Tunneling in low-dimensional and strongly correlated electron systems

by

Kelly R. Patton

(Under the direction of Michael R. Geller)

Abstract

It is well known that the tunneling density of states has anomalies (cusps, algebraic suppressions, and pseudogaps) at the Fermi energy in a wide variety of low-dimensional and strongly correlated electron systems. We propose that the origin of these anomalies is the infrared catastrophe associated with the sudden introduction of a new electron into a conductor during a tunneling event. A nonperturbative theory of the electron propagator is developed to correctly account for this infrared catastrophe. The method uses a Hubbard-Stratonovich transformation to decouple the electron-electron interactions, subsequently representing the electron Green's function as a weighted functional average of noninteracting Green's functions in the presence of space- and time-dependent external potentials. The field configurations responsible for the infrared catastrophes are then treated using methods developed for the x-ray edge problem.

Index words: Tunneling, infrared catastrophe, correlated electrons

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Dedication

To my parents.

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Chapter 1

Overview

In the conventional many-body theory treatment of tunneling, the low-temperature tunneling current between an ordinary metal and a strongly correlated electron system is controlled by the single-particle density of states (DOS) of the correlated system. Tunneling experiments are therefore often used as a probe of the DOS. In a wide variety of low dimensional and strongly correlated electron systems, including all 1D metals, the 2D diffusive metal, the 2D Hall fluid, and the edge of the sharply confined Hall fluid, the DOS exhibits anomalies such as cusps, algebraic suppressions, and pseudogaps at the Fermi energy. We proposed that the origin of these anomalies is the infrared catastrophe caused by the response of the host electron gas to the sudden introduction of a new particle that occurs during a tunneling event [1]. The infrared catastrophe is a singular screening response of a degenerate Fermi gas to a localized potential applied abruptly in time.

In the following chapters we will present a plausibility argument establishing the connection between the infrared catastrophe and tunneling into low-dimenstional and strongly correlated conductors. We then proceed to incorporated infrared catastrophe physics into a calculation of the tunneling DOS (or more precisely a Green's function) for a wide variety of systems. We do this in two stages. After a Hubbard-Stratonovich transformation of the interacting Green's function we single out the auxiliary field(s) that are responsible for the infrared catastrophe. In Chapters 1 and 2 we treat the most important of these fields in an approximate and exact way, obtaining qualitatively good results. In Chapter 3 we go beyond this so called "x-ray edge limit" to include a wider set of the Hubbard-Stratonovich field, obtaining quantitatively correct results.

Chapter 2

Tunneling and the infrared catastrophe

2.1 Introduction

The tunneling density of states (DOS) is known to exhibit spectral anomalies such as cusps, algebraic suppressions, and pseudogaps at the Fermi energy in a wide variety of low-dimensional and strongly correlated electron systems, including

(i) all 1D metals;[2, 3, 4, 5, 6, 7, 8]

(ii) the 2D diffusive metal;[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]

(iii) the 2D Hall fluid;[21, 22, 23, 24, 25, 26, 27, 28, 29, 30]

(iv) the edge of the confined Hall fluid.[31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63]

In this chapter we propose a unified explanation for these anomalies, develop a nonperturbative functional-integral formalism for calculating the electron propagator that captures the essential low-energy physics, and apply a simplified approximate version of the method to cases (i) and (iii).

We claim that the physical origin of the DOS anomalies in the above systems is the infrared catastrophe of the host electron gas caused by the sudden perturbation produced by an electron when added to the system during a tunneling event. This infrared catastrophe, a singular screening response of a conductor to a localized potential turned on abruptly in time caused by the large number of electron-hole pairs made available by the presence of a sharp Fermi surface, is known to be responsible for the singular x-ray optical and photoemission spectra of metals, [64, 65, 66] Anderson's related orthogonality catastrophe, [67, 68] and the Kondo effect [69, 70]. To understand the connection to tunneling, imagine the tunneling electron being replaced by a negatively charged, distinguishable particle with mass M. In the $M \to \infty$ limit, the potential produced by the tunneling particle is identical—up to a sign—to the abruptly turned-on hole potential of the x-ray edge problem, and an infrared catastrophe of the host electrons would be expected.

Tunneling of a real, finite-mass electron is different because it recoils, softening the potential produced. However, in the four cases listed above, there is some dynamical effect that suppresses recoil and produces a potential similar to that of the infinite-mass limit: In case (i) the dimensionality of the system makes charge relaxation slow; In case (ii), the disorder suppresses charge relaxation; In case (iii), the Lorentz force keeps the injected charge localized; and case (iv) is a combination of cases (i) and (iii). It is therefore reasonable to expect remnants of the infinite-mass behavior.

Our analysis proceeds as follows: We use a Hubbard-Stratonovich transformation to obtain an exact functional-integral representation for the imaginary-time Green's function

$$G(\mathbf{r}_{\rm f}\sigma_{\rm f}, \mathbf{r}_{\rm i}\sigma_{\rm i}, \tau_0) \equiv -\big\langle T\psi_{\sigma_{\rm f}}(\mathbf{r}_{\rm f}, \tau_0)\bar{\psi}_{\sigma_{\rm i}}(\mathbf{r}_{\rm i}, 0)\big\rangle_H, \tag{2.1}$$

and identify a "dangerous" scalar field configuration $\phi_{\rm xr}(\mathbf{r}, \tau)$ that causes an infrared catastrophe. Assuming $\tau_0 \geq 0$ and an electron-electron interaction potential $U(\mathbf{r})$,

$$\phi_{\rm xr}(\mathbf{r}, \tau) \equiv U\big[\mathbf{r} - \mathbf{R}(\tau)\big]\Theta(\tau)\Theta(\tau_0 - \tau), \tag{2.2}$$

where $\mathbf{R}(\tau) \equiv \mathbf{r}_{\rm i} + (\frac{\mathbf{r}_{\rm f}-\mathbf{r}_{\rm i}}{\tau_0})\tau$ is the straight-line trajectory connecting $\mathbf{r}_{\rm i}$ to $\mathbf{r}_{\rm f}$ with velocity $(\mathbf{r}_{\rm f} - \mathbf{r}_{\rm i})/\tau_0$. For the case of the tunneling DOS at point $\mathbf{r}_0$, we have $\mathbf{r}_{\rm i} = \mathbf{r}_{\rm f} = \mathbf{r}_0$ and

$$\phi_{\rm xr}(\mathbf{r}, \tau) = U(\mathbf{r} - \mathbf{r}_0)\Theta(\tau)\Theta(\tau_0 - \tau), \tag{2.3}$$

which is the potential that would be produced by the added particle in (2.1) if it had an infinite mass. Fluctuations about $\phi_{\rm xr}$ account for the recoil of the finite-mass tunneling electron. In this chapter we introduce an approximation where the fluctuations about $\phi_{\rm xr}$

are entirely neglected, which we shall refer to as the extreme x-ray edge limit. We expect the x-ray edge limit to give qualitatively correct results in the systems (i) through (iv) listed above; the effect of fluctuations will be addressed in Chapter 4.

The x-ray edge limit of our formalism can be implemented exactly, without further approximation, only for a few specific models. These include the 1D electron gas and 2D spin-polarized Hall fluid, both having a short-range interaction of the form

$$U(\mathbf{r}) = \lambda \delta(\mathbf{r}). \tag{2.4}$$

Here we will instead carry out an approximate but more generally applicable analysis of the x-ray edge limit for these two models, by resumming a divergent perturbation series caused by the infrared catastrophe. The exact implementation of the x-ray edge limit for these cases will be presented in Chapter 3.

2.2 Formalism

We consider a D-dimensional interacting electron system, possibly in an external magnetic field. The grand-canonical Hamiltonian is

$$\begin{aligned} H &= \sum_\sigma \int d^D r \, \psi_\sigma^\dagger(\mathbf{r}) \left[\frac{\Pi^2}{2m} + v_0(\mathbf{r}) - \mu_0 \right] \psi_\sigma(\mathbf{r}) \\ &+ \tfrac{1}{2} \sum_{\sigma\sigma'} \int d^D r \, d^D r' \, \psi_\sigma^\dagger(\mathbf{r}) \, \psi_{\sigma'}^\dagger(\mathbf{r}') \, U(\mathbf{r}-\mathbf{r}') \, \psi_{\sigma'}(\mathbf{r}') \, \psi_\sigma(\mathbf{r}), \end{aligned}$$

where $\mathbf{\Pi} \equiv \mathbf{p} + \frac{e}{c}\mathbf{A}$, and where $v_0(\mathbf{r})$ is any single-particle potential energy, which may include a periodic lattice potential or disorder or both. Apart from an additive constant we can write H as $H_0 + V$, where

$$H_0 \equiv \sum_\sigma \int d^D r \, \psi_\sigma^\dagger(\mathbf{r}) \left[\frac{\Pi^2}{2m} + v(\mathbf{r}) - \mu \right] \psi_\sigma(\mathbf{r}) \tag{2.5}$$

and

$$V \equiv \tfrac{1}{2} \int d^D r \, d^D r' \, \delta n(\mathbf{r}) \, U(\mathbf{r} - \mathbf{r}') \, \delta n(\mathbf{r}'). \tag{2.6}$$

H_0 is the Hamiltonian in the Hartree approximation. The single-particle potential $v(\mathbf{r})$ includes the Hartree interaction with the self-consistent density $n_0(\mathbf{r})$,

$$v(\mathbf{r}) = v_0(\mathbf{r}) + \int d^D r' \, U(\mathbf{r}-\mathbf{r}') \, n_0(\mathbf{r}'),$$

and the chemical potential has been shifted by $-U(0)/2$. In a translationally invariant system the equilibrium density is unaffected by interactions, but in a disordered or inhomogeneous system it will be necessary to distinguish between the approximate Hartree and the exact density distributions. V is written in terms of the density fluctuation $\delta n(\mathbf{r}) \equiv \sum_\sigma \psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r}) - n_0(\mathbf{r})$.

The Euclidean propagator (2.1) can be written in the interaction representation with respect to H_0 as

$$\begin{aligned} G(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0) &= -\frac{\langle T\psi_{\sigma_\mathrm{f}}(\mathbf{r}_\mathrm{f}, \tau_0)\bar{\psi}_{\sigma_\mathrm{i}}(\mathbf{r}_\mathrm{i}, 0)e^{-\int_0^\beta d\tau\, V(\tau)}\rangle_0}{\langle T e^{-\int_0^\beta d\tau\, V(\tau)}\rangle_0} \\ &= \mathcal{N} \int D\mu[\phi] \; g(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0|\phi), \end{aligned} \tag{2.7}$$

where

$$D\mu[\phi] \equiv \frac{D\phi \, e^{-\frac{1}{2}\int \phi U^{-1}\phi}}{\int D\phi \; e^{-\frac{1}{2}\int \phi U^{-1}\phi}} \quad \text{and} \quad \int D\mu[\phi] = 1. \tag{2.8}$$

Here $\mathcal{N} \equiv \langle T \exp(-\int_0^\beta d\tau \, V)\rangle_0^{-1}$ is a τ_0-independent constant, and

$$g(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0|\phi) \equiv -\big\langle T\psi_{\sigma_\mathrm{f}}(\mathbf{r}_\mathrm{f}, \tau_0)\bar{\psi}_{\sigma_\mathrm{i}}(\mathbf{r}_\mathrm{i}, 0) \, e^{i\int_0^\beta d\tau \int d^D r \, \phi(\mathbf{r},\tau)\, \delta n(\mathbf{r},\tau)}\big\rangle_0 \tag{2.9}$$

is a correlation function describing noninteracting electrons in the presence of a purely *imaginary* scalar potential $-i\phi(\mathbf{r}, \tau)$.

