PRESERVICE TEACHERS' UNDERSTANDINGS OF DIVISION AND PROPORTIONAL RELATIONSHIPS WITH QUANTITIES

by

IBRAHIM BURAK OLMEZ

(Under the Direction of Andrew Izsák)

ABSTRACT

The purpose of this study was to investigate how preservice teachers' understandings of multiplication and division support and constrain their understandings of ratios and proportional relationships in terms of quantities. Six preservice teachers were selected purposefully based on their performances in a previous course. An explanatory case study with multiple cases was used to make comparisons within and across cases. Two semi-structured interviews were conducted with each pair. The results revealed that preservice teachers' understandings of partitive and quotitive division, and reliance on only multiplicative relationships shaped in critical ways how they reasoned about ratios. Preservice teachers who did not explicitly identify different meanings for division were found to have difficulty maintaining one of two distinct perspectives on ratio. The results also showed that additive relationships such as repeated addition and the phrase "for every" might have constrained teachers from keeping the two perspectives distinct.

INDEX WORDS: Division, Multiplication, Ratios and proportional relationships, Preservice teachers

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IBRAHIM BURAK OLMEZ

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IBRAHIM BURAK OLMEZ

Major Professor: Andrew Izsák

Committee: Sybilla Beckmann-Kazez
Kevin C. Moore

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2014

DEDICATION

I dedicate this work to

My parents, Muhammet and Feride Olmez

My brother, Sener Bahadır

My sister, Elif Tugce, and

My wife, Safiye Bahar Olmez

For their endless love and support

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CHAPTER 1

INTRODUCTION

Lamon (2007) defines proportional reasoning as:

> [S]upplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities. (pp. 637-638).

It can be inferred from this definition that proportional reasoning is composed of different kinds of understanding. First, it requires learning the invariance relationship between two quantities, which involves the essential understanding of “attending to and coordinating two quantities” (Lobato & Ellis, 2010, p. 12). This, in turn, involves being familiar with the term “quantity.” According to Smith and Thompson (2007), “quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them- whether we have carried out those measurements or not- that makes them quantities” (p. 10). Teachers (and students) might have problems attending to quantities when they focus only on numbers, or they might concentrate on only one quantity at a time rather than two quantities. Noelting (1980) found that students first reason with only one quantity rather than focus on two quantities at the same time. In a parallel way, Sowder (1988) showed that students’ lack of coordinating two quantities and relationships directs them toward focusing on merely choosing numbers and operations. It can be concluded that coordinating two quantities is essential for learning ratios and porportional relationships.

Lobato and Ellis (2010) presented a second essential understanding of ratios and proportional relationships: A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit" (p. 12). For Lobato and Ellis, two types of ratio are composed units (or we will call them "batches" later) and multiplicative comparisons. For example, if we consider a mixture consisting of 3 mL of lavender oil and 2 mL of rose oil, then one composed unit or batch is thinking of them together as 3 mL of lavender oil and 2 mL of rose oil. Then, multiple batches can be composed from this initial batch as 6 mL and 4 mL, 9 mL and 6 mL, 3/7 mL and 2/7 and so on. Composed units or batches are regarded as one of the key steps in reaching a deep understanding of ratios and proportional relationships. Another way to form a ratio is through a multiplicative comparison of two quantities. Comparing quantities with questions about how many times greater/smaller is the amount of lavender oil than the amount of rose oil illustrates a multiplicative comparison, rather than asking about how much greater/smaller is the amount of lavender oil from the amount of rose oil, which indicates an additive comparison. Then, a proportion can be defined as "a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change" (Lobato & Ellis, 2010, p. 12). It can be concluded that reasoning about proportional relationships is "complex and involves understanding the following:

- Equivalent ratios can be created by iterating and/or partitioning a composed unit;
- If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and
- The two types of ratios- composed units and multiplicative comparisons- are related" (Lobato & Ellis, 2010, p.12)

Comparison problems and missing-value problems are two problem types which have been generally taught in schools and have been investigated in the literature related to ratios and proportional relationships (Lamon, 2007). In comparison problems, the values of quantities a, b, c, d are placed accordingly and students are asked to decide the order relation between the ratios a:b and c:d. In missing-value problems, three values of a, b, c, d are given and students are asked to obtain the unknown (missing) value. However, students are often encouraged to solve both types of problems by using rote numerical procedures such as cross-multiplication rather than by focusing on multiplicative relationships between two quantities. Using cross-multiplication or other rote procedures will orient students' attention toward numbers instead of quantities. Consequently, use of rote procedures for solving problems is not sufficient to have a solid understanding of proportional reasoning (e.g., Lamon, 2007; Lobato & Ellis, 2010). Izsák and Jacobson (2013) state that "A robust understanding of proportional relationships includes forming multiplicative relationships between two co-varying quantities and recognizing whether or not two co-varying quantities remain in the same constant ratio".

Background and Rationale

Ratios and proportional relationships are core concepts of elementary and secondary mathematics education (e.g., Kilpatrick, Swafford, & Findell, 2001; Lamon, 2007, Lesh, Post, & Behr, 1988; National Council of Teachers of Mathematics, 1989, 2000) and provide the foundations for diverse topics such as linear functions, slope, geometric similarity, probability in mathematics; speed, acceleration, power in physics; gas laws and calculation of chemical substances in chemistry; and population density and scale in map-making in geography (e.g., Ben-Chaim, Keret, & Ilany, 2012; Karplus & Peterson, 1970; Lobato & Ellis, 2010; Simon & Blume, 1994). Lesh et al. (1988) give utmost importance to proportional reasoning as the capstone of elementary mathematics and the cornerstone of high school mathematics. Lamon (2007) regards the concepts of ratios and proportions, together with fractions, as "the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites" (p. 629). Proportional reasoning is expected to develop gradually over time with adequate instruction, and many adults lack proportional reasoning (Lamon, 2007; Lobato & Ellis, 2010; Tourniaire & Pulos, 1985).

In contrast to the large body of research on the proportional reasoning of students (e.g., Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b), there is a small body of literature focusing on teachers' reasoning of proportionality. In these relatively few studies, teachers and prospective teachers are found to have constrained proportional reasoning and to perform poorly on proportional relationship tasks other than missing-value problems (e.g., Harel & Behr, 1995; Riley, 2010; Pitta-Pantazi & Christou, 2011). Teachers might have difficulty coordinating two quantities in a proportional relationship (e.g., Orrill & Brown, 2012) and, might not separate missing-value problems involving proportional relationships from ones that do not include such relationships (e.g., Fisher, 1988; Riley, 2010). In order to teach ratios and proportional relationships so that students develop a deep understanding of proportional reasoning, teachers have to possess such an understanding themselves. As a result, there is a critical need for studies that investigate teachers' understanding of ratios and proportional relationships.

Research Questions

I examine how preservice teachers' understanding of multiplication and division support and constrain their understanding of ratios and proportional relationships in terms of quantities. This study will address the following research questions:

1. What are preservice teachers' facility with multiplication in terms of quantities? Do they see multiplication as repeated addition or as multiplicative comparison? Do they use multiplication within measure spaces or between measure spaces? Do they rely on multiplicative relationships or additive relationships?

2. What are preservice teachers' facility with division in terms of quantities? Do they use partitive division or quotitive division? Are they clear about different meanings of division?
3. What are preservice teachers' facility with two perspectives on ratios? Can they reason with the variable parts perspective? Can they reason with the batch perspective? Can they keep these two perspectives distinct? Are they thinking numerically or quantitatively?

CHAPTER 2

LITERATURE REVIEW

Partitive and Quotitive Division

There are two primary meanings for division: partitive division and quotitive division (Greer, 1992). Even though the operation for v items divided into w groups is easily answered as division, its interpretation alters based on whether we are looking for the number of groups (how many groups) or for the size of each group (how many in each group). If we use the division operation $v \div w$ to get the number of groups when v items are divided by w items in each group, then it is called measurement or quotitive (how many groups) division. Or, if we regard the division $v \div w$ to get the number of items in each group when v items are shared by w groups equally, then it is called partitive (how many in each group) division. For example, when $6 \div 3$ is considered as 6 pizzas are divided by 3 pizzas in each box, then we will obtain the result of 2 boxes (the number of groups), indicating quotitive division. When $6 \div 3$ is interpreted as 6 pizzas are distributed evenly into 3 boxes, then the result of the division operation, 2 pizzas in each box, reflects the meaning of partitive division.

Similar definitions also exist in the literature (Fischbein, Deri, Nello, & Marino, 1985; Greer, 1992). Fischbein et al. defined partitive division as "an object or collection of objects is divided into a number of equal fragments or subcollections" (p. 7) and quotitive division as " one seeks to determine how many times a given quantity is contained in a larger quantity" (p. 7). However, there are assumptions in Fischbein's division models. Specifically, in partitive division, he assumed that the dividend is bigger than the divisor and the quotient and that the divisor is a whole number. In quotitive division, his only assumption was that the dividend is bigger than the divisor. I do not agree with the assumption located in their division models. I do not think that there is any constraint in defining partitive and quotitive division. For us, the meaning of the division in a problem asking *4 pizzas are divided by 10 pizzas in each box. How many boxes do you need?* reflects the interpretation of quotitive division and the meaning of the division in a problem like *4 pizzas are shared evenly among 10 boxes. How many pizzas do you need in each box?* implies partitive division regardless of the numbers in the problem. Hence, I consider the meanings of division, particularly of partitive and quotitive division, as independent from numbers being large, small, integer, fraction or whole number.

Conceptual Framework

The conceptual structure for this study is framed by Vergnaud's (1983, 1988) *multiplicative conceptual field* which places ratios and proportional relationships at the center of many interrelated topics such as multiplication, division, fractions, slope, trigonometry, ratios and proportions, and linear functions. Vergnaud divided multiplicative structures into three subgroups: isomorphism of measures, product of measures, and multiple proportion other than the product. In the isomorphism of measures, which analyzes direct proportions, Vergnaud explained the relationship between multiplication and division with respect to two measure spaces. Vergnaud defined the multiplication relationship between two measure spaces as functional and the relationship within two measure spaces as scalar. Similarly, Vergnaud defined "first-type division" as a scalar relationship within measure spaces, consistent with our definition of partitive division. For example, when the problem asks *6 pizzas need to be put into 3 boxes equally. How many pizzas will there be in each box?*, one of the measure spaces consists of 1 box and 3 boxes, and the other one is 6 pizzas and unknown pizzas. Hence, finding the number of pizzas for 1 box by dividing 3 boxes into 1 box and applying the same scalar, 3, to divide 6 pizzas by 3, will give the answer of 2 pizzas in each box, which is similar to partitive division. And, in "second-type division", for the problem of *6 pizzas are distributed equally into 2 pizzas in each box. How many boxes do we need?* requires connections between the two measure spaces, one with boxes and one with pizzas. However, in this case, the meaning of division is different from the meaning in the former problem. That is, it is expected to out 2 pizzas into each box and apply the function operator, 2, so that the answer will be 3 boxes, which is compatible with quotitive division. Vergnaud thought that this type of division is difficult for children since the division operator involves a unit such as pizza per box. Moreover, I think that definition of proportional reasoning is related to whole-number multiplication and division. The relationship between two ratios with the quantities a, b, c, and d can also be explained by multiplication and division types of partitive and quotitive division. We also know that the multiplier and multiplicand in the multiplication model, and the dividend, divisor, and quotient in the division model have different roles and meanings (e.g., Greer, 1992).

Teachers and Students' Reasoning of Division

Studies of the *multiplicative conceptual field* have found that prospective teachers have difficulty in understanding the meaning of division with fractions (e.g., Ball, 1990; Borko, Eisenhart, Brown, Underhall, Jones, & Agard, 1992; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). In one of these studies, Ball (1990) analyzed 19 prospective teachers' understanding of division with fractions in three different contexts. In one of them, Ball asked the teacher candidates to write a word problem for 1 ¾ ÷ ½ . Although 17 of 19 prospective teachers could find the correct answer, only 5 of them constructed an accurate word problem. Ball found that most prospective teachers in the study thought only in terms of partitive division and examined this as the possible reason for their difficulties with the meaning of division. Additionally, Ma (1999) used a similar task in her study by asking U.S. teachers to compute 1 ¾ ÷ ½ and to generate a word problem. As a result, while 9 of 23 U.S. teachers were able to use correct algorithm and attain to correct answer, only 1 of them could construct word problems for the division operation, as opposed to 65 of 72 Chinese teachers.

Although most research on the *multiplicative conceptual field* has reported teachers' difficulties understanding multiplication and division of fractions and decimals (Ball, 1990; Borko, Eisenhart, Brown, Underhall, Jones, & Agard, 1992; Graeber & Tirosh, 1988; Izsák, 2008; Izsák, Jacobson, de Araujo, & Orrill, 2012; Ma, 1999; Simon, 1993; Sowder, Philipp, Armstrong, & Schappelle, 1998), relatively small number of studies have focused on the whole number multiplication and division understanding of students and teachers (Byerley, Hatfield, & Thompson, 2012; Correa, Bryant, & Nunes, 1998; Fischbein, Deri, Nello, & Marino, 1985; Kaput, 1986; Lo & Watanabe, 1997; Simon, 1993).

Lo and Watanabe (1997) conducted a teaching experiment with a 5th grade student, Bruce. They gave him 12 quarters and 28 cubes for concrete representation and asked him to solve a problem, "Yesterday, I bought 28 candies with 12 quarters. Today, if I go to the same store with 15 quarters, how many candies can I buy?" (p. 218). Bruce's initial strategy to get the relationship between 28 candies and 12 quarters was to arrange 12 quarters into equal groups (quotitive division) where each had certain amount and 28 candies into equal groups (quotitive division), and to obtain the same number of groups for both the candies and the quarters. That is, he first constructed 4 groups of 3 quarters from 12 quarters (quotitive division), and then he attempted to distribute 28 candies into groups of 3 candies (quotitive division). He, however, should have shared 28 candies into 4 groups and to get 7 candies in each group (partitive division). He solved the problem correctly through trial and error, but his inability to shift from quotitive division to partitive division may inform us about the possible challenges of preservice teachers in our study. In another task, later in the experiment, Bruce was able to make a transition from quotitive division to partitive division, which might indicate that transitions between the types of division develop with a variety of tasks about ratios and proportions. A main finding of this study was that understanding of multiplication and division operations can have a great impact on the understanding of ratios and proportional relationships. As a result, Lo and Watanabe suggested that providing diverse ratio and proportion tasks will foster students' ratio and proportion concepts in addition to other concepts in the *multiplicative conceptual field*.

Fischbein, Deri, Nello, and Marino (1985) proposed the idea of primitive models. They gave 5th, 7th, and 9th grader students word problems involving multiplication and division and asked their opinion about the selection of the operation they would use. A main finding of this study was that students have one intuitive model for multiplication and two intuitive primitive models for division; repeated addition model for multiplication, and partitive division and quotitive division models for division, but the latter one is learned through instruction. In order to investigate the assertion of Fischbein et al. (1985), Kaput (1986) asked students from grades 4 to 13 to generate division problems. Results showed that while 81% of the problems reflected partitive division, only 17% of them were quotitive division, and there were no significant changes in use of partitive division with the age of the students. Hence, Kaput (1986) confirmed the result of Fischbein et al. that students' conceptualization of partitive division is more primitive than of quotitive division. Moreover, Vergnaud (1983) thought that younger children would have trouble with problems involving quotitive division because they are required to change the referents in the problems. For example, for the problem *Bahadır has $21 to buy* *cokes for $3 each. How many cokes can he buy?*, the result of the quotitive division will be 7 cokes. Thus, referent units will change from dollars to cokes. For Kaput (1986), younger students' difficulty with quotitive division might be another reason for thinking that partitive division is the primary model for students.

Correa, Bryant, and Nunes (1998) conducted experiments with young children between 5 and 7 years old and found that children are able to make inferences easier in tasks involving partitive division than in tasks involving quotitive division. Their result seemed to agree with Fishbein et al. They also found that children 6 years old are able to make inferences in tasks with quotitive division and conclude that instruction might not be necessary for children to develop the quotitive meaning for division, as opposed to Fishbein et al.'s finding. Correa et al. reported that children's initial understanding of division might be mainly composed of sharing, a form of partitive division, while quotitive division might require children to connect partitive division and repeated subtraction.

Byerley, Hatfield, and Thompson (2012) asked seven undergraduate Calculus 1 students to generate a problem involving division such as "division of 6 by ¾ ths" or "explain how a puppy is growing at a constant rate" (p. 3). Preliminary results of their study reported that college students did not demonstrate clear conceptual meanings for division. Specifically, students using partitive division were found to have trouble generating a problem asking for division by a fraction. In one of the cases, Jack could not see the multiplicative relationship in the division problem despite being good at quotitive division. Instead of realizing the multiplicative comparison between quantity A and quantity B in a division operation A ÷ B, he focused on *for every "a" units growth of quantity A, there are "b" units growth of quantity B*, which directed his attention toward additive thinking. Another student in the study interpreted her division operation numerically, ignoring meanings for division. Also, one student with strong quotitive division meanings tended to use additive explanations when she explained the idea of a proportional relationship. Based on these cases, Byerley et al. conclude that many Calculus students might not see the multiplicative relationship in a division operation.

Simon (1993) conducted a study with 33 prospective elementary teachers to investigate their understanding of division in terms of their knowledge of units and their conceptual knowledge. He gave teacher candidates problems to generate whole number division and division by fraction word problems. In a task asking prospective teachers to write different story problems for division of 51 by 4 for which answers of 12 ¾, 13, and 12, were appropriate, 74% of the prospective teachers wrote problems with partitive division and 17% of them with quotitive division, implying that the prospective teachers think primarily in terms of partitive division. This result is consistent with the studies of Graeber, Tirosh, and Glover (1986) and Tirosh and Graeber (1989) who found that preservice U.S. teachers generally consider division from the partitive model. Graeber et al. also asserted that thinking from the partitive model (primitive model) causes preservice teachers to apply the primitive model of partitive division by considering the divisor in an operation as only a whole number and the quotient as always less than the dividend. As a result, Graeber et al. argue that for this reason for teachers' believing that "division always makes smaller."

Another main finding of Simon (1993) was that prospective teachers were unable to make shifts in thinking between partitive and quotitive division because they had trouble generating a problem for the answer of 13 in Task 1 requiring quotitive division, as opposed to the answers of 12 ¾ and 12 requiring partitive division, and making flexible transitions among these problems. Moreover, Simon found evidence of numeric division among the prospective teachers showing lack of the meaning of division in terms of quantities. Hence, it can be concluded from Simon's study that prospective teachers do not understand the concept of division clearly, they have weak understanding of the relationship between partitive and quotitive division, and they mostly use units inappropriately.

It can be inferred from the results of these studies that teachers have trouble understanding and interpreting meanings for division, which might create a great potential for opening the door to deficiencies in ratios and proportional relationships. Another main finding from these studies is that teachers mostly regard multiplication as repeated addition and division as repeated subtraction rather than focusing on the multiplicative relationships of multiplication and division. Hence, I suspect in this study that teachers might lack facility with multiplicative concepts of division in terms of quantities. In particular, they might struggle with constructing partitive and quotitive division. They might also have trouble having both meanings for division. I hypothesize that teachers may consider division as only repeated subtraction, use only numeric division without clarity in the meaning of the operation, and not perceive the division in a problem context.

Theoretical Framework

The theoretical framework of this study is based on Beckmann and Izsák's (in press) mathematical analysis of ratios and proportional relationships combining multiplication, division, and proportional relationships with two perspectives on ratios.

Beckmann and Izsák considered the equation "$M \cdot N = P$" as the number of groups times the number of units in each whole group equals to the number of units in M groups (Table 1). Similar to the previous discussions, looking for "how many groups" in the equation is called quotitive division, getting "how many in each group" is interpreted as partitive division.

Table 1. *Overview of multiplicative relationships* (Adapted from Beckmann & Izsak, in press).

$$M \cdot N = P$$

(# of groups) **.** (# of units in each/one whole group) = (# of units in M groups)

$M \cdot N = x$ [Equation A] Unknown product, multiplication	$M \cdot x = P$ [Equation B] "How many in each group?" division	$x \cdot N = P$ [Equation C] "How many groups?" division
$x \cdot y = P$ [Equation D] Inversely proportional relationship	$x \cdot N = y$ [Equation E] "Variable number of fixed amounts" proportional relationship	$M \cdot x = y$ [Equation F] "Fixed numbers of variable parts" proportional relationship

Beckmann and Izsák extended previous literature by constructing parallels between the two meanings of division and the two meanings of ratios and proportional relationships. They defined two perspectives on ratios and proportional relationships which parallel the two meanings of division. First, "variable number of fixed amounts" (or simply, the batch) perspective is in the form of $x \cdot N = y$. In this perspective, A units of the first quantity and B units of the second quantity can be viewed as composed units or "batches," and the ratio A to B can consist of two quantities in which their amount is multiples of those fixed measurements (batches). Then, by regarding N as the value of B $\div$ A, N is derived from the equation $x \cdot N = y$ which is partitive (how many in each group) division. Second, the variable parts perspective is in the form $M \cdot x = y$. In this perspective, one can fix the number of "parts" for each of the two quantities with the condition that the number of parts stays the same but the size of each part can vary. Then, by regarding M as the result of the operation B $\div$ A, M is taken from the equation M $\cdot$ $x = y$, which indicates quotitive (how many groups) division.

Since double number lines fit with the batches perspective and strip diagrams are compatible with the variable parts perspective, the representations of preservice teachers also might reflect the opportunities and constraints they might have experienced.

