COLLEGE STUDENTS' DECISION MAKING WHEN SELECTING PROOF METHODS IN PROOF CONSTRUCTION

by

HYEJIN PARK

(Under the Direction of Jeremy Kilpatrick and AnnaMarie Conner)

ABSTRACT

This paper reports on three exploratory studies of college students' decision-making behaviors when choosing proof methods in the context of proof construction. The decision strategies that students used when making proof-method decisions and the constructs influencing their decisions were explored. Nine students (three for the first study, four for the second study, and two for the third study) participated in three studies, and the students in each study were taking a transition-to-proof class with a different instructor and during a different semester while each study was being conducted. For each study, I conducted interviews with the students, observed the students' transition classes, and examined their copies of class notes and homework. Based on the observations of the students' decision-making activities with proof tasks, their verbal reports while making proof-method decisions, and their responses after choosing methods, I found five decision strategies and eight constructs that contributed to their proof-method decisions across the three studies. For decision strategies, the students' decision acts differed depending on their familiarity with the problem statements. When a statement was familiar, the students immediately chose a method using the feature-matching strategy. When a statement was less familiar or unfamiliar, the students chose a method using one to three of the other four strategies. In particular, when they saw that more than two methods could be used to prove or disprove a statement, they used the comparison strategy to choose a method. For constructs, the students' knowledge about when to use proof methods and their orientations (e.g., beliefs about proof or proof-method preferences) were the primary influences on their decision making. However, there were other types of constructs affecting their decisions. More studies with a large number of students are needed to confirm that the strategies and constructs found in the three studies are indeed the most prevalent. It is also necessary to examine how mathematicians make proof-method decisions to see what differences may exist between mathematicians (experts) and students (novice provers) with respect to decision making about proof methods and also to help students make strategic proof-method decisions in proof construction.

INDEX WORDS: Transition-to-proof course, college students, decision making, proof methods, proof construction

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DEDICATION

I dedicate this dissertation to all of the student participants who participated in my three studies and who provided me with the opportunity to learn about their proof-method decision making in constructing proofs. Without their help, I would not have been able to complete my dissertation. Thanks to all the participants.

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CHAPTER 1

INTRODUCTION

Proof is a central activity in mathematics, and its multiple roles have received significant attention in mathematics education for students' learning of mathematics across most school grades. Dreyfus (1990) wrote, "Proving is one of the central characteristics of mathematical behavior" (p. 126). Proof is essential to almost all undergraduate-level mathematics courses. The Mathematical Association of America (MAA)'s Committee on the Undergraduate Program in Mathematics recommended that

> Students in all mathematics courses, whether or not for majors, should encounter elements of mathematical argument, precision, and justification. All mathematical science majors should learn to read, understand, analyze, and produce proofs, at increasing depth as they progress through a major. (MAA, 2015, p. 11)

Yet, it has long been established that proving is difficult for college students. A great deal of research on proof at the tertiary level has documented why students have trouble producing correct and normatively accepted proofs. Causes of students' difficulties include an inability to use definitions to structure proofs (Moore, 1994); a lack of decision-making strategies as to which facts or theorems to use when attempting to prove statements (Weber, 2001); an inability to unpack informal statements (Selden & Selden, 1995); inadequate knowledge about proof methods (Stylianides, Stylianides, & Philippou, 2007); and an acceptance of empirical arguments as proofs (Stylianides & Stylianides, 2009).

Many colleges and universities offer a “transition-to-proof course” (cf. Moore, 1994) as a bridge course to help students shift from computation-based to proof-based mathematics and to equip students for upper-level mathematics courses requiring mathematical reasoning and proving skills. The goals of the transition course are to introduce students to formal notation, rudiments of logic, methods of proof (contradiction, mathematical induction, etc.), and the mechanics of proof. Basic mathematical concepts such as sets, relations, and functions and their properties and other elementary work in number theory or combinatorics are commonly covered in this course. Students completing the course are expected to be able to read and write mathematical proofs. However, many students in upper-level mathematics courses reveal that they still have serious difficulty constructing proofs in spite of the fact that many of them have taken the transition course before taking the upper-level courses (Selden & Selden, 2013). Researchers began to be concerned about students’ learning to prove and to be interested in the transition course because many students are exposed to the concept of a mathematical proof for the very first time in this course. Studies on the transition course have largely focused on students’ views of proof, their ways of proof construction and validation, or their problems in constructing proofs (Baker & Campbell, 2004; Moore, 1994; Selden & Selden, 2003; Weber, 2010); or instructors’ pedagogical strategies when teaching proof in this context or alternative teaching practices to help student improve on proof-related activities (Alcock, 2010; Talbert, 2015). Marty’s (1991) study showed, in particular, the instructional effectiveness on students’ performance in later upper-level courses when focusing on proof techniques in teaching rather than on the mathematical content in the transition course. However, there has been little research on how students develop their knowledge of proof methods in that context and how they choose proof methods in proving activities after they are taught the methods through transition classes.

The ability to choose an appropriate proof method for a statement is not sufficient for producing its proof successfully, but it is definitely a necessary ability because it is closely connected to constructing the overall structure of the proof. Research is also needed as to whether students' choices of proof methods are appropriate and what their rationale for choosing certain methods for statements might be. In this dissertation, I describe transition-to-proof course students' proof-method decision-making behaviors when attempting to prove or disprove mathematical statements.

Research Questions

The overarching goal of the three qualitative studies that I conducted for this dissertation was to examine transition-to-proof course students' proof-method decisions in the context of proof construction. For this work, I particularly focused on students' decision making about six proof methods—direct proof, proof by contrapositive, proof by cases, proof by counterexamples, proof by contradiction, and proof by induction—that were basic proof techniques being introduced in transition classes. The focused two research questions that guided the three studies were as follows:

(1) What strategies do students use at the moment of proof-method decisions for mathematical statements in proof construction after they are taught the six proof methods in a transition-to-proof class?

(2) Why do students choose a particular proof method?

I also explored whether students can choose appropriate methods by using their decision strategies and whether students can prove or disprove statements with their selected methods.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I review the research that has been conducted on transition-to-proof courses (cf. Moore, 1994), focusing on students' proving behaviors in proof production, instructors' pedagogical strategies used when teaching proof and proof methods in that context, and on the six proof methods. Additionally, I discuss what is missing in the literature and what my present studies could provide.[1]

Transition-to-Proof Course Students' Acts in Proof Construction

A large volume of literature on proof construction (e.g., Moore, 1994; Weber, 2001) has examined competencies such as knowledge, strategies, beliefs, and dispositions—competencies that are necessary for success on proof construction but that students often lack—to find out why students failed to prove statements and what instruction would help students enhance their proving abilities. Additionally, some researchers have explored students' proving processes to characterize their proving behaviors while constructing proofs. Based on data from several empirical studies, Weber (2005) found three distinct approaches that undergraduates used when constructing proofs: first, the *syntactic* proof production, in which one uses a statement to structure a proof and draws inferences using its associated definitions and theorems; second, the *semantic* proof production, in which one uses examples of concepts to guide the inferences; and third, the *procedural* production, in which one uses an existing proof as a template when constructing a new proof. Alcock and Weber (2010a) noted that transition-to-proof course

[1] See reviews on learning and teaching of proof such as those written by Harel and Sowder (2007) or Stylianides, Stylianides, and Weber (2017) if interested in exploring the large body of work in that area.

students used syntactic and semantic reasoning in their proof production. Interestingly, one of two focus participants in their study consistently used a syntactic approach, whereas the other student used a semantic approach all the way through. Smith (2006) also investigated proof construction strategies that transition-to-proof course students used when attempting to prove. She found that, when attempting to prove after reading a problem statement, the lecture-based group students (receiving a traditional form of instruction) began looking for possible proof methods and chose a method focusing on surface features of the statement based on their past proving experiences with a similar type of problem, whereas the problem-based group students tried to understand the statement first (using examples) and then attempted to prove it using various proof strategies focusing on the concept involved in the statement. The lecture-based group students' ways of proving can be seen as a syntactic approach, and the problem-based students' ways of proving may be viewed as a semantic approach. Alcock (2010) remarked that students should be able to use both semantic and syntactic reasoning in their proof construction activities, not just one or the other. Alcock and Weber (2010b) also found out why transition-to-proof course students used examples in producing proofs. Their 11 student participants had four reasons for using examples: to understand a statement, to evaluate whether the statement was true, to construct counterexamples to disprove the statement, or to generate a proof of the statement.

From the literature review on this topic, I was able to understand transition-to-proof course students' proof construction behaviors in a broad sense and their behaviors involving the use of examples in attempting to prove or disprove statements in a narrow sense. However, assuming that depending on the statement, different methods of proof, theorems, definitions, or techniques need to be selected and used, I contend that students' decision-making acts involving the selection of which resources to use to prove or disprove the statement also need to be captured and examined as a part of their proving acts in proof construction activities. Weber (2001) stressed that students needed to have strategic knowledge of how to choose appropriate resources for a statement to be successful at proof construction. With a particular interest in proof methods, in my three studies, I investigated how students make decisions about which proof method to use for mathematical statements in the context of proof construction and why they make such decisions.

Teaching Proof and Proof Methods in Transition-to-Proof Courses

Teaching in most proof-based undergraduate-level mathematics courses involves a "definition-theorem-proof" (DTP) format of instruction, and this is a widely known traditional instruction type. Davis and Hersh (1981) stated that "a typical lecture in advanced mathematics ... consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (p. 151). DTP instruction can be characterized as follows:

> The instruction largely consists of the professor lecturing and the students passively taking notes, the material is presented in a strictly logical sequence, the logical nature (e.g., formal definitions, rigorous proofs) of the covered material is given precedent over its intuitive nature, and the main goal of the course is for the students to be capable of producing rigorous proofs about the covered mathematical concepts. (Weber, 2004, p. 116)

Studies about what teaching *actually* occurs when teaching proof in a transition-to-proof course (e.g., how instructors introduce and present proofs to students in that context) are scarce; however. Moore (1994) provided some descriptions of how an instructor taught proof in that context based on class observations and interviews. The instructor's pedagogy was DTP style instruction over all, but when presenting new concepts and proofs to students in lectures, the instructor used informal explanations of concepts and tried to provide details about the proofs to enhance students' understanding. In her interview study, Alcock (2010) identified teaching strategies that five transition course instructors were using when teaching proof. The majority of the pedagogical strategies that the instructors used aimed at developing students' structural thinking, so their predominant teaching acts were to provide proof-writing guidelines or rules to students as scaffolding. The instructors expected students to "generate a proof for a statement by using its form; that is, by introducing appropriate definitions and making deductions from either these or the statement itself according to the rules of logic" (p. 80). Alcock viewed this type of instruction as a *syntactic* approach to teaching. However, some of the instructors' teaching was example-based and emphasized generating examples and understating their meanings. She called this type of instruction a *semantic* approach to teaching.

Marty (1991) reported how he taught a transition class focusing on proof techniques using a class textbook by Solow (1982) entitled *How to Read and Do Proofs* and discussed the effectiveness of such instruction on students' performance in later advanced courses. In that class, he taught various proof techniques (e.g., the forward-backward technique and the choose-representative technique) including the six basic proof methods that are the focus of the present studies. In his report, he provided few details on how he introduced each technique to students but, while teaching the techniques, he focused on one technique at a time and discussed the technique with examples. He also had students practice using each technique with various problems and encouraged them to prove in multiple ways. Marty (1986), in another paper, described in greater detail how he taught students about proof techniques using Solow's textbook. For instance, with respect to the six proof methods, he suggested that students use mathematical induction if the conclusion of a statement was about the set of positive integers and that they use proof by contrapositive or contradiction if the conclusion involved the word *not* (p. 49).

From the literature review on this topic, only small snapshots of what happened in the context of a transition-to-proof course when teaching proof and proof methods were obtained. This means that more studies are required to see how transition-to-proof course instructors introduce proof and proof methods to students in this context and what proving experience students are actually able to gain to understand students' proving behaviors. Since the goal of the present three studies was to examine transition-to-proof course students' decision making about proof methods, my focus was not on examining instructors' introduction and presentation of proof and proof methods in that context. However, in the reports of the three studies, I provide descriptions of how my student participants were generally taught about proof methods (particularly about the six proof methods) in their transition classes based on my class observations.

Students' Understanding of Proof Methods and Their Proof-Method Preferences

Previous empirical studies on proof methods tended to focus on the identification of students' misunderstandings or difficulties when using particular methods such as proof by induction and indirect proof (proof by contrapositive or proof by contradiction) in the context of proof construction or proof validation. Some researchers (Thompson, 1992) directly asked students to describe verbally how to use methods in proving, thinking that if students were not able to describe them verbally, then they did not understand. Approaching the matter differently, a few studies (İmamoğlu & Toğrol, 2015; Mills, 2010) investigated students' flexibility in using different proof methods or types of methods that students use when proving mathematical statements. Participants in research in these areas were mostly secondary students, undergraduate mathematics majors, and prospective elementary and secondary teachers. Most of the mathematical statements used as tasks in interview settings were about elementary number theory, sets, or functions. For proof by induction, most task problems used in research were equality and inequality problems. In examinations of types of proof methods that students would use in proof construction, researchers provided participants specific directions such as "prove each of the following using any method" or asked students to determine whether the statement was true or false with a prompt such as "prove or disprove the statement." The findings produced by previous research regarding the six proof methods are briefly discussed in the following subsections.

Proof by contrapositive and proof by contradiction. Common findings from prior research on the two methods of indirect proof (Antonini & Mariotti, 2008; Brown, 2018; Harel & Sowder, 1998; Lin, Lee, & Wu Yu 2003; Reid & Dobbin, 1998; Stylianides, Stylianides, & Philippou, 2004; Thompson, 1996) were that students have difficulty understanding and using them (e.g., with respect to interpreting and formulating negations), dislike them, and do not find indirect proofs convincing. Antonini and Mariotti (2006) also noted that "current literature agrees on the fact that students show much more difficulties with indirect than direct proofs" (p. 65). In this sense, students find direct proofs more acceptable than indirect proofs and also prefer to use the direct method over the indirect method. However, Brown (2012; 2018) found that students sometimes preferred indirect proofs in certain problem situations.

Proof by counterexample. It is widely accepted that students do not believe that a single counterexample is sufficient to refute a universal claim. For instance, about 18% of secondary Australian students in the interview study by Galbraith (1981) reported this misunderstanding. Other findings about the counterexample method (Harel & Sowder, 1998; Stylianides & Al-

Mirani, 2010) are that students believe that it is possible that a proof and a counterexample can coexist; that students are not able to distinguish contradictions and counterexamples; and that proof by counterexamples is not convincing to students. Harel and Sowder (1998) also observed that while some students rarely used the counterexample method, other students often used it. However, the authors were not able to explain why this was so.

Proof by induction. In comparison to other types of proof methods, proof by induction has received considerable attention from researchers, and many empirical studies (Baker, 1996; Harel, 2001; Stylianides, Stylianides, & Philippou, 2007) have shown that students lack sufficient understanding of that method. Some findings are that students often accept a proof by induction as a convincing or valid argument based on its forms (appearance) regardless of the correctness of the reasoning or without understanding the induction method; that students believe that a proof by induction can be derived from a number of particular cases; and that students have difficulty understanding the meaning of the base and inductive steps. However, most prior studies on proof by induction placed little emphasis on when students use that type of method and how they make a decision to use that method for a statement.

Direct proof and proof by cases. Students' difficulties with direct proofs could be found mostly in studies exploring students' struggles when constructing proofs (Moore, 1994). However, these kinds of studies paid little attention to how students understand direct proof itself. Most reported other difficulties that students encountered when attempting to prove statements directly, not the difficulty of using the direct method itself. This could be because researchers were able to infer that students knew how to use that method when observing students' work of constructing direct proofs and thus felt no need to focus on it. In other words, for students in previous studies, understanding direct proof was not a problem at all. Studies on proof by cases are also rare. But some studies (İmamoğlu & Toğrol, 2015) show that students used that type of method in certain problem situations such as a statement involving integers.

From the literature review on the six proof methods, I was able to see what misunderstanding and difficulties students might have with each of the methods. However, overall, the previous studies focused on students' knowledge of "how" to use the methods, not on their knowledge of "when" to use them. Also, the literature on this topic does not address how students make a decision to use a particular method over others in proof construction. In their review work on proof, Stylianides, Stylianides, and Weber (2017) pointed out that this kind of research is required. In the present three studies, I examined what knowledge the student participants possessed with respect to when to use each proof method and also explored how students use that knowledge when making proof-method decisions.

CHAPTER 3

THEORETICAL FRAMEWORK

To provide guidance for the analysis of the three studies to address the focused research questions—What strategies do students use when choosing proof methods and why do they choose a particular proof method?—I drew on current empirical and theoretical work on proof, problem solving, and decision making. Selden and Selden's (1995) two theoretical constructs helped me determine where I should pay attention when analyzing students' decision making about proof methods in their proof construction activities. Theoretical and empirical findings from the decision-making literature about decision strategies and factors that influence decisions conducted in other contexts (e.g., Schoenfeld, 2011) provided me with information on how the student participants might behave at the moments of proof-method decisions and what constructs might affect their decisions.

Two Parts of Activities in Proof Construction

Selden and Selden (1995) divided proof construction activities into two parts: the formal rhetorical part and the problem-centered part. For a mathematical statement, the formal rhetorical part of proof writing involves activities such as choosing the "top-level" structure of its proof (called the *proof framework*), unpacking the statement, forming appropriate assumptions and conclusions according to the logical structure of the statement, and associating relevant definitions or theorems with the statement. Those activities do not require "a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985)" (Selden & Selden, 2013, p. 308). But the problem-centered part of proof writing requires creativity and depends on intuition about and deep understanding of the concepts involved and on genuine mathematical problem solving. Selden and Selden commented that, with sufficient practices/experience, the formal rhetorical part of proof writing can be largely procedural and that “being able to write a proof framework can be very helpful for students because it not only improves their proof writing, bringing it in line with accepted community norms, but also because it can reveal the nature of the problem(s) to be solved” (p. 309). However, they conjectured that if students cannot unpack the logical structure of informal statements, they will not able to construct proof frameworks. Selden and Selden’s (2009) study documented the benefit of the instruction in guiding undergraduate and beginning graduate students to first write the formal rhetorical part of proofs in their proving attempts, increasing their proving abilities. However, Stylianides, Stylianides, and Weber (2017) suspected that choosing the proof framework might not be completely procedural and might contain certain aspects of strategic decision making. Five students’ proving behaviors in Papadopoulos’s (2016) study supported Stylianides et al.’s claim. When attempting to decide which method to use among three methods—direct proof, proof by contrapositive, and proof by contradiction—after learning the methods with worked examples, the five students chose methods strategically based on brainstorming and experimentation. For instance, one participant, Rasa, chose proof by contradiction over direct proof and proof by contrapositive after comparing the methods in terms of which type of proving would be easier. I conducted the present three studies with the hypothesis that choosing a proof framework entails strategic decision making.

In my analysis, using Selden and Selden’s (1995) theoretical constructs, I focused on the areas in which the student participants engaged in the formal rhetorical aspects of proof writing, particularly when they were choosing proof methods for statements that might have guided them to write global structures of proofs of the statements that were equivalent to constructing "the first-level proof frameworks" (Selden, Benkhalti, & Selden, 2014).

Strategies Used to Make Decisions

Beach and Mitchell (1978) broadly classified decision strategies into three categories: *aided-analytic*, *unaided-analytic*, and *nonanalytic strategies*. Aided-analytic strategies involve the use of tools (e.g., pencil, calculator, a computer) and of prescriptive decision models. Unaided-analytic strategies do not use tools and include strategies such as the satisficing strategy[2] (Simon, 1955), the elimination by aspects strategy[3] (Tversky, 1972), and the lexicographic strategy[4] (Fishburn, 1974). Decision making along with these strategies is completely processed in the mind of the decision maker. These unaided-analytic strategies have been observed mostly in laboratory settings in studies involving task-based interviews. Strategies found in naturalistic settings (Zsambok, Beach, & Klein, 1992) such as feature/pattern matching[5], analogical reasoning[6], and mental simulation[7] also belong into the category of unaided-analytic strategies. Nonanalytic strategies include compliance with (cultural or social) conventions or habit. Decision strategies are also often characterized as either compensatory

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[2] A decision maker chooses the first option that meets or exceeds his or her minimum level of aspiration.
[3] A decision maker selects an attribute and eliminates all the options that do not possess the attribute, selects another attribute and eliminates options that do not possess that attribute, etc. This process is repeated until a single option remains.
[4] A decision maker chooses the option that possesses the most important attribute and eliminates other options. If two or more options have the same attribute, the decision maker selects the next most important attribute and eliminates options that do not possess that attribute.
[5] A decision maker judges a situation by "matching features of the situation against features stored in memory about previous situations or prototypes" or by "matching larger patterns in the situation against examples or prototypes stored in memory" (Zsambok et al., 1992, p. 24). Zsambok et al. called the first type of strategy *feature matching* and the latter type *holistic (pattern) matching*.
[6] A decision maker maps "the conceptual structure of one set of ideas (called a base domain) onto another set of ideas (called a target domain)" (Zsambok et al., 1992, p. 33).
[7] A decision maker "imagine[s] a sequence of events that might have plausibly results in the observed state of affairs" or mentally "evaluate[s] alternate hypotheses to see which makes the most sense" (Zsambok et al., 1992, p. 22).

(e.g., additive strategy[8] (Svenson, 1979)) or noncompensatory (e.g., elimination by aspects strategy) according to whether they permit compensability or not.

Schoenfeld (2011) explained that, depending on the circumstances in which one is situated, one performs one's decision making differently using either the schema-driven decision making strategy or the subjective expected values strategy. If the situation is familiar, the decision making is (relatively automatically) processed using the schemata stored in one's mind and based on prior experience. If the situation is not familiar, the decision making is processed using one's subjective expected values regarding possible options along with one's orientations. (An option that one puts more value on among the possible options is more likely to be chosen.) But Schoenfeld noted that "it is utterly implausible that anyone would actually compute any of these expected values before acting" (pp. 54–55), and so he used the computations of one's (approximate) subjective expected values as a tool for a post hoc explanation of why one would have made a certain decision. Beach and Mitchell (1978) included variations that represented approximations of the subjective expected value/utility strategy (Bernoulli, 1738) in which a decision maker makes a decision entirely in his or her head under the category of unaided-analytic strategies. Zsambok et al. (1992) also distinguished which strategy could be used in what situation. Whereas mental simulation and analogical reasoning strategies are used in either moderately familiar or unfamiliar situations, feature/pattern matching is used in both familiar and unfamiliar situations. This usage is related to the purposes for using each of the strategies. The purpose of the feature/pattern matching strategy is to identify situations, and the purposes of the other two strategies are to diagnose and assess situations.

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[8] A decision maker chooses the option for which the sum of the utilities of its attributes is the greatest among the sums of the utilities of other options' attributes.

Schoenfeld explained that decision making could reoccur in the process of implementing the selected choice in the situation if that choice did not work reasonably well in helping the decision maker reach a goal. In that process, one uses metacognitive skills (monitoring and self-regulation) to determine whether to go with the selected option or not. When analyzing the decision strategies that my participants used to choose proof methods, I first assessed whether problem statement situations were familiar or not to the participants based on their reactions to the problems. I then observed and identified their decision strategies, referring to strategies found in the decision-making literature addressed above, and also explored what strategies they used in a given situation.

Constructs Influencing Decision Making

Schoenfeld's (2011) theory of in-the-moment decision making modeled as a function of three constructs—resources, goals, and orientations—explains one's decisions and actions in goal-oriented activities such as teaching, problem-solving, and cooking. *Goals* are things that one wants to achieve in a particular situation in which one is situated. *Orientations* include one's "dispositions, beliefs, values, tastes, and preferences" (p. 29). *Resources* include everything that one can use in the situation to obtain the expressed goals such as one's knowledge (e.g., procedural knowledge, conceptual knowledge, and problem-solving strategies). Schoenfeld noted that the three constructs interact with one another at decision moments and claimed that one's decisions could be explained in minute detail using *only* these three constructs. That is, he viewed the three constructs play a key role in influencing one's decision making. To substantiate the theory, Schoenfeld used teaching episodes as instances and, with the theory, he explained teachers' in-the-moment teaching actions during instruction using a variety of data sources such as classroom observations, videotapes of teaching practices, teachers' journals, and teacher interviews. However, although Schoenfeld considered the impact of contexts (social and cultural factors) on one's knowledge and belief construction and one's behaviors and decisions, he did not include them in his theory.

