STUDENTS' CONSTRUCTION OF SPATIAL COORDINATE SYSTEMS

by

HWA YOUNG LEE

(Under the Direction of Leslie P. Steffe)

ABSTRACT

Coordinate systems are used as representational tools in the learning and doing of mathematics. However, in many mathematics curricula, coordinate systems are taken for granted and coordinate systems are often unnecessarily restricted to the two-dimensional case. Additionally, researchers rarely address how students might construct coordinate systems or the meanings students impute to coordinate systems.

In this study, I investigated how students construct and use coordinate systems in spatial contexts to quantitatively organize perceptual/sensorimotor space into representational space. Specifically, I conducted a constructivist teaching experiment to explore the mental operations and schemes involved in four ninth-grade students' construction of spatial coordinate systems. As the teacher-researcher of the teaching experiment, I designed tasks by means of which I asked students to locate objects in two- or three-dimensional perceptual/sensorimotor space and to coordinate units along multiple spatial dimensions. I also constructed second-order models that accounted for the students' mathematical activity and shifts in their reasoning through both on-going and retrospective analyses.

This dissertation reports results from the teaching experiment. In my analysis, I model the operations and schemes—frame of reference coordinating scheme and reversible decomposing scheme—that were involved in the students' construction and use of spatial coordinate systems. I also identify ways of reasoning that served as productive cognitive resources in the students' constructive activities: coordination of multiple images, logical multiplication, and levels of units coordination. The findings have important implications for teaching, curriculum development, and research in regards to students' learning and application of coordinate systems.

INDEX WORDS: Coordinate systems, Frames of reference, Spatial organization, Levels of units, Units coordination, Operations, Schemes, Radical Constructivism, Mathematical learning, Teaching experiment.

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HWA YOUNG LEE

B.S., Ewha Womans University, 2004

M.Ed., Ewha Womans University, 2009

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by

HWA YOUNG LEE

Major Professor:	Leslie P. Steffe
Committee:	Kevin C. Moore
	Edward A. Azoff

Electronic Version Approved:

Suzanne Barbour
Dean of the Graduate School
The University of Georgia
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DEDICATION

I dedicate this dissertation to my students who continuously pushed me to learn. My students—those who I had back in South Korea, the pre-service and in-service teachers I taught at the University of Georgia, and the students I will meet in the future—are the driving force in my work. In particular, I would not have been able to write this dissertation without the four ninth-grade students (Kaylee, Morgan, Craig, and Dan) who provided me the opportunity to learn from them how they construct spatial coordinate systems.

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CHAPTER 1

INTRODUCTION

In this study, I investigated four ninth-grade students' constructions of coordinate systems and their use in organizing space. In this introductory chapter, I present the background of study, state the research goal and research questions that guided the study, and explain why this inquiry is significant. Finally, I provide an overview of each of the subsequent chapters in this dissertation.

Background of Study

Starting in the spring of 2013, I participated in *Pathways to Algebra*, a research project led by Dr. Leslie Steffe in which we investigated students' algebraic reasoning. More specifically, we conducted a constructivist teaching experiment (Steffe & Thompson, 2000) to investigate the mental operations and schemes involved in high school students' proportional reasoning and their constructions of intensive quantities (e.g., Steffe, Liss, & Lee, 2014). It is during my participation in *Pathways to Algebra* in which I formulated my dissertation study. Being involved in this project influenced me to further think about students' constructions of representations and symbols and their use in quantitative and algebraic thinking.

Among the various types of graphical items that are conventionally used, I became interested in coordinate systems because they are commonly used in various domains of mathematics and can serve many purposes in doing mathematics. In the seventeenth century, Fermat and Descartes (separately) devised coordinate systems to solve problems of geometry using algebra (hence the development of analytic geometry) and opened ways to solve problems in various mathematical domains as well. The creation of coordinate systems was not a simple, one-time discovery, but built upon continuous work of mathematicians along the long line of ancient techniques of analysis (Katz, 2004). Now coordinate systems are used not only in mathematics but in various fields to represent objects (e.g., geometrical objects or relationships between quantities) in a systematic way.

Finding it important to attend to how students construct coordinate systems, I formed a research agenda to investigate high school students' constructions of coordinate systems, which I discuss next.

Research Goal and Research Questions

In this study, I conducted a constructivist teaching experiment (Steffe & Thompson, 2000) with four ninth-grade students to investigate their constructions of coordinate systems in two- or three-dimensional spatial contexts as organizations of perceptual/sensorimotor space into representational space. Here I use Piaget & Inhelder's (1967) distinction of perceptual/sensorimotor and representational space: Perceptual space refers to the space one constructs through perceptual activity on elements of raw material; sensorimotor space refers to the space abstracted from perceptual activity at the operational level, which is "perfectly organized and balanced at the level of action or behavior" but "still leaves the subject unable to imagine it or mentally to reconstruct it" (Laurendeau & Pinard, 1970, p. 11); representational space refers to the space one abstracts from perceptual space at the operational level and entails a symbolic function in which the individual could regulate spatial behavior in a systematic way.

I use *coordinate system* to refer to a system through which the individual quantitatively organizes (Piaget, Inhelder, and Szeminska, 1960) or coordinatizes points in the space being re-presented. By *frame of reference*, I refer to mental structures (e.g., axes) an individual constructs and superimposes onto perceptual/sensorimotor space through which the relative position of elements in that space can be gauged and re-presented qualitatively (Piaget and Inhelder, 1967). A quantitative organization of space presupposes a qualitative organization of space (Piaget et al., 1960). As such, a frame of reference is necessary to identify a coordinate system with respect to which the location of elements of perceptual space can be re-presented (Rock, 1992).

With these notions of coordinate system and frame of reference, the overarching research goal of this study was to investigate how the four ninth-grade students construct and use coordinate systems in spatial contexts to organize perceptual/sensorimotor space into representational space.

The first research question that guided the study is as follows.

a) How do the students construct and use coordinate systems when representing objects in two- or three-dimensional perceptual/sensorimotor space? More specifically, how do students construct frames of reference and coordinate measurements within those frames of reference to represent points in perceptual/sensorimotor space?

In relation to the first research question, I hypothesized that students' levels of units coordination and the relevant operations and schemes are involved in students' coordination of multiple frames of reference and measurements when constructing coordinate systems. Levels of units coordination refers to the different complexities of structures of units that students are able to coordinate and hold together mentally (Steffe

& Olive, 2010). To further explore students' coordination of multiple frames of reference, which I considered crucial in the investigation of the first research question, I formulated the second research question as follows.

b) How do the students count units within two- or three-dimensional spatial objects. Specifically, how do the students coordinate their frames of reference when asked to reason about spatial objects that entail arrays of units along two or three dimensions?

As the main teacher-researcher in the teaching experiment, through interactions with four ninth-grade students in their engagement in mathematical tasks over several months, I formulated and tested hypotheses of their ways of thinking and modeled the progress in their mathematical activity (Steffe & Thompson, 2000). Because I am aware that I do not have direct access to the students' ways of thinking, my ultimate goal in this study was to build viable second-order models of students' mathematical activity and document shifts in their ways of thinking. These models are never to be interpreted as one-to-one representations of students' thinking (Steffe & Thompson, 2000).

Rationale and Significance of Study

This study has significance and related implications in mathematics education in that it considers students' constructions of representations, contributes to research on the learning of Geometry and Algebra, and bridges gaps in extant literature investigating students' constructions of coordinate systems. In this section, I elaborate on each of these components to explain the rationale and significance of study.

Coordinate Systems as Representational Tools

Mathematical representations are considered important in the learning and doing of mathematics (Maher & Davis, 1990; Davis & Maher, 1990). According to Kaput (1987), "the root of phenomena of mathematics learning and application are concerned with representation and symbolization because these are at the heart of the content of mathematics and are simultaneously at the heart of cognitions associated with mathematical activity" (p. 22). In the Representation Standard in *Principles and Standards for School Mathematics*, the National Council of Teaching of Mathematics (2000) emphasized the importance of representations in school mathematics:

> *Instructional programs for prekindergarten through grade 12 should enable all students to*—Create and use representations to organize, record, and communicate mathematical ideas; Select, apply, and translate among mathematical representations to solve problems; Use representations to model and interpret physical, social, and mathematical phenomena. (p. 360)

Although important in the learning and doing mathematics, as von Glasersfeld (1987) argued, "[A] representation does not represent by itself—it needs interpreting and, to be interpreted, it needs an interpreter" (p. 216). In other words, mathematical representations do not contain any meaning and do not represent things in themselves until someone perceives them as a representation of something (Moore, 2014a; Moore, Paoletti, & Musgrave, 2013; Thompson, 1994b).

Coordinate systems are often used as representational tools in the learning and doing of mathematics. For example, in the Common Core State Standards for Mathematics, students are expected to use the Cartesian plane to investigate mathematical ideas in various grade levels in multiple domains of mathematics, such as in algebra, geometry, and statistics (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010).

Because of their prevalence in the presentation, learning, and teaching of mathematics, often coordinate systems are taken for granted (Lee & Hardison, 2017). Specifically, in curriculum, coordinate systems are restricted to the two-dimensional case and curricula documents rarely address *how* students might construct and use coordinate systems or the meanings students impute to coordinate systems and graphical items. For example, in the Common Core States Standards for Mathematics, the conventional Cartesian coordinate plane is "introduced" for the first time in 5th grade geometry as follows.

> Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010, p. 38)

After this "introduction" to a coordinate system, students are expected to make use of the coordinate plane as a tool to investigate other mathematical ideas in several domains throughout various grade levels, such as number systems, geometric figures, ratios and proportional relationships, and equations and functions (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010).

However, as NCTM (2000) stated,

> The fact that representations are such effective tools may obscure how difficult it was to develop them and, more important, how much work it takes to understand them. But as students move through the curriculum, the focus tends to be increasingly on presenting the mathematics itself, perhaps under the assumption that students who are old enough to think in formal terms do not, like their younger counterparts, need to negotiate between their naïve conceptions and the mathematical formalisms. (p. 68)

This study is important in that it addresses *how* students might construct and use coordinate systems and takes the students' roles in constructing coordinate systems seriously. This study also informs and supports considerations of the opportunities that mathematics educators can provide for students “to construct, refine, and use their own representations as tools to support learning and doing mathematics” (NCTM, 2000, p. 68).

More specifically, this study models the way students at different cognitive sophistications constructed coordinate systems and the mental operations and schemes involved in the students’ constructive activities. These models can inform teachers, teacher educators, and curriculum developers to better attend to students’ mathematical learning involving the use of coordinate systems. Second, the tasks and relevant findings of this study can provide insight for mathematics researchers and educators to allow more meaningful opportunities for secondary students to construct, refine, and use their own coordinate systems.

Coordinate Systems as a Means of Connecting Geometry and Algebra

Both geometry and algebra constitute a substantial part of high school mathematics and are often required for high school graduation (Stillman & Blank, 2009). Especially, Algebra has been described as a “demonstrable gateway to later achievement” (NMAP, 2008, p. 3) and a subject that all students should learn (NCTM, 2000). Although separated as two domains of mathematics, geometry and algebra are not mutually exclusive and are connected. Geometrical thinking and spatial reasoning can support algebraic reasoning by providing ways to interpret and describe physical environments, which serve as the source of abstractions of arithmetical or quantitative relationships. On the other hand, algebraic reasoning can enhance geometrical thinking and spatial reasoning by providing ways to abstract and formalize geometrical and spatial relationships.

The Geometry standpoint.

Kinach (2012) criticized that "tensions within the school curriculum have resulted in downplaying geometry and elevating either numeracy (primary level) or algebra (secondary level)" (p. 536). Furthermore, according to the National Mathematics Advisory Panel (2008), students' transitions from concrete or visual representations to internalized abstract representations in Geometry and Measurement are not clearly understood and needs to be addressed in research.

This study attends to this call for research in that it investigates how students construct coordinate systems to organize space from situations and problems that rely on perceptual/sensorimotor activity but allow for abstractions and coordination of measurements. Moreover, drawing from theoretical constructs from spatial cognition literature and focusing on students' constructions of coordinate systems to organize space, this study provides a stepping stone to connect research outside of mathematics education to better understand students' spatial reasoning and organization of space.

The Algebra standpoint.

According to Smith & Thompson (2008), content of algebra should depend on "ideas of coherence, representation, generalization, and abstraction" (p. 95). However, Smith and Thompson criticized the current way algebra is taught in schools:

> For too many students and teachers, mathematics bears little useful relationship to their world. It is first a world of numbers and numerical procedures (arithmetic), and later a world of symbols and symbolic procedures (algebra). What is missing is the linkage between numbers and symbols and the situations, problems, and ideas that they help us think about. (p. 95)

Moreover, NMAP (2008) explained, "There are many gaps in the current understanding of how students learn algebra and the preparation that is needed before they enter Algebra" (p. 32). As such, not only is there a gap between concrete, arithmetical mathematics in the early grades to abstract, symbolic mathematics in the secondary curriculum but there is also a gap in research documenting the ways students transition from the former to the latter.

This study provides ways to think about linking the situations and problems that coordinate systems help us think about. Put differently, the findings of this study can inform how coordinate systems might be used as meaningful representational tools to express, manipulate, and formalize arithmetical or quantitative relationships. Moreover, considering how students organize space, this study provides models of ways students abstract, generalize, and structure objects, which are key components of algebraic reasoning (Kaput, 2008; Smith & Thompson, 2008; Hackenberg, 2014).

Finally, by distinguishing the ways in which coordinate systems are used in mathematics (c.f., spatial organization vs. quantitative coordination), this study provides a tool for examining students' difficulties in understanding representations of quantitative relationships or geometrical objects in coordinate systems and encourages mathematics educators to attend to these different uses of coordinate systems to support students' balanced understanding of both uses.

Bridging Gaps in Extant Literature

In extant literature, researchers have given more attention to students' understandings of graphs of functions than coordinate systems (e.g., Herscovics, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Schwarz & Hershkowitz, 1999; Oertman, Carlson, & Thompson, 2008; Lloyd, Beckmann, & Cooney, 2010). In these studies, the coordinatized plane is assumed as an already given structure to be used in constructing graphs of functions. This study is important in that it adds to the limited body of research investigating how students construct and use coordinate systems.

Another significance of this study lies in the gap it bridges in extant literature on students' constructions of coordinate systems. There are earlier works exploring students' construction and use of coordinate systems to organize space like that of Piaget and colleagues (e.g., Piaget, Inhelder, & Szeminska, 1960; 1960; Piaget & Inhelder, 1967). There are also more recent works investigating how children or middle grades students construct coordinate planes and graphical representations of given situations (e.g., DiSessa, Hammer, Sherin, & Kolpakowski, 1991; Maverech & Kramarsky, 1997; Nemirovsky & Tierney, 2001; Moritz, 2003; Sarama, Clements, Swaminathan, & McMillen, 2003). Others have studied how college students understand covariation of two or more quantities with the use of coordinate systems (e.g., Oertman, Carlson, & Thompson, 2008; Moore, Paoletti, & Musgrave, 2013). However, I find three important elements missing in these studies.

First, there is little research on high school students' understandings of coordinate systems. The aforementioned studies investigated the constructions of coordinate systems of either younger students who have no to little formal instruction on coordinate systems or of college students whom were assumed to have already had ample experience working with coordinate systems in their past school experience. The lack of research on high school students' constructions of and reasoning about coordinate systems is problematic because these students are expected to use coordinate systems when exploring various mathematical concepts throughout school mathematics. Therefore, this study contributes to the literature by investigating high school students' constructions of coordinate systems and bridges current understandings of students' constructions of coordinate systems at various grade/age levels.

The second element that is often absent in the literature on students' constructions of coordinate systems is the emphasis on the students' *active* role in constructing coordinate systems. Even within research focusing on coordinate systems, some researchers provided students with pre-constructed, conventional coordinate systems. For example, Levenberg (2015) developed and experimented activities in which elementary, junior high school students and pre-service teachers engaged in "reading information presented by axes" (p. 48). Levenberg (2015) assumed axes and the graphs presented on the coordinate system as representing information, which the participants were expected to discover. Denying the existence of pre-made, ontological coordinate systems, this study provides insight for the kinds of coordinate systems students actively construct independently in various situations.

The third element I find lacking in extant literature on students' constructions of coordinate systems is modeling the processes in which students produce coordinate systems and graphical representations using those coordinate systems. Most investigations focused on documenting *what* students did but not necessarily on explaining *how* and *why* they may have done what they did. Except for the work of Piaget and colleagues, it is difficult to find research on the mental operations and schemes that are related in constructing coordinate systems in various situations.

The National Mathematics Advisory Panel (2008) identified research explaining mechanisms of learning as one of the areas of research needed. This study addresses this call by modeling the mental operations and schemes involved in students' constructions of coordinate systems.

Joshua, Musgrave, Hatfield, and Thompson (2015) proposed a framework of conceptualizing a frame of reference in terms of quantitative reasoning (Thompson, 2011). Within this framework, Joshua et al. discussed the mental actions that are involved in coordinating or combining multiple frames of reference and suggested that students' ability to think about measures within a frame of reference supports students in algebraic thinking. In contrast to Joshua et al.'s study, this study models students' reasoning within frames of reference and constructions of coordinate systems in spatial contexts. By considering mental operations that are involved in the construction and use of coordinate systems, identifying common mental operations invariantly involved in the construction of coordinate systems for both uses, i.e., spatial organization and quantitative coordination could inform us on how these different uses may relate to each other. The findings of this study may provide explanations for why students have difficulty constructing and understanding graphs of functions (e.g., shape thinking; Moore & Thompson, 2015).

Overview of Dissertation

In this chapter, I presented the background of this study; research goal and research questions; the theoretical constructs and hypotheses of the study; and the significance of this study. In Chapter 2, I will provide a more detailed review of the theoretical orientation and frameworks that guide this study and review related literature. In Chapter 3, I will discuss the methodology I used in this study, the constructivist teaching experiment, and present details about the teaching experiment I conducted. In

Chapter 4, I will present my findings from the initial interviews of the four participants. In Chapter 5, I analyze Morgan and Kaylee's activities in constructing coordinate systems when representing points in space in the North Pole Task and Fish Tank Task. In Chapter 6, I present findings from the Cubic Block Task with Kaylee and Morgan and discuss their different ways of coordinating units within three-dimensional objects. In Chapter 7, I present findings from the Floor Tile Task, Cubic Block Task, and Rectangular Prism Task with Craig and Dan to discuss their different ways of coordinating units within two- or three-dimensional objects. In Chapter 8, I analyze Craig and Dan's activities in constructing coordinate systems when representing points in space in the School Map Task, North Pole Task, and Fish Tank Task. Finally, in Chapter 9, I will summarize the findings and compare and contrast the activities across all four students. I will also discuss educational implications, and future research directions.

CHAPTER 2

THEORETICAL ORIENTATION, CONCEPTUAL FRAMEWORKS, AND LITERATURE REVIEW

Theoretical Orientation

In this section, I will discuss the principles of radical constructivism, address some of the criticism towards radical constructivism, explain the theoretical constructs that are used to model conceptual structures, and present how the theory of knowing orients this study.

Radical Constructivism as a Theory of Knowing

Basic principles of radical constructivism.

Radical constructivism emerged as a theory of knowing in the nineteen seventies. Von Glasersfeld problematized the fundamental assumptions taken in the traditional theories of knowledge of realist, traditional views and adopted from various sources the ideas that formed the backbone of radical constructivism. Like von Glasersfeld (1990) reflected in his exposition of radical constructivism, he “picked up relevant ideas (somewhat abbreviated and idealized)” (p. 20) way back from the doubts of the skeptics to the Italian Operational School and Piaget’s genetic epistemology. I will explain these relevant ideas as they come up in the overview.

The fundamental assumption grounded in the traditional epistemology von Glasersfeld (1990) found problematic was that there exists an objective, ontological reality, i.e., “a fully structured and knowable world” (p. 21). From this point of view, knowledge is considered a projection of the already existing world, to represent a world of "things-in-themselves" (p. 21) and it is the human subject's job to discover what this already-made reality is. Further, knowledge is true only when it matches to that already existing world.

Troubled by these traditional beliefs that lacked any consideration of the cognizing subject and the relationship between knowledge and reality in terms of the experience of the cognizing subject, von Glasersfeld developed a radical theory of knowing. Von Glasersfeld (1990) outlined the two basic principles of radical constructivism as the following:

> 1. Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognizing subject.
>
> 2. a. The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability.
>
> b. Cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality. (pp. 22–23)

Principle 1 emphasizes the active role of the individual in the construction of knowledge. Contrast to the traditional view of knowledge, Principle 1 puts the responsibility of knowledge building to the individual. As von Glasersfeld (1990) said, "it is we who are responsible for the world we are experiencing" and therefore the knower is responsible for what the knower constructs (p. 28).

The mobility of attention exemplified by the so-called psychological Cocktail Party Effect is what von Glasersfeld (1995) considered an indication that the cognizing subject indeed engages in active participation in the construction of knowledge. To illustrate, imagine you have stepped into a room full of people at a cocktail party. Despite a superfluous flow of raw, sensory material you receive through your senses, you still manage to pick up a conversation happening in the room. That is, you actively select and perceive of the sensory material that is necessary for you to understand a conversation occurring at the party. The Cocktail Party Effect describes the phenomenon in which we can willingly switch our attention from one sensory field to another. Von Glasersfeld (1995) reflected, “[T]he realization of this capability was an enormous encouragement to pursue the search for the active element in the perceiver and, ultimately, the builder of knowledge.” (p. 11)

When we consider knowledge as something actively built up by the individual (Principle 1), then knowledge requires experience of the cognizing subject from which knowledge can be built. Reflexively, through the construction of knowledge, the cognizing subject actively organizes his or her experiential reality (von Glasersfeld, 1995).

From the skeptics and other philosophers, von Glasersfeld (1990) adopted the view that “whatever ideas or knowledge we have must have been derived in some way from our experience, which includes sensing, acting, and thinking” (p. 20). Reflecting on his ability to speak in multiple languages and his activities in translating one language to another, von Glasersfeld (1995) explained that language not only contained different vocabulary or grammar but also “required another way of seeing, feeling, and ultimately another way of conceptualizing experience” (p. 3). Language entailed a conceptually different world and this served an indication for von Glasersfeld (1995) that human beings indeed have different experiential worlds.

Borrowing Vico’s (1710) slogan, “The human mind can know only what the human mind has made,” von Glasersfeld (1995) claimed, “What we make of experience constitutes the only world we consciously live in." (p. 1). Because we can only know what we experience, there is no way of checking an absolute truth of knowledge because it requires "access to such a world that does not involve our experiencing it" (von Glasersfeld, 1990, p. 20). Therefore, "cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality" (Principle 2b).

The second principle of radical constructivism is what makes this theory of knowing radical in that the view of the relationship between truth and knowledge is challenged. As von Glasersfeld (1989) stated:

> The revolutionary aspect of Constructivism lies in the assertion that knowledge cannot and need not be 'true' in the sense that it matches ontological reality, it only has to be 'viable' in the sense that it fits within the experiential constraints that limit the cognizing organism's possibilities of acting and thinking. (p. 2)

From the radical constructivist viewpoint, truth of knowledge is not something we can obtain; the only reality that an individual can perceive is his or her experiential reality. To know something is not a search for a picture-like iconic representation of ontological reality, but a way of organizing our experiential world. At best, knowledge is functional in the realm of being viable and fit within our experiential reality. The theory does not object the existence of ontological reality, but the point is that we have no way of knowing what that reality is. We can only know the boundaries of such reality through the limitations and constraints that we confront through our experiences.

Addressing criticism towards radical constructivism.

