LAGRANGIAN SUBMANIFOLDS OF PRODUCTS OF SPHERES

by

JOEL STANTFORD OAKLEY

(Under the direction of Michael Usher)

ABSTRACT

This work investigates certain Lagrangian submanifolds of products of spheres. In particular, we will study several constructions of "exotic" Lagrangian tori in $S^2 \times S^2$, and we will prove that they are all Hamiltonian isotopic. In the space $(S^2)^3$, we will investigate a Lagrangian submanifold that is diffeomorphic to $\mathbb{R}P^3$, and we will prove that it is nondisplaceable under Hamiltonian diffeomorphisms by showing that the homology of a certain chain complex (called the pearl complex) is non-trivial.

INDEX WORDS: Lagrangian submanifold, Symplectic manifold, Product of spheres, Hamiltonian, Nondisplaceable, Pearl complex

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Joel Stantford Oakley

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M.S., Mississippi State University, 2006

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Joel Stantford Oakley

Approved:

Major Professor: Michael Usher

Committee: David Gay
William H. Kazez
Gordana Matic
Robert Varley

Electronic Version Approved:

Julie Coffield
Interim Dean of the Graduate School
The University of Georgia
August 2014

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Contents

Chapter 1

Introduction and Symplectic Preliminaries

In this work, we will investigate certain Lagrangian submanifolds of products of spheres, beginning with some Lagrangian tori in $S^2 \times S^2$ in Chapter 2 and then moving on to a particular Lagrangian submanifold of $(S^2)^3$ in Chapters 3, 4, and 5. In order to study these objects, we must first establish some terminology. It is assumed that the reader is familiar with the fundamentals of smooth manifolds, and we begin by covering some definitions from symplectic topology.

Definition 1.1. A **symplectic vector space** is a pair (V, ω) consisting of a $2n$-dimensional real vector space V with a nondegenerate, skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. Given two symplectic vector spaces (V_1, ω_1) and (V_2, ω_2), a **linear symplectomorphism** from V_1 to V_2 is a linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi^* \omega_2 = \omega_1$.

Recall that nondegeneracy means that, for each non-zero vector $v_1 \in V$, there is a vector $v_2 \in V$ such that $\omega(v_1, v_2) \neq 0$. Skew-symmetry means that $\omega(v_1, v_2) = -\omega(v_2, v_1)$ for all $v_1, v_2 \in V$. A standard example of a symplectic vector space is $\mathbb{R}^{2n}$ with standard basis

$\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and the bilinear form given by

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

where $\{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n\}$ is the dual basis. In fact, a standard result says that every symplectic vector space of dimension $2n$ is linearly symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Definition 1.2. A **Lagrangian subspace** W of a symplectic vector space (V, ω) is a vector subspace $W \leq V$ such that $\dim(W) = \frac{1}{2}\dim(V)$ and $\omega\big|_{W \times W} = 0$.

Using the example $(\mathbb{R}^{2n}, \omega_0)$ from above, one example of a Lagrangian subspace of $\mathbb{R}^{2n}$ is $W = \operatorname{span}\{x_1, \ldots, x_n\}$.

Definition 1.3. The **Lagrangian Grassmannian**, denoted $\mathcal{L}(n)$, is the set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. That is,

$$\mathcal{L}(n) = \{W \leq \mathbb{R}^{2n} \mid \dim(W) = n \text{ and } \omega_0\big|_{W \times W} = 0\}.$$

It is well known that $\mathcal{L}(n)$ can be given the structure of a smooth manifold of dimension $n(n+1)/2$ whose fundamental group $\pi_1(\mathcal{L}(n))$ is isomorphic to $\mathbb{Z}$ (see Chapter 2 of [MS98] for example). An explicit isomorphism is provided by the Maslov index, which we will define shortly. First, we observe that the unitary group $U(n)$ acts transitively on $\mathcal{L}(n)$ with the stabilizer subgroup of a point being $O(n)$; and so $\mathcal{L}(n)$ is naturally diffeomorphic to $U(n)/O(n)$. Given a loop $\gamma : S^1 \to \mathcal{L}(n)$, we choose a path $U_\gamma : [0, 2\pi] \to U(n)$ corresponding to the path $t \mapsto \gamma(e^{it})$ via the identification $\mathcal{L}(n) \cong U(n)/O(n)$ and satisfying $\{\det(U_\gamma(0)), \det(U_\gamma(2\pi))\} \subset \{\pm 1\}$. It is not difficult to show that the Maslov index as defined below is independent of the choice of the path U_γ and depends only on $[\gamma] \in \pi_1(\mathcal{L}(n))$.

Definition 1.4. The **Maslov index** of $[\gamma] \in \pi_1(\mathcal{L}(n))$, denoted $\mu([\gamma])$, is defined to be the degree of the map

$$\begin{aligned} S^1 &\to S^1 \\ e^{it} &\mapsto \det\left((U_\gamma(t))^2\right). \end{aligned}$$

Definition 1.5. A **symplectic vector bundle** over a smooth manifold M is a pair (E, ω) consisting of a real rank-$2n$ vector bundle $E \to M$ and a smooth section $\omega : M \to \Lambda^2 E^*$ such that, for each $p \in M$, the fiber (E_p, ω_p) is a symplectic vector space. The section ω is called a **symplectic bilinear form** on E. Two symplectic vector bundles (E_1, ω_1) and (E_2, ω_2) over M are said to be **isomorphic** if there is a vector bundle isomorphism $\Psi : E_1 \to E_2$ covering the identity over M and satisfying $\Psi^*\omega_2 = \omega_1$.

Note that the condition $\Psi^*\omega_2 = \omega_1$ means that $\Psi\big|_{(E_1)_p} : \left((E_1)_p, (\omega_1)_p\right) \to \left((E_2)_p, (\omega_2)_p\right)$ is a linear symplectomorphism for each $p \in M$. A standard example of a vector bundle is the trivial vector bundle $M \times \mathbb{R}^{2n}$. If we endow $M \times \mathbb{R}^{2n}$ with the symplectic bilinear form given by $\omega_p = \omega_0$ for each $p \in M$, then it becomes a symplectic vector bundle called the trivial symplectic vector bundle over M.

Definition 1.6. A **symplectic trivialization** of a symplectic vector bundle (E, ω) over M is an isomorphism of symplectic vector bundles $\Psi : E \to M \times \mathbb{R}^{2n}$, where $M \times \mathbb{R}^{2n}$ is the trivial symplectic vector bundle over M.

Definition 1.7. A **symplectic manifold** (M, ω) is a $2n$-dimensional smooth manifold M together with a closed, nondegenerate differential 2-form ω called a **symplectic form**. If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, a **symplectomorphism** from M_1 to M_2 is a diffeomorphism $\varphi : M_1 \to M_2$ satisfying $\varphi^*\omega_2 = \omega_1$.

Recall that ω being closed means that $d\omega = 0$, and nondegeneracy means that (T_pM, ω_p) is a symplectic vector space for each $p \in M$. In other words, if M is a symplectic manifold, then the tangent bundle TM is a symplectic vector bundle with symplectic bilinear form ω. A standard example of a symplectic manifold is $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and symplectic form $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. Darboux's Theorem says that every symplectic manifold of dimension $2n$ is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Definition 1.8. A **Lagrangian submanifold** L in a symplectic manifold (M, ω) is an embedded submanifold $L \subset M$ such that $\dim(L) = \frac{1}{2}\dim(M)$ and $i^*\omega = 0$ for the inclusion $i : L \hookrightarrow M$.

Now that we have established definitions of symplectic manifolds and Lagrangian submanifolds thereof, we will define two important homomorphisms called the area and Maslov homomorphisms. Let L be a Lagrangian submanifold of the symplectic manifold (M, ω), and let $[u] \in \pi_2(M, L)$ be represented by a smooth map $u : (D^2, S^1) \to (M, L)$, where $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the closed unit disk and S^1 is the unit circle.

Definition 1.9. The **area homomorphism** $I_\omega : \pi_2(M, L) \to \mathbb{R}$ is defined by

$$I_\omega([u]) = \int_{D^2} u^*\omega.$$

Observe that the symplectic vector bundle u^*TM is trivial since its base is the disk D^2, and note that $\left(\left(u\big|_{S^1}\right)^* TL\right)_{e^{it}}$ is a Lagrangian subspace of $(u^*TM)_{e^{it}}$ at each point $e^{it} \in S^1$. Let $\tau : u^*TM \to D^2 \times \mathbb{R}^{2n}$ be a symplectic trivialization, and consider the loop $\gamma_{u,\tau} : S^1 \to \mathcal{L}(n)$ given by this trivialization, namely

$$\begin{aligned} \gamma_{u,\tau} : S^1 &\to \mathcal{L}(n) \\ e^{it} &\mapsto \mathrm{pr}_{\mathbb{R}^{2n}} \circ \tau\left(\left(\left(u\big|_{S^1}\right)^* TL\right)_{e^{it}}\right), \end{aligned}$$

where $\mathrm{pr}_{\mathbb{R}^{2n}} : D^2 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the obvious projection. It is a standard result that the Maslov index $\mu([\gamma_{u,\tau}])$ does not depend upon the symplectic trivialization, leading to the following definition.

Definition 1.10. The **Maslov homomorphism** $I_\mu : \pi_2(M, L) \to \mathbb{Z}$ is defined by

$$I_\mu([u]) = \mu([\gamma_{u,\tau}]),$$

where $\tau : u^*TM \to D^2 \times \mathbb{R}^{2n}$ is a symplectic trivialization. The number $I_\mu([u])$ is called the **Maslov index** of $[u]$.

In the late 1980s and early 1990s, the development of Lagrangian intersection Floer homology led to the study of a particular class of Lagrangian submanifolds called monotone. In particular, it was shown by Oh in [Oh93] and [Oh95] that Lagrangian intersection Floer homology can be defined for monotone Lagrangian submanifolds such that the Maslov homomorphism satisfies a certain restriction. All of the Lagrangian submanifolds studied in this paper will be monotone, which is defined as follows.

Definition 1.11. Let L be a Lagrangian submanifold of a symplectic manifold (M, ω). We say that L is **monotone** with monotonicity constant $\kappa > 0$ if the area and Maslov homomorphisms satisfy $I_\omega = \kappa I_\mu$.

One of the main results of this work (stated as Corollary 5.10) is to show that a certain monotone Lagrangian submanifold of $(S^2)^3$ is nondisplaceable under Hamiltonian diffeomorphisms, and so we now proceed to establish precisely what is meant by this.

Definition 1.12. Let (M, ω) be a symplectic manifold, and let $H : M \to \mathbb{R}$ be a smooth function. The vector field X_H defined by the equation $\omega(\cdot, X_H) = dH$ is called the **Hamiltonian vector field** associated to the **Hamiltonian** H. The flow of the Hamiltonian vector field X_H, denoted $\{\varphi_H^t \,|\, t \in \mathbb{R}\}$ and defined by

$$\frac{d}{dt}\varphi_H^t(p) = X_H\left(\varphi_H^t(p)\right) \text{ and } \varphi_H^0(p) = p$$

for all $p \in M$ and $t \in \mathbb{R}$, is called the **Hamiltonian flow** associated to H.

In the above definition, we assume that the vector field X_H is complete (in other words, the flow is defined for all $p \in M$ and $t \in \mathbb{R}$), an assumption that is not too restrictive since we usually consider compactly supported Hamiltonians $H : M \to \mathbb{R}$. It is also useful to consider time-dependent functions $H : M \times [0,1] \to \mathbb{R}$; we write $H_t = H(\cdot, t)$.

Definition 1.13. Let (M, ω) be a symplectic manifold, and let $H : M \times [0,1] \to \mathbb{R}$ be a smooth function. The time-dependent vector field X_{H_t} defined by the equation $\omega(\cdot, X_{H_t}) = dH_t$ is called the **time-dependent Hamiltonian vector field** associated to the **time-dependent Hamiltonian** H. The flow of the time-dependent Hamiltonian vector field X_{H_t}, denoted $\{\varphi_H^t \,|\, t \in [0,1]\}$ and defined by

$$\frac{d}{dt}\varphi_H^t(p) = X_{H_t}\left(\varphi_H^t(p)\right) \text{ and } \varphi_H^0(p) = p$$

for all $p \in M$ and $t \in [0,1]$, is called the **Hamiltonian flow** associated to H.

Again we assume in the above definition that the flow $\{\varphi_H^t\}$ is globally defined, and it is a standard result that φ_H^t is a symplectomorphism for all t (in both the time-dependent and time-independent cases).

Definition 1.14. A symplectomorphism $\varphi : M \to M$ is called a **Hamiltonian diffeomorphism** if $\varphi = \varphi_H^1$ for some (possibly time-dependent) Hamiltonian H. Two submanifolds $L_0, L_1 \subset M$ are said to be **Hamiltonian isotopic** if there is a Hamiltonian diffeomorphism φ such that $\varphi(L_0) = L_1$.

Definition 1.15. Let (M,ω) be a symplectic manifold, and let $A \subset M$. We say that A is **displaceable** under Hamiltonian diffeomorphisms if there is a Hamiltonian diffeomorphism φ such that $\varphi(A) \cap A = \varnothing$. Otherwise, we say that A is **nondisplaceable** under Hamiltonian diffeomorphisms.

We now proceed to define the language of Hamiltonian group actions. Throughout the remainder of this chapter, we let G denote a compact Lie group with Lie algebra $\mathfrak{g}$. Suppose that G acts on a symplectic manifold (M,ω) so that we have a diffeomorphism $\psi_g : M \to M$ for each $g \in G$. Of course, by the definition of a group action, we have $\psi_{g_1 g_2} = \psi_{g_1} \circ \psi_{g_2}$ for all $g_1, g_2 \in G$, and ψ_e is the identity on M if e is the identity element of G.

Definition 1.16. The Lie group G **acts on** M **by symplectomorphisms** if $\psi_g : M \to M$ is a symplectomorphism for all $g \in G$.

Given a group action G on M, each $\zeta \in \mathfrak{g}$ determines a vector field on M via the prescription

$$X_\zeta(p) = \left.\frac{d}{dt}\right|_{t=0} \psi_{\exp(t\zeta)}(p).$$

Definition 1.17. Suppose that G acts on M by symplectomorphisms; we say that action is **weakly Hamiltonian** if the vector field X_ζ is Hamiltonian for all $\zeta \in \mathfrak{g}$.

In other words, a group action of G on M by symplectomorphisms is weakly Hamiltonian if, for each $\zeta \in \mathfrak{g}$, there is a smooth function $H_\zeta : M \to \mathbb{R}$ such that $X_{H_\zeta} = X_\zeta$. In order to define a Hamiltonian group action, we first need to define the Poisson bracket.

Definition 1.18. Let $F : M \to \mathbb{R}$ and $H : M \to \mathbb{R}$ be smooth functions on a symplectic manifold (M,ω). The **Poisson bracket** of F and H is defined by

$$\{F,H\} = \omega(X_H, X_F) = dF(X_H),$$

where X_F and X_H are the Hamiltonian vector fields associated to F and H, respectively. If $\{F, H\} = 0$, then we say that F and H **Poisson commute**.

Another standard result says that the Poisson bracket defines a Lie algebra structure on the space of smooth real-valued functions on M, denoted $C^\infty(M)$.

Definition 1.19. Suppose that the action of G on M is weakly Hamiltonian so that, for each $\zeta \in \mathfrak{g}$, there is a smooth function $H_\zeta : M \to \mathbb{R}$ such that $X_{H_\zeta} = X_\zeta$. If the functions H_ζ can be chosen such that the map

$$\begin{aligned} &\mathfrak{g} \to C^\infty(M) \\ &\zeta \mapsto H_\zeta \end{aligned}$$

is a Lie algebra homomorphism, then the action is called **Hamiltonian**. If the action of G on M is Hamiltonian, then a **moment map** for the action is a map $\mathfrak{mom} : M \to \mathfrak{g}^*$ such that, for H_ζ defined by the prescription

$$H_\zeta(p) = \langle \mathfrak{mom}(p), \zeta \rangle = (\mathfrak{mom}(p))(\zeta),$$

the map $(\mathfrak{g} \to C^\infty(M)) : \zeta \mapsto H_\zeta$ is a Lie algebra homomorphism.

Of particular interest in the study of Hamiltonian group actions is the case in which the group acting is a torus. We write $\mathbb{T}^k$ for the k-dimensional torus $S^1 \times \cdots \times S^1$, and we note that its Lie algebra $\mathfrak{t}$ is abelian (*i.e.*, the Lie bracket vanishes). We identify $\mathfrak{t}$ with $\mathbb{R}^k$ via the map $\left(\zeta_1 \frac{\partial}{\partial \theta_1}, \ldots, \zeta_k \frac{\partial}{\partial \theta_k}\right) \mapsto (\zeta_1, \ldots, \zeta_k)$, and we identify $\mathfrak{t}^*$ with $\mathbb{R}^k$ using the standard Euclidean inner product. With such identifications, if the action of $\mathbb{T}^k$ on M is Hamiltonian, then a moment map for the action is a map $\mathfrak{mom} : M \to \mathbb{R}^k$. The following theorem (the proof of which can be found in Chapter 5 of [MS98]) demonstrates that the images of moment maps for Hamiltonian torus actions are well understood.

Theorem 1.20 (Atiyah, Guillemin–Sternberg)**.** *Let (M, ω) be a compact connected symplectic manifold, and suppose that $\mathbb{T}^k$ acts on M by symplectomorphisms. If the action is Hamiltonian with moment map $\mathfrak{mom} : M \to \mathbb{R}^k$, then:*

(1) the fixed points of the action form a finite union of connected symplectic submanifolds $C_1, \ldots, C_N$;

(2) the moment map $\mathfrak{mom}$ is constant on each C_j, and we write $\eta_j = \mathfrak{mom}\,(C_j)$;

(3) the image of $\mathfrak{mom}$ is the convex hull of $\{\eta_1, \ldots, \eta_N\} \subset \mathbb{R}^k$, i.e.,

$$\mathfrak{mom}\,(M) = \left\{ \sum_{j=1}^{N} \lambda_j \eta_j \;\middle|\; \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \geq 0 \right\}.$$

In the situation described by the above theorem, the image $\mathfrak{mom}\,(M)$ of the moment map is called the *moment polytope*. Recall that a group action is said to be *effective* if the only element of the group G that acts by the identity on M is the identity element of the group.

Definition 1.21. A **symplectic toric manifold** is a compact connected symplectic manifold (M, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}^k$ of dimension equal to half the dimension of M (*i.e.*, $\dim M = 2k$) and with a moment map $\mathfrak{mom} : M \to \mathbb{R}^k$ for the action.

By the work of Delzant in [De88], it is known that there are certain stringent restrictions on the polytopes that can arise as moment polytopes of symplectic toric manifolds, and moreover, the moment polytope determines the symplectic toric manifold (up to equivariant symplectomorphism). A similar classification for symplectic toric orbifolds has been carried out by Lerman and Tolman in [LT97].

Example 1.22. We consider $S^2 \times S^2$ as a subset of $\mathbb{R}^3 \times \mathbb{R}^3$ in the usual way:

$$S^2 \times S^2 = \left\{ (\vec{v}, \vec{w}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,\middle|\, |\vec{v}| = |\vec{w}| = 1 \right\}.$$

Then, where $\mathrm{pr}_1 : S^2 \times S^2 \to S^2$ and $\mathrm{pr}_2 : S^2 \times S^2 \to S^2$ are the projections onto the first and second factors, respectively, and where ω_{std} is the standard area form on S^2, we define a symplectic form Ω on $S^2 \times S^2$ by $\Omega = \frac{1}{2}\mathrm{pr}_1^* \omega_{\mathrm{std}} + \frac{1}{2}\mathrm{pr}_2^* \omega_{\mathrm{std}}$ (consistent with our conventions in Chapter 2). We write

$$R_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix},$$

and we let $\mathbb{T}^2$ act on $S^2 \times S^2$ by

$$\left(e^{i\theta_1}, e^{i\theta_2}\right) \cdot (\vec{v}, \vec{w}) = \left(R_{\theta_1}\vec{v}, R_{\theta_2}\vec{w}\right).$$

This action is effective and Hamiltonian with moment map $\mathfrak{mom} : S^2 \times S^2 \to \mathbb{R}^2$ given by $\mathfrak{mom}\,(\vec{v}, \vec{w}) = \left(-\frac{1}{2}\vec{v} \cdot \vec{e}_1, -\frac{1}{2}\vec{w} \cdot \vec{e}_1\right)$, where $\vec{e}_1$ is the first standard basis vector in $\mathbb{R}^3$. The fixed point set of the action is $\{(\vec{e}_1, \vec{e}_1), (-\vec{e}_1, \vec{e}_1), (\vec{e}_1, -\vec{e}_1), (-\vec{e}_1, -\vec{e}_1)\}$, and the moment polytope is

$$\mathfrak{mom}\left(S^2 \times S^2\right) = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, -\frac{1}{2} \leq x, y \leq \frac{1}{2} \right\}.$$

Thus, $S^2 \times S^2$ is a symplectic toric manifold with the above action and moment map. The fiber

$$\mathfrak{mom}^{-1}(0,0) = \left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \,\middle|\, \vec{v} \cdot \vec{e}_1 = 0, \vec{w} \cdot \vec{e}_1 = 0 \right\}$$

is the product of two "equatorial" circles and is often referred to as the "Clifford torus." The fiber over each interior point of the moment polytope is a Lagrangian torus, but the Clifford torus is the unique monotone Lagrangian torus fiber. Moreover, every fiber except the Clifford torus (including those over the boundary points of the polytope) is displaceable by Hamiltonian diffeomorphisms (by a rotation through angle π in both factors about the axis $\vec{e}_2$ for example). The Clifford torus, on the other hand, is nondisplaceable (as follows from the previous sentence and a result of [EP06] that states that at least one fiber must be nondisplaceable).

Given a symplectic manifold (M, ω), we say that the functions $f_1, \ldots, f_k : M \to \mathbb{R}$ are *independent* if their differentials $(df_1)_p, \ldots, (df_k)_p$ are linearly independent at each point p in some open dense subset of M. It can be shown that the coordinate functions of the moment map for a symplectic toric orbifold are independent and also pairwise Poisson commute. In the context of Example 1.22, the functions $(\vec{v}, \vec{w}) \mapsto -\frac{1}{2}\vec{v} \cdot \vec{e}_1$ and $(\vec{v}, \vec{w}) \mapsto -\frac{1}{2}\vec{w} \cdot \vec{e}_1$ are independent and Poisson commute. The following example can be seen as a generalization of Example 1.22 in the sense that it exhibits two independent Poisson commuting functions on $S^2 \times S^2$ (yet the functions are not the coordinate functions of a moment map for a Hamiltonian torus action).

Example 1.23. Let $(S^2 \times S^2, \Omega)$ be as in Example 1.22, and consider $G_1, G_2 : S^2 \times S^2 \to \mathbb{R}$ defined by

$$G_1(\vec{v}, \vec{w}) = |\vec{v} + \vec{w}|^2 \text{ and } G_2(\vec{v}, \vec{w}) = (\vec{v} + \vec{w}) \cdot \vec{e}_1.$$

The functions G_1 and G_2 are independent and Poisson commute (since one can easily see that G_1 is preserved under the Hamiltonian flow $\varphi^t_{G_2}$, which acts by simultaneous rotation of both factors about the axis $-\vec{e}_1$), and the fiber

$$(G_1 \times G_2)^{-1}(1, 0) = \left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \,\middle|\, |\vec{v} + \vec{w}|^2 = 1, (\vec{v} + \vec{w}) \cdot \vec{e}_1 = 0 \right\}$$

is a monotone Lagrangian torus. In fact, since $|\vec{v} + \vec{w}|^2 = 2 + 2\vec{v} \cdot \vec{w}$ for $(\vec{v}, \vec{w}) \in S^2 \times S^2$, it is easy to see that this fiber is exactly the torus T_{EP} studied in Chapter 2:

$$T_{EP} = \{(\vec{v}, \vec{w}) \in S^2 \times S^2 \mid (\vec{v} + \vec{w}) \cdot \vec{e}_1 = 0, \vec{v} \cdot \vec{w} = -1/2\},$$

and the torus T_{EP} was shown to be monotone in [EP09]. Moreover, by the results of [CS10] and [FOOO12] (and the equivalences exhibited in Chapter 2), the torus T_{EP} is nondisplaceable. Recall that the moment map $\mathfrak{mom} : S^2 \times S^2 \to \mathbb{R}^2$ in Example 1.22 has exactly one monotone nondisplaceable fiber, but in contrast the function $(G_1 \times G_2) : S^2 \times S^2 \to \mathbb{R}^2$ has two monotone nondisplaceable fibers since one can show that the fiber

$$(G_1 \times G_2)^{-1}(0,0) = \left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \,\middle|\, |\vec{v} + \vec{w}|^2 = 0, (\vec{v} + \vec{w}) \cdot \vec{e}_1 = 0 \right\}$$

is also monotone and nondisplaceable. In fact, it is clear that $(G_1 \times G_2)^{-1}(0,0)$ is exactly the anti-diagonal

$$\overline{\Delta} = \left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \,\middle|\, \vec{v} = -\vec{w} \right\},$$

which is monotone and is nondisplaceable by the results of [EP09].

Chapter 2 is dedicated to giving thorough descriptions of several monotone Lagrangian tori in $(S^2 \times S^2, \Omega)$ and showing that each of these tori is Hamiltonian isotopic to the torus T_{EP}. Analogous to Example 1.23, we now consider a system of 3 independent functions on $(S^2)^3$ that pairwise Poisson commute (but which are not the coordinate functions of a moment map for a Hamiltonian torus action).

Example 1.24. We write elements of $(S^2)^3$ as 3×3 matrices $\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ for $\vec{u}, \vec{v}, \vec{w} \in S^2$, and where ω_{std} is the standard symplectic form on S^2 and the map $\text{pr}_j : (S^2)^3 \to S^2$ is the (holomorphic) projection onto the j^{th} factor of $(S^2)^3$ for $j = 1, 2, 3$, we consider $(S^2)^3$ with the split symplectic form $\Omega = \text{pr}_1^* \omega_{\text{std}} + \text{pr}_2^* \omega_{\text{std}} + \text{pr}_3^* \omega_{\text{std}}$. We then consider functions

$H_1, H_2, H_3 : (S^2)^3 \to \mathbb{R}$ defined by

$$H_1 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = |\vec{u} + \vec{v} + \vec{w}|^2 ,$$
$$H_2 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = (\vec{u} + \vec{v} + \vec{w}) \cdot \vec{e}_1,$$
$$H_3 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \vec{u} \cdot (\vec{v} \times \vec{w}) .$$

It can be shown that the functions H_1, H_2, and H_3 are independent and Poisson commute. Indeed, it is not difficult to see that both H_1 and H_3 are preserved under the Hamiltonian flow $\varphi^t_{H_2}$ (which acts by simultaneous rotation of all 3 factors about the axis $-\vec{e}_1$), and one can also use the Jacobi identity for the cross product to show that the quantity $\vec{u}+\vec{v}+\vec{w}$ is preserved under the Hamiltonian flow $\varphi^t_{H_3}$ (and hence H_1 and H_2 are preserved under the flow $\varphi^t_{H_3}$ as well). We will show in Chapters 3, 4, and 5 that the fiber $(H_1 \times H_2 \times H_3)^{-1}(0,0,0)$ of the $\mathbb{R}^3$-valued function $H_1 \times H_2 \times H_3$ is monotone and nondisplaceable. Indeed, one can easily see that

$$(H_1 \times H_2 \times H_3)^{-1}(0,0,0) = \left\{ \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in \left(S^2\right)^3 \middle| \ \vec{u} + \vec{v} + \vec{w} = \vec{0} \right\},$$

which is exactly the Lagrangian L described in Chapters 3, 4, and 5.

Since the fiber $L = (H_1 \times H_2 \times H_3)^{-1}(0,0,0)$ in Example 1.24 is apparently analogous to the anti-diagonal $\overline{\Delta} \subset S^2 \times S^2$ from Example 1.23, one naturally wonders whether there is a monotone Lagrangian torus fiber $(H_1 \times H_2 \times H_3)^{-1}(x,y,z)$ that is analogous to the monotone Lagrangian torus $T_{EP} \subset S^2 \times S^2$ in Example 1.23. At the time of this writing, computations seem to indicate that the fiber $(H_1 \times H_2 \times H_3)^{-1}(1,0,0)$ should be monotone, but it turns out that the point $(1,0,0)$ is not a regular value of the map $H_1 \times H_2 \times H_3$ (for that matter, neither is $(0,0,0)$, but the fiber $(H_1 \times H_2 \times H_3)^{-1}(0,0,0)$ turns out to be a smooth submanifold of $(S^2)^3$ whereas the fiber $(H_1 \times H_2 \times H_3)^{-1}(1,0,0)$ is not a manifold). It would be interesting to know if the singular fiber $(H_1 \times H_2 \times H_3)^{-1}(1,0,0)$ is nondisplaceable, but the main focus of this work (in Chapters 3, 4, and 5) is to show that $L \subset (S^2)^3$ is nondisplaceable.

Chapter 2

Lagrangian Tori in $S^2 \times S^2$

In this chapter, which is self-contained for the most part, we will give several descriptions of "exotic" monotone Lagrangian tori in $S^2 \times S^2$, and we will prove that all of these tori are equivalent under Hamiltonian diffeomorphisms (closely following previous joint work with Usher in [OU13]). The term "exotic" was used by Entov and Polterovich in [EP09] to describe a monotone Lagrangian torus in $S^2 \times S^2$ that we denote T_{EP}. They used this term because they showed that there is no symplectomorphism $\varphi : S^2 \times S^2 \to S^2 \times S^2$ such that $\varphi(T_{EP})$ is equal to the more standard "Clifford torus," which is a product of equatorial circles $S^1_{eq} \times S^1_{eq} \subset S^2 \times S^2$.

Throughout this chapter, we consider $S^2 \times S^2$ as a subset of $\mathbb{R}^3 \times \mathbb{R}^3$ in the usual way:

$$S^2 \times S^2 = \left\{ (\vec{v}, \vec{w}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; |\vec{v}| = |\vec{w}| = 1 \right\}.$$

Then, where $\mathrm{pr}_1 : S^2 \times S^2 \to S^2$ and $\mathrm{pr}_2 : S^2 \times S^2 \to S^2$ are the projections onto the first and second factors, respectively, and where ω_{std} is the standard area form on S^2, we define a symplectic form Ω on $S^2 \times S^2$ by $\Omega = \frac{1}{2}\mathrm{pr}_1^* \omega_{\mathrm{std}} + \frac{1}{2}\mathrm{pr}_2^* \omega_{\mathrm{std}}$. In particular, with this convention, the sphere $S^2 \times \{\mathrm{point}\}$ has area 2π.

The first torus we define is the simplest to describe. Following [EP09], we define a torus

$$T_{EP} = \left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \,\middle|\, (\vec{v} + \vec{w}) \cdot \vec{e}_1 = 0, \vec{v} \cdot \vec{w} = -1/2 \right\}, \tag{2.1}$$

where here and throughout this chapter the vector $\vec{e}_1$ is the first standard basis vector in $\mathbb{R}^3$.

Next, we describe a torus in $S^2 \times S^2$ introduced by Chekanov and Schlenk in [CS10]. We begin with a curve Γ enclosing an area of $\frac{\pi}{2}$ and contained in the open upper half disk $\mathbb{H}\left(\sqrt{2}\right) = \{z \in \mathbb{C} \,|\, \mathrm{Im}\,(z) > 0, |z| < \sqrt{2}\}$ of radius $\sqrt{2}$. The curve $\Delta_\Gamma = \{(z, z) \in \mathbb{C}^2 \,|\, z \in \Gamma\}$ then lies in the diagonal of $B^2\left(\sqrt{2}\right) \times B^2\left(\sqrt{2}\right)$, where $B^2\left(\sqrt{2}\right) = \{z \in \mathbb{C} \,|\, |z| < \sqrt{2}\}$ is the open disk of radius $\sqrt{2}$. Then, we consider the torus Θ_{CS} in $B^2\left(\sqrt{2}\right) \times B^2\left(\sqrt{2}\right)$ given as the orbit of Δ_Γ under the circle action

$$e^{it} \cdot (z_1, z_2) = \left(e^{it} z_1, e^{-it} z_2\right).$$

In other words, we define

$$\Theta_{CS} = \left\{ \left(e^{it} z, e^{-it} z\right) \mid z \in \Gamma, t \in [0, 2\pi] \right\} \subset B^2\left(\sqrt{2}\right) \times B^2\left(\sqrt{2}\right),$$

and we then define a torus $T_{CS} \subset S^2 \times S^2$ to be the image of Θ_{CS} under a standard dense symplectic embedding $B^2\left(\sqrt{2}\right) \times B^2\left(\sqrt{2}\right) \hookrightarrow S^2 \times S^2$.

To describe the monotone Lagrangian torus in [FOOO12], one begins with a symplectic toric orbifold that is denoted $F_2(0)$ and whose moment polytope is

$$\Delta_{FOOO} = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, 0 \leq x \leq 2, 0 \leq y \leq 1 - \frac{1}{2}x \right\}$$

with exactly one singular point sitting over the point $(0, 1) \in \Delta_{FOOO}$. By replacing a neighborhood of the singular point in $F_2(0)$ with a neighborhood of the zero-section of the cotangent bundle T^*S^2, one obtains a manifold denoted $\hat{F}_2(0)$ that is shown to be symplectomorphic to $S^2 \times S^2$. Letting Θ_{FOOO} denote the fiber over the point $\left(\frac{1}{2}, \frac{1}{2}\right) \in \Delta_{FOOO}$, the monotone Lagrangian torus $T_{FOOO} \subset S^2 \times S^2$ is then defined to be the image of Θ_{FOOO} under a symplectomorphism $\hat{F}_2(0) \to S^2 \times S^2$.

In [AF08], it was shown that there is a (nondisplaceable) monotone Lagrangian torus Θ_{AF} in the cotangent bundle T^*S^2. Explicitly, we use the standard Riemannian metric on S^2 to identify T^*S^2 with TS^2, which we think of as a submanifold of $\mathbb{R}^3 \times \mathbb{R}^3$ via

$$T^*S^2 \cong TS^2 = \left\{ (\vec{p}, \vec{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \vec{q} \cdot \vec{p} = 0, |\vec{q}| = 1 \right\}.$$

Under this identification, the canonical 1-form on T^*S^2 is $\lambda = p_1 dq_1 + p_2 dq_2 + p_3 dq_3$, and we consider T^*S^2 with symplectic form $d\lambda$. We define

$$\Theta_{AF} = \left\{ (\vec{p}, \vec{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; |\vec{p}| = \frac{1}{2}, (\vec{p} \times \vec{q}) \cdot \vec{e}_1 = 0 \right\} \subset T^*S^2.$$

Where $D_1^*S^2 \subset T^*S^2$ denotes the open unit disk bundle, one can symplectically identify $D_1^*S^2$ with $(S^2 \times S^2) \setminus \Delta$, where $\Delta \subset S^2 \times S^2$ is the diagonal. We will provide such a symplectomorphism $\Phi_2 : (S^2 \times S^2) \setminus \Delta \to D_1^*S^2$ in Lemma 2.5 below, and although Albers and Frauenfelder did not explicitly consider their torus in T^*S^2 as a submanifold of $S^2 \times S^2$, we nonetheless write $T_{AF} = \Phi_2^{-1}(\Theta_{AF})$.

The final description of a torus in $S^2 \times S^2$ that we give (which will be denoted T_{BC}) was considered by Gadbled in [Ga13] and is based on a Lagrangian circle bundle construction of Biran in [Bi06]. First we recall the description of standard symplectic disk bundles from [Bi01], which is presented here with some modifications pertaining to normalization. Let (Σ, ω_Σ) be a symplectic manifold, and let $\pi_P : P \to \Sigma$ be a principal S^1-bundle with Chern class $\left[\frac{1}{\tau}\omega_\Sigma\right]$ for some $\tau > 0$. Also, let $\beta \in \Omega^1(P)$ be a connection 1-form on P normalized so that $d\beta = -\frac{1}{\tau}\pi_P^*(\omega_\Sigma)$. The *standard symplectic disk bundle* to Σ associated to the pair

(P,β) is the symplectic manifold $(\mathcal{D}_\tau(P), \omega_{\text{can}})$ defined as follows. Where $D\left(\sqrt{\tau/\pi}\right) = \left\{ z \in \mathbb{C} \,\middle|\, |z| < \sqrt{\tau/\pi} \right\}$ is the open disk of radius $\sqrt{\tau/\pi}$ in $\mathbb{C}$, the smooth manifold $\mathcal{D}_\tau(P)$ is defined by

$$\mathcal{D}_\tau(P) = \frac{P \times D\left(\sqrt{\tau/\pi}\right)}{(e^{i\theta} \cdot w, z) \sim (w, e^{i\theta} z)}.$$

Writing $\mathfrak{q} : P \times D\left(\sqrt{\tau/\pi}\right) \to \mathcal{D}_\tau(P)$ for the quotient map, $\omega_{\mathbb{C}}$ for the standard symplectic form on $\mathbb{C}$, and $\mathrm{pr}_P : P \times D\left(\sqrt{\tau/\pi}\right) \to P$ and $\mathrm{pr}_D : P \times D\left(\sqrt{\tau/\pi}\right) \to D\left(\sqrt{\tau/\pi}\right)$ for the obvious projections, the symplectic form ω_{can} on $\mathcal{D}_\tau(P)$ is defined by

$$\mathfrak{q}^* \omega_{\text{can}} = d\left(\left(\pi |z|^2 - \tau\right) \mathrm{pr}_P^* \beta\right) + \mathrm{pr}_D^* \omega_{\mathbb{C}}.$$

Note that the map $i_\Sigma : \Sigma \to \mathcal{D}_\tau(P)$ defined by $i_\Sigma(\pi_P(w)) = [(w,0)]$ gives an embedding of Σ into $\mathcal{D}_\tau(P)$ as the "zero-section" $\{[(w,0)] \,|\, w \in P\}$, and the symplectic form ω_{can} satisfies $(i_\Sigma)^* \omega_{\text{can}} = \omega_\Sigma$. Moreover, the projection $\mathrm{pr} : \mathcal{D}_\tau(P) \to \Sigma$ given by $\mathrm{pr}([(w,z)]) = \pi_P(w)$ gives $\mathcal{D}_\tau(P)$ the structure of a fiber bundle over Σ whose fibers are symplectic disks, each having area τ.

