written explanation		
LF		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“groups of copper”), N= 40 (“grams in one group”), and P= 200/7 (“grams copper in 5/7 groups”). Use of division (i.e. 5÷7) Identification of QDS Indicator: division symbol, “how many grams are in 1 group” Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gr gold in the math drawing part She was rated in category 3 since she used division and identified QDS.

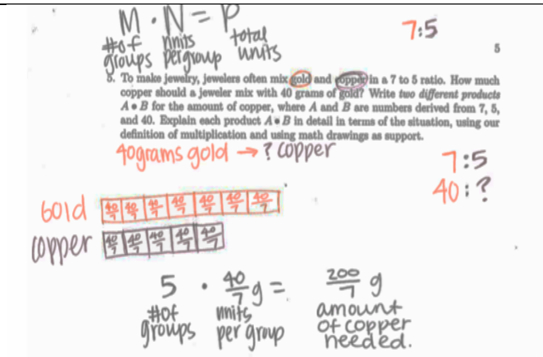
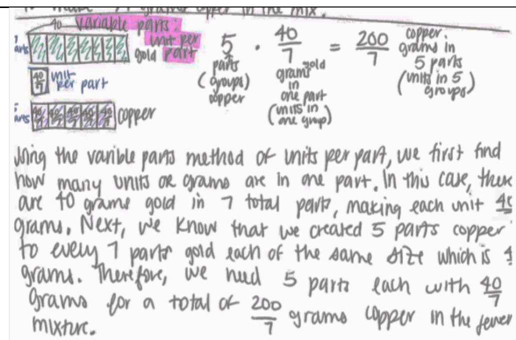
Appendix 2

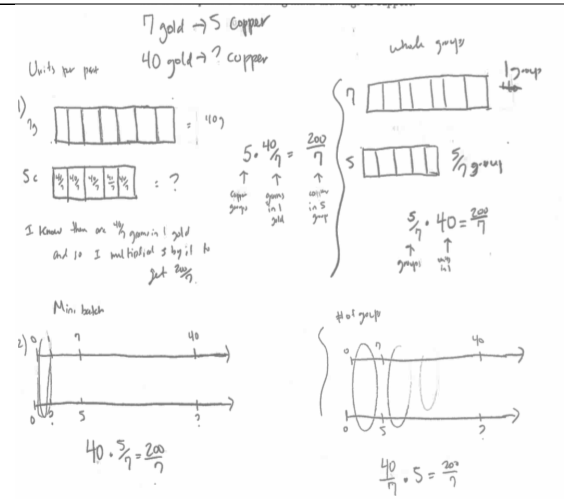
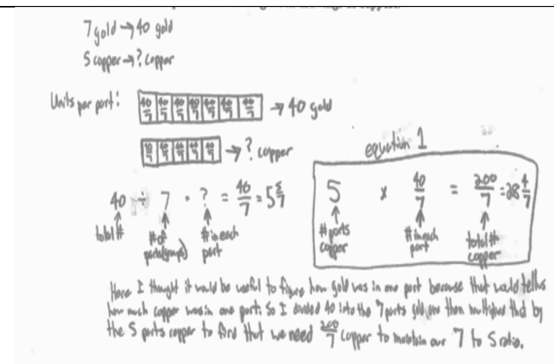
2. Variable Parts with Multiply One Part Method

2.1. Category 2

<table>
<tr><td colspan="3">
Perspective: Variable Parts
Method: Multiply one-part method
Category: Category 2
Criteria:
<u>Equation:</u>
the M and N have appropriate values given the method- M and N might be switched in position but values are appropriate
going through one part: M is 5, N is 40/7
whole group: M is 5/7 and N is 40
unit rate batch: M is 40 and N is 5/7
one batch: M is 40/7 and N is 5
<u>Math Drawing:</u>
show total amount of gold and copper
Variable parts perspective (going through one part, whole group): strip diagrams with correct number of parts with respect to the problem (i.e., copper is 5 parts and gold is 7 parts)
<u>Any part of solution (equation, explanation or math drawing)</u>: indicate 1 group and base units in 1 group (i.e. they need to identify the units as grams)
</td></tr>
<tr><td>BG</td><td>


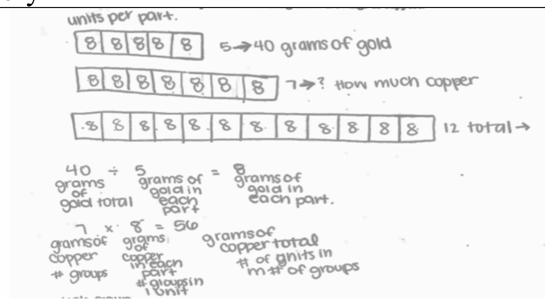
</td><td>
Include an equation by considering the M and N which have appropriate values given the method where M= 5 (“#equal groups”), N= 40/7 (“# in each group”), and P= 200/7 (“grams copper per 40 grams gold (grams copper in 5 groups)”).
No use of division (i.e. 40÷7)
No evidence for identification of PDS
Identification of MS with the equation
Use the mathematical drawing
▪ show total amount of gold and copper
▪ know 7 parts gold and 5 parts copper where
</td></tr>
</table>