Due to the emphasis on the active construction of knowledge by the cognizing subject, radical constructivism can be misunderstood as *solipsism*. However, in the radical constructivist's view, the active construction of knowledge does not mean that the individual can build anything as he or she likes. The construction of experiential reality does not refer to the construction of any reality of her or his liking. Individuals construct conceptual structures to organize their experiences and these structures meet their limits within reality through experiences. When the conceptual structures meet constraints the individual's prior conceptual structure and domain of experiences are modified and adapted to fit within the boundaries of reality. In this sense, "the function of cognition is adaptive" (von Glasersfeld, 1990, p. 23).

Often, radical constructivism is criticized for its lack of consideration of *social interaction* in the process of knowledge construction (e.g., Lerman, 1996). However, von Glasersfeld emphasized that knowledge is not only constructed in a way such that the conceptual structures are viable within our self-organized world but also within the social setting in which individuals interact. Emphasizing intersubjectivity, von Glasersfeld (1990) noted that "every individual's abstraction of experiential items is constrained (and thus guided) by social interaction and the need of collaboration and communication with other members of the group in which he or she grows up" (p. 26). Through collaboration and communication with others, the individuals learn if his or her concept is viable with others.

Finally, von Glasersfeld (1990) cautioned that "one cannot adopt the constructivist principles as an absolute truth, but only as a working hypothesis that may or may not turn out to be viable" (p. 23).

Modeling Conceptual Structures of the Cognizing Subject

To discuss *how* the cognizing subject constructs knowledge and *how* human subjects construct their experiential worlds, von Glasersfeld (1990; 1995) adopted ideas from the Italian Operational School and Piaget's genetic epistemology. Being part of The

Italian Operationist School led by Ceccato, von Glasersfeld engaged in operational analyses of concepts, in which concepts are defined in terms of operations that have to be carried out to build them (von Glasersfeld, 1995). From Piaget, von Glasersfeld (1990) adopted the adaptive function of cognition, the notion of fit, and the constructs of assimilation, accommodation, and equilibrium that explain the genesis of knowledge. In this section, I define scheme, operation, and concept, constructs that describe and model conceptual structures; and assimilation, accommodation, and abstraction, which describe the process through which one organizes conceptual structures.

Schemes and operations.

A *scheme* is a goal-directed basic sequence of events consisting of three parts: the subject's recognition of an experiential situation, a specific activity or operation that the subject associates with the situation, and the result or sequel of the activity in the situation (von Glasersfeld, 1980). *Operations* are the mental actions that are used in the activity of the scheme. More specifically, they are conceptual or internalized activities, "which can return to its starting point, and which can be integrated with other actions also possessing this feature of reversibility" (Piaget & Inhelder, 1967, p. 36).

Assimilation, accommodation, and abstractions.

Von Glasersfeld (1980) adopted Piaget's assimilation-accommodation-equilibrium model to explain the genesis of knowledge. Experiential elements are *assimilated* into existing schemes when the individual recognizes an experiential situation associated with the scheme. When the current schemes one has constructed do not meet constraints in his or her organization of experiences, then the individual's conceptual structures are at a state of *equilibrium*. A scheme is *accommodated* when an individual modifies or rearranges a current scheme. Such accommodations of schemes occur when experiential elements provide a perturbation because the current scheme does not solve the problem the experiential elements present.

A *functional accommodation* is a modification of a scheme that occurs within the context of the scheme being used. When a modification occurs outside the context of the scheme being used, and there is a general reorganization of a scheme, a *metamorphic accommodation* has occurred. It involves an interiorization and reorganization of specific operations and experiences (Steffe, 1991b) through which learning occurs.

Concepts are interiorized schemes in that they can be re-presented without having to carry out the activity that was involved through the process of abstraction of the operations involved in the construction of the scheme. Hackenberg (2010) explains a concept as the results of the schemes that people have abstracted or interiorized. Therefore, “when people have abstracted a concept from their schemes they can use the results of the schemes – and all the operations that went into producing those results – in assimilation.” (p. 387).

There are two types of abstractions involved in the construction of concepts that Von Glasersfeld (1991) adopts from Piaget: empirical and reflective abstractions. *Empirical abstraction* refers to the abstraction of figural patterns from sensory material. It is a simple abstraction of extracting several common perceptual or sensory properties from a set of objects. On the other hand, *reflective abstraction* refers to the abstraction of a higher level that uses the results of empirical abstraction. It takes the simple abstraction as units and compares, separates, and recombines them, which results in a “projection and adjusted organization on another operational level” (von Glasersfeld, 1991, p.58).

Therefore, reflective abstraction “takes place when the experiencing subject attends only the *mental* operations and abstracts them from whatever sensorimotor context that may have given rise to them” (von Glasersfeld, 1982, p.195).

Radical Constructivism as an Orienting Perspective

Ideas of radical constructivism has been used in areas such as literary studies, psychotherapy, and interpersonal management (von Glasersfeld, 1989). Von Glasersfeld (1989) outlined a few implications that the radical constructivist perspective has in educational practice and research. In this section I will summarize these implications in education in general, then discuss how these implications can be applied to mathematics education. Finally, I explain how radical constructivism guided this study.

Implications in education.

According to von Glasersfeld (1989), from the radical constructivist perspective, the researcher and educator’s interest “will be focused on what can be inferred to be going on inside the students’ head, rather than on overt ‘responses’” (p. 3). Here, there are two things to emphasize. First, that the researcher or educator is at best making inferences and second, that the focus is in the process through which one thinks, not solely in the responses students produce as a result of such thinking. Further, a radical constructivist’s teacher will take the stance that students are attempting to make sense of their experiential world; consequently, the teacher’s interest will be in “every instance where students deviate from the teacher’s expected path because it is these deviations that throw light on how the students, at that point in their development, are organizing their experiential world.” (von Glasersfeld, 1980, p. 3)

The knowing and learning of mathematics.

According to von Glasersfeld (1980), from the radical constructivist perspective, thinking about how human beings come to know mathematics requires investigating "their genesis as abstract entities in an experiential domain" (p. 16). Often the "theoretical infallibility of mathematical operations (von Glasersfeld, 1990, p. 25)" is used to show that mathematical reality is an objective ontological reality. However, such infallibility is due to the rules and system by which it was built; mathematics is still a human construction. It is important to note that the construction of experiential mathematical reality is not built simply or arbitrarily. As Steffe (1991b) explained, "mathematics learning is viewed as reflective abstraction in the context of scheme theory. In this view, mathematical knowledge is understood as coordinated schemes of action and operation" (p. 177).

From a radical constructivism viewpoint, mathematical knowledge concerns what students construct from their experiences. Thus, the goal in mathematics education is not to have the students correctly reproduce what the teacher is doing or find the "correct" answer. It is to help the students continuously construct an experiential mathematical reality. However, we do not want our students to construct anything they wish to or as in a solipsist's mind, out of his or her own imagination. It is important to provide opportunities for student so that the individual can construct their knowledge in a viable way so that their constructions and organization of mathematical experiences are successful ones. To be successful means that there is a connection with the prior mathematical knowledge the student has and that there is continuous reflection.

One purpose of the teaching experiment methodology is to “create situations that would allow the investigator to observe children at work and make inferences as to how they build up specific mathematical concepts” (Steffe, 1991 b, p. 17). Another goal is for the researcher to build *mathematics of children*, second-order models of *children’s mathematics*; children’s mathematics refers to the children’s first-order models of mathematics, which remain inaccessible to the researcher (Steffe, 1991b). The teaching experiment is a conjecture building process, to model children’s mathematics, which I elaborate further in Chapter 3.

Mathematical symbols and representations.

Following the radical constructivist orientation, symbols are constructed and given meaning to by the cognizing subject. As von Glasersfeld (1987) said, “A representation does not represent by itself—it needs interpreting and, to be interpreted, it needs an interpreter” (p. 216). Further, symbols are created through reflective abstraction and conceptualization (von Glasersfeld, 1991). Put differently, we formulate a symbol by associating a word—including other forms of notation—with a constructed conceptual structure, which when used, serves as an activation of bringing forth the abstracted experience but not necessarily having to re-present the whole conceptual structure. However, this association is not simply a stimulus-response relationship where the word calls up a direct response. Von Glasersfeld (1991) explained that “a word is used as a symbol, only when it brings forth in the user an abstracted generalized re-presentation, not merely a response to a particular situation” (p.51). First, the sound or the graphic marks should be recognized by the user; then they should be associated with an abstracted experience, the conceptual structure that it was derived from, so it brings forth an associated meaning as “re-presentation chunks of experience that have been isolated (abstracted)” (von Glasersfeld, 1991, p.52). Further, as the user becomes more proficient with the use of the word or symbol, they can “simply register the occurrence of the word as a kind of ‘pointer’ to be followed if needed at a later moment” (von Glasersfeld, 1991, p. 51) without carrying out the whole associated conceptual structure.

Coordinate systems as mathematical conventions are highly abstracted symbols that were developed throughout the history of mathematics to represent space and to study functions, relationships between quantities, and change. In 1637, Fermat and Descartes each devised a way of connecting geometry with algebra through constructing coordinate systems (Katz, 2004). The conventional Cartesian coordinate system was named after Descartes, from his system of finding constructible points by compass and straight-edge, using two lines to represent each dimension of the plane and assigning numbers to points in the plane in relation to these lines (Aczel, 2009). The creation of the Cartesian coordinate system shows us that coordinate systems were created, not discovered.

Graphical items—a coordinate system (structure) and the graph that is constructed within that coordinate system—can be viewed as symbolizations of mathematical ideas as well. Consistent with the radical constructivist perspective, for graphical items to be representations as symbols, they have to be constructed by the cognizing subject. That is, someone must perceive them as a representation of something and they do not contain meaning nor do they represent anything in and of themselves (Thompson, 1994b; Moore, 2014a; Moore, Paoletti, & Musgrave, 2013).

The radical constructivist theory of knowing orients this study in various ways. First, the theory influenced the formulation of my research questions: the research questions were formulated through the work with students as I engaged in experiencing their mathematical activities. Second, the theory influenced my stance on mathematical representations in general (as discussed above) and coordinate systems and frames of reference in particular (which I discuss in the next section). This perspective influenced the tasks I constructed and theoretical constructs I adopted in data analysis (which I discuss in the next section). Finally, the theory informed the methodology used in this study and the perspective I took in the teaching experiments as the teaching agent.

Theoretical Constructs and Literature Review

In the second part of Chapter 2, first, I discuss the notion of frames of reference and coordinate systems from three bodies of research—spatial cognition, Piagetian work, and quantitative reasoning—and review relevant literature. Second, I present my distinction between two uses of coordinate systems and review studies investigating students' constructions of coordinate systems. Third, drawing from existing studies and making distinctions of my own, I explain what I mean by spatial frames of reference and spatial coordinate systems. Finally, I discuss the theoretical construct of students' levels of units coordination and relevant operations and schemes.

Frames of Reference and Coordinate Systems in Literature

The notion of frames of reference has been used in multiple areas including spatial cognition, linguistics, physics, and mathematics. As Levinson (2003) explained, the notion of frames of reference is essential in the study of spatial cognition in various modalities such as vision, touch, and gesture. According to Levinson (2003), frame of reference is studied in multiple disciplines, such as philosophy, brain sciences, linguistics, developmental and behavioral psychology, vision theory, visual perception, and psycholinguistics. In this section, I provide an overview of the notion of frames of reference from three bodies of research—spatial cognition, Piagetian work, and quantitative reasoning—and review relevant literature. I also define what I mean by spatial frames of reference and spatial coordinate systems.

Frames of reference and spatial coordinate systems in spatial cognition literature.

Mental representations of space. In her exposition on how people think about space, Tversky (2003) explained that we think about space by constructing mental representations of space using frames of reference:

> Mental representations of space are constructions based on elements, the things in space, and the spatial relations among them relative to a reference frame…In human conceptions of space, the things in space are fundamental, and the qualitative spatial relations among them with respect to a reference frame form a scaffolding for mental spaces. Which elements or things are selected and which spatial relations are chosen as relevant depend on the space and the functions it serves us…Each of these spaces is represented schematically in terms of the things and spatial relations that are important for functioning within it. (pp. 66–67)

Tversky (2003) distinguished four types of mental representations of space; the space of the body, the space around the body, space of navigation, and space of graphics. According to Tversky, depending on the functions they serve, the activities that are invoked within them, and the elements involved, these spaces are conceptualized differently. The *space of the body* refers to the various parts constituting the body, which have different functions and sizes. Based on an empirical study about body stimuli response, Tversky (2003) claimed that “the mental representations of bodies are organized around significant body parts” (p. 69).

The *space around the body* refers to “the space of things that can be seen and often reached from the current position” (Tversky, 2003, p. 69). From a multitude of experiments with people naming directions of objects placed in various locations near the body, Tversky concluded that “the space around the body is conceived of three-dimensionally from a reference frame based on extensions of the three major body axes, head/feet, front/back, and left/right” (p. 71).

The *space of navigation* refers to “the space we explore, the space we inhabit as we move from place to place, typically a space too large to be seen at once” (Tversky, 2003, p. 72). Elements such as landmarks, paths, links, and nodes constitute the space of navigation and are organized relative to a reference frame, based on three different perspectives: the viewer, object, or environment.

Finally, the *space of graphics* refers to the external spaces created by humans “as tools to augment cognition” such as maps, drawings, and graphs; some used to represent space and others to “represent visually things that are not inherently visual” (Tversky, 2003, p. 76). The elements of this space can range from literal icons to abstract points and spatial relations such as distance and direction could represent literal relations or metaphoric relations. According the Tversky, “graphics take advantage of human capacity to reason about space, to estimate distances and direction, to mentally transform spatial arrays, and to infer function from structure” (p. 77).

Frame of reference. Levinson (2003) suggested that the idea of frame of reference has a long history. He used the puzzle Aristotle posed about a boat anchored to the bank of a river as an example:

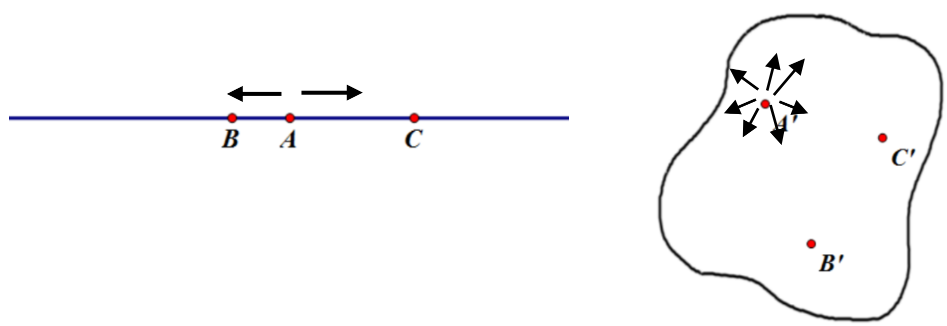
> If we think about the location of objects as places that they occupy, and places as containing the objects, then the puzzle is that if we adopt the river as frame of reference the boat is moving, but if we adopt the bank as frame, then it is stationary. (p. 24)

However, Levinson (2003) attributed the modern phrase and interpretation of frame of reference to the Gestalt theories of perception in the 1920s, and borrowed the Gestalt definition of frame of reference from Rock (1992): “A unit or organization of units that collectively serve to identify a coordinate system with respect to which certain properties of objects, including the phenomenal self, are gauged” (p. 404). Then what does a frame of reference entail?

Reference points. Sadalla, Burroughs, and Staplin (1980) defined spatial reference points as places within a region whose locations serve to define the location of adjacent places when building cognitive representations of large scale space. Drawing from their experiments with undergraduate students in identifying locations on a university campus, Sadalla et al. suggested that “spatial information is organized into conceptual units, with a number of locations cognitively located in relation to reference points.” (p. 527). Further, they claimed that spatial reference points “provide an organizational structure that facilitates the location of adjacent points in space” (p. 526).

Indeed, reference points are crucial bases for locating other objects in space. But they are not sufficient for that purpose. I agree with Levinson (2003) that solely focusing on reference points does not account for different frames of reference that one might take in locating objects and, “severely underplays the importance of coordinate systems in distinguishing frames of reference” (p. 25). Put differently, a frame of reference should entail more than a reference point. In order to locate a point in relation to a reference point, one will need a direction from the reference point to move towards the point and a description of the amount of movement needed from the reference point to the point.

Directionality. In one-dimensional space, any point in the space could be located in relation to a reference point in a rather simple manner. See Figure 2.1 (a) as an example. With a fixed point (A), in order to locate arbitrary points B or C in the one-dimensional space, one will need to know the direction in which to move—either on one side or the other of point A along the line—and a measurement of distance from the fixed point, A. In the one-dimensional case, direction is bidirectional: you can only move in one direction or in the opposite direction along the line.



(a) (b)

Figure 2.1. Locating a point in one- or two-dimensional space.

However, directionality in two-dimensional space is not as simple as in one-dimensional space. See Figure 2.1 (b) as an example in which arbitrary points B’ and C’ are located in relation to reference point A’. Different from one-dimensional space, there are infinitely many directions in which one could move. Depending on the direction one moves, the distance one will need to travel will differ. A similar analogy could be made about three-dimensional space. So, there could be multiple ways people can think about the location of one point in reference to another depending on the frame of reference they construct and the perspective they take when perceiving the space.

Types of Frames of Reference. Different types of frames of reference have been studied in spatial cognition literature. Drawing from various experimental data, Soechting and Flanders (1992) discussed how humans move in three-dimensional space. More specifically, they identified the frames of reference in which they claim sensory information is encoded and processed through the neural systems, to control and coordinate eye, head, and body movements. Although Soechting and Flanders do not explicitly define their notion of frame of reference, they distinguish different frames of reference by where they are fixed. Frames of reference could be fixed at various locations, such as the earth, a moving train (object), or the observer's eye. According to Soechting and Flanders, the processing of sensory information in various motor tasks of the body follows earth-fixed frames of reference with "one of the coordinate axes defined by the gravitational vertical …[and the other] defined by the sagittal horizontal axis" (p. 186).

Soechting and Flanders (1992) explained that there are two ways to represent a location of a point within a frame of reference: vectorially or through coordinate systems. Representing a location of a point within a frame of reference vectorially refers to defining an origin and assigning a direction and amplitude from the origin to each point. Another way to locate a point is to "define a coordinate system within the frame of reference by choosing a set of base vectors. Any point in the reference frame is now defined in terms of an amplitude along each of the base vectors (coordinate axes)" (p. 169). I interpret these two methods to be essentially the same, as both methods require assigning a notion of directionality and some amplitude from the origin to each point. However, the difference seems to lie in the way directionality is defined. In the first case, a single vector connecting the origin and point is used (e.g., polar coordinate system) whereas in the latter case, the single vector is now broken down into two or more base vectors that span the space (e.g., Cartesian coordinate system).

Other researchers distinguished three types of frame of reference used "for representing the spatial relationships among objects in the world" (Carlson-Radvansky & Irwin, 1993, p. 224). The three types are viewer-centered, environment-centered, and object-centered frames of reference. In a *viewer-centered frame of reference*, objects are represented relative to the viewer's perspective. This frame of reference could be centered at the perceiver's retina, or head, or body. Farah, Brunn, Wong, Wallace, and Carpenter (1990) explained that viewer-centered representations need to be adjusted if either the viewer or the object moves. In an *environment-centered frame of reference*, objects are represented relative to salient features of the environment, "such as gravity or prominent visual landmarks" (Carlson-Radvansky & Irwin, p. 224). Due to the stability of the environment, according the Farah et al., the representations induced from an environment-centered frame of reference do not change whenever the viewer moves. For example, "'left' is defined as to the left of the environmental midline, regardless of the position and orientation of the viewer and the objects in the environment" (Farah et al., p. 336). In an *object-centered frame of reference*, objects are represented with respect to an object and the axes intrinsic to the object. Farah et al. explained, "Because the object-centered frame of reference moves with the object, spatial representations in object-centered coordinates are stable over changes of the object's position and orientation with respect to the viewer and the environment" (p. 336). As such, the environment-centered and object-centered frames of reference are defined in an "absolute" sense in that there is no account for the observer of the environment or object. The environment or object that serve for the center of the frame of reference are taken as objective, stable entities independent of an observer.

Carlson-Radvansky and Irwin provided an example of the different frames of reference that could be used in describing objects in space, as illustrated in Figure 2.2.

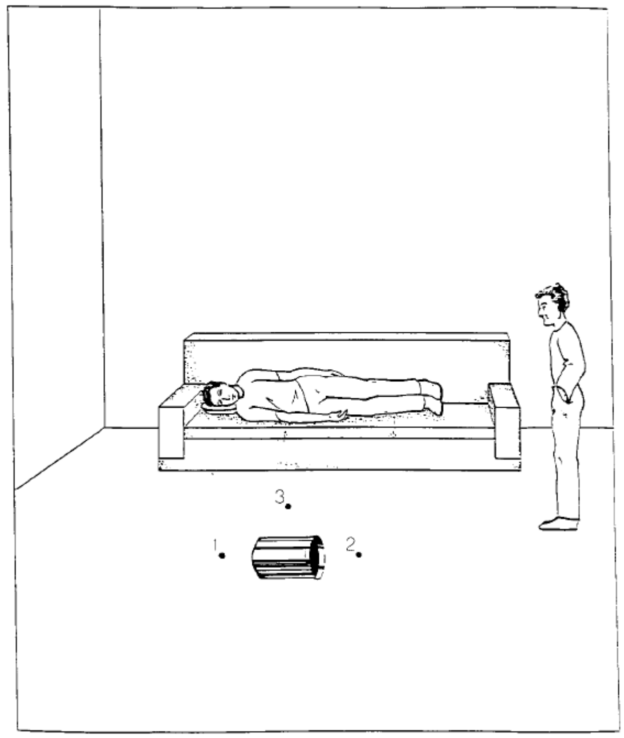


Figure 2.2. Figure 1 from Carlson-Radvansky & Irwin (1993, p. 225).

In their description of the drawing, Carlson-Radvansky and Irwin (1993) stated:

> Which object is “above” the trash can? From the perspective of the person lying on the couch, object 1 is above the trash can with respect to a viewer-centered frame, object 2 is above with respect to an object-centered reference frame, and object 3 is above with respect to an environment-centered reference frame. (p. 225)

Although subtle, one thing that is unclear in their description is whether Carlson-Radvansky and Irwin (1993) consider these frames of reference from an observer’s perspective or impute these frames of reference to the person lying on the couch. There is a difference between claiming that “object 1 is above the trash can with respect to a viewer-centered frame" and claiming that if the person lying on the couch says that object 1 is above the trash can, then he is describing the location of the object with respect to a viewer-centered frame of reference, since he chose the object that was above from his viewer's perspective. A similar analogy could be made for objects 2 and 3 in the situation. The difference lies in whether the observer of the objects is the "outside" observers of the context (such as the authors or readers) or the person lying on the couch. The uncertainty of the observer's role suggest again that in this framework, the environment or object that serve for the center of the frame of reference are taken as objective, stable entities independent of an observer.

Using this distinction between three different types of frame of reference, Farah et al. (1990) investigated the frames of reference (or what *they* also called spatial coordinate systems) used to code location when allocating attention to representations of space. From their study with neglect syndrome patients, Farah et al. suggested that attention is allocated to locations in space with respect to environment-centered and viewer-centered frames of reference. Carlson-Radvansky and Irwin (1993) investigated the frames of reference with respect to which spatial positions can be defined in perception and language. Specifically, they investigated how the spatial term "above" is interpreted when adults were asked to locate objects when perceptual cues for verticality were varied. From their experiments, Carlson-Radvansky and Irwin concluded that their subjects usually used an environment-centered frame of reference when describing the spatial relationship between objects. They explained, "on earth, the powerful influence of an environment-centered reference frame based on gravity most likely dominates, unless the reference object is made salient in some way" (p. 242).

