Pre-Service Teachers' Development of Bidirectional Reasoning

by

Teo Joseph Paoletti

(Under the Direction of Dr. Kevin C. Moore)

Abstract

Researchers have indicated students and teachers do not maintain productive function and inverse function meanings. Researchers have also indicated that students are capable of developing foundational meanings for mathematical ideas through quantitative and covariational reasoning. This dissertation reports results of an investigation into the ways in which students can leverage quantitative and covariational reasoning to develop meanings related to function and inverse function. Specifically, I examine two pre-service teachers' activities in a semester long teaching experiment in which they addressed tasks intended to support their quantitative and covariational reasoning. In my analysis, I identify a particular type of quantitative reasoning–bidirectional reasoning–and describe two manifestations of said reasoning: reasoning bidirectionally with respect to axes and reasoning bidirectionally with respect to quantities. I distinguish instances of students' activities exemplifying each manifestation and relationships between the manifestations. In addition to charactering the participants' bidirectional reasoning, I examine implications this reasoning had for their other mathematical meanings including function and function inverse. Namely, the participants' bidirectional reasoning had immediate implications for how they perceived aspects that I consider conventions common to school mathematics. These findings have important implications for the teaching and learning of pre-service teachers as well as for middle and high school students. I conclude with these implications as well as limitations of the study and directions for future research.

INDEX WORDS: Quantitative reasoning, Covariational reasoning, Function, Inverse function, Teaching experiment, Pre-service teachers

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by

Teo Joseph Paoletti

B.A., The College of New Jersey 2007

M.S., Texas A&M University 2010

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by

Teo Joseph Paoletti

Major Professor: Kevin C. Moore

Committee: Leslie P. Steffe
Sybilla Beckmann-Kazez

Electronic Version Approved:

Suzanne Barbour
Dean of the Graduate School
The University of Georgia
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Chapter 1: An Introduction

In this dissertation I characterize students' quantitative (Thompson, 2011; 2012) and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) that supported them in constructing and reasoning about relationships between quantities. I describe the students' development of bidirectional reasoning and their construction of bidirectional relationships, which supported them in understanding various representations as representing invariant relationships. Further, I characterize how the students engaging in this reasoning interplayed with their understanding of other mathematical ideas, including function, inverse function, and what I perceive to be conventions common to school mathematics. The results of this study have important implications for the design of curricular materials intended to support pre-service teachers', as well as middle and high school students', development of productive function and inverse function meanings.

Statement of the Problem

The current body of research indicates high school students, college students, pre-service teachers, and in-service teachers encounter persistent and widespread difficulties with both function and inverse function (Akkoc & Tall, 2005; Bayazit, 2011; Breidenback, Dubinsky, Hawks, & Nichols, 1992; Brown & Reynolds, 2007; Carlson, 1998; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Even, 1990; Leinhardt, Zaslavsky, & Stein, 1990; Lucus, 2005; Montiel, Vidakovic, & Kabael, 2008; Oehrtman, Carlson, & Thompson, 2008; P. W. Thompson, 1994c; Vidakovic, 1997; Vinner, 1983; Vinner & Dreyfus, 1989; Zaslavsky, 1997). Further, function and inverse function are major topics in middle, secondary, and post secondary education (e.g., the CCSSM (National Governors Association Center for Best Practices, 2010) includes function as a content strand in 8^{th} through 12^{th} grade). If pre-service teachers are

expected to support their future students in constructing sophisticated function and inverse function meanings, it is imperative that the pre-service teachers construct and maintain such meanings.

Adding to the complexity of students' function meanings, researchers have argued that conventions (from the researchers' perspectives) common to school mathematics can influence students' function understandings (Moore, Silverman, Paoletti, & LaForest, 2014; Moore et al., submitted). For example, Moore et al. (submitted) reported that when given the graph shown in *Figure 1*(a) and told that a hypothetical student claimed that this graph is such that "*x* is a function of *y*", nine of 25 students claimed the hypothetical student is incorrect and eight of the remaining 16 students claimed that the hypothetical student is correct only if the graph is turned so that the input is represented on the horizontal axis (e.g., as in *Figure 1*(b)).[1] As the authors argued, these 17 students' function meanings (consciously or unconsciously) required the quantity on the horizontal axis to represent the input when discussing if a graph represented a function.

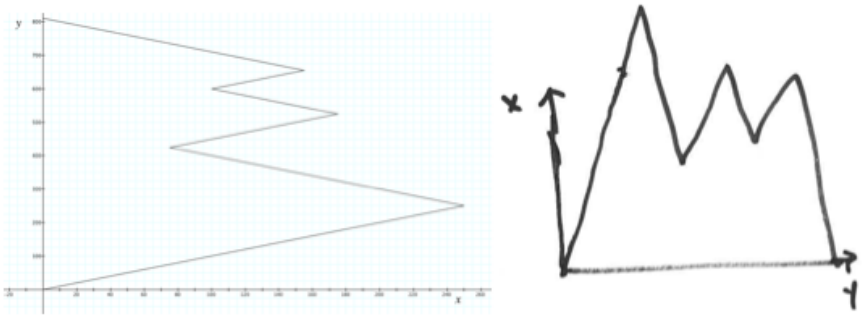


Figure 1: (a) A given graph and (b) a students' representation of the graph oriented so that it represents a function.

[1] The students in my pilot and main study were four of these eight students.

Researchers have indicated these same groups often hold compartmentalized meanings for inverse functions, typically related to executing specific actions in analytic or graphing situations (Bayazit & Gray, 2004; Brown & Reynolds, 2007; Carlson & Oehrtman, 2005; Engelke, Oehrtman, & Carlson, 2005; Kimani & Masingila, 2006; Lucus, 2005; Moore, 2014b; Paoletti, Stevens, Hobson, Moore, & LaForest, 2015). Additionally, researchers reported that when given a function representing the relationship between degrees Celsius and degrees Fahrenheit, a majority of pre-service teachers were unable to interpret a contextual meaning of the inverse function they constructed (Paoletti et al., 2015).

I interpret these results to indicate current curricular approaches to function and inverse function have not been effective in supporting students' development of meanings that can be foundational for success in undergraduate science, technology, engineering, and mathematics (STEM) courses. Confounding the issue, pre-service teachers often leave their undergraduate experiences to become teachers lacking foundational function and inverse function understandings. This is problematic if one accepts that teachers are expected to support students' development of productive function and inverse function understandings, and thus expected to support students' development of meanings that they, as teachers, do not hold. Hence, there is a need to better understand students' and teachers' function and inverse function meanings including identifying more productive approaches to the teaching and learning of function and inverse function.

I conjecture students' quantitative and covariational reasoning has the potential to provide a foundation on which to build sophisticated ways of thinking about function and inverse function. Researchers examining students' activities using a quantitative and covariational reasoning lens have provided important insights into students' learning of a variety of secondary mathematics

topics including rate of change, linear functions, and major calculus concepts (Carlson et al., 2002; Carlson, Larsen, & Jacobs, 2001; P. W. Thompson, 1994b). Despite a growing body of research indicating students are capable of developing productive meanings for various mathematical topics via quantitative and covariational reasoning before developing more formal mathematical understandings (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Johnson, 2012b), researchers have argued that both types of reasoning is largely absent in school mathematics (Carlson et al., 2002; Oehrtman, 2008; Smith III & Thompson, 2008; A. G. Thompson & Thompson, 1995; P. W. Thompson, 2008, 2011, 2013).

I extend this body of research by exploring students' ways of thinking about function and inverse function as they reasoned quantitatively and covariationally. By providing students with multiple opportunities to construct relationships between quantities involving numerous function classes, and relationships not defined by a particular function class, I intended to support the students in reconstructing their function and inverse function meanings based in their reasoning about relationships between quantities.

Background for the Investigation: A Exposition of Bidirectional Reasoning

When describing a student, Ann, conceiving of a relationship in the context of distance, time, and speed, A. G. Thompson and Thompson (1996) described Ann's images of motion involving distance increasing without automatically entailing time increasing; instead Ann understood she could calculate time *after* she knew the completed distance and average speed. A.G. Thompson and Thompson (1996) explained:

> Pat aimed to have Ann see increases in Rabbit's amount of distance and amount of time as happening simultaneously. His intention is depicted in Figure 4. The bi-directional relationship which constitutes covariation entails thinking of both quantities varying simultaneously *without* a necessary dependency. Rather, if one focuses first on distance, one can determine time; if one focuses first on time, one can determine distance. It was

> Pat's intention to have Ann understand motion as covariation of distance and time in this way — as a bi-directional, reversible relationship. (emphasis in original p. 8)

A.G. Thompson and Thompson were concerned with the student's conception of motion as composed of simultaneously changing amounts of distance and amounts of time, without any causal dependency; they defined an individual's conception of such a relationship as a *bidirectional relationship*.

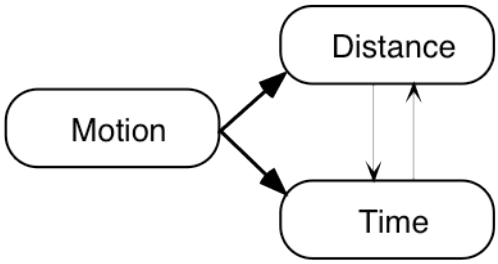


Figure 4. Pat's intention for Ann's understanding: Motion entails moving a distance, and it also entails some amount of time to move that distance. Distance and time become covarying quantities, as opposed to one being dependent upon the other.

Figure 2: Figure 4 from A. G. Thompson and Thompson (1996).

Although not describing bidirectional reasoning explicitly, Saldanha and Thompson (1998) were interested in a student's ability to reason about simultaneous continuous covariation. They defined simultaneous continuous covariation as someone maintaining a sustained image of two quantities changing in tandem where "one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 298). I infer that the authors' definition of simultaneous continuous covariation entails bidirectionality in that the student has a conception of a relationship between covarying quantities without causality or inherent input-output distinctions. In both Saldanha and Thompson's definition and A.G. Thompson and Thompson's example, the researchers' concerns did not foreground students' abilities to solve problems, answer questions, or relate numerical values. Instead they focused on broader conceptions and understandings of a situation and how

the student might come to understand the situation as constituted by quantities and a relationship between quantities.

Conceiving of a bidirectional relationship (i.e. *reasoning bidirectionally*) between quantities entails a process of constructing and re-constructing images and understandings of quantities and how they change in tandem so that the conceiver anticipates that two quantities' values exist and covary simultaneously (represented in *Figure 3*b) (Saldanha & Thompson, 1998; A. G. Thompson & Thompson, 1996). Hence, bidirectional reasoning involves understanding or anticipating that an invariant relationship between quantities exists regardless of the order in which one considers the quantities. Such an understanding stands in contrast with envisioning changes in one quantity *asynchronously producing* or *causing* changes in the other quantity (represented in *Figure 3*a). Bidirectional reasoning neither foregrounds nor excludes reasoning about input and output quantities or one quantity causing changes in the other quantity. In fact, and as I describe here, I conjecture reasoning bidirectionally has the potential to support students in reasoning about relationships between quantities in ways that are productive for their developing function and inverse function meanings.

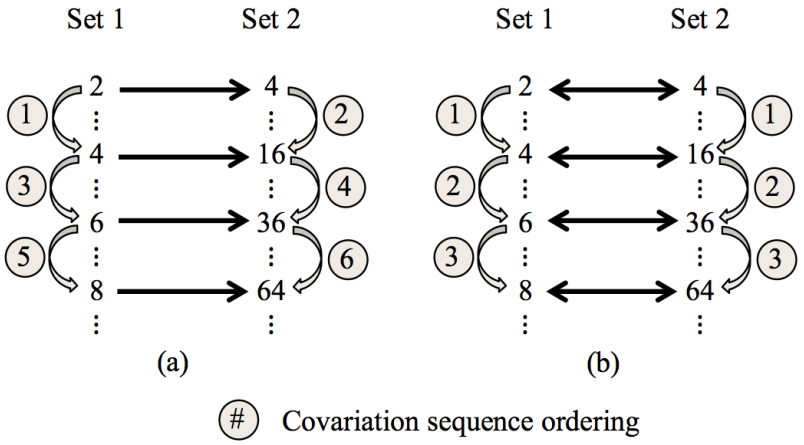


Figure 3: (a) A uni-directional and (b) bidirectional approach to function.

In order to reason bidirectionally, a student must construct quantities that can be related. Furthermore, when a student is presented problems and situations in which multiple quantities covary, she has the opportunity to construct covariational relationships between these quantities. Because of this, I investigated the students' schemes and mental operations as they reasoned quantitatively and covariationally, and explored how these schemes (and others) supported the students' reasoning bidirectionally and constructing bidirectional relationships between quantities.

Bidirectional reasoning in school mathematics.

It may or may not be obvious to the reader how bidirectional reasoning has the potential to be useful in school mathematics. To illustrate how such reasoning might influence a student's function and inverse function meanings, consider a rider on a Ferris wheel boarding the ride at ground level. There are many quantities a student can construct and reason about in this situation. For instance, a student could consider the relationship between distance from the ground and total arc distance traveled. She could understand that the two quantities covary simultaneously (e.g., understand the relationship as a bidirectional relationship), but she might choose to coordinate one quantity first when considering how to reason about and represent the relationship between these quantities. The student may consider changes in total arc distance first and imagine breaking the arc into equal lengths to represent equal changes in arc distance traveled (e.g., equally spaced points along the arc of the ride in *Figure 4*(b)), then using these equal lengths to examine how distance from the ground covaries with changes in total arc measure (e.g., comparing the magnitude of the vertical segments in *Figure 4*(c)).

The student could also first consider changes in distance from the ground and imagine breaking the height of the ride into segments of equal length (e.g., equally spaced points along

the vertical diameter in *Figure 5*(b)). She then could examine how total arc measure covaries with changes in distance from the ground (e.g., comparing the magnitude of the blue arc lengths created by the intersection of the horizontal segments and the circle in *Figure 5*(c)). Although it is necessary to discuss or imagine one quantity changing before discussing or imagining changes in the other quantity (e.g., for equal changes in total distance, distance from the ground increases, then decreases, etc.), bidirectional reasoning entails understanding this choice as arbitrary and not critical to the quantitative structure. An approach to function that draws on bidirectional reasoning emphasizes students constructing and representing relationships between quantities without an initial emphasis on dependency (Moore, 2014b; A. G. Thompson & Thompson, 1996).

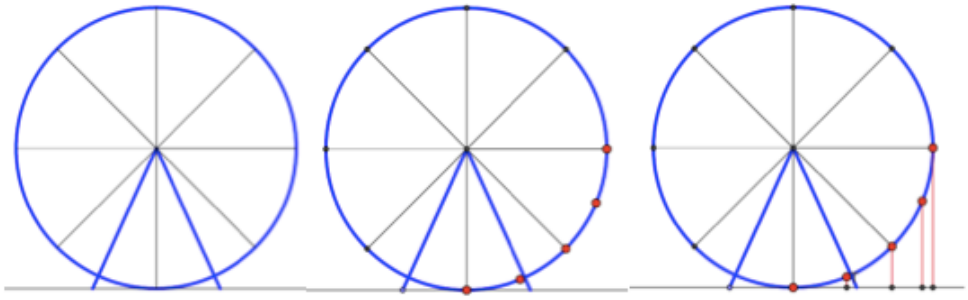


Figure 4: Coordinating distance from the ground for equal changes in total arc measure.

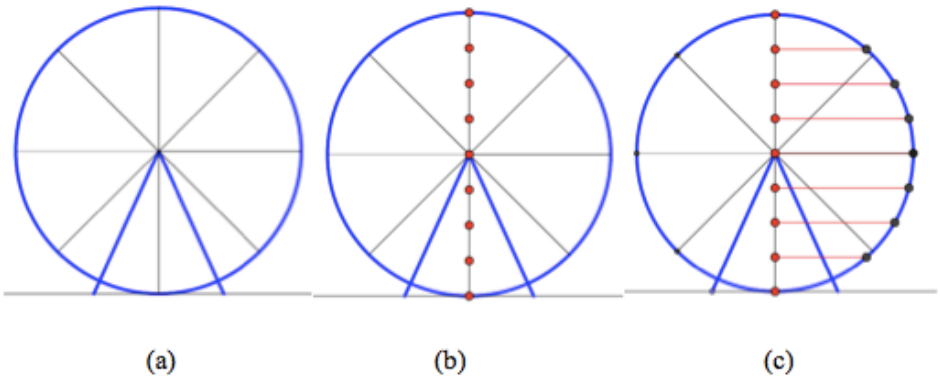


Figure 5: Coordinating total arc measure for equal changes in distance from the ground.

When representing the relationships described above, a student may create a graph similar to those represented in *Figure 6*. A student who understands the relationship between distance from the ground and total arc distance traveled as a bidirectional relationship recognizes that both graphs represent the same covariational relationship between the quantities. For instance, the student can interpret the graph as representing that starting from the beginning of the ride, for equal increases in total distance, the rider's distance from the ground increases at an increasing rate then increases at a decreasing rate (with respect to distance traveled) until distance from the ground reaches a maximum. After this, the rider's distance from the ground decreases at an increasing rate then decreases at a decreasing rate until the rider reaches the ground. The student could represent this relationship in a graph by marking equal increases on the total distance traveled axis and drawing segments perpendicular to this axis to the curve to represent distance from the ground (represented by solid segments in *Figure 6*a). The student could then represent the amounts of change of distance from the ground for equal changes in total distance traveled (represented by dashed segments in *Figure 6*a). Additionally, the student can interpret the graph as representing that starting from the beginning of the ride, for equal increases in the rider's distance from the ground, total arc distance increases at a decreasing rate then increases at an increasing rate until distance from the ground reaches a maximum. The student could represent this relationship by marking equal increases on the distance from the ground axis and drawing segments perpendicular to this axis to the curve to represent total arc distance traveled (represented by solid segments in *Figure 6*b), and segments representing the amounts of change of distance from the ground for equal changes in total arc distance traveled (represented by dashed segments in *Figure 6*b). As can be seen in this example, a student who engages in such reasoning is not restricted to reasoning about input on a specific axis (in contrast to the students

who did maintain such a requirement when describing a graph as a function as reported by Moore et al. (submitted)).

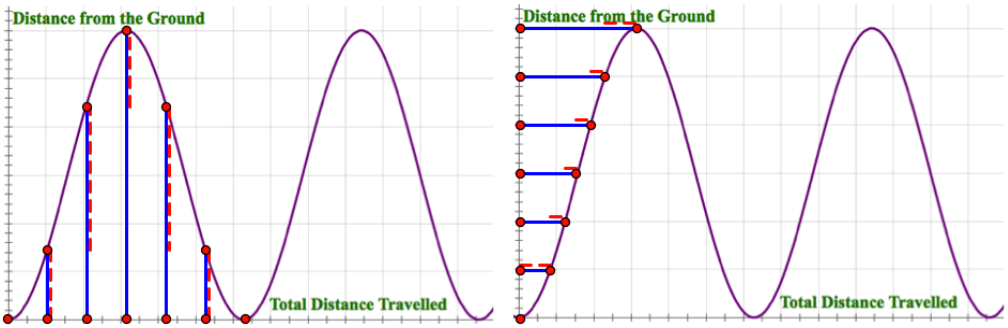


Figure 6: Two representations of the same relationship between quantities.

A student who understands a bidirectional relationship between quantities also has the potential to understand a way to interpret analytically defined rules without assumed input and output quantities. For instance, both graphs in *Figure 6* can be represented by the equation $g = -15\cos(d) + 15$, where d represents *total arc distance traveled in feet* and g represents *distance from the ground in feet*. A student who has constructed the relationship as a bidirectional relationship understands that this relationship can also be represented by the equations $(15 - g)/15 = \cos(t)$ or $t = \text{Arccos}(-(1/15)g + 15)$, where Arccos represents a multivalued function (e.g., $\text{Arcos}(1) = 2\pi n$ where n is an integer).[2] In all cases, a student who reasons bidirectionally can consider the possibility of g or t as representing the input quantity. Conceiving of a bidirectional relationship supports a student in considering all three equations as representing the same underlying covariational relationship; the same relationship the student understands the graphs in *Figure 6* as representing.

The above description illustrates an important connection between a student's bidirectional reasoning and various mathematics representations: a student's conceptualization of a

[2] For more on the distinction between Arccos(x) and arccos(x) as multivalued and single valued functions respectively, see http://mathworld.wolfram.com/InverseCosine.html.

bidirectional relationship as described above supports that student in developing ways of thinking that are not constrained by common school conventions (e.g., input is always represented on the horizontal axis, the output variable is always isolated in an equation). Rather, this type of reasoning has the potential to support a student in constructing and examining various, seemingly different, representations (e.g. graphs under different conventions, equations, tables) as representing the same relationship. An instructional approach that emphasizes bidirectional reasoning can give a student repeated opportunities to understand an invariance across these different representations; namely a quantitative and covariational relationship between quantities is the *something* that is the represented across these representations (Kline, 1974; Moore, Paoletti, & Musgrave, 2013a; Oehrtman et al., 2008; Smith III & Thompson, 2008; A. G. Thompson & Thompson, 1995; P. W. Thompson, 1994c, 2008, 2011, 2013).

By providing students with repeated opportunities to construct and reason about relationships between quantities, bidirectional reasoning can provide a foundation for students' simultaneous construction of sophisticated ways of thinking about function and inverse function. For instance, if a student constructs a (non-causal) relationship between two quantities (e.g., quantities A and B) that does not entail some inherent dependency between the quantities, the student can consciously choose one quantity to be the input of a relation (or function) (e.g., A input, B output) while anticipating that the inverse relation (or function) involves choosing the other quantity as the input (e.g., B input, A output). With respect to graphing relations and inverse relations in the Cartesian coordinate system, a student who understands relations in ways compatible with this can interpret a single graph as representing both a relation and its inverse. For example, a student could interpret the relationship between total distance traveled and distance from the ground described above by choosing to consider total distance traveled as the

input quantity and distance from the ground as the output quantity. Once chosen, the student understands the inverse relationship to have distance from the ground as the input quantity and total distance traveled as the output quantity. Whereas this understanding does not foreground function (or composition of function) as critical to developing inverse function meanings, I conjecture a student who develops understandings compatible with those described would have little difficulty making sense of the formal mathematical definition of inverse function that relies on composition of function.

Research Questions and Plan of Action

As mentioned above, and elaborated on in the next chapter, researchers have indicated that students', pre-service teachers', and in-service teachers' ways of thinking about function and inverse function generally do not meet the researchers' expectations. Although these researchers have made novel and significant contributions, I echo a call by a growing number of researchers (Carlson et al., 2002; Castillo-Garsow, Johnson, & Moore, 2013; Hackenberg, 2014; Moore, 2014a, 2014b; Moore et al., 2013a; Oehrtman et al., 2008; Smith III & Thompson, 2008; Steffe, Liss, & Lee, 2014; P. W. Thompson, 2011) in contending that more attention should be given to students' and teachers' conceptualization of quantitative relationships. I am interested in the ways in which students' conceptualization of quantitative relationships has the potential to be foundational for their ways of thinking about function and inverse function. In this study I engaged students in activities designed to engender bidirectional reasoning with the goal of exploring and describing the possible schemes and operations that constitute bidirectional reasoning. Specifically, I address the following questions:

a. What ways of reasoning do students engage in during activities intended to emphasize reasoning about relationships quantitatively, covariationally, and bidirectionally?
b. How do students ways of reasoning identified in addressing question (a) interplay with their function and inverse function meanings?

c. How do common school conventions interplay with understandings developed and described in (a) and (b)?

I investigated these questions using a combination of task-based clinical interviews (Clement, 2000; Goldin, 2000; Hunting, 1997) and a teaching experiment (Steffe & Thompson, 2000) with two students in the Spring 2014 semester.[3] The study consisted of three phases: one clinical interview per student at the beginning of the course, a teaching experiment consisting of 16 teaching episodes per student (15 with the students working together and one per student working individually), and two clinical interviews per student after the teaching experiment.

My goal in the pre-interviews was to explore the students' function and inverse function meanings as well as the extent to which these were constrained by what I perceive to be conventions common to school mathematics. Additionally, the interviews were intended to engage students in tasks in which they might (or might not) reason quantitatively, covariationally, or bidirectionally. Fellow researchers and I designed these tasks to provide baseline information regarding each research question.

Compatible with the first research question, my goal in the teaching experiment was to engage the students in activities designed to afford them opportunities to reason quantitatively, covariationally, and bidirectionally. I investigated each student's words and actions as she engaged in these problems and built models of each student's mathematics that viably explained her activity. As both students exhibited activity that I took to infer they were reasoning bidirectionally in some way, I explored the implications of this reasoning described in questions (b) and (c). I examined and described how the meanings each student maintained in regards to question (b) and (c) interplayed with her bidirectional reasoning. Although these three research

[3] See Chapter 3 for a more detailed description of the methodology and methods for my proposed study.

questions provided a start for my dissertation, as in any teaching experiment, more questions were raised both within and between teaching episodes that were documented and addressed throughout the teaching experiment.

Closing

In this chapter I provided the reader with an introduction to my dissertation study. In Chapter 2, I provide a description of my theory of knowing and learning, radical constructivism, which I leveraged both when designing tasks to promote the students' reasoning about relationships between quantities and when analyzing the students' activities. I then describe the definitions, constructs, and frameworks researchers have used when analyzing students' quantitative and covariational reasoning and present major findings from the research on students' quantitative and covariational reasoning. After this, I outline the major findings from the research on students' function and inverse function meanings and describe the frameworks researchers have created or used in their investigations of students' and teachers' function and inverse function meanings.

In my study, I use a combination of clinical interviews (Clement, 2000; Goldin, 2000; Hunting, 1997) and a teaching experiment (Steffe & Thompson, 2000) with two pre-service secondary education mathematics students. In Chapter 3, I provide a general description of each methodology and a more detailed description about how I used each methodology in this study. Additionally, I provide information about the students I worked with, the course in which the teaching experiment was situated, and an overview of the data I collected. I conclude Chapter 3 with a description of my data analyses efforts, which was an iterative process using open and axial techniques (Strauss & Corbin, 1998) and conceptual analyses (Steffe & Thompson, 2000; P. W. Thompson, 2000, 2008; von Glasersfeld, 1995).

In Chapter 4-5, I provide findings from a pilot study (Chapter 4) and my dissertation study (Chapter 5). I begin each chapter by characterizing the students' relevant meanings in an initial clinical interview. I describe the students' function and inverse function meanings as well as their quantitative and covariational reasoning. I then illustrate the students' activities as they engaged in constructing, reasoning about, and representing quantities and relationships between quantities and explain my models of the students' understandings that viably explain their activities. I characterize their general quantitative and covariational reasoning as well as their activities that I take to indicate their construction of bidirectional relationships and reasoning bidirectionally. With respect to the pilot study (Chapter 4), I conclude with how the results of this study informed my main dissertation study.

In Chapter 6, I elaborate on important findings from my dissertation study, including characterizations of the students' quantitative, covariational, and bidirectional reasoning. I also describe the implications this reasoning had for the students' function and inverse function meanings. For instance, by the end of the teaching experiment one student conceived a function and its inverse as representing the same invariant relationship but with a different quantity as the input, while the other student was uncertain if a function and its inverse represented the same relationship. I provide implications my study has for curriculum development and conclude the chapter with limitations of the study and areas for future research.

Chapter 2: Review of the Literature

My first goal in this chapter is to present an exposition of my theory of knowing and learning. I describe my understandings of two major principles of radical constructivism put forth by von Glasersfeld. I then describe terms researchers have used in adopting this theory of knowing and learning, including abstraction, scheme, assimilation, and accommodation, which provide tools that I used as I explored students' mental actions. After providing an exposition of radical constructivism, I provide constructs, frameworks, and definitions researchers' who ascribe to Piagetian and radical constructivists theories of knowing have found useful when describing students' quantitative and covariational reasoning. I discuss major findings from this literature base, including examples I take to indicate students engaging in reasoning bidirectionally.

As I am interested in how bidirectional reasoning has the potential to influence students' function and inverse function meanings, I highlight major findings from these literature bases and I describe frameworks researchers have created and used when studying these meanings. I conclude with a critique of the use of the modern function definition in school mathematics based on the findings from research examining students' quantitative and covariational reasoning and their function and inverse function meanings. Collectively, my goal in this chapter is to present an argument that an approach leveraging students' quantitative, covariational, and bidirectional reasoning has the potential to provide a more coherent foundation for students' function and inverse function meanings.

Theoretical Perspective

As a radical constructivist, I prescribe to the two principles of radical constructivism put forth by its originator, Ernst von Glasersfeld (1989a, 1995). von Glasersfeld (1995) separated

each of these principles into two parts as follows (where I have added the distinctions of (a) and (b) for reference purposes):

> 1(a). Knowledge is not passively received either through the senses or by way of communication;
> 1(b). knowledge is actively built up by the cognizing subject.
> 2(a). The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability;
> 2(b). cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality. (p. 51)

Before explaining these principles, I reiterate a statement made by von Glasersfeld (1984), "All communication and all understanding are a matter of interpretive construction on the part of the experiencing subject and, therefore, in the last analysis, I alone can take the responsibility for what is being said on these pages" (p. 2). What follows is my current interpretation of the principles of radical constructivism based on my reading of the works of von Glasersfeld and others, my conversations with peers and colleagues, and my experiences as a teacher working with students.

Although these four parts are best thought of as intricately intertwined, I attempt to explain each to elucidate my current understandings of these principles, although not in the order presented. The last of the four principles, 2(b), is what makes radical constructivism radical, as this principle rejects the notion that an individual can construct an ontological reality (von Glasersfeld, 1984, 1995). For if an individual were to come to know an ontological reality, she would have to be able to compare her experiential reality with what this representation is meant to represent. For this to happen, the individual would need to step outside of her own ways of perceiving and conceiving her world, which von Glasersfeld considered as an impossible task (von Glasersfeld, 1995).

An individual is unable to know an ontological reality; she can only interpret her world via the sensory signals she perceives. For example, scientists now conjecture many animals navigate using magnetism,[4] with certain species having the ability to sense the earth's magnetic field. These animals' experiential realities are markedly different than my own as they can sense something I cannot. Although this and other research indicates that the earth is surrounded by a magnetic field, this field is not part of my sensory experiences and is not part of my experiential world beyond my ability to read about it and use my understanding of it to explain related phenomena. This is just one example of the notion that individuals are unable to know any ontological reality outside of the constraints individuals may experience from such a reality (von Glasersfeld, 1984, 1995).

The question then arises, if an individual cannot construct their world as isomorphic to an ontic reality, what is she constructing? This question is addressed by the first principle (1(a) and 1(b)) of radical constructivism. This principle contends that an individual builds knowledge actively through her sensory experiences, (mental and physical) actions, and her reflection on these experiences and actions. While senses are critical to the construction of an individual's experiential reality, principle 1(a) emphasizes that an individual does not use her senses passively in such constructions. An individual's senses are constantly bombarded with massive amounts of sensory signals and it is up to her to isolate and interpret these signals, where many signals are either not perceived or ignored by the individual.[5] The individual pays attention to certain signals and interprets these signals according to her prior experiences in which she

[4] See http://news.discovery.com/animals/endangered-species/animal-navigation-magnetism-120709.htm for more information.

[5] The 'cocktail party effect' where an individual zones out the conversation she is currently in to listen to a conversation she deems more interesting is an example of how an individual is capable of ignoring certain signals (von Glasersfeld, 1995).

determined those signals to be indicative of something important related to that prior experience. It is through this process of re-presenting prior experiences with specific stimuli in a new situation that an individual *begins* to actively construct a stable reality (von Glasersfeld, 1995).[6]

von Glasersfeld has asserted that knowledge is adaptive in the biological sense. He posited that an individual retains knowledge that proves useful for making predictions about current experiences and adapts knowledge that does not prove useful for this purpose. von Glasersfeld compares this viability of knowledge to the notion of a species being fit or viable in Darwin's theory of evolution (1984, 1995). von Glasersfeld (1984, 1985) maintained the motto for evolution should be 'survival of the fit' rather than 'survival of the fittest', as either a species survives or does not. He argued, "The influence of the environment was and is, under all circumstances, limited to the elimination of non-viable variants. Hence, the environment can, at best, be held responsible for extinction, but never for survival" (von Glasersfeld, 1984, p. 6). von Glasersfeld contended, an individual's knowledge, like species, is constrained by the individual's environment but not caused by it per se. Further, just as an environment permits many viable species but only certain ones exist in that environment, an individual's environment permits many viable constructions, but she can only construct her world as she has happened to interpret it (von Glasersfeld, 1984, 1995).

Just as an individual cannot be confident her constructed reality is isomorphic to an ontological reality, a researcher cannot be confident his model of a student's mathematics is isomorphic to the student's understandings. A researcher using a radical constructivist lens maintains that someone else's knowledge is *fundamentally* unknowable to another person. The best a researcher can do is build models of the student's mathematics (i.e. the student's

[6] I emphasize *begins* as sensory material is not always needed when schemes are reflected to higher conceptual levels. I elaborate on this more in later sections.

mathematical realities) that provide viable explanations for the student's words and actions. Steffe (2013) noted, "Almost any situation can be organized as having something to do with logic or mathematics follows from the premise that logic or mathematics is not found in the situations but is put there by the assimilating individual" (p. 358). Hence, when exploring a student's mathematical reality, researchers need to be attentive to the ways in which the student is imbuing her logical or mathematical structures to the situation as the student conceives it.

Abstraction leading to schemes.

I maintain that an individual actively builds her knowledge of a stable reality through the process of abstraction. von Glasersfeld stated, "The living creature, be it fish, fowl, or human, thrives by abstracting regularities or rules from experience" (1989, p. 5). Piaget emphasized that abstract concepts, as well as all new knowledge, involved processes of abstraction (Oehrtman, 2008; Piaget, 2001; von Glasersfeld, 1991). According to Oehrtman (2008), Piaget maintained that an individual abstracts conceptual structures from her experiences through her actions and her coordination of actions on objects, physical or mental.

While abstraction is critical to my theory of learning, the process of abstraction is still somewhat mysterious in that there is still no general model on how such a process may be achieved (Simon et al., 2009; von Glasersfeld, 1991). Although all individuals are capable of making abstractions, the process by which an individual does so is still not widely understood. Before describing different types of abstraction later in this chapter, I describe other important terms in this theory of learning, specifically schemes, assimilation, and accommodation.

Schemes, assimilation, accommodation, and perturbation.

A *cognitive scheme* is one type of abstraction an individual makes from her experiences.[7] A scheme is a goal-directed mental activity consisting of three parts: recognition of a certain situation, a specific activity associated with that situation, and the expectation that the activity produces a previously experienced result (Hackenberg, 2010; Steffe & Olive, 2009; P. W. Thompson, 1994a; von Glasersfeld, 1995). When an individual recognizes new experiential material as an instantiation of a previous experience, she activates a scheme that has proved useful in her prior experiences (Steffe & Olive, 2009; von Glasersfeld, 1995). According to Hackenberg (2010), "This recognition is the result of assimilation, the basis for construction and modification of schemes" (p. 386). In the process of *assimilation*, an individual recognizes certain sensory material which indicates the activation of the scheme, while being unaware of or ignoring other available sensory material that does not fit into the schemes recognition template (von Glasersfeld, 1984, 1995). This means an individual may activate a scheme even if an observer views the situation as distinct from previous situations in which the individual activated that scheme.

Once a situation has been assimilated into a scheme, the activity of that scheme may be triggered. An individual can often imagine the scheme's activity or result without carrying out the activity, in which case the scheme is *anticipatory* (Steffe & Olive, 2009). This ability supports the individual in engaging in thought experiments, which von Glaserfield (1995) referred to as, "Perhaps the most powerful learning procedure in the cognitive domain" (p. 69). The activity (carried out or imagined) produces a result (experienced or imagined), which is again assimilated into the scheme so that the individual can compare this result to his expectation

[7] I reiterate that when I refer to both physical and cognitive experiences when I describe an individual's experiences.

of the result of the scheme, assessing whether the goal has been met by the activation of that scheme (Hackenberg, 2010). This assessment of the result of the scheme demonstrates that schemes, once constructed, are goal-directed rather than random or general activity.

While the above discussion may give the impression that assimilations only activate or trigger schemes, it is important to note that assimilation is also the source of schemes (Hackenberg, 2010; P. W. Thompson, 1994a). Although there are multiple ways schemes can emerge from assimilations, at the most basic level a scheme is a construction involving assimilation; during an experience, an individual must take perceived or imagined experience, consider this as similar (or identical) to a previous experience via re-presentation of the former experience and assimilate the two experiences to a scheme applicable to that experience. It is also important to note that an individual constantly tests a scheme and the scheme evolves with further experiences.

Steffe and Olive (2009) noted that schemes do not always progress from situation to activity to result. The researchers indicated that, in some cases, such as a baby's sucking scheme, the baby's activity of sucking triggers his goal of finding something to suck on (often his hand).[8] Steffe and Olive (2009) argued that a scheme's situation, activity, or result can be compared or related to the other two in some way, with the generated goal at the apex of the comparing or relating of the scheme's constitutive parts.

When the result of the activity of a scheme does not meet an individual's expectations this can cause a perturbation (von Glasersfeld, 1984, 1995). If the sensory material that activated the scheme is still available, she can review the material for characteristics that were initially disregarded or were not perceived in assimilation (von Glasersfeld, 1984, 1995). If the result of

[8] Or in the case of baby Teo, his blanket, his mom or dad's hand, or just about anything he can get his mouth on.

the activity is disappointing, she can adjust the conditions that trigger the scheme in the future, thus changing the recognition pattern of the scheme (von Glasersfeld, 1995). If the individual deems the unexpected result as positive, she may adjust her recognition pattern to include this new sensory material (von Glasersfeld, 1995). In either case, the unexpected result can lead to the individual making an *accommodation* in the scheme (Steffe & Olive, 2009; von Glasersfeld, 1984, 1995).

Researchers have distinguished between different types of accommodations; in this work I distinguish between *functional accommodations* and *metamorphic accommodations* (Steffe, 1991a, 1991b; P. W. Thompson, 2015). A functional accommodation is any modification to a scheme that occurs in the context of using the scheme (Hackenberg, 2010; Steffe, 1991a). P. W. Thompson (2015) indicated an understanding based in a functional accommodation "is an understanding that can be easily lost once the person's attention moves on and is typical of when a person makes sense of an idea for the first time" (p. 11).[9] Hence students may make functional accommodations to their schemes that appear (from the researcher's perspective) to have little lasting impact on the assimilatory structure of the scheme (e.g., the student may appear to 'revert' to their prior activity when engaging in a similar problem).

A metamorphic accommodation is an accommodation that occurs in no particular situation or activity and involves a general reorganization of a scheme (Steffe, 1991a, 1991b). Although a metamorphic accommodation does not occur in the context of using a scheme, such an accommodation often occurs "as a global **result** of such specific experiential episodes" (emphasis in original, Steffe, 1991a, p. 187). Hence students can experience metamorphic

[9] I note that P. W. Thompson's (2015) description (and my use) of the temporary nature of functional accommodations is compatible with Steffe, Cobb, and von Glasersfeld (1988) description of temporary modifications.

accommodations by engaging in goal-directed activity in which they experience functional accommodations multiple times.

Types of abstraction.

The most basic type of abstraction is an *empirical abstraction* (von Glasersfeld, 1991). Any abstraction of patterns that are composed of specific sensory material (e.g., smell, feel, taste) or motor signals (e.g., moving an object around in one's hand) is referred to as an empirical abstraction (von Glasersfeld, 1995). These abstractions are concerned with the product of activity rather than the activity itself. Moving beyond sensorimotor experiences, an individual is capable of making abstractions at the level of mental operations. One such type of abstraction is referred to as a *reflective abstraction*. The term reflective is used to emphasize that such abstractions happen when an individual's actions result in a concept moving to a higher conceptual level (von Glasersfeld, 1995). According to Thompson (1994a), reflective abstraction typically consists of two parts, "First is the re- construction of actions so as their activation is progressively less dependent on immediate experience. The second is an assimilation to higher levels of thought—to schemes of operations" (p. 188). Thompson related the re-construction of actions with learning and related assimilation to a higher level of thought with comprehending. Metamorphic accommodations can be the result of a reflective abstraction (Steffe, 1991b). The level of abstraction above reflective abstraction is referred to as *reflected abstraction*. Like reflective abstractions, reflected abstractions involve a coordination of actions or schemes as well as moving to a higher level of thought, but in this case, the individual is aware of the abstractions she has made.

Whereas both reflective and reflected abstractions are focused on the physical or mental activity the individual undertakes, meanings based in *pseudo-empirical abstractions* are

concerned with the product of the individual's physical or mental activity. The activity associated with a pseudo-empirical abstraction, like the activity associated with an empirical abstraction, only takes place if suitable sensorimotor material is available (Piaget, 2001; von Glasersfeld, 1995). However these abstractions are not constrained to sensorimotor operations and observables, as the individual has created generalizations from the product of and conditions for the actions or activity (Moore, 2014b; Piaget, 2001). Because of this, Piaget (2001) noted that a pseudo-empirical abstraction is a special case of reflective abstraction, "and not in any way a derivative of empirical abstraction" (p. 303). Further, Piaget (2001) argued pseudo-empirical abstractions are essential supports and auxiliaries for reflected abstractions. Piaget (2001), argued, "Without in any way ruling out their possible coexistence, the evolution of pseudo-empirical and reflected abstraction is thus characterized by this inversion in their proportions: Pseudo-empirical abstraction loses in relative value… while reflected abstraction gains" (p. 307).

In the context of this study, consider students' actions when engaging in problems related to inverse function. Multiple researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti et al., 2015) have indicated that students have activities to carry out when asked to work with or determine inverse functions in various representations or context. However, from the researchers' perspectives, the students' activities in the various representations and contexts are not always compatible (e.g., using a reciprocal to define the inverse function when given an analytic rule and reflecting over the line $y = x$ when graphing the inverse of a given function). These students have likely abstracted inverse function meanings via pseudo-empirical abstractions; when given appropriate material, the students can carry out activity that they abstracted as related to inverse function in previous experiences when working with similar given material. In contrast, a student who has inverse function meanings grounded in reflected or

reflective abstractions may understand a way to relate their actions in the various representations and contexts to a meaning that applies in each situation and context. For instance, the student may understand that if a function inputs quantity A and outputs quantity B then its inverse function inputs quantity B and outputs quantity A, leveraging this understanding across contexts and representations (the research examining students' inverse function meanings is described in greater detail below).

Summary and implications of my theoretical perspective.

My goal in this section was to give an exposition of radical constructivism (von Glasersfeld, 1995). To briefly summarize, an individual abstracts cognitive schemes as she constructs a stable reality. When a scheme produces an unexpected result, this causes a perturbation (von Glasersfeld, 1995). The perturbation has the potential to lead to an accommodation, which re-establishes the individual's equilibrium, constituting a cognitive change (von Glasersfeld, 1995).

In addition to describing schemes, assimilation, accommodation, and perturbation, I provide an outline of a theory of abstraction. This theory has important implications for the teaching and learning of mathematics. For instance, whereas reflective and reflected abstractions have the potential to support students in developing invariance at the level of mental operations, pseudo-empirical abstractions only support students in developing a sense of invariance in carrying out activity (Moore, 2014b). Oehrtman (2008) provided some important features abstraction theory has for instructional activity if we want students to develop understandings via higher-level abstractions that I used when designing activities for my dissertation study. He first suggested that the structure of understanding we intended our students to develop needs to be systematically reflected in the activities in which they are asked to engage. Secondly, he suggested that students should have repeated opportunities to coordinate their activity in ways

that help them attend to the feedback provided by the constraints of the context being explored. Finally, Oehrtman (2008) suggested a concept should not be treated in isolation, instead being extended through many applications and contexts. Hence, I used the constructivist's theory of knowing both as a means by which to discuss and analyze students' activities and as a guide for organizing instruction in order to effectively support students developing schemes via reflective and reflected abstractions.

Quantitative and Covariational Reasoning

Researchers who ascribe to Piagetian and radical constructivist theories of knowing and learning have developed definitions and frameworks they have found useful when describing and explaining students' mental processes and conceptual structures as the students constructed and reasoned about quantities and relationships between quantities (Carlson et al., 2002; Steffe, 1991c; P. W. Thompson, 1994a, 2011). In this section, I present constructs including definitions and frameworks created and/or used by these researchers investigating students' quantitative and covariational reasoning. These definitions do not claim to represent the ways students' reason quantitatively or covariationally, but instead represent my interpretation of one model a group of researchers have found effective in modeling students' mathematical activities. Then, I synthesize the research on quantitative and covariational reasoning, indicating emergent themes in the literature base.

Constructs, definitions, and theoretical frameworks.

I establish a system of working definitions for various terms related to students' quantitative and covariational reasoning so that the reader understands my use of this terminology. These definitions are referred to as working definitions as my understanding of these constructs is

always evolving through my continued research on student thinking and my interactions with colleagues and students.

Quantitative reasoning.

Steffe (1991c) described:

> Properties of concepts are introduced by the knowing subject's actions in the construction of the concepts. For example, the sweetness of sugar is introduced by the tasting individual just as the color of roses is introduced by the viewing individual. The idea that an activity introduced by the child in constructing a concept might be later abstracted as a property of the concept that could be called "quantity" is an important one. It changes the emphasis from viewing the sources of quantity as existing in what we take to be external to the child to taking quantity as emerging from the child's interactions with elements in his or her environment. (p. 62)

Building on Steffe's (1991c) description of quantity as emerging from an individual's interaction with his or her environment (i.e. as a conceptual entity rather than a physical entity), P. W. Thompson (1994a) defined, "A quantity is schematic: It is composed of an object or phenomena, a quality of the object or phenomena, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). An individual does not need to undertake or carry out the measurement process in order to construct a quantity; instead she must only imagine she is capable of making either gross comparisons or numerical measurements (Smith III & Thompson, 2008). Further, a sophisticated quantitative scheme is non-numeric in nature. For example, if a student has constructed the height and volume of water in a bottle as quantities, she may reason about these quantities without knowing exact (numeric) measures of height or volume of water in the bottle, instead relying on an awareness of a capability to measure these quantities' magnitudes.

Quantities are cognitive objects, constructed by an individual in order to make sense of her experiential world (Moore & Carlson, 2012; P. W. Thompson, 1994a, 2011; von Glasersfeld, 1995). This means conceptions of a quantity differ from individual to individual. For example,

whereas an individual from the Midwest United States is likely to have had experiences observing large amounts of grain, thus having the opportunity to construct amounts of grain as something that can be quantified, with a possible unit of measurement being a bushel, someone from an urban area in the Northeast is less likely to have such experiences. Without the necessary experiences with large amounts of grain (or grain more generally), an individual from the Northeast may have difficulty considering large amounts of grain as something to be quantified and understanding a bushel as a unit by which to quantify an amount of grain.

Initially, P. W. Thompson (1990, 1994b) defined *quantification* as the process by which one assigns a numerical value to a quantity through a process of direct or indirect measurement. Later in his first-order analysis, P. W. Thompson (2011) asserted that quantification is the process of deciding what it means to measure a selected attribute of an object or phenomena (e.g. the quantity), what aspect of that attribute is measured, and what this measurement means after it has been quantified. Measurement and quantification are not equivalent and an individual's conception of a quantity and its quantification are cyclic. Specifically, a quantity becomes a quantity for an individual in her attempts to develop a quantification process for it, the result of which is an internalized measurement process (Moore, 2012; P. W. Thompson, 2011). Further, an individual's conception of a quantity can evolve as she considers the quantification process, which can further impact her understanding of the quantity (P. W. Thompson, 1994a).[10]

Smith III and Thompson (2008) stated, "Some quantities arise directly from measuring things; others, like differences and ratios, arise from *quantitative operations*-operations on other quantities" (emphasis added, p. 23). Quantitative operations are defined as the mental operation by which two quantities are taken to produce a new quantity (Smith III & Thompson, 2008; P.

[10] I provide examples of this in my description of the results from the literature on students' quantitative and covariational reasoning.

W. Thompson, 1994a). For example, a multiplicative comparison of distance traveled and elapsed time can produce the quantity 'average speed' (P. W. Thompson, 1994a), and a multiplicative comparison of subtended arc length and the length of the radius of a circle can create the quantity 'radian angle measure' (Moore, 2013). Quantitative operations are distinct from numerical operations in that quantitative operations do not require specific values, instead having to do with the comprehension of a situation rather than an evaluation of a numerical relationship among measures (P. W. Thompson, 1994a, 2011).

When an individual conceives of a quantitative operation, they create a *quantitative relationship*, which is defined as the image of a quantitative operation where two quantities determine a third quantity (P. W. Thompson, 1994a). P. W. Thompson (2011) provided a first-order model of a conception of covariation concluding his description, "I can now represent a conception of two quantities' values covarying as $(x_e, y_e)= (x(t_e),y(t_e))$. I intend the pair (x_e, y_e) to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47).[11] I infer when Thompson described this image of covariation as a multiplicative object he intended to convey the object was composed of the Cartesian product of two sets (e.g., the two dimensional coordinate system, with the trace representing the covariational relationship being one possible subset–a multiplicative object– of the coordinate system).

When an individual creates a network of quantitative relationships, she creates a *quantitative structure* (P. W. Thompson, 1994a). P. W. Thompson (1994a) suggested that an individual's quantitative structure is emergent, where she constructs an image of a situation that lacks a well-developed structure of quantities. Through continual consideration and re-consideration of the

[11] I use *e* for typesetting purposes but Thompson (2011) used the lower-case Greek letter epsilon.

structure, quantities, and relationships, the individual constructs and refines her quantitative structure. Finally, *quantitative reasoning* is defined as the analysis of situations into quantitative structures which include a network of quantities and quantitative relationships (P. W. Thompson, 1994a). As such quantitative reasoning can involve both numerical and non-numerical reasoning (Johnson, 2012b), but the essence of quantitative reasoning is non-numerical, having more to do with the comprehension of the quantities in a situation and how they relate to each other (Smith III & Thompson, 2008).

Covariational reasoning.

P. W. Thompson (2011) maintained the aspects of quantitative reasoning that allows students to hold in mind invariant relationships among quantities that vary dynamically are not well understood. In order to conceive of a mathematics of variation, Thompson (2011) asserted students must imagine quantities whose values vary, stating, "It is easy to say what variation is, but it is nontrivial for students to develop the conceptual operations that allow them to see variation in quantities and quantitative situations" (p. 47). The literature describing students' covariational reasoning has begun to address the question of the conceptual operations that allow students to comprehend variation in quantities and quantitative situations.

Definitions and development.

Confrey and Smith (1991, 1995) described a covariational approach to function in which they viewed functions as, "The covariation between quantities. As one quantity changes in a predictable or recognizable pattern, the other also changes, typically in a differing pattern" (1991, p. 57). Confrey and Smith indicated that a student can describe a functional relationship by describing changes in consecutive x-values then the changes in the corresponding y-values. They used their definition of covariation to explore how this construct may help students develop

exponential function meanings by examining the ways students coordinate values in the columns of a t-table (Confrey & Smith, 1995). Confrey and Smith (1995) argued that students' covariational approach provided them with an understanding of exponential functions and claimed that the function concept in general would be better understood from a covariational perspective. The authors did not discuss students developing understandings of two quantities dynamically and simultaneously changing across a spectrum of values. Although students in Confrey and Smith's study may have developed such images, the authors did not explore this.

Whereas Confrey and Smith focused on students coordinating values presented in a table, Saldanha and Thompson (1998) were interested in exploring the imagistic foundations that supported students in 'seeing' quantities covary. They stated, "Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). They argued students' images of covariation are developmental, with students initially imaging one quantity changing, then the other, then back to the first, and so on. Later students develop images that allow them to imagine the two quantities changing in tandem.[12] Saldanha and Thompson described an operative image of covariation in which an individual is capable of imagining two quantities being tracked for some amount of time with the correspondence between the two quantities as an emergent property of this image. Based on their analysis of one student's activities as he engaged in a situation where he conceived quantities covarying and attempted to model this relationship, the authors concluded, "Understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (p. 303). By extending the examination of students' covariational reasoning to a continuum of states of covarying quantities, Saldanha and Thompson (1998) (as well as P.

[12] Students developing images of covariation is critical to their construction of bidirectional relationships. I will return to this in Chapter 6.

W. Thompson (1994b)) moved examining students' covariational reasoning beyond looking at how they coordinate values in a t-table, setting up the field for a more in-depth look at the mental actions students may use when constructing relationships between covarying quantities, including how students might come to conceive situations as entailing covariation.

Building on the work of those discussed above, Carlson et al. (2002) stated, "We define *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). While this definition is similar to that used by Confrey and Smith (1991, 1995), Carlson et al. (2002) were concerned with the ways students reasoned about these quantities changing across a continuum similar to Saldanha and Thompson (1998). Further, the authors concurred with Saldanha and Thompson's (1998) assertion that covariational reasoning is developmental, arguing that images of covariation can be defined according to levels, in which each level builds on the previous.

Mental actions and levels of covariational reasoning.

Carlson et al. (2002) presented a developmental framework that allows for a more fine-grained analysis of students' covariational reasoning. Their framework consists of five mental actions (MA1-MA5) (*Figure 7*) and five levels (L1-L5). The researchers developed the mental actions and levels by observing students' activities in the context of creating and interpreting graphs representing dynamic events (Carlson et al., 2002; Carlson et al., 2001). The researchers intended the framework be used in analyzing students' work on individual problems and not to classify a student generally at a certain covariational reasoning level (Carlson et al., 2002). Therefore, it is not problematic if a student exhibits L5 reasoning on one task, but only demonstrates L3 reasoning on a different task.

Mental Actions of the Covariation Framework

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., *y* changes with changes in *x*)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 7: The five mental actions from Carlson et al. (2002, p. 357).

To make these mental actions more tangible, I describe various ways students could exhibit these mental actions in the context of *the Bottle Problem*,[13] which was created by the Shell Center (Swan & Team, 1999) and modified by Carlson et al. (2002) (*Figure 8*). A student who exhibits MA1 describes the value of one quantity changing with changes in the other quantity. In the Bottle Problem a student may state that adding water to the bottle changes the volume of the water in the bottle and also changes the height of the water in the bottle. A student who exhibits MA2 recognizes that a change in one quantity causes a directional change in the other quantity.[14]

[13] Variations of the Bottle Problem presented here are used throughout the covariational reasoning literature, with a common variation asking students to work backwards to draw a bottle given a graph representing volume and height. I refer to both variations of problem simply as the Bottle Problem.

[14] I note that Carlson et al.'s (2002) framework assumes the changes in the input quantity to be increasing, although this was not always the case in this study.

A student may indicate that when the volume of the water in the bottle increases, the height increases as well.

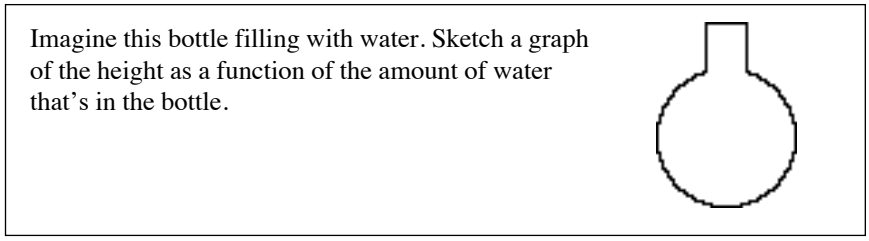
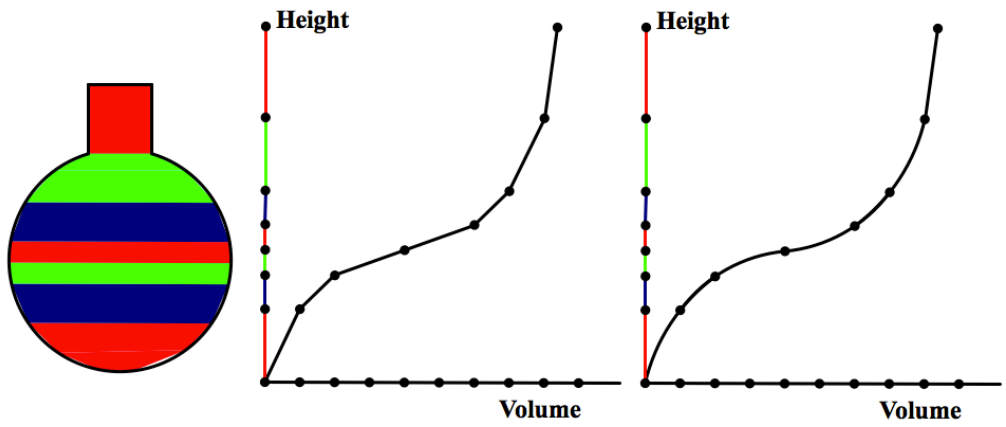


Figure 8: The Bottle Problem used by Carlson et al. (2002, p. 360).

When a student begins to focus on the amounts of change of one quantity while considering changes in the other, she exhibits MA3. For example, using the bottle in *Figure 9*, a student may consider equal changes of volume, then draw height tick marks getting progressively closer until the widest part of the bottle to represent that changes in height decrease as equal amounts of volume are added to the bottle (recreated in *Figure 9*a by different colored cross-sectional areas of the bottle representing equal amounts of volume). When a student is concerned with the average rate of change of one quantity over successive inputs of another, she exhibits MA4. A student engaging in MA4 may draw contiguous secant segments where the slope of each line is adjusted according to the rate at which the height and volume change as the water is poured into the part of the bottle under consideration, as represented in *Figure 9*b for the bottle represented in *Figure 8*. Finally, a student exhibits MA5 when she attends to the instantaneous rate of change of a relationship. A student exhibits such reasoning when she constructs smooth curves with correct concavity representing the relationship between volume and height while explicitly attending to the fact that she was considering smaller and smaller amounts of water being added to the bottle, represented in *Figure 9*c.



(a) (b) (c)

Figure 9: Representations of solutions to the Bottle Problem.

The levels of the covariation framework are designed such that each level supersedes the previous. For example, simply making statements that indicate MA4 reasoning is not enough to be classified as L4. Rather, a student exhibits L4 reasoning when she is able to unpack this reasoning, showing she is capable of MA1 through MA4. Although this may seem trivial, Carlson et al. (2002) presented a student who demonstrated MA5 reasoning, but was unable to unpack this reasoning in terms of the other mental actions. This led the authors to conclude the student did not exhibit L5 reasoning in that context.

The mental actions and levels described by Carlson et al. (2002) provide researchers with a tool to conduct fine-grained analysis of student thinking as well as provide insight into the ways students may develop their covariational reasoning abilities. As I am less interested in classifying students' levels on a task, but am necessarily interested in students' mental actions, including their ability to coordinate MA5, as well as MA1-4. I only focus on identifying mental actions (MA) in my data analysis.

Chunky and smooth variation.

Chunky and smooth variation (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013) is another paradigm researchers have used when building models of students' reasoning about quantities that vary in tandem. *Chunky variation* involves an individual imagining change occurring in completed chunks. A student considers variation occurring in a unit chunk repeatedly applied to the situation without attending to variation within the chunks (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). For instance, in the Bottle Problem, a student may imagine the height and volume of a bottle covarying as water is added to the bottle cup by cup. In this case, the student imagines the change occurring in amounts equal to the volume or size of the cup and is not concerned with how the height and volume change as the water in each cup is being poured into the bottle. As this example demonstrates, chunky thinkers attend to the discrete chunk values rather than an entire continuum of values. Some students exhibit an advanced form of chunky variation by indicating that the initial chunks can be divided into sub-chunks; however, the chunks always exist in their thinking. For example, Castillo-Garsow (2012) illustrated a student imagining breaking a day into minutes, then seconds, then microseconds, but at each level the student was concerned with these subdivisions as edges of reasoning, while not attending to the variation happening between these smaller and smaller chunks (Castillo-Garsow et al., 2013).

Contrasting with chunky variation, *smooth variation* involves students imagining a quantity continuously taking on varying values through the experiential or imagined flow of time (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). For example, in the Bottle Problem, an individual who imagines change happening smoothly might coordinate changes of height and volume while imagining water continuously being poured into the bottle from a spigot. When

reasoning smoothly, students imagine a quantity always in flux, with a beginning point (or magnitude) but no end point as once an endpoint is reached, progress stops and in the students imagery the quantity is no longer changing (Castillo-Garsow et al., 2013). Smooth reasoning does not rely on numerical values as obtaining this numerical information requires the student to imagine stopping in order to obtain a value. Presenting a researchable problem, Castillo-Garsow et al. (2013) argued that smooth thinking is a powerful root for students' images of variation, and that smooth thinkers are likely capable of chunky thinking while the reverse is not true.

The combination of smooth and chunky variation has been referred to as *continuous covariation* (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Saldanha & Thompson, 1998; P. W. Thompson, 2011). Continuous covariation is the recursive process in which a student conceives of a quantity varying smoothly over repeated chunks. This process is repeatable to smaller and smaller chunks with the awareness that within each chunk the quantity passes through each value in that chunk (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; P. W. Thompson, 2011). P.W. Thompson (2011) provided a first-order model of such an understanding in which an individual conceives of a quantity's, x's, value varying over conceptual time, t. The individual could then conceive of subdividing (or covering) the domain of t-values using intervals of size e, and consider the variation of x in these intervals (i.e. considering x_e as the set of x-values $(x(t), x(t + e)) = x(t_e)$). Repeating a quote from above, Thompson (2011) concluded his description, "I can now represent a conception of two quantities' values covarying as $(x_e, y_e) = (x(t_e), y(t_e))$. I intend the pair (x_e, y_e) to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47).

Static and emergent shape thinking.

Another construct researchers have found useful in describing the extent students reason (or quantitatively and covariationally in the context of graphing is *shape thinking* (Moore & Thompson, 2015, in preparation; Weber, 2012). Moore and Thompson (2015, in preparation) described students focusing on perceptual cues and global properties of shape in relation to learned facts as *static shape thinking*. They contrast this with *emergent shape thinking*, which they describe as students conceiving graphs in terms of an emergent trace constituted by covarying (projected) magnitudes.

The description of a student reasoning about a bidirectional relationship between total distance traveled and distance from the ground of a rider on the Ferris wheel (represented by *Figure 4*, *Figure 5*, and *Figure 6*) is compatible with emergent shape thinking. A student who reasons in this way constructs a relationship between covarying quantities in the situation, and constructs or reasons about the graph as representing an emergent trace capturing this relationship. In contrast, a student who engages in static shape thinking may treat the graph shown in *Figure 6* as representing a translation of a memorized shape (e.g. the conventional sine or cosine graph).

Static and emergent shape thinking are important given my interest in students' quantitative and covariational reasoning, particularly in situations in which I prompt students to graph relationships between covarying quantities. I concur with Moore and Thompson (in preparation), who conjectured quantitative and covariational reasoning has the potential to support students in constructing the *something* they represent using functions. For instance, Moore et al. (2013a) explored students' activities when working in the Cartesian and polar coordinate systems. They described the students' reasoning quantitatively and covariationally about graphs as emergent

traces to conceive of two different pictorial representations (e.g., linear functions in the Cartesian and polar coordinate systems) as representing the same invariant relationship. The students understood the quantitative and covariational relationship as the *something* represented by the different appearing graphs.

Results from research conducted in these paradigms.

I started this section by highlighting one insight researchers have provided by using a quantitative and covariational lens; namely Moore et al. (2013a) showed students reasoning quantitatively and covariationally to conceive of different appearing graphs in the polar and Cartesian coordinate systems as representing the same relationship between quantities. In this section I highlight other themes in the quantitative and covariational reasoning literature. Researchers have shown the importance of not taking for granted students' construction of and reasoning about quantities (Moore, 2013; P. W. Thompson, 1994a). Researchers have also indicated students' images or conceptions of a situation influence their construction of and reasoning with quantities they conceive of in a situation (Carlson et al., 2002; Ellis, 2007; Moore & Carlson, 2012; Rasmussen, 2001; Smith III & Thompson, 2008). Additionally, researchers have shown that reasoning quantitatively can support students in: developing foundational understandings before more formal mathematical ideas (Ellis et al., 2012; Johnson, 2012b); transferring their knowledge to novel situations (Carlson et al., 2001; Ellis, 2007; Lobato, Rhodehamel, & Hohensee, 2012); and leveraging continuous covariational reasoning to construct productive images of a situation (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Johnson, 2012b; Moore & Carlson, 2012; Moore et al., 2013a; Saldanha & Thompson, 1998; P. W. Thompson, 2011). I conclude with some examples of the various mathematical ideas, organized

by grade level, in which quantitative and covariational reasoning have proved effective in supporting students in developing productive mathematical meanings.

Concerning the importance of teachers and researchers not taking students' understanding of quantities and their quantification as a given, P.W. Thompson (1994a) presented an example of a student initially lacking foundational understandings of the concept of speed. Thompson described the activity of JJ, a 5th-grade student, as she constructed the relationship between distance and time as speed. JJ worked in a computer micro-world that depicted a race up and back on a track between a turtle and a rabbit. The program was designed so that the turtle could have a different constant speed for each leg of the trip while the rabbit's speed was constant on both legs. Entering the teaching experiment, JJ understood speed as a distance, or speed-length in one unit of time. She leveraged this understanding to determine the time it took for rabbit to travel 100 feet at 30 ft./s by multiplicatively comparing the total length with the number of speed lengths. However, this understanding did not support JJ in addressing other prompts the research posed, including predicting the constant speed rabbit needed to travel at to tie turtle traveling at different speeds on his way up and back. During the teaching experiment, JJ began to conceive of the situation as including two covarying quantities, distance and time, with both accumulating as the animals traveled. This supported JJ in abstracting speed as a ratio relating distance and time, hence constructing speed as a quantity unto itself; after this JJ answered certain questions posed by the researcher that she could not address before constructing speed as a quantity.

As another example of students' construction of quantities, Moore (2013a) focused on two college students' activities in a teaching experiment designed to engage them in constructing angle measure as a multiplicative comparison of the subtended arc length of a circle created by the rays forming the angle and the circumference of the circle. During activities conducted in a

pre-interview, neither student showed evidence of understanding a systematic way to measure an angle without a labeled protractor and both relied on degree angle measure in their solutions and explanations. Moore (2013a) described the students' activities as they constructed angle measure as a quantity composed of a multiplicative comparison between arc length and circumference that was independent of the size of the circle used to measure the arc and circumference. The students extended these understandings to degree measure, where 360 degrees rotate a circle, and to radian measure, where 2π radii rotate a circle. By constructing angle measure as a quantity composed of a multiplicative comparison of subtended arc length and circumference, the students constructed angle measure meanings that led to coherent and related degree and radian angle measure understandings. Collectively, P. W. Thompson (1994a) and Moore (2013) provided insights into the complex nature of quantities and quantitative reasoning, making it evident that teachers and researchers should not assume concepts like speed or angle measure are available to students in meaningful or productive ways, but rather need to be developed by the students through their experiences.

In addition to giving attention to students' construction of quantities, researchers have indicated the importance of students' mental images of a situation in their solution processes (Carlson et al., 2002; Ellis, 2007; Moore & Carlson, 2012; Rasmussen, 2001; Smith III & Thompson, 2008). For example, Moore and Carlson (2012) explored college students' problem solving abilities when working on novel problems. The authors indicated students' solutions (both correct and incorrect) stemmed from the students' mental images of the scenario. With respect to incorrect solutions, the students' mental images of a situation led them to create formulas that were *correct* with respect to their images of the situation; it was not until they re-constructed their images of the situation that the students re-evaluated their formulas and made

modifications to reflect modifications to their images of the situation. The authors also posited that students who maintained static mental images struggled to refine their understanding of a dynamic situation whereas students who imagined a situation in terms of dynamic quantities had opportunities to refine their understandings of emergent quantitative relationships.

Researchers have identified that another benefit of quantitative and covariational reasoning is its potential to make complex topics (e.g., certain function families) accessible to students before more formal definitions and notations are introduced (Ellis et al., 2012; Johnson, 2012b). For example, Ellis et al. (2012) explored three 8th-grade students reasoning as they modeled the height of tree that was growing at an exponential rate on a computer simulation. The authors indicated shifts in students' thinking; students started with an understanding of growth as repeated multiplication, then moved to coordinating the relationship between time and height of the tree. Without having learned the typical vocabulary and notation of exponential functions, the students developed their own notations in order to solve complex problems, with one student exhibiting an ability to use fractional exponents by the end of the teaching experiment. The authors noted that generally, students were capable of reasoning covariationally before constructing correspondence rules (e.g. $y = Ab^x$), indicating that a covariational approach to functional relationships may be available to students before a formal analytic approach.

Researchers have indicated students' quantitative and covariational reasoning may support them in transferring[15] their understandings from one context to another (Carlson et al., 2001; Ellis, 2007; Lobato et al., 2012). For example, Lobato et al. (2012) investigated students' abilities to transfer their linear function understandings to a novel context in two 7th-grade

[15] While researchers' definition of transfer varies, here I interpret transfer to refer to students' abilities to see a similar mathematical structure in apparently different situations (e.g. a constant rate of change in speed and gear situations).

classrooms. The researchers observed two classes over the course of a unit on linear function to explore the relationship between the aspects of linear functions and slope the teachers emphasized in their classroom conversations and students' abilities to interpret a novel problem context. The classes differed in their focus. The Class 1 teacher promoted a great deal of quantitative dialogue in her classroom, discussing the relationship between distance, time, and speed before introducing the concept of slope and its connection to these quantities. The Class 2 teacher emphasized a rise over run notion of slope by relating slope to the steepness of stairs. After the unit on linear functions, the researchers asked students from both classes to interpret the graph of a linear function meant to represent a hose filling a pool, a context neither class discussed. The researchers noticed clear differences in the ways the students in each class reasoned. All eight students from Class 1 noticed and acknowledged the quantities represented on the axes and discussed the meaning of the values of the points explicitly marked on the line. Additionally, five of the eight students from Class 1 explicitly reasoned quantitatively about the relationship between volume and time represented by the slope of the line. Conversely, students from Class 2 generally ignored the quantities on each axes and interpreted the graph in terms of the change in unit squares or boxes. Although the students were reasoning about quantities present (e.g., squares or boxes), these students were unable to interpret the slope of the line in terms of the quantities the researchers intended. The authors argued that students in both classes exhibited transfer and posited that students in a class focused on the relationships between quantities in contextualized situations are likely to have more success transferring their understandings to novel situations when compared to students in a class focused on physical descriptions of graphs.

Another emergent theme in the quantitative and covariational reasoning literature is that when students leverage continuous covariational reasoning they can achieve success in many different contexts (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Johnson, 2012b; Moore & Carlson, 2012; Moore et al., 2013a; Saldanha & Thompson, 1998; P. W. Thompson, 1994b, 2011). For example, Johnson (2012b) presented the work of one high school student, Hannah, who repeatedly engaged in quantitative and covariational reasoning when making sense of and modeling dynamic problem situations. When engaging in tasks, Hannah described the ways she interpreted the quantities and their variation including changes in magnitude and changes in the rates of change of the magnitudes. Hannah leveraged chunky and smooth images of change, using a combination of both to sketch a picture of a bottle from a given graph. Hannah reasoned about situations happening smoothly and she incorporated chunky thinking to produce numerical values either when she felt the situation suited it or to help her make a situation more concrete.

In addition to these general themes that emerged, researches have shown how quantitative and covariational reasoning can be used effectively in the teaching and learning of mathematics at many grade and age levels. Researchers have indicated young children are capable of reasoning covariationally (Piaget, Grize, Szeminska, & Bang, 1977; P. W. Thompson, 1994a). Piaget et al. (1977) described numerous instances of young children engaging with complex problem situations with the children reasoning both quantitatively and covariationally as they established relationships between quantities. For instance, students were presented with a physical system of springs, strings, and weights they were able to manipulate (*Figure 10*). The machinery involved a spring (x) attached to a string (y), which was wrapped around a pivot point and extended down (y') to a weight (z). Included in this system were two arrows, *Fa* and *Fb*, whose displacements were indicated on scales. Students had the opportunity to manipulate the

scale by adding weights and observing the situation, including any changes to the various quantities. They were then asked to make predictions about various relationships. When asked to discuss the relationship between y and y', 70% of children aged 7 and 8 recognized that $y + y'$ was constant. 96% of children aged 9 to 12 established this relationship. Further, there were indications of early forms of covariational reasoning in students' responses. For example, when discussing what happens as weight (z) is increased, one student stated, "The weight will stretch the spring, which will make the string here (y) smaller" (p. 61), exhibiting L2 reasoning in this situation using Carlson et al.'s (2002) framework.

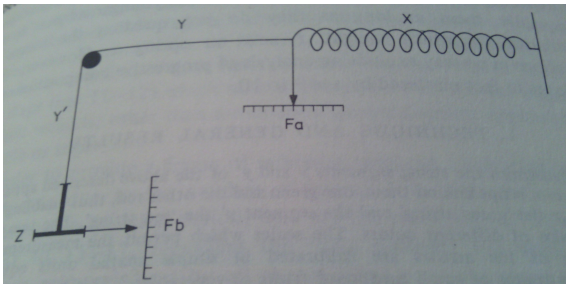


Figure 10: A representation of the spring system used by Piaget et al. (1977, p. 49).

Moving out of elementary school, researchers have indicated quantitative and covariational reasoning can play a foundational role to students' development of productive meanings for middle and high school mathematics topics including rate of change and linear functions (Ellis, 2007; Ellis & Grinstead, 2008; Johnson, 2012a, 2012b; P. W. Thompson, 1994a), quadratic functions (Ellis, 2011) exponential functions (Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis et al., 2012), angle measure (Moore, 2013), the unit circle (Moore, LaForest, & Kim, 2012), and trigonometric functions (Moore, 2014a). As one example, in a study of 8th-grade students, Ellis (2011) described the activity of six students as they worked on a task involving a dynamic sketch of a rectangle in which the length and width both increased at a constant predetermined ratio. Students constructed correspondence rules for the situations (e.g. $y = .5x^2$ where y is the

area, x is the height, and the width is always half as long as the height), and moreover they made meaningful connections between these rules and their graphical representations. For example, students predicted which of two quadratic equations had a 'narrower' graph based on the relationship between the height and area of the rectangles certain equations and graphs were meant to represent. These results support that quantitative and covariational reasoning can support students in constructing the *something* represented by graphs and equations, in this case the *something* being the relationship between height and area of a rectangle.

Moving to the collegiate level, researchers have shown that students can leverage quantitative and covariational reasoning in developing understanding of calculus concepts and differential equations (Carlson et al., 2002; Carlson et al., 2001; Rasmussen, 2001; P. W. Thompson, 1994b). For example, in the article in which Carlson and colleagues proposed the covariational reasoning framework (Carlson et al., 2002), the authors discussed their analysis of students' activity using the proposed framework. The researchers conducted interviews with six successful calculus students as they engaged in solving problems in dynamic situations. The authors noted that when students acted out problem contexts kinesthetically (e.g. using a hand or pencil against a book to imagine a ladder sliding against a wall), they were better suited to reason about changing magnitudes and instantaneous rates of change, exhibiting higher mental actions (MA3-MA5) more frequently than they exhibited on problems they did not act out. Overall, the authors claimed these students were capable of engaging L3 reasoning regularly, though most struggled to engage in L4 reasoning consistently, and had extreme difficulty engaging in L5 reasoning. The authors indicated that even high performing calculus students may struggle to reason covariationally, though it seems that many of the described mental actions should be available to them after completing their second calculus course.

In another article examining ideas of calculus, P. W. Thompson (1994b) explored students' development of the Fundamental Theorem of Calculus. Thompson presented a conceptual analysis of the ways students may come to understand a form of covariation in which they imagine a relationship where the distance traveled relative to an amount of elapsed time is imagined as a coordination of accruals of distance and accruals of time, which underlies the Fundamental Theorem of Calculus. His data indicated students struggled with this imagery and he hypothesized this was in part due to inadequacies in their function understandings, their rate of change schemes, or their inabilities to coordinate the necessary quantities. This article provided one possible explanation of how students could develop meanings for the Fundamental Theorem of Calculus as well as other calculus concepts from a quantitative and covariational perspective while addressing how complex and critical quantitative and covariational reasoning can be to developing calculus concepts.

Bidirectional Reasoning in the Literature

In addition to the few descriptions and examples of students' bidirectional reasoning (Saldanha & Thompson, 1998; A. G. Thompson & Thompson, 1996), I have identified other examples in the literature concerning students' quantitative and covariational reasoning and students' function understandings that I take to suggest students reasoning bidirectionally. Hackenberg (2010) discussed a form of bidirectional reasoning in her study concerning 6th-grade students reasoning with reverse multiplicative relationships. Bidirectional reasoning is apparent in her definition of reverse reciprocal relationships:

> If a student also conceives that the known height is 2/3 the size of the unknown height *and* that the unknown is 3/2 the size of the known, I would say that the student has constructed *reciprocal relationships* between the two quantities. Understanding these relationships means a student knows that if quantity A is 2/3 times the size of quantity B, then B must be 3/2 times the size of A. (p. 396)

Hackenberg described the activity of one student, Deborah, who exhibited the ability to reason with reverse reciprocal relationships by imagining either quantity A or B as the unit by which the other quantity could be measured. This ability should not be taken lightly as research indicates college students can struggle with varying the unit of measure and coordinating corresponding changes in a quantity's measure (Moore et al., 2012; P. W. Thompson, 2011; P. W. Thompson, Carlson, Byerley, & Hatfield, 2014). Hackenberg (2010) argued that Deborah coordinated two perspectives at once, allowing her to transform between two equivalent states. It is my contention that in her activity, Deborah was reasoning bidirectionally about the relationship between two quantities' magnitudes relative to one another. Although this activity was observed in only one student during the course of an 8-month teaching experiment, it indicates that middle school students are capable of reasoning in a bidirectional nature.

In another study of middle-school students, Dandola-Depaolo (2011) examined 7th-grade students' function and inverse function understandings by observing the students engaged in creating equations to match data presented in t-tables where the labels for the columns of the t-tables were represented by a square and a triangle. Compatible with covariational reasoning described by Confrey and Smith (1995), the students in Dandola-Depaolo's study reasoned covariationally about relationships represented on t-charts to create analytic function rules. Further, the students created corresponding inverse rules. Unsurprisingly, before the students were given instruction on how to determine the inverse function of an analytically defined inverse, the students did not switch the variables; if students initially devised a rule of the students wrote the inverse rule as $\square=\triangle-2$. Although Dandola-Depaolo did not take a quantitative perspective, I take the students' activities to indicate that the students' goal might have been to

maintain the relationship between box and triangle as they created the rule defining the inverse relationship.

In addition to these examples, there are instances in which researchers indicated students switched the order of the quantities they were reasoning with as they engaged in modeling dynamic situations (Carlson et al., 2002; Carlson, Larsen, & Lesh, 2003; Johnson, 2012a). For example, when students worked on the Bottle Problem, Carlson et al. (2002) stated, "Students have been observed switching the roles of the independent (volume) and dependent (height) variable several times in the context of discussing the thinking that they used to construct the graph for this task" (p. 361). Similarly, describing the work of a student, Mason, Johnson (2012a) noted, "In the first section … Mason considered the volume to be increasing faster than the height. In the second section… Mason considered the height to be increasing faster than the volume" (p. 48). Whereas in both cases the researchers had intended particular independent and dependent quantities, the students switched between the quantities as they saw fit. These results indicate that students may reason bidirectionally naturally as they attempt to make sense of dynamic situations.