Now if $\Lambda \subset \Sigma$ is a monotone Lagrangian submanifold, then, for any $r \in \left(0, \sqrt{\tau/\pi}\right)$, the submanifold

$$\Lambda_{(r)} = \{[(w,z)] \in \mathcal{D}_\tau(P) \,|\, |z| = r, \mathrm{pr}([(w,z)]) \in \Lambda\}$$

will be a monotone Lagrangian submanifold of $\mathcal{D}_\tau(P)$ which has the structure of a circle bundle over Λ. According to the main result of [Bi01], if (Σ, ω_Σ) is a complex hypersurface of a Kähler manifold (M,ω) that is Poincaré dual to the cohomology class $\left[\frac{1}{\tau}\omega\right]$, then the standard symplectic disk bundle $\mathcal{D}_\tau(P)$ symplectically embeds into M as the complement of an isotropic CW complex. By Proposition 6.4.1 of [BC09], there is typically a unique value of r for which a symplectic embedding $\mathcal{D}_\tau(P) \hookrightarrow M$ maps $\Lambda_{(r)}$ to a monotone Lagrangian submanifold of M, which we call the *Biran circle bundle construction* associated to Λ.

To define the torus which we denote $T_{BC} \subset S^2 \times S^2$, one works through the above construction with $(M, \omega) = (S^2 \times S^2, \Omega)$ and with Σ equal to the diagonal $\Delta \subset S^2 \times S^2$ (with $\tau = 2\pi$ in our conventions). Identifying Δ with S^2 in the obvious way, one takes the principal circle bundle P to be the unit circle bundle in the tangent bundle TS^2, namely

$$P = \left\{ (\vec{p}, \vec{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,\middle|\, |\vec{p}| = |\vec{q}| = 1, \vec{p} \cdot \vec{q} = 0 \right\}$$

with projection $\pi_P : (\vec{p}, \vec{q}) \mapsto \vec{q}$ and circle action given by

$$e^{it} \cdot (\vec{p}, \vec{q}) = ((\cos(t))\, \vec{p} + (\sin(t))\, \vec{q} \times \vec{p}, \vec{q}) \, .$$

Under the identification $\Delta \cong S^2$, we take $\Lambda = \{\vec{v} \in S^2 \,|\, \vec{v} \cdot \vec{e}_1 = 0\}$ to be an "equator" in Δ. We define T_{BC} to be the Biran circle bundle construction associated to Λ (an explicit symplectic embedding $\mathcal{D}_\tau(P) \hookrightarrow S^2 \times S^2$ will be given in the proof of Proposition 2.7).

It was shown in [Ga13] that the tori T_{BC} and T_{CS} are Hamiltonian isotopic, and it has been suspected by several authors that many (or all) of the above constructions led to Hamiltonian isotopic tori. Through joint work with Usher in [OU13], it has been demonstrated that this is indeed the case.

Theorem 2.1. *All of the tori T_{EP}, T_{CS}, T_{FOOO}, T_{AF}, and T_{BC} are Hamiltonian isotopic to each other.*

Remark 2.2. Since some of these tori are defined in terms of a symplectomorphism from some other symplectic manifold M to the standard $S^2 \times S^2$, it might appear that the existence of a Hamiltonian isotopy between any two of them depends upon the choice of symplectomorphism $M \to S^2 \times S^2$. However, according to 0.3.C of [Gr85], any symplectomorphism of $S^2 \times S^2$ is either a Hamiltonian diffeomorphism or can be written as the composition of a Hamiltonian diffeomorphism with the diffeomorphism $S^2 \times S^2 \to S^2 \times S^2$ that switches the two factors of $S^2 \times S^2$. Observe that T_{EP} is invariant under this latter diffeomorphism. Then, since each torus in Theorem 2.1 will be shown to be Hamiltonian isotopic to T_{EP}, it follows that there is in fact no dependence upon the choice of symplectomorphism $M \to S^2 \times S^2$.

The proof of Theorem 2.1 is given by Propositions 2.3, 2.6, and 2.7 below. While T_{EP} is given very explicitly as a submanifold of $S^2 \times S^2$, the same cannot be said of T_{FOOO}, which is instead described as a the image of a submanifold $\Theta_{FOOO} \subset \hat{F}_2(0)$ under a symplectomorphism $\hat{F}_2(0) \to S^2 \times S^2$. In [FOOO12], the proof that $\hat{F}_2(0)$ is symplectomorphic to $S^2 \times S^2$ makes it difficult to determine what the image of Θ_{FOOO} might be under such a symplectomorphism. Hence, most of our task in proving Proposition 2.3 will be to give a construction of the manifold $\hat{F}_2(0)$ that allows it to be symplectically identified with $S^2 \times S^2$ in an explicit way. Once the construction is complete, it will follow rather quickly that Θ_{FOOO} is mapped to T_{EP} under our symplectomorphism.

Similarly, most of the work in showing that T_{BC} is Hamiltonian isotopic to T_{EP} consists of giving an explicit construction of the standard symplectic disk bundle $\mathcal{D}_\tau(P)$ and determining a symplectic embedding $\mathcal{D}_\tau(P) \hookrightarrow S^2 \times S^2$. Having completed such constructions, it will be completely clear that T_{BC} is Hamiltonian isotopic to T_{EP} (in fact, we will see that the two are equal).

Proposition 2.3. *T_{AF} is equal to T_{EP}, and there is a symplectomorphism $S^2 \times S^2 \to S^2 \times S^2$ taking T_{FOOO} to T_{EP}.*

Before proving this proposition, we need to establish a couple of lemmata.

Lemma 2.4. *Where $B^4(2)$ is the open ball of radius 2 in the quaternions $\mathcal{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$, where $\mathbb{R}^3$ is identified with the pure imaginary quaternions, and where $0_{S^2} \subset T^*S^2$ is the zero-section, the map $\varphi_1 : B^4(2) \setminus \{0\} \to D_1^*S^2 \setminus 0_{S^2}$ defined by*

$$\varphi_1(\xi) = \left(-\frac{\xi^* k \xi}{4}, \frac{\xi^* j \xi}{|\xi|^2}\right)$$

is a symplectic double cover with $\varphi_1(\xi_1) = \varphi_1(\xi_2)$ if and only if $\xi_1 = \pm\xi_2$. Moreover, where $\vec{e}_1 \in \mathbb{R}^3$ is the first standard basis vector and $f_{T^*S^2}(\vec{p},\vec{q}) = |\vec{p}|$ and $g_{T^*S^2}(\vec{p},\vec{q}) = (\vec{p}\times\vec{q})\cdot\vec{e}_1$, we have

$$f_{T^*S^2}\circ\varphi_1(z_1+z_2j) = \frac{1}{4}(|z_1|^2+|z_2|^2) \text{ and } g_{T^*S^2}\circ\varphi_1(z_1+z_2j) = \frac{1}{4}(|z_1|^2-|z_2|^2)$$

for $z_1, z_2 \in \mathbb{C}$ with $0 < |z_1|^2+|z_2|^2 < 4$.

Proof. First, writing $\xi = z_1 + z_2j$, we observe that

$$\begin{aligned} g_{T^*S^2}\circ\varphi_1(z_1+z_2j) = g_{T^*S^2}\circ\varphi_1(\xi) &= \left(-\frac{\xi^*k\xi}{4}\times\frac{\xi^*j\xi}{|\xi|^2}\right)\cdot\vec{e}_1 = \left(\frac{1}{4}\xi^*i\xi\right)\cdot\vec{e}_1 \\ &= \left(\frac{1}{4}\left((|z_1|^2-|z_2|^2)i - \mathrm{Im}(2\bar{z}_1z_2)j + \mathrm{Re}(2\bar{z}_1z_2)k\right)\right)\cdot\vec{e}_1 \\ &= \frac{1}{4}(|z_1|^2-|z_2|^2) \end{aligned}$$

and also that

$$f_{T^*S^2}\circ\varphi_1(z_1+z_2j) = f_{T^*S^2}\circ\varphi_1(\xi) = \left|-\frac{\xi^*k\xi}{4}\right| = \frac{1}{4}|\xi|^2 = \frac{1}{4}(|z_1|^2+|z_2|^2),$$

which proves the second statement of the lemma and also makes clear that φ_1 has an appropriate codomain.

We then observe that $\varphi_1(-\xi) = \varphi_1(\xi)$, and we claim also that $\varphi_1(\xi_1) = \varphi_1(\xi_2)$ only if $\xi_1 = \pm\xi_2$. Indeed if $\varphi_1(\xi_1) = \varphi_1(\xi_2)$, then it follows that $|\xi_1| = |\xi_2|$ and also that

$$\begin{aligned} \left(\frac{\xi_1}{|\xi_1|}\right)^* j \left(\frac{\xi_1}{|\xi_1|}\right) &= \left(\frac{\xi_2}{|\xi_2|}\right)^* j \left(\frac{\xi_2}{|\xi_2|}\right), \\ \left(\frac{\xi_1}{|\xi_1|}\right)^* k \left(\frac{\xi_1}{|\xi_1|}\right) &= \left(\frac{\xi_2}{|\xi_2|}\right)^* k \left(\frac{\xi_2}{|\xi_2|}\right), \\ \left(\frac{\xi_1}{|\xi_1|}\right)^* i \left(\frac{\xi_1}{|\xi_1|}\right) &= \left(\frac{\xi_2}{|\xi_2|}\right)^* i \left(\frac{\xi_2}{|\xi_2|}\right). \end{aligned}$$

Then, writing $\mathcal{S}$ for the group of unit quaternions, it is well known that the map $\mathcal{S} \to SO(3)$ given by $\xi \mapsto \begin{pmatrix} \xi i \xi^* & \xi j \xi^* & \xi k \xi^* \end{pmatrix}$ is a surjective Lie group homomorphism with kernel $\{\pm 1\}$ (see Exercise 9-10 of [Lee03] for example); thus, it follows from the above displayed equations that $(\xi_1/|\xi_1|)^* = \pm (\xi_2/|\xi_2|)^*$ and hence that $\xi_1 = \pm \xi_2$. Moreover, the surjectivity of this Lie group homomorphism, when paired with the observation that $\left|-\frac{\xi^* k \xi}{4}\right| = \frac{1}{4}|\xi|^2$, implies that φ_1 is surjective. A routine computation shows that

$$\varphi_1^* \lambda = -\frac{y_1}{2} dx_1 + \frac{x_1}{2} dy_1 - \frac{y_2}{2} dx_2 + \frac{x_2}{2} dy_2,$$

from which it follows that

$$\varphi_1^*(d\lambda) = d\left(\varphi_1^* \lambda\right) = d\left(-\frac{y_1}{2} dx_1 + \frac{x_1}{2} dy_1 - \frac{y_2}{2} dx_2 + \frac{x_2}{2} dy_2\right) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2,$$

which of course is the standard symplectic form on $B^4(2)$. Then, since any symplectic map is an immersion, it follows that φ_1 is a symplectic double cover as claimed. □

Lemma 2.5. *Where $\Delta \subset S^2 \times S^2$ is the diagonal, the map $\Phi_2 : (S^2 \times S^2) \setminus \Delta \to D_1^* S^2$ defined by*

$$\Phi_2(\vec{v}, \vec{w}) = \left(\frac{\vec{v} \times \vec{w}}{|\vec{v} - \vec{w}|}, \frac{\vec{v} - \vec{w}}{|\vec{v} - \vec{w}|}\right)$$

is a symplectomorphism. Moreover, where $f_{T^*S^2}$ and $g_{T^*S^2}$ are as in Lemma 2.4, we have

$$f_{T^*S^2} \circ \Phi_2(\vec{v}, \vec{w}) = \frac{1}{2}|\vec{v} + \vec{w}| \quad \text{and} \quad g_{T^*S^2} \circ \Phi_2(\vec{v}, \vec{w}) = \frac{1}{2}(\vec{v} + \vec{w}) \cdot \vec{e}_1.$$

Proof. First, we observe that

$$\begin{aligned} g_{T^*S^2} \circ \Phi_2(\vec{v}, \vec{w}) &= \left(\frac{\vec{v} \times \vec{w}}{|\vec{v} - \vec{w}|} \times \frac{\vec{v} - \vec{w}}{|\vec{v} - \vec{w}|} \right) \cdot \vec{e}_1 = \left(\frac{(\vec{v} \times \vec{w}) \times \vec{v} - (\vec{v} \times \vec{w}) \times \vec{w}}{|\vec{v} - \vec{w}|^2} \right) \cdot \vec{e}_1 \\ &= \left(\frac{-(\vec{v} \cdot \vec{w})\vec{v} + \vec{w} + \vec{v} - (\vec{v} \cdot \vec{w})\vec{w}}{|\vec{v} - \vec{w}|^2} \right) \cdot \vec{e}_1 \\ &= \left(\frac{(\vec{v} + \vec{w})(1 - \vec{v} \cdot \vec{w})}{2 - 2\vec{v} \cdot \vec{w}} \right) \cdot \vec{e}_1 = \frac{1}{2}(\vec{v} + \vec{w}) \cdot \vec{e}_1, \end{aligned}$$

and the relationship

$$|\vec{v} - \vec{w}|^2 |\vec{v} + \vec{w}|^2 = 4|\vec{v} \times \vec{w}|^2 \text{ for } (\vec{v}, \vec{w}) \in S^2 \times S^2 \tag{2.2}$$

makes clear that $f_{T^*S^2} \circ \Phi_2(\vec{v}, \vec{w}) = \frac{1}{2}|\vec{v} + \vec{w}|$. Thus, we have proved the second statement of the lemma (which also makes clear that Φ_2 has an appropriate codomain).

To see that Φ_2 is a symplectomorphism, we observe that the vector fields

$$\begin{aligned} X_1(\vec{v}, \vec{w}) &= (\vec{v} \times \vec{w}, \vec{w} \times \vec{v}) & X_2(\vec{v}, \vec{w}) &= (\vec{v} \times (\vec{v} \times \vec{w}), \vec{w} \times (\vec{w} \times \vec{v})) \\ X_3(\vec{v}, \vec{w}) &= (\vec{w} \times \vec{v}, \vec{w} \times \vec{v}) & X_4(\vec{v}, \vec{w}) &= (\vec{v} \times (\vec{w} \times \vec{v}), \vec{w} \times (\vec{w} \times \vec{v})) \end{aligned}$$

give a basis for $T_{(\vec{v},\vec{w})}\left((S^2 \times S^2) \setminus \Delta\right)$ at each point $(\vec{v}, \vec{w})$ not in the anti-diagonal

$$\overline{\Delta} = \{(\vec{v}, \vec{w}) \in S^2 \times S^2 \mid \vec{v} = -\vec{w}\}.$$

We then compute that Ω evaluates on pairs as follows:

$$\begin{aligned} &\Omega(X_1, X_2) = \Omega(X_3, X_4) = |\vec{v} \times \vec{w}|^2, \\ &\Omega(X_1, X_3) = \Omega(X_1, X_4) = \Omega(X_2, X_3) = \Omega(X_2, X_4) = 0. \end{aligned}$$

Then, using the coordinate free formula for the exterior derivative of a one form, we will verify that $\Phi_2^* d\lambda$ evaluates on pairs in an identical manner to Ω. To that end, computing the commutators of the vector fields X_1, X_2, X_3, and X_4, one finds the following relationships:

$$
\begin{aligned}
[X_1, X_2] &= \frac{1}{2}|\vec{v} - \vec{w}|^2 X_1 = -[X_3, X_4], \\
[X_1, X_3] &= -2X_4, \\
[X_1, X_4] &= \frac{1}{2}|\vec{v} + \vec{w}|^2 X_3 = [X_2, X_3], \\
[X_2, X_4] &= (2\vec{v} \cdot \vec{w}) X_4.
\end{aligned}
$$

Moreover, another computation shows that $\Phi_2^*\lambda(X_1) = \frac{1}{2}|\vec{v} + \vec{w}|^2$ while $\Phi_2^*\lambda(X_j) = 0$ for $j \neq 1$ (note that (2.2) was used here to obtain the simplified form given for $\Phi_2^*\lambda(X_1)$), and yet another computation reveals that $X_j\left(\Phi_2^*\lambda(X_1)\right) = 0$ for $j \neq 2$ (since the quantity $\Phi_2^*\lambda(X_1) = \frac{1}{2}|\vec{v} + \vec{w}|^2$ is preserved under the flows of X_1, X_3, and X_4) while $X_2\left(\Phi_2^*\lambda(X_1)\right) = -2|\vec{v} \times \vec{w}|^2$. It then follows from (2.2) that

$$
\begin{aligned}
d\Phi_2^*\lambda(X_1, X_2) &= X_1\left(\Phi_2^*\lambda(X_2)\right) - X_2\left(\Phi_2^*\lambda(X_1)\right) - \Phi_2^*\lambda\left([X_1, X_2]\right) \\
&= 2|\vec{v} \times \vec{w}|^2 - \Phi_2^*\lambda\left(\frac{1}{2}|\vec{v} - \vec{w}|^2 X_1\right) \\
&= 2|\vec{v} \times \vec{w}|^2 - \frac{1}{4}|\vec{v} - \vec{w}|^2|\vec{v} + \vec{w}|^2 = |\vec{v} \times \vec{w}|^2, \\
d\Phi_2^*\lambda(X_3, X_4) &= X_3\left(\Phi_2^*\lambda(X_4)\right) - X_4\left(\Phi_2^*\lambda(X_3)\right) - \Phi_2^*\lambda\left([X_3, X_4]\right) \\
&= -\Phi_2^*\lambda\left(-\frac{1}{2}|\vec{v} - \vec{w}|^2 X_1\right) = \frac{1}{4}|\vec{v} - \vec{w}|^2|\vec{v} + \vec{w}|^2 = |\vec{v} \times \vec{w}|^2,
\end{aligned}
$$

and also that

$$
\begin{aligned}
d\Phi_2^*\lambda(X_1, X_3) &= X_1\left(\Phi_2^*\lambda(X_3)\right) - X_3\left(\Phi_2^*\lambda(X_1)\right) - \Phi_2^*\lambda\left([X_1, X_3]\right) \\
&= -\Phi_2^*\lambda\left(-2X_4\right) = 0, \\
d\Phi_2^*\lambda(X_1, X_4) &= X_1\left(\Phi_2^*\lambda(X_4)\right) - X_4\left(\Phi_2^*\lambda(X_1)\right) - \Phi_2^*\lambda\left([X_1, X_4]\right) \\
&= -\Phi_2^*\lambda\left(\frac{1}{2}|\vec{v}+\vec{w}|^2 X_3\right) = 0, \\
d\Phi_2^*\lambda(X_2, X_3) &= X_2\left(\Phi_2^*\lambda(X_3)\right) - X_3\left(\Phi_2^*\lambda(X_2)\right) - \Phi_2^*\lambda\left([X_2, X_3]\right) \\
&= -\Phi_2^*\lambda\left(\frac{1}{2}|\vec{v}+\vec{w}|^2 X_3\right) = 0, \\
d\Phi_2^*\lambda(X_2, X_4) &= X_2\left(\Phi_2^*\lambda(X_4)\right) - X_4\left(\Phi_2^*\lambda(X_2)\right) - \Phi_2^*\lambda\left([X_2, X_4]\right) \\
&= -\Phi_2^*\lambda\left((2\vec{v}\cdot\vec{w})X_4\right) = 0,
\end{aligned}
$$

and then (by continuity along the anti-diagonal $\overline{\Delta}$ where the vector fields X_j vanish) we see that $\Phi_2^* d\lambda = \Omega$ on $(S^2 \times S^2) \setminus \Delta$ as required. Finally, to see that Φ_2 is bijective, a routine check (using (2.2) and the fact that $4 - |\vec{v}+\vec{w}|^2 = |\vec{v}-\vec{w}|^2$ for $(\vec{v}, \vec{w}) \in S^2 \times S^2$) reveals that

$$
\Phi_2^{-1}(\vec{p}, \vec{q}) = \left(\sqrt{1-|\vec{p}|^2}\,\vec{q} - \vec{q}\times\vec{p}, -\sqrt{1-|\vec{p}|^2}\,\vec{q} - \vec{q}\times\vec{p}\right)
$$

defines the inverse for Φ_2. □

With Lemmata 2.4 and 2.5 proved, we are now ready to give a construction of the manifold $\hat{F}_2(0)$ and prove Proposition 2.3.

Proof of Proposition 2.3. First the fact that $T_{AF} = T_{EP}$ follows immediately from the definitions and from the computations of $f_{T^*S^2} \circ \Phi_2$ and $g_{T^*S^2} \circ \Phi_2$ in Lemma 2.5. Indeed, since

$$
|\vec{v}+\vec{w}| = \sqrt{2+2\vec{v}\cdot\vec{w}} \;\text{ for }\; (\vec{v}, \vec{w}) \in S^2 \times S^2,
$$

one can see from the definitions that $(\vec{v},\vec{w}) \in T_{AF}$ if and only if $f_{T^*S^2} \circ \Phi_2(\vec{v},\vec{w}) = \frac{1}{2}$ and $g_{T^*S^2} \circ \Phi_2(\vec{v},\vec{w}) = 0$, conditions that are equivalent to $|\vec{v}+\vec{w}| = 1$ and $(\vec{v}+\vec{w})\cdot\vec{e}_1 = 0$. These latter conditions are equivalent to $\vec{v}\cdot\vec{w} = -\frac{1}{2}$ and $(\vec{v}+\vec{w})\cdot\vec{e}_1 = 0$, which hold if and only if $(\vec{v},\vec{w})$ is an element of T_{EP}.

Since the preimage of the zero-section 0_{S^2} under Φ_2 is the anti-diagonal $\overline{\Delta} \subset S^2 \times S^2$, it follows from Lemmas 2.4 and 2.5 that the map $\Phi_2^{-1} \circ \varphi_1 : B^4(2) \setminus \{0\} \to (S^2 \times S^2) \setminus (\overline{\Delta} \cup \Delta)$ descends to a symplectomorphism

$$A : \frac{B^4(2) \setminus \{0\}}{\pm 1} \to (S^2 \times S^2) \setminus (\overline{\Delta} \cup \Delta)$$

which pulls back the function $(\vec{v},\vec{w}) \mapsto \frac{1}{2}|\vec{v}+\vec{w}|$ to the function $[(z_1,z_2)] \mapsto \frac{1}{4}(|z_1|^2 + |z_2|^2)$ and pulls back the function $(\vec{v},\vec{w}) \mapsto \frac{1}{2}(\vec{v}+\vec{w})\cdot\vec{e}_1$ to the function $[(z_1,z_2)] \mapsto \frac{1}{4}(|z_1|^2 - |z_2|^2)$.

Consequently, we may introduce the symplectic 4-orbifold

$$\mathcal{O} = \frac{(B^4(2)/\pm 1) \sqcup \left((S^2 \times S^2) \setminus \overline{\Delta}\right)}{[(z_1,z_2)] \sim A([(z_1,z_2)]) \text{ for } (z_1,z_2) \neq (0,0)}$$

since the fact that A is a symplectomorphism shows that the the symplectic forms on $(B^4(2)/\pm 1)$ and on $(S^2 \times S^2) \setminus \overline{\Delta}$ coincide on their overlap in $\mathcal{O}$. Moreover we have well-defined functions $F : \mathcal{O} \to \mathbb{R}$ and $G : \mathcal{O} \to \mathbb{R}$ defined by

$$F([(z_1,z_2)]) = \frac{1}{4}(|z_1|^2 + |z_2|^2) \qquad G([(z_1,z_2)]) = \frac{1}{4}(|z_1|^2 - |z_2|^2)$$
$$F(\vec{v},\vec{w}) = \frac{1}{2}|\vec{v}+\vec{w}| \qquad G(\vec{v},\vec{w}) = \frac{1}{2}(\vec{v}+\vec{w})\cdot\vec{e}_1$$

for $(z_1,z_2) \in B^4(2)$ and $(\vec{v},\vec{w}) \in (S^2 \times S^2) \setminus \overline{\Delta}$

One easily verifies that the map $(F+G, 1-F) : \mathcal{O} \to \mathbb{R}^2$ is a moment map for a symplectic toric action on the symplectic orbifold $\mathcal{O}$, with image equal to the polytope Δ_{FOOO}. The classification of toric orbifolds from [LT97] therefore implies that $\mathcal{O}$ is equivariantly symplectomorphic to the orbifold $F_2(0)$ from [FOOO12] (as $\mathcal{O}$ and $F_2(0)$ have identical moment polytopes and both have only one singular point, located at the preimage of $(0,1)$ under the moment map); accordingly we hereinafter implicitly identify $F_2(0)$ with $\mathcal{O}$. The manifold $\hat{F}_2(0)$ from [FOOO12] is then constructed by removing a neighborhood $\mathcal{U}$ of the unique singular point $[(0,0)]$ of $\mathcal{O}$ and gluing in its place a neighborhood $\mathcal{N}$ of 0_{S^2} in the cotangent bundle T^*S^2, using a symplectomorphism between $\mathcal{U} \setminus \{[(0,0)]\}$ and $\mathcal{N} \setminus 0_{S^2}$. While a particular choice of this symplectomorphism is not specified in [FOOO12], we have already constructed one that will serve the purpose, namely the map $\Phi_1 : (B^4(2) \setminus \{0\})/\pm 1 \to D_1^* S^2 \setminus 0_{S^2}$ induced on the quotient by the map φ_1 from Lemma 2.4. This gives a symplectomorphism between the manifold $\hat{F}_2(0)$ from [FOOO12] and the manifold

$$\frac{D_1^* S^2 \sqcup \left((S^2 \times S^2) \setminus \overline{\Delta}\right)}{(\vec{p}, \vec{q}) \sim \Phi_2^{-1}(\vec{p}, \vec{q}) \text{ for } (\vec{p}, \vec{q}) \in D_1^* S^2 \setminus 0_{S^2}}.$$

But of course the map Φ_2^{-1} then induces a symplectomorphism between this latter manifold and $S^2 \times S^2$.

There is an obvious continuous map $\Pi : \hat{F}_2(0) \to F_2(0)$ which maps the zero-section 0_{S^2} to the singular point $[(0,0)]$ and coincides with Φ_1^{-1} on $D_1^* S^2 \setminus 0_{S^2} \subset \hat{F}_2(0)$ and with the identity on $(S^2 \times S^2) \setminus \overline{\Delta} \subset \hat{F}_2(0)$; the torus $\Theta_{FOOO} \subset \hat{F}_2(0)$ is the preimage of the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ under the pulled-back moment map $((F+G) \circ \Pi, (1-F) \circ \Pi) : \hat{F}_2(0) \to \mathbb{R}^2$. In view of the expressions for the functions F, G on $(S^2 \times S^2) \setminus \overline{\Delta}$, it follows that Θ_{FOOO} is taken by our symplectomorphism $\hat{F}_2(0) \to S^2 \times S^2$ to

$$\left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \;\middle|\; \frac{1}{2}|\vec{v} + \vec{w}| + \frac{1}{2}(\vec{v} + \vec{w}) \cdot \vec{e}_1 = \frac{1}{2},\, 1 - \frac{1}{2}|\vec{v} + \vec{w}| = \frac{1}{2} \right\},$$

which is easily seen to be equal to the Entov-Polterovich torus

$$T_{EP} = \left\{ (\vec{v}, \vec{w}) \in S^2 \times S^2 \,\middle|\, (\vec{v} + \vec{w}) \cdot \vec{e}_1 = 0,\ \vec{v} \cdot \vec{w} = -\frac{1}{2} \right\}.$$

Thus, for our choice of symplectomorphism $\hat{F}_2(0) \to S^2 \times S^2$, we have $T_{FOOO} = T_{EP}$. □

Proposition 2.6. *There is a symplectomorphism $S^2 \times S^2 \to S^2 \times S^2$ taking T_{CS} to T_{EP}.*

Proof. We recall that T_{CS} is defined as $\psi \times \psi(\Theta_{CS})$, where

$$\psi : \left(B^2(\sqrt{2}), \omega_{\mathbb{C}}\right) \to \left(S^2 \setminus \{-\vec{e}_1\}, \frac{1}{2}\omega_{\text{std}}\right)$$
$$re^{i\theta} \mapsto \begin{pmatrix} 1 - r^2 \\ r\cos\theta\sqrt{2 - r^2} \\ r\sin\theta\sqrt{2 - r^2} \end{pmatrix}$$

is a symplectomorphism (shown by a standard computation) and

$$\Theta_{CS} = \{(e^{it}z, e^{-it}z) \mid z \in \Gamma, t \in [0, 2\pi]\} \subset B^2(\sqrt{2}) \times B^2(\sqrt{2})$$

for a curve $\Gamma \subset \mathbb{H}(\sqrt{2})$ enclosing area $\frac{\pi}{2}$ (the Hamiltonian isotopy class of T_{CS} is easily seen to be independent of the particular choice of Γ). Alternatively, Θ_{CS} is given as the orbit of the curve $\Delta_\Gamma = \{(z, z) \mid z \in \Gamma\}$ under the circle action

$$e^{it} \cdot (z_1, z_2) = (e^{it}z_1, e^{-it}z_2).$$

Another simple computation shows that

$$\psi\left(e^{it}re^{i\theta}\right) = R_t\,\psi(re^{i\theta}),\ \text{where } R_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \in SO(3),$$

from which it follows that

$$T_{CS} = \psi \times \psi(\Theta_{CS}) = \{(R_t\,\psi(z), R_{-t}\,\psi(z)) \mid t \in [0, 2\pi], z \in \Gamma\}.$$

In other words, T_{CS} is the orbit of the curve $\psi \times \psi\,(\Delta_\Gamma)$ under the following circle action, denoted ρ_{CS}, on $S^2 \times S^2$:

$$\rho_{CS}(e^{it}) \cdot (\vec{v}, \vec{w}) = (R_t\,\vec{v}, R_{-t}\,\vec{w})\,.$$

On the other hand, if we consider the smooth embedded curve $C \subset S^2 \times S^2$ parametrized by

$$\begin{aligned} [0, 2\pi] &\to S^2 \times S^2 \\ s &\mapsto \left(\begin{pmatrix} -\frac{\sqrt{3}}{2}\sin(s) \\ -\frac{\sqrt{3}}{2}\cos(s) \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2}\sin(s) \\ \frac{\sqrt{3}}{2}\cos(s) \\ \frac{1}{2} \end{pmatrix} \right), \end{aligned}$$

then we claim that the torus T_{EP} is the orbit of C under the following action of the circle on $S^2 \times S^2$:

$$\rho_{EP}(e^{it}) \cdot (\vec{v}, \vec{w}) = (R_t\,\vec{v}, R_t\,\vec{w})\,.$$

Indeed, T_{EP} is exactly the regular level set $(F_1 \times F_2)^{-1}\left(0, -\frac{1}{4}\right)$ for the $\mathbb{R}^2$-valued function

$F_1 \times F_2$, where $F_1 : (\vec{v}, \vec{w}) \mapsto -\frac{1}{2}(\vec{v} + \vec{w}) \cdot \vec{e}_1$ and $F_2 : (\vec{v}, \vec{w}) \mapsto \frac{1}{2}\vec{v} \cdot \vec{w}$. The Hamiltonian vector fields associated to the functions F_1 and F_2 are $X_{F_1}(\vec{v}, \vec{w}) = (\vec{e}_1 \times \vec{v}, \vec{e}_1 \times \vec{w})$ and $X_{F_2}(\vec{v}, \vec{w}) = (\vec{v} \times \vec{w}, \vec{w} \times \vec{v})$, respectively. We then observe that the curve C is the orbit of the point $\left(\begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}^\top, \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}^\top\right) \in T_{EP}$ under the Hamiltonian flow for F_2, and thus the torus T_{EP} is exactly the orbit of C under the Hamiltonian flow for F_1. Noting that the Hamiltonian flow for F_1 gives the circle action ρ_{EP}, we see that T_{EP} is the orbit of C under the action ρ_{EP} as claimed.

Next, we use the observation of Gadbled in [Ga13] that the actions ρ_{EP} and ρ_{CS} are conjugate in $SO(3) \times SO(3)$. Indeed a simple computation shows that

$$(R_t, R_t) = (\mathcal{P}_1, \mathcal{P}_2)^{-1} (R_t, R_{-t}) (\mathcal{P}_1, \mathcal{P}_2)$$

for $\mathcal{P}_1$ the identity and $\mathcal{P}_2 = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Hence, it follows that

$$(\mathcal{P}_1 \, \mathcal{P}_2) \left(\rho_{EP}(e^{it}) \cdot (\vec{v}, \vec{w})\right) = \rho_{CS}(e^{it}) \cdot ((\mathcal{P}_1, \mathcal{P}_2)(\vec{v}, \vec{w})), \tag{2.3}$$

and we define T'_{EP} to be the orbit of the curve $(\mathcal{P}_1, \mathcal{P}_2)(C)$ under the action of ρ_{CS}. Where $\Gamma' \subset S^2$ is the curve parametrized by $s \mapsto \begin{pmatrix} -\frac{\sqrt{3}}{2}\sin(s) & -\frac{\sqrt{3}}{2}\cos(s) & \frac{1}{2} \end{pmatrix}^\top$, we observe that $(\mathcal{P}_1, \mathcal{P}_2)(C)$ is the curve $\Delta_{\Gamma'} = \{(\vec{v}, \vec{v}) \in S^2 \times S^2 \mid \vec{v} \in \Gamma'\}$ in the diagonal of $S^2 \times S^2$. Where $D^2 \subset \mathbb{C}$ is the closed disk of radius 1, we observe that the disk $D' \subset S^2 \setminus \{-\vec{e}_1\}$ parametrized by

$$\begin{aligned} g : D^2 &\to S^2 \setminus \{-e_1\} \\ re^{i\phi} &\mapsto \begin{pmatrix} \frac{\sqrt{3}}{2} r \sin(\phi) \\ -\frac{\sqrt{3}}{2} r \cos(\phi) \\ \sqrt{1 - \frac{3}{4} r^2} \end{pmatrix} \end{aligned}$$

has boundary Γ', and a routine computation shows that

$$g^*\left(\frac{1}{2}\omega_{\text{std}}\right) = \frac{3r}{4\sqrt{4-3r^2}}\, dr \wedge d\phi.$$

Thus, D' has area

$$\int_{D'} \frac{1}{2}\omega_{\text{std}} = \int_{D^2} g^*\left(\frac{1}{2}\omega_{\text{std}}\right) = \int_0^{2\pi}\int_0^1 \frac{3r}{4\sqrt{4-3r^2}}\, dr\, d\phi = \int_0^{2\pi} \frac{1}{4}\, d\phi = \frac{\pi}{2},$$

which means that Γ' encloses a domain of area $\frac{\pi}{2}$ in $S^2 \setminus \{-\vec{e}_1\}$. It then follows that the curve $\psi^{-1}(\Gamma')$ encloses an area of $\frac{\pi}{2}$ since ψ is a symplectomorphism, and it is not difficult to see that $\psi^{-1}(\Gamma')$ also lies in $\mathbb{H}(\sqrt{2})$ since ψ maps $\mathbb{H}(\sqrt{2})$ to the hemisphere $\{\vec{v} \in S^2 \mid v_3 > 0\}$.