Although this distinction that Farah et al. (1990) and Carlson-Radvansky and Irwin (1993) used of frames of reference seemed plausible for many, it also brought confusion and controversy due to the lack of clear, definitive distinctions in certain cases (Tversky, 2003). For example, when a location of an object is addressed in relation to a person in space (e.g., the person lying on the couch in Figure 2.2 above), this person can be both a viewer or an object in the space depending on who the observer is. Therefore, Levinson (2003) adjusted the distinctions between reference frames to address the confusion and controversy and distinguished three types of frames of reference: the relative, intrinsic, and extrinsic (or absolute) reference frames.

In the *relative frame of reference*, the location of an object is described in relation to one of the participants, either the person describing the location or the person being addressed in the description. In the traditional distinction, it is similar to the viewer-centered frame of reference with the range of viewer extended to include the addressee. The *intrinsic frame of reference* is used to locate an object (which can also include a person) in relation to a specific object and requires participants to agree on the intrinsic sides (front/back, top/bottom, right/left) of the reference object. Finally, in the *extrinsic frame of reference* the location of an object is described in relation to something external to the space or a salient feature of the space (e.g., cardinal directions).

Three perspectives in generating descriptions of space. Drawing from multiple experiments of participants describing spatial environments they learned from maps, Taylor and Tversky (1996) distinguished different perspectives people take in generating descriptions of space: *gaze, route, and survey.* Further, they distinguished which of Levinson's (1996) three types of frames of reference each perspective corresponds to. In a *gaze perspective,* speakers adopt locations of objects taking an “outside viewpoint, as if their eyes were moving around the scene” (Taylor & Tversky, 1996, p. 375). According to Taylor and Tversky, because this type of description does not account for the listener’s perspective nor the listener’s motion within the environment but only for the speaker from a fixed outside viewpoint, the perspective is also termed “ego-oriented” (p. 375). This perspective is based on a *relative frame of reference* in that the origin of the frame of reference is at the person describing the location.

In a *route* perspective, the speaker describes the objects within the environment in relation to the listener and accounts for the changing viewpoint the listener might take from within the environment, as if the listener is following a route in the space. Since the location of objects are described in reference to a specific object (the listener) within the environment, this perspective uses an *intrinsic frame of reference*. Finally, the *survey perspective* refers to the viewpoint the speaker takes from viewing the environment from above the space using cardinal directions, like a map. Taylor and Tversky noted that this perspective would use an *extrinsic frame of reference*. They also explained that speakers can change perspectives, “for example, to take an addressee on a mental tour but describe locations of landmarks using the cardinal direction terms” (p. 377) and that “the choice of description perspective depends on characteristics of the environment themselves” (p. 384).

Taylor and Tversky (1996) differ from the aforementioned studies in that they accounted for different perspectives an observer can take. However, their distinction between gaze and survey perspectives lack clarity. The only main difference between gaze and survey descriptions in Taylor and Tversky’s (1996) framework seems to be in whether the speaker is outside or above the space and whether the spatial terms are described in relation to the objects within the space or in terms of cardinal directions. However, I view the survey perspective as a specific type of gaze perspective. Firstly, because both perspectives are taken from “outside” of the environment and secondly, because in an extrinsic frame of reference locations can also be described in terms of salient spatial features.

Although informed by their framework, different from Taylor and Tversky, I distinguish two perspectives, one embedded within the space (corresponding to Taylor and Tversky’s route perspective) and one taken from outside the space (corresponding to Taylor and Tversky’s gaze or survey perspective). Additionally, instead of attributing an intrinsic frame of reference centered from the outset to another person or object in space in the case of route perspective, I consider all types of spatial descriptions to be centered at the speaker/perceiver but entail different levels of *translation* of viewpoint to others, i.e., different levels of *decentration* involved. Here, I use Piaget and Inhelder’s (1967) use of decentration, “The passage from one centration to another” (p. 24). For example, in the route perspective, although the speaker describes the objects in relation to the changing viewpoint of the listener in the environment, this perspective still requires the speaker to imagine the listener’s viewpoint (through *decentering*) as if the speaker is the listener.

Relevance to my study and critique. The spatial cognition literature affords useful frameworks for defining frame of reference, distinguishing different types of frames of reference, and the relevant perspectives individuals take when representing space. However, in the aforementioned studies, researchers tended to assume an ontological space, which participants were expected to discover through embodied senses (e.g., through vision, sound, touch, and gestures). For example, in their experiments, Taylor and Tversky (1996) used pre-constructed maps and assumed "that subjects regarded the maps as representing environments rather than as marks on paper" (p. 387) and that "spatial environments have an objective reality" (p. 388). The maps they used were of places the participants have not been to, so they did not experience the environments personally but only through looking at maps constructed by the researchers.

Although I share the same goal of investigating individuals' mental representations of space, I view my dissertation study different from these studies because I take a different theoretical perspective. Oriented by the radical constructivist viewpoint, I do not assume an ontological space independent of an observer. Neither do I assume environments to have an objective reality which people are supposed to discover. Instead, I focus on investigating the processes by which individuals abstract and represent space.

Piaget and colleagues found sensory input as an important source for constructions of space but also emphasized the importance of investigating the process by which the human mind abstracts and operationalizes space: "the spatial organization of sensori-motor behavior results in new mental constructs, complete with their own laws" (Piaget & Inhelder, 1967, p. 3). Next, I will present an overview of Piaget's study of the child's conception of space and geometry.

Piaget's study of the child's conception of space and geometry.

Piaget and colleagues investigated children's conception of space and geometry through clinical interviews with children of various ages (approximately 2–12 years old) (Piaget, Inhelder, & Szeminska, 1960; Piaget & Inhelder, 1967). Piaget and Inhelder

(1967) emphasized, “the perception of space involves a gradual construction and certainly does not exist ready made at the outset of mental development” (p. 6).

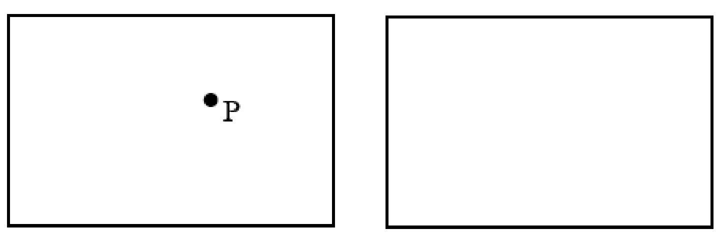
Perceptual, sensorimotor, and representational space. Piaget & Inhelder (1967) distinguished between perceptual space and representational space and explored how children at various developmental stages organized perceptual space and constructed space in a representational sense. Laurendeau and Pinard (1970) provided a summary of Piaget’s distinction between perceptual space and conceptual space, and within conceptual space, between sensorimotor and representational space. Perceptual space is the space perceived through perceptual activity on elements of raw material, whereas conceptual space is the space abstracted from perceptual space at the operational level. Perceptual space and conceptual space are not separate, as Laurendeau and Pinard explained:

> Between these two types of structures a reciprocal influence or functional interaction must operate; at all levels of development, the information provided by perception (or the mental image) serves as raw material for the intellectual action or operation, and, reciprocally, these intellectual activities exert an influence (direct or indirect) on perception, enriching and increasing the flexibility of its functioning with development. (p. 10)

At the conceptual level, space can be distinguished as sensorimotor or representational. Sensorimotor space is “a space which is practical and experienced, perfectly organized and balanced at the level of action or behavior, even though the absence of the symbolic function still leaves the subject unable to imagine it or mentally to reconstruct it” (Laurendeau & Pinard, 1970, p. 11). On the other hand, representational space entails a symbolic function, which leads one “to regulate his spatial behavior through a system of total representation of his displacements rather than according to simple motor expectations” (Laurendeau & Pinard, 1970, p. 12).

Qualitative and quantitative organization of space. Piaget, Inhelder, and Szeminska (1960) also distinguished between qualitative and quantitative organization of space. To organize space in a qualitative sense means to think about objects within the space topologically (e.g., proximity, separation, enclosure, or order) without a consideration of the scale or measurement of the intervals between the objects. When one accounts for the measurements of intervals between objects in the metric sense, then the space is organized in a quantitative sense. Here, Piaget et al. (1960) explained that the perception of space *develops* from qualitative to quantitative space and quantitative space presupposes a qualitative organization of space.

To illustrate, consider one of the tasks from Piaget et al.'s (1960) clinical interviews with children. Given two congruent sheets of rectangular paper (S and S' in Figure 2.3), children were asked to mark a point on the blank sheet of paper exactly where a mark (point P in Figure 2.3) was made on the other sheet of paper so when the two sheets of paper were superimposed on top of each other, the marks would line up. The children were given a ruler, stick, strips of paper, and length of thread as tools they could use.



Rectangle S   Rectangle S'

Figure 2.3. A model of rectangles S and S' and point P in Piaget et al.'s (1960) task.

Piaget et al. identified three stages in which children engaged in the task. The children in the first stage made a simple estimate, by looking at the point on the first sheet of paper. Without making use of any of the tools that they were given, the children in this stage made visual judgements about the location of the point based on topological features of the space. The children in the second stage started to make use of the ruler but attended to only one measurement. Among them, some children attempted to preserve the inclination of the ruler but this preservation was a visual one, and was not based on specific measurements carried out. As Piaget et al. explained, “This form of measurement is still one-dimensional but it shows a beginning of awareness that two dimensions are involved" (p. 155). The most sophisticated group of children attended to the measurement of this inclination. They become to understand that if a single measurement is to be used, then the slope of inclination must be maintained. “Gradually, they decompose its inclination and express it in terms of two separate measurements along different axes." (Piaget et al., 1960, p. 155). In the end, the children were able to take “both vertical and lateral dimensions into account…dissociate them and coordinate them operationally” (p. 159).

The purpose of asking children to copy the point on one paper to the other was to explore how children came to make use of measurements and coordinate them in two dimensions. As Piaget et al. (1960) stated, locating a point in two-dimensional space “involves logical multiplication of measurements as given by rectangular coordinates” (p. 154) with measurements oriented by the sides of the rectangular paper as axes. Here, I interpret logical multiplication to entail a recognition of a location of a point along one spatial dimension with the realization that the point has a specific location along the other spatial dimension. As a result of logical multiplication, the location of a point becomes a multiplicative location in that it is a product of a simultaneous coordination of its location along two separate dimensions. Logical multiplication does not necessarily involve the enactment of measuring activities. When it does involve a coordination of measurements then emerges a logical multiplication of measurements.

Relating their findings of children's measurements of length, Piaget et al. (1960) proposed that for children to coordinate two measurements and form a coordinate system, the child needs to coordinate between *subdivision* and *change of position*:

> Subdivision enters into the need to select which partial lengths should be measured out of all the possible straight lines which might be drawn from point P to any other point in the rectangle S. A system regulating the order and changes of position is implied by the necessity to realize the order of points given by the stationary lines of rectangles and successive positions of the ruler. A coordinate system, even though a qualitative one, requires the imposition of spatial order in two or more dimensions between the several elements, and also demands a systematic nesting of partial intervals between such elements. (p. 160)

Coordinate systems. From the above task, Piaget et al. (1960) explained that the changes of position needed for measurement of lengths from the sides of the rectangular paper need to emerge from the spatial context in which they occur. Then emerges a "general system embracing moving objects and stationary sites, as determined by reference points" (p. 164). This general system is what Piaget and Inhelder (1967) termed the "co-ordinates of Euclidean space" (p. 375) but also emphasized:

> However, a reference frame is not simply a network composed of relations of order between the various objects themselves. It applies equally to positions within the network as to objects occupying any of these positions and enables the relations between them to be maintained invariant, independent of potential displacement of the objects. Thus the frame of reference constitutes a Euclidean space after the fashion of a *container*, relatively independent of the mobile objects *contained* within it… (p. 376)

Therefore, Piaget and colleagues did not distinguish between reference frames, reference systems, and co-ordinates but used them interchangeably. In sum, for Piaget and colleagues, a reference frame or co-ordinate system is like a container, which entails the relations of order (qualitative relations) and distance (quantitative relations) between objects within the space and a simultaneous organization of all possible positions of any but no particular point within the space.

According to Piaget and Inhelder (1967), because of our posture and environment surrounding us, a natural coordination in space is that of horizontality and verticality. This claim seems consistent with the spatial cognition studies that claimed earth-fixed frames of reference were dominant in participants' mental representations of space (e.g., Seochting & Flanders, 1992; Carlson-Radvansky & Irwin, 1993). Further, Piaget and Inhelder (1967) explained that perceptual horizontality and verticality constitute the horizontal and vertical axes of the Cartesian coordinate system. However, starting to coordinate these two axes of verticality and horizontality does not necessarily mean that the child is aware of the Cartesian coordinate system. Moreover, as they stated, "Knowing nothing of the stages which led up to this transformation, the adult assumes that perception involves co-ordinate systems or vertical-horizontal relations right from the outset, when in fact such systems are extremely complicated and are only fully developed by the age of 8 or 9" (p. 4). This awareness comes only after through reflective abstractions of experience such that the space the coordinate system explains becomes representational (Piaget & Inhelder, 1967).

Relevance to my study and critique. Piaget and colleagues' work influenced this study in many ways. First, I adopted Piaget and colleagues' notion of representational space and extended their study of students' construction of representational space. Second, their locating tasks (point on rectangular paper and point in wired box) inspired the task design of the Locating Tasks used in this study. Third, the notion of logical multiplication served as an important construct in data analysis. Finally, Piaget et al.'s (1960) hypothesis that subdivision and change of position compose two important operations involved in coordinating measurements to form a coordinate system led me to consider students' progressive integration operations and levels of units coordination (which I elaborate on later in this chapter) and attend to students' various perspectives (as discussed alongside Taylor and Tversky's (1996) study in the spatial cognition literature) when engaging in tasks.

Although Piaget and colleagues contributed to the field by emphasizing the complexity of children's development of coordinate systems and investigating the mental actions involved in the development, they limited the possible coordination of measurements to that of vertical and horizontal distances. That is, Piaget et al. only considered orthogonal axes and these were implicitly cued by the sides of the paper in their task of copying a point on a rectangular sheet of paper to another. As they wrote, "[w]e make it easier by having the axes suggested by straight lines which form the outline of the area or space" (p. 153). Hence, a perpendicular coordination of two measurements was what they described as "[t]he final solution of the problem" (p. 155). In a similar experiment in which Piaget et al. (1960) extended their study with the rectangular sheets of paper to a three-dimensional case, Piaget et al. showed children two identical open boxes with a wire nailed in vertically to the wooden base of one of the boxes. A small bead was fixed to the top of the wire and the students were asked to make a copy of the location of the bead in the box to the second identical yet empty box. Again, in this task, I found it limiting in that the verticality of the visible wire could have acted as cues towards the coordination of horizontal and vertical distances.

Frames of reference and coordinate systems in quantitative reasoning literature.

Quantity and quantitative reasoning. Thompson (1993, 1994c, 2011) developed central constructs of a framework that drove and guided research on quantitative reasoning. Thompson (1994c) defined *quantity* as a conceptual entity an individual constructs as a measurable attribute of an object in conceptions of situations (Thompson, 2011). Thompson (2011) emphasized that a quantity is a mental construction; in other words, quantities do not exist outside of the mind of the perceiver. As Thompson, Carlson, Byerley, and Hatfield (2014) articulated, "Anyone's understanding of a quantity's size will be colored by his or her conception of the quantity being considered and by his or her understanding of how it might be measured" (p. 1). A quantity could be schematic in that the person recognizes an experiential situation, an object and a quality of the object which entails the quality's measurability; associates a specific activity or operation—a process by which to assign a numerical value to the quantity by use of an appropriate unit or dimension; and the result of the activity in the situation is a measurement of the quality of the object, either numerical or non-numerical (Thompson, 1994c). Quantitative reasoning refers to conceiving of and reasoning about quantities and relationships between quantities that arise from quantitative operations (Smith & Thompson, 2008).

Covariational reasoning and tight coupling. Attending to change in quantities, researchers have investigated students' covariational reasoning involved in conceptions of functions (Moore, Paoletti, Musgrave, 2013), intensity and rate of change (Saldanha & Thompson, 1998; Johnson, 2012), and trigonometry (Moore, 2010). Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002) defined covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Saldanha and Thompson (1998) described thinking covariationally as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 1). Saldanha and Thompson explained that covariation is to form a "multiplicative object" by a "tight coupling" (p. 7) of the two quantities which enables one to track "either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 2)

In order to investigate what conceptual operations are involved in reasoning about continuous covariation of quantities, Saldanha and Thompson conducted a teaching experiment with one eighth grade student. They hypothesized that having students engage in tasks requiring them to track two quantities simultaneously as beneficial for their "envisioning graphs as composed of points, each of which records the simultaneous state of two quantities that covary continuously" (p. 2). From the teaching experiment and the way the student engaged in the sequence of tasks, Saldahna and Thompson explained that although the student was able to operatively coordinate images of two individually varying quantities, there was not a "*tight coupling*" of the two quantities "so that one variation is not imagined without the other" and suggested that conceiving of graphs as "representing a continuum of states of covarying quantities" (p. 7) is not something to be taken as granted.

Related to these findings, Thompson (2011) summarized two aspects of students' construction of quantitative covariation as the following:

> The first is conceiving the quantities themselves and images of them that entail their values varying. The second is to conceptualize the multiplicative object made by uniting those quantities in thought and maintaining that unit while also maintaining a dynamic image of the situation in which it is embedded. This act, of uniting two quantities conceptually within an image of a situation that changes while staying the same, is nontrivial. Yet it is at the heart of using mathematics to model dynamic situations. (p. 48)

Thompson (2011) explained that this way of thinking is foundational for concepts of variable and function in calculus.

Framed quantity and frame of reference. Within this quantitative reasoning framework, Joshua, Musgrave, Hatfield, and Thompson (2015) defined a *framed quantity*, "which refers to when a person thinks of a quantity with commitments to unit, reference point, and directionality of comparison" (p. 37). Joshua et al. (2015) proposed that a framed quantity is well-defined, meaning that there are no extra qualifiers one needs to make sense of the quantity's measure value. Further, Joshua et al. (2015) explained, "conceptualizing frames of reference and quantitative reasoning are interrelated, with frames of reference providing an additional lens with which to look at quantitative reasoning." (Joshua et al., 2015, p. 32) and that thinking about measures within a frame of reference is a disposition that could aid students' algebraic thinking.

Joshua et al. (2015) defined *frame of reference* "to refer to a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 2). They distinguished a frame of reference from a coordinate system, which is the *product* of the mental activity involved in conceptualizing a frame of reference. Further, Joshua et al. (2015) offered a theoretical model of mental actions involved in conceptualizing measurable attributes within a *frame of reference*:

> An individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives

> meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both. (p. 32)

According to Joshua et al. (2015), when an individual works within one frame of reference, she works consistently with the same reference point, unit of measure, and directionality of comparison. Joshua et al. described coordinate systems as a tool for combining multiple frames of reference:

> As a further example [of combining multiple frames of reference], coordinate systems allows us (mathematicians, teachers, and students) to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference. … Students' acts of joining two or more number lines that represent measures of (one or more) quantities in different frames of reference, and anticipating that ordered pairs (or *n*-tuples) give information about the measures in relation to each other, is the heart of combining multiple frames of reference. (p. 35)

In their manuscript, Joshua et al. (2015) criticized that in the few extant literature about reasoning within frames of reference, mostly found in physics education, frames of reference are discussed as "objects external to a person reasoning with it" (p. 31) or "defined by the existence of a concrete object" (p. 36). Making the distinction between mental activity and the product of such activity, Joshua et al. emphasized the importance of focusing on the mental activity and the cognitive process of frames of reference. Put differently, a frame of reference is the cognitive process in which one constructs a framed quantity. However, Joshua et al.'s framework of frames of reference is restricted to the quantitative reasoning framework. Therefore, their emphasis on a frame of reference as a "set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 2) seems inevitable when the object that is being framed is a quantity, a conceptual entity an individual mentally constructs as a measurable attribute of an object. Therefore, I categorize their notion of frame of reference and coordinate system as *quantitative* and compare that with a *spatial* frame of reference and coordinate system. In the following, I will discuss two uses of coordinate systems, spatial and quantitative and then define the notions of frame of reference and spatial coordinate system I use in this study.

Spatial Coordinate Systems and Quantitative Coordinate Systems

A conceptual analysis.

Based on my conceptual analysis (Thompson, 2008), I distinguish between two related but different uses of coordinate systems in spatial organization and quantitative coordination (Lee, 2016; Lee & Hardison, 2016). The use of coordinate systems in *spatial organization* refers to their use to re-present space by establishing frames of reference and coordinating measurements to locate points within the space (e.g., a map). The use of coordinate systems in *quantitative coordination refers* to their use to coordinate sets of quantities in a representational space.

Spatial Organization. Following Piaget et al.'s (1960) distinction of qualitative and quantitative organization of space, I view spatial organization as an activity that may or may not involve the use of quantities. However, when I refer to a coordinate system used in spatial organization, it entails a coordinatization of space in a quantitative sense. When associating quantities to points in the space, I claim that it must satisfy two conditions: a) the associated quantities express a unique point or unique set of points (c.f., bipolar coordinate system), and b) any point in that space can be accounted for in this system of associating quantities. This relates to what Sayre and Wittman (2008) described as the properties of coordinate systems: orthogonality, span, equivalency, and value. Orthogonality refers to the property that the bases that span the system should be independent; span refers to the property that all coordinates expresses all possible points in the space; equivalency refers to the property that different coordinate systems can be used for the same situation; and value refers to the property that quantities can be labeled to the system (Sayre & Wittman, 2008).

Quantitative Organization. Coordinate systems can be used as a geometrical representation of the product of measure spaces and can thus provide a representational space that allows us to coordinate quantities in each space and construct graphs representing relationships between these quantities. These graphs are not necessarily projections of physical objects onto the space. For example, in the conventional Cartesian coordinate plane, when the horizontal axis represents time (time space) and the vertical axis represents the volume of water (volume space), the two-dimensional space that is made by the product of the two axes form a third space ({time×volume} plane) that is different but inherently connected to the two spaces that were coordinated to produce it. The collection of points (time, volume)—in each of the one-dimensional or the two-dimensional spaces do not represent the actual physical movement of the water or time but represents each quantity or the relationship between the two quantities of time and volume and how the volume changes as time changes, respectively.

I consider Tversky's (2003) discussion about the space of graphics relevant to my distinction of the two uses of coordinate systems. More specifically, I find Tversky's distinction of *literal relations from representations of visual things* versus *metaphoric relations from representations that are not inherently visual* to correspond to my distinction of the two uses of coordinate systems. Inferring that her notion of "visual" implies a perceptual, spatial element, the literal relations from representations of visual elements is compatible with my description of coordinate systems as representational spaces used for spatial organization. Analogously, metaphoric relations from representations that do not entail a spatial/visual element is compatible with my description of coordinate systems as representational spaces used for quantitative coordination.

Consider the two tasks in Figure 2.4. Task A is from a Pre-calculus textbook (Holliday, Cuevas, McClure, Carter, & Marks, 2006, p. 95) and Task B is from a Calculus textbook (Foerster, 2005, p. 112) both using a Ferris Wheel context.

31. **Entertainment** The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

(a) Task A (Holliday, Cuevas, McClure, Carter, & Marks, 2006, p. 95.

1. *Ferris Wheel Problem:* When you ride a Ferris wheel, your distance, $y(t)$, in feet from the ground, varies sinusoidally with time t, in seconds since the wheel started rotating. Suppose that the Ferris wheel has a diameter of 40 ft and that its axle is 25 ft above the ground (see Figure 3-8j). Three seconds after it starts, your seat is at its high point. The wheel makes 3 rev/min.

   a. Sketch the graph of function y. Figure out the particular equation for $y(t)$.
   b. Write an equation for $y'(t)$.
   c. When $t = 15$, is $y(t)$ increasing or decreasing? How fast?
   d. What is the fastest $y(t)$ changes? Where is the seat when $y(t)$ is changing the fastest?