CHAPTER 3

METHODOLOGY

Methods

The purpose of this study was to investigate the opportunities and constraints of preservice teachers in understanding multiplication, division, and ratios and proportional relationships in terms of quantities. "*Case study research* is a qualitative research approach in which researchers focus on a unit of study known as a bounded system" (Gay, Mills, & Airasian, 2008, p. 426). It is an appropriate method to use "when you are trying to attribute causal relationships- and not just wanting to explore or describe a situation" (Yin, 1993, p. 31). Since this study was based on explaining possible cause-effect relationships, *an explanatory case study* was used. Also, to have a general picture of preservice teachers' understanding of ratios and proportional relationships, multiple cases were selected to improve generalizability and external validity of the research (Gay, Mills, & Airasian, 2008). *An explanatory case study* with multiple cases provided an opportunity to make comparisons not only within each case but also comparison between cases.

Setting

The study was conducted with three pairs of preservice teachers from the middle grades program during the Fall 2012 semester at one large university in the Southeast. The preservice mathematics teachers in the study were being prepared to teach Grades 4-8. The middle grades teacher education program is an undergraduate, 4-semester program that includes coursework in two subject area emphases (from among mathematics, science, language arts, and social studies) and teaching methods related to middle grades curriculum and students. The program includes only a standard first semester calculus course but no more undergraduate mathematics courses. Other than the first calculus course, the preservice teachers had to take specialized content courses in the Department of Mathematics and methods courses in the College of Education. In Fall 2012, they had already taken paired content and method courses on number and operation whose main focus was multiplication, division, and fractions. In Spring 2012, the teacher candidates had completed one content and one method course on geometry, and in Fall 2012, at the time of the study, they were enrolled in paired content and method courses in algebra. The pairs in this study were enrolled in the algebra course.

The textbook for the middle grades content course was *Mathematics for Elementary Teachers 3rd edition* (Beckmann, 2011), and the teachers in the middle grades program were taught ratios and proportional relationships at the beginning of the algebra content course. The aim of this content course was to develop preservice teachers' understanding of multiplication, division, fractions, and ratios and proportions in ways consistent with the Common Core State Standards (Common Core State Standards Initiative, 2010). Specifically, the content course gave problem situations with quantities to preservice teachers and asked them to explain their solutions in group discussions, in homeworks, and in exams, rather than only solving problems. The course also addressed both the "multiple batches" perspective on ratios and the use of double number lines for representing quantities, and the "variable parts" perspective and the use of strip diagrams for representing quantities. At the beginning of the course, teachers said that they already knew rules and algorithms for fractions and proportional relationships, but had little or no experience with double number lines or other drawn representations.

Participants

The recruitment of three pairs of preservice teachers was based on their performance in a previous course on number and operation. Each pair consisted of two preservice teachers with similar performance in the previous course. As a result, the participants were selected purposefully by the instructor of the algebra content course.

Data Collection

Two semi-structured (e.g., Bernard, 1994, chapter 10) hour-long cognitive interviews with each pair of preservice teachers were videotaped. Preservice teachers were paid $25 for each hour. The first interviews were conducted after 3 weeks of instruction during which the two perspectives on proportional relationships were introduced. The second interviews took place nearly 3 weeks before the final exam when all topics of the course were covered. The tasks used in this study are shown in Table 2. The aim of these tasks was to ask the preservice teachers to reason quantitatively about proportional relationships, as expected in the Common Core State Standards for Mathematics (CCSSM; Common Core State Standards Initiative, 2010). The tasks in both interviews were constructed as a result of expert judgments, and these tasks were part of PROMPT-mini Project, which aimed to investigate how preservice teachers learn the web of ideas connecting multiplication, division, and proportional relationships in ways that will support their future students' learning.

The goal for the first interview was to explore the preservice teachers' facility the two definitions of ratio individually and in contrast to each other. In particular, I concentrated on the facility of each preservice teacher with types of division in terms of quantities, with keeping the two perspectives on ratio separate, and with interpretations of multiplication from within or between measure spaces. I specifically examined how the preservice teachers reasoned with quotitive division, partitive division, or only numeric division and looked for evidence that they used the two perspectives on ratios. The analysis of each pair proceeds chronologically to obtain evidence for their use of division and the perspectives on ratio that they used.

Table 2. *Interview Tasks*

	Interview 1
Task 1	A fragrant oil was made by mixing 3 mL of lavender oil with 2 mL of rose oil. What other amounts of lavender oil and rose oil can be mixed to make a mixture that has exactly the same fragrance?
Task 2	What does it mean to say that lavender oil and rose oil are mixed in a 3 to 2 ratio?
Task 3	If I give you some amounts of lavender oil and rose oil, how can you tell if they are mixed in a 3 to 2 ratio? For example, consider each of these mixtures: 12 mL of lavender oil, 8 mL of rose oil ; 21 mL of lavender oil, 12 mL of rose oil; 14 mL of lavender oil, 8 mL of rose oil
	Interview 2
Scenario	To make 14-karat gold jewelry, people mix pure gold with another metal such as copper to make a mixture that we'll call "jewelry-gold." Jewelry-gold is made by mixing pure gold with copper in the ratio 7 to 5.
Task 1	Please show a strip diagram to represent that ratio and explain how to interpret the strip diagram.
Task 2	Explain how to reason about a strip diagram to solve the following problems: How much pure gold will you need to mix with 23 grams of copper to make jewelry-gold?
Task 3	Explain how to reason about a strip diagram to solve the following problems: How much pure gold and copper will you need to make 65 grams of jewelry-gold?

Task 4	There was some jewelry-gold that was made by mixing pure gold and copper in a ratio 7 to 5. Then another 4 grams of pure gold was added to the jewelry-gold mixture. What can you say about this new mixture?

The goal for Interview 2 was to explore the students' understanding of ratio from the fixed numbers of variable parts perspective and their use of strip diagrams for reasoning about ratio problems. The analysis of each pair proceeds chronologically to obtain evidence for their use of division when responding to the tasks, in addition to their use of multiplication and division within measure spaces and between measure spaces. I also make inferences about progress in their ideas related to division from Interview 1 to Interview 2. I was particularly interested in the students' ability to think about the parts in a strip diagram as being variable, use of *how many in each group?* division to solve problems about ratio using a strip diagram, and reliance on additive or multiplicative relationships in order to keep two perspectives on ratios separate.

In each interview, a separate piece of paper was given to each participant for each task. The interviewer read the task, and preservice teachers worked together and explained their reasoning out loud. Each interview was video recorded using two cameras, one for capturing the interviewer and the pair, and one for capturing the written work of the pair. Then, two video files from two cameras were combined into one video file for resorted view (Hall, 2000) and were transcribed verbatim. Also, the written work of the pairs was collected. Hence, the data in this study consists of the two interviews for each pair, transcriptions of the interviews, and the written work of the participants.

Analysis

After the data were collected, I took multiple passes through the data, reviewing the transcripts side-by-side with the videos. I concentrated on talk, gesture, and inscription to gather evidence for the thinking of preservice teachers. To analyze the transcripts, I first constructed detailed summaries of each video and tried to identify mathematical ideas in preservice teachers' thinking. After finishing the summaries, I concentrated on getting a larger argument from the readings I have read so far and from the discussions with my advisor. In this process, I thought about the progress of the participants within each interview and across interviews. Then, I watched each interview by focusing on each research question separately. Finally, I came up with an idea that preservice teachers might have experienced difficulties with different meanings of division and the two perspectives on ratios. Based on this spine, I continued to watch the videos further and critiqued the differences and similarities within and across pairs. The following examples illustrate how I used data to make claims:

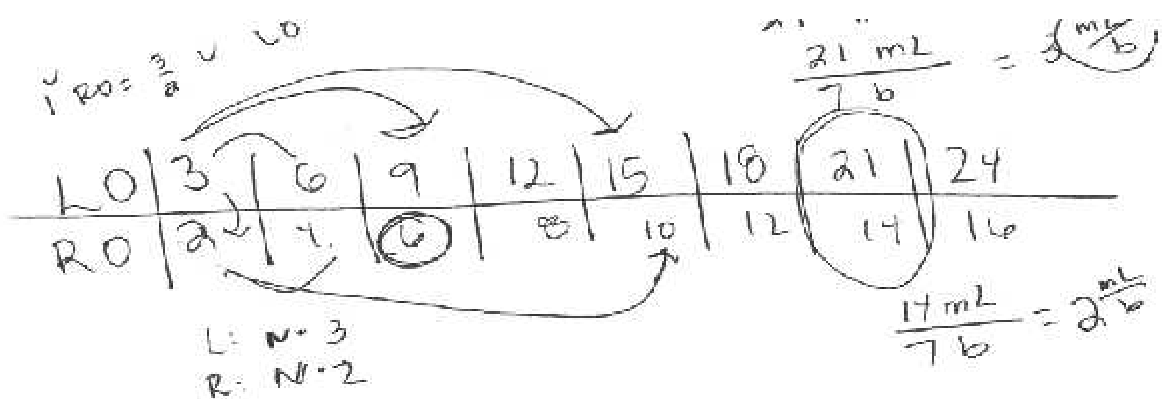


Figure 1: Amy's Drawing of Ratio Table

> Amy: I mean here I multiplied by whole numbers, you could also multiply by fractions. You could do like 3/2 to 1. And then ¾ to ½ and 3/10 to 1/5. And so even though these are smaller, you can go small or big but as along as its like 3 times the number of batches your making and 2 times the number of batches your making for lavender and rose oil, respectively, they should smell the same.

In this example, since the student divided the amount of lavender oil and rose oil by the number of batches, I inferred that she was using partitive division in addition to thinking from the batch perspective. She also seemed to rely on multiplicative relationships because she took multiples of the batches and explained the relationship between the batches in terms of multiplication both in the figure and the quote. On the other hand, the meaning which one student attributed for division is evident:

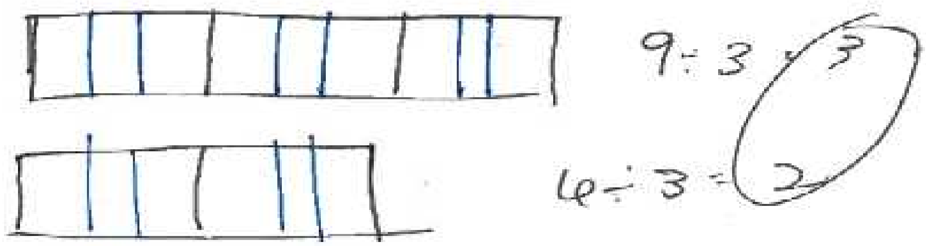


Figure 2: Lisa's Drawing of Strip Diagram

> Lisa: If you want to see like the ratio then you're going to need 9 to 6. And we know that there is 3 parts or 3 units in each part, so you divide the total by 3 numbers in each part so that would be divided by 3.

In this example, these data provided evidence that the student was using quotitive division to obtain the number of parts since she divided the total amount by the amount in each part. In addition, she appeared to combine quotitive division, associated with the batch perspective, with a sense of the variable parts perspective because she seemed to know that the number of parts stayed the same and the amount in each part could vary. Moreover, the following example is related to one student's emphasis on additive relationships through repeated subtraction:

> Tess: You know that for every 3 like you have 3, 3, 3, 3, 2, 2, 2, 2 if you take those away, they still smell the same as the first batch. Like if you take the previous 3's away that makes sense. Like from here we added 3, we added 3, we added 3 and then we added 2, we added 2, we added 2. So your doing the same thing every time. (lines 43-47)

It can be inferred from these data that this student might have perceived division as repeated subtraction. Her use of addition of batches in addition to the repeated subtraction were indicators of additive relationships.

Other than these examples, I will mostly refer to “keeping the two perspectives separate” and “combining the two perspectives” during the analysis. By “keeping the two perspectives separate” I mean that the student could use both perspectives, batches and variable parts, without mixing them. That is, it demonstrates using the batch perspective wording such as repeated addition of the batches with the batch perspective and using the variable parts perspective wording such as altering the amount in each part of a strip diagram with the variable parts perspective. For example, we will notice during the whole interviews that some students tended to use repeated addition of parts of a strip diagram when responding to the tasks instead of changing the amount inside each part of that strip diagram as in Figure 3:

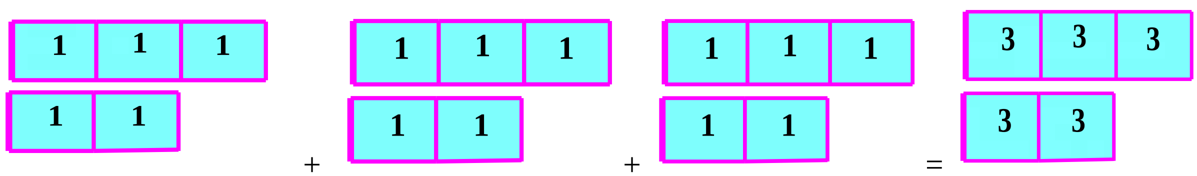


Figure 3: Mixing (Combining) the Two Perspectives

For these students, 9 mL of lavender oil and 6 mL of rose oil in a 3 to 2 ratio would be visually represented as left part of the equation, which demonstrates “not keeping the two perspectives separate” since parts of the strip diagram are repeated each time rather than increasing the amount in each part as right part of the equation. Hence, “not keeping the two perspectives separate” means “mixing or combining the two perspectives” in this study. Consequently, we hypothesize that while “keeping the two perspectives separate or distinct” suggests deeper understanding of proportional relationships, “combining the two perspectives” is more associated with weaker understanding of such relationships.

CHAPTER 4

FINDINGS

In this section, I present, in Table 3, a summary of all six preservice teachers' use of division and and their use of perspectives on ratio for each task shown in Table 2. In terms of understanding the background of the preservice teachers about their meaning for division, I give utmost importance to their use of division in Interview 1. Also, the tasks in this interview are more open to use of either perspective separately or combined since they do not ask specific use of one type of perspective or representation. However, the tasks in Interview 2 are based on understanding how preservice teachers interpret strip digrams and the variable parts perspective so that I expect to confront more with partitive division and the variable parts perspective during this interview.

I also present more detailed results suggesting that preservice teachers who have a clear understanding of both types of division could use both types of perspectives on ratio and could differentiate between the two perspectives. Moreover, I present that while preservice teachers who use mostly multiplicative relationships could keep the two perspectives separate, teachers who mostly rely on additive relationships could not differentiate them completely. In this section, I define "additive relationships" as including the relationships based on any type of addition such as repeated addition or subtraction of batches, simply adding or subtracting quantities in constructing ratios, and the use of the term, "for every", to imply addition repeatedly rather than multiplication. And I define "multiplicative relationships" as having a sense that when a quantity is multiplied or divided by a number, another quantity must also be multiplied or divided by the same number to keep the ratio between these two quantities the same. Hence, multiplication and division of batches by the same number, forming multiplicative comparisons within or between measure spaces among the quantities are indicators of multiplicative relationships.

Furthermore, I illustrate three cases including six teachers ranging from less to more proficient in their use of both types of division and perspective on ratios. Pair 1, Lisa and Tess, provided evidence that they were not clear about the different meanings of division and were not able to differentiate completely between the two perspectives. They also frequently used multiplicative relationships and additive relationships at the same time. Pair 2, Chip and Amber, performed better in using both types of division and perspectives on ratio than Pair 1. While Amber's use of division was mostly partitive and her use of perspective was primarily the batch, Chip was able to use both quotitive and partitive division, and both types of perspectives. However, Chip also had difficulty keeping the two perspectives distinct and occasionally combined multiplicative and additive relationships, similar to Amber. On the other hand, Pair 3, Amy and Paul, were able to use the batch and the variable parts perspectives and switch between the two perspectives. There was no indication that they mixed both perspectives. While Amy always relied on multiplicative relationships, Paul tended to use them mostly. In terms of division, Amy had a clear understanding of quotitive and partitive division. However, Paul always used partitive division.

Table 3. *Preservice Teachers' Use of division and Perspective in Each Task*

	Chip	Amber	Lisa	Tess	Amy	Paul
Interview 1						
Task 1	Repeated Subt. (Quotitive) and Partitive; Batch and Variable Parts	Partitive; Batch	Partitive and Numeric; Batch	Repeated Subt.; Batch	Partitive; Batch and Variable Parts	Partitive; Batch and Variable Parts
Task 2	Repeated Subt. (Quotitive) and Partitive; Variable Parts, Batch, Mixed	Partitive, Repeated Subt. (Quotitive); Variable Parts, Batch	Quotitive; Batch, Variable Parts, Mixed	Quotitive or Numeric; Batch, Variable Parts, Mixed	Skipped Task	Skipped Task
Task 3	Partitive or Numeric; Variable Parts	Partitive or Numeric; Batch, Variable Parts, Mixed	Partitive or numeric; No data	Quotitive; Batch	Quotitive; Batch	No data; No data
Interview 2						
Task 1	Partitive; Variable Parts, Mixed	Partitive; Variable Parts	Partitive; Variable Parts	No data; No data	No data; Variable Parts	Partitive; Variable Parts
Task 2	Partitive; Variable Parts	Partitive; Variable Parts	Partitive; Variable Parts	No data; No data	No data; Variable Parts	Partitive; Variable Parts
Task 3	Partitive; Variable Parts	Partitive; Variable Parts	Partitive; Variable Parts	Partitive; Variable Parts	Partitive; Variable Parts	Partitive; Variable Parts
Task 4	Partitive; Variable Parts and Batch	No data	Partitive; Variable Parts and Batch	Partitive; Variable Parts and Batch	Partitive; Variable Parts	Partitive; Variable Parts and Batch

Case 1: Lisa and Tess

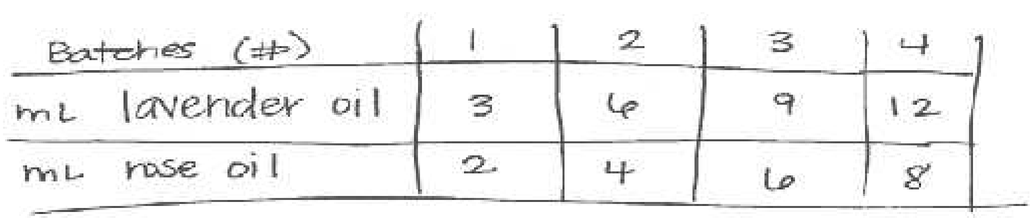
Summary for Lisa and Tess (Interview 1):

Lisa and Tess used both perspectives. They seemed to associate repeated addition with the batch perspective, and multiplication with the variable parts perspective. At the same time, students were not able to keep the two perspectives separate. That is, even though they were able to see the difference between the two perspectives, they both used mixed wording in explaining the distinction. It also appeared from the data that they mostly relied on additive relationships. I hypothesize one of the main reasons for such combination as that their consistent use of the term, “for every”, supported their attention to additive relationships. One of the students also confirmed that “for every” made her think additively. While Tess only used quotitive division, Lisa showed one use of quotitive division and otherwise used partitive division. In Task 3, the students were not clear about different meanings for division, which might explain their inability to completely differentiate between the two perspectives on ratio. Moreover, when the students used multiplication and division, they interpreted them within measure spaces.

Data and Analysis:

Task 1:

Both students started by making multiples of the original mixture supported by Lisa’s ratio table (Figure 4):



Batches (#)	1	2	3	4
mL lavender oil	3	6	9	12
mL rose oil	2	4	6	8

Figure 4: Lisa’s Drawing of Ratio Table

> Lisa: Okay we know for every 3mL of lavender oil in the mixture we have to have 2mL of rose oil and so we just use that those are the values that make up one batch of the mixture. So if we want double the initial batch then we just multiply each mL value of the initial ratio by the number of batches. So for two batches, or double the amount of the initial mixture, we multiply 3mL by 2 to get 6mL lavender oil. And then 2mL by 2 to get 4mL of rose oil, and then same for three and four batches. So it's kind of like additional copies. (lines 16-21)
>
> Tess: Yeah whatever your number of batches, you're just multiplying that by your first batch, so your original ratio to get your other ratios that are the same, (they're) equal. (lines 22-23)

These data demonstrated that the students discussed batches explicitly, in addition to their use of multiplication within measure spaces (e.g., the amount of lavender oil in one batch versus its amount in another batch; and the amount of rose oil in one batch versus another). Lisa used "for every" language to describe the relationship between the amounts of lavender oil and rose oil, which indicated an additive relationship, and she multiplied the original mixture by the batch numbers to obtain the larger batches, which indicated a multiplicative relationship. When the interviewer asked how they knew that that made the same fragrance, Lisa stated, "If you take each ratio and then divide it by the number of batches, then it will give you the initial ratio" (lines 29-30). Her division by batch number implied partitive division. A moment later, as a follow-up question, the interviewer asked whether there was another way to explain why the two mixtures would smell the same. Lisa thought that having 2 mL of rose oil for every 3 mL of lavender oil made the same smell, and Tess responded:

> Tess: You know that for every 3 like you have 3, 3, 3, 3, 2, 2, 2, 2 if you take those away, they're still the same as the first batch. Like if you take the previous 3's away that makes sense. Like from here we added 3, we added 3, we added 3 and then we added 2, we added 2, we added 2. So your doing the same thing every time. (lines 43-47)

Lisa's use of "for every" a second time, and Tess' emphasis on repeated addition and subtraction were the evidences of additive relationships. Moreover, Tess' wording in the quote above gave indication of her consideration of division as repeated subtraction. Then, the interviewer asked what would happen in the case of taking out a small portion, and Lisa appeared to use numeric division:

> Lisa: Because this is like one cup if you know 3 and 2 make one cup, then if your taking out a fourth of a cup then your divide by fourth. Then you're going to divide the one cup by fourth to get a fourth of a cup, so then you divide the amount of lavender oil which is three by four and then amount of rose oil by 4. (lines 69-72)

These data provided evidence for the batch perspective since Lisa considered one cup as "batch". In this task, while Lisa gave evidences of partitive division and numeric division, Tess presented evidence of repeated subtraction. Moreover, both students used the batch perspective. Additionally, when presenting the relationship between the amounts of lavender oil and rose oil, the students resorted to both multiplicative and additive relationships rather than only multiplicative ones.