Next we deform the contour of the functional integral by making the substitution $\phi \rightarrow i\phi_\mathrm{xr} + \phi$, leading to [71]

$$G(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0) = \mathcal{N} e^{\frac{1}{2}\int \phi_\mathrm{xr} U^{-1}\phi_\mathrm{xr}} \int D\mu[\phi] \, e^{-i\int \phi U^{-1}\phi_\mathrm{xr}} \, g(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0|i\phi_\mathrm{xr} + \phi). \tag{2.10}$$

This is an exact representation for the interacting Green's function, where ϕ now describes the *fluctuations* of the Hubbard-Stratonovich field about ϕ_xr.

2.3 DOS in the x-ray edge limit

As explained above, our method involves identifying a certain field configuration $\phi_{\rm xr}$ that would be the potential produced by the tunneling particle if it had an infinite mass. Apart from a sign change, this potential is the same as that caused by a localized hole in the valence band of an optically excited metal. As is well known from work on the x-ray edge singularities, such fields cause an infrared catastrophe in the screening response. Fluctuations about $\phi_{\rm xr}$ account for the recoil of a real, finite-mass tunneling electron.

In this section we introduce an approximation where these fluctuations are entirely neglected, which we shall refer to as the extreme x-ray edge limit. The approximation can itself be implemented in two different ways, perturbatively in the sense of Mahan [64] and exactly in the sense of Nozières and De Dominicis [65]. In Sec. 2.6, we apply the perturbative x-ray edge method to the 1D electron gas, including spin, and to the 2D spin-polarized Hall fluid. The x-ray edge method will be implemented exactly for these models in Chapter 3 . In the following analysis we assume $\tau_0 \geq 0$.

In the x-ray edge limit, we ignore fluctuations about $\phi_{\rm xr}$ in $g({\bf r}_{\rm f}\sigma_{\rm f}, {\bf r}_{\rm i}\sigma_{\rm i}, \tau_0 | i\phi_{\rm xr} + \phi)$, approximating it by $g({\bf r}_{\rm f}\sigma_{\rm f}, {\bf r}_{\rm i}\sigma_{\rm i}, \tau_0 | i\phi_{\rm xr})$ [72]. Then we obtain, from Eq. (2.10),

$$G({\bf r}_{\rm f}\sigma_{\rm f}, {\bf r}_{\rm i}\sigma_{\rm i}, \tau_0) = \mathcal{N} g({\bf r}_{\rm f}\sigma_{\rm f}, {\bf r}_{\rm i}\sigma_{\rm i}, \tau_0 | i\phi_{\rm xr}), \tag{2.11}$$

where, according to (2.9),

$$g({\bf r}_{\rm f}\sigma_{\rm f}, {\bf r}_{\rm i}\sigma_{\rm i}, \tau_0 | i\phi_{\rm xr}) = -\langle T \psi_{\sigma_{\rm f}}({\bf r}_{\rm f}, \tau_0) \bar{\psi}_{\sigma_{\rm i}}({\bf r}_{\rm i}, 0) \, e^{-\int \phi_{\rm xr} \, \delta n} \rangle_0. \tag{2.12}$$

Eqs. (2.11) and (2.12) define the interacting propagator in the x-ray edge limit. The local tunneling DOS at position ${\bf r}_0$ is obtained by setting ${\bf r}_{\rm i} = {\bf r}_{\rm f} = {\bf r}_0$ and $\sigma_{\rm i} = \sigma_{\rm f} = \sigma_0$, and summing over σ_0.

2.4 Infrared catastrophe and divergence of perturbation theory

To establish the connection between Eqs. (2.11) and (2.12) and the x-ray edge problem, we first calculate the tunneling DOS at $\mathbf{r}_0$ by evaluating (2.12) perturbatively in $\phi_{\rm xr}$,

$$
\begin{aligned}
&g(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 | i\phi_{\rm xr}) = G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0) \\
&+ \int d^D r_1 \, d\tau_1 \, \phi_{\rm xr}(\mathbf{r}_1, \tau_1) \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_1\sigma_0, \tau_0 - \tau_1) \, G_0(\mathbf{r}_1\sigma_0, \mathbf{r}_0\sigma_0, \tau_1) \\
&+ \int d^D r_1 \, d^D r_2 \, d\tau_1 \, d\tau_2 \, \phi_{\rm xr}(\mathbf{r}_1, \tau_1) \, \phi_{\rm xr}(\mathbf{r}_2, \tau_2) \\
&\times \bigg[G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_1\sigma_0, \tau_0 - \tau_1) \, G_0(\mathbf{r}_1\sigma_0, \mathbf{r}_2\sigma_0, \tau_1 - \tau_2) \, G_0(\mathbf{r}_2\sigma_0, \mathbf{r}_0\sigma_0, \tau_2) \\
&- \tfrac{1}{2} \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0) \, \Pi_0(\mathbf{r}_1, \mathbf{r}_2, \tau_1 - \tau_2) \bigg] + O\big(\phi_{\rm xr}^3\big),
\end{aligned}
\tag{2.13}
$$

Here $G_0(\mathbf{r}\sigma, \mathbf{r}'\sigma', \tau) \equiv -\langle T \psi_\sigma(\mathbf{r}, \tau) \bar{\psi}_{\sigma'}(\mathbf{r}', 0) \rangle_0$ is the mean-field Green's function associated with H_0, and

$$
\Pi_0(\mathbf{r}, \mathbf{r}', \tau) \equiv -\langle T \delta n(\mathbf{r}, \tau) \, \delta n(\mathbf{r}', 0) \rangle_0
$$

is the density-density correlation function, given by $\sum_\sigma G_0(\mathbf{r}\sigma, \mathbf{r}'\sigma, \tau) \, G_0(\mathbf{r}'\sigma, \mathbf{r}\sigma, -\tau)$. Now we use (2.3), and assume the short-range interaction (2.4). Then Eq. (2.13) reduces to

$$
\begin{aligned}
&g(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 | i\phi_{\rm xr}) = G_0(\mathbf{r}_0\sigma, \mathbf{r}_0\sigma, \tau_0) + \lambda \int_0^{\tau_0} d\tau_1 \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 - \tau_1) \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_1) \\
&+ \lambda^2 \int_0^{\tau_0} d\tau_1 \, d\tau_2 \bigg[G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 - \tau_1) \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_1 - \tau_2) \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_2) \\
&- \tfrac{1}{2} \, G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0) \, \Pi_0(\mathbf{r}_0, \mathbf{r}_0, \tau_1 - \tau_2) \bigg] + O\big(\lambda^3\big).
\end{aligned}
\tag{2.14}
$$

We evaluate (2.14) for the 1D electron gas and the 2D Hall fluid in the large $\tau_0\epsilon_{\rm F}$ limit using the low-energy propagators of Appendix A. $\epsilon_{\rm F}$ is the Fermi energy. In the 1D electron gas case this leads to

$$
\begin{aligned}
&g(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 | i\phi_{\rm xr}) = \\
&G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0) \bigg\{ 1 - 2\lambda N_0 \ln(\tau_0\epsilon_{\rm F}) + \lambda^2 N_0^2 \bigg[2\ln^2(\tau_0\epsilon_{\rm F}) - 2\ln(\tau_0\epsilon_{\rm F}) + \frac{\pi}{2}\tau_0\epsilon_{\rm F} \bigg] + \cdots \bigg\},
\end{aligned}
\tag{2.15}
$$

where we have used the asymptotic results given in Appendix B and have kept only the corrections through order λ^2 that diverge in the large τ_0 limit. Here N_0 is the noninteracting DOS per spin component at $\epsilon_{\rm F}$ [73]. These divergences are caused by the infrared catastrophe and the associated breakdown of perturbation theory at low energies. Similarly, for the Hall fluid we find

$$g(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 | i\phi_{\rm xr}) = G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0) \bigg\{ 1 - \lambda \bigg(\frac{1-\nu}{2\pi\ell^2} \bigg) \tau_0 + \frac{1}{2} \lambda^2 \bigg(\frac{1-\nu}{2\pi\ell^2} \bigg)^2 \tau_0^2 + \cdots \bigg\}, \tag{2.16}$$

where ν is the filling factor and ℓ is the magnetic length. The divergence in this case is stronger because of the infinite compressibility of the fractionally filled Landau level at mean field level.

2.5 Perturbation series resummation

A logarithmically divergent perturbation series similar to (2.15) occurs in the x-ray edge problem, where it is known that qualitatively correct results are obtained by reorganizing the series into a second-order cumulant expansion. Here we will carry out such a resummation for both (2.15) and (2.16).

In the electron gas case this leads to

$$g(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 | i\phi_{\rm xr}) \approx G_0(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0) \bigg(\frac{1}{\tau_0 \epsilon_{\rm F}} \bigg)^{\alpha} e^{\frac{\pi}{2} \lambda^2 N_0^2 \tau_0 \epsilon_{\rm F}}, \tag{2.17}$$

where

$$\alpha \equiv 2\lambda N_0 + 2\lambda^2 N_0^2. \tag{2.18}$$

The exponential factor in (2.17), after analytic continuation, produces a negative energy shift. However, this shift depends sensitively on the short-time regularization and is not reliably calculated with our method. α is equal to the x-ray absorption/emisssion edge exponent

$$2(\delta/\pi) + 2(\delta/\pi)^2$$

of Nozières and De Dominicis [65] (including spin) for a *repulsive* potential, with $\delta \equiv \arctan(\pi\lambda N_0)$ the phase shift at $\epsilon_{\rm F}$, when expanded to order λ^2.

In the Hall case

$$g(\mathbf{r}_0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0 | i\phi_{\rm xr}) \approx \frac{\nu - 1}{2\pi\ell^2} \, e^{\gamma(\nu-1)\tau_0}, \tag{2.19}$$

where

$$\gamma \equiv \frac{\lambda}{2\pi\ell^2} \tag{2.20}$$

is an interaction strength with dimensions of energy. This time the infrared catastrophe causes a positive energy shift and no transient relaxation. The energy shift in this case is predominantly determined by long-time dynamics and *is* physically meaningful.

2.6 Applications of the perturbative x-ray edge limit

The tunneling DOS is obtained by analytic continuation. We take the zero of energy to be the mean-field Fermi energy.

2.6.1 1D electron gas

For the 1D electron gas case we find a power-law

$$N(\epsilon) = \text{const} \times \epsilon^{\alpha}, \tag{2.21}$$

where the exponent α is given in (2.18). In (2.21) we have neglected the energy shift appearing in (2.17).