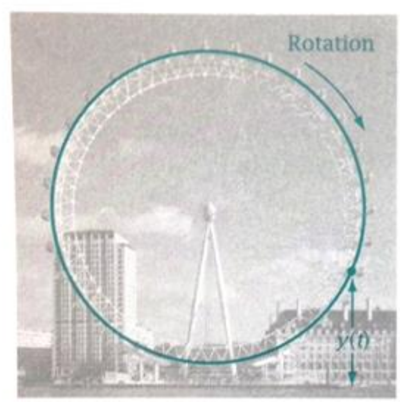


Figure 3-8j

(b) Task B (Foerster, 2005, p. 112)

Figure 2.4. Examples of two uses of coordinate systems.

In Task A, the problem asks students to find the coordinates of the car located at the loading platform and then in various positions, when the axle is located at the origin. In this case, problem solvers are asked to coordinate the location of each car within the single space in which the Ferris Wheel is situated. The coordinate system in this task is used for spatially organizing (imposed by the problem developer) the location of each car in reference to the position of the axle of the wheel; hence, an example of when coordinate systems are used for spatial organization. In Task B, the problem asks students to graph the relationship between the time passed since the wheel started rotating and the distance from the ground. So, problem solvers are expected to extract two quantities (time and distance) from the Ferris Wheel space and coordinate them by creating a new space wherein a graph is produced (Lee & Hardison, 2017). This new space does not entail the spatial situation from which the quantities were extracted. Therefore, I consider Task B as an example of when coordinate systems are used for quantitative coordination.

Literature on students' constructions of coordinate systems.

In this section I review literature on students' constructions of coordinate systems in order to present some notable findings and to discuss how my study is different from these studies.

Research on coordinate systems used in quantitative coordination. Most studies in the literature on students' constructions of coordinate systems focused on coordinate systems used in quantitative coordination. Among studies that investigated students' difficulties related to graphs of functions, a few identified the difficulties students had with elements of coordinate systems such as axes, scale, and coordination. For example, Herscovics (1989) studied cognitive obstacles students encounter while learning algebra.

In graphing, Herscovics (1989) identified that students struggle with constructing axes and scales, transitioning to continuous graphs, reading graphs from a global perspective, and interpreting meaning from graphs. According to Herscovics (1989):

> For dots on a line to represent both the relative order and the actual measure, the added notion of an 'interval scale' is essential. It is only when a line has been 'graduated' by the iteration of a given interval that it can become an axis and, hence, a scale on which data can be represented both as points and segments. (p. 70)

There are more recent works investigating how children construct graphical representations of given situations. Maverech and Kramarsky (1997) defined alternative conception as the knowledge that is different from what is to be learned and studied students' alternative conceptions of graphic representations. They asked 92 eighth-grade students to construct graphs representing four different situations regarding the relationship between success on tests and the amount of preparation time. After analyzing students' graphs, they identified three major students' alternative conceptions. Among these three they observed that some students constructed a series of graphs, each representing one factor from the given data. According to Mevarach and Kramarsky, some students were not able to coordinate the variables into one graph:

Graphs by their very nature represent interrelations between variables. Some students could not conceive simultaneously more than one factor. … Since these students understood that focusing on one factor is not sufficient for representing a varying situation, they constructed a series of graphs, each representing one factor from the relevant data. (p. 237)

Moritz (2003) conducted a task survey asking 133 students in grades 3, 5, 7, and 9 to construct a graph to represent temperature change over time for a given data set. Moritz (2003) categorized the students' responses of how they transformed the data given in table form into a coordinate graph into four levels—Nonstatistical, Single Aspect, Inadequate Coordinate, and Appropriate Coordinate graphs. The first three levels showed the difficulties students had when constructing graphs. For example, the Nonstatistical responses showed either only the context or the graph form without display of the data. Plotting points in two-dimensional space outside of the axes seemed to be challenging for some students. Single Aspect responses showed the given data along one dimension. This tendency in students' responses was similar to Mevarach and Kramarsky's (1997) observation of students difficulty in coordinating both variables into one graph. Inadequate Coordinate responses "showed bivariate data in two-dimensional space but inadequately showed either spatial variation or correspondence of values" (p. 226).

Nemirovsky and Tierney (2004) conducted a similar study as Moritz (2003). They conducted teaching experiments with individual students, small groups, or in a full classroom of third or fourth grade students. They investigated how children created representations for situations changing over time. In this work with students, after observing students' difficulty in expressing the start and end of the situation in time, Nemirovsky and Tierney wanted to know how they could encourage the students "to see conventional graphs as connected to their inventions and helpful to express their own ideas" (p. 39). They found that it was helpful to ask the students to explore organizing their lines from left to right which engendered their thinking of being on the right to mean a later moment in time. From their findings, Nemerovsky and Tierney suggested learning how to graph changing situations on paper "entails developing the capacity to 'direct seeing' (i.e., without intermediate inferences and calculations) events and qualities dwelling in symbolic expressions; a development that involves intricate experiences of seeing-as, recognizing-in, interpreting emptiness, and animating homogeneous spaces" (p. 46).

Maverech and Kramarsky (1997) and Nemirovsky and Tierney (2004) distinguished what was to be learned conventionally (i.e., how to graph in the Cartesian plane) with students' "alternative conceptions" (Maverech & Kramarsky) or "inventions" (Nemirovsky & Tierney). Moreover, Nemerovsky and Tierney assumed that there are certain "events and qualities dwelling in symbolic expressions" (p. 46) and expected students to "direct-seeing" these from graphs. I find this assumption to be problematic because, from my theoretical perspective, borrowing von Glasersfeld's (1987) words, "[A] representation does not represent by itself—it needs interpreting and, to be interpreted, it needs an interpreter" (p. 216). Taking this perspective, events and qualities do not dwell in symbolic expressions (or graphs in that manner) but are interpreted from or superimposed onto symbolic expressions by an active thinking agent.

Taking the perspective that mathematical representations do not contain any meaning and do not represent things in themselves until someone perceives them as a representation of something, Moore, Paoletti, and Musgrave (2013), conducted a teaching experiment with two secondary mathematics pre-service teachers, to investigate their ways of thinking when graphing relationships in the polar coordinate system. Moore et al. explained that the students engaged in covariational reasoning, which enabled them to make sense of graphing relationships of two varying quantities in the polar coordinate system, and to understand that although the pictorial shape of the graphs was different in the polar coordinate system and Cartesian coordinate system, the underlying covariational relationship remains invariant.

Research on coordinate systems used in spatial organization. In comparison to the body of research on coordinate systems used in quantitative coordination, there is a limited body of research on coordinate systems used in spatial organization. One study that I was able to identify was that of Sarama, Clements, Swaminathan, and McMillen (2003) whom also adopted theoretical constructs from Piaget and colleagues' work on children's conception of space and geometry.

Sarama et al. (2003) investigated the development of fourth-grade students' "concepts of rectilinear two-dimensional space, including grid structures, coordinates, and rectangles" (p. 287) as they are foundational to analytic geometry. From a case study with fourth-grade students, Sarama et al. observed difficulties or shifts in students' development of spatial structuring using two-dimensional grids. Sarama et al. defined *spatial structuring* as "the mental operation of constructing an organization or form for an object or set of objects in space" (p. 287). Sarama et al. conjectured that "related but different spatial structuring precedes meaningful use of grids and coordinate systems" (p. 288). Interpreting Piaget and Inhelder's (1967) notion of a coordinate reference system as a "container made up of a network of sites or positions" (p. 286), they explained, "From the simultaneous organization of all possible positions in three dimensions emerges the coordinate system" (p. 286).

From their study with the fourth-graders, Sarama et al. (2003) found that students' knowledge of grids and coordinate systems were related to students' "number sense, spatial-geometric relationship, and the ability to discriminate and integrate the two numbers constituting a coordinate pair and the two axes constituting a coordinate plane" (pp. 285 – 286). Based on these findings, Sarama et al. (2003) proposed a theory of development of grid and coordinate systems:

> Successful students mentally internalize the structure of grids as two-dimensional spaces, demarcated and measured with conceptual rulers (or number lines). They must integrate their numerical and spatial schemes to form a conceptual ruler (Clements et al., 1997; Steffe, 1991). They must then integrate conceptual rulers into two orthogonal number lines that define locations in that space…This integration is a distributive coordination; that is, one conceptual ruler must be taken as a mental object for input to another, orthogonal, conceptual ruler. We hypothesize that this is possible due to the recursive characteristic of the human cognition; we operate on an experience, in this case a conceptual ruler (or set of conceptual rulers), with the same scheme that generated each of the elements of this experience, or mental object. This cognitive activity is, then, analogous to a simultaneous unfolding of rows and columns in a matrix, in which both rows and columns are integrated with numeric schemes (Steffe, personal communication, October, 1995). (p. 313)

In addition to the integration of numerical and spatial schemes, Sarama et al. (2003) also conjectured that students' integration of intrinsic and extrinsic perspectives on geometry are crucial in the development of concepts of two-dimensional space in grid environments. The intrinsic perspective considers local properties of figures from a viewpoint of the logo turtle moving along figures, relative to its present location whereas the extrinsic perspective is when one looks down onto the figure in the plane, which they compare to a coordinate system. Although they do not explicitly account for whose perspective they are referring to, I interpreted their distinction of intrinsic and extrinsic perspectives as similar to my distinction of perspectives embedded within the space and taken from outside of the space, respectively.

I find Sarama et al.'s (2003) work important to the field in that it is one of the very few studies that investigate students' conceptions of grids and coordinate systems in spatial contexts after the work of Piaget colleagues. However, there are a few limitations of their study. First, similar to Piaget and Inhelder's (1967) experiments, Sarama et al.

limited their investigation to Cartesian-like coordinate systems. Second, in the instructional sessions they carried out with the fourth-grade students, the students were provided with pre-constructed grids superimposed onto real-world contexts such as streets and addresses and taxicab geometry. Rather than investigating what students independently construct in various situations, they designed activities so the students can learn how to "write coordinates like mathematicians do" (p. 291), similar to the approach Maverech and Kramarsky (1997) and Nemirovsky and Tierney (2004) took in their studies. Finally, the grids that students worked with were limited to those of integer coordinates.

In this study, I focus my investigation on students' constructions of coordinate systems in spatial contexts, similar to Piaget and colleagues' work on children's conception of space and geometry and Sarama et al. (2003)'s study of fourth-grade students' conceptions of two-dimensional grids. However, I consider my study different from theirs in that I investigate a different student population and do not limit students' constructions of coordinate systems to the Cartesian coordinate system. In the following section I explain how I use the term frame of reference and coordinate system.

My Notion of Spatial Frame of Reference and Spatial Coordinate System

Frame of reference.

I use *spatial frame of reference* (frame of reference hereafter) to refer to mental structures (e.g., axes) an individual constructs and superimposes onto perceptual or sensorimotor space through which the relative position of elements in that space can be gauged and re-presented (Piaget and Inhelder, 1967). As Tversky (2003) explained, "Mental representations of space are constructions based on elements, the things in space, and the spatial relations among them relative to a reference frame" (p. 66). So, a frame of reference is used to define qualitative spatial relations that "form a scaffolding for mental spaces" (Tverseky, 2003, p. 66). As such, a frame of reference can serve as a basis for a qualitative organization of space (Piaget et al., 1960).

I am in line with Piaget and Inhelder's (1967) and Joshua et al.'s (2015) emphasis that a frame of reference is something one constructs mentally, not objects inherent in space. Therefore, a frame of reference, or the mental structure that one constructs, is not inherent in perceptual or sensorimotor space but is abstracted from elements of the space the individual is re-presenting. Constructing a frame of reference is what I view compatible with Sarama et al.'s (2003) notion of spatial structuring: "the mental operation of constructing an organization or form for an object or set of objects in space" (p. 287).

Coordinate system.

Using a frame of reference, an individual can coordinatize points in the space he/she wants to re-present in a systematic way. By systematic I mean that each point in the perceptual/sensorimotor space is mapped onto (in a functional sense) the representational space induced by the coordinate system in a consistent manner. I use Piaget and Inhelder's (1967) notion of representational space, compatible with Tversky's (2003) notion of mental representations of space.

I refer to a *spatial coordinate system* (coordinate system hereafter) as a system through which the individual quantitatively organizes (Piaget et al., 1960) or coordinatizes points in the space being re-presented. This is done through defining a unit of measure and directionality and coordinating measurements using a frame of reference or multiple frames of reference. As Piaget et al. (1960) suggested, quantitative organization of space presupposes a qualitative organization of space. As such, a frame of reference is a necessary organization of references that collectively serve to identify a coordinate system with respect to which the location of elements of perceptual/sensorimotor space can be re-presented (Rock, 1992).

Students' Levels of Units Construct

Relating their findings of children's measurements of length, Piaget et al. (1960) explained, "A coordinate system, even though a qualitative one, requires the imposition of spatial order in two or more dimensions between the several elements, and also demands a systematic nesting of partial intervals between such elements" (p. 160). Here, the "imposition of spatial order" and systematic nesting of partial intervals between such elements" (p. 160) is consistent with Sarama et al's (2003) notion of conceptual ruler, which contains both the position and relative distance from zero. Sarama et al. (2003) suggested that for students to successfully internalize grid structures of two-dimensional spaces, they must integrate their numerical and spatial schemes to form number lines as conceptual rulers. Further, taking a conceptual ruler as input, the student needs to recursively operate on the conceptual ruler and distributively coordinate one conceptual ruler into another.

Drawing from these studies, I hypothesized that students' partitioning schemes and operations and their systematic nesting of partitioned intervals would be essential in their organization of space. Therefore, I use theoretical constructs such as levels of units, units-coordinating schemes and operations, and the distinction between a simultaneous coordination and a sequential coordination of units. Although my study is not focused on students' construction of fraction schemes, because the way students construct fractions can also provide insight for the ways they form nesting systems of parts, and because I considered the partitioning of segments (or any kind of geometrical object) to be informing in understanding students' ways of organizing space, these theoretical frameworks were used in the design of the initial interviews (Chapter 4) and in the analysis of results (Chapters 5, 6, 7, and 8). In this section, I unpack these theoretical constructs.

Based on a perspective of quantity as "emerging from the child's interactions with elements in his or her environment" (p. 62) and not existing external to the child, Steffe (1991a) specified the basic operation children use to generate quantity as the unitizing operation. He distinguished discrete and continuous quantity by the sensory material upon which the unitizing operation is used: for discrete quantity, the sensory material is numerical lots and for continuous quantity, the sensory material is continuous but segmented units. While Thompson (1993, 1994c) highlighted the difference between quantitative reasoning and numerical reasoning, Steffe viewed counting as a basic quantitative scheme. Steffe (2012) explained that counting schemes can be considered as measuring schemes because one is admitting a measurement process to the numerosity of composite wholes of some kind. Emphasizing that the units and their number systems that students construct are essential in students' measuring schemes, Steffe (2012) proposed the reorganization hypothesis that "children's continuous quantitative measuring schemes can be realized as accommodations of their discrete quantitative measuring schemes" (p. 35).

Whereas the unitizing operation was identified to be the basic operation children use to generate quantity, Steffe (2012) identified partitioning operations to be foundational in continuous quantitative measuring schemes. Coordinating discrete quantitative measuring schemes with Piaget, Inhelder, & Szeminska's (1960) analysis of length measuring schemes, Steffe explained that partitioning operations constitute a length measuring scheme, and the discrete quantitative measuring schemes are used as partitioning templates for partitioning the continuous object.

As Ulrich (2015) describes, the levels of units is the hierarchy of "the number of layers of embeddedness in the composite structures that a student is working with." (p. 3) For example, "*three levels of units* implies that a student is embedding units of 1 (the first level) in composite units (the second level), which are in turn embedded in units of units of units (the third level)" (p. 3). A student who is assimilating with three levels of units can use this embedded structure of three levels of units *prior to operating*, or in other words, *as given* (Steffe & Olive, 2010). When a student requires carrying out the operations to produce the three levels of units structure, but could reflect on the structure after carrying out the actions to produce it, then the three levels of units are coordinated *in activity* (Steffe & Olive, 2010). Steffe & Olive's (2010) distinction of "an experiential sequence of composite units rather than a unit containing that sequence that could be taken as input for further operating" (p. 92) captures the difference between a student who can produce the embedded layers of units in activity [experiential sequence of composite units] and a student who could operate with the embedded layers of units as given [take as input for further operating].

Taking the quantitative approach and construct of levels of units articulated in Steffe's (1991a, 2012) work, Hackenberg (2010) investigated middle school students' reasoning with reversible multiplicative relationships. Tillema (2012) conducted a teaching experiment with middle school students and investigated students' multiplicative reasoning related to combinatorial and spatial problems. Building off from Thompson's (1993) study on additive structures and Steffe's (Steffe, 1991a, 2012; Steffe & Olive, 2010) work on levels of units and students' construction of numbers, Ulrich (2012a, 2012b) investigated students' construction of quantities with positive or negative values and explored how students construct sums and differences as directed change quantities' values. In the following, I explain the partitioning schemes and operations and relevant fraction schemes and levels of units coordination that Steffe modeled through his work with children (Steffe & Olive, 2010).

Partitioning Schemes and Operations.

Equipartitioning Scheme. Sharing situations are often used to engender partitioning activities due to the social negotiation—making fair shares—that the situation entails (Steffe & Olive, 2010). The equipartitioning scheme is generally constructed when a student forms a goal to make fair shares of a continuous unit [segment]. The operations of the equipartitioning scheme include partitioning, disembedding, and iterating. More specifically, in equipartitioning a segment, a student can find one of the equal shares of the segment by *partitioning* the segment into equal parts using his number concept as a template (Steffe & Olive, 2010). After marking off one of the parts, the student *disembeds* the part from the whole without destroying the whole, which allows the student to take the one part as a unit by itself but also be aware that it is part of the whole at the same time. Further, to test if the one part is an equal share of the segment, the student can *iterate* the part to produce a segmented segment, which he can use to compare with the original segment. Here, disembedding and iterating are sequentially enacted and the parts are progressively integrated, producing two levels of units (Steffe, Liss, Lee, 2015). The sequentiality in the operations of disembedding and iterating emphasizes that iterating involves explicitly pulling a part from a stick and then iterating it. Further, "the units that are established by iterating are projected into the original stick," producing a unit of units structure (*two levels of units*).

Partitive Unit Fraction Scheme. In discussing the subdivision of areas and the concept of fractions, Piaget et al. (1960) set forth seven criteria for subdivision to be operational, which is a crucial element for the construction of fractions. Only when the child can subdivide a whole at the level of representation can the child construct the notion of fraction (Piaget et al., 1960). The equipartitioning scheme satisfies six of these seven criteria, which are that the student is aware of the continuous unit as a divisible whole, can partition the whole into a determinate number of parts, exhausts the whole, coordinates the number of partitions and the number of parts made by those partitions, is aware of making the parts in equal size, and finally, is aware that the whole remains invariant after the partitioning (Piaget, Inhelder, & Szeminska, 1960, pp. 309-311).

When the goal of the student who engaged in equipartitioning becomes to further establish a fractional relation between the part and the partitioned whole, a modification in the equipartitioning scheme can lead to the construction of a *partitive unit fraction scheme* (Steffe & Olive, 2010). The purpose of the partitive unit fraction scheme is "to partition a connected number, one, into so many equal parts, take one out of those parts, and establish a one-to-many relation between the part and the partitioned whole" (Steffe & Olive, 2010, pp. 101–102). Because the process of producing the parts through the enactment of the equipartitioning scheme involved disembedding and iterating, the segmented segment consists of identical parts, which can be constructed as iterating fraction units. So, a student who has constructed a partitive fraction scheme would establish that one of the parts of a stick shared equally among five people would amount to one out of five parts and that the one-fifth part could be used in iterating the part five times to produce the stick.

Equipartitioning Scheme vs. Equisegmenting Scheme. To illustrate the equipartitioning scheme in context, consider the problem in which the student is asked to imagine sharing a string equally among five people and to mark off the piece of string that one person would get (cf. Appendix A, Part II, item 1). If the student can project his number concept of five as a template into the segment, mark off one of the five parts, disembed the one part from the whole and iterate it to make a segmented segment of five of those parts to compare with the original segment, the student would be inferred to have constructed an equipartitioning scheme. Given the same context, Steffe & Olive (2010) explained that a child, Laura, "segmented her stick by transposing a unit from one site to another on a given stick when the unit that was being transposed was part of the stick being segmented" (p. 93).

In contrast to a child, Jason, who was inferred to have constructed an equipartitioning scheme, Laura did not disembed her estimate and iterate it. Although both students Jason and Laura used their number concepts as templates in finding one person's share, Jason activity was referred to as *partitioning,* which entails the breaking of the segment as one composite act whereas Laura's activity was described as *segmenting*, which entails the sequential breaking of the segment. The parts [units] that are produced from partitioning are *identical* in that they are results of a simultaneous partitioning, hence, they are maintained "as elements of the abstract composite unit (numerical concept) used in the partitioning" (Steffe & Olive, 2010, p. 121). On the other hand, the parts [units] that are produced from segmenting are considered *equivalent* in that they have the same length but because they are produced sequentially, they are experientially different and do not entail the abstract structure of an identical part.

Splitting Scheme. The difference between the equipartitioning scheme and the splitting scheme can be emphasized through the activation of the partitioning and iterating operations. Whereas the operations of partitioning and iterating are sequentially enacted in the equipartitioning scheme, the students' enactment of partitioning and iterating co-occur in the splitting scheme. The composition of partitioning and iterating is what Steffe & Olive (2010) refered to as the splitting operation. As such, the results of the equipartitioning scheme are considered the situation of the splitting operation (Steffe & Olive, 2010).

To illustrate the splitting scheme in context, consider a problem where a student is asked to make a string so that a given string is five times as long as the string to be made (cf. Appendix A, Part I, item 3). In the equipartitioning context, although the number concept of five is used as a template to partition the segment and make one mark, the partitioning and iterating of the marked off piece is carried out in succession. On the other hand, in the splitting context, in order to achieve the goal of making the requested string, the student needs to mentally construct a hypothetical string that could be iterated five times to produce a string the same length as the given string (Steffe & Olive, 2010). Steffe & Olive (2010) emphasized the importance of the hypothetical string because “a splitter produces an image of *some* stick (a hypothetical stick) and mentally sets it in relation to the unit stick in such a way that iterating the hypothetical stick produces the unit stick, prior to any observable action” (p. 102)

This situation requires the partitioning of the given string and the iteration of the hypothetical string to be enacted simultaneously. More specifically, the student needs to posit a hypothetical string as one of five equal parts of the given string [partitioning] but also as a part that had already been iterated to constitute the given string [iterating] (Steffe & Olive, 2010). As a result of these operations, a student who has constructed a splitting scheme can establish a multiplicative relation between the given string and one of its hypothetical parts prior to actually partitioning the given string. More specifically, the student knows that the given string is five times as long as the other string and that her string is one-fifth the length of the teacher’s string. Further, this student is “aware of the whole [string] as a unit [string] and of a hypothetical part of the unit [string] such that the unit [string] consisted of [five] iterations of the hypothetical part,” (Steffe & Olive, 2010, p. 102) producing a *three levels of units structure*.