		7 parts gold is 1 group which also 40 gold in the math drawing part ▪ mention "…we should mix up 40 grams of gold" in the explanation part.
BM	 	Include an equation by considering the M and N which have appropriate values given the method where M= 5 ("# of groups"), N= 40/7 ("units per group"), and P= 200/7 ("amount of the copper needed"). No use of division (i.e. 40÷7) No evidence for identification of PDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gold in the math drawing part ▪ show 40 grams gold–> ? copper
MS	 	Include an equation by considering the M and N which have appropriate values given the method where M= 5 ("parts- group copper"), N= 40/7 ("grams in one part- units in 1 group"), and P= 200/7 ("5 parts- units in 5 groups"). No use of division (i.e. 40÷7) No evidence for identification of PDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gold in the math drawing part ▪ know 40/7 is "grams in 1 part" in the equation part

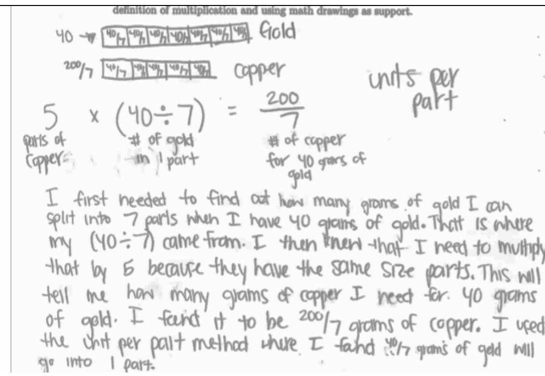
		▪ mention "…there are 40 grams gold in 7 total parts…" in the explanation part.
CS		Include an equation by considering the M and N which have appropriate values given the method where M= 5 ("copper groups"), N= 40/7 ("grams in 1 gold"), and P= 200/7 ("copper in 5 group"). No use of division (i.e. 40÷7) No evidence for identification of PDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 g gold in the math drawing part Even though the units are sloppy, especially N= 40/7 ("grams in 1 gold"), this study was rated in category 2 since there is an enough explanation which says "I know that three are 40/7 grams in 1 gold.". Thus she accepted 1 of 7 as 1 gold but still knows the total amount of 7 parts gold is 40 g.
PM		Include an equation by considering M and N where M= 5 ("# parts copper"), N= 40/7 ("# in each part"), and P= 200/7 ("total # copper") Use of division (i.e. 40÷7) Identification of PDS Indicator: division symbol, "how many grams are in 1 group" Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in 1 group which is 40 gold in the math drawing part

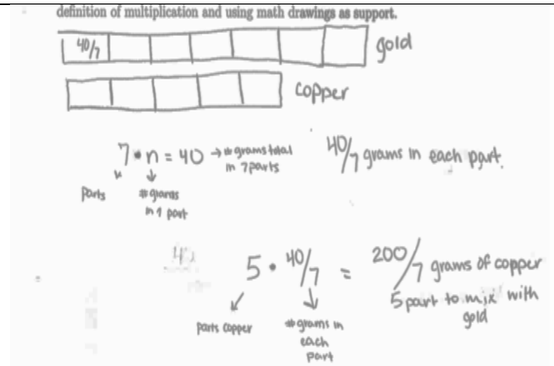
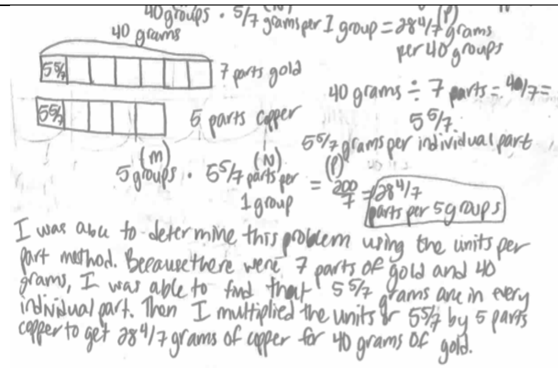
		▪ never used the "gram" P.S. His work was rated in category 2 even though he used the division, since he mostly left the grams off, which I think makes his work quite unclear.

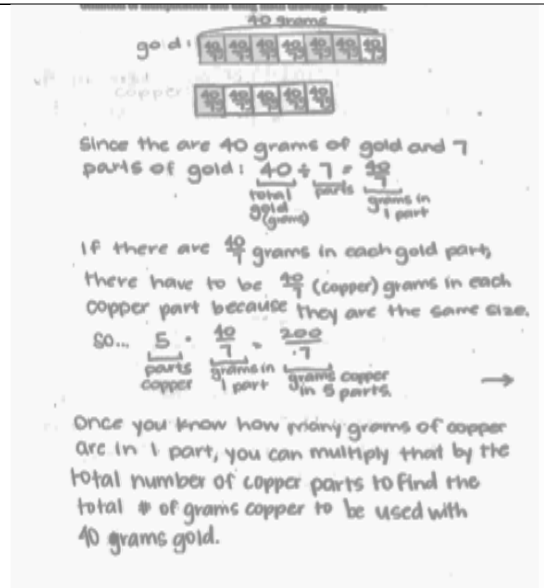
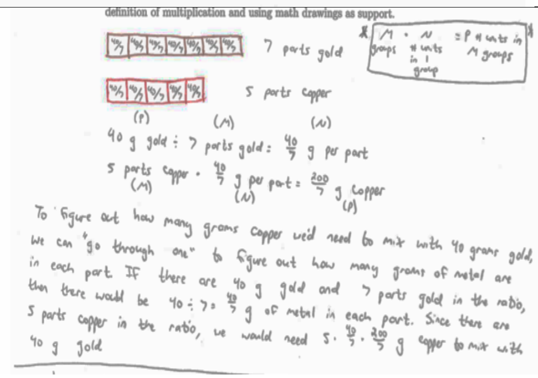
2.2. Category 1