The power-law DOS we obtain is qualitatively correct, although the value of the exponent certainly is not. The exponent will be modified by carrying out the x-ray edge limit exactly, and also presumably by including fluctuations about $\phi_{\rm xr}$. However, the fact that we recover the generic algebraic DOS of the Tomonaga-Luttinger liquid phase is at least consistent with our assertion that the algebraic DOS in 1D metals is caused by the infrared catastrophe.

We note that the x-ray edge approximation actually predicts a power-law DOS in clean 2D and 3D electron systems in zero field as well. However, in those cases there is no reason to expect the x-ray edge limit to be relevant: In those cases the electron recoil and hence fluctuations about $\phi_{\rm xr}$ are large.

2.6.2 2D Hall fluid

For the 2D Hall fluid we obtain

$$N(\epsilon) = \text{const} \times \delta\big(\epsilon - [1-\nu]\gamma\big). \tag{2.22}$$

The lowest Landau level is moved up in energy by an amount

$$\lambda \frac{1-\nu}{2\pi\ell^2}. \tag{2.23}$$

The DOS (2.22) is also qualitatively correct, in the following sense: The actual DOS in this system is thought to be a broadened peak at an energy of about $e^2/\kappa\ell$, with κ the dielectric constant of the host semiconductor, producing a pseudogap at $\epsilon_{\rm F}$. Of course the magnetic field dependence of the peak positions are different, but the physical system has a screened Coulomb interaction, and and here we obtain a hard "gap" of size (2.23) at $\epsilon_{\rm F}$. We speculate that the cumulant expansion gives the exact x-ray edge result for this model, but that fluctuations about $\phi_{\rm xr}$ will broaden the peak in (2.22).

2.7 Discussion

The results and (2.21) and (2.22) are consistent with our claim that the DOS anomalies in the 1D electron gas and the 2D spin-polarized Hall fluid are caused by an infrared catastrophe, similar to that responsible for the singular x-ray spectra of metals, the orthogonality catastrophe, and the Kondo effect. Anderson has gone even further and proposed that the effect we describe causes a complete breakdown of Fermi liquid theory in 2D electron systems, such as the Hubbard model, in zero field [74, 75, 76]. We are at present unable to address this question with the methods described here, which assume that the effects of fluctuations about $\phi_{\rm xr}$ are small.

In addition to providing a common explanation for a variety of tunneling anomalies, our method may provide a means of calculating the DOS in strongly correlated and low-dimensional systems, such as at the edge of the sharply confined Hall fluid investigated experimentally by Grayson *et al.*, [42] Chang *et al.*, [51] and by Hilke *et al.*, [53] where existing theoretical methods fail.

Chapter 3

Exact solutions of the x-ray edge limit

3.1 Introduction

To obtain quantitatively correct results it will be necessary to go beyond this "perturbative" x-ray edge limit. In Chapter 3 [77] we proposed and investigated a functional cumulant expansion method that includes field fluctuations away from $\phi_{\rm xr}(\mathbf{r},\tau)$, and treats field configurations close to $\phi_{\rm xr}(\mathbf{r},\tau)$ perturbatively as in Chapter 1 [1]. Although the improved method yields the exact DOS exponent for the important Tomonaga-Luttinger model, calculable by bosonization, we do not expect it to be generally exact in 1D. (Furthermore, the method fails in the presence of a strong magnetic field because of the ground state degeneracy.) In this paper we neglect fluctuations about $\phi_{\rm xr}(\mathbf{r},\tau)$ but treat that field configuration exactly (in the relevant long τ_0 asymptotic limit). This is accomplished by finding the exact low-energy solution of the Dyson equation for noninteracting electrons in the presence of $\phi_{\rm xr}(\mathbf{r},\tau)$, which we refer to as the Nozières-De Dominicis equation. We carry out this analysis for the 1D electron and 2D Hall fluid, both with short range interaction. To this end we obtain, for the first time, an exact solution of the Nozières-De Dominicis equation for the 2D electron gas in the lowest Landau level.

3.2 Formalism

In the x-ray edge limit, we ignore fluctuations about $\phi_{\rm xr}$, in which case

$$G(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0) \approx \mathcal{N} g(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0|i\phi_{\rm xr}). \tag{3.1}$$

Eq. (3.1) defines the interacting propagator in the x-ray edge limit. The local tunneling DOS at position $\mathbf{r}_0$ is obtained by setting $\mathbf{r}_\mathrm{i} = \mathbf{r}_\mathrm{f} = \mathbf{r}_0$ and $\sigma_\mathrm{i} = \sigma_\mathrm{f} = \sigma_0$, and summing over σ_0. In the remainder of this paper we will evaluate (3.1) for the 1D electron gas and the 2D Hall fluid, with a short-range interaction of the form

$$U(\mathbf{r}) = \lambda \delta(\mathbf{r}). \tag{3.2}$$

3.3 X-ray Green's function

The quantity $g(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0 | i\phi_\mathrm{xr})$ required in (3.1) is related to the Euclidean propagator,

$$G_\mathrm{xr}(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') \equiv -\frac{\langle T\psi_\sigma(\mathbf{r},\tau)\bar{\psi}_{\sigma'}(\mathbf{r}',\tau') e^{-\int \phi_\mathrm{xr}\delta n}\rangle_0}{\langle T e^{-\int \phi_\mathrm{xr}\delta n}\rangle_0},$$

according to

$$g(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0 | i\phi_\mathrm{xr}) = G_\mathrm{xr}(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}\tau_0, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}0)\, Z_\mathrm{xr}(\tau_0), \tag{3.3}$$

with

$$Z_\mathrm{xr}(\tau_0) \equiv \langle T e^{-\int_0^\beta d\tau \int d^D r\, \phi_\mathrm{xr}(\mathbf{r},\tau)\, \delta n(\mathbf{r},\tau)}\rangle_0. \tag{3.4}$$

We refer to $G_\mathrm{xr}(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau')$ as the x-ray Green's function which describes noninteracting electrons in the presence of a real-valued potential $\phi_\mathrm{xr}(\mathbf{r},\tau)$. It satisfies the Dyson equation

$$G_\mathrm{xr}(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau') = G_0(\mathbf{r}\sigma, \mathbf{r}'\sigma, \tau - \tau') + \int d^D\bar{r}\, d\bar{\tau}\; G_0(\mathbf{r}\sigma, \bar{\mathbf{r}}\sigma, \tau - \bar{\tau})\, \phi_\mathrm{xr}(\bar{\mathbf{r}}, \bar{\tau})\, G_\mathrm{xr}(\bar{\mathbf{r}}\sigma\bar{\tau}, \mathbf{r}'\sigma\tau'). \tag{3.5}$$

Here we have used that fact that the x-ray Green's function is diagonal in spin. For a calculation of the DOS we use the form (**??**), in which case (3.5) becomes

$$G_\mathrm{xr}(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau') = G_0(\mathbf{r}\sigma, \mathbf{r}'\sigma, \tau - \tau') + \lambda \int_0^{\tau_0} dt\; G_0(\mathbf{r}\sigma, \mathbf{r}_0\sigma, \tau - t)\, G_\mathrm{xr}(\mathbf{r}_0\sigma t, \mathbf{r}'\sigma\tau'), \tag{3.6}$$

where we have assumed the short-range interaction (3.2).

By using the linked cluster expansion and coupling-constant integration, Z_xr can be shown to be related to the x-ray Green's function by [65]

$$Z_\mathrm{xr}(\tau_0) = e^{n_0\lambda\tau_0} e^{-\lambda \sum_\sigma \int_0^1 d\xi \int_0^{\tau_0} d\tau\;\, G^\xi_\mathrm{xr}(\mathbf{r}_0\sigma\tau, \mathbf{r}_0\sigma\tau^+)}, \tag{3.7}$$

where $G^{\xi}_{\rm xr}(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau')$ is the solution of (3.6) with scaled coupling constant $\xi\lambda$.

There is no $\mathbf{r}_0$ dependence in the DOS for the translationally invariant models considered here and we can take $\mathbf{r}_0 = 0$.

3.4 1D electron gas

$G_{\rm xr}(0\sigma\tau, 0\sigma\tau')$ was calculated exactly in the large τ_0, asymptotic limit for the 3D electron gas in zero field by Nozières and De Dominicis [65]. Their result is actually valid for arbitrary spatial dimension D if the appropriate noninteracting DOS is used.

We take the asymptotic form of the noninteracting propagator as

$$G_0(\tau) \approx -\,\mathrm{P}\,\frac{N_0}{\tau}, \quad \text{with} \quad N_0 \equiv \frac{1}{\pi v_{\rm F}}. \tag{3.8}$$

N_0 is the noninteracting DOS per spin component at $\epsilon_{\rm F}$, and P denotes the principal part. The solution of (3.6) for this model with $\mathbf{r} = \mathbf{r}' = 0$ is

$$G_{\rm xr}(\tau_0) = -N_0 \cos(\delta_\lambda) \left[P \frac{\cos(\delta_\lambda)}{\tau_0} + \pi \sin(\delta_\lambda) \delta(\tau_0) \right] \left(\frac{a}{\tau_0} \right)^{2\delta_\lambda/\pi}, \tag{3.9}$$

and

$$Z_{\rm xr}(\tau_0) = \left(\frac{a}{\tau_0} \right)^{2(\delta_\lambda/\pi)^2}, \tag{3.10}$$

where δ_λ is the scattering phase shift of the electrons caused by the potential $\phi_{\rm xr}$ given by

$$\delta_\lambda = \arctan(N_0 \pi \lambda) \tag{3.11}$$

and a is a short time cut-off on the order of the Fermi energy.

Thus

$$G(\tau_0) \approx g(\tau_0 | i\phi_{\rm xr}) \sim \left(\frac{1}{\tau_0} \right)^{1+2\delta_\lambda/\pi+2(\delta_\lambda/\pi)^2} \tag{3.12}$$

which gives a DOS in the x-ray edge limit as

$$N(\epsilon) \sim \epsilon^{2\delta_\lambda/\pi+2(\delta_\lambda/\pi)^2}. \tag{3.13}$$

By expanding the exponent in (3.13) in powers of the coupling paramenter λ one recovers the perturbative x-ray result of Chapter 1 [1].

3.5 2D Hall fluid

Unlike the low-energy Dyson equation for the 1D electron gas, which is solvable by Hilbert transform techniques, there are no standard methods available to solve the corresponding integral equation for the Hall fluid. We were able to guess the exact analytic solution, aided by perturbation theory and by numerical studies carried out by expansion in a plane-wave basis followed by matrix inversion.

We assume the system to be spin-polarized and spin labels are suppressed. In the Landau gauge $\mathbf{A} = Bx\mathbf{e}_y$, the noninteracting propagator in the $|\tau| \gg \omega_c^{-1}$ limit is

$$G_0(\mathbf{r}, \mathbf{r}', \tau) = \Gamma(\mathbf{r}, \mathbf{r}') \, [\nu - \Theta(\tau)], \tag{3.14}$$

where ν is the filling factor satisfying $0 \leq \nu \leq 1$, and

$$\Gamma(\mathbf{r}, \mathbf{r}') \equiv \frac{1}{2\pi\ell^2} \, e^{-|\mathbf{r}-\mathbf{r}'|^2/4\ell^2} \, e^{-i(x+x')(y-y')/2\ell^2}. \tag{3.15}$$

First consider the case where $\mathbf{r}_i = \mathbf{r}_f = \mathbf{r}_0$. We can let $\mathbf{r}_0 = 0$ without loss of generality and at the origin (3.6) reduces to

$$G_{xr}(0\tau, 0\tau') = \frac{\nu - \Theta(\tau - \tau')}{2\pi\ell^2} + \gamma \int_0^{\tau_0} dt \, \big[\nu - \Theta(\tau - t)\big] \, G_{xr}(0t, 0\tau'), \tag{3.16}$$

where

$$\gamma \equiv \frac{\lambda}{2\pi\ell^2} \tag{3.17}$$

is an interaction strength with dimensions of energy.