Following Yackel and Cobb's (1996) perspective, I view mathematical learning as "both a process of active individual construction and a process of acculturation into the mathematical practices of wider society" (p. 460). In a mathematics classroom, an instructor plays a role as a representative of a mathematics community, and through instruction, students learn norms and teacher expectations (e.g., what counts as an acceptable mathematical proof in classrooms) and also, at the same time, they construct their personal knowledge, beliefs, and values. In this sense, I see proving as a socially embedded activity. Therefore, when examining students' behaviors on proof-related tasks, we should consider contexts in which students have engaged in similar proving activities while interacting with a community (instructors and classmates). Hemmi (2006) found that university students determined what constituted a proof by generalizing examples of proofs that they had seen in their mathematics classes. Participants (mathematics majors) in Weber's (2010) study acted in the same manner when determining whether arguments were (deductive) proofs. The types of proofs students found convincing were also influenced by what they had observed in classes. For instance, one participant, Lillian, in Brown's (2018) study affirmed that, for her, a proof by contradiction was less convincing than a direct proof. She explained that her instructors had told her classmates and her that they should "…not try the contradiction proof when there's always a direct proof… –use the contradiction as the last resort" (p. 6). That is, these studies documented that examples that instructors presented to students in lectures and instructors' verbal and written comments can influence students' proving acts and their beliefs about proof and proof methods. Brown noted that information written in the textbooks (instructional materials used when teaching proof and proof methods) could also affect students' views on proving with particular proof methods. For instance, textbook authors Barnier and Feldman (2000) stated:

> A proof by contradiction is often easier, since more is assumed true; you are able to assume both the hypothesis and the negation of the conclusion. On the other hand, a proof by contradiction is likely to be less elegant than a proof by contrapositive. In any case, for elegance and clarity, it is better to choose a direct proof over an indirect proof whenever possible. (p. 43)

If students read this excerpt, they might think that proof by contradiction is less elegant and that it is a better idea to use direct proof over proof by contrapositive and proof by contradiction if possible.

I hypothesized that mathematical norms, instructors, and textbooks could also affect students' proof-method decisions. The literature on decision making (e.g., Hsee & Weber, 1999; Zardo, Collie, & Livingstone, 2014) indicated that (social/cultural) contextual or other external factors influence people's decisions making. The findings from Herman's (2007) study also showed that college students considered teacher expectations in their decision making about problem-solving strategies. In her study, after they were taught three strategies—symbolic (solving by hand), graphical (using a graphing calculator to make a graph and analyzing the graph), and tabular (using a graphing calculator to make a table and analyzing numerical values in the table) methods—in an advanced algebra course, Herman asked the students at the end of the course to use the methods to solve six algebra problems that were similar to the problems of course assessments (exams and homework). She found that when solving the problems, among the three methods, her student participants did not initially use the tabular method at all and chose instead one of the other two methods, heavily relying on the symbolic method. From student interview and questionnaire response data, Herman found that the students' choice of method was influenced primarily by their perceptions of which method was more mathematical, of which method their instructors would value, and also, of which method was efficient. That is, both internal and external factors affected their decisions.

To find constructs that might influence my student participants' decisions about proof methods, on the basis of Schoenfeld's two constructs—knowledge and orientations—I examined data to determine what types of knowledge or orientations affected the participants' proof-method decisions. Since I directly asked the participants to choose proof methods for mathematical statements in the interviews (except in the first study), the goals were predetermined in the interview settings—the students were to find possible methods to prove or disprove the statements. Therefore, in my analysis, I paid little attention to goals as a construct affecting the students' proof-method decisions. However, I considered sociocultural factors that might influence their decisions.

Overview of the Three Studies

The three qualitative studies were designed with a similar format of data collection (interviewing student participants, observing participants' transition classes, and collecting participants' copies of class notes and homework), and the development of the interview protocols continued from the first through the third study. The data collection and analysis began with the first study conducted during the spring 2015 semester with three students, continued with the second study conducted during the spring 2016 semester with four students, and concluded with the third study conducted during the summer 2016 semester with two students.

The interview data used for the first study were not initially designed to explore students' decision making about proof methods, although some of the interview questions asked the students to explain their rationale for choosing particular methods for given problem statements. However, the findings from the first study with those data mostly covered students' decision-making behaviors when a problem situation was familiar and also provided guidance regarding how to go about exploring students' proof-method decision making in the context of proof construction. The problem statements used in the first study were problem situations with which students might be familiar. Based on the findings of the first study, in the next two studies, I intentionally included tasks whose problem situations contained situations with which students might be more familiar or less familiar, considering what problem statement types with respect to the six proof methods students had frequently had experience with in their transition classes. Also, based on what I learned from the first two studies, in the third study, I changed the directions when giving statements to students for their proof-method decisions activities from "prove the following statement" to "prove or disprove the following statement" to make students consider all six proof methods at their proof-method decision moments as possible methods.

In the next three chapters, I report the methods and findings of each study with details of how I developed the interview protocols from the first study to the third study.

The University Context: A Transition-to-Proof Course

I conducted the three studies at a large university in the southeastern United States. The mathematics department at USE (a pseudonym for the university) offered a transition-to-proof course entitled *Introduction to Higher Mathematics* every semester to prepare students for the mathematical reasoning and proof writing required for proof-based upper-level mathematics courses such as abstract algebra and real analysis. At USE, mathematics majors were required to take this course as a prerequisite to upper-level courses. This transition course was also a required course for secondary mathematics education majors before they entered the teacher education program. Two or three class sections with different instructors were usually open every spring and fall semester, and one class was usually offered in the summer semester. Course instructors were free to design the course and choose the textbooks they would use. Most instructors who had taught this course used Chartrand, Polimeni, and Zhang's (2013) *Mathematical Proofs: A Transition to Advanced Mathematics*. Some instructors used Gilbert and Vanstone's (2005) *An Introduction to Mathematical Thinking: Algebra and Number Systems* or Daepp and Gorkin's (2011) *Reading, Writing, and Proving: A Closer Look at Mathematics*. General topics covered in this course were mathematical logic, elementary set theory, and relations and functions with standard proof methods.

CHAPTER 4

THE FIRST STUDY

The purpose of the first study was to examine how and why the three students chose particular proof methods over others to use for mathematical statements that would be familiar. The problem statements used for the study were similar to class examples or textbook exercise problems that the students had seen or worked on before in their transition class. That is, the problem statements were routine types of problems. In this chapter, I describe how I conducted the first study and what data were used, and I report findings about the decision strategies students used when choosing methods and constructs influencing their decisions. I also report whether their selected methods were appropriate and whether they successfully proved or disproved the statements with the selected methods.

Methodology

Data source. The data used for the first study came from a larger study that I conducted during the spring 2015 semester at USE with three students who were taking a transition-to-proof class together with the same instructor, Dr. Burt (pseudonym). The larger study was designed to examine what perceptions of proof and proof methods students develop through a transition course and how their perceptions relate to their proof construction and validation. For the larger study, I observed their transition classes (50 minutes, three times per week over a 15-week semester), except on quiz or test days, and each participating student was asked to participate in four individual video-recorded semistructured interviews of varying lengths (30 to 80 minutes) over the semester and to submit copies of his or her homework and class notes. To answer the research questions, I purposefully concentrated on one particular interview of the four from the larger study. The focused interview occurred after the six proof methods had been introduced in the transition class. The six proof methods were the main proof techniques that the students learned in the class. At the time of the interviews, the students had practiced using the six methods in proving with various problems from the textbook assigned as homework for the transition class.

During the interview, the students were asked to give their general thoughts about proof methods, to describe the six proof methods regarding how and when the methods could be used, and then to prove or disprove six proof claims[9] (see Table 1), narrating their thought processes aloud as they progressed. I observed the students "thinking aloud" while working on the tasks using the verbal protocol methodology of Ericsson and Simon (1993). For the six tasks, when designing the larger study, I intentionally selected those that were similar to typical examples of the six proof methods (one typical example per method) found in class lectures or in the class textbook written by Chartrand et al. (2013) to check students' ability to choose an appropriate method according to problem situations after being taught the six methods and to observe their ability to use multiple methods for the same problem. (The first and third tasks were particularly selected for this purpose.) While students were working on the tasks during the interview, for each task, after the students had completed or attempted their proving or disproving work for it, I asked them to explain what they had done to prove or disprove the statement, what proof method they had chosen to use and why (I asked this question mostly right after the student had chosen a proof method), and if there were alternative methods that they could have used in addition to the method that they had used for proving or disproving it. Some interview questions used for the

[9] During the proving activities with the six tasks, my initial directions were "prove the following statement" when presenting each task to the students. When the students started suspecting that a statement might not be true, I supported their suspicion that the statement could be false.

interview were as follows (see Appendix A for the interview protocol used for the focused interview): Do you think that learning various proof methods is important? What is direct proof? When can you use direct proof? Why do you think this proof method is appropriate to use for this statement? Are there any other ways to prove this statement? I transcribed the interviews, and two students from Language Education and Mathematics Education reviewed them to verify their accuracy.

Table 1

Proof Claims

	Problem statement	Possible proof method
1	If x is an odd integer, then $9x + 5$ is even.	Direct proof, proof by contrapositive, proof by contradiction
2	If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd.	Proof by cases (Use direct proof for each case), direct proof
3	Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd.	Direct proof, proof by contrapositive, proof by contradiction
4	The real number $\sqrt{3}$ is irrational.	Proof by contradiction, direct proof
5	For every positive integer n, $n^2 + 5n$ is an odd integer.	Proof by counterexample[10]
6	For every positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.	Proof by induction

Participants. The three students were Camilla, Clay, and Max (pseudonyms). During the semester of the data collection, Max was a freshman, Clay was a junior in high school participating in a dual enrollment program at USE, and Camilla was a junior who had transferred from a junior college to USE in the fall of 2014 after a 4-year hiatus. The students were all pursuing a major in mathematics and were approximately at the same stages in their mathematics

[10] To show that the fifth statement is false, it is enough to provide a single counterexample. It does not require proving some other claim to show its falsity, but a student might show its falsity by proving that the statement, for every positive integer n, $n^2 + 5n$ is an even integer, is true. The student's choice of proof methods to make that argument would be acceptable.

programs. Before taking the transition class, Camilla and Clay had taken all three courses in the series of calculus courses, but Max was taking Calculus 3 (the third course in the calculus sequence) while taking the transition class. Other than the series of calculus courses, Camilla had taken three additional college-level mathematics courses—Linear Algebra, Differential Equations, and Foundations of Geometry (the last is a course for prospective teachers of secondary school mathematics)—that the other two participants had not yet taken. Whereas Clay and Max did not have proof-writing experience before the transition class, Camilla had some. However, all three learned about proof methods through the transition class that they were taking together.

Data analysis. For data analysis, focusing on one participant each time, I first carefully read through the interview transcripts and identified parts of the data in which the students talked about their general thoughts about proof methods and the six proof methods, separating these from parts in which they were working on the proof tasks. I then summarized their general views on proof methods, their perceived knowledge about the six proof methods, their past proving experiences with the methods, and their proof method preferences. For the sections of the interviews in which the students were working on the tasks, for each task, I identified specific segments where the students made decisions about proof methods for the problem, where they implemented the methods that they selected for the problem, and where they commented on the proof methods that they used for the problem after they completed their proofs, attempted to prove, or disproved with their selected methods. I then summarized how and why they chose particular proof methods for each task, paying attention to the moments of their proof method decisions. Using thematic analysis (Braun & Clarke, 2006), I grouped similar episodes together that were related to proof method decision strategies and gave them preliminary category names and definitions. When I coded new episodes, I created new categories or modified the names and definitions of the extant categories when appropriate. With the same analysis process, I developed categories about constructs that influenced their decisions. My codes and categories (themes) were both analytic and inductive, as I began with knowledge of people's decision making from the decision-making literature, but I remained open to (and found) other decision strategies or constructs discussed by my participants and those that were observable in their decision-making activities when choosing proof methods. Through this analysis process, I found three categories of the students' proof-method decision strategies and five categories of constructs that affected their decisions. Observations from class and excerpts from the students' class notes and homework were supplemental to the interviews, adding context to their responses in the interviews.

Learning the Six Proof Methods in Dr. Burt's Transition Class

Dr. Burt was a professor at USE who had taught a variety of courses from the undergraduate level to the graduate level over his 30 years with the university and who had taught the transition course multiple times during his career. In the transition class in which the first study took place, Dr. Burt used *Mathematical Proofs: A Transition to Advanced Mathematics* by Chartrand et al. (2013) as the main textbook. He covered the textbook from Chapter 1 to Chapter 9 over the semester, but some sections of the chapters were left out. His lectures, in large part, followed the structure of the textbook and its contents, and his lecture style could be characterized as the "Definition-Theorem-Proof" (DTP) style of instruction. Nevertheless, he frequently used examples or drawings as instantiations of mathematical concepts or for explanations of theorems depending on the topics that he taught. Most of the time, in his lectures, Dr. Burt wrote the contents of his lectures on the board as he verbally elaborated on them, and most students in the class copied that information in their notes. In this class, students were expected to produce deductive proofs based on definitions, previously proven results (theorems), rules of inference, and methods of proof.

Dr. Burt introduced various proof methods starting with Chapter 3 of the textbook. The order in which he introduced the six proof methods that this study focused on in lectures was direct proof, proof by contrapositive, proof by cases, proof by counterexamples, proof by contradiction, and proof by induction. This order corresponded to the order in which the textbook introduced the six methods, starting in Chapter 3 and ending in Chapter 6. When teaching the six proof methods (over approximately 5 weeks during the semester), Dr. Burt focused on one or two methods in each lesson, and for each method, he presented examples to show how to use the method and in what situations. Most of the examples he used in his lectures were similar to or the same as the textbook examples. He also sometimes used exercise problems as examples in lectures. Many times, while teaching direct proof and proof by contrapositive, when he presented proofs in class, he first discussed with students which of the two methods would be good to use in given proving situations and why, without just directly presenting the proofs using one of the methods. There was also an occasion when he evaluated direct, contrapositive, and contradiction proofs of a statement with students in terms of which proof method would be most efficient to use in that particular situation and why.

While students were learning about proof methods in class, through their weekly assignments, Dr. Burt had them practice using each method with the textbook exercise problems. Each assignment consisted of problems in which the proof methods taught that week were applicable. Since the textbook exercise problems in each chapter were organized by section, when a section was about a particular proof method and homework problems were under that section, students were not required to consider which proof method to use. The homework problems were selected by Dr. Burt from the even-numbered exercise problems in the textbook—problems for which the textbook did not provide answers, but most of the problems were similar to examples that Dr. Burt presented to students in his lectures in terms of the forms (structures) of the statements. Some homework problems asked students to prove the statements in multiple ways using different proof methods such as direct proof, proof by contrapositive, or proof by contradiction.

Results

Strategies used to choose proof methods. When deciding which proof method to use after looking at the problem statements, the three students used one or two strategies from among three types of decision strategies: the feature-matching strategy, the elimination strategy, and the exploration strategy. After choosing methods using those strategies, the students engaged in monitoring activities to determine whether they had selected appropriate methods as they were implementing the methods with the problems. In monitoring, if they reached an impasse or found that a problem statement was false when attempting to prove it with the selected method, their decision-making process was repeated at that point to find other possible methods for proving or disproving the statement using the strategies. Most of the time when they were familiar with the problem situations, using the feature-matching strategy, the students (instantly) made decisions about proof methods focusing on surface features of the statements such as the structures of the statements or keywords/phrases in the statements. The elimination strategy and the exploration strategy were observable on only a few occasions. Table 2 shows which strategies the students used for each task when making decisions about proof methods. In this section, I describe what each of the three strategies is and how the students used the strategies at the moment of their proof-method decisions. I also describe their monitoring activities after they chose a particular proof method.

Table 2

The Three Students' Decision Strategies Used in the Tasks

Task	Camilla	Clay	Max
1	FM	FM	FM
2	FM	FM	FM
3	FM	FM	FM
4	E	FM	E
5	FM+EP	FM+EP	FM
6	FM	FM	FM

Note. FM = the feature-matching strategy, E = the elimination strategy, and EP = the exploration strategy

The feature-matching strategy.

Description of the feature-matching strategy. When choosing which proof method he or she would use after looking at a target statement, the student (immediately) chose a particular proof method according to the (surface) features of the statement that he or she recognized as cues, such as its structure, a certain key phrase/word, a simpler assumption than the conclusion (particularly, when the statement was an implication), or its falsity. I labeled this decision strategy *the feature-matching strategy*, since the student (mentally) matched the features of the target statement with features of that type of statement associated with a particular proof method stored in memory and chose the particular proof method for the target statement based on these feature similarities.

How the students determined proof methods using the feature-matching strategy. For the six tasks, on most occasions, the three students immediately made their decisions to use particular proof methods using the feature-matching strategy according to the superficial features of the task statements that they recognized. Such quick proof-method decisions could be made because the tasks were routine problems, whose structure was familiar.

From their past proving experiences, all three students perceived that problem statements associated with proof by contrapositive included an implication and were hard to prove directly. With this in mind, when the third task (Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd) was given, all three students instantly chose the contrapositive method upon looking at the task statement, since the features of the statement that they recognized coincided with the features, based on their perception, of the type of problem that the contrapositive method was associated with. At the moment of their decision, they all recognized that the statement was an if-then statement and that direct proof would not work or would be difficult to use in that problem situation. When determining whether the statement could be easily proved directly, two participants, Clay and Max, mentally compared the algebraic expression $5x - 7$ in the assumption and the expression x in the conclusion in terms of complexity or inclusion relation. For instance, at the moment when Max chose the contrapositive method for the third task, he said,

> This one, I'd use contrapositive, because like I said before, that you have this statement, and this [pointing at the expression x in the conclusion] is a part of that statement [pointing at the expression $5x - 7$ in the assumption], and so, it's hard to go from the big thing to the small thing. So, that's why I think you have to go from the small.

Similarly, for the sixth task (For every positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$), since the task statement was quite similar to examples that he had proven previously by induction, Max chose the induction method as soon as he looked at the statement saying, "This is how I remember it being like." The two other students also chose the induction method in this manner.

The elimination strategy.

Description of the elimination strategy. When an appropriate proof method did not seem immediately apparent after looking at a statement, the student first eliminated proof methods that he or she judged not to fit into the problem situation since the methods were not associated with that type of statement; then, the student chose a proof method from those that remained after the elimination process. Until the student found a method, he or she repeated the elimination process. I labeled this decision strategy *the elimination strategy*.

How the students determined proof methods using the elimination strategy. Max and Camilla each used the elimination strategy one time when choosing a method for the fourth task. (The real number $\sqrt{3}$ is irrational.) For this task, they both chose the contradiction method after eliminating two other methods that they judged as not being suitable for the problem situation. Camilla's proof method decision process, until she decided to choose the contradiction method as a method for the problem, was as follows:

> So, in my head, I'm thinking, okay, we can't really do direct proof, because we're not going from one thing to another, and then…you can't really do…a contrapositive, because there really isn't a negation…, and then, it brings me to contradiction.

Since the features of the problem statement did not match the features of statement types that she associated with the direct method and the contrapositive method, she eliminated these two methods first and chose the contradiction method, which was the only one left (among her options) as a possible method. However, although she did not explicitly say this, she might have eliminated proof by cases and proof by induction as well from the options, according to her knowledge of problem types associated with the two methods that she had studied and remembered, in that she immediately chose those methods when working on other tasks. Max also chose the contradiction method in a similar way for the fourth task. When asked what method he would use for the task statement, he said, “That’s the thing that I don’t know about this one, because you can’t use any of the direct proof or contrapositive. Maybe, what you could use is like contradiction.” Since the task statement was not an implication, for him, this problem situation was not appropriate for applying the direct and contrapositive methods. Similar to Camilla, because Max immediately chose proof by cases, by counterexamples, and by induction when working on other tasks, he seemed to have eliminated those three methods in his mind at the moment of his proof-method decision for the fourth task, although he did not explicitly say that. That is, for Max, the contradiction method was the only method that he could choose since other proof methods did not seem appropriate to him in that problem situation.

The exploration strategy.

Description of the exploration strategy. In attempting to prove a statement, when the student felt that a statement might be false, he or she verified whether the statement was true or false using examples (by randomly picking some numbers and plugging the numbers into the statement) and then made decisions about proof methods according to this experimentation. When the result of the experiments showed that the statement was false, the student chose methods with which he or she could disprove the statement. But when the result of the experiments showed that the statement was true, the student chose methods with which he or she could prove the statement[11]. I labeled this decision strategy *the exploration strategy*.

How the students determined proof methods using the exploration strategy. Camilla used the exploration strategy one time, when working on the fifth task (For every positive integer n, $n^2 + 5n$ is an odd integer). For the task, when she suspected that the problem statement might be

[11] After verifying the truth or falsity of the statement, when choosing methods according to the test results, the student chose methods using the feature-matching strategy.

wrong, she tested the statement with a few numbers to confirm whether it was true or false and made her proof-method decisions on it later based on the test results. Once the outcomes of the experiments showed that the statement was false, she determined that it was appropriate to disprove the statement and chose three methods—proof by cases, direct proof (for proving each case), and proof by contradiction—as methods for disproving. Clay also proceeded in a similar manner when working on this task, but his choice of proof method was proof by counterexamples.

The monitoring activities.

Description of the monitoring activities. After choosing a method using one or two of the decision strategies, the student evaluated whether he or she had chosen the right method for a statement as he or she was monitoring his or her proving or disproving process with the selected method. The student sometimes verbally sketched how a proof might go with the method before starting proof writing to be certain that the method would work out successfully. In the proving or disproving process, if the selected method worked (or seemed to be working), the student would continue proving or disproving the statement with that method. If the student found that the statement was false or reached an impasse while proving with the selected method, he or she looked for another method that might work using the strategies. But, in this situation, if the student thought the selected method was the only possible method for the statement, he or she did not look for other methods and continued to attempt to prove with that method. I labeled those metacognitive behaviors *monitoring activities*.

How the students proceeded during the monitoring activities. The three students employed the monitoring activities as they were implementing the selected methods with the problems. For instance, when working on the first task (If x is an odd integer, then $9x + 5$ is even), Camilla's first choice of method was the direct method using the feature-matching strategy, but she reached an impasse while attempting to prove the task statement with that method. She paused for a few seconds at that moment, considered another method, and chose the contrapositive method to prove the statement using the feature-matching strategy. Camilla perceived that a statement that was in the form of an implication was a problem type associated with the direct method and the contrapositive method. However, she also perceived that if it was easy to move from the assumption to the conclusion, then that was a situation that called for the use of the direct method. In the reverse situation, the contrapositive method would be an appropriate choice. With this in mind, she chose the direct method at first, because she thought that she could easily go from the assumption to the conclusion, but when she was not able to prove the problem statement in that way, she considered another possible method—the contrapositive method—which, based on the features of statement types that she associated with that method, seemed appropriate. At the moment when she changed to the contrapositive method, she said, "it's hard to just move from a number [the expression x in the assumption] to that equation [the expression $9x + 5$ in the conclusion]." However, she reached an impasse again while attempting to prove it with the contrapositive method. Therefore, she reconsidered the direct method and decided to go back to using it. Camilla made this decision because, for her, direct proof and proof by contrapositive were the only methods that she associated with that type of problem statement. At that moment, she said, "Wait, why can we just [crossing out her proving work using the contrapositive method] … Okay, this is how I do the proof. I'm just like, wait, I can't do it this way. Alright, go back to the beginning."

Did the students choose appropriate methods for the tasks?

For the six tasks, with one exception, the three students chose appropriate methods for the tasks using the three decision strategies with the monitoring activities (see Table 3). However, there were some instances in which Camilla and Clay chose inappropriate methods at first. In the next section, I discuss in detail the constructs that affected their appropriate and inappropriate decisions about proof methods while working on the tasks.

Table 3

The Proof Methods That the Students Finally Selected for the Tasks

Task	Camilla	Clay	Max
1	Direct	Direct	Direct
2	Cases	Cases	Cases
3	Contrapositive	Contrapositive	Contrapositive
4	Contradiction	Contradiction	Contradiction
5	Cases, Direct, & Contradiction[12]	Counterexample	Counterexample
6	Induction	Induction	Induction

Constructs that influenced the students' choices of proof methods. One major construct that affected the students' proof-method decisions on the six tasks was their knowledge of when to use the six proof methods. In looking for and deciding on methods, the students largely relied on that type of knowledge. However, there were many other constructs that, paired with that knowledge, also influenced their decisions. Additionally, the constructs that influenced proof-method decisions differed from person to person and task to task. Across the three students, I identified five constructs contributing to their proof-method decisions: knowledge, orientations, control, authority, and behavior. In this section, I describe each construct and how it impacted their decision to choose a particular proof method on the given tasks.