Task 2:

As soon as the interviewer finished reading Task 2, Tess explained what a 3 to 2 ratio meant to her. The students explained further:

> Tess: I mean I think it means that for every 3mL of lavender oil you have 2mL of rose oil so like because there the ratio is 3 to 2 you always have to make when you're making that second batch when you add 3 of lavender you have to add 2 of rose oil for them to be the same. I don't know if that's….I mean that 's what it means. (lines 107-110)
>
> Lisa: I guess here if its not specified like mL's and this isn't specified to you can go the parts approach where for every 3 parts lavender oil you have 2 parts rose oil and then any volume quantity could represent. (lines 111-113)

That both students used of "for every" language and Tess's interpretation of it was based on addition provided further evidence for additive relationships. Lisa proposed a strip diagram to explain the 3 to 2 ratio. When the interviewer asked Lisa how her "for every" wording fitted into her "parts" approach, she replied:

> Lisa: The amount I mean like its trying to do repeated addition like the parts with the parts one, but it kind of you still have to have regardless of the volume quantity within each part, you still have to have 3 parts mixed in when you have 2 parts after its all mixed in. I am just kinda talking in circles.(lines 118-121)

These data suggested that there were elements of the batch perspective in Lisa's explanation. On the one hand, Lisa was talking about repeated addition of the parts which was compatible with the batch perspective. On the other hand, however, she was talking about keeping 3 parts and 2 parts fixed and changing the volume. Hence, Lisa's mixed wording combining the batch perspective with the strip diagram provided initial evidence that she struggled to keep the two perspectives separate.

Then, the interviewer continued with a follow-up question asking whether there were any visual representations about the previous discussion. The students then drew and discussed strip diagrams (Figure 5a, Figure 5b):

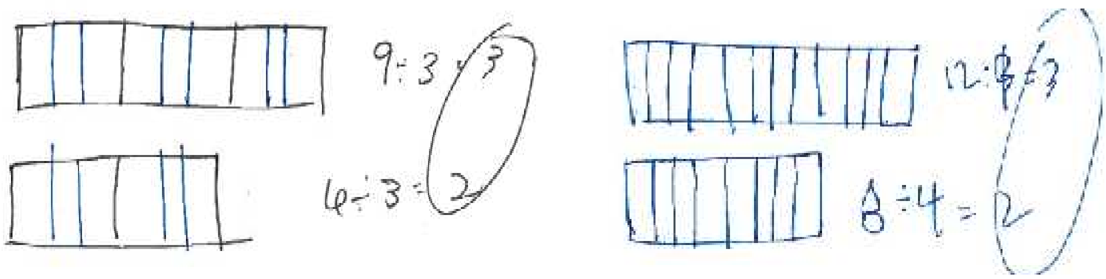


Figure 5: (a) Lisa's Drawing of Strip Diagram (b) Tess's Drawing of Strip Diagram (Blue dashes belong to Tess)

Int.1: Does that tell you something about the ratio?

Tess: Well the ratio like is always 3 parts to 2 parts even if you add, like this part is 3 cups, you're like make this 3 and then make this 3 and make this 3 and do the same thing down here. So you still have 3 to 2. You see how the strip diagram stayed 3 to 2 you just changed the amount in each part. So this would be 9 to 6 or like 9 over 6 but it's still 3 to 2 ratio.

Int.1: And how would you draw it if wasn't 3 cups in each but something else?

Tess: You could do 4. So now each part has 4 cups, we can say cups, so 4 times 4 times 4 or 4 plus 4 plus 4 is going to be 12 and then 8. So it's still the 3 to 2 ratio because our parts like our strip of 3 parts 2 parts stayed the same, just the numbers in each part, like the amount of parts in each unit. I don't really know how to …what I'm saying here.

Lisa: If you want to see like the ratio then you're going to need 9 to 6. And we know that there is 3 parts or 3 units in each part, so you divide the total by 3 numbers in each part so that would be divided by 3

Tess: Yup.

Lisa: equals 3 divided 3 equals 2, and that's your initial ratio.

Tess: and then you get the same thing here. You have 12 and 8, that doesn't look like an 8, and divide by 3 oh divide by 4 and you get 3 divide by 4 and you get 2 and then you have the same ratio as the beginning as when you started (lines 132-150).

These data demonstrated that Tess had a sense of the definition of the variable parts perspective because she seemed to know that while the number of parts stayed the same, the amount in each part could vary. These data also provided evidence of Lisa's use of quotitive division to obtain the number of parts to divide the total amount by the amount in each part. Later, Tess also appeared to repeat the same quotitive reasoning of Lisa by just dividing by 4 without explicit wording. Hence, Tess's words did not express explicitly the quotitive meaning of division at this point.

A moment later, the interviewer asked the students whether the expression, "for every 3 mL of lavender oil, there are 2mL of rose oil", was related to their strip digram drawings. The following was additional evidence that the students struggled to keep the two perspectives distinct:

> Int.1: So before you guys were saying for every 3 there are 2, for every 3mL of lavender there are 2mL of rose oil. Is that related to this or is that different? What do you think?
>
> Tess: I mean I think it's related to this, like its similar, but you can also explain it in a different way. Like how we've been explaining it with the parts, but still like, for like there's 3, for every 3 parts there is 2 parts. For every 3 parts lavender there is 2 parts rose oil, but it's different because whenever youwrote it in the last one each batch would increase in like the rose oil amount would increase but so would the lavender oil increase. So the ratio is the same, but here we are just changing the amount in each part. If that makes sense.
>
> Int.1: What is the wording for every, for every 3 there are 2. I'm wondering specifically about the for every part, tell me about that ,what does that mean to you or what do you think about that wording?
>
> Lisa: It makes me think about a repeated addition.
>
> Int.1: Say more about that
>
> Lisa: Like for every 3 that means anytime you add 3 you just add 2. Whereas here I mean I just always think about like with the first definition when you're using for every, you're altering both quantities that go in but the ratio remains constant. But then with the parts definition you alter the amount in each part.
>
> Tess: Like the first definition would be how many groups like in the first group there's 3 lavender 2 rose oil and in your second group there's 6 lavender 4 rose oil. So you added 3 and you added 2, and then from here, from the third group, you added 3 more and you added 2 more. But here you're looking at how many in each group, so you always have 3 groups and you always have 2 groups. We are just changing the amount in each group to add I mean whenever your adding lavender or rose it's always going to be the same ratio. And you can see that from the strip diagram, because we've stayed in our 3 to 2 ratio, we've just changed the amount in each part or in each group (lines 151-183).
>
> Lisa: I guess here if you say, if I were to look at this and you told me for every 3 parts there were 2 parts, I'm trying to think okay when I add 3 here then I'm going to add 2 here. Whereas, really you're increasing by the same amount.
>
> Int.1: Say more about that.
>
> Lisa: Like you're increasing each part has to be the same size, so when you alter the amount in each part, you're doing it by the same rate. Whereas here this always increases by 3 and this always

increases by 2. Here they have to increase by the same number. But the ratio still is constant because the parts are the set ratio.

These data suggested that although both students were able to see the difference between the two perspectives, they used mixed wording. Lisa, for example, mentioned about replicating the parts of a strip diagram as if each 3 parts of gold and 2 parts of copper is a batch, indicating the batch perspective wording in a strip diagram. Hence, use of mixed wording provided further evidence of Lisa's, and initial evidence of Tess's, difficulty keeping the two perspectives separate from each other. One main reason for such difficulty seemed to stem from their consistent use of "for every" language in their explanations. Lisa's confirmation that "for every" made her think in terms of repeated addition also supported my conjecture. It is evident in above exerpt that repeated addition, specifically "for every" language, directed the students' attention to additive relationships.

Furthermore, it appeared from the data that the students consistently adressed replicating of the parts of a strip diagram with the batch perspective and multiplying the amount in each part without changing the number of parts in a strip diagram with the variable parts perspective. As a result, they seemed to associate the use of repeated addition with the batch pespective and multiplication with the variable parts perspective. Lisa's saying that she always thought in terms of her "first definition", which was based on repeated addition, was additional evidence for why she was not able to keep the two perspectives separate. It was because she brought the batch perspective thinking with the repeated addition when she reasoned with a strip diagram.

For this task, it can be said that while Lisa used quotitive division, Tess was able to use quotitive or numeric division in their strip diagram drawings. They were able to use the batch and the variable parts perspectives and notice the difference between them. However, in their explanations, both relied mostly on additive relationships and these relationships caused them to use mixed wording, such as “for every”, with strip diagrams. As a result, they did not completely differentiate the two perspectives. Moreover, they appeared to use repeated addition with the batch (variable number of fixed parts) perspective and multiplication (multiples of the original mixture made in a 3 to 2 ratio) with the variable parts (fixed number of variable parts) perspective.

Task 3:

When the interviewer read the problems in Task 3, the following showed how Tess thought about the 12 mL of lavender oil and 8 mL of rose oil problem:

> Tess: Well I was going to divide by 3, I was just thinking to try to get it back to this original ratio, but you can’t. I mean if you divide by 3 that’s going to give you 4 and divide by 2 that’s going to give you 4. So I guess you could if you divide by, your lavender by 3 and your rose by 2 if you get this same number, then it is going to be like the same mixture. But if you don’t get the same number, then obviously you increased by a different amount. Does that make sense? Like. (lines 211-216)

These data did not make clear whether Tess was dividing to find the number of batches indicating quotitive division, or was using division to get the size per part, implying partitive division. Nevertheless, subsequent evidence made clearer how she was thinking of division in this context. One piece of evidence emerged from her consistent use of batch perspective in her explanations during the task and in her drawing (Figure 6b). Another evidence emerged in the case of 21 mL of lavender oil and 12 mL of rose oil problem. She gave an explanation similar to the previous problem (Figure 6a):

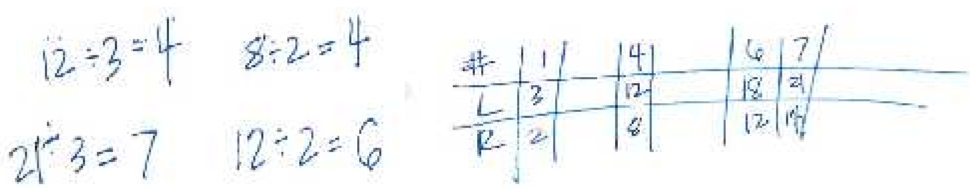


Figure 6: (a) Tess’s Division Representation (b) Tess’s Drawing of Ratio Table

> Tess: We know it’s the same, so if we do 21 divided by 3, what is that? 7? And then 12 divided by 2 is 6. So we know that this isn’t the same mixture because these two numbers aren’t constant. And then like for this the lavender oil it’s the seventh batch if you’re thinking about right here. But

when you see that it wasn't the seventh batch here, it was the sixth batch, so the mixture isn't the same. (lines 225-229)

Tess seemed to think that 21 mL lavender oil and 12 mL rose oil were not in a 3 to 2 ratio because they corresponded to different batches rather than the same batch. This provides strong evidence that she used quotitive division in this task. On the other hand, Lisa, proposed dividing the amount of lavender oil (12 mL) and rose oil (8 mL) by 4 to attain 3 to 2 ratio:

> Lisa: Yeah, well I was just saying you could divide, but essentially it's the same thing, but I was just saying if we could divide both values by a constant to get 3 to 2. So I saw it like you can divide 12 and 8 both by 4 to get 3 to 2. I knew that we couldn't divide these by a constant. (lines 241-244)

These data did not make clear what understanding of division Lisa used in this problem, but it was either partitive or numeric division. She seemed to see that she still got 3 to 2 ratio per batch, but what "the constant" stood for was not clear. When the interviewer asked them how they saw the similarity of their use of division, Lisa insisted that their reasoning was similar despite the interviewer's warning which told them that their arithmetic was different. This might have shown that Lisa had trouble understanding the difference between their use of division. This result implied that the students were not clear about different meanings for division, which may have contributed to keeping the two perspectives on ratios distinct.

In summary, in this task, Tess used the batch perspective and quotitive division. Lisa, however, tended to use partitive or numeric division showing that their understandings of division were different in this task. Both students used multiplication and division within measure spaces as well. Moreover, they provided evidence of multiplicative relationships.

Summary of Lisa and Tess (Interview 2):

Tess had difficulty thinking about the variable parts perspective with strip diagrams and also seemed to have a less developed understanding of partitive division, in comparison to Lisa. Tess's understanding of partitive division appeared to emerge when working on Task 3. Lisa, however, used partitive division throughout the interview. Both students used multiplication within and between measure spaces. The students could solve Task 4 only with prompting from the interviewer and they had trouble understanding that for every 4 grams of gold there would be 20/7 grams of copper. In this interview, they mostly used multiplicative relationships. In Task 4, they started to use additive relationships such as focusing on the difference between the two quantities. Coming back to additive relationships might have occurred as a result of the nature of the task asking addition of 4 grams.

Note: Differences between the two students in keeping the two perspectives might stem from their use of division. Recall that in Task 3 of Interview 1, Tess seemed to use a quotitive meaning for division, and Lisa seemed to use a partitive meaning or only a numeric meaning for division. Similarly, in Task 3 of Interview 2, Tess started to use partitive division reasoning by following Lisa. Hence, Tess's constraints with the partitive meaning for division might have led to constraints in thinking about the variable parts perspective.

Data and Analysis:

Task 1:

As soon as the interviewer finished reading the task, the students drew strip diagrams (Figure 7a, Figure 7b) and constructed the same equation (7/5)C = G using different reasoning:

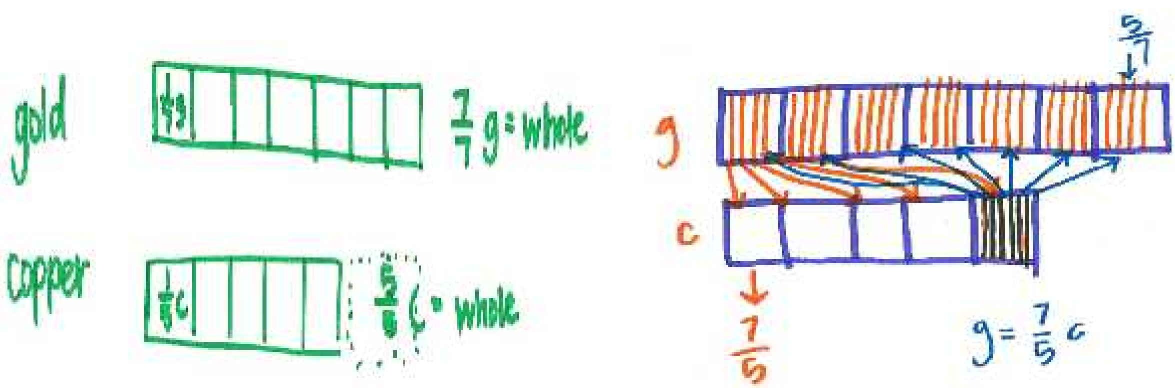


Figure 7: (a) Lisa's Drawing of Strip Diagram (b) Tess's Distribution Strategy

Lisa: Well I just drew the 7 parts for the gold and the 5 parts for the copper and I think about it that each part is 1/7 of G. And I'm using like G as my whole, all 7 of the 7 parts. And then I think of each of the 5 parts as 1/5 of the C using 5/5 C has my whole. And so when I'm trying to relate the two quantities, then I see that I need 5 plus like I drew like a dotted 2 more fifths to equal the amount of gold present. So I need 7 1/5 C parts to equal the amount of gold in the mixture. And then same concept when I'm relating gold and copper. I would need 5 of the 1/7 parts to equal the amount of copper in the mixture. So I would need 5/7 of the gold to equal the amount of copper. (lines 23-28)

Tess: I have 7 parts gold and 5 parts copper. And so I split just this 1 gold into 5 parts, and then I distributed 1 of these 5 parts into each of my 5 parts copper. It would be like, I would do it for all of them, but I just didn't show it. So I know that in one copper there's 7/5 gold. Because I have, I'm going to end up having, there's 1/5 here. There's 1/5 of this gold in here, but then I'm going to do it for every single one. So there's going to be 7/5 pieces in this one copper. So I'm going to have 7/5 gold for 1 copper. So right here whenever copper is 1, I'm going to have 7/5 gold. (lines 36-43)

Tess: And then to figure out how much copper or how much gold is in each copper, I broke this up into 5 pieces because there are 5 parts of copper. Because of the 7 to 5 ratio. So I want these each to be split evenly, so I know that I have an even, and there is equal parts in this one copper. So because there are 5 pieces, 5 parts copper I'm going to split each gold piece into 5 parts so that then I can distribute, I can just break up this one gold and put in each copper and do the same thing for each gold part that I have. So that there is an equal amount in each copper. (lines 62-68)

These data suggested that Lisa formed multiplication between measure spaces because she compared the copper parts with gold parts in the strip diagram visually and appropriately distinguished the multiplicative relationship across measure spaces. In addition, she used clear partitive division since she divided by the number of parts to find the amount in each part. Although Tess, inappropriately, used a distribution strategy by claiming that each part contained 7 mini parts and 5 same-sized mini parts, she seemed to make more an association of quantities between measure spaces rather than multiplying between measure spaces. Specifically, as a result of the distribution strategy, she deduced that "in one copper there's 7/5 gold" (line 39). The interviewer reminded her that although each gold part had 5 mini parts in her picture, each copper part had 7 same-sized mini parts, which was contradictory to Tess's beginning assumption stating that each part had the same size, an assumption consistent with the variable parts perspective. Tess then became confused and gave up using her distribution strategy. Even though her strategy was not accurate, she was able to get the same equation, G = (7/5)C, by using unit rate “for one part copper, there’s going to be 7/5 part gold” (line 179). When the interviewer asked Lisa how she thought about the distribution strategy of Tess, she responded:

> Lisa: Well that’s how like in EMAT we get the unit ratio is that how he is getting the unit ratio, but in my mind when I did it this way, I was getting my equation backwards. Because I was getting, I knew I was having 7/5 in this one. So my I was getting the equation 7/5 C equals G. I was getting 7/5 G because I was taking 7/5 of G and it was equaling my C. So I was getting my equations backwards, and it wasn’t and so that’s why I can’t, that I mean when you explain it, it makes sense. But like when I was doing it in my head, I was getting my equations backwards every time. (lines 111-118)

Lisa’s explanation was consistent with the assumption of the variable parts perspective having the amount in each part equal. In this task, Lisa used the variable parts perspective and partitive division. However, the data lacked evidence for Tess’s use of division and perspective on ratios.

Task 2:

In Task 2, Lisa presented clear examples of partitive division by finding out the number of grams in one part of copper:

> Lisa: Okay well if I just plugged in if I know that 23 grams is my whole for my copper, then each of my 5 parts is going to be 1/5 of 23 grams. And same concept you know I need 5 of these 1/5, I mean 7 of these 1/5 parts to find the equivalent amount of gold. So that would essentially mean the same thing as taking 1/5 of 23 7 times. (lines 199-203)

These data demonstrated Lisa’s understanding of partitive division because she seemed to know that “each part would be 1/5 of 23 grams” (line 214) and she needed “1/5 of 23 grams of copper 7 times” to obtain the amount of pure gold (lines 215-216). A moment later, when the interviewer asked Lisa to connect her partitive division reasoning with the equation, she said that “taking 1/5 C, which is 1/5 of 23 grams, times 7, which is essentially 7/5 times C” (lines 220-221). This quote provided additional evidence for Lisa’s multiplication between measure spaces. Strongest evidence for her multiplication across measure spaces came an exchange later when Tess asked more questions about Lisa’s partitive division reasoning:

Lisa: This is my whole. There's 7 parts for my whole for G. This is my whole, there's 5 parts for my whole C. So if I'm just like, when my I'm trying to find the amount or relate the amount of gold to copper. How much gold do I need to equal the amount of copper?, I only need 5 of this whole, five of my seven pieces. (lines 278-281)

Lisa: If we had 23 amount grams of gold is that what you said? We would have, so G you're amount of gold would be 23 and so each of these you know this would be 1/7 of 23. But I would only need these five, and so that's why I would only 5/7 of the amount of grams or the amount of gold to equal the amount of copper. Because I know that I have less copper in my mixture then I have gold. So I have to multiply it by a smaller fraction (lines 288-293)

Tess: Okay so you would do 1/7 of 23 5 times. Okay! Because you only need those 5 parts (lines 294-295)

These data showed Lisa's clarity in comparing the two quantities and partitive division reasoning, in contrast to Tess's effort to follow Lisa's thinking. Tess, on the other hand, did not seem to have the same perspective on the variable parts. For the 23 grams of copper problem in Task 2, Tess explained what her distribution strategy would be if she had used it:

Tess: I mean I like Lisa said, she's just is normally going to do it in the equation, and so would I. Like strip diagram not my favorite thing. I'd just much rather plug it in an equation and figure it out. But if I have five, I don't really know. I haven't thought about so if I had 5 of these but grams of copper but I really want 23 of them. Then the way I'm doing my strip diagram, I would make, I mean this is why I wouldn't do it this way, because I would have 23 of these boxes but I have 7 for every 5, so then I would have to add more of these gold and then split them each one up into 23 parts. And then distribute and so it would take forever. So that's why I would just plug it into my equation. But I do see how she would use a strip diagram here with her strip diagram to figure out like because like she said there's 23 grams of copper and so 1/5 of your amount of copper and then multiply it by 7 because that's equivalent to your gold. (lines 234-245)

These data suggested that Tess was constrained to consider one part in her strip diagram as 1 gram (Similar reasoning in lines 320-321). She also commented that she would not know how to do it with a strip diagram. Moreover, in line 265, she had trouble thinking of the entire strip diagram as one whole, in contrast to Lisa. These data suggested that she might have experienced constraints thinking about the variable parts perspective. In contrast to Tess's confusion in thinking about strip diagrams, Lisa seemed confident in her use of a strip diagram. An exchange later, the interviewer asked where Lisa saw 23 grams in her own picture. Lisa explained that she saw 23 grams in the strip diagram as "23 have to be contained equally among these 5 parts" (lines 304-305; similar reasoning in lines 308-309), which was consistent with the partitive meaning for division. Hence, in this task, Lisa presented a clear understanding of partitive division and the variable parts perspective, as opposed to Tess. Lisa also demonstrated multiplication between measure spaces. The students seemed to use multiplicative relationships and did not show indications of additive relationships. Even, they did not use the term, "for every", that they used frequently in Interview 1.