Iterative Fractional Scheme. Upon the emergence of the splitting operations, the student can construct an *iterative fractional scheme* because now the student can consider the whole as a unit consisting of hypothetical parts each of which can be iterated so many times to produce the whole (Steffe & Olive, 2010). Steffe & Olive (2010) explains that “[i]n this way of thinking, a unit fraction (a hypothetical unit part of the [segment]) becomes a fractional number freed from its containing whole” (p. 116) and available for use in the construction of improper fractions. The improper fraction, say, 6/5, can be comprehended as a composite unit containing the original string [unit] and another 1/5-part of the string [unit], forming a three levels of units structure.

Recursive Partitioning Scheme. The recursive partitioning scheme is generally constructed when a student, given partial results of a composition of two partitionings, forms a goal to produce all the parts—the numerosity of the full result—of the composition of two partitionings. The operations of the recursive partitioning scheme involve distributing partitioning across the continuous unit [segment] and holding the parts produced by the partitionings in one nesting structure in order to coordinate the number of units produced by each partitioning.

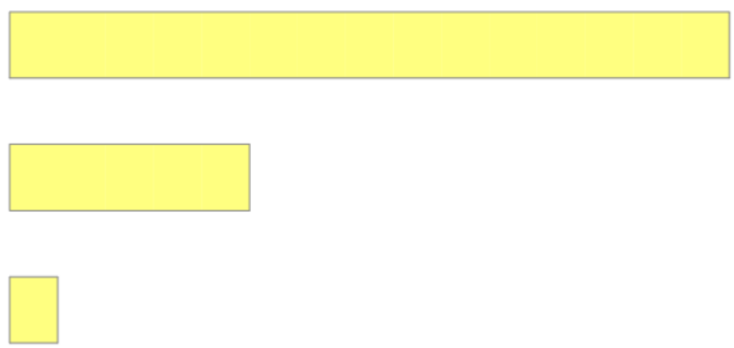


Figure 2.5. Sharing a long strip of candy among three people and then five people.

To illustrate recursive partitioning in context, imagine a long strip of candy is shared among three people and then one of those shares is shared again by five people (cf. Appendix A, item 2). Figure 2.5 illustrates a model of the candy and the sharing of it in this situation. The first rectangular length represents the original segment of candy being shared, which I refer to as the *whole*. The one below the whole represents one of the shares among three people sharing the original candy, a partial result of partitioning the whole into three equal parts. I will refer to this piece as the *first-level share* in that it is a part from the first sharing of the candy. The final piece in Figure 2.5 represents one of the shares among five people sharing the first-share, a partial result of partitioning the first-level share into five equal parts. I will refer to this piece as the *second-level share* in that it is a part from the second sharing of a part of the candy.

Given the partial results of the partitioning as shown in Figure 2.5, consider when asked the fractional amount of the second-level shares in comparison to the whole, the recursive partitioning scheme involves the student forming a goal to find how many of the second-level shares will make up the whole. The scheme involves the student's distribution of the partitioning that produced the first-share part and second-share part across each part (Figure 2.6).

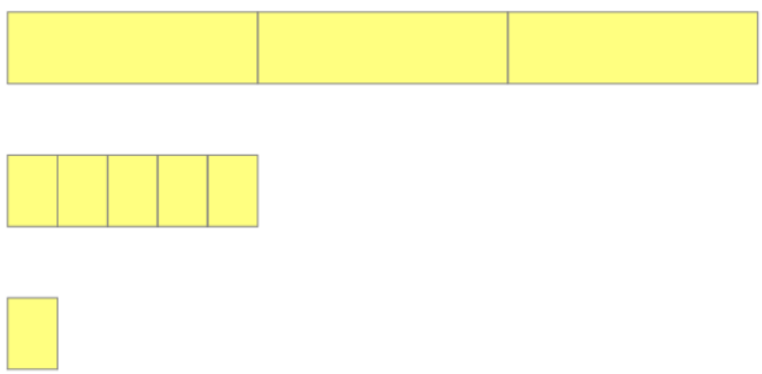


Figure 2.6. Distributing the partitioning that produced the parts across each part.

Further, the student needs to take the partition of three units as input to further partition each part into five parts. Using her unit-coordinating scheme and uniting each of the five units into a composite unit, as the result of recursive partitioning, the student can create a nested structure using the results of distributing the partitionings, which entails a composite unit [whole candy] containing three units of five units each—a three levels of units structure (Figure 2.7). Uniting refers to combining unitized units into composite units; hence, the uniting operation could be considered as a recursive implementation of the unitizing operation. Here, the uniting of the five units into a composite unit is crucial as Steffe & Olive (2010) explained, it is "an act of abstraction that distances the child from the [three] units and permits the child to regard the [three] units of [five] as if they were [three] singleton units while maintaining their composite quality" (p. 91).

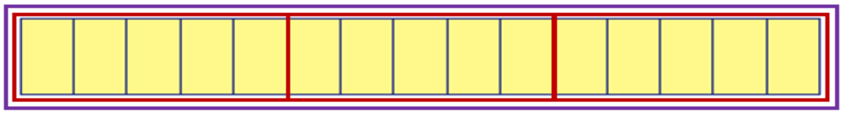


Figure 2.7. A unit of three units of five units structure.

Units-Coordinating Scheme. According to Steffe & Olive (2010), the *units-coordinating scheme* "is a multiplicative scheme that gets its name from the coordination of, to the observer, two composite units of units where one composite unit is inserted into each unit item of the other composite unit" (p. 91). In the context of the above example, the units-coordinating scheme is involved in finding the product of three and five, if a student inserts the unit of five units into each unit of three to produce three fives.

Here, I find it important to emphasize Steffe & Olive's (2010) explanation that the insertion of units are carried out mentally, prior to the actual activity of inserting units. When the insertion of units need to be carried out in activity, the coordination of the units becomes additive in that the composite units of five are sequentially added and progressively integrated to the preceding units. In this additive coordination of units, the student can disregard the composite unit of five after it is progressively integrated to the other units it was added to (Ulrich, 2016). On the other hand, in making a multiplicative coordination, there "involves an extra layer of complexity, in that the student must keep track of iterations of a composite unit," (p. 38) which involves the uniting of the five units into a composite unit. This distinction highlights the difference between a student who can reason with two levels of units as given and a student who can reason with three levels of units as given. A student with two levels of units would have difficulty in keeping track of the extra layer of complexity and in using the composite unit of units as material in operating. Two levels of units students typically lose track of either the three units within the original unit or the five units within each of the three units which limits them when solving the problem at hand.

Unit Fraction Composition Scheme. As Piaget et al. (1960) explained,

> [W]hen subdivision is operational and gives rise to true fractions, by which we imply a nesting system and not just a lot of juxtaposed pieces, the fractions themselves take on a dual character. They are parts of the original whole and they are also wholes in their own right, and as such they too can be subdivided further. (p. 310)

This sixth criterion for subdivision to be operational captures the recursive partitioning operations in that it includes the partition of partitioning (parts being subdivided further) and the embedding of parts in relation to the whole into a nested structure (parts of the original whole and wholes in their own right).

When this type of subdivision (recursive partitioning) gives rise to functions, we can focus our attention on the fractional amount of one of the second-level shares in relation to the original whole. In other words, we can consider the composition of unit fractions. In describing the *unit fraction composition scheme* Steffe & Olive (2010) explain:

> The goal of this scheme is to find how much a fraction of a unit fraction is of a fractional whole, and the situation is the result of taking a fractional part out of a fractional part of the fractional whole, hence the name “composition.” The activity of the scheme is the reverse of the operations that produced the fraction of a fraction, with the important addition of the subscheme, recursive partitioning. The result of the scheme is the fractional part of the whole constituted by the fraction of a fraction. (p. 61)

Returning to the aforementioned problem of sharing a share of a strip of candy, when asked what fraction of the entire candy is one of the five person's share [second-level share] out of the entire candy, the unit fraction composition scheme involves the student recognizing the goal to find one-fifth of one-third of the whole. Further, it involves the student using her recursive partitioning scheme to mentally construct a nested structure of three levels of units (cf. Figure 2.7) and using that unit structure as material in further operating (Steffe & Olive, 2010). The ability to hold this structure in mind allows the student to understand the multiplicative relationship between the number of pieces (the reverse of the operations that produced the parts) and, as a result, find that one share would be one-fifteenth of the entire candy.

Distributive Partitioning Scheme. The distributive partitioning scheme is generally constructed when the student forms a goal of sharing *n* items equally among *m* people. The activity of the scheme involves distributing partitioning operations on the *n* items into *m* parts and sharing one part from each of the *n* items to the *m* people, which is also referred to as distributive sharing. Here, the sharing goal evokes the students' reversible units-coordinating scheme in that there involves a coordination between the *n* units and *m* units in partitioning. As a result of the distributive sharing, the result of the scheme involves an establishment of the relation between one person's share and all of the items put together. More specifically, the student understands that the share of one person can be replicated *m* times to produce the whole of all *n* items (i.e., one person's share is one-*m*th of all *n* items). Further, if the *n* items were considered identical, then the student also understands that *n*-*m*ths of one item is equal to one-*m*th of all the *n* items together (Steffe & Olive, 2010); hence the distributive property. In a distributive partitioning scheme, the student can operate hypothetically and carry out the operations of the scheme mentally (Steffe, Liss, & Lee, 2014).

For example, given a task to share two cakes of different size equally among three people (cf. Appendix A, Part I, Item 6), if a student partitions each cake into three parts, shares one part from each of the two cakes to each of the three people, this indicates an enactment of the distributive sharing activity. Further, if she understands that one person's share can be replicated three times to produce the whole of the two cakes (or in other words, understands that one person's share is one-third of all of the cake), this would indicate an enactment of the distributive partitioning scheme. Further, if the two cakes are considered identical (cf. Appendix A, Part I, Item 5), then she would also know that one person's share amounts to two-thirds of one cake.

Distributive Partitioning and Commensurate Fractions. After producing six pieces in total, by distributing the partitioning across each cake, understanding that one person's share (two pieces from the six pieces) can be replicated three times to produce the whole of the two cakes requires a reorganization of the six pieces into a three composite units each containing two pieces, one piece each from each cake. Then, disembedding one of the three composite units each containing two units each, the student can further operate on the two levels of units to make a one-to-three comparison. These operations entail operating on a unit of units of units (Steffe & Olive, 2010). As such, the result of the distributive partitioning scheme entails the construction of two-sixths (two out of the six pieces) as commensurable to one-third.

This result derives from the identity of each of the three pieces that originated from the same cake. That is, each of the parts from the same cake are indistinguishable in that they are abstracted unit items such that any one of them could be iterated three times to make a cake equal in size to the cake it originated from. Hence, iterating three of one person's share would produce the equivalent amount of cake as the entirety of the two cakes. A student who lacks the uniting of pieces in one person's share, the iterability of one person's share, and the awareness of the identity of the parts from each cake usually focuses on the number of the pieces and not the sizes of each piece. Although such student establishes one person's share as two-sixths of all the cake because the person gets two out of six pieces, and can simplify the two-sixths to one-third using arithmetical calculations learned in school ("cancelling out"), the student's explanation does not entail the reverse relation between the size of the share and the whole (cf. see Brandon's case in Steffe, Liss, & Lee, 2014).

The case of sharing two identical cakes presents an additional complexity in that all of the pieces that are produced from partitioning each cake are all identical although half of the pieces are abstracted from a different cake material than the other half. If the student has constructed the parts from each cake as identical abstracted units, then she could posit two of any of the pieces within one of the cakes and establish that one person's share [two pieces] is two-thirds of one cake.

Levels of Units. As I discussed so far in this section, various sharing situations of continuous segments can engender partitioning operations. The result of partitioning produces parts of the continuous segments and the various ways that students structure the parts [units] and the ways that students evaluate the relation between the parts and the whole or other parts of the whole led to the discussion of different fraction schemes. The levels of units construct was used to model the different complexities of structures of units, such as the parts that were produced from partitioning, either mentally or in sensory-motor activities, and the ways that the students could hold these structures mentally and use for further operating, such as in constructing fraction schemes. The equipartitioning scheme requires the student to reason with two levels of units as given; whereas the recursive and distributive partitioning schemes requires three levels of units as given. When the student forms a goal to find the fractional amount of parts produced in the situations, the results of the equipartitioning scheme can be modified to the construction of a partitive fraction scheme and the recursive partitioning scheme can be used to construct a unit fraction composition scheme. Evaluating the results of distributive partitioning can involve the construction of commensurate fractions.

Summary of Chapter 2

In the first part of Chapter 2, I presented the theoretical perspective that orients my work in this dissertation study. Specifically, I provided an overview of radical constructivism and the central constructs used in this theory of knowing. In the second part of Chapter 2, first, I discussed the notion of frames of reference and coordinate systems from three bodies of research—spatial cognition, Piagetian work, and quantitative reasoning—and reviewed relevant literature. Second, I presented my distinction of two uses of coordinate systems and reviewed studies investigating students' constructions of coordinate systems. Third, drawing from existing studies and making distinctions with my own, I explained what I mean by spatial frames of reference and spatial coordinate systems. Finally, I discussed the theoretical notion of students' levels of units coordination and relevant constructs.

CHAPTER 3

METHODOLOGY AND METHODS

In order to explore the coordinate systems students construct when representing objects in two- or three-dimensional perceptual spaces and to model the mental operations and schemes students use in this process, I conducted a constructivist teaching experiment. In this chapter I first explain the constructivist teaching experiment methodology. Second, I provide an overview of methods specific to the teaching experiment I conducted for this study. Finally, I will outline the data analysis chapters.

The Constructivist Teaching Experiment Methodology

In a constructivist teaching experiment, the teacher-researcher engages in exploratory teaching to investigate students' ways of operating in constructing and understanding mathematical concepts. An ongoing process of forming and testing hypotheses of students' conceptual schemes and operations is carried out over an extended time period. Through this process, the teacher-researcher builds models of students' ways of thinking in the teaching in the moment of teaching and after. Each teaching episode includes a small group of students, selected through initial interviews, a teacher-researcher, witness/cameraperson, and mathematical tasks. In this section, I will discuss the historical and theoretical background, the goals and components, and analysis methods of the methodology.

The Development of a Constructivist Research Methodology

According to Steffe and Thompson (2000), the teaching experiment methodology emerged in a time when many of the research methods available for mathematics educators were borrowed from other fields such as psychology, with a strong reliance on classical experiment designs and psychometrics. There was a need for a model adequate for mathematics education. Moreover, these classical experiment designs and psychometrics aligned with the traditional realist perspective. Therefore, the teaching experiment methodology was developed to provide a means through which researchers could investigate student's construction of mathematical concepts and model how these constructions are used in further constructive activities (Steffe, & Thompson, 2000).

A multitude of studies adopted the constructivist teaching experiment methodology to investigate students' ways of thinking in constructing and understanding mathematical concepts. For example, Hackenberg (2010) conducted a teaching experiment with four sixth-grade students to investigate how they reasoned with reversible multiplicative relationships. Moore (2013) conducted a teaching experiment with two undergraduate pre-calculus students to investigate their angle measure concepts. Tillema (2013) conducted a teaching experiment with three eighth grade students to investigate their multiplication concepts.

The teaching experiment branched out into various forms. Although named as teaching experiment, some researchers used the teaching experiment in different ways. For example, Simon (1995) referred to his work with teachers a "whole-class, constructivist teaching experiment" (p. 114). Simon's (1995) teaching experiment is different from the teaching experiment articulated in Steffe and Thompson (2000) in that the goal of the methodology is to “develop a model of teacher decision making with respect to mathematical tasks” (p. 114).

Cobb, Confrey, DiSessa, Lehrer, and Schauble (2003) categorized the teaching experiment (Steffe & Thompson, 2000) methodology as one type of a design experiment. According to Cobb et al. (2003), a design experiment is conducted to develop theories to better understand complex learning ecologies “by designing its elements and by anticipating how these elements function together to support learning” (p. 9). The different types of design experiments include one-on-one design experiments with a small group of students; classroom experiments in collaboration with a teacher; pre-service and in-service teacher development experiments; school and school district restructuring experiments (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). The methodology I use in this study is the one aforementioned in Steffe and Thompson (2000), which I elaborate on next.

Goals of the Constructivist Teaching Experiment Methodology

According to Steffe and Ulrich (2013), there are two primary purposes of the constructivist teaching experiment. One goal of the teaching experiment is for the researcher to experience the way students learn and reason mathematically through teaching and interacting with students. Another goal is to construct second-order models (from the researcher’s perspective) of first-order models of the *mathematics of students* (which are the students’ mathematical concepts and operations). The two words that constitute the methodology, teaching and experiment align with these two goals: *Teaching* refers to the exploratory process through which the teacher-researcher experiences the students’ thinking and modifications in their ways of thinking.

Experiment refers to the continuous hypothesizing cycle of modeling students' ways of thinking in the moment of teaching and after (Steffe & Thompson, 2000).

Steffe and Thompson (2000) emphasized that the teaching experiment is not to simply explore current understandings of students as in Piagetian clinical interviews, but to explore the students' progress in their mathematical activity over an extended time period. Over this time period, an ongoing process of forming and testing hypotheses in the close work with students is carried out. Through this process the goal of the teacher-researcher in building second-order models of the students' mathematical thinking. Moore (2013) emphasized that the model is of the observer, not an objective model of the students' way of thinking:

> During a teaching experiment, a researcher aims to build viable models of students' mathematical understandings and document shifts in these understandings. These models may become more precise over time, but the models are *never* to be interpreted as one-to-one representations of the students' thinking. The researcher's mathematical understandings, the perspective that the researcher uses during the study (e.g., quantitative reasoning), and the researcher's learning goals for the students shape his models (Steffe & Thompson, 2000). (p. 233)

Components of a Teaching Experiment

The teaching experiment consists of a sequence of teaching episodes conducted over an extended time period, and each teaching episode "includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode" (Steffe & Ulrich, 2013, p.2). In this section, I discuss each of these components in detail. The teacher-researcher works with a research group of which the members are the witnesses of the teaching experiment and provide input in the retrospective analyses of the teaching sessions (Steffe & Thompson, 2000).

Teacher-Researcher.

Steffe (2002) discussed the three main roles of the researcher in the constructivist teaching experiment. First, the researcher has the role of the teacher, to create situations for students to engage in, ask critical questions to encourage students' thinking, and to encourage learning. Second, the researcher is a model builder whom through interactions with children in their engagement in mathematical tasks over an extended period of time, formulate and test hypotheses of children's way of thinking. Third, the researcher is to acknowledge the social considerations in the teaching experiment environment. Because the social interaction involved in cognitive construction is important, Steffe (2002) emphasized the researcher to build a playful attitude and confidence with the student (p. 178).

In the teaching episodes, the teacher-researcher who acts as the teaching agent has preplanned tasks and certain hypotheses of how students might engage in these tasks. The teacher-researcher has a certain goal when posing questions and providing tasks in a certain sequence. When actively interacting with the students, the teacher-researcher makes intuitive and analytical analyses based on the ways students engage in the tasks (Steffe & Thompson, 2000; Steffe & Ulrich, 2013). Thus, the course of the task and teaching episode may change in response to students' reactions and the hypotheses are revised and reformed. Together with the students' engagement in the tasks, the impromptu decision making the teacher-researcher carries out are critical, since they structure the constant forming and testing of hypotheses of the ways of students' thinking.

Witness-researcher.

Because the teacher-researcher is immersed in the teaching and learning process and because this process requires highly cognitive activity for the teacher-researcher, there is a possibility that the teacher-researcher can miss essential mistakes the students made or follow-up questions important to ask. He or she can also oversee small details of body movements or language of students which can be significant in understanding the students' way of thinking. Thus the witness, who is usually the cameraperson, can make comments or suggestions to the teaching agent when needed during the teaching episodes (Steffe & Thompson, 2000; Steffe & Ulrich, 2013). If the witness feels the need to interject, he or she asks the teacher-researcher beforehand and after confirming with the teacher-researcher, poses additional questions or helps clarifying the situation.

Students.

Students are selected through initial interviews carried out to understand the current mental operations and schemes the students can use in their ways of reasoning. The students are usually paired with another student or work individually with the teacher-researcher.

Analysis Methods.

The investigation of mental constructions, which the researcher does not have direct access to, is only possible by me making inferences from observing the physical, observable activities the students carry out. Therefore, the teacher-researcher focuses on the students' visual illustrations/inscriptions, verbal descriptions, and physical gestures. Data is collected from the video recordings of the teaching episodes, student work, and the feedback and field notes taken by the research group members.

The data is analyzed through an on-going analysis of the teaching episodes during the teaching experiment and a retrospective analysis after the conclusion of the teaching episodes (Steffe & Thompson, 2000). On-going analyses involves testing and formulating new hypotheses throughout the teaching episodes based on the ways students engaged in the tasks. When actively interacting with the students, the researchers make on-going intuitive and analytical analyses based on the ways students engage in the tasks (Steffe & Thompson, 2000; Steffe & Ulrich, 2013). Also, after each teaching episode, the teacher-researcher uses existing theoretical constructs or findings from previous teaching episodes to formulate new hypotheses or modify existing ones.

Once the teaching experiment is concluded, the teaching episodes are revisited in retrospective analyses through careful analysis of the videotapes and collected student work from the teaching episodes. Together with the mental records the researchers make through teaching and witnessing the interaction with students, the researcher interprets the interaction with the student and modifies or stabilizes the original interpretations from a prospective view in the retrospective analysis (Steffe & Thompson, 2000). Retrospective analyses of data involves identifying instances that would offer insights in building working models of students' ways of thinking. For such instances, the researcher performs a conceptual analysis (Thompson, 2008) in order to refine his or her models of the students' constructive activities.

Conceptual analysis is a way of analyzing and describing what students might understand when they know particular mathematical ideas in various ways (Thompson, 2008). Conceptual analysis can be done through building models of what particular mathematical ideas students know and what they can comprehend in specific situations;

describing ways of knowing that might be propitious or problematic to students' understanding of the mathematical ideas; or "analyzing the coherence, or fit, of various ways of understanding a body of ideas" (Thompson, 2008, p. 60).

Often, regular research meetings are held to obtain insight from other research members, to come to a consensus on the interpretations and analyses of the students' engagement in tasks, to plan subsequent teaching episodes and tasks, and to tests and reformulate hypotheses.

My Teaching Experiment

My goal in this study was to explore the coordinate systems students construct when locating objects in two- or three-dimensional spaces and to model the mental operations and schemes students use in this process. To achieve this research goal, I conducted a constructivist teaching experiment. As the main teacher-researcher in the teaching experiment, I interacted with four ninth-grade students in their engagement in mathematical tasks over an extended time. Through the interactions, I formulated and tested hypotheses of their ways of thinking and modeled the progress in their mathematical activity (Steffe & Thompson, 2000).

Overview: Timeline and Procedures

I conducted the teaching experiment over a two-year time span from mid-October 2013 through the end of April 2015. Table 1 outlines the timeline of research I followed in this study. In the summer of 2013, I obtained approval from the University of Georgia Institutional Review Board. After I selected two participants through initial interviews, I collected consent forms from the school district, school, participating students and their parents to participate in this study. I will provide a detailed description of the participant selection process in the next section.

Table 1. Timeline of Study

Month/Year	Activity
August, 2013	IRB approval
September, 2013	Preparation of initial interviews
October, 2013	Selection of participants through initial interviews
October, 2013 – February, 2014	19 teaching episodes with Kaylee and Morgan
March, 2014 – September, 2014	Retrospective analysis of work with first pair and planning of initial interviews and teaching episodes with second pair
October, 2014	Selection of participants through initial interviews
October, 2014 – April, 2015	31 teaching episodes with Craig and Dan
May, 2015 – December, 2015	Retrospective analysis of work with second pair

With the first pair of students, Morgan and Kaylee, I worked over five months, starting from October, 2013 through February, 2014 for a total of 19 teaching episodes. After completing the teaching episodes with Kaylee and Morgan, from March, 2014 through September, 2014 I engaged in retrospective analysis of the teaching episodes with Kaylee and Morgan. I also planned initial interviews and teaching episodes for a second pair of students. In late October, 2014, I selected the second pair of students, Craig and Dan. With Craig and Dan I worked over a 7-month time period, starting from October, 2014 through April 2015 for a total of 31 teaching episodes. After completing the teaching episodes with Craig and Dan, I engaged in retrospective analysis. I provide a description of the analysis process later in this chapter.