[12] Only after my interruption, after she had completed her disproving work with her selected methods, was Camilla able to start considering proof by counterexamples. Therefore, I ruled out the counterexample method as her selected method, although she was able to disprove the statement with one counterexample as soon as she considered that method.

Knowledge.

Description of the knowledge construct. A student's knowledge of when and how to use the six proof methods affected his or her proof-method decisions. I labeled this type of influence on the student's decision making *the knowledge construct*.

Knowledge of when to use the six proof methods. The students' knowledge of when to use the six proof methods was a primary construct affecting their proof-method decisions. This knowledge included their knowledge of the problem types associated with the six proof methods. The students decided to use a particular method based on the type of problem being considered. They constructed this knowledge by reflecting on their past proving experience with class examples or homework problems in which they had used certain proof methods. It is important to note that since the students had similar proving experiences—they were taking the transition class together with the same instructor—they possessed similar knowledge about a problem type per method. They also all perceived that there was an appropriate method for each type of problem situation, so the choice of method depended on the problem. For instance, for proof by cases, the three students all perceived that when a problem situation could be divided into parts (cases), proof by cases could be applicable. In particular, they knew that a problem involving integers could be divided into two cases: one case for even integers and the other case for odd integers. Therefore, proof by cases was appropriate for that type of problem. Based on their perception of the problem type, when the second task (If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd) was given, they all chose proof by cases recognizing that the task statement involved integers. At the moment when Max decided to use proof by cases for the task, he said, "This one, I would use proof by cases, because you're saying n is an element of $\mathbb{Z}$ [the set of integers], right? So, that must mean that n can be odd or even, either those two, right? So, that's why I'm thinking that you have to use two cases for this one."

However, there was an occasion when Camilla and Max spent a little longer determining to choose proof by contradiction because of their lack of knowledge about the problem type for proof by contradiction. When asked when she could use the contradiction method, Camilla said, "I can't think of a specific example when it would be just easier to do that." Nevertheless, she believed that there were times when the contradiction method would be appropriate. Max also was not able to provide an appropriate example of the contradiction method. He also did not possess solid knowledge about when it would be good to use the contradiction method. However, unlike Camilla and Max, Clay perceived several problem situations that seemed to call for the use of the contradiction method. He said that he could use the contradiction method "when a direct proof or [and] a contrapositive proof don't seem like they're going to work out, like both of them seem like they're not a possibility" and when "proving whether things are rational." He also added, "I see it [the contradiction method] a lot with irrational numbers," and "when Dr. Burt explained it [the contradiction method] to us, I remember I starred it in my notes, because he said that this is a very classic proof for showing something is irrational by contradiction. So, that's where I see it a lot." Thus, for him, a problem statement with irrational numbers was one problem type that likely required the contradiction method. Since he thought the contradiction method was a required method in this case, when the fourth task (The real number $\sqrt{3}$ is irrational) was given, Clay immediately chose the contradiction method for the problem based on its surface features. However, in Camilla and Max's case, since they lacked knowledge of a problem type associated with proof by contradiction, their decision behaviors were different from Clay's. They first had to eliminate the methods associated with the problem types that they knew and that did not fit into the target problem situation; then they were able to choose the contradiction method because the contradiction method was the only method that did not match any other problem type that they could think of.

Knowledge of how to use the six proof methods. Camilla's lack of knowledge about how to use the contradiction method caused her to decide inappropriately to use that method for the fifth task (For every positive integer n, $n^2 + 5n$ is an odd integer). In disproving the fifth task statement, she actually did not consider using the contradiction method at first, but later thought that she had used the contradiction method—"This is the contradiction"—when she showed that "for positive even integers, $n^2 + 5n$ is even, not odd." Since the statement was false, she thought that she could not avoid concluding that the contradiction method was appropriate and that using the contradiction in this way was reasonable. She did not know the way she used the contradiction method was wrong. In Max's case, although his knowledge of how to use the contradiction method was also fragile, it did not adversely affect his proof-method decision. For the fourth task (The real number $\sqrt{3}$ is irrational), he was able to choose the contradiction method because, for him, it was the only method that he could use in that problem situation. His lack of knowledge of how to use that method affected only his failure to prove the task statement by contradiction.

Orientations.

Description of the orientations construct. The student's preference for using the easiest method or the student's belief that a given statement would be true affected his or her proof-method decisions. I labeled this type of influence on the student's decision making *the orientations construct*.

How the orientation construct affected the students' proof-method decisions. Clay's preference for using the easiest method and Clay and Camilla's belief that given problem statements would be true had an impact on their proof-method decisions. However, the orientation construct did not affect Max's proof-method decisions.

Preference for using the easiest method for proving. Clay preferred to use the easiest method for proving, so he typically chose the method that he thought would be the easiest method when he considered possible methods in a problem situation. Of the three participants, only Clay demonstrated this type of decision-making behavior since he was the only one who could envisage using multiple proof methods for particular tasks. For instance, for the second task (If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd), Clay first chose proof by contradiction using the feature-matching strategy, since, for him, a statement that was in the form of an implication (an if-then statement) was one feature of the statement type associated with the contradiction method. However, when he employed that method, he reached an impasse, because he was not able to factor the expression $n^2 - n + 1$. He said, "There's not a really easy way of factoring it. I can't see it off the top of my head." Consequently, he abandoned proving it that way, looked for another method for the problem, found that proof by cases could work using the feature-matching strategy, and decided that he would go with that method, because he "saw that [proving it by] cases is *much easier*" than by contradiction. He evaluated proving it using the contradiction method as "too hard" and "not good." Other instances also show his tendency to look for the easiest method when making decisions about proof methods. For the third task (Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd), Clay immediately chose the contrapositive method not only because its statement features matched the features of statement types associated with the contrapositive method, but also because he perceived that, for that type of problem, using the contrapositive method made it much easier to prove the statement. When discussing alternative methods for the problem after he had completed his proof by contrapositive, Clay thought that direct proof could also be appropriate, but he stated that he would avoid using that method, because "[proving the statement by] direct proof seems like it would be very complicated." He said, for the statement, "the contrapositive just seems like, by far, the easiest." Therefore, he stuck to his original choice, proof by contrapositive, for that reason.

Belief that a statement would be true. Clay and Camilla's belief that a given problem statement would be true also affected their proof-method decisions, leading them not to consider methods for disproving each time the first choice of proof method needed to be made. For instance, for the fifth task (For every positive integer n, $n^2 + 5n$ is an odd integer), believing that the task statement was a true statement, Clay's first choice of method was the induction method (proof by induction), using the feature-matching strategy focusing on the key phrase "for every positive integer n." After he found that the base case when $n = 1$ was not true, he started believing that the statement might be false and determined to use the counterexample method to disprove the statement. That is, because his belief that the statement would be true, he did not initially consider the counterexample method when first looking for the appropriate method. Regarding the finding of the counterexample, he said, "I didn't even try." Camilla had a similar approach to this problem.

Control.

Description of the control construct. The student was not able to think of a proof method that could be used for a statement at an appropriate time while making a proof-method decision, although he or she knew what the method was and when it could be used. I labeled this type of influence on a student's decision making the *control construct*.

How the control construct affected the students' proof method decisions. One particular episode showed how the control construct affected Camilla's proof-method decisions. Camilla perceived that the counterexample method could be used when disproving a universal statement. She also perceived that a single counterexample was sufficient to show why the universal statement was false. However, while working on the fifth task (For every positive integer n, $n^2 + 5n$ is an odd integer), when she looked for a way to disprove the task statement after verifying its falseness with a few numbers, she was not able to think of the counterexample method as a method for disproving it. She therefore attempted to disprove it using other methods. After she had written out her argument explaining why the statement was false using other methods, I asked her if she could call the numbers counterexamples that she used to test whether the statement was true or false; her immediate response was "I didn't think of that…. Why didn't I use that?" She then quickly picked one of the numbers as a counterexample and was able to show that the statement was false in that way.

Authority.

Description of the authority construct. The student chose a proof method because the method was one that the transition class instructor used in similar problems in lectures. I labeled this type of influence on the student's decision making *the authority construct*.

How the authority construct affected the students' proof method decisions. On one occasion, the authority construct contributed to Max's proof-method decisions, but it was not possible to observe whether or not this construct influenced the other two students' proof-method decisions. For the third task statement (Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd), which could be proved in multiple ways, when asked if there were other ways he could prove it besides the method he chose, the contrapositive method, Max's response was "I don't think there is. Maybe there is; I have no idea, but *this is how we learned it. So, I think this is the way to do it*." Even though this is only one instance, this episode shows that a student's proof-method decision could rely on how a similar problem was proved by the instructor in class, not considering other possible proof methods. For this problem statement, Clay chose the contrapositive method over the direct method, because he believed that it was the best method for proving the statement. Camilla chose the contrapositive method, because she thought that it would be difficult to prove the statement directly. Neither of their decisions relied on how the instructor had proved this type of problem in class. Their decisions were made in their own ways.

Impulse.

Description of the impulse construct. The student hastily chose a proof method based on the superficial features of a problem statement at the moment of his or her proof-method decision without carefully reading the statement, causing him or her to choose an inappropriate method or to choose a method for a statement different from the intended statement. I labeled this type of influence on the student's decision making the *impulse construct*.

How the impulse construct affected the students' proof method decisions. While working on the tasks, Camilla often hastily chose proof methods, whereas the other two participants made their decisions prudently. Camilla's hurried selection process led her to choose inappropriate methods for the problems. For instance, for the third task (Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd), when asked what proof method she would use, she skimmed the statement and chose two methods—proof by contrapositive and proof by cases—right away based on the surface features of the problem statement without understanding or analyzing the statement. At that time, her recognition that it was a situation in which direct proof would not work and that it was a parity problem triggered her to pick those two methods. However, in implementing the two methods to the problem, she realized that using proof by cases in this problem situation was inappropriate. At that moment, she said, “You don’t really have to do proof by cases, because they’re telling us that x is odd and the negation of x is even, so, no, proof by cases.” Similarly, she initially chose a method for the fifth task statement (For every positive integer n, $n^2 + 5n$ is an odd integer) without reading it carefully. At first, she chose direct proof as she was misreading the phrase “for every positive integer n,” understanding instead “for every positive even integer n.” At the point where she realized that she was not proving what was asked, she said, “For every positive integer…oh, no, wait, why am I doing even?” and admitted, “I was trying to do it for even integers, because I misread that.” The hurried choices made by Camilla were not observed in the other two participants’ proof-method decision-making processes.

Can the students prove or disprove the statements using the methods that they selected?

For the most part, the three students successfully proved or disproved the tasks with their selected methods (see Table 4). However, all three failed to prove the fourth task (The real number $\sqrt{3}$ is irrational) even though they chose the appropriate method, the contradiction method. They failed to prove this statement because, in Camilla and Clay’s case, they were not able to determine how to arrive at a contradiction by relying on their memories of how a similar problem was proven by contradiction in their transition class. In Max’s case, he did not know how to use the contradiction method to prove that type of problem and he also recalled irrelevant resources that were not helpful in his proving of the problem. The students’ failure on the fourth task shows that although they may choose an appropriate method for a statement, a correct choice of method does not guarantee that they will successfully prove the statement with the method. For the fifth task, only Camila was not able to disprove the task statement successfully.

Her inability to do so was the result of her lack of knowledge of how to use the contradiction method for disproving a statement.

Table 4

Validity of the Students' Proofs for the Tasks

Task	Camilla	Clay	Max
1	Valid	Valid	Valid
2	Valid	-	Valid
3	Valid	Valid	Valid
4	Invalid	Invalid	Invalid
5	Invalid	Valid	Valid
6	Valid	Valid	Valid

Note. For the second task, Clay only verbally provided the outline of its proof.

Conclusions

The main findings from the first study indicate that when a problem statement was familiar, the students' decision was made using the feature-matching strategy. However, when the statement was highly familiar, their decisions were made immediately because the problem statements activated information stored in memory about which method they should use for that type of problem. When the statement was moderately familiar, they spent more time deciding on a method as they were matching features recognized in the statement with features stored in their memories associated with that type of problem and indicating which proof method would be appropriate. In this type of decision making, the students' knowledge of when to use proof methods, which included their knowledge of problem types associated with proof methods, was a key construct that influenced their decisions. When a problem statement was not familiar or was less familiar, the students chose a method using the elimination strategy. When they reached an impasse and felt that a statement might be wrong, they chose a method using the exploration strategy. Their knowledge of when to use proof methods still played a key role in their decisions in those two situations. The exploration strategy has not been observed in the decision-making literature. This could be because the tasks that I used for the study were context-specific; in other words, they were mathematical tasks requiring different types of strategies that are not applicable in other areas. However, Alcock and Weber's (2010b) study showed that students used examples to verify whether a statement was true and decide what type of proof to use to prove it. This aligns with why the three students used the exploration strategy in proof-method decisions. Schoenfeld (2011), in his theory, stated that, if a situation is not familiar, one would use the subjective expected values strategy in making decisions. However, in unfamiliar situations, the decision making of two of the students did not proceed in that way. They used the elimination strategy. That is, they made their decisions largely by relying on their knowledge of when to use a particular method. It could be that they were able to find only one possible option after eliminating unsuitable methods in that situation based on their knowledge, so that situation was not the type of situation in which they would compare methods to determine which one would be the best choice.

While working on the tasks, the main goal of the students was to prove the task statements, but the subgoal that they first established in order to reach the main goal was to find possible methods they could use for proving the task statements. They usually established this goal before I asked them which method should be used for the statements. With that goal in mind, the primary constructs that influenced the students' proof-method decisions were their knowledge about the methods and their orientations—their beliefs about problem statements and their proof-method preferences. This observation aligns with Schoenfeld's theory. However, unlike the theory, there were other constructs that affected the students' decisions such as *authority*, *control*, and *impulse* constructs. Schoenfeld claimed that his theory could explain one's actions or decisions in any goal-oriented well-practiced activities using the knowledge and orientations that one possesses. Based on that assertion, I suspect that the three additional constructs, which were not addressed in Schoenfeld's theory, could be explained by the fact that the three students were novice provers who had little experience with proving activities in which they were required to make decisions about proof methods. Studies about decision making in other areas (e.g., psychology or nursing) show that participants tend to make a choice based on the perceived view of an authority in the field (Pingle, 1997); experts were better than novices in terms of situation awareness with in-depth analysis (Randel & Pugh, 1996); and experts had better memory of past situations along with more rapid and reliable retrieval (Elstein, Schulman, & Sprafka, 1978). In the two studies that follow, I explored whether these three constructs contribute to the decision making of other transition-to-proof course students who took the transition class under different instructors. Also, since the proof tasks used in the first study were mostly familiar to the students, for the later studies, I included tasks that would be less familiar with a certain difficulty or complexity to see how students make decisions in these situations. However, the mathematical concepts involved in the task statements included only those covered in the class.

CHAPTER 5

THE SECOND STUDY

The main goal of the second study was to see if the three decision strategies and the five constructs found in the first study were observable from different students who were taking a transition-to-proof class together with a different instructor. The students in the first study and the students in the second study experienced the same or similar types of problems associated with the six proof methods since their respective instructors taught the six methods in a similar manner using the same class textbook. As in the first study, problem tasks given to the students for proof-method decision activities in the interviews for the second study were similar to class examples and textbook example/exercise problems that the students might have seen or worked on during the transition class. However, whereas the tasks used in the first study were similar to typical examples of the six proof methods that were used when introducing the methods in class (i.e., the students in the first study were mostly familiar with the tasks), in the second study, I included not only tasks that would be familiar but also tasks that would be less familiar to students.

Methodology

Data source. I conducted the second study during the 2016 spring semester at USE with four volunteer student participants. That semester, the students were taking a transition class together with the same instructor, Dr. Kent (pseudonym). The transition class met three times per week (50 minutes per class) over the semester, and the instructor taught six proof methods—direct proof, proof by contrapositive, proof by cases, proof by counterexamples, proof by contradiction, and proof by induction—over 5–6 weeks and within the first two months of the semester. For the second study, I conducted two video-recorded semistructured interviews (approx. 50–100 minutes) per participant[13], completed class observations while the students were learning the six methods in class, and collected copies of students' class notes and homework. The first interview was conducted after the six proof methods were introduced in the transition class. The second interview was conducted after the final class (before or after their final exam). During the two interviews, the students were asked to give their general thoughts about proof methods, to explain when and how to use the six proof methods, to describe their past proving experience with the methods, and to identify their proof-method preferences. They were also asked to prove or disprove proof claims[14] (5 tasks for the first interview; 6 tasks for the second interview) (see Table 5). Using verbal protocol methodology (Ericsson & Simon, 1993), I asked the students to think aloud while working on the tasks and observed their decision making about proof methods in the context of proof construction.

Table 5

Proof Claims

Interview	Task number	Problem statement	Possible proof method
1	1	If x is an odd integer, then $9x + 5$ is even.	Direct proof, proof by contrapositive, proof by contradiction
	2	If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd.	Proof by cases (Use direct proof for each case)
	3	Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd.	Direct proof, proof by contrapositive, proof by contradiction

[13] The four students' background information about their majors, undergraduate mathematics classes that they had taken before the transition class, and whether they had learned proof and proof methods before the transition class were collected from my initial study. I used this information for the second study.

[14] During the proving activities with the tasks, my initial directions were "prove the following statement" when presenting each task to the students. When the students started suspecting that a statement might not be true, I supported their suspicion that the statement could be false.

	4	The real number $\sqrt{5}$ is irrational.	Proof by contradiction, direct proof
	5	For any positive integer $n \geq 4$, $2^n < n!$	Proof by induction
2	6	For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3.	Use both proof by contrapositive and proof by cases; Use both proof by contradiction and proof by cases; Direct proof
	7	For every positive integer n, $n^2 + 3n$ is an odd integer.	Proof by counterexample
	8	For every nonnegative integer n, $7 \mid 3^{2n} - 2^n$.	Proof by induction; Direct proof
	9	Let $n \in \mathbb{Z}$. If $3n - 5$ is even, then $5n + 4$ is odd.	Use a lemma (If $3n - 5$ is even, then n is odd) and direct proof; Direct proof
	10	Let $n \in \mathbb{Z}$. If $n^2 \not\equiv n\ (mod\ 3)$, then $n \equiv 2\ (mod\ 3)$.	Use both proof by contrapositive and proof by cases; Use both proof by contradiction and proof by cases
	11	Let $n \in \mathbb{Z}$. If $4 \mid (n^2 - 1)$, then $4 \mid (n - 1)$.	Proof by counterexample

Note. During the interviews, the tasks were occasionally not given in this exact order.

I developed the first interview protocol by referring to the three students' responses in the first study before conducting the first interview and developed the second interview protocol after conducting the first interview by referring to the four students' responses in the first interview. When developing the first interview protocol, I initially set up a research goal to focus on the five proof methods that excluded the counterexample method to see how students use the five proof methods to construct proofs. But I changed the goal so that it included that method after the first interview for consistency with the first study. The eleven tasks used in the second study were the same as or similar to problems found in lectures or the class textbook. However, whereas the tasks used in the first interview were problems that students might be familiar with, the tasks used in the second interview were problems that students might be less familiar with.

The tasks used for the first interview were the same or similar to the tasks used in the first study. I selected them to compare the students who participated in the first study and in the second study with respect to their proof-method decision behaviors and to see if any similar decision behaviors would be observable. However, unlike the first study, in the first interview of the second study, I did not use a problem statement that could be disproved by the counterexample method according to my initial research goal. Also, I changed a problem type for the induction method from an equation problem to an inequality problem to see how students might react to that type of problem in their decision making. The equation and inequality problems were prototypical examples that could be proved using the induction method. In the second interview, the students were asked about the counterexample method (Allen talked about the counterexample method a little in his first interview without prompting), and I included two tasks related to that method. One of them came from the first study with one minor change. For the second interview, I also selected the tasks that included problem contexts and types that were not used in the first interview. Since the students in the first and second studies chose the induction method without hesitation for equality/inequality problems, I selected the divisibility problem as the eighth task of the second interview to see whether students in the second study also considered this type of problem as one in which the induction method could be used. Also, based on my observations of how the students from the first and second studies made decisions on the first and third tasks in the interviews, I specifically chose the ninth task for the second interview to see which method between the direct method and the contrapositive method the students would choose when both the algebraic expression in the assumption and in the conclusion of an implication were equally complicated.

Examples of interview questions used in the two interviews were as follows: What do you think about the roles of proof methods in proving? How would you explain direct proof? In the proving process, when do you usually consider which proof method is going to be used? What kind of proof methods do you consider at that moment? Why do you think this proof method is appropriate to use for this statement? What factors made you not use other proof methods to prove this statement? See Appendix B for the interview protocols used in the second study. All of the interviews were transcribed by a transcriber and me, and all were reviewed by two doctoral mathematics education students and me to check their accuracy.

Participants. The four student participants—Allen, Jaden, Larry, and Sammy (pseudonyms)—were all pursuing double-majors in mathematics and mathematics-related subjects. During the semester of the data collection, Larry was a freshman, Allen and Sammy were sophomores, and Jaden was a junior. Before the transition course, the students had all taken the first two calculus courses either in high school or college. Alongside the transition course that semester, Sammy and Larry were simultaneously taking the third calculus course, and Allen was taking an elementary differential equations course. Unlike the other participants, before the transition course, Jaden had already taken both the third calculus course and the differential equations course, and he had also taken a discrete mathematics course (for computer science). For three of these students, excluding Jaden, the transition class was the first class in which they started learning about mathematical proofs and about proof methods. Jaden had learned a little about proving and also remembered that he had learned two particular proof methods, mathematical induction and strong induction, when taking the discrete mathematics course.

Data analysis. The overall analysis process for the second study followed the analysis process of the first study using thematic analysis (Braun & Clarke, 2006). For the two interview transcripts per participant, I first separated the parts where the students discussed proof methods from the parts where the students were working on the proof tasks. For all the participants, I then summarized their general thoughts about proof methods, perceptions of how and when to use each of the six proof methods, past proving experiences with the methods, and proof-method preferences (participant by participant). When coding the data for the parts where the students were working on the proof tasks, with respect to their proof-method decision strategies, and contributing constructs on their proof-method decisions, respectively, I used the existing categories that emerged from the first study (three categories of decision strategies and five categories of contributing constructs on decisions), but I opened up the coding to new episodes that could be used to create new categories or to modify the existing categories. My class observation notes and copies of the students' class notes and homework were also used to give context to the students' responses in the interviews and to see what proving experience they had with the six proof methods through the class and what proof methods they had regularly used when proving various types of problems while working on their homework problems.

Learning the Six Proof Methods in Dr. Kent's Transition Class

Dr. Kent was a Senior Lecturer in the Mathematics Department at USE and had taught many lower-level courses along with some upper-level courses. He had taught the transition class three times before. His teaching style in the transition class was a traditional DTP format of instruction and was similar to Dr. Burt's instruction in the first study. For the transition class, he also used the textbook *Mathematical Proofs: A Transition to Advanced Mathematics* by Chartrand et al. (2013). He covered Chapter 1 to Chapter 10 (some sections of the chapters were excluded) in the textbook over the 15 weeks of the semester and introduced the six proof methods—direct proof, proof by contrapositive, proof by cases, proof by counterexamples, proof by contradiction, and proof by induction, in that order—while covering Chapter 3 through Chapter 6 over 5–6 weeks of the semester. He introduced one or two proof methods each week during that period. The order in which he introduced the six methods was the same as that of the textbook; although he largely followed the structure of the textbook, the contents of his lectures often expanded on the contents of the textbook. When teaching the methods, he provided various examples (mainly in calculus, elementary number theory, or elementary set theory) with explanations of how and when to use each of the methods. Most of the examples that he used in his lectures while teaching the methods were similar to examples in the textbook but were not exactly the same. He also provided extra handouts that included additional examples of proofs related to the methods with some tips on good opportunities for using particular methods in particular problem situations.

Over the semester, Dr. Kent assigned weekly homework that included practice problems and required problems. The practice problems were optional, and students were not required to hand them in, but he recommended doing them for practice. The required homework problems were problems that students were asked to hand in. For the required problems assigned while teaching the six proof methods over the 5–6 weeks, each week Dr. Kent assigned problems to which students could apply the methods taught during that week. Most of the required problems were even-numbered textbook exercises. While working on them, most of the time, students were not required to make decisions about which proof method to use, especially when the problems came from the textbook, since the textbook exercise problems per chapter were organized by section and since each section usually dealt with a certain proof method. For some of the problems that were not from the textbook, Dr. Kent also sometimes provided hints for proving them that included suggestions for using specific proof methods.