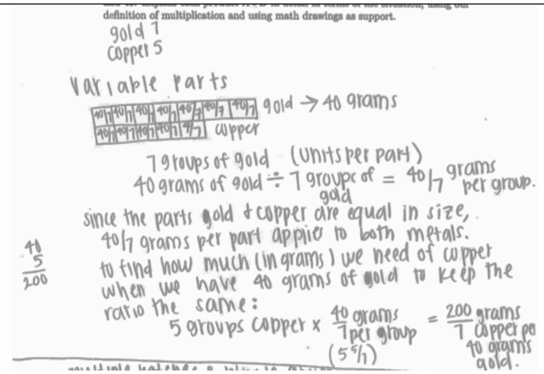
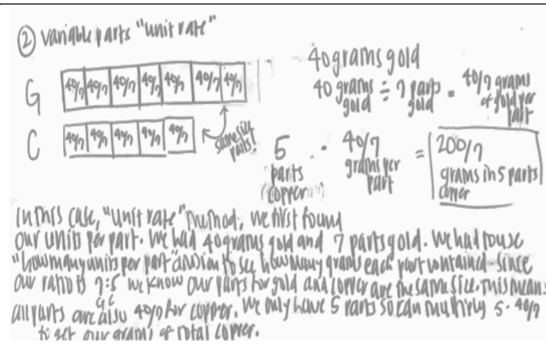
<table>
<tr><td colspan="3">Perspective: Variable Parts
Method: Multiply one-part method
Category: Category 1
Criteria:
Not meet the criteria for category 2</td></tr>
<tr><td>AA</td><td>
</td><td>Include an equation by considering M and N where M= 7(“grams of copper # group”), N= 8 (“grams of copper in each part # groups in 1 unit”), and P= 56(“grams of copper total # of units in # of groups”).
Use of division (i.e. 40÷5)
Identification of PDS
Indicator: division symbol
Identification of MS with the equation
Use the mathematical drawing
▪ show total amount of gold and copper
▪ know 5 parts gold and 7 parts copper where 5 parts gold is 1 group which also 40 grams of gold in the math drawing part
Even though she misread problem so interpret problem as 7 parts copper instead of 7 parts gold and 5 parts gold instead of 5 parts copper and solved accordingly so this work is appropriate for these numbers then use same criteria.</td></tr>
</table>

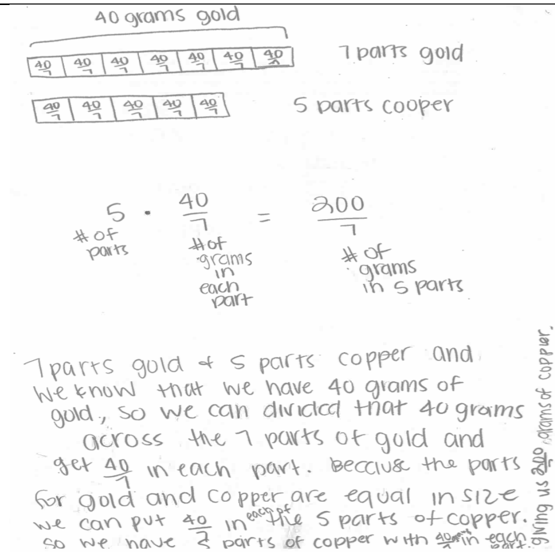
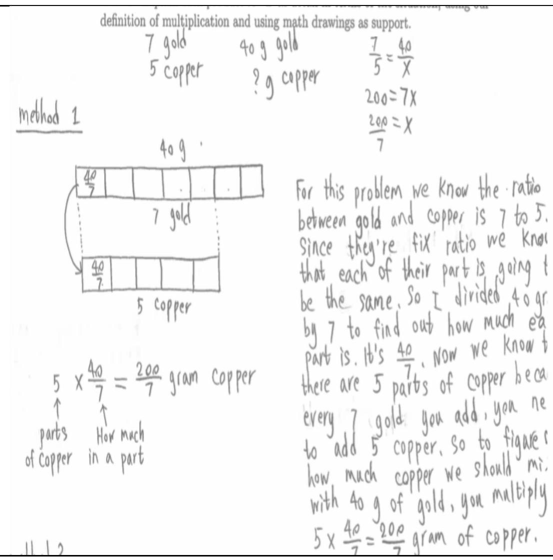
2.3. Category 3

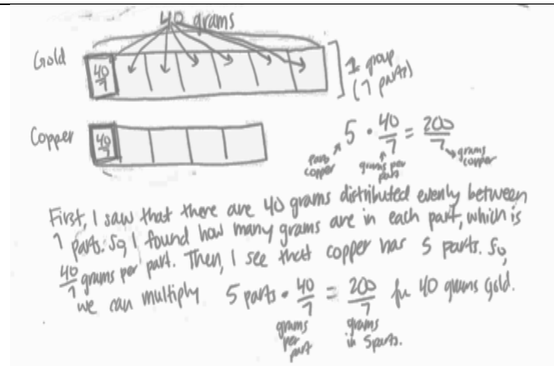
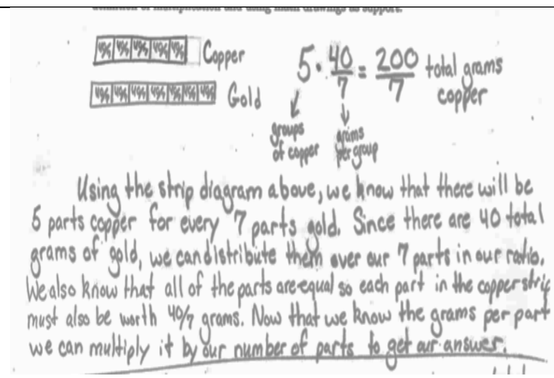
<table>
<tr><td colspan="3">Perspective: Variable Parts
Method: Multiply one-part method
Category: Category 3
Criteria:
Meet the criteria for category 2
Division</td></tr>
<tr><td>KA</td><td>
</td><td>Include an equation by considering M and N where M= 5(“parts of copper”), N= 40/7 (“# of gold in 1 part”), and P= 200/7 (“# of copper for 40 grams of gold”).
Use of division (i.e. 40÷7)
Identification of PDS
Indicator: division symbol, “how many grams are in 1 group”
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gold in the math drawing part
mention “…when I have 40 grams gold …” in the explanation part.</td></tr>
</table>