Summary of Research Examining Students' Function and Inverse Function Meanings

Although I am mainly concerned with students' quantitative, covariational, and bidirectional reasoning, I am also interested in how this reasoning may influence students' function and inverse function meanings. Hence, in this section I briefly synthesize the body of research on students' function and inverse function meanings. I first present results from researchers' examinations of students' function and inverse function meanings.[16] I then briefly summarize

[16] I note that researchers studying students' function meanings generally take the concept of function as a given, which I find problematic. However, my goal in this section is only to summarize the findings of these researchers.

some frameworks researchers have created or used when exploring and describing these meanings. I describe how each approach has moved the study of students' function and inverse function meanings forward and provide insights into how my approach to examining students' function and inverse function meanings is informed by and differs from the frameworks described.

Overview of results from examinations of students' function and inverse function meanings.

Although researchers have used a variety of frameworks to analyze students' activities in situations the researchers conceive of as involving function, there have been common findings across this research. Collectively, researchers have indicated that students hold disconnected function meanings (from the researchers' perspectives), both among the different representations of function and between their concept definitions and concept images (Akkoc & Tall, 2005; Leinhardt et al., 1990; Tall, 1988; Vinner, 1983; Vinner & Dreyfus, 1989). Some students exhibit strong preferences for explicit analytic rules when working with functions (Breidenback et al., 1992; Carlson, 1998; Even, 1990), which is especially problematic in certain situations: when functions are discontinuous or defined by more than one analytic formula (e.g. piecewise) (Carlson, 1998; Dubinsky & Harel, 1992; Vinner & Dreyfus, 1989), when functions are constant (Bayazit, 2011; Vinner & Dreyfus, 1989), and when functions are not defined by an analytic formula. Students are also often inclined to work with known or familiar functions (e.g. linear) regardless of the problem or situation (Ellis & Grinstead, 2008; Leinhardt et al., 1990; Schwarz & Hershkowitz, 1999; Zaslavsky, 1997).

When presented with graphical representations of functions (from the researchers' perspective), some students have a tendency to treat these as pictorial objects, treating a graph as

a visual object rather than representing the covariation of quantities (i.e. static shape thinking versus emergent shape thinking) (Moore & Thompson, 2015; P. W. Thompson, 1994b). I conjecture that shape thinking (Moore & Thompson, 2015, in preparation; Weber, 2012) may provide a way to explain researchers' findings that students often rely on the vertical line test when deciding if a graph represents a function (Akkoc & Tall, 2005; Breidenback et al., 1992; Dede & Soybas, 2011; Lucus, 2005; Montiel et al., 2008; Wilson, 1994). Additionally, mathematical conventions in graphical representations concerning the location of input and output or x and y (e.g., input or x is always represented on the horizontal axis) can play major roles in students' interpretation of graphs as representing functions (Breidenback et al., 1992; Dede & Soybas, 2011; Moore, 2014b; Moore et al., submitted).

Researchers exploring students' inverse function meanings have indicated that students and teachers often hold compartmentalized inverse function meanings, typically related to executing specific actions in analytic or graphing situations (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Paoletti et al., 2015). Researchers have noted students have ways to determine representations of inverse functions (e.g., determining an inverse function analytically) without connecting these techniques to function composition (Kimani & Masingila, 2006; Paoletti et al., 2015; Vidakovic, 1997). Moreover, researchers reported that when they presented students with a function that was meant to represent a context (e.g., the relationship between temperature measured in degrees Celsius and degrees Fahrenheit), pre-service teachers struggled to interpret the contextual meaning of the inverse function they constructed (e.g., interpreting the inverse function as representing the same relationship between temperature measured in degrees Celsius and degrees Fahrenheit) (Moore et al., 2014; Paoletti et al., 2015).

Researchers (Akkoc & Tall, 2005; Brown & Reynolds, 2007; Kimani & Masingila, 2006; Leinhardt et al., 1990; Paoletti et al., 2015; Tall, 1988; Vinner, 1983; Vinner & Dreyfus, 1989) have indicated that when tasked with both function and inverse function problems, students' activities are often concerned with certain procedural rules (e.g., using the vertical line test to decide if a graph represents a function, reflecting over the line $y = x$ to graph the inverse of a function, etc.) that lack connections the researchers deem as foundational (e.g., understanding the definition of inverse function via function composition). I conjecture such activity is indicative of the students having made pseudo-empirical abstractions in regards to function and inverse functions; their meanings in these situations are based on the product of their actions that proved effective for them in getting 'correct' answers in similar situations. As Moore (2014b) noted, "With regards to those situations when students' meanings seem to stem from the product of actions (e.g. pseudo-empirical abstractions), conceiving invariance across multiple representations in a coherent manner proves to be a difficult task" (p. 35). As an example of this, in my pilot study, a student, Ryan, reflected over the line $y = x$ when graphing the inverse of a function but used a reciprocal when determining the inverse of an analytic function, without observing an inconsistency between these two procedures. I take Ryan's activity to indicate he maintained meanings to determine inverse functions in various situations stemming from pseudo-empirical abstractions. Such meanings are in contrast to meanings students developed through their engaging in and reflecting on activities in a variety of contexts and representations (e.g., reflected or reflective abstractions). Without meanings developed through reflected or reflective abstractions, the terms 'function' and 'inverse' may only serve as pointers, directing the students to use the action they have internalized for that term in that representation (i.e. graph, equation, or table). Collectively, the results from the research on students' function and

inverse function meanings indicate there is a pressing need to explore approaches to the teaching and learning of these topics that support students' in constructing the foundational meanings for function and inverse function.

Frameworks for studying students' function and inverse function meanings.

In the sections that follow, I describe frameworks researchers have developed or used to obtain the findings regarding students' function and inverse function meanings described above. As my main interest in this study is students' bidirectional reasoning, and its relation to function and inverse function, my goal in these sections is to provide insights into these frameworks and explain how my approach builds on and varies from these approaches.

Concept image versus concept definition.

Some researchers exploring students' function meanings distinguished between students' concept definitions and concept images (Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989). Tall and Vinner (1981) defined a student's concept definition as, "A form of words used to specify that concept" (p. 152), and a student's concept image as, "The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). The "mental pictures" in a student's concept image include but are not limited to pictures, symbols, diagrams, and graphs (Tall, 1988; Vinner, 1983; Vinner & Dreyfus, 1989). Researchers using this framework indicated that occasionally the imagery a student used in one function situation was inconsistent or incongruent with imagery in another situation (Vinner, 1983; Vinner & Dreyfus, 1989). However, these inconsistencies were not problematic for the student, sometimes even if he or she was made explicitly aware of them (from the researchers' perspective).

Distinguishing between students' concept definition and concept image of function (Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989) proved foundational for researchers' study of students' function concepts. Although researchers often have students provide a definition of function, this framework supported researchers as they examined the students' concept images that the students relied on as they solved various problems. By distinguishing between students' definitions for a concept and the images they associated with that concept, researchers adopting the concept image construct have provided important insights into the distinctions that researchers' observe between these two constructs in students' activities. In this study, I am interested in the students' function and inverse function concept images (including how they relate these images to their concept definitions) and how their concept images develop throughout the study.

Vinner (1983) and Tall (1988) have indicated students rely on their concept images when working on problems not explicitly asking for the definition of function, positing generally students' concepts only require images and not formal definitions. However, researchers' distinguishing between students' function concept images and definitions do not provide detailed accounts of how students' develop specific concept images. Further, by taking the concept of function, including the modern correspondence definition of function, as a given, I conjecture researchers miss an opportunity to explore (and possibly support) students' understandings of function via their experiences modeling relationships between quantities. Hence, in my study, I intend to explore students' images (including but not limited to their function images) via their broader experiences representing relationships between quantities.

The APOS framework.

Introduced by Dubinsky and Harel (1992) and Breidenback et al. (1992), the APOS framework is concerned with the ways in which students conceive of functions as *actions*, *processes*, or *objects* and the students organization of these conceptions into *schemes* (APOS). For example, students who held an action view of function understood functions as static repeatable manipulations, physical or mental, which transform an input to an output. When a student was able to imagine a function acting on all elements in the domain producing the co-domain, the authors stated the function became a process to that student (Breidenback et al., 1992; Dubinsky & Harel, 1992). Students with a process conception of function imagined a dynamic transformation of quantities using a repeatable means whereby the function produced the same transformed output from an identical input. The researchers specified students who developed a process conception of function could come to see this process as something that could be transformed by some other action, at which point the function process had been encapsulated into an object for that individual (Breidenback et al., 1992; Dubinsky & Harel, 1992).

Researchers' have found this framework useful in describing students' conceptions of function ranging from seventh grade students relating tables of values and analytically defined function rules (Dandola-Depaolo, 2011) to second semester calculus students' understanding of parametric functions (Stalvey & Vidakovic, 2015). However, I find there to be some issues with the APOS framework, as well as researchers' use of the framework. One critique I have, related to the foundation of APOS being derived from students working in a computer programming course, is that the APOS framework emphasizes students' understanding of functions in terms of correspondence or rule-based relationships (e.g., functions as processes that changes the input to

an output, or functions as actions that inputs individual values into an equation). I contend this emphasis on inputs and outputs makes a great deal of sense in the computer programming context in which the APOS framework was developed, as well as in other important mathematical concepts (e.g., a formal treatment of transformations in the plane). However, this approach does not capture how students might conceive of relationships between covarying quantities without a natural dependency or input quantity as representing a function. My approach in this study does not emphasize inputs and outputs, and instead focuses on students' construction of relationships between quantities, and exploring how students can conceive of such relationships as being (or not being) functional. As such, I do not use the APOS framework for my examination of students' re-construction of their function meanings via their quantitative and covariational reasoning.

Multiple representations.

Researchers examining students' understandings of multiple representations of functions are typically concerned with students' activities with and understandings of tables, equations, and graphs, with an added focus on the connections students make between these three representations (Leinhardt et al., 1990; Lobato & Bowers, 2000).[17] This body of research has provided important insights into students' activities working with functions both within and across representations (e.g., exploring if students engage in activity that the researcher deems as compatible across graphical, analytic, and table representations).

For instance, Brenner et al. (1997) compared two pre-algebra classes, one class emphasized algebraic manipulations while the other focused on representing functions using graphs, tables, equations, and words. The researchers found that students in the second class were more likely to

[17] I direct the reader to Leinhardt et al. (1990) for an extensive review of the literature on students' understanding of function and multiple-representations.

use multiple representations when solving word problems and were more successful in obtaining correct solutions to these novel problems. As another example, Keller and Hirsch (1998) compared students' activities in a pre and post test from two calculus courses; in one course students used graphing calculators and in the other course they did not. The researchers indicated that students in both courses preferred to use tables and graphs in contextualized situations, while relying heavily on equations in non-contextualized situations. Further, they noted that students who had the opportunity to work with graphing calculators were more likely to use graphing solutions in both contextualized and decontextualized situations when compared to students in the course that did not use graphing calculators. Collectively, this research indicated that students are more successful when they leveraged multiple representations in their solution processes, while also describing the complexities that arise as students use multiple representations (e.g., their preference for certain representations in certain situations).

A critique of the multiple representation approach is that it assumes that students are 'seeing' something called function in the multiple representations. Referencing Thompson's (1994c) work, Lobato and Bowers (2000) argued, "The first concern is whether tables, graphs, and equations are multiple representations of *anything* to students. That is, students may learn to move among the representations, but not understand what is being represented" (emphasis in original, p. 4). Lobato and Bowers (2000) argued that while students develop activities or procedures to discuss and connect tables, equations, and graphs, the students may not be doing so in a way that allows them to abstract function as a coherent idea across these representations. One of my goals in this study was to explore the possibility of students' constructing relationships between quantities as the *something* that the students could represent via tables, graphs, and equations.

Property oriented approach.

Some researchers have focused on students' development of function understandings through specific properties. For instance, Slavit (1997) explored students developing function understandings from the specific to the more general through a process of abstracting various properties of functions. He conjectured this abstraction occurs as students explore specific function classes (e.g. linear, quadratic, and trigonometric) and their properties (e.g. x and y intercepts, rates of change, asymptotes, periodicity, etc.). Slavit (1997) posited that by exploring and considering various function types and their properties, students' property-oriented conceptions of specific types of functions allow them to develop a more general view of function. This general view continues to develop through exposure to novel functions and their properties. Slavit (1997) argued it would be beneficial for students to explore a wide variety of functions in their early experiences, including functions with discontinuities, finite domains, multivariable functions, as well as non-functional relations to prevent students from developing limited meanings from their early linear and quadratic function experiences.

The property-oriented approach is a viable way of explaining the way students develop their function understandings through their school experiences. Additionally, a researcher can use this framework as a way to analyze students' meanings as well as to provide insights into how to support students in developing more powerful function meanings. By examining the function properties or aspects that students construct via their experiences with a variety of function classes researchers can examine what generalizations students make across these classes. In this study, I leverage an approach similar to the property-oriented approach (Slavit, 1997), but I do not treat function as the paramount concept. I conjecture all of the above experiences can become part of students' relationship understandings and I am interested in examining how students re-

construct their function meanings via their experiences constructing relationships between quantities that represent functions and non-functions.

A genetic decomposition of inverse function.

Whereas there have been many frameworks used for exploring students' function meanings, there has been fewer systematic efforts to explore students' *inverse* function meanings. Vidakovic (1996) presented a genetic decomposition for inverse function (i.e. a description of how students might learn a concept, including possible methods for constructing their schemes). She proposed that students develop inverse function schemas in the following order: function, composition of functions, then inverse function. She conjectured students could develop inverse function meanings through their coordination of their function and function composition schema.

Whether implicitly or explicitly, other researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1997) examining students' inverse function meanings have maintained Vidakovic's (1996, 1997) emphasis on function composition. However, researchers have not yet shown that students develop inverse function meanings that entail reasoning about function composition. For example, Vidakovic (1997) noted some students "were able to define (in terms of switching x and y) and find (by switching x and y, or by trial and error) the inverse function without understanding the concept of the composition of two functions" (p. 191). Although students in Vidakovic (1997) were able to determine an analytic representation of the inverse function, they did relate this to the notion of function composition, a finding that has been echoed by other researchers (Brown & Reynolds, 2007; Kimani & Masingila, 2006).

By exploring the students' activities as they engaged in inverse function tasks, the researchers have provided me with important insights into what to expect in regards to students' inverse function meaning as they enter my study. However, my approach to students developing

inverse function meanings takes a markedly different approach. Rather than focusing on students' understanding of the formal mathematical definition of inverse function (i.e. inverse elements of a set under the operation composition of functions), I conjecture students can develop intuitive notions of inverse functions via reasoning about invariant relationships between quantities (i.e. the *something* students can reason about when representing functions).

History and Critique of the Function Definition in School Mathematics

Almost a century ago Thorndike et al. (1926) stated:

> For most high-school students any larger use of the function concept than that suggested in the mathematical graph is of doubtful value. It is one of those fundamentally powerful conceptions whose elaboration has been one of the half dozen significant achievements of the race, but to the high school student it is vague and tantalizing and stimulating rather than clarifying. (p. 82)

Thorndike and his colleagues' assertion that high school students find the function concept vague rather than clarifying has been echoed by the aforementioned body of research on students' function understandings. This has led many researchers to argue that students should have many opportunities to construct and work with both functional and non-functional mathematical situations before being introduced to the formal definition, procedures, and notation that they are expected to work with and understand (Carlson & Oehrtman, 2005; Dienes, 1964; Even, 1990; Even & Bruckheimer, 1998; Kline, 1974; Leinhardt et al., 1990; Oehrtman, 2008; P. W. Thompson, 1994c; Vinner, 1983). For example, Dienes (1964) indicated introducing formal notation and definitions has a deleterious effect for students, stating the "likely reason for the general ossification of mathematics in children's minds at an early stage is the rash use of symbols, i.e. *the introduction and manipulation of symbolism before adequate experience has been enjoyed of that which is symbolized*" (emphasis in original, p. 139).

Corroborating the statement by Thorndike et al. (1926), researchers examining students' function understandings have collectively indicated that students at all levels struggle to develop

understandings the researchers deem desirable. I conjecture these deficiencies may be directly associated with the function definition emphasized in school mathematics. School mathematics highlights a modern correspondence definition of function: a static view where each value of the domain is associated with a unique value in the codomain (Confrey & Smith, 1995; Ellis et al., 2012; P. W. Thompson, 1994c; Vinner, 1983). However, neither mathematicians nor mathematics educators have always agreed upon this definition. Those arguing for a more historical definition asserted that the formal definition ignores children's intuitions, which are rich with experiences of concrete, dynamic, and continuously changing quantities in their real life experiences (Leinhardt et al., 1990). Further, those arguing for a historical definition contended that great mathematicians of the past did not consider arbitrariness or univalence as critical to their function definitions (Even, 1990; Kline, 1970, 1974; Leinhardt et al., 1990; P. W. Thompson, 2011).

Even (1990) provided a description of the gradual development of the function definition to include univalence and arbitrariness.[18] She contended that 18th century mathematicians did not accept arbitrarily defined functions; it was not until Dirichlet introduced the now famous Dirichlet Function in the 19th century that arbitrariness gradually became accepted as integral to the function definition. Even (1990) indicated the univalence property of functions was a byproduct of mathematicians' attempts to keep certain functions (e.g. the square-root function) manageable. As univalence and arbitrariness were added to the function definition, intuitive concepts like motion, time, and infinitesimals were phased out of the concept in order to accommodate functions that were continuous everywhere and differentiable nowhere, or

[18] Univalence is the property that for each element in the domain there is a unique element in the range and arbitrariness refers to the fact that functions need not be defined by a 'nice' or known function rule.

continuous on the irrational numbers but discontinuous on the rational numbers (P. W. Thompson, 2011).

Mathematicians' modification of their function definition was paralleled in mathematics curricula. Kline (1974) maintained that curricula before the new mathematics movement referred to a function such as as a single-valued function and a function such as as a multiple-valued function.[19] Kline (1974) stated, "Such 'sloppiness' is not permitted in the modern texts" (p. 72), and provided a description of the 'new math' distinction between functions and (non-functional) relations, a distinction still presented to students today. He argued the term multi-valued function had the potential to be intuitive and meaningful for students whereas the term relation was ambiguous. Further, he argued that the ordered pair definition of function had the potential to create a paradox for students as this definition gives the impression that the number of pairs in a set is finite though it is impossible to enumerate all ordered pairs represented in a function such as $y = 3x$.

Kline (1974) continued his critique of the modern function definition, and 'new math' more generally, by arguing it stripped mathematics from its physical origins. He contended that many mathematical structures and concepts arose from physical situations and phenomena, and that stripping these physical relationships from mathematics is equivalent to keeping the rind while discarding the fruit. Specifically in regards to function, he argued:

> Functions are not sets of ordered pairs but relationships between real variables such as the height and time of flight of a ball thrown up into the air, the distance of a planet from the sun at various times of the year, and the population of a country over some period of years. Functions are laws of the universe and of society. (p. 115)

Kline contended that students could develop their own definition for function via their experiences with concrete examples, an argument similar to Slavit's (1997) property-oriented

[19] Note in both cases we assume that y is a function of x.

approach. Kline (1974) argued this approach, which emphasizes students constructing relationships between covarying quantities, would allow students' function meanings to evolve similar to the way in which the function definition had for mathematicians between the years 1700 and 1900.

Students' and teachers' lack of understanding of the need for the univalence and arbitrariness properties of function (Breidenback et al., 1992; Dandola-Depaolo, 2011; Dede & Soybas, 2011; Dubinsky & Harel, 1992; Even, 1990; Even & Bruckheimer, 1998; Leinhardt et al., 1990; Lucus, 2005; Vinner, 1983) corroborates Kline's critique of the utility of the modern definition of function in school mathematics. Addressing the property of univalence in her study of pre-service secondary teachers' subject matter knowledge, Even (1990) stated:

> Some serious questions are raised by the fact that, without prompting, none of the subjects could come up with a reasonable explanation for the need for the property of univalence. This requirement is, usually, presented to our students as one of the most important characteristics of functions and this is what many of them think. They know that it distinguishes between relations that are not functions and those which are. But, in many cases, they are not told why it is important to distinguished between these two groups. Many mathematics teachers do not explain what is it that you can do with functions that you cannot do with relations which are not functions. This approach may contribute to making mathematics look like an arbitrary collection of rules and definitions. (p. 531)

Similar to the rash use of symbols as described by Dienes (1964), Even argued the modern definition of function may make mathematics seem arbitrary or disconnected from students' experiences. Future and current mathematics teachers' general unawareness of the usefulness of key properties of the modern function definition raises questions about the usefulness of this definition in school mathematics. Poincaré raised the question of what makes a definition useful in the teaching and learning of mathematics:

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils. (Poincaré, 1908 qtd. in Tall 1988 p. 37)

Poincaré's statement gives credence to those who have criticized the modern definition of function on pedagogical grounds (Even, 1990; Even & Bruckheimer, 1998; Kline, 1970, 1974; P. W. Thompson, 1994c; Wilder, 1967). Using Poincaré's categorization of a good definition for the purposes of education, and the available body of research on students' and teachers' (lack of) understanding of this definition, we have to conclude that the modern function definition does not fall into the category of a 'good definition' for educational purposes as students and teachers are capable of repeating the definition of function without understanding why critical aspects of this definition are important.

Echoing those who have criticized the modern function definition on pedagogical grounds, Carlson (1998) argued the current approach to function requires students to make many connections and develop understandings without having ample opportunities to experience situations in which these connections and understanding are meaningful; without these experiences it is unreasonable to expect students to develop understandings of the modern function definition. As Moore (2012) states, a "definition is no more than sequence of words and symbols. What is most important is the cognitive structure that is evoked by a definition, and this cognitive structure is the product of years of experience" (p. 77).

Consistent with other researchers (Carlson, 1998; Carlson et al., 2002; Confrey & Smith, 1995; Kline, 1974; Oehrtman et al., 2008; Smith III & Thompson, 2008; P. W. Thompson, 1994c, 2008, 2011, 2013), I am not arguing against teaching function in secondary schools. Instead, my proposed approach (described in Chapter 1) is focused on students' reasoning about quantities and relationships between quantities to support them in developing function meanings grounded in coming to understand a function as a special type of relationship between covarying quantities. By foregrounding quantitative and covariational reasoning, the 'function-ness' of a

relationship emerges as a characteristic of the relationship based on how the student chooses to consider the quantities and certain properties of the relationship. Whereas this approach, which focuses on students' quantitative and covariational reasoning, does not start with the modern definition, I suspect it provides the foundations for the formal function definition.

Chapter 3: Subjects, Settings, Methodology, and Methods

In this chapter I address my methodology and methods to investigate students' reasoning. Recall, I organize my study around the research questions:

a. What ways of reasoning do students engage in during activities intended to emphasize reasoning about relationships quantitatively, covariationally, and bidirectionally?
b. How do students ways of reasoning identified in addressing question (a) interplay with their function and inverse function meanings?
c. How do common school conventions interplay with understandings developed and described in (a) and (b)?

In this chapter, I first describe the subjects and setting for my proposed study. I then provide an explanation of the proposed methodology and methods of data collection, including a description of the underlying theoretical principles that provided a basis for these methods. Finally, I describe the data analysis techniques I used in this study.

Setting and Subjects

I selected two students, Arya and Katlyn (pseudonyms), from the secondary mathematics education teacher preparation program at a large state institution in the southeast United States. The program consists of three consecutive semesters in which the students take paired content and pedagogy courses focusing on the teaching and learning of middle and high school mathematics. Arya and Katlyn were enrolled in the first pair of courses with the content course serving as the setting for the teaching experiment.

The content course met twice a week for seventy-five minutes per class. The course materials were designed to explore various secondary mathematics topics from a quantitative and covariational reasoning perspective. The development and use of a subset of the course materials are reported on elsewhere (e.g., angle measure (Moore, 2013), the unit circle (Moore et al., 2012), trigonometry (Moore, 2014b), the polar coordinate system (Moore et al., 2013a; Moore, Paoletti, & Musgrave, 2013b)). Although these materials had been used in this course previously,

the course was not designed to *require* specific tasks, giving the research team and I flexibility to deviate from the typical course sequence as required for the teaching experiment. The members of the researcher team (myself, Dr. Kevin C. Moore, Kevin L. LaForest, Irma E. Stevens, and Natalie F. L. Hobson) attended almost every class session, although certain members had to miss some sessions due to various scheduling conflicts.

Class time was spent with students working on a task in groups or pairs. Some time was spent each class having the groups or pairs report their activity and solutions to the whole class. The instructor (Dr. Moore) and teaching assistants (Irma Stevens and myself) facilitated these whole class discussions. The time working in groups or pairs provided an environment that was conducive to conducting the teaching experiment episodes (I worked with Arya and Katlyn with Kevin LaForest acting as an observer). Further, as the research team found necessary, we scheduled out of class episodes with Arya and Katlyn to further explore their thinking and reasoning. In all, the research team had great flexibility both in regards to content and to scheduling that allowed us to adapt the teaching experiment to fit our requirements.

I chose Arya and Katlyn on a volunteer basis from this course. Arya and Katlyn were juniors who had successfully completed a calculus sequence and at least two courses beyond this (e.g., linear algebra, differential equations). I chose Arya and Katlyn from a pool of four students based on their activities in the pre-interview and my observations of their activities from the initial class sessions. Specifically, Arya and Katlyn exhibited similar activities with respect to one another.[20] For instance, both students relied on switching techniques when finding inverse functions and both students insisted on using the horizontal axis as representing the input

[20] In addition to being similar to each other, their activities were similar to many students entering the course as described elsewhere (e.g., Moore et al. (submitted), Paoletti et al. (2015)).

quantity when interpreting a graph as a function (e.g., *Figure 1*b in Chapter 1 was Arya's work when responding to the hypothetical student).

Why pre-service teachers (PSTs)?

One reason for investigating PSTs' thinking was that they enter the EMAT 3700 course with ways of thinking developed over the course of their K-14 school experiences. Thus, working with PSTs provided me the opportunity to gain insights into the ways of thinking PSTs developed during their collective mathematics course work. Our previous experiences interviewing PSTs indicated that the PSTs often enter EMAT 3700 with unproductive and seemingly disconnected (from the researchers' perspective) meanings for a variety of topics including function and inverse function (see Moore et al. (submitted) and Paoletti et al. (2015) for more on this). The PSTs provided a unique opportunity to explore how engaging them in tasks meant to engender quantitative, covariational, and bidirectional reasoning interplayed with the ways of thinking that they brought into the course including how these prior meanings either supported or inhibited their quantitative, covariational, and bidirectional reasoning.

Another reason I chose to work with PSTs was to extend work that has provided insights into PSTs' quantitative and covariational reasoning and their learning of various concepts through this reasoning. For instance, previous researchers' have provided insights into PSTs' learning and understanding of angle measure, the unit circle, the polar coordinate system, and conventions common to school mathematics (Moore, 2013; Moore et al., 2012; Moore et al., 2013a, 2013b; Moore et al., 2014) through their reasoning quantitatively and covariationally. These researchers have indicated that PSTs are capable of engaging in these types of reasoning to develop productive meanings for a variety of topics. I intended to extend this body of research by

examining how to support PSTs' development of bidirectional reasoning as well as possible implications this reasoning can have for the PSTs' function and inverse function meanings.

Studying PSTs' ways of thinking is also important more generally because their content knowledge and pedagogical knowledge are important for their eventual instruction, and hence their students' learning. For instance, at the elementary school level, a teacher's mathematical knowledge for teaching has been correlated to high quality practice in their classroom (Hill et al., 2008) and to their students' achievement (Hill, Rowan, & Ball, 2005). Similarly, Lloyd and Wilson (1998) indicated that a teachers' strong function understandings supported his implementation of reform-oriented curriculum in productive ways (e.g., supported meaningful classroom dialogue with students) . Further, P. W. Thompson (2013) noted that students' meanings resemble their teachers' meanings, including the teachers' unproductive meanings that were not (explicitly) part of their instruction. Hence, Thompson conjectured that teachers should maintain strong meanings so that students might be afford better opportunities to construct productive meaning. By engaging PSTs in tasks that enhance their content knowledge, the PSTs have a chance to develop their content knowledge in ways that also support their pedagogical knowledge and their mathematical knowledge for teaching.

Clinical Interviews

I used clinical interviews as pre-interviews to explore the students' reasoning and to determine the meanings the students brought to the teaching experiment. I used clinical interviews as post-interviews to explore the students' reasoning and their meanings at the conclusion of the teaching experiment (the first a week after the last teaching episode and a second interview with each student three weeks after the last teaching episode). The pre- and post-interviews were semi-structured task-based clinical interviews (Clement, 2000; Goldin,

2000; Hunting, 1997). These interviews consisted of an individual subject working on preplanned tasks as she interacted with a researcher (Goldin, 2000). Clinical interviews give a researcher insight into students' meanings without intending to create shifts in their meanings. I used the information I inferred from the students' words and actions to create hypothetical models of the students' mathematics that viably explained the students' words and actions. These models were not considered isomorphic to the students' meanings; rather they were testable working hypotheses I continually adjusted as new observations were made so that the models remained viable explanations of the students' activities.

During clinical interviews, it was important for students to feel comfortable expressing their thinking in an environment that may feel unnatural to them. In order to make students feel comfortable, I emphasized that I was not looking for correct answers, but rather wanted the students to talk through their reasoning and actions (Hunting, 1997). I had students 'think aloud' as they worked through tasks, as this is one way researchers can procure information regarding the students' thinking (Clement, 2000; Hunting, 1997). I used minimal heuristics to probe student thinking as I did not intend to promote shifts in student thinking (Hunting, 1997). Although I did not intend to promote shifts in students' meanings in the clinical interviews, such shifts might have occurred as addressing tasks or my questions caused the students to reflect on their activity in such a way that raised perturbations for the students, leading to unintended shifts.

Clinical interviews in this study.

The research team and I designed the pre-interview tasks (Appendix A) to obtain information regarding the students' function and inverse function meanings, as well as their more general quantitative, covariational, and possibly bidirectional reasoning. We designed multiple questions in the interview that involved graphs where the input and output quantities were represented on

the vertical and horizontal axis respectively (e.g., Appendix A, Problem 13b, Problem 15c, etc.). Such tasks helped me explore if the students' reasoning was constrained by conventions common to school mathematics, as a student constrained by such conventions might be unable to make sense of a graph in which the input quantity is represented on the vertical axis. In contrast, a student who understands graphs as representing relationships between quantities might notice the unconventional nature of the axes while still being able to reason bidirectionally (e.g., consider either axes as representing the input quantity) to make sense of and reason about the relationship.

Additionally, we designed some of the tasks to explore students' function and inverse function meanings. For example, if a student was constrained to thinking of a function as a certain kind of pictorial object where a vertical line does not intersect the picture in two or more places at once (e.g., one meaning for the vertical line test), when I posed two graphs representing identical relationships (from my perspective) but with different axes orientations (e.g., Appendix A, Problem 9), the student might indicate that one graph represents a function whereas the other graph does not. Conversely, a student who understands function as a statement about a relationship with chosen input and output quantities might conceive that both graphs simultaneously represent a function and not a function. In addition to exploring students' function understandings, we designed some tasks to explore the students' inverse function meanings. For example, I posed numerous questions, both contextualized and decontextualized, which asked the students to graph, write, or interpret inverse functions. My analysis of the students' activities addressing these questions provided me baseline data regarding the students' understandings of these topics, which I compared to the students' emergent activity throughout the teaching experiment and in the post-interviews.

We designed the post-interview tasks (Appendix B) so the students worked similar problems as in the pre-interview with the goal of obtaining information regarding their function and inverse function meanings as well as their propensity to reason quantitatively, covariationally, and bidirectionally. By exploring these topics at the end of the teaching experiment, I had an opportunity to explore shifts in the students' thinking that had occurred through their activity and their reflection on their activity in the teaching experiment.

Teaching Experiment

The primary method I used in this study was a teaching experiment (Steffe & Thompson, 2000). Providing a way to apply the radical constructivist epistemology in research (von Glasersfeld, 1995), researchers have used teaching experiments to explore students' mathematical activity and build *models of students' mathematics* (Steffe & Thompson, 2000) in topics ranging from counting in elementary school (Steffe, 1988) to groups and isomorphism in abstract algebra (Larsen, 2009, 2013).

The teaching experiment emerged in mathematics education in the United States in the 1970's for several reasons (Steffe & Thompson, 2000). First, researchers in fields outside of mathematics education had begun to develop models of human understanding. While these models were not established for the purpose of educating students, they gave mathematics educators a starting point to develop models specific to mathematics education (Steffe & Thompson, 2000). Second, there was a large divide between the practices of research and teaching in mathematics education. Prior to 1970, mathematics education researchers typically used experiments with control and treatment groups that were largely unsuccessful (Steffe & Thompson, 2000), and did not translate to teaching practices. The teaching experiment provided

a natural way to connect teaching and research by building models to explain students' mathematical activities and their progress over time.

Teaching experiments serve as an exploratory tool, giving a researcher firsthand experiences with students' mathematics, allowing the researcher to explore the mathematical progress students make over an extended period of time (Steffe, 1991; Steffe & Thompson, 2000). It is important to again note that the word experience does not imply that the researcher has access to the students' realities, as these realities are independent of and unknowable to the researcher. The students' inherently unknowable body of understanding is referred to as the *students' mathematics* (Steffe & Thompson, 2000). The models of the students' mathematics that the researcher constructs using his interpretations of the students' words and actions are then referred to as the *mathematics of students* (Steffe & Thompson, 2000). The second-order models that a researcher constructs constitute epistemic students (Steffe, 2014). Steffe (2014) noted:

> I find it not possible to construct epistemic students without constructing living, experiential models of students' mathematics. Perhaps contrary to conventional wisdom in doing science in mathematics education, I have always found that building living, experiential models of students' mathematics through intensive and extensive interaction with them precedes the activity of constructing explanations. (p. 21)

Hence, in this study I engaged students in a variety of contexts and setting with the intention of creating living, experiential models of the two students ways and means of operating across tasks for the eventual purpose of constructing models of epistemic students.

Although a researcher begins a teaching experiment with a hypothesis he wishes to test (e.g., a student's bidirectional reasoning has the potential to support them in developing productive function and inverse function meanings), he also creates and tests many hypotheses during the teaching experiment to build models of students' mathematics, or the mathematics of students, with the students' activities possibly leading the researcher to abandoning his initial hypotheses to explore new conjectures created in the moment of interacting with the student (Steffe &

Thompson, 2000). One way a researcher builds these models is by examining students' 'essential mistakes', those that prevent students from making progress (Steffe & Thompson, 2000). As a researcher observes essential mistakes, he can use these mistakes to help make sense of and build models of the students' understandings. These mistakes are important for the researcher as they allow him to assess the viability of his models. If the researcher's current models do not explain a student's 'essential mistake', the researcher must revise his models in order to reinterpret previous activities as well as the new unexpected activity. If the researcher's models of the student's understanding explain the 'essential mistake' then it provides evidence supporting the viability of his models. Further, essential mistakes serve as constraints for the researcher, who attempt to devise ways of eradicating these mistakes (Steffe & Thompson, 2000).

A teaching experiment consists of a series of teaching episodes, each including a teacher-researcher, a witness, one or more students, and a method to record the episode (Steffe & Thompson, 2000). During the teaching episodes, the teacher-researcher poses problems to students in order to test or create hypotheses about students' mathematical understandings, allowing the researcher to develop models of the students' mathematics in this process (Steffe & Thompson, 2000). As students' mathematics is inaccessible to the researcher, he can use conceptual analyses (P. W. Thompson, 2008) in an attempt to make sense of the students' words and actions in order to construct viable models of the students' mathematics (Steffe, 1991; Steffe & Thompson, 2000). The researcher routinely tests his models for viability, as he is aware there is always a chance he has an untenable model of the students' thinking (Steffe & Thompson, 2000; Thompson, 1979). Although certainty is not an option, a researcher tests his models in his interactions with students by posing new problems to see if the students behave in ways compatible with his hypothesized models (Steffe & Thompson, 2000). It is through this iterative

process of constructing, testing, and re-constructing models of students' understandings that the researcher ensures his models are viable explanations of the students' understandings.

In addition to building models of the students' mathematics, another goal of the teacher-researcher is to foster student learning and development. The researcher infers learning has happened when students solve problems that they could not previously solve using their current schemes. Such learning may be the result of functional accommodations made in the activity of solving a problem. Whereas learning can be set in motion by an interaction, development refers to the processes that allow a student to overcome his or her essential mistakes; though set in motion by interactions, development does not occur in any specific interaction (Steffe & Thompson, 2000).[21] Development may be the result of metamorphic accommodations that occurred as a result of students addressing (and possibly reflecting on) a task or series of task (Steffe & Thompson, 2000).

In order to foster learning and development, a teacher-researcher introduces problems students may consider interesting, but have the potential to perturb the students' meanings if they engage in ways compatible with the researcher's hypothesized models of students' mathematics (Steffe, 1991a; Steffe & Thompson, 2000). For example, a researcher may pose a question that, from his perspective, a student can solve using her current schemes but in a new context that the student may not assimilate to her current schemes. The student may experience a perturbation until she comes to the realization that her previous scheme works in this new context. A researcher can also pose a problem that from his viewpoint is different than previous contexts, but that he conjectures the student will view as being the same as a previous context. In this case,

[21] It is important to note from the radical constructivist view point, while learning and development may occur during a teaching experiment, the teaching activity does not cause either, rather both are spontaneous independent contributions of the individual student (Steffe & Thompson, 2000; von Glasersfeld, 1995).

the student thinks she can solve the problem using her current schemes, only to discover these schemes ineffective in her solution process, which may lead to the student experiencing a perturbation (Steffe & Thompson, 2000). One goal of a researcher in posing such tasks is for the student to refine the assimilatory structure of their current schemes (i.e. make an accommodation) in such a way that the student accepts or rejects a broader set of input material for the activation of that scheme.

At times, a researcher may also hope to engender accommodations of a scheme in their students. As accommodations are difficult to engender, especially metamorphic accommodations, a teacher-researcher should use carefully formed hypotheses when deciding when to pose situations outside of students' current schemes. A researcher's continued interaction with students allows him to observe these accommodations over time. However, often a researcher's realization of accommodations a student makes over time does not occur until his retrospective analysis (Steffe & Thompson, 2000).

The researcher's ability to extract, maintain, and modify students' thinking provides some of the value of the teaching experiment (Steffe & Thompson, 2000). It is important for these changes to occur as well as for the researcher to corroborate the changes that transpired. When speaking of learning and development in a teaching experiment, the researcher is specifically looking at the changes in students' schemes. These changes must be observed more than once, and at different times, to ensure modifications have occurred. The aforementioned changes are not always those that the researcher expects or predicts, and are often only observed during the retrospective analysis. Whether in retrospective analysis or in action, it is important to note that in the teaching experiment the researcher is hoping to learn what changes students make and how these transformations are possible (Steffe & Thompson, 2000).

Teaching experiment in this study.

I conducted 16 teaching episodes in which the students worked in pairs with 14 occurring during class sessions and two occurring outside of class sessions. During the class sessions, I worked with one pair with Kevin LaForest serving as an observer.[22] During one of the out of class sessions, Dr. Moore and I acted as co-teacher-researchers and in the other Dr. Moore worked with Katlyn and I worked with Arya. In each case, Kevin LaForest acted as an observer. During the teaching experiment, certain activities were specifically designed to support investigating students' bidirectional reasoning including how to engender such reasoning (e.g., the *Graphing Sine/Arcsine Task, the Car Problem*, see Appendix C). Other activities were designed to engage students in reasoning quantitatively and covariationally, and we did not necessarily expect the students to reason bidirectionally as they engaged in these activities (e.g., periodicity and the PCS). I worked with the students during these latter tasks for two reasons: these tasks provided insight into the students' general quantitative and covariational reasoning and because we predicted that the students might engage in activity indicative of reasoning bidirectionally in unexpected or surprising ways. By working with the same students throughout the entire semester, I constructed and tested models of the their quantitative, covariational, and bidirectional reasoning.

Dr. Moore, the research team, and I met to discuss our observations after each teaching episode (and previous episodes) as well as plan for the next episodes, during which I took notes to record our observations. When needed, we met between teaching episodes to discuss instances from that week (or previous weeks) interactions that provided insights into the students' thinking. Both types of meetings supported my creation of hypothesized models of the students'

[22] I note that during one in-class teaching episode (Power Tower) I was inaccessible so Dr. Moore acted as the lead researcher and Kevin LaForest acted as the observer.

understandings so that I could adapt materials in order to test and further develop these models. I took notes of our thoughts and discussions that I used both for design of future class sessions and for retrospective analysis. Additionally, I referenced and expanded these notes as I watched (and re-watched) the videos of the episodes during my retrospective analysis.

Data Collection Summary

I participated in (either as interviewer or as observer with Irma Stevens as the interviewer) four one-on-one initial clinical interviews during the first week and a half of the course. In addition to providing valuable data concerning the students' meanings, I used the pre-interviews, along with the students' activities in class, to select Katlyn and Arya. Some criteria I used to select the pair included if they reasoned in similar ways to one another, if their reasoning was compatible with common ways of thinking we had observed in students in previous semesters, and if the students were adept at explaining their thinking. I worked with one pair from the beginning of the teaching experiment (about 2 weeks into the course) until the end of the semester (see *Figure 11* for a synopsis of my data collection efforts).

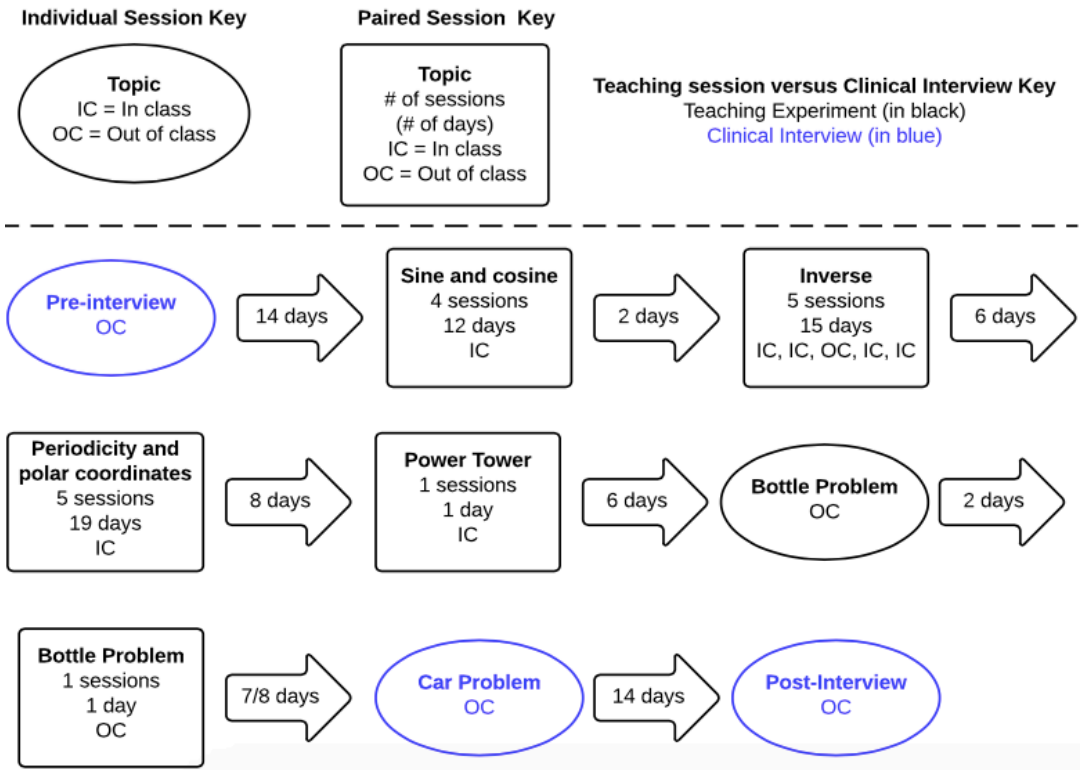


Figure 11: Data collection summary.

As seen in *Figure 11*, in addition to working with these students in class, we periodically met with them outside of class time. These out of class episodes varied both in format and purpose. I organized some of these episodes (e.g., the third inverse episode) so that the students continued to work together. During other episodes (e.g., the first bottle episode), Dr. Moore and I worked individually with one student each in order to further investigate how that student was thinking. Some activities in each setting were designed only to probe the students' meanings or test hypothesized models of the students' meanings, and other activities were designed to perturb students' ways and means of operating, possibly leading to the student making accommodations to her meanings. These out of class episodes provided us with added flexibility to support the students' development of bidirectional reasoning as well as to test our hypothesized models of the students' meanings. All episodes were videotaped so that the students' words, writing, and

actions were recorded. The recordings were analyzed throughout the semester to support and inform our instructional decisions in the teaching experiment.

I conducted follow up clinical interviews with Arya and Katlyn at the end of the semester after all teaching episodes. The goal of these interviews was to test my models of the students' meanings while also exploring shifts in each students' meanings that occurred during the semester. By exploring the students' meanings as they entered the study as well as when they left the study, I observed such shifts.

In summary, the study consisted of the following:

- Pre-interviews with four students (one interview per student)
- In class teaching experiment episodes with a pairs of students (14 episodes)
- Out of class teaching experiment episodes with a pair of students (two episodes)
- Out of class teaching experiment episodes with one student (one episode per student)
- Post clinical interviews for each students (two episodes per student)

The data corpus I collected from these episodes consisted of: videos of the students' activities in each episode, scans of the students' work in each clinical interview, pictures of the students' board work at the end of certain episodes, notes taken at meetings both after and between teaching episodes, and notes taken during my continual analysis.

By collecting data in this variety of contexts (e.g., clinical interviews and teaching experiments, working in groups and working individually), I had the flexibility to build and test models of the students' mathematics. These models addressed meanings developed as they engaged in activities intended to emphasize reasoning about relationships quantitatively, covariationally, and bidirectionally. The models also addressed how these meanings related to the students' function and inverse function meanings.

Data Analysis Overview and Procedure Summary

In order to analyze the data I conducted conceptual analyses to develop and refine models of the students' mathematics. As Thompson (2008) described, one purpose of a conceptual analysis is "building models of what students actually know at some specific time and what they comprehend in specific situations" (p. 60). With this goal in mind, I used the combination of clinical interviews and teaching experiment episodes to develop, test, and refine hypothesized models of the students' mathematics that explained their words and actions.

The recordings from both the clinical interviews and the teaching experiment were analyzed using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Initially, I analyzed the videos identifying instances of students' words and actions that provided insights into the students' meanings. Such instances supported my generation of tentative models of the students' mathematics that I could test by searching for corroborating or contradicting instances in the students' activity. When the students exhibited novel activity that contradicted my models, I revised my hypotheses to explain this activity including the possibility that this new activity indicated fundamental shifts in the meanings with which they operate (i.e. their *operative meanings*). Using this technique, data analysis was an iterative process of continually creating, refining, and adjusting hypotheses.

As can be seen in the description above, the analysis had both generative and convergent purposes (Clement, 2000). When analyzing data with generative purposes, a researcher is typically exploring unfamiliar behaviors where there is little existing research explaining these behaviors. My analysis also had convergent purposes. Analyzing with convergent purposes typically leads to a coded analysis of interviews, with the goal of providing reliable explanations of student's activities within a model or framework (Clement, 2000).

Specific to my study, I initially analyzed the data with generative purposes as I formed models of the students' mathematics. After constructing these models, I used a convergent analysis in an attempt to confirm or contradict these models. Further, the results from this convergent analysis led to changes in my proposed models, moving the analysis back to a generative purpose. This cycle happened numerous times in succession during the retrospective analysis (as well as during the ongoing analysis) before I developed models of the students' mathematics that viably explained their activity.

In order to create such models it was important I review the data carefully. I conducted multiple-levels of analysis during retrospective analysis. Initially, I watched the videos looking for instances that provided insight into students' thinking quantitatively, covariationally, or bidirectionally, as well as when they engaged in discussing function or inverse function, making notes of my observations of such instances. During this phase (and all phases) of analysis, I referred back to the notes made during the initial on-going analysis to inform my thinking regarding both the students' activities and our intentions with certain tasks. This initial analysis supported the development of preliminary conjectures concerning the students' ways and means of operating.

After this first analysis, I transcribed important instances of the data in which students' activities provided insights into their ways of thinking, including not only the students' and researchers' words but also descriptions of their actions, including their inscriptions and hand motions. As I transcribed, I expanded the notes from the first level of analysis to incorporate new observations that helped to support, refine, or reject my preliminary hypotheses. After transcriptions were completed, I re-analyzed key episodes to continue to create, test, refine, or reject hypotheses. In this level of analysis, I explicitly focused on looking across the activities of

one student throughout the teaching experiment as well as comparing the activities of the students. By exploring the activities of one student across the semester, I refined and adjusted my models of that student's mathematics that viably explained her activities across time. By using these models to compare the students' activities, I was able to describe ways of knowing that were propitious or deleterious to the students, both of which are purposes of a conceptual analysis (P. W. Thompson, 2008). Throughout the entire data analysis process I regularly worked with the research team to get additional insights and opinions regarding my interpretations of the students' words and actions. In this way, the research team helped to corroborate or contradict my models in ways that supported the construction of viable models.

Chapter 4: A Pilot Study on Students' Development of Bidirectional Reasoning

The goal of this chapter is to provide results from a pilot study exploring students' reasoning about relationships between quantities and possible implications of such reasoning. For the sake of brevity, I refer the reader to the methods and methodology chapter above for a general overview of how this study was conducted. Specific to the pilot study, I worked with two students, Ryan and Lauren (pseudonyms), in the Fall 2013 semester. I choose Ryan and Lauren from a pool of 13 students who volunteered to be part of the pre-interviews because each exhibited meanings consistent with each other (e.g., maintaining the input had to be represented on the horizontal axis) and because both students were adept at explaining their thinking aloud. I conducted one clinical pre-interview per student (see Appendix A for tasks), eight paired in-class teaching experiment episodes, one out-of-class teaching experiment episode per student (see Appendix D for these tasks), and one clinical post-interview per student (see Appendix B for tasks). For more detail regarding these episodes or the data analysis, please refer to the previous chapter.

Results

I first synthesize results from the pre-interviews, using the students' activities on particular problems as illustrative examples from which I inferred meanings the students maintained. I then describe their activities on various tasks from the teaching experiment in order to characterize their reasoning about relationships between quantities, especially in relation to their reasoning bidirectionally. I conclude with results from the post-interview episodes, including descriptions of shifts in their thinking relative to the pre-interviews. For brevity, I do not provide data from all interactions that informed my models of the students' thinking. Instead, I provide illustrative snapshots that provide insights into the ways of thinking each student engaged in over the course

of the study. My goal in providing these snapshots is to identify important findings that influenced and informed the dissertation study.

Results from pre-interviews.

I conducted clinical interviews to explore the students' meanings at the outset of the teaching experiment. Some interview tasks (Appendix A) were designed to evaluate, explore, and possibly perturb the students' function and inverse function meanings. Also, a subset of tasks was designed to more directly examine the students' quantitative and covariational reasoning. Due to time constraints of the interviews, I did not include tasks that prompted the students to construct a relationship and graph "from scratch." In this section, I describe each student's activity on certain problems that are pertinent to this pilot study.

Lauren.

Lauren's activity addressing the Celsius-Fahrenheit tasks provided me insights into her function meanings and her quantitative reasoning. I presented Lauren with the Celsius-Fahrenheit problems that gave her an analytically defined function rule (Appendix A, Problem 5) and graph (Appendix A, Problem 6) representing the relationship between temperatures measured in degrees Celsius and in degrees Fahrenheit. I asked her to represent the inverse function in each case and explain what these representations conveyed about the context of the problem (i.e. the relationship between Fahrenheit and Celsius measures). When initially approaching these problems, Lauren expressed discomfort with determining inverse functions analytically and graphically, and she used techniques that were idiosyncratic to the two representations.

For instance, when determining the rule for the inverse function of an analytically defined function (Problem 5), Lauren stated, "Does it have something to do with the derivative? Maybe

it's one over the derivative of this [*pointing to the function*], isn't it?" After this statement she wrote $C^{-1}(F) = 1/(C'(F))$, although she expressed discomfort with this solution. Lauren also experienced uneasiness with inverse functions as we moved to the graphical representation (Problem 6). She first identified a coordinate pair on the given line and switched the coordinate ordering to determine a point on the line representing the inverse function (e.g., the estimated point (10, 50) on the original line became the point (50, 10)).[23] With Lauren identifying the point (50,10) on the line representing the inverse function, I directed her to describe the meaning of this point in relation to the context. She stated the point meant 50-degrees Celsius equals 10-degrees Fahrenheit, which she immediately claimed was not true. This caused her to question her technique and left her perturbed about how to graph the "inverse". I took Lauren's activity to indicate that although she had procedures to 'find' inverse functions in both graphing and analytic context, she was unsure what the results of these procedures meant with respect to the contextual situation, leaving her unsure if these procedures were correct.

Later in the interview, Lauren exhibited a different technique for determining inverse functions analytically, switching the x and y variables and isolating y, which she used in numerous problems after the Celsius-Fahrenheit problems. Because of this, I returned her to the Celsius-Fahrenheit problems after she had responded to each interview task. Lauren identified that if the original function had an input quantity of degrees Fahrenheit and an output quantity of degrees Celsius, then "I think the inverse would be telling you how to go from Celsius to Fahrenheit." Lauren then wrote $F(C) = (9/5)C + 32$ as the inverse of $C(F) = (5/9)(F - 32)$, explaining that this formula inputted Celsius values and outputted Fahrenheit values. Because of

[23] This activity was consistent with her earlier activity where she switched x and y values in graphing situations. Also of note, she did not connect this with her work on the analytic Fahrenheit-Celsius problem at this point.

her new activity, I returned to her previous graphical solution and asked her to interpret a meaning for the point (50, 10) on the graph representing the inverse function. She responded by drawing, in blue, over her original red line and relabeling the axes in blue (*Figure 12*). She argued that switching the meaning of the axes maintained the relationship between degrees Celsius and degrees Fahrenheit while also providing a graph of the inverse function.

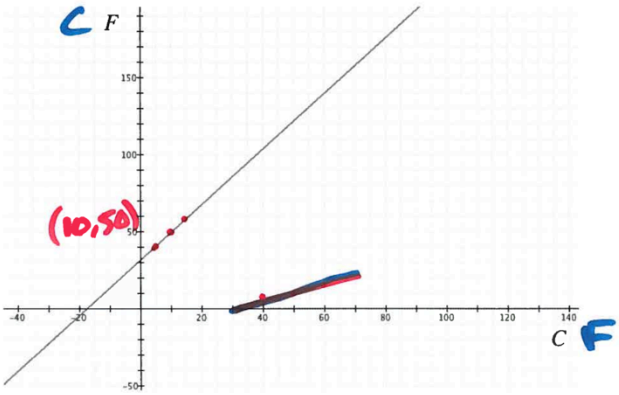


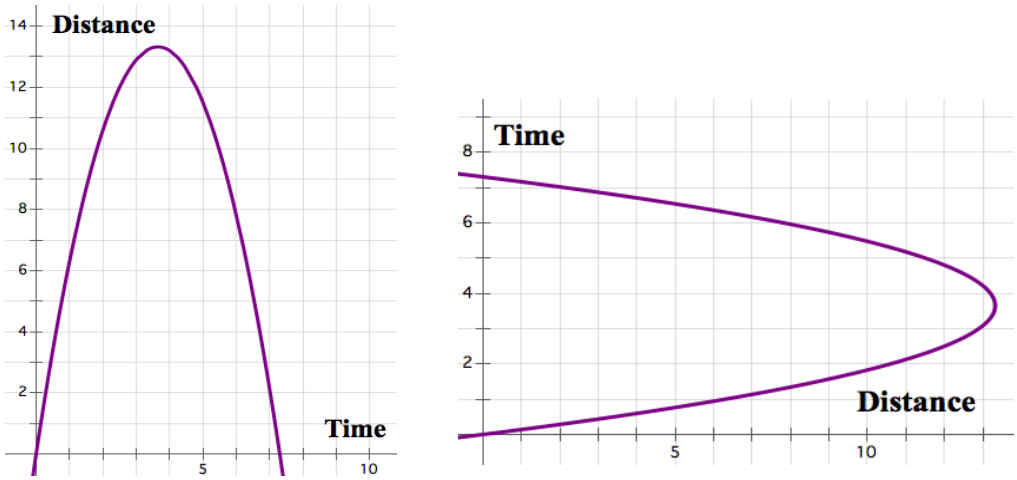
Figure 12: Lauren's work on the Celsius-Fahrenheit graph.

When initially approaching the Celsius-Fahrenheit tasks, Lauren did not discuss her solutions in terms of the quantities of the situation unless I prompted her to do so. Moreover, she expressed confusion and questioned her solution methods when I directed her to consider the quantities of the situation. When reconsidering the tasks, she maintained a focus on the quantities as she considered "switching" inputs and outputs, as opposed to solely reasoning about switching variable letters or the order of coordinate pairs. Lauren understood the inverse of the given function to be the function with switched input and output *quantities and associated values*, regardless of representation (e.g., analytically degrees Celsius became the input and graphically degrees Fahrenheit became the input).

As exhibited above, at times during the interview, Lauren's meanings were quantitative in nature; she interpreted graphs and equations as representing a relationship between quantities and

maintained a focus on these quantities throughout her solution process. Other times during the interview, her activity indicated that she drew on meanings that did not foreground reasoning about quantities and relationships between quantities, instead focusing on carrying out certain activities or on maintaining what I perceive to be conventions common to school mathematics in ways that did not suggest a quantitative focus. Furthermore, when she subsequently attempted to consider these meanings in relation to the quantities of a situation, she often experienced states of perturbation that I took to suggest an incompatibility between these meanings.

As another example in which Lauren exhibited conflicting meanings from my perspective, I gave her two graphs that we had designed to represent the same relationship but with different oriented axes (*Figure 13*). Lauren initially claimed the two graphs represented the same relationship between distance and time. When asked if each graph represented a function, Lauren claimed the left graph represented a function while the right graph did not. Lauren then questioned, "I mean if it's the same thing how could one, how could it be a function one way and not be a function the other way?" She experienced a perturbation stemming from this conclusion and she was unable to reconcile this perturbation on her own accord. I took Lauren's activity to indicate that she relied on representing the input quantity on the horizontal axis when determining if a graph represents a function.



(a) (b)

Figure 13: The Double Parabolas Problem.

Upon completion of the other interview tasks, I returned Lauren to the problem with the intention of Lauren considering that in both graphs, there is a unique distance for each time, and that for numerous distance values there is not a unique time. I intended that we have this discussion in order to see how she might relate this outcome to her previous conclusions about the "function-ness" of each graph. After discussing that for each time there is a unique distance in *Figure 13*a, Lauren said, "This one's [*Figure 13b*] saying like, if you look at distance as the input, then it's saying your at that distance at two separate times [*moving her marker up and down at a distance of 10*]." With Lauren identifying how she was interpreting *Figure 13b*, I continued to try to have her discuss what was the same between both graphs (Excerpt 1).

Excerpt 1: Lauren discussing the relationships represented in *Figure 13*.

1	TP:	[*referring to Figure 13a*] So but like at any given instant [*pause*]
2	Lauren:	You weren't at two places at once.
3	TP:	Okay, now is that true here [*pointing to Figure 13b*]?
4	Lauren:	Yes. [*pause*] Because at each time, if you did a horizontal line [*moving*

5		*her marker down the graph in a horizontal position*], you know.
6	TP:	Okay.
7	Lauren:	But I don't think horizontal line test is a thing, at least for testing a
8		function.
9	TP:	Okay, so we don't have some sort of horizontal line test to test if
10		something is a function?
11	Lauren:	I don't think so.
12	TP:	Okay, so but here you're saying it's [*pointing to Figure 13b*] not a
13		function, why again?
14	Lauren:	Because each input, the distance, doesn't have its own output being time.
15	TP:	Okay. So for each distance, there isn't a unique time?
16	Lauren:	Uh-huh.
17	TP:	Is that true here [*pointing to Figure 13a*]?
18	Lauren:	For each distance there isn't a unique time? [*pause*] Yes. [*long pause*]
19		You have a distance ten [*pointing to the value of 10 on the vertical axis*
20		*and moving horizontally to the corresponding values*], you were distance
21		ten at three seconds and again at, six-ish seconds.

Lauren used both graphs to consider whether for each time there is a unique distance and whether for each distance there is a unique time (lines 2, 4-5, 18-21), but she maintained the horizontal axis (distance) as representing the input quantity (lines 7-8, 14) when determining whether *Figure 13*b represented a "function". Over the course of the interview, and in the

context of making statements about whether a graph represents a "function," Lauren consistently (consciously or subconsciously) maintained to the horizontal axis as the input quantity.[24]

Summary of Lauren.

Lauren's activities described above represent examples that I took to indicate that she relied on meanings that were often in conflict with each other from my perspective. There were numerous instances where Lauren experienced a state of perturbation as she reasoned quantitatively to obtain a result only to realize this result conflicted with other meanings. Occasionally these other meanings relied on carrying out a procedure (e.g., switching variables) without being able to explain why she enacted these procedures (e.g., "Because somewhere, somewhere along the line in my math life, I've done that."). Other times the meanings that conflicted with her quantitative reasoning entailed a dependence on particular practices that are common in school mathematics (e.g., input is represented on the horizontal axis). In these cases, the conventions were not conventions to Lauren; they were *inherent* to her understandings of the situation (Moore et al., 2014; Moore et al., submitted). In both cases, Lauren was often left perturbed when attempting to reconcile the conflicts that she perceived in her meanings and activity.

Ryan.

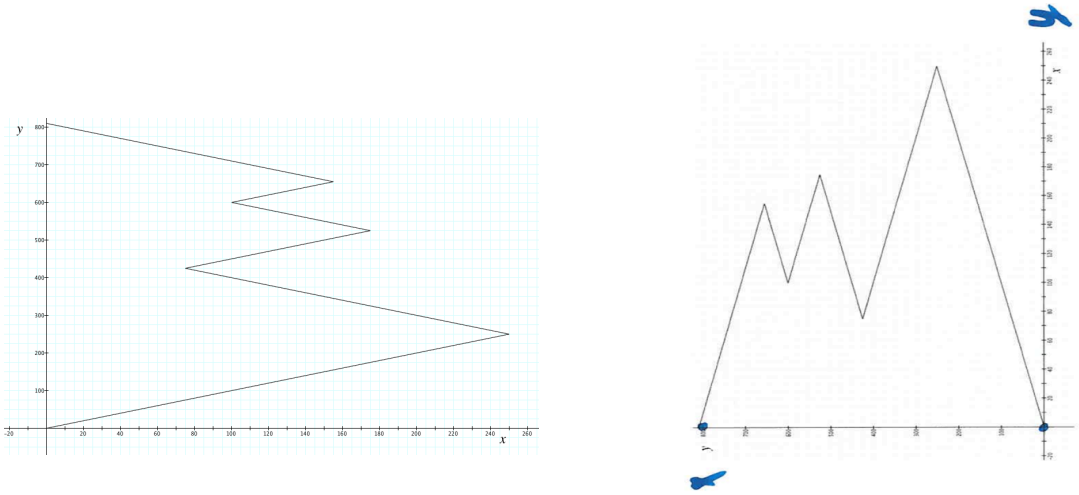
Like Lauren, Ryan often relied on meanings that were based in enacting certain procedures or maintaining what I perceive to be conventions. However, unlike Lauren, Ryan's meanings were such that he progressed through the interview without experiencing states of perturbation. For instance, he used two techniques for determining (or working with) inverse functions

[24] This meaning was reiterated in Problem 15 (part 3) when she rejected the student's work who labeled the *y*-axis as input stating "I think the *y* would come down here [*pointing to the horizontal axis*] and the *x* would go up here… Input typically goes on the *x*-axis [*motioning over the horizontal axis*]."

represented by graphs, equations, or tables throughout the interview. One technique involved switching the variables in an analytic equation and solving for the original output variable. In graphical or tabular representations, this technique corresponded to switching the x and y coordinates or values. The other technique involved writing the inverse function rule as the reciprocal of the function rule, which he also used in tabular situations (e.g., if $f(3) = 2$ then $f^{-1}(3) = 1/2$). In graphing situations, he used this same technique by drawing vertical asymptotes where the curve representing the function intersected the x-axis and by making rough sketches of the curve as if it were defined according to the reciprocal function rule (i.e., the point (x, y) in the original graph was mapped to the point $(x, 1/y)$ in the graph of the inverse function). After Ryan sketched two curves representing, to him, the inverse function, I asked why we had "two inverses," and he responded, "I do not know why we have two inverses." However, (a la Benny (Erlwanger, 1973)) he expressed confidence that there were two different techniques that produced a correct "inverse."

I took Ryan's Activity to indicate that, like Lauren, he held meanings for graphs and variables that relied on what I perceived to be conventions common to school mathematics. For instance, when told that a middle-school student insisted the graph shown in *Figure 14*a represented a function because 'x is a function of y," Ryan stated that the student was thinking of the horizontal and vertical axis as representing y-values and x-values, respectively. He drew these labels on the graph, and turned the paper 90-degrees counterclockwise (recreated in *Figure 14*b). With the graph re-labeled and the paper turned, he stated that the student was thinking of the graph in the rotated orientation and that the student was correct when *using this orientation* (recall that the given statement was "x is a function of y"). By re-labeling and rotating the graph, Ryan achieved the conventional orientation of x on the horizontal axis and y on the vertical axis,

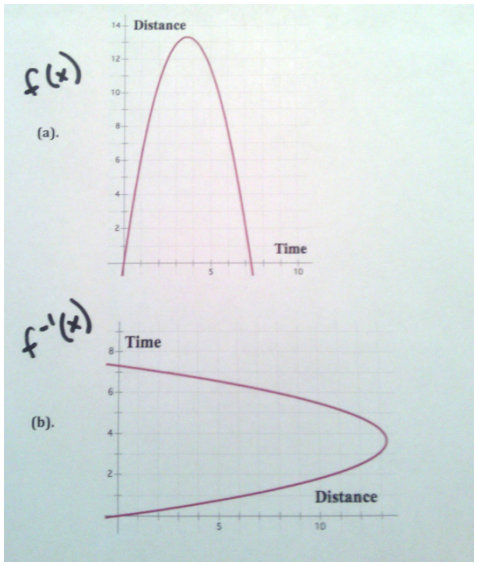
with x denoted as representing input values. Further, Ryan attributed both of these actions to the posed solution in order to make sense of the graph as representing a function. When discussing "function" throughout the interview, Ryan exhibited meanings that relied on the input quantity being represented by the variable x on the horizontal axis.



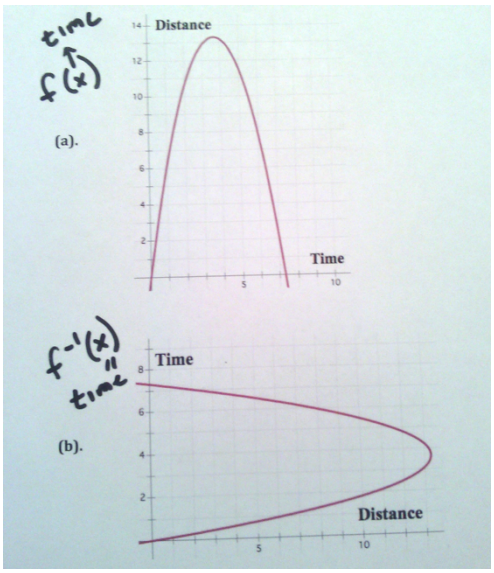
(a) (b)

Figure 14: (a) Graph from Problem 12 and (b) Ryan's work on Problem 12.

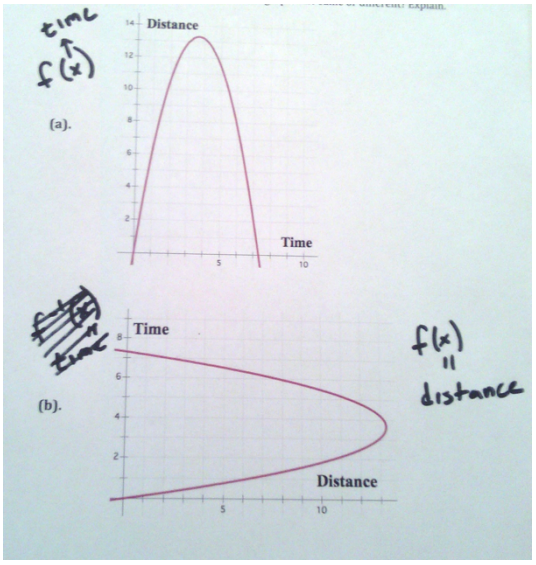
As another example of Ryan's activity during the interview, consider his response when discussing the two graphs shown in *Figure 13*. Ryan initially stated that the two graphs represented the same relationship and added that he thought the two graphs were "inverses" of one another, writing $f(x)$ next to *Figure 13*a and $f^{-1}(x)$ next to *Figure 13*b (this has been recreated in *Figure 15*.1). By labeling the two graphs as he had, I interpreted that x in $f(x)$ represented time and x in $f^{-1}(x)$ represented distance, although I was unsure if Ryan was interpreting the variables in this way. Intending to explore this, I drew Ryan's attention to the variables and quantities (Excerpt 2).



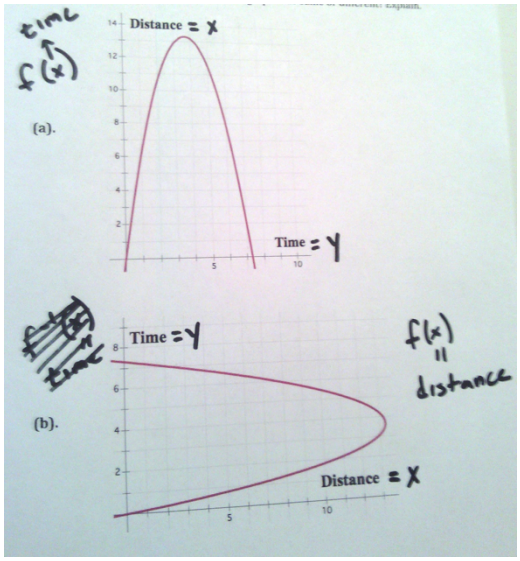
(1)



(2)



(3)



(4)

Figure 15.(1)-(4): Recreation of Ryan's work on Problem 9.

Excerpt 2: Ryan's explanation for why the two graphs in *Figure 13* are not inverses.

1	TP:	Let me ask you this then, there's no *x* in here really [*motioning over*
2		*Figure 15.*1*a*], I mean there's distance and time [*pointing to the labeled*
3		*axes*], so what is, you write *f* of *x* there [*pointing to the f* (*x*) *he had written*

4		*next to Figure 15.*1*a*]?
5	Ryan:	Ah, it's just *f* of time [*writes the word time above f* (*x*) *and draws an*
6		*arrow from the x indicating the x represents time, recreated in Figure*
7		*15.2a*].
8	TP:	Okay, and what about down here [*pointing to Figure 15.2b*]?
9	Ryan:	This would be, yeah I mean if it was an inverse it would be *f* of time
10		[*writes time below* $f^{-1}(x)$ *and draws an equal sign indicating x represents*
11		*time as shown in Figure 15.2b*]. But because, yeah I think I just
12		overcomplicated things [*crosses out* $f^{-1}(x)$ *and the word time*]. 'Cause I
13		think this [*pointing to Figure 15.2b*] is just a different function. That *f* of *x*
14		being distance [*writes f* (*x*) *with an equal sign indicating x represents*
		distance, recreated in Figure 15.3].

I took Ryan's activity to indicate that Ryan anticipated a function and its inverse to have the same input quantity with respect to the context of the problem and because *Figure 15*.1b had distance on the horizontal axis, which Ryan considered to represent the input axis (and the variable *x*), he questioned his claim that the graphs were inverses of each other (lines 8-12). To reconcile this discrepancy, he relabeled *x* as distance (*Figure 15*.3b), thus using *x* to represent time measures for one graph and distance measures for another graph, and claimed that the second graph did not represent the inverse function.

I later asked Ryan to further explain why the two different looking graphs represent the same relationship. He stated "Your change of… *x* and *y*. 'Cause you just flipped them." After stating this, in *Figure 15*.3b he re-labeled distance with the variable *x* and time with the variable *y* (shown in *Figure 15*.4b). Ryan then reconsidered the top graph in *Figure 15*, and maintained the

quantitative referent for each variable as he labeled the vertical distance axis in *Figure 15*.4b as x and the horizontal time axis as y. Rather than maintaining x on the horizontal axis as he had done previously, Ryan maintained the labeling of each quantity in both graphical representations (e.g., Distance = x, Time = y). When not concerned with the *function-ness* of the graphs, Ryan maintained the quantity referent for each variable and was not restricted to conventional variable-axis orientations.

Summary of pre-interviews.

During the pre-interviews, both Ryan and Lauren exhibited meanings that had common underlying themes. Both students' function meanings entailed representing the input on the horizontal axis. Even when I requested that they consider the vertical axis as representing the input quantity (e.g., Problem 12, Problem 15), the students engaged in activity that I took to indicate that each student imposed representing input on the horizontal axis as an inherent aspect of interpreting graphs (rather than a convention). Interestingly, while avoiding discussions that explicitly involved a focus on "function", the students focused on the quantities and relationships between quantities when explaining how two quantities were related. For instance, each student ensured that quantities were consistently represented by the same variable in contextualized situations (e.g., Lauren changing the C and F axes labels in the graph representing the inverse function to maintain the relationship between temperatures, Ryan maintaining time as the y variable and x as the distance variable). Both students exhibited activity to indicate they reasoned quantitatively, but also exhibited activity that indicated their thinking was constrained by certain conventions common to school mathematics. Because of this, I was curious how the students would respond to tasks I designed to possibly perturb their meanings and to engender bidirectional reasoning.

The Teaching Experiment

I characterize Lauren and Ryan's activity during tasks designed to engage constructing relationships between quantities via reasoning quantitatively, covariationally, and possibly bidirectionally. As the teaching experiment and the course did not begin concurrently, Ryan and Lauren had engaged in activities designed to engender meanings for angle measure, the unit circle, and the sine function rooted in quantitative and covariational reasoning, which are described elsewhere (Moore, 2013, 2014a; Moore et al., 2012). To briefly summarize, the students constructed the sine function as a covariational relationship between the angle measure measured in radians and vertical distance above the horizontal diameter measured in radii. The pair developed this understanding by examining how vertical distance changed with respect to equal changes in angle measure in a circular motion context. For instance, *Figure 16*a represents vertical distances (solid vertical blue segments) as well as the changes in vertical distances (dashed vertical blue segments) for equal changes of angle measure (green arcs) in the context of the unit circle. The students represented this relationship on a Cartesian plane by marking equal changes along the horizontal, θ, axis and translating the vertical distances and amounts of change of vertical distance to the appropriate angle measure, constructing the sine graph (*Figure 16*b). (For more on this, see (Moore, 2014a)).

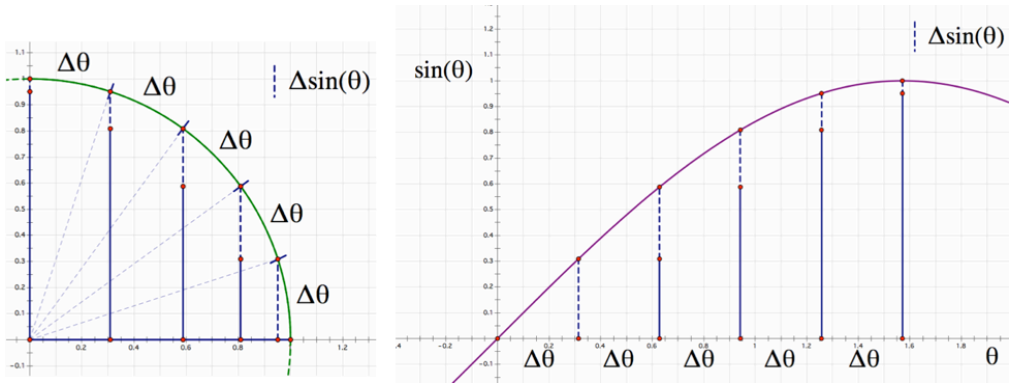


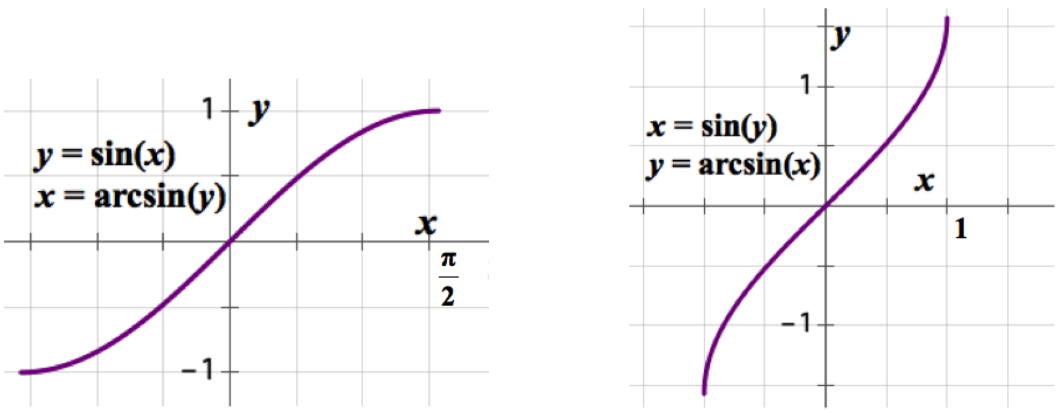
Figure 16: Figure 4 from Moore (2014).

The Graphing Sine/Arcsine Task.

After working with the sine function, we moved to the *Graphing Sine/Arcsine Task* (*Figure 17*). My primary goal of the task was that for the students to come to understand "sine" and "arcsine" as referring to the same relationship between covarying quantities (angle measure and directed vertical distance over a particular interval of angle measure values). I also intended for the students to understand that graphs in different orientations as simultaneously represent the same relationship and hence both of these functions (e.g., see *Figure 18*), with the only difference between the functions being how the quantities' values were defined with respect to the different input-output choices.

Graph 1:	Create a graph of the sine function with a domain of all real numbers. What is the range?
Graph 2:	Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function. What is the domain and range of this function?
Prompt 3:	Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not?

Figure 17: Graphing Sine/Arcsine Task.