Different from comparative studies, in which participants are given the identical learning tasks, the nature of my study required learning trajectories unique to each student and pair of students. Naturally, my work with the first pair of students informed my work with the second pair of students. Also, as the sole teaching agent in this study I decided it was more important to focus on one pair at a time to allow a more natural continuation of the teaching episodes, rather than working with two pairs simultaneously.

Site of Research

The site of this study was a rural high school in the southeastern United States. The research group selected the school as the research site for several reasons. First, because the school is a public high school, different from a more selective private school setting, we assumed that it was more likely to represent a typical body of students in the area. Second, the school was interested in the research project and the Principal was willing to have allow students to participate in a long-term research project with us. Finally, the proximity of the school was considered, as we needed to visit the school on a regular basis for an extended time.

At the time of the research project, the high school served grades 9–12 and had an enrollment of approximately 1500 students and employed approximately 90 teachers, with the math teacher population consisting approximately 17% of the teachers (High School Website, name omitted). The students who participated in the teaching experiment were enrolled in either one of the algebra classes or algebra/support classes. 15 out of 22 algebra classes were algebra/support classes, which were co-taught by a pair of a math teachers or a math teacher and another educational assistant. The algebra/support classes were year-long classes following a block schedule, while the other unsupported algebra classes ran for only one semester. Because they were not the main focus of the study, we did not collect or consider other information such as students' school grades, teachers' graduate degrees, years of teaching experience, or free or reduced lunch statistics.

Initial Interviews and Selection of Participants

Members of the research project conducted initial interviews with ninth-grade students who were recommended by the school's mathematics teachers as students to be at various mathematical levels and articulate in expressing their thinking. The research group developed the initial interview guide (Appendix A) in advance and used the same set of tasks for all potential participants of the study. Depending on the students' responses to the tasks, the order or number of tasks the interviewer asked the students to engage in differed.

The initial interviews served two main purposes. The first purpose of the initial interview was to understand the students' levels of units and the current schemes and operations the students could enact in partitioning and sharing situations. In investigating how students might superimpose frames of reference onto space and coordinate measurements in relation to their frames of reference, I hypothesized that the mental operations and schemes involved in coordinating units would be crucial for spatial organization. Here units to entail both the units constituting the frames of reference and the units of measurement that are induced from the frames of reference that are constructed in order to define qualitative spatial relations of elements of the space. Therefore, I used the construct of levels of units coordination (Steffe & Olive, 2000) for student selection.

This sampling type corresponds to Patton's (2002) construct of *theoretical sampling* in which participants are selected through theoretical constructs. Specifically, I decided to select two pairs of students to attend to differences in the sophistication level across the groups but with similar students within each pair. This sampling can be classified as what Patton (2002) describes as *maximum variation sampling* in which participants are selected so that the observations “cut across some range of variation” (cited in Glesne, 2010, p. 45). Therefore, after immediate analyses of the initial interviews conducted by the research group, for the first pair of students, I selected Kaylee and Morgan who both operated as if they could reason with three levels of units; Kaylee as given and Morgan in activity. For the second pair of students, I selected two students, Craig, whom operated as if he could reason with three levels of units in activity, and Dan, whom operated as if he could reason with two levels of units in activity. I present a detailed analysis of the initial interview with each student in Chapter 4.

The second purpose of the initial interview was to simply observe personalities or characteristics of students that could inform the pairing process. Although the main criterion of participant selection was based on a theoretical construct, among the group of students that were inferred to be compatible in terms of levels of units coordination, I considered two additional elements when pairing the students. First, I accounted for the students’ ability to articulate their thinking openly and clearly. Because the teaching experiment is based on the active interaction between the teacher-researcher and students and also between the students working together, it was important that they felt comfortable articulating their thinking to others. Second, I looked for students that would work well with each other, without having one student out-shadow the other at a level of confidence and articulation of thinking.

Kaylee and Morgan were both confident and articulate in sharing their thoughts and I learned that they were very close friends. On the other hand, Craig and Dan were both relatively shy but when I asked them to, they articulated their thinking well; based on my observations of their initial interviews, I decided that neither of them would take over the teaching episodes. Other criteria such as race, gender, mathematics achievement grades, and socioeconomic status were not considered as criteria for selecting participants.

Participant Description

Four ninth-grade students Kaylee, Morgan, Craig, and Dan participated in this study. From the initial interviews, I inferred Kaylee to have interiorized three levels of units, meaning that she could reason mentally with three levels of units as given; Morgan reasoned with three levels of units in activity, meaning that she had not yet interiorized three levels of units and had to carry out the activity in order to reason with three levels of units. At least, Craig operated as if he could reason with three levels of units in activity, and Dan reasoned with two levels of units in activity. I present a detailed description and analysis of the initial interviews for each student in Chapter 4.

Although criteria such as race, gender, mathematics achievement levels, and socioeconomic status were not of interest in this study, in this chapter, I will describe the participants in terms of some of these aspects, to the extent that I have knowledge of. These are things I learned mostly through my observations of students, casual conversations with the students before and after each teaching episode, and reflections recorded in my research journal. The purpose of these descriptions are to help the readers construct an image of each student that might help them distinguish each student from each other.

Morgan and Kaylee.

Morgan and Kaylee were both enrolled in ninth grade nonsupport mathematics classes together and were close friends. They said that they went to the beach together with family over the summer prior to starting the teaching experiment. Morgan and Kaylee were in similar classes and both participated in some of the same extracurricular activities such as choir and cheerleading club. Due to their familiarity with each other and their personal dispositions, both Morgan and Kaylee worked well with each other. There were some instances when the two were competitive in coming up with a quicker answer to a question I posed. However, most of the time Morgan and Kaylee did a good job in sharing their thoughts and listening to each other, even when I did not prompt them to do so. There were some instances when their strong collaboration in solving problems made it difficult for me to make a distinction between Morgan and Kaylee's original thinking.

I first met Morgan on October 22, 2013 in her initial interview. Morgan is a Caucasian female student and her mother was a math teacher at the same school. She seemed to be quite comfortable working with me and was articulate in expressing her thoughts out loud. She sometimes expressed her being tired due to having to come to school earlier than others to get a ride to school with her mother. Morgan was also involved in several extracurricular activities such as choir, cheerleading, and soccer and several times she explained that she was tired from practice. However, this did not seem to affect her engagement in the teaching episodes. She was eager to engage in the tasks and seemed to enjoy her time in our sessions.

I first met Kaylee on October 24, 2013. Because another researcher in the research group conducted Kaylee's initial interview, I saw her in our first teaching episode. Kaylee is a Caucasian female student and was tall for her age. Later from the video recording of her initial interview and my experience working with her, I learned that her speaking and physical actions were of quick tempo. Kaylee was not as talkative as Morgan was but she was articulate in expressing her thoughts when she felt necessary. Kaylee usually finished the tasks faster than Morgan did but also did a good job waiting for Morgan to finish her thinking. Similar to Morgan, Kaylee was eager to engage in the tasks and seemed to enjoy the time she spent in the sessions.

Craig and Dan.

Craig and Dan were both enrolled in ninth grade algebra/support classes but did not appear to be close friends like Morgan and Kaylee were. It did not seem as they knew each other before participating in the teaching experiment. They both liked to talk with me about school life before or after the teaching episodes but they did not talk to each other unless they were asked to do so. Both Craig and Dan had one thing in common: they both agreed that math was difficult and did not enjoy doing mathematics. Throughout the teaching episodes, I made extra effort to encourage both students to give them more confidence in their mathematical activities.

I first met Craig on September 5, 2014 in his initial interview. Craig is a Caucasian male student and wore glasses. Craig was an interesting student in that his verbal expressions and descriptions of situations were artistic. He tried to use fancy words but sometimes chose vocabulary that did not quite fit within the context. Craig appeared as a creative thinker but required ample time to think by himself. Usually, he needed me to repeat the description of a situation of the task or my questions until he fully grasped the context or question. He also was very articulate in expressing what he understood and what he did not understand. In the latter case, he asked questions until he fully grasped the problem context or question. Craig was quite competitive when working with Dan in that he wanted to share his thoughts as soon as he came up with a solution; when Dan came up with an answer before he did, he tried to come up with a solution that was different from Dan's. Craig preferred to work on his own; thus, sometimes it was challenging to get Craig and Dan to work together contrary to Morgan and Kaylee. Craig was deeply engaged in the tasks in most of the teaching episodes; usually when he was not in a good mood, he explained why he was out of focus that day.

I first met Dan on September 8, 2014 in his initial interview. Dan is a Caucasian male student on the football team and had a girlfriend who walked him to the room in which we met. Dan appeared to be very nervous and shy in our first meeting but soon warmed up in the following meetings. In some instances, Dan seemed unconfident and had a difficult time verbalizing his thoughts. However, he was more social than Craig was and was willing to work with Craig when I asked him to do so. Sometimes, when Craig shared his thoughts and Dan thought Craig obtained the "answer," Dan gave in saying things such as "he's right; I'm wrong." Although Dan seemed less confident in solving the tasks, Dan appeared to be willing to engage in the teaching episodes.

Structure of Teaching Episodes

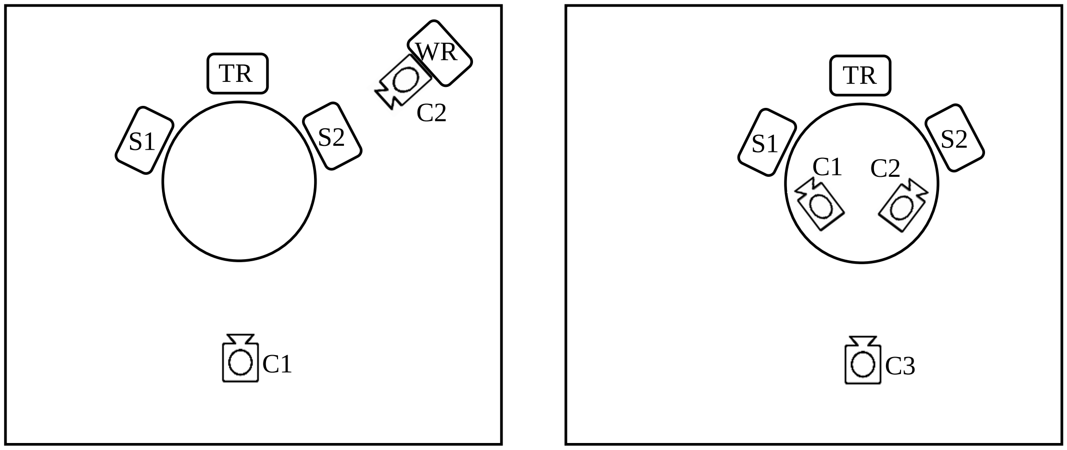
The teaching episodes were held twice a week for both pairs of students. In between teaching episodes, I took a week off approximately every other three weeks to allow time for analyzing data, planning subsequent episodes, and developing tasks. The teaching episodes with Morgan and Kaylee lasted for approximately 20–25 minutes held in faculty meeting rooms. The teaching episodes with Craig and Dan lasted for approximately 15–20 minutes held in a faculty break room. The reason why I worked with the second pair of students for a longer period was mainly due to the change in their bell schedule; I was given less time for each meeting.

In the work with the first pair of students, I acted as the teacher-researcher in the teaching episodes and there were one or more witnesses from my research group. In the work with the second pair of students, I was the sole teaching agent involved in the teaching episodes. The research group members met approximately once a week and participated in post-session analyses based on the video recordings and student work. The research meetings also served as a platform for brainstorming new tasks and discussing plans for upcoming teaching episodes. The research group members consisted of graduate students and Dr. Steffe, all of whom were familiar with the theoretical constructs that guided the study and the teaching experiment methodology.

For each initial interview and teaching episode, I constructed a general plan based on hypotheses and potential trajectories. The plan included a description of the situation of the task, the goal of the task, several questions to start off the session and some questions I could ask depending on various responses. However, I was also open to changing the course of the teaching episode, depending on the students' responses and reactions to the task. Therefore, the plan of episodes was semi-structured.

Physical Environment of Teaching Episodes

Figure 3.1 depicts the configuration of the room for each pair of students. As shown in Figure 3.1, the students sat on each side of me, the teacher-researcher (TR). In Morgan and Kaylee's case, there were two cameras setup in the room.



TR: Teacher-Researcher
WR: Witness-Researcher
S1/S2: Student 1/Student 2
C1: Stationary video camera for wide angle
C2: Handheld video camera operated by WR

(a) *Kaylee and Morgan's room*

TR: Teacher-Researcher
S1/S2: Student 1/Student 2
C1/C2: Video camera stationed on table for S1/S2
C3: Stationary video camera for wide angle

(b) *Craig and Dan's room*

Figure 3.1. Room configuration for teaching episodes.

As depicted in Figure 3.1 (a), I placed one camera (C1) on a tripod in front of the worktable to record a full view of the students and activities in a broader scope. The witness researcher (WR) handled another video camera (C2) while walking around the room to capture close-up activities of the students. The WR focused on students' hand motions or activities on paper to capture the process of their work.

In Craig and Dan's case, there were three cameras setup in the room. As depicted in Figure 3.1 (b), I placed one camera (C3) on a tripod in front of the worktable to record a full view of the students and activities at a broader scope. I placed two other video cameras (C1 and C2) on mini-tripods set on the table to capture Student 1 and Student 2's activities up-close, respectively. This configuration allowed more stable close-up recordings of both students' activities without relying on another researcher going back and forth from one student to another.

In most of the teaching episodes, I asked the students to work individually on the task until both students were finished organizing their thoughts and then asked each student to share their solutions. This was to observe each students' original activities that were carried out independently. In many cases, I also asked each student to compare, contrast, or critique their partner's solutions. This was to observe instances where one student might assimilate the solution or strategy the other student used in solving the task. Because an important aspect of the teaching experiment methodology is to engender and model shifts in students' schemes and operations, it was important to open opportunities for the students to learn from each other through interactions.

Data Sources

In addition to my in-the-moment observations of students' mathematical activities, the data sources I collected and compiled for analysis included video recordings and annotated transcripts; written excerpts, task artifacts, and written student work; and, my research journal.

Video recordings and annotated transcripts.

There were at least two video cameras recording each session, including both initial interviews and teaching episodes (see Figure 3.1 for configuration of rooms). For each session, I compiled all relevant video recordings into one video file, using a video editing software. When editing videos, in Kaylee and Morgan's case, I arranged the video from the wide-angle camera on the left and the video from the close-up camera on the right side of the frame. In Craig and Dan's case, I placed the video from the wide-angle camera at the top middle of the frame and the two videos from the close-up cameras below that, so the close-up views corresponded to the student sitting on that side in the wide view. These configurations allowed a simultaneous monitoring of the students' actions and verbalizations that were not necessarily captured in one camera but the other.

For the purpose of this study, I selected a total of 10 compiled videos for Morgan and Kaylee and a total of 20 compiled videos for Craig and Dan to analyze. The teaching episodes I did not select were exploratory sessions in which I asked students to engage in quantitative coordination, which I do not discuss in this dissertation. For the 30 teaching episodes I selected, I constructed annotated transcripts. Along with transcription of the dialogue, I also added screen captures or scanned student work when necessary, to illustrate the students' actions or the task artifacts that the students manipulated. Constructing these transcripts were helpful in that I was forced to re-live the interactions with my students and describe their actions at a micro-level. Through this activity I was able to observe things I did not notice before.

In the transcript documents, as I noticed patterns or irregularities in the data, I left notes of analyses. I also color-coded parts of the transcripts by highlighting events/themes that I found common across both students or the change/shifts in one students' thinking. In the earlier versions of the transcripts, I transcribed the students' and TR's dialogue verbatim but in the later versions of my transcripts, I was more selective in the transcribing process. Instead of transcribing the entire video and then identifying instances that provided insights into students' thinking, I first identified those instances after watching the video multiple times and constructing a map of major events in the teaching episode. Then, I transcribed the relevant parts to those instances, which served as Excerpts in this dissertation. I found the latter approach to be more productive in that it allowed me to see the bigger picture of the patterns/irregularities and changes in events in the teaching episode and saved me from having to transcribe portions of the teaching episodes that did not serve much purpose.

Written excerpts, task artifacts, and students' written work.

After each session with students, I collected all relevant material. These included the copy of the written excerpts I prepared for the teaching episode, task artifacts that were used in the teaching episode, and written student work. For task artifacts were not manipulated or difficult to preserve (e.g., fish tanks), I took pictures to store digital images of them. I scanned all written student work and filed both the digitalized and original work by date and pair for data analysis. These materials were used in describing the tasks, transcribing the teaching sessions, and analyzing students' mathematical activities.

Research journal.

I use the term journal loosely to mean record-keeping of various things in various formats. Sometimes I wrote in notepads by hand and sometimes I wrote in running documents saved as digital files on my computer. The things I kept record of changed as I transitioned from one stage to another in the teaching experiment.

After the teaching episodes, I wrote reflections in which I described the interactions between the students or with the students, explained the instructional decisions I made or questions I asked, and/or documented the various feelings and emotions I had in teaching episodes. One of the reasons I found it important to record these things was to attend to reflexivity, as cited by Glesne (2010), "an awareness of the self in the situation of action and of the role of the self in constructing that situation. (Bloor and Wood 2006, 145)" (p. 150). Indeed, the teaching episodes were co-

constructed by all participants, including myself. In addition, because I constantly formed and tested hypotheses and at times made impromptu decisions during the teaching episodes, it was important to keep record of these processes. Often, I added retrospective notes to the semi-structured written excerpts to denote the actual implementation of the teaching episode.

During the planning and analyzing processes, I kept notes from the research meetings to document the discussions we had in developing tasks, writing excerpts, and analyzing data. These notes were helpful in developing subsequent teaching episodes and in analyzing data.

Task Development and Overview of Tasks

Ongoing Task Development and Overview of Tasks.

The tasks that I developed during the teaching episodes and the trajectory of the teaching experiment emerged as I worked with the students. Although the tasks differed for each pair of students, there were two overarching task design principles that guided the task design. One was to provide tasks that I considered to be in the students' zones of potential construction (Steffe & D'Ambrosio, 1995), providing opportunities for the students to move forward towards constructing representational space (Piaget & Inhelder, 1967) by use of coordinate systems. The second was to embed tasks into situations that could be experientially real to the students (Gravemeijer & Doorman, 1999) and to provoke active engagement in the task. Because my focus was on students' constructive activities, I wanted to create situations that students could potentially engage in actively and enact or construct operations and schemes (von Glasersfeld, 1995).

Throughout the teaching experiment, I emphasized to the students that I was more interested in learning about how they thought when solving the problems and not whether they achieved a "correct" answer. When I asked students to explain their solutions, I asked the students to express their thoughts out loud to help me understand their ways of thinking. For each initial interview and teaching episode, I constructed excerpts to guide the session. These excerpts were general plans including a description of the situation of the task, the goal of the task, several questions to start off the session and some questions I could ask depending on various responses. However, I was also open to changing the course of the teaching episode, depending on the students' responses and reactions to the task. Therefore, the plan of episodes was semi-structured.

Table 2 provides an overview of the tasks and dates the students worked on the tasks. In this table, M, K, C, D each refer to Morgan, Kaylee, Craig, Dan, respectively. In the table I specify which student was present at the teaching episode in parentheses only when one student of the pair was present.

There were two types of tasks; Locating Tasks and Counting Spatial Objects Tasks. In the Locating Tasks, I asked students to describe the location of a point or the motion of a point in two or three-dimensional spaces in various situations. These tasks corresponded to the first research questions outlined in Chapter 1. To elaborate, through these tasks, I investigated how the students constructed and used coordinate systems when representing points or motion of points in two- or three-dimensional perceptual space. More specifically, I explored how students constructed frames of reference and coordinated measurements within those frames of reference to represent points in perceptual space.

Table 2. Overview of tasks and timeline for each pair of students.

Task Category	Task Name	Date: Morgan and Kaylee	Date: Craig and Dan
Initial Interview	Initial Interview Tasks	10/22/13 (M), 10/24/13 (M), 10/24/13 (K)	9/5/14 (C), 9/8/14 (D), 9/12/14 (D), 9/15/ 14 (C)
Locating Tasks	North Pole Task	11/7/13	1/23/15 (D), 1/26/15
	Fish Tank Task (cubic tank)	11/12/13, 11/14/13	1/30/15 (D), 2/2/15, 2/6/15, 2/9/15 (C)
	Fish Tank Task (cylindrical tank)	11/14/13, 11/19/13	2/20/15, 2/23/15
	School Map Task	n/a	12/8/14
Counting Spatial Objects Tasks	Cubic Block Task	11/21/13, 12/6/13	11/10/14, 11/14/14 (D)
	Floor Tile Task	n/a	11/17/14, 11/21/14, 12/1/14
	Brick Wall Task	n/a	12/12/14, 12/15/14

In the Counting Spatial Objects Tasks, I asked students to count arrays of units in two- or three-dimensional objects such as rectangular floors or cubic blocks. These tasks corresponded to the second research question: How do students coordinate units within two- or three-dimensional spatial objects? More specifically, I investigated how students coordinate their frames of reference when asked to reason about spatial objects that entail arrays of units along two or three dimensions.

As shown in Table 2, Morgan and Kaylee started with the Locating Tasks and ended with a Counting Spatial Objects Task (North Pole Task → Fish Tank Task → Cubic Block Task). On the other hand, Craig and Dan started with the Counting Spatial Objects Tasks and ended with the Locating Tasks (Cubic Block Task → Floor Tile Task

→ School Map Task → Brick Wall Task → North Pole Task→ Fish Tank Task). I elaborate on why each pair worked in a different sequence of tasks in the analyses chapters (Chapters 4–8) because the sequence of tasks emerged as the teaching episodes occurred and as I analyzed the data.

Data Analysis Techniques

In analyzing data, my goal was to build working models of ways students mentally construct frames of reference, coordinate measurements within those frames of reference (thus produce coordinate systems), and count arrays of units constituting two- or three-dimensional objects. Because I am aware that I do not have direct access to the students' ways of thinking, my goal in this study was to build viable second-order models of students' mathematical activity and to document shifts in their ways of thinking. These models are never to be interpreted as one-to-one representations of students' thinking (Steffe & Thompson, 2000).

The investigation of mental constructions, which I do not have direct access to, was only possible by me making inferences from observing the physical, observable activities the students carried out. Therefore, I concentrated on the students' visual illustrations, verbal descriptions, and physical gestures. These elements of our interactions were analyzed based on on-going and retrospective analyses, which I described in the teaching experiment methodology.

In terms of interpreting drawings that students produced, I took Piaget and Inhelder's (1996) account for drawings:

> A drawing is a representation, which means that it implies the construction of an image, which is something altogether different from perception itself, and there is no evidence that the spatial relationships of which this image is composed are on the same plane as those revealed by the corresponding perception. A child may

> see the nose above the mouth, but when he tries to conjure up these elements and is no longer really perceiving them, he is liable to reverse their order, not simply from want of skill in drawing or lack of attention but also and more precisely, from the inadequacy of the instruments of spatial representation which are required to reconstruct the order along the vertical axis. (p. 47)

On-going analysis.

On-going analyses involved testing and formulating new hypotheses during and throughout the teaching episodes based on the ways students engaged in the tasks. I inferred from students' engagement in the teaching episodes instances that corroborated the hypotheses or disproved and these inferences formulated new hypotheses on the students' ways of thinking.