BB		Include an equation by considering M and N where M= 5 (“parts copper”), N= 40/7 (“grams in each part”), and P= 200/7 (“grams of copper”). Use of division Identification of PDS Indicator: Use the multiplication with a missing factor (e.g., 7 •? = 40) Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts in the math drawing part ▪ know 40 is related to the gram since show 40/7 # grams in each part in the equation part
KC		Include an equation by considering M and N where M= 5 (“# groups”), N= 40/7 (“# grams in each group”), and P= 200/7 (“grams of copper”). Use of division (i.e. 40÷7) Identification of PDS Indicator: division symbol, “how many grams are in 1 group” Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 grams of gold in the math drawing part

<table>
<tr>
<td>LF</td>
<td>
</td>
<td>Include an equation by considering M and N where M= 5 (“# parts copper”), N= 40/7 (“grams in 1 part”), and P= 200/7 (“grams copper in 5 parts”).
Use of division (i.e. 40÷7)
Identification of PDS
Indicator: division symbol, “how many grams are in 1 group”
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 grams of gold in the math drawing part</td>
</tr>
<tr>
<td>JG</td>
<td>
</td>
<td>Include an equation by considering M and N where M= 5 (“parts copper”), N= 40/7 (“g per part”), and P= 200/7 (“g copper”).
Use of division (i.e. 40÷7)
Identification of PDS
Indicator: division symbol, “how many grams are in 1 group”
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts gold in the math drawing part
know 40 is related to the gram since show 40/7 grams per group in the equation part
mention “…if there are 40 g gold…” in the explanation part.</td>
</tr>
</table>

AH		Include an equation by considering M and N where M= 5 (“groups copper”), N= 40/7 (“# grams per group”), and P= 200/7 (“grams copper per 40 grams gold”). Use of division (i.e. 40÷7) Identification of PDS Indicator: division symbol, “how many grams are in 1 group” Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in 1 group which is 40 grams in the math drawing part
BK		Include an equation by considering M and N where M= 5 (“parts copper”), N= 40/7 (“grams per part”), and P= 200/7 (“grams in 5 parts copper”). Use of division (i.e. 40÷7) Identification of PDS Indicator: division symbol, “how many grams are in 1 group” Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in 1 group which is 40 grams (next to the math drawing and above the division) in the math drawing part

LM		Include an equation by considering M and N where M= 5 (“# parts copper”), N= 40/7 (“# in each part”), and P= 200/7 (“total # copper”) Mention division “…, so we can divide that 40 grams across the 7 parts of golf and get 40/7 in each part” Identification of PDS Indicator: how many grams are in 1 group Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in 1 group which is 40 grams of gold in the math drawing part
MU		Include an equation by considering the M and N which have appropriate values given the method where M= 5 (“parts of copper”), N= 40/7 (“how much in a part”), and P= 200/7 (“gram copper”). Mention division “So I divided 40 gr by 7 to find out how much each part is. It’s 40/7.” There is evidence for identification of PDS Indicator: how many grams are in 1 group Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in 1 group which is 40 grams of gold in the math drawing part

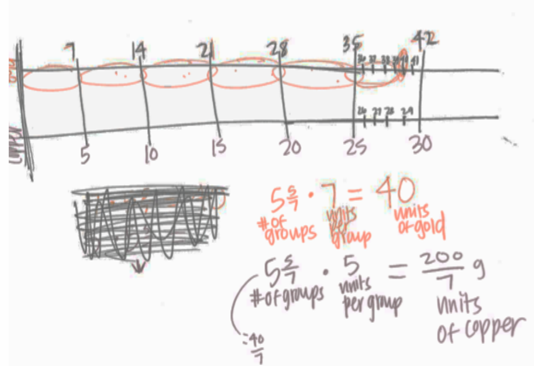
<table>
<tr>
<td>LB</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 5 (“parts copper”), N= 40/7 (“grams per part”), and P= 200/7 (“grams copper”).
Mention division (distributing 40 grams evenly among 7 parts)
There is evidence for identification of PDS
Indicator: how many grams are in 1 group
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts gold in 1 group which is 40 grams of gold in the math drawing part</td>
</tr>
<tr>
<td>CB</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 5 (“groups of copper”), N= 40/7 (“grams per group”), and P= 200/7 (“total grams copper”).
Use of division (i.e. 40÷7)
There is an evidence for identification of PDS
Indicator: We distribute them over 7 parts in our ratio
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
know 7 parts gold and 5 parts copper where 7 parts in the math drawing part
know 40 is related to the gram since show 40/7 grams per group in the equation part
mention “…Since there are 40 total grams of gold” in the explanation part.</td>
</tr>
</table>