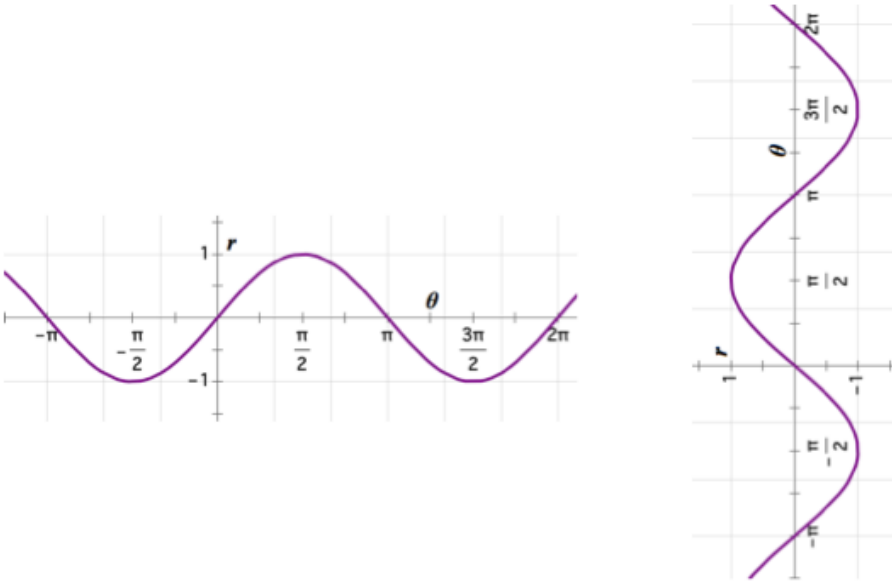


(a) (b)

Figure 18: Two representations of the same relationship.

Our initial discussion established that the arcsine function had an input of a directed vertical distance from the horizontal diameter measured in radii and an output of angle measure in

radians. Further, we established that the terms arcsine and inverse sine were synonymous. After this the students began to construct the graphs described in the task. Ryan and Lauren initially took a pointwise approach. They considered switching specific values in coordinate pairs to plot new points for the arcsine function. However as they discussed this, Ryan suggested turning the board 90-degrees counterclockwise, thus turning *Figure 19*a, to the orientation of *Figure 19*b. As the conversation continued, Ryan indicated that after rotating the graph 90-degrees, he needed to reflect the graph over the vertical axis (in the new orientation, *Figure 19*b) in order to obtain the arcsine graph. Such activity is consistent with his actions during the pre-interview; his goals included representing the input on the horizontal axis, in this case with the values on the horizontal axis oriented with positive values to the right of the origin.



(a) (b)

Figure 19: Recreation of the pairs (a) sine and (b) arcsine graph.

As Ryan discussed his strategy, Lauren remained quiet, apparently considering Ryan's actions and utterances. When Ryan completed his explanation, I asked Lauren if she understood how Ryan was approaching the problem. She responded, "I mean I understand why he did it, I

don't know if after you graph [the arcsine function] if that's what [the arcsine function]'s going to look like." Ryan added, "Yeah, that's how I am feeling to, I'm not sure if that's right." Thus, while the students had obtained what they thought might be a graph of the arcsine function by rotating and reflecting the graph of the sine function, they were unsure if the result was a graph of the arcsine function.[25] I conjecture both students were attempting to recall a graph for the arcsine function without success, leaving them questioning whether their hypothesized graph would "look like" the graph of the arcsine function.

With both students unsure of their proposed solution, the episode ended and the situation was left unresolved. We continued this activity at the beginning of the next episode and Lauren first spoke explicitly about the graph in terms of the quantities being represented. Ryan, on the other hand, initially focused more generally on the "function" as he referred to it. As an example of Lauren's focus on quantities, having sketched the two graphs recreated in *Figure 19*, she explained, "On this one [*pointing to Figure 19a*] the radians are the input and the vertical distance is the output, so on this one [*pointing Figure 19b*] the vertical distance became the input and the radians became the output." I then asked them to describe the covariational relationship represented by the two graphs to relate them. Working with *Figure 19*a, Lauren suggested considering equal changes in angle measure. Ryan began, "Zero to pi over two the *function* is…" (emphasis added). Lauren quickly interjected, "The vertical distance," expressing discomfort with Ryan's use of the term "function". Though brief, Lauren's quick interjection highlights her desire to remain focused on the quantities she conceived as being relevant to their graph.

Due to Lauren's quantitative focus, I conjectured that she might be reasoning about the relationship between angle measure and vertical distance in a way that allowed her to

[25] I note that the students did not raise the idea of univalence during their activity.

simultaneously see both the sine and arcsine function in either graphical representation of the relationship. To test this conjecture and gain further insights into her thinking, I asked Lauren to compare what they had termed the arcsine graph (*Figure 19*b) with the sine graph (*Figure 19*a). Lauren responded, "So the relationship is still the same. We can see that when going zero to pi over two [*motioning her marker up and down on the vertical, radian, axis in Figure 19b*] the distance is increasing at a decreasing rate just like it did here [*pointing to Figure 19a*]."[26] By maintaining an explicit focus on covarying quantities, Lauren conceived two different visual (and named) graphs as representing the same relationship.

With Lauren having conceived both graphs in terms of the same covariational relationship, I directed the students to the third prompt. After reading this prompt, Lauren suggested they could interpret their original graph (*Figure 19*a) as either the sine or arcsine function. Ryan interpreted Lauren's statement as her claiming that the vertical axis in *Figure 19*a represented angle measures and the following interaction ensued (Excerpt 3).

Excerpt 3: Lauren responding to Ryan.

1	Lauren:	But your theta is not one [*pointing to the 1 on the vertical axis in Figure*
2		*19a*]. Like you're not going to have a theta of one. You know what I
3		mean? Because your, you still have these values of, like this is still a value
4		of an angle [*motioning over the horizontal axis*] and this is still a value for
5		a vertical height [*motioning over the vertical axis*]. So I think it's, how
6		you read it.
7	TP:	So when you're saying "how you read it," what do you mean?
8	Lauren:	Like usually when you look at a, a graph on the *x* and *y* axis, you follow

[26] Lauren had previously established *Figure 19*a as conveying this relationship.

9	the *x*-axis, like your interested in an *x*-variable. So it's like where an *x* is,
10	like, like look at the *x*, look at your *x*-value to find *y*. Instead of doing that
11	we'll say look at your *y*-value to find *x*.

Lauren maintained the quantitative meanings of the axes in her interpretation (Lines 2-5), and conceived the graph (*Figure 19*a) as the arcsine function by interpreting the *y*-axis (or values) as the input you use to "find" *x* as the output (Lines 5, 7-10).[27] Whereas Lauren's actions predominantly focused on the graphs in terms of a relationship between quantities, Ryan's actions focused on labeling axes in order to maintain θ as the input variable for both the sine and arcsine function.

For instance, numerous times before the above interaction, when trying to address the third prompt, Ryan labeled the vertical axis in *Figure 19*a as θ, and the horizontal axis as $\sin(\theta)$. After the interaction in Excerpt 3, Ryan realized that sine of one was not equal to $\pi/2$ and he changed the label on the horizontal axis in *Figure 19*a to $\sin^{-1}(\theta)$ and the label on the vertical axis to θ. During this conversation, Lauren interjected, stating she did not like labeling the vertical axis as θ, as she preferred the variables representing the same quantity in each representation (i.e. r = vertical distance and θ = angle measure in both *Figure 19*a and *Figure 19*b). Ryan responded, "This [*pointing to his label on the horizontal axis in Figure 19a*] is just a variable so you just make the variables the same [*pointing to the* θ *in* $\sin^{-1}(\theta)$]."

Ryan was concerned with the way to name two functions using the same graph. His actions in this activity are compatible with those instances the pre-interview in which Ryan was concerned with function. In these instances, as in the interaction above, Ryan understood that a

[27] Issues of domain and range were not addressed during this interaction for the purpose of maintaining a focus on the graphs as representative of covariational relationships. Issues of domain, range, and univalence were raised in a later discussion.

function and its inverse have the same input variable (letter); he did not concern himself with a quantitative meaning for the input variable when discussing function. In this case, this understanding led to him maintaining as the input variable for the sine and arcsine function.

Although Ryan used his labeling to discuss one graph as the sine and arcsine functions using θ as the input variable in each case, Ryan's switching of variables made it difficult for him and Lauren to discuss the graph in compatible ways. For instance, as Ryan explained his new interpretation of *Figure 19*a as the arcsine function with θ on the vertical axis and $r = \sin^{-1}(\theta)$ on the horizontal axis, Lauren interjected, "variables are not magic letters…I think it's just important right now we stick with consistency."

Ryan agreed to comply with Lauren's request to "stick with consistency" with respect to quantitative variable referents, and he attempted to discuss the sine and arcsine relationship in each graph using θ and r as the inputs, respectively. Ryan considering two ways to interpret the graph recreated in *Figure 19*a, saying, "Alright theta is going to be our input [*pointing to* θ *in* $\theta = sin^{-1}(r)$ *written along horizontal axis*], that makes this be our output [*pointing to r in* $r = sin(\theta)$ *written along the vertical axis*]. Then she's talking r is going to be our input [*pointing to the r in the equation* $r = sin(\theta)$ *written along the vertical axis*], this becomes our output [*pointing to* θ *in* $\theta = \sin^{-1}(r)$ *written along the horizontal axis*]." Although Ryan's descriptions accurately described the relationship between inputs and outputs in the sine and arcsine functions, he relied heavily on referencing the equations $r = \sin(\theta)$ and $\theta = \sin^{-1}(r)$. Because of this, I was unsure if he was reasoning about quantities and a relationship between quantities or if his explanation stemmed from reasoning with the two equations. Unfortunately, the episode ended before I had the opportunity to question him further regarding the way he was making sense of the situation, and how he might have related the two graphs independent of the written equations.

Although time prevented further probing of Ryan's meanings, Lauren's actions indicated she was reasoning about one relationship in two ways; Lauren considered the two graphs as representing the same quantitative and covariational relationship but where choosing different input-output choices allowed her to discuss two different functions. For example, towards the end of the episode I asked if the sine graph (*Figure 19*a) was representing the inverse function to which Ryan responded, "I think it is the inverse function." Lauren responded, "I think it could be read as the inverse function." While subtle, Lauren's addition of *could be read* is telling and provides additional evidence that she was considering different possible input-output choices for the same relationship.

I was not as confident that Ryan had established such meanings. Although he was discussed different ways to interpret each graph, his initial desire was to maintain θ as the input quantity, using this variable to represent both angle measure and vertical distance. Further, his actions when describing the two graphs with consistent variable-quantity labels relied heavily on references to the analytic equations and I conjectured his reasoning might have been constrained to variables in an analytic equation. Because of this, I was unsure if he was reasoning about the graphs as representing an underlying relationship between quantities.

The polar coordinate system: Reasoning bidirectionally to interpret a relationship.

In order to explore Lauren and Ryan's reasoning, I continued to work with the pair on various activities in which they constructed and represented relationships between covarying quantities. The next series of activities involved working in the polar coordinate system (see Moore et al. (2013a, 2013b) for a more detailed description of this approach to the polar coordinate system). As an example task, we asked Ryan and Lauren to graph the equation $r = \theta^2$ in the polar coordinate system. While graphing their solution, the pair established that for equal

changes of θ, r increased at an increasing rate. In order to investigate if the pair would rely on similar conventions in the polar coordinate system as they had in the Cartesian coordinate system in the pre-interview (e.g., would interpreting $r = \theta^2$ with r as the independent variable be problematic), I asked how their description changed if they were to discuss equal changes of r. Initially, there was a long pause indicating both students were unsure how to approach the question. Then, without speaking, Lauren spontaneously drew what appeared to represent the function $y = x^2$, $x > 0$ in the Cartesian coordinate system (assuming conventional axes orientation). She marked equal changes along the vertical axis, drawing horizontal segments from the axis to the graph (*Figure 20*). Lauren then used these segments to examine the covariational relationship of the quantity on the horizontal axis with respect to the quantity on the vertical axis. When asked about her activity, Lauren indicated she was looking at equal changes in r rather than equal changes in θ.

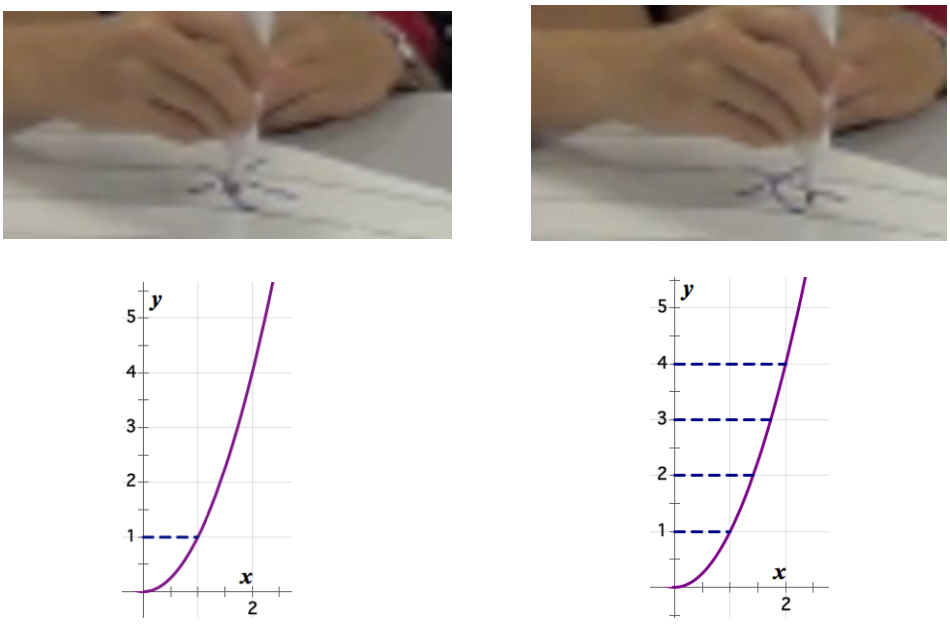


Figure 20: Lauren's spontaneous activity (top) and a recreation of her work (bottom).

From her actions, I infer that when Lauren drew $y = x^2$ in the Cartesian coordinate system she had related y with r and x with θ. Rather than solve this equation for x (or θ) and graph the

resulting equation with conventionally oriented axes to examine equal changes in the quantity represented on the horizontal axis, she was comfortable considering equal changes in the quantity represented on the vertical axis. I draw attention to her activity because it contrasts with her activity in the pre-interview in which she insisted on interpreting graphs according to the quantity represented on the horizontal axis. Instead, and consistent with her activity when addressing the Graphing Sine/Arcsine Task, her actions indicate she understood that the graph represented a relationship in which either quantity could be considered first, which I take to indicate a shift in her thinking. Further, this was the first time Lauren spontaneously represented equal changes along the vertical axis in the Cartesian coordinate system, indicating reasoning bidirectionally with respect to the axes (i.e., understanding either axis can represent the input quantity) had become an operative part of her thinking.

The Bottle Problem: Reasoning about the axes bidirectionally.

Like Lauren, Ryan exhibited a propensity to spontaneously reason bidirectionally as the teaching experiment progressed. As an example that occurred later in the semester, I provided Ryan and Lauren a picture of a bottle and asked them to create a graph representing the relationship between the height and volume of water in a bottle as water was poured into the bottle. Ryan immediately drew Cartesian coordinate axes and labeled the vertical and horizontal axes as height and volume respectively. Ryan then drew equally spaced horizontal segments perpendicular to the height axis (i.e., used the vertical axis as representing the independent quantity).

The pair's attempt to draw the graph using equal changes along the vertical axis created a perturbation for Ryan. Specifically, when describing a portion of the bottle between six and a

half and seven inches where the bottle was getting thinner, the following conversation ensued (Excerpt 4).

Excerpt 4: Lauren and Ryan's conversation surrounding concavity.

1	Lauren:	The volume is still increasing but at a slower rate [*Ryan begins sketching*
2		*in a concave down curve*]. And it's slowing down as you approach the
3		seven, this line [*indicating the segment on the picture of the bottle where*
4		*the bottle stops getting thinner, at approximately 7 inches in height*], so
5		it's curved like this [*making a concave up motion with her hand*] isn't it?
6		[*Lauren notices Ryan had drawn a concave down curve*] Wait, no I think
7		it's concave up [*again making a motion with her hand to indicate a*
8		*concave up graph*].
9	Ryan:	Up?
10	Lauren:	Yeah.
11	Ryan:	[*Ryan erases his initial curve and draws a concave up curve*] Like that?
12		Well let's think, um-
13	TP:	Now why would it be concave up?
14	Lauren:	Your volume, as you go up in height between these two heights [*pointing*
15		*to heights of 6 ½ and 7 inches on the vertical axis*]. As your height is
16		increasing [the rate of change of] your volume is decreasing slower and
17		slower.

In this interaction Lauren's description of the graph as concave up (lines 1-8, 10) was a description of a shape of the graph. Her explanation of why the graph should have this concavity relied on her image of how volume changed as she moved between two height values (lines 14-

17). Specifically, she imagined the height of the water in the bottle increasing and, noticing that the bottle was getting thinner as she moved up, she imagined the volume increasing at a decreasing rate in this section.

Lauren's description of the relationship tacitly treated height, on the vertical axis, as the input quantity (e.g., considering changes in volume as height increases). However, when Lauren described the volume as "increasing but at a slower rate", Ryan drew a concave down curve. I conjecture Ryan's concave down curve can be explained by Ryan's tendency to rely on the input quantity being represented on the horizontal axis; a concave down curve represented the quantity on the vertical axis increasing at a decreasing rate with respect to increases in the quantity on the horizontal axis. When Lauren stated she thought the graph was concave up, Ryan experienced a perturbation (lines 9, 11-12). As Lauren provided her explanation (lines 4-7), Ryan drew concave down and concave up curves on the whiteboard then marked equal changes horizontally to examine the covariational relationship represented by each curve (*Figure 21*). Ryan used these sketches to develop an understanding of the covariational relationship represented by concave up and concave down curves with equal changes along the vertical axis. For instance, after the interaction in Excerpt 4, the pair drew in a concave up curve representing the relationship between the height values of 6 ½ and 7 inches. When I asked him to justify this, Ryan reiterated that, "The volume is increasing at a slower rate," and explaining the shape of the curve Ryan motioned over the top curve recreated in *Figure 21*b and stated, "Cause the amounts of change, you can see it is…decreasing," using his index finger and thumb to indicate the amounts of change were getting smaller as he moved up the concave up graph. This reasoning led him to agree with Lauren's statement that the graph should be concave up when the bottle got thinner.

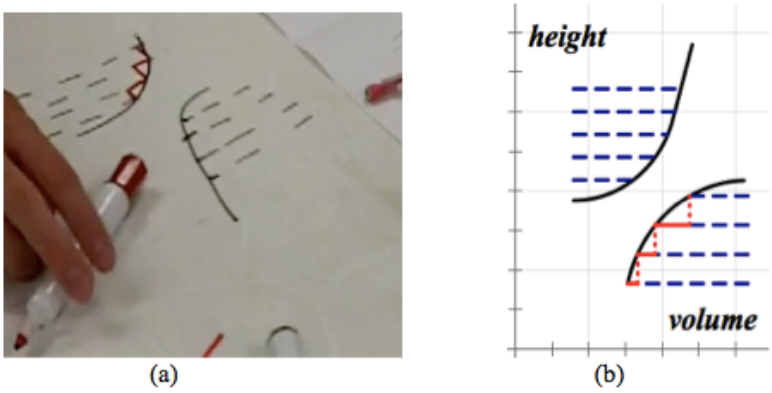


Figure 21: (a) Ryan's work when making sense of the concavity of the graph, (b) a recreation of his work.

After creating the graph for a given bottle, I gave the pair a graph with height on the vertical axis and volume on the horizontal axis (the same orientation they choose to use in the first problem) to explore how they interpreted a graph in order to sketch the cross section of a bottle. As before, they used equal changes in height to determine the shape of the graph. For instance, when describing the relationship for a concave down part of the curve Ryan stated, "So we see that, for, I guess, equal changes of height [*moving his marker along the vertical, height, axis*], volume is increasing at an increasing rate." When asked what this meant in terms of the bottle Ryan responded, "The bottle would get wider." As when constructing a graph from the bottle, the pair's activity indicated they were thinking of equal changes of height. However, overall during this prompt, Ryan and Lauren did not need to explicitly denote the covariational relationship (e.g., draw horizontal segments from the vertical axis to determine if volume was increasing at an increasing or decreasing rate for equal changes in height). Instead, Ryan and Lauren looked at the shape of the graph (i.e. concave up, concave down) and inferred what this shape meant for the bottle (i.e. getting thinner, wider). I hypothesize that the pair had abstracted shape-relationship pairs (e.g., (concave up, thinner), (concave down, wider)) during their activity drawing a graph from a given bottle and used these associations in this case due to the fact that

the axes orientation was the same between their created graph and the graph I asked them to interpret. This hypothesis is supported by their actions described below.

I provided the pair with a different graph in which height was represented on the horizontal axis. In this case, Ryan immediately marked equal changes of height along the horizontal axis, and drew vertical segments to the graph to represent volumes at each height. He also represented changes of volume at each height (his work shown in *Figure 22*) and used these to accurately draw a bottle representing the relationship between height and volume represented by this graph. I conjecture that since this graph had a novel axes orientation, the pair could not rely on the shape-relationship pairs they abstracted in their prior activity, and instead had to re-engage in representing how volume varied for equal changes in height for this axis orientation.



Figure 22: Ryan and Lauren's work when height is on the horizontal axis.

This interaction shows how Ryan and Lauren spontaneously leveraged covariational reasoning to make sense of the relationship between height and volume in a novel axis orientation. Specifically, when first creating the graph for a pictured bottle, the pair represented equal changes of height along the vertical axis to establish visual cues (e.g., a concave up or concave down curve) that corresponded to changes in the shape of the bottle (e.g., getting thinner or wider). Once they had created this graph, they worked backwards to draw a bottle by using the products of the reasoning they exhibited when solving the first problem as the axes had the same orientation. However, once the axes orientation changed, the pair returned to representing

changes of volume for equal changes of height in the graph in order to determine the bottle's shape based on the relationship between volume and height. Once these rules were established, they again had no difficulty drawing a correct bottle. As can be seen in this example, both Ryan and Lauren leveraged covariational reasoning to represent a relationship between quantities, using either axes as representing the quantity (height) they coordinate first.

I note that the pair always used height as the input quantity. I conjecture this was because of their mental images of the situation, which is critical to students' activity when modeling relationships between quantities (Moore & Carlson, 2012). When creating a graph, both Ryan and Lauren imagined breaking the bottle up into sections with equal height, then considered how much water filled each section. For instance, Ryan stated, "Cause like I'm just kinda viewing as, for each section of height, I'm looking at the volume." Similarly, Lauren expressed her preference for using height as the independent quantity stating, "I mean when you look at a bottle, like I'm not going to look right here [*pointing to the middle of the bottle*] and know how much volume it has," which indicated her preference to select known height values (e.g., the point halfway up the bottle) and consider changes in volume between these points.

This type of reasoning is a different manifestation of bidirectional reasoning. When working in the Graphing Sine/Arcsine Task, my goal was for the students to consider a relationship using either *quantity* as the input, regardless of axes orientations. For compatible reasoning in this context, the pair would have needed to consider either volume or height as the input quantity when describing the relationship. However, their mental images of the situation relied on considering how volume changed for equal changes of height. As such, in this case the pair did not necessarily reason bidirectionally with respect to the situational quantities, but they did reason bidirectionally with respect to the axes. That is, they were comfortable representing the

input quantity on both the horizontal and vertical axis. This activity demonstrates shifts in both students' meanings from their pre-interviews where each student showed a strong preference for representing the input quantity on the horizontal axis.

Implications for function meanings: The Skydiver Problem.

Both students continued to show flexibility when interpreting graphs with either axis representing the input quantity during the rest of the teaching episodes and in the (individual) post-interviews. For instance, in the *Skydiver Problem* (*Figure 23*) both Ryan and Lauren reasoned covariationally to create a graph representing the relationship between the quantities in the problem. During this problem I presented each student with a different representation of the situation than the one he or she created by drawing a graph representing (to me) the same relationship but with the opposite axes orientation. In response, Lauren quickly stated, "It's the same relationship," and described the same covariational relationship in each graph to establish the two graphs represented the same relationship as she had in the Graphing Sine/Arcsine Task. Like Lauren, Ryan reasoned covariationally to explain how the two graphs (the one he produced and the one I produced) represented the same relationship.

> **Skydiver Problem:**
> A skydiver jumps from a plane and accelerates towards the ground until he reaches a terminal velocity. He continues at this velocity for a short period of time before opening his parachute. His velocity quickly drops and he returns to the ground at a constant speed. Create a graph representing the skydivers distance from the ground and velocity.

Figure 23: The Skydiver Problem.

Of additional interest, both Ryan and Lauren addressed 'function' more generally during the *Skydiver Problem*. When I asked whether each graph represented a function, both Ryan and Lauren identified that in each graph, if distance was the input and velocity was the output the graph represented a function whereas the reverse was not true. This activity was novel relative to

their activity in the pre-interviews where each required the input to be represented on the horizontal axis. In each case, this led to a conversation surrounding the importance of function. When I asked Ryan why function was "a big deal," he responded, "Hmm [*long pause*] I don't really know why, like it's just something we've always been told, it can't be a function if it has two outputs, or it can't be a function if it doesn't pass the vertical line test... And I guess it's just because if you have a person who wants to, actually find out… like where I'm going to be at 60 mph, you want to know the exact spot [*using his marker to indicate two height values that corresponded to traveling at 60 mph*]." Ryan acknowledged that the single-valued property of function leads to less ambiguity in regards to getting a single output value for a specific input value. However, Ryan's hesitation (indicated by the long pause) indicates that although this aspect of function was emphasized in his prior experiences, he was unsure as to why this was important.

Like Ryan, Lauren was hesitant regarding the importance of function, but took her argument a step further. When I asked why it mattered if a graph represented a function, Lauren responded, "Yeah, I don't know. We've been talking about functions for a long time and I don't know... Like I understand what makes a function a function, and that changing your inputs and outputs is… what matters. But, what does it matter if it is a function or not a function. Like why is that, why are we hung up on that?" After this, and like Ryan, Lauren expressed that when a graph represents a function there is less ambiguity when inquiring about an output value for a specific input value, although she continued to question why it was important if a relationship represented a function or not, which echoes other researchers (Breidenback et al., 1992; Dandola-Depaolo, 2011; Dede & Soybas, 2011; Dubinsky & Harel, 1992; Even, 1990; Even & Bruckheimer, 1998; Leinhardt et al., 1990; Lucus, 2005; Vinner, 1983) who have noted that

students and teachers often do not understand the need for the univalence and arbitrariness properties of function.

Through these statements and the conversations surrounding them, both Lauren and Ryan exhibited an understanding of graphs as representing relationships between quantities. Although each identified relationships that were functions and those that were not, both were comfortable with a situation that did not represent a function. Although these results may seem surprising, it is important to note that the students experienced a sustained sequence of activities that focused on establishing and representing relationships between covarying quantities, without (much) regard for formal aspects of function (i.e. univalence). Each student had opportunities to establish function meanings that were quantitative in nature; their function meanings relied on interpreting a relationship using chosen input and output quantities rather than a reliance on a certain quantity being represented on the horizontal axis, as observed in their pre-interviews.

Conclusions and Implications of the Pilot Study

In any teaching experiment, the goal of researcher is to learn what changes students make and how these transformations are possible (Steffe & Thompson, 2000). The results of this study show transformations in both Ryan's and Lauren's meanings from the pre-interviews to the post-interviews. Before examining some of these shifts, I first discuss how these transformations were possible. Specifically, during the teaching experiment, I conjecture much of Ryan and Lauren's shifts can be explained by their repeatedly engaging in constructing, reasoning about, and representing relationships between quantities in various ways.

Developing bidirectional reasoning and its two manifestations.

These results shows that both Ryan and Lauren developed and spontaneously used bidirectional reasoning as a way to make sense of and represent relationships between covarying

quantities. However, I never exposed Lauren and Ryan to the term bidirectional reasoning nor made them aware my goal was for them to develop such reasoning. Rather, through repeated opportunities to construct and represent relationships, the pair began to show flexibility when creating and interpreting graphs meant to represent these relationships. Each student abstracted meanings that allowed him or her to reason bidirectionally while representing these relationships. For instance, in the latter parts of the teaching experiment, and in the post-interviews, Ryan and Lauren consistently referred to an underlying relationship when comparing graphs with different axes orientations (e.g., the Skydiver problem).

The students' bidirectional reasoning manifested itself in two ways. One way the students exhibited reasoning bidirectionally was when they reasoned about a relationship by considering either quantity as the input in a relationship, while maintaining an understanding that in either case they were still discussing the same underlying quantitative relationship. Whereas Lauren exhibited such meanings early in the teaching experiment (i.e. the Graphing Sine/Arcsine Task), both students had developed such meanings by the post-interviews (i.e. the Skydiver Problem). Another way the students engaged in bidirectional reasoning was with respect to the axes. As the teaching experiment progressed, the pair showed a propensity to spontaneously reason about the input quantity on either axis (e.g., Lauren in the polar coordinate system, both on the Bottle Problem). In these cases, that student no longer maintained the input had to be represented on the horizontal axis as both had in the pre-interview.

General implications of this study.

A combination of the two manifestations of bidirectional reasoning can help explain the shifts in Ryan and Lauren's thinking during and after the teaching experiment. Specifically, the second manifestation allows an individual to break from common school conventions when

creating or interpreting graphs. In the pre-interview both Ryan and Lauren maintained that the input quantity had to be represented on the horizontal axis. During the teaching experiment, Ryan and Lauren exhibited shifts as they began to consider representing the input on either axis as well as an awareness of representing input on the horizontal as being a choice rather than a requirement, both of which I attribute to their emerging bidirectional reasoning abilities.

Lauren and Ryan constructing certain conventions as conventions (in combination with their development of bidirectional reasoning) had implications for their function meanings. Specifically, during the pre-interviews, when making a statement about if a graph represented a function, they relied on interpreting the horizontal axis as representing the input. However, as they developed reasoning bidirectionally, the pair began to discuss the *'function-ness'* of a relationship by examining two different input-output choices.

The two types of bidirectional reasoning supported the students in considering graphs with different axes orientations and/or with different input-output choices as representing the same underlying relationship, explaining the shifts in their thinking over the course of the study. More generally, bidirectional reasoning may have important implications for the teaching and learning of function. For instance, during the Skydiver Problem although both students acknowledged that when a graph represented a function there was less ambiguity when it came to answering certain questions, neither was confident as to why this property was especially important. Rather, by establishing *'function-ness'* as a special property of certain relationships, the students had an opportunity to consider their function understandings through the lens of more general relationship understandings. This may provide evidence that a focus on relationships could prove foundational to students' function understandings.

Implications of the pilot study for the dissertation.

The results of this study had important implications for my dissertation study. First, I observed Lauren and Ryan engaging in two forms of bidirectional reasoning. In the first form of bidirectional reasoning the students constructed a relationship between quantities such that they could consider either quantity as the input (or independent) quantity (i.e. bidirectional reasoning with respect to the relationship). In the second form of bidirectional reasoning the students reasoned about only one quantity as the input quantity but understood that this quantity could be represented on either axis (i.e. bidirectional reasoning with respect to the axes). However, I did not distinguish between these two forms of bidirectional reasoning in my own thinking as I engaged Lauren and Ryan during this study. Because of this, I am unsure if the pair had in fact reasoned about the sine relationship bidirectionally or if they were just using the axes bidirectionally to describe both the sine and arcsine functions (or relations). In my dissertation study, it was important that I pay explicit attention to how the students were reasoning about the quantities to distinguish between the different types of bidirectional reasoning.

The pilot study also gave important implications for the design of tasks in my dissertation study. For instance, I observed that certain situations may have supported the students in spontaneously engaging in reasoning bidirectionally. Situations that naturally broke from conventions by changing coordinate systems (e.g., polar coordinates) and situations where students interpreted a certain quantity as naturally represented on the vertical axis (e.g., height of a bottle, height of a skydiver) supported students in using the axes in novel ways. Additionally, situations where the students had some personal experience to refer to (e.g., skydiving) supported students in reasoning about relationships that both were and were not functions, which assisted their re-consideration of their function meanings. Hence, in my dissertation study I intended to

design tasks that promote students thinking about certain mathematical ideas (e.g., conventions, function, inverse function) in novel ways so that they could possibly re-organize their meanings for these ideas.

Chapter 5: Results

Pre-Interview Design

We designed the pre-interview tasks to provide insights into the students' function and inverse function meanings as well as their quantitative and covariational reasoning (I described the pre-interview task design in Chapter 3, see Appendix A for the tasks). Rather than characterizing all of the students' activities from which I inferred insights into their meanings, I provide illustrative examples which were compatible with each student's activities across the interview.

Results from Arya's Pre-Interview

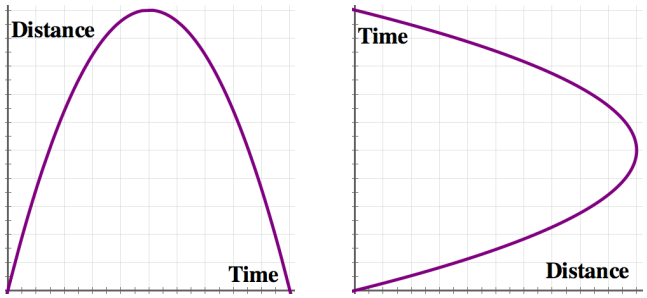
Arya's activities during the pre-interview provided insights into her meanings for multiple mathematical ideas and her reasoning in a variety of contexts. I inferred function and inverse function meanings she maintained including the extent to which these meanings relied on practices common to school mathematics. Additionally, Arya reasoned quantitatively and covariationally during certain prompts, which provided me insights into her reasoning in this regard entering the teaching experiment.

Arya's reasoning about graphs as representing functions.

Arya reasoned that the horizontal axis represented the input quantity, even with tasks that we designed to explicitly identify the vertical axis as representing the input quantity (e.g., Problems 9, 12, 13, 15). For example, when addressing Problem 9, the *Double Parabolas Problem* (recreated in *Figure 24*), Arya started by saying, "I don't like this graph [*pointing to Figure 24*b *throughout*]. Because it doesn't make much sense. You can't change time [*motioning along the time axis*] based on distance [*motioning along the distances axis*], right? … So that's hard, but this is what it's saying. As you change your distance, your, the time is moving forward." Arya

later stated that the two graphs were showing the same (pointwise) relationship, indicating she was reasoning quantitatively about coordinates as representing time and distance pairs. I asked Arya if each graph represented a function and she responded that *Figure 24*a represented a function while *Figure 24*b did not. I asked her if this related to her dislike of *Figure 24*b and she responded, "I mean this [*pointing to Figure 24*b] isn't a function, and those things are ugly to work with anyways… you can't increase, you can't change time. So increasing distance really doesn't change the time. So the graph [*Figure 24*b] doesn't make sense really." Hypothesizing Arya was perturbed by her interpretation of one graph representing a function and the other not after first claiming that the graphs represented the same relationship, I asked, "Do they still represent the same relationship?" Arya responded, "They represent the same relationship between time and distance I guess… It's so ugly. I don't work much with non-functions… I don't really know anything about things that aren't functions."

Although Arya understood the two graphs represented the same (pointwise) relationship, she interpreted the two relationships differently based on her restricting the input quantity to the horizontal axis. Further, in this case, Arya imposed a 'causal' quality to the quantity on the horizontal axis; she conceived that the quantity represented on the horizontal axis caused changes to the quantity represented on the vertical axis. In the case of *Figure 24*b, this conflicted with how she understood distance and time to be related experientially; to her, time does not change "based on" distance, implying the relationship had a casual quality differing from the relationship she conceived with respect to the graph.



Problem: Are these graphs the same or different? Explain.

Figure 24: The Double Parabolas Problem.

As another example in which Arya maintained that the horizontal axis represented the input quantity, despite our posing tasks in ways that we intended to define the vertical axis as representing a function's input, consider her activity when addressing a hypothetical student's claim that "x is a function of y" with respect to the graph in *Figure 25*a (Appendix A, Problem 12), Arya claimed, "x being a function of something y… We think y changes and that affects x, right? … How we would graph what she's saying is, if x was a function of y, we'd put x on this [*vertical*] axis and y on this [*horizontal*] axis." Arya then drew another graph (*Figure 25*b) and claimed the student was correct in saying "x is a function of y" for her drawn graph, but not for the given graph. Here (and in other tasks) Arya's meanings for functions and their graphs required the input quantity to be represented on the horizontal axis and she often maintained that this quantity "changes" or "affects" the quantity on the vertical axis. Further, and related to requiring the input on the horizontal axis, Arya often referred to the graph (or "it") as being or not being a function (e.g., "I mean this [*pointing to Figure 24*b] isn't a function"); consistent with P.W. Thompson's (1994b) observation that his students "were oriented to 'the function' (often meaning the visual object called its graph)," (p. 268), Arya treated graphs as a pictorial object with an associated property of function-ness, rather than considering one relationship that

can be conceived of in two different ways in each graph and considering whether each of these did or did not represent a function.

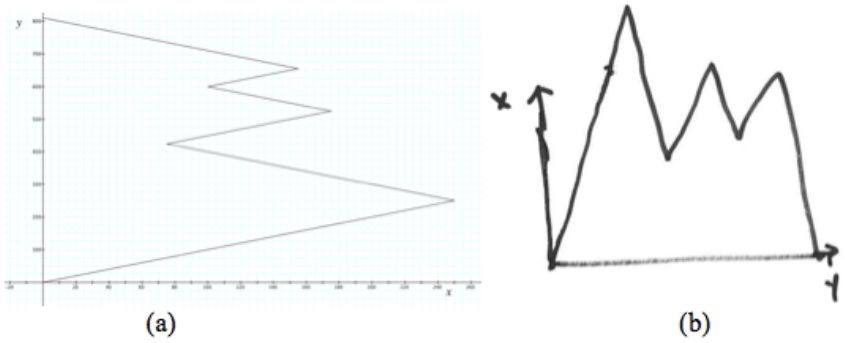


Figure 25: (a) The graph given and (b) Arya's representation of how the graph would need to be oriented to represent a function.

Arya's inverse function meanings conflicting with her reasoning quantitatively.

Arya primarily understood inverse function in terms of 'switching' techniques. In graphical situations, she switched the coordinate ordering; the point (a, b) in the graph representing the original function became the point (b, a) in the graph representing the inverse function (e.g., Problems 1a-d, 6). In analytical situations, Arya switched variable letters and (typically) solved for the dependent variable of the original equation (as she conceived it) to represent the inverse function; the equation $y = mx + b$ became $x = my + b$ and then $y = (x - b)/m$ (e.g., Problems 2, 5, 6, 11, 14).

Arya used her 'switching' techniques as she addressed the contextualized Celsius-Fahrenheit problems and she experienced a complexity in her thinking as she attempted to relate her switching techniques to a contextual meaning for her constructed inverse functions. For instance, for the analytic rule (Appendix A, Problem 5), Arya determined the inverse of $C(F) = (5/9)(F - 32)$ to be $C^{-1}(F) = (9/5)F + 32$ by switching variables. After this, Arya indicated there was a second way to "write" the inverse rule, saying, "This is how to find Celsius [*pointing to* $C(F) =$

(5/9)(F – 32)], so the inverse would be, how to find Fahrenheit." I asked her, "Would you write an equation [to represent that relationship]? Or like does this equation [*pointing to* $C^{-1}(F)$ = (9/5)F + 32] represent that?" Arya examined her inverse equation, $C^{-1}(F) = (9/5)F + 32$, and responded, "No, this equation is, um, is close I guess, but it's, what? [*pause*] I messed myself up. So, I don't know… I mean I know that's what it does [*implying the inverse function should determine degrees Fahrenheit*]. Just looking at the equation [$C^{-1}(F) = (9/5)F + 32$], I don't know how that would work." Arya then reiterated that if the original rule inputted degrees Fahrenheit and outputted degrees Celsius, the inverse rule inputted degrees Celsius and outputted degrees Fahrenheit, although she argued that this was not the relationship represented by $C^{-1}(F) = (9/5)F + 32$ as she understood this rule as inputting degrees Fahrenheit, leaving her perturbed as to how to interpret the contextual meaning of the inverse function.

Arya experienced a similar complexity when addressing the Celsius-Fahrenheit graph (Appendix A, Problem 6). She switched the coordinate points to determine the graph representing the inverse function of the given graphed function. She approximated the point (20, 70) on the line representing the function and switched coordinate pairs to obtain the point (70, 20) on the line representing the inverse function. I asked what each of these points meant in context and Arya stated that (20, 70) represented, "twenty degrees Celsius is seventy degrees Fahrenheit," and (70, 20) represented, "seventy degrees Fahrenheit is twenty degrees Celsius." I asked Arya if both statements were true in context and she responded, "Yeah, if they're equal in one direction, they're equal in the other direction." I then pointed to the axes labels on the graph and asked, "And then what about these labels here, C and F?" Arya responded, "Oh gosh… So seventy degrees. Aw! That's dumb. That's not true. But! [*pause*] So I've been doing this wrong the whole time?" Arya then identified that the 70 in (70, 20) represented 70 degrees Celsius with

the axes as labeled, adding, “That doesn’t make sense this way. Seventy degrees Celsius… It’s not twenty degrees Fahrenheit… So those labels don’t really make sense.” I asked, “Could you change [the axes labels] so they do make sense for the inverse?” Arya responded by labeling the horizontal axis *F* and the vertical axis *C* (see *Figure 26* for her work) but immediately noted with the new axes labels, “The graphs would be telling you the same thing.” I asked what her drawn line representing the inverse function represented with the original axes and Arya responded, “It’s telling you the opposite… seventy degrees Celsius would be twenty degrees Fahrenheit which is the opposite… But [the relationship is] actually exactly opposite… and that’s what we want on the inverse.”

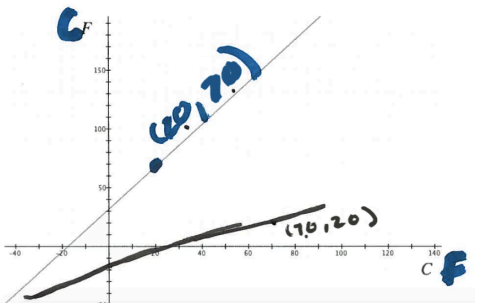


Figure 26: Arya’s work on the Celsius-Fahrenheit graph.

When addressing both the contextualized graphing and analytic situations, Arya experienced perturbations as her quantitative meanings (i.e. maintaining the relationship between degrees Celsius and degrees Fahrenheit) was incompatible with her switching techniques (i.e. switching the variables and switching coordinate pairs while maintaining the quantitative referent for each axis). In both cases, she decided the latter activity was correct, concluding the inverse functions represented the “opposite” relationship when compared to the original function.

Because I conjectured Arya was reasoning about inverse functions in these two distinct ways, I returned to these prompts after she addressed all other prompts to further explore her thinking. Arya re-explained that to maintain the relationship between quantities for her graph of the

inverse function, she represented degrees Fahrenheit (F) on the horizontal axis and degrees Celsius (C) on the vertical axis (see blue in *Figure 26*). I asked her how this related to her analytic rule, and she wrote the rule as $F(C) = (9/5)C + 32$, which differed from the rule she had previously determined in that she did not switch the variables. I posed that a hypothetical student argued that $F(C) = (9/5)C + 32$ represented the same relationship as $C(F) = (5/9)(F - 32)$ and that Arya had only solved for F. Arya responded, "Right, you're just saying, but still in this equation [*pointing to* $C(F) = (5/9)(F - 32)$, *pausing*]. But you flip 'em. What'd I do wrong! I feel like I did something wrong again!" For the next thirteen minutes, Arya experienced a sustained state of perturbation as she attempted to relate her quantitative meanings with her switching variables technique. She was unsuccessful, and concluded, "Just solving for F [*pointing to the equation* $F(C) = (9/5)C + 32$] is not the same thing as finding the inverse. You do have to switch the input and output. Like that's what an inverse is."

In both the graphical and analytical Celsius/Fahrenheit problems, Arya considered two inverse function understandings: (1) maintaining the relationship between the quantities, which was manifested by switching the quantitative referents of the axes in the graphical prompt and by solving for the previously dependent quantity (F) in the analytic prompt; and (2) switching variables or switching coordinate pairs to obtain what she conceived as a different relationship. Occasionally, Arya referred to the activities involved in her switching techniques as switching inputs and outputs. I conjecture this was due to her understanding that the input was always represented by x (on the horizontal axis) and the output was always represented by y (on the vertical axis). This conjecture is supported by the fact that Arya did not consider $F(C) = (9/5)\ C + 32$ as being the result of switching the input and output of $C(F) = (5/9)(F - 32)$. Instead, Arya

understood switching input and output to mean switch the variables or coordinate ordering; Arya was unsure of the meaning of the result of this activity.

Arya engaging in quantitative and covariational reasoning.

Arya exhibited some activities that I took to indicate she had the capacity to reason about graphs as representing relationships between covarying quantities. For example when addressing the Bottle Problem (Appendix A, Problem 8), she initially conjectured, "These are the same graphs," later adding, "It's not the same graph, obviously they look different, but it's the same function graphed two different ways." I asked, "So are [the graphs] representing the same glass?" Arya responded, "Yeah. Right? 'Cause this is… as height increases [*motioning over horizontal axis in top graph*], volume increases [*motioning over vertical axis in top graph*] and this is as height increases [*motioning over vertical axis in bottom graph*]… the volumes still increases [*motioning over horizontal axis in bottom graph*]" (MA2). Arya compared the graphs by first considering height in each graph, with height represented on the vertical axis in the second graph, which indicated the potential of her reasoning bidirectionally with respect to axes. Further, her language describing the relationship (e.g., "As height increases…") indicated she might be imagining change happening smoothly. I took her activity this case, and in others (e.g., the Double Parabolas problem), to indicate Arya might engage in smooth reasoning.

Summary of Arya's pre-interview.

Arya maintained meanings for functions and their graphs that relied on the horizontal axis as representing the input quantity and that graphs (as pictorial objects) were or were not functions. Her inverse function meanings were rooted in switching variables or the ordering of coordinate pairs, and she was left unsure what the result of this procedure meant in a contextualized

situation. Finally, Arya reasoned about graphs as representing quantitative and covariational relationships when she was not concerned with formal ideas of function or inverse function.

Results from Katlyn's Pre-Interview

Katlyn exhibited activities similar to Arya. I briefly summarize these activities, focusing this section more on Katlyn's actions that differed from those of Arya. As one example, Katlyn made (functional) accommodations to her function meanings *during* the interview to resolve experienced perturbations.

Katlyn's reasoning about graphs as representing functions.

Like Arya, throughout most of the interview, Katlyn maintained that the horizontal axis represented the input quantity (e.g., Problems 7, 9, 12, 13, 15). For example, when I told Katlyn that a student claimed the graph *Figure 25*a (Appendix A, Problem 12) was a function because "*x* is a function of *y*", Katlyn stated, "It's backwards. Because *x* is a function of *y*… I think of that as the graph like kinda this way [*turns paper 90-degrees counterclockwise*]. So like *y*… if this is our [*motioning over the horizontal y-axis on the turned paper*] horizontal that's true. Because for every *y* there is one unique *x* but for every *x* there is not one unique *y*." When I asked if the student was correct in making his statement about the *given* graph, Katlyn responded, "No, she's incorrect because it's like backwards… that's not what we're looking at here… If that was correct, then the *x* would be… here [*indicating on the vertical axis*] and the *y* would be here [*indicating on the horizontal axis*]." Katlyn then drew the graph in *Figure 27*, stating the student's statement was correct for that graph but not for the given graph. To Katlyn, a statement about a graph representing a function required (possibly unconsciously) that the input quantity was represented on the horizontal axis.

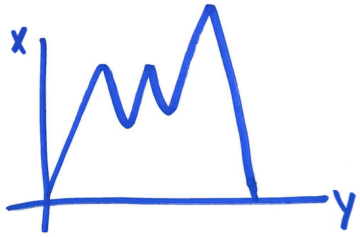


Figure 27: Katlyn's representation of x as a function of y.

Katlyn's inverse meanings.

I started the interview with a vague question ("When I bring up the term inverse, what comes to mind?") to see how Katlyn responded when I raised the term "inverse." She described "inverse" as the reciprocal and wrote $f^{-1}(x) = 1/(f(x))$. I then provided her the first prompt (Appendix A, Problem 1a) and Katlyn drew a line that was a reflection of the given line over the x-axis, indicating that her drawn line went in the "opposite direction." Katlyn created graphs for the second and third prompts that were also reflections over the x-axis, indicating the graphs went in the "opposite direction." However, Katlyn took a different approach to the fourth prompt. She wrote an expression for the given line, $x + 10$, and then determined the inverse equation, $f^{-1}(x) = 1/(x + 10)$. She made a table and plotted the points she found in the table, connecting them with a straight line (*Figure 28*). Although Katlyn used her reciprocal meaning to determine the inverse rule, I infer from Katlyn's graphing $f^{-1}(x) = 1/(x + 10)$ as a line that she anticipated that the graphs of a function and its inverse have a similar shape (e.g., the inverse of a line is a line, the inverse of a curve is a curve). Further she did not perceive any inconsistency between the inverse rule she obtained and the inverse graph she drew, nor did she conceive an inconsistency between her approach to this graph and the other graphs.

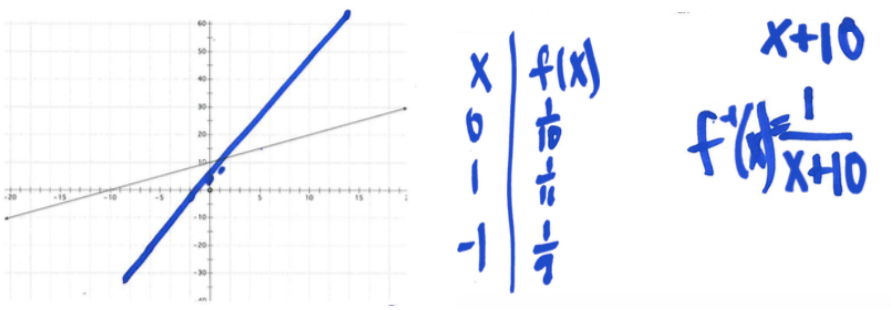


Figure 28: Katlyn's work for Problem 1(d).

Katlyn used reciprocals (e.g., $f^{-1}(x) = 1/(f(x))$ to determine inverse functions for analytically defined functions in Problems 2, 3, 4 (Appendix A). After reading the first Celsius-Fahrenheit problem (Appendix A, Problem 5), Katlyn said, "Oh crap… That's where you replace all the *C* of *F*s with *F* and then *F*s with *C* of *F*s." She then produced the inverse rule $C^{-1}(F) = (9/5)F + 32$, explaining, "I don't remember where in my math brain I'm thinking of this and it could be totally wrong, but I remember switching *x* and *y* at some point, but I don't… remember what it was for, but I feel like in this problem that's what it calls for… When I saw the two different, like Celsius and Fahrenheit, I feel like I've done that before, and I feel like we switched them." Katlyn then wrote $C^{-1}(F) = 1/((5/9)F - (160/9))$, to check if this produced the same result she obtained when she switched variables. She identified the two rules were not the same and was left unsure which equation was correct.

With Katlyn unsure how to proceed, I asked her to address to the graphical Celsius-Fahrenheit prompt. Like Arya, Katlyn drew a graph by switching the ordering of coordinate pairs. She identified that the point (10, 50) on the given line represented that 10 degrees Fahrenheit corresponded to 50 degrees Celsius. Interpreting the point (50, 10) on her line representing the inverse, Katlyn said, "My whole reasoning in this entire process right here is switching *x* and *y*, is switching *C* and *F* which is how I came up with this graph. So I don't necessarily know what… the new graph would stand for." Once Katlyn had recalled switching

techniques to define inverse functions, Katlyn maintained this activity as critical as she continued to use procedures related to switching variables or coordinate pairs in both graphing and analytic prompts. Further, and like Arya, Katlyn was uncertain how to interpret the results of her switching techniques in the Celsius-Fahrenheit problems, indicating she was not attentive to the quantities in the context when she engaged in switching techniques.

Katlyn's quantitative and covariational reasoning.

Katlyn exhibited actions during the pre-interview that I took to indicate she reasoned quantitatively and covariationally. For example, when addressing the student's work in Problem 13(b) (Appendix A), Katlyn recognized that the student had constructed the correct pointwise relationship between x and y. Although she first claimed the student had correctly graphed $x = 1/3\ y$, relying on the horizontal axis as representing the input, she eventually concluded that the student had correctly graphed the relationship defined by $y = 3x$.[28]

As another example (that had important implications for her activity later in the teaching experiment), Katlyn drew on a quantitative meaning for inverse sine when addressing Problem 15 (Appendix A). Katlyn understood the inverse sine function as inputting a sine value and producing "the angle whose sine is" that value. Katlyn drew on this understanding when interpreting the work of a student who represented the input of the inverse sine function on the vertical axis (Problem 15c) (Excerpt 5).

Excerpt 5: Katlyn addressing Problem 15c.

1	Katlyn:	It's correct, but it's not, you know it's like not common. I always, I
2		always see [the input] there [*pointing to the horizontal axis*]. So I'm like

[28] I note that when asked to assign the student a grade, she deducted points for the unconventional nature of his graph.

really trying to step out of that and see it here [*pointing to the vertical axis*], like the input there… I guess this is what this graph is saying. The angle whose sine is one is pi over two, which is right [*motioning over the points she is referring to*]. Angle whose sine is negative one, negative pi over two, which I think is right. But is that the inverse?

After this, Katlyn indicated that the student was representing $\sin^{-1}(y) = x$, and that he had not switched variables from the graph of the sine function in Problem 15(a) that was represented by $y = \sin(x)$. Unable to relate her quantitative inverse sine meanings with her switching variables technique, Katlyn was unsure as to the correctness of the hypothetical student's solution.

In later situations, Katlyn exhibited activities that I took to indicate she considered sine inverse as distinct from the other inverse function situations. For instance, after addressing Problem 15 (and all other prompts), I returned to the Celsius-Fahrenheit problems to examine if she would interpret this problem in terms of the quantities. Katlyn stated she had no expectation for what the inverse of the relationship between degrees Celsius and degrees Fahrenheit was meant to represent in context, adding, "I guess more so with the, with the sine and cosine… Just because like, like sine inverse of something you can always apply it to the angle whose sine is. It has an application to me."

In addition to reasoning quantitatively, Katlyn also exhibited activity that I took to indicate she was imagining quantitates varying smoothly (e.g., Problem 1c, 8, 13). For instance, in Problem 8 she described the bottle represented by the top graph saying, "The volume is slower to increase at the beginning then gets really fast at the end," and she described the relationship represented by the bottom graph saying, "The height increases really rapidly." Katlyn's words indicate that she was imagining the quantities changing actively (e.g., "increases" and "gets

really fast") which I take to indicate Katlyn might have conceived of these graphs by leveraging smooth images of change.

Katlyn's (functional) accommodation in the Double Parabolas Problem.

When initially addressing the Double Parabolas Problem, Katlyn described that in *Figure 24*a, each time corresponded to one distance, and in *Figure 24*b (some) distances corresponded to multiple times. However, she was uncertain if the two graphs were "the same," concluding, "I don't think that this [*Figure 24*b] is a function. But then again, then they can't be the same because this [*Figure 24*a] is a function." At this point, Katlyn was restricted to considering a function's input on the horizontal axis (as described earlier) and described the graph (i.e., "this" as a pictorial object) as being or not being a function.

I returned to this problem after she addressed all other prompts, conjecturing she was close to conceiving of a way to consider the two graphs as representing the same relationship. After Katlyn described that in *Figure 24*b there was not a unique time for each distance, I asked if that statement was true in *Figure 24*a. Katlyn agreed, and I pointed to *Figure 24*b and questioned, "If for each time there is a unique distance, couldn't we say that that's a function?" Katlyn responded, "I don't think so because distance is, it's not like specified which one is the input. Like [if] distance was the input, then no." I asked, "Could you input a distance of one [in *Figure 24*a]?" Katlyn responded, "If I input a distance one up there, then you would get two times again… So like distance as the input, makes [*Figure 24*a] not a function." Katlyn continued, "That's what I'm going with, that's what I think. So like inputting distance [*33-second pause, looking between the two graphs*]. So in order to make [*Figure 24*b] a function, time would have to be the input." As the conversation continued, Katlyn re-explained two different ways to interpret each graph (Excerpt 6).

Excerpt 6: Katlyn describing different ways to interpret each graph.

1	Katlyn:	When you input time this is a function [*motioning over the vertical axis*
2		*in Figure 24*b]. But when you input distance [*motioning over horizontal*
3		*axis in Figure 24*b], it's not a function. So like I automatically just
4		inputted distance, assumed this was the input, and I think that's not a
5		function.
6	TP:	Okay.
7	Katlyn:	Yeah, but when you input time it is a function [*pointing to vertical axis in*
8		*Figure 24*b]. Yes.
9	TP:	Can you go through the same thing up here [*pointing to Figure 24*a]?
10	Katlyn:	Uh-huh, um so when I input time [*pointing to horizontal axis in Figure*
11		*24*b], it is a function. But when I input distance, it is not a function.
12	TP:	So that's true in both?
13	Katlyn:	Yes.
14	TP:	Okay so is this graph a function [*Figure 24*a] and this graphs not a
15		function [*Figure 24*b]?
16	Katlyn:	What's the input?
17	TP:	So you're saying what's the input?
18	Katlyn:	Right, that's what I want to know.
19	TP:	Okay.
20	Katlyn:	Cause I feel like from there, then you can determine it.

Katlyn initially conceived that the quantity represented on the horizontal axis, distance, was the input (lines 3-5) to conclude, " [the graph]'s not a function." In this excerpt, she considered a

novel way to interpret *Figure 24*b with time, on the vertical axis, as the input (lines 1-2, 6-7) to conclude, "[The graph] is a function." Further, she described *Figure 24*a as being or not being a function depending on the input quantity (lines 9-10), concluding that defining the input quantity was the critical when determining if a graph was a function (lines 16, 18, 20). I inferred Katlyn made an (functional) accommodation to her function meanings as she emphasized the importance of defining a quantity to represent the input to determine if the graph was a function. Although Katlyn considered different possible input and output quantities on both the horizontal and vertical axis, she continued to refer to each graph as being or not being a function; I was unsure the extent to which this activity influenced Katlyn's understanding that a graph *is* a function versus a graph *representing* a function depending on the quantity she considered as representing the input.

Summary of Katlyn's pre-interview.

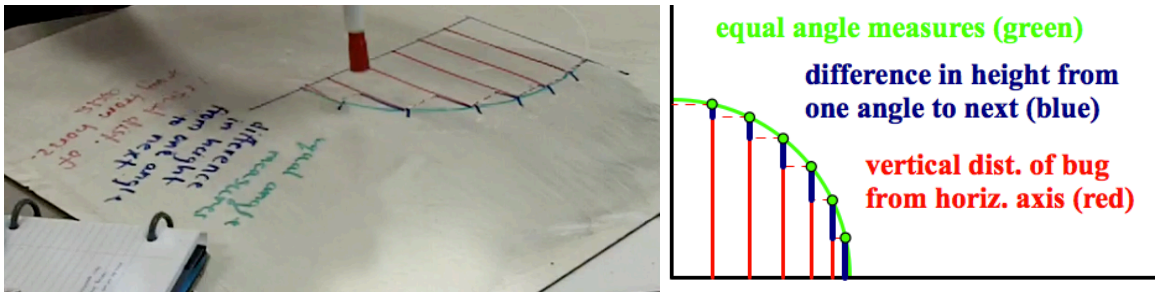
Katlyn, like Arya, maintained meanings for function and inverse function that were based in procedures or restricted by practices common to school mathematics; she treated graphs essentially as pictorial objects that were or were not functions. Katlyn also reasoned quantitatively and covariationally when she was not concerned with formal ideas of function and inverse function. Unlike Arya, Katlyn maintained a quantitative meaning for inverse sine but was unsure how this meaning related to her switching techniques. Additionally, Katlyn made an (at least functional) accommodation at the end of the interview regarding the importance of identifying input and output quantities when determining if a graph "is" or "is not a function," which I conjectured had the potential to support her in re-constructing her function and inverse function meanings such that they were less reliant on treating the graph as a pictorial object during the teaching experiment.

Sine Episodes: Pertinent Results

Fourteen days after I conducted the pre-interviews, I began the teaching experiment. During the first four episodes, Arya and Katlyn engaged in modeling covarying quantities in a circular motion context to construct the sine (and later cosine) relationship. Specifically, the pair constructed and graphed the relationship between the vertical distance above the horizontal diameter (a directed distance) and the angle measure swept out by a bug as it revolved on a fan starting at the 3 o'clock position (see Appendix C for the Bug on a Fan Problem). As their activity was consistent with students' activities reported elsewhere (Moore, 2012, 2014a), I do not describe their activities in depth. Instead, I characterize the meanings the students developed that were important to their later activity.

At the beginning of the second episode,[29] Katlyn and Arya set out to systematically represent the relationship between the bugs vertical distance above the horizontal diameter and its angle measure swept out. They first represented the relationship on a drawing of the fan, concentrating on the bug's first quarter rotation. They drew a quarter circle and marked seven "equal angle measures" in green. They then drew in red segments representing "vertical distance…of the bug… at that angle," and blue segments representing the "difference in height moving from one angle to the next," providing a written description of each quantity on their board (*Figure 29*).

[29] In the first session we introduced terminology related to the covariational framework presented by Carlson et al. (2002) including "direction of change" (MA2) and "amount of change" (MA3) after the students' initial attempts to represent the relationship between angle measure and vertical distance.



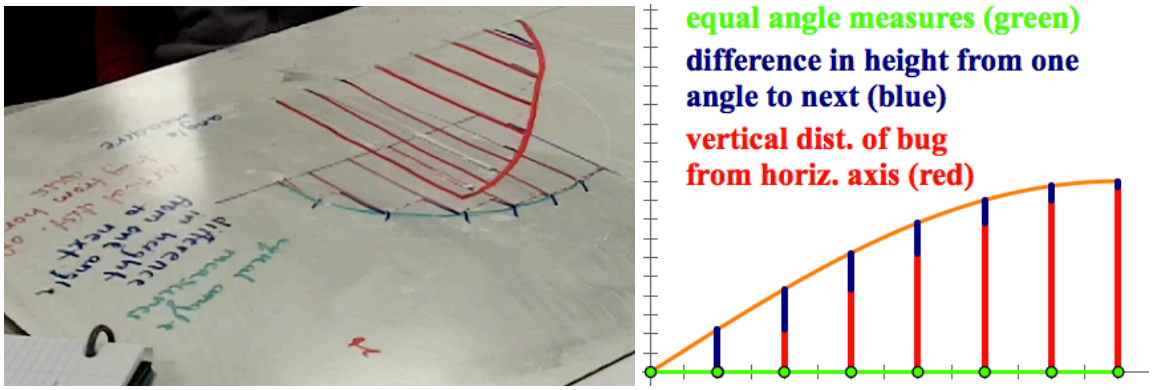
(a) (b)

Figure 29: (a) The pair's work and (b) a recreation of their work.

After describing and representing the quantities in a picture of the situation, the pair set out to graph the relationship between angle measure and height of the bug. Katlyn suggested they use green when labeling tick marks on the horizontal axis as this axis represented angle measures. Attempting to obtain numerical values to put on the horizontal axis, Arya questioned, "What are our angles?" Katlyn responded, "Well, there are seven." Arya then labeled seven tick marks along the horizontal axis (*Figure 30*) and the pair worked non-numerically for the remainder of their activity.

Because the students spontaneously color coordinated equal changes in angle measures in their graph and their picture of the situation, I asked them to describe how the other quantities (e.g., vertical distance, changes of height) they identified in the situation translated to their graph. Arya drew a red vertical segment at the first green tick mark and said, "That would be our… first one [*pointing to the first red segment from the 3 o'clock position in the drawing of the situation*]." The pair continued drawing red segments at each tick mark, with each segment increasing in length but by less each time, because, as Katlyn noted, "The difference in height is slowly decreasing as your angle measure increases [*sweeping over the quarter circle*]" (MA3-4). After drawing these segments in the coordinate system, the pair identified how the blue

"difference in height" segments translated from the situation to the graph, with their final representation in *Figure 30*a.



(a) (b)

Figure 30: (a) The pairs graph and (b) a recreation of their graph.

I then asked the pair if anything in the picture of the situation (e.g., a circle) corresponded to the trace of their graph (e.g., the curve) (Excerpt 7).

Excerpt 7: The pair describing what the trace of the curve represents.

1	TP:	What about this, that you kind of drew in red here [*motioning over the*
2		*curve in red in Figure 30(a) and recreated in orange in Figure 30(b)*].
3		That curve. Does that correspond to [anything in the picture of the
4		situation]?
5	Arya:	No, it's the relationship.
6	Katlyn:	Relationship.
7	TP:	Okay.
8	Katlyn:	Yeah.
9	Arya:	So this is something new we're graphing [*motioning over the curve*]-
10	Katlyn:	We just didn't have a fourth color.

11	TP:	Okay.
12	Arya:	How those two relate to each other.

The students' activities here (and their activities moving forward) indicated that they understood that the curve represented a relationship composed of two projected magnitudes, with these magnitudes contextually representing angle measures and vertical distances above the horizontal diameter (lines 4-5, 8-12). The pair recognized the trace of this curve as representable by the sine function (i.e. $y = \sin(\theta)$).

The Graphing Sine/Arcsine Task

Two days after the last sine episode, I began a series of five teaching episodes concerning the arcsine function or relationship (four in class, one out of class). During the first teaching episode, the pair used the arcsine (or inverse sine,[30] the terminology was used interchangeably) function to "cancel out" the sine function to determine specific angle measures for a given vertical distance above the horizontal diameter (e.g., if $\sin(A) = 0.9$ then they "took the inverse sine of both sides" to obtain $A = \arcsin(0.9)$). The pair had not discussed the arcsine function in terms of covariation. In the remaining four teaching episodes, I asked the students to address the *Graphing Sine/Arcsine Task* (*Figure 31*) for the sine and arcsine functions, as well as other decontextualized and contextualized functions.

[30] We chose to use arcsine as this word has the potential to help the students understand that the function's output is the arc length in radians for a given vertical distance above the horizontal diameter.

Graph 1:	Create a graph of the sine function with a domain of all real numbers. What is the range?
Graph 2:	Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function. What is the domain and range of this function?
Prompt 3:	Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not?

Figure 31: The Graphing Sine/Arcsine Task.

Task design.

We designed the Graphing the Sine/Arcsine Task with the intention of supporting the students in reasoning about the vertical axis as representing the input quantity and relating such reasoning to the particular functions (or relationships) under consideration. We asked the students to create a graph (Graph 1) of the sine function as well as a graph (Graph 2) of the arcsine function, explicitly requesting the pair use covariational talk when creating and justifying the arcsine graph as one of our goals was for the students to leverage reasoning covariationally to understand the conventional sine and arcsine graphs as representing an invariant relationship between angle measure and vertical distance. In the third prompt, we asked the students to alter Graph 1 so that it represented the arcsine function (or relationship). I hypothesized that if the students understood their previously constructed graphs as representing an invariant relationship between quantities, then they might reason about Graph 1 as representing both the sine and arcsine functions given appropriate angle measure restrictions. I also conjectured that the third prompt might generate a perturbation, as reasoning about the graphs as representing an invariant relationship would require the students to consider representing the input quantity on the vertical axis.

The Graphing Sine/Arcsine Task Day One: Constructing Inverse Function Meanings via Quantitative Reasoning

For the first prompt, the pair constructed the sine graph as they had in previous episodes, describing the covariational relationship from π to $3\pi/2$ accurately when I prompted them to do so. The pair then drew a set of axes identifying these were for Graph 2. I asked the pair how they planned to proceed. Consistent with her activity in the pre-interview, Arya responded, "Just switch the points. Is that okay? Switch the input and output for the points and just graph those points." Arya then described the point ($\pi/2$, 1) in Graph 1 being mapped to (1, $\pi/2$) in Graph 2. I asked Katlyn, "What do you think about that strategy?" The following conversation ensued (Excerpt 8).

Excerpt 8: Arya and Katlyn's plan to draw the arcsine function.

1	Katlyn:	I'm just seeing a bunch of vertical lines. Like I'm just seeing it all go
2		[*motioning vertically over the axes of Graph 2*], like 'cause they're all
3		going to be one or zero, or negative one. I don't know. I'm trying to think
4		about it, maybe I'm just thinking about it wrong. [*TP starts to interject,*
5		*she continues*] Okay, so what like, what she's saying is like, this is one
6		[*labeling 1 on the horizontal axis in Graph 2*], then this is pi over two
7		[*labeling $\pi/2$ on the vertical axis in Graph 2*]-
8	Arya:	And that's just one of our points.
9	Katlyn:	But it's going to be, but then like, what about not pi over, yeah three pi
10		over two? 'Cause this would be there like that [*marks the points* (*1, $\pi/2$*)
11		(*1, $3\pi/2$*)]. Is that right? Then pi is going to be in the middle, then that's
12		going to be here [*marking a point at* (π, *0*)].

13	TP:	Okay.
14	Arya:	So why not? Is it supposed to look? I don't know.
15	Katlyn:	Cause it's not-
16	TP:	So let's create that graph-
17	Katlyn:	A function, it's not a function.

Katlyn questioned Arya's technique for determining the inverse as Katlyn imagined that the graph resulting from switching all abscissa and ordinate values in Graph 1 resulted in a graph such that each abscissa has multiple associated ordinates and therefore did not represent a function (lines 1-3, 17). Katlyn's response is reasonable as the prompt asked the pair to draw the arcsine (or inverse sine) *function* and her imagined graph did not correctly address the question. Whereas Katlyn's activity indicated she envisioned a graph that resulted from switching the abscissa and ordinate values of all coordinate pairs, Arya had not yet constructed an image of the completed graph (line 14).

Conjecturing Katlyn's imagined graph not representing a function might be roadblock for constructing and discussing the graph that resulted from switching the coordinate values of every point in Graph 1, I suggested, "Let's not worry about that for now." Katlyn acquiesced and drew in the curve for Graph 2 (see *Figure 32*). Arya then stated, "I feel like this is wrong too." Once Arya saw the completed graph, she expressed an objection similar to Katlyn's; the graph that resulted from switching the coordinate values of each coordinate of Graph 1 did not result in a graph that the students conceived as representing a function (e.g., the students maintained that the horizontal axis represents the input quantity).

With both students claiming that their graph was not correct, I asked them to relate Graph 1 and Graph 2. Katlyn responded, "I guess like what it came to was… switch the input and the

output. So here [*pointing to Graph 1*] we, if we inputted pi [*pointing to π on horizontal axis*], we got out zero [*tapping marker on the point* (π, 0)]. So here [*pointing to Graph 2*] we input zero [*tapping the origin*] and get out pi [*moving the marker up to the point* (0, π)]." Katlyn described a way to relate the two graphs by focusing pointwise on the coordinate pairs in each graph, interpreting the horizontal axis as representing the input quantity for each graph. Notably, Katlyn considered switching coordinate values without addressing the other possible outputs for an input of 1 in Graph 2.

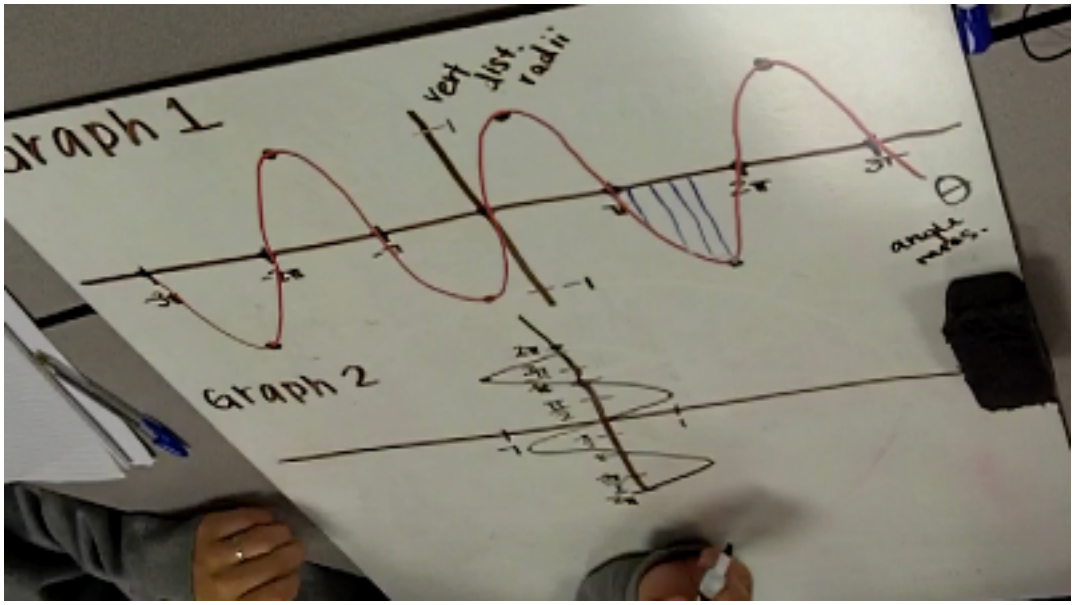


Figure 32: The pairs arcsine graph.

To this point, each student's activity was consistent with her pre-interview with respect to understanding the graphs as representing the same pointwise relationship, implicitly assuming the horizontal axis represented the input quantity, and determining the inverse of a graphed function by switching coordinate pairs. In order to examine if (and if so, how) the students associated their activities to the quantities related by the sine function, I asked the pair to indicate quantitative referents for their axes in Graph 2. Katlyn responded, "I don't even know what we're talking, like I'm just switching the two. So I mean. I guess this… is theta [*labeling the*

vertical axis of Graph 2 θ] and then this is… vertical distance [*labeling the horizontal axis of Graph 2 vert. distance, Figure 33*].” Consistent with her switching techniques in the pre-interview, I infer from Katlyn’s initial uncertainty that, until my question, her reasoning was focused on switching coordinate pairs without a concern for the quantities each coordinate was representing. When I asked the pair to label the axes in Graph 2, the quantities represented by each coordinate became relevant. Katlyn understood the points in Graph 1 represented (*angle measure*, *vertical distance*) and when switching the coordinate pairs she also switched the meaning of the abscissa and ordinate to (*vertical distance*, *angle measure*), which was consistent with her understanding of inverse sine inputting a sine value and outputting “the angle whose sine is” that value.

Before I had the opportunity to examine how each was thinking about the quantities represented on the axes of Graph 1 and Graph 2, Arya described restricting Graph 2 from $-\pi/2 \leq \theta \leq \pi/2$ on the vertical axis so that the graph represented a function. I responded, “Would that correspond to a certain part up here [*in Graph 1*]?” Both students indicated this corresponded to restricting the horizontal values from $-\pi/2 \leq \theta \leq \pi/2$ in Graph 1, drawing vertical dashed segments at these values on Graph 1 (*Figure 33*).

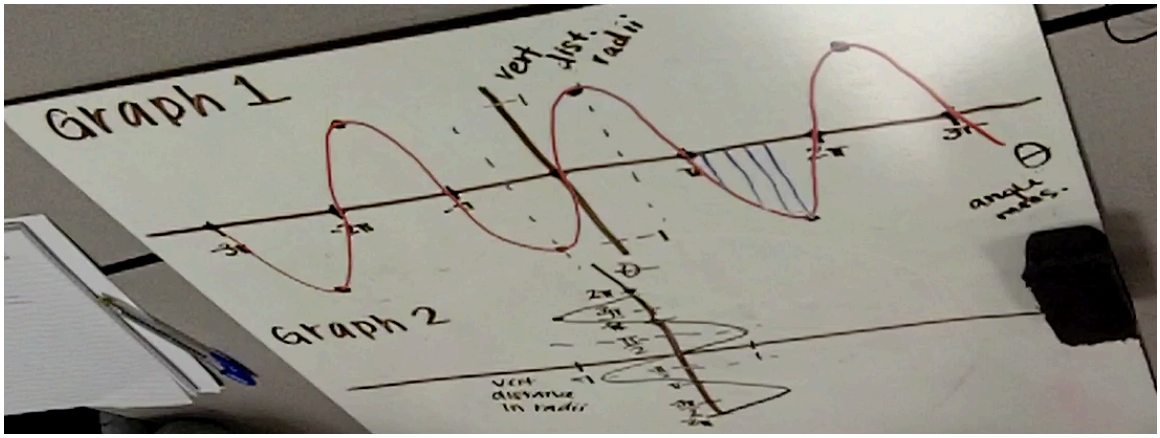


Figure 33: Restricting the Arcsine relationship.

Because of the way the pair represented the interval in each graph, I conjectured Arya and Katlyn understood Graph 1 and Graph 2 as representing the same (at least pointwise) relationship between angle measure and vertical distance. However, I was unsure if their interpretation of each graph relied on the horizontal axis representing the input quantity. To investigate this, I asked them to describe the relationship represented by each graph (Excerpt 9).

Excerpt 9: Arya and Katlyn relating the two graphs.

1	TP:	So in terms of what these two graphs are representing, are they
2		representing-
3	Arya:	So this is theta [*pointing to the vertical axis on Graph 2*], this is [*pointing*
4		*to the horizontal axis on Graph 2*]. So this represents [*pause*] oh yeah.
5	TP:	What's up?
6	Arya:	[*pause*] I get, it's, it's the same relationship. So this is the relationship
7		[*pointing to Graph 1*] between theta and vertical distance and this
8		[*pointing to Graph 2*] is the relationships between, it's just opposite, this
9		is [*pointing to Graph 1*] how angle gives vertical distance, and this is
10		vertical distance [*pointing to Graph 2*], how that gives us angle. But it's
11		the same. You're still graphing the relationship between the two.

Arya's explanation supported my conjecture that Arya understood each graph as representing the same (at least) pointwise relationship between angle measure and vertical distance; both graphs provided the same angle measure-vertical distance pairs (lines 6, 11). At the same time, to Arya, angle measure inherently represented the input quantity in Graph 1 and vertical distance inherently represented the input quantity in Graph 2 (e.g., how quantity A "*gives*" quantity B). Arya reasoned that the two graphs represented the dependency of one quantity on the other

differently (e.g., the graphs represented different uni-directional relationships between the quantities); the order in which she considered the quantities in each graph was different. Thus, although Arya claimed that the two graphs represented the same angle measure-vertical distance pairs, I was not convinced either student had considered the two graphs as representing the same (simultaneous, bidirectional) covariational relationship, an important goal of the task.

At this point, Katlyn had been quiet for an extended period, and thus I asked her to comment on the conversation. Katlyn said, "[Graph 2] just looks so wrong… this [*pointing to Graph 2*], this function… it's not a function." Katlyn did not consider Graph 2, as a whole, to represent a function, so Graph 2 could not be correct. I asked Katlyn to identify specific outputs for a given input in Graph 2 to explore whether she was reasoning about Graph 1 and Graph 2 as representing the same pointwise relationship (Excerpt 10).

Excerpt 10: Arya and Katlyn discussing coordinate points in Graph 1 and Graph 2.

1	TP:	At a vertical distance of half [*pointing to Graph 2*], I want to know what
2		angle value.
3	Katlyn:	Well we could be here, here, here, here, here, [*Katlyn points to the various*
4		*points on Graph 2 where the angle measure corresponds to a vertical*
5		*distance of 0.5*]-
6	Arya:	That's why we, exact[ly]. But it's the same for this [*pointing to Graph 1*].
7	TP:	[*to Arya*] What do you mean?
8	Arya:	You get one half, like [*pause*] these angle measures give us one half. And
9		so all these different, where's one half [*moves to value of 0.5 on vertical*
10		*axis of Graph 1*]? All these different angle measures [*motioning*
11		*horizontally at a vertical distance of 0.5*] give us one half [*she points to*

12	*specific points on Graph 1 with a vertical distance of 0.5*].