Finally, taking the curve Γ in Chekanov and Schlenk's construction to be the curve $\psi^{-1}(\Gamma')$, the corresponding torus $T_{CS} \subset S^2 \times S^2$ is exactly the orbit of the curve

$$\psi \times \psi\,(\Delta_\Gamma) = \Delta_{\Gamma'} = (\mathcal{P}_1, \mathcal{P}_2)(C)$$

under the action of ρ_{CS}; in other words, $T_{CS} = T'_{EP}$. Now, by (2.3) and the fact that T_{EP} is the orbit of C under the action ρ_{EP}, it is clear that T'_{EP} is nothing more than the image of T_{EP} under the map $(\mathcal{P}_1, \mathcal{P}_2)$, and thus $T_{CS} = T'_{EP} = (\mathcal{P}_1, \mathcal{P}_2)\,(T_{EP})$. Since $(\mathcal{P}_1, \mathcal{P}_2)$ is a symplectomorphism (a Hamiltonian diffeomorphism in fact), the desired result has been obtained. □

Proposition 2.7. *The tori T_{BC} and T_{EP} are equal.*

Proof. We begin by giving a clear construction of the torus T_{BC}, including describing the standard symplectic disk bundle $\mathcal{D}_\tau(P)$ (for an appropriate choice of τ and P to be given below) as well as giving an explicit symplectic embedding $\mathcal{D}_\tau(P) \hookrightarrow S^2 \times S^2$. Observe that the diagonal $\Delta \subset S^2 \times S^2$ is a complex hypersurface that is Poincaré dual to the cohomology class $\left[\frac{1}{2\pi}\Omega\right]$ since the intersection numbers of Δ with $S^2 \times \{\text{point}\}$ and with $\{\text{point}\} \times S^2$

are both 1 while

$$\int_{S^2\times\{\text{point}\}} \frac{1}{2\pi}\Omega = 1 \text{ and } \int_{\{\text{point}\}\times S^2} \frac{1}{2\pi}\Omega = 1.$$

Note that $\left(\Delta, \Omega\big|_\Delta\right)$ is symplectomorphic to $(S^2, \omega_{\text{std}})$ in an obvious way, and we identify them in this manner hereafter. By the main result of [Bi01], there is a symplectic embedding $\mathcal{D}_{2\pi}(P) \hookrightarrow S^2 \times S^2$, where $\pi_P : P \to \Delta$ is a principal S^1-bundle over Δ with Chern class $\left[\frac{1}{2\pi}\Omega\big|_\Delta\right]$. In other words, under the identification of Δ with S^2, the principal S^1-bundle $\pi_P : P \to S^2$ should have Chern class $\left[\frac{1}{2\pi}\omega_{\text{std}}\right]$.

We choose P to be the unit circle bundle in the tangent bundle TS^2, namely

$$P = \left\{ (\vec{p}, \vec{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; |\vec{p}| = |\vec{q}| = 1, \vec{p}\cdot\vec{q} = 0 \right\}$$

with projection $\pi_P : (\vec{p}, \vec{q}) \mapsto \vec{q}$ and circle action given by

$$e^{it} \cdot (\vec{p}, \vec{q}) = \left((\cos(t))\,\vec{p} + (\sin(t))\,\vec{q}\times\vec{p}, \vec{q} \right).$$

Note that

$$T_{(\vec{p},\vec{q})}P = \left\{ \left(\vec{a}, \vec{b}\right) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \vec{a}\cdot\vec{p} = 0, \vec{b}\cdot\vec{q} = 0, \vec{a}\cdot\vec{q} + \vec{p}\cdot\vec{b} = 0 \right\},$$

and observe that the 1-form β defined by $\beta_{(\vec{p},\vec{q})}\left(\vec{a}, \vec{b}\right) = \frac{1}{2\pi}\vec{a}\cdot(\vec{q}\times\vec{p})$ is a connection 1-form for P. A routine computation shows that, for $\left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right) \in T_{(\vec{p},\vec{q})}P$, one has

$$d\beta_{(\vec{p},\vec{q})}\left(\left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right)\right) = \frac{1}{2\pi}\left(2\vec{q}\cdot(\vec{a}_1\times\vec{a}_2) - \vec{p}\cdot\left(\vec{b}_1\times\vec{a}_2 + \vec{a}_1\times\vec{b}_2\right)\right)$$

and that $d\beta = -\frac{1}{2\pi}\pi_P^*(\omega_{\text{std}})$ as required. Then, where $D_{\sqrt{2}}S^2$ is the radius-$\sqrt{2}$ disk bundle

$$D_{\sqrt{2}}S^2 = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; |\vec{y}| = 1, |\vec{x}| < \sqrt{2}, \vec{x}\cdot\vec{y} = 0 \right\}$$

in TS^2, we note that the map $\Psi : \mathcal{D}_{2\pi}(P) \to D_{\sqrt{2}}S^2$ defined by

$$\Psi\left(\left[(\vec{p},\vec{q}), re^{i\theta}\right]\right) = \left((r\cos\theta)\,\vec{p} + (r\sin\theta)\,\vec{q}\times\vec{p}, \vec{q}\right)$$

is a diffeomorphism mapping the "zero-section" $\{[(\vec{p},\vec{q}),0] \mid (\vec{p},\vec{q}) \in P\}$ of $\mathcal{D}_{2\pi}(P)$ to the zero-section of $D_{\sqrt{2}}S^2$. Recall that the symplectic form ω_{can} on $\mathcal{D}_{2\pi}(P)$ satisfies

$$\mathfrak{q}^*\omega_{\mathrm{can}} = d\left(\left(\pi|z|^2 - 2\pi\right)\mathrm{pr}_P^*\,\beta\right) + \mathrm{pr}_D^*\,\omega_{\mathbb{C}} = d\left(\left(\pi r^2 - 2\pi\right)\mathrm{pr}_P^*\,\beta + \left(\frac{r^2}{2} - 1\right)d\theta\right),$$

where $\mathfrak{q} : P \times D\left(\sqrt{2}\right) \to \mathcal{D}_{2\pi}(P)$ is the quotient map, $\omega_{\mathbb{C}}$ is the standard symplectic form on $\mathbb{C}$, and $\mathrm{pr}_P : P \times D\left(\sqrt{2}\right) \to P$ and $\mathrm{pr}_D : P \times D\left(\sqrt{2}\right) \to D\left(\sqrt{2}\right)$ are the obvious projections. Where $\eta \in \Omega^1\left(D_{\sqrt{2}}S^2 \setminus 0_{S^2}\right)$ is given by

$$\eta_{(\vec{x},\vec{y})}\left(\vec{a},\vec{b}\right) = \left(\frac{1}{2} - \frac{1}{|\vec{x}|^2}\right)\vec{a}\cdot\left(\vec{y}\times\vec{x}\right),$$

we claim that $d\left(\Psi^*\eta\right) = \omega_{\mathrm{can}}$ away from the "zero-section," a fact that can be verified by showing that

$$\mathfrak{q}^*\Psi^*\eta = \left(\pi r^2 - 2\pi\right)\mathrm{pr}_P^*\,\beta + \left(\frac{r^2}{2} - 1\right)d\theta,$$

which in turn follows from a routine computation. Thus, we may symplectically identify $\mathcal{D}_{2\pi}(P)$ with $D_{\sqrt{2}}S^2$, where the symplectic form on $D_{\sqrt{2}}S^2$ is given by $d\eta$ away from 0_{S^2}.

Now, where $\overline{\Delta} \subset S^2 \times S^2$ is the anti-diagonal, we define $\Phi_3 : D_{\sqrt{2}}S^2 \to \left(S^2 \times S^2\right) \setminus \overline{\Delta}$ by

$$\Phi_3\left(\vec{x},\vec{y}\right) = \left(\left(1 - \frac{|\vec{x}|^2}{2}\right)\vec{y} + \sqrt{1 - \frac{|\vec{x}|^2}{4}}\,\vec{x}, \left(1 - \frac{|\vec{x}|^2}{2}\right)\vec{y} - \sqrt{1 - \frac{|\vec{x}|^2}{4}}\,\vec{x}\right).$$

It is not difficult to check (using the identity $|\vec{v} - \vec{w}|^2 = \left(2 - |\vec{v} + \vec{w}|\right)\left(2 + |\vec{v} + \vec{w}|\right)$ for

$(\vec{v}, \vec{w}) \in S^2 \times S^2$) that the smooth map $(S^2 \times S^2) \setminus \overline{\Delta} \to D_{\sqrt{2}}S^2$ defined by

$$(\vec{v}, \vec{w}) \mapsto \left(\frac{\vec{v} - \vec{w}}{\sqrt{2 + |\vec{v} + \vec{w}|}}, \frac{\vec{v} + \vec{w}}{|\vec{v} + \vec{w}|} \right)$$

is an inverse to Φ_3, and so Φ_3 is a diffeomorphism. Note that Φ_3 maps the zero-section of $D_{\sqrt{2}}S^2$ to the diagonal $\Delta \subset S^2 \times S^2$. Where $\Phi_2 : (S^2 \times S^2) \setminus \Delta \to D_1^* S^2$ is the symplectomorphism defined in Lemma 2.5, we consider the composition $\Phi_2 \circ \Phi_3$ (restricted to the complement of the zero-section of $D_{\sqrt{2}}S^2$), which (as one can easily compute) is given by

$$\Phi_2 \circ \Phi_3 (\vec{x}, \vec{y}) = \left(\left(\frac{|\vec{x}|^2}{2} - 1 \right) \left(\vec{y} \times \frac{\vec{x}}{|\vec{x}|} \right), \frac{\vec{x}}{|\vec{x}|} \right).$$

Moreover, where λ is the canonical 1-form on T^*S^2, one computes that $(\Phi_2 \circ \Phi_3)^* \lambda = \eta$, which implies that

$$(\Phi_2 \circ \Phi_3)^* \, d\lambda = d\eta$$

on the complement of the zero-section of $D_{\sqrt{2}}S^2$. Since Φ_2 is a symplectomorphism, this implies that Φ_3 is a symplectomorphism on the complement of the zero-section, and hence globally by continuity.

To complete the construction of T_{BC}, we consider the monotone Lagrangian "equatorial" circle $\Lambda = \{\vec{v} \in S^2 \,|\, \vec{v} \cdot \vec{e}_1 = 0\}$ in $S^2 \cong \Delta$, and the Biran circle bundle construction associated to Λ is the image under Φ_3 of the unit circle bundle over Λ. (The radius 1 is the radius necessary to guarantee monotonicity according to Proposition 6.4.1 of [BC09] after adjusting for differences in normalization). More explicitly, we consider the circle bundle

$$\Lambda_{(1)} = \left\{ (\vec{x}, \vec{y}) \in D_{\sqrt{2}}S^2 \,\middle|\, |\vec{x}| = 1, \vec{y} \cdot \vec{e}_1 = 0 \right\} \subset D_{\sqrt{2}}S^2,$$

and T_{BC} is defined to be $\Phi_3 \left(\Lambda_{(1)} \right)$. Writing $(\vec{v}, \vec{w}) = \Phi_3 (\vec{x}, \vec{y})$ for $(\vec{x}, \vec{y}) \in D_{\sqrt{2}}S^2$, we observe that

$$(\vec{v} + \vec{w}) \cdot \vec{e}_1 = \left(2 - |\vec{x}|^2\right) \vec{y} \cdot \vec{e}_1$$

and

$$\vec{v} \cdot \vec{w} = \left(1 - \frac{|\vec{x}|^2}{2}\right)^2 - \left(1 - \frac{|\vec{x}|^2}{4}\right) |\vec{x}|^2 = 1 - 2|\vec{x}|^2 + \frac{|\vec{x}|^4}{2}.$$

Since $|\vec{x}| < \sqrt{2}$, it follows that $(\vec{v}, \vec{w}) \in T_{EP}$ if and only if $\vec{y} \cdot \vec{e}_1 = 0$ and $|\vec{x}| = 1$, conditions which are equivalent to $(\vec{x}, \vec{y}) \in \Lambda_{(1)}$. Hence, $T_{EP} = \Phi_3\left(\Lambda_{(1)}\right) = T_{BC}$ as claimed. □

Chapter 3

A Lagrangian $\mathbb{R}P^3$ in $\left(S^2\right)^3$

The tori listed in Theorem 2.1 are of interest not only because they are distinct from the Clifford torus $S^1_{eq} \times S^1_{eq} \subset S^2 \times S^2$ (as shown in [EP09]) but also because they are nondisplaceable under Hamiltonian diffeomorphisms (as shown in [FOOO12] and [CS10]). As noted in Example 1.23, the torus $T_{EP} \subset S^2 \times S^2$ defined in (2.1) can be viewed as a regular fiber of the $\mathbb{R}^2$-valued function $G_1 \times G_2 : S^2 \times S^2 \to \mathbb{R}^2$, where G_1 and G_2 are defined by

$$G_1\left(\vec{v}, \vec{w}\right) = \left|\vec{v} + \vec{w}\right|^2 \text{ and } G_2\left(\vec{v}, \vec{w}\right) = \left(\vec{v} + \vec{w}\right) \cdot \vec{e}_1.$$

We also noted in Example 1.23 that the anti-diagonal

$$\overline{\Delta} = \left(G_1 \times G_2\right)^{-1}(0,0) = \left\{ \left(\vec{v}, \vec{w}\right) \in S^2 \times S^2 \,\middle|\, \vec{v} = -\vec{w} \right\}$$

is a monotone nondisplaceable Lagrangian submanifold.

For the remainder of this work, we will investigate a particular Lagrangian submanifold $L \subset \left(S^2\right)^3$ that is analogous to $\overline{\Delta} \subset S^2 \times S^2$ (as noted in the discussion following Example 1.24). Writing elements of $\left(S^2\right)^3$ as 3×3 matrices $\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ for $\vec{u}, \vec{v}, \vec{w} \in S^2$, we will consider

$$L = \left\{ \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in \left(S^2\right)^3 \,\middle|\, \vec{u} + \vec{v} + \vec{w} = \vec{0} \right\}.$$

Where ω_{std} is the standard symplectic form on S^2 and the map $\text{pr}_j : \left(S^2\right)^3 \to S^2$ is the (holomorphic) projection onto the j^{th} factor of $\left(S^2\right)^3$ for $j = 1, 2, 3$, we consider $\left(S^2\right)^3$ with the split symplectic form $\Omega = \text{pr}_1^* \omega_{\text{std}} + \text{pr}_2^* \omega_{\text{std}} + \text{pr}_3^* \omega_{\text{std}}$. In this chapter, we will show that L is a Lagrangian submanifold that is diffeomorphic to $\mathbb{R}P^3$, and we will describe the relative homotopy group $\pi_2 \left(\left(S^2\right)^3, L \right)$.

Consider the Lie group $SU(2)$, which we think of as

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Identifying $SU(2)$ with the group of unit quaternions

$$\mathcal{S} = \left\{ \xi = \alpha + \beta j \,\middle|\, \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1 \right\},$$

we observe that each of $\xi i \xi^*$, $\xi j \xi^*$, and $\xi k \xi^*$ is a pure imaginary quaternion and may therefore be thought of as a column vector in $\mathbb{R}^3$. More explicitly, for $\xi = \alpha + \beta j$, one easily computes that

$$\begin{aligned} \xi i \xi^* &= \left(|\alpha|^2 - |\beta|^2\right) i - i \left(\alpha\beta - \bar{\alpha}\bar{\beta}\right) j - \left(\alpha\beta + \bar{\alpha}\bar{\beta}\right) k \\ &= \begin{pmatrix} |\alpha|^2 - |\beta|^2 \\ -i \left(\alpha\beta - \bar{\alpha}\bar{\beta}\right) \\ -\left(\alpha\beta + \bar{\alpha}\bar{\beta}\right) \end{pmatrix} \end{aligned}$$

and also that

$$\begin{aligned}
\xi j \xi^* &= i\left(\bar{\alpha}\beta - \alpha\bar{\beta}\right) i + \frac{1}{2}\left(\alpha^2 + \beta^2 + \bar{\alpha}^2 + \bar{\beta}^2\right) j - \frac{i}{2}\left(\alpha^2 + \beta^2 - \bar{\alpha}^2 - \bar{\beta}^2\right) k \\
&= \begin{pmatrix} i\left(\bar{\alpha}\beta - \alpha\bar{\beta}\right) \\ \frac{1}{2}\left(\alpha^2 + \beta^2 + \bar{\alpha}^2 + \bar{\beta}^2\right) \\ -\frac{i}{2}\left(\alpha^2 + \beta^2 - \bar{\alpha}^2 - \bar{\beta}^2\right) \end{pmatrix}, \\
\xi k \xi^* &= \left(\alpha\bar{\beta} + \bar{\alpha}\beta\right) i + \frac{i}{2}\left(\alpha^2 - \beta^2 - \bar{\alpha}^2 + \bar{\beta}^2\right) j + \frac{1}{2}\left(\alpha^2 - \beta^2 + \bar{\alpha}^2 - \bar{\beta}^2\right) k \\
&= \begin{pmatrix} \alpha\bar{\beta} + \bar{\alpha}\beta \\ \frac{i}{2}\left(\alpha^2 - \beta^2 - \bar{\alpha}^2 + \bar{\beta}^2\right) \\ \frac{1}{2}\left(\alpha^2 - \beta^2 + \bar{\alpha}^2 - \bar{\beta}^2\right) \end{pmatrix}.
\end{aligned}$$

Then, we consider the map $\Phi : SU(2) \to SO(3)$ defined by

$$\Phi(\xi) = \begin{pmatrix} \xi i \xi^* & \xi j \xi^* & \xi k \xi^* \end{pmatrix}, \tag{3.1}$$

which is well known (see Exercise 9-10 of [Lee03] for example) to be a surjective Lie group homomorphism with kernel

$$\ker \Phi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

(or $\ker \Phi = \{\pm 1\}$ if we think of Φ as having domain $\mathcal{S}$).

Next, we consider a group action of $SU(2)$ on S^2 defined by

$$\xi \cdot \vec{u} = \Phi(\xi)\vec{u} \tag{3.2}$$

for $\xi \in SU(2)$ and $\vec{u} \in S^2$. Since Φ is a homomorphism, it is easy to see that (3.2) gives a legitimate group action:

$$(\xi_1\xi_2)\cdot\vec{u} = \Phi(\xi_1\xi_2)\,\vec{u} = \Phi(\xi_1)\,\Phi(\xi_2)\,\vec{u} = \xi_1\cdot(\xi_2\cdot\vec{u}).$$

Proposition 3.1. *Using the action given by* (3.2)*, the group* $SU(2)$ *acts on* S^2 *by holomorphic automorphisms.*

Proof. First, we identify $\mathbb{C}P^1$ with S^2 by the map

$$\begin{aligned}\varphi &: \mathbb{C}P^1 \to S^2\\ [w:z] &\mapsto \begin{pmatrix} \frac{|w|^2-|z|^2}{|w|^2+|z|^2} & i\frac{w\bar{z}-\bar{w}z}{|w|^2+|z|^2} & \frac{w\bar{z}+\bar{w}z}{|w|^2+|z|^2}\end{pmatrix}^\top,\end{aligned}$$

which is holomorphic with respect to the standard complex structures on $\mathbb{C}P^1$ and S^2. Indeed, when working in the usual affine charts for $\mathbb{C}P^1$, the map φ is exactly the inverse of the usual (orientation preserving) stereographic projections from $-\vec{e}_1$ and $\vec{e}_1$:

$$\begin{aligned}\varphi([w:1]) &= \begin{pmatrix} \frac{|w|^2-1}{|w|^2+1} & i\frac{w-\bar{w}}{|w|^2+1} & \frac{w+\bar{w}}{|w|^2+1}\end{pmatrix}^\top = \begin{pmatrix} \frac{|w|^2-1}{|w|^2+1} & \frac{2\operatorname{Im}(w)}{|w|^2+1} & \frac{2\operatorname{Re}(w)}{|w|^2+1}\end{pmatrix}^\top,\\ \varphi([1:z]) &= \begin{pmatrix} \frac{1-|z|^2}{1+|z|^2} & i\frac{\bar{z}-z}{1+|z|^2} & \frac{\bar{z}+z}{1+|z|^2}\end{pmatrix}^\top = \begin{pmatrix} \frac{1-|z|^2}{1+|z|^2} & \frac{-2\operatorname{Im}(z)}{1+|z|^2} & \frac{2\operatorname{Re}(z)}{1+|z|^2}\end{pmatrix}^\top.\end{aligned}$$

Next, we recall that $SU(2)$ acts on $\mathbb{C}P^1$ by holomorphic automorphisms in an obvious way by thinking of $[w:z]$ as a column vector; writing $\xi = \alpha + \beta j$ and noting that

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\begin{pmatrix} w \\ z\end{pmatrix} = \begin{pmatrix} \alpha w + \beta z \\ -\bar{\beta} w + \bar{\alpha} z\end{pmatrix},$$

we set

$$\xi\cdot[w:z] = [\alpha w + \beta z : -\bar{\beta} w + \bar{\alpha} z].$$

Then, a routine computation shows that

$$\varphi\left(\xi \cdot [w : z]\right) = \xi \cdot \varphi\left([w : z]\right),$$

which completes the proof of the proposition.

□

Corollary 3.2. *The Lie group $SU(2)$ acts on $\left(S^2\right)^3$ by holomorphic symplectomorphisms.*

Proof. Using the action given by (3.2), we let $SU(2)$ act diagonally on $\left(S^2\right)^3$:

$$\xi \cdot \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \Phi(\xi) \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \tag{3.3}$$

for $\vec{u}$, $\vec{v}$, $\vec{w} \in S^2$. Since the standard complex structure on $\left(S^2\right)^3$ is the split structure and $SU(2)$ acts on each factor holomorphically, the result follows immediately from Proposition 3.1 along with the fact that Φ has image $SO(3)$, which acts on $\left(S^2\right)^3$ by symplectomorphisms.

□

Recall that the Lie algebra $\mathfrak{su}(2)$ consists of 2×2 skew-Hermitian matrices, a basis for which is given by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Using this basis, we identify $\mathbb{R}^3$ with $\mathfrak{su}(2)$ by the prescription

$$\vec{\zeta} = \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix}^\top \mapsto \zeta = \zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3\sigma_3. \tag{3.4}$$

Lemma 3.3. *Given nonzero $\zeta \in \mathfrak{su}(2)$, the matrix $\Phi\left(\exp\left(t\zeta\right)\right) \in SO(3)$ acts by right-handed rotation about the axis $\vec{\zeta}/|\vec{\zeta}|$ through angle $2t|\vec{\zeta}|$.*

Proof. First, we observe that

$$\exp\left(t|\vec{\zeta}|\sigma_1\right) = \begin{pmatrix} e^{i|\vec{\zeta}|t} & 0 \\ 0 & e^{-i|\vec{\zeta}|t} \end{pmatrix},$$

and hence

$$\Phi\left(\exp\left(t|\vec{\zeta}|\sigma_1\right)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(2|\vec{\zeta}|t\right) & -\sin\left(2|\vec{\zeta}|t\right) \\ 0 & \sin\left(2|\vec{\zeta}|t\right) & \cos\left(2|\vec{\zeta}|t\right) \end{pmatrix},$$

which acts by right-handed rotation about the axis $\vec{e}_1$ through angle $2t|\vec{\zeta}|$. Since $SO(3)$ acts transitively on S^2, we may choose $A \in SO(3)$ such that $A\left(\vec{\zeta}/|\vec{\zeta}|\right) = \vec{e}_1$; then, since $\Phi : SU(2) \to SO(3)$ is surjective, we may choose $\xi \in SU(2)$ such that $\Phi(\xi) = A$. Writing

$$\xi = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

the fact that $(\Phi(\xi))^\top \vec{e}_1 = (\Phi(\xi))^{-1} \vec{e}_1 = A^{-1}\vec{e}_1 = \vec{\zeta}/|\vec{\zeta}|$ implies the equations

$$\frac{\zeta_1}{|\vec{\zeta}|} = |\alpha|^2 - |\beta|^2,\ \frac{\zeta_2}{|\vec{\zeta}|} = i\left(\bar{\alpha}\beta - \alpha\bar{\beta}\right),\ \frac{\zeta_3}{|\vec{\zeta}|} = \alpha\bar{\beta} + \bar{\alpha}\beta.$$

Then, observe that

$$\xi^{-1}\sigma_1\xi = \begin{pmatrix} i\left(|\alpha|^2 - |\beta|^2\right) & 2i\bar{\alpha}\beta \\ 2ia\bar{\beta} & -i\left(|\alpha|^2 - |\beta|^2\right) \end{pmatrix} = \begin{pmatrix} i\frac{\zeta_1}{|\vec{\zeta}|} & \frac{\zeta_2}{|\vec{\zeta}|} + i\frac{\zeta_3}{|\vec{\zeta}|} \\ -\frac{\zeta_2}{|\vec{\zeta}|} + i\frac{\zeta_2}{|\vec{\zeta}|} & -i\frac{\zeta_1}{|\vec{\zeta}|} \end{pmatrix} = \frac{1}{|\vec{\zeta}|}\zeta,$$

from which it follows that

$$\xi^{-1}\left(t|\vec{\zeta}|\sigma_1\right)\xi = t|\vec{\zeta}|\xi^{-1}\sigma_1\xi = \frac{t|\vec{\zeta}|}{|\vec{\zeta}|}\zeta = t\zeta.$$

Then, using the fact that $\exp\left(\xi^{-1}\left(t|\vec{\zeta}|\sigma_1\right)\xi\right) = \xi^{-1}\exp\left(t|\vec{\zeta}|\sigma_1\right)\xi$, we compute that

$$
\begin{aligned}
\Phi\left(\exp\left(t\zeta\right)\right) &= \Phi\left(\exp\left(\xi^{-1}\left(t|\vec{\zeta}|\sigma_1\right)\xi\right)\right) \\
&= \Phi\left(\xi^{-1}\exp\left(t|\vec{\zeta}|\sigma_1\right)\xi\right) \\
&= \Phi\left(\xi^{-1}\right)\Phi\left(\exp\left(t|\vec{\zeta}|\sigma_1\right)\right)\Phi\left(\xi\right) \\
&= A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(2|\vec{\zeta}|t\right) & -\sin\left(2|\vec{\zeta}|t\right) \\ 0 & \sin\left(2|\vec{\zeta}|t\right) & \cos\left(2|\vec{\zeta}|t\right) \end{pmatrix} A,
\end{aligned}
$$

which acts by right-handed rotation about the axis $\vec{\zeta}/|\vec{\zeta}|$ through angle $2t|\vec{\zeta}|$.

□

Recall also that $\mathfrak{su}(2)$ has an inner product given by $\langle \eta, \zeta \rangle = \frac{1}{2}\,\mathrm{trace}\left(\bar{\eta}^\top \zeta\right)$, which identifies $\mathfrak{su}(2)^*$ with $\mathfrak{su}(2)$. With this convention and our previous identification of $\mathbb{R}^3$ with $\mathfrak{su}(2)$, we have the following proposition:

Proposition 3.4. *The action of* $SU(2)$ *on* $\left(S^2\right)^3$ *given by* (3.3) *is Hamiltonian with moment map given by*

$$
\begin{aligned}
\mathfrak{mom} : \left(S^2\right)^3 &\to \mathfrak{su}(2)^* \\
\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} &\mapsto -2\left(\vec{u} + \vec{v} + \vec{w}\right).
\end{aligned}
$$

Proof. Given $\zeta \in \mathfrak{su}(2)$, the infinitesimal action determines a vector field X_ζ on $\left(S^2\right)^3$ defined by

$$
X_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \left.\frac{d}{dt}\right|_{t=0} \Phi\left(\exp\left(t\zeta\right)\right)\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}.
$$

By Lemma 3.3, $\Phi\left(\exp\left(t\zeta\right)\right) \in SO(3)$ acts by right-handed rotation about the axis $\vec{\zeta}/|\vec{\zeta}|$

through angle $2t|\vec{\zeta}|$ (or by the identity if $\vec{\zeta} = \vec{0}$), and so it follows quickly that

$$X_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = 2 \begin{pmatrix} \vec{\zeta} \times \vec{u} & \vec{\zeta} \times \vec{v} & \vec{\zeta} \times \vec{w} \end{pmatrix}.$$

On the other hand, if we define $H_\zeta : (S^2)^3 \to \mathbb{R}$ by

$$H_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \left\langle \mathfrak{mom} \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}, \zeta \right\rangle,$$

then it is easy to compute that $H_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = -2\left(\vec{u} + \vec{v} + \vec{w}\right) \cdot \vec{\zeta}$. Then, given a tangent vector $\begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \end{pmatrix} \in T_{\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}} \left(S^2\right)^3$, we observe that

$$dH_\zeta \begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \end{pmatrix} = -2\left(\vec{x} + \vec{y} + \vec{z}\right) \cdot \vec{\zeta}$$

while

$$\begin{aligned} \Omega\left(\begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \end{pmatrix}, X_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}\right) &= 2\Omega\left(\begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \end{pmatrix}, \begin{pmatrix} \vec{\zeta} \times \vec{u} & \vec{\zeta} \times \vec{v} & \vec{\zeta} \times \vec{w} \end{pmatrix}\right) \\ &= 2\left(\vec{u} \cdot \left(\vec{x} \times \left(\vec{\zeta} \times \vec{u}\right)\right) + \vec{v} \cdot \left(\vec{y} \times \left(\vec{\zeta} \times \vec{v}\right)\right) + \vec{w} \cdot \left(\vec{z} \times \left(\vec{\zeta} \times \vec{w}\right)\right)\right) \\ &= 2\left(-\vec{x} \cdot \vec{\zeta} - \vec{y} \cdot \vec{\zeta} - \vec{z} \cdot \vec{\zeta}\right) = -2\left(\vec{x} + \vec{y} + \vec{z}\right) \cdot \vec{\zeta}, \end{aligned}$$

which shows that the Hamiltonian vector field X_{H_ζ} is exactly X_ζ.

Finally, given $\zeta, \eta \in \mathfrak{su}(2)$, we will show that $H_{[\zeta,\eta]} = \{H_\zeta, H_\eta\}$ and thus that the map $\zeta \mapsto H_\zeta$ is a Lie algebra homomorphism as required. A routine computation reveals that

$$[\zeta, \eta] = \zeta\eta - \eta\zeta = 2\left(\vec{\zeta} \times \vec{\eta}\right),$$

and so

$$H_{[\zeta,\eta]}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} = -4\left(\vec{u}+\vec{v}+\vec{w}\right)\cdot\left(\vec{\zeta}\times\vec{\eta}\right).$$

On the other hand, we have

$$\begin{aligned}\{H_\zeta, H_\eta\}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= dH_\zeta\left(X_{H_\eta}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}\right) = \Omega\left(X_{H_\eta}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}, X_{H_\zeta}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}\right)\\
&= \Omega\left(2\begin{pmatrix}\vec{\eta}\times\vec{u} & \vec{\eta}\times\vec{v} & \vec{\eta}\times\vec{w}\end{pmatrix}, 2\begin{pmatrix}\vec{\zeta}\times\vec{u} & \vec{\zeta}\times\vec{v} & \vec{\zeta}\times\vec{w}\end{pmatrix}\right)\\
&= 4\left(\vec{u}\cdot\left((\vec{\eta}\times\vec{u})\times\left(\vec{\zeta}\times\vec{u}\right)\right)+\vec{v}\cdot\left((\vec{\eta}\times\vec{v})\times\left(\vec{\zeta}\times\vec{v}\right)\right)\right.\\
&\qquad\left.+\vec{w}\cdot\left((\vec{\eta}\times\vec{w})\times\left(\vec{\zeta}\times\vec{w}\right)\right)\right)\\
&= 4\left(-\left(\vec{u}\cdot\left(\vec{\zeta}\times\vec{\eta}\right)\right)-\left(\vec{v}\cdot\left(\vec{\zeta}\times\vec{\eta}\right)\right)-\left(\vec{w}\cdot\left(\vec{\zeta}\times\vec{\eta}\right)\right)\right)\\
&= -4\left(\vec{u}+\vec{v}+\vec{w}\right)\cdot\left(\vec{\zeta}\times\vec{\eta}\right),\end{aligned}$$

which shows that $H_{[\zeta,\eta]} = \{H_\zeta, H_\eta\}$ and completes the proof of the proposition. □

Proposition 3.5. *Where* $\mathfrak{mom} : \left(S^2\right)^3 \to \mathfrak{su}(2)^*$ *is as in Proposition 3.4, the subset*

$$L = \mathfrak{mom}^{-1}\left(\vec{0}\right) = \left\{\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}\in\left(S^2\right)^3 \;\middle|\; \vec{u}+\vec{v}+\vec{w}=\vec{0}\right\}$$

is a Lagrangian submanifold of $\left(S^2\right)^3$ *that is diffeomorphic to* $\mathbb{R}P^3$*.*

Proof. First, we exhibit an explicit diffeomorphism $\Upsilon : L \to SO(3)$, and we recall that $SO(3)$ is well known to be diffeomorphic to $\mathbb{R}P^3$. Set

$$\Upsilon\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\left(\vec{u}\times\vec{v}+\sqrt{\frac{3}{2}}\vec{u}\right) & \frac{2}{3}\left(\vec{v}\times\vec{w}+\sqrt{\frac{3}{2}}\vec{v}\right) & \frac{2}{3}\left(\vec{w}\times\vec{u}+\sqrt{\frac{3}{2}}\vec{w}\right)\end{pmatrix}. \tag{3.5}$$

If $\vec{u}+\vec{v}+\vec{w}=\vec{0}$, then it follows quickly that

$$3+2\vec{u}\cdot\vec{v}+2\vec{u}\cdot\vec{w}+2\vec{v}\cdot\vec{w} = \left|\vec{u}+\vec{v}+\vec{w}\right|^2 = 0,$$

which in turn yields

$$1 + 2\vec{v} \cdot \vec{w} = 3 - 2 + 2\vec{v} \cdot \vec{w} = 3 + 2\vec{u} \cdot (-\vec{u}) + 2\vec{v} \cdot \vec{w} = 3 + 2\vec{u} \cdot (\vec{v} + \vec{w}) + 2\vec{v} \cdot \vec{w} = 0.$$

Thus, we have $\vec{v} \cdot \vec{w} = -1/2$, and a similar argument shows that $\vec{u} \cdot \vec{w} = \vec{u} \cdot \vec{v} = -1/2$ as well. It then follows that

$$\begin{aligned} |\vec{u} \times \vec{v}|^2 &= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 = 1 - \frac{1}{4} = \frac{3}{4}, \\ |\vec{v} \times \vec{w}|^2 &= |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 = 1 - \frac{1}{4} = \frac{3}{4}, \\ |\vec{w} \times \vec{u}|^2 &= |\vec{w}|^2 |\vec{u}|^2 - (\vec{w} \cdot \vec{u})^2 = 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned} \tag{3.6}$$

Moreover, assuming that $\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in L$, we obtain the equations

$$\begin{aligned} \vec{u} \times \vec{v} &= \vec{v} \times (-\vec{u}) = \vec{v} \times (\vec{v} + \vec{w}) = \vec{v} \times \vec{w}, \\ \vec{v} \times \vec{w} &= \vec{w} \times (-\vec{v}) = \vec{w} \times (\vec{u} + \vec{w}) = \vec{w} \times \vec{u}, \\ \vec{w} \times \vec{u} &= \vec{u} \times (-\vec{w}) = \vec{u} \times (\vec{u} + \vec{v}) = \vec{u} \times \vec{v}. \end{aligned} \tag{3.7}$$

Then, combining (3.6) and (3.7), a quick computation reveals that

$$\left(\Upsilon\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}\right)^{\top} \left(\Upsilon\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using again the equations (3.6) and (3.7), one can also compute that

$$\begin{aligned}
&\frac{2}{3}\left(\vec{v}\times\vec{w}+\sqrt{\frac{3}{2}}\vec{v}\right)\times\frac{2}{3}\left(\vec{w}\times\vec{u}+\sqrt{\frac{3}{2}}\vec{w}\right)\\
&\qquad=\frac{4}{9}\left(\sqrt{\frac{3}{2}}\left(\vec{v}\times\vec{w}\right)\times\vec{w}+\sqrt{\frac{3}{2}}\vec{v}\times\left(\vec{w}\times\vec{u}\right)+\frac{3}{2}\vec{v}\times\vec{w}\right)\\
&\qquad=\frac{4}{9}\left(\sqrt{\frac{3}{2}}\left(-\vec{v}-\frac{1}{2}\vec{w}\right)+\sqrt{\frac{3}{2}}\left(-\vec{w}-\frac{1}{2}\vec{v}\right)+\frac{3}{2}\vec{v}\times\vec{w}\right),
\end{aligned}$$

and thus we see (again using (3.6) and (3.7)) that

$$\begin{aligned}
\det\left(\Upsilon\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}\right)&=\frac{2}{3}\left(\vec{u}\times\vec{v}+\sqrt{\frac{3}{2}}\vec{u}\right)\cdot\left(\frac{2}{3}\left(\vec{v}\times\vec{w}+\sqrt{\frac{3}{2}}\vec{v}\right)\times\frac{2}{3}\left(\vec{w}\times\vec{u}+\sqrt{\frac{3}{2}}\vec{w}\right)\right)\\
&=\frac{2}{3}\left(\vec{u}\times\vec{v}+\sqrt{\frac{3}{2}}\vec{u}\right)\cdot\frac{4}{9}\left(\sqrt{\frac{3}{2}}\left(-\vec{v}-\frac{1}{2}\vec{w}\right)+\sqrt{\frac{3}{2}}\left(-\vec{w}-\frac{1}{2}\vec{v}\right)+\frac{3}{2}\vec{v}\times\vec{w}\right)\\
&=\frac{8}{27}\left(\frac{3}{2}\left|\vec{u}\times\vec{v}\right|^2+\frac{3}{2}\left(\frac{1}{2}+\frac{1}{4}\right)+\frac{3}{2}\left(\frac{1}{2}+\frac{1}{4}\right)\right)=1.
\end{aligned}$$

Thus, we see that $SO(3)$ is an appropriate codomain for Υ. To verify that Υ is a diffeomorphism, we will show that its inverse $\Upsilon^{-1}: SO(3)\to L$ is given by

$$\Upsilon^{-1}\begin{pmatrix}\vec{a} & \vec{b} & \vec{c}\end{pmatrix}=\begin{pmatrix}\sqrt{\tfrac{3}{2}}\left(\tfrac{2}{3}\vec{a}-\tfrac{1}{3}\vec{b}-\tfrac{1}{3}\vec{c}\right) & \sqrt{\tfrac{3}{2}}\left(\tfrac{2}{3}\vec{b}-\tfrac{1}{3}\vec{a}-\tfrac{1}{3}\vec{c}\right) & \sqrt{\tfrac{3}{2}}\left(\tfrac{2}{3}\vec{c}-\tfrac{1}{3}\vec{a}-\tfrac{1}{3}\vec{b}\right)\end{pmatrix}.$$

Given $\begin{pmatrix}\vec{a} & \vec{b} & \vec{c}\end{pmatrix}\in SO(3)$, it is easy to compute that

$$\begin{aligned}
\left|\sqrt{\frac{3}{2}}\left(\frac{2}{3}\vec{a}-\frac{1}{3}\vec{b}-\frac{1}{3}\vec{c}\right)\right|^2&=1,\\
\left|\sqrt{\frac{3}{2}}\left(\frac{2}{3}\vec{b}-\frac{1}{3}\vec{a}-\frac{1}{3}\vec{c}\right)\right|^2&=1,\\
\left|\sqrt{\frac{3}{2}}\left(\frac{2}{3}\vec{c}-\frac{1}{3}\vec{a}-\frac{1}{3}\vec{b}\right)\right|^2&=1,
\end{aligned}$$

and it is also plain to see that

$$\sqrt{\frac{3}{2}}\left(\frac{2}{3}\vec{a}-\frac{1}{3}\vec{b}-\frac{1}{3}\vec{c}\right)+\sqrt{\frac{3}{2}}\left(\frac{2}{3}\vec{b}-\frac{1}{3}\vec{a}-\frac{1}{3}\vec{c}\right)+\sqrt{\frac{3}{2}}\left(\frac{2}{3}\vec{c}-\frac{1}{3}\vec{a}-\frac{1}{3}\vec{b}\right)=\vec{0}.$$

Hence, L is an appropriate codomain for Υ^{-1}. The computation that

$$\Upsilon^{-1}\circ\Upsilon\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}=\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}$$

is straightforward using (3.7) and the fact that $\vec{u}+\vec{v}+\vec{w}=\vec{0}$. Noting that $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, and $\vec{c}\times\vec{a}=\vec{b}$ for $\begin{pmatrix}\vec{a} & \vec{b} & \vec{c}\end{pmatrix}\in SO(3)$, it is also straightforward to compute that

$$\Upsilon\circ\Upsilon^{-1}\begin{pmatrix}\vec{a} & \vec{b} & \vec{c}\end{pmatrix}=\begin{pmatrix}\vec{a} & \vec{b} & \vec{c}\end{pmatrix}.$$

Thus, we have shown that Υ is a diffeomorphism as required.