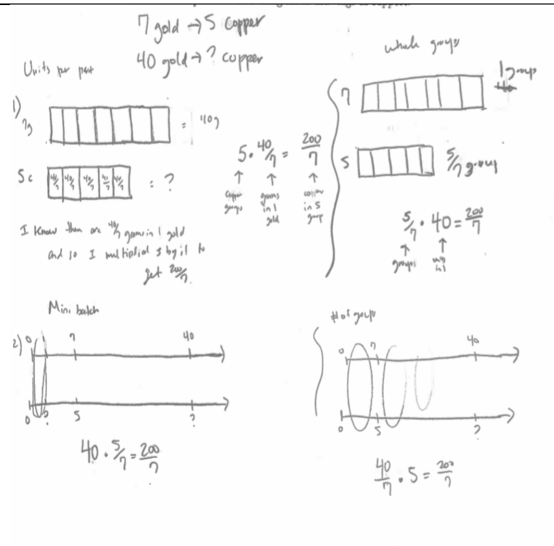
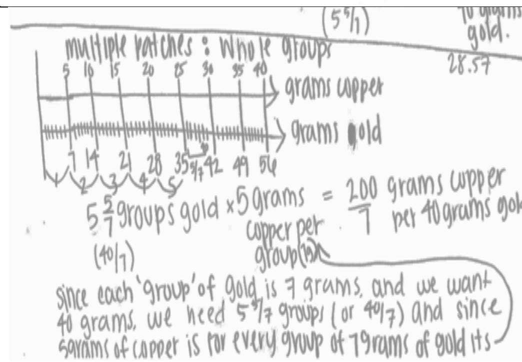
Appendix 3

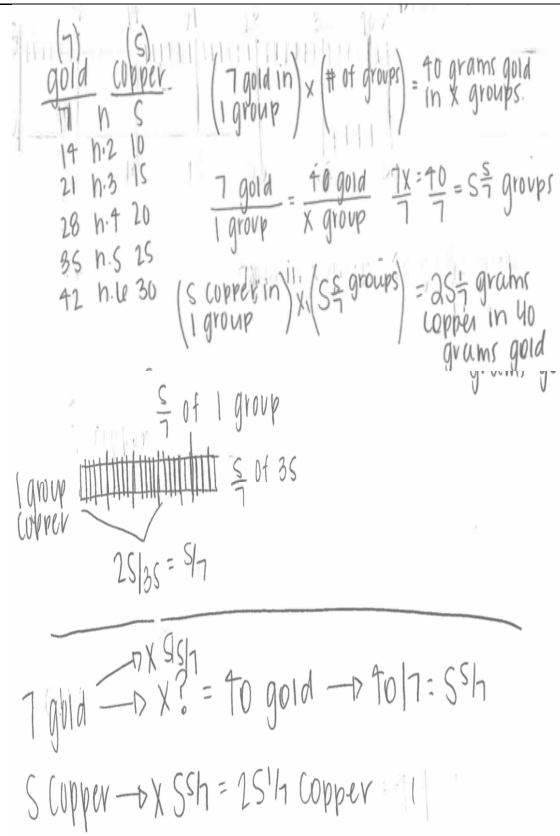
3. Multiple Batches Perspective with Multiply One Part Method

3.1. Category 2

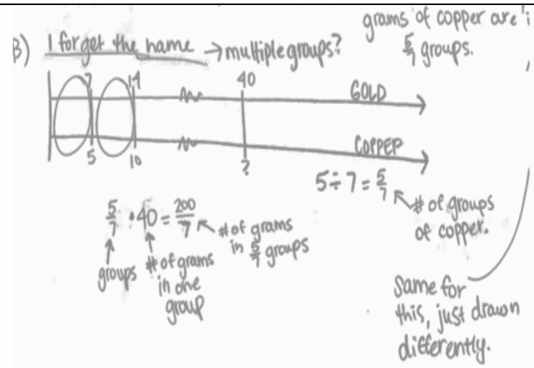
Perspective: Multiple Batches
Method: Multiply one batch method
Category: Category 2
Criteria:
Equation:
the M and N have appropriate values given the method- M and N might be switched in position but values are appropriate
going through one part: M is 5, N is 40/7
whole group: M is 5/7 and N is 40
unit rate batch: M is 40 and N is 5/7
one batch: M is 40/7 and N is 5
Math Drawing:
show total amount of gold and copper
Multiple batches perspective: DNL (or strip diagram) indicate target amount (e.g., tick mark for 40 grams gold and ? grams copper or 40 grams copper and ? grams gold)
Multiple Batches Perspective
one batch: DNL indicate initial batch (e.g., tick mark for 7 grams gold and 5 grams copper)
unit rate batch: DNL indicate unit-rate batch (e.g., tick mark for 1 gram gold and 5/7 gram copper OR 1 gram copper and 7/5 gram gold)
Any part of solution (equation, explanation or math drawing): indicate 1 group and base units in 1 group (i.e. they need to identify the units as grams)

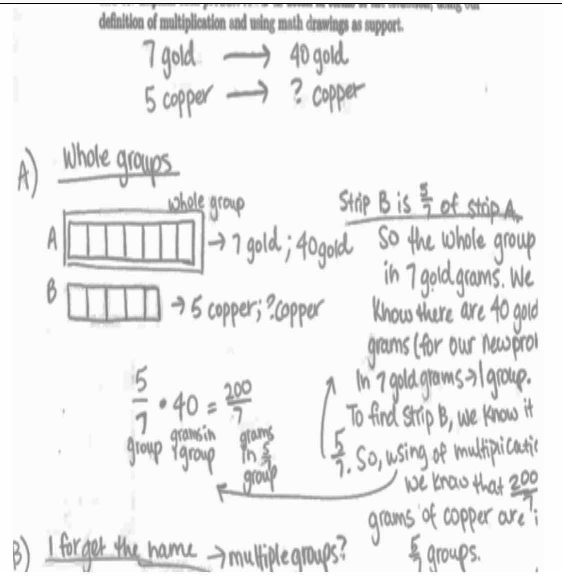
BM		Include an equation by considering the M and N which have appropriate values given the method where M= 40/7 ("#of groups"), N= 7 ("units per group"), and P= 40("units of gold") and M=40/7 ("#of groups"), N= 5 ("units per group"), and P= 200/7 ("units of copper") No use of division (i.e. 40÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ DNL indicated target amount (e.g., tick mark for 40 for gold) ▪ DNL indicated initial batch (e.g., tick mark for 7 for gold and 5 for copper) ▪ Used "grams" in her previous solution ▪ She provided only equation and math drawing (without explanation) so this is okay