After Katlyn stated that Graph 2 represented multiple vertical distance outputs for an angle measure input of 0.5 (lines 3-5), Arya claimed Katlyn's statement translated to Graph 1 by considering the angle measure value of 0.5 as the input on the vertical axis (lines 8-12). Conceiving that both graphs represented the same pointwise relationship between angle measure and vertical distance when interpreting Katlyn's argument, Arya spontaneously engaged in activity in which she considered values on the vertical axis as (tacit) input values; the input values were tacit in the sense that Arya was not explicitly considering 'function' as she engaged in reasoning about specific output values for a given input value.

Katlyn continued to question Graph 2, as, to her, Graph 2 did not represent a function. Arya argued they restricted Graph 2 so that it represented a function. Arya then drew a circle (see *Figure 34*), and using a vertical distance of 0.5, Arya motioned to a location in the first (recreated in *Figure 34* in green) and second (drawn in red) quadrant that both had a vertical distance of 0.5 to explain why there was more than one angle measure output for a given vertical distance input. Arya referred back to the quantitative context in which she constructed the sine relationship to justify why there were multiple angle measures associated with a given vertical distance.

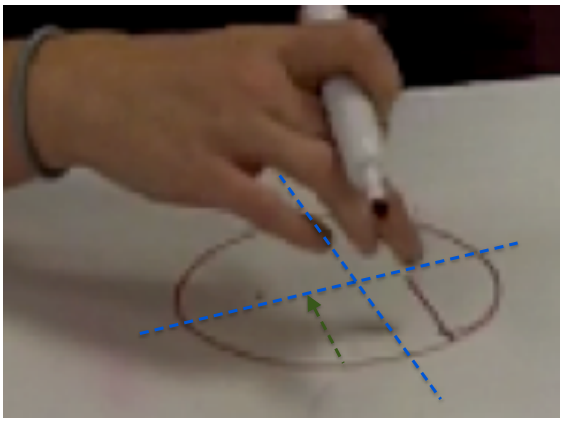


Figure 34: Arya using a circle to reason about the arcsine relationship.

Katlyn expressed that she did not conceive of Graph 2 as a correct representation of the arcsine function, so I asked Katlyn to represent the arcsine function. She created two graphs; the first had a range from $-\pi/2$ to $\pi/2$ and the second had a range from $\pi/2$ to $3\pi/2$ (*Figure 35*). Katlyn identified that either of these graphs represented the arcsine function as in both graphs every vertical distance input value from negative one to one corresponded to only one angle measure. As the conversation continued, Arya noted, "This [*motioning over the entire curve in Graph 2*] as a whole is not a function, but if we're just taking this section [*indicating the interval* $-\pi/2 \leq \theta \leq \pi/2$]… it is a function." Katlyn added, "These [*pointing to her restricted graphs*] are just pieces of [Graph 2]." By conceiving of Graph 1 and Graph 2 as representing the same pointwise relationship, Katlyn understood her restricted "pieces of" the curve in Graph 2 were representing the same relationship as the associated portion of the entire curve in Graph 2.

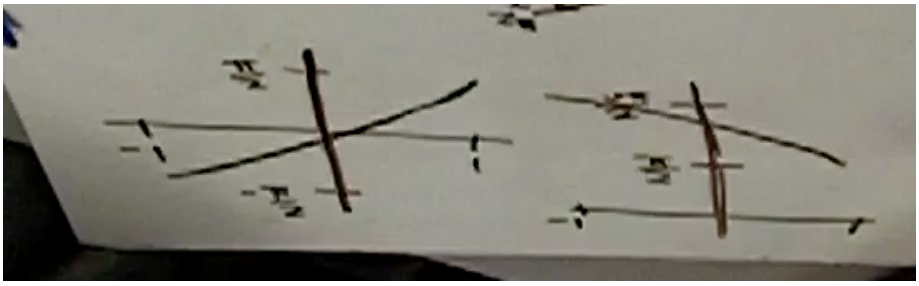


Figure 35: Katlyn's restricted graphs.

At this point, both students understood the arcsine function as a relationship that inputted a vertical distance and outputted an angle measure over specified intervals such that for each vertical distance input from negative one to one there was exactly one corresponding angle measure. Although Arya had (tacitly) reasoned about the vertical axis as representing the input in Graph 1, each student's reasoning assumed angle measure was the input quantity in Graph 1, which represented the sine function, and that vertical distance was the input quantity in Graph 2, which represented the arcsine function; neither student was reasoning about Graph 1 or Graph 2 as simultaneously representing both the sine and arcsine functions.

Constructing the arcsine relationship: When conventions meet quantitative reasoning.

Each student understood both graphs represented the same pointwise relationship between angle measure and vertical distance (i.e. each student understood that each pair of angle measure and vertical distance values were in both graphs). However, I conjectured each student understood the graphs as representing *something* differently, namely which quantity represented the input.[31] In order to test this conjecture and to provide an opportunity for the students to reorganize their understandings of the relationship(s) represented by each graph, I asked them to consider the third prompt: Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not? (*Figure 31*).

Initially, the pair considered different 'turns' of Graph 1 but they were unable to determine a single turn that transformed Graph 1 to Graph 2. As I considered that discussing 'turns' did not (necessarily) require the pair to consider the relationships the graphs represented, instead allowing them to focus on the perceptual shape of the graph, I asked, "Could we leave this graph [*pointing to Graph 1*] to look like this, but use it somehow to represent the arcsine function?" Katlyn responded (Excerpt 11).

Excerpt 11: Arya and Katlyn addressing the third prompt.

1	Katlyn:	I was just thinking like rewriting the, like rewriting the equation. So like
2		if, if the equation of this graph is y equals sine theta [*writes* $y = \sin(\theta)$
3		*above Graph 1*].
4	TP:	Okay.

[31] To this point in their activity the students had not addressed the covariational relationship(s) represented by each graph.

5	Katlyn:	Then we could re-write the equation to just be [*pause*]. What am I saying?
6		So like [*writes* $\sin^{-1}(y) = \theta$*, see Figure 36*], is like what I'm thinking. I
7		don't, I just think of like re-writing the equation. So then like, now we're
8		looking at the y [*pointing to the y in* $\sin^{-1}(y) = \theta$], so we go to one
9		[*motioning to 1 on the vertical axis in Graph 1*] and then we're like okay
10		well at one which angle is, or which angle's sine is one? And then we can
11		see that it's all of those [*moving horizontally from 1 on the vertical axis*
12		*and pointing to the 3 locations in Graph 1 with a vertical distance of 1*].

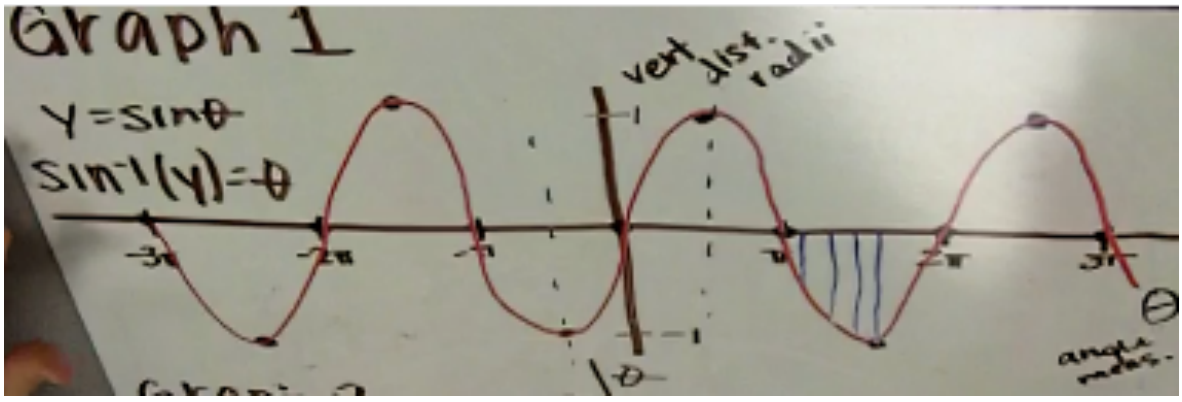


Figure 36: Graph 1 with both the sine and arcsine equations.

Katlyn interpreted Graph 1 as the arcsine function (lines 5-12), using her quantitative meaning for inverse sine as "the angle whose sine is" and the equations $y = \sin(\theta)$ and $\sin^{-1}(y) = \theta$ to justify this shift in interpretation. When addressing the third prompt Katlyn considered the vertical axis as (explicitly) representing the input quantity, which was novel reasoning in this teaching episode.

I asked Arya if she understood Katlyn's argument, and Arya stated, "I understand what she's saying. I don't know if that, can you do that?" Because I did not allow Arya to rotate the graph and because her meanings for graphs required the horizontal axis to represent the input quantity, she questioned whether Graph 1 represented the arcsine function. Katlyn replied, "I don't think

so," erasing the inverse sine equation she had written. I asked Katlyn to rewrite her sine inverse equation and the following interaction ensued (Excerpt 12).

Excerpt 12: Arya and Katlyn describing Graph 1 as the inverse sine function.

1	TP:	So if we have this equation [*pointing to* $\sin^{-1}(y) = \theta$] how are we
2		interpreting that in terms of the graph [*motioning towards Graph 1*]?
3	Katlyn:	Well there I would look at, you would look at *y* first [*pointing to y in* $\sin^{-1}(y) = \theta$].
5	Arya:	Yea [*pointing to the vertical axis in Graph 1*]. But it's still your *y*, 'cause
6		it's still on your, 'cause it's, it's where we have it. So what just? That
7		works, right? Or could you just solve for [*pause*]-
8	TP:	So, what are you thinking Arya?
9	Katlyn:	For theta, that's what I did-
10	Arya:	No that's what you did.
11	Katlyn:	That's what I did.
12	Arya:	Okay, yeah, that works.
13	TP:	So why does this work [*pointing to* $\sin^{-1}(y) = \theta$]?
14	Arya:	'Cause you didn't change-
15	Katlyn:	Because we're trying to get theta as our, [*Arya begins to speak, Katlyn*
16		*continues*] as our answer. Like as our, like if we're switching the input and
17		output, I guess is what I'm thinking. So we want theta to be our answer,
18		'cause then originally theta was our input [*pointing to the horizontal axis*
19		*in Graph 1*] but now we want it to be our output. So if we take the sine
20		inverse of *y* [*pointing to both sides of* $y = \sin(\theta)$] and we're left with theta

21		on this side [*pointing to θ in* $\sin^{-1}(y) = \theta$], then we want to look at *y* first
22		[*pointing to y in* $\sin^{-1}(y) = \theta$] and say which angles sine is one, or negative
23		one, or zero [*motioning to these values on the vertical axis in Graph 1*].
24		And then you'd get [*pointing to values on the horizontal axis*
25		*corresponding to a vertical distance of 0*] your theta.
26	Arya:	Yeah, yeah, I think that works.

Consistent with her activity in the pre-interview, when working with the sine and arcsine functions Katlyn considered switching the quantities she perceived as input and as output (lines 3-4, 9, 15-25). Whereas in the pre-interview, Arya relied on switching techniques when determining inverse functions, in this instance she claimed that Katlyn's interpretation of the vertical axis as representing the input to the arcsine function in Graph 1 "works" (lines 5-7, 12, 26). I conjectured Arya made an accommodation to her meanings for graphs by considering the vertical axis as representing the input quantity.

Because Katlyn led much of the aforementioned interactions, I questioned the extent that Arya understood Katlyn's justification. I stated, "So, Arya, you were making some faces earlier though like… you didn't like something." Arya responded, "I think that makes sense," and she explained how to interpret Graph 1 with the vertical axis representing the input. However, shortly after this, Arya questioned, "Then what's this [*pointing to Graph 2*], what would you call this graph? … So would you still say [Graph 2] is sine inverse? … Then you can't just re-call. The graphs aren't the same. You know what I'm saying?" I conjecture prior to considering Katlyn's reasoning, Arya associated equations and graphs using picture-name pairs (e.g. (Graph 2, arcsine function), (Graph 1, sine function)). Although she was able to explain Katlyn's reasoning in her own words, to Arya, Graph 1 and Graph 2 could not both be represented by the

rule $\sin^{-1}(y) = \theta$ because there is only one picture or graph that corresponds to this equation; having two graphs associated with a particular function name was problematic to her. This activity is consistent with her activity in the pre-interview, in which Arya at times treated graphs as pictorial objects.

As Arya continued to question what to "call" each graph in order to maintain picture-name pairs, Katlyn argued that Graph 1 and Graph 2 represented the same (at least pointwise) relationship between angle measure and vertical distance values, so it was mathematically appropriate to use $\sin^{-1}(y) = \theta$ to represent both graphs. By defining both Graph 1 and Graph 2 according to the same analytic rule, Katlyn reasoned in a way that supported her considering either quantity as the input on either axis and both graphs as representing the same functions.

Considering Katlyn's reasoning, Arya indicated that the equation $\sin^{-1}(y) = \theta$ was "one way to write the relationship." I then asked the students if $y = \sin(\theta)$ is another viable analytic rule to define the curve in Graph 2. Arya responded, "Man. [*sighs*] Yea… Based on [Katlyn's] logic you can… It still bothers me." After this, Arya interpreted Graph 2 as representing $y = \sin(\theta)$ by considering θ, on the vertical axis, as the input quantity. As with Katlyn, I noted that it was Arya's interpretation of the analytic rules might have dictated the quantity she considered as input and output, with this supporting her in conceiving either quantity (and either axes) as representing the input to a function.

Arya's words and body language led me to conjecture she was not convinced that her reasoning was mathematically correct, indicating Arya perceived an inconsistency between her system of meanings and her current activity. When I asked about their activity, both students expressed that they were unsure if their reasoning was correct. As the conversation continued Katlyn addressed why she was uncomfortable, saying, "You're used to inputting this [*motioning*

over the horizontal axis in Graph 2]." Arya interjected, "You want, *y* is generally your, [*pause*] output… it's like this [*pointing to the horizontal axis in Graph 2*] is your input, this [*pointing to the vertical axis in Graph 2*] is your output, so it's weird to have it like that [*pointing to the equations near Graph 1*]… Because you're inputting your *y* [*pointing to y in* $\sin^{-1}(y) = \theta$ *near Graph 1*], you're inputting something on [the vertical axis of Graph 1]… your horizontal axis is usually your input." Arya was uncomfortable with representing the input on the vertical axis as well as with labeling the input as *y*. I asked if the students were more comfortable if I labeled vertical distance as *v* instead of *y* and both students indicated they were more comfortable when the input quantity was represented on horizontal axis. As each student maintained meanings prior to this teaching episode in which she assumed the input was represented on the horizontal axis, each was uncomfortable with considering the input as represented on the vertical axis.

As the interaction continued, Arya persistently questioned the 'correctness' of interpreting the vertical axis as the input axis, but was unable to satisfactorily determine an incorrect aspect of their arguments. For example, she claimed, "I don't see anything mathematically incorrect. I don't see it." Arya then argued that each graph represented "vertical distance and angle measure, they're showing the relationship between the two." Conceiving of each graph as representing the relationship between angle measure and vertical distance (i.e. reasoning quantitatively) supported Arya in interpreting both graphs with either axis representing the input quantity.

Considering function in relation to their earlier reasoning.

I conjectured that Arya and Katlyn had made (at least functional) accommodations to their meanings for interpreting graphs by reasoning about graphs as representing relationships between quantities, regardless of the axes on which these quantities were represented. In the pre-interview each student leveraged function meanings that did not support her in describing graphs

similar to Graph 2 as representing a function so I decided to explore how their current reasoning might influence their meanings for interpreting graphs as representing functions. I thus returned the students to Graph 2 without the domain restrictions and asked if Graph 2 represented a function. Arya indicated she was considering Graph 2 as representing the arcsine relationship and said, "I think it can't be a function because… there's no way to know which output you could have." Katlyn agreed that, in this case, Graph 2 did not represent a function.

Previous to my raising "function," each student described Graph 2 as being representable by $y = \sin(\theta)$, but when asked if Graph 2 represented a function both students assumed the input was represented on the horizontal axis, with Arya describing the graph (or "it") as a whole, not being a function. In an attempt to perturb each student's ways of thinking about Graph 2 in regards to "function," I provided a hypothetical student's response (Excerpt 13).

Excerpt 13: Arya responding to a hypothetical student arguing Graph 2 is a function.

1	TP:	What if a student says, "Well I was looking at this graph [*pointing to*
2		*Graph 2*] and using that equation [*pointing to* $y = \sin(\theta)$]"?
3	Arya:	[*long pause*] The graph's still not a function, I don't know. Yeah, I guess
4		you could do that [*inaudible*].
5	TP:	So you just, Arya, you just said the graph's not a function. But then you
6		said, you can do that?
7	Arya:	I don't know [*laughs*]. The graph's not a function. You can still use this
8		graph to, **this** [*voice raises, points to* $y = \sin(\theta)$] is a function.

When considering the hypothetical student's response, I conjecture Arya realized an inconsistency between her earlier reasoning about Graph 2 as representing the sine relationship and her conclusion that Graph 2 did not represent a function (lines 3-4, 7-8). Attempting to

resolve this inconsistency, she argued, "[$y = \sin(\theta)$] represents a function because… for every input [*pointing to* θ *in* $y = \sin(\theta)$] you get a single output [*pointing to* y *in* $y = \sin(\theta)$]… You can use [Graph 2] to find those things [*underlining the equation* $y = \sin(\theta)$], to show that relationship. So this is a function [*pointing to the equation* $y = \sin(\theta)$], but this is not [*pointing to* $\sin^{-1}(y) = \theta$]. No this, I mean [*pointing to* $\sin^{-1}(y) = \theta$ *then motioning over the whole graph*]. This whole [graph]… is not." As Arya attempted to reach a state of equilibrium, her language changed from describing a graph as being a function (function as a property the graph as object) to describing graphs as representing relationships (which could be functions). Specifically, Arya argued if she interpreted Graph 2 as representing the relationship defined by $y = \sin(\theta)$ then Graph 2 was a function but if she interpreted Graph 2 with vertical distance as the input then Graph 2 was not a function. A complexity arose when Arya attempted to represent Graph 2 with $\sin^{-1}(y) = \theta$, as she understood $\sin^{-1}(y) = \theta$ as representing a function; Arya did not have a way to define an analytic rule to represent Graph 2 in its entirety (e.g., a notation such as $\text{Sin}^{-1}(y) = \theta$ or $\text{Arcsin}(y) = \theta$ to represent the multi-valued function).

Before I had the opportunity to question Arya, Katlyn addressed the hypothetical student's solution, stating, "When I look at this [*pointing to* $y = \sin(\theta)$], I see theta as my input so I want to put it on this x-axis [*motioning over the horizontal axis*]… for it to be the input." Although Katlyn was the first to describe interpreting each graph using both analytic rules, when I raised the idea of "function," Katlyn continued to impose that the horizontal axis represented the input quantity.

After Katlyn's statement, Arya continued, "I don't want to say, it's so ugly to call that [*pointing to Graph 2*] a function." I asked Arya to be explicit about what she meant by "it's a function" to ensure she discussed the relationship she was considering. Arya stated she was

considering θ as her input and y as her output in $y = \sin(\theta)$, and this meant that "[Graph 2]'s a function… Oh but it's so, but *it's* not a function" (*emphasis* added). Shortly after this, Arya again described two ways to interpret Graph 1 (i.e. as the arcsine relationship with vertical distance as the input and as the sine relationship with angle measure as the input), concluding, "Does that make more sense than saying like a graph itself is a function or not a function?" I asked what the "key idea" was when interpreting Graph 2 using both equations with the intention of supporting Arya in explicitly describing her understandings of graphs and functions. Arya stated, "When the inputs and outputs were [*pause*] different… So are you starting with theta, or are you starting with your vertical distance?"

By explicitly addressing which quantity she considered as the input and output, Arya again conceived of a way to interpret Graph 2 as representing a function (i.e. the sine function). Previous to this interaction, to Arya, graphs either were or were not functions (e.g. "*it's* not a function"), an understanding that was supported by her assumption that the input quantity was always represented on the horizontal axis. Arya's reasoning in this episode problematized this meaning. Arya now understood defining an input quantity as critical to determining if a graph represented a function and questioned whether it made sense to refer to a graph, as a whole, as a function.

Katlyn was quiet for most of Arya's description of Graphs 1 and 2 as simultaneously representing the sine function and arcsine relationship, and followed Arya's argument above by saying, "I'm just really frustrated." I asked Katlyn to explain her frustration. Katlyn initially responded by describing different ways to interpret Graph 1, stating, "If we're looking at it here [*motioning over Graph 1*] and vertical distance. If we input and it's on this axis [*motioning over vertical axis on Graph 1*]… is that what you're saying?" Katlyn then described, "When you input

vertical distance, then you can get out multiple [*motioning over Graph 1*], infinite numbers of angle measures that have that vertical distance, so it's not a function." After this, Katlyn described Graph 2 as representing a function by considering θ as the input, saying, "For every input theta [*pointing to the vertical axis in Graph 2*], we're getting one unique output, y… It just doesn't look like [a function]." I conjectured Katlyn applied her function definition (a unique output for each input) to each graph as she considered the quantity on the vertical axis as representing the input quantity. Katlyn considering the vertical axis as representing as the input was not novel, however, Katlyn needed to re-engage in this reasoning as she attempted to reconcile a perturbation that arose once I raised "function," as Graph 2 did not fit her expectations for what a function "look[ed] like." I also note that Katlyn continued to make statements such as, "*it's* not a function" (*emphasis* added), which may indicate she was drawing on meanings such that graphs as a whole were or were not functions, which inhibited her from reconciling her perturbation.

Each student had reasoned about the relationship between angle measure and vertical distance to interpret Graph 1 and Graph 2 as both being or not being (representations of) a function depending on her choice of input and output. Although Katlyn engaged in this reasoning, she continued to question the mathematical correctness of this conclusion. In one such instance, Arya responded to Katlyn, saying, "Yea but this [*motioning as if drawing the whole curve in Graph 2*] is the whole relationship, it's just not a function." Katlyn responded, "Okay so, whoa, see you just blew my mind. 'Cause you said relationship, not function. This [*Graph 2*] is not a function this is a relationship." Katlyn's reaction supports my conjecture that she was experiencing a sustained perturbation during the previous conversations concerning Graph 2 in relation to "function." Katlyn remained quiet during much of the interaction, likely because of

this perturbation, making it difficult for me to infer meanings from her activity. Further, as this interaction occurred at the end of the teaching episode, I was unable to further explore how Katlyn was differentiating between functions and relationships in her thinking.

Summary of the Graphing Sine/Arcsine Task day one.

Katlyn and Arya started by graphing the arcsine relationship (or function) by switching the abscissa and ordinate of points of the graph on the sine function. They understood that the sine and arcsine relationships represented the same pointwise relationship between quantities with the distinction being which quantity they considered to be the input of each function. As they differentiated between the two functions during their initial activity, they consistently assumed the horizontal axis represented the input quantity and described graphs as being (or not being) functions.

I conjecture each student re-constructed her understandings of interpreting graphs by moving between Graph 1 and Graph 2. As they did this, they considered that the only difference between the two graphs was which axis each quantity was located, and that this difference did not affect the underlying relationship. The students considered either the horizontal or vertical axis as representing the input quantity multiple times, while understanding that the other choice was also mathematically viable (i.e. each student reasoned *bidirectionally with respect to the axes*, at least in activity). However, engaging in this reasoning was not trivial, as it required each student to re-consider if it was critical to represent the input on the horizontal axis. Further, when I posed questions concerning "function," each student had to re-construct their reasoning bidirectionally with respect to the axes to consider how this reasoning related to "function." By the end of the teaching episode, both students described ways of interpreting both Graph 1 and Graph 2 as being and not being (representations of) functions. However, both students also expressed that

this reasoning was at odds with their other mathematical meanings throughout the episode. As such, one goal I had for the next episode was to explore if (and if so how) the students continued to reason in this way, and if so, how this thinking may have influenced other mathematical meanings, including function.

I noted that during much of their activity, each student referenced the equations $y = \sin(\theta)$ and $\theta = \sin^{-1}(y)$ as they discussed the different input and output quantities. I was unsure the extent to which these equations were critical to their interpretations of the graphs. Hence, another goal I had for the next episode was to explore if (and if so how) the students were using the equations to support or hinder their reasoning about relationships represented by the graphs.

The Graphing Sine/Arcsine Task Day Two: Examining the Students' Reasoning

Four days after the first Graphing Sine/Arcsine Task episode, in an out-of-class teaching episode, Dr. Moore and I worked together as teacher-researchers with Kevin LaForest acting as an observer. We intended to investigate how Arya and Katlyn approached the Graphing Sine/Arcsine Task (*Figure 31*) after allowing them time to reflect on their prior activity. I intended to explore if and how they interpreted Graph 1 and Graph 2 as representing both the sine and arcsine relationships. Further, as the pair often referred to the analytic rules when describing both graphs, I erased these from their board to explore if (and if so how) they reasoned about these analytic rules (*Figure 37* shows their board at the outset of the episode).

In the prior teaching episode, the pair worked with the pointwise relationship between angle measure and vertical distance without further describing the covariational relationship between the quantities. I hypothesized that considering the covariational relationships represented by the graphs had the potential to further support the students in conceiving the graphs as representing the same (pointwise and covariational) relationships. Furthermore, I intended to investigate if

engaging in, and possibly reflecting on, their activity from the prior episode influenced their function and inverse function meanings. For instance, I hypothesized that the students would interpret Graph 2 as representing the sine function by considering θ, on the vertical axis, as the input.

Discussing function at the outset.

I asked the pair for their recollections from the previous episode and Arya responded, "We were talking about, what the definition of a function is, I guess… At first this [*pointing to Graph 1*] was our sine graph and our [*pointing to Graph 2*] arcsine graph. And we were saying, how you'd look at that [*motioning over the curve in Graph 2*] to call it a function, can you call [Graph 2] a function or whatever?" Arya recalled her activity from the prior episode and questioned whether it made sense to call a "graph a function." Addressing Arya's question, Katlyn raised the distinction of relationship versus function that she made at the end of the previous teaching episode. Katlyn stated, "I don't think it's a function. I think it's a relationship. I don't know if that's different but it's different to me." Referencing Graph 2 throughout, Katlyn described this difference, "This a relationship between when we put in vertical distance… like the vertical distance of one we get… an infinite number of angle measures in radii. So that I feel is like, that's a relationship… it's not a function." Consistent with their initial activity in the previous episode, both students described the graphs as *being* functions (e.g., "it's not a function," "call it a function") and assumed that the horizontal axis represented the input quantity in each graph.

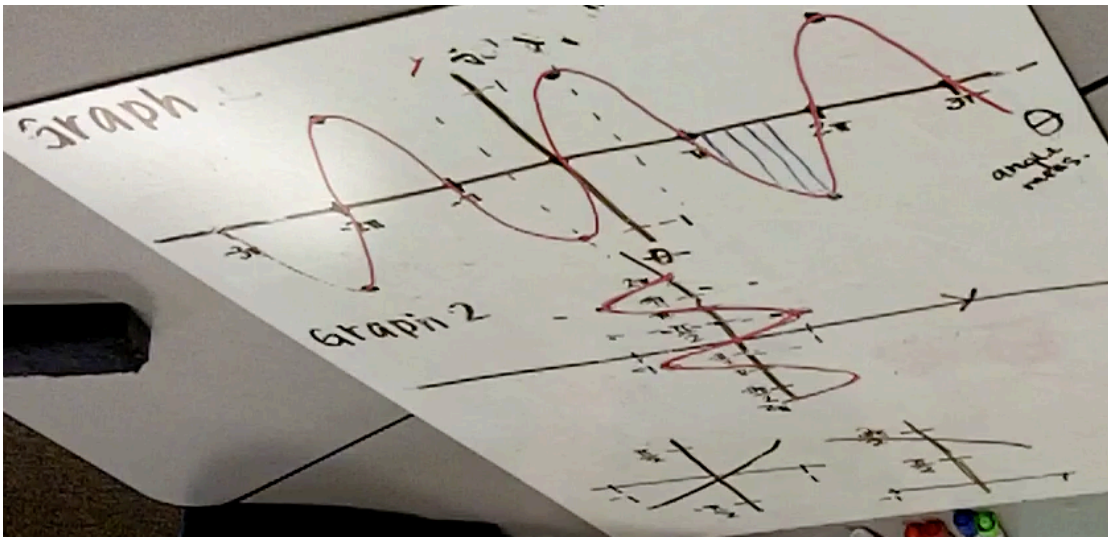


Figure 37: The pair's board work from the previous episode with the equations erased.

I directed them to the third prompt to explore if both students would continue to assume the input was represented on the horizontal axis and if Arya would describe her understanding of Graph 2 as representing or not representing a function. After reading the prompt there was a long pause, then Katlyn responded (Excerpt 14).

Excerpt 14: Katlyn addressing the third prompt.

1	Katlyn:	I think what we said was that if we change the, what we're inputting and
2		what we're outputting. So like here [*pointing to the Graph 1 throughout*]
3		like what this shows is like y being vertical distance equals sine theta
4		[*writes equation* $y = \sin(\theta)$ *above Graph 1*] but if we wanted to change it
5		to… the arcsine relationship then we would have the [*writes* $\sin^{-1}(y) = \theta$].
6		Have like arcsine y equals theta, so like now… we're inputting y
7		[*motioning over the vertical axis in Graph 1*]. And so like this, this is
8		labeled y [*rewriting the label y on the vertical axis*]. And we're getting
9		out theta [*pointing to the θ label on horizontal axis on Graph 1*].

As in the previous episode, Katlyn wrote both the sine and arcsine equations and referenced these when describing two ways to interpret Graph 1 (lines 2-9). Whereas Katlyn started the episode by assuming the input was represented on the horizontal axis, when addressing the third prompt, she interpreted the vertical axis of Graph 1 as representing the input quantity. Additionally, I note that, in this case, Katlyn referred to "arcsine relationship" as she wrote the rule $\sin^{-1}(y) = \theta$. One possible explanation for this, in relation to Katlyn's distinction between function and relationship, and her perturbation in the prior episode, is that Katlyn did not conceive of Graph 2 as being represented by the function $\sin^{-1}(y) = \theta$. By distinguishing, between $\sin^{-1}(y) = \theta$ and $\text{Sin}^{-1}(y) = \theta$ in her own thinking, and reasoning with the latter rule in this case, Katlyn experienced less difficulty describing the functions or relationships Graph 2 represented.

Both students were still experiencing a complexity as they drew on meanings, exhibited in the pre-interview, such that graphs either were or were not functions. For example, when Dr. Moore asked Katlyn to use similar reasoning as she had in Excerpt 14 for Graph 2, Katlyn responded, "And you want us to make it the sine graph? … I don't think it can happen." I questioned, "You don't think it can happen?" Katlyn responded, "I mean you can do the same thing, but then… it's not a function. I don't know. But, I guess if like we're switching our axes then that's okay 'cause it's like that would be our input [*pointing near the* θ *in* $y = \sin(\theta)$]… and then this [*pointing to the horizontal axis in Graph 2*] would be our output." Katlyn continued, "Clicking now, that's okay because um, theta is now our input [*pointing to* θ *in* $y = \sin(\theta)$]… and so now all of our outputs [*Arya traces curve in Graph 2 with her finger*] are unique. Yeah so that works." Katlyn's initial response was based on her understanding of graphs (or "it"), as a whole, being or not being functions (i.e. "it's not a function"). By instead attending to the quantities represented by the graph, Katlyn considered a way to interpret Graph 2 (or, more appropriately,

the relationship underlying Graph 2) as representing a function. This was in contrast to her activity when she imposed that the horizontal axis represented the input quantity when describing a graph (or "it") as a function.

Because Katlyn had led much of the conversation, I asked Arya to interpret Katlyn's reasoning. Consistent with her activity in the prior teaching episode, Arya described Graph 2 as representing a function by interpreting the input on the vertical axis. Arya paid explicit attention to which quantity (or axis) she considered as representing the input quantity (or axis).

Covariational relationship.

Both students stated the two graphs were showing the same "relationship" although they had not yet discussed this relationship as more than as a pointwise pairing of angle measures and vertical distances. To explore their understanding of the graphs in terms of a covariational relationship between the quantities, I moved the conversation to the covariational relationship they established in Graph 1 in the prior episode. Arya and Katlyn described that for equal changes in angle measure from π to $3\pi/2$, vertical distance decreased at a decreasing rate (see *Figure 32*) (MA1-4). Because they had claimed that the graphs represent "the same relationship," I asked how they related this to Graph 2 (Excerpt 15).

Excerpt 15: Describing a covariational relationship in Graph 2.

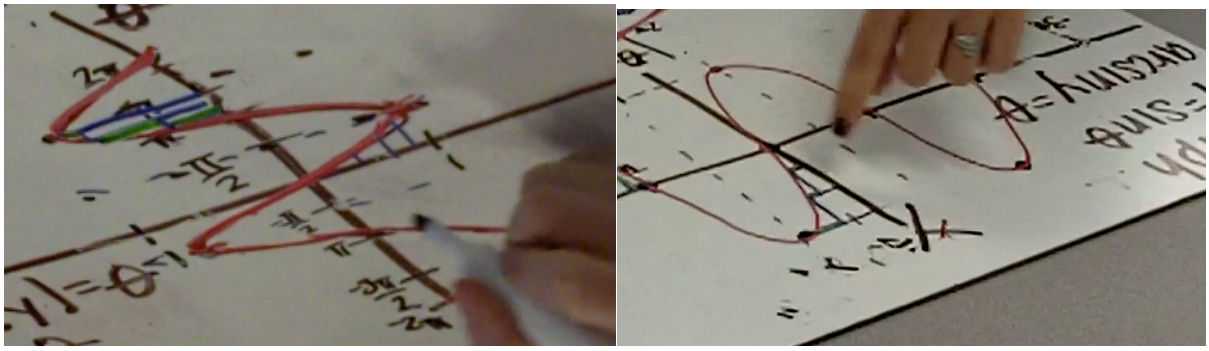
1	TP:	Now is there anything that would correspond to that here [*pointing to*
2		*Graph 2*]?
3	Arya:	[*pause*] Well, which way?
4	Katlyn:	You can, you can draw the same lines if that's what you're saying. I just
5		yeah, you can.
6	TP:	So what do you mean, when you [*to Arya*] said 'which way,' what do you

7		mean?
8	Arya:	Well do you want, [*pause*] are we still talking about our sine [*pointing to*
9		*Graph 1*]?
10	Katlyn:	Are we still inputting angle measure? I guess that's what I want to know.
11	Arya:	Yeah. Yeah.

Rather than assuming the input was represented on the horizontal axis, or rotating the graph, Arya and Katlyn focused on representing a relationship compatible with that represented in Graph 1 (lines 3-5, 8-11). Whereas when previously describing the covariational relationship between quantities they considered equal changes of the input quantity, which was inherently represented on the horizontal axis, they now considered the possibility that the input quantity was represented on the vertical axis.

Both students described that they were representing the "same thing", drawing vertical distance (in blue) and changes in vertical distance (in green) for equal changes of angle measure from π to $3\pi/2$ along the vertical axis in Graph 2 (*Figure 38*a). After they considered angle measure as the input in both graphs, I asked, "You were sort of implying… there was another way we could have looked it?" Katlyn identified, "For equal changes of vertical distance," and Arya drew equally spaced segments from the horizontal axis (in blue) in Graph 2 to represent this (*Figure 38*a). Katlyn used Arya's segments to conclude that for the interval $0 \leq \theta \leq \pi/2$, "With equal changes in vertical distance… the angle measure is… increasing at an increasing rate" (MA1-4). I asked what was "analogous to" this in Graph 1 in order to investigate how Arya and Katlyn reasoned about equal changes in vertical distance in Graph 1. Katlyn drew equal changes of vertical distance along the vertical axis in Graph 1 (*Figure 38*b), and described,

"Same thing… As you go through equal changes in vertical distance, your angle measure is… increasing at an increasing rate" (MA1-4).



(a) (b)

Figure 38: Representing changes of vertical distance in (a) Graph 2 and (b) Graph 1.

Arya and Katlyn argued that both graphs represented the same covariational relationships between angle measure and vertical distance as they understood that the curves in Graph 1 and Graph 2 represented the same relationship between two projected magnitudes representing angle measures and vertical distances from the horizontal diameter, with the distinction being which axis corresponded to which quantitative referent and associated magnitude. Further, each student anticipated that the covariational relationship in a certain interval for an input quantity on one graph was maintained on the same interval for the same input quantity on the other graph. Their reasoning supported them in understanding the graphs as representing invariant pointwise and covariational relationships that they interpreted with either quantity as representing the input on either axis (i.e. reasoned bidirectionally with respect to the axes).[32]

[32] I note that each student reasoned in this way with respect to the graph. Neither student addressed how they could represent this relationship in a circular motion context (e.g., described a way to imagine coordinating changes in angle measure for equal changes of vertical distance in a circular motion context).

Understanding the input on the horizontal axis as a convention.

To investigate if reasoning covariationally supported the students in distinguishing between Graph 1 and Graph 2 as being or not being functions versus Graph 1 and Graph 2 representing relationships that were or were not functions, I asked if Graph 2, "Can be a function?" Arya responded, "[Graph 2] typically represents this [*horizontal axis*] input." Katlyn added, "Yeah so I guess I want to know what you're inputting. 'Cause like if we're inputting theta, then that's okay." In activity, both students relied on function meanings that gave precedence to defining input and output quantities and considering what the graph "represents."

I conjectured drawing Katlyn and Arya's attention to what was 'typical,' and therefore what was atypical, had the potential to support their reflecting on their prior activities (Excerpt 16).

Excerpt 16: Katlyn and Arya reasoning about Graph 2 as a function.

1	TP:	But if we're talking about something being typical, then we must also be
2		able to talk about something-
3	Arya:	So if you want to say, it just depends on what, yeah, what you're calling
4		your input and output. So if you want to say this is your input [*pointing to*
5		*the label θ on the vertical axis on Graph 2*] then call it a function yes.
6		That's what it would be.
7	Katlyn:	Yes.
8	TP:	So it could be?
9	Arya:	It could be. [*pause*] I hate to say that.
10	Katlyn:	I know, me too. It's scaring me.

Rather than focusing on the *graph* or "it" unquestionably being a function or not and assuming the horizontal axis represented the input quantity, the students considered either

quantity (or axis) as the input when discussing Graph 2 as representing a function (lines 4-7). However both students continued to express their discomfort with this (e.g., lines 9, 10).

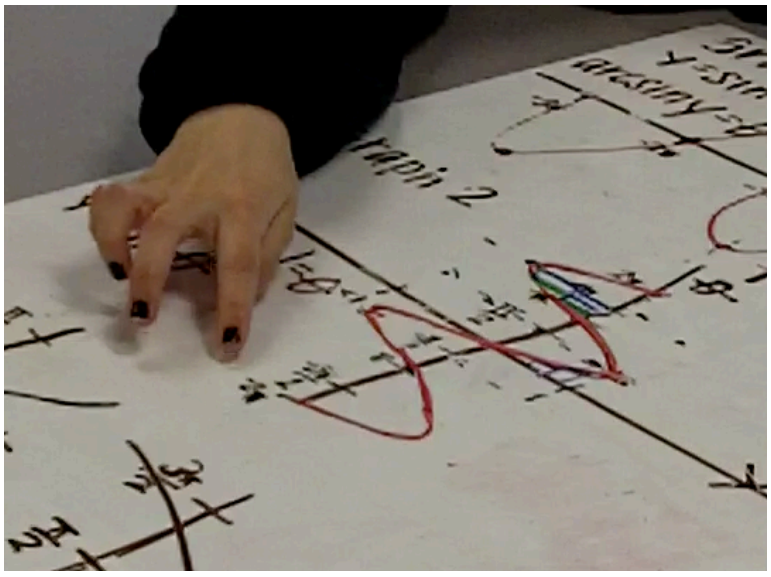
As the conversation continued, Katlyn wavered between describing a graph, as a whole, as a function (assuming input on the horizontal axis) and reasoning about input and output quantities. In contrast, Arya's thinking remained focused on the quantities and relationships. For instance, after Katlyn described the two graphs differently by interpreting each graph with the horizontal axis as representing the input, I asked, "So, are the graphs the same?" Arya responded, "Visually, no, but they give the same information. So, mathematically, they're equivalent." Arya understood the graphs as perceptual objects or shapes were different but the mathematical relationship the two different perceptual objects represented were mathematically equivalent.

I asked the pair how they would respond if a hypothetical student graphed $y = \sin(\theta)$ as drawn in Graph 2 to explore each student's thinking. Responding to this, Katlyn turned the graph to view the input quantity on the horizontal axis to check if the student accurately represented the sine function, indicating she wanted to use a memorized shape for the sine function to evaluate the student's response. Dr. Moore realized the turned graph did not represent the conventional sine graph as the orientation of the horizontal axis was reversed; negative values were directed to the right of the origin. Dr. Moore rotated the graph (see *Figure 39*) and asked them if a hypothetical student was correct if he produced this graph for $y = \sin(\theta)$ (Excerpt 17).

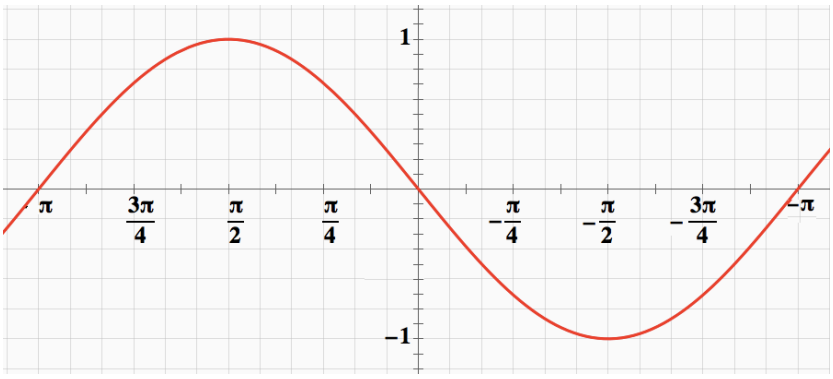
Excerpt 17: Considering a novel representation of $y = \sin(\theta)$.

1	Arya:	From my perspective that's good, that's what we have up there [*pointing*
2		*to Graph 1*].
3	Katlyn:	But this, yeah I know, but this is negative [*pointing to the right side of the*
4		*θ axis on the now rotated graph*].

5	Arya:	Ohhh, oh. I see what you're saying. Oh gosh. [*As if talking to the fictitious*
6		*student*] Put your negatives on the right side.
7	Katlyn:	I guess like in my opinion, this would just have to be really well labeled-
8	Arya:	I'd probably be grading it from sight, and be like nope, but-
9	Katlyn:	Yeah, exactly, because I'm like no this is wrong [*motioning over the*
10		*curve*]. Because it's supposed to go up first [*motioning her finger*
11		*indicating her drawing the conventional sine function*, *laughs*].
12	Arya:	But, [*TP begins to talk but Arya continues*] for arguments sake, that's
13		mathematically the right relationship.
14	Katlyn:	Yeah it is.
15	Arya:	You just can't argue with that.
16	Katlyn:	You can't! You're just like, is it labeled that well? Okay fine. But no.



(a)



(b)

Figure 39: (a) The turned graph and (b) a recreation of the turned graph.

Arya and Katlyn both recognized the unconventional nature of the axes orientation (lines 3-6), but acknowledged that the hypothetical student correctly graphed the sine relationship (lines 12-15). They understood that representing the relationship between quantities, and not the picture they associated with the function rule, was critical (lines 7-16). However, each student expressed she was uncomfortable with the unconventional axes (line 5-6, 16), which may have been the result of the student's current reasoning being fundamentally different than the reasoning they relied on prior to the teaching experiment (e.g., interpreting graphs as pictorial objects, assuming the input was represented on the horizontal axis).

To explore how the students were reasoning about this new graph (*Figure 39*) in relation to 'function,' I asked, "Say… the student produces this graph orients it like this [*pointing the graph as shown in Figure 39b*] and then says this graph doesn't represent a function." Arya responded, "What's your input and what's your output?" Katlyn added, "Would be my question." The pair then argued the hypothetical student would be correct if he was using vertical distance as his input. Each student focused on input and output quantities when asked to determine if a graph represented a function. Hence, I conjectured each student understood that defining input and output quantities was paramount to describing a graph as representing a function.

Returning to inverse function.

We decided to explore if (and if so how) the students' activities might have further influenced their understandings of the sine and arcsine functions as inverse functions of each other *on an appropriate interval*. Dr. Moore asked if he restricted the domain in the normally oriented Graph 1 (e.g., *Figure 37*) to $-\pi/2 \leq \theta \leq \pi/2$, "Would we still call that the sine curve?" Katlyn responded, "It's an interval of the sine curve." The conversation continued (Excerpt 18).

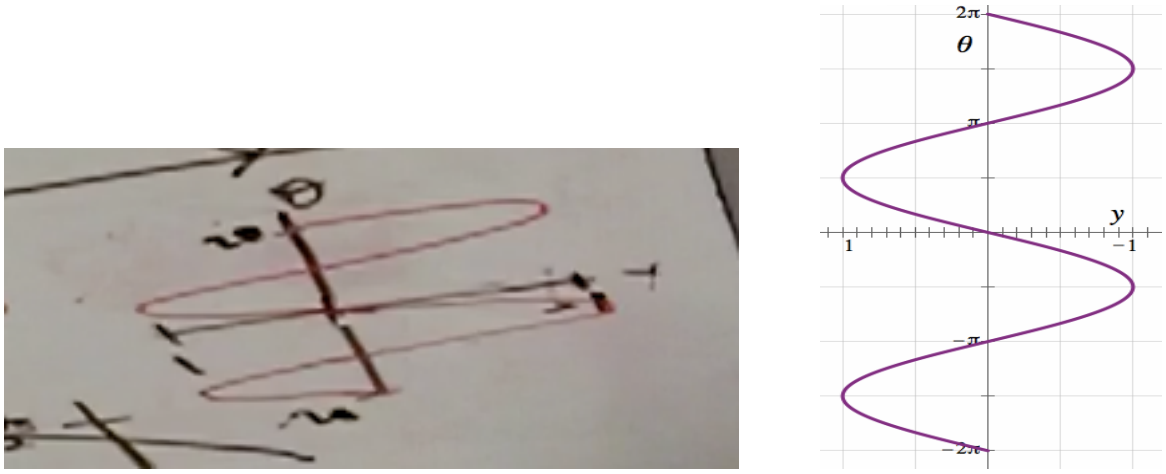
Excerpt 18: Katlyn and Arya describing the sine and arcsine relationships as 'the same.'

1	DM:	Could we call it anything else?
2	Katlyn:	[*long pause*] Well, it's the same thing down here [*pointing to the interval*
3		$-\pi/2 \leq \theta \leq \pi/2$ *on Graph 2*] so I think you could call it, [*pause*] the arcsine.
4	Arya:	It also shows the arcsine.
5	Katlyn:	That's crazy.

I take Katlyn's comment, "That's crazy," to indicate she was engaging (or had engaged) in novel reasoning. Conjecturing this realization had the potential to influence Katlyn's inverse function meanings, Dr. Moore asked, "What were you commenting on, 'that's crazy'?" Katlyn responded, "That it's like, arcsine is in sine… but it's like you just, not the same input. That just kinda really freaks me out a little bit, 'cause like I see it… It's like in the sine curve. It's a part of it." Katlyn exhibited a novel understanding that the arcsine curve was "in the sine curve," which I take to indicate she understood the graph *as representing two functions*, both of which stemmed from the same relationship (i.e. the relationship between vertical distance and angle measure). However, Katlyn's words and body language (e.g., "really freaks me out," "that's crazy") indicate this reasoning was so distinct from her previous experiences that she had no compatible ways of thinking about functions and their graphs.

Reasoning about the relationship versus the perceptual object.

After the conversation above, Arya emphasized the importance of paying attention to the underlying relationship regardless of how this relationship was represented. She argued, "You could even come up with a different way and it would be the same thing." Arya then drew another unconventional coordinate system with angle measure on the vertical axis, vertical distance on the horizontal axis, with negative values to the right of the origin on the horizontal axis. Arya stated, "You can do that. And it's the same relationship and the same graph [*apparently anticipating a graph*]. It's just whatever… you want to label your axes. And that's why it's so important to label your axes." Katlyn then represented the sine relationship on Arya's coordinate system indicating this graph represented "the same relationship" between quantities as the other graphs they constructed (*Figure 40*). I then drew another unorthodox graph, where the origin was located at the coordinate point (1, π/2), as in *Figure 41*, and each student argued that it did not matter how I oriented my coordinate system as long as the curve accurately represented the relationship between angle measure and vertical distance.



(a) (b)

Figure 40: The pair's work for sine on spontaneously created unconventional axes.

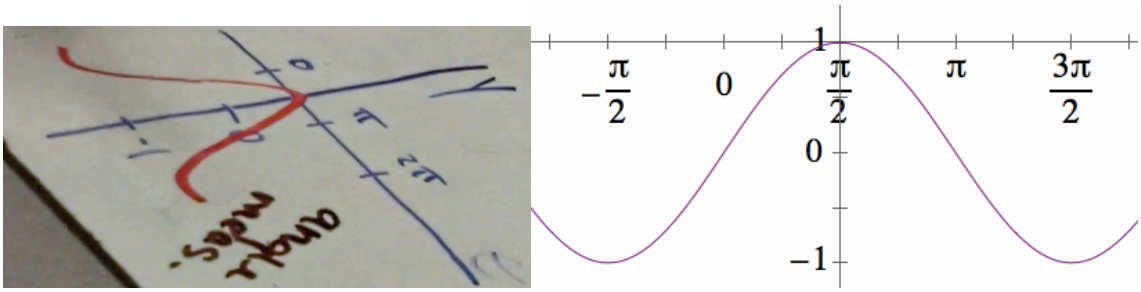


Figure 41: Another unconventional graph of the sine relationship.

Summary of the Graphing Sine/Arcsine Task day one and two.

By the end of the second teaching episode, both students reasoned about Graph 1 and Graph 2 as representing the same quantitative and covariational relationship between angle measure and vertical distance, first considering the points in each graph as representing the same angle measure vertical distance pairs, and later considering the graphs as representing an invariant covariational relationship. This reasoning supported them in considering different graphs (e.g., Graph 1 and Graph 2) and different rules ($\sin^{-1}(y) = \theta$ and $y = \sin(\theta)$) as representing an invariant relationship, with their choice of input quantity differentiating how to name the relationships.

Engaging in this reasoning had implications for both students' function meanings; rather than describing a graph (or "it") as a whole being or not being a function, the students re-organized their function meanings such that they emphasized the importance of identifying input and output quantities before describing a relationship as being a function; the students understood a graph represents relationships that may be functions as compared to the graph being a function. Additionally, and related to the students' understanding of graphs 'being' functions, by the end of the teaching episode, neither student was relying on name-graph pairs when considering the relationships represented by graphs.

I conjecture each of these accommodations to the students' function meanings occurred hand-in-hand with the students' developing understanding of representing the input on the

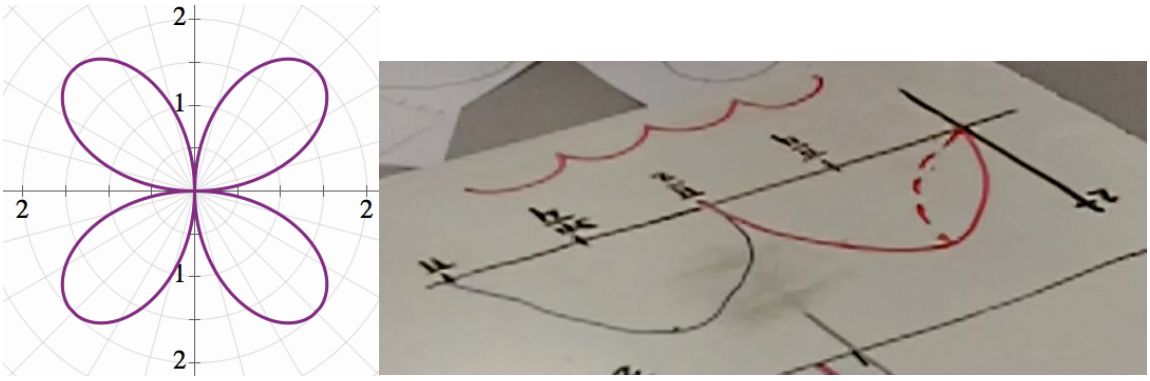
horizontal axis as a convention; by the end of the episode each student understood that either the horizontal or vertical axis had the potential represented the input quantity and that this choice did not influence the underlying relationship (i.e. each student repeatedly reasoned bidirectionally with respect to the axes). An unexpected byproduct of their development of this understanding was that they spontaneously created and represented relationships in other unconventional coordinate systems, accurately representing the sine and arcsine relationships in each of these coordinate systems.

Selected Results from the Polar Coordinate System Episodes

I intended to give the students the opportunity to repeatedly engage in quantitative and covariational reasoning in a variety of contexts and representational systems. Therefore, I engaged them in five teaching episodes over the course of 19 days in which they constructed the polar coordinate system and represented relationships in this system via their reasoning quantitatively and covariationally. As much of the students' activities were compatible with students' activities described elsewhere (Moore et al., 2013a, 2013b), I do not describe the students' activities during these teaching episodes in depth. Instead, I present selected findings that provide insights the students' quantitative and covariational reasoning, and their understanding of graphs as representing emergent traces.

I gave Katlyn and Arya the graph in *Figure 42*a and asked them to determine the analytic rule that corresponded to the graph. Arya immediately indicated, "You just don't know where it's, like you want to see it drawn, like draw itself, if that makes sense…. [determining an analytic rule] would be a lot easier… if I knew how it was traced out." As the pair analyzed the graph in *Figure 42*a, they identified that for $0 \leq \theta \leq \pi/2$, r was "increasing" until $\pi/4$ and "then decreasing" back to zero. The pair moved to considering $\pi/2 \leq \theta \leq \pi$ and Arya indicated,

"There's a couple things that could happen there." Each student made hand motions over the curve as if imagining the curve being traced out, tracing the curve in the first quadrant, then the second, indicating that that r again increased and then decreased back to zero for $\pi/2\ \theta\ \pi$ as well as for the third and fourth quadrants. The students represented the relationship they described in the Cartesian coordinate system (from $0 \leq \theta \leq \pi$ on the graph and a sketch from $0 \leq \theta \leq 2\pi$ below that graph *Figure 42*b) and stated that the curve was defined by $r = 2|\sin(2\theta)|$.



(a) (b)

Figure 42: The graph of $r = |\sin(2\theta)|$ in (a) polar coordinates and (b) Cartesian coordinates.

With both students confident they had accurately found the rule that defined the graph in *Figure 42*a as $r = 2|\sin(2\theta)|$, I asked the pair what would change if the rule was $r = 2\sin(2\theta)$. Arya indicated that in the polar coordinate system, "A lot of things would change." The pair then identified that in the Cartesian coordinate system, r-values were reflected over the horizontal axis from $\pi/2 \leq \theta \leq \pi$. As the pair considered how this relationship translated to the polar coordinate system, they eventually stated that the curve defined by $r = 2\sin(2\theta)$ traced in the first, fourth, third, then second quadrant (whereas the curve defined by $r = 2|\sin(2\theta)|$ traced in the first, second, third, then fourth quadrant), with Arya adding, "I kind of have to watch it go to know what graph it is." I conjectured the pair understood the curve as an emergent trace representing

covarying quantities, with the same pictorial object being representable by two different traces (and therefore two different rules).

The Power Tower Activity

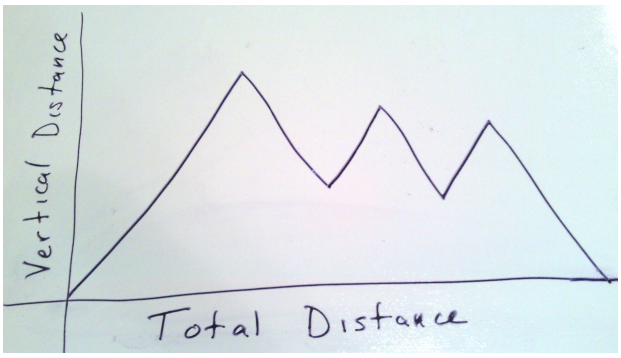
Eight days after the culminating polar coordinate system teaching episode, Dr. Moore engaged Katlyn and Arya in the Power Tower Activity,[33] to give the students another opportunity to reason quantitatively and covariationally in a new context. The Power Tower Activity begins with the students watching a video of an amusement park ride,[34] the Power Tower (*Figure 43*a), shooting riders up a vertical tower before allowing the riders to drop back towards the ground and then shooting them back up. This up-and-down process repeats twice more before returning the riders to the ground (for a more detailed description of this activity and its origins see Moore et al. (2014)). As the video played on a loop, Dr. Moore asked the pair to sketch a graph of a rider's total distance traveled (heretofore referred to as distance) and the rider's distance from the ground (heretofore referred to as height), with height represented on the vertical axis (for a correct sketch see *Figure 43*b).

[33] As mentioned in the methods section, I was inaccessible during this particular teaching session so Dr. Moore acted as the primary teacher-researcher and Kevin LaForest served as the observer.

[34] See http://bit.ly/1zmVVdS for a video of the ride.



(a)



(b)

Figure 43: (a) The Power Tower (b) a correct representation of a rider's total distance and vertical distance.

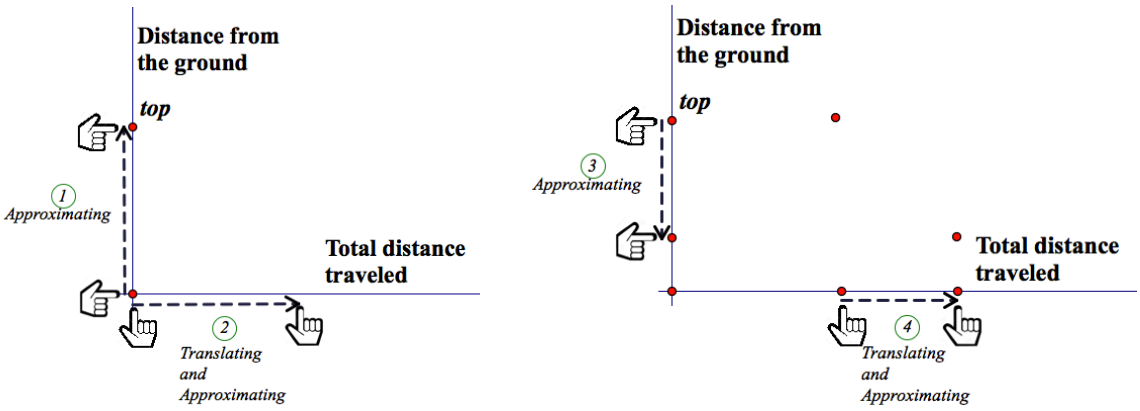
Our goal was that the students engage in constructing and re-constructing the quantities in the situation such that they developed refined quantitative structures that included the covariational relationship between distance and height. Because of our past experiences using this task, we conjectured the students might engage in reasoning about the quantities in ways that supported them in considering either quantity as the independent quantity (without concern of single-valued-ness). That is, and similar to Saldanha and Thompson's (1998) description of covariation, the students might first imagine one quantity changing, then coordinate changes in the other quantity, then back to the first, and so on. We conjectured this process might support the students in developing images that allowed them to imagine the two quantities changing in tandem such that they considered neither quantity as dependent on the other quantity (i.e. construct a bidirectional relationship between distance and height). Additionally, we conjectured by asking them to represent a relationship in which neither quantity increases in a linear relationship with respect to time, the students might conceive of a covariational relationship parametrically with

respect to conceptual or experiential time, compatible with P. W. Thompson's (2011) first-order model of an understanding of continuous covariation.

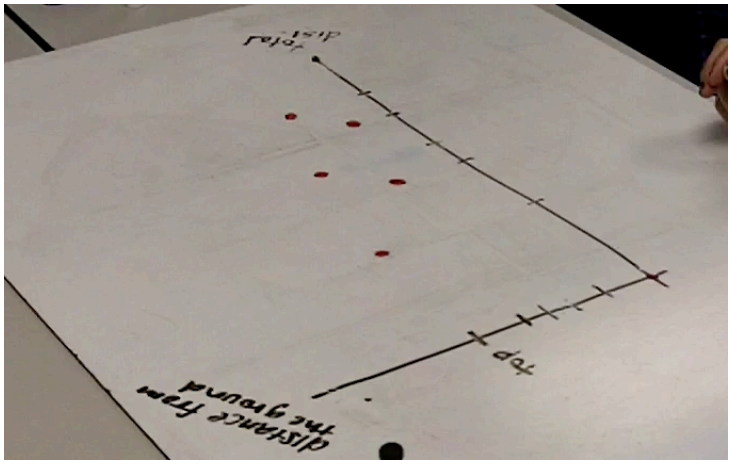
The students' activities.

The pair first questioned if they were given an exact value for the maximum height of the tower. Dr. Moore suggested they use a fictitious value but they decided to use the non-numeric label of "top" for the maximum height. After labeling their axes (see *Figure 44*c), the pair identified landmark moments where the ride changed directions. Arya stated, "They go all the way to the top [*pointing to the tick mark labeled 'top' on the vertical axis*], and then that's where [*pointing to the location on the horizontal axis equally far from the origin as the point labeled 'top'*], how far they traveled first." Determining a precise magnitude, Katlyn used her fingers to measure the length from the origin to the tick mark labeled 'top' along the vertical axis and approximated this same length from the origin along the horizontal axis (*Figure 44*a).

Next, the pair marked a point on the vertical axis to represent the rider's height where the first drop ended. Katlyn measured the length along the vertical axis from 'top' to this new point and translated this length to the horizontal axis starting at the previous point they had marked (*Figure 44*b). They continued this process of estimating minimum and maximum heights of the rider by placing ticks on the vertical axis and measuring the length between consecutive ticks on this axis. They then coordinated this change in height to an equivalent change in distance by translating the length measured as the change in height to the last point on the horizontal axis (MA3). Each student understood that a (chunk of) change in height corresponded to an equal (chunk of) change in distance (in magnitude), with the result of this coordination producing the graph in *Figure 44*c. At this point in their activity, I did not interpret their activity to imply they had considered how the two quantities co-varied within these chunks.



(a) (b)



(c)

Figure 44: Arya and Katlyn's (a)-(b) activity coordinating changes in distance from the ground to changes in total distance traveled and (c) the resultant graph.

The pair then spontaneously sought to determine how to connect the points; as they did so they considered the speed of the rider. For instance, Katlyn stated, "They look like they shoot up," and described the rider's speed at a later moment as "way slower." Arya added, "So it stops here [*pointing to the first point on the plane representing the maximum height of the rider*]. So it's gotta [*motioning from origin to the first point on the plane*], it's got to speed up and then slow down." After drawing a concave down curve from the origin to the first point corresponding to the rider's maximum height, Arya stated, "Okay, so if we think of this as our

total distance [*pointing to the label 'total distance' on the horizontal axis*]…We're covering distance pretty quickly here [*pointing to the beginning of the curve*], or we're covering height here, so this is at a pretty fast rate [*motioning her finger from the origin over the beginning of their curve*]." Although Arya initially referred to the quantity on the horizontal axis as "total distance," neither student explicitly referred to the quantity represented on the horizontal axis during the next three minutes of creating and justifying their graph (*Figure 45*), instead focusing on height and the "rate" at which it changed. Dr. Moore intended to explore how the pair was coordinating quantities and using the rider's speed. Focusing on the first part of their graph, Dr. Moore asked, "You mentioned rates and faster and slower, what's going on there?" The pair responded (Excerpt 19).

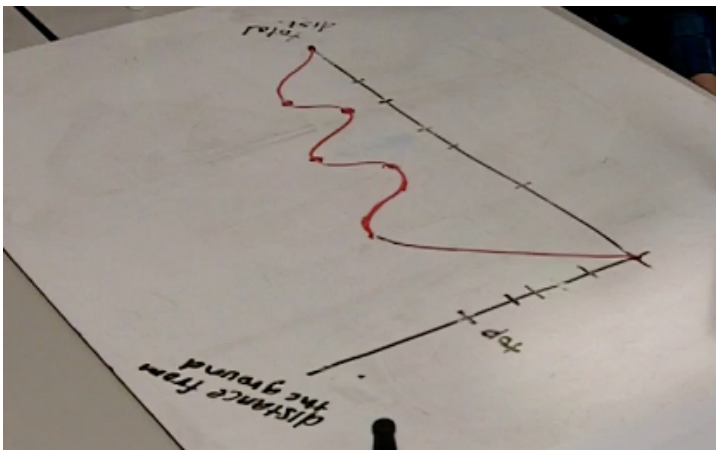


Figure 45: The pairs completed graph.

Excerpt 19: The pair discussing speed in their graph.

1	Katlyn:	So you start off going, like from rest going extremely fast [*motioning from*
2		*origin up the curve*]. But then once, in order to stop you have to slow
3		down. And so that's why it kind of gets less steep at the top [*motioning*
4		*towards the top of the curve*]. And then you're at a rest here [*pointing to*

5		*the first point they drew*], so you're going to start going really, really, fast
6		but in order to stop again [*moving from first point on plane representing a*
7		*maximum to second point on plane representing a minimum*], at the semi-
8		bottom, it's the same thing.
9	Arya:	Like the slope represents the speed.

As they watched the video, Katlyn (lines 1-8) and Arya (line 9) were attentive to the rider's velocity changing throughout the trip, imagining the "slope" of their curve as representing the rider's speed. Thus they (tacitly) coordinated the rider's height, speed and (tacit or experiential) time (lines 1-2, 5-6) when constructing the smooth curve. This hypothesis is supported by the fact that their graph accurately represents the rider's time and height.

After the pair constructed the smooth curve, Dr. Moore asked them to re-explain how they plotted their initial points to determine if the pair realized the inconsistency he perceived in their activities. Addressing this, Katlyn re-engaged in measuring changes in height along the vertical axis and translating these to changes in total distance on the horizontal axis (e.g., *Figure 44*(a) and (b)). Justifying this activity, Arya said the vertical axis "can kind of represent our tower," and leveraged this image to argue, "Each time they travel that's part of the total distance. So every height traveled as they go to the next height you have to add that distance traveled to the total distance." Both students then described total distance as an accumulation of the rider's changes of height as he went up and down the ride. As in their initial activity, both students first coordinated completed changes in height then coordinated those changes to equal completed changes in distance; they were not concerned with how height and distance were covarying *within* the interval.

In order to have them further consider their activity when 'filling-in' the graph, Dr. Moore asked, "So talk to me about this first trip, you said things are speeding up slowing down, can you break it up in terms of amounts of change?" Arya stated, "For equal changes in total distance [*Katlyn taps along the horizontal axis indicating equal changes*], the change in distance from the ground [*Arya points to the 'distance from the ground' label on the vertical axis*] is decreasing [*Katlyn taps locations along the curve*]" (MA1-3). Katlyn provided a similar explanation (i.e. engaged in MA1-3), with both of their explanations accurately describing their *created graph* (see *Figure 46* for their work representing the covariational relationship in their graph).

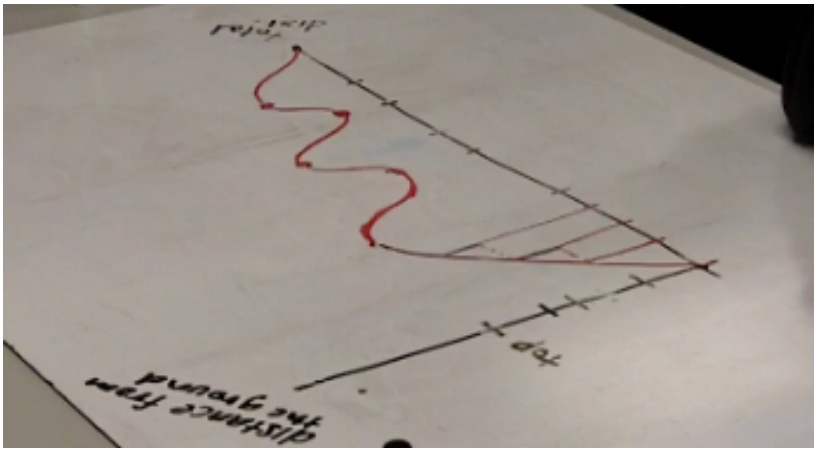


Figure 46: Using covariational reasoning to describe the relationship represented in their graph.

Summary of their activity so far.

When initially representing the relationship between height and total distance, the pair leveraged an image of the situation in which the vertical axis represented the height of the tower, and they (accurately) coordinated that a change of a chunk of height corresponded to an equal change of a chunk of distance. They imagined the rider's height and distance changing at landmark locations of the ride (e.g., when the ride changed directions) as they coordinated height first. As they connected the points, they leveraged an image of the rider's speed through one

complete ride to construct a graph composed of smooth curves. Further, neither student observed a contradiction as they moved between these two descriptions. After constructing a graph composed of smooth curves, they *accurately* described the relationship between distance and height *as depicted by their graph*.

Returning to the problem.

Dr. Moore asked the students to elaborate on how they were interpreting speed, conjecturing this might support the students in conceiving an inconsistency in their activities. Arya identified speed as "rate of change." to which Dr. Moore asked, "What's the quantities between?" Katlyn responded that speed is a coordination of "distance over time." Arya then questioned their graph (Excerpt 20).

Excerpt 20: Arya questioning their graph.

1	Arya:	Do we have to do distance over time? Like this graph [*pointing to the*
2		*graph in Figure 46*] doesn't have to include time necessarily so.
3	Katlyn:	It's like one [complete ride]. Like it's not, it doesn't really. Does it really
4		matter?
5	Arya:	But it kind of does. Because the way you changed the rate that the
6		distance changes is over, distance changes by time.
7	Katlyn:	But I mean, even if you said this was like *t* equals zero [*pointing to origin*]
8		and *t* equals [*pointing to where the graph intersects the horizontal axis*
9		*again*], like if it takes it thirty seconds for them to reach, I don't know.
10		Does it really matter when we're talking about total distance?
11	Arya:	Does time matter?
12	Katlyn:	Yeah.

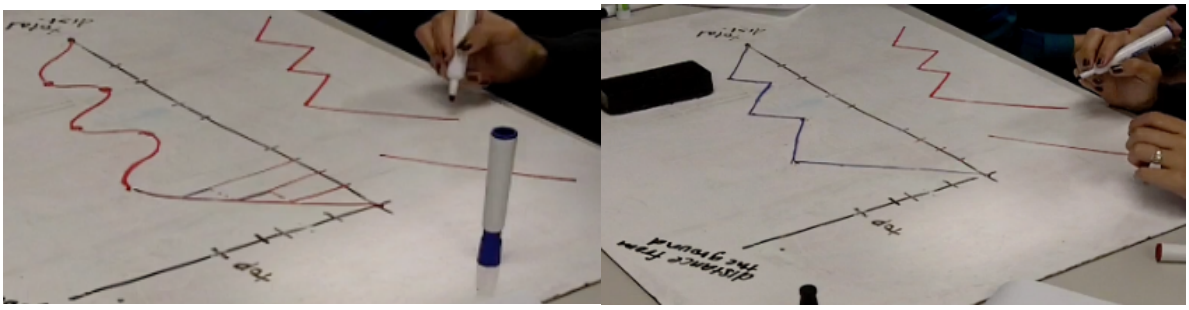
13	Arya:	Yeah, but that's how we figured this out [*pointing to their graph*]. What I
14		was thinking is like how else could you do [*motioning along the first part*
15		*of the curve*], is there another way?
16	Katlyn:	To get less steep is that what you mean?
17	Arya:	Yeah, like or [*pause*], could you, is there a different relationship?

Until this point, Katlyn and Arya had not maintained attention to the quantity they represented along the horizontal axis; in this instance Arya argued that they had represented time on the horizontal axis when drawing smooth curves (lines 3-12). Further, Arya realized that their graph did not (directly) represent time (lines 1-2) and began to question whether they accurately represented the relationship between height and distance (line 13-15, 17). Yet, immediately after Excerpt 20, Arya argued their created graph was correct.

Because the students were only momentarily perturbed, Dr. Moore returned the pair to considering equal changes in height, but this time choosing a change of height that did not correspond to a completed up or down trip (i.e. chunk); he asked how much total distance changed if the rider went up ten feet, which corresponds to only a portion of a completed up trip. Katlyn responded, "They've traveled ten feet total." Dr. Moore then asked, "What if we rise up another ten feet?" Arya responded, "We're at twenty feet and we've traveled twenty feet" (MA1-3). Each student continued to coordinate specified (numeric or magnitude) changes in height with equivalent changes in distance (though still in completed but smaller chunks).

Dr. Moore then asked, "So let me reverse it now. Let's say total distance increases by ten. So I start at the bottom, total distance increases by ten, what do we know?" Katlyn responded, "Our distance from the ground is also ten." Dr. Moore asked about another change of total distance of ten and the pair argued the rider might go up five and then down five. After assuming the ride

did not change direction for this increase, the students engaged in novel reasoning as they coordinated that for an additional change in distance there was an equal change in height. Katlyn then claimed their graph, "Is so not right then," and indicated the graph was composed of linear segments. Arya agreed, stating, " [The graph] doesn't include speed." Katlyn sketched a new graph below their original graph (*Figure 47*a), and each student described that any change in distance corresponded to an equal change in height when justifying this graph.



(a) (b)

Figure 47: (a) A second sketch for the Power Tower and (b) their completed work.

After this activity, both students reasoned that for a given change in either quantity, there was an equal change in the other quantity (assuming the ride did not change direction in the interval of change). Although they had reasoned that a change in height corresponded to an equal change in distance (over completed down or up portions of the trip), this reasoning did not initially carry into their activity coordinating a change in distance to a change in height, particularly when considering what happened *within* the chunks they initially created. Instead the pair coordinated height, speed, and (experiential or tacit) time as they watched the rider complete one ride when drawing the curve *within* the chunks. Whereas the students' coordination of the quantities in the situation changed depending on if they were reasoning about completed chunks of change or within those chunks of change, by the end of the episode, they conceived of an invariant, bidirectional, relationship between distance and height.

Summary and implications.

The pair constructed a bidirectional relationship between height and distance of a rider on the Power Tower. However, constructing this relationship was not trivial. During their construction and re-construction, the pair reasoned about different relationships (and different quantities) depending on which quantity (and axis) they considered first. They initially coordinated that for a (chunk of) change in height there was an equivalent (chunk of) change in distance to determine specific points on their graph. As they moved to connecting these points, they constructed a smooth curve as they coordinated height, speed, and (tacit or experiential) time.

It was only through explicitly addressing changes in distance for changes in height, and then coordinating changes in height for changes in distance that the pair conceived of an invariant relationship regardless of the direction in which they coordinated the quantities. Their activity was compatible with Saldanha and Thompson's (1998) description of a student constructing a covariational relationship by coordinating changes in one quantity, then the other, then back to the first. Through this process they constructed a bidirectional relationship between height and distance that entailed anticipating that for any change (in magnitude) of either quantity, there was an equivalent change (in magnitude) of the other quantity, with neither quantity *causing* changes to the other quantity (Moore & Paoletti, 2015; A. G. Thompson & Thompson, 1996).

In addition to giving a detailed example of students constructing a bidirectional relationship, this example provides evidence of the complexities students can encounter as they coordinate quantities. Namely, the pair conflated quantities as they considered change happening in completed chunks and coordinating changes within those chunks, coordinating equal changes of height corresponded to equal changes of distance in the former case and only attending to changes of height (with respect to tacit or experiential time) in the latter case. Realizing this

inconsistency was not trivial, as the students did not conceive of a conflict until Dr. Moore asked them to consider and re-consider the relationship by coordinating each quantity first. Specifically, the students did not reason bidirectionally with respect to the axes to interpret the graph as representing an invariant relationship *until* they constructed the relationship between height and distance as a bidirectional relationship by reasoning about amounts or rates of change.

Implications of constructing a sophisticated quantitative structure.

Once the students had constructed a bidirectional relationship, they leveraged this understanding when addressing related prompts about the relationship between distance and height of a rider on the ride. For instance, when asked to draw a graph for a tower with double the height, Katlyn immediately labeled a new 'top' by doubling the magnitude of the length from the origin to top along the vertical axis and then drew a segment conveying the same rate of change as their previous graph to represent the relationship in this new situation (*Figure 48*). Each student understood that the underlying covariational relationship between height and distance did not change if the height of the ride doubled; regardless of the height of the tower, an equal change of quantity A corresponded to an equal change of quantity B. When Dr. Moore asked how the graph changed if the speed of the ride doubled, both students indicated their original graph (*Figure 47*b) represented this scenario. Once the pair had constructed a bidirectional relationship between height and distance, they had no difficulties addressing changes to the posed problem situation.

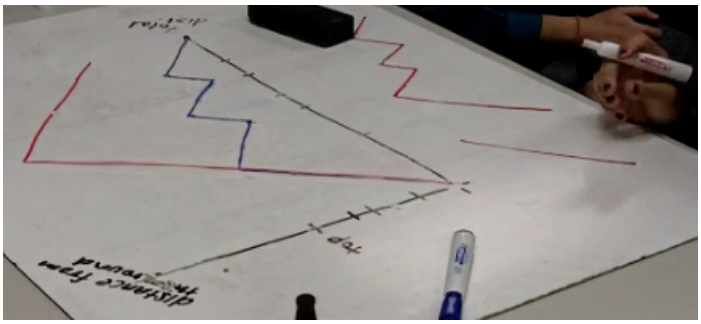


Figure 48: The pair's graph for a tower with twice the height of the original.

The Bottle Problem

Arya and Katlyn had worked together in all prior episodes, so I intended to explore how each student represented a relationship in a novel situation without the other. Hence, six days after the Power Tower teaching episode, I worked with Arya and Dr. Moore worked with Katlyn with Kevin LaForest acting as an observer as we engaged the students in the Bottle Problem.

Task design.

Originally designed by the Shell Center (Swan & Team, 1999), researchers have used the Bottle Problem to explore students' quantitative and covariational reasoning (Carlson et al., 2002; Carlson et al., 2003; Johnson, 2012a) with Carlson and colleagues providing a fine-grained analysis of the mental actions students may engage in as they reason about the relationship between height and volume (described in Chapter 2). In our version of the Bottle Problem, each teacher-researcher started the episode by providing the student a picture of a physical bottle and asking her to imagine the bottle filling with liquid. We asked the student what quantities she imagined in the situation, eventually asking her to represent the relationship between height and volume of liquid in the bottle as it filled.