Results

Strategies used to choose proof methods. The three proof-method decision strategies—the feature-matching strategy, the elimination strategy, the exploration strategy—and the monitoring activities found in the first study were observed in the second study. However, two more proof-method decision strategies—*the comparison strategy* and *the mental simulation strategy*—were observed in the second study. When making decisions about proof methods on the tasks, the four students used four to five of those five strategies (see Table 6). Overall, for a *familiar* problem situation, the four students (immediately) chose a particular proof method for a problem statement using the feature-matching strategy. For a *less familiar* problem situation, most of the time they used the comparison strategy and chose a proof method from among possible methods that they could consider for a problem statement based on its features after evaluating the possible methods to determine which method would be best for the statement. They often used the comparison strategy together with another strategy, the mental simulation strategy, when evaluating two or more possible methods. They mentally simulated proving with each of the possible methods and picked the method that seemed to be the best option. However, in the less familiar problem situation, Larry also used the exploration strategy three times, and the other three students used this strategy once. However, they all used the exploration strategy in attempting to prove with methods that they had already selected using other strategies and when they felt a statement might be false. Exceptionally, Jaden also used the elimination strategy once. For each task, after choosing a method using those strategies, in applying the selected method to the statement with the goal of proving or disproving the statement, all the students engaged in monitoring activities to see whether they had chosen the right method for the statement until they reached the goal. In this section, I present how the students used each of the five strategies in decision making about proof methods for the tasks and their monitoring activities after choosing methods. I also include descriptions of the comparison strategy and the mental simulation strategy since they emerged from the data of the second study.

Table 6

The Students' Decision Strategies Used in the Tasks

Interview	Task	Allen	Jaden	Larry	Sammy
1	1	FM	C	FM	FM
	2	FM	FM	FM	FM
	3	FM	FM	C+MS	FM
	4	FM	E	FM	FM
	5	C+MS	FM	FM	FM
2	6	C+MS+FM	C+MS+FM	C+MS+FM	C
	7	C+ EP+FM	FM+EP	FM+EP	FM+EP
	8	FM	C+MS	C+MS	FM
	9	C+FM	C	C+MS+EP+FM	C+MS+FM
	10	FM	FM	FM+MS	C+MS+FM
	11	C+MS+FM	C+FM	C+EP	C

Note. FM = the feature-matching strategy, C = the comparison strategy, EP = the exploration strategy, E = the elimination strategy, and MS = the mental simulation strategy

The feature-matching strategy.

How the students decided on proof methods using the feature-matching strategy. When a problem statement was familiar, the four students all (immediately) chose a particular method using the feature-matching strategy, as did the three students in the first study. For instance, using that strategy for the fourth task (The real number $\sqrt{5}$ is irrational), three of the students, excluding Jaden in this problem, chose the contradiction method, focusing on the word *irrational* in the statement. At the moment of deciding on a method for the task, Larry said:

> This is kind of like I was saying that I'd like to use contradiction when I see something that says, "prove that this is isn't"…When I see irrational, I just think it's not rational [writing "not rational" on the paper], is my first thought.

The negative meaning of the word triggered him to recall a problem type associated with the contradiction method. Similarly, when the fifth task statement (For any positive integer $n \geq 4$, $2^n < n!$) was given, using the feature-matching strategy, three students, with the exception of Allen, immediately chose the induction method, focusing on the features of the task statement, such as the key phrase "for any positive integer $n \geq 4$" or the forms of algebraic expressions with symbols in the statement. In a similar way, Sammy and Allen also quickly chose the induction method for the eighth task (For every nonnegative integer n, $7|3^{2n} - 2^n$). At the moment that he made his proof-method decision for the eighth task, Sammy said:

> Maybe, this one, I should use induction for…because *my key phrase for every non-negative integer* n which, essentially, means natural numbers. So, you'd start at n greater than or equal to 1.

The elimination strategy.

How the students decided on proof methods using the elimination strategy. Jaden was the only participant who used the elimination strategy in decision making about proof methods while working on the tasks. He used the elimination strategy in much the same way that Camilla and Max had used it in the first study. For instance, for the fourth task (The real number $\sqrt{5}$ is irrational), Jaden chose the contradiction method and explained as follows:

> No definitions, so that kind of x's out direct proof…. No implication, so that kills contrapositive, because contrapositive, you know, just from the definition of contrapositive, and there is no proof by cases here…You can't make two separate cases from this statement. (mumbles) So, the only one that's left is contradiction.

After eliminating other methods that did not fit the problem situation according to his perceived features of statement types associated with other methods in mind, he chose the contradiction method because it was the only method left among his options.

The exploration strategy.

How the students decided on proof methods using the exploration strategy. The four students used the exploration strategy with examples, known facts, or partial proofs of a statement when confirming whether the statement was true or false. However, they used this strategy in attempting to prove the statement with initially selected methods and when they suspected that the statement might be false. Also, across the eleven tasks, whereas Larry used the exploration strategy three times, the other three students used the strategy only one time, respectively. The seventh task statement (For every positive integer n, $n^2 + 3n$ is an odd integer) was the one for which all of the students used the exploration strategy when making proof-method decisions. For this statement, in Larry's case, when he started doubting that the statement might be false in attempting to prove it by cases, he paused for a few seconds and checked the parity of the expression $n^2 + 3n$ using parity facts learned in his transition class to verify the truth or falsity of the statement. He factored the expression $n^2 + 3n$ to $n(n + 3)$ and claimed that n and $n + 3$ had the "opposite parity," and so, "one of them is even, and one of them is odd"; therefore, $n^2 + 3n$ was "not an odd integer." Based on the result of this experiment, his proof-method decision was re-made, and he chose the counterexample method to disprove the statement. The other three students also used the exploration strategy in a similar way. But when verifying whether the seventh task statement was true or false, Allen and Jaden used a partial proof of the statement that they wrote in their papers, and Sammy used examples.

The comparison strategy.

Description of the comparison strategy. When asked what proof method they would use after looking at a statement or when a statement was given, the students searched possible methods they could use in that problem situation based on features of the statement that they considered the most important in proof-method decision making and (mentally) evaluated which proof method among the possible methods would be the most appropriate to use in that situation or which one they preferred to use. This was the second important attribute that they considered in proof-method decision-making. Based on the evaluations, the students chose one method over other possible methods. I labeled this decision behavior *the comparison strategy*.

How the students decided on proof methods using the comparison strategy. The comparison strategy was a newly emerged strategy noted during observations of the four students, especially when problem statements were *less* familiar. For instance, for the ninth task (Let $n \in \mathbb{Z}$. If $3n - 5$ is even, then $5n + 4$ is odd), Sammy chose direct proof over proof by contrapositive after considering both methods. When looking for a method for the task, he first noticed that "there was an implication," and so he considered both the direct method and contrapositive method at that moment focusing on that feature, because he perceived that when a statement was an implication, it was a situation in which those two methods could be applicable. However, he also perceived that when the implication had an easy assumption to make, it was a situation where the direct method would be the best fit. If the implication had a complicated assumption, it was a situation where the contrapositive method would be the best method to use. According to these perceptions, when deciding which of the two methods to use for the task, still focusing on features of the statement, he compared the complexity of the algebraic expression in the assumption and in the conclusion of the statement. However, since the direct method and the contrapositive were equally appropriate for him, he chose a method between the two methods using his second most important attribute (which method he liked better). His decision to use the direct method was made based on his proof-method preference. He liked the direct method better than the contrapositive method because, for him, it was a “simple” and an “easy” method. After deciding to use the direct method, he said, “I told you, it’s my favorite.” Jaden also chose direct proof over proof by contrapositive for the task in the same manner. This type of decision-making behavior was observed many times when no particular method came to mind and when the student could see more than two proof methods as possible methods for the problem statements.

The mental simulation strategy.

Description of the mental simulation strategy. For a proof method that the student was considering as a possible method for proving a statement, the student performed a mental simulation of proving with that method to see whether he or she could prove it using that method or if the method was necessary for proving the statement before making a decision to use that method. If the simulation seemed to work (easy), the student decided to use that method. If not, the student did not choose that method. I labeled this decision behavior *the mental simulation strategy*. This strategy was usually used with the comparison strategy, especially when evaluating (more than two) possible methods in order to pick one of them.

How the students decided on proof methods using the mental simulation strategy. The mental simulation strategy was also a newly emerged strategy documented during observations of the four students in the second study. In decision making, the students sometimes used the mental simulation strategy to see if they could (easily) prove the statement with a method that they were considering rather than directly choosing that method. That is, this was a way for them to assess the feasibility of proving with a method that they were considering before actually choosing the method for the statement. For example, Jaden used the mental simulation strategy in making a proof-method decision for the eighth task statement (For every nonnegative integer n, $7|3^{2n} - 2^n$). For the statement, he first considered proof by cases, but he abandoned proving that way, estimating that there would be a great deal of work with six different cases if he went with that method. So, he looked for another method and chose the induction method. Regarding his decision on the induction method, Jaden said, "I went through the base case, and I went through the inductive hypothesis in my brain, and it looks like it's going to be okay. So, we'll try that." Since the mental proving simulation with the induction method seemed easier than proof by cases, Jaden decided to use the induction method for the problem.

Monitoring activities.

How the students proceeded during the monitoring activities. After selecting methods for the problem statements, when implementing the methods, all students engaged in monitoring activities. However, there were two occasions when the students' decision making occurred again that were not observed in the first study. The first one was in proving a statement with their initially selected methods when they saw that a deduced statement obtained in proving might need another method to proceed; on that occasion, the students chose one more method that was appropriate for the deduced statement. For instance, for the sixth task (For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3), in applying their first selected method, the contrapositive method, to the problem statement, Jaden and Larry recognized that there would be two cases as they were writing the contrapositive of the task statement and interpreting the assumption of the contrapositive. After that recognition, they immediately decided to use proof by cases using the feature-matching strategy. For his choice on proof by cases, Jaden said, "Well, once I did the contrapositive statement, it just read 'if n is not a multiple of 3,' and so, there are two different cases when n isn't a multiple of 3." Since he perceived that "any time you have remainders, any time you're dividing, you have remainders, and so, you just have to do the cases where there are remainders," it automatically occurred to Jaden to choose proof by cases when he encountered the contrapositive of the statement having that feature. Larry described his decision to use proof by cases in that problem situation in a similar manner.

The second occasion was, in monitoring, when they came to an impasse in attempting to prove a statement using the initially selected method. On those occasions, the students sometimes chose another method that they had considered along with the initial method using the comparison strategy but had not chosen at the beginning. For the sixth task, unlike Jaden and Larry, Allen first chose the direct method after considering both the direct method and the contradiction method as possible methods for the problem. When he reached an impasse while proving the problem using the direct method, he decided to use the contradiction method instead, which he had considered at the beginning but had not chosen at that time.

Did the students choose appropriate methods for the tasks?

For the most part, the students chose appropriate methods for the tasks (see Table 7). However, there were several instances in which the students chose inappropriate methods in the process of finding methods for the problem statements, although they were eventually able to choose appropriate methods. Also, there were some instances in which their final choices in methods were inadequate. In the next section, I discuss what constructs affected both their inappropriate and appropriate proof-method decisions on the problem statements.

Table 7

The Proof Methods That the Students Finally Selected for the Tasks

Task	Allen	Jaden	Larry	Sammy
1	Direct	Direct	Direct	Direct
2	Cases	Cases	Cases	Cases

3	Direct	Contrapositive	Contrapositive	Contrapositive
4	Contradiction	Contradiction	Contradiction	Contradiction
5	Induction	Induction	Induction	Induction
6	Contradiction & Cases	Contrapositive & Cases	Contrapositive & Cases	Direct
7	Counterexample	Cases	Counterexample	Counterexample
8	Induction	Induction	Cases	Induction
9	Direct & Cases	Direct	Direct & Cases	Direct & Cases
10	Contradiction & Cases	Contrapositive & Cases	Contrapositive & Cases	Contrapositive & Cases
11	Direct & Cases	Contrapositive & Cases	Counterexample	Direct

Constructs that influenced the students' proof-method decisions.

Three more constructs—the *class*, *task*, and *intervention*—that affected proof-method decisions were found in the second study. However, as in the first study, the main construct that influenced the students' proof-method decisions was *still* their knowledge of when to use proof methods, which is a part of the *knowledge* construct. When they were not able to determine which method would be the most appropriate to use among the possible methods, the students chose a method influenced by the *orientations* construct—their preferences for using the easiest method or their beliefs about proof, for example. The *control* construct was also observable in the second study. One student, Jaden, was not able to recall adequate knowledge of a statement type associated with a particular method at the appropriate time, causing him to struggle when attempting to choose a method. The *authority* construct was observed in the second study as well. One student considered which form of proof would meet his transition class instructor's expectations as an acceptable and convincing argument when choosing a proof method among possible methods with which he could make that type of argument. But the *impulse* construct, which was found in the first study, was not observable in the second study. In other words, the students in the second study did not make hasty decisions about proof methods. On the whole, the constructs that influenced their decisions differed from individual to individual and from task to task. In this section, I describe how each construct influenced the four students' decision making about proof methods for the tasks and also include descriptions of the three constructs that emerged in the second study.

Knowledge.

Knowledge of when to use the six proof methods. Similar to the students' knowledge in the first study, the four students' knowledge of when to use the six proof methods was a central construct influencing their proof-method decisions on the tasks. For instance, all four students understood that possible problem types in which they could use proof by cases were parity or divisibility problems where the assumption of a statement could be divided into different scenarios. According to this understanding, whenever they encountered those problem situations while working on the tasks (e.g., for the second, sixth, seventh, eighth, ninth, tenth, and eleventh tasks), the students immediately opted to use proof by cases or considered whether they could use that method. A concrete problem that they had (successfully) worked on before with a particular proof method and that they remembered also affected their proof-method decisions as part of their knowledge of when to use the particular method, especially when they found that a target problem was similar to the concrete problem. For example, for the eleventh task (Let $n \in \mathbb{Z}$. If $4|(n^2-1)$, then $4|(n-1)$), Sammy chose the direct method because that method had worked out successfully when he was proving a similar problem in the past. When asked why he chose the direct method right after he picked that method, he said,

> I really remembered a problem that I was just working on the other day that this was similar to, and I can factor this phrase right here [pointing at the expression n^2-1 in the assumption] to be, n minus 1 times n plus 1. And since I'm assuming that I can go from there

[pointing at the assumption] to here [pointing at the conclusion], so I was gonna go direct by doing that.

Knowledge of how to use the six proof methods in proving. There was one instance in which one student, Sammy, was not sure whether he could say he used the contradiction method because of his lack of knowledge of how to use that method. For the ninth task (Let $n \in \mathbb{Z}$. If $3n - 5$ is even, then $5n + 4$ is odd), Sammy did not initially consider proof by contradiction to prove it, but when his proof revealed that when n was even, then $3n - 5$ was odd, he thought that *"n* cannot be even, because it contradicts our assumption [$3n - 5$ is even]," and started considering the contradiction method. Since he never assumed that *"*$5n$ plus 4 was even at any point" in his proof, he thought that he did not follow "any traditional contradiction" according to his procedural knowledge of how to use the contradiction method, but he was uncertain about this argument, not being able to conclude whether he had used that method or not.

Orientations.

How the orientations construct affected the students' proof-method decisions. In addition to the students' beliefs that given problem statements were true statements and to their preference for using the easiest method or for a method that did not take much time (observed in the first study), several other types of beliefs and preferences influencing the students' proof-method decisions emerged from the observations of and interviews with the four students in the second study.

Likelihood of success. The students' (subjective) expectation of success with a method that they were considering affected their decision of whether to use that method. If their expectation was positive, the students chose that method; if not, they considered other methods. Such outcome expectations closely relate to the mental simulation strategy. As an example, for the eighth task (For every nonnegative integer n, $7|3^{2n} - 2^n$), Jaden was initially not sure whether he could use the induction method. Therefore, he performed a mental simulation of how proving with that method might go. After the simulation, he decided to use that method because the simulation gave him a positive expectation that proving it by induction would work out. At the decision moment, he said, “I went through the base case, and I went through the inductive hypothesis in my brain, and it looks like it’s going to be okay. So, we’ll try that.”

Preference for using a method that is the most comfortable. While working on the tasks, Allen and Sammy often chose methods they felt most comfortable using. For example, for the ninth task (Let $n \in \mathbb{Z}$. If $3n - 5$ is even, then $5n + 4$ is odd), Sammy initially chose the direct method over other possible methods, because it was his “favorite” method. He liked the direct method the best, because he felt the most comfortable using it. Allen also preferred to use a method with which he felt comfortable. For the third task (Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd), Allen also chose the direct method over two other methods, the contrapositive method and the contradiction method, because he was “more comfortable with the idea” of direct proof than he was with the other two methods.

Preference for using the easiest method or the method that took less work. When there were multiple ways to prove a statement, the students mostly chose or would prefer to choose a method with which they could easily prove the statement or do less work. For example, for the first task (If x is an odd integer, then $9x + 5$ is even), Jaden chose the direct method over the contrapositive method and the contradiction method, because he felt that “it would be a little easier just to do direct proof” and that there would be no need for him “to switch anything or do anything extra” that required work when going with either the contrapositive method or the contradiction method. He also added that whereas proving the statement directly would be “just shorter," proving it by contrapositive or by contradiction would "add a little more time." For this task, Allen also thought that the contrapositive and contradiction methods would "technically" work; however, he felt that the direct method was "probably the easiest" and that the other two methods were "a little bit more convoluted."

Belief that a statement would be true. When deciding which method to use at the beginning after looking at the problem statements, most often, the four students did not consider a method with which they could disprove a statement, believing that the given statements were true statements. For instance, for the seventh task (For every positive integer n, $n^2 + 3n$ is an odd integer), Sammy initially did not consider the counterexample method, which was a method he could use to disprove a statement, until he found that the task statement was false. For this problem, he first chose the induction method, because of the key phrase "for every positive integer n" and because of his belief that the task statement was true. The other three students' approaches to the seventh task were similar to Sammy's, although the methods that they initially chose for proving the statement differed. For most of the tasks, their proof-method decisions were made based on this belief.

Beliefs about (academic) proof in mathematics. How a student viewed proof in the field of mathematics also impacted his proof-method decisions. For example, for the ninth task (Let $n \in \mathbb{Z}$. If $3n - 5$ is even, then $5n + 4$ is odd), when attempting to justify his claim that the parity of n should be odd if $3n - 5$ is even, Sammy first attempted to prove the claim directly using parity results from class (i.e., an odd number times an odd number was an odd number), but he switched gears and selected proof by cases to prove it. His rationale for choosing proof by cases at that moment was that he believed the following:

> When you're, maybe, we're kind of just doing math for fun or in some context if I'm trying to tell you this, if we're having a conversation, then I feel like it would be easier to say, "okay, we both know that an odd number times an odd number is odd," but if I'm trying to be really academic and show exactly the complete and full process, I should do this as fully as I can.

He believed that (academic) proof should explain in detail why something is true. Such beliefs prompted him to use proof by cases to better explain the conjecture rather than direct proof, citing the class results and saying, "We did this in class. You should believe me."

Control.

How the control construct affected the students' proof-method decisions. All four students possessed the knowledge that one problem statement type for the contradiction method was a problem involving an irrational number. However, at the proof-method decision moment for the fourth task (The real number $\sqrt{5}$ is irrational), Jaden was not able to recall that information, whereas the other three students were. Therefore, Jaden had some trouble choosing a method for the task. He was eventually able to choose the contradiction method as he was eliminating methods that he thought did not fit the problem situation. Yet even after choosing the contradiction method, he was still uncertain about his choice. The one problem type associated with the contradiction method that he was able to recall at that time was not the type of this particular problem statement. He said, "Typically, with proof by contradiction, [it] is almost always an implication statement. So, it kind of threw me off a little bit that it was only one statement."

Authority. [15]

How the authority construct affected the students' proof-method decisions. When determining which method to use for the tasks, one student, Jaden, wanted to choose a method with which he could make a proof that would meet his transition class instructor's expectations for mathematical proofs. For instance, for the seventh task statement (For every positive integer n, $n^2 + 3n$ is an odd integer), he chose proof by cases to prove it but found that the statement was false when he attempted to prove the statement using that method when n was even. At that point, he thought that "since it said for 'every,' and since this is false [when n was even], the whole thing [the statement] is false." But he continued proving for the other case, when n was odd, to show that the statement was false for all the cases. When asked if there might be another way that he could show that this statement was false, Jaden briefly considered the counterexample method. But he determined that showing that the statement was false using proof by cases—not just providing a few counterexamples—would be "better" because it would "give more credibility" and take the instructor's standard into account. His additional comment on why he preferred to use proof by cases over proof by counterexamples was as follows:

> That's just from Dr. Kent's [the transition class instructor's] standards. So, *I just know that this* [pointing at his proof by cases] would satisfy him. So, I'm just like "Okay, this is probably the best way to do it," because it doesn't leave any room for what if…no room for ambiguity or anything. It just covers everything; shows you definitions; shows you exactly how I got there in both instances.

In a like manner, to prove the second task statement (If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd), Jaden preferred using proof by cases to using proof by contrapositive since he was not sure

[15] I extended the authority category to include one theme that emerged from the data of the second study—The student chose a proof method over other possible methods with which he could have made a proof that would have met his transition class instructor's expectations as an acceptable and convincing argument.

whether its contrapositive proof would "suffice" as an acceptable argument in the transition class based on proofs that he had seen in class. Looking at the sketch of the contrapositive proof that he had jotted down, he said it showed that the root of the equation $n^2 - n + 1 = 0$ was "not going to be an integer. That's going to be some irrational number," and so it proved the statement. However, he thought that although his contrapositive argument was true, since the argument was too algebraic, the argument might not be an acceptable proof in class, so he determined that it was not appropriate to use the contrapositive method for the problem.

Task.

Description of the task construct. A particular proof method that the student had (successfully) used in the previous task during the interview affected the student's proof-method decision on a later task. I labeled this type of influence on the student's decision making *the task construct.*

How the task construct affected the students' proof-method decisions. There were some instances in which the task construct had an effect on proof-method decisions. For example, for the seventh task statement (For every positive integer n, $n^2 + 3n$ is an odd integer), Sammy did not initially consider the counterexample method at the moment of the proof-method decision. He chose the induction method to prove it at first. After finding that the base case was not true and verifying that the statement was false with a few more examples, he eventually disproved it using the counterexample method. However, for the tasks given after the seventh task, when contemplating which method to use on them, Sammy started considering the counterexample method as well. When searching for a method for the tenth task statement (Let $n \in \mathbb{Z}$. If $n^2 \not\equiv n \ (mod\ 3)$, then $n \equiv 2 \ (mod\ 3)$), he first briefly considered the contrapositive method, focusing on its features. However, he then considered the counterexample method saying, "Let me think about counterexamples first, since that was something I overlooked last time, and see if I can find something simple."

Class.

Description of the class construct. When deciding on a proof method for a problem statement in an interview setting, the student transferred his past practices in the transition class to the interview setting and made a decision about proof methods in a manner similar to that employed in the class. I labeled this type of influence on the student's decision making *the class construct*.

How the class construct affected the students' proof-method decisions. When the tasks were given, most of the time the four students expected the task statements to be true and also did not initially consider the counterexample method at their proof-method decision moments. This had to do with how they had used the counterexample method in the transition class. The three students, excluding Allen, reported that, in the transition class, they had used the counterexample method only when their transition class instructor explicitly asked them to find counterexamples. Therefore, except for this situation, for them, problem statements were supposed to be true, so they considered the use of other methods to prove problems, ruling out the counterexample method. Regarding not considering the counterexample method at the decision moments, Larry said,

> I think I'm just in a bad habit of not checking to see if they're [statements are] right or wrong, because normally, in class [the transition class], we only ever get something that says, "Prove this," or if he [the instructor of the transition class] gives us a statement and our automatic assumption is that is true, and we need to prove it, unless he says, "Give a counterexample."

That is, following their past practices experienced in the transition class, while working on the tasks during the interviews, since the tasks given in the interviews did not specifically ask the participants to find counterexamples[16], they automatically assumed, at the beginning, that the task statements were going to be true and ignored the counterexample method at their proof-method decision moments.

Intervention.