CS		Include an equation by considering the M and N which have appropriate values given the method where M= 40/7, N= 5, and P= 200/7 No use the division (i.e. 40÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ DNL indicated target amount (e.g., tick mark for 40 for gold) ▪ DNL indicated initial batch (e.g., tick mark for 7 for gold and 5 for copper) ▪ Used "grams" in her previous solution
AH		Include an equation by considering the M and N which have appropriate values given the method where M= 40/7 ("groups gold"), N=5 ("copper per group"), and P= 200/7("grams copper per 40 grams gold") No use the division (i.e. 40÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ DNL indicated target amount (e.g., tick mark for 40 grams of gold) ▪ DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper)

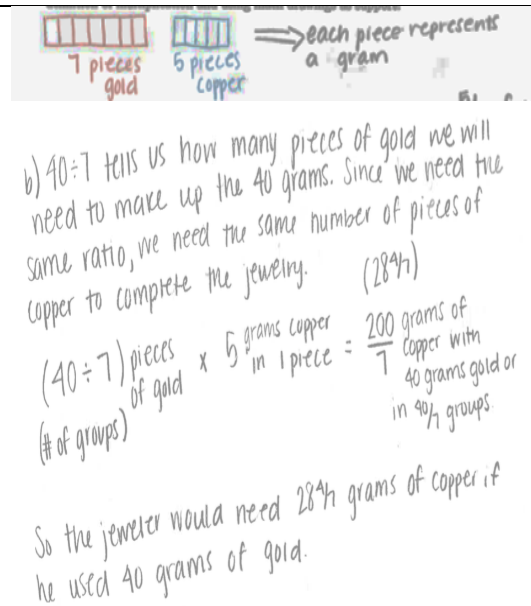
MW		Include an equation by considering the M and N which have appropriate values given the method where M= 40/7 (“groups gold”), N=5 (“copper per group”), and P= 200/7(“grams copper per 40 grams gold”) Use the division “7× # = 40” There is evidence for identification of QDS (since 7•? =40) Indicator: Use the multiplication with a missing factor (e.g., 7 •? = 40) Identification of MS with the equation Use a table ▪ Not show total amount of gold and copper ▪ not indicate target amount (e.g., tick mark for 40 grams of gold) ▪ indicate initial batch (e.g., tick mark for 7 gold and 5 copper) She reversed the multiplier and multiplicand but still her indication for 40/7 “groups” and 5 “copper in 1 group” are reasonable.

3.2. Category 1

Perspective: Multiple Batches **Method:** Multiply one batch method **Category:** Category 1 **Criteria:** Not meet category 2		
JP		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“groups”), N= 40 (“# grams in one group”), and P= 200/7 (“# grams in 5/7 groups”). Use of division (i.e. 5÷7) Identification of QDS Indicator: division Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ DNL indicated target amount (e.g., tick mark for 40 gold) ▪ DNL indicated initial batch (e.g., tick mark for 7 gold and 5 copper) Mention “gram” in the equation part JP used the one batch method, but he did not provide an equation with respect to his method. Since for the one batch method, the equation should be 40/7• 5 by considering M and N. The solution and method are not consistent.

JP		Include an equation by considering the M and N which have appropriate values given the method where M= 5/7 (“group”), N= 40 (“grams in 1 group”), and P= 200/7 (“grams in 5/7 group”). No use of division (i.e. 5÷7) No evidence for identification of QDS Identification of MS with the equation Use the mathematical drawing show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold is 1 group which also 40 gold in the math drawing part ▪ know 40 is “grams in 1 group” in the equation part ▪ mention “…We know there are 40 gold grams…” in the explanation part. Since she mentioned “So the whole group in 7 gold grams” instead of mentioning 7 parts stand for 40 gr gold, this work was rated in Category 1.