Next, we show that L is the orbit of the point

$$C=\begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}\\ 0 & 0 & 0\end{pmatrix}\in\left(S^2\right)^3$$

under the $SU(2)$ action given by (3.3). Since $\vec{0}\in\mathfrak{su}(2)^*$ is a fixed point of the coadjoint action, its preimage $L=\mathfrak{mom}^{-1}\left(\vec{0}\right)$ is invariant under the action of $SU(2)$. Noting that $C\in L$, it follows that the orbit of C is contained in L.

On the other hand, given a point $\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}\in L$, we want to show that there is some $\xi\in SU(2)$ such that $\xi\cdot\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}=C$. Since $SO(3)$ acts transitively on S^2, we may choose a matrix $A_1\in SO(3)$ such that $A_1\vec{u}=\vec{e}_1$. Writing $A_1\vec{v}=\begin{pmatrix}v_1' & v_2' & v_3'\end{pmatrix}^\top$ and

$A_1\vec{w} = \begin{pmatrix} w_1' & w_2' & w_3' \end{pmatrix}^\top$, we have

$$A_1 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \begin{pmatrix} 1 & v_1' & w_1' \\ 0 & v_2' & w_2' \\ 0 & v_3' & w_3' \end{pmatrix}.$$

The fact that $A_1 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ is an element of L implies that

$$1 + v_1' + w_1' = 0,\ v_2' + w_2' = 0,\ v_3' + w_3' = 0.$$

By the fact that $A_1\vec{w} \in S^2$, we see that

$$\begin{aligned} 1 &= (w_1')^2 + (w_2')^2 + (w_3')^2 \\ &= (-1 - v_1')^2 + (-v_2')^2 + (-v_3')^2 \\ &= 1 + 2v_1' + (v_1')^2 + (v_2')^2 + (v_3')^2 \\ &= 2 + 2v_1', \end{aligned}$$

which implies that $v_1' = -1/2$, and similarly $w_1' = -1/2$. Thus, we may write

$$A_1 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & v_2' & -v_2' \\ 0 & v_3' & -v_3' \end{pmatrix}.$$

Using the fact that $A_1\vec{v} \in S^2$, we obtain $(v_2')^2 + (v_3')^2 = 3/4$, and so it is easy to verify that the matrix

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}v_2' & \frac{2}{\sqrt{3}}v_3' \\ 0 & -\frac{2}{\sqrt{3}}v_3' & \frac{2}{\sqrt{3}}v_2' \end{pmatrix}$$

is an element of $SO(3)$. Furthermore, one readily sees that

$$A_2 A_1 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}v_2' & \frac{2}{\sqrt{3}}v_3' \\ 0 & -\frac{2}{\sqrt{3}}v_3' & \frac{2}{\sqrt{3}}v_2' \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & v_2' & -v_2' \\ 0 & v_3' & -v_3' \end{pmatrix} = C.$$

Since the map $\Phi : SU(2) \to SO(3)$ is surjective, we may choose $\xi \in SU(2)$ such that $\Phi(\xi) = A_2 A_1$, and thus we have

$$\xi \cdot \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = A_2 A_1 \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = C$$

as required. Hence, we see that L is exactly the orbit of C under the action of $SU(2)$ as claimed.

Since L is the orbit of C under the $SU(2)$ action, we have

$$T_{\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}} L = \left\{ X_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \,\middle|\, \zeta \in \mathfrak{su}(2) \right\},$$

where $X_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ is defined as in the proof of Proposition 3.4. Observe now that

$$H_\zeta \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \left\langle \mathfrak{mom} \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}, \zeta \right\rangle = 0$$

for all $\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in L$ and $\zeta \in \mathfrak{su}(2)$. In particular, this implies that H_ζ is constant on

$L = \mathfrak{mom}^{-1}\left(\vec{0}\right)$ for all $\zeta \in \mathfrak{su}(2)$. Then, since $X_\zeta = X_{H_\zeta}$, we see that

$$\begin{aligned}\Omega\left(X_\eta\left(\vec{u} \;\; \vec{v} \;\; \vec{w}\right), X_\zeta\left(\vec{u} \;\; \vec{v} \;\; \vec{w}\right)\right) &= \Omega\left(X_\eta\left(\vec{u} \;\; \vec{v} \;\; \vec{w}\right), X_{H_\zeta}\left(\vec{u} \;\; \vec{v} \;\; \vec{w}\right)\right) \\ &= dH_\zeta\left(X_\eta\left(\vec{u} \;\; \vec{v} \;\; \vec{w}\right)\right) = 0\end{aligned}$$

for all $\left(\vec{u} \;\; \vec{v} \;\; \vec{w}\right) \in L$ and $\eta, \zeta \in \mathfrak{su}(2)$. Hence, L is isotropic and thus Lagrangian since $\dim(L) = 3 = \frac{1}{2}\dim\left(\left(S^2\right)^3\right)$. □

Remark 3.6. There are diffeomorphisms $L \to SO(3)$ that are significantly simpler than Υ as defined in (3.5). In particular, one quite simple one will be given in the proof of Lemma 3.7. However, the map Υ ends up providing the necessary genericity to work with the pearl complex described in Section 5.1.

Writing $D^2 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ with $S^1 = \partial D^2 = \{z \in \mathbb{C} \,|\, |z| = 1\}$, we consider a map $u_D : (D^2, S^1) \to \left(\left(S^2\right)^3, L\right)$ defined by

$$u_D\left(re^{it}\right) = \begin{pmatrix} \frac{2r}{1+r^2}\cos(t) & \frac{2r}{1+r^2}\cos\left(t + \frac{4\pi}{3}\right) & \frac{2r}{1+r^2}\cos\left(t + \frac{2\pi}{3}\right) \\ \frac{2r}{1+r^2}\sin(t) & \frac{2r}{1+r^2}\sin\left(t + \frac{4\pi}{3}\right) & \frac{2r}{1+r^2}\sin\left(t + \frac{2\pi}{3}\right) \\ \frac{1-r^2}{1+r^2} & \frac{1-r^2}{1+r^2} & \frac{1-r^2}{1+r^2} \end{pmatrix}.$$

Lemma 3.7. *The map u_D is holomorphic, and the homotopy class $\left[u_D\big|_{S^1}\right]$ is the unique nontrivial element of $\pi_1(L)$.*

Proof. To show that u_D is holomorphic, it suffices to show that each map $\mathrm{pr}_j \circ u_D : D^2 \to S^2$ is holomorphic. By composing each map $\mathrm{pr}_j \circ u_D$ with the (holomorphic) stereographic projection from the point $-\vec{e}_3$, it becomes clear that each such map is holomorphic. Indeed, we have the stereographic projection given by

$$\varphi_{(-\vec{e}_3)} : S^2 \setminus \{-\vec{e}_3\} \to \mathbb{C}$$
$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \mapsto \frac{p_1 + ip_2}{1 + p_3},$$

and a simple computation shows that the compositions $\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_j \circ u_D$ are given by

$$\begin{aligned}
\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_1 \circ u_D \left(re^{it}\right) &= re^{it}, \\
\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_D \left(re^{it}\right) &= re^{i\left(t+\frac{4\pi}{3}\right)}, \\
\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_D \left(re^{it}\right) &= re^{i\left(t+\frac{2\pi}{3}\right)},
\end{aligned}$$

each of which is holomorphic. Therefore, u_D is holomorphic.

Although we have already established a diffeomorphism $\Upsilon : L \to SO(3)$, we now give a different diffeomorphism to establish the lemma. Consider $\chi : L \to SO(3)$ defined by

$$\chi \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} \left(\vec{v} + \frac{1}{2}\vec{u}\right) & \vec{u} & \frac{2}{\sqrt{3}} \left(\vec{v} \times \vec{u}\right) \end{pmatrix},$$

with inverse given by

$$\chi^{-1} \begin{pmatrix} \vec{a} & \vec{b} & \vec{c} \end{pmatrix} = \begin{pmatrix} \vec{b} & -\frac{1}{2}\vec{b} + \frac{\sqrt{3}}{2}\vec{a} & -\frac{1}{2}\vec{b} - \frac{\sqrt{3}}{2}\vec{a} \end{pmatrix}.$$

It is quite straightforward to show that χ is a diffeomorphism, and so the proof is omitted.

Noting that

$$\begin{aligned}
\chi \circ u_D\left(e^{it}\right) &= \chi \begin{pmatrix} \cos(t) & \cos\left(t+\frac{4\pi}{3}\right) & \cos\left(t+\frac{2\pi}{3}\right) \\ \sin(t) & \sin\left(t+\frac{4\pi}{3}\right) & \sin\left(t+\frac{2\pi}{3}\right) \\ 0 & 0 & 0 \end{pmatrix} \\
&= \chi \begin{pmatrix} \cos(t) & -\frac{1}{2}\cos(t)+\frac{\sqrt{3}}{2}\sin(t) & -\frac{1}{2}\cos(t)-\frac{\sqrt{3}}{2}\sin(t) \\ \sin(t) & -\frac{\sqrt{3}}{2}\cos(t)-\frac{1}{2}\sin(t) & \frac{\sqrt{3}}{2}\cos(t)-\frac{1}{2}\sin(t) \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sin(t) & \cos(t) & 0 \\ -\cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

we see that $\chi \circ u_D\big|_{S^1}$ represents the unique nontrivial element of $\pi_1(SO(3))$, and the lemma then follows immediately from the fact that χ is a diffeomorphism. □

We also consider maps $u_{S_j} : S^2 \to (S^2)^3$, for $j = 1, 2, 3$, defined by

$$\begin{aligned}
u_{S_1}(\vec{p}) &= \begin{pmatrix} \vec{p} & -\frac{1}{2}\vec{e}_1 - \frac{\sqrt{3}}{2}\vec{e}_2 & -\frac{1}{2}\vec{e}_1 + \frac{\sqrt{3}}{2}\vec{e}_2 \end{pmatrix}, \\
u_{S_2}(\vec{p}) &= \begin{pmatrix} \vec{e}_1 & \vec{p} & -\frac{1}{2}\vec{e}_1 + \frac{\sqrt{3}}{2}\vec{e}_2 \end{pmatrix}, \\
u_{S_3}(\vec{p}) &= \begin{pmatrix} \vec{e}_1 & -\frac{1}{2}\vec{e}_1 - \frac{\sqrt{3}}{2}\vec{e}_2 & \vec{p} \end{pmatrix},
\end{aligned}$$

which are easily seen to be holomorphic.

Lemma 3.8. *The Hurewicz homomorphism* $\pi_2\left((S^2)^3, L\right) \to H_2\left((S^2)^3, L\right)$ *is an isomorphism, and the relative homotopy classes* $[u_D], [u_{S_j}] \in \pi_2\left((S^2)^3, L\right)$ *satisfy the relationship*

$$2[u_D] = [u_{S_1}] + [u_{S_2}] + [u_{S_3}].$$

Proof. According to Section 4.2 of [Ha01], the absolute and relative Hurewicz homomorphisms fit into a commutative diagram coming from the homotopy and homology long exact sequences:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \pi_2(L) & \longrightarrow & \pi_2\left((S^2)^3\right) & \longrightarrow & \pi_2\left((S^2)^3, L\right) & \longrightarrow & \pi_1(L) & \longrightarrow & \pi_1\left((S^2)^3\right) & \longrightarrow & \cdots \\
 & & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 & & \downarrow h_5 & & \\
\cdots & \longrightarrow & H_2(L) & \longrightarrow & H_2\left((S^2)^3\right) & \longrightarrow & H_2\left((S^2)^3, L\right) & \longrightarrow & H_1(L) & \longrightarrow & H_1\left((S^2)^3\right) & \longrightarrow & \cdots .
\end{array}$$

Both h_1 and h_5 are isomorphisms since the corresponding groups are trivial, and h_4 is an isomorphism since it corresponds to abelianization and $\pi_1(L) \cong \mathbb{Z}/2\mathbb{Z}$ is already abelian. Moreover, the Hurewicz theorem implies that h_2 is an isomorphism since $(S^2)^3$ is simply connected. Then, by the five lemma, it follows that h_3 is an isomorphism as claimed.

To prove the second statement of the lemma, we consider the holomorphic submanifolds

$$\begin{aligned}
V_{1,2} &= \left\{ \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in (S^2)^3 \;\middle|\; \vec{u} = \vec{v} \right\}, \\
V_{1,3} &= \left\{ \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in (S^2)^3 \;\middle|\; \vec{u} = \vec{w} \right\}, \\
V_{2,3} &= \left\{ \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix} \in (S^2)^3 \;\middle|\; \vec{v} = \vec{w} \right\},
\end{aligned} \tag{3.8}$$

none of which intersect L, and thus there are well-defined intersection numbers $B \cdot [V_{1,2}]$, $B \cdot [V_{1,3}]$, and $B \cdot [V_{2,3}]$ whenever $B \in H_2\left((S^3)^2, L\right)$. In particular, we compute the following intersection data for the classes $[u_D]$ and $\left[u_{S_j}\right]$:

$\cdot$	$[V_{1,2}]$	$[V_{1,3}]$	$[V_{2,3}]$
$[u_D]$	1	1	1
$[u_{S_1}]$	1	1	0
$[u_{S_2}]$	1	0	1
$[u_{S_3}]$	0	1	1

Indeed, it is not difficult to show that the map u_D intersects each of the submanifolds $V_{1,2}$, $V_{1,3}$, and $V_{2,3}$ transversely in a single point, and the intersection numbers are positive since the maps and submanifolds are all holomorphic. The remainder of the intersection numbers in the above table are likewise easy to compute.

Next, a portion of the homotopy exact sequence reads

$$0 \longrightarrow \pi_2\left(\left(S^2\right)^3\right) \longrightarrow \pi_2\left(\left(S^2\right)^3, L\right) \longrightarrow \pi_1(L) \longrightarrow 0,$$

which implies that $2[u_D]$ is contained in the image of the map $\pi_2\left(\left(S^2\right)^3\right) \to \pi_2\left(\left(S^2\right)^3, L\right)$ since $\pi_1(L) \cong \mathbb{Z}/2\mathbb{Z}$. Since the above sequence is exact and the classes $[u_{S_1}]$, $[u_{S_2}]$, and $[u_{S_3}]$ generate $\pi_2\left(\left(S^2\right)^3\right)$, we may write

$$2[u_D] = a_1[u_{S_1}] + a_2[u_{S_2}] + a_3[u_{S_3}]$$

for some integers a_1, a_2, a_3. Then, by the above intersection data, we obtain the equations

$$a_1 + a_2 = 2,$$
$$a_1 + a_3 = 2,$$
$$a_2 + a_3 = 2,$$

from which we see that $a_1 = a_2 = a_3 = 1$, completing the proof of the lemma. □

Let $\mathcal{F}_{\text{ab}}(D, S_1, S_2, S_3)$ denote the free abelian group generated by formal variables D, S_1, S_2, and S_3, and consider the group homomorphism $\mathfrak{F} : \mathcal{F}_{\text{ab}}(D, S_1, S_2, S_3) \to \pi_2\left(\left(S^2\right)^3, L\right)$ defined by

$$\mathfrak{F}(D) = [u_D], \mathfrak{F}(S_1) = [u_{S_1}], \mathfrak{F}(S_2) = [u_{S_2}], \mathfrak{F}(S_3) = [u_{S_3}].$$

Proposition 3.9. *The group homomorphism $\mathfrak{F}$ is surjective with kernel generated by $2D - S_1 - S_2 - S_3$. Thus, $\mathfrak{F}$ descends to a group isomorphism*

$$\frac{\mathcal{F}_{\mathrm{ab}}(D, S_1, S_2, S_3)}{\langle 2D - S_1 - S_2 - S_3 \rangle} \cong \pi_2\left(\left(S^2\right)^3, L\right).$$

Proof. First, recall that a portion of the homotopy long exact sequence reads

$$0 \longrightarrow \pi_2\left(\left(S^2\right)^3\right) \longrightarrow \pi_2\left(\left(S^2\right)^3, L\right) \xrightarrow{\partial} \pi_1(L) \longrightarrow 0,$$

and note that $\partial[u_D] = \left[u_D\big|_{S^1}\right]$ is the unique nontrivial element of $\pi_1(L)$ by Lemma 3.7. Now given $B \in \pi_2\left(\left(S^2\right)^3, L\right)$, we consider two possibilities. If $B \in \ker\partial$, then by exactness and the fact that the classes $[u_{S_1}]$, $[u_{S_2}]$, and $[u_{S_3}]$ generate $\pi_2\left(\left(S^2\right)^3\right)$, we may write

$$B = a_1[u_{S_1}] + a_2[u_{S_2}] + a_3[u_{S_3}]$$

for some integers a_1, a_2, a_3. On the other hand, if $B \notin \ker\partial$, then $\partial B = \partial[u_D]$, which implies that $B - [u_D] \in \ker\partial$ and hence that

$$B - [u_D] = a_1[u_{S_1}] + a_2[u_{S_2}] + a_3[u_{S_3}]$$

for some integers a_1, a_2, a_3. In such a case, it follows that

$$B = a_1[u_{S_1}] + a_2[u_{S_2}] + a_3[u_{S_3}] + [u_D].$$

Thus, we see that $\pi_2\left(\left(S^2\right)^3, L\right)$ is generated by the classes $[u_D]$, $[u_{S_1}]$, $[u_{S_2}]$, and $[u_{S_3}]$, and thus $\mathfrak{F}$ is surjective.

Next, we note that $2D - S_1 - S_2 - S_3 \in \ker \mathfrak{F}$ by the definition of $\mathfrak{F}$ and Lemma 3.8. On the other hand, if $a_0 D + a_1 S_1 + a_2 S_2 + a_3 S_3 \in \ker \mathfrak{F}$, then we have

$$a_0 [u_D] + a_1 [u_{S_1}] + a_2 [u_{S_2}] + a_3 [u_{S_3}] = 0.$$

It then follows that

$$a_0 \partial [u_D] = \partial \left(a_0 [u_D] + a_1 [u_{S_1}] + a_2 [u_{S_2}] + a_3 [u_{S_3}] \right) = \partial (0) = 0,$$

which implies that $a_0 = 2k$ for some integer k. Then, using Lemma 3.8 once more, we write

$$\begin{aligned} 0 &= a_0 [u_D] + a_1 [u_{S_1}] + a_2 [u_{S_2}] + a_3 [u_{S_3}] \\ &= 2k [u_D] + a_1 [u_{S_1}] + a_2 [u_{S_2}] + a_3 [u_{S_3}] \\ &= (k + a_1) [u_{S_1}] + (k + a_2) [u_{S_2}] + (k + a_3) [u_{S_3}], \end{aligned}$$

which can only hold if $a_1 = a_2 = a_3 = -k$ since the generators $[u_{S_1}]$, $[u_{S_2}]$, and $[u_{S_3}]$ have no dependence relation. Thus, we have

$$a_0 D + a_1 S_1 + a_2 S_2 + a_3 S_3 = 2kD - kS_1 - kS_2 - kS_3 = k (2D - S_1 - S_2 - S_3),$$

which shows that $2D - S_1 - S_2 - S_3$ generates $\ker \mathfrak{F}$, completing the proof of the proposition.

□

Remark 3.10. Using the isomorphism given by Proposition 3.9, we will often refer to elements of $\pi_2 \left((S^2)^3, L \right)$ by a representative element in $\mathcal{F}_{\text{ab}} (D, S_1, S_2, S_3)$. In particular, we will frequently use D and S_j in place of $[u_D]$ and $\left[u_{S_j} \right]$, respectively.

Chapter 4

Holomorphic Disks with Lagrangian Boundary

The goal of this chapter is to describe certain holomorphic disks $u : (D^2, S^1) \to \left(\left(S^2\right)^3, L \right)$. In particular, we will be interested in those disks with Maslov index 2; and we will shortly show that such disks can only represent one of 3 classes in $\pi_2 \left(\left(S^2\right)^3, L \right)$.

Proposition 4.1. *The Lagrangian $L \subset \left(S^2\right)^3$ described in Proposition 3.5 is monotone with monotonicity constant π, and the class $D \in \pi_2 \left(\left(S^2\right)^3, L \right)$ has Maslov index 6 while each of the classes S_j has Maslov index 4.*

Proof. By the proof of Proposition 3.9, the classes D, S_1, S_2, S_3 generate $\pi_2 \left(\left(S^2\right)^3, L \right)$. Where $c_1 \left(\left(S^2\right)^3 \right)$ is the first Chern class of $T \left(S^2\right)^3$, we recall that

$$I_\mu \left(S_j\right) = 2 \left\langle c_1 \left(\left(S^2\right)^3 \right), S_j \right\rangle = 4.$$

On the other hand, since the standard area of S^2 is 4π, we have

$$I_\Omega \left(S_j\right) = \int_{S^2} u_{S_j}^* \Omega = \int_{S^2} \omega_{\text{std}} = 4\pi,$$

which shows that $I_\Omega(S_j) = \pi I_\mu(S_j)$ for $j = 1, 2, 3$. Then, since we have the relation $2D = S_1 + S_2 + S_3$, it follows that

$$2I_\Omega(D) = I_\Omega(2D) = I_\Omega(S_1 + S_2 + S_3) = \pi I_\mu(S_1 + S_2 + S_3) = \pi I_\mu(2D) = 2\pi I_\mu(D),$$

which implies that $I_\Omega(D) = \pi I_\mu(D)$. Since the classes D, S_1, S_2, S_3 generate $\pi_2\left(\left(S^2\right)^3, L\right)$, we have $I_\Omega = \pi I_\mu$, and so L is monotone with monotonicity constant π as claimed.

To compute the Maslov index of D, we first compute that

$$\begin{aligned}
I_\Omega(D) &= \int_{D^2} u_D^* \Omega \\
&= \int_{D^2} u_D^* \left(\mathrm{pr}_1^* \omega_{\mathrm{std}} + \mathrm{pr}_2^* \omega_{\mathrm{std}} + \mathrm{pr}_3^* \omega_{\mathrm{std}}\right) \\
&= \int_{D^2} u_D^* \mathrm{pr}_1^* \omega_{\mathrm{std}} + \int_{D^2} u_D^* \mathrm{pr}_2^* \omega_{\mathrm{std}} + \int_{D^2} u_D^* \mathrm{pr}_3^* \omega_{\mathrm{std}} \\
&= \int_{\mathrm{pr}_1 \circ u_D(D^2)} \omega_{\mathrm{std}} + \int_{\mathrm{pr}_2 \circ u_D(D^2)} \omega_{\mathrm{std}} + \int_{\mathrm{pr}_3 \circ u_D(D^2)} \omega_{\mathrm{std}} \\
&= 2\pi + 2\pi + 2\pi \\
&= 6\pi,
\end{aligned}$$

where the penultimate equality above follows from the fact that each map $\mathrm{pr}_j \circ u_D$ is an embedding with image the hemisphere $\{\vec{p} \in S^2 \mid \vec{p} \cdot \vec{e}_3 \geq 0\}$, which has standard area 2π. Then, since L is monotone with monotonicity constant π, it follows that $I_\mu(D) = 6$. □

Proposition 4.2. *If $B \in \pi_2\left(\left(S^2\right)^3, L\right)$ has a holomorphic representative and satisfies $I_\mu(B) = 2$, then $B = D - S_j$ for some $j \in \{1, 2, 3\}$.*

Proof. Using the fact that the classes D, S_1, S_2, S_3 generate $\pi_2\left(\left(S^2\right)^3, L\right)$ and the relation $2D = S_1 + S_2 + S_3$, we note that B can be written in the form

$$B = a_0 D + a_1 S_1 + a_2 S_2 + a_3 S_3$$

for $a_0 \in \{0,1\}$ and integers a_1, a_2, a_3. If $a_0 = 0$, then it follows that $I_\mu(B) = 4(a_1 + a_2 + a_3)$, which contradicts $I_\mu(B) = 2$. Thus, we write

$$B = D + a_1 S_1 + a_2 S_2 + a_3 S_3,$$

and it follows that $2 = I_\mu(B) = 6 + 4(a_1 + a_2 + a_3)$, which in turn implies

$$a_1 + a_2 + a_3 = -1. \tag{4.1}$$

Since B has a holomorphic representative, its intersection numbers with the holomorphic submanifolds $V_{1,2}$, $V_{1,3}$, and $V_{2,3}$ given in (3.8) must be nonnegative. In other words, we must have

$$\begin{aligned}
1 + a_1 + a_2 &= (D + a_1 S_1 + a_2 S_2 + a_3 S_3) \cdot [V_{1,2}] = B \cdot [V_{1,2}] \geq 0, \\
1 + a_1 + a_3 &= (D + a_1 S_1 + a_2 S_2 + a_3 S_3) \cdot [V_{1,3}] = B \cdot [V_{1,3}] \geq 0, \\
1 + a_2 + a_3 &= (D + a_1 S_1 + a_2 S_2 + a_3 S_3) \cdot [V_{2,3}] = B \cdot [V_{2,3}] \geq 0,
\end{aligned}$$

which combine with (4.1) to yield the inequalities

$$a_1 \leq 0,\ a_2 \leq 0, a_3 \leq 0.$$

Then, again using (4.1) and the fact that a_1, a_2, a_3 are integers, one quickly sees that exactly one of a_1, a_2, a_3 must be -1 with the remaining two coefficients being 0. In other words, we must have $B \in \{D - S_1, D - S_2, D - S_3\}$. □

We now proceed to classify holomorphic representatives $u : (D^2, S^1) \to \left((S^2)^3, L\right)$ of the classes $D - S_1, D - S_2, D - S_3$. To that end, we write $B_j = D - S_j$, and we consider maps

$u_{B_j} : (D^2, S^1) \to \left((S^2)^3, L \right)$ given by

$$u_{B_1}\left(re^{it}\right) = \begin{pmatrix} 0 & \frac{2\sqrt{3}r}{3+r^2}\cos(t) & \frac{2\sqrt{3}r}{3+r^2}\cos(t+\pi) \\ 0 & \frac{2\sqrt{3}r}{3+r^2}\sin(t) & \frac{2\sqrt{3}r}{3+r^2}\sin(t+\pi) \\ -1 & \frac{3-r^2}{3+r^2} & \frac{3-r^2}{3+r^2} \end{pmatrix},$$

$$u_{B_2}\left(re^{it}\right) = \begin{pmatrix} \frac{2\sqrt{3}r}{3+r^2}\cos(t) & 0 & \frac{2\sqrt{3}r}{3+r^2}\cos(t+\pi) \\ \frac{2\sqrt{3}r}{3+r^2}\sin(t) & 0 & \frac{2\sqrt{3}r}{3+r^2}\sin(t+\pi) \\ \frac{3-r^2}{3+r^2} & -1 & \frac{3-r^2}{3+r^2} \end{pmatrix},$$

$$u_{B_3}\left(re^{it}\right) = \begin{pmatrix} \frac{2\sqrt{3}r}{3+r^2}\cos(t+\pi) & \frac{2\sqrt{3}r}{3+r^2}\cos(t) & 0 \\ \frac{2\sqrt{3}r}{3+r^2}\sin(t+\pi) & \frac{2\sqrt{3}r}{3+r^2}\sin(t) & 0 \\ \frac{3-r^2}{3+r^2} & \frac{3-r^2}{3+r^2} & -1 \end{pmatrix}.$$

Proposition 4.3. *Each of the maps u_{B_j} is holomorphic and satisfies*

$$I_\Omega\left([u_{B_j}]\right) = 2\pi,\ I_\mu\left([u_{B_j}]\right) = 2,\ \textit{and}\ [u_{B_j}] = B_j = D - S_j.$$

Proof. We prove the proposition only for $j = 1$, the other two cases being nearly identical. First, to see that u_{B_1} is holomorphic, one need only show that each map $\mathrm{pr}_j \circ u_{B_1}$ is holomorphic. It is clear that $\mathrm{pr}_1 \circ u_{B_1}$ is holomorphic since it is constant; then by composing with the holomorphic stereographic projection from $-\vec{e}_3$ (as in the proof of Lemma 3.7), one obtains

$$\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_{B_1}\left(re^{it}\right) = \frac{\sqrt{3}}{3} re^{it},$$
$$\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_{B_1}\left(re^{it}\right) = \frac{\sqrt{3}}{3} re^{i(t+\pi)},$$

each of which is holomorphic. Therefore u_{B_1} is holomorphic as claimed.

Next, using the fact that $\mathrm{pr}_1 \circ u_{B_1}$ is constant, we compute that

$$\begin{aligned} I_\Omega([u_{B_1}]) &= \int_{D^2} u_{B_1}^* \Omega \\ &= \int_{D^2} u_{B_1}^* \left(\mathrm{pr}_1^* \omega_{\mathrm{std}} + \mathrm{pr}_2^* \omega_{\mathrm{std}} + \mathrm{pr}_3^* \omega_{\mathrm{std}}\right) \\ &= \int_{D^2} u_{B_1}^* \mathrm{pr}_1^* \omega_{\mathrm{std}} + \int_{D^2} u_{B_1}^* \mathrm{pr}_2^* \omega_{\mathrm{std}} + \int_{D^2} u_{B_1}^* \mathrm{pr}_3^* \omega_{\mathrm{std}} \\ &= \int_{D^2} 0 + \int_{\mathrm{pr}_2 \circ u_{B_1}(D^2)} \omega_{\mathrm{std}} + \int_{\mathrm{pr}_3 \circ u_{B_1}(D^2)} \omega_{\mathrm{std}} \\ &= 0 + \pi + \pi \\ &= 2\pi, \end{aligned}$$

where the penultimate equality above follows from the fact that both $\mathrm{pr}_2 \circ u_{B_1}$ and $\mathrm{pr}_3 \circ u_{B_1}$ are embeddings with image $\{\vec{p} \in S^2 \mid \vec{p} \cdot \vec{e}_3 \geq 1/2\}$, which has standard area π. By Proposition 4.1 and the definition of monotonicity, it follows that $I_\mu([u_{B_1}]) = 2$.

Next, by Proposition 4.2, it must be the case that $[u_{B_1}] = D - S_j$ for some $j \in \{1, 2, 3\}$. By computing and comparing the intersections

$$[u_{B_1}] \cdot [V_{1,2}] = 0,\ [u_{B_1}] \cdot [V_{1,3}] = 0,\ [u_{B_1}] \cdot [V_{2,3}] = 1,$$

with

$$\begin{aligned} &(D - S_1) \cdot [V_{1,2}] = 0,\ (D - S_1) \cdot [V_{1,3}] = 0,\ (D - S_1) \cdot [V_{2,3}] = 1, \\ &(D - S_2) \cdot [V_{1,2}] = 0,\ (D - S_2) \cdot [V_{1,3}] = 1,\ (D - S_2) \cdot [V_{2,3}] = 0, \\ &(D - S_3) \cdot [V_{1,2}] = 1,\ (D - S_3) \cdot [V_{1,3}] = 0,\ (D - S_3) \cdot [V_{2,3}] = 0, \end{aligned}$$

one quickly sees that the only possibility is that $[u_{B_1}] = D - S_1$ as claimed. □

We now present a slight generalization of the maps u_{B_j} that will prove useful shortly. We consider

$$S^1\left(TS^2\right) = \{(\vec{p},\vec{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, \vec{p}\cdot\vec{q} = 0, |\vec{p}| = |\vec{q}| = 1\},$$

the unit circle bundle of the tangent bundle TS^2. Given a point $(\vec{p},\vec{q}) \in S^1\left(TS^2\right)$, we observe that the matrix $\begin{pmatrix}\vec{p} & \vec{q}\times\vec{p} & \vec{q}\end{pmatrix}$ is an element of $SO(3)$. In particular, this means that the map

$$\begin{aligned} u_{\vec{p},\vec{q}} : \left(D^2, S^1\right) &\to \left(\left(S^2\right)^3, L\right) \\ z &\mapsto \begin{pmatrix}\vec{p} & \vec{q}\times\vec{p} & \vec{q}\end{pmatrix} u_{B_1}(z) \end{aligned} \tag{4.2}$$

is holomorphic since u_{B_1} is holomorphic and since $SU(2)$ acts on $\left(S^2\right)^3$ holomorphically through multiplication by elements of $SO(3)$. In other words, there is some $\xi \in SU(2)$ such that $\Phi(\xi) = \begin{pmatrix}\vec{p} & \vec{q}\times\vec{p} & \vec{q}\end{pmatrix}$, and then

$$\begin{pmatrix}\vec{p} & \vec{q}\times\vec{p} & \vec{q}\end{pmatrix} u_{B_1}(z) = \Phi(\xi)\, u_{B_1}(z) = \xi\cdot u_{B_1}(z)$$

so that $u_{\vec{p},\vec{q}}$ is holomorphic as claimed. Since L is an orbit of the $SU(2)$ action, we also see clearly that $u_{\vec{p},\vec{q}}\left(S^1\right) \subset L$. Note that $u_{\vec{e}_1,\vec{e}_3}$ is identically u_{B_1}, and moreover, the map $u_{\vec{p},\vec{q}}$ represents the class B_1 since $SU(2)$ is connected.

We also consider maps $\mathfrak{s}_{12} : \left(S^2\right)^3 \to \left(S^2\right)^3$ and $\mathfrak{s}_{13} : \left(S^2\right)^3 \to \left(S^2\right)^3$ defined by

$$\begin{aligned} \mathfrak{s}_{12}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{v} & \vec{u} & \vec{w}\end{pmatrix}, \\ \mathfrak{s}_{13}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{w} & \vec{v} & \vec{u}\end{pmatrix}. \end{aligned}$$

It is easy to see that $\mathfrak{s}_{12}$ and $\mathfrak{s}_{13}$ are holomorphic and that $u_{B_2} = \mathfrak{s}_{12}\circ u_{B_1}$ and $u_{B_3} = \mathfrak{s}_{13}\circ u_{B_1}$. Additionally, given $(\vec{p},\vec{q}) \in S^1\left(TS^2\right)$, the maps $\mathfrak{s}_{12}\circ u_{\vec{p},\vec{q}}$ and $\mathfrak{s}_{13}\circ u_{\vec{p},\vec{q}}$ are holomorphic and represent the classes B_2 and B_3, respectively.