Description of the intervention construct. The researcher's intervention during the student's decision moment affected the student's proof-method decision. I labeled this type of influence on the student's decision making *the intervention construct*.

How the intervention construct affected the students' proof-method decisions. There was one moment when Larry changed his proof-method decision because of my intervention. When working on the eleventh task (Let $n \in \mathbb{Z}$. If $4|(n^2-1)$, then $4|(n-1)$), Larry's first choice of method was direct proof, thinking that there would be less work when proving it directly than when proving it by contrapositive. After he made this decision, I asked him if proof by contradiction was also possible in this problem situation; I wanted to see how he would react to the idea of using the contradiction method in that problem situation. He considered the method momentarily at that point and thought that it might be possible to use it. He then changed his mind and decided to use the contradiction method for the task, ruling out the direct method, which he had selected to use. When asked why he changed his method at that point, he said, "You pushed me…well, I thought about it and, so, now, I'm going to. Let's see how it works out." Although I had not intended to have him use the contradiction method to prove the problem, he interpreted my question in that way, changed his mind, and used the contradiction

[16] I purposefully did not direct students to "find counterexamples" or "disprove a statement," because such a statement would give them a direct hint to use the counterexample method, which would have defeated the purpose of the study.

method. But once he reached an impasse in proving it that way, he went back to his first choice of method, the direct method, to prove it.

Could the students prove or disprove the statements using the methods that they selected?

Among the eleven tasks that the students worked on using their selected proof methods, three to four of their arguments per participant were incomplete or invalid (see Table 8). When counting the number of valid arguments that the students made, I excluded the tasks that asked only to choose proof methods because of the time limit of the interviews. Although I do not discuss in detail the reasons for their failure on some tasks since that was not the focus of this study, there were only three occurrences (one occurrence per student) in which the students made inappropriate proof-method decisions. Consistent with the findings of the first study, these results also show that choosing appropriate proof methods does not ensure students' success in proof construction. However, we might at least say that the instruction provided in the transition class with emphasis on proof methods was effective in that students' decision making about proof methods was successful for the most part for the types of statements used in this study as experimental tasks.

Table 8

Validity of the Students' Proofs for the Tasks[17]

Interview	Task	Allen	Jaden	Larry	Sammy
1	1	Valid	Valid	Valid	Valid
	2	Invalid (two minor algebraic errors)	Valid	Valid	Valid
	3	Valid	Valid	Valid	Valid
	4	Incomplete	Invalid	Incomplete	Incomplete
	5	Valid	Valid	Valid	Valid
2	6	Valid	Valid	Valid	-
	7	Valid	Valid	Valid	Valid
	8	-	Invalid	Incomplete	-
	9	Valid	Invalid (two algebraic errors)	Invalid (one minor algebraic error)	Valid
	10	-	Valid	-	Incomplete
	11	Incomplete	Invalid	Valid	Invalid

Conclusions

Depending on their familiarity with the problem situation, the students in the second study used one to four of the five strategies—the feature-matching strategy, the comparison strategy, the exploration strategy, the elimination strategy, and the mental simulation strategy—per statement when making proof-method decisions. Jaden used all five strategies across the eleven problem statements over the two interviews, and the other three students also used all of them except the elimination strategy. When a problem statement was highly familiar, the students in the second study immediately chose methods using the feature-matching strategy, as

[17] When interview times were limited, for some of the tasks, I asked the students only to determine which method they would use and did not ask them to prove the statements. However, in that situation, they often wrote or verbally described outlines of proofs with methods that they selected. I marked this occurrence as (-). When the students were stuck while working on a task and stated that they did not know what to do for their proofs with the selected methods, I discussed proof methods with them that they had used for the task or talked about the difficulties that they had encountered in proving and asked them to move on to the next task, leaving their written proving work incomplete. In this case, I marked their arguments as *incomplete,* not evaluating them as invalid arguments. For their completed arguments, if the arguments were logically correct without any algebraic errors, I evaluated them as *valid* arguments, but if the arguments were not logically correct or included algebraic errors, I evaluated them as *invalid* arguments.

did the students in the first study. The students still used the feature-matching strategy when a problem statement was moderately familiar but, in this problem situation, they spent more time matching the features of the statement that they identified with features of statement types associated with proof methods stored in their minds. The other four strategies were used in less familiar or unfamiliar problem situations. However, the students often used the comparison strategy together with the mental simulation strategy. The purpose of their use of the mental simulation strategy was generally to evaluate the feasibility of proving with a proof method that they were considering, but when they used this strategy with the comparison strategy, they were attempting to mentally evaluate possible proof methods (more than two) and choose the one that seemed to be the best option (e.g., in terms of proving efficiently with less work). Unlike other strategies, the comparison strategy involved the comparative trait since, when using that strategy, the students saw more than two proof methods as possible options and, therefore, the comparison feature was involved in their decision to choose one of the options. To understand why the comparison strategy and the mental simulation strategy were observed in the second study, but not in the first study, I hypothesized that this was because of the tasks that I used for the second study. Some of the tasks in which the students used those two strategies were tasks that were more complex and that seemed less familiar to the students, causing them to be unable to choose a particular proof method with certainty. Beach and Mitchell (1978) noted that characteristics (unfamiliarity, ambiguity, complexity, and instability) of decision problems influenced an individual's selection of which decision strategy to use to make decisions. The students in the second study used the exploration and elimination strategies in a way similar to that of the students in the first study.

In the students' proof-method decision-making activities in the second study, seven constructs—*knowledge*, *orientations*, *authority*, *control*, *class*, *task*, and *intervention*—contributed to their decisions. As in the first study, the knowledge construct, particularly the students' knowledge of when to use proof methods constructed based on their past proving experiences with proof methods, was a primary construct influencing the students' decisions about proof methods. This result showed how important it is, with respect to students' proving experience with proof methods and instruction on proof methods, to help students build robust knowledge of when to use proof methods and of how to choose an appropriate proof method in a given situation. However, the orientations construct was also influential in the students' proof-method decisions together with the knowledge construct. Unlike the first study, the second study provided information on the different types of beliefs or preferences that influenced the students' proof-method decisions, such as the students' beliefs about proof and about the likeness of success. When there were multiple methods that the students could use for the problem statements, the orientation construct sometimes played a key role in their decision to choose a certain method over other possible methods. In this situation, it was observed that the students used the knowledge construct to gather possible proof methods for the statements. It was also observed in the second study that the control construct had a significant impact on the students' proof-method decisions. Whether or not students could recall appropriate resources at an appropriate time (at the moment of the proof-method decision) seemed to be of critical importance. I think that this type of ability could be developed with more practice/experience. The authority construct was observed but only during the proof work of one student. However, the authority construct showed that a student's consideration of which form of proof would be acceptable and convincing to his transition class instructor at the moment of the proof-method decision could influence his or her choice of a proof method. The student would choose a method with which he or she could make that type of argument and thus meet the instructor's expectations. As I discussed in the result section, there was one construct—*impulse*—that was observed in the first study but not in the second study. Based on my observations of the students' decision behaviors across both two studies, I hypothesized that the impulse construct might be related to the characteristics of the student.

The other three constructs—*class*, *task*, *intervention*—were the constructs that were newly observed in the second study. The intervention construct seemed to relate to the authority construct in that the students tended to prefer using methods suggested by an instructor or a researcher whom they considered more knowledgeable than they were. The task construct showed that the students tended to find the patterns of the problems as they were working on them and that these patterns could be a part of their knowledge of problem types associated with proof methods. The class construct showed that a learned pattern of use regarding a particular method in a certain situation with repeated practice in the transition class seemed to become not only a part of the students' knowledge of when to use that proof method but also a habit of using that method only in that situation, influencing their proof-method decisions and causing them not to consider that method in other problem situations. However, the task construct showed that the students' habitual proof-method decision behaviors connected to a particular proof method could be changed when they experienced that method in other proving situations.

Consistent with the tasks in the first study, the tasks used in the second study were routine types of problems. However, I used a greater variety of problem types, including problems that would be less familiar and more challenging. Therefore, I was able to observe different decision behaviors among the students who were given those problem situations. In the next study conducted with other transition-to-proof course students taught by a different instructor, I continued to explore how students behaved when making decisions about proof methods according to problem situations that were both familiar and unfamiliar and why they chose a certain proof method.

CHAPTER 6

THE THIRD STUDY

The main goal of the third study was to see whether the decision strategies and constructs identified in the first two studies were detectable with students taking a transition class taught by a different instructor. The context of the transition class in which the third study occurred was different from the contexts of the two classes in which the first two studies occurred in terms of instruction time spent teaching the six proof methods. Whereas the two instructors in the first two studies used the same textbook and taught the six proof methods over a considerable amount of time, method by method, with many examples per method, the instructor in the third study used a different textbook and taught the six proof methods in a shorter period of time with fewer examples, except when teaching proof by induction.

Methodology

Data source. I conducted the third study with two volunteer students who were taking a transition-to-proof class under Dr. Tait (pseudonym) during the 2016 summer semester at USE. The transition class met daily (38 days of class, 60 minutes per class) during the summer semester. For the study, I developed two sets of interview protocols based on the interview protocols used for the first two studies. During the semester, using the two protocols, I conducted two video-recorded semistructured interviews (approximately 90–100 minutes each) per student and observed the transition class, particularly when the six proof methods—direct proof, proof by contrapositive, proof by cases, proof by counterexamples, proof by contradiction, and proof by induction—were introduced, to see how students learned about the methods through the class.

As in the first two studies, I also collected copies of the students' class notes and homework as supplements to see what students learned about proof methods throughout the course and to give context to the students' responses during the interviews.

The first and second interviews were conducted during the semester at different times. The first interview occurred one week after the students were taught the six proof methods in class; the second interview took place after the final class (before or after their final exam). During those interviews, the students were asked to offer their general thoughts about proof methods, to provide descriptions of the six proof methods, to explain their proof method preferences from among the six methods, to discuss past proving experience with the six methods (before or during the transition class), and to prove or disprove two sets of tasks (one set per interview) (see Table 9). Each set included six proof claims. The tasks were made by adapting or revising problem statements found in various resources such as the tasks used in the previous two studies, the class examples and two class textbooks (Daepp & Gorkin, 2011; Rosenthal, Rosenthal, & Rosenthal, 2014) that Dr. Tait used when teaching the six proof methods, as well as other textbooks (Hammack, 2013; Solow, 1982; Taylor & Garnier, 2014), including the textbook (Chartrand et al., 2013) that the two instructors from the first two studies used when teaching the six methods in class. However, when forming the six tasks for each interview, I purposefully included problem situations with which students might be more familiar or less familiar. Some tasks used for the first two studies were reutilized for the third study to see if there would be any consistency in students' proof-method decisions for certain types of problems across the three studies. However, within the scope of the contents covered by the transition class that the two students took, I also included different problem types and contexts that had not been used in the previous studies, such as a statement involving an existential quantifier or the word *unique*, and a statement about sets or functions to see how the students would react to these problem situations when making decisions about proof methods.

Table 9

Proof Claims

Interview	Task number	Statement	Possible proof methods
1	1	The sum of any two consecutive positive integers is odd.	Direct proof, proof by contrapositive, proof by induction, proof by contradiction
	2	If m and b are real numbers with $m \neq 0$, then the function $f(x) = mx + b$ is one-to-one.	Direct proof; proof by contrapositive; proof by contradiction
	3	For every integer $n \geq 2$, if $x_1, \dots, x_n$ are real numbers strictly between 0 and 1, then $$(1 - x_1)(1 - x_2) \cdots (1 - x_n) > 1 - x_1 - x_2 - \cdots - x_n.$$	Proof by induction
	4	Let $x, y \in \mathbb{Z}$. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$.	Use both proof by contrapositive and proof by cases
	5	The real number $\sqrt{5}$ is irrational.	Proof by contradiction, direct proof
	6	For every positive integer n, $n^2 + 3n$ is an odd integer.	Proof by counterexample
2	7	For all $n \in \mathbb{N}$, $7 \mid 3^{2n} - 2^n$.	Proof by induction, Direct proof
	8	If $(A \cap B) = \emptyset$, then $A \subseteq B^c$.	Direct proof, Proof by contrapositive
	9	If x is an odd integer, then $9x + 5$ is even.	Direct proof, proof by contrapositive, proof by contradiction
	10	The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$.	Proof by contradiction, Direct proof
	11	There exists an integer n such that $n^3 - n + 1$ is even.	Proving the negation of the problem statement either by direct proof or by cases; Or, disproving the problem statement by contradiction
	12	Let $n \in \mathbb{Z}$. If $4 \mid (n^2 - 1)$, then $4 \mid (n - 1)$.	Proof by counterexample

Note. During the interviews, the tasks were not given to the participants in this exact order.

Using verbal protocol analysis (Ericsson & Simon, 1993), while working on the tasks during the interviews, I asked the students to think aloud, and I observed their decision-making process about proof methods. For each task, I asked them which proof method they decided to use, why they chose it, and what proof methods they considered before deciding to use a certain method. I usually asked those questions right after they had chosen a method. When time permitted, after they had completed their proving or disproving work on the tasks using their selected methods, I asked them if they could prove or disprove each task in a different way using another method and asked them which method they preferred to use—the method that they had already used or the method that they were considering—and why. In addition, to see whether the students chose proof methods for homework problems in the same manner that they made decisions for the tasks during the interviews, at the end of each interview, the students were also asked how they made decisions about which method to use when working on two homework problems that I selected from among their assigned homework problems and why they made those decisions. While engaged in this activity, the students were given copies of their proofs of the problems that they had submitted to me on the days when they submitted their homework to their instructor to help them recall their proof-method decisions for the problems. The first homework problem selected (Let n be an integer. Prove that if n^2 is divisible by 3, then n is divisible by 3) was one of the tasks used for the second study (the task numbered six), although the wording used in the two problem statements differed. That particular homework problem was purposefully selected to see if the students in the third study made proof-method decisions for this problem in a manner similar to that of the students in the second study. In the data analysis, I used the students' responses during this activity for reference purposes only in order to find consistencies with respect to their proof-method decision behaviors after analyzing their main decision-making activities with the tasks.

A transcriber completed the first round of interview transcriptions, and I reviewed the transcripts for precision. Sample interview questions were as follows: What are the roles of proof methods? How would you describe direct proof? When can you use this method? When do you usually consider which proof method is going to be used? What kinds of proof methods do you consider at that moment? What proof method did you use to prove this statement? Why did you choose this proof method over other proof methods? Can you explain to me how you chose the proof method when proving each homework problem? See Appendix C for the interview protocols used in the third study.

Participants. The two student participants were Matt and Kassie (pseudonyms). When the third study was conducted, Kassie was a freshman, and Matt was a junior; however, Matt was a transfer student who had taken two years of coursework at another college. Both participants entered the transition course after taking three consecutive calculus courses and a differential equations course, and both had the intention of majoring in mathematics or a mathematics-related field. However, Matt had taken two more courses, linear algebra and discrete mathematics, which Kassie had not taken yet. Whereas the transition class was the first class in which Kassie was taught about proof methods, Matt had already gained experience with direct proof, proof by contradiction, and proof by induction through previous classes, although he did not know what they were called until he was introduced to the three methods in the transition class.

Data analysis. The analysis process for the third study exploring the decision strategies that these two students used when making decisions about proof methods and constructs followed the same process that the two previous studies had followed by focusing on the interview data. For each interview, participant by participant, I first separated the parts where the students talked about proof methods from the parts where they worked on the proof tasks and summarized them in terms of their general views about proof methods, their perceptions of the six proof methods (particularly about when and how to use them), their proving experiences with the six proof methods (before or during the transition class), and their proof-method preferences. I then coded the parts where the students made proof-method decisions for the tasks using existing categories of decision strategies and constructs that had emerged in the two previous studies. However, I opened up new themes as needed to create new categories unique to the third study's data or to modify existing categories.

Learning the Six Proof Methods in Dr. Tait's Transition Class

Dr. Tait was a professor at USE and had taught a variety of undergraduate and graduate courses during a period of 16 to 17 years. He had taught the transition class 5 to 6 times before. Dr. Tait's teaching was also a DTP style of instruction similar that of the instructors in the first two studies. Unlike the other two instructors, though, in his transition class, Dr. Tait did not spend a great deal of time teaching the six proof methods. He spent approximately 7-8 days (about two and a half weeks of a regular semester) teaching the proof methods. However, he spent much more time introducing and practicing the induction method (almost half of the 7-to-8-day period). Also, whereas, for their classes, the first and second instructors used the same textbook by Chartrand et al. (2013), Dr. Tait used a different textbook for the class. The main class textbook was *Reading, Writing, and Proving: A Closer Look at Mathematics* by Daepp and Gorkin (2011), and he covered the material from Chapter 1 through Chapter 23 in class. He used another textbook, *A Readable Introduction to Real Mathematics,* by Rosenthal et al. (2014), as a supplement. He used this textbook when presenting additional examples of the induction method.

To a large extent, Dr. Tait's lectures followed the structure and contents of the class textbook. Most of the examples that he used in lectures were also from the textbook. The order in which he introduced five of the six methods in class was as follows: proof by contrapositive, proof by contradiction, direct proof, proof by cases, and proof by induction. Proof by counterexample was not discussed in class, but it was introduced as in a textbook reading assignment. The order in which the textbook introduced the six proof methods was slightly different. Proof by contradiction was introduced right after introducing direct proof, and proof by counterexample was introduced after introducing proof by cases. But the textbook introduced four methods—direct proof, proof by contradiction, proof by cases, and proof by counterexample—in one chapter (Chapter 5, titled "Proof Techniques") all together. Dr. Tait introduced direct proof and proof by cases together in one lecture. Each of the other three methods was introduced on a different day. Whereas he used one example per method when introducing direct proof, proof by contrapositive, and proof by contradiction, he provided more examples when introducing proof by cases (three examples) and proof by induction (five examples).

Over the semester, he assigned two types of assignments: reading assignments and written homework assignments. For the reading assignments, students were asked to read one or two chapters of the class textbook a day before the lecture that covered the chapters. The written homework assignments consisted of short-answer, long-answer, and practice problems and were assigned weekly over the semester, but the students were not asked to hand in the practice problems. Most of the homework problems assigned after covering chapters on the six methods were from the textbook exercise problems and required students to choose which method to use on their own. However, some problems specified that particular methods should be used—the induction method or the counterexample method, for example.

Results

Strategies used to choose proof methods. *No* additional proof-method decision strategies were found based on the data gathered from the two students, Kassie and Matt. While working on the tasks, Kassie made her proof-method decisions using all five of the strategies—the feature-matching strategy, the comparison strategy, the exploration strategy, the elimination strategy, and the mental simulation strategy—found in the two previous studies. Matt used all of them except the elimination strategy. For each task, they used one to three of the strategies. When a problem situation was *familiar*, they chose a particular method immediately using the feature-matching strategy. But when the situation was *less familiar*, they typically chose a method using one to three of the other four strategies. In this situation, both Matt and Kassie often used the comparison strategy together with the mental simulation strategy. Kassie also often used the exploration strategy but Matt used that strategy only twice. That is, Kassie frequently made her proof-method decisions after confirming whether the problem statements were true or false. Kassie also used the elimination strategy once to narrow down her choices. After choosing a method for each task, Kassie and Matt also engaged in the monitoring activities to assess whether they had chosen the right method. Table 10 shows which strategies the students used for each task when making proof-method decisions. I report how the two students used each of the five decision strategies when making decisions in each subsection below.

Table 10

The Students' Decision Strategies Used for the Tasks

Interview	Task	Matt	Kassie
1	1	C+MS	C+EP
	2	C+MS	C+EP+MS
	3	FM	FM
	4	C+MS	C+MS+FM
	5	FM	C+MS
	6	FM+EP	FM+EP
2	7	C+MS	C+MS+EP
	8	C+MS	C+EP+MS
	9	C+MS	C+MS+EP
	10	FM	E+EP
	11	EP+FM	C+EP
	12	C+MS+FM	C+EP

Note. FM = the feature-matching strategy, C = the comparison strategy, EP = the exploration strategy, E = the elimination strategy, and MS = the mental simulation strategy

The feature-matching strategy.

How the students decided on proof methods using the feature-matching strategy. When a statement seemed to be familiar, the two students made decisions about proof methods using the feature-matching strategy. For instance, for the third task statement (For every integer $n \geq 2$, if $x_1, \ldots, x_n$ are real numbers strictly between 0 and 1, then $(1 - x_1)(1 - x_2) \cdots (1 - x_n) > 1 - x_1 - x_2 - \cdots - x_n$)), Kassie chose proof by induction because features that she identified in the statement matched with features of statement types associated with that method based on her perception. At the decision moment, she said, "I'm going to use induction because you're doing an inequality again, and also, like your n starts at 2." She elaborated more on what made her choose that method after completing her proof using that method as follows:

> I saw that it had a base, and it was a product, and then, you were proving an inequality, and so, that's why I did induction other than like other methods. … I noticed there was a base

> step [pointing at "$n \geq 2$" in the statement], and this [pointing at "$x_1, \ldots, x_n$" in the statement] was a sequence.

Matt had similar perceptions regarding the usage of the induction method. Therefore, for the same task, Matt also chose the induction method based on its superficial features. At the moment of his proof-method decision for the task, he said,

> Oh, this is probably going to be induction. I can kind of look at it and see that, because…we have the little fancy sequence here...inequality…another fancy sequence. And, so, that usually says it's induction. n is greater than or equal to 2; every integer…I'm thinking induction.

Later, regarding his choice, he also added, "I saw kind of a bounded thing…and when I see inequalities, especially, I think induction, because it's kind of hard to prove it otherwise."

The elimination strategy.

How the students decided on proof methods using the elimination strategy. While working on the tasks, Kassie, on one occasion, used the elimination strategy in making a decision on which method to use. She used this strategy to screen out proof methods that seemed to be inappropriate for a problem situation and thus to narrow down her options. For example, Kassie immediately eliminated the contrapositive method for the tenth task statement (The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$) after reading it, saying, "It's not really an if-then statement, so I'm not going to use the contrapositive." She then had only two methods—direct proof and proof by contradiction—remaining among her options. She was not able to screen out one of the two methods because she was not sure whether the statement was true or false. She did not verbally state which methods she screened out, other than proof by contrapositive, but since she had identified specific features of statement types associated with proof by induction and proof by cases and since she immediately had chosen those methods in other tasks, she seemed to be eliminating them as well right after looking at the statement.

The exploration strategy.

How the students decided on proof methods using the exploration strategy. Among the twelve tasks used in this study over the course of two interviews, Matt used the exploration strategy two times, but Kassie used that strategy most of the time (she used the exploration strategy all of the time when working on the tasks during the second interview). In Kassie's case, she usually used the exploration strategy at the beginning after reading a problem statement and paying close attention to the problem's prompt—"prove or disprove the following statement," which Matt paid little attention to. Kassie always suspected that the problem statement might not be true due to the prompt, and so, before choosing a method, she usually wanted to verify whether the statement was true or false, and decided on a method according to the results of the experiments. When verifying, she used various tools other than examples, such as Venn diagrams, known parity facts, and partial proofs. As examples of her use of the exploration strategy for making decisions about proof methods, for the seventh task (For all $n \in \mathbb{N}$, $7 | 3^{2n} - 2^{n}$), Kassie considered direct proof or proof by contradiction as possible methods for the problem. To make a choice between these two methods, she verified the task statement using examples and chose the direct method because she confirmed that the statement was true through the experimentation. For her, direct proof was a method for proving a statement and proof by contradiction was a method for disproving a statement. When discussing why she decided to use direct proof after she attempted to prove it with that method, she said, "because I think this [the problem statement] is true," and demonstrated why she thought the statement was true as follows:

> Just like some natural numbers…so, just plugging in like some examples…if n is 0, 0 is divisible by 7…1, it's 7, which is divisible by 7. 2, 81 minus 4 which is 78, wait, 77, which is divisible by 7.

However, there was one moment when she was not able to make a decision as to which method to use, because she was not able to verify whether a statement was true or false. This happened when she was working on the tenth task (The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$). When asked why she could not make a choice about proof methods for the task, she said, "I don't know if this is true or false, so I can't do direct proof or contradiction. I can't choose one yet." For the problem, at least, she thought that if the problem statement turned out to be true, she would use direct proof for proving the statement, but, if not, she would use proof by contradiction for disproving it. However, since she was not able to confirm whether the statement was true or false at that point, she did not make a decision about it.