The time arguments of $G_{xr}(0\tau, 0\tau')$ on the left side of Eq. (3.16) can assume the 12 possible orderings $k = 1, 2, \cdots, 12$ defined in Fig. 3.1; the right side produces terms with two or more different orderings $k', k'', \cdots$. We therefore seek a solution of the form

$$G_{xr}(0\tau, 0\tau') = \sum_k A_k \, W_k(\tau, \tau') \, f_k(\tau), \tag{3.18}$$

where $W_k(\tau, \tau')$ is unity if τ and τ' have ordering k and zero otherwise; an explicit form for $W_k(\tau, \tau')$ is given in Appendix C. The functions $f_k(\tau)$ are chosen to reflect the fact that

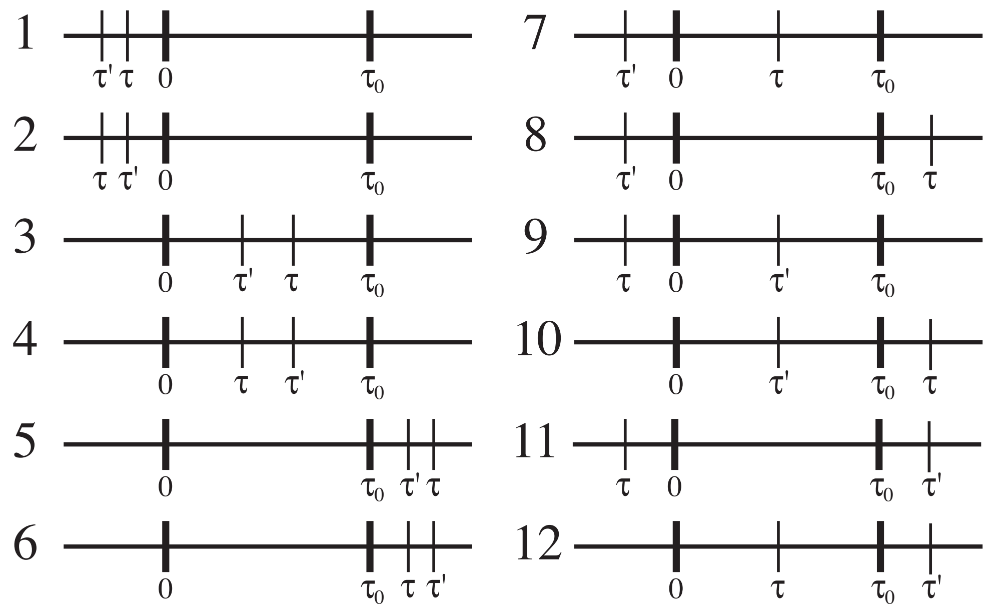

Figure 3.1: The 12 possible time orderings $k = 1, 2, \cdots, 12$ of $G_{\rm xr}(0\tau, 0\tau')$. τ_0 is assumed to be nonnegative.

an electron accumulates an additional phase $\gamma\,\Delta\tau$ while in the presence of $\phi_{\rm xr}$ for a time $\Delta\tau$, whereas a hole acquires a phase $-\gamma\,\Delta\tau$. The 12 unknown coefficients A_k (which depend parametrically on τ_0 and τ') are obtained by solving the 12 linearly independent equations resulting from the decomposition of (3.16) into distinct time orderings $k = 1, 2, \cdots, 12$. The result is

$$
\begin{aligned}
G_{\rm xr}(0\tau, 0\tau') = {} & \frac{1}{2\pi\ell^2}\left(\frac{1}{1-\nu+\nu e^{-\gamma\tau_0}}\right) \\
& \times\Big[(\nu-1)\big(W_1+W_5\big)+\nu e^{-\gamma\tau_0}\big(W_2+W_6\big)+(\nu-1)e^{-\gamma(\tau-\tau')}\,W_3 \\
& +\nu e^{-\gamma\tau_0}e^{\gamma(\tau'-\tau)}\,W_4+(\nu-1)e^{-\gamma\tau}\,W_7+(\nu-1)e^{-\gamma\tau_0}\,W_8+\nu e^{-\gamma(\tau_0-\tau')}\,W_9 \\
& +(\nu-1)e^{-\gamma(\tau_0-\tau')}\,W_{10}+\nu\,W_{11}+\nu e^{-\gamma\tau}\,W_{12}\Big].
\end{aligned} \tag{3.19}
$$

As a side note, the solution for general $\mathbf{r}_{\rm i}$ and $\mathbf{r}_{\rm f}$ is obtained using the same method and when both τ and τ' are in the interval $(0, \tau_0)$, the result is

$$
\begin{aligned}
G_{\rm xr}(\mathbf{r}\tau, \mathbf{r}'\tau') = {} & [\nu-\Theta(\tau-\tau')]\big[\Gamma(\mathbf{r},\mathbf{r}')-2\pi\ell^2\,\Gamma(\mathbf{r},0)\,\Gamma(0,\mathbf{r}')\big] \\
& +2\pi\ell^2\,\Gamma(\mathbf{r},0)\,\Gamma(0,\mathbf{r}')\left[\frac{(\nu-1)\,\Theta(\tau-\tau')+\nu e^{-\gamma\tau_0}\,\Theta(\tau'-\tau)}{1-\nu+\nu e^{-\gamma\tau_0}}\right]e^{-\gamma(\tau-\tau')};
\end{aligned} \tag{3.20}
$$

the other cases follow similarly. At the origin (3.20) reduces to

$$
G_{\rm xr}(0\tau, 0\tau') = \frac{1}{2\pi\ell^2}\left[\frac{(\nu-1)\,\Theta(\tau-\tau')+\nu e^{-\gamma\tau_0}\,\Theta(\tau'-\tau)}{1-\nu+\nu e^{-\gamma\tau_0}}\right]e^{-\gamma(\tau-\tau')}. \tag{3.21}
$$

Finally

$$
G_{\rm xr}(0\tau_0, 00) = \frac{1}{2\pi\ell^2}\left(\frac{\nu-1}{1-\nu+\nu e^{-\gamma\tau_0}}\right)e^{-\gamma\tau_0}. \tag{3.22}
$$

Using Eq. (3.7) we obtain

$$
Z_{\rm xr} = e^{\nu\gamma\tau_0}(1-\nu+\nu e^{-\gamma\tau_0}), \tag{3.23}
$$

therefore

$$
g(\mathbf{r}0\sigma_0, \mathbf{r}_0\sigma_0, \tau_0|i\phi_{\rm xr}) = \frac{\nu-1}{2\pi\ell^2}\,e^{\gamma(\nu-1)\tau_0}. \tag{3.24}
$$

This is identical to what we obtained in Chapter 1 [1]. The tunneling DOS is therefore

$$
N(\epsilon) = \text{const}\times\delta\big(\epsilon-[1-\nu]\gamma\big). \tag{3.25}
$$

3.6 Discussion

In this chapter, we have carried out an exact treatment of the x-ray edge limit introduced in Chapter 1 [1], for the same models considered there. Whereas the 1D electron gas result (3.13) would be expected, the DOS of the 2D Hall fluid remains gapped as in Chapter 1 [1]. A generalization of our method that accounts for fluctuations about $\phi_{\rm xr}$, and that can be used in a magnetic field, will be needed to recover the actual pseudogap of the Hall fluid [21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Chapter 4

Beyond the x-ray edge limit

4.1 Introduction

In the previous chapters we introduced an exact functional-integral representation for the interacting propagator, and developed a nonperturbative technique for calculating it by identifying a "dangerous" scalar field configuration of the form (2.2), and then treating this special field configuration by using methods developed for the x-ray edge problem. All other field configurations were ignored, thereby reducing the tunneling problem to an x-ray edge problem. Nonetheless, qualitatively correct results were obtained for the 1D electron gas and the 2D Hall fluid using this approach. In this chapter we attempt to go beyond this so-called x-ray edge limit by including fluctuations about $\phi_{\rm xr}$. We find that by including fluctuations through the use of a simple functional cumulant expansion, a qualitatively correct DOS is obtained for electrons with short-range interaction and no disorder in one, two, and three dimensions. We also show that and when applied to the solvable Tomonaga-Luttinger model, the low-energy fixed-point Hamiltonian for most 1D metals, the exact DOS is obtained.

4.2 General formalism and cumulant expansion method

Following Chapter 1 [1], we use a Hubbard-Stratonovich transformation to write the exact Euclidean propagator

$$G(\mathbf{r}_{\rm f}\sigma_{\rm f}, \mathbf{r}_{\rm i}\sigma_{\rm i}, \tau_0) \equiv -\big\langle T\psi_{\sigma_{\rm f}}(\mathbf{r}_{\rm f}, \tau_0)\bar{\psi}_{\sigma_{\rm i}}(\mathbf{r}_{\rm i}, 0)\big\rangle_H, \tag{4.1}$$

in the form

$$G(\mathbf{r}_{\rm f}\sigma_{\rm f}, \mathbf{r}_{\rm i}\sigma_{\rm i}, \tau_0) = \mathcal{N}\,\frac{\int D\phi\, e^{-\frac{1}{2}\int \phi U^{-1}\phi} g(\mathbf{r}_{\rm f}\sigma_{\rm f}, \mathbf{r}_{\rm i}\sigma_{\rm i}, \tau_0|\phi)}{\int D\phi\, e^{-\frac{1}{2}\int \phi U^{-1}\phi}} \tag{4.2}$$

where

$$g(\phi) \equiv -\big\langle T\psi_{\sigma_{\rm f}}(\mathbf{r}_{\rm f},\tau_0)\bar{\psi}_{\sigma_{\rm i}}(\mathbf{r}_{\rm i},0)\, e^{i\int \phi(\mathbf{r},\tau)\,\delta n(\mathbf{r},\tau)}\big\rangle_0 \tag{4.3}$$

is a noninteracting correlation function in the presence of a purely imaginary scalar potential $i\phi(\mathbf{r},\tau)$, and $\mathcal{N} \equiv \langle T \exp(-\int_0^\beta d\tau\, V)\rangle_0^{-1}$ is a constant, independent of τ_0. Eq. (4.2) is an exact expression for the interacting Green's function.