Task 3:

In Task 3, Lisa continued using partitive division and thinking from the fixed number of variable parts perspective by drawing a strip diagram (Figure 8a). She could see the total number of parts and suggested dividing 65 total grams by 12 to find that "each part is 65/12 of the mixture" (lines 372-373).

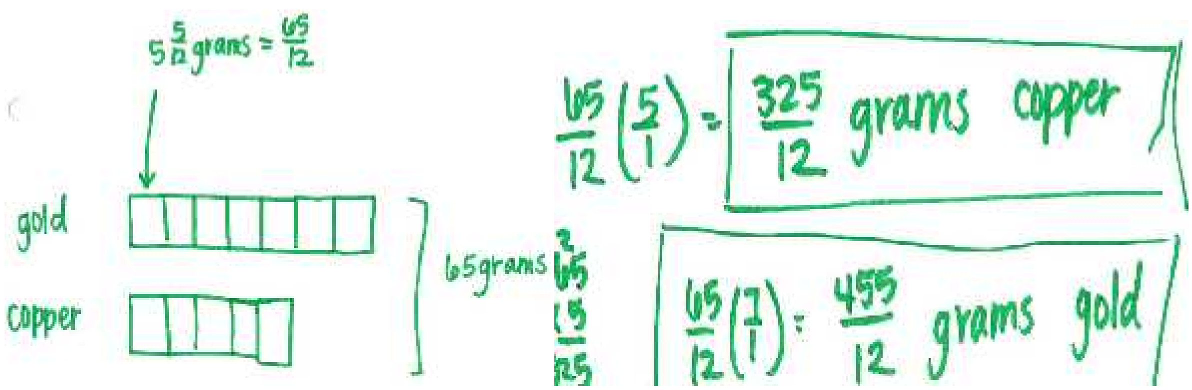
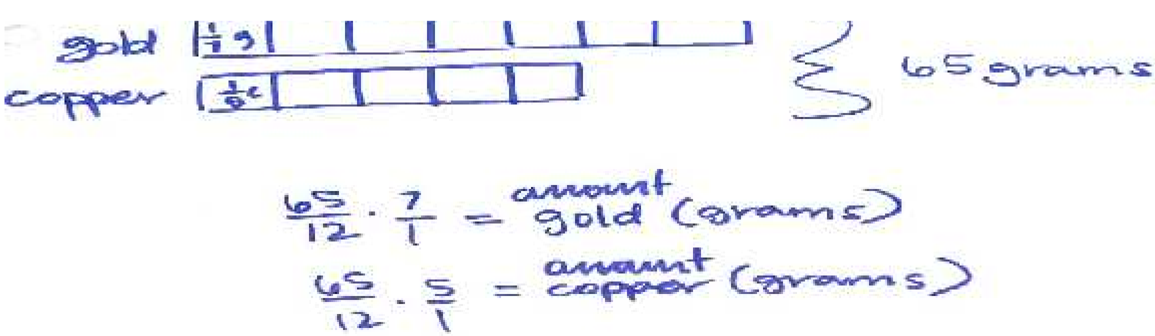


Figure 8: (a) Lisa's Drawing of Strip Diagram



(b) Tess's Drawing of Strip Diagram

Similarly, Tess drew a strip diagram (Figure 8b) and applied the same reasoning as Lisa for the first time in this interview:

> Tess: Right, because there's 7, there's 2 more parts gold then there is copper. So even though you can look at these parts as the same. There's 12 parts and they're all the same size. There's still 2 more gold even if all these parts equal the same size the same measurement 5 and 5/12 or 65/12. Because you're going to have 5 65/12 copper and you're going to have 7 65/12 gold. So there is still more gold. (lines 374-378; similar reasoning in lines 403-404)

That Tess divided 65 by 12 parts demonstrated her emerging use of partitive division and the variable parts perspective she first used in the previous task. She also said that as long as there was the same amount in each part, there would be 7/5 more gold than copper, which was consistent with her emerging use of the variable parts perspective. In this task, both students used partitive division and the variable parts perspective.

Task 4:

In Task 4, Tess knew that adding 4 grams of gold would change the ratio. Tess initially thought that they would have to know the amount of the mixture in order to find how much copper should be added for 4 grams of gold, and Lisa was not sure how to begin. In addition, they also thought that the problem could not be solved because 4 grams was a fixed amount, but parts could change:

> Lisa: Because there's not necessarily, this isn't necessarily equal to one gram each. So if I were to add on 4 parts then because this can change. The rate, like the amount of gram in each part can change. Whereas this is a constant value. Four grams is always added, so it's a constant value regardless of the amount, like the amount in each part can change. (lines 427-431)

> Tess: Because there's already more gold than copper. So like if you add 4 here and 4 here, it's going to equal like the same. (lines 456-457)

> Lisa: Because then at that point you're adding to the total. You're adding like the same amount of mixture of both to the total. So that just increases the amount of grams present in the total mixture and it's constant for both. (lines 458-460)

> Tess: It is not changing the ratio of 7 to 5. You're right. (lines 461)

These data reflected constraints in the students' thinking about the variable parts perspective. They thought incorrectly that adding the same amount to both gold and copper amounts did not change the ratio, indicating an additive relationship. After the partitive division examples, the interviewer asked the students again how they thought about a 7 to 5 ratio in a strip diagram with different amounts such as each part with 17 grams. Lisa continued relying on an additive relationship:

> Lisa: And when I added my own two. Like when I was trying to find the equivalency, when I added my two, I could do that because they're the same. And when I took away these two, or don't account for these two, that would equal because they're the same. Whereas if we had 11 and 9, you couldn't do that (lines 524-527)

Then, the interviewer moved on helping the students notice that whatever the amount of the mixture, for every addition of 4 grams of gold there must be 20/7 grams of copper to preserve the 7 to 5 ratio. With such an aim, first, the interviewer asked whether in the case of each part's being 17 grams, the ratio will be 7 to 5 or not. The students answered "yes" (line 520), because "you still have 7 parts and 5 parts" (line 522). Then the interviewer continued by asking what if 7 parts of gold equaled 4 grams, rather than adding 4 grams, then how much copper would they need. For this question, Lisa answered quickly as "4/7 times 5" (line 532). This reasoning was similar to the partitive division she used in Task 2 and Task 3. Then, the interviewer went back to asking how much copper needs to be added in the case of 7 grams of gold and 5 grams of copper with the addition of 4 grams of gold. The students found the correct answer of 20/7 grams of copper the second time, by subtracting 5 total grams of copper from 55/7 grams of initial copper. After the interviewer asked whether their arriving at 20/7 grams of copper twice was a coincidence, Tess started to see the connection between addition of grams of gold and copper, but Lisa still did not. Tess explained the relationship:

Tess: Yeah, we said that there, okay this equation we have 4/7 gold, correct? Each of these parts is 4/7, because this whole is 4. To find the copper divide by 5 that's going to give you 5 parts. So that's 20/7 grams copper for every 4 grams gold. So if we're adding 4 grams, we're going to add 20/7, because that was for our, for every 4 grams we add 20/7 copper. So if we're adding 4 grams, we're going to add that right. Is this the same? (lines 590-595)

Despite Lisa's reaching the first solution by using the variable parts perspective (same partitive division reasoning during the interview), Lisa and Tess's second solution was also driven by the batch perspective. As a result, they could solve the two problems with the help of the interviewer. Tess started to see the connection when the two solutions had the same result, indicating that Tess understood correctly that for every 4 grams of gold there must be 20/7 grams of copper. However, whether she understood that this relationship held regardless of the amount of the initial mixture was unclear. We can conclude that they both got the answer of 20/7 by conducting two separate solutions with prompting from the interviewer.

We conjecture the situation in this task that the students' constraints in thinking from the variable parts perspective might have caused them to not solve Task 4 and not see the connection clearly. Addition of 4 grams might have constrained thinking from the variable parts perspective because as in Interview 1, they might have connected additive relationships to the batch perspective and multiplicative relationships to the variable parts perspective. In this task, both students used partitive division and the variable parts perspective. Although there was no explicit wording for batch, the discussion was driven from the batch perspective. The students started to use additive relationships. It might have stemmed from the difficulty of the task or its nature focusing on addition of 4 grams.

Case 2: Chip and Amber

Summary for Chip and Amber (Interview 1):

While Amber provided evidence primarily of partitive division, Chip used both quotitive and partitive division during the interview. They also provided strong evidence for thinking of the batch and the variable parts perspectives. Both students tended to use partitive division with the variable parts perspective, and quotitive division with the batch perspective. While Chip was able to notice the difference between the two perspectives by saying that "for every" language was not suitable for thinking involving strip diagrams, he had difficulty relating his thinking based on the batch perspective into the strip diagram involving the variable parts perspective. As a result, he was not able to keep the two perspectives separate. It seemed that the use of additive and multiplicative relationships caused him to combine the two perspectives, as opposed to keeping them distinct.

Moreover, Amber used partitive division with lack of units during the interview, which might imply difficulty thinking in terms of quantities. In terms of the use of perspective, she used both perspectives but she could not differentiate between the two perspectives, as similar to Chip. In the case of strip diagram drawings, she brought batch perspective thinking into the problem by putting batch numbers inside of each part of the strip digrams and had trouble explaining her thinking. As a result, she mixed the two perspectives when responding to problems involving strip diagrams, indicating her not being able to keep the two perspectives separate. As Chip, Amber also relied on both multiplicative and additive relationships. When the students used multiplication and division, their thinking was based on within measure spaces.

Data and Analysis:

Task 1:

As soon as the interviewer presented Task 1, Chip started to draw a double number line, and both students demonstrated whole number multiples of 3 and 2 on the line (Figure 9a). They talked about batches explicitly and used whole number multiples as batch numbers to obtain larger batches:

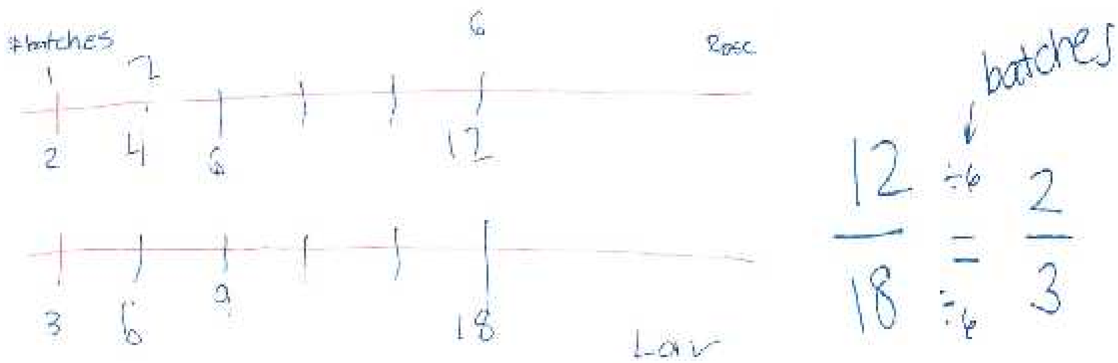


Figure 9: (a) Chip's Drawing of Double Number Line (b) Amber's Division by Batches

> Chip: We can do like we are in class, like this is one batch. Like number of batches up here. So that's 1 and that's 2. And then if we get all … Say this is the sixth batch, then all we're going to do is our initial ratio 2 to 3 times that 6. So 2 times 6 is twelve, and then the 3 times 6 would be 18 and you can keep going as high as you want to. (lines 53-56)
>
> Amber: Yeah I was going to explain it the same way. If you were only giving 2 to 3, and you want us to figure out the seventh batch, umm you would just multiply the number of batches by each number in the original ratio. (lines 60-62)

These data gave initial evidence for their multiplication within measure spaces (e.g., the amount of lavender oil in one batch versus another; the amount of rose oil in one batch versus another) by whole numbers and for their use of multiplicative relationships in their explanations. Additional evidence came a moment later when the interviewer asked how they knew those multiples have the same fragrance:

> Amber: This will simplify down to 2 to 3 to 2/3
> Chip: Yeah
> Int.1: So tell me more about that. What do you mean by simplify down?
> Amber: Like

> Chip: You can reduce that to the same original 2/3 fraction that you have to start with.
> Int.1: So you are basically saying those are equivalent fractions?
> Chip: Yeah, They are proportional fractions. As long as you can make it into a fraction, say 2/3, so rose to lavender is 2/3, or something like that. So then as long as 12/8 is equivalent to 2/3. So anything that is equivalent to that 2/3 fraction, ratio, whatever you want to call it, will be good enough. (lines 65-74)

The interviewer indicated that Chip and Amber were considering the same fragrance with the condition of being simplified into the 3 to 2 ratio, which was also consistent with their use of multiplicative relationships. An exchange later, the strongest evidence for Chip's use of multiplicative relationships came when the interviewer asked for more information about being in the same fragrance. Chip said explicitly that there was "like a multiplicative relationship between them" (line 78). In terms of division, their interpretation of multiples of the original mixture as "batches" and multiplying the original mixture by the batch numbers (e.g., the numbers 2 and 6 indicate batch number in Figure 9a), and Amber's explicit drawing (Figure 9b), implied that they also simplified the multiples into 2/3 by dividing by the number of batches to get the number in each batch, which was consistent with partitive division. The following also provided evidence for Chip's use of quotitive division, when the interviewer asked them to explain the same fragrance without using the words ratio, fraction or proportion:

> Chip: There's this for every 3, I think a good way to explain this is for every 3mL of lavender you have 2 mL of rose. So even if you have a huge tub of it, if you can separate it out to where 3 lavender goes to 2 rose and just keep separating it out, then eventually get to where there is none left. You don't have any leftovers but you just have a bunch of groups of the 2 to 3 like oil, then that would mean that it would smell the same.(lines 87-91)

Because Chip aimed to find the number of groups when he divided the total amount by the number in each group and his specific use of the words "a bunch of groups," it can be concluded that Chip had a sense of quotitive division in addition to his use of repeated subtraction. Another situation evident in this quote was Chip's transition from multiplicative relationship to an additive one with the help of "for every" language because it seemed that "for every" wording caused him to focus on subtraction of batches. A moment later, the interviewer repeated the same question in a different way by asking why larger batches would smell the same as the first batch. This time, Amber responded that they " dumped " the first batch in the same whole number "more times" (line 116). Chip also commented:

> Chip: Because you are doing the same thing every time, you are adding the same amount of each every time. So once you add the same amount of each over and over again and get up to six batches that you have up to 30 total cups in there, it mixes all together and makes this smell. So then all of those 30 cups are going to end up being that same fragrance. So then whatever you take out, regardless of what it is, I mean it's all mixed together at this point, so those 30 cups are now this mixed fragrance. (lines 119-124)

Amber's "dump" language and Chip's emphasis on repeated addition were evidence to support that the students relied on additive relationships at this time of the interview. Then, the interviewer continued by asking whether there were other ways to show that lavender oil and rose oil had the same fragrance. In response, Chip proposed drawing of a strip diagram in Figure 10 (Amber drew it following his explanation), where we are about to see the use of partitive division and the variable parts perspective:

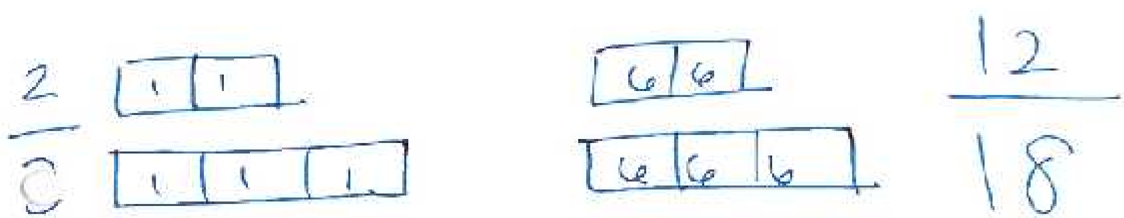


Figure 10: Amber' Drawing of Strip Diagram Based on Chip' Ideas

> Chip: I mean you could do it like 2 pieces, 3 pieces and then just say how much is in one piece. So that's 2/3 original right there. But now if we do that same thing and just say one of those pieces is like….Say in this one, say there were, that is one cup. One of these pieces each has one cup in it. Now in this next picture, we still have the same 2 to 3 ratio, we still have 2 parts of the rose and three parts of the lavender but now each part is worth six cups instead of one cup. So 2/3 is equivalent to 12/8 or 12/18. (lines 135-140)

These data demonstrated that Chip could also use the variable parts perspective, and his use of "how much is in one piece?"(line 135) provided initial evidence of partitive division.

From the video recording, it can be understood more clearly that Chip implied division of 2 by 2 and 3 by 3 to get 1 in each part and of 12 by 2 and 18 by 3 to get 6 in each part. Hence, for Task 1, it can be concluded that the students started with demonstrations of whole number multiplicative relationships within measure spaces and continued with additive ones. While Chip used both types of division and perspectives, Amber presented the partitive division and the batch perspective.

Task 2:

The question in Task 2 was to understand what lavender oil and rose oil mixed in a 3 to 2 ratio meant for the students. Chip responded to this question from both perspectives. First, he said it meant "as long as there is 3 parts of the lavender to every 2 parts of the rose, then there in that 3 to 2 ratio" (lines 148-149). Then, he repeated his previous explanation about the 3 to 2 ratio:

> Chip: Like I said earlier that 3 to 2 ratio just means for every 3 lavender you have 2 rose. So if you have 50 things laying on the table, if you can break them up, if you have 25 rose and 25 lavender, which wouldn't work out, but if you can just pull 3 of the lavender and 2 of the rose and place them together, if you can just keep doing that and splitting them all up, and if you get to a point where there is none left and you have made like 10 groups of that 3 to 2. Than that's what that 3 to 2 ratio means.(lines 155-161)
>
> Chip: So we just break it up, 3 lavender 2 rose here, and then that's a group. So then that's five. So now you have 45 left. Out of those 45 you go 3, 2 again and now you are down to 40. You just keep doing that till you get all the way down to zero. And you get down to zero there is nothing left, but you've done that splitting up making a group 10 times. (lines 184-187)

These data suggested that Chip was consistently using "for every" language and repeated subtraction to explain what the 3 to 2 ratio implied. His wording, "10 groups of that 3 to 2", indicated reasoning consistent with quotitive division because his goal was to obtain the number of groups. It can also be inferred that on the one hand, he was implying a multiplicative relationship by taking whole number multiples of the batches, and on the other hand, he was using an additive relationship by taking subtractions of these batches. Hence, we can conclude that Chip used both multiplicative and additive relationships simultaneously. Furthermore, Chip's "for" every" language related to the parts of a strip diagram might be an initial evidence of mixed wording if he implied replicating the parts of that strip diagram. In contrast, Amber explained her thinking about the 50 total cups example of Chad by drawing a strip diagram (Figure 11a):

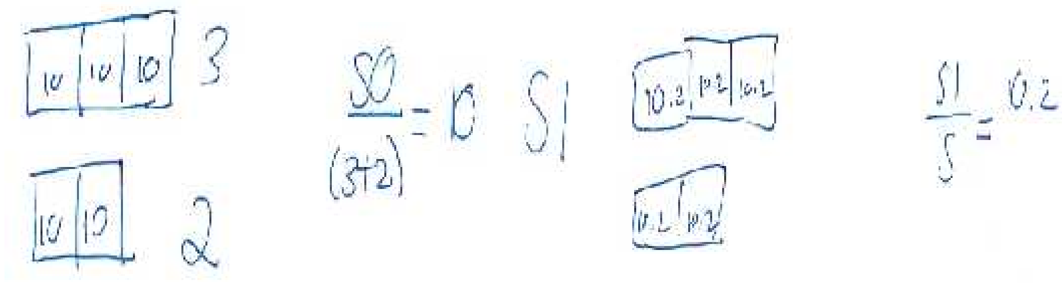


Figure 11: (a) Amber's Drawing of Strip Diagram (b) Chip's Drawing of Strip Diagram

> Amber: So if you have 50 total cups of whatever you can break it and do 50 divided by the 3 plus 2 to get 10 in each thing. Like Chip was saying, if you have 51, that wouldn't work out. (lines 167-168)

Amber's operation in the figure and her explanation for finding the amount in each part provided evidence of partitive division and the variable parts perspective. This was the second evidence for her use of partitive division. When the interviewer asked whether having 51 cups total was possible, Chip gave a clear partitive division example (Figure 11b) by stating that there would be 10.2 cups in each group in that case. At this time of the interview, Chip said that he did not see the strip diagram as a good representation of his thinking in the situation of repeated subtraction (taking the total amount away until there was nothing leftover):

> Chip: Okay so this isn't really a great representation of it, this picture, but I'm saying if we have 50 cups, like just say their individual different cups. Fifty cups on the table, and we know that we have a 3 to 2 ratio. (lines 180-182)

The interviewer followed up on Chip's thinking:

> Int.1: Are you relating that to that, that drawing there, the strip diagram?
> Chip: I would say that drawing there is not really what I was talking about. The way I'm talking about it would be like the individual batches.
> Int.1: Could you relate it to that drawing?