Now consider the moduli space $\widetilde{\mathcal{M}}(B_j)$ of parametrized holomorphic disks $u : (D^2, S^1) \to \left((S^2)^3, L\right)$ representing the class $B_j \in \pi_2\left((S^2)^3, L\right)$, which is a smooth manifold according to the following lemma.

Lemma 4.4. *The moduli space $\widetilde{\mathcal{M}}(B_j)$ is a smooth manifold of dimension* 5.

Proof. By Corollary 3.2, the group $SU(2)$ acts on $(S^2)^3$ by holomorphic automorphisms, and by the proof of Proposition 3.5, the Lagrangian L is an orbit of the $SU(2)$ action. Thus, the pair $\left((S^2)^3, L\right)$ is $SU(2)$-homogeneous in the parlance of [EL14]. It then follows from Lemma 3.2.1 of [EL14] that $\widetilde{\mathcal{M}}(B_j)$ is a smooth manifold. The dimension of $\widetilde{\mathcal{M}}(B_j)$ is given by the formula

$$\dim\left(\widetilde{\mathcal{M}}(B_j)\right) = I_\mu(B_j) + \frac{1}{2}\dim\left((S^2)^3\right),$$

which yields $\dim\left(\widetilde{\mathcal{M}}(B_j)\right) = 2 + 3 = 5$ since B_j has Maslov index 2. □

We let G denote the group of holomorphic automorphisms of the disk D^2, and following closely the conventions of Appendix A of [BC12], we have

$$G = \left\{\sigma_{\theta,\alpha} \,|\, \theta \in (-\pi, \pi], \alpha \in \operatorname{Int}\left(D^2\right)\right\},$$

where

$$\sigma_{\theta,\alpha}(z) = e^{i\theta}\frac{z+\alpha}{1+\bar{\alpha}z}.$$

Noting that $\frac{1+\bar{\alpha}}{1+\alpha} \in S^1 \backslash \{-1\}$ whenever $\alpha \in \operatorname{Int}(D^2)$ and writing log for the standard principal complex logarithm, we see that

$$\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right) = i\theta$$

for some $\theta \in (-\pi, \pi)$. In particular, this implies that $-i\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right) \in (-\pi, \pi)$ whenever $\alpha \in \operatorname{Int}(D^2)$, a fact we use in the statement of the following theorem.

Theorem 4.5. *Suppose that* $\Psi : S^1(TS^2) \times \mathrm{Int}(D^2) \to \widetilde{\mathcal{M}}(B_1)$ *is defined by*

$$\Psi(\vec{p}, \vec{q}, \alpha) = u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1},$$

where $u_{\vec{p},\vec{q}}$ *is as in* (4.2) *and* $\theta_\Psi(\alpha) = -i\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$*. Then* Ψ *is a diffeomorphism, and the maps* $\mathfrak{s}_{12} \circ \Psi : S^1(TS^2) \times \mathrm{Int}(D^2) \to \widetilde{\mathcal{M}}(B_2)$ *and* $\mathfrak{s}_{13} \circ \Psi : S^1(TS^2) \times \mathrm{Int}(D^2) \to \widetilde{\mathcal{M}}(B_3)$ *are diffeomorphisms.*

Before proving the above theorem, we will establish a few related lemmata.

Lemma 4.6. *If* $\sigma : D^2 \to D^2$ *is a holomorphic automorphism with* $\sigma(1) = 1$*, then* $\sigma = \sigma_{\theta_\Psi(\alpha),\alpha}$ *for some* $\alpha \in \mathrm{Int}(D^2)$*, where* $\theta_\Psi(\alpha) = -i\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$ *as in the statement of Theorem 4.5. Conversely, given any* $\alpha \in \mathrm{Int}(D^2)$*, we have* $\sigma_{\theta_\Psi(\alpha),\alpha}(1) = 1$*.*

Proof. Since $\sigma \in G$, we have $\sigma = \sigma_{\nu,\alpha}$ for some $\nu \in (-\pi, \pi]$ and $\alpha \in \mathrm{Int}(D^2)$. This implies that

$$1 = \sigma(1) = \sigma_{\nu,\alpha}(1) = e^{i\nu}\frac{1+\alpha}{1+\bar{\alpha}},$$

which in turn yields

$$e^{i\nu} = \frac{1+\bar{\alpha}}{1+\alpha}.$$

Therefore, we see that

$$i\nu = \log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right),$$

which implies that $\nu = -i\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right) = \theta_\Psi(\alpha)$. Hence, $\sigma = \sigma_{\theta_\Psi(\alpha),\alpha}$ as required. The converse is a simple computation. □

Given a positive integer p, we write

$$\widetilde{\mathcal{M}}_p(B_j) = \widetilde{\mathcal{M}}(B_j) \times T_p,$$

where $T_p \subset (S^1)^p$ is the open set consisting of all tuples of points $\vec{z} = (z_1, \ldots, z_p)$ with the property that all of the z_k are distinct and additionally are cyclically ordered (with respect to the usual orientation of S^1) if $p \geq 3$. By Lemma 4.4, it is clear that $\widetilde{\mathcal{M}}_p(B_j)$ is a smooth manifold of dimension $5 + p$.

Observe that G acts on $\widetilde{\mathcal{M}}(B_j)$ and $\widetilde{\mathcal{M}}_p(B_j)$ in obvious ways:

$$\sigma \cdot u = u \circ \sigma^{-1} \text{ and } \sigma \cdot (u, z_1, \ldots, z_p) = \left(u \circ \sigma^{-1}, \sigma(z_1), \ldots, \sigma(z_p)\right).$$

We then write $\mathcal{M}(B_j) = \widetilde{\mathcal{M}}(B_j)/G$ and $\mathcal{M}_p(B_j) = \widetilde{\mathcal{M}}_p(B_j)/G$, and we note that $\mathcal{M}(B_j)$ and $\mathcal{M}_p(B_j)$ are smooth manifolds of dimension 2 and $2 + p$, respectively.

Next, since $SU(2)$ acts on $(S^2)^3$ by holomorphic automorphisms and L is an orbit of the $SU(2)$ action, it follows that $SU(2)$ acts on $\widetilde{\mathcal{M}}(B_j)$ as well. Given $\xi \in SU(2)$ and $u \in \widetilde{\mathcal{M}}(B_j)$, we define $\xi \cdot u : (D^2, S^1) \to \left((S^2)^3, L\right)$ by

$$(\xi \cdot u)(z) = \xi \cdot (u(z)),$$

where the action on the right hand side above is that given by (3.3). Since $SU(2)$ is connected, the map $\xi \cdot u$ obviously represents the class B_j whenever u does, and so $\xi \cdot u \in \widetilde{\mathcal{M}}(B_j)$ as required. It is easy to see that this action descends to yield actions of $SU(2)$ on $\mathcal{M}(B_j)$ and $\mathcal{M}_p(B_j)$ by

$$\xi \cdot [u] = [\xi \cdot u] \text{ and } \xi \cdot [u, z_1, \ldots, z_p] = [\xi \cdot u, z_1, \ldots, z_p], \tag{4.3}$$

respectively.

Lemma 4.7. *Given $[u_0] \in \mathcal{M}(B_j)$, the stabilizer of $[u_0]$ under the action given by (4.3) is a 1-dimensional subgroup of $SU(2)$.*

Proof. We roughly follow the proofs of Lemma 3.4.1 and Lemma 3.4.2 in [EL14]. First, we observe that map $\mathrm{ev} : \mathcal{M}_1(B_j) \to L$ defined by $\mathrm{ev}([u, z]) = u(z)$ is equivariant, and we claim that ev is in fact a local diffeomorphism. Since $\mathcal{M}_1(B_j)$ and L are both 3-dimensional, it is sufficient to prove that ev has no critical values. Assume to the contrary that $P \in L$ is a critical value, in which case $\xi \cdot P$ is also critical for all $\xi \in SU(2)$. Since $SU(2)$ acts transitively on L, this implies that every point of L is a critical value, contradicting Sard's Theorem. Hence, ev is a local diffeomorphism as claimed.

Next, we claim that the stabilizer of $[u_0, z_0]$ is a 0-dimensional subgroup of $SU(2)$. Consider the diagram

$$\begin{array}{ccc} SU(2) & \xrightarrow{\mathfrak{o}^{([u_0,z_0])}} & \mathcal{M}_1(B_j) \\ & \searrow^{\mathfrak{o}^{(u_0(z_0))}} & \downarrow \mathrm{ev} \\ & & L, \end{array}$$

where $\mathfrak{o}^{([u_0,z_0])}(\xi) = \xi \cdot [u_0, z_0]$ and $\mathfrak{o}^{(u_0(z_0))}(\xi) = \xi \cdot (u_0(z_0))$. The above diagram commutes since the map ev is equivariant, and the stabilizer of $[u_0, z_0] \in \mathcal{M}_1(B_j)$ is given by

$$\mathrm{Stab}([u_0, z_0]) = \left(\mathfrak{o}^{([u_0,z_0])}\right)^{-1}([u_0, z_0])$$

while the stabilizer of $u_0(z_0) \in L$ is given by

$$\mathrm{Stab}(u_0(z_0)) = \left(\mathfrak{o}^{(u_0(z_0))}\right)^{-1}(u_0(z_0)).$$

Since the diagram commutes and $[u_0, z_0] \in \mathrm{ev}^{-1}(u_0(z_0))$, it follows that

$$\begin{aligned} \mathrm{Stab}([u_0, z_0]) &= \left(\mathfrak{o}^{([u_0,z_0])}\right)^{-1}([u_0, z_0]) \\ &\subset \left(\mathfrak{o}^{([u_0,z_0])}\right)^{-1}\left(\mathrm{ev}^{-1}(u_0(z_0))\right) \\ &= \left(\mathfrak{o}^{(u_0(z_0))}\right)^{-1}(u_0(z_0)) \\ &= \mathrm{Stab}(u_0(z_0)). \end{aligned}$$

It is not difficult to see that $\operatorname{Stab}(u_0(z_0))$ is 0-dimensional based on the definition of the action given in (3.3). In fact, one can easily compute that

$$\operatorname{Stab}(u_0(z_0)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

and hence $\operatorname{Stab}([u_0, z_0])$ is also 0-dimensional. Moreover, the map $\mathfrak{o}^{(u_0(z_0))}$ is a local diffeomorphism, and thus $\mathfrak{o}^{([u_0,z_0])}$ is also a local diffeomorphism since the diagram commutes and ev is a local diffeomorphism.

Next, we consider the diagram

$$\begin{array}{ccc} SU(2) & \xrightarrow{\mathfrak{o}^{([u_0,z_0])}} & \mathcal{M}_1(B_j) \\ & \searrow^{\mathfrak{o}^{([u_0])}} & \downarrow \mathfrak{f} \\ & & \mathcal{M}(B_j), \end{array}$$

where $\mathfrak{o}^{([u_0])}(\xi) = \xi \cdot [u_0]$ and $\mathfrak{f}([u, z]) = [u]$. This diagram also commutes since $\mathfrak{f}$ is equivariant. Moreover, the stabilizer of $[u_0]$ is given by

$$\operatorname{Stab}([u_0]) = \left(\mathfrak{o}^{([u_0])}\right)^{-1}([u_0]) = \left(\mathfrak{o}^{([u_0,z_0])}\right)^{-1}\left(\mathfrak{f}^{-1}([u_0])\right),$$

which is 1-dimensional since the fibers of $\mathfrak{f}$ are 1-dimensional and $\mathfrak{o}^{(u_0(z_0))}$ is a local diffeomorphism. □

Recalling the identification of $\mathfrak{su}(2)$ with $\mathbb{R}^3$ given by (3.4), we consider $\zeta \in \mathfrak{su}(2)$ with $|\vec{\zeta}| = \frac{1}{2}$. By Lemma 3.3, the matrix $\Phi(\exp(t\zeta))$ acts by right-handed rotation about the axis $2\vec{\zeta}$ through angle t. Then, given a point $\vec{q} \in S^2$, it is clear that the map $e^{it} \mapsto \Phi(\exp(t\zeta))\,\vec{q}$ is well-defined and parametrizes a loop in S^2.

Lemma 4.8. *Suppose that* $\zeta \in \mathfrak{su}(2)$ *satisfies* $\vec{\zeta} = \frac{1}{2}\vec{e}_3$ *and that* $w : D^2 \to S^2$ *is holomorphic and satisfies*

$$w \circ \delta\left(e^{it}\right) = \Phi\left(\exp\left(t\zeta\right)\right)\vec{q}$$

for some orientation preserving diffeomorphism $\delta : S^1 \to S^1$ *and some* $\vec{q} \in S^2 \setminus \{\pm\vec{e}_3\}$*. Then*

$$w\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p}\cdot\vec{e}_3 \geq \vec{q}\cdot\vec{e}_3\}.$$

Proof. First, observe that

$$w \circ \delta\left(e^{it}\right) = \begin{pmatrix} \cos\left(t\right) & -\sin\left(t\right) & 0 \\ \sin\left(t\right) & \cos\left(t\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{q},$$

and so we have

$$w\left(D^2\right) \supset w\left(S^1\right) = \{\vec{p} \in S^2 \,|\, \vec{p}\cdot\vec{e}_3 = \vec{q}\cdot\vec{e}_3\}.$$

Now, assume that the result is not true, and choose some point $\vec{x} \in S^2$ with $\vec{x}\cdot\vec{e}_3 > \vec{q}\cdot\vec{e}_3$ and $\vec{x} \notin w\left(D^2\right)$. Let $\varphi_{\vec{x}} : S^2 \setminus \{\vec{x}\} \to \mathbb{C}$ denote a holomorphic stereographic projection from $\vec{x}$, and observe that the image of S^1 under the map $\varphi_{\vec{x}} \circ w$ is a circle in $\mathbb{C}$ (of finite radius since $\vec{x} \notin w\left(S^1\right)$). Suppose that $\varphi_{\vec{x}} \circ w\left(S^1\right)$ has center a and radius r, and let $f_{a,r} : \mathbb{C} \to \mathbb{C}$ be given by $f_{a,r} : z \mapsto \frac{1}{r}\left(z - a\right)$. Then the map $f_{a,r} \circ \varphi_{\vec{x}} \circ w$ is holomorphic and maps S^1 onto S^1, and it then follows from the maximum modulus principle that $f_{a,r} \circ \varphi_{\vec{x}} \circ w\left(D^2\right) \subset D^2$. By composing with $\left(f_{a,r} \circ \varphi_{\vec{x}}\right)^{-1}$, one quickly sees that

$$w\left(D^2\right) \subset \{\vec{p} \in S^2 \,|\, \vec{p}\cdot\vec{e}_3 \leq \vec{q}\cdot\vec{e}_3\}.$$

In particular, this implies that $\vec{e}_3 \notin w\left(D^2\right)$, and so we can consider the map $\varphi_{\vec{e}_3} \circ w$, where

$$\begin{aligned}\varphi_{\vec{e}_3} : S^2 \setminus \{\vec{e}_3\} &\to \mathbb{C} \\ \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} &\mapsto \frac{p_1 - ip_2}{1 - p_3}\end{aligned}$$

is a holomorphic stereographic projection from $\vec{e}_3$. Writing $\vec{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^\top$, we observe that

$$\varphi_{\vec{e}_3} \circ w \circ \delta\left(e^{it}\right) = \frac{\left(q_1 \cos\left(t\right) - q_2 \sin\left(t\right)\right) - i\left(q_1 \sin\left(t\right) + q_2 \cos\left(t\right)\right)}{1 - q_3} = \frac{q_1 - iq_2}{1 - q_3} e^{-it}.$$

Writing $f : z \mapsto \frac{1-q_3}{q_1 - iq_2} z$, we have a holomorphic map $f \circ \varphi_{\vec{e}_3} \circ w : D^2 \to D^2$, and the map $f \circ \varphi_{\vec{e}_3} \circ w \circ \delta : S^1 \to S^1$ has degree -1. Then, using the fact that the standard area form on $\mathbb{C}$ is $rdr \wedge d\theta = d\left(\frac{1}{2} r^2 d\theta\right)$, it follows from Stokes' Theorem, the degree theorem, and the fact that $\delta : S^1 \to S^1$ is an orientation preserving diffeomorphism that

$$\begin{aligned}\int_{D^2} \left(f \circ \varphi_{\vec{e}_3} \circ w\right)^* d\left(\frac{1}{2} r^2 d\theta\right) &= \int_{S^1} \left(f \circ \varphi_{\vec{e}_3} \circ w\big|_{S^1}\right)^* \left(\frac{1}{2} r^2 d\theta\right) \\ &= \int_{S^1} \left(f \circ \varphi_{\vec{e}_3} \circ w\big|_{S^1}\right)^* \left(\frac{1}{2} d\theta\right) \\ &= \int_{S^1} \delta^* \left(f \circ \varphi_{\vec{e}_3} \circ w\big|_{S^1}\right)^* \left(\frac{1}{2} d\theta\right) \\ &= \int_{S^1} \left(f \circ \varphi_{\vec{e}_3} \circ w \circ \delta\right)^* \left(\frac{1}{2} d\theta\right) \\ &= \deg\left(f \circ \varphi_{\vec{e}_3} \circ w \circ \delta\right) \int_{S^1} \frac{1}{2} d\theta \\ &= -\pi,\end{aligned}$$

which contradicts the fact that the nonconstant holomorphic map $f \circ \varphi_{\vec{e}_3} \circ w$ should have positive area. Thus, it must be the case that

$$w\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p}\cdot\vec{e}_3 \geq \vec{q}\cdot\vec{e}_3\}$$

exactly as claimed. ☐

Lemma 4.9. *Suppose that* $u : (D^2, S^1) \to \left(\left(S^2\right)^3, L\right)$ *is holomorphic and represents the class* B_j*, and write*

$$u(1) = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}.$$

Suppose also that $\zeta \in \mathfrak{su}(2)$ *satisfies* $\vec{\zeta} = \frac{1}{2}\vec{e}_3$ *and that there is some orientation preserving diffeomorphism* $\delta : S^1 \to S^1$ *such that*

$$u \circ \delta\left(e^{it}\right) = \Phi\left(\exp\left(t\zeta\right)\right)\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}.$$

Then exactly one of the following must hold:

(i) $\vec{u} = -\vec{e}_3$ *and* u *is a reparametrization of* u_{B_1}*.*

(ii) $\vec{v} = -\vec{e}_3$ *and* u *is a reparametrization of* u_{B_2}*.*

(iii) $\vec{w} = -\vec{e}_3$ *and* u *is a reparametrization of* u_{B_3}*.*

Proof. First we will show that $-\vec{e}_3 \in \{\vec{u}, \vec{v}, \vec{w}\}$, and we will then deal with the statement regarding u being a reparametrization of u_{B_j}. Assume to the contrary that $-\vec{e}_3 \notin \{\vec{u}, \vec{v}, \vec{w}\}$, and we will obtain a contradiction in each of several possible cases.

First, we consider the case that $-\vec{e}_3 \notin \{-\vec{u}, -\vec{v}, -\vec{w}\}$ in addition to the assumption $-\vec{e}_3 \notin \{\vec{u}, \vec{v}, \vec{w}\}$. We consider the holomorphic maps $\mathrm{pr}_1 \circ u$, $\mathrm{pr}_2 \circ u$, and $\mathrm{pr}_3 \circ u$, and we observe that

$$\mathrm{pr}_1 \circ u\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p} \cdot \vec{e}_3 \geq \vec{u} \cdot \vec{e}_3\},$$
$$\mathrm{pr}_2 \circ u\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p} \cdot \vec{e}_3 \geq \vec{v} \cdot \vec{e}_3\},$$
$$\mathrm{pr}_3 \circ u\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p} \cdot \vec{e}_3 \geq \vec{w} \cdot \vec{e}_3\},$$

by Lemma 4.8. In particular, this implies that

$$\int_{D^2} (\mathrm{pr}_1 \circ u)^* \omega_{\mathrm{std}} \geq 2\pi \left(1 - \vec{u} \cdot \vec{e}_3\right),$$
$$\int_{D^2} (\mathrm{pr}_2 \circ u)^* \omega_{\mathrm{std}} \geq 2\pi \left(1 - \vec{v} \cdot \vec{e}_3\right),$$
$$\int_{D^2} (\mathrm{pr}_3 \circ u)^* \omega_{\mathrm{std}} \geq 2\pi \left(1 - \vec{w} \cdot \vec{e}_3\right),$$

which in turn yields

$$\begin{aligned} I_\Omega([u]) &= \int_{D^2} u^*\Omega \\ &= \int_{D^2} (\mathrm{pr}_1 \circ u)^* \omega_{\mathrm{std}} + \int_{D^2} (\mathrm{pr}_2 \circ u)^* \omega_{\mathrm{std}} + \int_{D^2} (\mathrm{pr}_3 \circ u)^* \omega_{\mathrm{std}} \\ &\geq 2\pi \left(3 - (\vec{u} + \vec{v} + \vec{w}) \cdot \vec{e}_3\right) = 2\pi \left(3 - \vec{0} \cdot \vec{e}_3\right) = 6\pi. \end{aligned}$$

By monotonicity, it follows that $I_\mu([u]) \geq 6$, contradicting u representing the class B_j since $I_\mu(B_j) = 2$.

Next, we consider the case that $\vec{u} = \vec{e}_3$, which implies that $\vec{v} \cdot \vec{e}_3 = \vec{w} \cdot \vec{e}_3 = -\frac{1}{2}$. By Lemma 4.8, we see that

$$\mathrm{pr}_2 \circ u\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p} \cdot \vec{e}_3 \geq \vec{v} \cdot \vec{e}_3\},$$
$$\mathrm{pr}_3 \circ u\left(D^2\right) \supset \{\vec{p} \in S^2 \,|\, \vec{p} \cdot \vec{e}_3 \geq \vec{w} \cdot \vec{e}_3\},$$

which yields

$$\int_{D^2} (\mathrm{pr}_2 \circ u)^* \omega_{\mathrm{std}} \geq 2\pi (1 - \vec{v} \cdot \vec{e}_3) = 3\pi,$$
$$\int_{D^2} (\mathrm{pr}_3 \circ u)^* \omega_{\mathrm{std}} \geq 2\pi (1 - \vec{w} \cdot \vec{e}_3) = 3\pi.$$

It then follows that

$$\begin{aligned} I_\Omega([u]) &= \int_{D^2} u^*\Omega \\ &= \int_{D^2} (\mathrm{pr}_1 \circ u)^* \omega_{\mathrm{std}} + \int_{D^2} (\mathrm{pr}_2 \circ u)^* \omega_{\mathrm{std}} + \int_{D^2} (\mathrm{pr}_3 \circ u)^* \omega_{\mathrm{std}} \\ &\geq 0 + 3\pi + 3\pi \\ &= 6\pi, \end{aligned}$$

leading to a contradiction as above. The cases of $\vec{v} = \vec{e}_3$ and $\vec{w} = \vec{e}_3$ lead to similar contradictions. Hence, we must have $-\vec{e}_3 \in \{\vec{u}, \vec{v}, \vec{w}\}$ as claimed.

Suppose now that $\vec{u} = -\vec{e}_3$ so that $\vec{v} \cdot \vec{e}_3 = \vec{w} \cdot \vec{e}_3 = \frac{1}{2}$. Applying Lemma 4.8 again, we see that

$$\begin{aligned} I_\Omega([u]) &= \int_{D^2} u^*\Omega \\ &= \int_{D^2} (\mathrm{pr}_1 \circ u)^* \omega_{\mathrm{std}} + \int_{D^2} (\mathrm{pr}_2 \circ u)^* \omega_{\mathrm{std}} + \int_{D^2} (\mathrm{pr}_3 \circ u)^* \omega_{\mathrm{std}} \\ &\geq 0 + \pi + \pi \\ &= 2\pi \end{aligned}$$

with equality if and only if $\mathrm{pr}_1 \circ u$ is constant and both $\mathrm{pr}_2 \circ u$ and $\mathrm{pr}_3 \circ u$ have area π, conditions that must hold since u represents B_j and $I_\Omega(B_j) = 2\pi$ by monotonicity. Note that there is some point $\vec{x} \in S^2$ with $\vec{x} \cdot \vec{e}_3 < \frac{1}{2}$ and $\vec{x} \notin \mathrm{pr}_2 \circ u(D^2)$, since otherwise we would have $\mathrm{pr}_2 \circ u\,(D^2) = S^2$ and hence

$$\int_{D^2} (\mathrm{pr}_2 \circ u)^* \, \omega_{\mathrm{std}} \geq 4\pi,$$

a contradiction. By composing with a holomorphic stereographic projection from $\vec{x}$ and applying the maximum modulus principle as in the proof of Lemma 4.8, we can conclude that

$$\mathrm{pr}_2 \circ u \left(D^2\right) = \left\{ \vec{p} \in S^2 \;\middle|\; \vec{p} \cdot \vec{e}_3 \geq \frac{1}{2} \right\}.$$

In particular, $-\vec{e}_3 \notin \mathrm{pr}_2 \circ u\,(D^2)$, and so we can consider the map $\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u$, where $\varphi_{(-\vec{e}_3)} : S^2 \setminus \{-\vec{e}_3\} \to \mathbb{C}$ is a holomorphic stereographic projection as in the proof of Lemma 3.7. Similarly, one can show that $-\vec{e}_3 \notin \mathrm{pr}_3 \circ u\,(D^2)$, and so we can also consider the map $\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u$. Writing $\vec{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^\top$ and $\vec{w} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}^\top$, we have

$$u \circ \delta\left(e^{it}\right) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ -1 & v_3 & w_3 \end{pmatrix}.$$

We then compute that

$$\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u \circ \delta\left(e^{it}\right) = \varphi_{(-\vec{e}_3)} \begin{pmatrix} v_1 \cos(t) - v_2 \sin(t) \\ v_1 \sin(t) + v_2 \cos(t) \\ v_3 \end{pmatrix} = \frac{v_1 + i v_2}{1 + v_3} e^{it}$$

and similarly that

$$\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u \circ \delta\left(e^{it}\right) = \frac{w_1 + i w_2}{1 + w_3} e^{it}.$$

Noting that $v_3 = w_3 = \frac{1}{2}$, we have

$$\left|\frac{v_1 + iv_2}{1 + v_3}\right| = \sqrt{\frac{v_1^2 + v_2^2}{(1 + v_3)^2}} = \sqrt{\frac{1 - v_3^2}{(1 + v_3)^2}} = \frac{\sqrt{3}}{3},$$

and similarly

$$\left|\frac{w_1 + iw_2}{1 + w_3}\right| = \frac{\sqrt{3}}{3}.$$

Thus, we may write

$$\frac{v_1 + iv_2}{1 + v_3} = \frac{\sqrt{3}}{3} e^{i\phi}$$

for some $\phi \in [0, 2\pi)$, and since $w_1 = -v_1$ and $w_2 = -v_2$, we have

$$\frac{w_1 + iw_2}{1 + w_3} = -\frac{v_1 + iv_2}{1 + v_3} = \frac{\sqrt{3}}{3} e^{i\phi} e^{i\pi}.$$

Where $\sigma_\phi : D^2 \to D^2$ is the automorphism $\sigma_\phi : z \mapsto e^{i\phi} z$, it follows from the proof of Proposition 4.3 that

$$\begin{aligned} \frac{3}{\sqrt{3}} \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u \circ \delta\left(e^{it}\right) &= e^{i\phi} e^{it} = \frac{3}{\sqrt{3}} \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_{B_1} \circ \sigma_\phi\left(e^{it}\right), \\ \frac{3}{\sqrt{3}} \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u \circ \delta\left(e^{it}\right) &= e^{i\phi} e^{i\pi} e^{it} = \frac{3}{\sqrt{3}} \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_{B_1} \circ \sigma_\phi\left(e^{it}\right). \end{aligned} \tag{4.4}$$

Since δ is a diffeomorphism, we see that the holomorphic maps

$$\begin{aligned} \left(\frac{3}{\sqrt{3}} \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u\right) &: D^2 \to D^2, \\ \left(\frac{3}{\sqrt{3}} \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u\right) &: D^2 \to D^2, \end{aligned}$$

are injective when restricted to S^1 and are therefore holomorphic automorphisms of the disk.

Likewise the maps

$$\left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_{B_1} \circ \sigma_\phi\right) : D^2 \to D^2,$$
$$\left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_{B_1} \circ \sigma_\phi\right) : D^2 \to D^2,$$

are automorphisms of the disk, and thus there are automorphisms $\sigma_2, \sigma_3 \in G$ such that

$$\begin{aligned}\left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u\right) \circ \sigma_2 &= \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_{B_1} \circ \sigma_\phi\right), \\ \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u\right) \circ \sigma_3 &= \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_{B_1} \circ \sigma_\phi\right).\end{aligned} \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\sigma_2\left(e^{it}\right) = \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u\right)^{-1} \circ \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_{B_1} \circ \sigma_\phi\right)\left(e^{it}\right) = \delta\left(e^{it}\right),$$
$$\sigma_3\left(e^{it}\right) = \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u\right)^{-1} \circ \left(\frac{3}{\sqrt{3}}\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_{B_1} \circ \sigma_\phi\right)\left(e^{it}\right) = \delta\left(e^{it}\right),$$

which in turn implies that $\sigma_2 = \sigma_3$. Writing $\sigma = \sigma_2 = \sigma_3$, it follows from (4.5) that

$$\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u \circ \sigma = \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_2 \circ u_{B_1} \circ \sigma_\phi,$$
$$\varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u \circ \sigma = \varphi_{(-\vec{e}_3)} \circ \mathrm{pr}_3 \circ u_{B_1} \circ \sigma_\phi,$$

and thus we have

$$\mathrm{pr}_2 \circ u = \mathrm{pr}_2 \circ u_{B_1} \circ \left(\sigma_\phi \circ \sigma^{-1}\right),$$
$$\mathrm{pr}_3 \circ u = \mathrm{pr}_3 \circ u_{B_1} \circ \left(\sigma_\phi \circ \sigma^{-1}\right).$$

Since both $\mathrm{pr}_1 \circ u$ and $\mathrm{pr}_1 \circ u_{B_1}$ are constant with value $-\vec{e}_3$, we also have

$$\mathrm{pr}_1 \circ u = \mathrm{pr}_1 \circ u_{B_1} \circ \left(\sigma_\phi \circ \sigma^{-1}\right).$$

It follows that u is a reparametrization of u_{B_1} in the case that $\vec{u} = -\vec{e}_3$, and the proofs for the cases of $\vec{v} = -\vec{e}_3$ and $\vec{w} = -\vec{e}_3$ are extremely similar. □

We are now ready to prove Theorem 4.5.

Proof of Theorem 4.5. We only prove the statement of the theorem for the moduli space $\widetilde{\mathcal{M}}(B_1)$; the proofs for the other two moduli spaces are very similar. First, we show that Ψ is surjective. Suppose that $u \in \widetilde{\mathcal{M}}(B_1)$, and consider the (unparametrized) disk $[u] \in \mathcal{M}(B_1)$. By Lemma 4.7, the stabilizer of $[u]$ is a 1-dimensional subgroup of $SU(2)$. Let $\zeta \in \mathfrak{su}(2)$ be a generator of the stabilizer subgroup of $[u]$ satisfying $|\vec{\zeta}| = \frac{1}{2}$. We then have $\exp(t\zeta) \cdot [u] = [u]$ for all $t \in \mathbb{R}$. In particular, this implies that, for each $t \in [0, 2\pi]$, there is an automorphism $\sigma_t \in G$ such that

$$\exp(t\zeta) \cdot u = u \circ \sigma_t.$$

We define $\delta : S^1 \to S^1$ by $\delta\left(e^{it}\right) = \sigma_t(1)$ so that

$$\exp(t\zeta) \cdot u(1) = u \circ \delta\left(e^{it}\right),$$

and we claim that δ is a diffeomorphism. Note that $\Phi(\exp(2\pi\zeta))$ is the identity element of $SO(3)$ (according to Lemma 3.3) and hence $\sigma_{2\pi} = \sigma_0$ is the identity in G; therefore δ is well-defined.

Now assume that $\sigma_s(1) = z_0 = \sigma_t(1)$ for some $s, t \in [0, 2\pi)$. Consider $[u, z_0] \in \mathcal{M}_1(B_1)$, whose stabilizer subgroup $\mathrm{Stab}([u, z_0]) \subset SU(2)$ is a 0-dimensional subgroup according to the proof of Lemma 4.7. More precisely, the proof of Lemma 4.7 implies that

$$\operatorname{Stab}([u, z_0]) \subset \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Observe that

$$\begin{aligned} \exp(t\zeta) \cdot [u, z_0] &= [\exp(t\zeta) \cdot u, z_0] = [u \circ \sigma_t, z_0] = [u, (\sigma_t)^{-1} z_0] = [u, 1] \\ &= [u, (\sigma_s)^{-1} z_0] = [u \circ \sigma_s, z_0] = [\exp(s\zeta) \cdot u, z_0] = \exp(s\zeta) \cdot [u, z_0], \end{aligned}$$

and hence $(\exp(s\zeta))^{-1} \exp(t\zeta) \in \operatorname{Stab}([u, z_0])$, which implies that $\exp(t\zeta) = \pm\exp(s\zeta)$. It follows that $\Phi(\exp(t\zeta)) = \Phi(\pm\exp(s\zeta))$; by Lemma 3.3, the matrix $\Phi(\exp(t\zeta))$ acts by right-handed rotation about axis $\vec{\zeta}/|\vec{\zeta}|$ through angle t while the matrix $\Phi(\pm\exp(s\zeta))$ acts by right-handed rotation about axis $\vec{\zeta}/|\vec{\zeta}|$ through angle s. Since $s, t \in [0, 2\pi)$, it follows that $s = t$, and hence δ is injective. Since S^1 is compact, we know that δ is an embedding, and it must be surjective since its image is a compact connected subset of S^1 that is homeomorphic to S^1. Therefore δ is a diffeomorphism as claimed.

Moreover, we can assume without loss of generality that δ is orientation preserving. Indeed, if δ as defined above is orientation reversing, then we instead choose $-\zeta$ as generator of the stabilizer subgroup of $[u]$. We would then have $\exp(t(-\zeta)) \cdot u = \exp(-t\zeta) \cdot u = u \circ \sigma_{-t}$ so that the diffeomorphism $\delta' : e^{it} \mapsto \sigma_{-t}(1)$ is orientation preserving with

$$\exp(t(-\zeta)) \cdot u(1) = u \circ \delta'\left(e^{it}\right).$$

Thus, we assume henceforth that δ as initially defined above is orientation preserving.