Together with the way students engage in the tasks these hypotheses tested and reformulated throughout the teaching experiment guided the trajectory of the teaching episodes. For example, in case of Morgan and Kaylee, the students were very enthusiastic in sharing their thoughts and building on from each other's ideas. After finding difficulty in understanding how the two students were independently engaging in the tasks, in the research meeting, my research group members and I decided to have the two students work separately on two different computers or separately on paper first and then share their thinking. The analyses not only guided the way I worked with the students and how we had them work with each other but also guided our formulation of tasks. Based on the way the two students engaged in earlier tasks, I designed tasks that would help me further test what we hypothesized from their earlier activities.

Retrospective analysis.

Retrospective analyses of data involved identifying instances that would offer insights in building working models of students' spatial organization. For such instances, I transcribed the video and performed a conceptual analysis (Thompson, 2008) in order to refine my models of the students' constructions of coordinate systems. Weekly research meetings were used to come to a consensus on the interpretations and analyses of the students' engagement in tasks and in testing and reformulating hypotheses.

Overview of Data Analysis Chapters

In Chapter 4, I will present my findings from the initial interviews of the four participants. In Chapter 5, I analyze Morgan and Kaylee's activities in constructing coordinate systems in locating points in space in the North Pole Task and Fish Tank Task. In Chapter 6, I present findings from the Cubic Block Task with Kaylee and Morgan and discuss their different ways of coordinating units within three dimensional objects. In Chapter 7, I analyze Craig and Dan's activities in constructing coordinate systems in locating points in space in the School Map Task, North Pole Task, and Fish Tank Task. In Chapter 8, I present findings from the Floor Tile Task, Cubic Block Task, and Rectangular Prism Task with Craig and Dan to discuss their different ways of coordinating units within three dimensional objects. Finally, in Chapter 9, I will summarize the findings and compare and contrast the activities across all four students. I will also discuss educational implications, limitations of study, and future research directions.

CHAPTER 4

INITIAL INTERVIEWS

In this chapter, I present the initial interviews that were conducted with the four participants. First, I discuss the background of the initial interviews, including the goal and the tasks of the initial interview. Second, I present an analysis and findings from the initial interviews with each student. Finally, I explain how the results of the initial interview guided my teaching experiment.

Background of Initial Interviews

Before entering the teaching experiment, I conducted an initial interview with each participant. In constructing the initial interviews, I adopted partitioning and units-coordinating tasks that were developed and used in investigating students' constructions of number sequences and fractional schemes (Steffe & Olive, 2010). Because I was interested in studying how students might produce coordinated systems of measurements in organizing space and how they might structure multiple spatial dimensions in reasoning, the initial interview was designed to investigate the students' current partitioning schemes and levels of units coordination. Although my study is not focused on students' construction of fraction schemes, because the way students construct fractions can also provide insight for the ways they form nesting systems of parts, the initial interviews also included questions involving finding the fractional amount of parts in relation to the whole.

In the analysis of the initial interviews, I focused on students' partitioning schemes and operations, the levels of units of coordination, units-coordinating schemes and operations, and the distinction between a simultaneous coordination and a sequential coordination of units. An account of said theoretical constructs are elaborated in Chapter Two.

Although the overarching goal of the initial interview was the same for all four students, as I specified above, the tasks that were used in the initial interviews were slightly different for each pair of students (cf. Appendix A). Craig's and Dan's initial interviews were similar to Kaylee's and Morgan's with an exception of starting with the equi-partitioning task and adding a units coordination task series. These tasks were added because I was looking for participants reasoning with two levels of units as given but not three. Each student independently participated in the initial interviews. Each session lasted for approximately 20–30 minutes. The number of sessions and the time admitted for each session differed for each student, depending on various situations that I explain for each student.

Kaylee's Initial Interview

Kaylee's initial interview, held on October 24, 2013, was conducted by another interviewer from my research group at the same time that I interviewed Morgan. Kaylee's initial interview was relatively short compared to the other participants because her actions and verbal expressions, when observable, were very quick and she did not require as many follow-up questions as did the other students. Also, for most of the time, Kaylee was able to solve the tasks mentally, which means that, in these cases, she did not engage in observable sensory-motor activity. In the following sections, I discuss Kaylee's engagement in each task and present my analysis of her partitioning schemes and levels of units coordination .

Distributive Partitioning

Sharing two cakes of same size and flavor.

After the interviewer presented two equal sized cake models of the same color (cf. Figure A.2 in Appendix A), he explained that the two cakes were of the same flavor and size and covered the cakes with a handkerchief. When the interviewer asked her how she would share the two cakes equally among three people, Kaylee said that she would split both cakes into three sections and give each person two sections. When asked what fraction of one cake one person would get, Kaylee answered two-thirds. In explaining how she knew it was two-thirds, Kaylee said, “Because each one is split into three sections and if one person gets two, then they’d have two.” Her explanation showed that Kaylee considered the two cakes as interchangeable, so one of the three parts of one of the two cakes could be used as if it were one of three equal sized parts of the other cake. Moreover, the parts were indistinguishable in that any one of the three parts of a cake could be used in iteration to reconstitute the whole cake. Therefore, Kaylee was aware that each part from the partitioned cakes could be substituted for any other part. In other words, she treated the parts as being of identical size relative to the cakes although the parts were physically distinct as were the cakes they originated from. In that sense, one could say that the parts of the cake were identical as abstracted unit items; they were intentionally made to be of equal size using abstract numerical units comprised by her concept, three (Steffe & Olive, 2010).

Further, when the interviewer asked what fraction of all the cake one person would get, Kaylee said it would be one-third. When asked how she knew it was one-third, Kaylee first explained that one person would get two out of six pieces, which simplifies to one-third. As such, Kaylee's explanation suggested that she has reconstituted the cake pieces into a unit of three units [number of people] each of which contained two units [two parts of the cake]. The interviewer asked Kaylee if there was another way she could explain why one person's share was one-third. Kaylee explained "because there's three people. And so if you're one person, then you would have one-third of what's split up." This explanation corroborated that Kaylee has constructed a composite unit containing three composite units each of which contained two composite units; in other words, a three levels of units structure.

As demonstrated in her engagement in the task, Kaylee was able to carry out the operations mentally, with the two cakes hidden under the handkerchief. As such, the three levels of units structure seemed to be available for her as given. After describing the distributive sharing activity, Kaylee understood that one person's share amounted to two-thirds of one cake and one-third of both cakes. Therefore, I inferred that Kaylee used an assimilatory distributive partitioning scheme in solving the two identical cakes situation, indicative of reasoning with three levels of units as given.

Sharing two cakes of different size and flavor.

This time the interview presented Kaylee with two different sized cake models of different color (cf. Figure A.3 in Appendix A). The interviewer explained that the two cakes were of the different flavor and size and covered the cakes with a handkerchief. Then, the interviewer asked Kaylee how she might find one-third of all of the cake. In solving this task, Kaylee went through four different phases. In the first phase, Kaylee suggested a strategy that assumed the big cake was twice as big as the small cake. The strategy entailed splitting the big cake into half, producing three pieces of the same size. This indicated that Kaylee was aware that one-third of the cake meant producing three equal pieces; therefore, an indication of a coordination between the number of pieces and the size of each piece. Her assumption of the big cake being twice as big as the small cake also indicated her awareness of the size of the pieces in her coordination of number of pieces to produce. However, Kaylee pointed out that she did not know if the big one was actually twice as big as the other one.

The interviewer explained that they did not know the relationship between the sizes of the two cakes, which led to Kaylee's second attempt. Kaylee clarified with the interviewer "How we can split equally among three people?" which corroborated that Kaylee understood finding one-third of the cake was equivalent to sharing the cake equally among three people. Again, this indicated that Kaylee was aware that finding one-third of the cake meant producing three equal pieces. However, Kaylee said she did not know how to do so. It seemed as though she was not able to assimilate the new situation using her distributive partitioning scheme.

To see if having physical models of the cakes in her visual field might evoke her enactment of her distributive partitioning scheme, the interviewer uncovered the cakes, to which reached Kaylee's third phase in solving the task. Looking at the cake models, Kaylee explained "We can split this [pointing to the big cake] into four and that [pointing to the small cake] in two, but I don't know if those would be equal and then each person get two." Again, Kaylee attempted to partition the cakes so that the result of partitioning would result in a number of pieces that were a multiple of three, this time looking at the cake models. However, again, Kaylee acknowledged that she didn't know if the pieces would be equal.

The interviewer repeated Kaylee's explanation that each person would get two pieces and was about to redirect her attention to equal shares when Kaylee demonstrated an assimilation of the situation using her distributive partitioning scheme and further enactments using the scheme. Excerpt 4.1 illustrates this fourth phase in Kaylee's reasoning about finding one-third of two cakes of different size and flavor. I, K, and W each refer to the interviewer, Kaylee, and the witness, respectively.

Excerpt 4.1. Kaylee finds one-third of two cakes of different size and flavor.

I: And each person gets two. Okay, what if we also want…[*interrupted by K*].

K: Oh!

I: Oh?

K: I can split this [*points to the big cake*] into three and this [*points to the small cake*] into three and each person gets a smaller piece [*again pointing to the small cake*] and a big piece [*pointing to the big cake*], if they're split equally.

I: Oh, okay, that's nice, that's nice.

W: Can you cut the cakes using the knife?

K: [*Holding the butter knife to cut the cake*] I don't know if this will be equal but I'll try. [*Lays butter knife onto the big cake as to gauge and mark where to make the first cut*].

I: What are you thinking when you're trying to make sure that they're equal?

K: That I don't want someone to get a smaller piece and not be equal. [*Cuts the big cake into three pieces and takes them apart. Then, cuts the small cake in a similar manner by making marks with the knife first and then making the cuts. Next, after taking the small cake apart, she puts one piece of the small cake along with one piece of the big cake as shown in Figure 4.1*.]

Figure 4.1a. Kaylee cut the two cake models each into three pieces and puts them together into three piles.

[*Continued*.]

I: Okay, can you hand us one of each of our shares?

K: [*Passes out each person's share consisting of one small (yellow) piece and one big (brown) piece*.]

I: Okay, so what fraction of all of the cake does one person get? What fraction of all of the cake do you have?

K: Of both cakes?

I: Yeah.

K: Two-thirds, or, one-third.

I: Okay, and how do you know it's one-third?

K: Cuz if we put these [*pointing to each person's share*] back together that would be three thirds, so I have one.

At the beginning of Excerpt 4.1., Kaylee exclaimed "Oh!" very excitedly as if she realized a way to solve the problem at hand. Then, Kaylee enacted the distributive sharing actions she used in the first case of sharing two cakes of same size and flavor, indicated by her comment that she would split each cake into three and that each person would get one piece from each cake. Although the witness and interviewer asked Kaylee to carry out the distributive sharing actions, because she was able to articulate her plan of actions before carrying out the activity, I believe that the enactment of the actual activity was unnecessary. Further, as demonstrated in the end of Excerpt 4.1, Kaylee understood that one person's share would consist of one-third of all the cake because "if we put these back together, that would be three thirds, so I have one." Therefore, although it took her some time to assimilate the situation using her distributive partitioning scheme, I inferred that Kaylee used her distributive partitioning scheme in solving the task of sharing two cakes of different size and flavor equally among three people. This was another indication that Kaylee had interiorized three levels of units in operating. Yet another indication came in her solving the next task.

Recursive Partitioning

After Kaylee opened a candy strip (cf. Figure A.1 in Appendix A), the interviewer asked Kaylee to imagine the strip of candy with her eyes closed. With her eyes closed, the interviewer asked Kaylee to imagine cutting off one person's share when trying to share this candy equally among three people. After confirming with Kaylee that she had a picture of the cut in her mind, the interviewer asked Kaylee to open her eyes and to show where she might cut off one person's share. In response, Kaylee made a trifold of the candy to assure that she was making an equal share of three pieces in the candy strip. This was demonstrated in her explanation of her actions "I can fold it in to make sure that they are equal." After making several adjustments in her folding, Kaylee cut off one of the three parts as one person's share. Her folding of the candy into three equal parts indicated her projection of a number three template into the entire candy. This activity was similar to the way Kaylee gauged the size of the cake parts with the knife prior to cutting. Therefore, such mental equipartitioning of the continuous units (cake or candy strip) produced identical parts.

Then the interviewer asked Kaylee to imagine the one person's share [first-level share] she just cut off from the candy, with her eyes closed. Further, he asked Kaylee to imagine cutting off one piece when she wanted to share the one person's share among a total of five people [second-level share]. Note I referred to the sharing results as first-level and second-level share to distinguish the steps of sharing in Chapter 2. Kaylee then opened her eyes and was ready to make the cut she imagined. The interviewer asked Kaylee to first explain what she was thinking about before doing anything to the candy. Kaylee explained that she wanted to fold the candy a different way and carried out the folding of the candy to produce five equal parts. After adjusting her five-fold to make it more accurate, Kaylee ripped off one section.

When the interviewer asked her what amount of all the candy is the little piece she just ripped off, Kaylee looked at the pieces on the table for approximately two seconds and answered “one-fifteenth.” The interviewer asked her how she knew that it was one-fifteenth, and Kaylee explained “Because it’s one-fifth [points to the second-level share] of one-third [points to the first-level share] so if you were to multiply that, it would be one-fifteenth. Since there are three sections.” In her explanation of why one person’s share was one-fifteenth of the entire candy, Kaylee demonstrated an awareness of the fractional amount of each part produced by each partitioning and also composed the unit fractions. It also seemed as if Kaylee enacted her recursive partitioning scheme and units-coordinating scheme, which are hinted by her quick response, mentioning of multiplying, and pointing to the “three sections” referring to the original candy. Being able to enact these schemes mentally served in the construction of a unit fraction composition scheme. These observations served as another indication that Kaylee has interiorized three levels of units in operating.

Splitting

When Kaylee was asked to make her string, given the interviewer's wax string was five times as long as her string, she initially folded the interviewer's string in order to make an equal fold of five parts into the string. Kaylee tried to make an accurate five-fold over several attempts. Because our focus was not on the conciseness of the folds but more on the way she wanted to make her string, the witness in the room asked if she could make an estimate. In response to the witness's prompt, Kaylee laid the string straight on the table and placed her fingers along the string while making an estimation and moving that estimation along the string several times. Once she came to an estimate that she felt confident with, Kaylee picked up a new string, laid it next to the interviewer's string, and cut off a string the length of her estimation. When the interviewer asked how she would check whether or not her estimate was a good estimate, Kaylee explained "I can make four others this length then put them next to it to see if it's the same length." When the witness asked her how much the piece of string was out of all of the string, Kaylee answered "one-fifth."

Although Kaylee carried out folding of the original candy, it was apparent that Kaylee had split the teacher's string into five parts prior to carrying out the activity of folding, by the way her folding was intended to produce five parts. Moreover, when asked to make an estimate, Kaylee simultaneously partitioned and iterated a hypothetical string (represented by the gap between her fingers) in order to make the estimate of her string. Then, without prompting, she picked up another string and cut off a string the same length of her estimate, indicating that she was aware of the hypothetical string she was using in operating. Further, Kaylee established the multiplicative relation between her string and the original string, indicated by her response one-fifth as the amount of the string in relation to all of the string. Therefore, I inferred that Kaylee operated as if she had constructed an assimilatory splitting scheme.

Summary of Kaylee's Initial Interview

As discussed in the analysis of Kaylee's initial interview, Kaylee operated as if she had constructed a splitting scheme, a recursive partitioning scheme, and a distributive partitioning scheme that she used in assimilation. Therefore, I inferred that Kaylee operated as if she had interiorized three levels of units, meaning that she could coordinate three levels of units as given.

Morgan's Initial Interview

I conducted Morgan's initial interview over two sessions, one on October 22 and the second on October 24, 2013. In contrast to Kaylee's initial interview, Morgan's initial interview often consisted of her carrying out observable sensory-motor activities in solving the tasks. In the following sections, I will discuss Morgan's engagement in each task and present my analysis of her partitioning schemes and levels of units coordination.

Distributive Partitioning

Sharing two cakes of same size and flavor.

When asked to share two equal sized cakes of the same flavor (cf. Figure A.2 in Appendix A) hidden under a handkerchief equally among three people—her two friends and herself—Morgan initially said that she would let her two friends have all the cake. When I asked her again to equally share the cakes among three people, Morgan mentioned "this is kind of like a pie chart, you can put it into three parts equally" making a peace sign in the air to demonstrate the pie chart. This indicated a projection of her concept, three, into the cake. However, next, Morgan asked if she could combine the cakes by putting them together and “just make a peace sign.” Her comment indicated that her partitioning of the cake using her concept, three, was not distributed to each cake. Rather, she intended to partition both cakes combined together.

Because I wanted to know if she could distribute, or had already implicitly distributed, the partitioning across each cake, I told Morgan that she could not put the cakes together because the frosting and decorations will get mixed up. As a response, Morgan said, “somebody wouldn’t get an equal amount.” I then asked Morgan if it was difficult to share the cakes equally keeping them separate, which she replied to that it was. This comment indicated that she was yet to distribute the partitioning operations to each of the cakes. So, I uncovered the cake models and gave her a butter knife to carry out the cutting activity, to find if seeing the material and carrying out the sensory-motor activity might evoke distributing the partitioning into three parts across both cakes.

Morgan sat in silence looking at the cake models as they were positioned side by side on the plate. Then, she made one mark with the butter knife on each cake as shown in Figure 4.2 (a) and (b) as if she was gauging where to cut the cakes. Following her marking the cakes, she cut along the marks she made and distributed the cake pieces into three piles, each representing what one person would get, as shown in Figure 4.1 (c). As shown in Figure 4.1 (c), each piece she cut off from each cake constituted one person’s share and the remaining of each cake each constituted one person’s share, making three shares in total.

(a)

(b)

(c)

Figure 4.1b. Morgan first sharing of two cakes among three people.

When I asked Morgan how she would know this sharing was fair, she said that she could compare the sizes but was not confident how to do so. Her not being able to explain how the sharing was fair suggested that Morgan was not aware of the relative sizes of the pieces in relation to each cake when she was making the cuts. This corroborated that Morgan's cutting of the cake did not entail a projection of three into each cake, although her cutting off one piece from each cake suggested that Morgan used her concept of three when making the shares, more likely projected on the two cakes imagined put together.

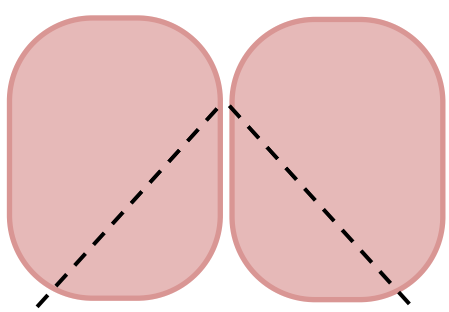


Figure 4.2. A model of Morgan's partitioning of the two cakes.

Figure 4.2 shows how I understood Morgan's cutting and sharing activity she demonstrated in Figure 4.1. The pink figures represent the two cakes and the black dashed line segments represent the cut she made with the butter knife. As modeled in Figure 4.2, although the cakes were not attached, the way Morgan made the cuts in the cake resembled her earlier strategy of putting the cakes together and making a peace sign.

Figure 4.3. Morgan demonstrates what she means by combining the cake and making a peace sign.

Because I interpreted her sharing activity as such, I further prompted Morgan to see if my questions would perturb and activate a reorganization of her partitioning of all the cake into distributing the partitioning to each cake. For instance, I pointed out that each person got a different number of pieces and asked her if there was a way to share the cakes so that everybody received the same number of pieces. I also asked her to imagine each cake being at two different corners in the room. These questions seemed to confuse her more, so I asked her to show what she meant by combining the cakes and making the peace sign. Morgan combined the two playdoh cake models together to demonstrate making a circle and putting a peace sign in the circle as shown in Figure 4.3, which corroborated my model of Morgan's earlier partitioning activity as demonstrated in Figure 4.2. However, Morgan acknowledged that it did not work when she couldn't put the cakes together.

The witness then intervened and asked how much of the smaller pieces that she took off from each cake was of one cake. To clarify the question, the witness took the pieces apart in Figure 4.3 and pointed to the smaller piece and asked what fraction that was in reference to one cake. The following excerpt starts with Morgan's response to the witness's question. In the following and subsequent excerpts from Morgan's initial interview, I, M, and W each refer to the interviewer, Morgan, and the witness, respectively.

Excerpt 4.2. Morgan finds the fractional amount of the smaller piece in reference to one cake.

M: That's a sixth of the cake.
I: This is a sixth of…?
M: [*Points to all the pieces on her plate, taps on the cake 6 times as demonstrated by the red arrows in Figure 4.4*]. Yeah.

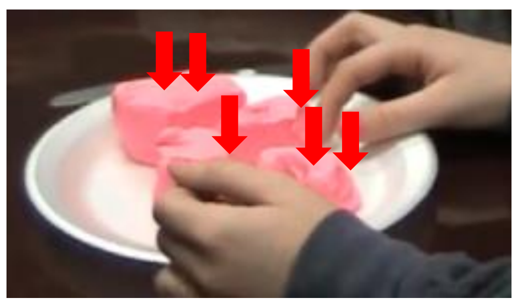


Figure 4.4. Morgan's tapping on the cake pieces six times.

Figure 4.5. The three piles of equal sharings Morgan produced.

[*Continuation*]
I: Okay.
M: So… If I put that in half and put that in half, they'll each have two pieces, right? [*cuts the two bigger pieces into half with her butter knife*.] Then,

that can get two, that can get two, and that gets two. [*Separates her pieces into three piles as shown in Figure 4.5 and smiles*.]

I: That was really good. Okay, so you're saying that this is one person's share, right? [*Points to one of the piles*.]

M: Mm-hmm.

I: So, what fraction is one person's share out of all of the cake?

M: A third.

I: One-third?

M: Yes.

I: Okay, then do you remember how we had two separate cakes at the beginning? What amount would this cake [*referring to one pile of pieces*] be out of one of those cakes?

M: Half? Or… [*Pauses for approximately 5 seconds looking into the air, then touches the pieces with her hand for another 9 seconds*.] Well, a little bit more than a half of the cake, I guess. But I don't know how that would work if I combined them.

Because Morgan explained that one small piece of the cake was one-sixth of all the cake and the way she tapped on the cake pieces as shown in Figure 4.4, again, it was possible that Morgan had already mentally partitioned each cake into three pieces when producing the pieces as she did in Figure 4.1. However, based on several indicators other than the ones I already noted before, I inferred that the witness's question in asking Morgan the fractional amount of one small piece triggered a distribution of the partitioning into each cake *after* she has made the cuts of the cake as shown in Figure 4.1. The first thing that led to such hypothesis was that, although the witness asked the fractional amount of the small piece in comparison to one cake, Morgan replied that it was one-sixth of all the cake. As such, after making the cuts, which produced six equal pieces, Morgan noticed that one small piece was one out of six pieces in total. Further, even though she partitioned each remaining pieces into halves, it did not seem like she reflected on the result of that partitioning. Therefore, it was likely that Morgan was not aware that her second set of partitioning [halving the two bigger pieces] resulted in partitioning each cake into three pieces each. Rather than the partitioning being anticipatory, the pieces were produced *after* carrying out cutting the cakes multiple times, likely using a halving strategy.