Amber: Well you would just have set up all the cups that you're pulling, you could just set them up to look like this. So you just keep going. You pull 3 out and pull 2 out, and then pull 3 out and 2 out, and then it would look similar to that. So you could see the ratio.

Int.1: So you sort of pull them one at a time and you would see that 3 2 thing like that.

Amber: Yeah then it would go 45 40. We'd just have ten 3 to 2s.

Chip: Yeah (lines 188-197)

This piece of data indicated that Chip seemed to know that his taking away idea was suitable to the batch perspective, which was different from the variable parts perspective in the strip diagram. However, when the interviewer asked them to relate Chip's taking away example to the strip diagram drawing, Chip preferred to be quiet. This might show that Chip had difficulty transferring his batch perspective thinking into the strip diagram, a representation tool of the variable parts perspective. This also might show that Chip felt constrained switching between the perspectives. Even though performing better than Chip in relating her batch perspective thinking into the strip diagram, Amber's explanation was not enough to connect the two perspectives because it appeared to be driven by the batch perspective.

The data also demonstrated her use of repeated subtraction, implying a sense of quotitive division, as in Chip's case. In summary, in Task 2, Chip and Amber provided evidence for multiplicative and additive relationships. While use of division indicated multiplicative relationships, "for every" language and repeated subtraction were evidence for additive relationships. The students used partitive division and the variable parts perspective. They used repeated subtraction in a way of supporting their sense of quotitive division and the batch perspective. Both students had difficulty in making a shift from the batch perspective to the variable parts perspective when asked to relate the "taking away" idea into a strip diagram.

Task 3:

For Task 3, Chip and Amber discussed simplifying fractions, as in the first task. They both divided by 4 to reduce 12 mL of lavender oil and 8 mL of rose oil to 3 to 2 ratio (Figure 12a). However, this time, they did not use the word "batch" explicitly:

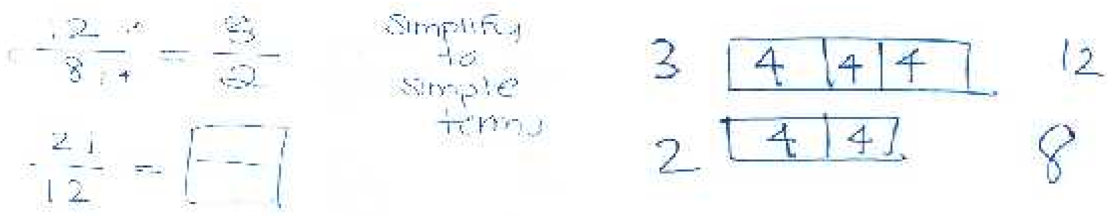


Figure 12: (a) Amber's Division Operation (b) Amber's Drawing of Strip Diagram

Amber: Just by simplifying the fraction, if it simplifies down into 3 to 2.
INT1: So like … just show me one for example
Amber: Hold on.
Chip: Divide by 4. That one works because you're reducing the fraction to a 3 to 2. That one doesn't work [pointing to 21 mL LO and 12 mL RO]
Amber: This one wouldn't work. [pointing to 21 mL LO and 12 mL RO]
INT1: So by work or not work you mean?
Chip: It doesn't simplify to that 3 to 2 ratio. (lines 203-210)

The exerpt showed that their use of division when reducing fractions might have been either partitive or only numeric division. Using the same method, the students noticed that 21 mL of lavender oil and 12 mL of rose oil would not work. Another evidence we took from these data was their use of the multiplicative relationship, since both Chip and Amber used division in reducing fractions. A moment later, when the interviewer asked if there was any other way to solve the problems without working with fractions, Chip quickly proposed drawing a strip diagram (Figure 12b). The figure (Amber drew the diagram based on his explanation) and the following were evidence for his use of the variable parts perspective and also partitive division for the problems including 12 mL of lavender oil and 8 mL of rose oil, and 21 mL of lavender oil and 12 mL of rose oil:

> Chip: I mean you could redraw your picture of like the 12 to 8, like the 3 pieces and then 2 pieces and then just say each piece is worth 4, I guess it would be. (lines 223-224)
>
> Chip: Well if we are looking at picture like this, 21 and 12 there is not going to be a number that can go into those boxes to where they're all going to be equal pieces. Its not going to simplify down to where like 3 times 4 is equal this, and 2 times 4 is equal to that. There's no number to where 3 times what gives you 21 and 2 times what equals 12, there's no number that can fit both of those equations. (lines 231-235)

Amber, on the other hand, gave an explanation using the batch perspective for the 12 mL of lavender oil and 8 mL of rose oil problem, even though she created a strip diagram (Figure 12b) following Chip, and her diagram was based on the variable parts perspective:

> Amber: You're just creating this 4 times. So you are doing 3 times 4 equals 12 and 2 times 4 equals 8. So you're still keeping the 3 to 2 ratio. You're just adding the 3 to 2 four times to get to 12/8ths so (inaudible) all the same. You're just increasing the number of batches of the 3 to 2. (lines 226-229)

These data implied that Amber put batch numbers into each part of a strip diagram, indicating a mixing of the two perspectives. Hence, Amber was not able to keep the two perspectives separate. Since Amber thought primarily in terms of the batch perspective, she might have had trouble thinking from the variable parts perspective when she confronted with the strip diagram. Note that during the whole interview, only Chip proposed strip diagram drawings as another way of representation. As a result, she brought batch perspective into thinking of strip diagrams which caused her to combine the two perspectives as opposed to keep them separate. Then, for the 21 mL of lavender oil and 12 mL of rose oil problem, she came back to the batch perspective. She basically told that the ratio 21 to 12 would smell more lavendery because of multiplying 3 x 7 = 21 and 2 x 6 = 12. According to her, since lavender oil and rose oil corresponded to 7th and 6th batches, the mixture would not be in the same ratio, and lavender oil would have been dumped 3 ml more, indicating an additive relationship. In summary, both students used division in reducing fractions. Chip and Amber provided evidences of partitive division. Whereas Chip used the variable parts perspective, Amber used both perspectives by mixing them. Despite her drawing of the strip diagram with a sense of the variable parts perspective, she put the batch numbers inside of the strip diagram. We conclude from this evidence that Amber had difficulty keeping the two perspectives distinct since she was mainly thinking from the batch perspective and her mostly reliance on the additive relationships throughout the interview.

Summary for Chip and Amber (Interview 2):

Chip's difficulty with keeping the two perspectives separate continued in this interview. He consistently used multiplicative and additive relationships, as similar to Amber. There was no evidence for the students' use of multiplication in terms of between measure spaces. For the type of division, Amber's use of partitive division developed throughout the interview in addition to her use of the variable parts perspective. She started to use partitive division with lack of units but improved as the interview moved. On the contrary, Chip used partitive division and the variable parts perspective throughout the interview. Chip solved Task 4 by taking each part first to be 1 gram and then to be 10 grams, made the connection that for every 4 grams of gold there were 2.8 grams of copper regardless of the amount of initial mixture. Despite his success in seeing the connection between two solutions, these solutions were based on the batch perspective. He agreed that his understanding emerged within the problem, and he generalized the solutions by writing an algebraic equation. On the other hand, even though Amber could solve the problems when each part contained 1 gram and 10 grams with little help from Chip, she could not see the connection between the two solutions and did not make any conclusion. She even checked her arithmetic to make sure that for every 4 grams of gold, there were 2.8 grams of copper.

Data and Analysis (Chip and Amber):

Task 1:

As soon as the interviewer finished reading Task 1, Chip gave an explanation of what a ratio of 7 to 5 meant for him that paralleled his explanation of what a 3 to 2 ratio meant in second task of Interview 1:

> Chip: So if we have a ratio of 7 to 5 that means that we have 7 parts pure gold for every 5 parts copper, and those parts can be anything. I guess if we're talking about gold and copper that can be like, you can use like grams or something. It has to be like a set unit. All it means to say a ratio 7 to 5 is for 7 of these parts gold, then there's 5 of those same like equal parts in copper. So this can either be one gram, and there can just be 7 one gram pieces of pure gold and then 5 one gram pieces of copper. Or it could be each one of these pieces is a thousand grams, but then you would just have 7 one thousand gram pieces as opposed to 5 of the one thousand gram pieces. (lines 22-30)

These data suggested his keeping "for every" language from Interview 1 in a sense of supporting additive relationship between the amount of gold and copper. As in interview 1, he might have considered parts of the strip diagram as batches which are replicated, implying mixed perspective. Moreover, these data demonstrated his thinking in terms of the variable parts perspective and clear use of units, though we were at the beginning of the interview yet. He seemed aware that while the number of parts stayed the same, their size could change with the condition that each part contained the same amount. A moment later, Amber and Chip explained what a 7 to 5 ratio meant for them in a strip diagram by giving examples of the multiples of the original mixture and simplifying to 7 to 5 (Figure 13), with the indication of division and multiplication within measure spaces:

Figure 13: Amber's Division Within Measure Space

Amber: Well it just always has to stay in the ratio, so if you had 14 grams then you would have 10 grams of copper. Our ratio is 7 to 5. So you could just do 14, 10, 21, 15 and that still simplifies down to 7 to 5 and just would divide by 3. (lines 32-34)

Chip: Yeah pretty much the 7 to 5, if we look at that as like a division. Well then whatever you do to the top, if we do that at the bottom, then its still going to be in the same ratio. So like that's what she's saying right here 5 times 2 is 10 well then 7 times 2 is 14 which means that those are still in the same ratio as the 7 to 5 original. (lines 44-48)

These data were insufficient for concluding whether Amber divided multiples of the 7 and 5 by the batch number to reduce them to the 7 to 5 ratio. Unfortunately, when Amber was thinking what to label the 3, Chip interrupted her silence and explained that the 3 as the grams in each part. After his explanation, Amber started using partitive division for the example of 21 grams of gold and 15 grams of copper:

Amber: Yeah, so if we were doing 21 and 15, keeping the 7 to 5 ratio, that means each thing would have to be 3, because 3 times 7 would be 21 and 3 times 5 would be 15. (lines 37-39)

Her wording lacked explicit units, similar to her use of partitive division in Interview 1. This might suggest constraints in explicit meanings for partitive division and the variable parts perspective.

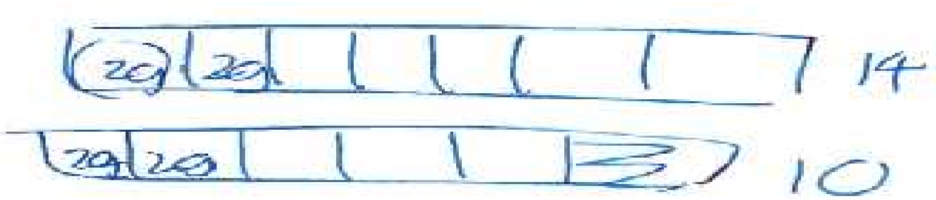


Figure 14: Amber's Drawing of Strip Diagram

When the interviewer asked the students to talk more about the 7 to 5 ratio, Chip used "for every" language a second time in this interview by saying that for every 7 grams of gold there would be 5 grams of copper (lines 65-66). An exchange later, the interviever asked a similar question, as in Interview 1, about how the "for every" wording fitted with the strip diagram:

Int.1: How does that, if you say for every 7 grams there's 5 grams copper how does that fit with the strip diagram, or do you see that fitting with the strip diagram?

Chip: Well I think you can fit it with the strip diagram. Like if you go, well in this one for every 7 grams, well now you have 14 grams so that means you have 2 sets of that. So for every 7 grams now you have that twice.

Amber: Yeah so if we were doing 14 and 10, that means it would still be 7 but 2, 3, 4, 5, 6. Oops. That was bad.

Chip: Not quite equal

Int.1: I know it is hard to draw.

Amber: And these would all be 2 grams to get the 14 out of the 10, which is still the 7 and the 5. Its still has the 7 and the 5, it's just what changes in it to get a different amount but still has the same ratio

Int.1: Does that fit with for every 7 grams of gold there are 5 grams of copper?

Chip: Not necessarily in the strip diagram, like that wording doesn't directly apply to the, that really I guess to say you need to say for every

Amber: Group? For every group (inaudible). I don't know.

Chip: I don't know. That wording doesn't directly relate to the strip diagram like we were talking about. For every 7 parts of pure gold. So if you take whatever amount of pure gold you have and divide it into 7 equal parts, well then for every 7 of those parts then there's 5 parts of the same equal value of the copper. This is more relating to one situation like in our strip diagram, then obviously each part would be 1 gram. But to just talk about it in general terms of the strip diagram, then you need to say for every 7 pieces of some unit of pure gold, then there's 5 of that same piece of the copper, like 5 equivalent pieces of copper.

Int.1: Let me here Amber's thoughts.

Amber: I think you can kind of relate it to this, because you're still saying for every 7 grams of pure gold you have 5 grams copper. Well if you have 14 grams of pure gold, that's just 7 twice. You know?

These data provided additional evidence for Chip's difficulty keeping the two perspectives separate. His consideration of the strip diagram with 14 grams of gold and 10 grams of copper as "2 sets of that [pointing to 7 parts and 5 parts of the strip diagram]" implied replication of 7 parts and 5 parts of strip diagram twice rather than filling the amount in each part. Hence, Chip could not differentiate between the two perspectives completely. The main reason for such constraint seemed to be the influence of "for every" language on directing Chip's attention to additive relationships even though he knew that "for every language" elicited different thinking than the thinking that accompanies the strip diagram. Hence, Chip's mentioning about replicating of the parts of a strip diagram indicated his tendency to mix the two perspectives. Recall that in the previous interview, he had difficulty relating his taking away method (based on the batch perspective) to the strip diagram (based on the variable parts perspective).

As a summary of the first task, the students used simplification in a sense of division and presented evidence for their thinking of whole number multiplication and division within measure spaces. After Chip explained what dividing by 3 meant in simplification, Amber started to use partitive division and the variable parts perspective, but had difficulty with identifying appropriate units as in Interview 1. At one point, Chip gave an indication that he was able to use partitive division and the variable parts perspective, but consistent use of "for every" language appeared to constrain his use of the variable parts perspective and caused him, to have difficulty keeping the two perspectives separate, as in Interview 1. Amber also experienced the same difficulty due to the use of additive relationships.

Task 2:

When the interviewer read the task, the students divided 23 grams of copper by the number of parts of copper and multiplied that result by the number of parts of gold to determine the total grams of gold. The meanings for their use of partitive division and the variable parts perspective in addition to their comparison of the amounts multiplicatively and additively are evident in the following exerpts and figures (Figure 15a, Figure 15b):

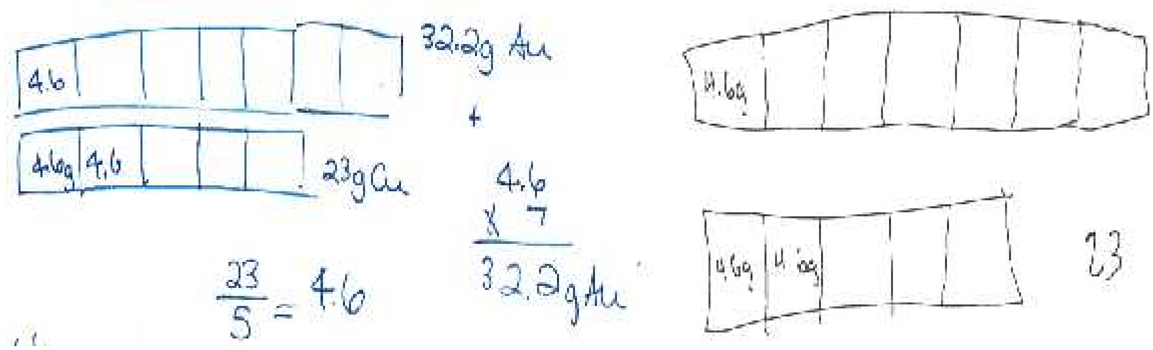
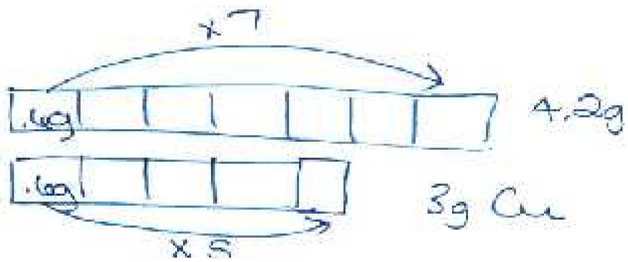


Figure 15: (a) Amber's Drawing of Strip Diagram (b) Chip's Drawing of Strip Diagram



(c) Chip's Drawing of Strip Diagram for 3 Grams of Copper

Amber: Okay so if it tells us that we have 23 grams of copper and our ratio is 7 to 5. Well we know the copper part is with the 5. So I did 23 divided by 5 to find out that there was 4.6 grams in each box. So I know, and the boxes are the same, so then I took 4.6 and multiplied it by 7 to figure out how many grams of gold I would need.

Chip: Which all goes back to the equivalent pieces you want an equal piece of copper to match up with this equal piece of pure gold. So that's why you found 1 piece of copper is 4.6 grams. Well then that, in our strip diagram directly states that one piece of pure gold is also 4.6 grams. And then we know we have 7 pieces of pure gold for the 5 pieces of copper. So then you just do the 4.6 plus 4.6 plus 4.6 up to the 7 pieces and then. Which is just 4.6 times 7. (lines 152-162)

These data demonstrated both students' use of partitive division and the variable parts perspective. They both continued to use division and multiplication within measure spaces. While Amber provided a multiplicative relationship by multipliying the amount in each part of the strip diagram by the total number of parts, Chip had a sense of an additive relationship due to repeated addition of the amount in each part. Additional evidence came a moment later when the interviewer asked them to consider 3 grams of copper (Figure 15c):

Chip: Like then you pretty much do the exact same thing we did last time. We have 3 grams of copper split into 5 equal parts, so we need to figure out how much one part is worth. Which would be 3/5 of a gram or .6 of a gram. So then you have 3/5, 3/5, 3/5, 3/5, 3/5, which adds up to 3 total. And

> since we know the copper parts are equivalent to the gold parts then we have 3/5 of a gram per box of pure gold, and we have 7 boxes of pure gold. So 3/5 times 7 is whatever that is, 21/5. So we have 21/5, which you can simplify to 4 and 1/5 grams of pure gold for 3 grams of copper. (lines 168-175)

On one side, Chip was using partitive division by dividing the total amount of copper to get the amount in each copper part, which indicates multiplicative relationship. However, on the other side, he added the amount in each part copper to get the total amount of copper, which implies an additive relationship. Consequently, Chip demonstrated both additive and multiplicative relationships. In this task, both students used partitive division and the variable parts perspective. Amber's use of partitive division was apparently improving from the previous task, because she developed the use of units in her explanations. Both students continued using multiplication and division within measure spaces. Whereas Amber demonstrated multiplicative relationship, Chip combined additive and multiplicative relationships.