Now, choose a matrix $A \in SO(3)$ such that $A\vec{\zeta} = \frac{1}{2}\vec{e}_3$, and write $C_1 = A\left(u(1)\right)$ so that $u(1) = A^{-1}C_1$. Furthermore, we choose $\xi_A \in SU(2)$ with $\Phi\left(\xi_A\right) = A$. Then observe that

$$\begin{aligned}
\left(\xi_A \cdot u\right) \circ \delta\left(e^{it}\right) &= \Phi\left(\xi_A\right)\left(u \circ \delta\left(e^{it}\right)\right) \\
&= A\left(\exp\left(t\zeta\right) \cdot u\left(1\right)\right) \\
&= A\Phi\left(\exp\left(t\zeta\right)\right) u\left(1\right) \\
&= A\Phi\left(\exp\left(t\zeta\right)\right) A^{-1}C_1 \\
&= \Phi\left(\exp\left(t\frac{\vec{e}_3}{2}\right)\right) C_1,
\end{aligned}$$

where the final equality follows Lemma 3.3 and from the fact that $A\Phi\left(\exp\left(t\zeta\right)\right)A^{-1}$ acts by right-handed rotation about the axis $\vec{e}_3$ through angle t. Noting that

$$\left(\xi_A \cdot u\right)(1) = \Phi\left(\xi_A\right)\left(u(1)\right) = A\left(u(1)\right) = C_1,$$

it follows from Lemma 4.9 and the hypothesis that u represents the class B_1 (and hence so does $\left(\xi_A \cdot u\right)$) that $C_1 = \begin{pmatrix} -\vec{e}_3 & \vec{v} & \vec{w} \end{pmatrix}$ and that $\left(\xi_A \cdot u\right)$ is a reparametrization of u_{B_1}. Then, we choose a matrix $B \in SO(3)$, namely a rotation about the axis $\vec{e}_3$, such that

$$BC_1 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Writing $C_2 = BC_1$ and choosing $\xi_B \in SU(2)$ such that $\Phi\left(\xi_B\right) = B$, it follows that

$A^{-1}B^{-1}C_2 = u(1)$ and thus that

$$\begin{aligned}
(\xi_B\xi_A \cdot u) \circ \delta\left(e^{it}\right) &= \Phi\left(\xi_B\xi_A\right)\left(u \circ \delta\left(e^{it}\right)\right) \\
&= BA\left(\exp\left(t\zeta\right) \cdot u\left(1\right)\right) \\
&= BA\Phi\left(\exp\left(t\zeta\right)\right) u\left(1\right) \\
&= BA\Phi\left(\exp\left(t\zeta\right)\right) A^{-1}B^{-1}C_2 \\
&= \Phi\left(\exp\left(t\frac{\vec{e}_3}{2}\right)\right) C_2
\end{aligned}$$

similarly to above. Likewise, we observe that

$$(\xi_B\xi_A \cdot u)\left(1\right) = \Phi\left(\xi_B\xi_A\right)\left(u(1)\right) = BA\left(u(1)\right) = C_2,$$

and so it follows from Lemma 4.9 that $(\xi_B\xi_A \cdot u)$ is a reparametrization of u_{B_1}. In other words, we have $(\xi_B\xi_A \cdot u) = u_{B_1} \circ \sigma^{-1}$ for some $\sigma \in G$. Moreover, we note that

$$u_{B_1} \circ \sigma^{-1}\left(1\right) = (\xi_B\xi_A \cdot u)\left(1\right) = C_2 = u_{B_1}\left(1\right),$$

which implies that $\sigma^{-1}\left(1\right) = 1$ since u_{B_1} is injective. Then by Lemma 4.6, it follows that $\sigma = \sigma_{\theta_\Psi(\alpha),\alpha}$ for some $\alpha \in \operatorname{Int}\left(D^2\right)$, where $\theta_\Psi\left(\alpha\right) = -i\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$ as in the statement of Theorem 4.5. Thus, we have $(\xi_B\xi_A \cdot u) = u_{B_1} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}$, which implies

$$u = \left(\xi_B\xi_A\right)^{-1} \cdot \left(u_{B_1} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}\right).$$

In other words, we have

$$\begin{aligned}
u(z) &= (\xi_B \xi_A)^{-1} \cdot \left(u_{B_1} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(z)\right) \\
&= \Phi\left((\xi_B\xi_A)^{-1}\right) u_{B_1}\left(\left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(z)\right) \\
&= A^{-1}B^{-1}u_{B_1}\left(\left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(z)\right) \\
&= \begin{pmatrix} \vec{p} & \vec{q}\times\vec{p} & \vec{q} \end{pmatrix} u_{B_1}\left(\left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(z)\right) \\
&= u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(z),
\end{aligned}$$

where $(\vec{p},\vec{q}) \in S^1(TS^2)$ are chosen so that $\begin{pmatrix} \vec{p} & \vec{q}\times\vec{p} & \vec{q} \end{pmatrix} = A^{-1}B^{-1}$ in $SO(3)$. Thus, we have shown that

$$u = u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1} = \Psi\left(\vec{p},\vec{q},\alpha\right),$$

and so Ψ is surjective.

Now to show that Ψ is injective, suppose that $\Psi(\vec{p},\vec{q},\alpha) = \Psi(\vec{x},\vec{y},\beta)$, which combines with Lemma 4.6 to yield

$$\begin{aligned}
\begin{pmatrix} -\vec{q} & \frac{\sqrt{3}}{2}\vec{p}+\frac{1}{2}\vec{q} & -\frac{\sqrt{3}}{2}\vec{p}+\frac{1}{2}\vec{q} \end{pmatrix} &= \begin{pmatrix} \vec{p} & \vec{q}\times\vec{p} & \vec{q} \end{pmatrix} u_{B_1}(1) \\
&= u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(1) \\
&= u_{\vec{x},\vec{y}} \circ \left(\sigma_{\theta_\Psi(\beta),\beta}\right)^{-1}(1) \\
&= \begin{pmatrix} \vec{x} & \vec{y}\times\vec{x} & \vec{y} \end{pmatrix} u_{B_1}(1) \\
&= \begin{pmatrix} -\vec{y} & \frac{\sqrt{3}}{2}\vec{x}+\frac{1}{2}\vec{y} & -\frac{\sqrt{3}}{2}\vec{x}+\frac{1}{2}\vec{y} \end{pmatrix}.
\end{aligned}$$

From the above computation, we immediately see that $\vec{q} = \vec{y}$, which then quickly implies that $\vec{p} = \vec{x}$. It then follows that

$$u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1} = u_{\vec{x},\vec{y}} \circ \left(\sigma_{\theta_\Psi(\beta),\beta}\right)^{-1} = u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\beta),\beta}\right)^{-1},$$

and hence we have $\sigma_{\theta_\Psi(\alpha),\alpha} = \sigma_{\theta_\Psi(\beta),\beta}$ since $u_{\vec{p},\vec{q}}$ is injective. In particular, this implies that

$$\frac{1+\bar{\alpha}}{1+\alpha}\alpha = \sigma_{\theta_\Psi(\alpha),\alpha}(0) = \sigma_{\theta_\Psi(\beta),\beta}(0) = \frac{1+\bar{\beta}}{1+\beta}\beta,$$

which implies that $|\alpha| = |\beta|$ since $\left|\frac{1+\bar{\alpha}}{1+\alpha}\right| = 1$ and $\left|\frac{1+\bar{\beta}}{1+\beta}\right| = 1$. Using this fact and simplifying the above equation, one quickly obtains $\alpha = \beta$. Thus, Ψ is injective as required.

Regarding smoothness of Ψ, we refer the reader primarily to Chapter 3 and Appendix B of [MS04]. In short, for sufficiently large integers k and p, the moduli space $\widetilde{\mathcal{M}}(B_1)$ is an embedded submanifold of $W^{k,p}\left((D^2, S^1), \left((S^2)^3, L\right)\right)$, which is a Banach manifold consisting of continuous maps $(D^2, S^1) \to \left((S^2)^3, L\right)$ that are represented by $W^{k,p}$-functions in local coordinate charts. Given a smooth $u \in W^{k,p}\left((D^2, S^1), \left((S^2)^3, L\right)\right)$, the tangent space at u consists of $W^{k,p}$-sections of the bundle pair $\left(u^*T(S^2)^3, \left(u\big|_{S^1}\right)^* TL\right) \to (D^2, S^1)$:

$$T_u W^{k,p}\left((D^2, S^1), \left((S^2)^3, L\right)\right) = W^{k,p}\left((D^2, S^1), \left(u^*T(S^2)^3, \left(u\big|_{S^1}\right)^* TL\right)\right).$$

Now observe that $W^{k,p}\left((D^2, S^1), \left((S^2)^3, L\right)\right)$ is a Banach submanifold of the Banach space $W^{k,p}(D^2, \mathbb{R}^9)$ since $(S^2)^3 \subset \mathbb{R}^9$ in a natural way. Then note that the map

$$(\vec{p}, \vec{q}, \alpha, z) \mapsto u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}(z)$$

is smooth, from which it follows that Ψ is smooth when viewed as a map to $W^{k,p}(D^2, \mathbb{R}^9)$.

Now consider a path $\gamma : t \mapsto (\vec{p}(t), \vec{q}(t), \alpha(t))$ in $S^1(TS^2) \times \operatorname{Int}(D^2)$ so that $\frac{d}{dt}\big|_{t=0} \gamma(t)$ is a tangent vector in $T_{(\vec{p}(0),\vec{q}(0),\alpha(0))}(S^1(TS^2) \times \operatorname{Int}(D^2))$. Then observe that

$$\Psi_*\left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t)\right)(z) = \left.\frac{d}{dt}\right|_{t=0} \left(u_{\vec{p}(t),\vec{q}(t)} \circ \left(\sigma_{\theta_\Psi(\alpha(t)),\alpha(t)}\right)^{-1}(z)\right).$$

It is then not difficult to see that Ψ_* is injective. For example, one obtains 3 linearly independent vectors in

$$\Psi_*\left(T_{(\vec{p}(0),\vec{q}(0),\alpha(0))}\left(S^1\left(TS^2\right) \times \operatorname{Int}\left(D^2\right)\right)\right) \subset W^{k,p}\left(\left(D^2, S^1\right), \left(u^*T\left(S^2\right)^3, \left(u\big|_{S^1}\right)^* TL\right)\right)$$

by choosing paths γ_1, γ_2, and γ_3 that fix α and vary $(\vec{p}, \vec{q})$ in 3 distinct directions within $S^1(TS^2)$ – note that each such section will be non-vanishing at $1 \in D^2$. Then, one can obtain 2 more linearly independent vectors by choosing paths γ_4 and γ_5 that fix $\vec{p}$ and $\vec{q}$ while varying α in 2 distinct directions within $\operatorname{Int}(D^2)$ – note that each of these sections will vanish at $1 \in D^2$ according to Lemma 4.6. Thus, we see that Ψ_* is injective, and so Ψ is an immersion.

Finally, since $\Psi : S^1(TS^2) \times \operatorname{Int}(D^2) \to \widetilde{\mathcal{M}}(B_1)$ is a smooth bijective immersion, it follows from the Inverse Function Theorem that Ψ^{-1} is also smooth, and thus Ψ is a diffeomorphism as claimed. $\square$

We now consider the subgroup $H \subset G$ of automorphisms fixing -1 and 1. According to Appendix A of [BC12], this is a 1-dimensional subgroup consisting of the elements $\sigma_{0,\beta}$ with $\beta \in (-1, 1)$:

$$H = \{\sigma_{\theta,\beta} \in G \,|\, \theta = 0, \beta \in (-1, 1)\}.$$

As a subgroup of G, H also acts on $\widetilde{\mathcal{M}}(B_j)$ by the prescription

$$\sigma_{0,\beta} \cdot u = u \circ (\sigma_{0,\beta})^{-1}. \tag{4.6}$$

Proposition 4.10. *The subgroup $H \subset G$ acts on $S^1(TS^2) \times \mathrm{Int}(D^2)$ by the prescription*

$$\sigma_{0,\beta} \cdot (\vec{p}, \vec{q}, \alpha) = \left(\vec{p}, \vec{q}, \frac{\alpha + \beta + \alpha\bar{\alpha} + \alpha\beta}{1 + \bar{\alpha} + \bar{\alpha}\beta + \alpha\bar{\alpha}\beta}\right),$$

and the diffeomorphisms Ψ, $\mathfrak{s}_{12} \circ \Psi$, and $\mathfrak{s}_{13} \circ \Psi$ from Theorem 4.5 are equivariant with respect to this action and the action given by (4.6).

Proof. Using the fact that $\bar{\beta}_1 = \beta_1$ and $\bar{\beta}_2 = \beta_2$ for $\beta_1, \beta_2 \in (-1, 1)$, we compute that

$$\begin{aligned}\sigma_{0,\beta_1} \circ \sigma_{0,\beta_2}(z) &= \sigma_{0,\beta_1}\left(\frac{z + \beta_2}{1 + \beta_2 z}\right) = \frac{\dfrac{z + \beta_2}{1 + \beta_2 z} + \beta_1}{1 + \beta_1 \dfrac{z + \beta_2}{1 + \beta_2 z}} = \frac{z + \beta_2 + \beta_1 + \beta_1\beta_2 z}{1 + \beta_2 z + \beta_1 z + \beta_1\beta_2} \\ &= \frac{z(1 + \beta_1\beta_2) + (\beta_1 + \beta_2)}{(1 + \beta_1\beta_2) + (\beta_1 + \beta_2)z} = \frac{z + \dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}}{1 + \dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} z} = \sigma_{0, \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}}(z).\end{aligned}$$

Now, on the one hand, using the fact that $\beta_1, \beta_2 \in (-1, 1)$, we compute (leaving out some details) that

$$\begin{aligned}\sigma_{0,\beta_1} \cdot (\sigma_{0,\beta_2} \cdot (\vec{p}, \vec{q}, \alpha)) &= \sigma_{0,\beta_1} \cdot \left(\vec{p}, \vec{q}, \frac{\alpha + \beta_2 + \alpha\bar{\alpha} + \alpha\beta_2}{1 + \bar{\alpha} + \bar{\alpha}\beta_2 + \alpha\bar{\alpha}\beta_2}\right) \\ &= \left(\vec{p}, \vec{q}, \frac{\beta_1 + \beta_2 + \alpha + \alpha\bar{\alpha} + \alpha\beta_1 + \alpha\beta_2 + \alpha\beta_1\beta_2 + \alpha\bar{\alpha}\beta_1\beta_2}{1 + \beta_1\beta_2 + \bar{\alpha} + \bar{\alpha}\beta_2 + \alpha\bar{\alpha}\beta_2 + \bar{\alpha}\beta_1 + \alpha\bar{\alpha}\beta_1 + \bar{\alpha}\beta_1\beta_2}\right)\end{aligned}$$

On the other hand, using the first computation of the proof, we have

$$\begin{aligned}(\sigma_{0,\beta_1} \circ \sigma_{0,\beta_2}) \cdot (\vec{p}, \vec{q}, \alpha) &= \sigma_{0, \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}} \cdot (\vec{p}, \vec{q}, \alpha) \\ &= \left(\vec{p}, \vec{q}, \frac{\alpha + \dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} + \alpha\bar{\alpha} + \alpha\dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}}{1 + \bar{\alpha} + \bar{\alpha}\dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} + \alpha\bar{\alpha}\dfrac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}}\right) \\ &= \left(\vec{p}, \vec{q}, \frac{\beta_1 + \beta_2 + \alpha + \alpha\bar{\alpha} + \alpha\beta_1 + \alpha\beta_2 + \alpha\beta_1\beta_2 + \alpha\bar{\alpha}\beta_1\beta_2}{1 + \beta_1\beta_2 + \bar{\alpha} + \bar{\alpha}\beta_2 + \alpha\bar{\alpha}\beta_2 + \bar{\alpha}\beta_1 + \alpha\bar{\alpha}\beta_1 + \bar{\alpha}\beta_1\beta_2}\right),\end{aligned}$$

which shows that H acts on $S^1(TS^2) \times \mathrm{Int}(D^2)$ exactly as stated.

To show that Ψ is equivariant, we first note that a routine computation yields

$$(\sigma_{\nu,\gamma})^{-1}(z) = e^{-i\nu}\frac{z-\gamma e^{i\nu}}{1-\bar{\gamma}e^{-i\nu}z} \tag{4.7}$$

for $\sigma_{\nu,\gamma} \in G$. Now, where $\theta_\Psi(\alpha) = -i\log\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$ as in the statement of Theorem 4.5, we compute that

$$\begin{aligned}\theta_\Psi\left(\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\right) &= -i\log\left(\frac{1+\dfrac{\bar{\alpha}+\beta+\alpha\bar{\alpha}+\bar{\alpha}\beta}{1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta}}{1+\dfrac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}}\right)\\ &= -i\log\left(\frac{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}{1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta}\right)\end{aligned}$$

for $\alpha \in \mathrm{Int}(D^2)$ and $\beta \in (-1,1)$. Then taking $\gamma = \frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}$ and $\nu = \theta_\Psi(\gamma)$, we compute that

$$\begin{aligned}&\left(\sigma_{\theta_\Psi\left(\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\right),\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}}\right)^{-1}(z)\\ &\quad= \frac{1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\cdot\frac{z-\dfrac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\cdot\dfrac{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}{1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta}}{1-\dfrac{\bar{\alpha}+\beta+\alpha\bar{\alpha}+\bar{\alpha}\beta}{1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta}\cdot\dfrac{1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\cdot z}\\ &\quad= \frac{(1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta)\,z-(\alpha+\alpha\bar{\alpha}+\beta+\alpha\beta)}{(1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta)-(\bar{\alpha}+\alpha\bar{\alpha}+\beta+\bar{\alpha}\beta)\,z},\end{aligned}$$

while on the other hand we have

$$\begin{aligned}
\left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}\circ\left(\sigma_{0,\beta}\right)^{-1}(z) &= \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}\left(\frac{z-\beta}{1-\beta z}\right)\\
&= \frac{1+\alpha}{1+\bar{\alpha}}\cdot\frac{\dfrac{z-\beta}{1-\beta z}-\alpha\cdot\dfrac{1+\bar{\alpha}}{1+\alpha}}{1-\bar{\alpha}\cdot\dfrac{1+\alpha}{1+\bar{\alpha}}\cdot\dfrac{z-\beta}{1-\beta z}}\\
&= \frac{(1+\alpha+\alpha\beta+\alpha\bar{\alpha}\beta)\,z-(\alpha+\alpha\bar{\alpha}+\beta+\alpha\beta)}{(1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta)-(\bar{\alpha}+\alpha\bar{\alpha}+\beta+\bar{\alpha}\beta)\,z}.
\end{aligned}$$

Thus, we see that

$$\left(\sigma_{\theta_\Psi\left(\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\right),\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}}\right)^{-1} = \left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}\circ\left(\sigma_{0,\beta}\right)^{-1},$$

which implies that

$$\begin{aligned}
\Psi\left(\sigma_{0,\beta}\cdot(\vec{p},\vec{q},\alpha)\right) &= \Psi\left(\vec{p},\vec{q},\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\right)\\
&= u_{\vec{p},\vec{q}}\circ\left(\sigma_{\theta_\Psi\left(\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\right),\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}}\right)^{-1}\\
&= u_{\vec{p},\vec{q}}\circ\left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}\circ\left(\sigma_{0,\beta}\right)^{-1}\\
&= \sigma_{0,\beta}\cdot\left(u_{\vec{p},\vec{q}}\circ\left(\sigma_{\theta_\Psi(\alpha),\alpha}\right)^{-1}\right)\\
&= \sigma_{0,\beta}\cdot\Psi\left(\vec{p},\vec{q},\alpha\right).
\end{aligned}$$

Thus, we have shown that Ψ is equivariant. The above equation also holds when replacing Ψ with $\mathfrak{s}_{12}\circ\Psi$ or $\mathfrak{s}_{13}\circ\Psi$, thus completing the proof. □

Corollary 4.11. *The diffeomorphisms Ψ, $\mathfrak{s}_{12} \circ \Psi$, and $\mathfrak{s}_{13} \circ \Psi$ from Theorem 4.5 descend to diffeomorphisms*

$$\begin{aligned}
\Psi : \frac{S^1\left(TS^2\right) \times \operatorname{Int}\left(D^2\right)}{H} &\to \frac{\widetilde{\mathcal{M}}(B_1)}{H}, \\
\mathfrak{s}_{12} \circ \Psi : \frac{S^1\left(TS^2\right) \times \operatorname{Int}\left(D^2\right)}{H} &\to \frac{\widetilde{\mathcal{M}}(B_2)}{H}, \\
\mathfrak{s}_{13} \circ \Psi : \frac{S^1\left(TS^2\right) \times \operatorname{Int}\left(D^2\right)}{H} &\to \frac{\widetilde{\mathcal{M}}(B_3)}{H},
\end{aligned}$$

which (abusing notation) we still denote Ψ, $\mathfrak{s}_{12} \circ \Psi$, and $\mathfrak{s}_{13} \circ \Psi$.

Proof. This follows immediately from Theorem 4.5 and Proposition 4.10. □

It will be useful for us to parameterize the space $\left(S^1\left(TS^2\right) \times \operatorname{Int}\left(D^2\right)\right)/H$, which we do with the following proposition.

Proposition 4.12. *Where $\alpha_\Gamma : (-\pi, \pi) \to \operatorname{Int}\left(D^2\right)$ is given by*

$$\alpha_\Gamma\left(\phi\right) = \frac{i \sin\left(\phi\right)}{2 + 2\cos\left(\phi\right) - i\sin\left(\phi\right)},$$

we have a diffeomorphism $\Gamma : S^1\left(TS^2\right) \times (-\pi, \pi) \to \left(S^1\left(TS^2\right) \times \operatorname{Int}\left(D^2\right)\right)/H$ given by

$$\Gamma\left(\vec{p}, \vec{q}, \phi\right) = [\vec{p}, \vec{q}, \alpha_\Gamma\left(\phi\right)].$$

Proof. We will show that Γ has an inverse given by the formula

$$\Gamma^{-1}\left([\vec{p}, \vec{q}, \alpha]\right) = \left(\vec{p}, \vec{q}, -i\log\left(\frac{|\alpha|^2 + 2\alpha + 1}{|\alpha|^2 + 2\bar{\alpha} + 1}\right)\right).$$

It is extremely straightforward to compute that

$$\begin{aligned}\frac{|\alpha_\Gamma(\phi)|^2+2\alpha_\Gamma(\phi)+1}{|\alpha_\Gamma(\phi)|^2+2\overline{\alpha_\Gamma(\phi)}+1} &= \frac{1+\cos(\phi)+i\sin(\phi)}{1+\cos(\phi)-i\sin(\phi)} \\ &= \frac{1+e^{i\phi}}{1+e^{-i\phi}} = e^{i\phi}\frac{e^{-i\phi}+1}{1+e^{-i\phi}} = e^{i\phi},\end{aligned}$$

from which it follows that

$$-i\log\left(\frac{|\alpha_\Gamma(\phi)|^2+2\alpha_\Gamma(\phi)+1}{|\alpha_\Gamma(\phi)|^2+2\overline{\alpha_\Gamma(\phi)}+1}\right) = \phi,$$

and hence that $\Gamma^{-1}\circ\Gamma$ is the identity on $S^1(TS^2)\times(-\pi,\pi)$.

On the other hand, given $\alpha\in\mathrm{Int}(D^2)$, we note that

$$\frac{|\alpha|^2+2\alpha+1}{|\alpha|^2+2\bar{\alpha}+1} = e^{i\tau}$$

for some $\tau\in(-\pi,\pi)$ so that

$$-i\log\left(\frac{|\alpha|^2+2\alpha+1}{|\alpha|^2+2\bar{\alpha}+1}\right) = \tau,$$

and we then compute that

$$\begin{aligned}\cos(\tau) &= \mathrm{Re}\left(\frac{|\alpha|^2+2\alpha+1}{|\alpha|^2+2\bar{\alpha}+1}\right) = \frac{1}{2}\left(\frac{|\alpha|^2+2\alpha+1}{|\alpha|^2+2\bar{\alpha}+1}+\frac{|\alpha|^2+2\bar{\alpha}+1}{|\alpha|^2+2\alpha+1}\right), \\ \sin(\tau) &= \mathrm{Im}\left(\frac{|\alpha|^2+2\alpha+1}{|\alpha|^2+2\bar{\alpha}+1}\right) = \frac{i}{2}\left(\frac{|\alpha|^2+2\alpha+1}{|\alpha|^2+2\bar{\alpha}+1}-\frac{|\alpha|^2+2\bar{\alpha}+1}{|\alpha|^2+2\alpha+1}\right).\end{aligned}$$

Using this, it follows from a straightforward computation that

$$\begin{aligned}\alpha_{\Gamma}(\tau) &= \frac{i\sin(\tau)}{2+2\cos(\tau)-i\sin(\tau)} \\ &= \frac{\alpha-\bar{\alpha}}{2+\alpha+3\bar{\alpha}+2\alpha\bar{\alpha}},\end{aligned}$$

and so we have shown that

$$\Gamma\circ\Gamma^{-1}([\vec{p},\vec{q},\alpha]) = \left[\vec{p},\vec{q},\frac{\alpha-\bar{\alpha}}{2+\alpha+3\bar{\alpha}+2\alpha\bar{\alpha}}\right].$$

Then, writing $\beta = -\dfrac{\alpha+\bar{\alpha}+2\alpha\bar{\alpha}}{2+\alpha+\bar{\alpha}}$, another routine computation reveals that

$$\begin{aligned}\sigma_{0,\beta}\cdot(\vec{p},\vec{q},\alpha) &= \left(\vec{p},\vec{q},\frac{\alpha+\beta+\alpha\bar{\alpha}+\alpha\beta}{1+\bar{\alpha}+\bar{\alpha}\beta+\alpha\bar{\alpha}\beta}\right) \\ &= \left(\vec{p},\vec{q},\frac{\alpha-\dfrac{\alpha+\bar{\alpha}+2\alpha\bar{\alpha}}{2+\alpha+\bar{\alpha}}+\alpha\bar{\alpha}-\alpha\cdot\dfrac{\alpha+\bar{\alpha}+2\alpha\bar{\alpha}}{2+\alpha+\bar{\alpha}}}{1+\bar{\alpha}-\bar{\alpha}\cdot\dfrac{\alpha+\bar{\alpha}+2\alpha\bar{\alpha}}{2+\alpha+\bar{\alpha}}-\alpha\bar{\alpha}\cdot\dfrac{\alpha+\bar{\alpha}+2\alpha\bar{\alpha}}{2+\alpha+\bar{\alpha}}}\right) \\ &= \left(\vec{p},\vec{q},\frac{\alpha-\bar{\alpha}}{2+\alpha+3\bar{\alpha}+2\alpha\bar{\alpha}}\right).\end{aligned}$$

Thus, we have shown that

$$\Gamma\circ\Gamma^{-1}([\vec{p},\vec{q},\alpha]) = \left[\vec{p},\vec{q},\frac{\alpha-\bar{\alpha}}{2+\alpha+3\bar{\alpha}+2\alpha\bar{\alpha}}\right] = [\vec{p},\vec{q},\alpha],$$

completing the proof. □

Of particular interest are evaluation maps $\text{ev}_1 : \widetilde{\mathcal{M}}(B_j)/H \to L$ and $\text{ev}_{-1} : \widetilde{\mathcal{M}}(B_j)/H \to L$ defined by

$$\text{ev}_1([u]) = u(1) \quad\text{and}\quad \text{ev}_{-1}([u]) = u(-1).$$

Proposition 4.13. *Suppose that* $u_{\vec{p},\vec{q}}$ *is as in* (4.2), Γ *is as in Proposition 4.12, and* Ψ, $\mathfrak{s}_{12} \circ \Psi$, *and* $\mathfrak{s}_{13} \circ \Psi$ *are as in Corollary 4.11. Then we have*

$$\begin{aligned}
\mathrm{ev}_1 \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) &= u_{\vec{p},\vec{q}}\left(1\right), \\
\mathrm{ev}_{-1} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) &= u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right), \\
\mathrm{ev}_1 \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) &= \mathfrak{s}_{12} \circ u_{\vec{p},\vec{q}}\left(1\right), \\
\mathrm{ev}_{-1} \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) &= \mathfrak{s}_{12} \circ u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right), \\
\mathrm{ev}_1 \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) &= \mathfrak{s}_{13} \circ u_{\vec{p},\vec{q}}\left(1\right), \\
\mathrm{ev}_{-1} \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) &= \mathfrak{s}_{13} \circ u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right).
\end{aligned}$$

Proof. Following the definitions of the maps in view, we observe that

$$\Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) = u_{\vec{p},\vec{q}} \circ \left(\sigma_{\theta_\Psi(\alpha_\Gamma(\phi)),\alpha_\Gamma(\phi)}\right)^{-1},$$

and then a simple computation show that

$$\theta_\Psi\left(\alpha_\Gamma\left(\phi\right)\right) = -i \log\left(\frac{2 + 2\cos\left(\phi\right) - i\sin\left(\phi\right)}{2 + 2\cos\left(\phi\right) + i\sin\left(\phi\right)}\right).$$

Now, using (4.7) with $\nu = \theta_\Psi\left(\alpha_\Gamma\left(\phi\right)\right)$ and $\gamma = \alpha_\Gamma\left(\phi\right)$, we compute (with details omitted) that

$$\begin{aligned}
\left(\sigma_{\theta_\Psi(\alpha_\Gamma(\phi)),\alpha_\Gamma(\phi)}\right)^{-1}\left(z\right) &= e^{-i\theta_\Psi(\alpha_\Gamma(\phi))} \frac{z - \alpha_\Gamma\left(\phi\right) e^{i\theta_\Psi(\alpha_\Gamma(\phi))}}{1 - \overline{\alpha_\Gamma\left(\phi\right)} e^{-i\theta_\Psi(\alpha_\Gamma(\phi))} z} \\
&= \frac{2z + 2z\cos\left(\phi\right) + i\left(z-1\right)\sin\left(\phi\right)}{2 + 2\cos\left(\phi\right) + i\left(z-1\right)\sin\left(\phi\right)}.
\end{aligned}$$

It follows that

$$\left(\sigma_{\theta_\Psi(\alpha_\Gamma(\phi)),\alpha_\Gamma(\phi)}\right)^{-1}\left(1\right) = 1$$

and also that

$$\left(\sigma_{\theta_\Psi(\alpha_\Gamma(\phi)),\alpha_\Gamma(\phi)}\right)^{-1}(-1) = \frac{-2-2\cos(\phi)-2i\sin(\phi)}{2+2\cos(\phi)-2i\sin(\phi)}$$
$$= -\frac{1+e^{i\phi}}{1+e^{-i\phi}} = -e^{i\phi}\frac{e^{-i\phi}+1}{1+e^{-i\phi}} = -e^{i\phi} = e^{i(\phi+\pi)}.$$

The proposition then follows immediately from the definitions of ev_1 and ev_{-1}. □

We now provide bases for $T_{(\vec{p},\vec{q},\phi)}\left(S^1\left(TS^2\right)\times(-\pi,\pi)\right)$ and $T_{\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}}L$ that will be useful later for proving transversality of certain evaluation maps. We define

$$\begin{aligned}
\mathcal{V}_1(\vec{p},\vec{q},\phi) &= \frac{\sqrt{3}}{2}(-\vec{q},\vec{p},0),\\
\mathcal{V}_2(\vec{p},\vec{q},\phi) &= \frac{\sqrt{3}}{2}\left(\vec{0},\vec{q}\times\vec{p},0\right),\\
\mathcal{V}_3(\vec{p},\vec{q},\phi) &= \left(\vec{q}\times\vec{p},\vec{0},0\right),\\
\mathcal{V}_4(\vec{p},\vec{q},\phi) &= \left(\vec{0},\vec{0},1\right),
\end{aligned} \tag{4.8}$$

which are easily seen to provide a basis for the tangent space $T_{(\vec{p},\vec{q},\phi)}\left(S^1\left(TS^2\right)\times(-\pi,\pi)\right)$. We provide three different bases for $T_{\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}}L$ that will correspond to the maps in Proposition 4.13. Define

$$\begin{aligned}
\mathcal{X}_1\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{u}\times(\vec{v}\times\vec{w}) & \vec{v}\times(\vec{w}\times\vec{u}) & \vec{w}\times(\vec{u}\times\vec{v})\end{pmatrix},\\
\mathcal{X}_2\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{v}\times\vec{w} & \frac{1}{2}\vec{w}\times\vec{v} & \frac{1}{2}\vec{w}\times\vec{v}\end{pmatrix},\\
\mathcal{X}_3\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{0} & \vec{w}\times\vec{v} & \vec{v}\times\vec{w}\end{pmatrix},
\end{aligned} \tag{4.9}$$

and define

$$\begin{aligned}
\mathcal{Y}_1\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}-\vec{u}\times(\vec{v}\times\vec{w}) & -\vec{v}\times(\vec{w}\times\vec{u}) & -\vec{w}\times(\vec{u}\times\vec{v})\end{pmatrix},\\
\mathcal{Y}_2\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\tfrac{1}{2}\vec{w}\times\vec{u} & \vec{u}\times\vec{w} & \tfrac{1}{2}\vec{w}\times\vec{u}\end{pmatrix},\\
\mathcal{Y}_3\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{w}\times\vec{u} & \vec{0} & \vec{w}\times\vec{u}\end{pmatrix},
\end{aligned} \tag{4.10}$$

and finally

$$\begin{aligned}
\mathcal{Z}_1\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}-\vec{u}\times(\vec{v}\times\vec{w}) & -\vec{v}\times(\vec{w}\times\vec{u}) & -\vec{w}\times(\vec{u}\times\vec{v})\end{pmatrix},\\
\mathcal{Z}_2\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\tfrac{1}{2}\vec{u}\times\vec{v} & \tfrac{1}{2}\vec{u}\times\vec{v} & \vec{v}\times\vec{u}\end{pmatrix},\\
\mathcal{Z}_3\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix} &= \begin{pmatrix}\vec{v}\times\vec{u} & \vec{u}\times\vec{v} & \vec{0}\end{pmatrix}.
\end{aligned} \tag{4.11}$$

Proposition 4.14. *At each point $(\vec{p}, \vec{q}, \phi) \in S^1(TS^2) \times (-\pi, \pi)$, we have*

$$\begin{aligned}
(\mathrm{ev}_1 \circ \Psi \circ \Gamma)_* \mathcal{V}_j &= \mathcal{X}_j,\\
(\mathrm{ev}_1 \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma)_* \mathcal{V}_j &= \mathcal{Y}_j,\\
(\mathrm{ev}_1 \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma)_* \mathcal{V}_j &= \mathcal{Z}_j,
\end{aligned}$$

for each $j \in \{1, 2, 3\}$ and

$$\begin{aligned}
(\mathrm{ev}_1 \circ \Psi \circ \Gamma)_* \mathcal{V}_4 &= \begin{pmatrix}\vec{0} & \vec{0} & \vec{0}\end{pmatrix},\\
(\mathrm{ev}_1 \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma)_* \mathcal{V}_4 &= \begin{pmatrix}\vec{0} & \vec{0} & \vec{0}\end{pmatrix},\\
(\mathrm{ev}_1 \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma)_* \mathcal{V}_4 &= \begin{pmatrix}\vec{0} & \vec{0} & \vec{0}\end{pmatrix}.
\end{aligned}$$

Moreover, if $\phi = 0$, then we have

$$(\mathrm{ev}_{-1} \circ \Psi \circ \Gamma)_* \mathcal{V}_1 = -\mathcal{X}_1,$$
$$(\mathrm{ev}_{-1} \circ \Psi \circ \Gamma)_* \mathcal{V}_2 = -\mathcal{X}_2,$$
$$(\mathrm{ev}_{-1} \circ \Psi \circ \Gamma)_* \mathcal{V}_3 = \mathcal{X}_3,$$
$$(\mathrm{ev}_{-1} \circ \Psi \circ \Gamma)_* \mathcal{V}_4 = \mathcal{X}_3,$$

and

$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma)_* \mathcal{V}_1 = -\mathcal{Y}_1,$$
$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma)_* \mathcal{V}_2 = -\mathcal{Y}_2,$$
$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma)_* \mathcal{V}_3 = \mathcal{Y}_3,$$
$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{12} \circ \Psi \circ \Gamma)_* \mathcal{V}_4 = \mathcal{Y}_3,$$

and finally

$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma)_* \mathcal{V}_1 = -\mathcal{Z}_1,$$
$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma)_* \mathcal{V}_2 = -\mathcal{Z}_2,$$
$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma)_* \mathcal{V}_3 = \mathcal{Z}_3,$$
$$(\mathrm{ev}_{-1} \circ \mathfrak{s}_{13} \circ \Psi \circ \Gamma)_* \mathcal{V}_4 = \mathcal{Z}_3.$$

Proof. Using Proposition 4.13, an explicit formula is given for each function whose derivative appears in the statement of this proposition. One can then easily verify each equation by hand or with the aid of a computer algebra system. □

Chapter 5

Nondisplaceability of the Lagrangian

5.1 The Pearl Complex

Here we briefly describe the pearl complex as defined in [BC07] and [BC09]. Let $L \subset M$ be a connected, closed, monotone, Lagrangian submanifold of a connected, closed, symplectic manifold (M, ω). Assume also that the minimal Maslov number of L is at least 2; more precisely, where $I_\mu : \pi_2(M, L) \to \mathbb{Z}$ is the Maslov homomorphism, we require that

$$\min \{ I_\mu(A) > 0 \,|\, A \in \pi_2(M, L) \} \geq 2.$$

Writing $H_2^D(M, L)$ for the image of the Hurewicz homomorphism $\pi_2(M, L) \to H_2(M, L)$, we consider the group ring $\widetilde{\Lambda} = (\mathbb{Z}/2\mathbb{Z})\,[H_2^D(M, L)]$, whose elements may be thought of as "polynomials" in the formal variable T with coefficients in $\mathbb{Z}/2\mathbb{Z}$. More explicitly, we write $P(T) \in \widetilde{\Lambda}$ as

$$P(T) = \sum_{A \in H_2^D(M,L)} a_A T^A,$$

where $a_A \in \mathbb{Z}/2\mathbb{Z}$, only finitely many of the coefficients a_A are non-zero, and the "polynomials" are subject to the obvious addition and multiplication rules (including $T^0 = 1 \in \mathbb{Z}/2\mathbb{Z}$).

Given a commutative $\widetilde{\Lambda}$-algebra $\mathcal{R}$ with a unit $1_{\mathcal{R}} \in \mathcal{R}$ and with structural morphism $q : \widetilde{\Lambda} \to \mathcal{R}$, we will describe the pearl complex with coefficients in $\mathcal{R}$.

Consider a Morse function $f : L \to \mathbb{R}$ and a Riemannian metric ρ so that the pair (f, ρ) is Morse-Smale, and let J be an almost complex structure compatible with ω. Letting $\mathrm{Crit}\,(f) \subset L$ denote the set of critical points of f, it has been shown in [BC07] and [BC09] that for a generic choice of the triple (f, ρ, J), there is a chain complex

$$\mathcal{C}\,(L; f, \rho, J; \mathcal{R}) = \left((\mathbb{Z}/2\mathbb{Z})\,\langle \mathrm{Crit}\,(f)\rangle \otimes \mathcal{R}, d^{\mathcal{R}}\right)$$

called the pearl complex with coefficients in $\mathcal{R}$.