Matt used the exploration strategy twice while working on the tasks. On one occasion, he used this strategy right after reading a problem statement, and the other time he used the strategy in attempting to prove a problem statement with a method that he had already selected, just as the students in the first two studies had done. The first occasion happened when he was working on the eleventh task (There exists an integer n such that $n^3 - n + 1$ is even). After reading the task statement, he suspected that the statement might be false, verified whether the statement was true or false using parity facts he knew, and immediately considered the counterexample method once he found that the statement was false. At the moment of his initial decision, he said,

> So, this [the statement] I know not to be true. I'm pretty sure, pretty sure. Just because any odd times an odd, let me think if it's right; odd times odd times odd—this is still odd. That's odd, odd minus odd is still odd, plus an odd number is odd. So, I guess I'll just pick some value to show it's wrong…0.

The comparison strategy.

How the students decided on proof methods using the comparison strategy. Kassie and Matt used the comparison strategy many times when making proof-method decisions while working on the twelve tasks (Matt: 7 times, Kassie: 9 times out of 12). They used the comparison strategy whenever they were not able to immediately determine which method to use. For instance, for the seventh task (For all $n \in \mathbb{N}$, $7|3^{2n} - 2^{n}$), Matt first considered all six of the proof methods as possible methods. But he chose the induction method over the other five methods based on his evaluation because he believed that he could prove the problem much more easily or with less work. He determined that proving the problem with the other five methods would be "annoying" or "a pain," but, for the induction method, he said, "[the induction method is] much easier for me to conceptualize." Similar to Matt, for the eighth task statement (If $(A \cap B) = \emptyset$, then $A \subseteq B^{c}$), Kassie considered direct proof and proof by contrapositive as possible methods based on its features. However, she decided to use direct proof because she felt that proving the problem by contrapositive would be more difficult than proving it directly. When discussing why she chose the direct method over the contrapositive method, she stated,

> I didn't do the contrapositive, because proving that something is a subset is easier than proving something isn't a subset. And, the intersection is not empty. It would just change a whole lot about the problem. So, I didn't want to use the contrapositive….

The mental simulation strategy.

How the students decided on proof methods using the mental simulation strategy. While working on the twelve tasks, Matt and Kassie often used the mental simulation strategy to determine whether to use a method that they were considering or to evaluate which method would be good to use along with the comparison strategy and then make a choice. For instance, for the ninth task statement (If x is an odd integer, then $9x + 5$ is even), Matt chose the direct method over the contrapositive method because he saw that proving the statement by contrapositive would be "a little bit more annoying," based on his mental proving simulation. For this decision, he added,

> For this one, honestly, I saw this…and if you had to go contrapositive, you'd have to say $9x$ plus 5 is equal to 2 times some integer. And then you'd have to kind of go backwards and say "well, x has to be odd," and so you'd have to subtract things and divide things and it could work. It could not show what you wanted it to show because I know for a fact that 5 over 9 is not an integer anymore.

For the same task, Kassie also used the mental simulation strategy in a similar way. She also chose the direct method over the contrapositive method after taking a few seconds to think about which method of proving would be easier. For this decision, she stated, "I thought it would be easier to suppose the x be odd [the assumption of the original statement] rather than this statement be odd [the assumption of the contrapositive of the original statement]. So, that's why I took out the contrapositive."

The monitoring activities.

How the students proceeded during the monitoring activities. Just as the students in the first two studies did, after choosing methods for the problem statements, Kassie and Matt engaged in the monitoring activities as they were applying their selected methods to the statements to see whether they had chosen appropriate methods to reach their goals of proving or disproving the statements. For example, in proving the fourth task statement (Let $x, y \in \mathbb{Z}$. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$) using the contrapositive method that she had selected, Kassie made one more decision when she saw that there were options in the assumption of its contrapositive. Once she identified this, she immediately decided to use proof by cases for the contrapositive using the feature-matching strategy focusing on that feature. At this decisive moment, she said, "Since the if statement [the assumption of the contrapositive] can be broken into two parts, I'm going to break this up into cases, actually." Later, when discussing the methods that she had used for the task, she provided more explanations of how she came up with the idea of using proof by cases in the middle of proving the statement by contrapositive.

> Once I wrote out the contrapositive and noticed this "or" statement [pointing at the assumption of the contrapositive that she wrote on the paper; "$5|x$ or $5y$"], because a "or" statement, only one of them has to be true. That means since there was an "or" statement that I could separate into cases, so that's why. So, [proof by] cases was like the last one [the last method that I chose], but I still proved the contrapositive to be true.

Did the students choose an appropriate method?

Most of the time, Matt and Kassie chose appropriate methods in their final choices (see Table 11), but there were a few times when their final choices were inappropriate. In the next section, I discuss what constructs affected their proof-method decisions and caused them to make appropriate or inappropriate choices. These constructs also explain why the students sometimes chose different methods for the same problem statements.

Table 11

The Proof Methods That the Students Finally Selected for the Tasks

Task	Matt	Kassie
1	Direct	Direct
2	Contrapositive	Contradiction
3	Strong Induction	Induction
4	Contrapositive	Contrapositive & Cases
5	Contradiction	Contradiction
6	Counterexample	Counterexample
7	Induction	Direct
8	Direct	Direct
9	Direct	Direct
10	Direct & Cases	Direct or Contradiction (No decision was made)
11	Contrapositive & Cases	Contradiction
12	Contrapositive & Cases	Counterexample

Constructs that influenced the students' proof-method decisions. *No* additional constructs were found based on the data gathered from the two students. Although some of the constructs (the *class*, *control*, and *authority* constructs) and some types of beliefs or preferences subsumed under the *orientation* construct observed in the previous two studies were not seen in the third study, the *knowledge* and *orientation* constructs were still found to be the main constructs influencing the students' proof-method decisions.

Knowledge.

How the knowledge construct affected the students' proof-method decisions. Like the knowledge of the students in the previous studies, Kassie's and Matt's knowledge of how and when to use proof methods was the primary construct contributing to their choice of proof method. However, their knowledge of various mathematical concepts and the related proving skills also had an impact on their decisions. This new subtheme emerged from the third study.

Knowledge of when to use the six proof methods. Both Matt and Kassie constructed their knowledge of when to use the six methods based on their past proving experiences with the methods. However, since Matt already had more experience with some of the six proof methods in other mathematical contexts through the courses that he had taken before the transition class, he was more familiar with a greater variety of problem types/situations associated with the methods than was Kassie. However, their knowledge of when to use the methods was somewhat similar, since they had taken the same transition class. For instance, both Matt and Kassie perceived that when a statement was true and when it was an implication, either direct proof or proof by contrapositive could be used. However, there was a situation in which proving by contrapositive would be "easier" or "better," especially when an assumption of the implication was more difficult than its conclusion. They saw that proof by cases could be used when a problem involved multiple options (or conditions). In particular, they perceived that a problem involving integers and a divisibility problem were problem types that proof by cases was associated with. They all also perceived that problems involving summations, inequality, sequences, and series were problem types connected to proof by induction. For proof by counterexamples, both Kassie and Matt specifically connected the method to a problem type explicitly asking them to find counterexamples.

Relying to a great extent on this type of knowledge, Kassie and Matt made their proof-method decisions for the problem statements during the interviews, and most of them were appropriate. However, there was one instance in which, because of his lack of knowledge of when to use proof by contrapositive, Matt made an inappropriate proof-method decision, choosing that method in an unsuitable problem situation. Although their understanding of methods was often similar, Matt and Kassie had somewhat different views about proof by contradiction. Whereas Matt saw proof by contradiction as a method for proving, Kassie saw it as a method for disproving. Therefore, while working on the tasks, when they found that something was wrong, whereas Matt only considered the counterexample method, Kassie considered both proof by contradiction and proof by counterexamples as possible methods. The students' knowledge of how similar problems were proven with particular methods in the past also affected their proof-method decisions. For instance, for the fifth task (The real number $\sqrt{5}$ is irrational), Matt immediately chose the contradiction method once he looked at the statement saying, "I know this one. I can't think of what we did though. (mumbling) I remember doing this proof in class. It was square root of 2, though. It's the same thing, though. I can't remember what we did, though. So, contradiction." Similarly, when reporting her decision on direct proof for the first task (The sum of any two consecutive positive integers is odd), Kassie stated, "It looked very similar to something we would've proven early on in the class, so I was like 'direct proof.'"

Knowledge of how to use the six proof methods in proving. Kassie and Matt had correct knowledge of how to use the six proof methods with the exception of proof by contradiction. The following instances show how their lack of knowledge of how to use proof by contradiction caused them to choose or not to choose that method. For the second task statement (If m and b are real numbers with $m \neq 0$, then the function $f(x) = mx + b$ is one-to-one), Kassie chose proof by contradiction over direct proof, thinking that she could prove it more easily using that method by making $m = 0$. She said,

> Because I thought it [proving by contradiction] would be simpler to…like, I knew the statement was true, and if I let m equal 0, I'd only be dealing with the b, because your $0x$ term would cancel out. I mean, would not cancel; it would be 0. So, I just thought it would be easier to do it that way.

She thought that proving it by contradiction was assuming that $m = 0$ and f(x) was one-to-one. For the same task, like Kassie, Matt also thought that proof by contradiction might be a possible method for proving it at first. However, applying the method to the problem incorrectly, he was not able to get what he was looking for to prove the problem, and so, he ruled out the contradiction method and chose another method, the contrapositive method, for the problem. His report of what he thought about using the contradiction method for proving the problem was:

> …I was thinking you could do a contradiction; technically, you can I think. You say, assume that m is equal to 0, and then, you kind of say, "well, this has to be f of x is one to one." So, you say, "well, m which is 0…$x1$ plus b equals $mx1$ plus b …." So, you haven't really shown anything useful…that 0 out, that 0 out… b equals b, well, we know…. So, that's why I didn't go by this method, because it's kind of trivial to show.

Knowledge of a mathematical concept involved in a statement and its related proof methods. There was one moment when Kassie was not able to make a decision as to which method to use, because she had no idea of how to proceed with a problem. This happened when she was working on the tenth task (The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$). After assuming that "let m be a real # [number] so $1 < m < 2$," she was not able to proceed any further. She admitted, "I didn't know where to go," based on the assumption that she wrote and said, "we did do one similar problem that was proving something that had a unique solution [in the transition class], but, I can't really remember how to solve it." For this problem, in Matt's case, he was able to choose a method focusing on features of the statement, but he did not know how to use that method to show the uniqueness of the solution and failed to prove it.

Orientations.

Likelihood of success. Like the students from the second study, particularly when using the mental simulation strategy, Matt's and Kassie's subjective expectation outcomes regarding their success in proving a problem statement with a method that they were considering, based on their mental proving simulations, often influenced their proof-method decisions about whether to use that method. For instance, when there were multiple ways that he could prove a statement, Matt chose a method which seemed to offer him more success in proving the statement. For the second task (If m and b are real numbers with $m \neq 0$, then the function $f(x) = mx + b$ is one-to-one), Matt chose proof by contrapositive, avoiding choosing direct proof, because he thought that he could work better with the assumption of the contrapositive than the assumption of the original statement. For this decision, he said,

> I decided to go by contrapositive, because I know I can do a lot more with this as my assumption, that one to one part, instead of just having m is not equal to 0, because that's kind of a weak assumption. It's kind of the idea of why the second principle of induction is strong, because a lot of things, you kind of go forward with.

Preference for using the easiest method or the method that took less work. Matt and Kassie sometimes chose a method with which they could easily prove a statement or that took less work to prove. For instance, for the first task (The sum of any two consecutive positive integers is odd), Matt thought that both the direct method and the induction method would work for the problem, but he chose the direct method over the induction method, because he felt that proving it directly would be "the less tedious way to go" and a faster approach. For his decision, he added,

> Yeah, this [proving by induction] just takes longer. That's why I chose direct proof, because induction, you have to write out a lot of stuff, but direct proof, your only assumption is what you start off with. You don't have to assume that, show that $P(1)$ is true and then assume this is true and then move on…. It's too much time. So, I just went with direct proof. But, you can show it that way.

Similarly, Kassie chose proof by contrapositive over direct proof for the fourth task (Let $x, y \in \mathbb{Z}$. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$), because she felt that proving it by contrapositive would be "shorter" and "easier" than proving it directly, recognizing that there would be "more cases" when proving it directly. However, most of the time, she consistently considered proof by contrapositive first as a possible method while working on the tasks, particularly when the task statements were implications because she preferred to use that method. She said, "I prefer to do the contrapositive because it usually makes it easier to solve the proof."

Belief that a statement would be true. The students' beliefs about a problem statement—that a statement would be true or difficult—affected their proof-method decisions. As examples of this subtheme, for the second task statement (If m and b are real numbers with $m \neq 0$, then the function $f(x) = mx + b$ is one-to-one), when attempting to prove it, Kassie said, "I'm going to, I think it's right. *I'm going to suppose that this is a true statement* and that the function is one to one." A similar type of decision behavior was also exhibited by Matt. However, Matt had another view about problem statements. Matt confessed that, when working with a problem statement, he normally assumed that the problem statement was going to be difficult, and so, he habitually did not consider direct proof first, since he viewed that method as a method for proving an easy problem. He said,

> If I'm thinking direct proof first, I'm assuming the problem is going to be easy, and I don't want to assume the problem is going to be easy [laughs]. Then, if I assume the problem is going to be easy and if it's actually hard, then I'll get discouraged, and I don't want to have to deal with that. So, *I assume that it is going to be harder*.

With this mindset, while working on the tasks (particularly during the second interview), when direct proof and proof by contrapositive might both be applicable, in general, Matt considered or chose proof by contrapositive first. He considered or chose direct proof only when proof by contrapositive did not seem to work or when using it would require too much work.

Task.

How the task construct affected the students' proof-method decisions. There was one instance when Kassie first considered a method that she had used for previous tasks during the interview when looking for a method for the target task. When discussing the methods that she had used for the fourth task statement (Let $x, y \in \mathbb{Z}$. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$), Kassie admitted that she had first considered using direct proof as a possible method for it, because the problem statement was similar to other task statements that she had worked on before the fourth task during the interview. At that point, she said, "Well, the first thing I thought was direct proof, just because the other questions were similar to…"

Intervention.

How the intervention construct affected the students' proof-method decisions. There was one occasion when the intervention construct affected Matt's proof-method decision. For the twelfth task statement (Let $n \in \mathbb{Z}$. If $4|(n^2 - 1)$, then $4|(n - 1)$), Matt chose proof by contrapositive and proof by cases as methods for proving it, thinking that the statement was going to be true. As a result, while working on this, he did not consider a method with which he could disprove the statement until I asked him whether the statement was true. After my interruption, he started suspecting that the statement might be false and said, “If I show that thing is true for all of them [all of the cases], then I’m done, but, if not, then there was probably some counterexample I probably could’ve used and probably should’ve used to notice.” He then considered the counterexample method and looked for counterexamples to disprove the statement.

Impulse.

How the impulse construct affected the students’ proof-method decisions. When choosing a method for the sixth task (For every positive integer n, $n^2 + 3n$ is an odd integer), Matt failed to examine the statement carefully at the beginning. Ignoring the phrase “for every positive integer n” and only focusing on the part “$n^2 + 3n$ is an odd integer” in the statement, Matt made his initial proof-method decision. He translated that part as the form of an implication, “$\forall n \in n^2 + 3n \rightarrow n = 2k + 1$,” and chose the contrapositive method to prove it. Later, when he found that he had overlooked some parts of the statement in attempting to prove the statement by contrapositive, he confessed, “Essentially what I did at first with contrapositive is *I wasn’t looking at the full statement*. I looked at just the second part of the statement, and I was like, ‘you can prove this by itself,’ which you can’t really.”

Could the students prove or disprove the statements using the methods that they Selected?

Among the twelve tasks, the two students, Matt and Kassie, were able to perform successfully on only five to six tasks (see Table 12). This result shows that the students’ choices of appropriate methods for the tasks did not guarantee their success on proof construction. That is, though such selection ability is necessary to construct proofs, other proving skills are still required. I do not discuss causes of the students' failure on the tasks in this paper, but causes included making a wrong assumption by failing to translate the informal statement to the formal statement appropriately, lacking knowledge about how to use the contradiction method, misunderstanding the problem statement, and making the wrong contrapositive of a statement.

Table 12

Validity of the Students' Proofs for the Tasks[18]

Interview	Task	Matt	Kassie
1	1	Valid	Invalid
	2	Valid	Invalid
	3	Invalid	Invalid
	4	Invalid	Valid
	5	Invalid	Incomplete
	6	Valid	Valid
2	7	Invalid	Incomplete
	8	Invalid	Valid
	9	Valid	Valid
	10	Invalid	Incomplete
	11	Invalid	Valid
	12	Valid	Valid

[18] In attempting to prove the fifth, seventh, and tenth tasks, Kassie was stuck at certain points and admitted that she did not know where to go. Since the main goal of the study was to explore the students' decision making about proof methods, at such moments, I asked her to stop at the points where she was stuck, to discuss proof methods that she had used for the tasks, and then to move on to the next task. I evaluated her arguments for these three tasks as *incomplete*. For the third, eighth, and tenth tasks, Matt made his verbal arguments with diagrams or wrote out some parts of the arguments, but since his verbal arguments for the tasks were not logically valid, I evaluated them as *invalid* arguments.

CHAPTER 7

CONCLUSIONS AND IMPLICATIONS

The main research questions that guided my analysis of the data of the three studies were the following: (1) What strategies do students use at the moment of proof-method decisions for mathematical statements after they are taught the six proof methods in their transition class? (2) Why do students choose a particular proof method? Across the three studies, I found that the participants used four to five proof-method decision strategies and that eight constructs influenced their proof-method decisions.

Students' Decision Strategies Used When Choosing Proof Methods

The nine students across the three studies showed similar decision behaviors when choosing proof methods for mathematical statements. Depending on their familiarity or lack of familiarity with a problem situation, the students used one to five of the following five decision strategies—the feature-matching strategy, the elimination strategy, the exploration strategy, the comparison strategy, and the mental simulation strategy—when making decisions about which method to use. Except for one case in which some of the students used the exploration strategy based on partial proofs of statements that they wrote down as a judgement tool to confirm whether the statements were true or not, these five strategies were "unaided-analytic" (Beach & Mitchell, 1978) types of strategies in that the students' decision making was entirely processed in their heads at the decision moments. For the three studies, I particularly focused on the students' initial proof-method decision moments and, based on how the nine students made proof-method decisions at those moments across the three studies, I describe the five strategies as follows:

- The feature-matching strategy: For a statement, the student chooses a particular proof method according to its (sometimes superficial) features, such as its structure, a particular key word or phrase in it, or its apparent falsity. The method is chosen because the features match features of statement types associated with the particular method that he or she perceives. Depending on the degree of feature similarity, the students make decisions immediately or after taking the time required to match features. If the features of the target statement closely match those of statement types associated with a particular proof method stored in memory, the student considers the statement a familiar problem situation and makes a quick decision to use that particular method for the target statement.
- The elimination strategy: For a statement, the student chooses a proof method (with some uncertainty) after eliminating proof methods that he or she judges not to fit the problem statement based on his or her understanding of features of statement types associated with other proof methods. The student selects one feature associated with a certain method that he or she perceives and eliminates that method as an option for the statement if the problem statement does not have that feature. The student then repeats this process, selecting another feature associated with another method, and uses this strategy until only one or two proof methods remain.
- The exploration strategy: For a statement, the student chooses a method after exploring whether the statement is true or false. The student uses diagrams, examples, known facts, or partial proofs of the statement to determine the truth or falsity of the statement. From these experiments, if the statement turns out to be true, the student chooses a method with which he or she can prove the statement. If not, the student chooses a method with which

he or she can disprove the statement. After confirming the truth or falsity of the statement, when choosing methods, the student uses one to three of the other four strategies (often the FM strategy).

- The comparison strategy: For a statement, the student selects possible methods based on the features of the statement that he or she considers the most important in proof-method decision making. The student then chooses one method over other possible methods according to subjective expectations regarding the appropriateness or efficiency of using that method in the statement or according to subjective preferences for a particular method among the possible methods. If the method selected through this process does not work well in proving or disproving the statement, the student sometimes considers a method other than the one initially selected among the possible options.
- The mental simulation strategy: For a statement, the student chooses a proof method that he or she is considering as a possible method after simulating proving with that method in his or her head and when the simulation shows that the method might help the student reach his or her goal of proving. The student sometimes uses this strategy along with the comparison strategy to evaluate possible proof methods (when there are more than two methods that could be used in the problem situation) and to choose one of them.

The students used these five strategies with different purposes at their proof-method decision moments. The feature-matching, elimination, and comparison strategies were mainly used to select (possible) proof methods, the exploration strategy was used to adjudicate the problem situation in terms of its truth or falsity and to narrow down the method choices, and the mental simulation strategy was used to assess methods being considered for use before actually applying them to statements. The elimination and comparison strategies also involved narrowing down options. The four strategies, excluding the exploration strategy, were also similar in many ways to decision strategies found in the decision-making literature where the strategies were employed when participants were engaged in different types of activities (e.g., selecting a college and choosing a house). For instance, while engaging in an apartment-choosing activity involving 6 or 12 options, college-age subjects in Payne's (1976) study used decision strategies such as the "elimination-by-aspects" strategy (Tversky, 1972) that is a strategy used to screen out inferior options until only the best option remains. This strategy allowed them to eliminate some of the options quickly. The student participants in the present studies seemed to use a similar strategy. When a problem statement was less familiar, the participants first eliminated methods that did not appear appropriate for the statement and chose a method that was not eliminated by this process. That is, how Payne's participants used the elimination-by-aspects strategy was similar to how the participants in the present studies used the elimination strategy. Gray's (1975) study showed that elementary school students used variations of expected value strategies when selecting academic tasks involving arithmetic problems varying in difficulty. The students wanted to choose a problem that would give them more success in problem solving. I observed a similar approach when the participants in the present studies used the comparison and mental simulation strategies. However, I did not find the exploration strategy in the decision-making literature. As I discussed in Chapter 4, I hypothesized that the exploration strategy is a context-specific strategy that students would use in proof-method decision making activities that are a part of the work involved in proof construction. Some student participants in other studies (e.g., Alcock & Weber, 2010b; Hanusch, 2015) showed that they chose a method in a way similar to the exploration strategy in proof construction. However, the exploration strategy was used only a few times across the three studies. Also, with some exceptions, the students mostly used the exploration strategy in the middle of their proving attempt with their initially selected method when they began to have doubts about the truth of the problem statements. In other words, they did not use the exploration strategy at the moment of their initial proof-method decisions for the problem statements. According to Smith's (2006) study, transition-to-proof course students who received problem-based instruction tended to use examples more often to understand statements at the problem recognition stage. However, these behaviors were observed only a few times with transition course students who had received lecture-based instruction in Smith's study. That group of students tended to immediately choose proof methods based on the features of a statement after reading the statement in proof-construction activities. Such behaviors were also frequently observed in the present three studies, possibly because all the student participants also were taking a lecture-based transition class. Therefore, I hypothesize that the frequency of the use of the exploration strategy could be related to the type of instruction that the students had received.

I identified common patterns regarding students' use of the five strategies when making proof-method decisions. For a *familiar* problem situation, the students most often made decisions using the feature-matching strategy. In this situation, most of the time, they quickly identified the surface features of a problem statement while looking at the statement, which brought a particular method to mind based on their understanding of the features of statement types associated with that method, and they immediately chose the method based on these feature similarities. The other four strategies were used when a problem situation was *less familiar* or *unfamiliar*. In this type of problem situation, with the comparison strategy, the students selected possible methods (usually two to three methods) and chose the most appropriate method among the possible methods. But they often used the mental simulation strategy along with it, especially when evaluating possible methods. Some of the students used either the elimination strategy or the exploration strategy in this situation as well, but the exploration strategy was usually used with one or two other strategies in the process of seeking proof methods. This use could have occurred because the main function of the exploration strategy was to verify the truth or falsity of the statement and narrow down options, not to select methods as discussed before. Also, I found that the students generally focused on the surface features of the problem statement when they were selecting methods using the feature-matching, comparison, and elimination strategies. Based on the surface features that they identified, they immediately chose a particular method, selected several possible methods, or eliminated inappropriate methods. The problem situations in which the comparison, mental simulation, and feature-matching strategies were all used together were those in which decision making occurred twice. In those situations, the students usually chose a method using the comparison and mental simulation strategies at the beginning before applying them and then chose another method using the feature-matching strategy when they recognized that a statement deduced in proving a problem statement with the selected method might require another method and that the features of the statement seemed to resemble those of a statement type associated with a particular method (e.g., proof by cases). Schoenfeld (2011) claimed that if a problem situation is familiar, people choose an option using the schema-driven strategy, and that if the situation is not familiar, they choose an option using a form of the subjective expected values strategy. The characteristics of the first strategy are associated with the feature-matching strategy, and those of the second strategy are associated with the comparison and mental simulation strategies. However, the student participants in the three studies used the other two types of strategies—the elimination strategy and the exploration strategy—in unfamiliar situations as well. That is, the three studies showed not only that the students used different strategies depending on their familiarity with the problem statements but also that the students used various strategies to choose methods in less familiar or unfamiliar problem situations.