Task 3:

For Task 3, Chip and Amber divided 65 grams of jewelry gold by 12 parts to obtain the amount in each part, which implied partitive division. Seeing the whole (total number of parts) apparently helped them to solve the problem. While the only evidence for Amber' use of partitive division and variable parts perspective was her written work (Figure 16a), Chip again preferred to explain his strip diagram in addition to his written work (Figure 16b):

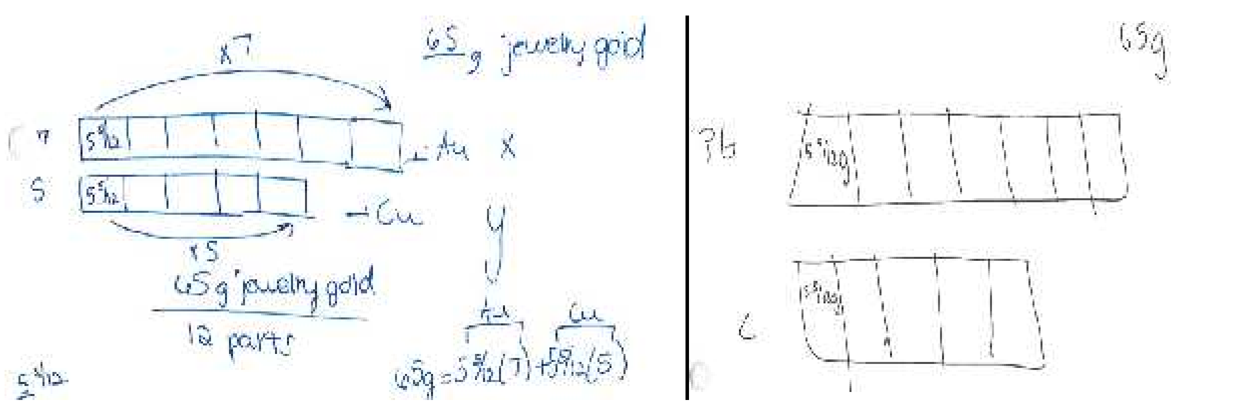


Figure 16: (a) Amber's Drawing of Strip Diagram (b) Chip's Drawing of Strip Diagram

Chip: Okay now we have equal parts and we have 7 parts for pure gold, and then for copper we have 5 parts. So we're taking 65 grams and each part is equivalent. So if we take 65 and divide it by the total number of parts we have, which would be 12, so if we do 65 divided by 12, 5 and ½ I think 12, 24, 36, 48, 60. No it's not 5 and 1/2. It's 5 and 5/12. So that 5 and 5/12 gives us what number of grams goes in each box. So 5 and 5/12 grams per each box of our strip diagram. (lines 182-187)

Chip: So then we have equivalent parts like we already know, and then you just do 5 and 5/12 times 7 to get the total number of gold and then 5 and 5/12 times the 5 pieces of copper that you have to get that total number of copper. And that's numbers I don't feel like working with. (lines 189-192)

These data provided evidence that Chip continued to use multiplication and division within measure spaces in addition to his use of partitive division with the variable parts perspective. At this time, he only used multiplicative relationships. When the interviewer asked whether they could write an expression about 65 grams of jewelry-gold, the students appropriately wrote 65 = (5 5/12)(7+5) (Figure 17a & 17b) and used distributive property of multiplication over addition:

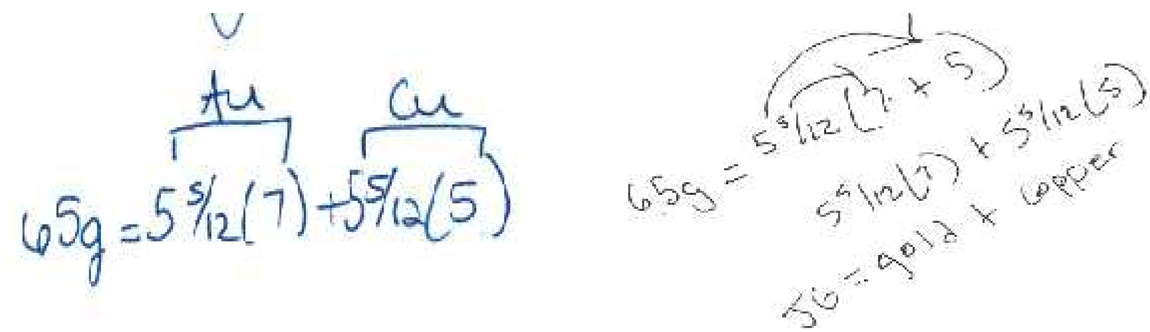


Figure 17: (a) Amber's Expression of the Equation (b) Chip's Expression of the Equation

Chip: We have 65 total grams, and we know that the gold plus the copper is equal to the 65 grams of the jewelry gold or whatever. So I just took what we found a minute ago, which is 5 and 5/12 grams per piece in the gold. So I represented the gold by the 7 pieces and then the copper by the 5 pieces and then if you just create the equation 5 and 5/12 times the 7 plus 5, but obviously in math you would do the parenthesis first and that would just equal 12. But we already did that over here, because we know there's 12 parts. (lines 205-211)

Amber: Well if we had, if we did 5 and 5/12 times 7 plus 5 and 5/12 times 5, this would equal 65, and it does, 'cause 12 times 5 and 5/12 is 65. It can be easily seen from this part. (lines 214-216)

In this task, both students used partitive division and the variable parts perspective again. Also, Chip showed only the use of multiplicative relationships, as opposed to the previous task.

Task 4:

In Task 4, the students saw quickly that the ratio 7 to 5 changed due to the addition of 4 grams. Then, as a follow-up question, the interviewer asked whether the new ratio would be 11 to 5 in the case of adding 4 grams of gold. Amber doubled 11 and 5 and decided that it would not:

> Amber: Well the 22 to 10 is keeping the 11 to 5 ratio the same. You’re just multiplying it by 2. But this has 12 more grams of the pure gold in it than this. Whereas this one it only had 4 more to get from this to this. The 7, you added the 4. But it completely changes the ratio for anything but 1 gram. (lines 256-259)

It seemed that her explanation was based on additive relationship, because she relied on the difference between the ratios to understand whether the ratio was preserved rather than looking at the multiplicative relationship between them. A moment later, the interviewer asked how much copper was needed to balance the mixture in the ratio 7 to 5, Chip used partitive division and the variable parts perspective (Figure 18) similar to his reasoning in Task 2 and Task 3 in this interview:

Figure 18: Chip’s Division of 11 by 7 Parts of the Strip Diagram

> Chip: Like we talked about that would be the example when the 11 to 5 ratio would be true, because each box was only one gram. But then for that 11 to 5 ratio you need to get it back to a 7 to 5 ratio. Which means you have 11 grams up here, but they need to still be broken down into 7 boxes and then you can apply whatever that box like amount is to the 5 boxes of this. So if you divide 11 by 7, so now you have 11/7 grams per box in you’re strip diagram. And if you wanted it still be that same 7 to 5 ratio, then all you do is you have 11 to 7 grams in each one of the 1, each one of the 7 pieces and then it’s 11/7ths grams in each of the 4 copper or the 5 copper pieces. So then if you just add these numbers up, its 11/7 times 5 pieces equal to 50, which is 55/7. So if you had 4 point, or you had 5 grams originally, so if you add 2.8 grams then that would be the same as adding the 4 grams. (lines 267-277)

In this quote, Chip basically assumed 1 gram per box and divided 11 grams of gold by 7 boxes to find grams per box of pure gold in the new mixture. Then, he multiplied the result by 5 to get new weight of copper and finally, subtracted it from initial 5 grams of copper, and obtained the result in a decimal form, 2.8 grams of copper, which was approximately equal to 20/7. Hence, his division of 11 by the number of parts indicated evidence of partitive division with clear use of units. In this problem, Amber seemed also to apply correct calculations and could solve the problem despite some confusion with the arithmetic (Figure 19):

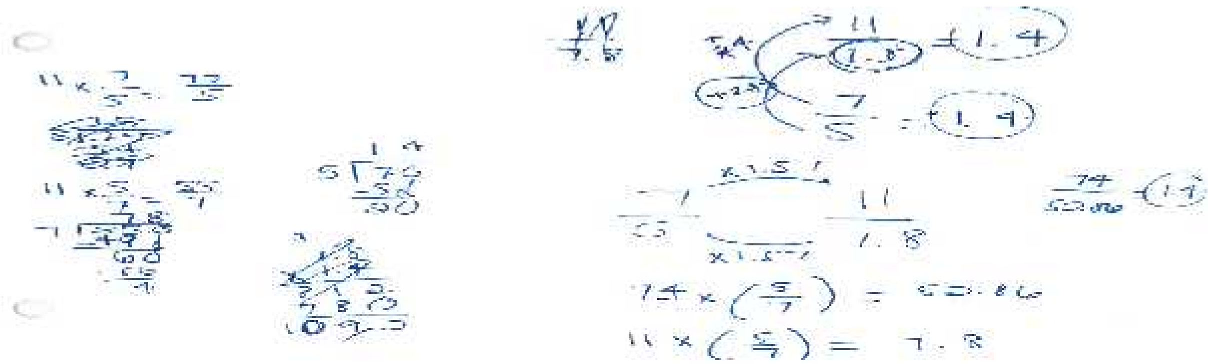


Figure 19: Amber' Arithmetic Without Drawing of Strip Diagram

At this time of the interview, Amber only explained her arithmetic (Figure 19) without any evidence of partitive division and the variable parts perspective. In the figure, she wrote 11 x 5/7 instead of 11/7 x 5 but the data were not sufficient to decide how she thought about the multiplication. Then, the interview moved on to the problem what would happen if there were 10 grams in each box. For this problem, Chip again did the same calculation and got the same answer second time, 2.8 grams of copper. As soon as Chip got the same answer, he saw the connection between the two calculations:

> Chip: So really what that says is for whatever number you have, if you add 4 grams of gold, then you're adding 2.8 grams of copper. (lines 374-375)

It appeared from the data that Chip thought that with any amount of the initial mixture, he would have to add 2.8 grams of copper when adding 4 grams of gold. When the interviewer asked specifically whether they needed to know the initial amounts of gold and copper to solve the problem, Chip responded quickly:

> Chip: No because we already proved that. If whatever if you're adding 4 grams to gold then you're adding 2.8 grams to copper, whatever you started with. As long as whatever you started with is in the 5/7 or 7/5 ratio it doesn't matter what it was. Just add 4, then add 2.8.(lines 436-439)

This quote suggested that he was quite sure that addition of 2.8 grams of copper with 4 grams of gold did not depend on the initial mixture. A moment later, when the second interviewer asked whether this understanding emerged during the interview, Chip answered that he figured it out while he was working on the problem. This was evidence that Chip could see the connection between the addition of 4 grams of gold with respect to 20/7 grams of copper during the interview. On the other hand, for 10 grams in each box problem, Amber had trouble seeing 2.8 grams of copper from 52.8 grams of copper (Figure 20a), but Chip explained the expressions clearly (Figure 20b).

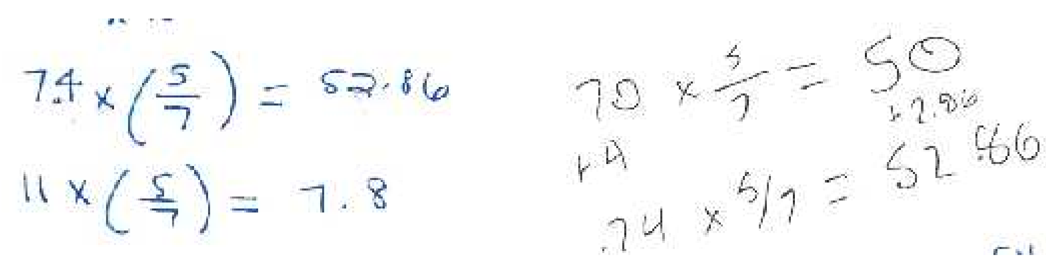


Figure 20: (a) Amber's Arithmetic (b) Chip's Clarification of Amber's Confusion

Amber, even, was not sure whether for every 4 grams of gold, there needed to be 2.8 grams of copper, so she checked her arithmetic by using a calculator. Hence, these data suggested that Amber could not solve Task 4, as opposed to Chip. Finally, when the interviewer asked the students to relate the solution of this to other amounts through an equation, Chip, but not Amber, generalized the solution by constructing an equation in which *x* represented the amount of gold and *y* represented the amount of gold (lines 488-495; Figure 21).

Figure 21: Chip's Algebraic Equation

Although Chip was able to write the equations by generalizing the quantities, the data lacked evidence to conclude that he interpreted multiplication in terms of between measure spaces or not. In summary of this task, Chip continued to use partitive division and the variable parts perspective, but his solutions for 1 gram in each part and 10 grams in each part seemed to driven by the batch perspective. Although Chip was able to solve the task, he could not see initially that for every 4 grams of gold there must be 2.8 grams of copper. His understanding about this connection emerged after he obtained the same solution for 1 gram and 10 grams in each part problems. On the other hand, the data lacked evidence for Amber to conclude whether she used partitive division and the variable parts perspective, because she mostly seemed to be interested in arithmetic. While Amber's written work indicated multiplicative relationships, her reliance on addition in keeping the same ratio indicated an additive relationship. When we compare Chip's performance with Pair 1's performance in this task, it can be concluded that Lisa and Tess could not connect the two solutions to the same extent that Chip did. Although Chip's understanding of the connection had emerged during the interview, he was able to solve this task without prompting from the interviewer.

Case 3: Paul and Amy

Summary for Paul and Amy (Interview 1):

The students were able to use both perspectives. Amy also seemed to keep the two perspectives separate. Additionally, in both students’ explanations, there was no indication that they mixed the two perspectives, which indicated that Amy and Paul could completely differentiate between them. While Amy used both partitive and quotitive division, Paul only used partitive division during the interview. Students mostly used partitive division with the variable parts perspective, and Amy used quotitive division with the batch perspective. Amy provided evidence that she has a clear understanding and identification of both types of division, as opposed to Lisa and Tess. Amy and Paul mostly used multiplicative relationships. Amy was the only participant who used multiplicative comparison explicitly and used multiplication between measure spaces during Interview 1.

Data and Analysis:

Task 1:

As soon as the interviewer finished reading Task 1, Amy generated multiples of the original mixture using a ratio table (Figure 22):

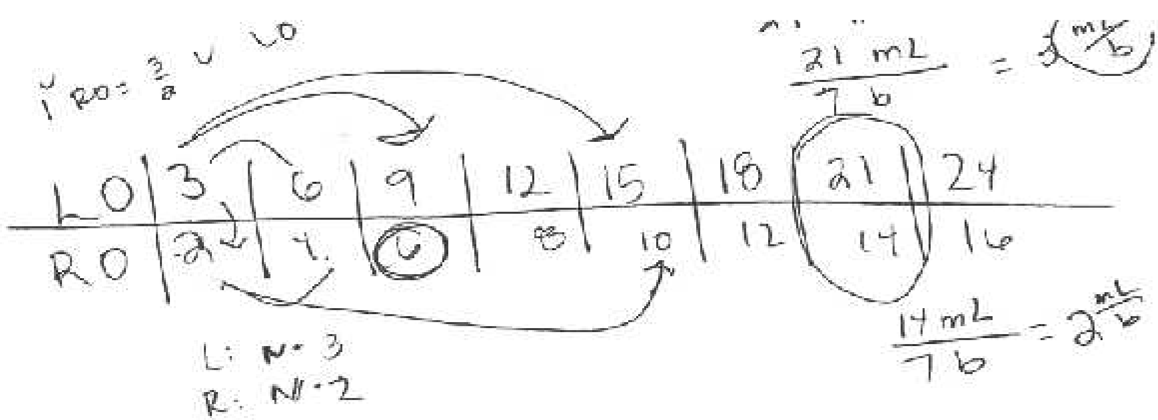


Figure 22: Amy’s Drawing of Ratio Table

Amy: I just did since its 3mL of lavender oil and 2mL of rose oil that, they are in sets of 3 and 2. So I, just to find out what other amounts of the oil can be mixed to make the exact same fragrance each time, we want to make more, we had to multiply these by the same number. So like 2, 3 times 2 and 2 times 2 or 5 three times 5 and 2 times 5. To make the same smell it can be any number and recipes or batches and multiply that number times 3 to find how much, how many mL of lavender oil multiply it by 2 to find out how many mL of rose oil. (lines 32-38)

Amy: Because we kept these in the same ratio, like no matter which of these little sets of numbers you choose, they simplify to 3 and 2. So if you take this set and divide them each by 7, they simplify to 3 and 2 so they are all the same ratio, so they will all smell the same. (lines 42-45)

These data provided initial evidence that Amy used multiplicative relationships rather than additive ones when constructing larger batches. The strongest evidence came a moment later when the interviewer asked whether there was any way to explain smelling the same fragrance:

Amy: I guess you could do this where you have like this, like if this number is 3/2 ,no if this number is 2/3 of this number, so if the amount of rose oil is 2/3 the amount of lavender oil you used, it will smell the same. Or vice versa, if the amount of lavender oil is 3/2 the amount of rose oil, it will smell the same. (lines 54-58)

Amy's multiplicative comparison between the two quantities demonstrated her deep understanding of multiplicative relationships. She was also the only one of the preservice teachers who made explicit multiplicative comparisons between measure spaces. Paul, on the other hand, started making multiples of the original mixture by drawing a double number line (Figure 23):

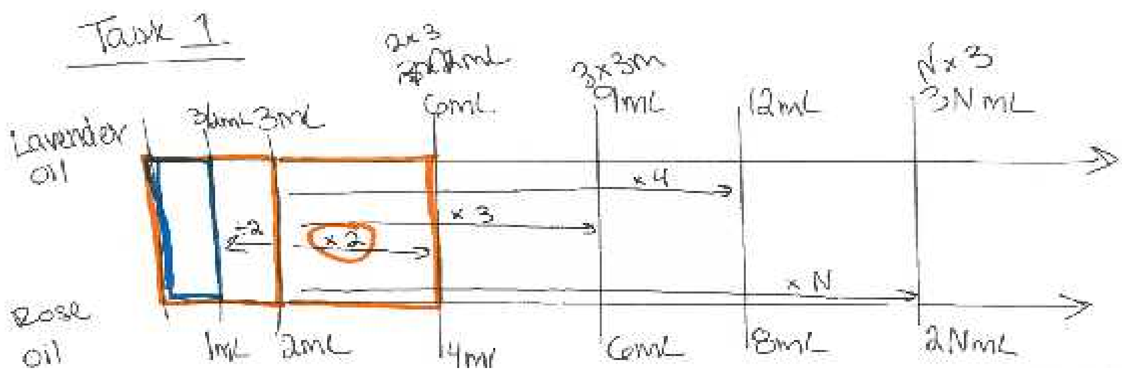


Figure 23: Paul's Drawing of Double Number Line

Paul: Like the first part is like, say you want to think like it's a recipe or something like that. This is one recipe for it, and then to go to the next one which is like the times 2 part, you know if you get to 6mL you actually have 2 of the recipe, like 2 times. So in reality when you look at this it is like 3 times 2 mL and or 2 times 2mL and you're able keep going up. And also like right here you are able to see this is actually literally half of like the original recipe. So if you want you can go down. (lines 73-79)

These data provided evidence that Paul interpreted the replication of the batches using multiplication instead of repeated addition. In addition to Figure 23, when the interviewer asked why the two mixtures with the same ratio would smell the same, he explained by saying that half of the brownie tasted the same as the other half when he divided a brownie into half, which implied partitive division. After the interviewer asked if there was any other way to show the solution of Task 1, he suggested a strip diagram drawing (Figure 24) and made the following explanation:

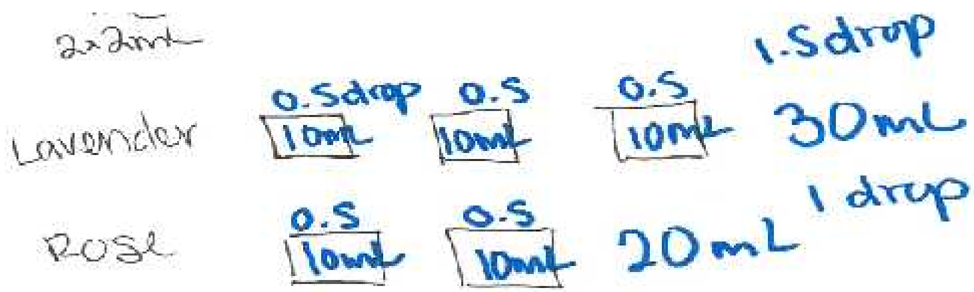


Figure 24: Paul's Drawing of Strip Diagram

Paul: And you can do the strip diagram type of thing, too, if you wanted to in which whatever you put in one of these boxes has to go into every single one of the other boxes. So say if you put 10 mL in the first box, that means that you have to put 10mL, 10mL for every single part, in which then you would end up with like 30 total and 20 total. And that could also be like hey, let's put half a drop in each box. So say you want to put half a drop, you have to put 0.5, 0.5 for every single one of these. And then you get like 1.5 drops to 1 drop. If you want to, you can do it like that. So you don't even have to use the mL anymore. (lines 138-145)

Paul: Well like I mean how you can see this, for like the 1/10 for instance, you have 3 groups of 1/10, like 3 parts of the 1/10, like the whole you know. And then you have like here which can actually be written like 2/10s. You have like 2 parts where it's 1/10 of the whole. You can like separate each of those, you have 1/10 here, 1/10 here, 1/10 here and then that gives you 3/10. And the same as right here where you have 1/10, 1/10. That's 2/10 so. (lines 168-173)

These data indicated that Paul had a sense of partitive division and the variable parts perspective. It also appeared from the data that his explanations made use of the multiplicative relationships. Then, Amy explained the difference between their thinking with respect to the two perspectives, indicating her ability to explicitly identify distinctions between the two perspectives:

> Amy: Well this is like I did the number of batches, like so if I did half a batch, ½ times this. But he did like okay so changing the batches you're changing your parts. So each part has got a half and then this would now instead of representing half a batch would be half a mL in each one, which ends up being the same because its 3/2 of lavender and one whole rose oil. (lines 181-185)

Furthermore, Amy continued making multiples of the original mixture using her ratio table (Figure 22) by using "simplification". She used simplification to divide by the number of batches, indicating partitive division similar to Chip and Amber. When the second interviewer asked what she meant by simplification, she said that she thought more in terms of division. A moment later, the interviewer asked Amy to compare her thinking with the batch perspective to her thinking with the variable parts perspective for 21 mL of lavender oil and 14 mL of rose oil (In Figure 22, she had made simplification to get the amount in each batch). She responded by drawing a strip diagram (Figure 25), providing initial evidence that she could keep the two perspectives separate:

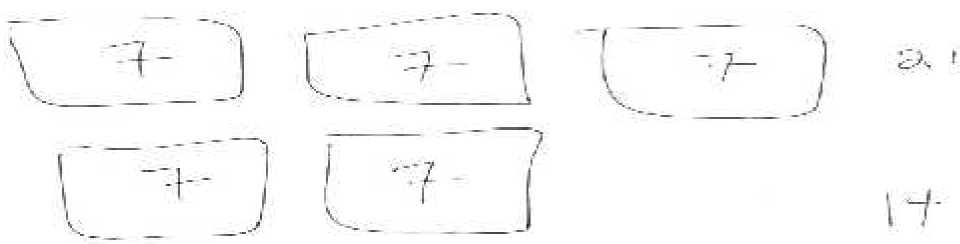


Figure 25: Amy's Drawing of Strip Diagram

> Amy: So if you knew that this was 21 and this was 14, then you would kind of be simplifying to see if like x times 14 has to equal … No. No. No. 2x has to equal 14, and 3x has to equal 21. And so then you would be simplifying to find out the 7 here. (lines 260-263)

These data suggested how Amy could shift her perspective from the batch to the variable parts and her ability to keep these two perspectives separate. In this task, the students used both perspectives and partitive division. There was initial evidence that Amy was able to keep the two perspectives separate. Moreover, both students seemed to rely on a multiplicative relationship. Amy used multiplication between measure space and made a multiplicative comparison explicitly.