After each student had constructed a graph from a bottle, we provided her with a new graph and asked her to draw a bottle from the given graph. As each student had previously engaged in activity that suggested her reasoning bidirectionally with respect to the axes, we intended to

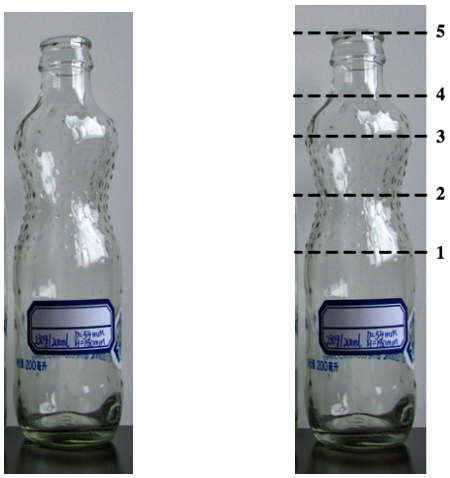
explore if (and if so how) the students leveraged this reasoning if given a graph with the opposite axes orientation than the one they used when constructing their graph. In order to accomplish this, we prepared two graphs for each student, one with the horizontal axis labeled volume and the other with the horizontal axis labeled height, with the intention of providing each student a graph from which to draw a bottle with the opposite axes orientation of the graph they produced.

After each student completed both parts of the problem, we intended to ask the students to switch boards with each other and produce a bottle from the peer-produced graph and a graph from the peer produced bottle. We anticipated that if either student had reasoned in a way that resulted in their producing an incorrect graph or bottle from our perspective, creating a bottle/graph from their partners graph/bottle then comparing the results and the original bottle/graph would support the students in reflecting on inconsistencies.

Arya's Activity in the Bottle Problem

Arya identified height and volume as two quantities she imagined if a bottle was filling with liquid; I asked her to describe how she imagined the two quantities changing for a given bottle (*Figure 49*a). She described volume as, "If you're pouring water in, just how much water changes over time as your filling it up," and height as, "As the water increases… like how much [*sliding her finger vertically up the bottle*] of the bottle is filled." After this, I asked her to graph the relationship between volume and height "as we fill the bottle." Arya stated, "Well as the bottle is being filled [*moving her finger vertically up the bottle*], so I guess that's like kinda like height … And like how much water [*motioning horizontally over a cross section of the bottom of the bottle*] is kinda like the volume in that space. So it's increasing as you [*moving her finger up the bottle*] are filling it up, the height and the volume" (MA2). Arya's activity indicated she was using cross sectional areas of the bottle to consider the volume of the liquid in a specific height

interval. I infer from her last statement that her imagery of the situation did not entail a dependence of one quantity on the other; Arya imagined the two quantities increasing as she imagined the experience of watching the bottle fill.



(a) (b)

Figure 49: (a) Arya's bottle and (b) a recreation with tick marks to represent landmark points.

I asked if the graph would be linear, "Or what's sort of happening?" Addressing this, Arya drew landmark locations (recreated with dashed segments and numbers for the sake of the reader in *Figure 49*b) on the bottle to identify heights where the nature of the bottle's curvature changed. Arya identified that from the bottom to tick 1, "the volume is like constant," which I took to mean that volume was increasing at a constant rate with respect to height (MA3). Continuing, she described, "The increase has changed [*motioning from tick 1 to tick 2*], [volume]'s still increasing but its change is decreasing" (MA3). Moving from tick 2 to tick 3, Arya stated the cross sections of volume contained more water as she moved up the bottle, as the bottle was getting wider. Finally she concluded that volume increased by less for changes of height from tick 3 on because the bottle was getting thinner. Although Arya initially described the quantities simultaneously covarying in terms of directional change (MA2), as she moved to

describing the relationship at a more fine-grained level (i.e. MA3), she coordinated how volume changed for changes in height.

I asked Arya to sketch a graph of the relationship she described and she drew axes, labeling the horizontal axis height, and the vertical axis volume. She marked a point on the horizontal axis that corresponded to the maximum height of the bottle, which she labeled as 180 based on a number on the label of the bottle. Arya identified she intended to represent the constant change she imagined in the situation and drew a straight segment. When I asked why the rate was constant, Arya responded, "For every change in height, the amount of water in the bottle is increasing the same" (MA3). Arya then drew a concave down curve to corresponded to the relationship from tick 1 to tick 2, and continued by sketching curves on the plane corresponding to the other sections (see *Figure 50* for her completed work) without providing a description of why she drew the graph in this way.

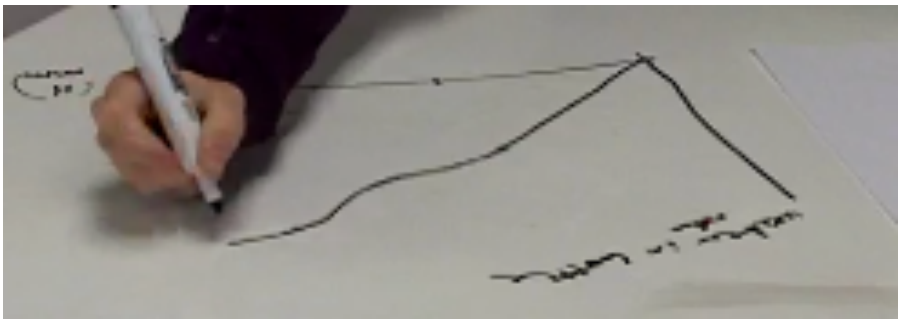


Figure 50: Arya's completed sketch for the Bottle Problem.

To examine how Arya was moving from the situation to her graph, I highlighted her tick 1 on the bottle in blue and asked where this tick was represented on the graph. Arya identified tick 1 on the picture of the bottle corresponded a point on her horizontal axis and drew a vertical segment from the horizontal axis to the point where the curve first changed concavity.[35] Arya

[35] I provided this height a hypothetical value of 90 and Arya noted this on the horizontal axis. However, in her activity moving forward neither of us used specific values.

then argued the magnitude of this segment represented the volume of the water in the bottle at tick 1 (*Figure 51*) and engaged in compatible activity for tick 2 (in red) and tick 3 (in green). Arya understood the points on her curve as representing the relationship between two projected magnitudes representing the height and volume of liquid in the bottle as liquid entered the bottle.

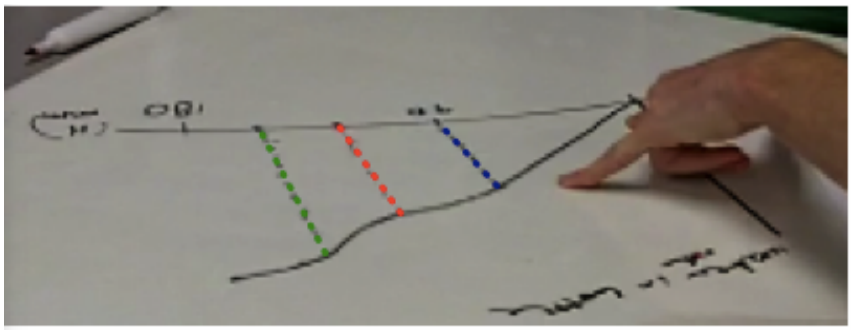


Figure 51: Arya's graph representing heights and volumes at specific values.

Arya had not yet described how she associated a concave down curve with a widening bottle. I asked her to justify the curvature of her graph. After re-addressing the straight segment representing the relationship from the bottom to tick 1, Arya moved to the picture of the bottle and continued, "And then once we get here [*between tick 1 and tick 2*], our bottle starts getting smaller. So as you're going up [*adds two tick marks between tick 1 and tick 2*], the amount of water in those [*indicating the space between successive tick marks from tick 1 to 2*] is getting smaller between the different changes… The amount of water is still increasing 'cause the bottle is holding more water… So the volume is increasing… But the change is decreasing" (MA1-3). Consistent with her previous activity, Arya relied on her image of the volume as a cross sectional area to identify that for equal changes of height between ticks 1 and 2 the volume increased at a decreasing rate.

After this, I asked Arya to explain how the relationship she just described was represented in her graph to investigate how she translated from the bottle to the graph. Arya marked equal changes along the horizontal axis between the blue and red segments to verify she represented

the relationship she described when working with the bottle. When describing the other sections of the bottle, Arya's activity was consistent with her activity between tick 1 and tick 2. Arya relied on landmark heights and the chunks of volume between these landmarks to describe the relationship between height and volume of water filling the bottle. In order to justify her curve, she divided these chunks into sub-chunks on the picture of her bottle to describe the covariational relationship *within* each chunk, representing these equal changes on the graph to ensure it matched with her description of the relationship.

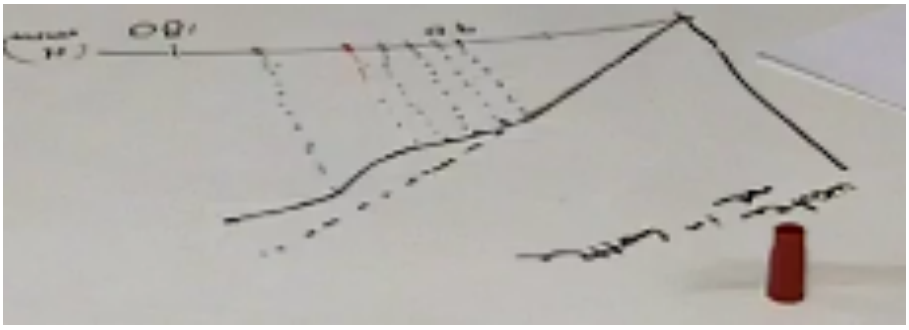
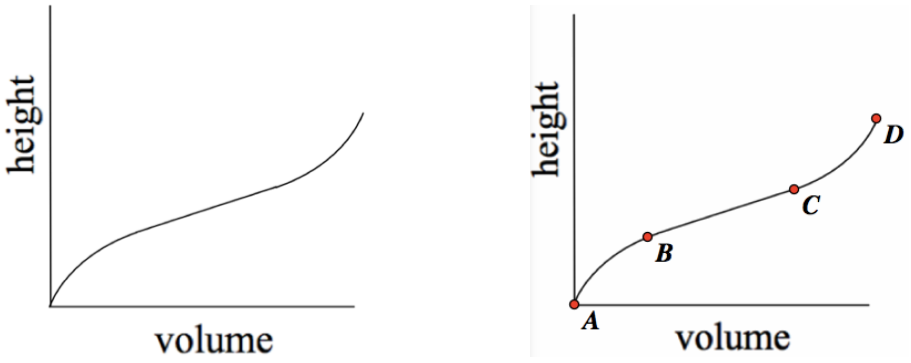


Figure 52: Arya representing equal changes between tick 1 and tick 2.

Moving from graph to bottle.

Because Arya constructed a graph in which she represented height on the horizontal axis, I gave Arya a graph with height represented on the vertical axis (*Figure 53*). Looking at the graph, without explicitly acknowledging which quantity was represented on which axis, Arya sketched a bottle that became thinner as height increased. I conjectured Arya assumed the axes were oriented with height on horizontal axis and interpreted the graph (from A to B) as representing volume increasing at a decreasing rate with respect to height. Arya then concluded that the curve was straight between points B and C. As she considered how to draw the next portion of her bottle, Arya looked at the curve, then back to the picture of her bottle, then back to the curve, and said, "Oh!… [*erases her bottle*] I've been reading this graph [*laughs*] … This [is] the volume

[*pointing to horizontal axis*] and that's the height [*pointing to the vertical axis*]." Arya continued (Excerpt 21).



(a) (b)

Figure 53: (a) The height and volume graph I gave to Arya and (b) the graph with added reference points for readability.

Excerpt 21: Arya describing a perturbation stemming from the changed axes.

1	Arya:	So as height increases [*moves her finger vertically up the height axis, then*
2		*turns the graph 90-degrees counterclockwise*]. You messed me up here.
3		As volume [*4 second pause*]. Okay, maybe I can still do it this way [*turns*
4		*paper back to its original orientation*]. The height is [*13 second pause,*
5		*grabs picture of original bottle, Figure 49*].
6	TP:	So talk me through what you're trying to figure out now, so I know.
7	Arya:	Um, so the change in the height [*motioning over the curve in Figure 53*
8		*oriented as shown*], if I think of it as. I may have to turn this to think of it
9		different. For equal changes of volume, the change of the height is
10		decreasing. Right? But what does that look like? The change in height,
11		'cause here [*looking at prior work, when given a picture of a bottle*], okay.
12		For equal changes in volume. I don't know how to look at that well. I'm

13	going to have to look at this different.

Interpreting the curve by considering equal changes of the quantity on the horizontal axis, Arya made true statements regarding how height covaried for equal changes in volume (lines 9-11). However, she was unsure how to translate these statements to a picture of a bottle (line 10). Arya's imagery was constrained to interpreting the relationship by coordinating height first (lines 8-9, 12-13) in order to draw a bottle from the given graph.

As she continued to interpret the relationship represented in the graph, Arya said, "So for equal changes in [*turns the paper 90-degrees, then back*] height [*motioning from the vertical, height, axis horizontally to the curve throughout*], the volume, okay [*pause*], is increasing at an increasing rate [*pointing from A to B*]… And then it's straight, and then it's at a decreasing rate [*motioning from C to D*]." Arya accurately described the relationship from my perspective, however, she was not confident that the curve was linear from B to C. To investigate this, she turned the paper 90-degrees counterclockwise and drew segments to represent volumes at equal changes in height (represented in *Figure 54*), which led her to conclude the curve was linear from B to C.

Arya spontaneously reasoned with the vertical axis as representing the input quantity with the goal of coordinating equal changes of the height to determine how volume varied. I conjecture her activity stemmed from her inability to imagine how the relationship she inferred from the graph by coordinating volume first translated to a picture or cross section of a bottle. Although she turned her paper and drew vertical segments to investigate the relationship represented by the curve (as shown in *Figure 54*), based on her initial activity motioning horizontally from the vertical axis, I conjectured she did not *need* the graph oriented in this way. Later in the episode, I tested this conjecture asking, "How would the justification change if the paper was oriented that

way [*motioning towards the paper oriented normally*]?" Arya responded, "The justification doesn't change… I just have to draw my lines sideways."

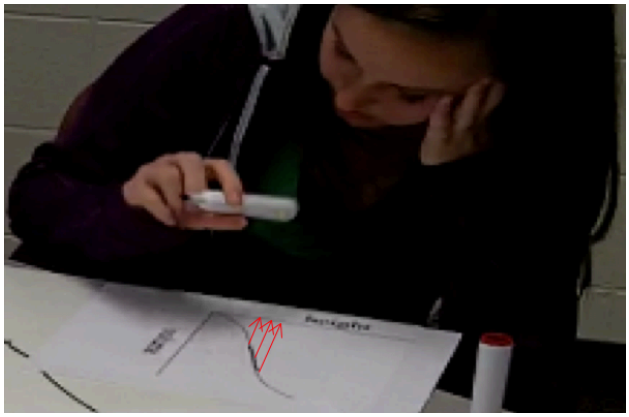


Figure 54: Arya using the axes bidirectionally.

After describing how volume varied for equal changes of height in each section of the graph, Arya drew the cross section of a bottle (*Figure 55*). I asked Arya to explain how she sketched the bottle and she represented how volume varied for equal changes in height in the graph (*Figure 56*a and b). Arya used the segments she drew to conclude that from A to B, "The change in volume is getting bigger… so it should be holding more water as you go up the bottle" (MA1-3). After describing this relationship in the graph, Arya moved to the representation of the bottle, engaging in MA1-3, to ensure that she accurately represented the relationship she intended.

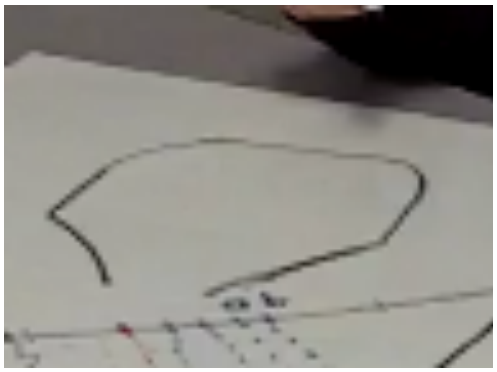
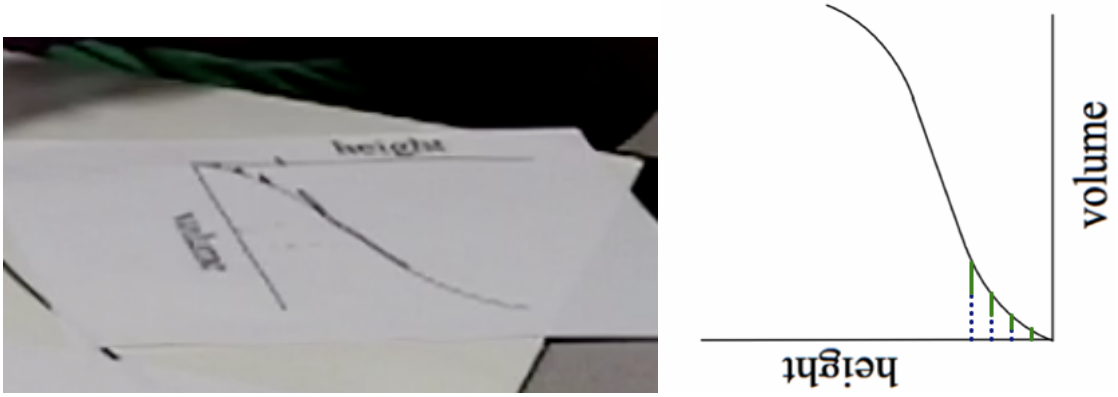


Figure 55: Arya's final bottle sketch.



(a) (b)

Figure 56: (a) Arya's work when representing equal changes of height and (b) a recreation of her work oriented in the same way she had worked with it.

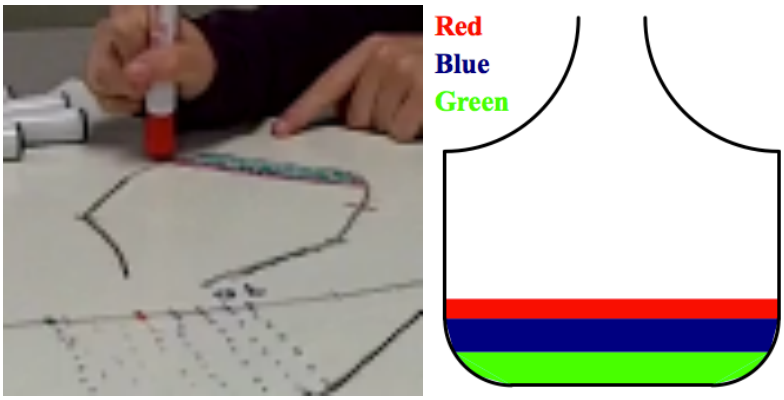
Arya described and represented the relationship between height and volume in both the graph and the bottle from B to C and from C to D. In both the graph and the bottle, Arya coordinated how volume changed for equal changes of height; Arya's imagery of the situation left her restricted to reasoning about the relationship between height and volume uni-directionally. That is, although Arya understood height and volume simultaneously covaried directionally (MA1-2), her imagery of the situation required her to coordinate height first in order to consider the relationship in terms of amounts of change (MA3) or rates of change (MA4-5).

Spontaneous change in perspective.

Arya finished describing how she drew the bottle from the graph and I intended to engage her in a new line of questioning concerning how the graph would change if the curvature of the bottle changed slightly (e.g. if the top of her bottle was still thinning but the sides of the bottle were concave down instead of concave up). Before I did so, Arya engaged in spontaneous activity (Excerpt 22).

Excerpt 22: Arya's spontaneous activity after identifying the orientation did not matter.

1	TP:	Okay. And then [*Arya points to her drawing of the bottle and starts to*
2		*mumble*]. Keep going. What were you thinking?
3	Arya:	I just want to see this [*motioning over the graph oriented as in Figure 53*].
4		I have a hard time visualizing for equal changes in volume. I guess an
5		equal change of volume, this [*using her fingers to indicate an interval*
6		*near the bottom of the bottle in Figure 55*] would be getting smaller, so
7		our change, okay. That makes sense.
8	TP:	So, what are you thinking now?
9	Arya:	So, for equal changes in volume. I just wouldn't be able to draw that, but I
10		can kind of see it.
11	TP:	So explain to me how you're seeing it.
12	Arya:	So we have this much water [*shading in an area at the bottom of the bottle*
13		*in green, Figure 57a and b*]. Right? And then we want that much water
14		again. It's going to be-
15	TP:	Maybe in a new color.
16	Arya:	Yeah. This is hard to see. So that much water would be about, since it's
17		further, smaller thickness of line, right? 'Cause we're, we're holding more
18		'cause it's further out. So [*sketching in a thinner layer in blue, Figure 57a*
19		*and b*] some water. And so our height changed that much [*using her*
20		*fingers to indicate the height of the green layer*] that time. But that time
21		[*indicating the height of the blue layer*], our height got smaller and our
22		volume is, but that volume is the same.



(a) (b)

Figure 57: (a) Arya's depiction of height changes for equal changes of volume and (b) a recreation of her work.

Arya spontaneously considered how height varied for equal changes of volume (MA1-3) (lines 3-7, 9-10), coordinating the quantities in the situations in a way that she previously indicated she was unable to imagine (e.g., Excerpt 21). I asked her to draw one more layer of volume, which she did in red, with her final work in *Figure 57*. She engaged in compatible reasoning for the neck of the bottle, representing that for equal changes of volume, the height increased at an increasing rate (*Figure 58*).

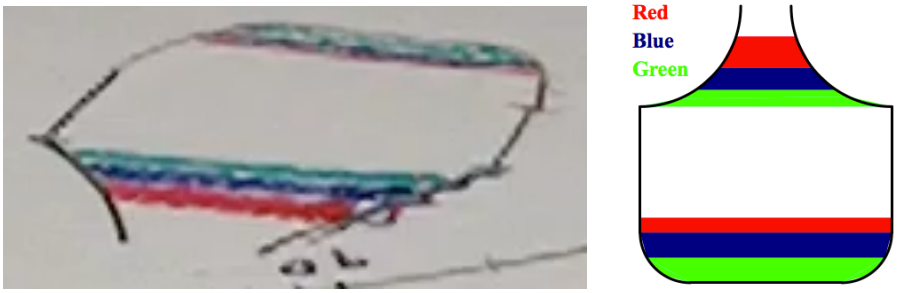


Figure 58: Arya's representation and a recreation of equal changes of volume for the top of the bottle.

I did not expect for Arya to engage in this reasoning unprompted, however, she set this goal for herself. Arya intended to reason about the relationship in the situation by representing and

coordinating changes in height for equal changes in volume, a relationship she already described through her interpretation of the graph. She successfully did this by constructing a new image of the relationship in the situation such that she represented equal changes of volume using layers of equal area to consider how the corresponding heights varied. Through this re-conceptualization of the relationship and her images of the situation, Arya constructed a bidirectional relationship between volume and height of water in the bottle such that she could coordinate either quantity first, understanding throughout that one quantity did not depend on the other.

Summary and implications of Arya's activity.

By constructing and re-constructing her images of the situation, Arya engaged in quantitative and covariational reasoning in ways that indicated both manifestations of bidirectional reasoning. Arya initially indicated the two quantities covaried at MA2. As she developed a more refined understanding of the relationship (e.g., MA3-5), she reasoned about the relationship uni-directionally, always coordinating equal changes of height and then changes of volume. Arya interpreted the vertical axis as representing the input to coordinate how volume varied for equal changes of height, which was compatible with her earlier bidirectional reasoning with respect to the axes (i.e. she understood that the vertical axis was a choice for representing the quantity she coordinated first).

Prior to interpreting the vertical axis as the input, Arya described the relationship between height and volume in the graph by using equal changes of volume. However, she could not imagine a bottle based on the relationship she inferred. Arya spontaneously set out to coordinate volume first in the situation, a goal she accomplished as she refined her images of the quantities and relationships she conceived of in the situation. Through this process, which was compatible

with Saldanha and Thompson's (1998) description of a students' developing imagery of continuous covariation (e.g., moving from considering one quantity then the second then back to the first), she constructed a bidirectional relationship between volume and height of liquid in terms of amounts of change (MA3) or rates of change (MA4-5) (i.e. reasoned bidirectionally with respect to the quantities in terms of amounts of change (MA3) or rates of change (MA4-5)). I note throughout her activity, Arya understood the two quantities simultaneously covaried directionally (MA1-2). Although this was not the first time she reasoned bidirectionally with respect to the axes or constructed a relationship between quantities bidirectionally, this was the first time she exhibited both types of reasoning in the same activity, in this case with her bidirectional reasoning with respect to the axes supporting her conception of a bidirectional relationship between quantities.

Katlyn's Activity in the Bottle Problem

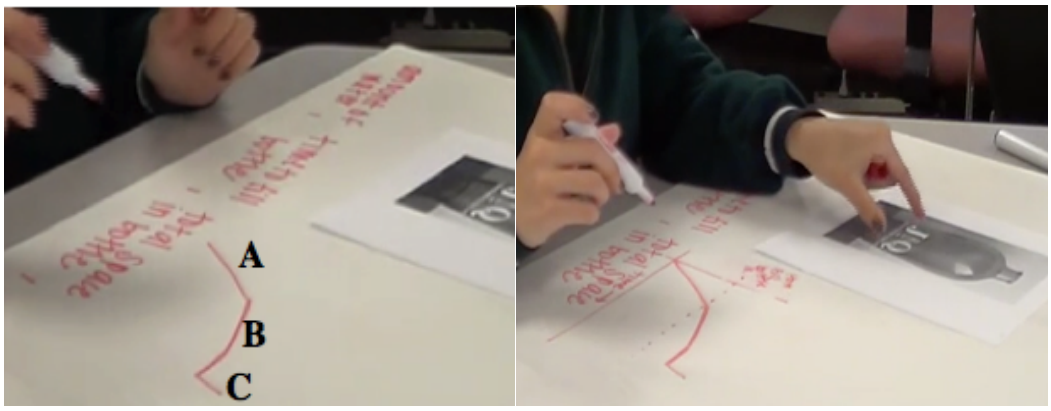
Dr. Moore provided Katlyn a picture of a bottle (*Figure 59*a) and asked her, "Imagine your filling this bottle with liquid.... Talk to me about some of the different… quantities in the situation, how you envision that occurring." Katlyn responded, "It's going to fill up really quickly… it's just going to progressively slow down [*moving her finger up the bottle*], if the water stream is constant." Katlyn then drew the graph in *Figure 60*a by imagining breaking the bottle into three areas, which I indicate by A, B, and C in *Figure 59*b. Justifying her graph, Katlyn said, "The stream is constant, then like 'cause it's [*motioning over* (A) *in Figure 59b*] so much skinnier at the bottom it's going to fill up faster [*tracing over the segment A in Figure 60a*] at the bottom. Then once it gets like a little wider it'll be slower [*tracing over segment B in Figure 60a*], and even more slow as we get closer to the top [*tracing over segment C in Figure*

60a].” Katlyn added axes to her graph (see *Figure 60*b), labeling the horizontal and vertical axes time and “amount of water,” respectively.



(a) (b)

Figure 59: (a) Katlyn’s bottle and (b) a recreation of the sections Katlyn initially reasoned about.



(a) (b)

Figure 60: (a) Katlyn’s rough sketch of the relationship and (b) her sketch with added axes.

After this, when describing the relationship she was imagining, Katlyn stated, “So like you’re increasing the amount of water but… if like the stream is constant then the amount… would be constant but it’s like it’s going to fill the bottle faster.” Whereas I inferred Katlyn was using “amount of water” to mean the height of liquid in the bottle when she constructed her graph, as

she described what "amount of water" meant a complexity arose as she considered the quantity "amount of water" in different ways; Katlyn had not clearly delineated what she meant by "amount of water" in her own thinking.

After providing her description of "amount of water" compatible with total volume, Katlyn (accurately) argued (B) contained more volume than (A) and realized this was not the relationship she represented in her graph. Hence, she erased the label "amount of water" and labeled the vertical axis "how full bottle is". Although Katlyn accurately represented the height of the liquid in the bottle as it filled at a constant rate, what she meant by "how full bottle is" was vague to the researcher (and I conjecture was vague in her own thinking).

Katlyn constructing a volume and height graph.

Due to the confusion that arose by her use of "amount of water," Dr. Moore asked Katlyn to describe how she imagined volume and height to investigate how she was conceiving of the situation in terms of these quantities (Excerpt 23).

Excerpt 23: Dr. Moore asking Katlyn to describe volume and height.

1	DM:	So as we're pouring in, we have some volume of water in the bottle right?
2		And what's happening as we're pouring in?
3	Katlyn:	That is increasing. The volume of water is increasing.
4	DM:	And then we're also going to have the height of water in the bottle. So
5		what would that, what would I be referring to when I say the height of the
6		water in the bottle? What would I be referring to?
7	Katlyn:	So like at some time maybe you're at like [*motioning her marker across*
8		*the bottle at the bottom*] one inch. Then you're at [*motioning up the bottle*
9		*and then across the bottle again*] two inches. Then you're at [*repeats her*

10 *motions*] three inches.

Having identified how she was imagining both quantities (lines 1-3, 7-10), Katlyn described that as water was added to the bottle the height was "increasing" (MA2). Dr. Moore asked Katlyn to "create a graph with those two quantities in mind," and Katlyn questioned, "I guess I'm like not even thinking about time anymore. You just want height and volume?" Whereas it appeared Katlyn reasoned about the relationship between height and volume (e.g., both are increasing), her question as well as her activity in Excerpt 23 (and consistent with her drawing of the graph in *Figure 60*) indicate she imagined both quantities changing with respect to time without an explicit consideration of how they varied in tandem; as her understanding of the question evolved, Katlyn had to remove time as a quantity she was directly representing.

After Dr. Moore confirmed he was asking for a graph of height and volume, Katlyn drew equally spaced height tick marks on the bottle (recreated in *Figure 61* (a) and (b) with added numbers for the tick marks and letters to represent the cross sectional areas for referencing). Katlyn identified that for equal changes of height, volume increased because the bottle widened (MA2). Katlyn conceived each cross sectional area of volume as containing more volume than the previous, stating that the amount of volume in (B) was twice the amount of volume in (A) and the amount of volume in (C) was three times the amount of volume in (A) (in contrast to considering that that the volume in (A) and (B) combined was three times the volume in (A)).[36] To clarify how Katlyn conceived of volume, Dr. Moore asked, "If I'm talking about volume, you're imagining volume referring to what?" Katlyn indicated she was comparing volumes in consecutive intervals of height rather than considering the total volume of liquid in the bottle at a certain height.

[36] To this point in her activity she had not referenced that the bottle thinned after tick 4.



Figure 61: A recreation of Katlyn's bottle with added letters representing volumes and numbers representing heights.

Dr. Moore intended Katlyn create a graph representing total volume so he stated, "When the height is here [*indicating the 3rd tick mark*], when I say volume I'm referring to the amount that's [*motioning over* (A)*,* (B)*, and* (C)] in the whole thing." Dr. Moore then asked Katlyn to create a new graph using this volume interpretation. Katlyn argued, "It'd be like **the amount of water in the bottle** [*voice raises*] but… that's pretty small down here [*indicating in* (A)] than like, with the amount of water in the bottle when it's at like four for example … that's a lot more [*motioning over* (D)]." Katlyn then sketched a concave up curve that accurately represented the volume changing with respect to height (presuming she represented volume on the vertical axis).

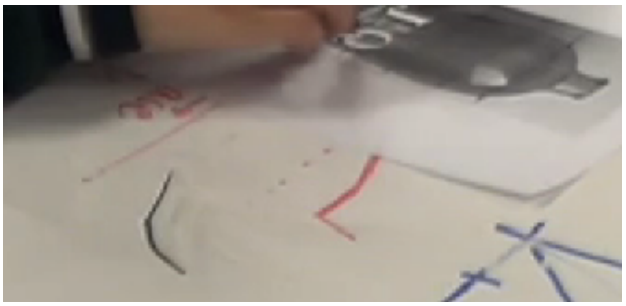


Figure 62: Katlyn's initial volume with respect to height sketch (in black, without axes).

To explore how Katlyn was reasoning about volume, Dr. Moore asked Katlyn to "say a little more." She responded by reiterating that the consecutive volumes (A), (B), and (C) increased. She then compared her bottle to an imagined cylindrical bottle, describing how the graphs representing the relationship would differ for her imagined and pictured bottles (Excerpt 24).

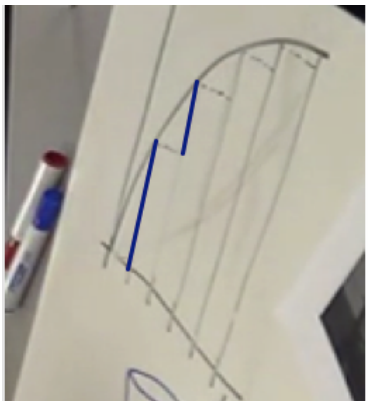
Excerpt 24: Katlyn comparing her bottle with a cylindrical bottle.

1	Katlyn:	If this was like a, like a water bottle straight, straight up and down
2		[*mumbling*]. You know what I'm saying, straight up and down where if it
3		was just a bunch of these [*indicating the cross section* (A)], then, then it
4		would just look like this [*indicating a straight segment she drew earlier*].
5	DM:	Oh, so if it was straight up and down then it would-
6	Katlyn:	Right, right, then it would look like this [*motioning over a straight*
7		*segment*]. But because it's like smaller at the bottom is like what I'm
8		thinking. So like how many of these [*indicating taking* (A) *and translating*
9		*it up the bottle*] can I get like in, like, that's a lot. This [*pointing at* (A)] is
10		a lot smaller than what I'd have at the [*pointing to the wider part of the*
11		*bottle*], it would be like [*drawing concave up curve*].

Katlyn reasoned that in a cylindrical bottle, for equal changes in height there were equal changes in volume (MA1-3) (lines 1-4). Because the amounts of volume between successive heights were increasing in her bottle (e.g., (B) contained more volume than (A)), Katlyn reasoned that for equal changes of height, volume increased at an increasing rate with respect to her bottle (MA1-3) (lines 6-11).

Although Katlyn made accurate statements regarding the relationship between total volume and height, she was not confident in her conclusion. For instance, after Excerpt 24, Dr. Moore

asked, “You’re saying there’s a lot less here [*pointing to* (A)] than here [*pointing to* (C)]. What are you using that to [do]?” Katlyn responded, “That just continues to go back to the volume’s increasing but like, how fast? Faster? Does it go up? [*8-second pause*] I don’t know. This is really making me mad.” After this, Katlyn constructed a graph (*Figure 63*a), but it was unclear to the researcher if Katlyn was reasoning about total volume or consecutive volumes as she constructed this graph. To investigate this, Dr. Moore motioned from the origin to the vertical segment from the first tick mark on the horizontal axis and asked, “How much volume do I gain?” Katlyn identified that the vertical segment at the first tick mark on the horizontal axis (recreated in blue in *Figure 63*a) represented the volume in (A). Dr. Moore motioned from tick 1 to tick 2 on the horizontal axis on her graph and asked, “How much volume do we gain?” Katlyn highlighted the change in volume from tick 1 to 2 in blue (recreated in *Figure 63*a) and represented this change on her bottle by drawing vertical segments from where tick 1 met the bottle to tick 2 and shading the part of the bottle represented in red in *Figure 63*b. She argued, “That would just be like the difference of these two [*motioning from* (B) *to* (A)]. So like if [(A)] is one and [(B)] is one-and-a-half then you gain a half.” Katlyn reverted to thinking compatible with her comparing the relative volume of consecutive cross sections; she accurately represented the volume of water and amount of change of volume of water *within each cross section* with respect to height in the graph (*Figure 63*a) and in the situation (*Figure 63*b).





(a) (b)

Figure 63: A recreation of (a) Katlyn's graph and (b) Katlyn's bottle.

Intending to support Katlyn in re-considering total volume, Dr. Moore requested she shade parts of the bottle for corresponding parts of the graph. He first asked her to represent in the bottle what corresponded to the blue segment at the first tick mark on her graph in *Figure 63*a, and, as before, she indicated this segment corresponded to (A). He then asked her to represent in the bottle whatever corresponded to the to the black segment that went from the second tick mark on the horizontal axis to her curve in *Figure 63*a. Katlyn stated, "Black is like this whole thing," and motioned over (A) and (B). Next, Dr. Moore asked what the blue segment that represented the amount of change from tick 1 to tick 2 represented in the bottle. Katlyn responded by again shading in the difference in area between the consecutive cross sections, represented in red in *Figure 63*b. Although Katlyn described the entire segment at tick 2 on her graph as representing total volume at tick 2, when describing the segment representing the amount of change from tick 1 to tick 2, she reverted to reasoning with volume in consecutive cross sections.

Hypothesizing Katlyn was close to identifying an inconsistency in her use of volume, Dr. Moore questioned, "So just that part?" Katlyn responded, "I think so. Because like that's [*pause*]. Well, I mean [*pause*]. Yea, I don't know I'm [*puts hand on head*]." Dr. Moore again asked Katlyn what the black segment at tick 2 corresponded to and she reiterated that this represented

the total volume in (A) and (B). He then covered the blue segment representing the amount of change from tick 1 to tick 2 and asked what the remaining black segment represented (i.e. the black segment from the horizontal axis at tick 2 equal in length to the segment from tick 1 to the curve). Katlyn responded (Excerpt 25).

Excerpt 25: Katlyn realizing and resolving an inconsistency in her thinking.

1	Katlyn:	How much is that? That's just this part [*indicating* (A)].
2	DM:	Okay, so then that [*indicating the entire black segment*] has to be?
3	Katlyn:	That has to be this whole thing. I don't get it. [*pause*] I mean, it makes
4		sense. So then this has to be, then this graph is wrong.
5	DM:	So what makes you say that?
6	Katlyn:	'Cause this [*pointing to* (B)] is so big compared to this [*pointing to* (A)]. It
7		has to be like [*drawing red segment starting at the same place but longer*
8		*than her original amount of change segment from tick 1 to tick 2, see*
9		*Figure 64*]. [The red segment corresponding to (B)] is going to have to be
10		like double [the blue segment corresponding to (A)].

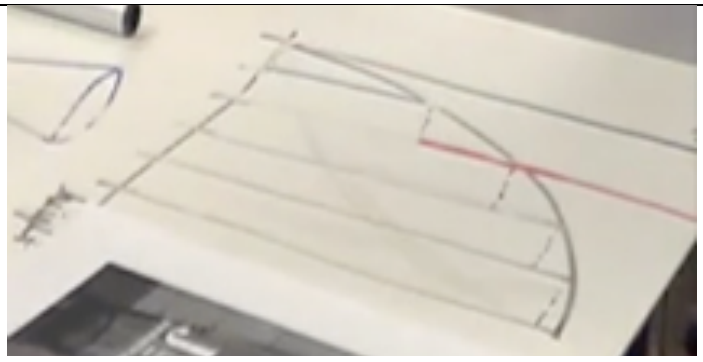
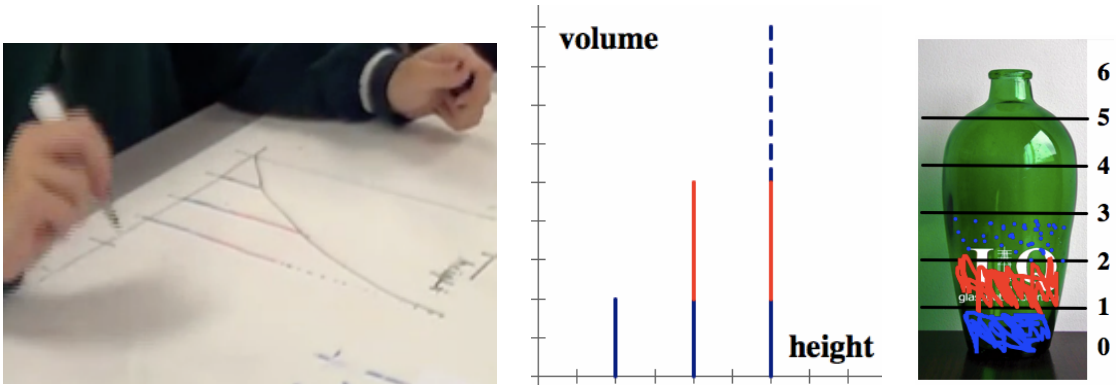


Figure 64: Katlyn adjusting her graph.

As she re-considered how her representation of amounts of change of volume in her graph and in the bottle related, Katlyn realized her graph did not correctly represent total volume (lines

3-4). Once she realized this, she had no difficulty accurately describing the change in volume from tick 1 to tick 2 on her horizontal axis as representing the volume in (B) rather than the difference between the volumes in (A) and (B). Katlyn refined her images of the quantities in the situation as she considered what she intended to represent.

Katlyn drew a new set of axes, with height and volume on the horizontal and vertical axis, respectively, and marked equal changes along the horizontal axis. She drew a solid blue segment at tick 1 on the horizontal axis and indicated this segment represented the volume in (A). At tick 2 on the horizontal axis, she drew a solid blue segment congruent to the segment at tick 1, then added a longer solid red segment, indicating the solid blue segment represented the volume in (A) and the solid red segment represented the volume in (B). Katlyn repeated this process, representing the volume in (C) by adding a dashed blue segment longer than the solid red segment (see *Figure 65*a-b). Katlyn represented this relationship in her bottle by color-coding cross sectional areas (e.g., (A) in solid blue, (B) in solid red, (C) in dotted blue, recreated in *Figure 65*c) She concluded this interaction by drawing a concave up curve (see *Figure 65*a). Katlyn constructed a graph that accurately represented total volume with respect to height with a conception of volume compatible with total volume; she reasoned about the magnitudes of segments she constructed as representing amounts of volume at specific height intervals, understanding that each added segment corresponded to a cross sectional amount of volume added to the total volume. Her completed graph is shown in *Figure 66*.



(a) (b) (c)

Figure 65: Katlyn representing total volume with respect to height (a) in a graph (b) recreated in a graph and (c) recreated in the bottle.

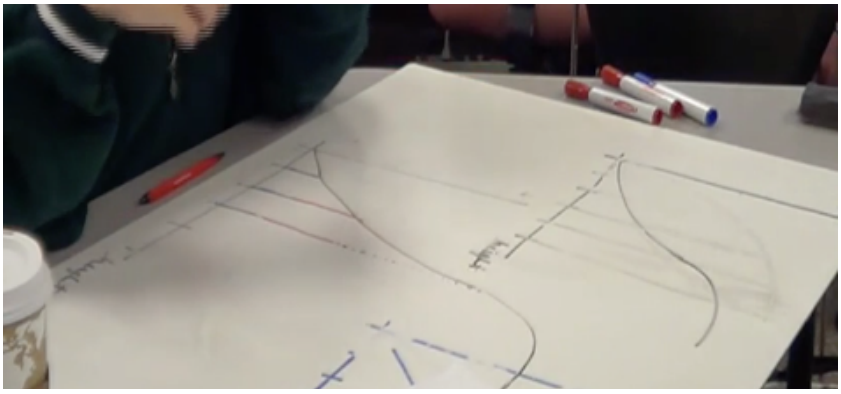
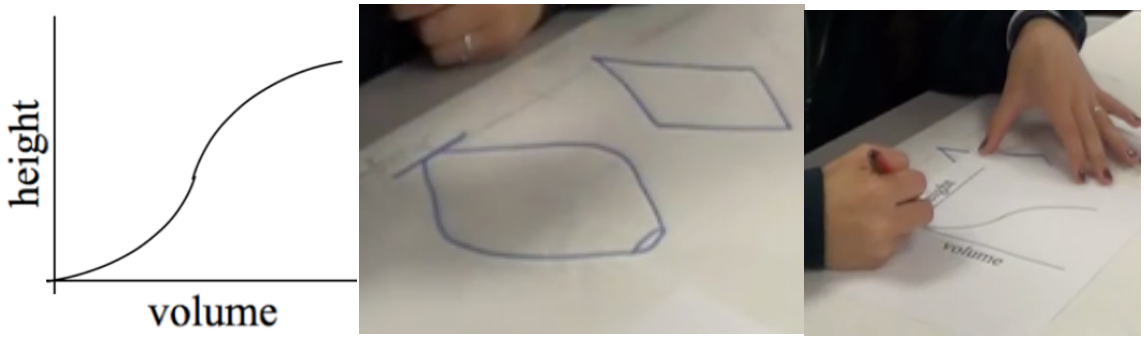


Figure 66: Katlyn's final graph.

Moving from graph to bottle.

After her activity above, Dr. Moore asked Katlyn to sketch a bottle that corresponded to a graph that represented volume on the vertical axis (*Figure 67*a). Katlyn initially stated the bottle was shaped like a "rounded diamond," sketching the bottle in *Figure 67*b. I conjecture when drawing this bottle, and like Arya, Katlyn was not attentive to the quantities on each axis and instead generalized that the particular curvatures corresponded to the bottle getting wider or narrower (e.g., concave up means wider, concave down means narrower). This conjecture is

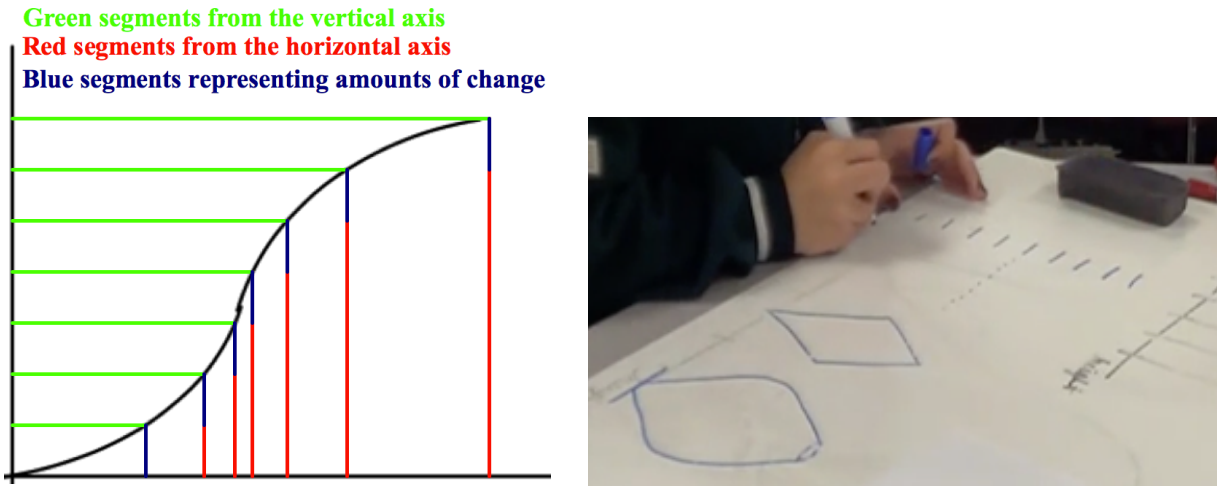
supported by the fact that her bottle accurately represented the relationship depicted by the graph if the horizontal axis represented height.



(a) (b) (c)

Figure 67: (a) Katlyn's graph, (b) Katlyn's initial conjecture, and (c) Katlyn turning the graph to engage in prior reasoning.

Dr. Moore asked, "So why that?" Katlyn then turned the paper 90-degrees (see *Figure 67*c), marked equal changes along the now horizontal axis, and drew vertical segments from these equal changes to the curve (recreated in green in *Figure 68*a). She then turned the paper back to the given orientation and drew vertical segments to the now horizontal axis from where the equally spaced segments drawn from the now vertical axis intersected the graph (recreated in red in *Figure 68*a). Katlyn then drew equally spaced tick marks on her board to represent equal changes of height for a new sketch of a bottle. It was unclear from the researcher's perspective if Katlyn intended to use this sketch to confirm her initial bottle or if she anticipated that she was constructing a bottle with a different shape.



(a) (b)

Figure 68: (a) A recreation of Katlyn's graph and (b) Katlyn represents equal changes of height for her bottle.

Katlyn represented amounts of change from one (now) vertical segment to the next (recreated in blue in *Figure 68*a) and as she examined the segments, she said, "I'm just looking to see… how's it changing with equal changes in height." I conjecture that although Katlyn initially represented equal changes of height along the now vertical axis, in this moment, Katlyn was not attentive to which quantity was on which axis and assumed height was represented on the horizontal axis. She coordinated the blue segments recreated in *Figure 68*a represented changes of volume and then argued the bottle was getting thinner until halfway up the bottle because the changes of volume were decreasing for successive equal changes of height (MA1-3). Katlyn's statements about the bottle were accurate for the relationship she *inferred* from the graph.

Dr. Moore asked Katlyn to explain why she drew the segments as she had to examine how she was using the segments. Katlyn responded, "I guess for me it's easy to see, hold on a second. [*pause*] Oh, see, maybe I should be looking at these [*using her fingers to indicate the space along the horizontal axis between consecutive red segments in Figure 68*a]." Katlyn identified

that the consecutive changes along the horizontal axis between the red segments represented "the amount of volume that we need at that height to fill it up." She then erased her bottle, and considered changes in volume (on the horizontal axis) for equal changes of height (on the vertical axis), drawing a bottle (*Figure 69*).

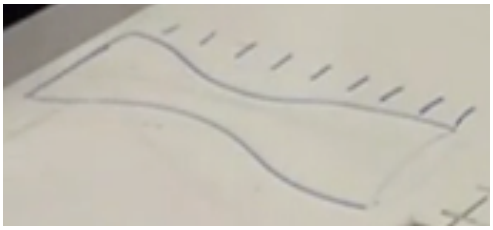


Figure 69: Katlyn's sketch of a bottle given a graph.

Katlyn justified her bottle by motioning over the consecutive changes in volume along the horizontal axis between red ticks (*Figure 68*a) and describing that for equal changes in height, the changes in volume decreased until the middle of the bottle, then increased after the middle of the bottle (MA1-MA4). She understood that for equal changes of height, if volume increased at a decreasing rate then the bottle thinned and if volume increased at an increasing rate then the bottle widened, regardless of the axis on which volume or height were represented.

Summary and implications of Katlyn's activity.

Katlyn considered and coordinated many quantities during this episode (e.g., "amount of water," "how full the bottle is," height, time, total volume, cross sectional volume). She refined her images of the quantities as she re-considered the question Dr. Moore posed. Initially, she considered height and volume (co)varying independently of the other, instead both varying with respect to time (MA2). As her understanding of the relationship between height and volume continued to evolve (i.e. in terms of amounts of change (MA3) or rates of change (MA4-5)), Katlyn always coordinated how volume varied for equal changes of height. By the end of the

episode, and like Arya, Katlyn reasoned about her curve as representing two projected magnitudes representing the height and total volume of liquid in the bottle.

Bottle Problem Day Two

For the second Bottle Problem episode, we had the students work individually to recreate graphs from the bottles in order to refresh their memories of their previous activity as well as to examine if they exhibited compatible activity with the previous episode.[37] After this, I intended to ask the pair to compare each other's work, as they did not have this opportunity during the previous episode. Since the students created accurate bottles and graphs, I conjectured this conversation would be short, but I also considered this as a productive way to have the pair start working together again. Finally, I planned to ask the pair to graph volume and height of liquid in a bottle in related situations (e.g., water leaving the bottle via evaporation or drinking) to explore how they reasoned as they constructed a relationship with similar quantities in novel situations.

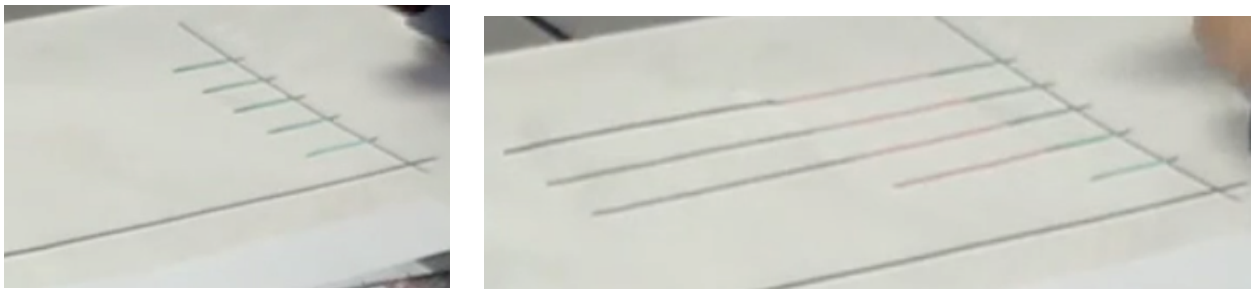
Arya's initial activity.

Consistent with her activity in the previous episode, Arya constructed a graph from her given bottle using equal changes of height, representing height on the horizontal axis. After sketching and justifying her graph, I asked her about the equal changes of volume and she described that when the bottle thinned, equal increases of volume corresponded to increasing changes in height. She represented this in the bottle by using layers to represent equal changes of volume and on the graph by marking equal changes along the vertical, volume, axis. Arya understood ways to coordinate either volume or height first when representing the relationship between volume and height of liquid in a bottle in both the picture of the bottle and in the graph.

[37] As before, I worked with Arya, Dr. Moore worked with Katlyn, and Kevin LaForest acted as the observer.

Katlyn's initial activity.

Because Katlyn used different versions of volume in the prior episode, we were unsure which version of volume she would reason with as she re-engaged in creating a graph from a bottle. Katlyn described that the volume in the first cross section contributed to the total volume at all other heights. She marked equal changes of height, along the horizontal axis, and drew a vertical green segment to represent the volume at the first equal change of height. She then translated this segment to the other equally spaced tick marks on the horizontal axis (*Figure 70*a). Katlyn continued by marking a red segment to represent the volume contained in the second height interval starting where the green segment at the second tick mark on the horizontal axis ended, and translated this to the successive segments, and again for the third height interval in black (*Figure 70*b). Compatible with her activity at the end of the last episode, Katlyn understood the graph as representing two projected magnitudes, representing the height and total volume of liquid in the bottle.



(a) (b)

Figure 70: Katlyn's activity graphing volume for equal changes of height.

As Katlyn provided a description of her graph showing that volume increased at an increasing rate for equal changes of height, she motioned over cross sectional areas shaded in the picture of the bottle. When asked to describe what these areas represented, Katlyn responded, "I'm just imagining that it's going to take more, like if we take the amount of water that we put

in here [*indicating section* (A) *in Figure 61*], and like put it in here [*indicating section* (C)], then we'd end up with something that wasn't… at the same height I guess is how I'm thinking about it" (MA3). Katlyn then described that if the volume in (A) was put into (B), "Then it would probably give you three-quarters of the height or something like that. So… if you change the volume equally then the [change of] heights would like decrease." Katlyn represented the relationship on the graph by marking in equal changes along the vertical axis and drawing horizontal segments from these to the curve (in green, *Figure 71*). She then used dashed segments to represent the amounts of change of height and argued, "These horizontal green lines" represented that "the height is increasing" and the "dashed lines" indicated the changes in height are "getting smaller" (MA1-3). In the previous episode, Katlyn always first coordinated height when reasoning about the relationship in the situation or graph. In this case, Katlyn constructed a new image of the situation when she imagined that equal changes of volume filled less height as the bottle widened, spontaneously considering how to coordinate changes of height for equal changes of volume in the situation. Katlyn re-constructed her images of the relationship between volume and height such that she constructed the relationship between height and volume as a bidirectional relationship, reasoning bidirectionally with respect to the axes to represent this relationship in her graph.

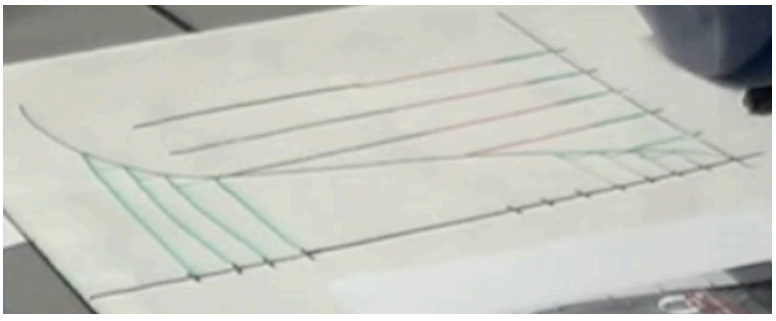
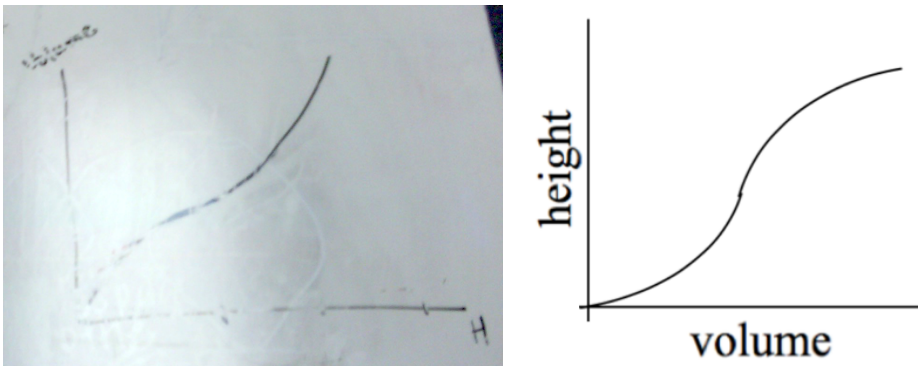


Figure 71: Katlyn's graph representing height for equal changes of volume.

Comparing their boards.

After each student had constructed graphs, we had Arya and Katlyn compare and contrast their board work from the previous episode to explore how the students would interpret each other's work as well as to have them re-engage as a pair. This interaction was brief as both students had constructed accurate graphs and bottles. Of interest, both students initially indicated Arya's constructed graph (*Figure 72*a) was "backwards" when compared to Katlyn's given graph (*Figure 72*b). Katlyn then noticed the change of axes labels and stated that the graphs were "the same." Arya agreed and addressing how she knew the graphs were the same stated, "As the height is increasing [*motioning from the origin along the height axis in both graphs*], our volume is increasing at a decreasing rate" (MA5). By focusing on the underlying covariational relationship represented by each graph, the students described the two different pictorial objects as representing the same relationship.



(a) (b)

Figure 72: (a) Arya's created graph and (b) Katlyn's given graph.

Addressing water leaving the bottle.

We asked the students to work together to sketch a graph representing the height and volume of liquid in the bottle we originally provided Katlyn (*Figure 61*) as water evaporated from the bottle to explore if (and, if so, how) the students reasoned about the relationship between height

and volume in a modified situation. Having the students work on Katlyn's board from earlier (*Figure 71*), they drew a new set of axes and labeled the horizontal axis height and the vertical axis volume. The pair then set out to construct the curve with Arya noting they should start at "full volume, full height." Katlyn then argued, "It's going to look backwards… we can literally just travel this way instead [*motioning over her original curve from the top-right most point back to the origin*]." Indicative of imagining change happening smoothly, Katlyn motioned from the top right most point to the origin and then further elaborated that if they started at the top of the bottle then moved down some height, the volume at that height was equal to the volume at that height when the bottle was filling.

To investigate if using the same curve for a new context created a perturbation for Katlyn or Arya, I asked, "So then is the situation the same? You're ending up with the same graph." Katlyn responded, "No, I just want to draw little arrows... we're going this way now [*draws an arrow on the curve pointing towards the origin, Figure 73*]." Katlyn understood that the same pictorial curve represented the appropriate measures of height and volume of liquid in the bottle, regardless if water was entering or leaving the bottle; she distinguished between the two scenarios by adding an arrow that indicated the direction the curve traced out in the latter situation. I conjecture that Katlyn's goal was to represent the dynamic nature of the situation (e.g., her imagined experience of the situation) and adding the arrow supported her in representing this aspect of the situation.

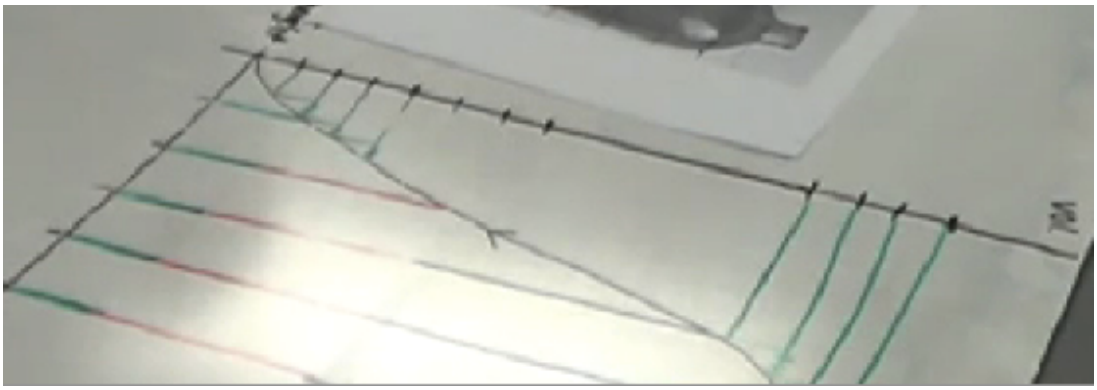


Figure 73: Katlyn's original graph with an added arrow to indicate water leaving the bottle.

When Katlyn added the arrow to her graph, she introduced experience or (at least implicit) time to her graph. I conjectured this opened the possibility that she began to conflate quantities and possibly reason with the graph as representing height with respect to time as water evaporated at a constant rate (i.e. a situation in which evaporated volume and time were proportional). I asked if their graph changed if I were taking sips out of the bottle rather than the water evaporating, a situation (from my perspective) that had the potential to make it apparent if the students were reasoning with time as one of the quantities represented on an axis. Katlyn described interpreting each sip as an equal change of volume (along the vertical axis) to coordinate how height varied for each sip, tracing along the curve from the top right point down as she did so. I asked, "What if I wasn't a consistent sipper?" Katlyn said, "You can still follow [the curve]." Arya agreed and added, "There's no time." Neither student was reasoning with time as one of the quantities explicitly represented on an axis. Instead, both students understood that changing the (imagined) experience of taking sips did not affect the relationship under consideration; each student understood the graph as an emergent trace representing the relationship between height and volume as water left the bottle. Underlying this understanding was the students' conception of points on the curve representing two projected magnitudes representing the volume and height of liquid in the bottle.

Dr. Moore asked the pair to describe the covariational relationship between height and volume as water left the bottle. Katlyn described considering equal changes of height starting at the right most point on the horizontal axis, and both students argued that for equal decreases of height (moving towards the origin) volume was decreasing (MA2). Arya then indicated that volume decreased at an increasing rate, explaining that if changes in volume were decreasing as the height increased then the changes in volume were increasing as height decreased. After considering this for a moment, Katlyn agreed and argued that for equal changes of height, volume was "decreasing by a greater amount every time." The pair then accurately described the covariational relationship for equal changes of volume starting at the top most point on the vertical axis. The students again exhibited their understanding of the relationship between height and volume of liquid in the bottle as water left the bottle as a bidirectional relationship, reasoning bidirectionally with respect to the axes to interpret the relationship. Further, this context supported the students in engaging in novel covariational reasoning as they coordinated how one quantity varied according to equal *decreases* of the other quantity.

Takeaways from the Students' Activities Addressing the Bottle Problem

Consistent with Saldanha and Thompson's (1998) and Carlson et al.'s (2002) description of students' conceptions of covariation as emergent, the students initially understood the relationship bidirectionally at MA2 (i.e. both quantities increased simultaneously as imagined time increased). As they refined their images of the relationship, the students considered how the two quantities covaried by first coordinating changes of height, then changes of volume, then back to changes of height. Through this process, each student developed and refined her images of the quantities and relationships in the situation, eventually supporting her in conceiving of a way to coordinate how height varied for equal changes in volume. Each student understood that

neither quantity caused changes to the other, instead that the two quantities covaried simultaneously, constructing the relationship between volume and height as a bidirectional relationship in terms of amounts of change (MA3) or rates of change (MA4-5) (Moore & Paoletti, 2015; A. G. Thompson & Thompson, 1996).

Constructing the relationship bidirectionally supported the students in addressing how height and volume covaried as water left the bottle. The students engaged in novel covariational reasoning as they interpreted how each quantity varied for equal *decreases* in the other quantity. The students coordinated that if quantity A increased at an increasing rate for equal increases of quantity B then quantity A decreased at a decreasing rate, or "by less each time," for equal decreases of quantity B. Whereas previous researchers have always explored situations in which one quantity is always increasing (cf.Carlson et al., 2002; Carlson et al., 2001; Carlson et al., 2003; Ellis et al., 2012; Johnson, 2012b; A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994b), and Carlson et al.'s (2002) framework implicitly assumes the independent quantity is increasing, Arya's and Katlyn's activities provides insights into a more sophisticated way students can conceive of two quantities covarying. Namely the students considered how one quantity covaried with respect to increases and decreases in the other quantity.[38]

Of additional interest was Katlyn's addition of an arrow to indicate the direction in which the graph was traced out. In previous instances (e.g., determining an analytic rule that corresponded to a static picture in the polar coordinate system, interpreting the sine function), the pair exhibited understandings of graphs as emergent traces. Similarly, in this case, the students understood the two quantities covaried and that this relationship between these two quantities

[38] I note the students engaged in this reasoning *after* they had constructed the relationship with both quantities increasing. I am unsure the extent to which this activity supported the students in considering equal decreases of each quantity.

(and therefore curve representing this relationship) did not change if water was entering or leaving the bottle. In this case, the students intended to denote something additional in their graph, namely the direction in which the curve was being traced as they imagined the experience of water leaving the bottle, which they did by adding an arrow to their graph.

Collectively, I take the students' activities above to indicate that they maintained conceptions of the quantities and a relationship between the quantities that were compatible with Thompson's (2011) first-order model of a conception of covariation. Specifically, the students understood that both height and volume covaried according to (experienced or tacit) time, and conceived of the coordinate point as multiplicative object (i.e. the Cartesian product) that was produced by uniting two quantities simultaneously (e.g., they conceived of their coordinate points as $(h_e, v_e) = (h(t_e), v(t_e))$ with $h(t_e)$ and $v(t_e)$ representing the fact that height and volume vary according to experiential (or implicit) time (t_e)).

Implications of Bidirectional Reasoning: Returning to The Graphing Sine/Arcsine Task Days Three and Four

When addressing the Bottle Problem, the Power Tower Activity, and the Graphing Sine/Arcsine Task, the students reasoned bidirectionally. This reasoning had implications for their other mathematical meanings, which was apparent during the post interviews. Before describing the students' activities in post-interviews, I describe implications for the students' inverse function meanings that were related to their reasoning bidirectionally with respect to the axes by returning to their activity during the latter two Graphing Sine/Arcsine Task teaching episodes. At the outset of Graphing Sine/Arcsine Task Day Three, I asked the pair to address the same prompts in the Graphing Sine/Arcsine Task (*Figure 31*) for the function implicitly defined by the equation $y = x^3$ as I was interested in exploring if the students would engage in similar

reasoning as they had in the previous episode (e.g., reasoning that a function and its inverse represent the same quantitative and covariational relationship) or as they had in the pre-interview (e.g., Katlyn exhibited activity that I took to indicate she maintained meanings for sine and arcsine that she did not apply to other inverse functions and Arya indicated that a function and its inverse *did not* represent the same relationship). I was prepared to raise a context if the students reasoned about the inverse function in ways that did not support them understanding a function and its inverse as representing an invariant relationship.

Constructing a decontextualized inverse function: Their initial activity.

After drawing a sketch of $y = x^3$, the pair attempted to create Graph 2.[39] Katlyn said, "If we're changing input and output, we would say x equals y cubed," and wrote $x = y^3$. Arya agreed and stated this was equivalent to $y = x^{1/3}$ with Katlyn writing this on their board (see *Figure 74*).[40] Arya then tried to imagine and describe the graph of $y = x^{1/3}$ without plotting points. After approximately 30 seconds, I asked, "What if we tried to connect it to what we did with sine?" Responding to this, in Graph 2 Katlyn labeled the horizontal axis y, the vertical axis x, and the pair reasoned covariationally (MA2-5), examining how y varied as "x gets more negative" in both graphs to deduce the curvature of the graph in the third quadrant in Graph 2. I asked them to label some values on Graph 2 to ensure they were clear about how they were orienting their axes (*Figure 74*). Consistent with their activity in the first two Graphing Sine/Arcsine Task episodes, the students' goal was to maintain the relationship represented by the function when representing the inverse function.

[39] During their activity, they did not label the graphs as Graph 1 and Graph 2. For the sake of the reader, I use these labels for the top and bottom graphs, respectively.

[40] The students used $\sqrt[3]{x}$ or $\sqrt[3]{y}$ throughout their activity, but I use $x^{1/3}$ or $y^{1/3}$ for typesetting.

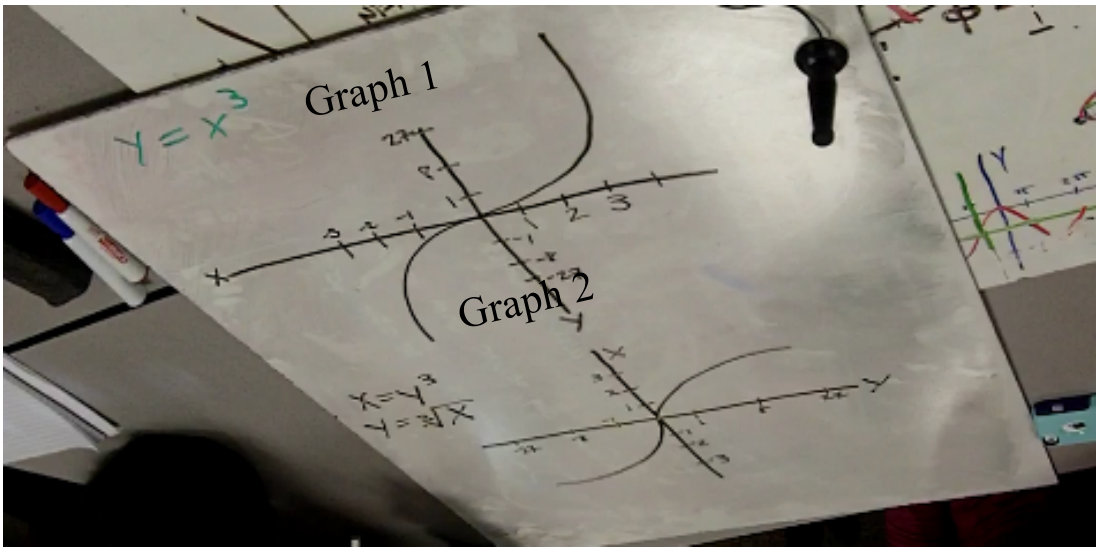


Figure 74: The pairs inverse graph.

The pair then described and represented the graphs as representing the same covariational relationship for equal changes of y in both graphs, but had not attended to the equation $y = x^{1/3}$ that Katlyn had written near Graph 2. I conjectured this equation may be problematic for them as I interpreted their graph, labeled as it was, as the relationship defined by $x = y^{1/3}$. Rather than directly raising this inconsistency, I asked, "What if we had to write an equation… for [Graph 2]?" Katlyn pointed to the equation $y = x^{1/3}$ but then both students realized this equation did not define their graph as labeled. Arya changed the axes labels in Graph 2 so that their curve was defined according to $y = x^{1/3}$ (i.e. labeling the vertical axis y and the horizontal axis x (in blue)). Katlyn wrote the equation $x = y^{1/3}$ in red indicating this equation was "what we had originally" (*Figure 75*).

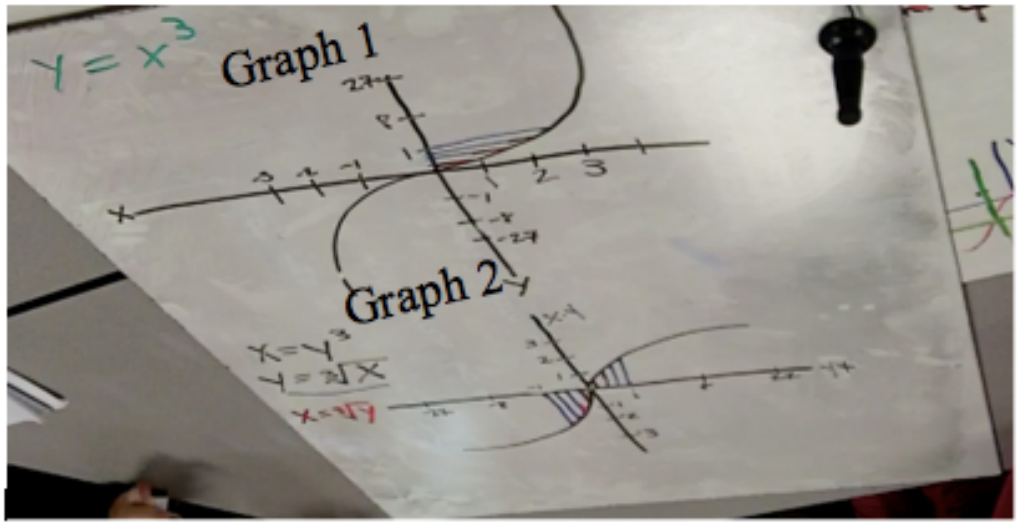


Figure 75: The pair's graph with the axes on Graph 2 labeled as both x and y.

After this, Arya stated that, with their new axes labeling (Graph 2 with x on the horizontal axis), Graph 2 "is not the same information" as Graph 1, arguing that the covariational relationship between x and y represented in Graph 1 was not maintained if she labeled the horizontal axis x and the vertical axis y in Graph 2. The variables were tied to the covariational relationship Arya understood the graph as representing; she did not consider using the variables arbitrarily to consider Graph 1 and Graph 2, with her new axes labels, as representing the same relationship.

With both students uncertain if the relationship should be maintained in Graph 1 and Graph 2, I asked them to describe the process they used to determine $y = x^{1/3}$. They indicated they switched the variables and solved. I then wrote $y = \sin(\theta)$ and $\theta = \arcsin(y)$ and the pair noticed an inconsistency as they identified that they did not switch the variables when defining the arcsine relationship but they did switched the variables in their current activity. After this, Katlyn stated, "So like inverse is like switching [*Arya points to equation* $y = x^{1/3}$] the input and the output, which is this [*Katlyn points to* $y = x^{1/3}$]. But here [*pointing to* $x = y^{1/3}$] we just solved for… x." Compatible with their activity during the pre-interview, each student relied on

switching techniques to determine the inverse function (e.g., pointing to $y = x^{1/3}$); because of this the students questioned their activity from the previous episodes when working with the sine and arcsine relationships as they had not switched variables. The students experienced a sustained state of perturbation at this point in the teaching episode. I conjectured when discussing switching the variables, the students were not attentive to the quantities the variables represented, and thus were not aware that the switching variables technique entails using variables arbitrarily (for the purpose of maintaining conventions). Before I had a chance to explore the students' reasoning in this regard, a class conversation began.

Conjecturing the students were experiencing a perturbation, when we returned from the class conversation, I asked, "So what are we thinking about now?" Katlyn stated that $y = x^{1/3}$ is the correct inverse because "we're switching input and output," and "I don't know why we decided to do [$x = y^{1/3}$]." Katlyn again argued that switching the variables was critical to defining an inverse function. When Katlyn reasoned about the sine and arcsine relationships, she considered switching the quantity she considered as input and output (e.g., "the angle whose sine is") while maintaining the quantitative referent for each variable. However, when addressing the decontextualized function, Katlyn understood they needed to switch the variables to determine the inverse function without concern for what the variables represented (e.g., without understanding variables as arbitrary).

After Katlyn's argument, Arya pointed to the graphs and equations (i.e. $y = x^{1/3}$ and $x = y^{1/3}$) and indicated, "The picture's going to be the same. But it's what you call your axes that are different. And, it changes the relationship." Arya understood that the graphs of $y = x^{1/3}$ and $x = y^{1/3}$ have the same pictorial representation if x and y were represented on the horizontal axis, respectively. Further, she understood that the two equations did not represent the same

covariational relationship between x and y (e.g., she understood that for $x > 0$, in $y = x^{1/3}$, y increased at an decreasing rate for equal changes of x, and in $x = y^{1/3}$, y increased at an increasing rate of equal changes of x).

I questioned the students about how their current activity was compatible with their activity with the sine and arcsine relationships and Arya again indicated they did not switch the variables when working with the sine and arcsine relationships. Arya then argued, "We just solved. Those are [*pointing to* $\arcsin(y) = \theta$ *and* $y = \sin(\theta)$], that's the same equation… written two different ways. It's saying the same thing. This [*pointing to* $y = \arcsin(\theta)$] is saying something else." Arya then described $y = \sin(\theta)$ and $\theta = \arcsin(y)$ as "inverse function[s] but they're not inverse equations." After this, she covered her face in frustration and said, "But with arcsine… you are getting theta [as the output]. So I don't understand, so I'm confused here." What Arya meant by inverse equation in the second case was vague from my perspective. However, it is clear that Arya did not conceive of $y = \arcsin(\theta)$ and $\arcsin(y) = \theta$ as representing the same relationship or function, which would have required her to reason about changing the quantitative referents of each variable (i.e. in $y = \arcsin(\theta)$, θ represents vertical distance above the horizontal axis and y represents angle measure).

Arya experienced a sustained perturbation from this point forward in the teaching episode. Specifically, Arya maintained two inverse function understandings that she conceived of as being incompatible: (1) maintaining the relationship between the quantities in the contextualized situations, with the distinction between a function and its inverse being the choice of the input quantity (i.e. her quantitative inverse function understanding) and (2) switching variables to obtain an inverse function that defined what she perceived as a different relationship than the original function (i.e. her switching variables understanding).

Contextualizing the analytic rule.

Both students were unable to relate their switching variables technique to their understanding that a function and its inverse represent the same relationship. As such, I decided to contextualize the function rule by claiming the equation $y = x^3$ represented the volume of a cube for a given side length, writing $V = s^3$, with the caveat that we could have negative volume. By starting with a decontextualized function then introducing a context for this function, I intended to support the students in considering a way to relate their decontextualized function ($y = x^3$) and its inverse ($y = x^{1/3}$) to the contextualized function ($V = s^3$) and its inverse ($s = V^{1/3}$ or $V = s^{1/3}$) possibly via reasoning about variables as arbitrary.

Responding to this, Arya labeled the horizontal axis side length and the vertical axis volume in Graph 1, then moving to Graph 2, Arya said, "That would make me choose [$x = y^{1/3}$]." Arya added, "[$x = y^{1/3}$] would show the same relationship. But is that what an inverse, now I'm confused by my own definition of inverse." I suggested we might need to "throw out" some things they knew about "inverse" in order to support the students in re-considering their inverse function understandings. After this, Arya labeled Graph 2 in a way that maintained the relationship between volume and side length (recreated in *Figure 76*). Conjecturing they would not switch the variables to represent the inverse rule, I pointed to Graph 2 and asked, "What would the equation be?" Arya pointed to $x = y^{1/3}$ and Katlyn responded "cube root of V equals s." I repeated, "cube root of V equals s," and Katlyn immediately replied, "No, but that's not right." Like Arya, Katlyn experienced a perturbation as she oscillated between her quantitative inverse function understanding (i.e. maintaining the relationship between volume and side length, and maintaining the same quantitative referents for variables) and her switching techniques (i.e. the inverse of $V = s^3$ is $V = s^{1/3}$).