For a statement, after choosing a method using one to four of the five strategies, the students engaged in monitoring activities to evaluate whether they had chosen the right or appropriate method as they were observing their proving or disproving work with the selected method. The monitoring activities continued until they reached their goal of proving or disproving the statement with their selected method. When they found that the selected method did not work well, the decision-making process began anew, and the students chose another method using the strategies again. The monitoring activities played an important role in helping the student participants make successful proof-method decisions. Schoenfeld (2013) saw this metacognitive aspect of activities as “a major component of decision making” (p. 19).

Why Students Chose a Particular Method

To understand why the student participants decided to use a particular method over others for a mathematical statement, focusing on the parts of the interview data in which they provided rationales for proof-method decisions, I found 8 constructs that influenced their decisions across the three studies (5 constructs were found in the first and third study, and 7 constructs were found in the second study; see Table 13). Many of these constructs are reflected in the decision-making literature across disciplines (Juliusson, Karlsson, & Garling, 2005; Pingle, 1997) and in the mathematics education literature (Schoenfeld, 1985). Across the three studies, the *knowledge* and *orientations* constructs were commonly observed and played an important role in the students’ proof-method decision making for mathematical statements. These findings align with Schoenfeld’s (2011) claim that these two constructs were significant components of decision making. However, Schoenfeld also considered a goal that a decision maker would establish given a situation as a construct that would contribute to decision making and explained that three constructs—the knowledge, orientations, and goal constructs—interacted with one another when making a decision. In the present three studies, given problem statements, the student participants first established the goal (sometimes, as a subgoal to reach the main goal of proving or disproving the statements) of choosing a proof method by themselves (this was the case for the students in the first study) or by prompt (this was the case for the students in the second and third studies purposefully designed to study students' proof-method decision making) and, with that goal in mind, they brought several types of knowledge and orientations to the fore when they explored the task statements. However, since the goal had little effect at the moments of the students' proof-method decisions, the goal construct was not considered as a deciding factor for the students' proof-method decision making in the present studies. The findings of the three studies also indicate that there are other constructs affecting the students' decision making, in addition to Schoenfeld's three.

Table 13

Constructs That Influenced the Students' Decision Making in the Three Studies

	The First Study	The Second Study	The Third Study
Constructs	Knowledge [of] • When to use the six proof methods • How to use the six proof methods Orientations • Preference for using the easiest method or the method that took less work • Belief that a statement would be true Control	Knowledge [of] • When to use the six proof methods • How to use the six proof methods Orientations • Likelihood of success • Preference for using a method that is the most comfortable • Preference for using the easiest	Knowledge [of] • When to use the six proof methods • How to use the six proof methods • A mathematical concept involved in a statement and its related proof methods Orientations • Likelihood of success • Preference for

	Authority Impulse	method or the method that took less work • Belief that a statement would be true • Beliefs about (academic) proof in mathematics Control Authority Task Class Intervention	using the easiest method or the method that took less work • Belief that a problem statement would be true or difficult Impulse Task Intervention

The nine students' proof-method decisions across the three studies were largely influenced by their knowledge of when to use proof methods. Only on a few occasions were their decisions influenced by their knowledge of how to use proof methods. Therefore, we can see the importance of the knowledge of when to use proof methods for successful proof-method decision making. The nine students had developed this type of knowledge from their past proving experience with proof methods and with certain problem statement types. This knowledge was mostly constructed through the transition class that they were taking. Some students who took other proof-based courses before taking the transition-to-proof course seemed to have more resources available to help them decide when to use which method. The students who had more experience with methods in various problem situations possessed greater knowledge of when to use methods according to the problem situations. Although the focus of his study was not on proof methods, Weber (2001) emphasized that students needed to have this type of strategic knowledge—for example, when or when not to use certain theorems, strategies, and domain-specific proof techniques for success on proof construction. In Weber's study, whereas doctoral students had this strategic knowledge, undergraduates lacked that knowledge.

The present three studies showed that various types of orientations affected the students' proof-method decisions. The *orientations* construct included several subthemes. The common subthemes observed across all three studies were related to the students' problem expectations and their proof-method preferences. Their subjective expectations of proving success when using certain proof methods were observed in the second and third studies. However, their beliefs about proof in the field of mathematics were documented only in the second study. These expectations, preferences, and beliefs usually affected the students' decision making when the students saw more than two possible methods for a problem statement and were attempting to choose one of them. Yet even when there was only one option, their subjective expectations of proving success when using that particular method affected their decisions on whether to use that method or not. In this situation, the students chose that particular method when their expectations were positive. This finding showing the influence of the orientations construct on decision making also aligns with the findings of studies conducted by Ennis and her colleagues (1991, 1992). They showed that secondary school physics teachers' curriculum decisions were influenced by the teachers' beliefs or value orientations—for example, their beliefs about students and the school context.

The *impulse* construct was observed in the first and third studies. Based on observations of the nine students' decision behaviors, this construct seems to relate to personal characteristics of students. Two of the nine students sometimes made hasty decisions, not looking at problem statements carefully and quickly picking out some surface features of the statements. Such behavior led them to make inappropriate proof-method decisions. Similar decision behavior was also found in studies of hasty decision makers. Some studies (Hatfield-Eldred, Skeel, & Reilly, 2015) showed that impulsive choices are related to working memory capacity. The *control* construct affected some of the students' decision making in the present studies, as they were not able to recall an appropriate resource (knowledge of statement types associated with a particular proof method or a particular proof method itself) at an appropriate time and thus made inappropriate proof-method decisions or decisions based on uncertainty. Schoenfeld (1985) noted that how well a solver "managed" resources or strategies at his or her disposal was a main determinant of his or her success or failure at problem-solving.

The *class* construct showed that the students' proof-method decision making could be influenced by their habitual behaviors learned in a social context (mathematics classrooms). Three of the four students in the second study did not initially consider a method, proof by counterexamples, as a possible option with which they could disprove a problem statement at proof-method decision moments while working on the tasks in the interview settings, because they had used that method only when their transition class instructor asked them to use it with directions such as "find counterexamples" and because they did not see those kinds of directions in the interview tasks. This learned behavior, which was the result of repeated practice with that particular method in class, seemed to become habitual behavior that caused them not to consider that method in other proving situations unless they could see similar directions. That is, the class construct showed that repeated behavior could also have a significant effect on a student's proof-method decision making. Similar findings can be found in studies about habitual behaviors and decision making conducted in other disciplines. For instance, studies by Aarts, Verplanken, and van Knippenberg (1998) and by Klöckner and Matthies (2004), in which people's decision making about travel mode choices were explored, found that habit strength influenced decisions that people made. In other words, the extent to which a particular mode of transport was habitually chosen in the past directly influenced the likelihood that that particular mode would be chosen in the future. Klöckner and Matthies noted that if people had weak habits, they tended to make more deliberate norm-based decisions.

The *authority* and *intervention* constructs showed that the students sometimes made proof-method decisions relying on knowledgeable people's (instructors' or researchers') proof-method decisions and believing that their proof-method decisions were correct. Similar decision behavior was observed in previous studies. For instance, second-year nursing major undergraduates in Baxter and Rideout's (2006) study made their decisions in clinical settings relying on the responses of the tutors, whom they believed to be knowledgeable people. Tsui's (2003) study showed that, when planning lessons, novice teachers followed rules or guidelines established by people with authority, whereas expert teachers acted with autonomy, making their own decisions. Therefore, these two *authority* and *intervention* constructs seemed to be more observable with novice decision makers. The *authority* construct also showed that a student's proof-method decision could be influenced by the student's views of teacher expectations about proofs. In the second study, one student was particularly affected by his transition course instructor in terms of what proofs would be acceptable and convincing to the instructor at his proof-method decision moments. That is, the student desired to choose a proof method with which he could make proofs that would satisfy the instructor. This behavior was similar to that of student participants in Herman's (2007) study, who considered which strategy their instructors would value the most and chose that strategy, just as the participant in the second study did. The *task* construct could be associated with the *knowledge* construct in that problem situations in which this construct affected the students' proof-method decisions were the types of problems in which the students made decisions based on their past experiences by discerning patterns of method usage and features similar to those seen in the previous task statements that they had worked on. Therefore, this construct also shows the influence of past experience with proof methods and with certain problem types on decision making. The task construct also showed that the students' habit of using a certain method only in a particular situation could be changed by having them experience a different problem situation with respect to the method that they had not experienced before.

Connections Between Decision Strategies and Constructs

In a broad sense, each of the five decision strategies seems to have to do particularly with the *knowledge* and *orientation* constructs. When using the feature-matching, comparison, and elimination strategies, the students focused on features of the problem statements that they recognized. Depending on how closely the features of the problem statements matched features of statement types associated with proof methods stored in their minds, they used one of the three strategies to find methods for the statements. The features of statement types associated with proof methods were a part of their knowledge of when to use the proof methods. In this sense, these three strategies are related to the *knowledge* construct, particularly its subtheme—knowledge of when to use proof methods.

When making proof-method decisions using the mental simulation and comparison strategies, the students chose a proof method with which they expected that they could successfully prove a problem statement, with which they could easily prove the statement with less work, make a proof that would be acceptable in the field of mathematics, or prove the statement comfortably without any trouble. Such expectations seemed to be based on the orientations (beliefs and preferences about proof and proof methods) that the students had. Thus, in this sense, the mental simulation and comparison strategies are related to the *orientation* construct. However, the mental simulation strategy is also related to the *knowledge* construct, particularly regarding its subtheme—knowledge of how to use proof methods—in that when choosing a method using the mental simulation strategy, the students performed mental simulations of proving with a method that they were considering for a problem statement to judge the feasibility of using that method to reach their goals of proving or disproving the problem statement. Therefore, this strategy required that the students know how to use that method to proceed with the mental proving simulation. Without that knowledge (regardless of whether the knowledge was fragile), the students could not use this strategy.

The exploration strategy is also related to the *knowledge* construct, particularly with respect to its subtheme—knowledge of when to use proof methods. When using this strategy, after confirming whether a problem statement was true or false and using their knowledge of when to use proof methods, the students narrowed down their options. If the statement turned out to be true, they considered only methods with which they could prove the statement as possible options. In the reverse situation, they considered only methods with which they could disprove the statement as possible options. However, the exploration strategy is also related to the *orientation* construct, particularly with respect to its subtheme—belief that a statement would be true. The students who approached the problems with a belief that the problem statements were going to be true did not use the exploration strategy at the beginning when looking for methods after reading the statements. These students used the exploration strategy only when they found that the statements might be false in attempting to prove with initially selected methods. However, the students who came to the problems with some suspicion that the statements might be false chose a method using the exploration strategy at the beginning along with other strategies. Therefore, *overall*, the five strategies are all concerned with the *knowledge* construct.

But the mental simulation, exploration, and comparison strategies are also related to the *orientations* construct.

Proof-Method Decision-Making Model

Based on how the participants, who were novice provers, made proof-method decisions, I constructed a diagram illustrating the decision-making process with decision strategies and constructs (see Figure 1). The diagram explains the decision-making process as follows. A student examines a problem statement, chooses a proof method using one to five decision strategies, and monitors whether the method allows him or her to reach the goal of proving or disproving the statement. If the method does not work, this decision making cycle repeats. If the method seems to work, the student continues proving or disproving with that method. If the student recognizes that an additional method might be needed for a statement deduced from the original statement in proving, decision making also reoccurs at that time. When choosing a method using the strategies that the student is able to come up with, the eight constructs affect not only his or her strategy selection but also overall proof-method decision making.

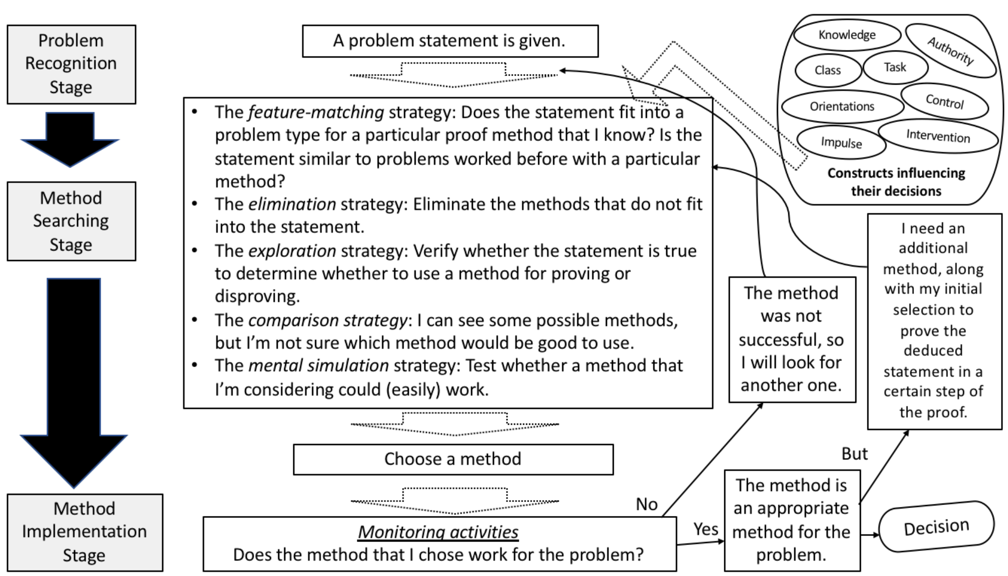

Figure 1. Novice provers' proof-method decision model with strategies and constructs.

Limitations

There are several limitations that could affect the conclusions of the present studies. The first is generalizability. I conducted the three studies with a small number of transition-to-proof course students. Further studies using a larger number of students are needed to confirm the decision strategies and constructs that emerged in the three studies. The second limitation is related to methods. I identified the students' proof-method decision strategies and the constructs that influenced their choice of methods using their "think-aloud" verbal reports and my observations of their behaviors while the students were engaging in proof-method decision-making activities with the tasks. However, the students may not have included everything that they thought about before making their decisions regarding which method to use when reporting their thought processes. Also, I judged the students' level of familiarity with the problem tasks based on their reactions to the tasks when they were reading the task statements. It might have been good to ask the students to describe their familiarity with the tasks using a scale of 0 through 10 (or 100) to more accurately determine whether they were less familiar or unfamiliar with the problem situations faced.

Implications

Implications from the results of the three studies. Overall the three studies showed that the knowledge and orientation constructs made the most significant contribution to the students' proof-method decisions and that those constructs also seemed to relate to their proof-method decision strategies. If instructors want to understand why their students make a decision to use a particular proof method for a given statement, they could ask them about their knowledge and orientations with respect to proof methods to be able to better understand their proof-method decision behaviors. However, since the occasions when the orientation construct influenced the proof-method decisions were when the knowledge construct also influenced the proof-method decisions along with the orientation construct, it seems that the knowledge construct had the greatest influence on students' proof-method decisions. Since the students constructed their knowledge of when to use which method based on their prior proving experience with methods, I can also say that what past proving experience with proof methods students have is crucial for their future proof-method decisions in the context of proof construction.

In the first and second study, when I presented students problem statements to work on, for each statement, my initial directions asked the students to read the statement and prove it. Most of the student participants in the first two studies did not initially consider proof methods with which they could disprove statements, believing that given statements would be true because of the prompt. Such behavior seemed to relate to how they were taught proof methods in their transition class. In class, problems in which they were asked to prove statements involved problem statements that were true and which required the students to use one or two of the five proof methods (There were a few exercise problems assigned asking students to prove a statement in multiple ways using direct proof, proof by contrapositive, or proof by contradiction.), excluding the counterexample method. For problems requiring the use of the counterexample method, the textbook authors used the prompt "disprove the statement" or "show that the statement is false." Their class instructors also used a similar prompt in class which asked them to use the counterexample method. Therefore, the participants in the first two studies used the counterexample method only when such directions were given and when that was their experience with that method. Whereas the first instructor assigned homework problems for a chapter of the textbook titled "Prove or Disprove," where students were required to consider all of the six proof methods for proving or disproving statements as possible methods, the second instructor skipped that chapter. Thus, the participants in the first study had a little more experience with situations in which they needed to consider all of the methods. However, overall, because of repeated practice using the counterexample method only when the above directions were given and using other methods when asked to prove the statement, most of the participants in both of the first two studies did not consider the counterexample method initially at proof-method decision moments in interview settings. However, in the third study, when presenting problem statements to student participants in the interviews, I provided the directions “prove or disprove the following statement.” Unlike participants in the first two studies, two student participants in the third study considered all six proof methods as possible methods for statements at their proof-method decision moments. These findings tell us at least two things. First, if a study is designed to explore students’ decision making about proof methods and if the study aims to have students consider all proving and disproving proof methods as possible options for statements, the use of the problem prompt “prove or disprove the following statement” will allow the researcher to reach the goal of study. Second, transition-to-proof course instructors might need to consider that if they teach proof by counterexample with a problem type such as “disprove the statement” or “provide counterexamples,” their students will most likely not consider this method for use in other problem situations. This approach might prevent students not only from expanding their ability to verify the truth or falsity of a statement by themselves but also from increasing their use of the counterexample method.

Suggestions for course. Across the three studies, the students’ proof-method decisions were successful for the most part. Based on this finding, I can say that teaching proof methods with multiple examples helps students build their knowledge of when to use proof methods and develop their ability to choose an appropriate method according to problem situations.

Papadopoulos' (2016) study also showed the effectiveness of using worked examples on students' learning of proof methods and on building their strategic knowledge of when to use a particular proof method. However, I observed that many of the student participants across the three studies had a rather fragile knowledge of when to use proof by contradiction. Many of them had a limited knowledge of this method and also did not completely understand it. This observation seems to be related to the amount of past proving experience that the students had with the contradiction method. The instructors in the first two studies provided their students with various examples when teaching proof methods in lectures but provided relatively few examples when teaching proof by contradiction. Also, they did not assign many exercise problems that required the use of and practice with that method, compared to the number of problems assigned for practice with other proof methods. The instructor in the third study provided only one example when introducing proof by contradiction and did not provide students many exercise problems as well. The three studies also showed that because of the students' lack of knowledge of when to use the contradiction method, the students used that method inappropriately or had trouble making decisions about whether the contradiction method would be appropriate in a given situation. Also, some of the participants did not know how to use proof by contradiction, leading them to make incorrect decisions because they expected that the contradiction method would work out for problem statements. These results show that transition-to-proof course instructors need to spend more time on instructing proof by contradiction with enough examples and exercises to help students understand that method and build robust knowledge of when and how to use the method, enabling them to make appropriate proof-method decisions when it comes to using that method.

The National Council of Teachers of Mathematics (2000) asserts that students across all K-12 grades should be able to “select and use various types of reasoning and methods of proof” (p. 56). I think that having students prove a mathematical statement in different ways using different methods is one way to build such ability and increase their proficiency and flexibility in using multiple methods. Rittle-Johnson and Star’s (2007) study showed the effectiveness of this kind of activity on students’ learning of mathematics. They compared two groups of seventh-grade students who participated in an activity in which they were asked to compare multiple solution methods for algebra linear equation problems and who participated in an activity in which they were asked only to reflect on single-solution methods for the problems (one method per problem) and found that, after participating in these activities, the first group of students gained more procedural and conceptual knowledge of and flexibility in using multiple methods. However, to support this kind of activity with proving, teachers are required to have robust knowledge of proof and proof methods and of how to implement this type of activity in mathematics classrooms. Leikin (2009) called that type of proving activity multiple proof tasks (MPTs). She had prospective teachers engaged in MPTs and found that they were effective at developing the teachers’ knowledge and beliefs about learning and teaching of proof in classrooms. Although not all students in transition-to-proof course students are future teachers, I contend that including MPTs in a transition class when teaching proof methods not only helps develop students’ flexibility in using multiple proof methods but also gives students who are prospective teachers ideas about how to teach proof and proof methods in mathematics classrooms.

Directions for Future Research

This study yielded a research-based model for novice proof-method decision-making. Additional research can build on this model to explore both how introduction to proof courses might be improved and how students make proof-method decisions as they progress through subsequent coursework. The findings of the three studies showed that the students, who had been taught six proof methods, used 4 to 5 decision strategies and were affected by 5 to 8 constructs at their proof-method decision moments. Little is known about course instructors' expectations of how students will choose proof methods in proof construction. These expectations are important in building on this study's findings in terms of whether students actually learn to make decisions in ways aligned with their instructor's expectations. One direction for future research would investigate how instructors expect students to choose proof methods for mathematical statements?

The goals of transition-to-proof courses are important but not widely understood. However, instructors of subsequent courses tend to expect students to have an understanding of proof methods arising from this course. Thus, an implicit goal of the course is that students learn to use six proof methods. However, based on a personal conversation with the third instructor (Dr. Tait), I was able to learn that the instructor did not consider teaching proof methods as important as teaching mathematical concepts—that was his rationale for not spending too much time on teaching proof methods in his transition class. Unlike Dr. Tait, Dr. Burt and Dr. Kent, the first two instructors, spent relatively more instruction time on proof methods with a number of examples and exercises. According to my observations, overall, students in the first two studies seemed to have greater knowledge of when and how to use proof methods than did students in the third study, whose knowledge turned out to be the major influence on students' success at choosing appropriate proof methods for mathematical statements in the three studies. That is, to help students choose the right proof methods, instruction with a focus on teaching proof methods seemed to be more beneficial than instruction with a focus on teaching mathematical concepts. Future research should explore why the instructors had different perspectives on teaching proof methods and how transition-to-proof courses might be improved in order to support students' success at choosing proof methods and even further their success at constructing proofs on their own.

The results of this study apply to transition-to-proof students taught in a lecture-based format. Future research might investigate whether decision strategies that transition-to-proof course students use when making proof-method decisions are beneficial when they make proof-method decisions in proof construction activities while taking subsequent upper-level courses (e.g., abstract algebra and real analysis). Do students choose proof methods in later courses in a similar manner? If not, how do they make decisions differently and why? Additionally, I suspect that students receiving problem-based instruction might use the decision strategies that emerged in the three studies differently or that they might use different strategies. Therefore, an exploration of the similarities and differences in the proof-method decision-making behaviors of students receiving lecture-based instruction and of those receiving problem-based instruction is needed. The decision strategies and constructs from this study form a basis for future studies about student proof-method decision-making in a variety of contexts.

REFERENCES

Aarts, H., Verplanken, B., & van Knippenberg, A. (1998). Predicting behavior from actions in the past: Repeated decision making or a matter of habit? *Journal of Applied Social Psychology*, *28*(15), 1355–1374.

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. In F. Hitt, D. Holton, & P. Thompson (Eds.), *Research in collegiate mathematics education VII* (pp. 63–92). Providence, RI: American Mathematical Society.

Alcock, L., & Weber, K. (2010a). Referential and syntactic approaches to proving: Case studies from a transition-to-proof course. In F. Hitt, D. Holton, & P. Thompson (Eds.), *Research in collegiate mathematics education VII* (pp. 93–114). Providence, RI: American Mathematical Society.

Alcock, L., & Weber, K. (2010b). Undergraduates' example use in proof construction: Purposes and effectiveness. *Investigations in Mathematics Learning*, *3*(1), 1–22.

Antonini, S., & Mariotti, M. A. (2006). Reasoning in an absurd world: Difficulties with proof by contradiction. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 65–72). Prague, Czech Republic: Charles University.

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM Mathematics Education*, *40,* 401–412.

Baker, D., & Campbell, C. (2004). Fostering the development of mathematical thinking: Observations from a proofs course. *Primus*, *14*, 345–353.

Baker, J. D. (1996, April). *Students' difficulties with proof by mathematical induction*. Paper presented at the Annual Meeting of the American Educational Research Association. New York, NY.

Barnier, W., & Feldman, N. (2000). *Introduction to advanced mathematics* (2nd ed.). Upper Saddle River, NJ: Prentice Hall.

Baxter, P. & Rideout, E. (2006). Second-year baccalaureate nursing students' decision making in the clinical setting. *Journal of Nursing Education*, *45*(4), 121–127.

Beach, L. R., & Mitchell, T. R. (1978). A contingency model for the selection of decision strategies. *Academy of Management Review*, *3*(3), 439–449.