3.3. Category 3

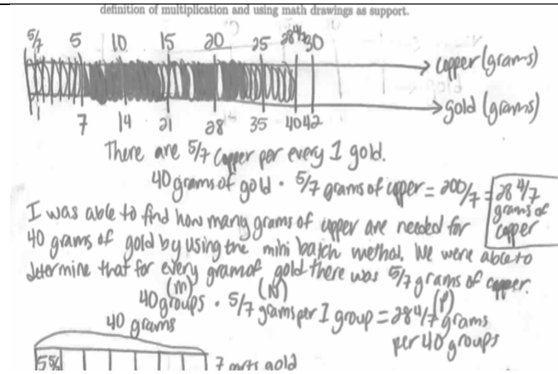
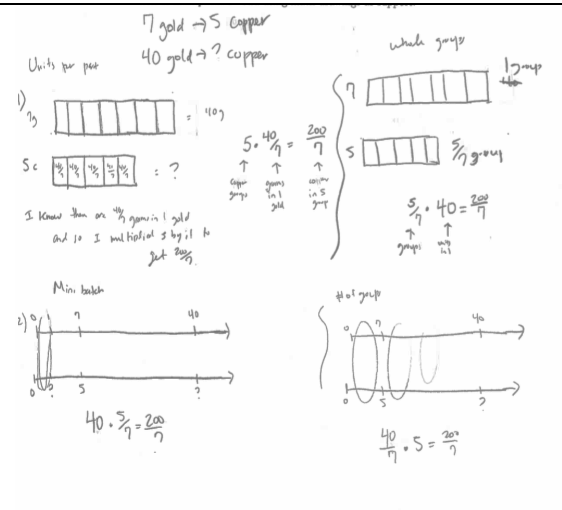
Perspective: Multiple Batches **Method:** Multiply one batch method **Category:** Category 3 **Indicators:** Meet category 2 Division		
MJ		Include an equation by considering the M and N which are N= 40/7 (“pieces of gold-# of groups”), M= 5(“grams copper in 1 piece”), and P= 200/7 (“grams of copper with 40 grams gold or in 40/7 groups”). M and N were switched in position Use of division (i.e. 40÷7) Identification of QDS Identification of MS with the equation Not use the appropriate mathematical drawing ▪ show total amount of gold and copper ▪ know 7 parts gold and 5 parts copper where 7 parts gold in the math drawing part ▪ mention “…we will need to make up the 40 grams…” in the explanation part.

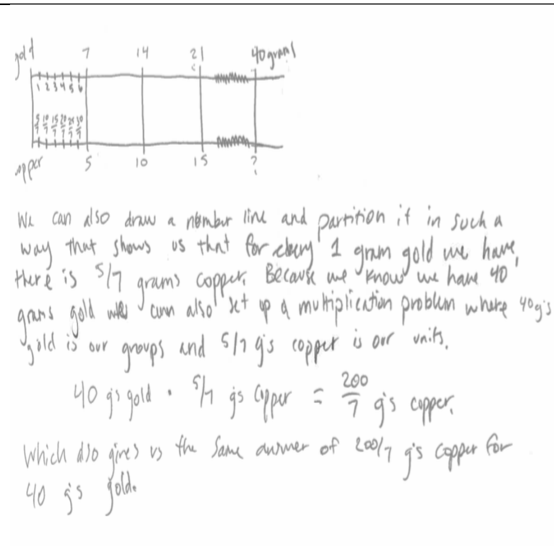
Appendix 4

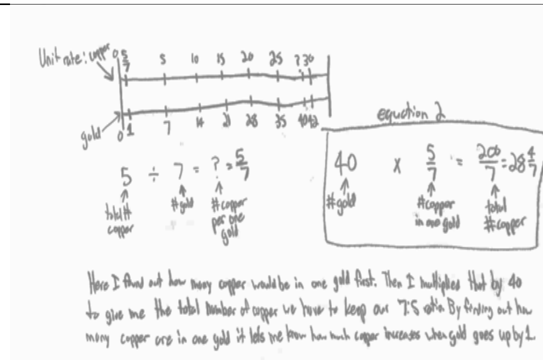
4. Multiple Batches Perspective with Multiply Unit-Rate Batch Method

4.1.1. Category 2

Perspective: Multiple Batches
Method: Multiply unit-rate batch method
Category: Category 2
Criteria:
<u>Equation:</u>
the M and N have appropriate values given the method- M and N might be switched in position but values are appropriate
going through one part: M is 5, N is 40/7
whole group: M is 5/7 and N is 40
unit rate batch: M is 40 and N is 5/7
one batch: M is 40/7 and N is 5
<u>Math Drawing:</u>
show total amount of gold and copper
Multiple batches perspective: DNL (or strip diagram) indicate target amount (e.g., tick mark for 40 grams gold and ? grams copper or 40 grams copper and ? grams gold)
Multiple Batches Perspective
one batch: DNL indicate initial batch (e.g., tick mark for 7 grams gold and 5 grams copper)
unit rate batch: DNL indicate unit-rate batch (e.g., tick mark for 1 gram gold and 5/7 gram copper OR 1 gram copper and 7/5 gram gold)
<u>Any part of solution (equation, explanation or math drawing)</u>: indicate 1 group and base units in 1 group

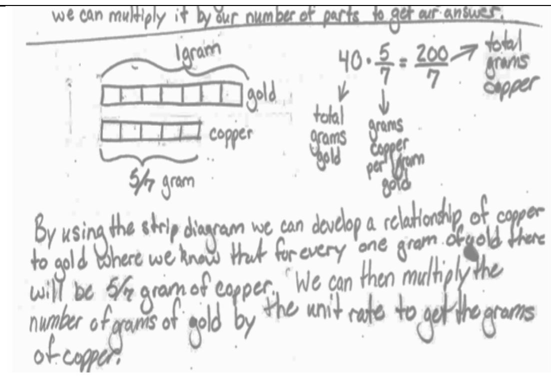
<table>
<tr><td>KC</td><td></td><td>Include an equation by considering the M and N which have appropriate values given the method where M= 40 (“groups”), N= 5/7 (“grams of copper”), and P= 200/7 (“grams of copper”).
Not use the division (i.e. 5÷7)
No evidence for identification of PDS
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
DNL indicated target amount (e.g., tick mark for 40 grams of gold)
DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper)</td></tr>
<tr><td>CS</td><td></td><td>Include an equation by considering the M and N which have appropriate values given the method where M= 40, N= 5/7, and P= 200/7
Not use the division (i.e. 5÷7)
No evidence for identification of PDS
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
DNL indicated target amount (e.g., tick mark for 40 gold)
DNL indicated initial batch (e.g., tick mark for 7 gold and 5 copper)
Mention “gram” in the previous solution</td></tr>
</table>