Task 3:

The interviewer skipped Task 2 and passed on to Task 3. In Task 3, Paul found a common denominator for 5/3 and 3/2 and got the following equivalent fractions, 10/6 and 9/6. He concluded that 5 mL lavender oil and 3 mL rose oil were not in a 3 to 2 ratio, because the mixture had a stronger lavender smell than the original mixture made in a 3 to 2 ratio. When the interviewer asked Amy what it meant for two mixtures to be in the same ratio, Amy used "for every" language:

> Amy: Well I did it so that I kind of thought of it like for every 3 units you're using 2 units. And so if I'm going to make a certain number of batches, if I'm still using that number of batches times 3 for lavender oil and that number of batches times 2 for rose oil. So that's where these didn't line up. Because to make 7 batches. I'm not then using the same amount of rose oil. So they are not the same ratio. (lines 396-400)

This was the first time that Amy used "for every" wording during the interview. In contrast to previous groups, her use of the wording was based on multiplication rather than addition, because she talked about multiples of the batches instead of addition of the batches. A moment later, Amy drew a ratio table (Figure 26) and by taking multiples of the original mixture, she quickly decided which ones were in a 3 to 2 ratio and which ones were not. When the second interviewer asked whether her simplifying strategy worked here, she gave evidence of quotitive division, for the first time during the interview:

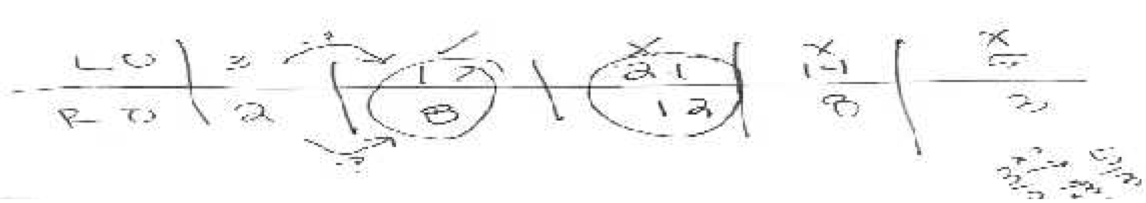


Figure 26: Amy's Drawing of Ratio Table

> Amy: I mean I think it's still kind of simplifying, your just going about it differently, because that's what I was, when I think of this I'm thinking of simplifying 12 over 3 to be 4 batches, but I mean I guess that's not really simplifying. (lines 409-411)

These data supported that she had quotitive understanding of division. Since she could differentiate the meaning of simplification as partitive division and perceive quotitive division not as simplification, she seemed to have a clear understanding of division. It can be concluded that across different tasks she used each meaning of division appropriately, indicating that she had a strong understanding of both types of division. She also used multiplication within measure space to determine which mixtures were in a 3 to 2 ratio. In this task, the data lacked evidence about Paul's use of division and perspective.

Summary for Paul and Amy (Interview 2):

Amy and Paul used both perspectives and partitive division throughout the interview. Amy seemed to keep the two perspectives separate. The data were not sufficient to conclude that Paul could keep the two perspectives separate. Nevertheless, it was evident that in keeping them separate, both students performed better than Chip and Amber in Pair 2, and Lisa and Tess in Pair 1. The data were sufficient to conclude that Amy was able to explicitly identify the distinction between the two perspectives on ratio. Additionally, there were occasions when the students interpreted multiplication between measure spaces. Amy also used multiplicative comparisons similar to the previous interview. Amy and Paul mostly relied on multiplicative relationships instead of additive ones. However, by the end of the interview, Paul started to use additive relationships and the batch perspective. For Task 4, both students solved the task and saw the connection that for every 4 grams of gold there were 20/7 grams of copper regardless of the amount in the initial mixture. When the interviewer asked why this connection made sense, Paul returned to the batch perspective. Amy could quickly solve Task 4 by using the variable parts perspective. In contrast to Chip, Amber, Lisa, and Tess, Pair 3 could initially solve the task by making the connection between the addition of grams of gold and copper. Recall that while Chip could solve Task 4, his understanding about the connection had emerged during the interview; Amber, Lisa, and Tess could not solve the task and not understand the connection completely.

Data and Analysis:

Task 1:

As soon as the interviewer finished reading Task 1, Paul drew a strip diagram (Figure 27). Their work provided evidence for the students' use of the variable parts perspective, but Paul did not make units explicit when discussing partitive division:

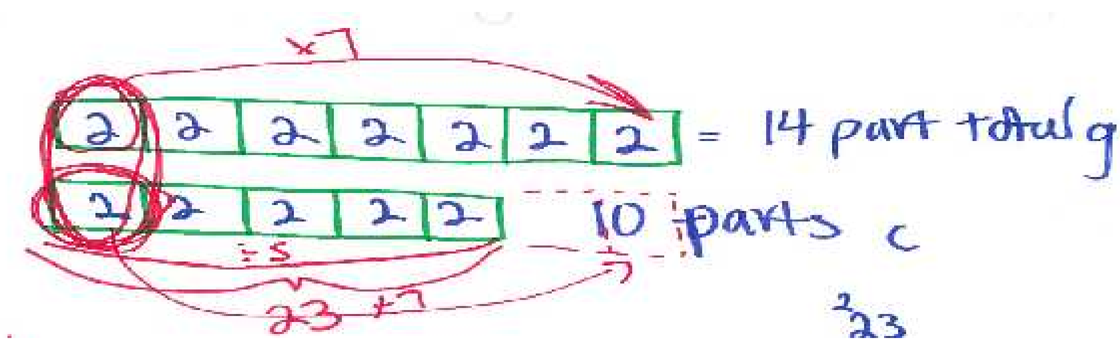


Figure 27: Paul's Drawing of Strip Diagram

Paul: So, I mean, if you have, I guess if you were gonna keep going. If you have like, fourteen parts of gold, like total. You know that you had two in every single part, because you divide 14 by seven, since you have seven like little compartments for each of it, and then, from there you know these parts are the same as this. (lines 44-47)

Amy: Well, they all have to, like he was saying, have the same amount per part. So, like, this size total is like five sevenths of this size. And vice versa like, this size is like seven fifths of this total size. (lines 72-74)

That Amy's comparison of the size of gold with copper as seven fifths demonstrated multiplicative comparisons between measure spaces. She had used similar multiplicative comparison reasoning during Interview 1. At this point, the interviewer moved on to other tasks.

Task 2:

In Task 2, Amy solved the 23 grams of copper problem by writing an algebraic equation G = (7/5)C and putting the copper value into the equation (Figure 28):

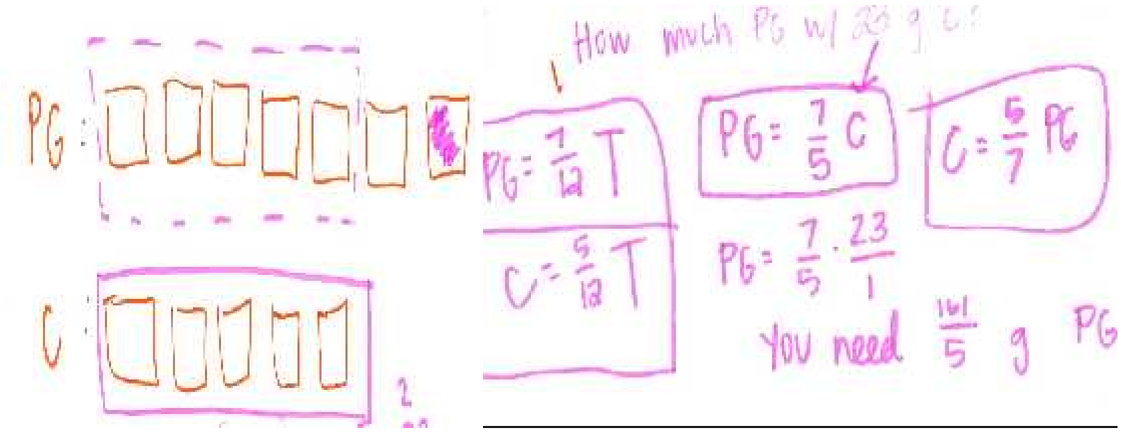


Figure 28: Amy's Drawing of Strip Diagram and Algebraic Equation

> Amy: So, my pure gold is always going to be seven fifths the amount … like because here we're looking at it kinda from of this perspective, where like this is one amount of copper, and this amount is always going to be seven fifths of this amount. And so, when this whole amount is 23, I need seven fifths of that to be my pure gold.
> Int.1: Uh huh. And how are you getting that seven fifths?
> Amy: uhh So this is five parts and this is seven parts. So if you were to multiply these five parts by seven fifths you'd be adding 2, and that would create this. (lines 107-115)

A moment later, Amy explained the reason for choosing to write an algebraic equation instead of using partitive division:

> Amy: Well, because 23 doesn't really divide that nicely into my five parts, I decided to look at it more holistically, I suppose. And that this was always seven fifths of this amount, and so I set up this little equation and hopefully did my math right and find I need a hundred and six one fifth grams of pure gold with 23 grams of copper. (lines 95-98)

These data demonstrated that when constructing the equation, she continued between measure space reasoning by comparing the sizes of G and C. On the other hand, Paul reasoned with partitive division when using a strip diagram (Figure 29):

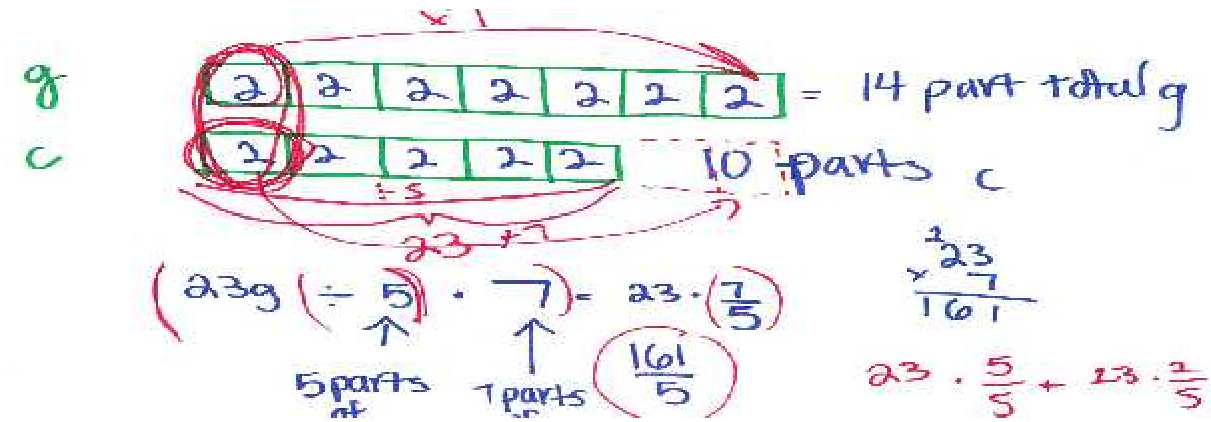


Figure 29: Paul's Drawing of Strip Diagram

> Paul: So you get that 23 fifths huh because you divide it by five, then multiply by seven, so because these right here are the exact same, like these two right here are the exact same size. So we multiply this times seven to find out that you end up ah … say you multiply it by seven then you got … the easiest way for me to figure it out is like 23 times seven over five and just multiply it out and get 161 fifths. So you kind of like do where you find where one part is both are equal the same, you multiply it by seven. (lines 128-133)

These data provided evidence that he reasoned with partitive division, dividing by 5 parts to get the amount in 1 part, and multiplying the result by 7. However, as in the previous task, Paul did not explicitly express units in his explanation. When the interviewer noticed Paul's use of 7/5 in multiplication (Figure 29), she asked him whether he saw multiplying it by seven fifths or not:

> INT1: So are you seeing that multiplying it by seven fifths, you just sort of automatically saw
> BS: yes
> INT1: as the same as divide by the five and then multiply by seven.
> BS: Yes, cause when you have the commutative, it's both multiplication and division, it can be both ways and still get the same answer, so you can multiply it by seven first and divide by five, still get the same exact answer. (lines 155-164)

These data were not sufficient to claim that Paul also interpreted multiplication between measure spaces but it seemed that he might have had a sense of it. At this point, Amy explained the difference between her and Paul's thinking:

> Amy: Uh huh. Basically he was explaining … he, I think, since all these parts are the same size, he went through one part and then multiply that times seven, whereas I just looked at the amount of pure gold per amount of copper. (lines 169-171)

After Amy's explanation of her thinking, Paul explained his understanding of the distinction. He thought that Amy used distribution 23**.**(5/5) + 23**.**(2/5) even though Amy did not agree with him. He commented:

> Paul: I think yours is more … like I mean … well it comes more abstractly you think of it this way, I kinda figure it this way. For her, she's kinda doing … 23 times five over five, and then times or kinda like an extra little …you know what I'm saying … we're like … having multiply by one but then she's also adding like 23 times two over five … in a way … it's very complicated like she's doing the seven fifths … like five over five … that's multiplying it by one … so you have this little whole right here … and then adding on to that the extra two. (lines 187-193)

Paul's use of distribution provided additional evidence that he multiplied between measure spaces because he compared the whole sizes of gold and copper in the strip diagram across parts (Figure 29). In summary, the students used the variable parts perspective and provided evidence for multipliying across measure spaces. While Paul used partitive division, Amy preferred to write an algebraic equation since 5 does divide 23 evenly. In multiplication and division, both students relied on multiplicative relationships rather than additive ones.

Task 3:

In Task 3, Amy started using partitive division reasoning by drawing a diagram (Figure 30a):

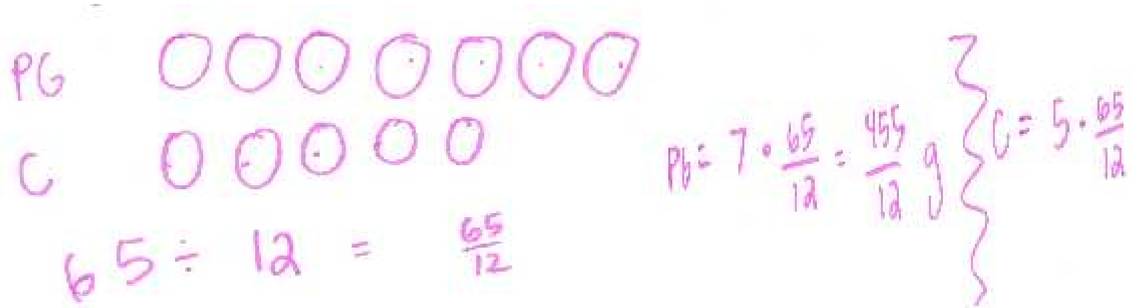
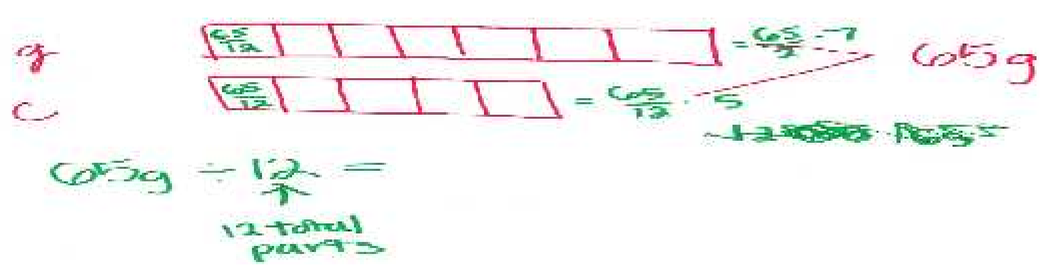


Figure 30: (a) Amy's Drawing of Diagram



(b) Paul's Drawing of Strip Diagram

Amy: I started again with seven parts and five parts, and we want 65 grams to be our sum or our product. So divide it by the twelve total parts to find out how much per each of these parts would be and multiply, which is 65 twelfths and so 65 twelfths times seven parts is gonna be how many grams of pure gold you need, and sixty five twelfths per part times five parts is gonna be how many grams of copper you need. (lines 232-236)

These data demonstrated Amy's use of partitive division in addition to her clear use of units. Similarly, Paul continued using partitive division (Figure 30b):

Paul: Um yes. I have … you have seven plus five total parts. So if you have like five cups. Or a cup is each part, you'd have twelve cups. So um, since you have twelve parts and since all of them are the same size, I just found out how much it is for one part. So you divide 65 by all twelve little parts to find out how much is in one part … sixty five twelfths … then you just multiply by however many parts you have … so you have five parts copper multiply it by five … cause you have five groups of sixty five twelfths … and the same for the gold … you have seven groups of sixty five twelfths (lines 240-246)

It seemed that Paul's use of units developed with respect to the previous tasks in this interview. In summary, the students used the same partitive division reasoning and the variable parts perspective for this task. They also continued to rely on multiplicative relationships in their explanations.

Task 4:

In Task 4, Amy and Paul knew that adding 4 grams of gold would change the 3 to 2 ratio. Amy was able to solve the problem using the variable parts perspective and partitive division reasoning (Figure 31). She clearly explained:

Figure 31: Amy's Drawing of Strip Diagram

> Amy: Sure. So I took my four grams that have been added to the pure gold, divided those among the seven parts, so that's four sevenths per part and then you need to multiply four sevenths times five, which equals two and six sevenths grams of copper, and add that with the four grams of pure gold to keep it in the same mixture, ratio, recipe. 'Cause right now it has too much gold in it. (lines 297-301)

These data provided evidence that Amy could solve the task using the variable parts perspective and partitive division. In previous groups (Chip and Amber, Lisa and Tess), there was no student who was able to solve this task initially and see the connection between the addition of grams of gold and copper by using this perspective. Recall that Chip could solve the problem with the batch perspective, and he had said that his understanding of the connection that for every 4 grams of gold there are 20/7 grams of copper regardless of the initial mixture emerged during the interview. Amber had trouble with arithmetic and seemed not able to solve the task. Similarly, Lisa and Tess could not solve the task and then the interviewer helped them solve it.

When we come to Paul, he could not find a way to solve the problem, initially. Then, he seemed to repeat Amy's solution:

> Paul: Well, like what she was doing where you have four grams … spread it over the seven so it would be four sevenths of a gram in each. So you have four sevenths over here in this small little one. And so to get, to fill up every single one you need to multiply by five so that'd be twenty seventh grams total of copper that's needed to raise it up to the gold, the fourteen carat gold. (lines 151-155)

These data were not sufficient to say that Paul also solved the task with the variable parts perspective since he might have followed Amy's solution, but at least he seemed to have a sense of it. A moment later, the interviewer asked whether the 20/7 grams of copper depended on the initial mixture. Both students agreed it did not.