The region of function space that contributes to the functional integrals in (4.2) is controlled by the width of the Gaussian, which in the small U limit becomes strongly localized around $\phi = 0$. By expanding (4.3) in powers of ϕ and doing the functional integrals term by term, one simply recovers the standard perturbative expansion for $G(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0)$ in powers of U. Therefore, it will be necessary to go beyond a perturbative expansion for $g(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0|\phi)$. We evaluate $g(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0|\phi)$ approximately, using a second-order functional cumulant expansion. Such an expansion amounts to a resummation of the most divergent terms in the perturbation series when $\phi = \phi_{\rm xr}$ and the infrared catastrophe occurs. Indeed, one can view our resulting expression for $g(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0|\phi)$ as a functional generalization of Mahan's "perturbative" result for a similar correlation function.[64] Furthermore, for field configurations far from $\phi_{\rm xr}$, the cumulant expansion will yield a result that is, by construction, exact through second order in U. After carrying this out we obtain

$$g(\phi) \approx G_0(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0)e^{\int C_1(\mathbf{r}\tau)\,\phi(\mathbf{r},\tau)+\int C_2(\mathbf{r}\tau,\mathbf{r}'\tau')\,\phi(\mathbf{r},\tau)\,\phi(\mathbf{r}',\tau')}, \tag{4.4}$$

where

$$C_1(\mathbf{r}\tau) = \frac{g_1(\mathbf{r}\tau)}{G_0(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0)} \tag{4.5}$$

and

$$C_2(\mathbf{r}\tau,\mathbf{r}'\tau') = \frac{g_2(\mathbf{r}\tau,\mathbf{r}'\tau')}{G_0(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0)} - \frac{g_1(\mathbf{r}\tau)\,g_1(\mathbf{r}'\tau')}{2\,[G_0(\mathbf{r}_{\rm f}\sigma_{\rm f},\mathbf{r}_{\rm i}\sigma_{\rm i},\tau_0)]^2}. \tag{4.6}$$

Here

$$g_n(\mathbf{r}_1\tau_1,\cdots,\mathbf{r}_n\tau_n) \equiv -\frac{i^n}{n!}\big\langle T\psi_{\sigma_{\rm f}}(\mathbf{r}_{\rm f},\tau_0)\bar{\psi}_{\sigma_{\rm i}}(\mathbf{r}_{\rm i},0)\,\delta n(\mathbf{r}_1\tau_1)\,\cdots\,\delta n(\mathbf{r}_n\tau_n)\big\rangle_0 \tag{4.7}$$

is the coefficient of ϕ^n appearing in the perturbative expansion of (4.3), as in

$$g(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0|\phi) = G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0) + \sum_{n=1}^{\infty} \int g_n(\mathbf{r}_1\tau_1, \cdots, \mathbf{r}_n\tau_n)\, \phi(\mathbf{r}_1, \tau_1) \cdots \phi(\mathbf{r}_n, \tau_n). \quad (4.8)$$

The cumulants C_1 and C_2 in terms of G_0 are

$$C_1(\mathbf{r}\tau) = -i \sum_\sigma \frac{G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}\sigma, \tau_0 - \tau)\, G_0(\mathbf{r}\sigma, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau)}{G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0)} \quad (4.9)$$

and (suppressing spin for clarity)

$$C_2(\mathbf{r}\tau, \mathbf{r}'\tau') = \frac{1}{2G_0(\mathbf{r}_\mathrm{f}, \mathbf{r}_\mathrm{i}, \tau_0)^2} \Big\{ [G_0(\mathbf{r}, \mathbf{r}', \tau - \tau') G_0(\mathbf{r}_\mathrm{f}, \mathbf{r}_\mathrm{i}, \tau_0)$$
$$- G_0(\mathbf{r}, \mathbf{r}_\mathrm{i}, \tau) G_0(\mathbf{r}_\mathrm{f}, \mathbf{r}', \tau_0 - \tau')][\mathbf{r} \leftrightarrow \mathbf{r}', \tau \leftrightarrow \tau'] \Big\} \quad (4.10)$$

The functional integral in (4.2) can now be done exactly, leading to

$$G(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0) = \mathcal{A}(\tau_0)\; G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0)\; e^{-S(\tau_0)}, \quad (4.11)$$

where

$$\mathcal{A} \equiv \mathcal{N}\, \frac{\int D\phi\; e^{-\frac{1}{2}\int \phi (U^{-1} - 2C_2)\phi}}{\int D\phi\; e^{-\frac{1}{2}\int \phi U^{-1}\phi}} = \mathcal{N}\big[\,\det(1 - 2C_2 U)\big]^{-\frac{1}{2}}, \quad (4.12)$$

and

$$S \equiv \frac{1}{2}\int_0^\beta d\tau\, d\tau' \int d^D r\, d^D r' \rho(\mathbf{r}, \tau)\, U_\mathrm{eff}(\mathbf{r}\tau, \mathbf{r}'\tau')\, \rho(\mathbf{r}', \tau'). \quad (4.13)$$

Here

$$\rho(\mathbf{r}, \tau) \equiv -i\, C_1(\mathbf{r}\tau) = -\sum_\sigma \frac{G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}\sigma, \tau_0 - \tau)\, G_0(\mathbf{r}\sigma, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau)}{G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{f}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0)}, \quad (4.14)$$

and

$$U_\mathrm{eff}(\mathbf{r}\tau, \mathbf{r}'\tau') \equiv \big[U^{-1}(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau') - 2C_2(\mathbf{r}\tau, \mathbf{r}'\tau')\big]^{-1} \quad (4.15)$$

is a screened interaction.

Because spin-orbit coupling has been neglected in H, the noninteracting Green's function is diagonal in spin, and

$$\rho(\mathbf{r}, \tau) = -\frac{G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{i}, \mathbf{r}\sigma_\mathrm{i}, \tau_0 - \tau)\, G_0(\mathbf{r}\sigma_\mathrm{i}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau)}{G_0(\mathbf{r}_\mathrm{f}\sigma_\mathrm{i}, \mathbf{r}_\mathrm{i}\sigma_\mathrm{i}, \tau_0)}\; \delta_{\sigma_\mathrm{i}\sigma_\mathrm{f}}. \quad (4.16)$$

Eq. (4.11) is the principal result of this work.

4.3 Charge spreading interpretation

We interpret (4.11) as follows: S is the Euclidean action [78] for a time-dependent charge distribution $\rho(\mathbf{r},\tau)$. We shall show that $\rho(\mathbf{r},\tau)$ acts like a charge density associated with an electron being inserted at $\mathbf{r}_\mathrm{i}$ at $\tau=0$ and removed at $\mathbf{r}_\mathrm{f}$ at τ_0. This charge density interacts via an effective interaction $U_\mathrm{eff}(\mathbf{r}\tau,\mathbf{r}'\tau')$ that accounts for the modification of the electron-electron interaction by dynamic screening [79]. Our result can therefore be regarded as a variant of the intuitive but phenomenological "charge spreading" picture of Spivak [80] and of Levitov and Shytov [15]. However, here the dynamics of $\rho(\mathbf{r},\tau)$ is completely determined by the mean-field Hamiltonian, and has the dynamics of essentially noninteracting electrons.

First consider the integrated charge,

$$Q(\tau) \equiv \int d^D r\ \rho(\mathbf{r},\tau) = -\frac{\int d^D r\ G_0(\mathbf{r}_\mathrm{f},\mathbf{r},\tau_0-\tau)G_0(\mathbf{r},\mathbf{r}_\mathrm{i},\tau)}{G_0(\mathbf{r}_\mathrm{f},\mathbf{r}_\mathrm{i},\tau_0)}. \tag{4.17}$$

Using exact eigenfunction expansions for the Green's functions we obtain

$$\begin{aligned} Q(\tau) = &-\left[\sum_\alpha \Phi^*_\alpha(\mathbf{r}_\mathrm{f})\Phi_\alpha(\mathbf{r}_\mathrm{i})e^{-(\epsilon_\alpha-\mu)\tau_0}[n_\mathrm{F}(\epsilon_\alpha-\mu)-1].\right]^{-1}\sum_\alpha \Phi^*_\alpha(\mathbf{r}_\mathrm{f})\Phi_\alpha(\mathbf{r}_\mathrm{i})e^{-(\epsilon_\alpha-\mu)\tau_0} \\ &\times\left\{[n_\mathrm{F}(\epsilon_\alpha-\mu)-1]^2\Theta(\tau_0-\tau)\Theta(\tau)+n_\mathrm{F}(\epsilon_\alpha-\mu)[n_\mathrm{F}(\epsilon_\alpha-\mu)-1]\left[\Theta(-\tau)+\Theta(\tau-\tau_0)\right]\right\} \end{aligned} \tag{4.18}$$

where n_F is the Fermi distribution function and the Φ_α are the single-particle eigenfunctions of H_0. In the zero temperature limit

$$\begin{aligned} &Q(\tau) = \\ &-\frac{\sum_\alpha \Phi^*_\alpha(\mathbf{r}_\mathrm{f})\Phi_\alpha(\mathbf{r}_\mathrm{i})e^{-(\epsilon_\alpha-\mu)\tau_0}\left\{(N_\alpha-1)^2\Theta(\tau_0-\tau)\Theta(\tau)+N_\alpha(N_\alpha-1)\left[\Theta(-\tau)+\Theta(\tau-\tau_0)\right]\right\}}{\sum_\alpha \Phi^*_\alpha(\mathbf{r}_\mathrm{f})\Phi_\alpha(\mathbf{r}_\mathrm{i})e^{-(\epsilon_\alpha-\mu)\tau_0}(N_\alpha-1)}, \end{aligned} \tag{4.19}$$

where N_α is the ground-state occupation number of state α, which in the absence of ground state degeneracy takes the value of 0 or 1. In this case (4.19) reduces to

$$Q(\tau) = \Theta(\tau_0-\tau)\Theta(\tau). \tag{4.20}$$

When the sum rule (4.20) holds, the net added charge, as described by $\rho(\mathbf{r},\tau)$, is unity (in units of the electron charge) for times between 0 and τ_0, and zero otherwise. This behavior correctly mimics the action of the field operators in (4.1).

At short times, $\tau \ll \tau_0$, the charge density is approximately

$$\rho(\mathbf{r},\tau) \sim -G_0(\mathbf{r},\mathbf{r}_{\mathrm{i}},\tau) \tag{4.21}$$

which is localized around $\mathbf{r} = \mathbf{r}_{\mathrm{i}}$. As time evolves this distribution relaxes. Then as τ approaches τ_0 the charge density again becomes localized around $\mathbf{r} = \mathbf{r}_{\mathrm{f}}$,

$$\rho(\mathbf{r},\tau) \sim -G_0(\mathbf{r}_{\mathrm{f}},\mathbf{r},\tau_0-\tau). \tag{4.22}$$

A plot of $\rho(\mathbf{r},\tau)$ for the 1D electron gas is given in Fig. 4.1.

The dynamics of $\rho(\mathbf{r},\tau)$ can be shown to be governed by the equation of motion

$$[\partial_\tau + H_0(\mathbf{r})]\,\rho(\mathbf{r},\tau) = -\delta(\mathbf{r}_{\mathrm{f}}-\mathbf{r})\delta(\tau_0-\tau') + \delta(\mathbf{r}-\mathbf{r}_{\mathrm{i}})\delta(\tau). \tag{4.23}$$

This can be seen by noting that the noninteracting Green's function satisfies

$$[\partial_\tau + H_0(\mathbf{r})]\,G_0(\mathbf{r},\mathbf{r}',\tau,\tau') = -\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \tag{4.24}$$

Then, using the definition (4.14) one can obtain (4.23). Again, we stress that $\rho(\mathbf{r},\tau)$ is describing the dynamics of noninteracting electrons, governed by the mean-field Hamiltonian H_0.