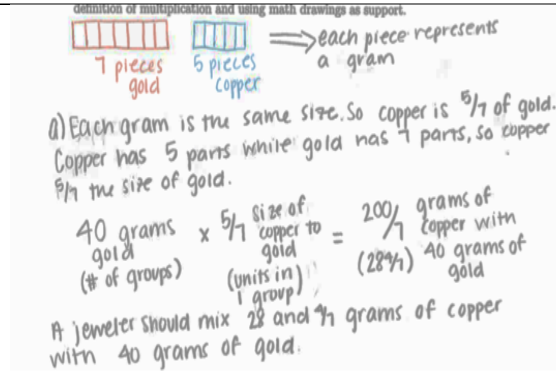
<table>
<tr>
<td>AG</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 40, N= 5/7, and P= 200/7
Not use the division (i.e. 5÷7)
No evidence for identification of PDS
Identification of MS with the equation
Use the mathematical drawing

show total amount of gold and copper
DNL indicated target amount (e.g., tick mark for 40 grams of gold)
DNL indicated initial batch (e.g., tick mark for 7 grams of gold and 5 grams of copper)
</td>
</tr>
<tr>
<td>PM</td>
<td>
</td>
<td>Include an equation by considering the M and N which have appropriate values given the method where M= 40(“# gold”), N= 5/7 (“# copper in one gold”), and P= 200/7 (“total # copper”).
Use of division (i.e. 5÷7)
Identification of PDS
Indicator: division, how many in 1 group
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper

DNL indicated target amount (e.g., tick mark for 40 gold)
DNL indicated initial batch (e.g., tick mark for 7 gold and 5 copper)
Never mention “gram”

P.S. For the multiply one unit rate batch method, his work in category 2 even though he used the division, since he mostly left the grams off, which I</td>
</tr>
</table>

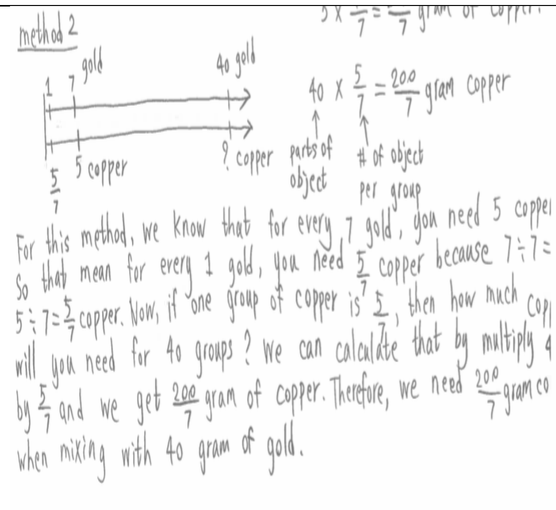
		think makes his work quite unclear.
CB		Use the variable parts perspective with multiply total amount method in the mathematical math drawingand use of the multiple batches perspective logically in the mathematical solution With respect to the variable parts perspective, the future teacher determined the number of the group and then iterated 40 times by considering multiple batches perspective with multiply unit-rate batch method. Thus, the future teacher mixed both methods and wrote the equation with respect to other perspective. Include an equation by considering the M and N which have appropriate values given the method where M= 40 (“total grams gold”, N= 5/7 (“grams copper per gram gold”), and P= 200/7 (“total grams copper”) Not use the division (i.e. 5÷7) No evidence for identification of PDS Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ Strip did not indicate target amount (e.g., tick mark for 40 grams of gold) but the equation included 40 is the total grams gold. Strip indicated 7 parts as gold and 5 parts as copper

<table>
<tr>
<td>MJ</td>
<td>
</td>
<td>Use the variable parts perspective with multiply total amount method in the mathematical math drawing and use of the multiple batches perspective logically in the mathematical solution
With respect to the variable parts perspective, the future teacher determined the number of the group and then iterated 40 times by considering multiple batches perspective with multiply unit-rate batch method. Thus, the future teacher mixed both methods and wrote the equation with respect to other perspective.
Include an equation by considering the M and N which have appropriate values given the method where M= 40 (“grams gold-# of groups”), N= 5/7 (“units in 1 group”), and P= 200/7 (“grams of copper with 40 grams of gold”)
Not use the division (i.e. 5÷7)
No evidence for identification of PDS
Identification of MS with the equation
Use the mathematical drawing
show total amount of gold and copper
Strip did not indicate target amount (e.g., tick mark for 40 grams of gold) but the equation included 40 is the total grams of gold.
Mention 40 grams in the explanation part
Strip indicated 7 parts as gold and 5 parts as copper</td>
</tr>
</table>

4.2. Category 1

Perspective: Multiple Batches **Method:** Multiply unit-rate batch method **Category:** Category 1 **Criteria:** Not meet category 2
NO STUDY

4.3. Category 3

Perspective: Multiple Batches **Method:** Multiply unit-rate batch method **Category:** Category 3 **Indicators:** Meet category 2 Division		
MU		Include an equation by considering the M and N which have appropriate values given the method where M= 40(“# gold”), N= 5/7 (“# copper in one gold”), and P= 200/7 (“total # copper”). Use of division (i.e. 5÷7) Identification of PDS Indicator: division Identification of MS with the equation Use the mathematical drawing ▪ show total amount of gold and copper ▪ DNL indicated target amount (e.g., tick mark for 40 gold) ▪ DNL indicated initial batch (e.g., tick mark for 7 gold and 5 copper) Mention “gram” in the explanation part