In order to describe the differential $d^{\mathcal{R}}$, we must first define the moduli spaces of "pearly trajectories" between critical points of f. Let $\gamma_t : L \to L$ denote the time t *negative* gradient flow of f, and let $W_f^s\,(x)$ and $W_f^u\,(x)$ denote the stable and unstable manifolds, respectively, of a critical point $x \in \mathrm{Crit}\,(f)$ taken with respect to γ_t, namely

$$W_f^s\,(x) = \left\{p \in L \,\middle|\, \lim_{t\to\infty} \gamma_t\,(p) = x\right\} \text{ and } W_f^u\,(x) = \left\{p \in L \,\middle|\, \lim_{t\to-\infty} \gamma_t\,(p) = x\right\}.$$

Given $x, y \in \mathrm{Crit}\,(f)$, we consider the space of gradient trajectories from x to y:

$$\widetilde{m}\,(x, y) = W_f^s\,(y) \cap W_f^u\,(x)\,.$$

Note that $\mathbb{R}$ acts on $\widetilde{m}\,(x, y)$ by $t \cdot p = \gamma_t\,(p)$, and define $m\,(x, y) = \widetilde{m}\,(x, y)\,/\mathbb{R}$. By standard Morse theory arguments, the space $m(x, y)$ is a compact 0-dimensional manifold whenever $|x| - |y| = 1$. Here, the notation $|x|$ denotes the Morse index of the critical point x.

Writing $\mathbb{R}_+$ for the positive real numbers, we define $Q_{f,\rho} \subset L \times L$ to be the image of the embedding

$$(L \setminus \mathrm{Crit}\,(f)) \times \mathbb{R}_+ \hookrightarrow L \times L,$$
$$(p, t) \mapsto (p, \gamma_t\,(p))\,.$$

Additionally, for each non-zero homology class $A \in H_2^D\,(M, L)$, we consider the moduli space $\widetilde{\mathcal{M}}\,(A, J)$ of parametrized J-holomorphic disks $u : (D^2, S^1) \to (M, L)$ representing the homology class A. Let H denote the group of automorphisms of D^2 that fix -1 and 1, and note that H acts on $\widetilde{\mathcal{M}}\,(A, J)$ in an obvious way (just as in (4.6)). Then given a sequence of non-zero homology classes $\mathbf{A} = (A_1, \ldots, A_\ell)$ with $A_1, \ldots, A_\ell \in H_2^D\,(M, L)$, we write

$$\mathcal{M}\,(\mathbf{A}, J) = \widetilde{\mathcal{M}}\,(A_1, J)\,/H \times \cdots \times \widetilde{\mathcal{M}}\,(A_\ell, J)\,/H.$$

Since each element of H fixes -1 and 1, we have a well-defined evaluation map

$$\mathrm{ev}_{\mathbf{A}} : \mathcal{M}\,(\mathbf{A}, J) \to L^{2\ell},$$
$$([u_1], \ldots, [u_\ell]) \mapsto (u_1\,(-1)\,, u_1\,(1)\,, \ldots, u_\ell\,(-1)\,, u_\ell\,(1))\,.$$

Given $x, y \in \mathrm{Crit}\,(f)$, we then define

$$\mathcal{P}\,(x, y, \mathbf{A}; f, \rho, J) = \mathrm{ev}_{\mathbf{A}}^{-1}\left(W_f^u\,(x) \times (Q_{f,\rho})^{(\ell-1)} \times W_f^s\,(y)\right)$$

as the moduli space of pearly trajectories from x to y.

Given a sequence $\mathbf{A} = (A_1, \ldots, A_\ell)$ as above, we write $I_\mu\,(\mathbf{A}) = \sum_{j=1}^{\ell} I_\mu\,(A_j)$. By choosing the data (f, ρ, J) generically, one can ensure that, for every sequence $\mathbf{A}$ and pair of critical points $x, y \in \mathrm{Crit}\,(f)$ satisfying $I_\mu\,(\mathbf{A}) + |x| - |y| - 1 \leq 1$, the evaluation map $\mathrm{ev}_{\mathbf{A}}$

is transverse to $W_f^u(x) \times (Q_{f,\rho})^{(\ell-1)} \times W_f^s(y)$. Under such assumptions, the moduli space $\mathcal{P}(x, y, \mathbf{A}; f, \rho, J)$ is either empty or a smooth manifold of dimension

$$\dim(\mathcal{P}(x, y, \mathbf{A}; f, \rho, J)) = I_\mu(\mathbf{A}) + |x| - |y| - 1.$$

Moreover, in the case that $\mathcal{P}(x, y, \mathbf{A}; f, \rho, J)$ is 0-dimensional, it is in fact compact and so consists of a finite number of points. Where $\#_2(FS)$ denotes the mod 2 count of a finite set FS, the differential $d^{\mathcal{R}}$ is defined by the equation

$$\begin{aligned} d^{\mathcal{R}}(x \otimes 1_{\mathcal{R}}) = & \sum_{\substack{y \in \text{Crit}(f) \\ |x|-|y|=1}} \#_2(m(x,y))\, y \otimes 1_{\mathcal{R}} \\ & + \sum_{\substack{y \in \text{Crit}(f), \mathbf{A}=(A_1,\ldots,A_\ell) \\ I_\mu(\mathbf{A})+|x|-|y|-1=0}} \#_2(\mathcal{P}(x, y, \mathbf{A}; f, \rho, J))\, y \otimes q\left(T^{A_1+\cdots+A_\ell}\right). \end{aligned} \tag{5.1}$$

According to [BC07] and [BC09], $d^{\mathcal{R}}$ as above satisfies $d^{\mathcal{R}} \circ d^{\mathcal{R}} = 0$, and the resulting homology of $\mathcal{C}(L; f, \rho, J; \mathcal{R})$ is denoted $QH_*(L; \mathcal{R})$. Moreover, by Theorem A in [BC09], the isomorphism class of the homology $QH_*(L; \mathcal{R})$ is independent of the choice of the generic triple (f, ρ, J), and there is an isomorphism $QH_*(L; \mathcal{R}) \to HF_*(L; \mathcal{R})$, where $HF_*(L; \mathcal{R})$ is the Lagrangian Floer homology as described in Section 3.2.g of [BC09].

5.2 Pearl Complex Computation

Here we consider again the specific case of the Lagrangian $L \subset (S^2)^3$ as described in Chapters 3 and 4. The goal of this section is to show that the homology $QH_*(L; \mathcal{R})$ as described in Section 5.1 is non-vanishing for an appropriate choice of coefficients $\mathcal{R}$. To that end, we begin by describing a Morse function on $S^3 \subset \mathbb{R}^4$. We write elements of S^3 as vectors $\vec{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^\top$ and the standard basis vectors of $\mathbb{R}^4$ as $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and $\vec{e}_4$.

Proposition 5.1. *The function $\widetilde{h} : S^3 \to \mathbb{R}$ defined by $\widetilde{h}(\vec{x}) = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2$ is Morse and has the following critical points:*

(i) the points $\vec{e}_1$ and $-\vec{e}_1$, which have index 0,

(ii) the points $\vec{e}_2$ and $-\vec{e}_2$, which have index 1,

(iii) the points $\vec{e}_3$ and $-\vec{e}_3$, which have index 2,

(iv) the points $\vec{e}_4$ and $-\vec{e}_4$, which have index 3.

Moreover, with respect to the standard metric on S^3, the function $\widetilde{h}$ is Morse-Smale with stable and unstable submanifolds given by

$$\begin{aligned}
W^s_{\widetilde{h}}(\vec{e}_1) &= \{\vec{x} \mid x_1 > 0\}, & W^u_{\widetilde{h}}(\vec{e}_1) &= \{\vec{e}_1\}, \\
W^s_{\widetilde{h}}(-\vec{e}_1) &= \{\vec{x} \mid x_1 < 0\}, & W^u_{\widetilde{h}}(-\vec{e}_1) &= \{-\vec{e}_1\}, \\
W^s_{\widetilde{h}}(\vec{e}_2) &= \{\vec{x} \mid x_1 = 0, x_2 > 0\}, & W^u_{\widetilde{h}}(\vec{e}_2) &= \{\vec{x} \mid x_3 = 0, x_4 = 0, x_2 > 0\}, \\
W^s_{\widetilde{h}}(-\vec{e}_2) &= \{\vec{x} \mid x_1 = 0, x_2 < 0\}, & W^u_{\widetilde{h}}(-\vec{e}_2) &= \{\vec{x} \mid x_3 = 0, x_4 = 0, x_2 < 0\}, \\
W^s_{\widetilde{h}}(\vec{e}_3) &= \{\vec{x} \mid x_1 = 0, x_2 = 0, x_3 > 0\}, & W^u_{\widetilde{h}}(\vec{e}_3) &= \{\vec{x} \mid x_4 = 0, x_3 > 0\}, \\
W^s_{\widetilde{h}}(-\vec{e}_3) &= \{\vec{x} \mid x_1 = 0, x_2 = 0, x_3 < 0\}, & W^u_{\widetilde{h}}(-\vec{e}_3) &= \{\vec{x} \mid x_4 = 0, x_3 < 0\}, \\
W^s_{\widetilde{h}}(\vec{e}_4) &= \{\vec{e}_4\}, & W^u_{\widetilde{h}}(\vec{e}_4) &= \{\vec{x} \mid x_4 > 0\}, \\
W^s_{\widetilde{h}}(-\vec{e}_4) &= \{-\vec{e}_4\}, & W^u_{\widetilde{h}}(-\vec{e}_4) &= \{\vec{x} \mid x_4 < 0\}.
\end{aligned}$$

Proof. First, a routine computation reveals that the gradient vector field of $\widetilde{h}$ (with respect to the standard Riemannian metric on S^3) is given at $\vec{x} \in S^3$ by

$$\operatorname{grad}_{\widetilde{h}}(\vec{x}) = \begin{pmatrix} 2\left(1 - \widetilde{h}(\vec{x})\right) x_1 \\ 2\left(2 - \widetilde{h}(\vec{x})\right) x_2 \\ 2\left(3 - \widetilde{h}(\vec{x})\right) x_3 \\ 2\left(4 - \widetilde{h}(\vec{x})\right) x_4 \end{pmatrix}.$$

It is simple to compute that this vector field vanishes at each of the purported critical points, and so they are indeed critical points. On the other hand, if $\vec{x} \in S^3$ is not one of the known critical points, then at least two of its entries must be nonzero (for example, $x_1 \neq 0$ and $x_2 \neq 0$). In any case, it follows that at least one of the entries of the vector $\mathrm{grad}_{\widetilde{h}}(\vec{x})$ will be non-zero since $\widetilde{h}(\vec{x})$ has a well-defined value (and cannot be simultaneously 1 and 2 for instance). It follows that the critical points listed in the statement of the proposition are indeed the only ones.

To show that each critical point has the index claimed, it suffices to work in local coordinates. We exhibit here the proof in the case of $\vec{e}_3$, and the proofs for the other critical points are very similar. In a small neighborhood of $\vec{0} \in \mathbb{R}^3$, the map

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}} y_1 \\ y_2 \\ \sqrt{1 - \left(\frac{1}{2} y_1^2 + y_2^2 + y_3^2\right)} \\ y_3 \end{pmatrix}$$

parametrizes a neighborhood of the critical point $\vec{e}_3 \in S^3$ with $\vec{0}$ mapping to $\vec{e}_3$. Composing this parametrization with $\widetilde{h}$ yields the map

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \frac{y_1^2}{2} + 2y_2^2 + 3\left(1 - \left(\frac{1}{2} y_1^2 + y_2^2 + y_3^2\right)\right) + 4y_3^2 = 3 - y_1^2 - y_2^2 + y_3^2,$$

which clearly shows that $\vec{e}_3$ is a Morse critical point of index 2.

Now consider the negative gradient vector field of $\widetilde{h}$, which is given by

$$-\mathrm{grad}_{\widetilde{h}}(\vec{x}) = \begin{pmatrix} 2\left(\widetilde{h}(\vec{x}) - 1\right)x_1 \\ 2\left(\widetilde{h}(\vec{x}) - 2\right)x_2 \\ 2\left(\widetilde{h}(\vec{x}) - 3\right)x_3 \\ 2\left(\widetilde{h}(\vec{x}) - 4\right)x_4 \end{pmatrix},$$

and we let γ_t denote the time t flow of this vector field. We write $\vec{a}$ for an arbitrary point in S^3, and we consider the curve $\vec{x}^{(\vec{a})} : \mathbb{R} \to S^3$ given by $\vec{x}^{(\vec{a})}(t) = \gamma_t(\vec{a})$. By the definition of the flow of a vector field, we have

$$\frac{d}{dt}\vec{x}^{(\vec{a})}(t) = \frac{d}{dt}\begin{pmatrix} \vec{x}_1^{(\vec{a})}(t) \\ \vec{x}_2^{(\vec{a})}(t) \\ \vec{x}_3^{(\vec{a})}(t) \\ \vec{x}_4^{(\vec{a})}(t) \end{pmatrix} = \begin{pmatrix} 2\left(\widetilde{h}\left(\vec{x}^{(\vec{a})}(t)\right) - 1\right)\vec{x}_1^{(\vec{a})}(t) \\ 2\left(\widetilde{h}\left(\vec{x}^{(\vec{a})}(t)\right) - 2\right)\vec{x}_2^{(\vec{a})}(t) \\ 2\left(\widetilde{h}\left(\vec{x}^{(\vec{a})}(t)\right) - 3\right)\vec{x}_3^{(\vec{a})}(t) \\ 2\left(\widetilde{h}\left(\vec{x}^{(\vec{a})}(t)\right) - 4\right)\vec{x}_4^{(\vec{a})}(t) \end{pmatrix} \text{ and } \vec{x}^{(\vec{a})}(0) = \vec{a}. \tag{5.2}$$

Now suppose that $\vec{a} \in S^3$ satisfies $a_1 > 0$. Since $\widetilde{h}(\vec{x}) \geq 1$ for all $\vec{x}$, it follows from (5.2) that $\vec{x}_1^{(\vec{a})}(t) > 0$ for all $t \geq 0$. In particular, this implies that $\lim_{t\to\infty} \gamma_t(\vec{a}) = \lim_{t\to\infty} \vec{x}^{(\vec{a})}(t)$ cannot be any critical point other than $\vec{e}_1$. It follows that $\{\vec{x} \,|\, x_1 > 0\} \subset W^s_{\widetilde{h}}(\vec{e}_1)$. On the other hand, if $\vec{a}$ satisfies $a_1 < 0$, then a similar argument shows that $\{\vec{x} \,|\, x_1 < 0\} \subset W^s_{\widetilde{h}}(-\vec{e}_1)$. Moreover, if $\vec{a}$ satisfies $a_1 = 0$, then it follows from (5.2) that $\vec{x}_1^{(\vec{a})}(t) = 0$ for all t, and hence $\vec{a} \notin W^s_{\widetilde{h}}(\vec{e}_1)$ and also $\vec{a} \notin W^s_{\widetilde{h}}(-\vec{e}_1)$. Thus, we have shown that $W^s_{\widetilde{h}}(\vec{e}_1)$ and $W^s_{\widetilde{h}}(-\vec{e}_1)$ are exactly as claimed in the statement of the proposition.

Next, suppose that $\vec{a}$ satisfies $a_1 = 0$ so that $a_2^2 + a_3^2 + a_4^2 = 1$. From the definition of $\widetilde{h}$, one can then easily see that $\widetilde{h}(\vec{a}) \geq 2$. If we suppose also that $a_2 > 0$, then it follows from (5.2) that $\vec{x}_2^{(\vec{a})}(t) > 0$ for all $t \geq 0$ (since $\vec{x}_1^{(\vec{a})}(t) = 0$ for all t and $\widetilde{h}(\vec{x}) \geq 2$ for all $\vec{x}$ with $x_1 = 0$). This implies that $\lim_{t\to\infty} \gamma_t(\vec{a}) = \lim_{t\to\infty} \vec{x}^{(\vec{a})}(t)$ cannot be any critical point other than

$\vec{e}_2$, and so

$$\{\vec{x} \mid x_1 = 0, x_2 > 0\} \subset W^s_{\widetilde{h}}(\vec{e}_2) .$$

On the other hand if $\vec{a}$ satisfies $a_1 = 0$ and $a_2 < 0$, then a similar argument shows that

$$\{\vec{x} \mid x_1 = 0, x_2 < 0\} \subset W^s_{\widetilde{h}}(-\vec{e}_2) .$$

Moreover, if $\vec{a}$ satisfies $a_1 \neq 0$, then we already know that $\vec{a} \notin W^s_{\widetilde{h}}(\vec{e}_2)$ and $\vec{a} \notin W^s_{\widetilde{h}}(-\vec{e}_2)$ since it must be in either $W^s_{\widetilde{h}}(\vec{e}_1)$ or $W^s_{\widetilde{h}}(-\vec{e}_1)$. If $\vec{a}$ satisfies $a_2 = 0$, then it follows from (5.2) that $\vec{x}^{(\vec{a})}_2(t) = 0$ for all t, and hence $\vec{a} \notin W^s_{\widetilde{h}}(\vec{e}_2)$ and also $\vec{a} \notin W^s_{\widetilde{h}}(-\vec{e}_2)$. Thus, we have shown that $W^s_{\widetilde{h}}(\vec{e}_2)$ and $W^s_{\widetilde{h}}(-\vec{e}_2)$ are exactly as claimed in the statement of the proposition.

Similar arguments work to show that the other stable submanifolds are as claimed, and by working with the ordinary gradient flow instead of the negative gradient flow, the exact same arguments will work to prove that the unstable submanifolds (of the negative gradient flow) are as claimed in the proposition. The proofs of these remaining cases are omitted. Proving that $\widetilde{h}$ is Morse-Smale with respect to the standard Riemannian metric is done by going through all cases. Here we present a few examples and leave the remaining cases as exercises for the interested reader.

First, we note that $W^s_{\widetilde{h}}(\vec{e}_1)$ is transverse to every unstable submanifold. This is obvious since $W^s_{\widetilde{h}}(\vec{e}_1)$ is an open subset of S^3, and the same holds for $W^s_{\widetilde{h}}(-\vec{e}_1)$.

Next, we will show that $W^s_{\widetilde{h}}(\vec{e}_2)$ is transverse to every unstable submanifold. Note that $W^s_{\widetilde{h}}(\vec{e}_2)$ does not intersect $W^u_{\widetilde{h}}(\vec{e}_1)$, $W^u_{\widetilde{h}}(-\vec{e}_1)$, or $W^u_{\widetilde{h}}(-\vec{e}_2)$, and so there is nothing to prove in these cases. If $\vec{x} \in W^s_{\widetilde{h}}(\vec{e}_2) \cap W^u_{\widetilde{h}}(\vec{e}_2)$, then we have $\vec{x} = \vec{e}_2$, and a basis for $T_{\vec{e}_2}S^3$ is given by $\vec{e}_1 \in T_{\vec{e}_2}W^u_{\widetilde{h}}(\vec{e}_2)$ and $\vec{e}_3, \vec{e}_4 \in T_{\vec{e}_2}W^s_{\widetilde{h}}(\vec{e}_2)$. Thus, $W^s_{\widetilde{h}}(\vec{e}_2)$ is transverse to $W^u_{\widetilde{h}}(\vec{e}_2)$. If $\vec{x} \in W^s_{\widetilde{h}}(\vec{e}_2) \cap W^u_{\widetilde{h}}(\vec{e}_3)$ or $\vec{x} \in W^s_{\widetilde{h}}(\vec{e}_2) \cap W^u_{\widetilde{h}}(-\vec{e}_3)$, respectively, then we have $x_1 = 0$ and $x_4 = 0$, and thus a basis for $T_{\vec{x}}S^3$ is given by the vectors $\vec{e}_4$, $\begin{pmatrix} 0 & x_3 & -x_2 & 0 \end{pmatrix}^\top \in T_{\vec{x}}W^s_{\widetilde{h}}(\vec{e}_2)$ and $\vec{e}_1 \in T_{\vec{x}}W^u_{\widetilde{h}}(\vec{e}_3)$ or $\vec{e}_1 \in T_{\vec{x}}W^u_{\widetilde{h}}(-\vec{e}_3)$, respectively. Hence, $W^s_{\widetilde{h}}(\vec{e}_2)$ is transverse to both

$W^u_{\widetilde{h}}(\vec{e}_3)$ and $W^u_{\widetilde{h}}(-\vec{e}_3)$. Since $W^u_{\widetilde{h}}(\vec{e}_4)$ and $W^u_{\widetilde{h}}(-\vec{e}_4)$ are both open subsets of S^3, there is nothing to prove regarding the transversality of $W^s_{\widetilde{h}}(\vec{e}_2)$ with these unstable submanifolds. Hence, we have shown that $W^s_{\widetilde{h}}(\vec{e}_2)$ is transverse to all of the unstable submanifolds.

The remainder of the cases are similar and are left to the reader. □

We recall that S^3 can be identified with the group of unit quaternions $\mathcal{S}$ via the correspondence $\vec{x} \mapsto x_1 + x_2 i + x_3 j + x_4 k$, and so we may think of the map Φ defined in (3.1) as a map $S^3 \to SO(3)$. We observe that Φ is invariant with respect to the action of the antipodal map on S^3, and so the following proposition follows rather easily from Proposition 5.1.

Proposition 5.2. *The map $\Phi : S^3 \to SO(3)$ induces a Riemannian metric g on $SO(3)$ from the standard Riemannian metric on S^3, and the map $\widetilde{h} : S^3 \to \mathbb{R}$ descends to a map $h : SO(3) \to \mathbb{R}$ satisfying $\widetilde{h} = h \circ \Phi$. With respect to the induced metric, the function h is Morse-Smale with the following critical points:*

(i) the point $\Phi(\pm\vec{e}_1)$, which has index 0,

(ii) the point $\Phi(\pm\vec{e}_2)$, which has index 1,

(iii) the point $\Phi(\pm\vec{e}_3)$, which has index 2,

(iv) the point $\Phi(\pm\vec{e}_4)$, which has index 3.

Additionally, the stable and unstable submanifolds of these critical points are given by:

$$W_h^s(\Phi(\pm\vec{e}_1)) = \Phi\left(W_{\widetilde{h}}^s(\vec{e}_1)\right) = \Phi\left(W_{\widetilde{h}}^s(-\vec{e}_1)\right),$$

$$W_h^u(\Phi(\pm\vec{e}_1)) = \Phi\left(W_{\widetilde{h}}^u(\vec{e}_1)\right) = \Phi\left(W_{\widetilde{h}}^u(-\vec{e}_1)\right),$$

$$W_h^s(\Phi(\pm\vec{e}_2)) = \Phi\left(W_{\widetilde{h}}^s(\vec{e}_2)\right) = \Phi\left(W_{\widetilde{h}}^s(-\vec{e}_2)\right),$$

$$W_h^u(\Phi(\pm\vec{e}_2)) = \Phi\left(W_{\widetilde{h}}^u(\vec{e}_2)\right) = \Phi\left(W_{\widetilde{h}}^u(-\vec{e}_2)\right),$$

$$W_h^s(\Phi(\pm\vec{e}_3)) = \Phi\left(W_{\widetilde{h}}^s(\vec{e}_3)\right) = \Phi\left(W_{\widetilde{h}}^s(-\vec{e}_3)\right),$$

$$W_h^u(\Phi(\pm\vec{e}_3)) = \Phi\left(W_{\widetilde{h}}^u(\vec{e}_3)\right) = \Phi\left(W_{\widetilde{h}}^u(-\vec{e}_3)\right),$$

$$W_h^s(\Phi(\pm\vec{e}_4)) = \Phi\left(W_{\widetilde{h}}^s(\vec{e}_4)\right) = \Phi\left(W_{\widetilde{h}}^s(-\vec{e}_4)\right),$$

$$W_h^u(\Phi(\pm\vec{e}_4)) = \Phi\left(W_{\widetilde{h}}^u(\vec{e}_4)\right) = \Phi\left(W_{\widetilde{h}}^u(-\vec{e}_4)\right).$$

Proof. According to the discussion preceding Proposition 3.1, the map $\Phi : S^3 \to SO(3)$ is a two-fold covering map. Given vectors $X_1, X_2 \in T_P SO(3)$ and $\vec{x} \in \Phi^{-1}(P)$, there are unique vectors $\widetilde{X}_1, \widetilde{X}_2 \in T_{\vec{x}} S^3$ satisfying $\Phi_* \widetilde{X}_1 = X_1$ and $\Phi_* \widetilde{X}_2 = X_2$. We then define g by the equation

$$g(X_1, X_2) = \left\langle \widetilde{X}_1, \widetilde{X}_2 \right\rangle,$$

noting that this definition is independent of the choice made since $\Phi^{-1}(P) = \{\pm\vec{x}\}$ and since the antipodal map $a : S^3 \to S^3$ is an isometry satisfying $\Phi \circ a = \Phi$. Similarly, we can define $h(P) = \widetilde{h}(\vec{x})$ for any $\vec{x} \in \Phi^{-1}(P)$ since $\Phi^{-1}(P) = \{\pm\vec{x}\}$ and since $\widetilde{h}(\vec{x}) = \widetilde{h}(-\vec{x})$. It is clear that $h \circ \Phi = \widetilde{h}$. From the above constructions, it is easy to see that

$$\mathrm{grad}_h(P) = \Phi_* \mathrm{grad}_{\widetilde{h}}(\vec{x}),$$

where the right hand side is independent of the chosen $\vec{x} \in \Phi^{-1}(P)$ and so well-defined. The remaining statements of the proposition then follow immediately from Proposition 5.1. □

Now, where g is defined as in the proof of Proposition 5.2 and where Υ is defined as in (3.5), we define a Riemannian metric ρ on the Lagrangian $L \subset (S^2)^3$ by $\rho = \Upsilon^* g$. Furthermore, where h is as in the proof of Proposition 5.2, we define $f : L \to \mathbb{R}$ by $f = h \circ \Upsilon$.

Corollary 5.3. *Let ρ and f be as above. With respect to the Riemannian metric ρ, the map $f : L \to \mathbb{R}$ is Morse-Smale with the following critical points:*

(i) the point $C_0 = \Upsilon^{-1}(\Phi(\pm\vec{e}_1))$, which has index 0,

(ii) the point $C_1 = \Upsilon^{-1}(\Phi(\pm\vec{e}_2))$, which has index 1,

(iii) the point $C_2 = \Upsilon^{-1}(\Phi(\pm\vec{e}_3))$, which has index 2,

(iv) the point $C_3 = \Upsilon^{-1}(\Phi(\pm\vec{e}_4))$, which has index 3.

Additionally, the stable and unstable submanifolds of these critical points are given by:

$$\begin{aligned}
W_f^s(C_0) &= \Upsilon^{-1}\left(W_h^s(\Phi(\pm\vec{e}_1))\right),\\
W_f^u(C_0) &= \Upsilon^{-1}\left(W_h^u(\Phi(\pm\vec{e}_1))\right),\\
W_f^s(C_1) &= \Upsilon^{-1}\left(W_h^s(\Phi(\pm\vec{e}_2))\right),\\
W_f^u(C_1) &= \Upsilon^{-1}\left(W_h^u(\Phi(\pm\vec{e}_2))\right),\\
W_f^s(C_2) &= \Upsilon^{-1}\left(W_h^s(\Phi(\pm\vec{e}_3))\right),\\
W_f^u(C_2) &= \Upsilon^{-1}\left(W_h^u(\Phi(\pm\vec{e}_3))\right),\\
W_f^s(C_3) &= \Upsilon^{-1}\left(W_h^s(\Phi(\pm\vec{e}_4))\right),\\
W_f^u(C_3) &= \Upsilon^{-1}\left(W_h^u(\Phi(\pm\vec{e}_4))\right).
\end{aligned}$$

Proof. This follows immediately from Proposition 5.2, the definitions of f and ρ, and the fact that Υ is a diffeomorphism. $\square$

As part of the pearl complex computation, we need to know how many (negative) gradient flow lines there are between critical points whose indices differ by 1. Let C_0, C_1, C_2, and C_3 be the critical points of f as in Corollary 5.3. Since the pair (f, ρ) is Morse-Smale, the space of gradient trajectories

$$\widetilde{m}\left(C_j, C_{j-1}\right) = W_f^s\left(C_{j-1}\right) \cap W_f^u\left(C_j\right)$$

is 1-dimensional. Moreover, we recall that $\mathbb{R}$ acts on $\widetilde{m}\left(C_j, C_{j-1}\right)$ via the negative gradient flow of f, and we write $m\left(C_j, C_{j-1}\right) = \widetilde{m}\left(C_j, C_{j-1}\right)/\mathbb{R}$, noting that this space is a compact 0-dimensional manifold.

Lemma 5.4. *The mod* 2 *count of* $m\left(C_j, C_{j-1}\right)$ *is* 0 *for* $j = 1, 2, 3$. *In particular, we have*

$$\#_2\left(m\left(C_1, C_0\right)\right) = 0,\ \#_2\left(m\left(C_2, C_1\right)\right) = 0,\ \ \textit{and}\ \ \#_2\left(m\left(C_3, C_2\right)\right) = 0.$$

Proof. We only prove that $\#_2\left(m\left(C_2, C_1\right)\right) = 0$ since the other cases are very similar. To prove this result, it suffices to show that $\widetilde{m}\left(C_2, C_1\right)$ consists of 2 disjoint arcs. Furthermore, it follows quickly from the definitions and Corollary 5.3 that we need only show that

$$W_h^s\left(\Phi\left(\pm\vec{e}_2\right)\right) \cap W_h^u\left(\Phi\left(\pm\vec{e}_3\right)\right)$$

consists of 2 disjoint arcs in $SO(3)$. Using Proposition 5.2, this means that we need to show that

$$\Phi\left(W_{\tilde{h}}^s\left(\vec{e}_2\right)\right) \cap \Phi\left(W_{\tilde{h}}^u\left(\vec{e}_3\right)\right)$$

consists of 2 disjoint arcs. It follows rather quickly that

$$\begin{aligned}\Phi\left(W_{\tilde{h}}^s\left(\vec{e}_2\right)\right) \cap \Phi\left(W_{\tilde{h}}^u\left(\vec{e}_3\right)\right) &= \Phi\left(\Phi^{-1}\left(\Phi\left(W_{\tilde{h}}^s\left(\vec{e}_2\right)\right)\right) \cap \Phi^{-1}\left(\Phi\left(W_{\tilde{h}}^u\left(\vec{e}_3\right)\right)\right)\right) \\ &= \Phi\left(\left(W_{\tilde{h}}^s\left(\vec{e}_2\right) \cup W_{\tilde{h}}^s\left(-\vec{e}_2\right)\right) \cap \left(W_{\tilde{h}}^u\left(\vec{e}_3\right) \cup W_{\tilde{h}}^u\left(-\vec{e}_3\right)\right)\right).\end{aligned}$$

By Proposition 5.1, we see that $\left(W^s_{\tilde{h}}(\vec{e}_2) \cup W^s_{\tilde{h}}(-\vec{e}_2)\right) \cap \left(W^u_{\tilde{h}}(\vec{e}_3) \cup W^u_{\tilde{h}}(-\vec{e}_3)\right)$ consists of the 4 following disjoint arcs in S^3:

$$W^s_{\tilde{h}}(\vec{e}_2) \cap W^u_{\tilde{h}}(\vec{e}_3) = \{\vec{x} \,|\, x_1 = 0, x_4 = 0, x_2 > 0, x_3 > 0\},$$
$$W^s_{\tilde{h}}(\vec{e}_2) \cap W^u_{\tilde{h}}(-\vec{e}_3) = \{\vec{x} \,|\, x_1 = 0, x_4 = 0, x_2 > 0, x_3 < 0\},$$
$$W^s_{\tilde{h}}(-\vec{e}_2) \cap W^u_{\tilde{h}}(\vec{e}_3) = \{\vec{x} \,|\, x_1 = 0, x_4 = 0, x_2 < 0, x_3 > 0\},$$
$$W^s_{\tilde{h}}(-\vec{e}_2) \cap W^u_{\tilde{h}}(-\vec{e}_3) = \{\vec{x} \,|\, x_1 = 0, x_4 = 0, x_2 < 0, x_3 < 0\}.$$

Since $\Phi(-\vec{x}) = \Phi(\vec{x})$, it is easy to see that the image under Φ of the 4 arcs above is a pair of arcs in $SO(3)$ (the first and fourth arcs above have the same image under Φ and likewise for the second and third arcs). In other words, the space $\Phi\left(W^s_{\tilde{h}}(\vec{e}_2)\right) \cap \Phi\left(W^u_{\tilde{h}}(\vec{e}_3)\right)$ in $SO(3)$ consists of the following 2 disjoint arcs:

$$\Phi\left(W^s_{\tilde{h}}(\vec{e}_2) \cap W^u_{\tilde{h}}(\vec{e}_3)\right) = \Phi\left(W^s_{\tilde{h}}(-\vec{e}_2) \cap W^u_{\tilde{h}}(-\vec{e}_3)\right),$$
$$\Phi\left(W^s_{\tilde{h}}(\vec{e}_2) \cap W^u_{\tilde{h}}(-\vec{e}_3)\right) = \Phi\left(W^s_{\tilde{h}}(-\vec{e}_2) \cap W^u_{\tilde{h}}(\vec{e}_3)\right).$$

One can easily see that these 2 arcs are disjoint by noting that $\Phi\big|_{\{\vec{x}\,|\,x_2>0\}}$ is injective. □

Now let $B_j = D - S_j$ (for $j \in \{1, 2, 3\}$) as in Proposition 4.3, and let J be the standard complex structure on $(S^2)^3$. Following the notation of Section 5.1, we write

$$\mathcal{M}(\mathbf{A}, J) = \widetilde{\mathcal{M}}(\mathbf{A}, J)/H = \widetilde{\mathcal{M}}(B_j)/H$$

for the (length 1) sequence $\mathbf{A} = (B_j)$. As in Section 5.1, for each such $\mathbf{A}$, we consider the map $\mathrm{ev}_{\mathbf{A}} : \mathcal{M}(\mathbf{A}, J) \to L \times L$ defined by

$$\mathrm{ev}_{\mathbf{A}}([u]) = (u(-1), u(1)) = (\mathrm{ev}_{-1}([u]), \mathrm{ev}_1([u])).$$

Observe that the submanifold $W_f^u(C_2)$ is not compact (since it is diffeomorphic to $\mathbb{R}^2$), but it does have an obvious compactification, which we denote $\overline{W_f^u(C_2)}$ and whose boundary is given by

$$\partial\left(\overline{W_f^u(C_2)}\right) = W_f^u(C_1) \cup W_f^u(C_0).$$

This also gives a compactification of $W_f^u(C_2) \times W_f^s(C_3)$, which we denote $\overline{W_f^u(C_2) \times W_f^s(C_3)}$ and whose boundary is given by

$$\partial\left(\overline{W_f^u(C_2) \times W_f^s(C_3)}\right) = \left(W_f^u(C_1) \cup W_f^u(C_0)\right) \times \{C_3\}. \tag{5.3}$$

Lemma 5.5. *Let* $\mathbf{A} = (B_j)$ *for some* $j \in \{1,2,3\}$. *Then the map* $\mathrm{ev}_{\mathbf{A}}$ *is transverse to* $W_f^u(C_2) \times W_f^s(C_3)$, *and the resulting moduli space of pearly trajectories*

$$\mathcal{P}(C_2, C_3, \mathbf{A}; f, \rho, J) = \mathrm{ev}_{\mathbf{A}}^{-1}\left(W_f^u(C_2) \times W_f^s(C_3)\right)$$

consists of exactly one point. In particular, we have

$$\#_2\left(\mathcal{P}(C_2, C_3, \mathbf{A}; f, \rho, J)\right) = 1.$$

Moreover, the image of the map $\mathrm{ev}_{\mathbf{A}}$ *does not intersect* $\partial\left(\overline{W_f^u(C_2) \times W_f^s(C_3)}\right)$, *and so the number* $\#_2\left(\mathcal{P}(C_2, C_3, \mathbf{A}; f, \rho, J)\right)$ *is invariant under small perturbations of the data* (f, ρ, J).

Proof. We prove the result only for the case of $\mathbf{A} = (B_1)$ since the other two cases are very similar. By Corollary 4.11 and Proposition 4.12, the map $\Psi \circ \Gamma$ is a diffeomorphism from $S^1(TS^2) \times (-\pi, \pi)$ to $\mathcal{M}(\mathbf{A}, J)$. Thus, to prove the desired result, it suffices to show that the image of the map

$$\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma : S^1\left(TS^2\right) \times (-\pi, \pi) \to L \times L$$

intersects $W_f^u(C_2) \times W_f^s(C_3)$ transversally in a single point and has empty intersection with $\partial\left(\overline{W_f^u(C_2) \times W_f^s(C_3)}\right)$.