> Amy: Well, I'm thinking if you … I don't think it matters as long as it is in ratio before, no matter if it was like … a tenth of a gram as long as it was in ratio, and you add the four you add the two and six sevenths gram of copper. (lines 365-367)
>
> Paul: Like ah, since you add another four grams, it means it was already in equilibrium, so it's kinda like if you drew like a little scale or something like that, you'd have, originally you were here, like the equilibrium is five to seven … and then you added … you have the five over here and the original seven plus four grams. And then to add that to make it equal you have to make it back to like

the seven plus four grams and the five plus twenty sevenths. Just kind of make it back in equilibrium. (lines 371-376)

Paul: Like say this over here is the four grams. Kind of a tug of war, you know. So you have like four grams pulling this way, to make it equal you need the twenty sevenths grams to balance it out. (lines 385-387)

These data demonstrated both students' clear explanations about the independence of 20/7 grams of copper with the initial mixture. However, it also appeared from the data that Paul might have implied emphasis toward additive relationships because he mentioned adding some amount to both sides of the scale to keep a balance. At this point, the second interviewer asked Paul how he interpreted 5 parts and 7 parts in a strip diagram. The following was Paul's response to this question:

Paul: Well, it's like the five, which is the copper, and this is the gold. So … I mean they are not technically equal, but in a way … in the manner … every five you add on you have to add seven on. (lines 410-412)

This was further evidence showing his use of an additive relationship, because he used "for every" language. Moreover, the data remained unclear to conclude that Paul was thinking of replicating the parts of the strip diagram by using "for every". A moment later, the interviewer reminded them of the concept of the variable parts perspective and pushed them to think whether this perspective was useful here. While Paul did not speak, Amy repeated the explanation she gave previously based on the variable parts perspective:

Amy: Yeah because if I had made like a punch or something where it was like seven parts of water and five parts of juice, I could have used like seven teaspoons of water and five teaspoons of juice, or like, seven gallons and five gallons. But regardless, as long as I was …even if it was a little bit in a cup before, if I'm adding four more grams of water, I have to add five sevenths of that amount of juice to the mixture. So same with this. (lines 441-446)

Then, the interviewer asked the students why that made sense. This time, the only difference in Amy's explanation was that she thought the units of these parts could be altered. However, Paul gave a different answer indicating his use of the batch perspective (Figure 32):

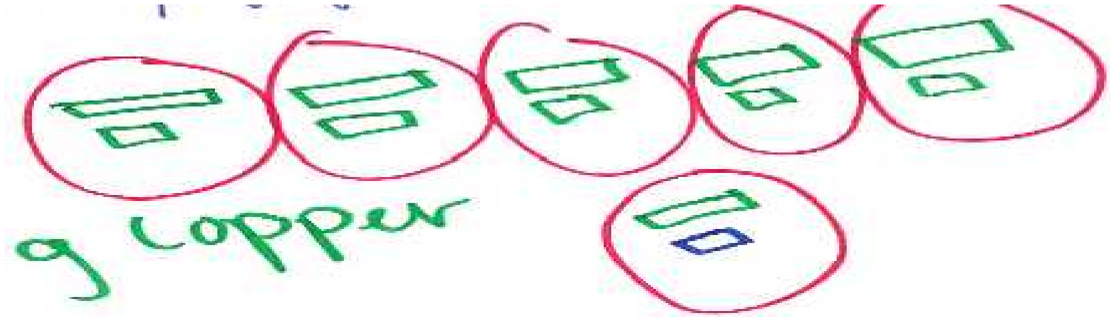


Figure 32: Paul's Drawing of Strip Diagram indicating the Batch Perspective

Paul: The way I think of it is to … beforehand you already have however many groups of these you have, so you have the same … so you have like five groups of these strips already, beforehand. Then you add four grams, well that makes a whole new group of the strip. So you have this strip already filled up, so you have to fill this strip before it can be equal. (lines 462-466)

Paul: So like Amy was saying, you have like 5 cups are already where each cup has the five to seven ratio of gold to copper, well then you added the four grams well you have to (kinda) complete of this [pointing to the last batch in the figure] to make another like little group of it. (lines 472-474)

These data suggested that Paul started to use the batch perspective instead of continuing with the variable parts perspective. We conjecture that shifting from the variable parts perspective to the batch perspective might reflect difficulty in thinking from the variable parts perspective. It might have stemmed from his use of an additive relationship toward the end of this task. At this time, the interviewer asked Paul whether he was taking a different perspective than he was before:

Paul: I mean slightly, but I do not know kind of (line 508)

Paul: I am trying to think. I mean I still think of it like if you are adding four, you separated seven different groups (lines 512-513)

Paul: and then you have the four sevenths and then you have to multiply that by five. But I also think of it as like you are starting a new little group, so you have to (lines 517-518)

These data suggested that Paul was able to view from the two perspectives. Moreover, the data remained insufficient to evaluate whether his drawing of strip diagrams inside the batches stood for the variable parts perspective. If this was the case, then Paul would mixed the two perspectives. At this point, the data were not clear for such a claim. After his talking about the use of the two perspectives, the interviewer asked Amy if Paul's explanation was consistent with how she was thinking about it. Amy commented:

> Amy: No. I am doing like the whole, he is doing batches and then adding that last little bit and I am doing ummm like what the batches are composed or what the parts are composed of versus making copies of it. (lines 526-528)

According to Amy, she used the variable parts perspective while Paul used the batch perspecive. These data showed that she was able to explicitly identify the distinction between the two perspectives. Also, it can be inferred that she was able to see the batch perspective, but she preferred to use the variable parts perspective. These data gave us strong evidence that Amy could keep the two perspectives separate. Additional evidence came a moment later when the interviewer asked the same problem with regards to 70 grams of gold and 50 grams of copper:

> Amy: So before like since this sevenths, this is still circle, before these are each ten grams per part but then in addition to that, now going back (adding if we are adding) four grams total of these so to keep them in the same ratio I need to find out how that, those four grams split up among these parts and then know that that will then be the same amount on each of these five parts versus like making like groups of them. (lines 534-538)
>
> Amy: And then so seventy four divided by 7 will be… so seventy four sevenths per part, so 10 and four sevenths. And that holds true down here ten and four sevenths times five should equal fifty two and six sevenths. (lines 547-549)
>
> Amy: Right so this could have been like seventy drops and fifty drops and then it would just be seventy drops plus four grams pure gold and then fifty drops plus two and six seventh grams. I mean the only thing we know is that the addition was in grams. It doesn't really matter what the unit was before as long as it is in the same ratio. (lines 558-561)

It appeared from the data that Amy, again, could see both perspectives but used the variable parts perspective without doing any calculation since the initial mixture was not important. Additionally, she said that 70 drops of gold plus 4 grams of gold and 50 drops of gold plus 20/7 grams of copper had the same ratio no matter what the initial mixture was. In summary, it can be concluded that Amy was able to keep the two perspectives separate and see the connection at the beginning of the task, by saying that for every 4 grams of gold there must be 20/7 grams of copper regardless of the amount of the initial mixture. Although Paul saw the same connection and solved the task with the variable parts perspective, the data lacked evidence to conclude whether he could keep the two perspectives separate or not. It was evident in this task that both students performed better than Chip, Amber, Lisa, and Tess in keeping the two perspectives separate.

Cross-case Analysis

Three cases involving six preservice teachers were presented in the previous sections by following the order of pairs from less to more proficient use of conceptual meanings of division, and two perspectives on ratios. I present the cross-case analysis in a tree diagram (Figure 33).

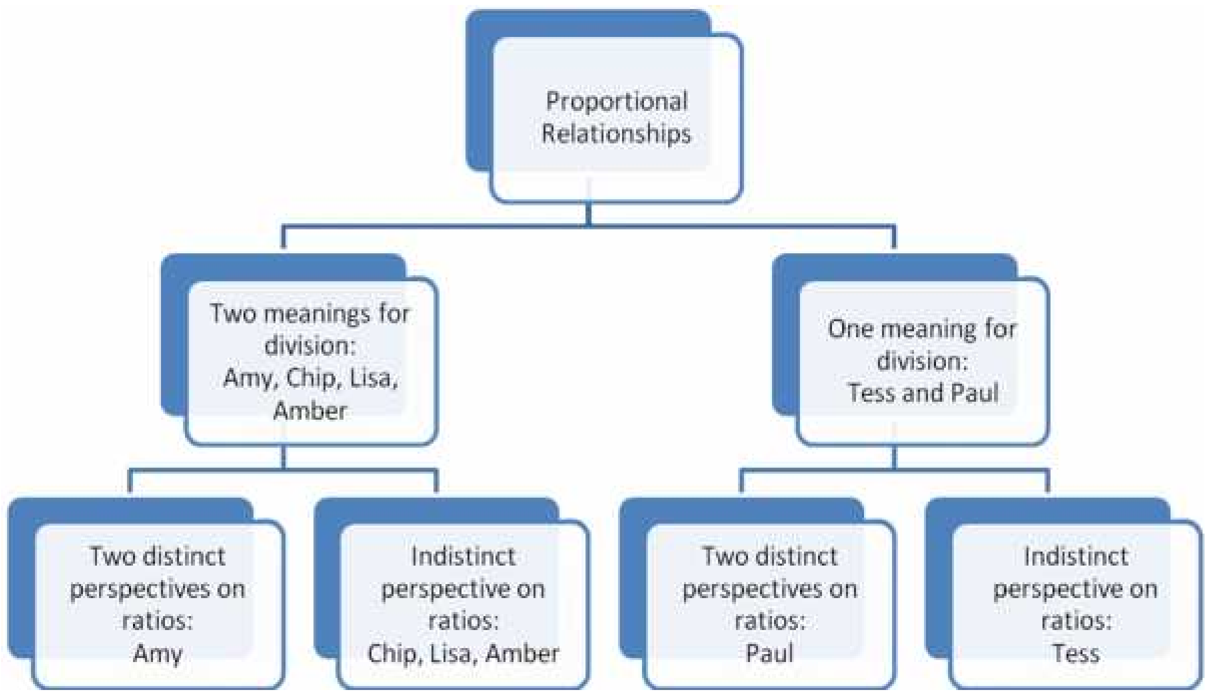


Figure 33: Tree Diagram on Proportional Relationships

The least proficient pair was Pair 1, Lisa and Tess. While Lisa primarily used partitive division, she rarely used quotitive division. Tess, on the other hand, used only quotitive division. Later in Interview 2, Tess followed Lisa's reasoning of partitive division, which implies her less developed understanding of partitive division. These preservice teachers seemed not to have robust understanding of both types of division, because they were not able to realize the difference between partitive and quotitive division. Moreover, Lisa and Tess could not keep the two perspectives on ratios separate despite their use of both perspectives. Instead, they mixed the two perspectives when responding to the tasks related to strip diagrams. In addition to the mixed perspective, they consistently used both multiplicative and additive relationships rather than relying on only multiplicative ones. The emphasis on “for every” language might have directed their attention toward additive relationships such as repeated addition and subtraction of batches, and the difference between the two quantities. Their difficulty of solving the task involving addition of 4 grams might have stemmed from their reliance on additive relationships because the nature of the task itself was based on addition. Furthermore, Lisa and Tess mostly used multiplication within measure spaces, but they made comparison between measure spaces in Interview 2 and constructed algebraic equations accurately after they drew strip diagrams.

Pair 2, Chip and Amber, was more proficient than Pair 1 in terms of use of division and two perspectives on ratios. While Chip used both types of division, Amber’s primary division was partitive in addition to her use of quotitive division once. As opposed to Chip, Amber’s use of units in partitive division reasoning and the variable parts perspective improved throughout the Interview 2. The data lacked evidence to support whether they explicitly identified the distinction between different meanings of division. What the data did make clear was that both students were not able to keep the two perspectives distinct. They consistently used mixed wording like Pair 1 when responding to the tasks related to strip diagrams. In one of these tasks, Amber also put batch numbers inside the parts of a strip diagram, which indicates mixing the two perspectives. Regarding the relationships between the quantities, both Chip and Amber relied heavily on multiplicative and additive relationships. The emphasis of the students on “for every” appeared to direct their attention toward additive relationships. Furthermore, they used only multiplication and division within measure spaces.

The most proficient pair was Pair 3, Paul and Amy. While Amy had conceptual meanings for both types of division, Paul used only partitive division. As opposed to Amy who showed clear use of units, Paul improved his use of units throughout the interviews. Amy provided evidence that she was able to explicitly identify the distinction between the two meanings for division. Moreover, Amy was able to keep the two perspectives on ratios separate. Paul (except the end of Task 4 in Interview 2) and Amy did not demonstrate any indication of mixing the two perspectives such as using mixed wording or putting batch numbers inside of strip diagrams. Amy could shift from the batch perspective to the variable parts perspective without mixing the two. Hence, both students seemed to differentiate the two perspectives. They mostly used partitive division with the variable parts perspective and quotitive division with the batch perspective. Furthermore, both students used multiplicative relationships. Through the end of Interview 2, only Paul started to provide demonstrations of additive relationships in parallel with his start using the batch perspective. These preservice teachers used multiplication and division between measure spaces in addition to within measure spaces. Amy was the only participant who used multiplicative comparison and between measure spaces in Interview 1. After drawing of strip diagrams, they constructed proper algebraic equations.

CHAPTER 5

DISCUSSION AND CONCLUSIONS

Conclusions

The findings including six preservice teachers from the middle grades program suggest new lines of research for ratios and proportional relationships. Past research documented students and teachers' consistent difficulties with different meanings for division but, there was no study in the literature investigating the relationship between students or teachers' understanding of division and their use of the two perspectives on ratio, which are key concepts of the *multiplicative conceptual field*.

One of my main results is that preservice teachers who did not maintain explicitly different meanings for division tended to have difficulty maintaining distinct perspectives on ratios. Whereas Amy could identify partitive and quotitive meanings for division, Lisa and Tess were not able to discriminate between both types of division. Although Chip was clear about different meanings for division, the data remained insufficient to conclude whether he could identify these meanings for division, as was the case for Amber and Paul. Moreover, the tree diagram (Figure 33) highlighted evidence of Amy and Paul's use of the two distinct perspectives, as opposed to Pair 1 and Pair 2. Despite Paul's understanding only the partitive meaning for division, Paul and Amy were able to differentiate between the two perspectives completely. Amy could also make transitions between both perspectives when responding to the tasks related to strip diagrams.

Another main result, parallel to the previous one, is that preservice teachers' ability to keep the two perspectives separate depends on how much they relied on multiplicative and additive relationships in addition to their ability to identify explicitly different meanings for division. In the case of Chip, Lisa, and Amber, where teachers did not attend usually to multiplicative relationships, they had a harder time keeping the two perspectives separate despite their ability to use both types of division. In the case of Amy and Paul, where teachers relied mostly on multiplicative relationships, they performed better in differentiating between the two perspectives even if they used only one type of division. It can be concluded that preservice teachers relying on multiplicative relationships between the quantities in a ratio could keep the two perspectives on ratios distinct.

On the other hand, preservice teachers relying on additive relationships tended not to differentiate the two perspectives depending on the extent to which they used these relationships. Emphasis on additive relationships such as repeated addition and subtraction of the batches, focusing on the difference between the quantities in a ratio, and the presence of the phrase "for every" might have constrained the teacher candidates from keeping the two perspectives separate. In addition to the evidence confirming that "for every" caused additive thinking, the presence of the phrase was also associated with mixing the two perspectives, as opposed to keeping them separate. The teacher candidates Chip and Lisa, consistently used the term "for every" when responding to the tasks involving strip diagrams. Using expressions like "for every 3 parts of lavender oil there are 2 parts of rose oil" or "for every 7 parts of gold there are 5 parts of copper" directed the participants' attention to replicate these parts through repeated addition. With such an aim, they considered 3 parts of lavender oil and 2 parts of rose oil as one batch (or 7 parts of gold and 5 parts of copper as one batch) and imagined additional copies of this batch. Therefore, they could not attempt to change the amount in each part of the strip diagrams, one core assumption of the variable parts perspective. Bringing the batch perspective thinking (replication of batches) into the strip diagrams caused these participants to mix the two perspectives. The result of this study suggests that reliance on additive relationships such as repeated addition and the use of “for every” might have constrained teachers from keeping the two perspectives distinct. Future studies should continue to examine whether this preliminary result extends to other teachers’ and students’ performance on ratios and proportional relationships.

Sowder et al. (1998) reported that the concept of ratio is crucial in shifting from additive to multiplicative reasoning, because a ratio requires multiplicative comparison of two quantities. Similarly, the results I have presented suggest that an emphasis on multiplicative relationships is critical in keeping the two perspectives on ratios distinct, and in ensuring a robust understanding of ratios and proportional relationships. Some studies in the literature have concluded that students’ additive reasoning was not sufficient to reason with situations involving multiplicative relationships (Harel et al., 1994; Noelting, 1980a, 1980b; Simon & Blume, 1994b). Vergnaud (1983), and Lobato and Ellis (2010) also emphasized the necessity of multiplicative comparison instead of additive comparison between the two quantities to reason proportionally.

Repeated addition through replication or iteration is sometimes referred to as “build-up strategies,” which consist of forming ratios by extending the initial ratio with addition (Piaget, Grize, Szeminska, & Bang, 1968, as cited in Lamon, 2007). Lamon (2007) did not consider the “build-up strategies” as proportional reasoning due to their lack of emphasis on the constant ratio between the proportionally related quantities. My results are consistent with Lamon’s (2007) study in the way that these strategies might have constrained proportional reasoning by causing the mixing of the two perspectives on ratios. I also found that “for every” language had a great potential for directing the participants’ attention to additive relationships such as repeated addition and subtraction. Further studies should continue to examine the effects of additive relationships, “build-up strategies”, and “for every” language on ratios and proportional relationships.

We have defined multiplicative relationships as establishing a constant ratio with the basis of multiplication such that the two quantities must be multiplied with the same number. In this study, I also examined preservice teachers’ multiplicative relationships in terms of within and between measure spaces. The result suggests that all six teachers discussed multiplying within measure spaces, as their primary approach. However, Pair 2 and Pair 3 demonstrated evidence of multiplication between measure spaces only after they drew strip diagrams, and these participants were able to acquire accurate equations with such reasoning. Hence, strip diagrams and the variable parts perspective seemed to provide a more accessible approach for multiplying between measure spaces and constructing algebraic equations than the batch perspective for some preservice teachers. This result forms an initial evidence for Beckmann and Izsák’s (in press) question asking whether the variable parts perspective is more helpful in forming multiplicative relationships between measure spaces. Future studies should continue to investigate the relationship between multiplication across measure spaces and the two perspectives on ratios. In particular, the role that multiplication within and between measure spaces plays in keeping the two perspectives separate should be analyzed fully.

Furthermore, future studies are needed to examine how multiplication within and between measure spaces affect additive and multiplicative relationships. Orrill and Brown (2012) reported that the teachers’ high reliance on additive relationships such as “build-up strategies” rather than multiplicative ones might occur due to their difficulty in reasoning between measure spaces. My study did not seem to support this hypothesis, because Lisa and Tess, half of the participants using multiplication between measure space, provided evidence of between measure space reasoning despite their consistent use of additive relationships. However, there should be further studies to investigate the hypothesis of Orrill and Brown (2012) in order to have a more accurate picture of their hypothesis.

Next, I note that while teachers with partitive division were inclined to use the variable parts perspective throughout the interviews, teachers with quotitive division tended to use the batch perspective. In cases where teacher candidates understood mostly quotitive division, they tended to use the batch perspective and had difficulty using the variable parts perspective. Some of them even explicitly said that they mostly thought from the batch perspective and strip diagrams were not their favorite tool. Since their primary perspective is the batch perspective, they experienced persistent difficulties when working with strip diagrams and this might open the door to mix the two perspectives such as putting batch numbers inside of strip diagrams or replicating strip diagrams. In cases where teacher candidates used mostly partitive division, they were better able to use the variable parts perspective and the batch perspective than candidates who primarily used quotitive division.

Another result of the study suggests that preservice teachers were more facile with quotitive division than reported in past research. In these few studies of understanding of division, preservice teachers generally were supported by the understanding of partitive division (Ball, 1990; Graeber, Tirosh, & Glover, 1986; Tirosh, & Graeber, 1989; Ma, 1999; Simon, 1993). Nevertheless, the percentage of teacher candidates using quotitive division appeared to increase in this study. While four preservice teachers used both types of division, one used only partitive and one used only quotitive division. Ma (1999) evaluated one reason for the failure of U.S. teachers to understand the meaning of division with fractions as their lack of knowledge about connections and links. With the possible effects of reform movements until now, this study revealed that preservice teachers tended to use both partitive and quotitive division. At the same time, most of these teachers still struggled to make connections between the meanings of division, as Ma (1999) suggested. In other words, these teachers were found to have trouble identifying explicitly the distinction between the two meanings of division.

A number of studies documented that students fell back on additive strategies when problems started to become more complex (Hart, 1984; Karplus, Pulos, & Stage, 1983b; Tournaire, & Pulos, 1985). As consistent with this finding, there was a moment in this study when Paul came back to the batch perspective during Interview 2 when the interviewer pushed him to further explain his reasoning. For the first time in this interview, Paul started to use such thinking by repeating the strip diagram rather than changing the amount in each part. Hence, future studies should investigate whether there is a tendency among teachers (and students) to return to the batch perspective when a problem is difficult.

Implications

An important implication of this study is that mathematics courses for future teachers of middle grades programs should be designed to deliberately support proportional reasoning. In particular, these courses should include all the topics in the *multiplicative conceptual field* such as multiplication, division, ratios and proportional relationships. This study suggested that explicit identification of the meanings for both types of division is critical to keeping the two perspectives on ratios separate, which is key aspect of the robust understanding of proportional relationships. Similarly, this study demonstrated the urgent need for emphasis on multiplicative relationships and prevention of future teachers from focusing on additive relationships. Therefore, the mathematics courses of future teachers should address, at least, both meanings of division, both perspectives on ratios, and multiplicative relationships between proportionally related quantities.

Future Research

In this study, I found that multiplicative relationships are necessary for understanding proportional relationships. During the study, I also consistently mentioned multiplication within and between measure spaces. Further studies should investigate what role multiplication within and between measure spaces play in proportional relationships, specifically in keeping the two perspectives separate.

In a further study I would like to conduct the same study with larger samples and with larger number of interviews for each pair in order to make generalizations more efficiently. I would also like to analyze each participant’s homework and exams in their enrolled course.

Concluding Comment

This study answered several important questions related to the affordances and constraints of prospective middle grades teachers in terms of multiplication, division, and ratios and proportional relationships with quantities. Specifically, this study suggests that mathematical ideas of partitive and quotitive division, and formation of multiplicative relationships are necessary to keep the two perspectives on ratios separate. However, this study also suggests that the ability to keep them separate is complex, so these ideas might not be sufficient to support this ability completely. For example, mixing the two perspectives might tend to go along with a lack of attention to units or it may indicate that preservice teachers are not thinking about the diagrams as quantitative representations.

This study also raised further questions about multiplicative relationships and the two perspectives on ratios. I believe that the results of this study are especially valuable because the understandings of future teachers were analyzed with respect to core topics in middle grades mathematics.

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