Although $\rho(\mathbf{r},\tau)$ has many properties that make it reasonable to interpret as the charge density associated with the added and subsequently removed electron in the Green's function (4.1), one should not take this interpretation too literally. For instance, the sum rule (4.20) only holds in the zero temperature limit and in the absence of ground state degeneracy. Also, as will be seen below in Sec. 4.4.3, $\rho(\mathbf{r},\tau)$ may even be complex valued.

4.4 Applications of the cumulant method

In the following examples we will assume electrons with a short-range interaction U, no disorder, and no magnetic field. In Sec. 4.4.1 we show that our method correctly predicts a constant DOS near the Fermi energy in 2D and 3D, and in Sec. 4.4.2 we obtain a power-law DOS in 1D, in qualitative agreement with Luttinger liquid theory.[2, 3, 4, 5, 6, 7, 8] Finally, in Sec. 4.4.3 we use our method to calculate the DOS for the solvable Tomonaga-Luttinger model, obtaining the exact DOS exponent.

4.4.1 2D and 3D electron gas: Recovery of the Fermi liquid phase

The sum rule (4.20) allows one to determine the energy dependence of the DOS in D dimensions, asymptotically in the low-energy limit, as follows: In the absence of disorder, a droplet of charge injected into a degenerate Fermi gas with velocity $v_{\rm F}$ will relax to a size of order $\ell \sim v_{\rm F}\tau$ after a time τ. Approximating $\rho(\mathbf{r},\tau)$ to be of uniform magnitude in a region of size ℓ and vanishing elsewhere, the sum rule then requires the magnitude or ρ to vary as ℓ^{-D}. The interaction energy of such a charge distribution (assuming a short-range interaction) is

$$E = \frac{U}{2}\int d^D r\, [\rho(\mathbf{r})]^2 \sim \frac{U}{\ell^D}, \tag{4.25}$$

which varies with time as τ^{-D}. The action accumulated up to time τ_0 therefore scales as

$$S \sim \frac{U}{\tau_0^{D-1}} \quad \text{if} \quad D \geq 2 \tag{4.26}$$

or

$$S \sim U \ln \tau_0 \quad \text{if} \quad D = 1. \tag{4.27}$$

The cases (4.26) and (4.27) are dramatically different: In 2D and 3D the action vanishes a long times, and the propagator (4.11) is therefore not appreciably affected by interactions. The resulting DOS is energy independent at low energies, and the expected Fermi liquid behavior is recovered. In 1D, however, the action diverges logarithmically, leading to an algebraic DOS.

4.4.2 1D electron gas: Recovery of the Luttinger liquid phase

The scaling argument of the previous section showed that the DOS in the 1D electron gas with short-range interaction is algebraic, as expected. In this section we calculate the associated exponent.

We proceed in two stages. Initially we keep only the first cumulant C_1, and then afterwards we discuss the effect of C_2. In 1D it is possible to calculate the action (4.13) exactly in the long-time asymptotic limit at the first-cumulant level. Setting $x_{\rm f} = x_{\rm i} = 0$, we have [see (4.11)]

$$G(\tau_0) = \text{const} \times G_0(\tau_0)\, e^{-S(\tau_0)}. \tag{4.28}$$

Considering a local interaction of the form $U(x-x') = U_0\lambda\delta(x-x')$ the action is

$$S(\tau_0) = \frac{U_0\lambda}{2}\int_0^\beta d\tau \int_{-\infty}^{\infty} dx\, \big[\rho(x,\tau)\big]^2. \tag{4.29}$$

By linearizing the spectrum around the Fermi energy, the zero-temperature propagator at low energies is

$$G_0(x,\tau) \;=\; \frac{1}{\pi}\,\mathrm{Im}\left(\frac{e^{ik_{\rm F}x}}{x+iv_{\rm F}\tau}\right) = \frac{x\sin k_{\rm F}x - v_{\rm F}\tau\cos k_{\rm F}x}{\pi(x^2+v_{\rm F}^2\tau^2)}. \tag{4.30}$$

The charge density (4.14) in this case is

$$\rho(x,\tau) \;=\; \frac{v_{\rm F}\tau_0}{\pi}\,\frac{x\sin k_{\rm F}x - v_{\rm F}(\tau_0-\tau)\cos k_{\rm F}x}{x^2+v_{\rm F}^2(\tau-\tau_0)^2}\,\frac{x\sin k_{\rm F}x - v_{\rm F}\tau\cos k_{\rm F}x}{x^2+v_{\rm F}^2\tau^2}. \tag{4.31}$$

Fig. (4.1) shows the charge density $\rho(x,\tau)$ as it spreads in time from its initially localized position. By a lengthy but straightforward calculation it can be shown that

$$S(\tau_0) = \frac{3}{8}\frac{U_0\lambda}{v_{\rm F}\pi}\ln\left(\frac{\tau_0}{a}\right), \tag{4.32}$$

where a is a microscopic cutoff. This leads to a power-law decay of the interacting propagator as

$$G(\tau) \sim \frac{1}{\tau^\alpha}, \tag{4.33}$$

where

$$\alpha = \frac{3}{8}\frac{U_0\lambda}{v_{\rm F}\pi} + 1 \tag{4.34}$$

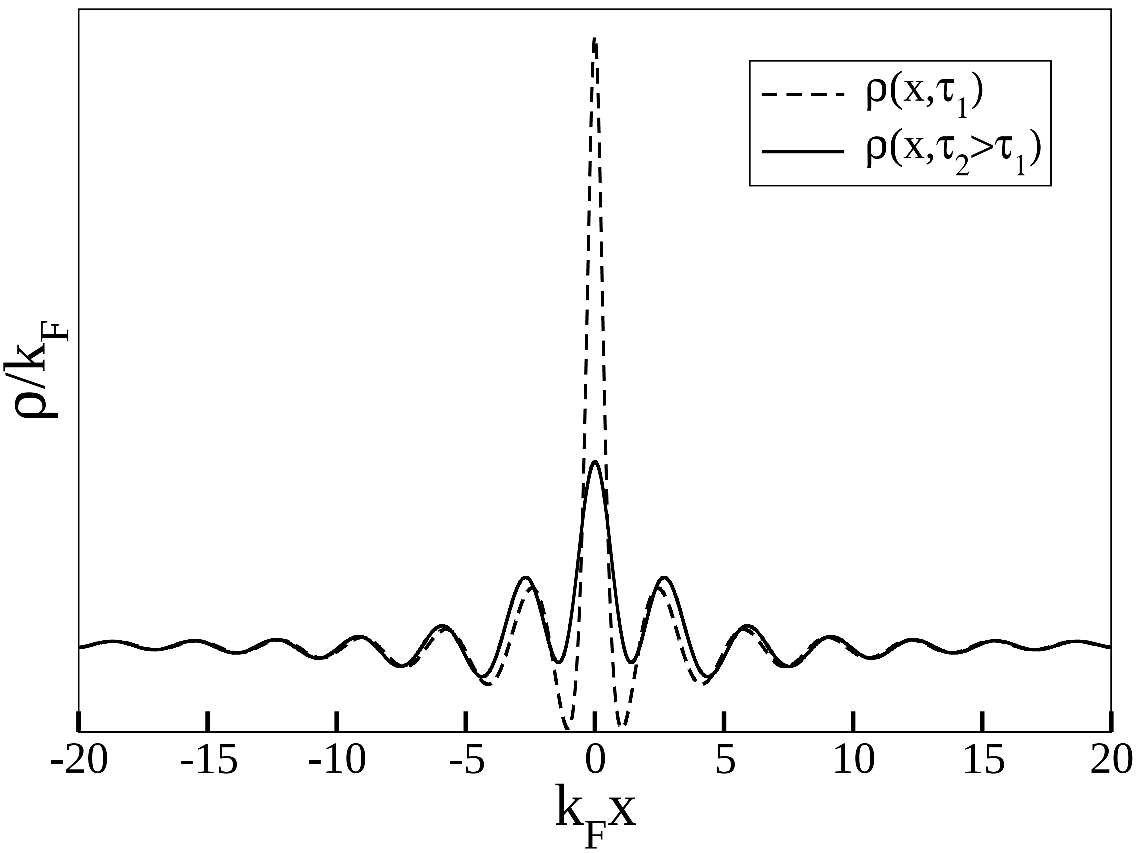

Figure 4.1: Charge density $\rho(x, \tau)$ for the 1D electron gas at two times, showing Friedel oscillations and gradual spreading.

is the propagator exponent. The DOS exponent δ, defined as

$$N(\epsilon) = \text{const} \times \epsilon^{\delta}, \tag{4.35}$$

is given in this case by

$$\delta = \frac{3}{8} \frac{U_0 \lambda}{v_{\rm F} \pi}. \tag{4.36}$$

The effects of the second cumulant C_2 are now straightforward to understand: In addition to introducing a slowly varying prefactor $\mathcal{A}$, whose only τ_0 dependence comes from the time dependence of the screening in (4.15), the second cumulant screens the bare interaction and does not prevent the logarithmic divergence of the action, but it does modify the DOS exponent. It is interesting, however, that in the large U limit, the effective interaction becomes independent of U, a clear indication of nonperturbative behavior.

It is illustrative to compare (4.36) to the prediction of the perturbative x-ray edge limit of Chapter 2, where one neglects all field configurations in (4.2) except $\phi_{\rm xr}$. There we found (for this same short-range interaction model),

$$\delta = 2 \, \frac{U_0 \lambda}{v_{\rm F} \pi} \tag{4.37}$$

to leading order in U. The DOS exponents (4.36) and (4.37) are in qualitative agreement, but the inclusion of fluctuations about $\phi_{\rm xr}$ in (4.36) softens the exponent by almost a factor of four, as one might expect.

The exact DOS exponent is not known for this model. In the next section we apply the cumulant method to the Tomonaga-Luttinger model, for which the exact propagator can be calculated using bosonization.