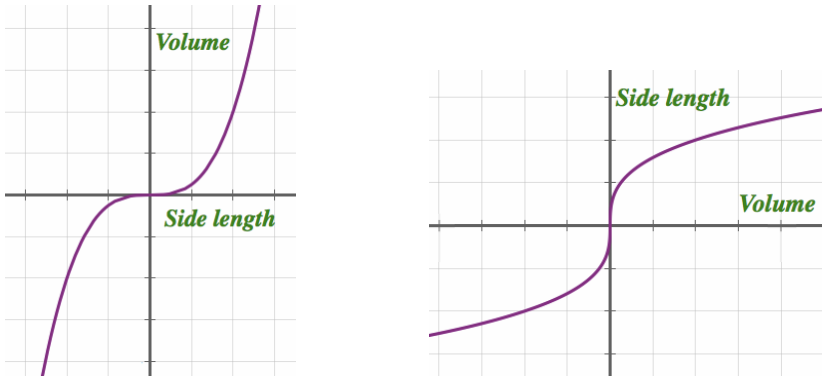


Figure 76: A recreation of their board work with the labels volume and side length.

As the conversation continued, each student attempted to reconcile her two inverse function understandings. For instance, Arya understood $V = s^{1/3}$ represented a new relationship; by switching the variables (and not considering switching variable referents), Arya concluded (reasonably) that in context the inverse "equation" did not maintain the relationship represented by the original function. In an attempt to support the students in re-considering their inverse function meanings, I asked the pair why they switched variables. Arya responded, "I don't know," and Katlyn added, "No, I just remember doing that, that's just our definition… you like switched x and y and solved for y again because in standard position y is [on the vertical axis on Graph 2] and x is [on the horizontal axis on Graph 2]."

I considered Katlyn's argument concerning "standard position" of x and y as a way to support the students in connecting their switching variables and quantitative inverse function understandings, so I drew the students' attention to the possibility of using variables to arbitrarily represent quantities values. I wrote the equations $y = \sin(x)$ in blue and $y = \arcsin(x)$ in red along with Cartesian coordinate axes next to each (recreated in *Figure 77*a). I asked the students to identify the variable and quantity each axis in each graph would represent if they were going to graph each rule. Responding to this, Katlyn used the variables x and y arbitrarily to define angle

measure and vertical distance to maintain representing the input on the horizontal axis and by the variable x in each graph (see *Figure 77*b).

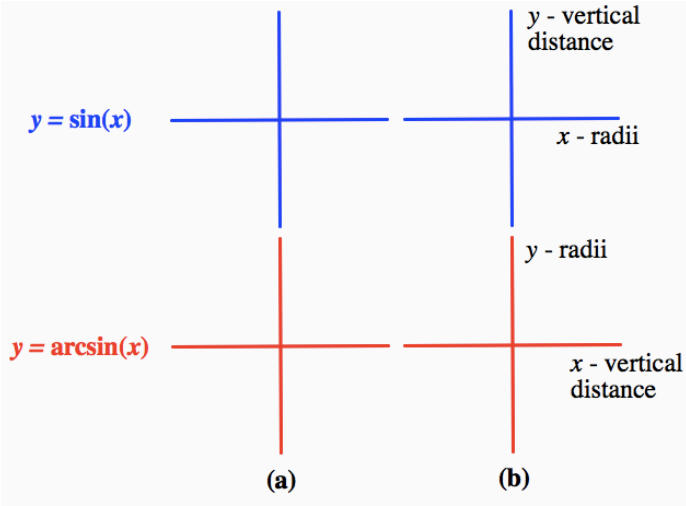


Figure 77: Recreated equations and axes.

Arya objected to Katlyn's activity stating, "You didn't do it wrong, but you changed what x and y mean… You can't use them [*pointing to the x in both* $y = \sin(x)$ *and* $y = \arcsin(x)$] to say the same thing and then change what they mean." Arya reasoned that if the variables were meant to represent quantities values, then "there's no reason" to switch the variables when determining the inverse function; to Arya, switching the meaning of the variables was unnecessary if a function and its inverse represented the same relationship. However, I considered understanding using the variables arbitrarily as a way to conceive compatibility between switching the variables and their quantitative inverse function understandings, so I continued to probe the students by returning them to the volume and side length example. I asked what x represented in the inverse function, if the equation $y = x^3$ corresponded to $V = s^3$. Katlyn responded, "Volume," and Arya interjected, "Oh so when you switch it… your saying something that doesn't make sense… And then you make it make sense." I conjectured Arya was considering changing the quantitative referents of the variables to maintain the relationship, so I asked how they could interpret $y = x^{1/3}$

as the inverse of $V = s^3$. Both students argued if they wanted to maintain the same quantitative relationship then y represented side length and x represented volume. Arya again expressed her dissatisfaction, questioning, "Why do we do that?" I asked, "Any guesses why we do that?" Arya responded (Excerpt 26).

Excerpt 26: Arya conjecturing a reason for switching and solving.

1	Arya:	No, I don't know why we do that. It doesn't make sense. [*pause*] Just
2		because we want this to be our input [*pointing to x in the equation* $y =$
3		$x^{1/3}$] and that to be our output [*pointing to y in* $y = x^{1/3}$]? I feel like that's
4		really the only [*pause*]. That's, that's going to mess me up.
5	TP:	So what are you thinking now?
6	Arya:	It's just, [*pause*] so that this [*pointing to the horizontal axis in Graph 2*],
7		you can call your input x and your output y.
8	TP:	What is?
9	Arya:	Why you switch them. It's **just** [*voice raises*] so you can call your input x
10		and your output y.

Arya hypothesized the reason for switching the variables to define an inverse function was to represent the input and output quantities by x and y, respectively, in the analytic rule (lines 1-4, 9-10) as well as to represent x and y on the horizontal and vertical axis, respectively, in the graph (lines 6-7). I conjecture that Arya's activity in the previous episodes representing the input quantity on the vertical axis or by y supported Arya in making this conclusion. That is, Arya constructed representing the input quantity by x on the horizontal axis as conventions that did not need to be maintained and leveraged this understanding when reaching her conclusion.

After Excerpt 26, Katlyn expressed was unsure what inverse functions "mean" or "represent." I asked, "Did you ever know what they represented?" Katlyn responded, "No, I just knew that you switched x and y." I asked, "Let's says… I gave you volume, and you gave me back side length… What would we expect the inverse rule to do?" Both students responded that the inverse rule inputted side length and outputted volume and that Graph 2 represented this relationship when labeled as in *Figure 76*. However, shortly after this, when moving back to the decontextualized function, both Arya and Katlyn argued that $x = y^{1/3}$ was not "the inverse" but rather just a manipulation of $y = x^3$. I take the students' activities to indicate they were still experiencing a perturbation as they attempted to reconcile their inverse function understandings.

After this Katlyn described reasoning about the inverse function in the volume-side length context by using the variables arbitrarily, arguing, "We've just been saying like we need to switch them in the equation [*pointing to* $y = x^{1/3}$] but like, we're like switching them in real life." Katlyn then reasoned that in order to maintain the relationship between volume and side length in the inverse rule she had to switch the quantitative referents when switching the variables (i.e. s represented volume and V represented side length). In the moment, Katlyn understood that in a contextualized situation (e.g., sine and arcsine, volume and side length), a function and its inverse represented the same relationship between quantities and that she switched the quantitative referents of each variable in order to maintain the relationship.

During the teaching episode, I conjectured both students understood that the function and its inverse represented the same relationship in a contextualized situation. Thus, after Katlyn's statement, I asked, "Do we want the relationship between volume and side length to stay the same?" Neither gave a clear response and Arya questioned, "You have all the information… Why would you do it this way [*pointing to Graph 2*]?" Arya understood she did not need to

produce a new graph if the graphs represented the same relationship. After I asked the students if they were ever asked to interpret the vertical axis as the input in their prior experiences, Arya, sounding frustrated, responded, "Yeah but… if they gave me [Graph 1] and wanted me to find the information, with [the vertical axis] as the input I certainly could… I could turn my head and look at [Graph 1 with the vertical axis as input] and understand what that means. There's still no reason for [Graph 2]. So [Graph 2], like when you switch, then you're saying something new." I conjectured that Arya considered switching the variables while maintaining the relationship under consideration as nonsensical, leading her to reason that the inverse function must represent a different relationship than the original function. However, the episode ended before I had the opportunity to further explore Arya's thinking.

Summary of the episode.

The students reasoned bidirectionally with respect to the axes when drawing Graph 2 with the goal of maintaining the relationship between x and y. However, when the students realized Graph 2 did not match the rule they obtained using their switching technique, each student experienced a sustained state of perturbation. As the students considered their activity representing the sine and arcsine relationships in relation to determining the inverse analytic rule and graph of a decontextualized function, each student considered two inverse function understandings that she understood as incompatible.

After I contextualized the function rule, each student continued to waver between her two inverse function understandings, so I worked with them in ways that raised the idea of using the variables arbitrarily to support them in relating these two understandings. Katlyn considered using the variables arbitrarily by switching the quantitative referent of each variable so that a function and its inverse represented the same relationship in a contextualized situation. Although

Arya seemed to understand using the variables arbitrarily, she reasoned that a new inverse rule and graph were unnecessary if the inverse represented the same relationship as the original function, leaving her questioning if a function and its inverse were meant to represent the same relationship.

In the next episode, I intended to investigate if Katlyn would continue to reason about a function and its inverse as representing the same relationship. Further, I planned to investigate if (and if so how) Arya was considering the relationship represented by a function and its inverse to explore if (and if so how) she had resolved her perturbation.

The Graphing Sine/Arcsine Task day four.

To begin this in-class episode, I gave the pair their board work from the prior episode and asked what they remembered. Katlyn described how she constructed the graph by maintaining the relationship between x and y, but then reasoned that the graph of the inverse of $y = x^3$ did not maintain the relationship between the quantities and was unsure if the relationship between volume and side length was maintained in the inverse function. Responding to this, Arya stated, "I don't like it but," and described that if in the original function ($V = s^3$) you are inputting side length (s) and outputting volume (V) then in the inverse function rule ($V = s^{1/3}$), "s is now your volume and V is now your length". I asked the pair how they would label Graph 2 with variables and both Arya and Katlyn argued the horizontal axis was "volume or s" and the vertical axis was "side length, V."

As in the previous episode, when given time to consider her inverse function meanings, Katlyn reverted to her switching understanding. Further, she initially did not understand a way to relate this procedure to the contextualized volume and side length situation. In contrast, I conjectured Arya reflected on her activity to make accommodations to her inverse function

meanings that supported her in relating her switching variables technique with her quantitative understanding by considering switching the quantitative referents of each variable when she switched the variables in a contextualized situation. Although she understood how this procedure maintained the relationship represented by a function and its inverse, Arya still considered the procedure cumbersome, later arguing, "It's stupid, but it's the same [*covers her face as if frustrated*]… if you give them a meaning you're reassigning things, which is stupid. But you can do it, it doesn't change the fact that they're the same."

I questioned Arya about her frustration here and in the previous episode and she described two ways she considered interpreting contextualized inverse functions: (1) a function and its inverse represented the same relationship, when switching the variables she switched the quantitative referents of each variable and (2) switching variables defined a new relationship that "take[s] away all the meaning [of the original function] and say[s] that they're opposite." Although Arya described a way to maintain the relationship under consideration by switching the quantitative referents when switching the variables, she considered this procedure and reasoning as so unnecessary and cumbersome that she continued to consider her second interpretation as a viable way to interpret the result obtained by switching variables.

Shortly after the interaction above, Arya questioned, "Why would you rename them [*referring to the variables*]?" Discussing this question, Katlyn interjected stating, "Because like this is secretly y [*pointing to x in* $y = x^{1/3}$] and this is like secretly x [*pointing to y in* $y = x^{1/3}$]." Here and in the conversation that continued, the students reasoned that when switching the variables, they "secretly" made the x variable y and vice versa. In activity, Katlyn leveraged this reasoning to resolve the discrepancy she perceived between switching the variables and maintaining the relationship between quantities. However, shortly after this interaction, Katlyn

exclaimed, “It’s not good, there’s no conclusion!” Whereas in some cases, Katlyn made functional accommodations to alleviate her perceived discrepancy between switching the variables and maintaining the relationship between quantities, in other cases, Katlyn concluded that her activity in one situation did not correlate with her activity in another situation. As such, I was unsure if Katlyn understood a way to relate switching variables and maintaining the relationship by switching quantitative referents.

Towards the end of the conversation, when discussing switching variables, Arya indicated “mathematicians of times past” switched variables to maintain the conventional x-horizontal, y-vertical axis orientation and that, “They didn’t mind changing the meanings.” I took this (as well as her prior) activity to indicate Arya understood switching quantitative referents of variables to maintain the relationship between quantities in a contextualized situation and understood switching the variables as a procedure based in maintaining the convention of representing x and y on the horizontal and vertical axis, respectively.

Summary of the Graphing Sine/Arcsine Task day three and four.

Both students experienced sustained states of perturbation in regards to their inverse function meanings during the Graphing Sine/Arcsine Task day three and four. In the pre-interview, neither student maintained the relationship between degrees Celsius and degrees Fahrenheit when determining the inverse of a function that represented this relationship. In contrast, when representing the arcsine function (or relationship), the students leveraged reasoning bidirectionally with respect to the axes to maintain the relationship they understood the sine function (or relationship) as representing. Critical to their activity (and related to their bidirectional reasoning with respect to the axes), the students constructed (in the moment of considering the sine and arcsine functions) representing the input on the horizontal axis was a

convention that did not need to be maintained. However, as they represented the inverse of a decontextualized cubic function during the next two teaching sessions, both students perceived an inconsistency in understanding inverse functions in terms of maintaining a particular relationship and switching variables. By the end of these episodes, Arya understood a function and its inverse represented the same relationship, and that switching the quantitative referents of each variable was necessary if one wished to follow particular conventions while maintaining the relationship under consideration. Developing this understanding was not trivial as Arya needed to reflect on her inverse function meanings in order to bring certain aspects to the surface (e.g., the inverse defines a new relationship than the original) in order to re-conceptualize how a function and its inverse function are related (e.g., a function and its inverse represent an invariant relationship). Because of this, I was unsure if Arya would carry this understanding forward in later sessions and the post-interviews.

I conjecture (in retrospective analysis) that Katlyn maintained (at least) two meanings for inverse functions, one particular to contextualized functions (e.g., sine and arcsine, volume and side length) in which she maintained the relationship between quantities and another for decontextualized functions in which switching variables was critical to defining the inverse function. I was unsure if Katlyn had conceived of a way to connect these two understandings or if she considered the two types of situations as being "unrelated." Hence, I was unsure how Katlyn would address questions concerning inverse function moving forward. I intended to explore both students' inverse function or relationship meanings in the post interviews as they addressed the Car Problem, Cylinder Problem, and other tasks.

The Car Problem

About a week after the Bottle Problem episodes (Arya seven days after and Katlyn eight days after), we engaged each student in an individual task-based semi-structured clinical interview addressing the Car Problem, with Dr. Moore and I acting as co-teacher-researchers. Initially designed and used by Saldanha and Thompson (1998) and further developed by Whitmire (2014) and Thompson (personally communication),[41] the Car Problem shows a dynamic image of Homer driving along a road from beginning (Beg) to end (End) with two towns represented, Springfield and Shelbyville, neither of which is on the road (*Figure 78*). After asking the students what quantities they imagined in the situation, we intended to ask each student to represent the relationship between Homer's distance from Springfield and distance from Shelbyville. The applet allowed us to show or hide segments representing Homer's distance from each city in the situation, graph, or both, in order to explore how the students considered these segments in relation to their created graphs (see *Figure 80* for a screenshot of this with distance from Shelbyville on the horizontal axis).

[41] I received permission to use this task and applet (including reproduced images) from Thompson (personal communication). Any inquiries about the task should be directed to him.

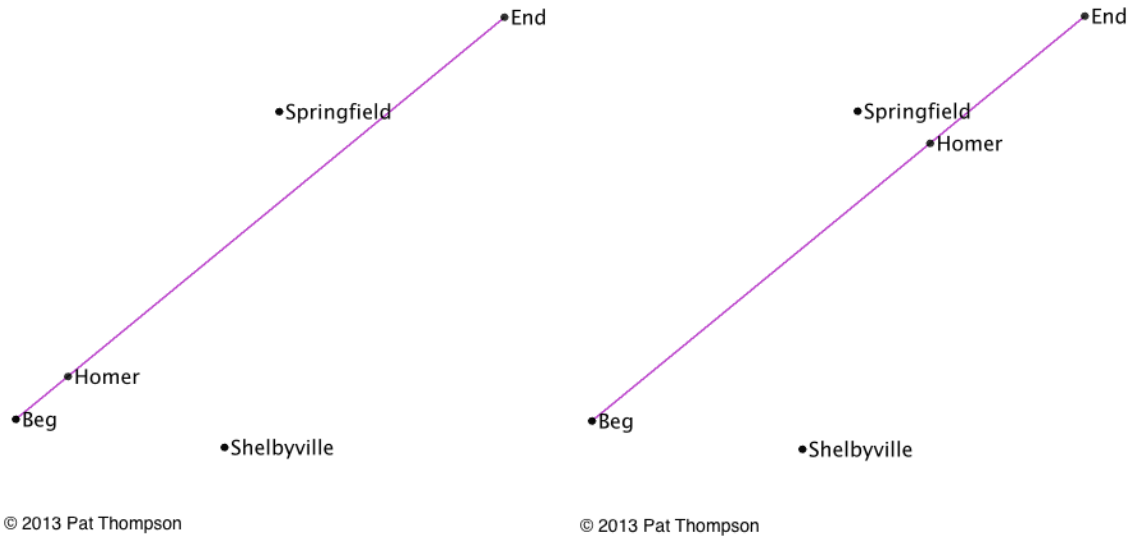
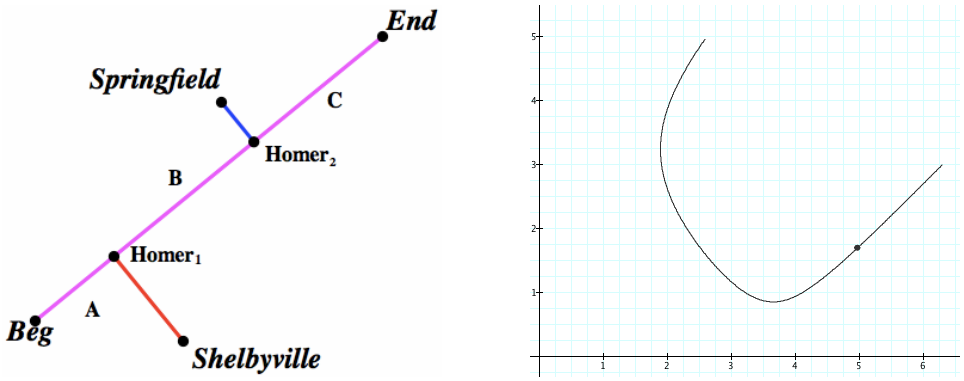


Figure 78: Homer moving along the road from beginning to end.

The directional relationship between the distances from the cities varies according to Homer's location on the road.[42] There is one location in which Homer is at a minimum distance from each city (see $Homer_1$ and $Homer_2$ in *Figure 79*a). As Homer travels along segment A (Beg to $Homer_1$), the distance from each city is decreasing; as Homer travels along segment B ($Homer_1$ to $Homer_2$), the distance from Shelbyville increases and distance from Springfield decreases; and as Homer travels along segment C ($Homer_2$ to End), the distance from each city increases. *Figure 79*b is an accurate resultant graph (with distance from Shelbyville represented on the horizontal axis).

[42] We did not ask (nor expect) the students to consider the variation beyond directional variation as this requires a great deal of physical and mental activity (e.g., carefully drawing numerous concentric circles to examine equal changes in one quantity then coordinating the changes in the other quantity) that we conjectured would distract from insights more pertinent to the study.



(a) (b)

Figure 79: (a) A recreation with added letters for segments and (b) the resultant graph.

As one modification that we made to the task, we had two versions tracing out the relationship between Homer's distance from the two cities as Homer moved along the road, one with distance from Shelbyville represented on the horizontal axis and one with distance from Springfield represented on the horizontal axis. After the student created her graph, we presented the applet with opposite axes orientation to explore how the student related her graph with the graph on the computer.

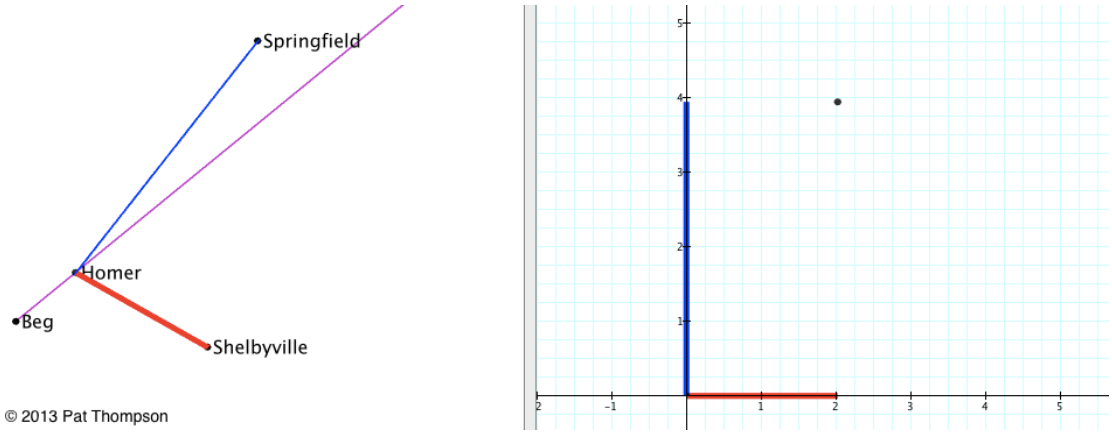


Figure 80: A screenshot of the program showing the situation and the graph.

In addition to investigating the students' construction of quantities and relationships in a dynamic situation, I intended to explore the students' function and inverse function meanings. I conjectured that the students would reason that the resultant graph did not represent a function

with either quantity on the graph as the input. I intended to ask if the students conceived of any functional relationships in the situation, conjecturing the students would focus on defining input and output quantities as they addressed questions regarding function, possibly supporting their description of functional relationships in the situation (e.g., relationships with time or total distance as the input and a distance from either city as the output).

I intended to raise the idea of inverse function or relationship by asking the students to describe if their created graph and the graph on the computer (with the opposite axis orientation) represented inverse functions or relationships. I conjectured when describing inverse functions or relationships Arya would describe two cases, one with distance from Springfield as the input and the other with distance from Shelbyville as the input. I was unsure how Katlyn would reason about a function or relationship and its inverse as she seemed to be experiencing a perturbation at the end of the Graphing Sine/Arcsine Task teaching episodes. I conjectured these questions would allow me to explore if (and if so how) Katlyn had re-constructed her inverse meanings (e.g., did she maintain the relationship between a quantity and its inverse in contextualized situations?).

Arya's Activity in the Car Problem

After watching the animation and identifying various distances in the situation (e.g., distance from beginning, distance from end, distance from each city), I asked Arya to graph the relationship between distance from Shelbyville and distance from Springfield. Arya drew axes and labeled them as shown in *Figure 81*. After a pause Arya drew, from right-to-left, a segment getting closer to the horizontal and vertical axes (recreated by (1) in *Figure 81*). Describing this segment, Arya claimed, "We start off… far from Springfield and pretty close to Shelbyville [*pointing to Beg. on computer screen then traces along road*]. Then… you're getting closer to

Shelbyville for a little ways and closer to Springfield as we're moving along the road" (MA1-2). After describing this relationship using the animation, Arya moved to her graph and marked points on each axis to confirm the graph was representing that Homer's distance from each city was decreasing (indicated by (2) and (3) in *Figure 81*).

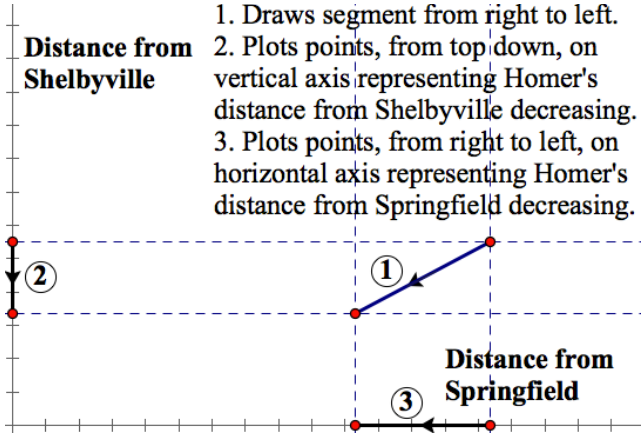
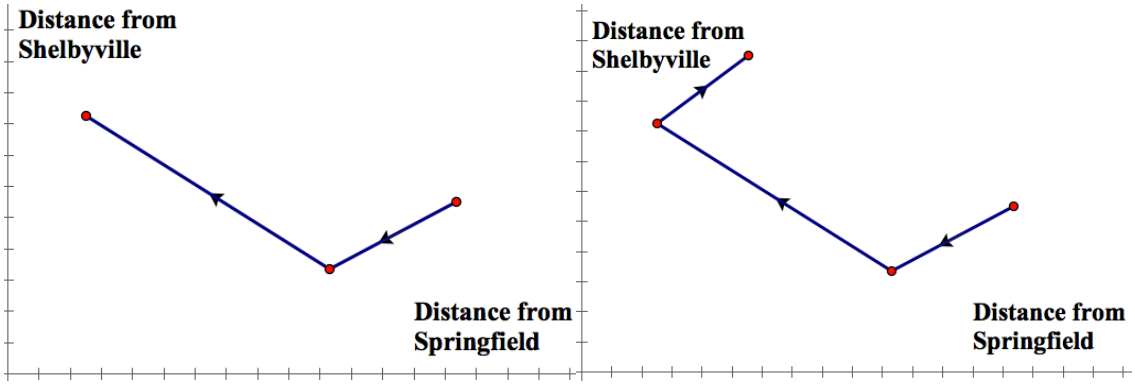


Figure 81: Arya's using the axes to verify she was graphing the relationship accurately.

Arya then watched Homer cover segment B and stated, "We're moving away from Shelbyville after that and closer to Springfield" (MA1-2). She drew in a segment from right to left (recreated in *Figure 82*a) representing this. Arya continued, "And then, we move away from Springfield again, and away from Shelbyville. [*laughs*] And so this, okay. Away from Springfield away from Shelby [*draws segment from left to right, recreated in Figure 82b*]. So I don't-" (MA1-2). I take Arya's laughter and her hesitation after drawing the latter segment to indicate she was uncomfortable when drawing a segment that 'wrapped back on itself' and made the graph no longer 'appear' to represent a function. Conjecturing she was uneasy with her graph, Dr. Moore questioned, "So you don't what?" Arya responded, "I mean that's just what happened, and we're moving this way [*draws an arrow on the first segment she drew to represent that the graph traced from the right most point*]."

Arya understood her curve as representing the relationship between Homer's distance from Springfield and distance from Shelbyville; her goal was to represent this relationship regardless of the appearance of the resultant graph, although the appearance did make her uneasy. Arya conceived of the coordinate points on her graph as representing two projected covarying magnitudes, as indicated by her careful attention to the axes when drawing the first segment (as in *Figure 81*). Further, and similar to their activity when addressing the Bottle Problem, Arya added the arrow to represent an additional aspect of the situation in her graph, namely the direction in which the graph was traced as Homer traveled along the road.



(a) (b)

Figure 82: A recreation of Arya's activity on the Car Problem with the arrows indicating the direction in which Arya drew each segment.

Of additional interest, Arya did not always coordinate the same quantity first. She coordinated that the distance from Shelbyville was decreasing for segments A and B, and then considered how distance from Springfield varied. For the last segment, she first coordinated that distance from Springfield increased, and then considered how distance from Shelbyville varied. I conjecture neither distance took precedence in Arya's thinking and she anticipated the two quantities simultaneously covaried (i.e. understood the directional relationship (MA2) as bidirectional) with the trace of the curve representing this covariation.

After this, I provided Arya the applet with distance from Shelbyville represented on the horizontal axis on the graph (see *Figure 80*). Arya watched Homer's location move on the road as well as the point representing Homer's distance from each city move on the graph. As she did this, she described how Homer's distance from each city was changing for each part of his trip (e.g., along segments A, B, C). Arya then identified that the relationship she described was represented by both her created graph and the animated graph, which I produced on the screen. I asked if the two graphs were "showing the same thing?" Arya responded, "Yeah. I mean [the graph on the computer]'s more to scale or whatever… I mean what's happening is the same… The relationship should be the same." Arya understood that the underlying relationship between the two distances did not depend on which quantity was represented on which axis; she anticipated that the graphs represented the same relationship (i.e. anticipated reasoning bidirectionally with respect to the axes).

Considering 'function' in relation to the Car Problem.

Conjecturing Arya was initially uneasy with her graph 'wrapping back on itself,' I explored how Arya might describe the situation in relation to 'function'. I asked, "Could you talk about anything in this situation in terms of things being functions?" Arya responded, "If you take either like the distance from Springfield or the distance from Shelby as your input you're going to have more than one output in some places… So neither are really functions." In contrast to the pre-interview, Arya considered the quantity represented on the horizontal and vertical axis as the possible input quantity (e.g., she did not describe the graph, as a whole, as representing or not representing a function); she reasoned that in both cases, certain input values corresponded to more than one output value, leading her to conclude, "Neither are… functions."

Intending to investigate if Arya conceived of any functional relationships in the situation, I asked, “Is there anything else we could have asked you about that may or may not have represented a function?” Arya described considering either distance from a city “and how it travels over total distance.” She then more specifically considered Homer’s total distance traveled and distance from Shelbyville, and argued if Homer’s total distance traveled was the input and distance from Shelbyville was the output, then the relationship was a function, “Cause we have a distance [from Shelbyville] for every total distance. The other way wouldn’t be a function… If we’re using… the distance from Shelbyville as our input, we have two total distances as our output.” Arya made a similar argument for total distance and distance from Springfield. Compatible with her activity discussing the relationships between the distances from each city, Arya considered using both quantities under consideration as the input quantity to successfully make conclusions regarding the ‘function-ness’ of each relationship.

Considering Arya’s addition of an arrow to her graph to represent the direction in which the graph was traced out as Homer traveled might support Arya in reasoning about the trace of her graph as being defined parametrically, Dr. Moore asked, “What if your input was total distance traveled and your output was two dimensional? … So it’s like a pair of values. Where that pair is your distance from Springfield and your distance from Shelbyville… What do you think about that case?” As she considered this, we discussed two possible conceptions of total distance: (1) Homer only traveled from beginning to end, so if a trip up was 100 miles then Homer’s maximum total distance traveled was 100 miles, and (2) Homer traveled back and forth along the road accumulating total distance throughout, so if a trip up was 100 miles then traveling up and back was a total distance of 200 miles and there was no maximum total distance traveled. Arya stated that in either case the relationship represented a function if she considered total distance as

the input. I then asked about the inverse relationship (from my perspective) for each version of total distance with the ordered pair as the input. Arya argued if we used the second conception of total distance then, "We're going to have more than one total distance because we're just moving back and forth across those input pairs so we're going to hit those more than once… for each input that could correspond to a number of distances," concluding with this conception of total distance as the output the relationship was not a function. I asked about her first conception of total distance (e.g., one trip up) and Arya traced along the curve, then responded, "Each input value, that corresponds to one total distance as you move along. That's good."

Throughout her activity, Arya considered different input quantities and how those related to a given output quantity, leveraging this reasoning to consider if a relationship (and its inverse relationship) was functional, regardless of if this relationship was explicitly represented by a graph or equation. I conjecture that because Arya had multiple opportunities to construct, describe, and graph both functional and non-functional relationships, she abstracted function as a property defining a characteristic of input and output quantities of a relationship, regardless of where these quantities were represented on a graph (or if they were even represented on a graph). Further, although her activity indicated she was uneasy with the graph 'wrapping back on itself,' Arya argued representing the relationship accurately was paramount, which was in contrast to her activity in the pre-interview when working with graphs that did not appear to represent functions (e.g., *Figure 24*b), or when she indicated "I don't work much with non-functions… I don't really know anything about things that aren't functions."

Arya's inverse function meanings.

Intending to continue to explore Arya's other mathematical meanings in relation to her reasoning about relationships between quantities, I raised the idea of "inverse." I asked her if her

graph and the graph on the computer were "inverses of each other," conjecturing she would describe the relationships in terms of inputs and outputs (e.g., the graphs represented inverse relationships of each other if the horizontal axis represented the input quantity in each graph). Arya responded (Excerpt 27).

Excerpt 27: Arya discussing inverse relationships in the Car Problem.

1	Arya:	Well I mean, what did they use as their input? 'Cause I kinda like, I
2		started using [distance from Springfield] as my input. Because that was
3		easier for me to see so that wouldn't really be, I was graphing the same
4		thing just in different directions. So, they used different axes. Depends
5		on what they used at their input, whether you'd call it an inverse
6		relationship or not, I guess.
7	TP:	If you want to call it an inverse relationship, it would depend on?
8	Arya:	Yeah, if we're using the horizontal axis as our input, then yeah they
9		graphed the inverse
10	TP:	Okay. Now, but say they used the horizontal axis as their input and you
11		used [*pointing to vertical axis, representing distance from Springfield*
12		*on her graph*]-
13	Arya:	This. Then as we were graphing, we were graphing the same
14		relationship, it just looks different.

Arya understood defining input and output quantities as critical when making statements about inverse relationships (lines 1-6, 8-9, 12-14). Arya argued that if she assumed both graphs represented input on the horizontal axis then, "They graphed the inverse" (lines 8-9). Further, Arya described that she coordinated distance from Springfield, represented on the vertical axis of

her graph, first for (most) of her activity (lines 1-6), and argued if the hypothetical student also coordinated distance from Springfield first, then she and the student graphed the same relationship, not inverse relationships (lines 13-14). By reasoning about relationships and their inverses in terms of identifying input and output quantities, Arya explicitly addressed that the horizontal axis did not have to represent the input quantity. Both of these meanings were novel relative to her pre-interview when she implicitly assumed that horizontal axis represented the input quantity, and questions about inverse called her attention to engaging in certain activities tied to certain representations or context.

Katlyn's Activity in the Car Problem

After watching the animation and describing quantities she imagined in the situation (e.g., distance from beginning, distance from end), I asked Katlyn to represent the relationship between distance from Springfield and distance from Shelbyville. Once Katlyn understood these quantities compatibly with what I intended,[43] she drew axes and labeled her axes as shown in *Figure 83*. Katlyn identified, "Distance from Springfield. That's going to be really high at first. Distance from Shelbyville is going to be really low at first," and indicated a point in the first quadrant corresponding to this initial condition. After some consideration, Katlyn drew a curve similar to that recreated in *Figure 83*. She described, "He gets closer to Springfield [*starting at the top left most point and tracing along the curve*]… he continues to get really, really, close to Springfield… Then [distance from Springfield]'s going to increase again [*motioning over the point where the curve gets closest to the horizontal axis*], 'cause he's getting further away from [Springfield]" (MA1-2). When coordinating how the distance from Springfield varied, she

[43] Katlyn spent about 14 minutes creating a graph representing a relationship I did not intend. Specifically, I asked her to represent the relationship between the "distance from the two cities," and she took "distance from the two cities" to be one quantity, creating a graph representing distance along the path and minimum distance from either city.

apparently imagined the distance from Shelbyville increasing, as she claimed that Homer was "always moving away from Shelbyville."

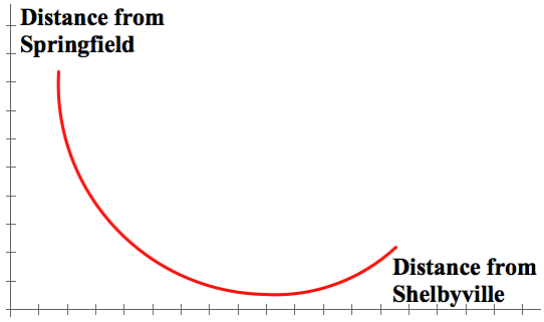


Figure 83: A recreation of Katlyn's Car Problem graph.

After reaffirming that Katlyn envisioned that the distance from Shelbyville was increasing throughout Homer's trip, I revealed segments representing the distances from each city in the animation and asked her how these segments related to her graph. As Katlyn considered this, she indicated that the distance from Shelbyville decreased as Homer moved along segment (A) in *Figure 79*. She subsequently drew a new set of axes and the curve recreated in *Figure 84*. As Katlyn traced along the curve from the top left endpoint to the bottom right endpoint, she described, "He starts far away from both, but then he gets closer to Shelbyville and to Springfield [*tracing until the point closest to the vertical axis*]. And then he starts getting further away from Shelbyville and closer to Springfield [*tracing until the point closest to the horizontal axis*]. And then he gets further away from Springfield and further away from Shelbyville at the end [*traces the rest of the curve*]" (MA1-2). Whereas Katlyn initially only coordinated how distance from Springfield varied (implicitly as distance from Shelbyville increased), in this case Katlyn switched the order in which she coordinated the quantities (e.g., "closer to Shelbyville and to Springfield," "further away from Springfield and further away from Shelbyville"). I take this, along with her phrasing, to suggest that she understood the two quantities as simultaneously covarying (i.e. understood the directional relationship (MA2) as bidirectional).

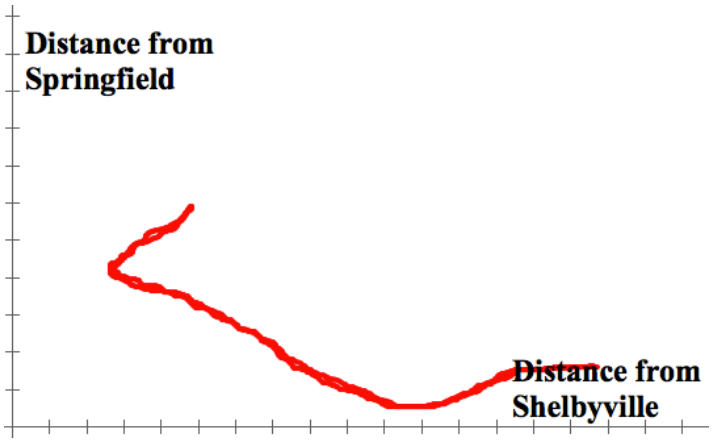


Figure 84: A recreation of Katlyn's sketch.

After Katlyn provided a similar description of Homer's return trip, I directed her to consider the graph on the computer with distance from Shelbyville on the vertical axis showing the point traced out on the curve as Homer traveled along the road as well as the segments on each axis representing Homer's distance from each city (i.e. as in *Figure 80* but with the opposite axis orientation). Katlyn stated, "[The axis orientation] doesn't matter. It's showing the same thing. This point [*pointing to the dynamic point on the graph on the computer*] is kind of following the curve that I drew [*tracing along her curve from the top left endpoint to the bottom right endpoint*]. And then this, these two quantities [*pointing to the red and blue segments on the axes on the graph on the computer*] are just changing with respect to how far away he is from each city." Katlyn indicated that both her graph and the graph on the computer represented the same projected magnitudes representing distances from each city, but in different orientations. Katlyn understood both her graph and the graph on the computer as representing an emergent trace that represented the same relationship between quantities; the axes on which the quantities were represented did not affect the relationship under consideration (i.e. reasoned bidirectionally with respect to the axes).

Katlyn's discussing the "inverse relationship."

As Katlyn described the two graphs as "the same thing," I conjectured this was an opportunity to explore if (and if so how) she connected these two graphs to the notion of "inverse." I asked if the graph on the computer represented the "inverse relationship" of her produced graph and she responded, "I hate that word, maybe." She expressed that she was confused as to what "inverse" meant, and then elaborated, "I did different axes than them, but I'm not necessarily inputting something and outputting something I guess is what I'm thinking." Katlyn identified that she did not conceive the graph in terms of "inputting" and "outputting" values, which supports my conjecture that she understood the two quantities as simultaneously covarying. I continued the conversation (Excerpt 28).

Excerpt 28: Katlyn discussing "inverse."

1	TP:	So if it were the inverse what would sort of need to be happening there?
2	Katlyn:	You would need like [*pause*]. Um so like, if I am this distance away from
3		Shelbyville that implies that I am this distance away from Springfield and
4		so then you'd have to [*pause*] switch them. For whatever reason. Switch
5		them and then [*pause*] and then you have to switch the meaning. I hate, oh
6		my gosh, this just stresses me out. What does that mean?

As we asked Katlyn about her statement "switch them," she did not directly address what she meant by this, instead stating, "In order to have an inverse you have to have a function. This is just a relationship so I don't really think it needs an inverse."

Katlyn appeared to be constrained to thinking about inverse in terms of some switching technique (lines 3-4). While she did not elaborate on her statement regarding switching the meaning (lines 4-5), this may be indicative of her considering switching in relation to her activity

with the sine and arcsine, and volume and height in which she switched the quantitative referents of variables to maintain the relationship between quantities in other contextualized situations. However, Katlyn's words and actions indicate she was still experiencing a perturbation in regards to what inverses represent and how they relate to her prior activities. Before we had a chance to explore this, Katlyn argued that in order to define a function and its inverse, she needed to define input and output quantities for each and because she did not conceive of either distance from Springfield or Shelbyville as the input quantity, she did not further discuss her graph and the graph on the computer as "inverses."

Katlyn's function meanings.

With Katlyn bringing up the idea of function, I redirected my questioning by vaguely asking, "So, is this a function?" Katlyn identified that there were two "mile marker[s]" along the road on the computer-generated representation at which Homer was ten miles from Springfield and from Shelbyville, concluding, "If you want to know how far you are away from one of the two cities, and there's two different places on your path where you're that far, like, you don't know, which one they're talking about." Hypothesizing Katlyn was reasoning about "places on the path" as an implicit parameterized output, we asked Katlyn to explicitly describe the input and output quantities that she had in mind when considering whether her graph represented a function. As she responded, she identified that there were multiple occasions that two points on her curve corresponded to the same distance from Springfield, stating, "So if I input a distance away from [Springfield]… I could get two outputs of points along my graph… So input distance from [Springfield], output… a point on your travels." When asked to describe a relationship in the situation as a function, Katlyn (tacitly) reasoned with "points on my travels" or "places on the

path," to accurately conclude that there were two "points on her travels" that corresponded to the same distance from Springfield.

To investigate this I drew and labeled a coordinate system (*Figure 85*), and asked Katlyn if this was how she was reasoning about the relationship she was describing as not being a function. She agreed and justified this stating, "Because there's going to be multiple points on your graph… where you're the same distance from Springfield [*pointing to the horizontal axis*] but you're at different point on your path [*pointing to the vertical axis*]." I asked, "What if we were to look at this the other way?" Responding, Katlyn sketched the graph (in red in *Figure 85*) by starting at the point closest to the horizontal axis while describing how the distance from Springfield increased and then decreased as she moved up and to the left. She then argued, "It is a function when you're inputting points along your path from beginning to end. It is not a function when you're inputting the distance you are from Springfield." Katlyn made accurate statements regarding the relationships in *Figure 85* by describing the different input-output options, however, she continued to describe the graph as being or not being (rather than representing or not representing) a function when describing the different relationships (e.g., "it is a function when… it is not a function when…"). Compatible with her activity in the pre-interview, Katlyn was still drawing on meanings for graphs and functions such that graphs either were or were not functions.

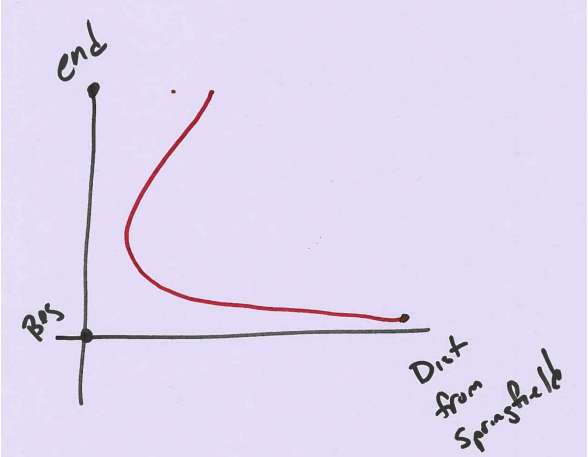


Figure 85: One relationship Katlyn was reasoning about when discussing function.

Katlyn's use of "points on the path" was consistent with her construction of the graph and her activity when addressing questions regarding inverse function. Namely, she conceived of distance from Springfield and distance from Shelbyville as varying according to the "points on the path" and (implicitly) parameterized the relationship between the distances from each city to conceive of these two quantities as covarying *depending* on Homer's position on the path.

After describing the relationship between Homer's position on the path and distance from Springfield, I returned to her graph representing the relationship between the distances from each city and asked her to engage in a similar discussion for this graph. Katlyn responded, "You're not really inputting and outputting anything. I guess you're just showing… traveling along my path, I'm this far away from Springfield and this far away from Shelbyville… So I don't really think it… matters." Consistent with her reasoning about the distances from each city as dependent on Homer's position on the path, Katlyn did not conceive of the relationship between distance from Springfield and Shelbyville as being a 'functional' relationship (i.e. she did not conceive of a dependency between these two quantities).

I asked Katlyn if the relationship inputting Homer's position on the path and outputting a two dimensional output, (distance from Shelbyville, distance from Springfield) was a function.

Katlyn identified, "Well that's what [*Figure 84*] shows, right?" Katlyn stated that for any of Homer's positions on the path there was only one point on her curve so this relationship was a function with Homer's position on the path as input. However, Katlyn indicated she was uncomfortable with this reasoning, explaining, "I understand, like, what I've been drawing this whole time is like, how I'm traveling on like this purple path. But I don't, I never thought of that as my input, but it really is." Although Katlyn was (implicitly) coordinating Homer's position on the path first when she constructed the relationship and her graph, she was not explicitly aware of this until I asked her to consider a one dimensional input and a two dimensional output. Although my question about a one dimensional input and a two dimensional output may have supported Katlyn in considering that she was coordinating Homer's position on the path first, her activity did not suggest she was explicitly considering a function that mapped one dimensional values to two dimensional values. Instead, she may have been mapping Homer's position on the path in the image of the situation to the points on her curve representing a relationship composed of two projected magnitudes.

I asked Katlyn if the relationship inputting the two dimensional point and outputting points on the path (i.e. the inverse relationship) was a function if we only considered one trip from beginning to end. Katlyn initially indicated this relationship was not a function, but then continued, "If it was just like distance from Shelby[ville] that you're inputting, then you would get out multiple distances on the path. If you were inputting distance from Springfield you'd get out multiple distances on your path. But like together? Hmm, that's a good question." Katlyn's uncertainty supports my conjecture that she was not explicitly reasoning about a two dimensional input and one dimensional output. However, as she considered this, Katlyn identified that there were two points on her curve corresponding to a certain distance from Shelbyville, but these two

points had different distances from Springfield. She leveraged this to accurately conclude that the relationship with a two dimensional input was a function.

Katlyn understood the graphs she constructed (e.g., distances from each city, distance from Springfield and points on the path) as representing emergent traces of relationships between covarying quantities. Katlyn understood the axes on which these quantities were represented did not change the underlying relationship represented by the graphs (i.e. anticipated reasoning bidirectionally with respect to the axes). Further, Katlyn constructed and described multiple relationships and their inverses (e.g., distance along the path and distance from Springfield, distance along the path and the two dimensional output), including if each of the relationships were functions. However, Katlyn did not always conceive of these relationships (or functions) as being inverses as she did not always perceive of input (or independent) and output (or dependent) quantities in the situation.

Clinical Post-Interviews

I engaged each student in a clinical interview 14 days after she completed the Car Problem, with Dr. Moore present for Arya's interview. The interview tasks were designed to explore the students' meanings for various ideas in relation to both the pre-interview and the teaching episodes (e.g., function and inverse function, quantitative, covariational, and bidirectional reasoning). I waited two weeks after the previous episode in order to explore what meanings or ways of thinking each student had developed during the teaching experiment. For brevity's sake, I do not analyze each student's responses to all prompts but instead provide analyses of their activities on specific tasks that provided insights into their more general understandings and shifts from the pre-interview (see Appendix B for all post-interview tasks). I first describe the design of selected tasks, and then present the students' activities on selected tasks that provided

insights into their meanings. Throughout, I make comparisons to their activities during the pre-interview in order to highlight particular shifts in their meanings.

Task design.

The first task I posed to each student was the Cylinder Problem (Appendix B, Problem 1), which asked the students to graph the relationship between surface area and height of a dynamic cylinder with a constant radius and varying height (see *Figure 86*). As the animation played, a point on a slider moved back and forth, and the students had the option to stop the animation and use this point to vary the animation. I designed this problem to give me another opportunity to explore the students' activities as they constructed and represented a relationship between covarying quantities when given a video of a dynamic situation.

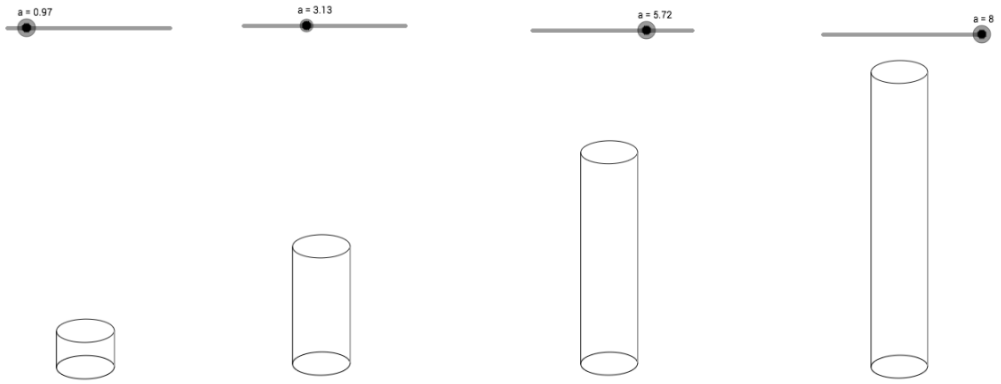


Figure 86: Screenshots from the Cylinder Problem.

I had prepared two additional prompts for the students. First, I intended to show the student an emergent graph with the opposite axes orientation than the graph they produced (see *Figure 87* for screenshots with the graph having height on the horizontal axis) for the purpose of investigating how the students described their created graph and the graph on the computer in relation to one another. As the students had repeated opportunities to reason about graphs with different axes orientations as representing the same relationship, I intended to investigate if the

students continued to reason in this way three weeks after the last teaching episode (i.e. if this had become part of their operative meanings).

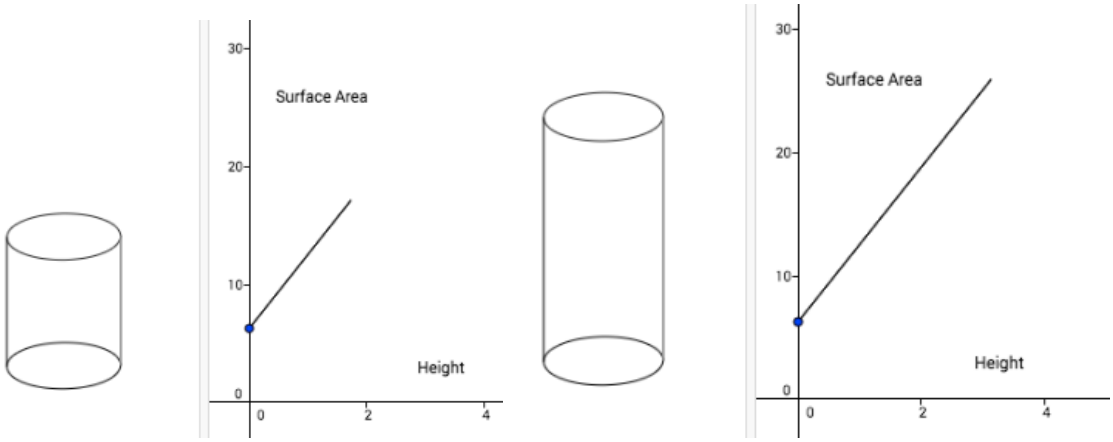


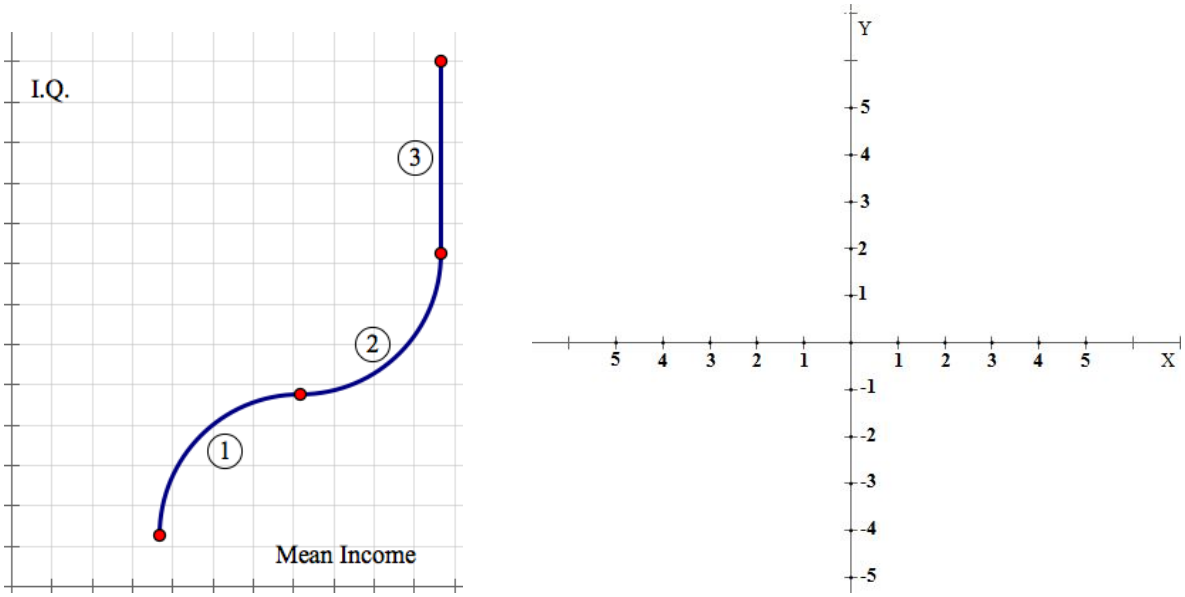
Figure 87: The cylinder with dynamic curve.

Second, because I conjectured each student would construct a formula,[44] I intended to ask the students to derive the "inverse formula". I conjectured Arya would describe the inverse as the formula that maintained the relationship between quantities but with a different quantity as the (implicit or explicit) input quantity. I was unsure how Katlyn would approach this problem, but conjectured Katlyn's approach might depend on her conception of the relationship as entailing input and output quantities. I considered this as an opportunity to more directly investigate Katlyn's inverse function meanings by having her describe the inverse of a function represented by both a graph and function rule.

Following the Cylinder Problem, we designed other prompts to explore the students' understanding of graphs as representing relationships between quantities and their meanings for function and inverse function. I included the Sine Inverse task with the first prompt being identical to a prompt from the pre-interview (Appendix B, Problem 6). I adapted the second

[44] I had prepared a prompt (Appendix B, Problem 1c) that provided the formula if they did not derive a formula, but this prompt was not needed as each student derived a formula spontaneously.

prompt slightly to explore how Arya and Katlyn responded to a hypothetical student whose thinking was reliant on interpreting the horizontal axis as the input, as they were during the pre-interview. I also included multiple prompts to explore the students' reasoning about graphs and functions in relation to conventions common to school mathematics. The Mean Income/I.Q. problem (Appendix B, Problem 3, graph in *Figure 88*a), and the Stay-Positive Axis problem (Appendix B, Problem 5, graph in *Figure 88*b) are two examples of this. The fourth prompt of the Mean Income/I.Q. problem asked if the graph represented mean income as a function of I.Q.. In the pre-interview each student imposed that the horizontal axis represented the input values of a function. For the post-interview, I conjectured that each student would rely on function meanings that emphasized defining inputs and outputs including their respective axes.



(a) (b)

Figure 88: (a) the Mean Income/I.Q. Graph and (b) the Stay-Positive Axis graph.

Along the same vein, the Stay-Positive Axis problem asked students to "Graph $y = x$ on the coordinates below." However, the coordinate system differs from conventional Cartesian coordinates as the x-values are only positively directed. I intended to examine if the students

noticed this change in convention and how they graphed the relationship $y = x$ in this coordinate system (e.g., would they attempt to re-write the horizontal axis to match conventions or would they represent the given relationship on the given axes). Additionally, I planned to ask each student if the graph represented x as a function of y to investigate how she approached questions regarding function in this unfamiliar coordinate system (e.g., would they use the horizontal line test to conclude x is not a function of y, or would they interpret the statement in terms of inputs and outputs to conclude x is a function of y).

I also included questions designed to explore implications of the students' development of bidirectional reasoning that we did not directly discuss in the teaching experiment. For instance, the Two Rates of Change problem (Appendix B, Problem 9) asked the students determine the rate of change of the relationship graphed in *Figure 89*. Although we did not discuss slope or rate of change explicitly in the teaching experiment, I conjectured each student was capable of reasoning bidirectionally (with respect to the axes and/or the quantities) to describe the graph as representing either a rate of change of four or one-fourth depending on which quantity she considered the input (e.g., did the students ask the question, "Rate of change of what with respect to what?").

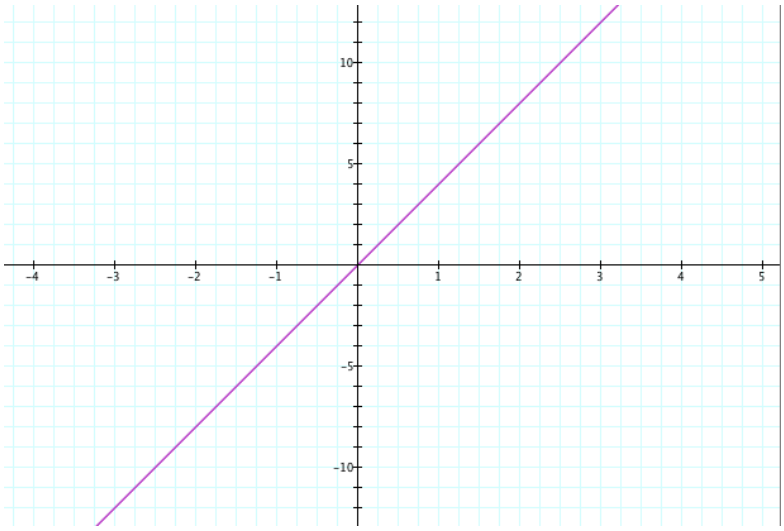


Figure 89: The Two Rates of Change graph.

Arya's Activity the Cylinder Problem

After watching the animation, Arya identified many quantities that were changing (e.g., height, volume) and not changing (e.g., radius of the cylindrical base, area of the cylindrical base). I asked her to represent the relationship between surface area and height. Arya attempted to recall a formula but was unable to do so. She then said, "As height increases surface area is increasing obviously, but I don't know… It's at the same rate so it's linear, right?" (MA1-2). Although she expressed she was not confident that the graph was linear, she sketched a linear graph to represent the relationship between surface area and height (*Figure 90*).

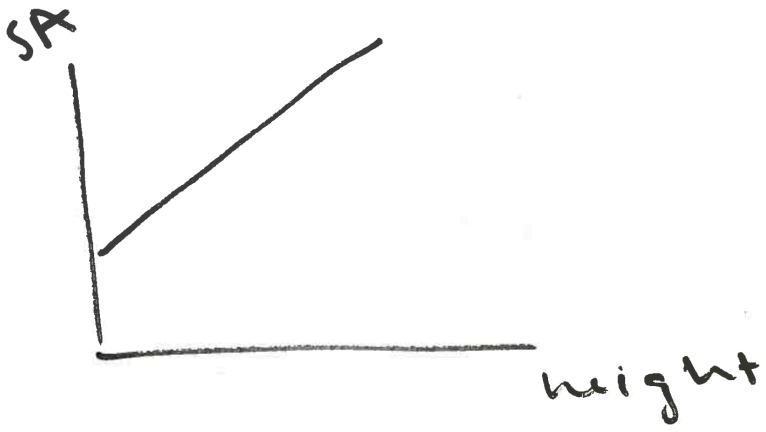


Figure 90: Arya's surface area/height graph.

Arya then described imagining surface area in terms of "unwrapping" the cylinder and defined surface area, "Circumference times the height plus… two times the area of the base." She then wrote the equation $SA = 2\pi rh + 2\pi r^2$. I suggested we use a hypothetical value of one for r, but Arya stated, "r is not changing… that's a constant so, just don't even think about that as a variable… it's not varying." She then concluded that because r was constant, the surface area formula (SA) was a linear with respect to h.

Intending to determine if Arya would use the situation to further justify the linear relationship, Dr. Moore asked, "Why does that make sense?" Arya responded by moving the animation from a height of zero to one and indicated, "We have some surface area." After then moving the slider from height one to two, she said, "So we've changed by the same surface area." Moving from a height of two to three, Arya described, "I could cut off the different heights and all the surface area, all the pieces would be the same." Arya justified that for equal changes of height, surface area increased by equal amounts or at a constant rate (MA3-4). I infer the "pieces" Arya referred to were strips wrapping around the cylinder, not including the bases.

Because Arya did not use the situation in this way when creating the graph, Dr. Moore asked, "Were you envisioning that earlier or did that just come to you?" Arya responded, "That's a way to show it, I wasn't thinking of it exactly that way but that kind of justifies what I was thinking." I infer that Arya's initial image for the relationship was based on her intuitive notion of the two quantities simultaneously covarying (i.e. understood the directional relationship (MA2) as bidirectional). As Arya refined her justification and image of the situation, she systematized a way to consider the two quantities covarying, and she coordinated changes in surface area for equal changes of height to describe and justify the linear relationship.

I then showed Arya the animation with the graph that represented surface area on the horizontal axis and asked what this graph represented. Arya described, "As we change surface area how height changes. It doesn't matter how we want to think of it… You could think of changes in surface area and how it changes height but that's weird to think about, 'cause really you change height." Arya understood the two graphs as representing the same relationship (i.e. reasoned bidirectionally with respect to the axes) and she anticipated that she could consider either quantity as the input quantity (i.e. anticipated constructing the relationship bidirectionally in terms of amounts of change (MA3) or rates of change (MA4-5)). Experientially she considered height as a more natural quantity to coordinate first, which might have been supported by the numerical value associated with the point on the slider appearing to represent the height of the cylinder.

Addressing prompts regarding the inverse relationship.

I asked Arya "to write the inverse" of her equation and she solved her equation for h, obtaining $h = (SA - 2\pi r^2)/(2\pi r)$. She indicated the inverse equation was "inputting surface area, outputting height." Arya then added a caveat, "It doesn't really matter either way… You'd have to solve for this [*pointing to h in* $h = (SA - 2\pi r^2)/(2\pi r)$] to do the math so, I mean, it doesn't matter which one you're inputting [*motioning over both equations*]. If you, if you're changing surface area, you can still solve for h using this formula [*pointing to* $SA = 2\pi rh + 2\pi r^2$] but it ends up being this [*pointing to* $h = (SA - 2\pi r^2)/(2\pi r)$], so it doesn't matter I think." Arya understood that a function and its inverse represent the same invariant relationship between quantities and she anticipated she could consider either quantity as the input in either equation despite the equations being written in a way that imply a particular input quantity.

As the conversation continued, Arya indicated $h = (SA - 2\pi r^2)/(2\pi r)$ was similar to the graph on the computer, "If you think of [the horizontal axis] as your input axis." Arya then added that either equation represented either graph. I infer from Arya's capacity to consider two graphs and two equations as each representing an invariant relationship that Arya understood that although there were certain conventions regarding input and output for graphs and equations, maintaining these was not critical when representing relationships between quantities.

To explore how Arya was making sense of the switching technique she used in the pre-interview and during the Graphing Sine/Arcsine Task episodes, I returned to this problem after she had addressed all other prompts and asked about a hypothetical student who determined the inverse of $SA = 2\pi rh + 2\pi r^2$ to be $SA = (h - 2\pi r^2)/(2\pi r)$. She questioned the rule produced by switching the variables, "Is that the relationship between height and surface area? …You changed the relationship to something that's not true." Arya further described, "They switched their variables and solved like we were taught to do in school. They followed the system… But that is just for the x and y axes, you switch your variables so that you can graph them on the same axes and have the different graphs." Arya understood the technique of switching variables to be for the purposes of maintaining certain graphing conventions (e.g., x-input-horizontal axis, y-output-vertical axis). Arya also described that for a student to make sense of switching the variables in relation to the context, "You define SA as height and h as surface area… Then it holds the situation. Why would you do that?" Consistent with her activity at the end of the Graphing Sine/Arcsine Task episodes, Arya understood a function and its inverse represented the same relationship, and considered switching the variables as confusing and unnecessary in context as it added the complexity of using variables arbitrarily.

Katlyn's Activity in the Cylinder Problem

After describing the quantities she conceived of in the situation, I asked Katlyn to represent the relationship between height and surface area. She initially identified surface area changed as height changed (MA1), then further described the surface area as the area of the top and bottom circular enclosures plus the area of a rectangle that wrapped around the cylinder, making hand motions as if unfolding the cylinder. Katlyn wrote the expression $2\pi r^2 + 2\pi rh$, sketched a linear graph (*Figure 91*), but then stated, "I don't know if that's right." As she considered this, Katlyn stated that the area of the circular enclosures were constant, pointing to $2\pi r^2$ in her expression, adding, "The only thing that you're changing is your height… 'Cause like the circumference of your cylinder isn't changing… I feel like that's a linear relationship." Katlyn used a combination of her image of the situation (e.g., unwrapping a cylinder) and her derived expression to conclude the expression defining surface area, $2\pi r^2 + 2\pi rh$, was linear with respect to h.

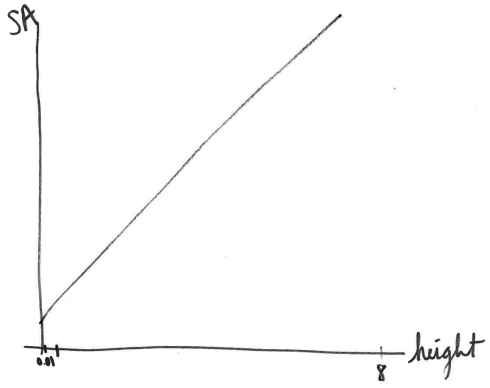


Figure 91: Katlyn's surface area/height graph.

Although Katlyn had accurately sketched a graph of the relationship between height and surface area, her activity focused on constructing and interpreting an expression that defined the relationship. Intending to support Katlyn in reasoning about the situation to describe the relationship, I moved the slider to specific height values (e.g., heights of one, two, and three). Katlyn described that from a height of one to a height of two the area of the rectangle doubled,

and from a height of one to a height of three the area of the rectangle tripled. Katlyn understood that the area of the rectangle wrapping around the cylinder was proportional to the height of the rectangle, using this to justify the linear relationship she produced in her graph and expression. Whereas in activity, I conjectured Katlyn was reasoning about the covariational relationship represented in the situation, during retrospective analysis I conjecture Katlyn could have been focused on the relationship she inferred from her derived expression (e.g., since r was not changing, $2\pi rh$ was proportional to h).

Conjecturing Katlyn was reasoning about the covariational relationship, I then showed Katlyn the dynamic graph with surface area on the horizontal axis. Katlyn described, "[the graph on the computer]'s the same… So as like the height is increasing, surface area is also increasing linearly [*tracing along the line on the computer screen as the animation plays*]. And then as the height is decreasing surface area is also decreasing [*tracing back along the line on the screen as the animation continues*]" (MA1-2). Katlyn coordinated how surface area varied as height increased and decreased (i.e. reasoned uni-directionally with height as the input), reasoning bidirectionally with the axes as she considered the relationship on the graph on the computer by coordinating height, on the vertical axis, first.

Conjecturing Katlyn was capable of reasoning about surface area as the input, but that, like Arya, she considered height to be a more natural input quantity, I asked, "Is there another way to read [the graph]?" Katlyn responded, "As surface area increases, height increases, and as surface area decreases, height decreases.... whatever happens to one is like happening to the other one." Like Arya, Katlyn's preference was to coordinate height first, but she understood this was not the only option. Katlyn anticipated she could construct the relationship as a bidirectional relationship

in terms of amounts of change (MA3) or rates of change (MA4-5) as she understood the quantities simultaneously covaried (e.g., changing height did not *cause* changes to surface area).

Katlyn addressing "inverse."

I was intentionally vague as I asked Katlyn "for the inverse" of her function rule (she had added '*SA* =' to her expression $2\pi r^2 + 2\pi rh$ prior to my question). Katlyn switched the variables and solved for *SA* (*Figure 92*). I asked Katlyn to explain her solution strategy (Excerpt 29).

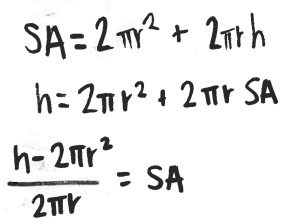


Figure 92: Katlyn's work determining "the inverse."

Excerpt 29: Katlyn describing "inverse" in the Cylinder Problem.

1	Katlyn:	It's funny that you say that, 'cause I'm tutoring two girls and we were
2		doing inverses yesterday. And I don't, and I still can't explain why we
3		do this. I was trying to think of a way to explain it to them, and I didn't
4		know the answer. Um [*pause*]. Because that's what I've been told to do
5		for six years. I don't know, I really don't know the answer to your
6		question.
7	TP:	Okay. So you said you were just tutoring someone on this?
8	Katlyn:	Yeah, and… they were just like, 'well how do I do it?' And so I told
9		them, like you have to make sure the… function is one-to-one so like
10		for every… input there's only one output and for every output there is
11		only one input. All that nonsense that doesn't, I don't really know why

12		we do that. But that's what has to happen before you can switch your
13		input and output and then solve. So, why do we do this? I don't know.
14		But I know this is what the answer is and I. Yeah, I don't know.
15	TP:	Okay and so this is the answer [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?
16	Katlyn:	Yes. Yeah, yeah.
17	TP:	But, you're sort of also acting like there's something you're not
18		comfortable with about it.
19	Katlyn:	I just don't know what it means, like I don't, why do I care about this
20		[*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]?
21	TP:	So say a little bit more what do you mean you don't know what this
22		[*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$] means?
23	Katlyn:	I don't know what it means. I know [$SA = (h - 2\pi r^2)/(2\pi r)$] is the
24		inverse, for surface area of a cylinder. That is all I know. Why is it the
25		surface area? What does it, what does the inverse for surface area mean?
26		I guess I'm thinking like. [*pause*] Okay, it reminds me of that time that
27		we were doing like volume of a cube being like side-squared and then
28		we switched the two and then I was like, okay so now, *s* means volume
29		and *V* means side[length]. So now does here, [*pause*] surface area mean
30		height and height mean surface area? Or did we just not finish the
31		problem in class to conclude about what, I don't, I don't remember. I
32		have no idea why we do this.
33	TP:	So, you're starting to say here [*pointing to* $SA = (h - 2\pi r^2)/(2\pi r)$]. If, if
34		*SA*… represented height, and *h* represented surface area?

35	Katlyn:	Well, it wouldn't make any sense. Because then it would just be the
36		same. Like if you multiplied [$SA = (h - 2\pi r^2)/(2\pi r)$] all back out you
37		would get [$SA = 2\pi r^2 + 2\pi rh$], I guess. And so like I'm attributing [$SA =$
38		$(h - 2\pi r^2)/(2\pi r)$] to be the same thing where this is now height [*pointing*
39		*to SA in* $SA = (h - 2\pi r^2)/(2\pi r)$] and this is now surface area [*pointing to*
40		*h in* $SA = (h - 2\pi r^2)/(2\pi r)$]. That doesn't make any sense. We might as
41		well have kept it that way. [*pause*] That's probably not right then cause
42		it has to mean, it has to mean something different.

Throughout her activity, Katlyn indicated she understood that switching variables was critical to defining the inverse function (lines 1-6, 8-14, 35-42), but she was unable to provide a justification for why this activity was critical (lines 3-6, 10-15, 23-25). Although she described "switching input and output" (lines 12-13), her activity indicated that she understood this to mean switching variables without explicitly paying attention to the quantitative referents for the variables. Katlyn recalled switching the quantitative referents when discussing the relationship between side length and volume of a cube and accurately described how she maintained the relationship between volume and side length by switching the quantitative referents (lines 26-30). However, in this case when describing the relationship between height and surface area, she rejected this reasoning (lines 40-42). Similar to Arya's objecting during the Graphing Sine/Arcsine Task episodes, Katlyn considered switching variables to be unnecessary and cumbersome if a function and its inverse were meant to represent the same relationship, leaving her to conclude that the inverse function must define a different relationship than the original function (lines 34-42). Hence, as Katlyn considered her two inverse function meanings, switching variables and switching the quantitative referents of each variable to maintain the

relationship between quantities and switching the variables while maintaining the quantitative referents to define a new relationship, she concluded that the latter meaning must be correct.

Comparing the students' activity addressing the Cylinder Problem.

In the pre-interview, both students understood the inverse of a contextualized function represented a different relationship than the original function. Katlyn's activity in the teaching experiment was not enough to support her in making metamorphic accommodations to this understanding. In contrast, Arya understood an inverse function represented the same relationship as the original, and re-conceptualized her switching technique by considering switching the quantitative referent of each variable to maintain the relationship represented by the function and its inverse.

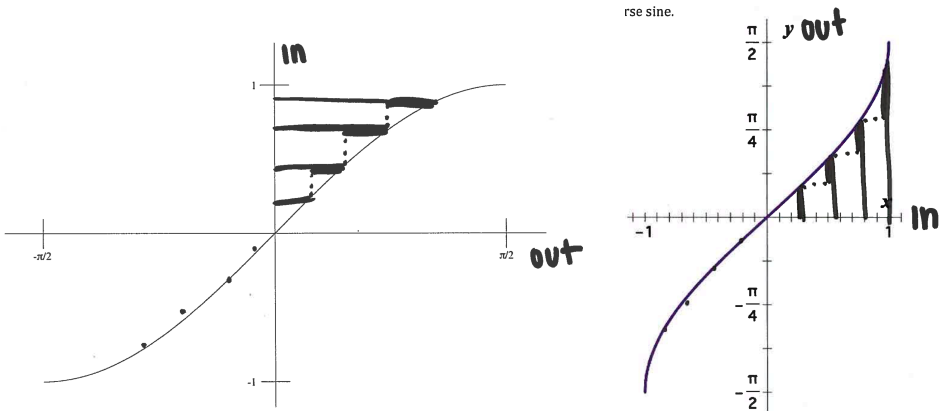
Although the two students had different understanding of how their activity related to inverse function, both students anticipated that height and surface area simultaneously covaried and one quantity did not depend on the other (i.e., understood the relationship as a bidirectional relationship directionally (MA2)), without having to develop refined images of how to represent this in a representation of the situation (i.e. developing images for the relationship in terms of amounts of change (MA3) or rates of change (MA4-5)). Additionally, each student reasoned bidirectionally with respect to the axes to interpret her constructed graph (or the graph on the computer) with either quantity as the input.

Students' Activities on Other Prompts

In this section I present the students' activities on prompts that provided insights into their understandings at the end of the teaching experiment. I specifically identify activities that stood in contrast with their activity in the pre-interview, explaining how I interpret these shifts in terms of shifts in the students' mathematical meanings that occurred during the teaching experiment.

Addressing function in relation to conventions common to school mathematics.

When addressing the Sine Inverse prompts (Appendix B, Problem 6a-b), Arya and Katlyn described that a hypothetical student's graph (Problem 6a) represented the inverse sine function if the quantity represented on the vertical axis (i.e., vertical distance relative to the unit circle) was considered the input quantity. When given a second hypothetical student's graph of the inverse sine function (Problem 6b), Arya and Katlyn claimed that both graphs represented angle measure "increasing at an increasing rate" as vertical distance increased from zero to one, with Katlyn identifying how to represent associated amounts of change (e.g., changes in angle measure increasing for equal successive changes in vertical distance) in both graphs (MA1-3 see *Figure 93* for her work).



(a) (b)

Figure 93: Katlyn's work on the inverse sine prompts.

By the end of the teaching experiment, both students understood the inverse sine function as representing the same quantitative and covariational relationship between vertical distance and angle measure as the sine function; both students understood the graphs were representing the same functions. This reasoning stood in contrast to the students' activities in the pre-interview

when neither student considered the hypothetical student's work represented in *Figure 93*a as being correct.

Arya and Katlyn exhibited compatible activity in tasks that I designed to explore their function meanings in relation to conventions common to school mathematics in the post-interview. When addressing the fourth prompt in the I.Q./Mean Income problem (Appendix B, Problem 3), both students argued that if I.Q. was the input and mean income was the output, the graph represented a function, and if mean income was the input and I.Q. was the output than the graph did not represent a function. Rather than imposing that the input was represented on the horizontal axis as each had in the pre-interview (e.g., re-drawing the graph so that y was on the horizontal axis when addressing if a graph represented "x as a function of y"), each student interpreted the same relationship in two ways, applying her function definition (i.e. a unique output for each input) to make accurate conclusions regarding if each relationship was or was not a function with the different input quantities.

The students' activities addressing the Stay-Positive Axis Problem provided me additional insights into their function meanings in relation to conventions common to school mathematics. Arya immediately identified the unconventional axes and produced the graph seen in *Figure 94*a. Initially, Katlyn drew $y = x$ as if the axes were conventional; after identifying the unconventional axes, she adjusted her graph to represent $y = x$ with respect to these axes (see *Figure 94*b). Whereas in the pre-interview, the students described disliking representations that were unfamiliar to them (e.g., *Figure 24*b in the Double Parabolas Problem) or attempted to re-draw a graph in a more familiar way (e.g., redrawing the graph as shown in *Figure 1*b), during the post-interview, neither student attempted to change the axes to match conventional Cartesian

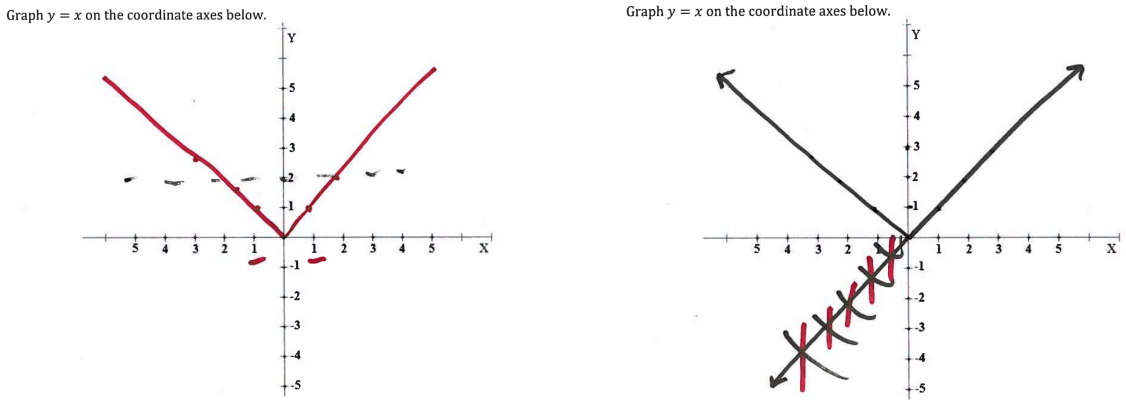
coordinates. Instead, each student's goal was to accurately represent the relationship defined by $y = x$ on the given axes.