Suppose that $(\vec{p}, \vec{q}, \phi) \in (\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma)^{-1}\left(W_f^u(C_2) \times W_f^s(C_3)\right)$, and note that this implies that $\mathrm{ev}_1 \circ \Psi \circ \Gamma(\vec{p}, \vec{q}, \phi) = C_3$ since $W_f^s(C_3) = \{C_3\}$. In particular, one can compute that

$$C_3 = \Upsilon^{-1}(\Phi(\pm\vec{e}_4)) = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix},$$

and by Proposition 4.13, it follows that

$$\begin{pmatrix} -\vec{q} & \frac{\sqrt{3}}{2}\vec{p} + \frac{1}{2}\vec{q} & -\frac{\sqrt{3}}{2}\vec{p} + \frac{1}{2}\vec{q} \end{pmatrix} = u_{\vec{p},\vec{q}}(1) = \mathrm{ev}_1 \circ \Psi \circ \Gamma(\vec{p}, \vec{q}, \phi) = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix}.$$

From this equation, it quickly follows that

$$\vec{q} = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \text{ and } \vec{p} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{5.4}$$

and we assume that they are as such for the remainder of this proof. The assumption that $(\vec{p}, \vec{q}, \phi) \in (\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma)^{-1}\left(W_f^u(C_2) \times W_f^s(C_3)\right)$ implies that $\mathrm{ev}_{-1} \circ \Psi \circ \Gamma(\vec{p}, \vec{q}, \phi) \in W_f^u(C_2)$. By Proposition 4.13, this means that $u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right) \in W_f^u(C_2)$. Using the diffeomorphism $\Upsilon : L \to SO(3)$, it follows from Corollary 5.3 that $\Upsilon \circ u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right) \in W_h^u(\Phi(\pm\vec{e}_3))$. It is not difficult to check by hand or using a computer algebra system that

$$\Upsilon \circ u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right) = \Phi \begin{pmatrix} \frac{1}{\sqrt{6}}\cos\left(\frac{\phi}{2}\right) \\ \frac{1}{\sqrt{6}}\cos\left(\frac{\phi}{2}\right) \\ \sqrt{\frac{2}{3}}\cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{pmatrix},$$

from which it follows that

$$\begin{pmatrix} \frac{1}{\sqrt{6}}\cos\left(\frac{\phi}{2}\right) \\ \frac{1}{\sqrt{6}}\cos\left(\frac{\phi}{2}\right) \\ \sqrt{\frac{2}{3}}\cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{pmatrix} \in \Phi^{-1}\left(W_h^u\left(\Phi\left(\pm\vec{e}_3\right)\right)\right).$$

By Propositions 5.1 and 5.2, we have

$$\Phi^{-1}\left(W_h^u\left(\Phi\left(\pm\vec{e}_3\right)\right)\right) = W_{\tilde{h}}^u\left(\vec{e}_3\right) \cup W_{\tilde{h}}^u\left(-\vec{e}_3\right) = \{\vec{x} \,|\, x_4 = 0, x_3 \neq 0\},$$

and so it must be the case that $\phi = 0$. Thus, we have shown that

$$\left(\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma\right)^{-1}\left(W_f^u\left(C_2\right) \times W_f^s\left(C_3\right)\right) = \left\{\left(\vec{p}, \vec{q}, 0\right)\right\},$$

where $\vec{p}$ and $\vec{q}$ are as in (5.4).

To show that $\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma$ is transverse to $W_f^u\left(C_2\right) \times W_f^s\left(C_3\right)$, we begin by defining $C = \mathrm{ev}_{-1} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, 0\right)$, and we will show that

$$T_C W_f^u\left(C_2\right) = \mathrm{span}\{\mathcal{X}_1\left(C\right), \mathcal{X}_2\left(C\right)\},$$

where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined as in (4.9). Indeed, we compute that

$$C = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix},$$

from which it follows that

$$\mathcal{X}_1(C) = \begin{pmatrix} 0 & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & 0 \\ -\frac{\sqrt{6}}{4} & 0 & \frac{\sqrt{6}}{4} \end{pmatrix} \text{ and } \mathcal{X}_2(C) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

On the other hand, we note that $C = \Upsilon^{-1} \circ \Phi(\vec{a})$ for

$$\vec{a} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \in W^u_{\tilde{h}}(\vec{e}_3),$$

and routine computations show that

$$\left(\Upsilon^{-1} \circ \Phi\right)_* \begin{pmatrix} \frac{\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{8} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & 0 \\ -\frac{\sqrt{6}}{4} & 0 & \frac{\sqrt{6}}{4} \end{pmatrix} = \mathcal{X}_1(C) \quad \text{for} \quad \begin{pmatrix} \frac{\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{8} \\ 0 \\ 0 \end{pmatrix} \in T_{\vec{a}} W^u_{\tilde{h}}(\vec{e}_3),$$

and also that

$$\left(\Upsilon^{-1}\circ\Phi\right)_*\begin{pmatrix}\frac{1}{4}\\ \frac{1}{4}\\ -\frac{1}{4}\\ 0\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\end{pmatrix}=\mathcal{X}_2\left(C\right) \text{ for } \begin{pmatrix}\frac{1}{4}\\ \frac{1}{4}\\ -\frac{1}{4}\\ 0\end{pmatrix}\in T_{\vec{a}}W^u_{\tilde{h}}\left(\vec{e}_3\right).$$

Since $W^u_f\left(C_2\right)=\Upsilon^{-1}\circ\Phi\left(W^u_{\tilde{h}}\left(\vec{e}_3\right)\right)$, it follows from the above computations that

$$T_C W^u_f\left(C_2\right)=\operatorname{span}\{\mathcal{X}_1\left(C\right),\mathcal{X}_2\left(C\right)\}$$

exactly as claimed.

Now, noting that $\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\left(\vec{p},\vec{q},0\right)=\left(C,C_3\right)$, we use the splitting

$$T_{\left(C,C_3\right)}\left(L\times L\right)=T_C L\times T_{C_3}L,$$

and it follows from the definition of $\mathrm{ev}_{\mathbf{A}}$ that

$$\left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)_*=\left(\left(\mathrm{ev}_{-1}\circ\Psi\circ\Gamma\right)_*,\left(\mathrm{ev}_1\circ\Psi\circ\Gamma\right)_*\right).$$

Then, it follows from Proposition 4.14 that

$$\begin{aligned}\left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)_*\mathcal{V}_1&=\left(-\mathcal{X}_1\left(C\right),\mathcal{X}_1\left(C_3\right)\right),\\ \left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)_*\mathcal{V}_2&=\left(-\mathcal{X}_2\left(C\right),\mathcal{X}_2\left(C_3\right)\right),\\ \left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)_*\mathcal{V}_3&=\left(\mathcal{X}_3\left(C\right),\mathcal{X}_3\left(C_3\right)\right),\\ \left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)_*\mathcal{V}_4&=\left(\mathcal{X}_3\left(C\right),\begin{pmatrix}\vec{0} & \vec{0} & \vec{0}\end{pmatrix}\right),\end{aligned}$$

and it is then easy to see that

$$\left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)_* T_{(\vec{p},\vec{q},0)}\left(S^1\left(TS^2\right)\times(-\pi,\pi)\right)+T_{(C,C_3)}\left(W^u_f\left(C_2\right)\times W^s_f\left(C_3\right)\right)=T_{(C,C_3)}\left(L\times L\right)$$

since we have already shown that

$$T_C W^u_f\left(C_2\right)=\mathrm{span}\{\mathcal{X}_1\left(C\right),\mathcal{X}_2\left(C\right)\}.$$

Thus, we have shown that $\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma$ is transverse to $W^u_f\left(C_2\right)\times W^s_f\left(C_3\right)$ as required.

Finally, we suppose that $(\vec{p},\vec{q},\phi)\in\left(\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma\right)^{-1}\left(\partial\left(\overline{W^u_f\left(C_2\right)\times W^s_f\left(C_3\right)}\right)\right)$, which by (5.3) implies that $\mathrm{ev}_1\circ\Psi\circ\Gamma\left(\vec{p},\vec{q},\phi\right)=C_3$. As before, this forces $\vec{p}$ and $\vec{q}$ to be defined by (5.4), and we also must have $\mathrm{ev}_{-1}\circ\Psi\circ\Gamma\left(\vec{p},\vec{q},\phi\right)\in W^u_f\left(C_1\right)\cup W^u_f\left(C_0\right)$ by (5.3). By Proposition 4.13, this implies that $u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right)\in W^u_f\left(C_1\right)\cup W^u_f\left(C_0\right)$. Then, just as before, it follows that

$$\begin{pmatrix}\frac{1}{\sqrt{6}}\cos\left(\frac{\phi}{2}\right)\\ \frac{1}{\sqrt{6}}\cos\left(\frac{\phi}{2}\right)\\ \sqrt{\frac{2}{3}}\cos\left(\frac{\phi}{2}\right)\\ \sin\left(\frac{\phi}{2}\right)\end{pmatrix}\in\Phi^{-1}\left(W^u_h\left(\Phi\left(\pm\vec{e}_2\right)\right)\cup W^u_h\left(\Phi\left(\pm\vec{e}_1\right)\right)\right).$$

By Propositions 5.1 and 5.2, we have

$$\begin{aligned}\Phi^{-1}\left(W^u_h\left(\Phi\left(\pm\vec{e}_2\right)\right)\cup W^u_h\left(\Phi\left(\pm\vec{e}_1\right)\right)\right)&=W^u_{\tilde{h}}\left(\vec{e}_2\right)\cup W^u_{\tilde{h}}\left(-\vec{e}_2\right)\cup W^u_{\tilde{h}}\left(\vec{e}_1\right)\cup W^u_{\tilde{h}}\left(-\vec{e}_1\right)\\&=\{\vec{x}\,|\,x_3=0,x_4=0\},\end{aligned}$$

from which it follows that $\cos\left(\frac{\phi}{2}\right)=0=\sin\left(\frac{\phi}{2}\right)$, a contradiction. So, the map $\mathrm{ev}_{\mathbf{A}}\circ\Psi\circ\Gamma$ does not intersect $\partial\left(\overline{W^u_f\left(C_2\right)\times W^s_f\left(C_3\right)}\right)$. Invariance of the number $\#_2\left(\mathcal{P}\left(C_2,C_3,\mathbf{A};f,\rho,J\right)\right)$ under small perturbations of the data (f,ρ,J) follows from the fact that the map $\mathrm{ev}_{\mathbf{A}}$ is transverse to $W^u_f\left(C_2\right)\times W^s_f\left(C_3\right)$ and does not intersect $\partial\left(\overline{W^u_f\left(C_2\right)\times W^s_f\left(C_3\right)}\right)$. □

Similarly to the discussion preceding Lemma 5.5, we note that $W_f^u(C_1) \times W_f^s(C_2)$ is not compact but admits an obvious compactification, which we write as

$$\overline{W_f^u(C_1) \times W_f^s(C_2)} = \left(W_f^u(C_1) \cup W_f^u(C_0)\right) \times \left(W_f^s(C_2) \cup W_f^s(C_3)\right). \tag{5.5}$$

Lemma 5.6. *Let* $\mathbf{A} = (B_j)$ *for some* $j \in \{1,2,3\}$*. Then the map* $\mathrm{ev}_{\mathbf{A}}$ *satisfies*

$$\mathrm{ev}_{\mathbf{A}}^{-1}\left(\overline{W_f^u(C_1) \times W_f^s(C_2)}\right) = \varnothing.$$

In particular, the map $\mathrm{ev}_{\mathbf{A}}$ *is transverse to* $W_f^u(C_1) \times W_f^s(C_2)$*, and we have*

$$\#_2\left(\mathcal{P}\left(C_1, C_2, \mathbf{A}; f, \rho, J\right)\right) = \#_2\left(\mathrm{ev}_{\mathbf{A}}^{-1}\left(W_f^u(C_1) \times W_f^s(C_2)\right)\right) = 0,$$

which is invariant under small perturbations of the data (f, ρ, J)*.*

Proof. Again we only prove the result for the case of $\mathbf{A} = (B_1)$ with the proofs of the other cases being very similar. Since $\Psi \circ \Gamma$ is a diffeomorphism from $S^1(TS^2) \times (-\pi, \pi)$ to $\mathcal{M}(\mathbf{A}, J)$, we need only show that the image of the map

$$\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma : S^1\left(TS^2\right) \times (-\pi, \pi) \to L \times L$$

does not intersect $\overline{W_f^u(C_1) \times W_f^s(C_2)}$. By (5.5), we need to show that the image of the map $\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma$ does not intersect $\left(W_f^u(C_1) \cup W_f^u(C_0)\right) \times \left(W_f^s(C_2) \cup W_f^s(C_3)\right)$.

First, we observe that $W^s_{\tilde{h}}(\vec{e}_3) \cup W^s_{\tilde{h}}(\vec{e}_4)$ can be parametrized by

$$\vec{x}_s : (-\pi, \pi] \to S^3$$
$$t \mapsto \begin{pmatrix} 0 \\ 0 \\ \cos\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) \end{pmatrix},$$

and since $W^s_f(C_2) \cup W^s_f(C_3) = \Upsilon^{-1}\left(\Phi\left(W^s_{\tilde{h}}(\vec{e}_3) \cup W^s_{\tilde{h}}(\vec{e}_4)\right)\right)$, one can easily compute that $W^s_f(C_2) \cup W^s_f(C_3)$ is parametrized by

$$\Upsilon^{-1} \circ \Phi \circ \vec{x}_s : (-\pi, \pi] \to L$$
$$t \mapsto \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{\cos(t)+\sin(t)}{\sqrt{6}} & \frac{2\cos(t)-\sin(t)}{\sqrt{6}} & -\frac{\cos(t)-2\sin(t)}{\sqrt{6}} \\ \frac{\cos(t)-\sin(t)}{\sqrt{6}} & \frac{\cos(t)+2\sin(t)}{\sqrt{6}} & -\frac{2\cos(t)+\sin(t)}{\sqrt{6}} \end{pmatrix}.$$

Now assume that $(\vec{p}, \vec{q}, \phi) \in (\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma)^{-1}\left(\overline{W^u_f(C_1) \times W^s_f(C_2)}\right)$, and by Proposition 4.13 and (5.5), it follows that

$$\begin{pmatrix} -\vec{q} & \frac{\sqrt{3}}{2}\vec{p} + \frac{1}{2}\vec{q} & -\frac{\sqrt{3}}{2}\vec{p} + \frac{1}{2}\vec{q} \end{pmatrix} = u_{\vec{p},\vec{q}}(1) = \mathrm{ev}_1 \circ \Psi \circ \Gamma(\vec{p}, \vec{q}, \phi) \in W^s_f(C_2) \cup W^s_f(C_3).$$

Using the above parametrization of $W^s_f(C_2) \cup W^s_f(C_3)$ and the projection $\mathrm{pr}_1 : (S^2)^3 \to S^2$ onto the first factor, we see that

$$\vec{q} = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{\cos(t)+\sin(t)}{\sqrt{6}} \\ -\frac{\cos(t)-\sin(t)}{\sqrt{6}} \end{pmatrix}$$

for some $t \in (-\pi, \pi]$. For any such choice of $\vec{q}$ as above, we compute using Proposition 4.13 that

$$\mathrm{pr}_1 \circ \mathrm{ev}_{-1} \circ \Psi \circ \Gamma(\vec{p}, \vec{q}, \phi) = \mathrm{pr}_1 \circ u_{\vec{p},\vec{q}}\left(e^{i(\phi+\pi)}\right) = -\vec{q} = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ -\frac{\cos(t)+\sin(t)}{\sqrt{6}} \\ \frac{\cos(t)-\sin(t)}{\sqrt{6}} \end{pmatrix}. \tag{5.6}$$

On the other hand, similarly to above, we can parametrize $W^u_{\tilde{h}}(\vec{e}_2) \cup W^u_{\tilde{h}}(\vec{e}_1)$ by

$$\begin{aligned} \vec{x}_u : (-\pi, \pi] &\to S^3 \\ \tau &\mapsto \begin{pmatrix} \sin\left(\frac{\tau}{2}\right) \\ \cos\left(\frac{\tau}{2}\right) \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

and since $W^u_f(C_1) \cup W^u_f(C_0) = \Upsilon^{-1}\left(\Phi\left(W^u_{\tilde{h}}(\vec{e}_2) \cup W^u_{\tilde{h}}(\vec{e}_1)\right)\right)$, one can easily compute that $W^u_f(C_1) \cup W^u_f(C_0)$ is parametrized by

$$\begin{aligned} \Upsilon^{-1} \circ \Phi \circ \vec{x}_u : (-\pi, \pi] &\to L \\ \tau &\mapsto \begin{pmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{\cos(\tau)+\sin(\tau)}{\sqrt{6}} & \frac{-2\cos(\tau)+\sin(\tau)}{\sqrt{6}} & \frac{\cos(\tau)-2\sin(\tau)}{\sqrt{6}} \\ \frac{\cos(\tau)-\sin(\tau)}{\sqrt{6}} & \frac{\cos(\tau)+2\sin(\tau)}{\sqrt{6}} & -\frac{2\cos(\tau)+\sin(\tau)}{\sqrt{6}} \end{pmatrix}. \end{aligned}$$

Since we have assumed that $(\vec{p}, \vec{q}, \phi) \in (\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma)^{-1}\left(\overline{W^u_f(C_1) \times W^s_f(C_2)}\right)$, it must be the case that $\mathrm{ev}_{-1} \circ \Psi \circ \Gamma(\vec{p}, \vec{q}, \phi) \in W^u_f(C_1) \cup W^u_f(C_0)$. Using the above parametrization and the projection pr_1, this implies that

$$\mathrm{pr}_1 \circ \mathrm{ev}_{-1} \circ \Psi \circ \Gamma\left(\vec{p}, \vec{q}, \phi\right) = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{\cos(\tau)+\sin(\tau)}{\sqrt{6}} \\ \frac{\cos(\tau)-\sin(\tau)}{\sqrt{6}} \end{pmatrix}$$

for some $\tau \in (-\pi, \pi]$, which contradicts (5.6) regardless of the choices of t and τ. Hence, it must in fact be the case that $\left(\mathrm{ev}_{\mathbf{A}} \circ \Psi \circ \Gamma\right)^{-1}\left(\overline{W_f^u\left(C_1\right) \times W_f^s\left(C_2\right)}\right)$ is empty as required. Invariance of the number $\#_2\left(\mathcal{P}\left(C_1, C_2, \mathbf{A}; f, \rho, J\right)\right)$ under small perturbations of the data (f, ρ, J) follows from the fact that the map $\mathrm{ev}_{\mathbf{A}}$ does not intersect $\overline{W_f^u\left(C_1\right) \times W_f^s\left(C_2\right)}$. □

Let $\mathcal{R}$ denote the field with 4 elements. More explicitly, where $\left(\mathbb{Z}/2\mathbb{Z}\right)[X]$ is the ring of polynomials in the variable X with coefficients in $\mathbb{Z}/2\mathbb{Z}$, we have

$$\mathcal{R} = \frac{\left(\mathbb{Z}/2\mathbb{Z}\right)[X]}{\langle X^2 + X + 1\rangle}.$$

The four elements of $\mathcal{R}$ can be represented by the elements $0, 1, X, X+1$, and we refer to the elements by these representatives henceforth. According to Lemma 3.8, we see that $H_2^D\left(\left(S^2\right)^3, L\right)$ is isomorphic to $\pi_2\left(\left(S^2\right)^3, L\right)$, which is in turn isomorphic to

$$\frac{\mathcal{F}_{\mathrm{ab}}\left(D, S_1, S_2, S_3\right)}{\langle 2D - S_1 - S_2 - S_3\rangle}$$

by Lemma 3.9. Let $\mathcal{R}^* = \{1, X, X+1\}$ denote the multiplicative group of units of $\mathcal{R}$, and define a group homomorphism $\varphi_q : \mathcal{F}_{\mathrm{ab}}\left(D, S_1, S_2, S_3\right) \to \mathcal{R}^*$ by requiring

$$\varphi_q(D) = 1,\ \varphi_q\left(S_1\right) = 1,\ \varphi_q\left(S_2\right) = X,\ \varphi_q\left(S_3\right) = X + 1. \tag{5.7}$$

Lemma 5.7. *The homomorphism* $\varphi_q : \mathcal{F}_{\mathrm{ab}}(D, S_1, S_2, S_3) \to \mathcal{R}^*$ *descends to a homomorphism*

$$\frac{\mathcal{F}_{\mathrm{ab}}(D, S_1, S_2, S_3)}{\langle 2D - S_1 - S_2 - S_3 \rangle} \to \mathcal{R}^*,$$

which we still denote φ_q.

Proof. This simply amounts to the computation

$$\begin{aligned}\varphi_q(2D - S_1 - S_2 - S_3) &= (\varphi_1(D))^2 (\varphi_q(S_1))^{-1} (\varphi_q(S_2))^{-1} (\varphi_q(S_2))^{-1} \\ &= (1)^2 (1)^{-1} X^{-1} (X+1)^{-1} = (1)(1)(X+1)(X) = 1,\end{aligned}$$

the penultimate equality following from the fact that $X^{-1} = X + 1$ and $(X+1)^{-1} = X$. □

Following the notation from Section 5.1, we write $\widetilde{\Lambda} = (\mathbb{Z}/2\mathbb{Z})\left[H_2^D\left((S^2)^3, L\right)\right]$. Based on our previously established conventions, we think of elements of $\widetilde{\Lambda}$ as "polynomials" in the formal variable T with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Identifying $H_2^D\left((S^2)^3, L\right)$ with

$$\frac{\mathcal{F}_{\mathrm{ab}}(D, S_1, S_2, S_3)}{\langle 2D - S_1 - S_2 - S_3 \rangle}$$

as discussed above, we define a map

$$\begin{aligned} q : \widetilde{\Lambda} &\to \mathcal{R} \\ \sum_{A \in H_2^D\left((S^2)^3, L\right)} a_A T^A &\mapsto \sum_{A \in H_2^D\left((S^2)^3, L\right)} a_A \varphi_q(A). \end{aligned} \tag{5.8}$$

Lemma 5.8. *The map* q *above is a ring homomorphism, and defining multiplication by*

$$P(T) \cdot Z = q(P(T))\, Z$$

for $P(T) \in \widetilde{\Lambda}$ *and* $Z \in \mathcal{R}$ *makes* $\mathcal{R}$ *into a* $\widetilde{\Lambda}$*-algebra with structural morphism* q.

Proof. The fact that q is a ring homomorphism is trivial since $\varphi_q : H_2^D\left(\left(S^2\right)^3, L\right) \to \mathcal{R}^*$ is a group homomorphism. This also makes it clear that $\mathcal{R}$ is a $\widetilde{\Lambda}$-module with the multiplication defined in the statement of the lemma. It is also a trivial calculation to verify that this module structure is compatible with the standard multiplication of $\mathcal{R}$, and so $\mathcal{R}$ is a $\widetilde{\Lambda}$-algebra with structural morphism q. □

Theorem 5.9. *The homology $QH_*(L;\mathcal{R})$ of the pearl complex with coefficients in $\mathcal{R}$ is non-trivial.*

Proof. We refer the reader back to Section 5.1 for a brief description of the chain complex $\mathcal{C}(L; f, \rho, J; \mathcal{R})$ and its differential $d^{\mathcal{R}}$, which is defined in (5.1). We will show that $d^{\mathcal{R}}(C_2 \otimes 1) = 0$ and that $C_2 \otimes 1$ is not in the image of $d^{\mathcal{R}}$.

Observe that L has minimal Maslov number 2 by Propositions 3.9 and 4.1; and moreover, the only Maslov index 2 classes with holomorphic representatives are B_1, B_2, and B_3 according to Proposition 4.2. Since non-constant holomorphic disks have positive area and hence positive Maslov index by monotonicity of L (see Proposition 4.1), it follows that $\mathbf{A} = (B_j)$ for some $j \in \{1,2,3\}$ whenever $\mathbf{A}$ is a sequence of non-zero homology classes $\mathbf{A} = (A_1, \ldots, A_\ell)$ whose entries have holomorphic representatives and which satisfies $I_\mu(\mathbf{A}) = 2$.

We claim that the only terms appearing in the second sum of (5.1) in the case of $x = C_2$ are those with $y = C_3$ and $\mathbf{A} = (B_j)$ for some $j \in \{1,2,3\}$. Indeed, if a sequence of non-zero homology classes $\mathbf{A} = (A_1, \ldots, A_\ell)$ and a critical point $y \in \text{Crit}(f)$ satisfy

$$I_\mu(\mathbf{A}) + |C_2| - |y| - 1 = 0,$$

then it must be the case that $I_\mu(\mathbf{A}) = |y| - 1$. If $|y| < 3$, then we have $I_\mu(\mathbf{A}) < 2$, which implies that the moduli space $\mathcal{P}(C_2, y, \mathbf{A}; f, \rho, J)$ is empty. If $|y| \geq 3$, which implies $y = C_3$, then we must have $I_\mu(\mathbf{A}) = 2$. This in turn implies either that $\mathbf{A} = (B_j)$ or else that the corresponding moduli space $\mathcal{P}(C_2, y, \mathbf{A}; f, \rho, J)$ is empty. In other words, the moduli space

$\mathcal{P}(C_2, y, \mathbf{A}; f, \rho, J)$ is empty unless $y = C_3$ and $\mathbf{A} = (B_j)$ for some $j \in \{1, 2, 3\}$, proving the claim at the start of this paragraph.

We can then compute that

$$\begin{aligned} d^{\mathcal{R}}(C_2 \otimes 1) = \#_2 (m(C_2, C_1))\, C_1 \otimes 1 \\ + \#_2 (\mathcal{P}(C_2, C_3, (B_1); f, \rho, J))\, C_3 \otimes q\left(T^{B_1}\right) \\ + \#_2 (\mathcal{P}(C_2, C_3, (B_2); f, \rho, J))\, C_3 \otimes q\left(T^{B_2}\right) \\ + \#_2 (\mathcal{P}(C_2, C_3, (B_3); f, \rho, J))\, C_3 \otimes q\left(T^{B_3}\right). \end{aligned}$$

According to Lemma 5.4, the first term above is 0, and by Lemma 5.5 and the definition of q in (5.8), we obtain $d^{\mathcal{R}}(C_2 \otimes 1) = C_3 \otimes \varphi_q(B_1) + C_3 \otimes \varphi_q(B_2) + C_3 \otimes \varphi_q(B_3)$. Then, by the definition of φ_q in (5.7), we note that

$$\begin{aligned} \varphi_q(B_1) = \varphi_q(D - S_1) = \varphi_q(D)(\varphi_q(S_1))^{-1} = (1)(1)^{-1} = 1, \\ \varphi_q(B_2) = \varphi_q(D - S_2) = \varphi_q(D)(\varphi_q(S_2))^{-1} = (1)(X)^{-1} = X + 1, \\ \varphi_q(B_3) = \varphi_q(D - S_3) = \varphi_q(D)(\varphi_q(S_3))^{-1} = (1)(X + 1)^{-1} = X, \end{aligned}$$

from which it follows that

$$d^{\mathcal{R}}(C_2 \otimes 1) = C_3 \otimes (\varphi_q(B_1) + \varphi_q(B_2) + \varphi_q(B_3)) = C_3 \otimes (1 + X + 1 + X) = C_3 \otimes 0 = 0.$$

By an argument similar to the one given above, one can easily see that the moduli space $\mathcal{P}(C_3, y, \mathbf{A}; f, \rho, J)$ is empty for all choices of $\mathbf{A} = (A_1, \ldots, A_\ell)$ and $y \in \mathrm{Crit}(f)$. It then follows from Lemma 5.4 that $d^{\mathcal{R}}(C_3 \otimes 1) = 0$. Another similar argument shows that the moduli space $\mathcal{P}(C_1, y, \mathbf{A}; f, \rho, J)$ is empty unless $y = C_2$ and $\mathbf{A} = (B_j)$ for some $j \in \{1, 2, 3\}$. Then, using Lemma 5.4 and Lemma 5.6, we obtain $d^{\mathcal{R}}(C_1 \otimes 1) = 0$.

By yet another argument similar to the one given above, one can see that the moduli space $\mathcal{P}(C_0, y, \mathbf{A}; f, \rho, J)$ is empty unless one of the following two conditions holds:

(i) $y = C_1$ and $\mathbf{A} = (B_j)$ for some $j \in \{1, 2, 3\}$,

(ii) $y = C_3$ and $I_\mu(\mathbf{A}) = 4$.

Since we additionally have $|C_0| = 0$, it follows that

$$\begin{aligned} d^{\mathcal{R}}(C_0 \otimes 1) &= \sum_{\substack{y \in \mathrm{Crit}(f), \mathbf{A} = (A_1, \ldots, A_\ell) \\ I_\mu(\mathbf{A}) + |C_0| - |y| - 1 = 0}} \#_2 \left(\mathcal{P}(C_0, y, \mathbf{A}; f, \rho, J) \right) y \otimes q\left(T^{A_1 + \cdots + A_\ell}\right) \\ &= \sum_{\mathbf{A} = (B_j)} \#_2 \left(\mathcal{P}(C_0, C_1, \mathbf{A}; f, \rho, J) \right) C_1 \otimes q\left(T^{B_j}\right) \\ &\quad + \sum_{\substack{\mathbf{A} = (A_1, \ldots, A_\ell) \\ I_\mu(\mathbf{A}) = 4}} \#_2 \left(\mathcal{P}(C_0, C_3, \mathbf{A}; f, \rho, J) \right) C_3 \otimes q\left(T^{A_1 + \cdots + A_\ell}\right), \end{aligned}$$

from which it is clear that $C_2 \otimes 1$ is not in the image of $d^{\mathcal{R}}$ (since we have already shown that $d^{\mathcal{R}}(C_1 \otimes 1) = d^{\mathcal{R}}(C_3 \otimes 1) = 0$).

Finally, we give a remark regarding the genericity of the data (f, ρ, J). While our choice of data (f, ρ, J) may not necessarily be sufficiently generic to guarantee that all of the relevant moduli spaces are cut out transversally (for instance, we have not verified transversality for $\mathcal{P}(C_0, C_3, \mathbf{A}; f, \rho, J)$ with $I_\mu(\mathbf{A}) = 4$), the fact that (f, ρ) is Morse-Smale and the results of Lemmata 5.5 and 5.6 indicate that, for a small perturbation of the data which guarantees sufficient genericity, all of the computations performed in the proof of this theorem will still hold true. In particular, we will still have $d^{\mathcal{R}}(C_2 \otimes 1) = 0$, and it will still be the case that $C_2 \otimes 1$ is not in the image of $d^{\mathcal{R}}$. Thus, we see that the homology $QH_*(L; \mathcal{R})$ is nonvanishing as claimed. □

Corollary 5.10. *The Lagrangian submanifold* $L \subset (S^2)^3$ *is nondisplaceable under Hamiltonian diffeomorphisms.*

Proof. Assume to the contrary that $\vartheta : (S^2)^3 \to (S^2)^3$ is a Hamiltonian diffeomorphism such that $L \cap \vartheta(L)$ is empty, and let $F : (S^2)^3 \times [0,1] \to \mathbb{R}$ be a time-dependent Hamiltonian function generating ϑ (so that ϑ is the time-1 flow of the time-dependent Hamiltonian vector field X_{F_t}). In order to define $HF_*(L;\mathcal{R})$ as in Section 3.2.g of [BC09], one considers the path space

$$\mathcal{P}_0(L) = \left\{ \gamma \in C^\infty\left([0,1], (S^2)^3\right) \,\middle|\, \gamma(0) \in L, \gamma(1) \in L, [\gamma] = 1 \in \pi_1\left((S^2)^3, L\right) \right\},$$

and one then considers the subset $\mathcal{O}_F \subset \mathcal{P}_0(L)$ consisting of orbits of the Hamiltonian flow associated to F. In our case, the subset $\mathcal{O}_F$ is empty since $L \cap \vartheta(L) = \varnothing$. Then, where $p : \pi_1(\mathcal{P}_0(L)) \to H_2^D\left((S^2)^3, L\right)$ is the natural epimorphism, one considers the regular, abelian cover $\widetilde{\mathcal{P}}_0(L))$ associated to $\ker(p)$, and one defines $\widetilde{\mathcal{O}}_F$ to be the set of lifts $\tilde{\gamma}$ of orbits $\gamma \in \mathcal{O}_F$. Since $\mathcal{O}_F$ is empty in our case, the space $\widetilde{\mathcal{O}}_F$ will be empty as well. Then, assuming that $\mathcal{R}$ is a commutative $\widetilde{\Lambda}$-algebra, one defines the Floer chain complex to be the $\mathcal{R}$-module

$$(\mathbb{Z}/2\mathbb{Z})\langle \widetilde{\mathcal{O}}_F \rangle \otimes \mathcal{R},$$

and the Lagrangian Floer homology $HF_*(L;\mathcal{R})$ is defined to be the homology of this chain complex. (The differential for the chain complex involves moduli spaces of certain holomorphic strips – see Section 3.2.g of [BC09] for more details.) Since $\widetilde{\mathcal{O}}_F$ is empty in our case, it follows that $HF_*(L;\mathcal{R}) = \{0\}$. However, by Theorem A of [BC09], there is an isomorphism $QH_*(L;\mathcal{R}) \to HF_*(L;\mathcal{R})$, and we have shown in Theorem 5.9 that $QH_*(L;\mathcal{R})$ is nontrivial, a contradiction. Thus, it must be the case that L is nondisplaceable as claimed. □

Chapter 6

Bibliography

[AF08] Peter Albers and Urs Frauenfelder. *A nondisplaceable Lagrangian torus in T^*S^2*. Communications on Pure and Applied Mathematics, Volume 61 (2008), Issue 8, 1046 – 1051.

[Bi01] Paul Biran. *Lagrangian Barriers and symplectic embeddings*. Geometric and Functional Analysis, Volume 11 (2001), Issue 3, 407 – 464.

[Bi06] Paul Biran. *Lagrangian non-intersections*. Geometric and Functional Analysis, Volume 16 (2006), Issue 2, 279 – 326.

[BC07] Paul Biran and Octav Cornea. *Quantum Structures for Lagrangian Submanifolds*. arXiv:0708.4221.

[BC09] Paul Biran and Octav Cornea. *Rigidity and Uniruling for Lagrangian Submanifolds*. Geometry & Topology, Volume 13 (2009), Issue 5, 2881 – 2989.

[BC12] Paul Biran and Octav Cornea. *Lagrangian Topology and Enumerative Geometry*. Geometry & Topology, Volume 16 (2012), Issue 2, 963 – 1052.

[CS10] Yuri Chekanov and Felix Schlenk. *Notes on monotone Lagrangian twist tori.* Electronic Research Announcements in Mathematical Sciences, Volume 17 (2010), 104 – 121.

[De88] Thomas Delzant. *Hamiltoniens périodiques et images convexes de l'application moment.* Bulletin de la Société Mathématique de France, Volume 116 (1988), Issue 3, 315 – 339.

[EP06] Michael Entov and Leonid Polterovich. *Quasi-states and symplectic intersections.* Commentarii Mathematici Helvetici, Volume 81 (2006), Issue 1, 75 – 99.

[EP09] Michael Entov and Leonid Polterovich. *Rigid subsets of symplectic manifolds.* Compositio Mathematica, Volume 145 (2009), Issue 3, 773 – 826.

[EL14] Jonathan David Evans and Yanki Lekili. *Floer Cohomology of the Chiang Lagrangian.* arXiv:1401.4073.

[FOOO12] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Toric Degeneration and Nondisplaceable Lagrangian Tori in* $S^2 \times S^2$. International Mathematics Research Notices, Volume 2012, Issue 13, 2942 – 2993.

[Ga13] Agnès Gadbled. *On exotic monotone Lagrangian tori in* $\mathbb{C}P^2$ *and* $S^2 \times S^2$. Journal of Symplectic Geometry, Volume 11 (2013), Issue 3, 343 – 361.

[Gr85] Mikhail Gromov. *Pseudo holomorphic curves in symplectic manifolds.* Inventiones mathematicae, Volume 82 (1985), Issue 2, 307 – 347.

[Ha01] Allen Hatcher. *Algebraic Topology.* (2001)

[Lee03] John M. Lee. *Introduction to Smooth Manifolds.* Springer-Verlag New York, Inc. (2003)

[LT97] Eugene Lerman and Susan Tolman. *Hamiltonian torus actions on symplectic orbifolds and toric varieties.* Transactions of the American Mathematical Society, Volume 349 (1997), Issue 10, 4201 – 4230.

[MS98] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology, Second Edition.* Oxford University Press, Inc., New York. (1998)

[MS04] Dusa McDuff and Dietmar Salamon. *J-holomorphic Curves and Symplectic Topology.* AMS Colloquium Publications. (2004)

[OU13] Joel Oakley and Michael Usher. *On certain Lagrangian submanifolds of* $S^2 \times S^2$ *and* $\mathbb{C}P^n$. arXiv:1311.5152.

[Oh93] Yong-Geun Oh. *Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs, I.* Communications on Pure and Applied Mathematics, Volume 46 (1993), Issue 7, 949 – 993.

[Oh95] Yong-Geun Oh. *Addendum to "Floer cohomology of Lagrangian intersections and pseudo-holomorphic Discs, I".* Communications on Pure and Applied Mathematics, Volume 48 (1995), Issue 1, 1299 – 1302.