Bernoulli, D. (1738). Specimen theoriae novae de mensura sortis [Exposition of a new theory on the measurement of risk]. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, *5*, 175–192.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Brown, S. A. (2012). Making jumps: An exploration of students' difficulties interpreting indirect proofs. In S. A. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (Vol 2. pp. 7–16). Portland, OR: SIGMAA on RUME.

Brown, S. A. (2018). Are indirect proofs less convincing? A study of students' comparative assessments. *Journal of Mathematical Behavior*, *49*, 1–23.

Chartrand, G., Polimeni, A., & Zhang, P. (2013). *Mathematical proofs: A transition to advanced mathematics* (3rd ed.). Upper Saddle River, NJ: Pearson Education.

Daepp, U., & Gorkin, P. (2011). *Reading, writing, and proving: A closer look at mathematics*. S. Axler & K. A. Ribet (Eds.). New York, NY: Springer.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience.* New York, NY: Viking Penguin.

Dreyfus, T. (1990). Advanced mathematical thinking. In P. Nesher, & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 113–134). New York, NY: Cambridge University Press.

Elstein, A. S., Shulman, L. S., & Sprafka, S. A. (1978). *Medical problem solving: An analysis of clinical reasoning*. Cambridge, MA: Harvard University Press.

Ennis, C. D., & Zhu, W. (1991). Value orientations: A description of teachers' goals for student learning. *Research Quarterly for Exercise and Sport*, *62*, 33–40.

Ennis, C. D., Chen, A., & Ross, J. (1992). Educational value orientations as a theoretical framework for experienced urban teachers' curricular decision making. *Journal of Research and Development in Education*, *25*, 156–163.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: Verbal reports as data*. Cambridge, MA: MIT Press.

Fishburn, P. (1974). Lexicographic order, utilities and decision rules: A survey. *Management Science*, *20*, 1442–1471.

Galbraith, P. L. (1981). Aspects of proving: A clinical investigation of process. *Educational Studies in Mathematics*, *12*(1), 1–28.

Gilbert, W. J., & Vanstone, S. A. (2005). *An introduction to mathematical thinking: Algebra and number systems*. Upper Saddle River, NJ: Pearson Education.

Gray, C. A. (1975). Factors in students' decisions to attempt academic tasks. *Organizational Behavior and Human Performance*, *13*, 147–164.

Hammack, R. (2013). *Book of proof*. Richmond, VA: Author.

Hanusch, S. E. (2015). *The use of examples in a transition-to-proof course* (Doctoral dissertation). Retrieved from https://digital.library.txstate.edu/handle/10877/5739

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234–283). Providence, RI: American Mathematical Society.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory* (pp. 185–212). Westport, CT: Ablex.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching on learning* (pp, 805–842). Greenwich, CT: Information Age Publishing.

Hatfield-Eldred, M. R., Skeel, R. L., & Reilly, M. P. (2015). Is it random or impulsive responding? The effect of working memory load on decision-making. *Journal of Cognitive Psychology*, *27*(1), 27–36.

Hemmi, K. (2006). *Approaching proof in a community of mathematical practice* (Doctoral dissertation). Stockholm University, Stockholm, Sweden. Retrieved from: http://www.diva-portal.org/smash/get/diva2:189608/FULLTEXT01.pdf

Herman, M. (2007). What students choose to do and have to say about use of multiple representations in college algebra. *Journal of Computers in Mathematics and Science Teaching, 26*(1), 27–54.

Hsee, C. K., & Weber, E. U. (1999). Cross-national differences in risk preference and lay predictions. *Journal of Behavioral Decision Making*, *12*, 165–179.

İmamoğlu, Y., & Toğrol, A. Y. (2015). Proof construction and evaluation practices of prospective mathematics educator. *European Journal of Science and Mathematics Education*, *3*(2), 130–144.

Juliusson, E. Á, Karlsson, N, & Garling, T. (2005). Weighing the past and the future in decision making. *European Journal of Cognitive Psychology*, *17*(4), 561–575.

Klöckner, C. A., & Matthies, E. (2004). How habits interfere with norm-directed behavior: A normative decision-making model for travel model choice. *Journal of Environmental Psychology*, *24*, 319–327.

Leikin, R. (2009). Multiple proof tasks: Teacher practice and teacher education. In F. Lin, F. Hsieh, G. Hanna, & M. de Villers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education.* (Vol. 2, pp. 31–36). Taipei, Taiwan: National Taiwan Normal University.

Lin, F-L., Lee, Y-S., & Wu Yu, J-Y. (2003). Students' understanding of proof by contradiction. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Annual Meeting of the International Group for Psychology in Mathematics Education* (Vol. 4, pp. 443–449). Honolulu, HI: University of Hawaii.

Marty, R. H. (1986). Teaching proof techniques. *Mathematics in College* (Spring/Summer), 46–53.

Marty, R. H. (1991). Getting to eureka! Higher order reasoning in math. *College Teaching*, *39*(1), 3–6.

Mathematical Association of America (2015). *2015 CUPM Curriculum Guide to Majors in the Mathematical Science*. Retrieved from https://www.maa.org/sites/default/files/pdf/CUPM/pdf/CUPMguide_print.pdf

Mills, M. (2010). Which path to take? Students' proof method preferences. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education.* Raleigh, NC: SIGMAA on RUME.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*, 249–266.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Papadopoulos, D. (2016). *Transitioning to proof with worked examples* (Doctoral dissertation). Retrieved from https://idea.library.drexel.edu/islandora/object/idea%3A7041

Payne, J. W. (1976). Task complexity and contingent processing in decision making: An information search and protocol analysis. *Organizational Behavior and Human Performance*, *16*, 366–387.

Pingle, M. (1997). Submitting to authority: Its effect on decision-making. *Journal of Economic Psychology*, *18*, 45–68.

Randel, J. M., & Pugh, H. L. (1996). Differences in expert and novice situation awareness in naturalistic decision making. *International Journal of Human-Computer Studies*, *45*, 579–596.

Reid, D., & Dobbin, J. (1998). Why is proof by contradiction so difficult? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the*

Psychology of Mathematics Education (Vol. 4, pp. 41–48). Stellenbosch, South Africa: University of Stellenbosch.

Rittle-Johnson, B. & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, *99*(3), 561–574.

Rosenthal, D., Rosenthal, D., & Rosenthal, P. (2014). *A readable introduction to real mathematics.* Cham, Switzerland: Springer.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (2011). How we think: A theory of goal-oriented decision making and its educational applications. New York, NY: Routledge.

Schoenfeld, A. H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast*, *10*(1), 9–34.

Selden, A., & Selden, J. (2003). Validations of proofs written as texts: Can undergraduate tell whether an argument proves a theorem*? Journal for Research in Mathematics Education, 36*, 4–36.

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *Montana Mathematics Enthusiast*, *10*, 303–334.

Selden, J., Benkhalti, A., & Selden, A. (2014). An analysis of transition-to-proof course students' proof constructions with a view towards course redesign. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 246–259). Denver, CO: SIGMAA on RUME.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*(2), 123–151.

Selden, J., & Selden, A. (2009). Understanding the proof construction process. In F. Lin, F. Hsieh, G. Hanna, & M. de Villers (Eds.), *Proceeding of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education.* (Vol. 2, pp. 196–201). Taipei, Taiwan: National Taiwan Normal University.

Simon, H. A. (1955). A behavioral model of rational choice. *Quarterly Journal of Economics*, *69*(1), 99–118.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory course. *Journal of Mathematical Behavior*, *25*, 73–90.

Solow, D. (1982). *How to read and do proofs*. New York, NY: Wiley.

Stylianides, A. J., & Al-Murani, T. (2010). Can a proof and a counterexample coexist? Students' conceptions about the relationship between proof and refutation. *Research in Mathematics Education*, *12*(1), 21–36.

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics, 55*, 133–162.

Stylianides, A. J., & Stylianides, G. J. (2009). Proof constructions and evaluations. *Educational Studies in Mathematics*, *72*, 237–253.

Stylianides, G., Stylianides, A., & Philippou, G. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education, 10*(3), 145–166.

Stylianides, G., Stylianides, A., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.). *Compendium for Research in*

Mathematics Education (pp. 237–266). National Council of Teachers of Mathematics: Reston, VA.

Svenson, O. (1979). Process descriptions of decision making. *Organizational Behavior and Human Performance*, *23*, 86–112.

Talbert, R. (2015). Inverting the transition-to-proof classroom. *PRIMUS*, *25*(8), 614–626.

Taylor, J., & Garnier, R. (2014). *Understanding mathematical proof*. Boca Raton, FL: Chapman & Hall/CRC Press.

Thompson, D. R. (1992). *An Evaluation of a New Course in Precalculus and Discrete Mathematics* (Doctoral dissertation). University of Chicago, Chicago, IL.

Thompson, D. R. (1996). Learning and teaching indirect proof. *Mathematics Teacher*, *89*(6), 474–482.

Tsui, A. B. M. (2003). Characteristics of expert and novice teachers. In C. A. Chapelle & S. Hunston (Eds.), *Understanding expertise in teaching: Case studies of second language teachers* (pp. 22–41). Cambridge, United Kingdom: Cambridge University Press.

Tversky, A. (1972). Elimination by aspects: A theory of choice. *Psychological Review*, *79*, 281–299.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, *23*(2), 115–133.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior*, *24*, 351–360.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning*, *12*, 306–336.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.

Zardo, P., Collie, A., & Livingstone, C. (2014). External factors affecting decision-making and use of evidence in an Australian public health policy environment. *Social Science & Medicine*, *108*, 120–127.

Zsambok, C. E., Beach, L. R., & Klein, G. (1992). *A literature review of analytical and naturalistic decision making* (Prepared under contract N66001-90-C-6023 for the Naval Command, Control and Ocean Surveillance Center, San Diego, CA). Fairborn, OH: Klein Associates.

APPENDIX A

Interview Protocol Used in the First Study [19]

Learning proof methods

- What types of proof methods did you learn from the course? *How did you learn them? How did your instructors teach these proof methods to students? Tell me about the class. Do you think that learning various proof techniques is important? Tell me why with your proving experience.*
- How would you describe direct proof? When can you use direct proof? When do you use direct proof versus proof by contradiction? How about proof by contradiction? *Ask the first two questions for the other four proof methods—proof by cases, proof by contradiction, proof by counterexamples, and proof by induction.*

******GO TO ACTIVITIES******

Now, I will give you several different statements. I want to know what ideas you have to prove each statement. Let's start with this statement.
For each statement, ask:

- Read and prove the statement. (For their work, I will provide pencil and paper.) Please think aloud while proving the statements.
- After proving the statement, would you explain how you proved it? Why do you think this proof method is appropriate to use for this statement? (*Ask this question right after students choose a proof method for the statement.)* Is there any other ways to prove this statement? If so, why didn't you use another possible way of proving to prove this statement?

If x is an odd integer, then $9x + 5$ is even.
If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd.
Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd.
The real number $\sqrt{3}$ is irrational.
For every positive integer n, $n^2 + 5n$ is an odd integer.
For every positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

[19] The first study focused on one particular interview conducted in a larger study. The student participants' background information, such as their majors, their past proving experiences, and the undergraduate-level mathematics courses that they had taken before taking a transition-to-proof course, was obtained in another group of interviews that were part of the larger study.

After all statements,

- Which of the proof methods that you used for these statements do you like the best and the least? Why?
  If there are students who are majoring in mathematics education, ask them which proof methods they think students should learn in school mathematics?

Closing:
Any other questions you want me to ask? Thank you for coming in for this interview. Have a great day!

APPENDIX B

Two Interview Protocols Used in the Second Study[20]

Interview 1

- What types of proof methods did you learn in this class?
- What do you think about proof methods in general?
- How would you explain each proof method?
- When learning proof methods, what difficulties did you have? Why did it not make sense to you?
- Among those proof methods, which method do you prefer? Why?
- When can you use this proof method? Can you give me an example of when this proof method would possibly be used?

****** GO TO ACTIVITIES******

Now, I will give you several different statements. I want you know what ideas you have to prove each statement. Let's start with this statement.

For each statement, ask:

- Read and prove the statement. (For their work, I will provide pencil and paper.) Please think-aloud while proving these statements.

After proving each statement,

- Would you explain how you proved it?
- Why do you think this proof method is appropriate to use for this statement? (*Ask this question right after students choose a proof method for the statement.)*
- Are there any other possible ways to prove this statement? If so, why did you choose this proof method over other proof methods to prove this statement?

If x is an odd integer, then $9x + 5$ is even.
If $n \in \mathbb{Z}$, then $n^2 - n + 1$ is odd.
Let $x \in \mathbb{Z}$. If $5x - 7$ is even, then x is odd.
The real number $\sqrt{5}$ is irrational.
For any positive integer $n \geq 4$, $2^n < n!$

[20] The student participants' background information, such as their majors, their past proving experiences, and the undergraduate-level mathematics courses that they had taken before taking a transition-to-proof course, was obtained in another set of interviews conducted during my initial study, before the second study.

After all statements,

- Which of the proof methods that you used for these statements do you like the best and the least? Why?

Interview 2

Thank you for participating in the last interview. Today, we will continue discussing proof methods and do some proving activities.

- What do you think were the main ideas of this course?
- Tell me some proving experiences that you have had in this course, both successful and unsuccessful experiences.
  If a student addresses his or her difficulties with proof methods in proving, ask
  - Can you tell me a bit more as to what problems you had in using the proof method you mentioned in your proving process?
  - In your proving experiences, have you ever encountered a problem that you were not sure which proof method should be used for proving something? If so, can you tell me the moment that this situation happened?
- Through the class, you learned various proof methods. What do you think about the roles of proof methods in proving? Is it important to know proof methods? Why?
- In the proving process, when do you usually consider which proof method is going to be used? Why do you usually consider this at that moment? What kind of proof methods do you consider at that moment?
- Can you use various proof methods to prove a mathematical statement? If so, do you usually try to prove the statement by using all possible methods, or do you choose a proof method among all possible proof methods to use?
  If a student says that he or she is the latter case, ask
  - Tell me what you mostly consider when choosing a proof method among the possible proof methods to use.
- Last time, some people mentioned about proof by counterexample. What do you think about proof by counterexample? When can you use this proof method? Can you give me an example of when this proof method would possibly be used?
- How do you decide whether you can use a direct proof?
  Ask this question again for proof by contrapositive, proof by cases, proof by counterexample, proof by contradiction, and proof by induction.

******GO TO ACTIVITIES******

We will do activities similar to those we did last time. I will ask you to prove six different statements. I am interested in which proof method you would use for each statement. Let's start with this statement.

For each statement, *ask:*

- Read and prove the statement. (For their work, I will provide a pencil and paper.) Please think-aloud while proving the statement.

After proving each statement,

- Would you explain how you proved it?
- Why do you think this proof method is appropriate to use for this statement? (*Ask this question right after students choose a proof method for the statement.)* When did you

choose this proof method in your proving process? Can you tell me your thought process in selecting this proof method?
If a student is not sure what proof method could be used to prove the statement, leading to not completing the proof for the statement, ask
    - What proof methods are you considering to prove this statement? Why?
- Are there any other possible ways to prove this statement? If so, why did you choose this proof method over other proof methods to prove this statement? What factors made you not use other proof methods to prove this statement? Can you also try to prove it with a different proof method?

For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3.
For every positive integer n, $n^2 + 3n$ is an odd integer.
For every nonnegative integer n, $7|3^{2n} - 2^n$
Let $n \in \mathbb{Z}$. If $3n - 5$ is even, then $5n + 4$ is odd.
Let $n \in \mathbb{Z}$. If $n^2 \not\equiv n \ (mod\ 3)$, then $n \equiv 2 \ (mod\ 3)$.
Let $n \in \mathbb{Z}$. If $4|(n^2 - 1)$, then $4|(n - 1)$.

After all statements,
- Which of the proof methods that you used for these statements do you like the best and the least? Why?

Closing:
Thank you for coming in for this interview. Have a great day!

APPENDIX C

Two Interview Protocols Used in the Third Study

Interview 1

Open Interview – Thank you for coming today and participating in this study. Today, we're going to talk about your proving experiences with proof methods in MATH 3200 and do some proving activities. Before starting the interview, there is one thing that I want to ask you. As I'm going to be talking to other participants as well over the next few days, I'd appreciate it if you didn't talk to anyone about what I ask you today until everyone has had the chance to be interviewed. That way, no one will have the opportunity to think about these things ahead of time.

- Tell me a little bit about yourself, including what you are majoring in now or in what you are planning to major.
- Why did you take MATH 3200? What did you expect to learn from MATH 3200?
  If there is a student retaking this course, *ask*
  - What problems did you have when you took this course before?
- What undergraduate mathematics classes have your taken before MATH 3200?
- Are there undergraduate-level mathematics courses in which you have had to prove mathematical statements? If so, tell me about your proving experiences.
  If a student addresses a proof method when talking about his or her proving experiences, ask
  - How did you learn the proof method from that course?
  - What other proof methods did you learn from that course?
- What types of proof methods did you learn in MATH 3200?
- What do you think about proof methods in general? What are the roles of proof methods?
- How would you describe direct proof? When can you use this method? Can you give me some examples of when this proof method would possibly be used? You can create your own statements, which would be proved using this method, if you want. When learning this method, what difficulties did you have? Why did it not make sense to you? *Ask the same questions for proof by contrapositive, proof by contradiction, proof by cases, proof by counterexample, and proof by induction.*
  After asking these questions for each of the methods, ask the following questions (I'll

  prepare cards with the written names of the six proof methods).
  - How would you rate those six proof methods in terms of their difficulty of usage in proving? Explain why you ranked them like this.
  - How would you categorize these six proof methods?

- Among the six proof methods we discussed, which methods do you like the best and the least? Why? When proving, which proof method do you prefer to use? Why?

GO TO ACTIVITIES

Let's move on to the proving activities. I'm going to give you six different statements to ask you to prove or disprove. Please think aloud while proving the statements to let me in on your thought processes so that I might see how you make proof-method decisions. Let's start with this statement. (For their work, I will provide a pencil and paper.)

Prove or disprove the following statement: The sum of any two consecutive positive integers is odd.

Prove or disprove the following statement: If m and b are real numbers with $m \neq 0$, then the function $f(x) = mx + b$ is one-to-one.

Prove or disprove the following statement: For every integer $n \geq 2$, if $x_1, \dots, x_n$ are real numbers strictly between 0 and 1, then $(1 - x_1)(1 - x_2) \cdots (1 - x_n) > 1 - x_1 - x_2 - \cdots - x_n$.

Prove or disprove the following statement: Let $x, y \in \mathbb{Z}$. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$.

Prove or disprove the following statement: The real number $\sqrt{5}$ is irrational.

Prove or disprove the following statement: For every positive integer n, $n^2 + 3n$ is an odd integer.

For each statement, ask

- Read and prove or disprove the statement.

After proving each statement,

- Do you think you proved the statement? Would you explain how you proved it? What proof method did you use to prove this statement? (*Ask this question right after students choose a proof method for the statement.)* When did you decide to use this proof method? What proof methods did you consider before deciding to prove it with this method? Why did you choose this proof method over other proof methods?

  If a student says he or she was not sure what proof method could be used to prove the statement, leading to not completing the proof for the statement, ask
  - What proof methods did you consider using to prove this statement? Why?
  - What problems did you encounter when using this method to prove?

After students complete their proofs for the statements and answer the above questions for each statement, if time permits, ask

- What other proof methods can you use to prove these statements other than the one that you used? Can you try to prove this statement using a different method?
  After a student completes the proof using the different method, ask

  - Do you think that this way of proving (his or her first proof) is better than this way (his or her second proof)? Why?
  - *Present other ways of proving the statement to the students, and ask*
    This might be another possible way to prove this statement. What do you think about this way of proving compared to your way of proving this statement?

If time permits,

Let's talk about these homework problems that you did for your class assignment. Can you explain to me how you chose a proof method when proving each problem? Why do you think the method you chose was appropriate to use for this problem? What other proof methods can you use to prove the problem other than the one that you used?

At the close of the interview – Are there any questions that you wanted me to ask that were not asked? Thank you for coming in for this interview. Have a great day!

Interview 2

Thank you for participating in this second interview. Today, we will continue discussing your proving experiences using the six proof methods – direct proof, proof by contrapositive, proof by cases, proof by contradiction, proof by induction, and proof by counterexample – and do some proving activities.

- What do you think were the main ideas of this course? Did you learn what you expected to learn from the course? Tell me what aspects of the course you found difficult. How would you evaluate your performance in this course?
- Tell me some proving experiences that you have had in this course.
  If a student addresses his or her difficulties with the proof methods in proving, ask
    - Can you tell me a bit more about what problems you had in using the proof method in your proving process?
    - In your proving experiences, have you ever encountered a problem in which you were not sure which proof method should be used for proving something? If so, can you tell me about your experiences when this situation happened?
- What learning materials (or resources) did you usually use while taking the course? Maybe, your class notes? What else?
  For a student who was often absent from the class, ask
    - Did you ask a classmate for a copy of his or her notes when you had to be absent from class? Or, did you study by yourself using the textbook to cover the lecture when you were absent?
- Through the class, you learned various proof methods. Do you think it is important to know proof methods? Why?
- Among the six proof methods (*use cards with the written names of the proof methods)*, which methods do you like the best and the least? Why? Which method do you prefer to use? Why?
- Let's talk about the homework a bit. Overall, what difficulties did you often encounter while doing your homework? How did you figure out such issues? Is there a moment in which you had an issue using the six proof methods when you did your homework?
- Consider your proving experiences. In the proving process, when do you usually consider which proof method is going to be used? Why do you usually consider this at that moment? What kind of proof methods do you consider at that moment in general? Why?
- Do you think a mathematical statement can be proved using various proof methods? Why? Do you usually try to prove the statement by using all possible methods, or do you choose a proof method among all possible proof methods to use? Why?
  If a student says that he or she is the latter case, ask
    - Tell me what you mostly consider when choosing a proof method over other possible proof methods to prove the statement.

******GO TO ACTIVITIES******

We will do activities similar to those we did last time. I will ask you to prove or disprove six different statements. Please think aloud while working on the statements to let me in on your thought processes so that I might see how you make proof-method decisions. Let's start with this statement. (For their work, I will provide a pencil and paper.)

Prove or disprove the following statement: For all $n \in \mathbb{N}$, $7 \mid 3^{2n} - 2^{n}$.

Prove or disprove the following statement: If $(A \cap B) = \emptyset$, then $A \subseteq B^{c}$.

Prove or disprove the following statement: If x is an odd integer, then $9x + 5$ is even.

Prove or disprove the following statement: The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$.

Prove or disprove the following statement: There exists an integer n such that $n^3 - n + 1$ is even.

Prove or disprove the following statement: Let $n \in \mathbb{Z}$. If $4 \mid (n^2 - 1)$, then $4 \mid (n - 1)$.

For each statement, ask

- Read and prove or disprove the statement.

After proving each statement,

- Do you think you proved the statement? Would you explain how you proved it?
- What proof method did you use to prove this statement? (*Ask this question right after students choose a proof method for the statement.)* When did you decide to use this proof method? What proof methods did you consider before deciding to prove it with this method? Why did you choose this proof method over other proof methods?

  If a student says he or she was not sure what proof method could be used to prove the statement, leading to not completing the proof for the statement, ask
    - What proof methods did you consider using to prove this statement? Why?
    - What problems did you encounter when using this method to prove?

After students complete their proofs for the statements and answer the above questions for each statement, if time permits, ask

- What other proof methods can you use to prove these statements other than the method that you used for each statement? Can you try to prove this statement using a different method?
  After a student completes the proof using the different method, ask
    - Do you think that this way of proving (his or her first proof) is better than this way (his or her second proof)? Why?
    - *Present other ways of proving the statement to the students, and ask*
      This might be another possible way to prove this statement. What do you think about this way of proving compared to your way of proving this statement?

If time permits,

Let's talk about these homework problems that you did for your class assignment. Can you explain to me how you chose a proof method when proving each problem? Why do you think the method you chose was appropriate to use for this problem? What other proof methods can you use to prove the problem other than the one that you used?

At the close of the interview – Are there any questions that you wanted me to ask that you were not asked? Thank you for coming in for this interview.

The six proof methods will be written on cards, and the cards will be used when discussing the six methods during the interviews. Also, students will be asked to rank their proof method preferences using the cards.

Direct Proof	**Proof by Contradiction**
Proof by Cases	**Proof by Contrapositive**
Proof by Induction	**Proof by Counterexample**