I conjecture that by having repeated opportunities to construct, represent, and reason about relationships between quantities with the quantities represented on different axes in different graphs (e.g., Graphing Sine/Arcsine Task, the Bottle Problem, the Cylinder Problem), the students abstracted that representing the input on the horizontal axis is a convention that did not need to be maintained. Further, reasoning about relationships represented in unconventional coordinate systems (e.g., different Cartesian coordinates during Graphing the Sine/Arcsine Task Day Two, polar coordinates) supported the students in developing understandings that representing relationships between quantities was paramount when creating or interpreting graphs, rather than maintaining certain conventions. I conjecture these understandings supported and were supported by the students' reasoning bidirectionally with respect to the axes and quantities.

When I asked each student if their graph showed, "x as a function of y," both students initially responded that this was not true.[45] However, as they further considered their graphs, each student identified there were two points on their curve for certain y input values (i.e. $y > 0$). Each student then identified that the same coordinate pair defined the two points on the curve (e.g., the two points on the curve with a y-value of two were (2, 2) and (2, 2)). The student then applied her function definition (e.g., a unique output for each input) to conclude that x was a function of y, with some coordinate pairs appearing twice. For instance, Arya explained "If we input y, there's really only one x-output, they're just in two different places." Similarly, Katlyn argued, "You're only getting one output, twice." Arya and Katlyn no longer relied on certain

[45] Interestingly, neither student referred to the equation $y = x$ when describing the graph as representing or not representing a function.

perceptual features of a graphical representation (e.g., if a line perpendicular to the axis representing the input quantity crossed the graph twice) as they had in the pre-interview. Instead, each student considered the values that constituted those points when deciding if a graph represented a function. Further, each student understood graphs as representing relationships; she understood that defining input and output quantities and considering if each input value was mapped to exactly one output value was critical to determining if a relationship was a function. I conjecture the students' activities discussing relationships as functions, especially relationships that they had not explicitly represented on a graph (e.g., distance from a city and points on the path in the Car Problem) likely supported this reasoning.



(a) (b)

Figure 94: (a) Arya's and (b) Katlyn's work.

Addressing rate of change.

Each student reasoned bidirectionally with respect to the axes and quantities during the teaching experiment and each student engaged in reasoning compatible with this when addressing the Two Rates of Change task (Appendix B, Problem 9). Specifically each student initially described the rate of change represented by the curve as four, writing the equation $y = 4x$ (assuming x was represented on the horizontal axis and y on the vertical axis). After Arya

determined this equation, I asked, "What if a student looked at it and said, 'No the change is one-fourth'?" Arya responded (Excerpt 30).

Excerpt 30: Arya describing the rate of change in two ways.

1	Arya:	The rate of change of what? So, this is also x equals one fourth y [*writes* x
2		$= (1/4)y$]. So as we change y, x is changing by a fourth.
3	TP:	So you can say it's either four or one-fourth?
4	Arya:	You'd have to define your variable, I mean you'd have to define your
5		[*pause*]. One is one fourth and one is four. You can't say one is both.
6		Does that make sense?
7	DM:	So like if I said one is four, that means I'm saying what really?
8	Arya:	[*pause*] What? That your change in y so, is four every time. Yeah.
9	DM:	And if I said my change is a fourth.
10	Arya:	Then your change in x is a fourth every, for every change in y.

Arya understood the graph as simultaneously representing a relationship that could be interpreted as having a rate of change of four or one-fourth depending on which quantity she coordinated first (lines 1-2, 8, 10). Further, she described that the rate of change represented how much quantity A changed for equal changes of quantity B (MA1-3).

Katlyn engaged in similar reasoning. After she determined the formula $y = 4x$, I asked, "Is there another way you could look at it?" Katlyn responded, "If I'm inputting y," and wrote the rule $x = (1/4)y$. She then explained the rate of change is "four when you're inputting x, and one-fourth if you're inputting y." As in the pre-interview, each student initially assumed that the input was represented on the horizontal axis. However, in this interview, each student anticipated that

there was another option; she understood that the rate of change of the relationship depended on what quantity, variable, or axis (or a combination of the three) she coordinated first.

Explaining the Students' Activities in the Post-Interviews

Relative to the pre-interviews, each student exhibited novel meanings that I conjecture she constructed during her experiences in the teaching experiment. When discussing function and rate of change, each student emphasized the importance of defining input and output quantities (e.g., Problem 3, 5, 6). Additionally, I asked the students to represent relationships with unconventional (relative to school mathematics) axes (e.g., Problem 2, 4, 5, 8) and each student worked with these representations without perturbation. Each student identified that regardless of convention, accurately representing a relationship between quantities was paramount when assessing a graphs mathematically correctness.

As previously described, I conjecture by having multiple opportunities to reason bidirectionally with respect to the axes and by constructing multiple bidirectional relationships, each student had the opportunity to abstract meanings for function, rate of change, and interpreting graphs that supported her in understanding the *something* that these different representations of relationships represented (P. W. Thompson, 1994c). Further, reasoning bidirectionally with respect to axes and quantities supported the students in understanding that defining input and output quantities was critical when defining a function, describing the rate of change of a function or relationship, or interpreting a graph.

The main distinction between the student's activities in the post interviews was when addressing "inverse" in the Car Problem and the Cylinder Problem. Namely, while both students accurately described multiple relationships and their inverses (from my perspective), their understanding of how these descriptions related to inverse functions and relationships differed. I

conjecture that during the four Graphing Sine/Arcsine Task episodes, Arya re-conceptualized her inverse function understandings such that she understood that a function and its inverse represented the same relationship; she made accommodations to her switching the variables understanding by considering switching the quantitative referents of each variable so that the relationship represented by the resultant analytic rule represented the same relationship as the original rule. Arya leveraged this understanding when describing inverse relationships in the Car Problem and in the Cylinder Problem.

In contrast, Katlyn often did not understand her activity describing a relationship and its inverse as being related to inverse functions (or relationships). During the Graphing Sine/Arcsine Task episodes, Katlyn made sense of switching the variables in relation to maintaining the relationship between quantities in activity and recalled this activity in the post interview (e.g., Excerpt 29). However, Katlyn did not make metamorphic accommodations to her inverse function meanings that supported her in understanding switching the variables in relation to maintaining the relationship between quantities in all cases (e.g., Excerpt 29). By the end of the teaching experiment, Katlyn was still perturbed as to what inverse functions represented, and conjectured that a function and its inverse were not meant to represent the same relationship between quantities.

Chapter 6: Conclusions and Discussion

In this chapter, I describe and summarize the students' activities throughout the teaching experiment in the context of my research questions. Recall the questions that I set out to address were:

a. What ways of reasoning do students engage in during activities intended to emphasize reasoning about relationships quantitatively, covariationally, and bidirectionally?
b. How do students ways of reasoning identified in addressing question (a) interplay with their function and inverse function meanings?
c. How do common school conventions interplay with understandings developed and described in (a) and (b)?

I first describe the students' general quantitative and covariational reasoning. I then characterize how this reasoning was foundational to the students' construction of quantitative structures and their bidirectional reasoning. I describe two forms bidirectional reasoning (with respect to the axes and with respect to the quantities) and how these two forms of reasoning interplayed with one another.

After addressing the students' quantitative, covariational, and bidirectional reasoning, I describe how this reasoning interplayed with the students' other mathematical meanings, including the students' understanding of graphs as well as their function and inverse function meanings. Throughout, I discuss students' reasoning and mathematical meanings in relation to what I perceive to be conventions common to school mathematics (and the extent that the students perceived them to be conventions). After addressing the students' activities as related to the research questions, I describe implications these results have for curriculum development. I conclude with limitations of the study and areas for future research.

The Students' General Quantitative and Covariational Reasoning

As bidirectional reasoning is a particular form of quantitative and covariational reasoning, in this section I characterize the students' quantitative and covariational reasoning that provided

foundations for their bidirectional reasoning. I first describe the non-numeric nature of the students' reasoning and how this went hand-in-hand with their understanding of curves as representing projected magnitudes. I then describe the students' covariational reasoning and their coordination of chunky and smooth images of change to engage in continuous covariation. I conclude this section by describing how the students' images of the quantities and relationships played a crucial role in their constructing and representing these relationships.

Reasoning non-numerically and understanding curves as representing projected magnitudes.

Both students worked non-numerically throughout the teaching experiment, which is the essence of quantitative reasoning (Johnson, 2012b; P. W. Thompson, 2011). In some instances, the students asked for or began to use specific values that they deemed as important landmarks with respect to the problem context (e.g., asking for a value of the height of the tower in the Power Tower Activity, Arya using numbers on the bottle in the Bottle Problem). However, in these cases, the students either decided they did not need specific values or did not use the values in a way that was critical to their reasoning or their representation of the relationship beyond identifying landmarks. Instead, the students anticipated that they could obtain measurements (either physically or fictitiously) for the varying quantities while understanding that these measurements were not critical to their reasoning about the covariational relationships between the quantities. By repeatedly engaging in situations that asked them to construct relationships between quantities without providing specific values, the students developed ways of reasoning that supported them in constructing and representing these relationships without the need for specific values; the students' ways of reasoning entailed them developing images of quantities as

increasing or decreasing as well as changing at a constant, increasing, or decreasing rates without their images necessitating specific values.

Additionally, consistent with emergent shape thinking as described by Moore and Thompson (2015, in preparation), the students developed ways of reasoning about graphs as representing these relationships by conceiving of curves as representing two projected covarying magnitudes representing these varying quantities values or magnitudes. For example, when addressing the Bottle Problem, each student considered subdividing the height of the bottle into parts (e.g., *Figure 49*b, *Figure 61*). She then coordinated total volume for these changes of height by comparing the amount of volume within each cross section of height to describe and compare changes of total volume as height increased (MA1-3). As each student moved to translating this relationship to her graph, she marked segments of equal length along the horizontal axis, with the length of a segment from the origin to a tick on this axis representing the height of the bottle from the bottom to that tick on the picture of the bottle. She then represented the volume at that height by drawing a vertical segment from this tick (e.g., *Figure 52*, *Figure 65*), adjusting the size of consecutive segments to represent the covariational relationship she intended (e.g., volume increasing at an increasing or decreasing rate with respect to height) (MA1-4).

As a second example, consider the students' initial activities during the Power Tower Activity. The students coordinated that for any change of height there was an equal change in total distance (MA3). As they moved to representing this on the graph, the students approximated changes of height along the vertical axis, motioning up when measuring an increase in height and motioning down when measuring a decrease in height. The students then translated these changes of height to changes in total distance by translating these same lengths measured along the vertical axis to the horizontal axis starting at the last point they plotted on

this axis (e.g., as recreated in *Figure 44*a-b). As the students engaged in these activities, they plotted points that corresponded to projected magnitudes representing the height and total distance of the rider on the Power Tower.

As a third example, consider Arya's careful attention to the axes as she addressed the Car Problem (e.g., recreated in *Figure 81*). After using the situation to coordinate Homer's direction of change (e.g., Homer's distance from each city was decreasing) (MA1-2), Arya plotted two points on her graph and connected these with a segment, moving from the top right to the bottom left point. Intending to verify that this segment represented the relationship she intended, Arya motioned as if drawing horizontal and vertical lines to the vertical and horizontal axes. She then traced from where these imagined lines intersected the axes towards the origin (corresponding to the direction in which she traced out the segment) to verify that the two points on her segment represented Homer's distance from each city decreasing (MA1-3) (e.g., recreated by 2 and 3 in *Figure 81*). Arya anticipated that the points on her segment represented covarying projected magnitudes representing Homer's distance from Springfield and Shelbyville and examined the axes to verify that she represented each of these magnitudes decreasing.

As the students reasoned with curves as representing two projected magnitudes, they were aware that they could measure the magnitudes of the quantities using different units (e.g., feet or radii in the Bug on a Fan Problem, feet or a unit equivalent in magnitude to the maximum height of the tower in the Power Tower Activity), and their actions indicated that they understood that the choice of unit did not affect the magnitude of the quantity or the underlying covariational relationship. Although I did not directly examine the students' levels of thinking with magnitudes as described by P. W. Thompson et al. (2014), their capacity to reason about non-

numerical magnitudes proved foundational for much of the students' activities throughout the teaching experiment.

On the students' covariational reasoning.

During the pre-interview, each student exhibited activities that I took to indicate she was reasoning about graphs as representing relationships between covarying quantities (or magnitudes). For instance, when addressing the Bottle Problem in the pre-interview each student made accurate statements regarding the directional covariation of height and volume of liquid in a bottle (MA1-2). However, neither student exhibited activities to indicate she understood ways to coordinate the amount of change of one quantity with respect to the other quantity (MA3). Because of this, neither student was successful in drawing an accurate bottle for the given graph.

During the Bug on a Fan Problem, we introduced a way to systematically consider changes of one quantity for equal changes of another quantity based on the framework described by Carlson et al. (2002). Arya and Katlyn considered equal changes of angle measure to systematically describe and represent the relationship between distance above the vertical axis and angle measure of a bug moving in a circular motion on a fan, and understood this as the sine relationship if the vertical distance was measured in a unit magnitude equivalent to the radius of the circle (i.e. they reasoned covariationally to construct the sine function as representing a relationship between covarying quantities, see Excerpt 7).

After completing the Bug on a Fan Problem, Katlyn and Arya repeatedly engaged in systematically considering changes in one quantity for equal changes of the other quantity when describing relationships between covarying quantities. Moreover, because they often did so without prompting from the teacher-researcher (e.g., coordinating volume and changes of volume for equal changes of height in the Bottle Problem as in Excerpt 21 and Excerpt 25), I

conjecture the students developed ways of reasoning (and the activities associated with this reasoning) about covariation. That is, they became reflectively aware of particular ways of thinking about covariation and assimilated situations where they conceived of two quantities covarying to those ways of reasoning.

As Arya and Katlyn continued to construct and represent relationships between quantities, they exhibited activity I took to indicate both students imagined (and coordinated) change happening both smoothly and in chunks, which is referred to as *continuous covariation* (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Saldanha & Thompson, 1998; P. W. Thompson, 2011). As an example, consider each student's activities in the Bottle Problem. Suggesting a smooth image of change, Arya initially described the height and volume of water in the bottle increasing *as water was being poured* into the bottle. She then considered landmark heights where the bottle changed shape and the relationship between height and volume between these landmark heights. As she considered the relationship as the bottle's curvature varied, Arya sub-divided each section of height into sub-chunks and described how the volume changed from one sub-chunk to the next, anticipating that she could break each sub-chunk into smaller chunks (MA1-3). Similarly, once Katlyn understood volume as total volume, she coordinated how volume changed for equal changes of height in her bottle, leveraging chunky images of change to construct an accurate graph (MA1-3). However, Katlyn's words and actions indicated that she understood the quantities as smoothly increasing within these chunks. For instance, she used hand motions to describe the bottle changing in shape as height increased, then describing how volume covaried as height increased smoothly.

In these cases, each student reasoned via a recursive process in which she first considered the quantities covarying simultaneously and smoothly, leveraging this image to conceive of and

make statements about the directional covariation of quantities (MA2). Then, as the student attempted to describe the relationship in more detail (e.g., make statements about amounts of change), she drew on chunky images of change to consider how (chunks of) one quantity changed as the other quantity changed by equal chunks (MA3). Each student was capable of breaking her chunks into smaller chunks with the anticipation that within a chunk each quantity passed through every value in that chunk.

In addition to providing examples of students engaging in continuous covariational reasoning, the students' activities provide empirical evidence for the parametric nature of covariational reasoning as described by P. W. Thompson's (2011) first-order model. Recall, P. W. Thompson (2011) described, "I can now represent a conception of two quantities' values covarying as $(x_e, y_e)= (x(t_e), y(t_e))$. I intend the pair (x_e, y_e) to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47).[46] Arya and Katlyn constructed multiple relationships in which they coordinated two quantities as each varied with respect to (experiential or cognitive) time, rather than one quantity being dependent on the other. For example, in both the Bottle Problem and the Car Problem, I take each student's activities to indicate she reasoned about the two quantities under consideration as simultaneously covarying with respect to a (often implicit) third quantity. In the Bottle Problem, each student made references to time, and I conjecture each student imagined the height and volume of liquid in the bottle changing as (imagined) time elapsed (e.g. Katlyn's graph in *Figure 60*). In the Car Problem, each student conceived of Homer's position on the road as varying with respect to (experiential) time and parameterized Homer's distances from Springfield and Shelbyville in her thinking according to one, or possibly both, of these

[46] I use *e* for typesetting purposes but Thompson (2011) used the lower-case Greek letter epsilon.

quantities. In each case, I take the students' activities to indicate that she conceived of the relationship between two quantities as covarying simultaneously with respect to a third quantity, often (experiential or cognitive) time (t_e).

Students' images of quantities and relationships between quantities in a situation.

Consistent with the findings of Moore and Carlson (2012), the students' images of the quantities and relationship between the quantities were critical to their representation of the relationships. With respect to the importance of a student's conception of quantities, consider Katlyn's activities when addressing the Bottle Problem. Katlyn conflated quantities (or possibly worked with an ill-defined quantity) as, from the researchers' perspectives, she moved between two conceptions of volume (total volume and volume in consecutive cross sectional areas) without realizing she was doing so. For instance, Katlyn accurately coordinated and compared segments on her graph as representing total volume and changes of total volume (*Figure 63*a), but as she described how this relationship was represented in the bottle, her description of changes in volume became changes in cross sectional volume (e.g., her shading in *Figure 63*b). It was not until Katlyn identified how to represent total volume and changes in total volume in both her graph and the picture of the bottle that Katlyn no longer conflated total volume with volume in consecutive cross sectional areas.

Katlyn's activities in the Car Problem is one example in which the students accurately represented the situation they conceived of in the situation, with this conceived relationship being different that the relationship the researchers perceived. Katlyn initially imagined Homer's distance from Shelbyville always increasing, which she accurately represented in her graph (e.g., she represented that the distance from Springfield decreased then increased as the distance from

Shelbyville monotonically increased, *Figure 83*). Once Katlyn conceived that Homer's distance from Shelbyville decreased, she appropriately represented this in her graph (e.g., *Figure 84*).

Throughout the teaching experiment, the students described relationships they conceived of in a situation and their representations were always consistent with these images. Collectively, these examples provide additional evidence for the importance of taking students' understanding of quantities and their associated images of the relationships between quantities seriously (Moore, 2013; Moore & Carlson, 2012; P. W. Thompson, 1994a, 1994b, 2011).

Developing Rich Quantitative Structures and Reasoning Bidirectionally

Recall A. G. Thompson and Thompson (1996) defined, "The bi-directional relationship which constitutes covariation entails thinking of both quantities varying simultaneously *without* a necessary dependency" (p. 8). Compatible with this description of a bidirectional relationship, Saldanha and Thompson (1998) defined simultaneous continuous covariation as an individual maintaining a sustained image of two quantities changing in tandem where "one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 298). With the goal of supporting students in constructing bidirectional relationships via simultaneous continuous covariational reasoning (i.e. reasoning bidirectionally), I asked students to represent relationships between quantities, that from my perspective, did not have a natural input quantity or necessary dependency, often using dynamic representations (i.e. videos and applets) to support the students conceiving that the two quantities were changing in tandem.

As the students engaged in constructing and representing relationships in these situations, I noticed nuances in their reasoning. Specifically, I distinguished between the students' reasoning bidirectionally with respect to the axes and their reasoning bidirectionally with respect to the

quantities that constitute the phenomenon or situation they were reasoning about and representing. Reasoning bidirectionally with respect to the quantities involved the students constructing images of quantities and relationships between quantities that allowed them to coordinate either quantity first with the "immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha & Thompson, 1998, p. 298). I note that the students understood and anticipated directional (MA2) bidirectional relationships between quantities; it was through their attempts to refine their images and develop more sophisticated quantitative structures that they exhibited activities indicative of bidirectional reasoning with respect to the quantities in terms of amounts of change (MA3) or rates of change (MA4-5). Reasoning bidirectionally with respect to the axes *is* reasoning bidirectionally with respect to quantities, with these quantities being projected magnitudes whose pairing form a point. As such, reasoning bidirectionally with respect to the axes has the added complexity of using a representational system to represent something (possibly) independent of that system (e.g., a relationship between two quantities that constitute some physical phenomenon). The students developed these two types of reasoning hand-in-hand as the teaching experiment progressed.

In the discussion that follows I characterize four instantiations of students' activities in which I inferred the students did or did not engage in reasoning bidirectionally with respect the axes and the quantities.[47] I use the conventions: U means uni-directionally, B means bidirectionally, and the ordered pairs represent the students activities (with respect to the axis, with respect to the quantities) (*Figure 95*).

[47] The claims I make regarding the students' reasoning in certain situations are not meant to indicate the students could not reason in ways other than those described. Instead, I only intend to classify what I inferred from my observation of the students' activities as they constructed and represented relationships between quantities.

	Uni-directionally with respect to the axes	*Bidirectionally with respect to the axes*
Uni-directionally with respect to the quantities	*(U, U)*	*(B, U)*
Bidirectionally with respect to the quantities	*(U, B)*	*(B, B)*

Figure 95: Four instantiations of students' activities reasoning bidirectionally.

Throughout the pre-interview, I took the students' activities to indicate they were reasoning uni-directionally with respect to the axes; each student maintained meanings for interpreting graphs in which the horizontal axis (consciously or unconsciously) had to represent the input quantity, especially as she considered questions concerning "function" (e.g., interpreting the two graphs in *Figure 24*, interpreting a hypothetical student's statement that "*x* is a function of *y*" for the graph in *Figure 25*a). Additionally, when addressing the Double Parabolas Problem (*Figure 24*) Arya noted, "You can't change time based on distance," indicating she understood their to be a necessary dependence of distance on time. I take these activities to be indicative of her engaging in (U, U).

By repeatedly reasoning about different appearing graphs as simultaneously representing the same relationship between quantities (e.g., angle measure and vertical distance for the sine and arcsine relationships, volume and side length of a cube), each student constructed ways of reasoning that supported her in coordinating the quantity on either axis first (i.e. constructed reasoning bidirectionally with respect to the axes). I note that during their activities addressing the Graphing Sine/Arcsine Task, the students did not consider a way to coordinate changes of angle measure for changes of height in a circular motion context, the context in which they constructed the sine relationship.[48] Hence, when working with the sine and arcsine relationships (or functions), the students reasoned bidirectionally with respect to the axes while only

[48] I am not arguing the students could not construct such images. I did not ask them to do so, nor did they do so spontaneously.

exhibiting activities indicating their reasoning uni-directionally with respect to the quantities of the situation, which I take as indicative of (B, U).

When attempting to coordinate the quantity represented on the vertical axis first, I conjecture each student needed to bring her meanings for interpreting graphs to the forefront of her thinking. As each student did this, she began to consider and reflect on what was critical (e.g., the relationship) and what was not critical (e.g., the axes orientation and the order in which the axes are considered) when constructing or interpreting a graph. Each student's understandings related to interpreting graphs became objects of reflection as she engaged in breaking from conventions she previously held as critical to the mathematics. I conjecture this process supported the students in abstracting meanings for constructing and interpreting graphs at a higher level. As the teaching experiment progressed, each student's goal when constructing or interpreting curves was to appropriately represent the relationship between quantities she conceived of in a situation, regardless of axes orientation (e.g., *Figure 40*, *Figure 41*).

In addition to operationalizing reasoning about the axes bidirectionally, Arya and Katlyn constructed multiple bidirectional relationships with respect to the quantities that constituted the phenomenon or situation they were reasoning about and representing during the teaching experiment (e.g., height and total distance in the Power Tower, volume and height in the Bottle Problem). In each of these tasks, the students' initial activities were consistent with (U, U) when interpreting the relationship in terms of amounts of change (MA3) or rates of change (MA4-5). In some cases, the students refined their images of the situation so that they reconsidered the relationship by first coordinating equal changes in the quantity they initially considered second (i.e. engaged in (U, B)). For example, during the Power Tower Activity, the students first described the relationship by coordinating changes of height. As the students re-constructed their

images of the situation, they coordinated that for any change in total distance there was an equal change in height, constructing the relationship between height and distance as a bidirectional relationship.[49] After they refined their images of the situation such that they understood the relationship between height and distance as a bidirectional relationship, the pair coordinated either axis first when describing the relationship between height and distance, with the covariational relationship remaining invariant regardless of which axis they coordinated first (i.e. engaged in (B, B)).

In other cases, after constructing a relationship and graph by engaging in (U, U) in terms of amounts of change (MA3) or rates of change (MA4-5), the student interpreted the relationship represented by the graph by reasoning bidirectionally with respect to the axes (i.e. engaging in (B, U)). This reasoning then supported the students in developing novel images of the situation. For example, in the Bottle Problem, the students constructed the relationship using images of volume covarying for changes in height and constructed a graph representing this relationship with height represented on the horizontal axis (e.g., engaged in (U, U)). Then, when given a graph representing height on the vertical axis, each student drew a bottle by first coordinating changes in height. In Arya's case, she also interpreted the relationship represented in the graph by first coordinating changes of volume and then changes in height (i.e. engaged in (B, U)), initially indicating she was unable to imagine how the relationship she inferred translated to an image of the situation. With the goal of first coordinating changes of volume, Arya re-constructed her images of the situation such that she was able to coordinate changes of volume first in an image of the situation. In this case, Arya leveraged bidirectional reasoning with respect

[49] Prior to this, the students engaged in (B, U) with the relationship they described changing (from the researcher's perspective) depending on which axis they coordinated first.

to the axes to support her developing quantitative structures such that she could represent a bidirectional relationship between quantities (i.e. engaged in (B, B)).

Regardless of the process in which they did so, when the students constructed a bidirectional relationship between quantities, they understood ways to coordinate either quantity first, with the persistent awareness that the two quantities simultaneously covaried (Moore & Paoletti, 2015; Saldanha & Thompson, 1998; A. G. Thompson & Thompson, 1996). Although the pair did not develop images that supported them in constructing every relationship they represented in the teaching experiment as a bidirectional relationship (nor did I expect or prompt them to), by the post-interview, each student exhibited activity that indicated she anticipated that she *could* construct relationships bidirectionally in terms of amounts of change (MA3) or rates of change (MA4-5) without actually representing the relationship by coordinating either quantity first in terms of amounts or rates of change. Further, she anticipated that the quantity she coordinated first *could* be represented on either axis. For instance, in the Cylinder Problem, each student coordinated height first, representing height on the horizontal axis. When given a graph representing surface area on the horizontal axis, both students indicated that they could continue to coordinate height (on the vertical axis) first, or they could coordinate surface area first (regardless of which axis represented surface area). The students' anticipation is a hallmark of bidirectional reasoning, as they anticipated that regardless of the order in which quantities were considered, the relationship would remain invariant.

Implications for the Students' Other Mathematical Meanings

Consistent with P. W. Thompson's (2011) first-order model, as the students constructed relationships between covarying quantities, they (often) implicitly coordinated these quantities with respect to a third quantity, typically experienced or cognitive time, t or t_e. They understood

that the two quantities covaried with respect to this third quantity, with graphs representing the relationships between quantities representing emergent traces of the two quantities covarying as the third quantity varied over its domain, D_t. If we represent the values that two quantities can take on with x and y then we can represent each quantity changing with respect to a third quantity as $x(t)$ and $y(t)$. If we consider these two quantities to covary simultaneously as described by Thompson, we can represent this multiplicative object by $\{(x(t), y(t)) \mid t \in D_t\}$.

When reasoning bidirectionally with respect to the axes the students understood that the relationship could be interpreted with either quantity or axis as representing the quantity they coordinated first or they considered as the input quantity. Further, regardless of which quantity was coordinated first or considered the input, the students understood the relationship between covarying quantities was invariant regardless of which axis represented which quantity; they understood $\{(x(t), y(t)) \mid t \in D_t\} = \{(y(t), x(t)) \mid t \in D_t\}$ assuming graphs represent coordinates using the convention (horizontal axis, vertical axis). Katlyn's activity addressing the inverse sine prompts in the post-interview (i.e. *Figure 93*), in which she conceived of and represented the same relationship between angle measure and vertical distance in both graphs is one example indicative of such understanding two different appearing graphs as representing the same quantitative and covariational relationship.

From my perspective, as each student constructed bidirectional relationships between quantities, they reasoned about a (possibly functional) relationship and its (possibly functional) inverse relationship. However, the students did not always relate their activities constructing and reasoning about bidirectional relationships to their function and inverse function meanings. In the sections that follow I examine how the students' emerging bidirectional reasoning interplayed with their function and inverse function meanings (i.e. addressing the second

research question guiding this study). Because the students' function and inverse function meanings were initially constrained by what I perceive to be conventions, I address the interplay between these meanings simultaneously (i.e. addressing the third research question guiding this study).

Graphs as objects versus graphs as representing relationships between quantities.

Consistent with students' activities reported elsewhere (Leinhardt et al., 1990; Moore & Thompson, 2015, in preparation; Oehrtman et al., 2008; P. W. Thompson, 1994b), in the pre-interview, Arya and Katlyn treated graphs as pictorial objects with mathematical properties essentially being properties of these objects. Moore and Thompson (in preparation) and Weber (2012) termed such an understanding–that of graph-as-wire–as static shape thinking (versus emergent shape thinking as previously described). Multiple times in the pre-interview, each student described "graphs" as being or not being functions (e.g., the Sideways Mountain problem, the Double Parabolas problem, Excerpt 6), with "function" being a property of the graph-as-wire. Further, I conjecture such an understanding went hand-in-hand with the students understanding that the input was always represented on the horizontal axis. If a graph either *is* or *is not* a function, then maintaining the convention of representing the input on the horizontal axis is critical to determining if a graph is a function so that there is not the contradiction of a graph both being and not being a function (e.g., *Figure 24*).

When first addressing Prompt 3 in the Graphing Sine/Arcsine Task (*Figure 31*), both students exhibited activity consistent with understanding graphs as being functions. Arya questioned if Graph 1 could be interpreted as the arcsine relationship as she understood that the horizontal axis represented the input quantity. Although Katlyn was comfortable describing Graph 1 as representing the arcsine relationship, she objected to referring to Graph 2 as

representing a function because each input value on the horizontal axis was mapped to more than one output value, on the vertical axis. As the students continued to address Prompt 3, they considered what was important when creating or interpreting a graph (e.g., the relationship between quantities) and what was not crucial (e.g., which axis these quantities were located on), eventually coming to understand that representing the input quantity on the horizontal axis was not critical to representing relationships between quantities (e.g., if x is a function of y, then this is true regardless if the relationship is graphed as $\{(x(t), y(t)) \mid t \in D_t\}$ or $\{(y(t), x(t)) \mid t \in D_t\}$).

I intentionally and repeatedly provided the students opportunities to construct, describe, or interpret graphs that represented the same relationship but with different quantities on the horizontal and vertical axes (e.g., the Bottle Problem, the Car Problem, the Cylinder Problem), with Cartesian graphs with different conventions (e.g., representing the sine and arcsine relationship as in *Figure 39*, *Figure 40*, and *Figure 41*), or with different coordinate systems altogether (e.g., the polar coordinate system). As they engaged in these activities, the students focused on the relationship represented by the graphs, consistently arguing that, although pictorially different, the graphs represented the same relationships. Compatible with the findings of others (Moore et al., 2013a), by focusing on the underlying quantitative and covariational relationship, the students maintained ways to understand these seemingly different graphs as representing the same quantitative and covariational relationships.

Further, and possibly more interestingly, this reasoning supported the students in interpreting the *same* pictorial object as representing different relationships or situations. For instance, in the polar coordinates teaching episodes, the students interpreted one graph (*Figure 50*a) as representing two different analytically defined rules (e.g., $y = 2\sin(2\theta)$ and $y = 2|\sin(2\theta)|$) based on their understanding of the covariational relationships each rule defined. As another example,

in the Bottle Problem, the students interpreted one graph (*Figure 73*) as representing two different situations: water entering and water leaving a bottle. In these cases, the students reasoned about graphs as representing emergent traces between covarying quantities, with the same pictorial object representing multiple possible traces.

Conceiving of graphs as representing relationships between quantities had implications for the students' function meanings. During the Graphing Sine/Arcsine Task episodes, as they considered both Graph 1 and Graph 2 as simultaneously representing the sine and arcsine relationships (or functions) each student emphasized the importance of identifying the input and output quantities when discussing if a graphed relationship represents a function. By repeatedly identifying input and output quantities when discussing graphs and functions (e.g., Excerpt 12, Excerpt 15, Excerpt 16), the students re-constructed their function meanings so that graphs (as a whole) did or did not represent functions but instead relationships between covarying quantities with defined input and output quantities did or did not represent functions (e.g., The students' activities addressing the Mean Income/I.Q. problem, and the Stay-Positive Axis problem). Each student understood that when constructing the relationship between two quantities, x and y, and graphing this relationship (i.e. $\{(x(t), y(t)) \mid t \in D_t\}$), they could define D_x and D_y as the domain of x and y, respectively, over the domain of t (D_t). They could then choose either quantity as representing the input to a function defining the relationship between quantities, with the domain of the other quantity becoming the codomain of the function, which may possibly need to be restricted in order for the relationship to represent a function (e.g., if x is considered the input quantity, then $y = f(x) \ \forall \ \{(x, y) \mid x \in D_x, y \in D_y\}$ with D_y restricted such that there is a unique y for each x; if y is considered the input quantity, then $x = g(y) \ \forall \ \{(x, y) \mid x \in D_x, y \in D_y\}$ with D_x restricted such that there is a unique x for each y). Further, I conjecture that this understanding

influenced their understanding of related ideas, such as rate of change. For instance, each student anticipated a linear graph could be interpreted with two constant rates of change depending on which variable or axis was considered to represent the input (e.g., four or one-fourth in the Two Rates of Change problem, Excerpt 35).

Researchers (Lobato & Bowers, 2000; P. W. Thompson, 1994c) have questioned whether multiple representations of functions from the researchers' perspectives (e.g., tables, graphs, and equations) are multiple representations of anything to students. Further, Moore and Thompson (in preparation) conjectured constructing and reasoning about emergent relationships between quantities can support students in conceiving of the something that is representable via multiple representations. The students' activities near the end of the teaching experiment provide evidence of their assimilating questions about "function" as questions about relationships between quantities they conceived of in a situation, supporting Moore and Thompson's (in preparation) conjecture. For example, when I asked each student if there were any relationships that could be described as a function in the Car Problem, both students referenced quantities and relationships between quantities in the situation such that for every value of a chosen input quantity there was only one value of the chosen output quantity. Arya described restricting Homer's position on the path to three separate pieces (e.g., segments A, B, and C, in *Figure 86*a) so that the relationships between distances from Springfield and Shelbyville would represent functions in each of these restricted domains. Katlyn described considering Homer's position on the path as the input and either distance from Springfield or Shelbyville as the output as for each position on the path there was only distance from either city. In both cases, the students were describing the emergent relationship between quantities as the *something* that has some attribute related to function.

Whether it be using concept definition and concept image (Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989), the APOS framework (Breidenback et al., 1992; Dandola-Depaolo, 2011; Dubinsky & Harel, 1992; Stalvey & Vidakovic, 2015), or a multiple representations approach (Brenner et al., 1997; Keller & Hirsch, 1998; Leinhardt et al., 1990; Lobato & Bowers, 2000), previous researchers examining students' function meanings have taken the function concept as paramount. This approach is sensible when we consider how little attention is given to non-functions as reflected in Arya's comment when addressing the Double Parabolas Problem in the pre-interview, "I don't work much with non-functions… I don't really know anything about things that aren't functions". However, this study takes a markedly different approach to function by making function a specific class of quantitative and covariational relationships. By the post-interview, both students reasoned about functional and non-functional relationships as representing relationships between quantities without perturbation. By engaging in multiple situations, both functional and non-functional, the students re-constructed their function meanings such that they understood functions are a special type of relationship between quantities in which each value of a chosen input quantity corresponds to exactly one value of a chosen output quantity. Hence this study adds a major contribution to the research on students' function meanings by providing empirical evidence of the potential benefits of having students construct function meanings via their experiences with functions and non-functions, an approach called for by many researchers (Carlson & Oehrtman, 2005; Dienes, 1964; Even, 1990; Even & Bruckheimer, 1998; Kline, 1974; Leinhardt et al., 1990; Oehrtman et al., 2008; Slavit, 1997; P. W. Thompson, 1994c; Vinner, 1983).

On the students' inverse function meanings.

During the teaching experiment, each student constructed multiple relationships as bidirectional relationships; in each case the student understood that the two quantities (A and B) varied "simultaneously *without* a necessary dependency" (A. G. Thompson & Thompson, 1996, p. 8). In these cases, each student understood that A and B covaried, anticipating that she could consider quantity A as the input and quantity B as the output with the awareness that she could also consider quantity B as the input and quantity A as the output. The major difference between the students was their understanding of how this reasoning related to her inverse function meanings. In the sections that follow, I characterize each student's inverse function meanings as the teaching experiment progressed.

Arya's inverse function meanings.

During the initial interview, Arya relied on switching techniques when determining inverse functions and she understood that a function and its inverse did not represent the same relationship (e.g., interpreting the inverse functions she derived in the Celsius/Fahrenheit problems as representing the "opposite" relationship of the given relationship). As Arya addressed the prompts in the Graphing Sine/Arcsine Task for various functions and relationships (e.g., sine and arcsine, $y = x^3$ and $y = x^{1/3}$, $V = s^3$ and $V = s^{1/3}$), she explicitly described two possible meanings for a function and its inverse in a contextualized situation: (1) a function and its inverse represented the same relationship (when switching the variables she switched the quantitative referents of each variable) and (2) switching variables defined a new relationship in which you "take away all the meaning [of the original function] and say that they're opposite."

Arya understood these two meanings to be incompatible and questioned which meaning was mathematically correct numerous times during the Graphing Sine/Arcsine Task episodes.

Although she considered using variables arbitrarily as a way to relate her two meanings, Arya questioned this as, to her, this procedure was cumbersome and concealed the contextual meaning of the inverse function *if* it represented the same relationship as the original function. However, by the end of the teaching experiment, Arya understood that a relationship and its inverse (regardless if the relationship or its inverse were functions in the formal sense) represented the same quantitative and covariational relationship; to Arya, the distinction between a function and its inverse was her choice of input and output quantities. For example, in the Car Problem Arya described that if both her and a hypothetical student coordinated distance from Springfield first, regardless if they represented distance from Springfield on the same axis, then they had graphed the same relationship and not inverse relationships. As a second example, when determining the inverse of the (implicitly defined) function rule she constructed when addressing the Cylinder Problem, Arya chose not to switch the variables instead re-writing the formula to imply a new input and output dependency (e.g., she determined the inverse of the function defined by $SA = 2\pi rh + 2\pi r^2$ to be the function defined by $h = (SA - 2\pi r^2)/(2\pi r)$).

Compatible with prior research (Kimani & Masingila, 2006; Paoletti et al., 2015; Vidakovic, 1997), Arya initially exhibited activities for determining inverse functions in various representations without relying on composition of functions nor understanding that a function and its inverse represent an invariant relationship between quantities. However, by the end of the teaching experiment, Arya leveraged her function meanings described above to make accommodations to her inverse function meanings. Arya came to understand that when constructing the relationship between two quantities, x and y, and graphing this relationship (i.e. $\{(x(t), y(t)) \mid t \in D_t\}$), she could define D_x and D_y as the domain of x and y. If she considered x the input quantity, then $y = f(x) \ \forall \ \{(x, y) \mid x \in D_x, y \in D_y\}$ with D_y restricted such that there is a unique

y for each x; if she considered y the input quantity, then $x = g(y) \ \forall \ \{(x, y) \mid x \in D_x, y \in D_y\}$ with D_x restricted such that there is a unique x for each y. Arya related this to her inverse function meanings by conceiving that g and f, with their restricted domains and ranges, were inverse functions; Arya understood $f^{-1}(y) = g(y)$ and $g^{-1}(x) = f(x)$.

Katlyn's inverse function meanings.

Like Arya, Katlyn used switching techniques when addressing questions concerning inverse function in the pre-interview. However, Katlyn did not use a switching technique when addressing the inverse sine prompts (Appendix A, Problem 15). Instead she exhibited a meaning for inverse sine that she maintained throughout the teaching experiment. Specifically, Katlyn understood inverse sine as representing "the angle whose sine is." When addressing the Graphing Sine/Arcsine Task, Katlyn leveraged this quantitative meaning for arcsine to describe both Graph 1 and Graph 2 as representing the sine and arcsine relationships. Towards the end of these episodes, Katlyn indicated that she understood, "Arcsine is in sine… it's like… just not the same input." Collectively, I took Katlyn's activities to indicate that she understood the sine and arcsine as representing the same relationship, with the distinction between these two functions being her choice of input and output quantities (while maintaining the quantitative referents of the variables).

However, while still addressing the prompts in the Graphing Sine/Arcsine Task, I changed the function under consideration and Katlyn reverted to switching techniques when determining inverse functions (e.g., Excerpt 22). After this, and like Arya, Katlyn considered two possible meanings for inverse functions, one that involved maintaining the relationship between quantities and the other in which a function and its inverse represented different relationships. As an example of her considering the former meaning, when describing the function that defined the

relationship between volume and side length of a cube and its inverse, Katlyn argued, "We've just been saying like we need to switch them in the equation [*pointing to* $y = x^{1/3}$] but like, we're like switching them in real life." She then argued that in $V = s^{1/3}$ (which she understood as defining the inverse function of $V = s^3$), V represented side length and s represented volume. During these activities, Katlyn conceived that a function and its inverse represented the same relationship relative to a contextualized situation, and she considered switching the quantitative referents of the variables to maintain this relationship. However, Katlyn did not exhibit activities compatible with this in the moment understanding later in the teaching experiment.

In the post-interview, when describing the inverse of the relationship between height and surface area in the Cylinder Problem, Katlyn used a switching technique to find the inverse of the rule she derived, indicating that this rule no longer represented the relationship between height and surface area. Katlyn recalled her activity switching the quantitative referents of the variables (e.g.,

Excerpt 29) and argued that if she switched the quantitative referents then the inverse rule would represent the same relationship as the original rule, concluding, "That doesn't make any sense. We might as well have kept it that way [*indicating she could have kept the original rule*]. That's probably not right then cause it has to mean, it has to mean something different." Similar to Arya's reasoning during the Graphing Sine/Arcsine Task episodes, Katlyn considered switching the quantitative referents of the variables, but understood this process to be unnecessary and cumbersome, leading her to reject the notion that a function and its inverse represent the same relationship. Hence, like Arya, Katlyn was capable about reasoning about a relationship and its inverse (from the researcher's perspective) as representing an invariant relationship between quantities without perturbation. However, when the formal notion of

inverse function (or relationship) was raised, Katlyn relied on meanings constructed during her prior school experiences in which switching variables was paramount to determining an inverse function.

Takeaways from the students' inverse function meanings.

Like the research on function, researchers examining students' inverse function meanings have generally taken the formal mathematical definition of inverse function involving function composition as paramount (Brown & Reynolds, 2007; Kimani & Masingila, 2006; Vidakovic, 1996, 1997). Critical to this is students already maintaining sophisticated function understandings (see Vidakovic's (1996) genetic decomposition). In contrast, in this study, I took markedly different approach by examining students developing relationship (regardless of function-ness) and inverse relationship (regardless of function-ness) meanings simultaneously.

Katlyn provides an example of the complexities that arise when attempting to support a student in reconsidering (her possibly unproductive) meanings that were constructed repeatedly as a means for assimilating her school experiences. Although capable of reasoning about a relationship and its inverse relationship (from the researchers' perspective), Katlyn did not always understand this reasoning as being related to her inverse function meanings. Instead, her prior and current (e.g., tutoring students) experiences reinforced the importance of switching techniques when determining inverse functions, although she questioned the meaning of the rules obtained using these methods. Katlyn's activities in the teaching experiment exemplifies how difficult it can be for students to re-consider their ways of understanding that have been formed and repeatedly reinforced during their prior school experiences.

In contrast, Arya made significant accommodations to her inverse function meanings over the course of the teaching experiment. However, developing these understanding was not trivial.

Arya reorganized her meanings for interpreting graphs (e.g., which axis could represent the input quantity), her meanings for variables (e.g., changing the quantitative meaning of *x* depending on the function under consideration), and her inverse function meanings (e.g., a function and its inverse represent something different). Compatible with the findings of others (Ellis et al., 2012; Johnson, 2012b), Arya's activities provides insights into a how a student's reasoning quantitatively and covariationally can provide foundational supports in her understanding more formal mathematical definitions or concepts (e.g., the definition of inverse functions under the operation of function composition).

A productive meaning for inverse function.

Collectively the students' activities provide insights into productive meanings for inverse relationships (or functions); a student with productive inverse relationship meanings understands that a relationship (regardless of function-ness) and its inverse represent the same relationship between quantities. Compatible with P.W. Thompson's (2011) first-order model of a conception of covariation, a student who understands the relationship between two covarying quantities, *x* and *y*, and a graph of this relationship (i.e. $(x(t), y(t) \mid t \in D_t)$) can understand D_x and D_y as the domain of *x* and *y*, respectively, over the domain of *t* (D_t). Then the student understands that she can consider either quantity as the input. If she considers *x* as the input quantity then she can represent the relationship as $y = F(x) \ \forall \ \{(x, y) \mid x \in D_x, y \in D_y\}$.[50] The student understands that the inverse relationship is defined as $x = F^{-1}(y) \ \forall \ \{(x, y) \mid x \in D_x, y \in D_y\}$. The student understands she may need to restrict the domain of *x* (D_x), the domain of *y* (D_y), or both in order for the original relationship, its inverse, or both, to represent functions (i.e. $y = f(x) \ \forall \ \{(x, y) \mid x \in R_x, \ y \in R_y\}$ and

[50] Compatible with the distinction between arcsin(x) and Arcsin(x), I use *F* to represent the relationship (regardless of function-ness) defined by *F* and *f* to represent a function defined by restricting *F*'s domain, range, or both.

$x = f^{-1}(y) \ \forall \ \{(x,y) \mid x \in R_x,\ y \in R_y\}$ where $R_x \subseteq D_x,\ R_y \subseteq D_y$ represent the restricted domains of x and y).

A student who maintains such an understanding of inverse relationships and functions understands they are using the variables arbitrarily for the purpose of maintaining conventions (e.g., the variable x represents the input quantity, the input quantity is represented on the horizontal axis) when engaging in switching techniques to determine or represent the inverse of a given relationship or function (e.g., switching the variables and solving with an analytically defined function, or switching the abscissa and ordinate of a graphed function). The student understands the result of these techniques does not change the underlying relationship represented by the relationship and its inverse; maintaining the relationship represented by the relationship or function and its inverse is paramount to their system of meanings.

Suggestions for Curriculum and Instruction

In this study, I provided findings that have important insights for the design of curriculum and for instruction. In the sections that follow, I first describe the implications these findings can have for the teaching and learning of function, inverse function, and related ideas in middle and high school mathematics, including a description of potential features of tasks that were important in the students' activities and their eventual learning. I then describe two areas in which I conjecture the students created an intellectual need for a mathematical concept that I did not directly address in the teaching experiment that have potential implications for curriculum and instruction.

Teaching and learning of function, inverse function, and related ideas.

By giving students multiple opportunities to represent relationships between covarying quantities they conceived of in (dynamic) situations, the goal of the students' activities was to

represent the relationships they conceived regardless of the final appearance of their graphs; the relationships were the *something* that the students reasoned about when discussing function (Lobato & Bowers, 2000; Moore & Thompson, 2015, in preparation; P. W. Thompson, 1994c). Additionally, by engaging in these activities, the students had the opportunity to abstract function meanings that emphasized the importance of identifying input and output quantities when describing relationships as functions. Hence, this study provides evidence that curriculum that emphasizes students constructing relationships between covarying quantities without a natural or deterministic input quantity has the potential to support the students in developing foundations for productive function meanings, meanings which many students and teachers are currently lacking (Akkoc & Tall, 2002; Breidenback et al., 1992; Carlson, 1998; Dubinsky & Harel, 1992; Even, 1990; Even & Bruckheimer, 1998; Leinhardt et al., 1990; Oehrtman et al., 2008; P. W. Thompson, 1994c; Vinner, 1983; Vinner & Dreyfus, 1989).

This study also provides insights into a way for curriculum designers to support students in developing inverse function meanings. Namely, the students in this study constructed relationships that entailed (from the researcher's perspective) a function and its inverse. By repeatedly constructing and reasoning about a relationship and its inverse relationship the students were able to anticipate that relationships (regardless of function-ness) can be interpreted in (at least) two ways. Researchers examining students' inverse function meanings have generally started with the formal mathematical definition of inverse function, which emphasizes composition of functions (Kimani & Masingila, 2006; Lucus, 2005; Vidakovic, 1996, 1997). These results indicate these researchers may have missed an opportunity to explore students' development of notions foundational for inverse function via their first constructing and

reasoning about the relationships without defined input and output quantities, and then considering issues of input, output, and function composition.

In addition to function and inverse function, curriculum designers could leverage students' reasoning bidirectionally with respect to axes and quantities when attempting to support students in constructing productive meanings for ideas related to function including rate of change, the derivative, and implicit differentiation. For instance, a student who conceives of relationships as bidirectional are attentive to identifying input and output quantities when describing the rate of change (or derivative) of a relationship (e.g., the students activity addressing the Two Rates of Change Problem). Hence, these students would not assimilate statements concerning rate of change as being about the perceptual features of the graph (e.g., slantiness), but instead about the relationship between covarying quantities.

Task design principles that support students developing productive function and inverse function meanings.

There were certain features of the tasks we provided the students that I conjecture were important to the students reasoning bidirectionally as well as their reconstruction of their function and inverse function meanings. Specifically, we attempted to use situations that would be familiar and accessible to the students, with most tasks including dynamic videos or applets that we intended to support their construction of quantities and their development of images of the quantities and the relationships between the quantities (i.e., support their quantitative and covariational reasoning). We intended for the students to focus on representing relationships between covarying quantities, regardless of which axis these quantities were represented or the general appearance of the graph. To maximize the opportunities for the students to reason *co*variationally, we intentionally did not ask students to reason with time as one of the two

quantities they considered, as students can reason uni-variationally when time is one of the quantities under consideration (Leinhardt et al., 1990).

Curriculum designers could provide students multiple opportunities to represent relationships with multiple axes orientations and with different coordinate systems in order to de-emphasize the appearance of graphs, and instead focus on graphs in terms of their representing emergent traces of covarying quantities, a call similar to that made by Moore et al. (2013). Through their engaging in these activities, the students in this study had opportunities to reflect on their meanings for interpreting graphs (and for functions and inverse functions) such that they eliminated some elements that were previously critical to their meanings (e.g., representing the input quantity on the horizontal axis) while re-considering aspects of their meanings that were critical (e.g., representing relationships accurately, identifying input and output quantities).

Students' activities that created intellectual needs.

In addition to the immediate implications for curriculum described above, there a few interactions in which the students engaged in activity that raised (or had the potential to raise) an intellectual need for the students. Harel (2007) described, "The term *intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate" (emphasis in original, p. 13). Multiple times during the teaching experiment the students' activities created an intellectual need that was addressed in their later activities. However, there were two instances in which I did not identify an intellectual need until my retrospective analysis. Specifically, the students created a need for a way to parametrically define relationships between quantities as well as for notational systems to represent multi-valued functions.

By engaging in multiple situations in which two quantities covaried with respect to a third quantity (e.g., position on a path, experiential time, cognitive time), each student developed ways of reasoning about covarying quantities that was parametric in nature, compatible with P. W. Thompson's (2011) first-order description of covariation. Further, as we provided students with different experiential situations that produced the same pictorial graph, the students developed a need to represent the difference between the two situations in the same curve; they accomplished this by adding arrows to indicate the direction in which the graph was traced as the situation progressed (e.g., with respect to experiential time). The students' activities lead me to conjecture that tasks that result in students constructing the same curve for two different experiential situations can provide students with an intellectual need for parametrically defined functions.

The students' activities also raised an intellectual need for a way to define multi-valued functions or relationships with analytic rules, something currently omitted from school mathematics. Consider the students' activities as they described their constructed Graph 1 and Graph 2 representing the conventional sine and arcsine relationships, respectively (*Figure 33*) as representing and not representing functions. During their activities, each student made multiple references to the analytic rules they associated with each graph (e.g., Graph 1 and $y = \sin(\theta)$, Graph 2 and $\sin^{-1}(y) = \theta$). For instance, Arya argued, "[$y = \sin(\theta)$] represents a function because… for every input [*pointing to* θ *in* $y = \sin(\theta)$] you get a single output [*pointing to y in* $y = \sin(\theta)$]… So this is a function [*pointing to the equation* $y = \sin(\theta)$], but this is not [*pointing to* $\sin^{-1}(y) = \theta$]. No this, I mean [*pointing to* $\sin^{-1}(y) = \theta$ *then motioning over the whole graph*]. This whole [graph]… is not." I conjecture as Arya engaged in this activity, a complexity arose as she attempted to describe Graph 2, which she was associating with $\sin^{-1}(y) = \theta$, as not being a function. Arya understood that the graph represented that for each input value there were

multiple associated output values and hence did not represent a function, but also understood that the analytic rule $\sin^{-1}(y) = \theta$ did represent a function. Hence, I conjecture at this moment in the teaching experiment, Arya (as well as Katlyn) had an intellectual need for a notation by which to define the multi-valued arcsine relationship with an analytic rule (e.g., $\theta = \text{Arcsin}(y)$ or $\text{Sin}^{-1}(y) = \theta$).

Limitations and Directions for Future Research

In this study, I provided the field with a fine-grained analysis of students' reasoning as they constructed relationships between covarying quantities, including their bidirectional reasoning. I also describe implications for this reasoning, including the students' function and inverse function meanings. However, I recognize there were limitations to this study, many of which provide avenues for future research.

Limitations of the study.

In this study, I investigated two college students reasoning abilities and understandings as they constructed and represented relationships between covarying quantities. As every student constructs his or her own knowledge, the results I presented are only intended to provide insights into the models that I constructed through building and testing hypotheses of two students' reasoning and understandings as they engaged in such situations and reconstructed or made accommodations to their function and inverse function meanings. Further, my models of each student's mathematics were influenced by the paired nature of most of the teaching episodes. Namely, each student participated to a different extent in each teaching episode making it difficult to construct models if a particular student did not participate frequently in a particular interaction.

Although I conjecture the results of this study have the potential to have major implications for middle school and high school curriculum designers, Arya and Katlyn were college students who had already developed a variety of function and inverse function meanings through their prior school experiences. Whereas these results are useful for the development of curricular materials for pre-service teachers, and college students more generally, I would not maintain that students who have not been formally presented with function and inverse function in their schooling experiences would exhibit activities consistent with the students in this study (e.g., experience similar successes or difficulties).

Another limitation is related to the students' images of change. Researchers have found it useful to differentiate between students who imagine change happening smoothly and chunkily (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). Throughout this study (e.g., starting at the pre-interview and in the sine episodes), both students exhibited activity that indicated she was imagining change happening smoothly in certain situations. Further, all students who volunteered to take part in the study exhibited smooth reasoning in the initial interview so I was unable to explore differences between Arya, Katlyn, and students who show more of a propensity to construct chunky images. As such, I am unsure how students who show a propensity to construct chunky images of change would respond to a series of activities that are designed to emphasize reasoning about relationships quantitatively, covariationally, and bidirectionally.

Another limitation that I only considered in my retrospective analysis is that I did not explore (in detail) how the students' reasoning about bidirectional relationships may (or may not) have influenced their activities in situations in which they conceived of a 'natural' or 'deterministic' input quantity. One example where this did arise was when Katlyn addressed 'function' in the Car Problem. Specifically, Katlyn conceived that the distances from the two cities were

determined by Homer's "position on the path". Because of this, Katlyn would not describe the relationship between the distances from each city as being a function as, to her, there was nothing 'functional' about the relationship. Whereas in this case, Katlyn conceived of two quantities represented in her graph as being determined by a third quantity, I did not observe the students engaging in a situation where the considered one of the two quantities they were representing as causing or determining changes in the other quantity. Hence, I am uncertain how Arya and Katlyn would respond to situations in which they conceived of a natural or deterministic quantity.

Directions for future research.

In this study, I described the students' activities as they engaged in two types of bidirectional reasoning. Their activities seemed to indicate that as they operationalized reasoning bidirectionally with respect to the axes, they were more likely to reason bidirectionally with respect to the quantities in the situation. However, this observation may have been a byproduct of the order of the tasks. For instance, I am unsure if the students would have constructed the relationship between volume and height bidirectionally before they had engaged in the Graphing Sine/Arcsine Task and operationalized reasoning bidirectionally with respect to the axes. Hence, future researchers may be interested in further exploring students' bidirectional reasoning, including if operationalize reasoning bidirectionally with respect to the axes supports students in constructing bidirectional relationships between quantities or if the reverse could also be true (i.e. reasoning bidirectionally with respect to quantities supports students in operationalizing reasoning bidirectionally with respect to axes).

Although this study provides important insights into college students reconstructing or making accommodations to their function and inverse function meanings via their reasoning

about relationships between quantities, a productive line of inquiry is to engage students who had not been formally introduced to ideas of function and inverse function (i.e. middle school students) in a similar instructional sequence to explore how such a sequence may support (or need to be adapted to support) these students' development of function and inverse meanings. If researchers are successful in engendering such meanings, it would be productive to explore implications for these meanings (e.g., How does this understanding support the students in developing productive ways of thinking about rate of change or the derivative?).

I conjecture that both students' propensity to construct smooth images of change supported their activities as they constructed, reasoned about, and represented relationships between covarying quantities. Echoing the call of Castillo-Garsow et al. (2013), I conjecture it would be productive for researchers to compare how students who show both a propensity to primarily construct chunky and smooth images of change prior to a teaching experiment engage in tasks asking them to represent relationships between quantities that vary smoothly from the researcher's perspective. Such examinations would support the field in making comparisons concerning how students' images of change influence their activities as they construct, reason about, and represent relationships between quantities.

As part of our task design, we intentionally provided the students with multiple relationships in which we did not conceive of a natural or deterministic quantity. The students often conceived of these situations in compatible ways, considering the quantities as covarying rather than one quantity causing or determining changes in the other quantity. An open question for future research is whether constructing bidirectional relationship supports students in reasoning about deterministic relationships as well as if reasoning about deterministic relationships supports or inhibits students constructing bidirectional relationships.

In *Suggestions for Curriculum and Instruction*, I described features of the task design that I conjectured supported the students' construction of bidirectional relationships between quantities. I also provided two examples in which I conjecture the students created an intellectual need for a mathematical idea (from the researcher's perspective): parametrically defined functions and notation for multi-valued functions. Future research could support or refute these conjectures. For instance, I conjecture examinations of the potential for activities similar to those used in the teaching experiment to engender such intellectual needs and the possible paths that would support the students in constructing these mathematical objects would be productive for the field.

Multiple researchers have examined middle through college students' understanding of variables (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Kuchemann, 1978; Trigueros & Ursini, 1999; Warren, 1999). For instance, Knuth et al. (2005) and Kuchemann (1978) examined students' understandings of variables as objects, as representing a specific value, or as representing multiple possible values. This study provides an additional aspect of variables future researchers may be interested in exploring, namely students' understanding of using variables to arbitrarily represent quantities in a situation. Developing such an understanding was critical to Arya connecting her switching techniques to her quantitative meaning for inverse. As I did not realize the importance of this until my retrospective analysis, I did not further explore this understanding, and future research exploring such an understanding of variables would be productive.

Finally, as another possible avenue for future research concerning students' bidirectional reasoning, consider the following anecdote. After the teaching experiment, we engaged Arya, Katlyn, and their classmates in a series of lessons on exponential functions. Two semesters later,

when discussing her thinking about teaching logarithms to her students, Arya described teaching logarithms in a way similar to how we had covered them after the teaching experiment had ended. I objected claiming we did not cover logarithms in the course. Arya refuted, arguing that when we explored exponential functions we were also studying logarithmic functions. Arya's inverse meanings supported her in constructing exponential and logarithmic functions hand-in-hand. Compatible with Arya's simultaneous construction of exponential and logarithmic functions, researchers may be interested in exploring students' construction of other function classes (e.g., quadratic and square root functions, trigonometric and inverse trigonometric functions) via their reasoning about bidirectional relationships between quantities.

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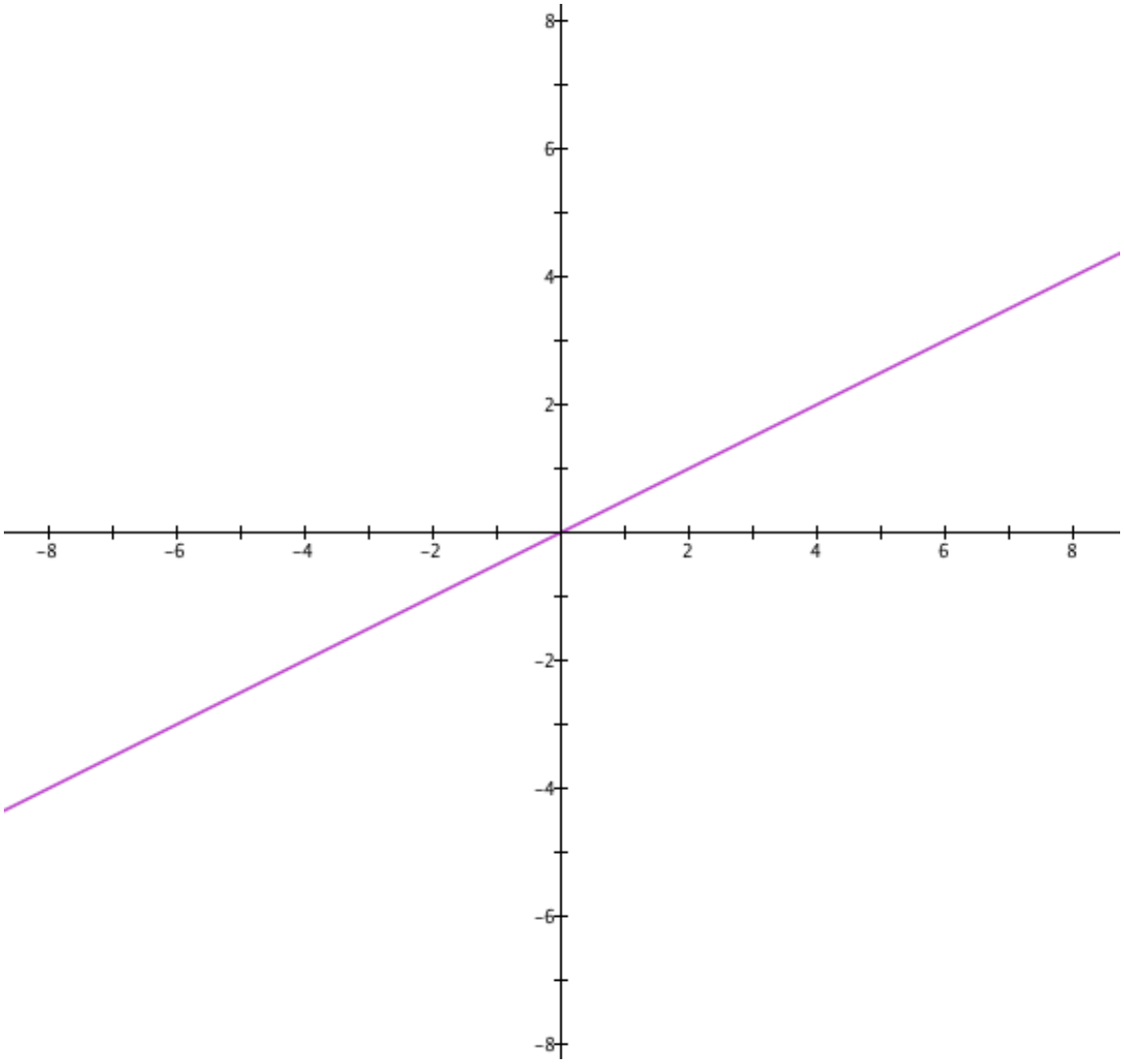
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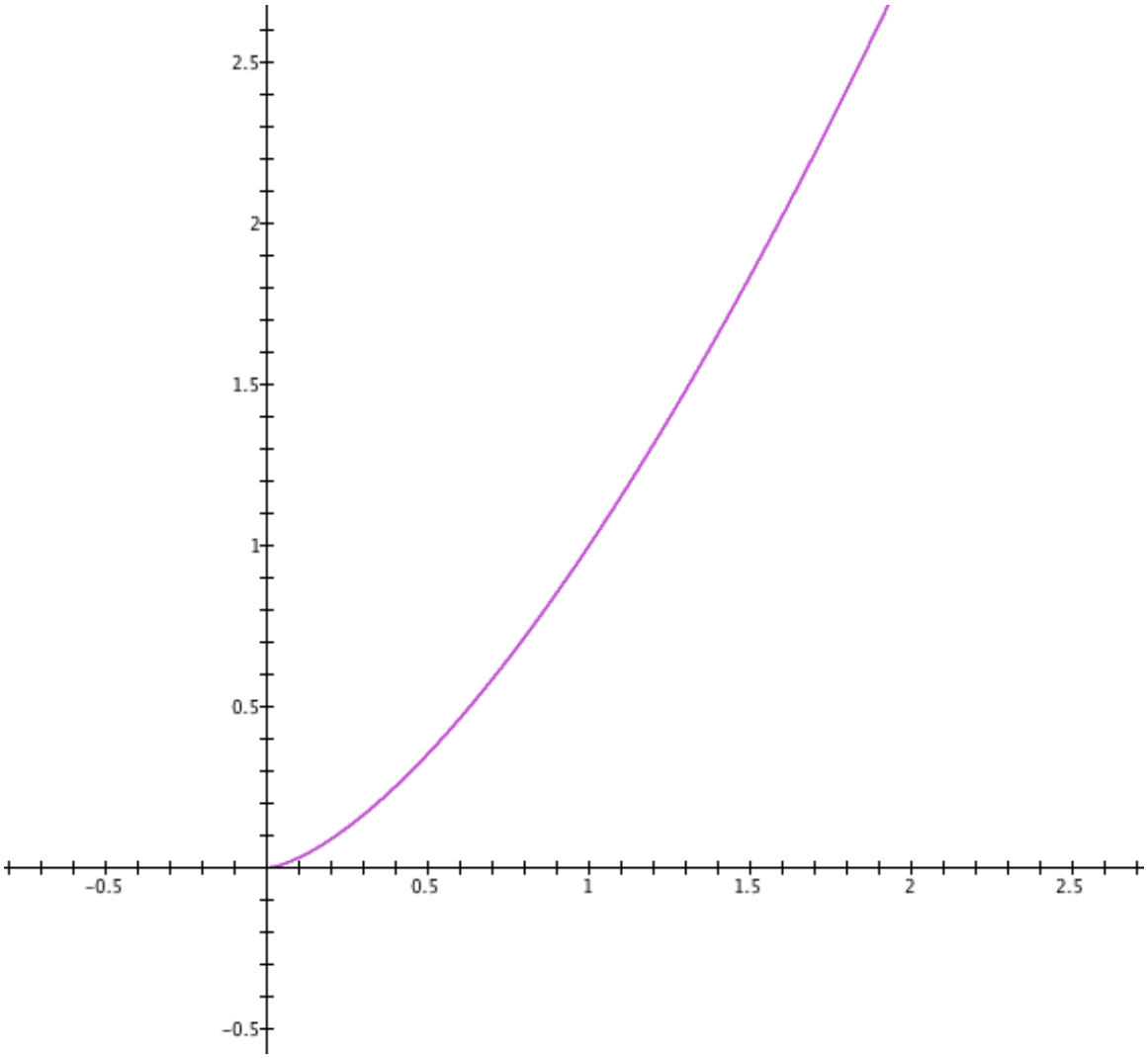
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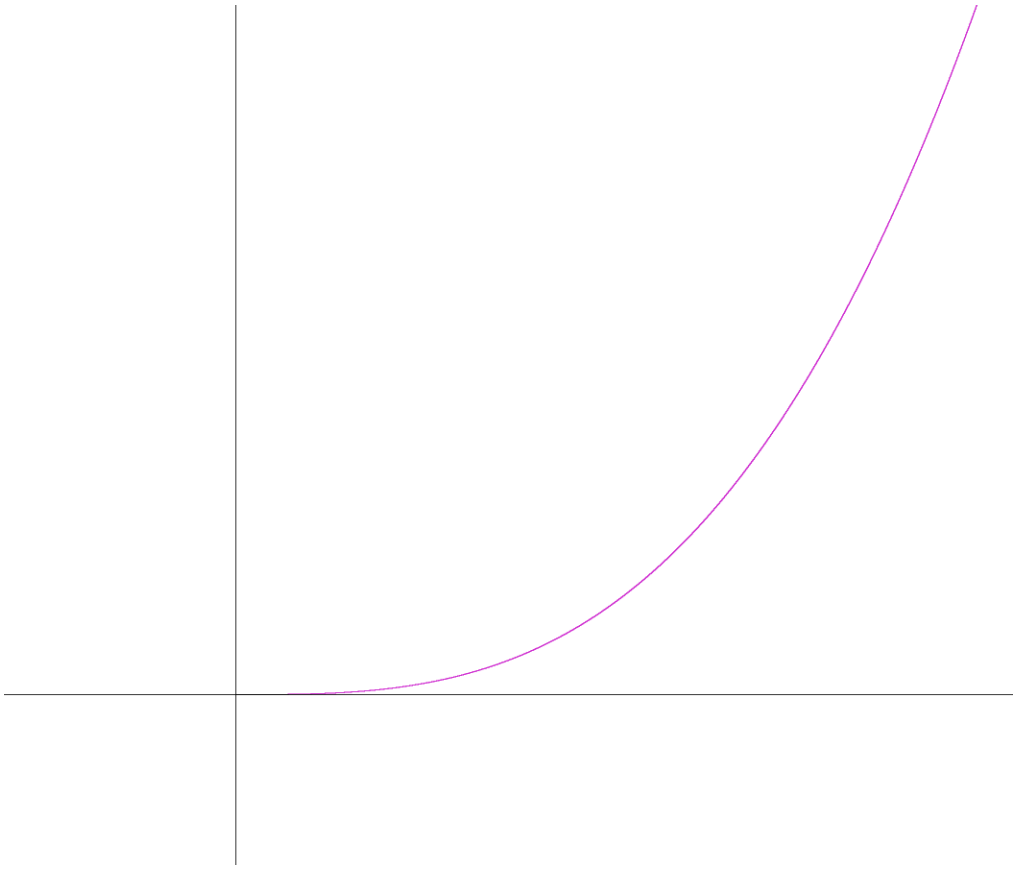
Appendix A: Pre-interview Tasks

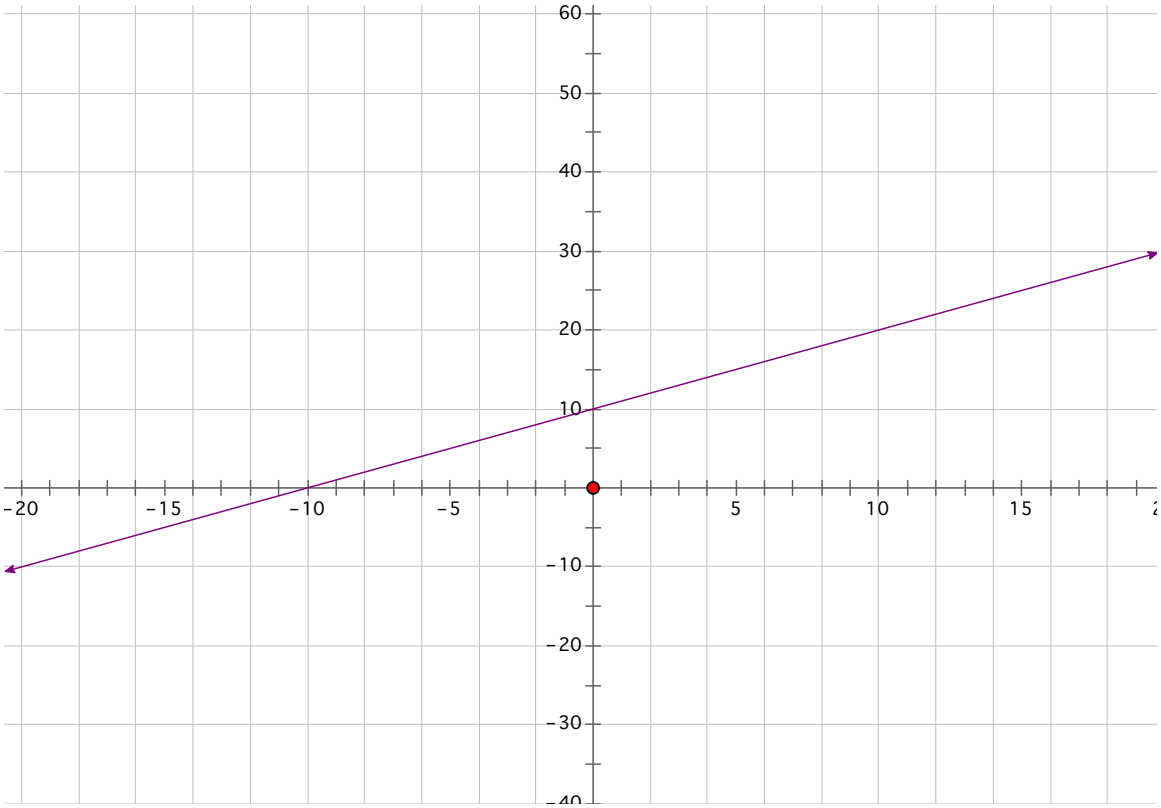
Problem 1:

Given the following graphs of functions, graph their inverse functions.




2.5
2
1.5
1
0.5
-0.5
-0.5
0.5
1
1.5
2
2.5




60
50
40
30
20
10
−20
−15
−10
−5
5
10
15
−10
−20
−30

Problem 2:

For each pair of functions, determine if they are each other's inverse.

- $f(x) = x + 1$, $g(x) = \frac{1}{x+1}$
- $f(x) = x + 1$, $g(x) = x - 1$
- $f(x) = x^2$, $g(x) = \sqrt{x}$
- $f(x) = x^2$, $g(x) = x^{-2}$

Problem 3:

Suppose that $f(x)$ is a one-to-one function whose inverse function is $f^{-1}(x) = (x+1)^3 - 5x^2 + 2$. Find a value of x so that $f(x) = 1$.

Problem 4:

For the function $f(x) = x^3 - x$, answer the following:

a. What is $f(2)$?
b. What is $f^{-1}(6)$?
c. How would you convince a student you have found the correct answers?

Problem 5:

The following function C relates measures in Celsius, $C(F)$, with measures in Fahrenheit, F.

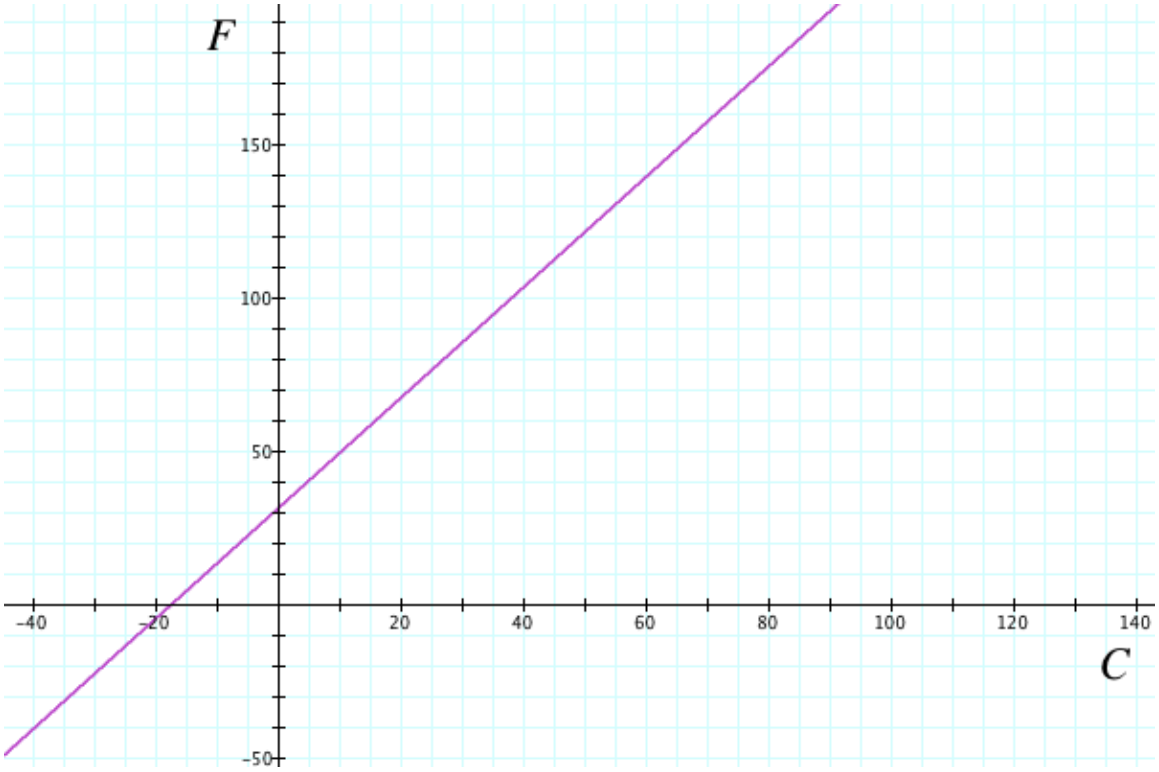
$$C(F) = \frac{5}{9}(F - 32)$$

Determine the inverse function.

Problem 6:

The following is a graph that relates temperatures in Celsius to temperatures in Fahrenheit.