4.4.3 Tomonaga-Luttinger model

We consider the spinless or $U(1)$ Tomonaga-Luttinger model. The noninteracting spectrum is

$$\epsilon_k = \mu + v_{\rm F}(\pm k - k_{\rm F}), \tag{4.38}$$

where the upper sign refers to the right branch and the lower to the left one. The interaction is

$$V = \frac{1}{2}\int dx\; \delta n_i(x)\, U_{ij}\, \delta n_j(x), \tag{4.39}$$

$$\delta n_\pm(x) \equiv \lim_{a\to 0} :\psi_\pm(x+a)\psi_\pm(x):, \tag{4.40}$$

where the normal ordering is with respect to the noninteracting ground state. The matrix U has the form

$$\mathbf{U} = \begin{pmatrix} U_4 & U_2 \\ U_2 & U_4 \end{pmatrix}. \tag{4.41}$$

We want to calculate

$$G_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}) = -\mathcal{N}\,\langle T\psi_\pm(x_\mathrm{f},\tau_\mathrm{f})\bar\psi_\pm(x_\mathrm{i},\tau_\mathrm{i})\, e^{-\int d\tau V(\tau)}\rangle. \tag{4.42}$$

Make a Hubbard-Stratonovich transformation of the form

$$e^{-\frac{1}{2}\int \delta n_i\, U_{ij}\, \delta n_j} = \frac{\int D\phi_-\, D\phi_+\; e^{-\frac{1}{2}\int \phi_i U_{ij}^{-1}\phi_j}\; e^{i\int \phi_i\, \delta n_i}}{\int D\phi_-\, D\phi_+\; e^{-\frac{1}{2}\int \phi_i U_{ij}^{-1}\phi_j}}, \tag{4.43}$$

which leads to

$$G_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}) = \mathcal{N}\,\frac{\int D\phi_-\, D\phi_+\; e^{-\frac{1}{2}\int \phi_i U_{ij}^{-1}\phi_j}\;\; g_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}|\phi_-,\phi_+)}{\int D\phi_-\, D\phi_+\; e^{-\frac{1}{2}\int \phi_i U_{ij}^{-1}\phi_j}}, \tag{4.44}$$

where

$$g_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}|\phi_+,\phi_-) \equiv -\Big\langle T\psi_\pm(x_\mathrm{f}\tau_\mathrm{f})\bar\psi_\pm(x_\mathrm{i}\tau_\mathrm{i})\, e^{i\int_0^\beta d\tau \int dx\, \phi_i(x,\tau)\, \delta n_i(x,\tau)}\Big\rangle_0. \tag{4.45}$$

The correlation function (4.45) can also be written as

$$g_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}|\phi_+,\phi_-) = g_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}|\phi_\pm)\cdot Z_\mp[\phi_\mp], \tag{4.46}$$

where

$$Z_\pm[\phi_\pm] \equiv \Big\langle T e^{i\int_0^\beta d\tau \int dx\, \phi_\pm(x,\tau)\, \delta n_\pm(x,\tau)}\Big\rangle_{0,\pm}, \tag{4.47}$$

and

$$g_\pm(x_\mathrm{f}\tau_\mathrm{f}, x_\mathrm{i}\tau_\mathrm{i}|\phi_\pm) = -\Big\langle T\psi_\pm(x_\mathrm{f}\tau_\mathrm{f})\bar\psi_\pm(x_\mathrm{i}\tau_\mathrm{i})\, e^{i\int_0^\beta d\tau \int dx\, \phi_\pm(x,\tau)\, \delta n_\pm(x,\tau)}\Big\rangle_{0,\pm}. \tag{4.48}$$

Next we cumulant expand both (4.47) and (4.48) to second order. For (4.47)

$$Z_\pm[\phi_\pm] \approx e^{\frac{1}{2}\int d\tau d\tau' \int dx dx' \, \Pi_\pm(x-x',\tau-\tau')\phi_\pm \phi'_\pm} \tag{4.49}$$

Where $\Pi_\pm$ is the noninteracting density-density correlation function

$$\Pi_\pm(x,\tau) \equiv -\big\langle T\delta n_\pm(x,\tau)\,\delta n_\pm(0)\big\rangle_{0,\pm}, \tag{4.50}$$

which can be written as

$$\Pi_\pm(x,\tau) = G_{0,\pm}(x,\tau)\, G_{0,\pm}(-x,-\tau). \tag{4.51}$$

The noninteracting chiral propagator is

$$G_{0,\pm}(x,\tau) = \pm\frac{1}{2\pi i}\;\frac{e^{\pm i k_{\rm F} x}}{x \pm i v_{\rm F}\tau}. \tag{4.52}$$

For (4.48)

$$g_\pm(x_{\rm f}\tau_{\rm f}, x_{\rm i}\tau_{\rm i}|\phi_\pm) \approx G_{0,\pm}(x_{\rm f}\tau_{\rm f}, x_{\rm i}\tau_{\rm i}|\phi_\pm) e^{\int dx d\tau\, C_{1,\pm}\phi_\pm + \int dx dx' d\tau d\tau'\, C_{2,\pm}\phi_\pm\phi'_\pm} \tag{4.53}$$

where

$$C_{1,\pm}(x,\tau) = -i\frac{G_{0,\pm}(x_{\rm f},x,\tau_0-\tau)\, G_{0,\pm}(x,x_{\rm i},\tau),}{G_{0,\pm}(x_{\rm f},x_{\rm i},\tau_0)} \tag{4.54}$$

and $C_{2,\pm}$ for this model reduces to

$$C_{2,\pm}(x-x',\tau-\tau') = \frac{1}{2}\,\Pi_\pm(x-x',\tau-\tau'). \tag{4.55}$$

Now we solve for $U_{\rm eff}$, defined by

$$\int dx'' d\tau'' \mathbf{U}_{\rm eff}^{-1}(x-x'',\tau-\tau'')\mathbf{U}_{\rm eff}(x''-x',\tau''-\tau') = \delta(x-x')\delta(\tau-\tau')\mathbb{1}, \tag{4.56}$$

where

$$U^{\rm eff}_{ij}(x,\tau) = \big[U^{-1}_{ij}(x,\tau) - \Pi_\pm(x,\tau)\delta_{i+}\delta_{j+} - \Pi_\mp(x,\tau)\delta_{i-}\delta_{j-}\big]^{-1} \tag{4.57}$$

and $\mathbb{1}$ is a 2×2 identity matrix. To achieve this we Fourier transform (4.56). This reduces (4.56) to a matrix equation which gives

$$\mathbf{U}_{\rm eff}(k,\omega) = \left[\begin{pmatrix} U_4 & U_2 \\ U_2 & U_4 \end{pmatrix}^{-1} - \begin{pmatrix} \Pi_\pm(k,\omega) & 0 \\ 0 & \Pi_\mp(k,\omega) \end{pmatrix}\right]^{-1}. \tag{4.58}$$

The $++$ or $--$ component is

$$U_{\text{eff}}(k,\omega)=\frac{U_4-(U_4^2-U_2^2)\Pi_{\mp}(k,\omega)}{1-U_4\Pi_t(k,\omega)+(U_4^2-U_2^2)\Pi_+(k,\omega)\Pi_-(k,\omega)} \tag{4.59}$$

where

$$\Pi_{\text{t}}(k,\omega)=\Pi_+(k,\omega)+\Pi_-(k,\omega)=-\frac{1}{\pi}\,\frac{k^2}{(\omega+ik)(\omega-ik)}. \tag{4.60}$$

The effective interaction for right movers is (with $v_{\text{F}}=1$)

$$U_{\text{eff}}(k,\omega)=U_4\,\frac{(\omega+ik)(\omega-iuk)}{(\omega+ivk)(\omega-ivk)}, \tag{4.61}$$

where

$$u\equiv 1+\frac{(U_4/2\pi)^2-(U_2/2\pi)^2}{(U_4/2\pi)} \tag{4.62}$$

and

$$v\equiv\sqrt{\left(1+\frac{U_4}{2\pi}\right)^2-\left(\frac{U_2}{2\pi}\right)^2}. \tag{4.63}$$

The action can be written as

$$S=\frac{1}{2}\int\frac{dk}{2\pi}\frac{d\omega}{2\pi}\;\rho_{\pm}(-k,-\omega)\,U_{\text{eff}}(k,\omega)\,\rho_{\pm}(k,\omega). \tag{4.64}$$

The chiral tunneling charge density is

$$\rho_{\pm}(x,\tau)=-\frac{G_{0,\pm}(x_{\text{f}},x,\tau_0-\tau)\,G_{0,\pm}(x,x_{\text{i}},\tau)}{G_{0,\pm}(x_{\text{f}},x_{\text{i}},\tau_0)}. \tag{4.65}$$

We now specialize to the DOS case where $x_{\text{i}}=x_{\text{f}}=0$ and assuming right movers we set $\rho_+\equiv\rho$. The tunneling charge density is

$$\rho(x,\tau)=\frac{v_{\text{F}}\tau_0}{2\pi}\frac{1}{(x+iv_{\text{F}}\tau)[x+iv_{\text{F}}(\tau-\tau_0)]}, \tag{4.66}$$

which satisfies

$$\int dx\;\rho(x,\tau)=\Theta(\tau)\,\Theta(\tau_0-\tau). \tag{4.67}$$

Fourier transforming, we find that

$$\rho(k,\omega)=\frac{1}{i\omega-v_{\text{F}}k}\big[\big(e^{i\omega\tau_0}-1\big)\,\Theta(k)+\big(e^{i\omega\tau_0}-e^{v_{\text{F}}k\tau_0}\big)\Theta(-k)\big] \tag{4.68}$$

and

$$\rho(k,\omega)\,\rho(-k,-\omega) =$$
$$\frac{1}{(\omega+ik)^2}\Bigg[\Big(1-e^{i\omega\tau_0}e^{-k\tau_0}+e^{-k\tau_0}-e^{-i\omega\tau_0}\Big)\Theta(k)+\Big(1-e^{i\omega\tau_0}+e^{k\tau_0}-e^{-i\omega\tau_0}e^{k\tau_0}\Big)\Theta(-k)\Bigg]. \tag{4.69}$$

The action therefore is

$$\begin{aligned} S(\tau_0) =& \frac{1}{2}\int \frac{dk}{2\pi}\frac{d\omega}{2\pi}\,\frac{(\omega+ik)(\omega-iuk)}{(\omega+ivk)(\omega-ivk)}\,\frac{U_4}{(\omega+ik)^2} \\ &\times\Bigg[\Big(1-e^{i\omega\tau_0}e^{-k\tau_0}+e^{-k\tau_0}-e^{-i\omega\tau_0}\Big)\Theta(k)+\Big(1-e^{i\omega\tau_0}+e^{k\tau_0}-e^{-i\omega\tau_0}e^{k\tau_0}\Big)\Theta(-k)\Bigg]. \end{aligned} \tag{4.70}$$

The action can be written as $S = S_> + S_<$ where

$$S_> = \frac{U_4}{8\pi^2}\int_{0^+}^{\infty} dk \int_{-\infty}^{\infty} d\omega\,\frac{(\omega-iuk)}{(\omega+ivk)(\omega-ivk)(\omega+ik)}\Big(1-e^{i\omega\tau_0}e^{-k\tau_0}+e^{-k\tau_0}-e^{-i\omega\tau_0}\Big) \tag{4.71}$$

and

$$S_< = \frac{U_4}{8\pi^2}\int_{-\infty}^{0^-} dk \int_{-\infty}^{\infty} d\omega\,\frac{(\omega-iuk)}{(\omega+ivk)(\omega-ivk)(\omega+ik)}\Big(1-e^{i\omega\tau_0}+e^{k\tau_0}-e^{-i\omega\tau_0}e^{k\tau_0}\Big). \tag{4.72}$$