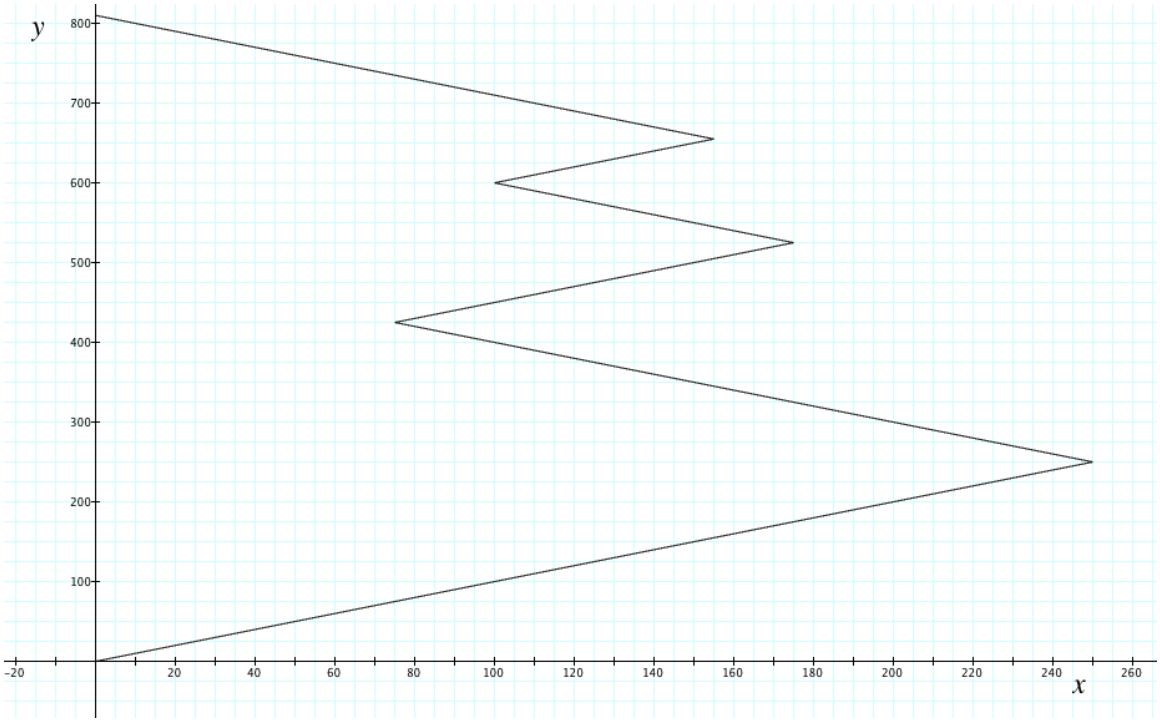
Determine a graph of the inverse relation.



Identify a point on the graph of the graphed inverse function and explain what the coordinate values for the point represent.

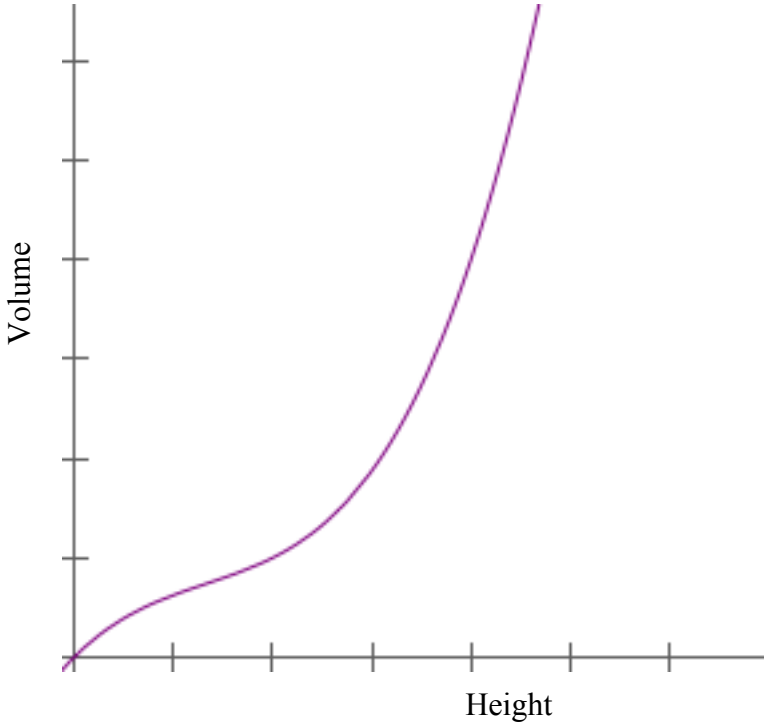
Problem 7:

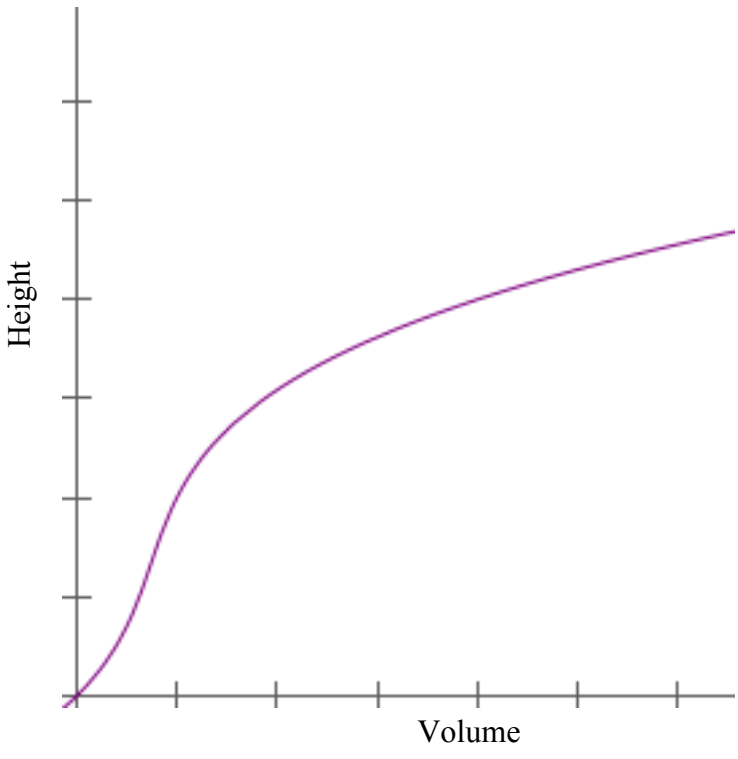
Is the following graph a function?



Problem 8:

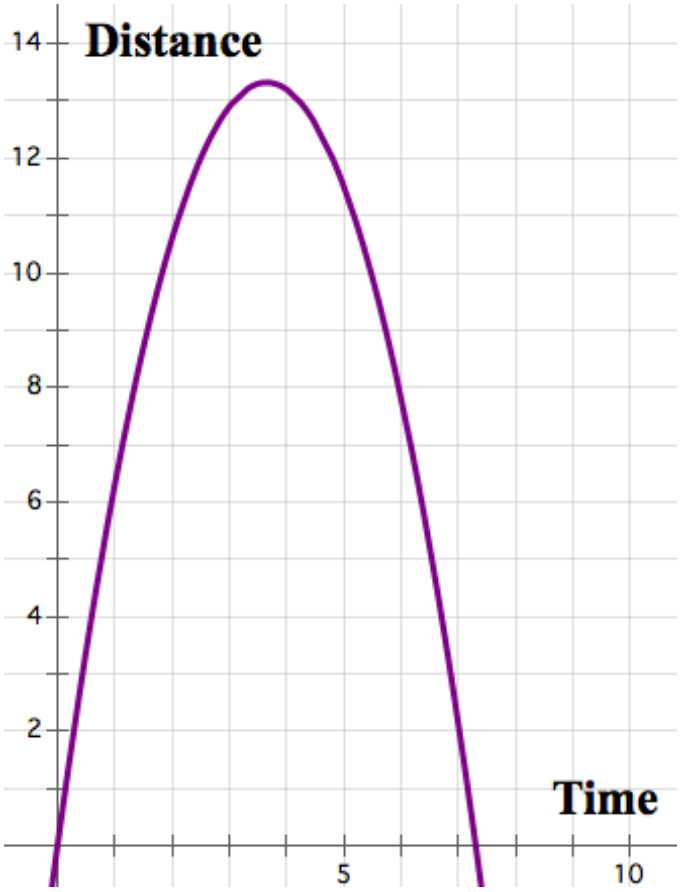
The following two graphs represent relationships between the height of water in a glass and the volume of water in the glass. Describe the two glasses and how they compare.

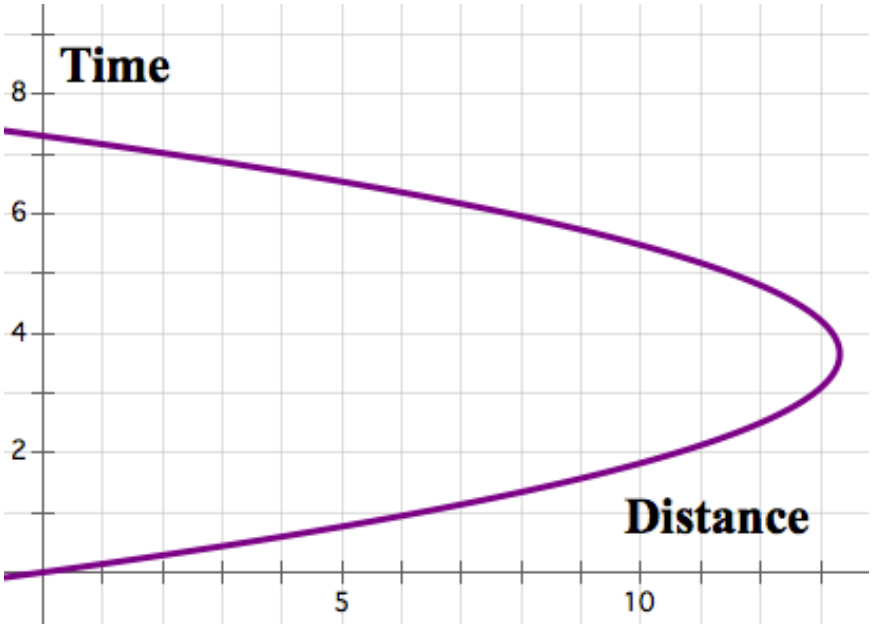




Problem 9:

There are two graphs presented below. Are these graphs the same or different? Explain.





Problem 10:

Graph the relationship $3x=y$.

Graph the relationship $3y=x$.

Graph the relationship $-3y + x = 0$.

Problem 11:

Give the inverses of the following:

$y = x + 1$

$x = y + 1$

$f(y) = y + 1$

$r = \theta + 1$

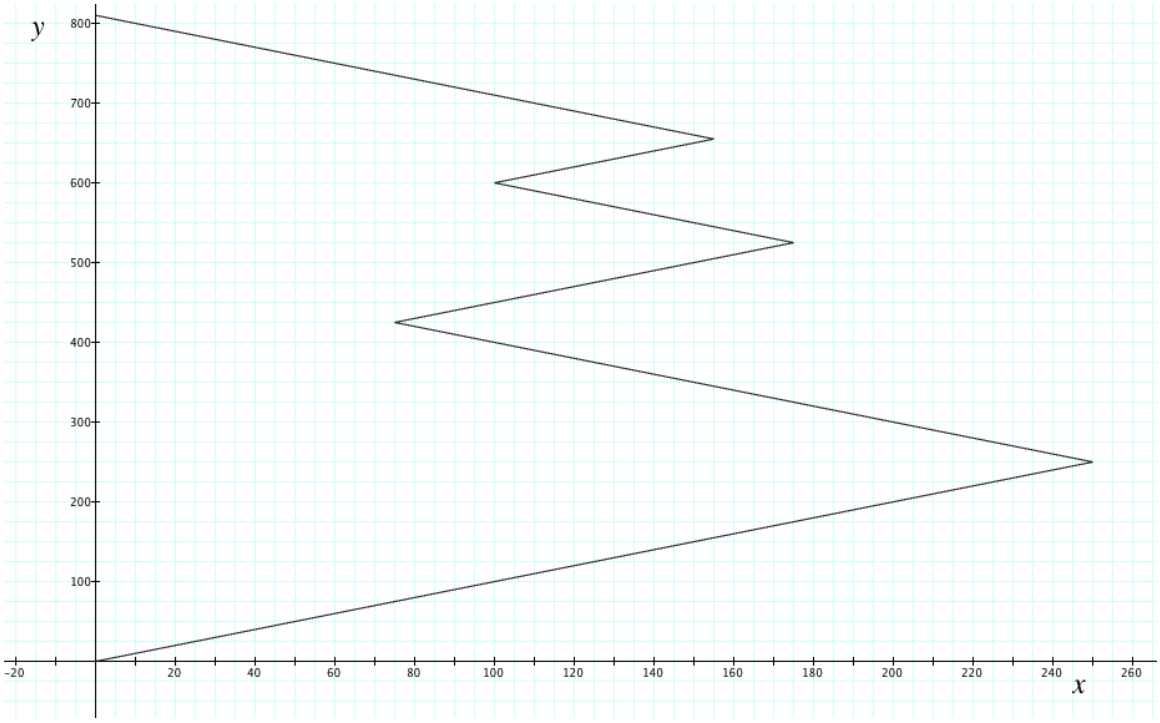
$r(\theta) = \theta + 1$

$a = b + 1$

$a + b = 1$

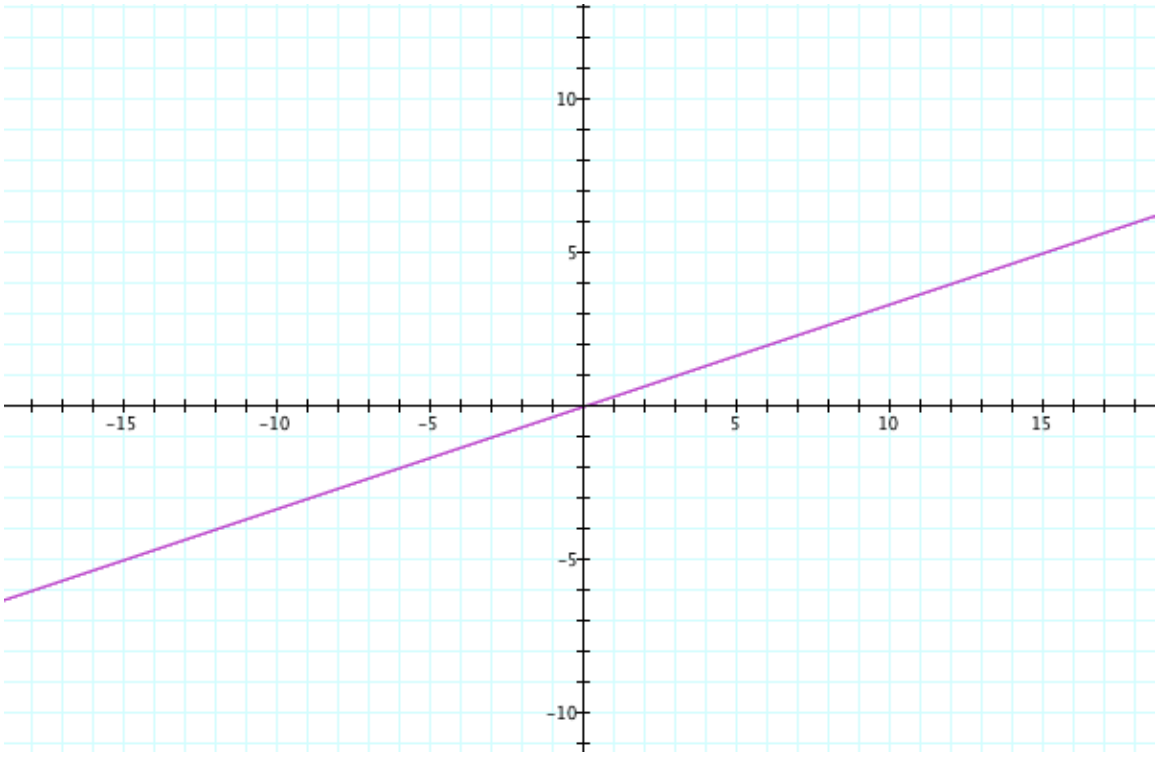
Problem 12:

Is the following graph a function?



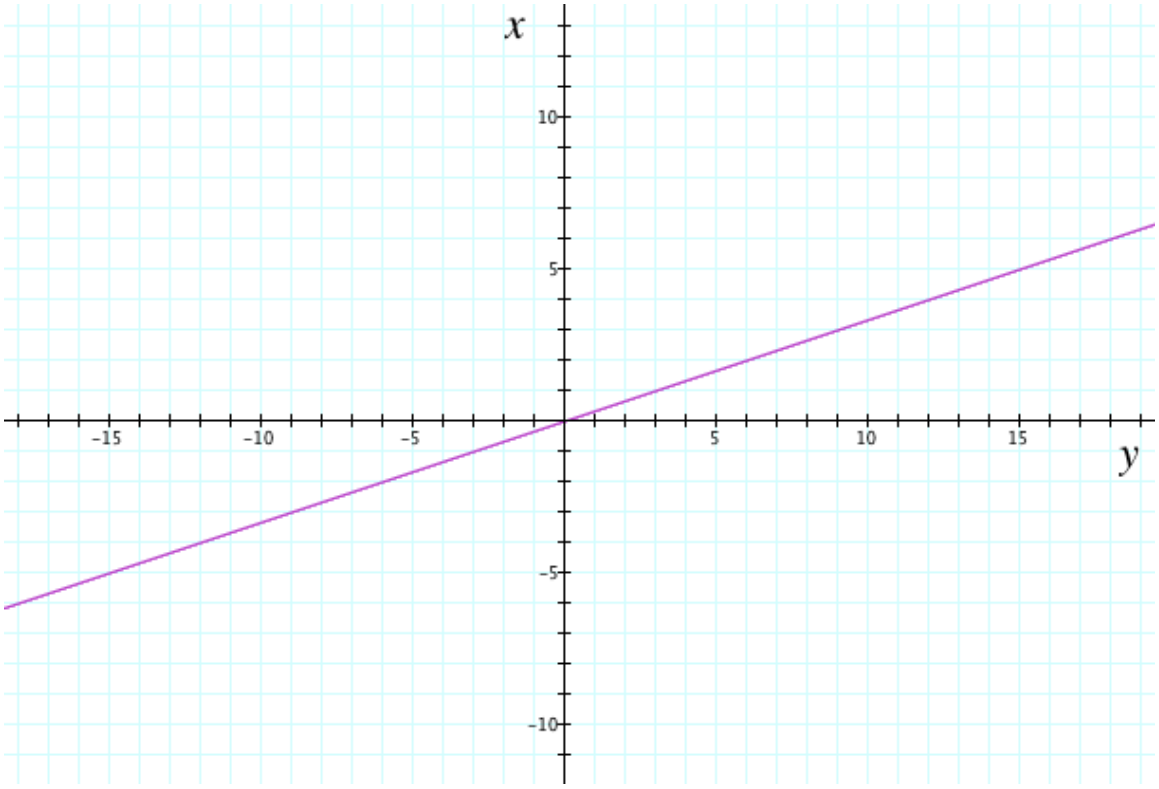
Problem 13(a):

You are working with a student who happens to be graphing $y=3x$. He provides the following graph. How might he be thinking about the situation?



Problem 13(b):

What if, while working with the student, he did the following:



How would you respond to the student?

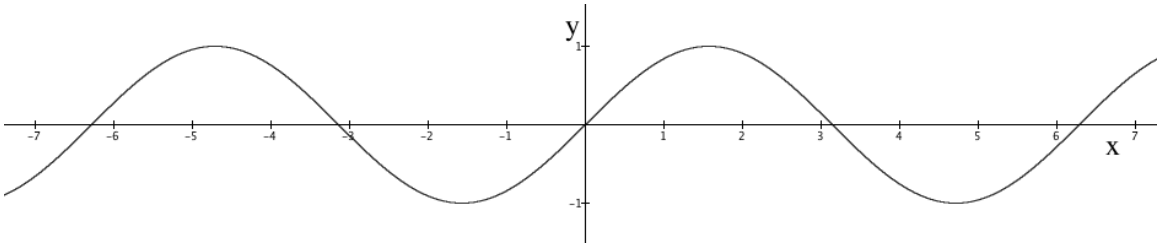
Problem 14:

To find the inverse of $y = 2x+1$, a student said, "I switched x and y to get $x = 2y +1$." How would you respond to this student?

To find the inverse of $y = 2x+1$, another student said, "I solved for x to get $x = \frac{1}{2}y - \frac{1}{2}$." How would you respond to this student?

Problem 15(a):

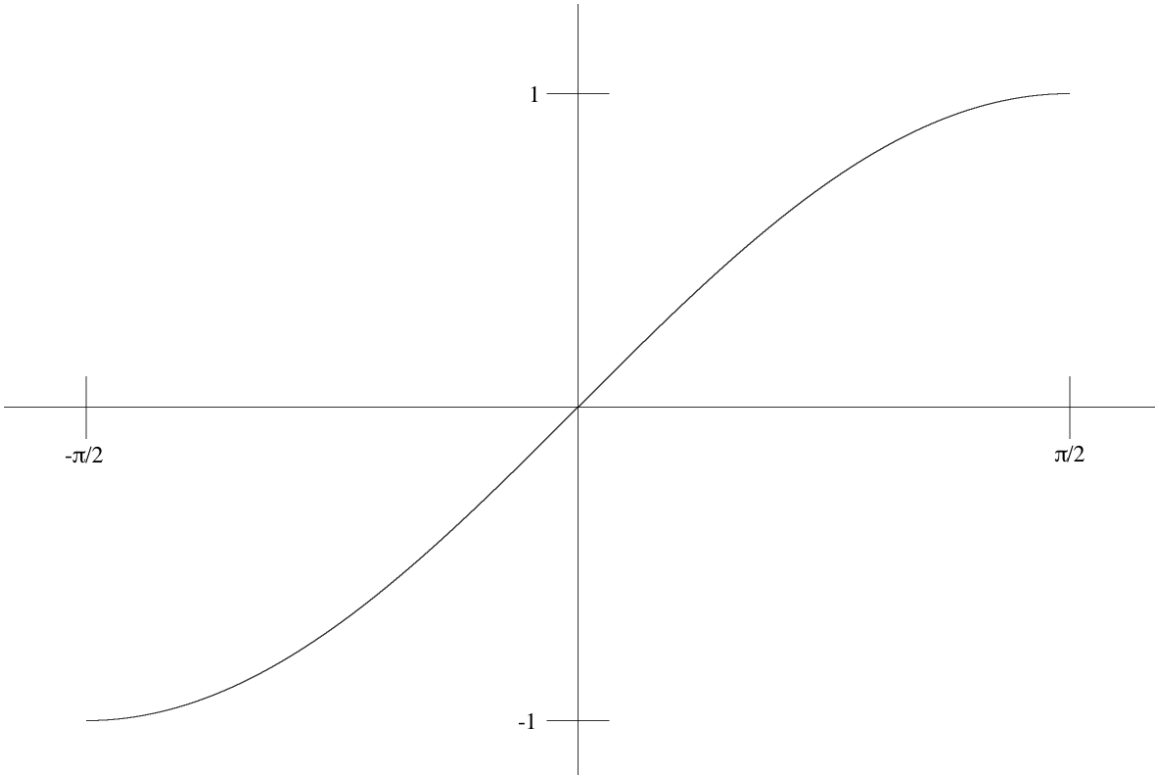
The following graph is a graph of the sine function. Determine a graph of the inverse sine function.



When working with a student, how would you explain graphing the inverse sine function?

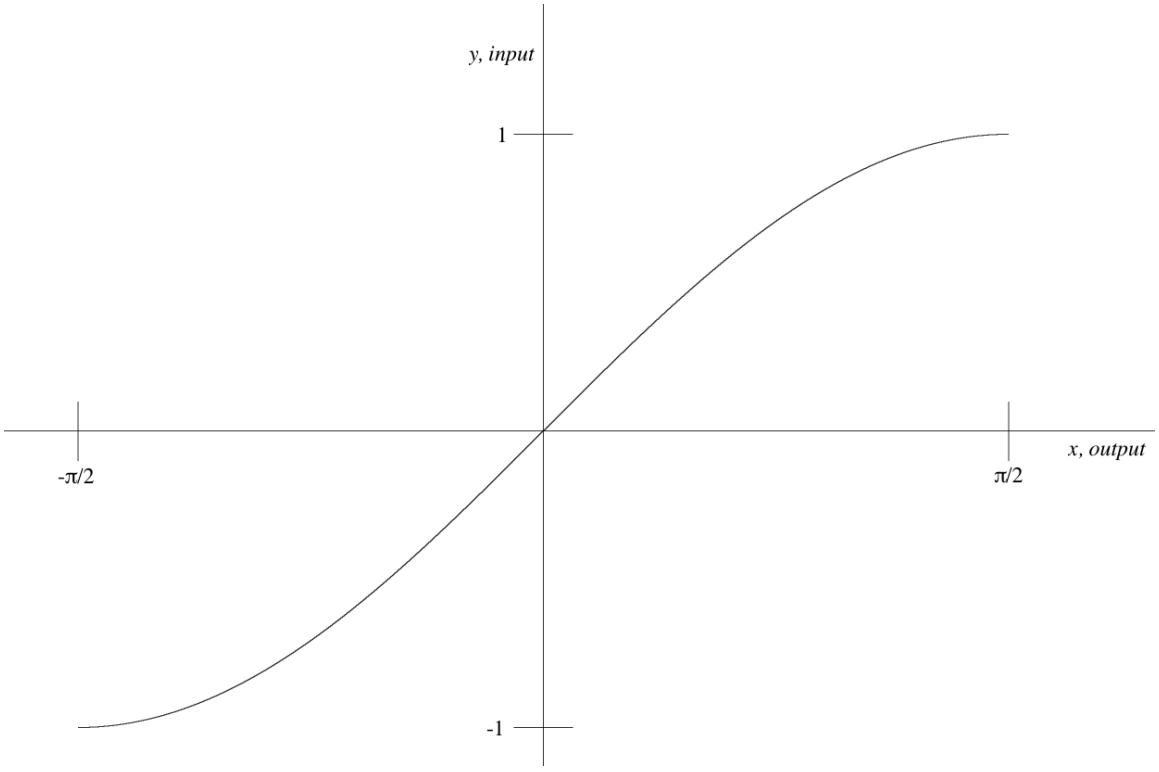
Problem 15(b):

A student came to you with this graph, claiming it was a graph of the inverse sine function. What would you say to that? Could that be true?



Problem 15(c):

When questioning the students from the previous problem, they claim, "Well, because we are graphing the inverse of the sine function, we just think about *x* as the output and *y* as the input." When giving this explanation, they add the following labels to their graph.



What do you think about the students' graph? Is it correct? Why or why not? How would you respond to the students?

Problem 16:

Use the table of values to answer the following:

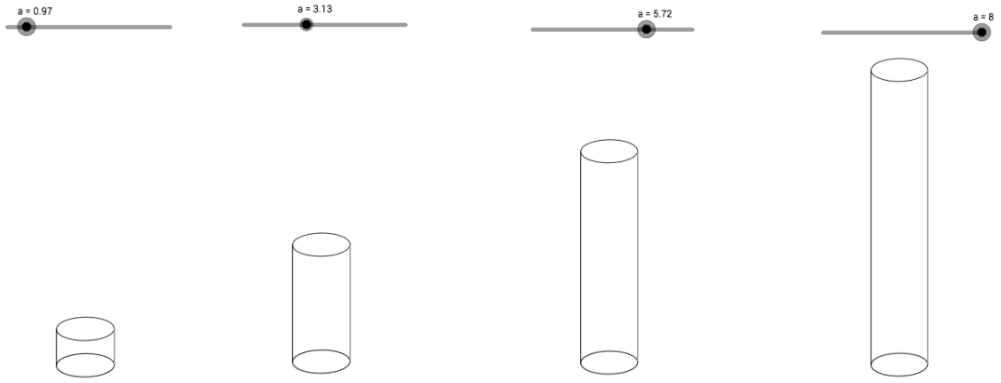
x	$f(x)$
0	*1*
1	*4*
2	*3*
3	*5*
4	*2*
5	*0*

$f^{-1}(3) =$

Appendix B: Post Interview Protocol

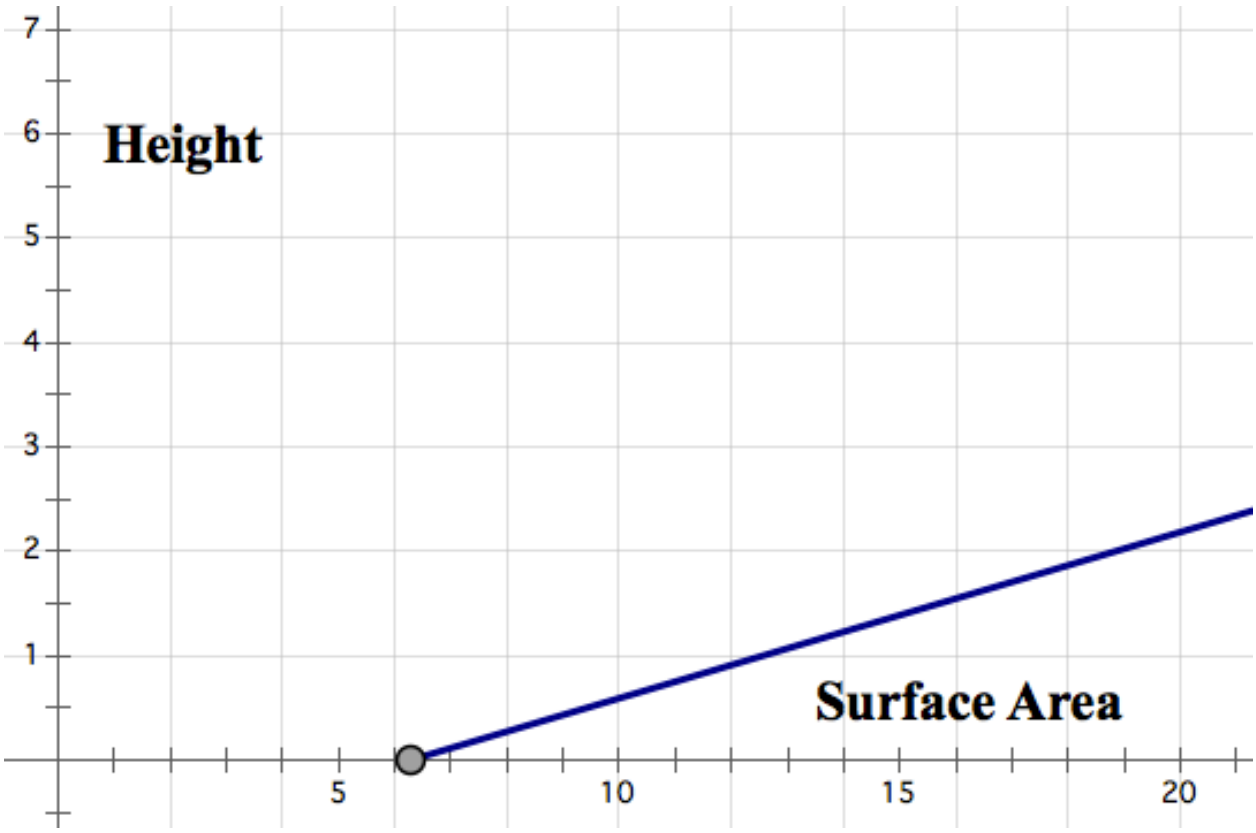
Problem 1a:

Represent the relationship between surface area and height of the cylinder.



Problem 1b:

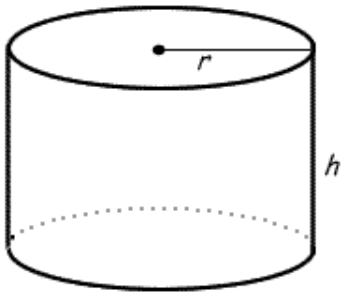
The following is a graph that relates the surface area of the cylinder and height of the cylinder described above. Use covariation talk to describe the relationship.



Discuss and graph the inverse of the relationship you just described.

Problem 1c:

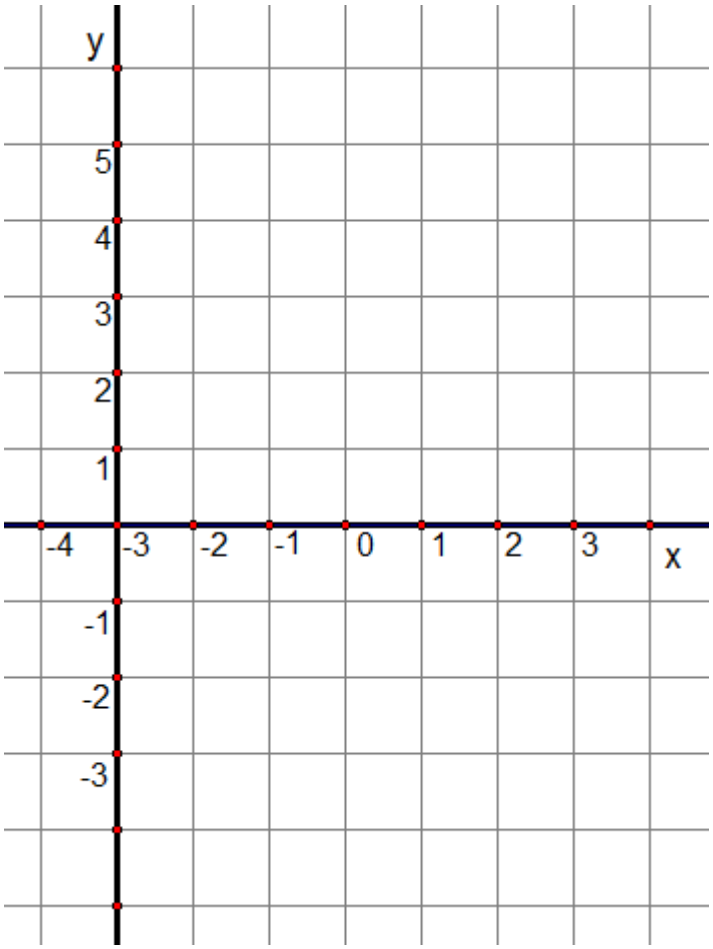
The formula for the surface area of a cylinder is defined as $SA = 2\pi rh + 2\pi r^2$ where r is the radius of the base and h is the height.



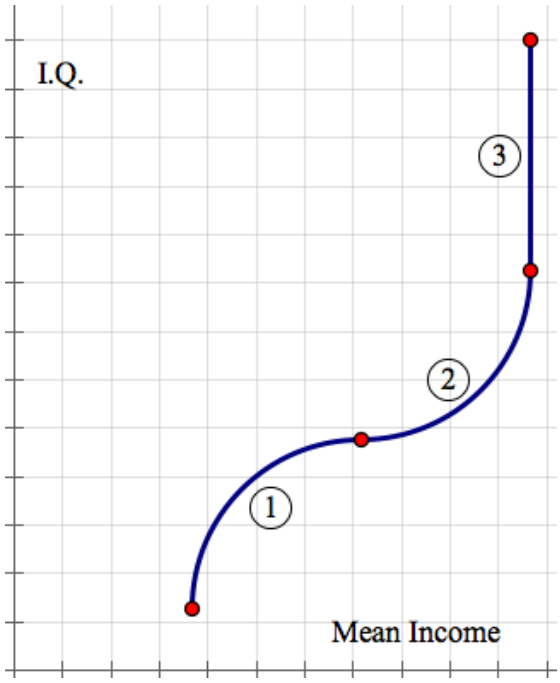
a. Write the inverse function of your function from part (a).

Problem 2:

Graph the relationship defined by $y = 3x + 1$ on the axes below.



Problem 3:



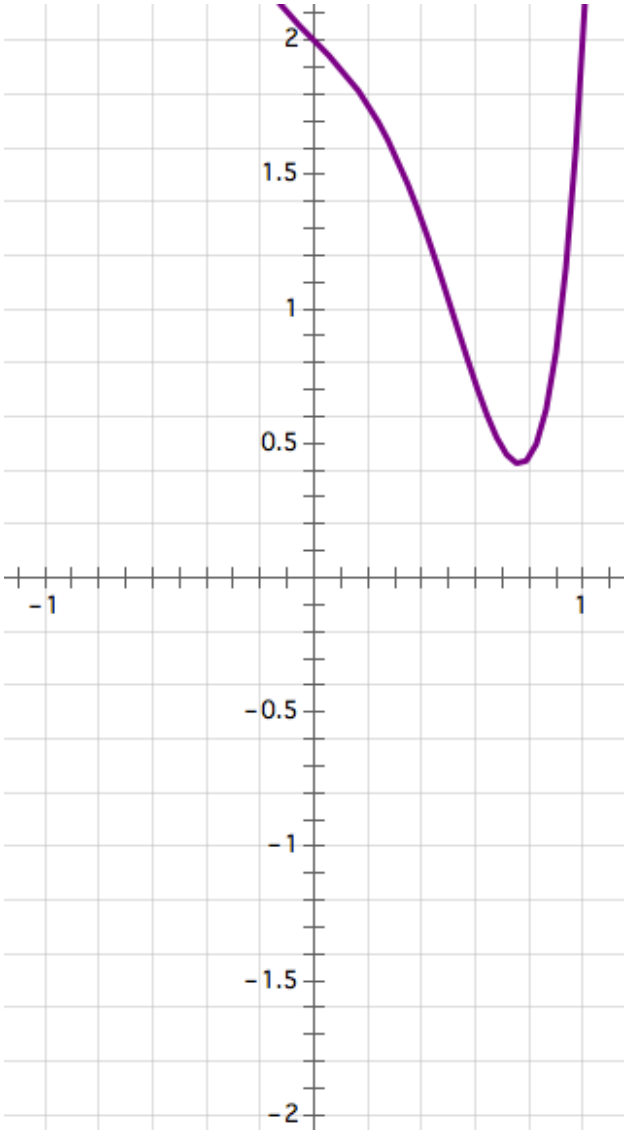
The graph above shows the relationship between the mean income of adults in the United States and their I.Q, broken up into 3 sections.

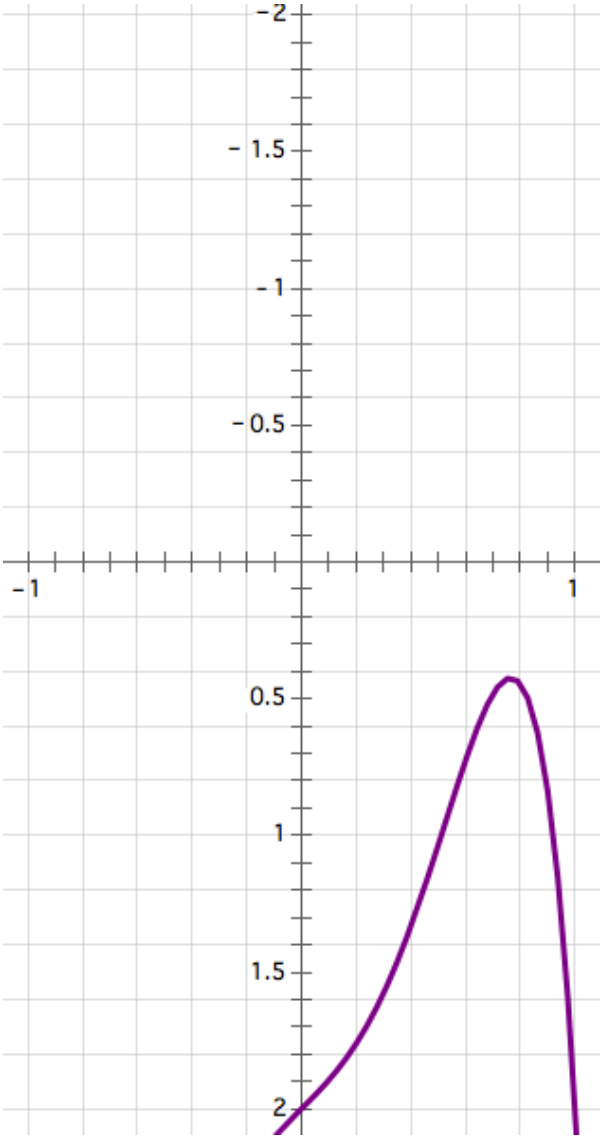
Assess the validity of the following claims by students and justify your assessment:

a.) In section 1, I.Q. increases at an increasing rate with respect to mean income
b.) In section 2, mean income increases at a decreasing rate with respect to I.Q.
c.) At some point the rate of change of I.Q. with respect to income is constant.
d.) Mean income is a function of I.Q.

Problem 4:

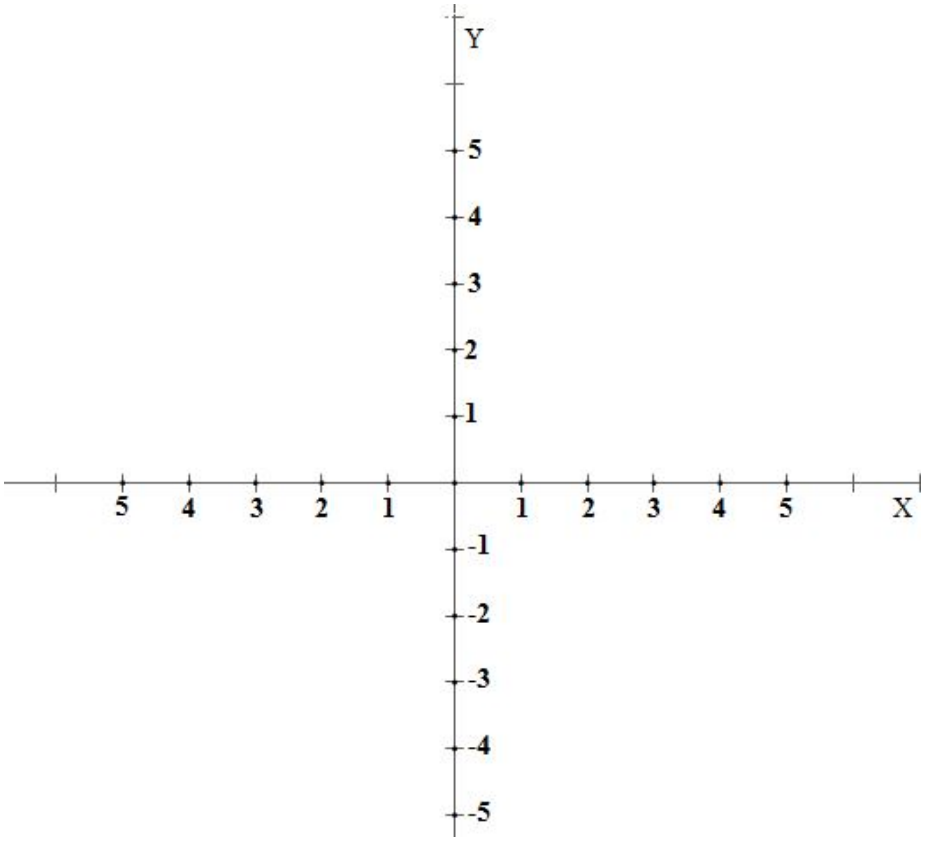
There are two graphs presented below. Are these graphs the same or different? Explain.





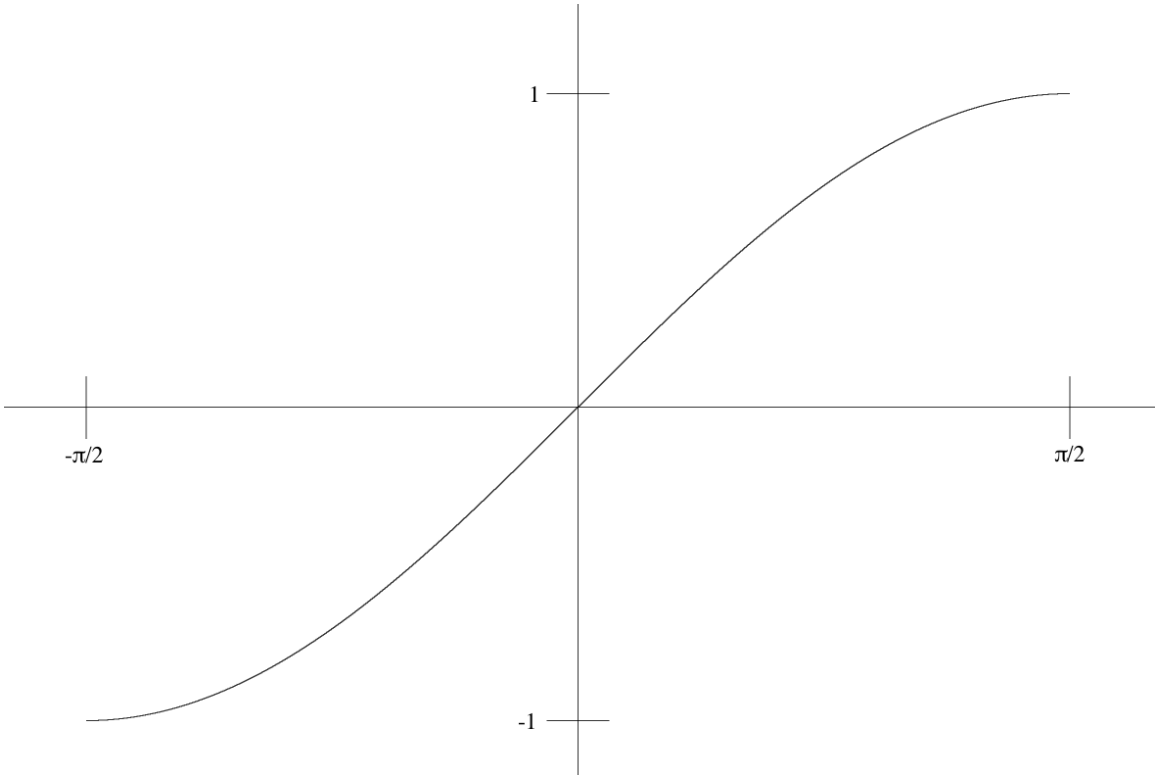
Problem 5:

Graph $y = x$ on the coordinate axes below.



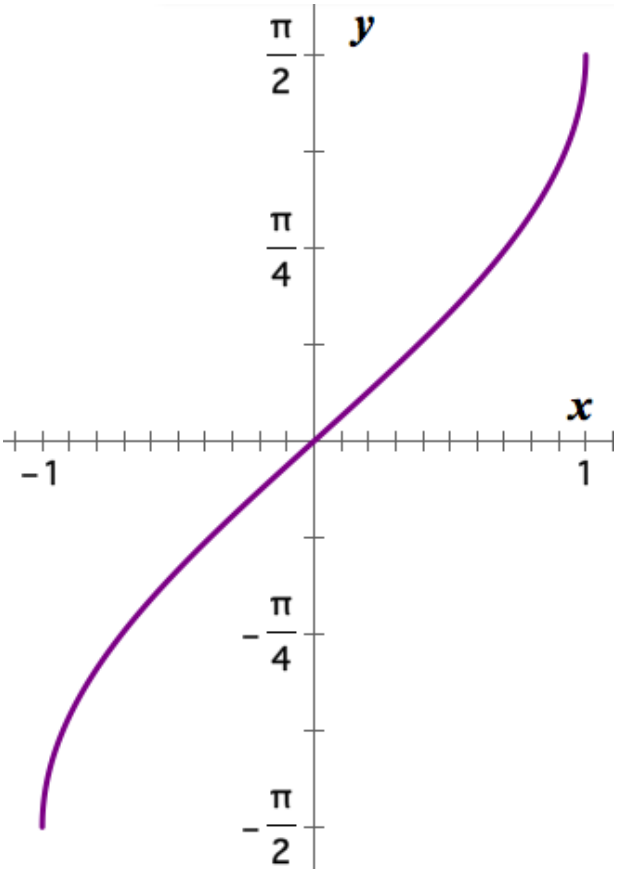
Problem 6a:

A student came to you with this graph, claiming it was a graph of the inverse sine function. What would you say to that? Could that be true?



Problem 6b:

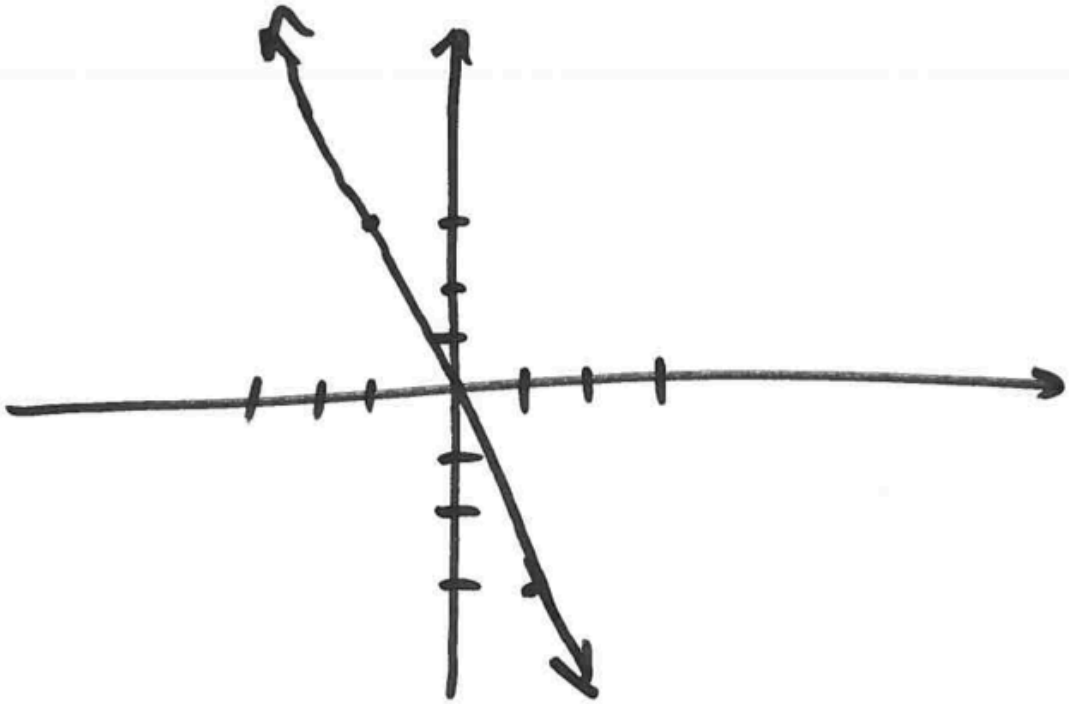
Another student claims that the previous student has to be wrong because the graph below is the graph of the inverse sine.



How would you respond to this student?

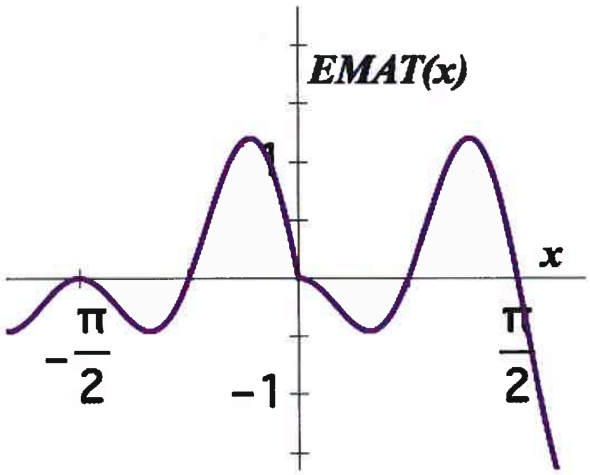
Problem 7:

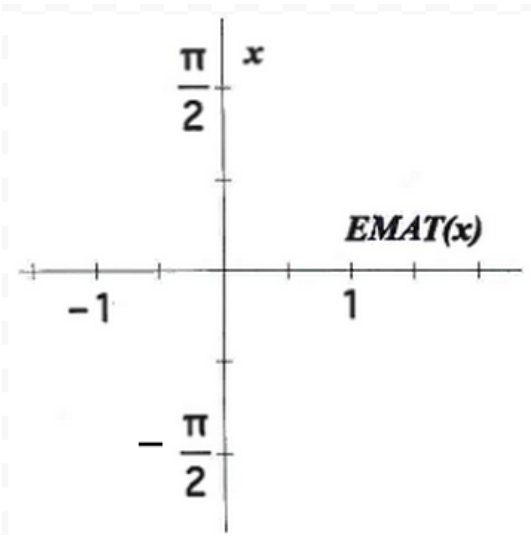
You are working with a student who happens to be graphing $y = 3x$. He provides the following graph. How might he be thinking about the situation?

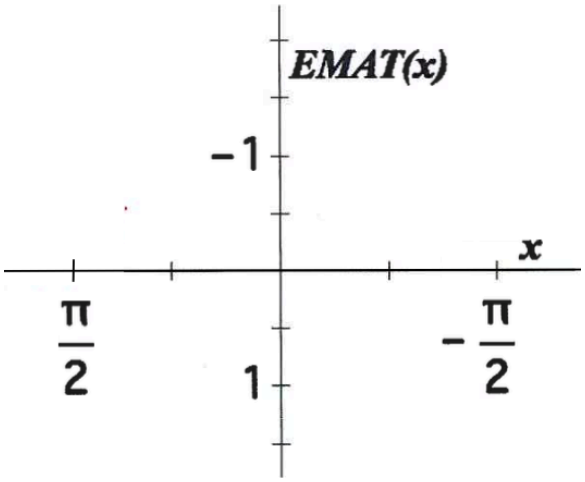


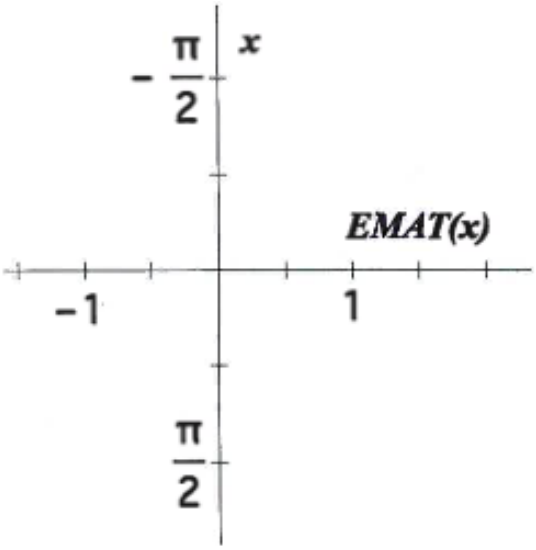
Problem 8:

Below is a graph of the EMAT function. Graph this function on the provided axes.





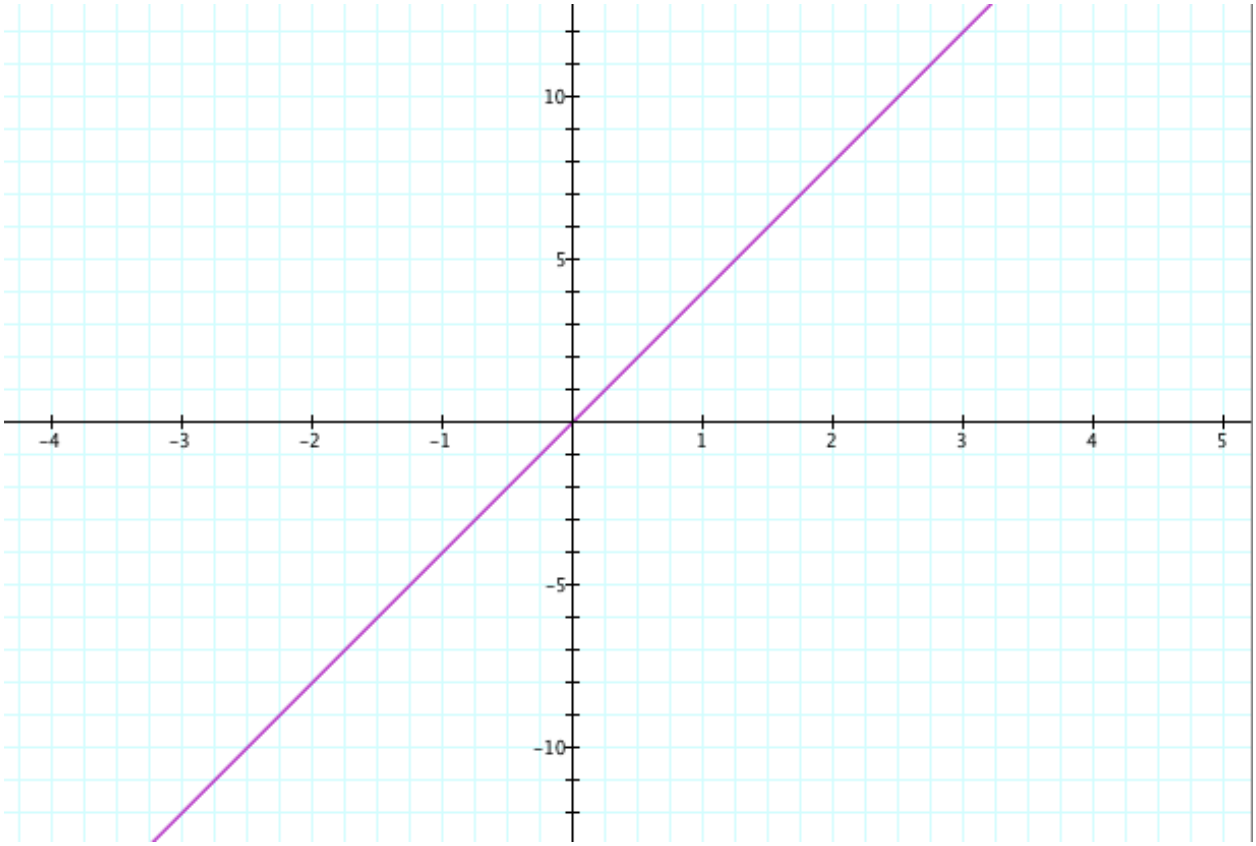




After producing each graph, let the vertical axis represent *k* and the horizontal axis be denoted by *y*. Write at least one analytic relationship between *k* and *y* for each graph.

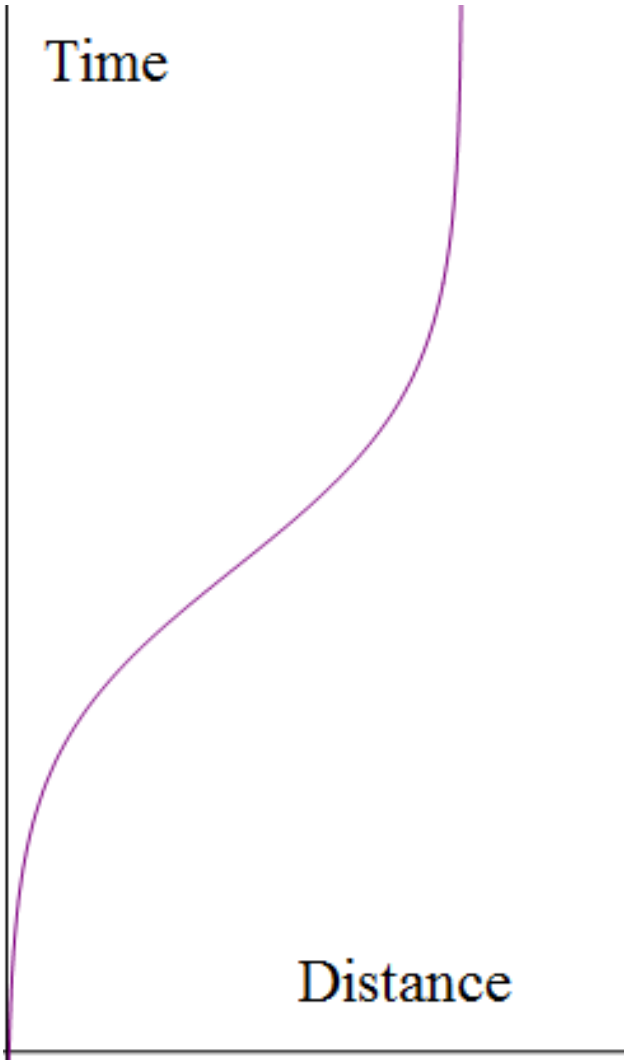
Problem 9:

What is the rate of change of the graph below?



Problem 10:

The below graph represents the distance and time relationship for some object. Identify when the object is speeding up, slowing down, and moving at a constant speed.



Final Prompts:

A student is working on a graphing problem and asks, "Why does the y-value need to be on the vertical axis and the x-value on the horizontal axis?" What would you tell this student?

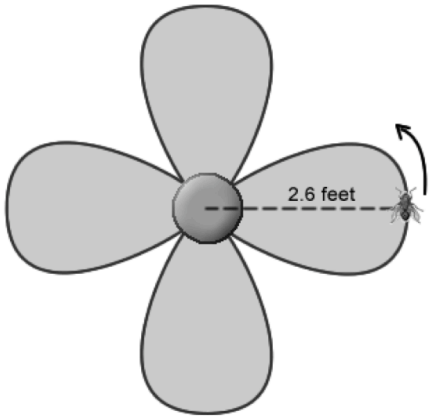
What do you think about these interview tasks? What do you think our intention is for these tasks? [Ask the students to give specific examples]

Appendix C: Teaching Experiment Tasks

Bug on a Fan Problem:[51]

Constructing the sine function:

Imagine that a bug sits on the end of a blade of a fan as the blade revolves counter clockwise.



1. What quantities might you track to represent the location of the bug as it rotates around the center of the fan?
2. How is the bug's trip around the fan related to angle measure?
3. Create a graph to illustrate how the bug's vertical distance above the horizontal diameter co-varies with the measure of the angle swept out by the bug's fan blade (measured in radians).
4. As the angle measure increases from 0 radians to $\pi/2$ radians, we concluded that:
    a. The vertical distance of the fly above the horizontal diameter increases,
    b. The change of vertical distance decreases for successive equal changes of angle measure,
    c. And the rate of change of the vertical distance with respect to angle measure decreases.
5. Discuss:
    a. Directional change.
    b. Amounts of change (illustrate on diagram and graph).
    c. Rates of change.

[51] See Moore (2014) for more on this prompt.

Graphing Sine/Arcsine Task:

Graph 1:	Create a graph of the sine function with a domain of all real numbers. What is the range?
Graph 2:	Using **covariation talk**, create and justify a graph of the arcsine (or inverse sine) function. What is the domain and range of this function?
Prompt 3:	Can you alter (**do not draw a new graph**) Graph 1 such that it represents the graph of the arcsine function? Does this graph convey the same relationship as the second graph? How so or how not?

The Power Tower:



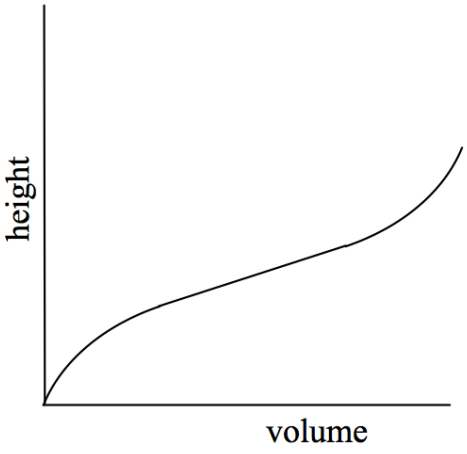
Power Tower is a popular ride at Cedar Point, an amusement park in Sandusky, OH. Watch the task's video to see what happens on the ride. Graph the distance from the ground of an individual (vertical axis) versus the individual's total distance traveled (horizontal axis). Assume that the person's feet were touching the ground at the beginning of the video. Give a justification for why your graph looks the way it does.

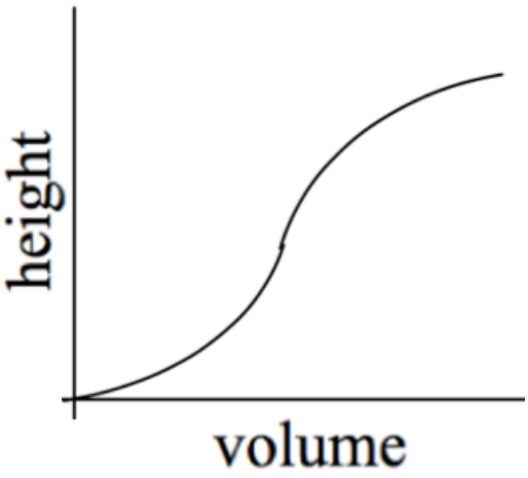
The Bottle Problem:

- Draw a height and volume graph that corresponds to the given bottle.
- Draw a cross-section of a bottle that would produce the graph when filled with water.



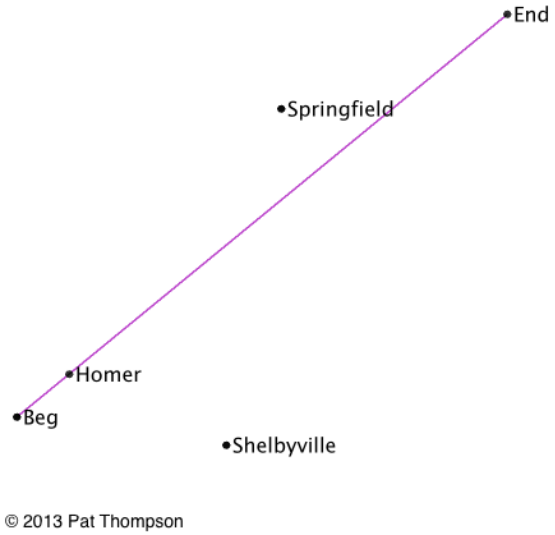






The Car Problem:

Graph the relationship between Homer's distance from Springfield and distance from Shelbyville as Homer travels along the road.



1. *Beginning*: What quantities could you explore/consider in this graph?
2. *When graph is created*: Choose a point and tell me what it represents on the graph and relative to the event.
3. *When shown graph*: (If they do not bring up the fact that it wraps back on itself or if they do not seem to have a problem with it) What about a student who says that the graph cannot wrap back around?
4. *When shown graph*: Is it a function?

Appendix D: Pilot Study Post Interview Tasks

Deep Sea Diver:

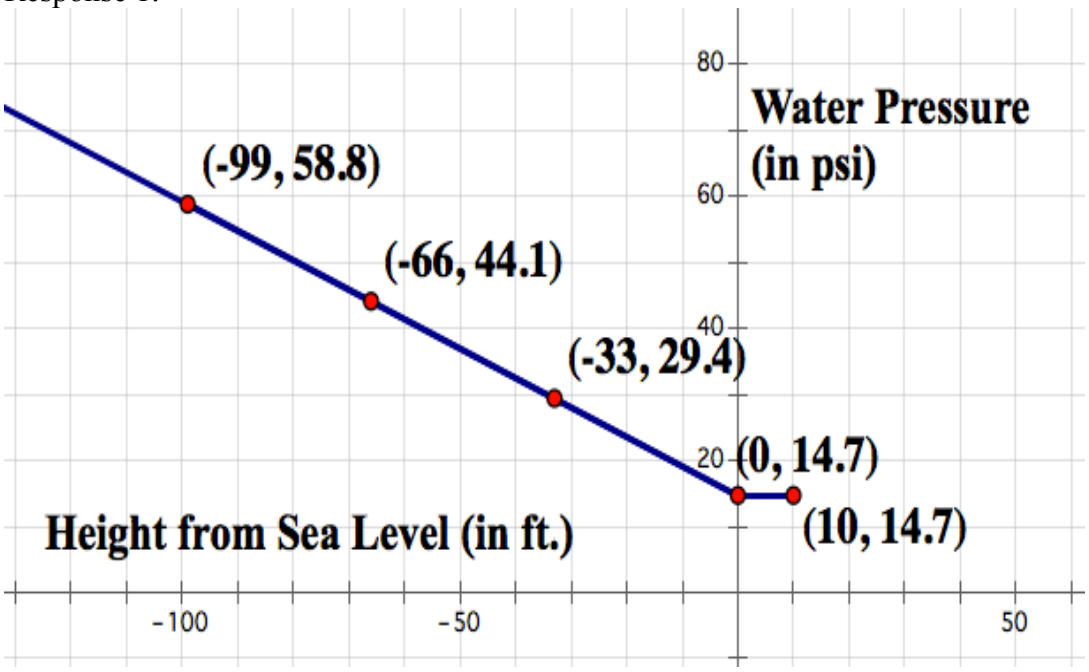
Students are given the following prompt:

> From (http://www.onr.navy.mil/focus/ocean/water/pressure1.htm):
>
> Even though we do not feel it, 14.7 pounds per square inch (psi) of pressure are pushing down on our bodies as we rest at sea level. Our body compensates for this weight by pushing out with the same force. Since water is much heavier than air, this pressure increases as we venture into the water. For every 33 feet down we travel, one more atmosphere (14.7 psi) pushes down on us. For example, at 66 feet, the pressure equals 44.1 psi, and at 99 feet, the pressure equals 58.8 psi.
>
> A deep-sea diver starts on a boat 10 feet above sea level and enters the water, descending to an unknown depth. Assuming there is no measureable difference in pressure between sea level and 10 feet above sea level, create a graph showing the relationships between the divers depth and water pressure.

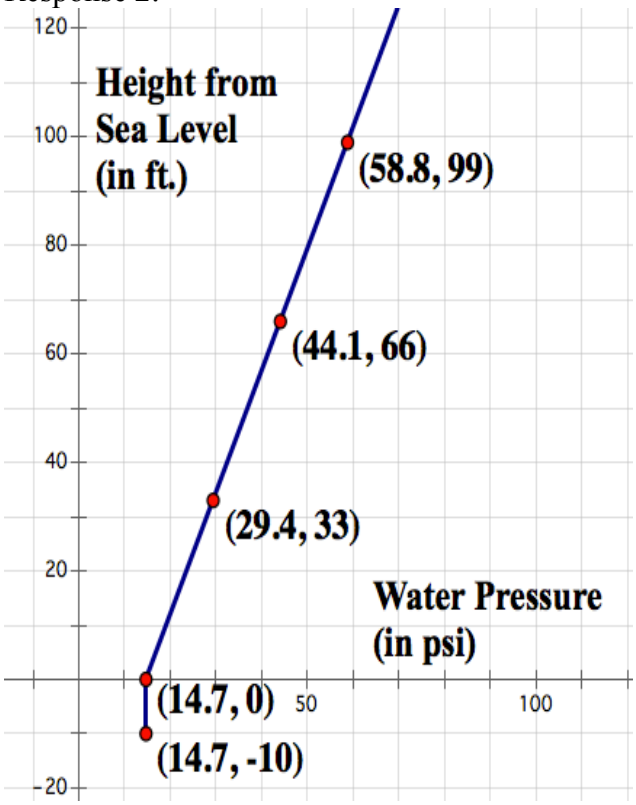
Produce a graph representing this relationship.

Evaluate the following student produced graphs. Explain how the student may be imagining the situation or how the student may have been thinking when creating their graph.

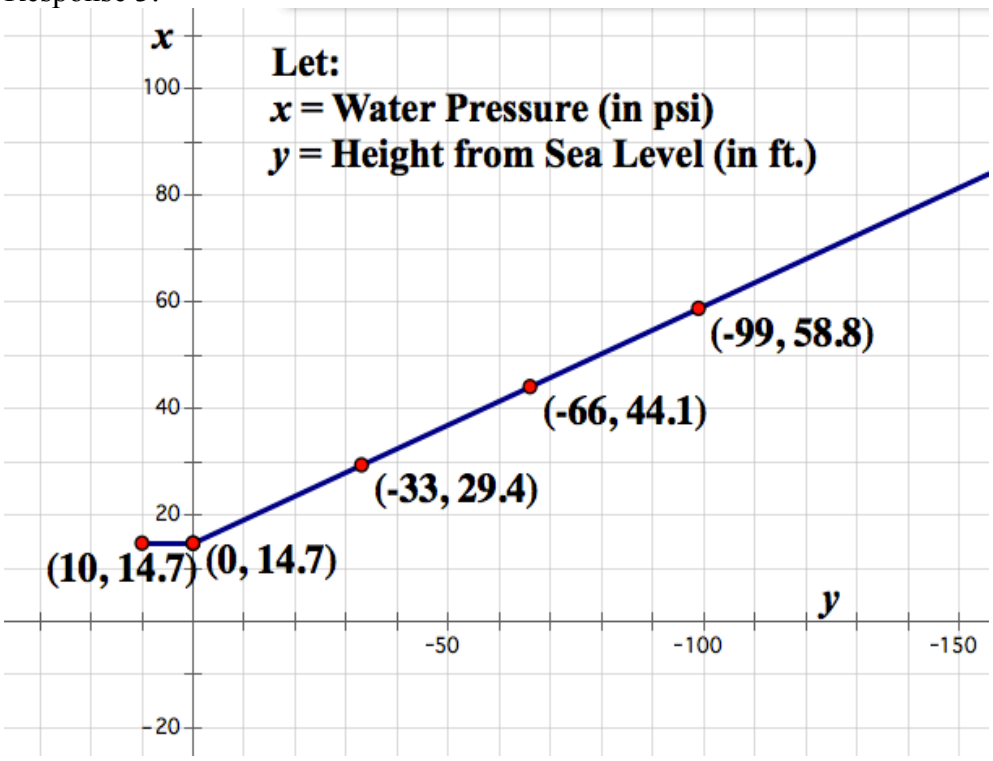
Response 1:



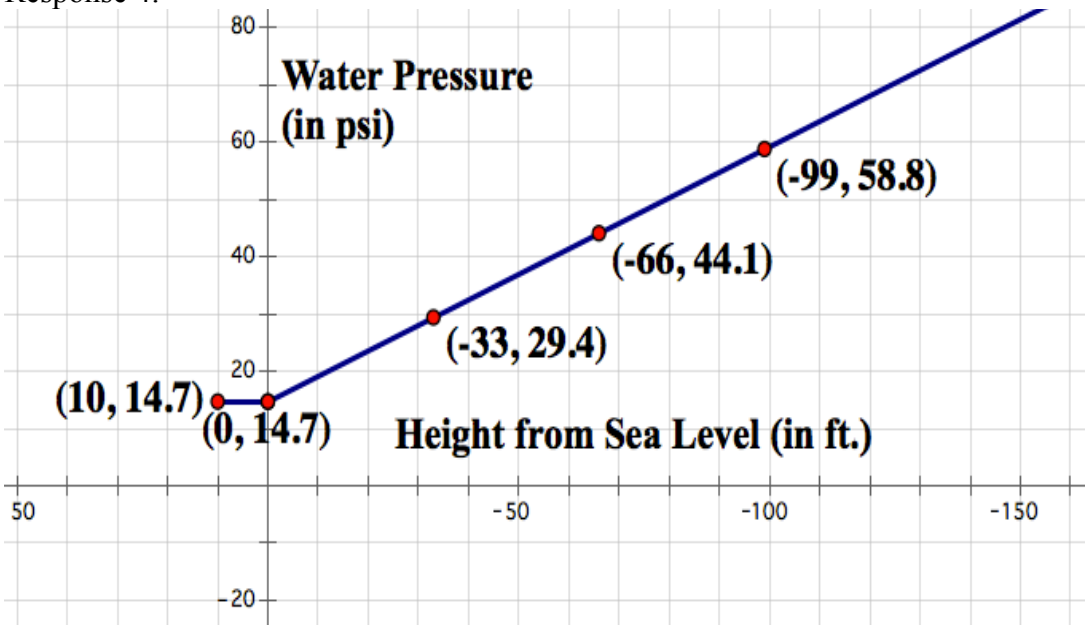
Response 2:



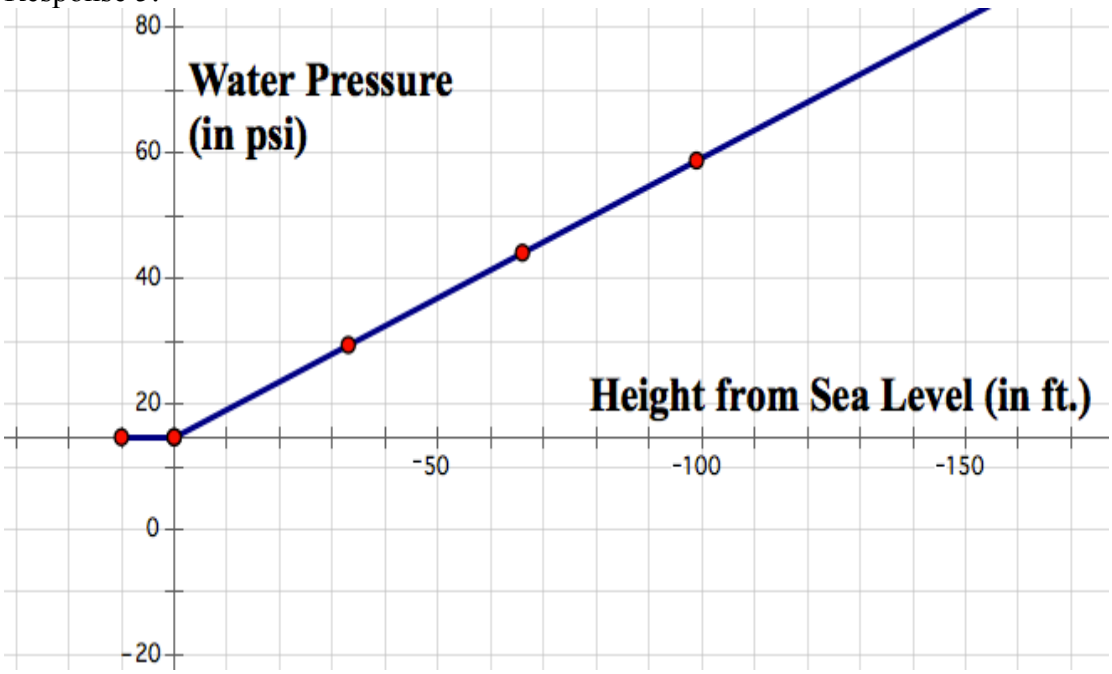
Response 3:



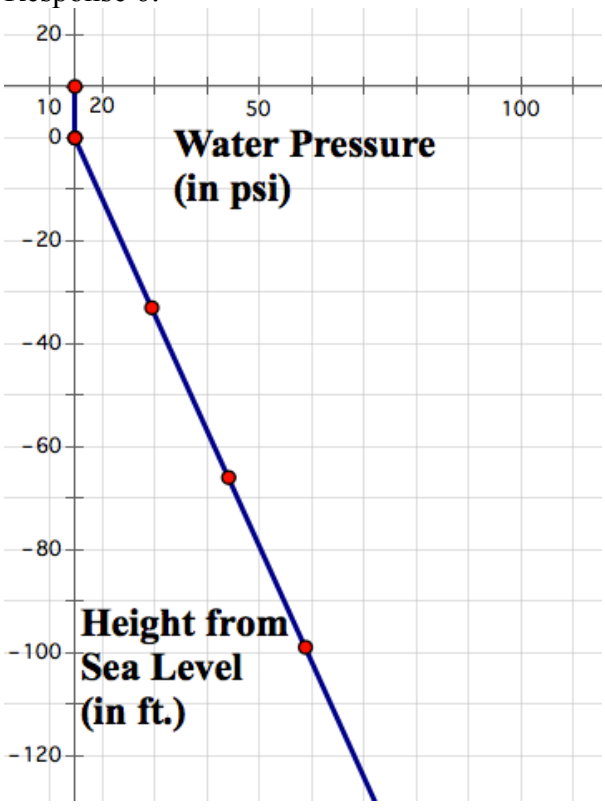
Response 4:



Response 5:



Response 6:



Skydiver Problem:

A skydiver jumps from a plane and accelerates towards the ground until he reaches a terminal velocity. He continues at this velocity for a short period of time before opening his parachute. His velocity quickly drops and he returns to the ground at a constant speed. Create a graph representing the skydivers distance from the ground and velocity.

Bungee Jumper Problem:[52]

Consider the video of the bungee jumper. Represent the relationship between a jumper's vertical velocity and his distance from the top of the tower.

http://www.youtube.com/watch?feature=player_detailpage&v=zG22qQydPVQ#t=75

[52] Pat Thompson was the first to suggest the use of this task. I thank him for this.

Car Racing Problem:

Consider the video of the cars racing. Choose one car and graph the relationship between that car's distance from the starting line and its distance from the finish line.

http://www.youtube.com/watch?feature=player_detailpage&v=Wf6G3_gIvoY#t=17