$S_> = S_<$ under change of coordinates $k \to -k$ and $\omega \to -\omega$, so $S = 2S_>$. To proceed we need the large-τ_0 asymptotic result

$$I(\tau_0) \equiv \int_{0^+}^{\infty} dk\,\frac{e^{-k\tau_0}}{k} \longrightarrow -\ln\tau_0 \tag{4.73}$$

where the additive constant, not shown explicitly, is cutoff dependent. These lead, in the large τ_0 limit, to

$$S = \frac{U_4}{4\pi}\left[\frac{2(1+u)}{(1+v)(1-v)} - \frac{u+v}{v(1-v)}\right]\ln(\tau_0) \tag{4.74}$$

or

$$S = \frac{U_4}{4\pi}\left[\frac{v-u}{v(1+v)}\right]\ln(\tau_0). \tag{4.75}$$

Finally, we obtain

$$S = \delta\ln\tau_0 + \text{const} + O(1/\tau_0) \tag{4.76}$$

and

$$N(\epsilon) = \text{const} \times \epsilon^{\delta}, \tag{4.77}$$

where

$$\delta = \frac{U_4}{4\pi}\,\frac{v-u}{v(1+v)} = \frac{(u-v)(1-v)}{2v(1+u)} \tag{4.78}$$

$$= \frac{1+\frac{U_4}{2\pi v_{\rm F}} - \sqrt{\left(1+\frac{U_4}{2\pi v_{\rm F}}\right)^2 - \left(\frac{U_2}{2\pi v_{\rm F}}\right)^2}}{2\sqrt{\left(1+\frac{U_4}{2\pi v_{\rm F}}\right)^2 - \left(\frac{U_2}{2\pi v_{\rm F}}\right)^2}}. \tag{4.79}$$

This is in exact agreement with the bosonization result

$$\delta = \frac{g+g^{-1}}{2} - 1 \tag{4.80}$$

with

$$g = \sqrt{\frac{1+\frac{U_4}{2\pi v_{\rm F}} - \frac{U_2}{2\pi v_{\rm F}}}{1+\frac{U_4}{2\pi v_{\rm F}} + \frac{U_2}{2\pi v_{\rm F}}}}. \tag{4.81}$$

Why does the cumulant method give the exact result for this model? The answer is that a second order cumulant expansion of the form used here is exact for free bosons, which are the exact eigenstates of the Tomonaga-Luttinger model [81].

4.5 Discussion

Our principal result (4.11) suggests that the dominant effect of interaction on the low-energy DOS in a variety of strongly correlated electron systems is to add a semiclassical time-dependent charging energy contribution to the total potential barrier seen by a tunneling electron, as in Ref. [15]. The energy is computed according to classical electrostatics with a dynamically screened two-particle interaction. In 2D and 3D the added charge is accommodated efficiently and reaches a zero-action state at long times. In 1D the added charge leads to diverging action, and hence suppressed tunneling.

The robustness of our method has not been fully explored, although it is known to fail qualitatively in systems with ground state degeneracy, such as in the quantum Hall fluid. We believe the cause of this failure to be the non-satisfiability of the sum rule (4.20) in such situations.

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[71] This amounts to deforming a one-dimensional integral at each point $\mathbf{r}, \tau$ in space-time from the real line into the complex z plane along the line Im $z = \phi_{\rm xr}(\mathbf{r}, \tau)$.

[72] This approximatioin corresponds to regarding ϕ as small, Taylor expanding about $i\phi_{\rm xr}$, and keeping only the zeroth-order term in the series. However, it is unclear whether such an expansion is meaningful here.

[73] For a free electron gas with parabolic dispersion in D dimensions

$$N_0 = \frac{D S_D \epsilon_{\rm F}^{D-1}}{2(\pi \hbar v_{\rm F})^D},$$

where S_D is the D-dimensional unit sphere volume and $v_{\rm F}$ is the Fermi velocity.

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[78] Note that S is not a functional of ρ, but rather is a function of τ_0.

[79] Equivalently, $U_{\rm eff}(\mathbf{r}\tau, \mathbf{r}'\tau')$ is the inverse (scalar) photon propagator in the system.

[80] B. Z. Spivak, unpublished.

[81] This interesting observation was pointed out to us by Giovanni Vignale.

Appendix A

Low-engergy noninteracting propagator

For a noninteracting electron system with single-particle states $\phi_\alpha(\mathbf{r})$ and spectrum ϵ_α, the imaginary-time Green's function defined in Eq. (4.1) is given by

$$G_0(\mathbf{r}\sigma,\mathbf{r}'\sigma',\tau) = \delta_{\sigma\sigma'} \sum_\alpha \phi_\alpha(\mathbf{r})\phi^*_\alpha(\mathbf{r}')e^{-(\epsilon_\alpha-\mu)\tau}\bigg(\big[n_{\rm F}(\epsilon_\alpha-\mu)-1\big]\,\Theta(\tau)+n_{\rm F}(\epsilon_\alpha-\mu)\,\Theta(-\tau)\bigg), \tag{A.1}$$

where $n_{\rm F}(\epsilon) \equiv (e^{\beta\epsilon}+1)^{-1}$ is the Fermi distribution function. In this appendix we shall evaluate the diagonal elements $G_0(\mathbf{r}\sigma,\mathbf{r}\sigma,\tau)$ for two models in the large $|\tau|$, asymptotic limit.

A.1 1D electron gas

The first model is a translationally invariant electron gas at zero temperature in 1D (the derivation we give is actually valid for any dimension D). In the limit of large $\epsilon_{\rm F}|\tau|$ we obtain

$$G_0(\mathbf{r}\sigma,\mathbf{r}\sigma,\tau) \to -\frac{N_0}{\tau}, \tag{A.2}$$

where N_0 is the noninteracting DOS per spin component at the Fermi energy $\epsilon_{\rm F}$ [73]. It will be necessary to regularize the unphysical short-time behavior in Eq. (A.2). The precise method of regularization will not affect our final results of interest, such as exponents, which are determined by the long-time behavior. We will take the regularized asymptotic propagator to be

$$G_0(\mathbf{r}\sigma,\mathbf{r}\sigma,\tau) \approx -\,\mathrm{Re}\,\frac{N_0}{\tau+i/\epsilon_{\rm F}}. \tag{A.3}$$

When possible, we will let the short-time cutoff in (A.3) approach zero, in which case (A.3) simplifies to

$$G_0(\mathbf{r}\sigma,\mathbf{r}\sigma,\tau) \approx -\,\mathrm{P}\,\frac{N_0}{\tau}, \tag{A.4}$$

where P denotes the principal part. The results (A.3) and (A.4) are valid for any spatial dimension D; the only D dependence appears in the value of N_0 for a given $\epsilon_{\rm F}$.

A.2 2D Hall fluid

The second model we consider is a 2D spin-polarized electron gas in the lowest Landau level at zero temperature with filling factor ν. In the gauge $\mathbf{A} = Bx\mathbf{e}_y$,

$$\phi_{nk}(\mathbf{r}) = c_{nk}\, e^{-iky}\, e^{-\frac{1}{2}(x/\ell - k\ell)^2}\, H_n(x/\ell - k\ell), \tag{A.5}$$

where $c_{nk} \equiv (2^n n! \pi^{\frac{1}{2}} \ell L)^{-\frac{1}{2}}$. Here $\ell \equiv \sqrt{\hbar c/eB}$ is the magnetic length and L is the system size in the y direction. The spectrum is $\epsilon_n = \hbar\omega_{\rm c}(n + \frac{1}{2})$, with $\omega_{\rm c} \equiv eB/m^* c$ the cyclotron frequency (m^* is the band mass). In the lowest Landau level $0 < \nu \leq 1$,

$$\phi_k = (\pi^{\frac{1}{2}} \ell L)^{-\frac{1}{2}}\, e^{-iky}\, e^{-\frac{1}{2}(x/\ell - k\ell)^2}. \tag{A.6}$$

At long times $\tau \gg \omega_{\rm c}^{-1}$ we find

$$G_0(\mathbf{r}, \mathbf{r}, \tau) = \frac{\nu - \Theta(\tau)}{2\pi\ell^2}. \tag{A.7}$$

Appendix B

Asymptotic evaluation of time integrals

Here we note the asymptotics used to obtain (2.15):

$$\int_0^{\tau_0} d\tau \ \mathrm{Re}\left(\frac{1}{\tau_0 - \tau + i/\epsilon_{\mathrm{F}}}\right) \mathrm{Re}\left(\frac{1}{\tau + i/\epsilon_{\mathrm{F}}}\right) \approx \frac{2}{\tau_0} \ln(\tau_0 \epsilon_{\mathrm{F}}), \tag{B.1}$$

$$\int_0^{\tau_0} d\tau \, d\tau' \ \mathrm{Re}\left(\frac{1}{\tau_0 - \tau + i/\epsilon_{\mathrm{F}}}\right) \mathrm{Re}\left(\frac{1}{\tau - \tau' + i/\epsilon_{\mathrm{F}}}\right) \mathrm{Re}\left(\frac{1}{\tau' + i/\epsilon_{\mathrm{F}}}\right) \approx \frac{2}{\tau_0} \ln^2(\tau_0 \epsilon_{\mathrm{F}}), \tag{B.2}$$

and

$$\int_0^{\tau_0} d\tau \, d\tau' \left[\mathrm{Re}\left(\frac{1}{\tau - \tau' + i/\epsilon_{\mathrm{F}}}\right)\right]^2 \approx \frac{\pi}{2} \tau_0 \epsilon_{\mathrm{F}} - 2 \ln(\tau_0 \epsilon_{\mathrm{F}}). \tag{B.3}$$

In these expressions we have retained all terms, including subdominant contributions, that diverge in the $\tau_0 \epsilon_{\mathrm{F}} \to \infty$ limit.

Appendix C

Time ordering functions

Let

$$
\begin{aligned}
W_1(\tau,\tau') &\equiv \Theta(-\tau)\,\Theta(-\tau')\,\Theta(\tau-\tau') \\
W_2(\tau,\tau') &\equiv \Theta(-\tau)\,\Theta(-\tau')\,\Theta(\tau'-\tau) \\
W_3(\tau,\tau') &\equiv W(\tau)\,W(\tau')\,\Theta(\tau-\tau') \\
W_4(\tau,\tau') &\equiv W(\tau)\,W(\tau')\,\Theta(\tau'-\tau) \\
W_5(\tau,\tau') &\equiv \Theta(\tau-\tau_0)\,\Theta(\tau'-\tau_0)\,\Theta(\tau-\tau') \\
W_6(\tau,\tau') &\equiv \Theta(\tau-\tau_0)\,\Theta(\tau'-\tau_0)\,\Theta(\tau'-\tau) \\
W_7(\tau,\tau') &\equiv W(\tau)\,\Theta(-\tau') \\
W_8(\tau,\tau') &\equiv \Theta(\tau-\tau_0)\,\Theta(-\tau') \\
W_9(\tau,\tau') &\equiv \Theta(-\tau)\,W(\tau') \\
W_{10}(\tau,\tau') &\equiv \Theta(\tau-\tau_0)\,W(\tau') \\
W_{11}(\tau,\tau') &\equiv \Theta(-\tau)\,\Theta(\tau'-\tau_0) \\
W_{12}(\tau,\tau') &\equiv W(\tau)\,\Theta(\tau'-\tau_0),
\end{aligned}
$$

where $\Theta(t)$ is the Heaviside step function and W (with no subscripts) is the a window function, defined as

$$W \equiv \Theta(\tau_0-\tau)\Theta(\tau). \tag{C.1}$$