According to Piaget et al. (1960), the halving strategy is an intuitive partitioning strategy children use when asked to divide continuous units without necessarily anticipating the result of the halving. Secondly, Morgan said that one person's share in comparison to one cake would be a little more than a half of the cake. This corroborates my hypothesis that Morgan was not aware that the result of her partitioning resulted in partitioning each cake into three pieces. This also indicated that now Morgan considered that the pieces she produced were equal amounts, but the pieces were not identical numerical parts in that one of them could be iterated thrice to constitute a cake partitioned into three equal parts. If the pieces were identical numerical parts as was Kaylee's, there has to be a unitizing of the pieces into abstract units which would have allowed Morgan to operate mentally on the cakes. Further, the pieces would have been treated as indistinguishable in that any one of the three parts of a cake could be used in iteration to reconstitute the whole cake. However, these elements in Morgan's reasoning were not observable. Therefore, although prompted by the witness's question, in the moment, Morgan carried out distribution of the partitioning, it is difficult to attribute this activity as an independent and anticipatory distribution of partitioning.

Sharing three cakes of same size and flavor.

The witness suggested a new situation in which we asked Morgan to share three cakes equally among four people. After I showed her three cake models that we assumed were the same size and flavor, the witness asked Morgan to give one cake to each of the three researchers in the room. After Morgan passed out one cake to each of us, the witness asked her to share the three cakes equally among the four people (the three researchers and Morgan) in the room.

Morgan put the cakes back onto the plate in front of her and stared at the cakes for approximately 17 seconds. Then, she picked up the butter knife and cut each cake into four pieces as I modeled in Figure 4.6. The pink figures represent the three cakes and the black dashed line segments represent the cut she made with the butter knife.

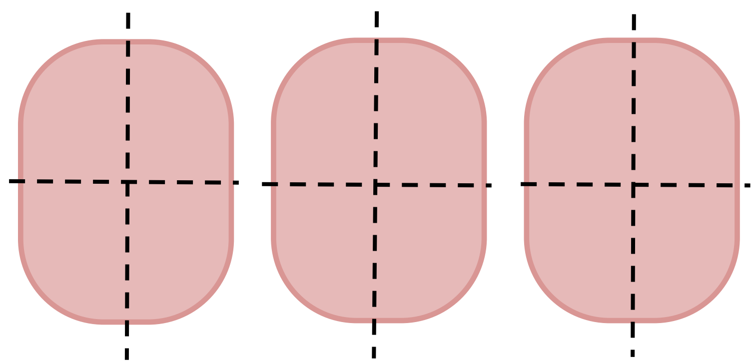


Figure 4.6. Diagram modeling Morgan's partitioning of each of the three cakes.

Starting with one of the cakes cut into four pieces, Morgan placed one piece from the cake in front of each person, saying "if I give you one, then I give you one, him one, and I get one." She repeated the distribution of the pieces for the remaining two cakes, one cake at a time and grinned as if she realized how to share the three cakes equally among four people successfully. Curious to know how she would explain her sharing process, I asked Morgan to explain how she cut the cake. The following excerpt starts with Morgan's response to that question.

Excerpt 4.3. Morgan explains how she shared the three cakes equally among four people.

M: I cut them into one fourths to where we can each get a fourth of each cake.
I: Cool. Okay, so this is your share, right? [*Points to M's share in front of her*.] So, what fraction of all of the cake we started would this share be?
M: Umm.. [looks up in the air, looks back down at the cake for about five seconds], One fourth, right? Yeah… [Moves finger in the air as if pointing to each person.]

I: One fourth? Why do you think it’s one fourth?
M: Because I have three out of the 12 and then I simplified that, so I got one fourth.
W: How much fraction is that out of one cake? How much of one cake is that?
M: Three fourths.
I: That was really fast. How did you know?
M: Well, there were four sections in one cake, and so I had three, so I just put three-fourths.

The relatively long pause of 17 seconds and Morgan’s intent staring at the cake models suggest that Morgan could have been partitioning the cakes figuratively in a trial and error manner. It is likely that this trial and error was carried out using her halving strategy repeatedly. Her execution of the physical distribution of each piece one at a time and her satisfactory smile at the end of her activity also suggest that the partitioning of the three cakes was a novel task to Morgan. Further, the distribution of the pieces was in contrast to that of Kaylee.

In Kaylee’s case, when sharing two cakes equally among three people, after (mentally) partitioning each cake into three pieces each, Kaylee said she would give each person two pieces each, which indicated that she regarded the pieces as identical. On the other hand, Morgan went through each cake one at a time, instead of distributing three pieces from one cake to one person as Kaylee explained she would with the two cakes. This corroborated my earlier hypothesis that Morgan’s pieces of the partitioned cake were equal in size but not yet constructed as identical abstracted numerical units. However, different from the earlier case of sharing two cakes, in reflecting on her activity, Morgan was aware that she had cut each cake into fourths and that each person would get a fourth of each cake. Further, Morgan knew that her share was one-fourth of all the cake and three-fourths of one cake. Morgan’s ability to reflect on her partitioning activity, switch her focus from the composite unit of all the cake to a unit of one cake, and to coordinate those units demonstrated progress from the previous problem.

Based on the way Morgan was able to successfully engage in a distributive sharing activity of three cakes to four people, as she carried them out in activity, one could say that her distributive partitioning operations were latent. In other words, it is possible that she had constructed such operations but did not have the opportunity to organize it in a way that the result of the partitioning would allow her to share the cakes equally among a certain number of people that was different from the number of cakes. However, it seems important to note that this particular context of sharing three cakes among four people was more suited to her halving strategy. That is, halving each cake repeatedly was conducive to her goal of sharing it among four people. Therefore, it is likely that in this particular context, using her halving strategy, Morgan enacted a distributive sharing activity on the model cakes and, as a result, a distributive partitioning scheme emerged. Although she was now aware that her partitioning produced a fourth of one cake and the fractional amount of one person's share in relation to all the cake and one cake respectively, I did not have enough evidence to claim that she had constructed a distributive partitioning scheme that could be used across various distributive situations.

Sharing two cakes of different size and flavor.

Next, I placed two cake models of different size and color that represented different flavors. After covering the two cakes with the handkerchief, I asked Morgan to share the two cakes equally among three people. Morgan sat looking at the handkerchief for approximately five seconds and said "I would cut them up into six pieces then y'all can get two pieces of each cake." The witness asked Morgan if there was a way to make a smaller number of cuts. Morgan responded,

> "I could just cut them into three, instead of putting the slice in the half. If I was going vertically instead of like the peace sign and so instead of doing that I could just do it vertically and hope that they're all the same amount."

By the way Morgan referred to the peace sign and "putting the slice in half," it seemed as though her first response of cutting each cake into six pieces rooted from her recalling the way she shared the two cakes of equal size among three people. That is, she remembered producing six pieces in total from halving the cuts she had made based on her peace sign strategy. Nonetheless, partitioning each cake into an equal number of pieces that would allow her to achieve her goal of sharing the cakes equally among three people was enacted. When redirected to make a smaller number of cuts, without much hesitation, Morgan said that she could cut each cake vertically into three pieces, which suggested that Morgan used her distributive partitioning operations in assimilation. I asked Morgan to demonstrate her new idea of cutting each cake into three pieces and Morgan produced a partition of each cake as shown in Figure 4.7.

Figure 4.7. Morgan's cuts of each of two cakes of different size and flavor into three equal pieces.

Next, I asked Morgan what the witness's share would be, to which she responded by distributing one piece from each of the cake in front of the three researchers in the room. Instead of showing what just the witness's share would be, Morgan distributed all of the pieces exhausting all of the cake. After Morgan distributed all the cake, I asked her what amount of all the cake my share would be. Morgan first counted the number of pieces and simplified two out of six pieces to one-third. When asked to explain differently, Morgan said that one person's share would consist of one-third of all the cake because "if there's three people and you want to split it evenly and so you each want to get one of the third." Although Morgan was aware that each person had the same amount of cake from each cake, she did not consider the unequal sizes of the pieces that each person had, by the way she simplified two out of six to one-third. Moreover, different from Kaylee, Morgan's justification of one person's share being one-third of all the cake did not entail iterating one person's share to make all of the cake. Therefore, although she could produce a three levels of units structure, it did not seem as if she could use it as input for further reasoning.

Based on her engagement in the cake sharing tasks, I inferred that Morgan had constructed distributive partitioning operations as she engaged in the activity of sharing the cakes and pseudo-empirical abstractions (von Glasersfeld, 1991) of the activity led to an emergence of a distributive partitioning scheme in the particular context of sharing three cakes equally among four people. In the cake sharing tasks, the involvement of the witness and interviewer seemed to have triggered reorganizations of her reasoning. Further, I did not observe other indicative activities to impute a generalized scheme that involved a sequence of situations where she made an independent use of the distributive partitioning scheme. Therefore, I could not infer that she constructed a distributive partitioning scheme that she could use in assimilation across various situations.

Recursive Partitioning

The task in Appendix A, Part I, Item 2 was presented to Morgan in a similar fashion as how it was presented to Kaylee in her initial interview. After Morgan closed her eyes and was asked to make the first cut of the candy when sharing equally among three people, I prompted her to show me the first cut on the candy model. Morgan sat for approximately 14 seconds and said she would just cut in the middle but then changed her mind after confirming that she had to share the strip of candy among three people. Morgan placed her finger on the candy strip as shown in Figure 4.8. Morgan's placement of her finger did not quite amount up to one-third of the strip of candy from my perspective.

Figure 4.8. Morgan marks one person's share when sharing the candy equally among three people.

Then, Morgan asked if she had to make only one cut. When referring to "cut" I intended to mean to make one person's share. However, in retrospect, my use of the word "cut" led her to think of having to make all three pieces with only one cut. This issue was later caught and cleared by one of the witnesses in the room. Before this was clarified, Morgan tried to fold the candy and find a way to make one cut to produce three pieces of the candy at the same time. It is likely that Morgan got distracted from trying to produce the pieces with only one physical cut of the candy at the beginning of the task, which may have led to the long pauses in her cutting activity.

Assuming she had one person's share in her mind, I proceeded to ask her to imagine sharing that one person's piece equally among five people and asked Morgan what fraction the mini-piece [second-level share] would be out of the entire candy. Morgan responded that it would be "just one-fifth." After I asked her to explain why it was one-fifth and asked her to make the cuts, one of the witnesses in the room finally realized that Morgan and I were talking past each other and that the situation was not made clear for Morgan. The following excerpt starts with the witness clarifying the question of making one "cut" and how Morgan found the fractional amount of the mini-piece in relation to the entire strip of candy. In the following excerpt, CP refers to the witness who also acted as the cameraperson to distinguish him from the other witness.

Excerpt 4.4. Morgan finds the fractional amount of the mini-piece of candy in relation to the entire strip of candy.

CP: So, you don't necessarily need to make everybody's pieces. Just one person's piece.
M: Oh, just one person's piece! Okay. [*Cuts off one person's share*.]
I: Alright, so... Okay, and then from this piece, I asked you to share this equally among five people, right?
M: This right here? Okay. [*Folds the candy into half, again into half, and again into half…and pinches the folds, while looking carefully at the candy, as if she's trying to count the pieces made out of the folds*.]
W: Why don't you just make a cut without folding? Just make an estimate. Just make a cut.
M: [*Opens the candy up and makes one cut as shown in Figure 4.9, comparable to what I viewed to be one-fifth of the candy*.]

Figure 4.9. Morgan cuts off a piece and then a mini-piece from the candy strip.

[*Continued*.]

W: So, that's going to be one out of five, right?
I: So, this is going to be one out of the…
M: The five, yeah.
I: So, my question was, what fraction would this be [*pointing to the smallest piece*]?
M: Out of the entire thing? [*points to each piece on the table referring to the entire candy*.]
I: Yes, out of the entire thing.
M: Okay, [*after approximately four seconds looking at the candy*] one-fifteenth.
I: How did you get that?
M: Because, [*puts the left-over from the one-third (the four-fifths of the one-third) piece next to the leftover candy*] so, if I made these all into three, then I have one out of the three, and then I cut up to five [*uses hand as if she's cutting the one*-*third piece into five pieces*], so five times three equals fifteen, and this is only one out of that 15.
I: What did you mean by 5 times 3?
M: Well, there are five sections in each thing. So, I would just add up all fives together to get fifteen.

In the end of Excerpt 4.4, Morgan demonstrated distributing the partitioning that produced one of the three pieces into the entire strip of candy and then the partitioning that produced one of the five pieces into each three pieces. Finally, she coordinated the units in activity in order to find fifteen pieces in total. However, Morgan's explanation seemed different from Kaylee's in some notable ways.

When solving the same task, Kaylee abstracted the fractional amounts from each of the pieces she produced based on her understanding of the inverse relation between the size of a share in comparison to the number of people sharing (e.g., one-third of the cake is equivalent to sharing the cake equally among three people). Then, Kaylee also established that the mini-piece was one-fifth of one-third, enacting her unit fraction composition scheme. As such, Kaylee's one-fifteenth was mentally constructed using her recursive partitioning and multiplicative units coordinating schemes in assimilation.

On the other hand, in Morgan's case, even after the witness and interviewer directed her attention to the mini-piece being one out of five of the middle-sized piece [the one cut off as one-third of the entire candy], Morgan mentally produced all pieces by distributing the partitioning that engendered each piece to the remaining of the strip, one at a time. First, she distributed the partition that produced the middle-sized piece to the remaining of the entire candy, resulting in three units. Then, Morgan distributed the partition that produced the one out of five pieces to each middle-sized piece. Each of these three units containing the five units were progressively integrated, demonstrated by her comment "I would just add up all fives together to get fifteen." her referring to the mini-piece as one out of five and the one-third piece as one out of three suggests a partitive fractional scheme,

Although subtle, the difference between Kaylee's and Morgan's reasoning about the one-fifteenth seems to be in the ability to hold the three levels of units structure as given. Kaylee's one-fifteenth was embedded in a structure of a unit containing three units, each of which contained five units. The activities that Morgan demonstrated to carry out sequentially seemed to be enacted in one fell swoop by Kaylee. Morgan's one-fifteenth was obtained as one out of fifteen equal sized pieces produced through partitioning the candy pieces recursively one at a time. Therefore, based on her observable activities, at best I could only infer that Morgan was able to recursively partition the candy in activity. Further, Morgan's reflection of the result of the partitioning did not seem to entail an interiorized three levels of units structure.

Splitting

Placing a wax string in front of her, I first asked Morgan to make her string when the given string—my string—was five times as long as her string. At first, Morgan asked if she could add more string to my string. I told Morgan that she could use any of the material, placing a pile of wax strings in front of her. Morgan then wanted to clarify the question again. Touching my string, she asked “so this is yours, right?” I confirmed that the string she was given was my string and repeated that my string was five times as long as her string. Morgan then asked if she could make my string smaller, to which I responded that she could use mine to find hers. As such, it seemed as if Morgan first thought about making my string longer but after I repeated the question, she became aware that her string had to be shorter than mine. After moving her right index finger along my string, Morgan placed her finger as shown in Figure 4.10. Then, pointing to the part with her other hand, she said “mine’s that long.”

Figure 4.10. Morgan marks on interviewer’s string how long her string is.

When asked to explain why she made her estimation of the length of her string where she did, at first, Morgan made a general comparison that her string had to be shorter than mine. When further questioned how she would check if my string was really five times longer than hers, Morgan explained how she would “measure [her string] out.”

Morgan moved her finger as if she was making a mark where the first mark was (Figure 4.10) and moved her hand to her left along the string once as if she was copying the part she marked. However, she did not mention pulling the piece out or specify how many times she would copy it to check if it was a good estimation. It seemed as though Morgan was focused on copying the length of her estimation rather than focused on the number of units in relation to the size of the parts and the whole.

When the witness asked her how much of my given string was of her string, Morgan replied, “Five. Like, there would be five of mine” as she moved the tip of a marker along the string as if she was iterating her piece five times. Further, when the witness asked Morgan what fraction her string was of mine, she replied it was one-fifth. As such, after the witness asked Morgan questions that required her to specifically account the inverse relation between the partitioning and iterating, Morgan acknowledged that there were five of her strings in my string and that her string was one-fifth of my string. Therefore, it is difficult to impute an independent construction of a splitting scheme to Morgan. At best, I could infer that Morgan has engaged in splitting *after* she partitioned the string into five equal pieces and *after* she was specifically asked to think about the relation between the partitioning and iterating.

Summary of Morgan’s Initial Interview

As discussed in the analysis of Morgan’s initial interview, in many events of the initial interview, Morgan seemed to reflect on the result of her activities *after* carrying them out and being redirected or prompted by the interviewer or witness. For instance, when sharing two same sized and flavored cakes equally among three people, not until the witness asked the fraction of one small piece out of one cake did she further partition each remaining pieces into halves. Morgan then acknowledged that she produced six pieces in total but she did not seem to be aware that the further halving activity resulted in partitioning each cake into three pieces. In other words, Morgan did not demonstrate observable activities to impute an independent and anticipatory distribution of partitioning to each cake. Then, when redirected by the witness to a context more appropriate for using her halving strategy, Morgan reflected on her partitioning activity and reasoned about the amount of one person's share in relation to one cake and all the cake, respectively. Similarly, in the splitting task, not until after the witness asked Morgan questions that required her to specifically account for the inverse relation between the partitioning did she acknowledge that there were five of her strings in my string and that her string was one-fifth of my string.

As demonstrated in the report of Morgan's activities and my analysis, Morgan's interactions with the interviewer or witness were implicated in her continuation of her activities in her initial interview. Therefore, it is difficult to claim that Morgan solved the tasks independently. Nonetheless, Morgan demonstrated enactments of several operations when her attention was redirected by the interviewer or witness. More specifically, in the cake sharing tasks, when given the context of sharing three cakes equally among four people, Morgan demonstrated using distributive partitioning operations and successfully partitioned each cake into fourths, likely using her halving strategy. Also, in the recursive partitioning task, she seemed to enact recursive partitioning operations to find how many mini-pieces in total would fit into the entire candy. In that process, Morgan used her whole number multiplication, although the result of the units coordination seemed more additive than multiplicative in that the composite units of five were sequentially added through progressive integration. Finally, in the splitting task, she seemed to enact the splitting operation. However, although these operations were carried out in activity, the reflection of the results of the operations was not strong enough to impute an independent construction of schemes to Morgan.

In addition, Morgan's observable activities suggested that the pieces produced from partitioning were of equal size but not necessarily identical unit items that were abstracted from the objects representing the pieces. For instance, when sharing cakes of same size and flavor, Morgan consistently carried out the physical activity of distributing one piece from each cake, one at a time. This contrasted with Kaylee's distribution in that Kaylee said she would distribute two pieces each to each person. As such, for Kaylee, the cakes were indistinguishable and the pieces were abstracted unit items that could be replaced by any one of them and iterated so many times to produce one whole cake; it did not matter which cake the piece originated from. On the other hand, for Morgan, the cake pieces were equal in size but the material of the pieces were different—they originated from different cakes. This is why it may have been difficult for Morgan to posit two pieces into one cake when finding the fractional amount of one person's share in relation to one cake when sharing two cakes equally among three people; Morgan thought it would be a little more than a half.

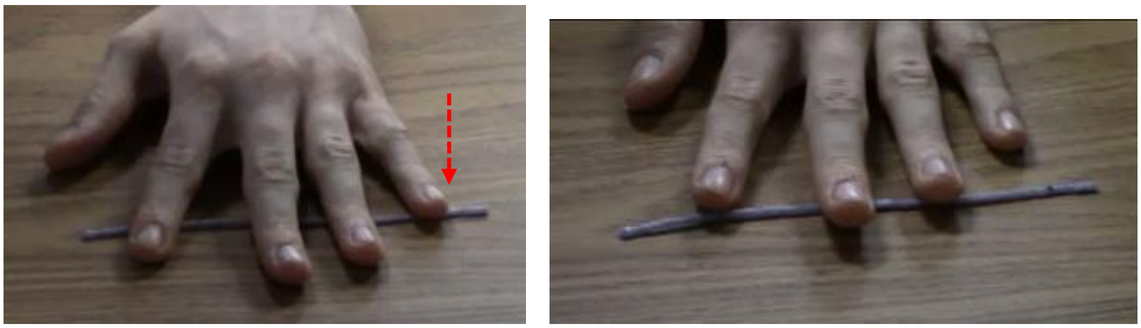
Another example is in her recursive partitioning. Although she was aware of the one-third pieces being of the same size, the way she explained inserting five units into each third piece, one at a time, suggested that each of the five units that were projected into the one-third units were progressively integrated, likely because each one-third piece was not identical—they were different pieces of paper strips. Therefore, although her activity of putting these three strips each of which contained five strips together produced a three levels of units structure, the result of her recursive partitioning did not entail an interiorized structure of three levels of units constituted by abstract unit items. Therefore, from the analysis of the initial interview sessions with Morgan, I inferred that Morgan operated as if she could utilize the operations that produce three levels of units in activity but not coordinate three levels of units as given.

Craig’s Initial Interview

I conducted Craig’s initial interview over two sessions, one on September 5 and the second on September 15, 2014. Similar to Morgan’s initial interview, Craig’s initial interview often consisted of him carrying out sensory-motor activities in solving the tasks. In the following sections, I will discuss Craig’s engagement in each task and present my analysis of his partitioning schemes and levels of units coordination.

Equi-partitioning Task

When I asked Craig to mark off one person’s share when sharing the piece of wax string placed in front of him equally among five people, Craig placed his hand on the wax string as shown in Figure 4.11 (a). He explained that he wanted to use his fingers “to measure it out” and marked with a pen where his little finger ended, indicated by the red dashed arrow in Figure 4.11(a). But after making the mark and taking his hand off of the string, he said that his mark was too small and that the share was probably for more people. In other words, he evaluated that the share was not a fair share because it was too small. So, Craig tried again, using his fingers as a template, as shown in Figure 4.11 (b), saying that he wanted to “make it like five rulers.” This indicated that Craig was using his number concept, five, as a partitioning template. As shown in Figure 4.11, it seemed as though Craig's four fingers indicated the places to mark and the space in between his four fingers and outside of his index finger and little finger were the segments. However, the amount he would need to spread out his fingers to make fair shares did not seem to be well coordinated yet.



(a) (b)

Figure 4.11. Craig marking off one person's share on the wax string using his fingers as a template for partitioning.

Because the mark he made earlier was still on the string, I rolled the wax string over so that he could start with the side of the string with no marks on it. He looked at the string and without placing his hand over the string again, Craig made a new mark as shown in Figure 4.12.

Figure 4.12. Craig's second mark of one person's share on the wax string.

The red arrow in Figure 4.12 points to where Craig made the mark on the string and the dashed segments are copies of the length of the part along the wax string. As shown in Figure 4.12, his estimation marked a little less than one-fourth the length of the entire string.

When I asked Craig how he would check if the marked piece was a fair share, Craig used his thumb and index finger as if he were to hold the length of the part and moved it along the string four times. There was a little bit of wax string left in the end. Next, Craig said that the mark "would probably be in between the first mark and the most recent mark I made." As such, Craig iterated the part he marked off and evaluated how to adjust his mark to make it a fair share. Although he did not explicitly say he would pull out the part and iterate the part four more times to check if it fit in the original string five times, by the way he marked a hypothetical piece with his fingers, and the way he said that the mark should be in the middle of his first mark and the second mark after he counted four times and a little bit left, I inferred that Craig intended to find whether his mark would fit five times into the original string.

Moreover, by the way Craig used his fingers as a template of five to project onto the string, it was apparent that his goal was to partition the string into five equal parts. Therefore, from his observable activities, I inferred Craig to have enacted an equi-partitioning scheme. This suggested that the unit [entire wax string] contained five units [five iterations of the marked part] and that Craig has interiorized two levels of units in operating.

Additive Units Coordination Task

After laying two pieces of pipe cleaner strings on the table as shown in Figure 4.13, I explained that the yellow string and the purple string were each 14 cm and 29 cm long, respectively. Then, I asked Craig how much more string was needed to make the yellow string as long as the purple string.

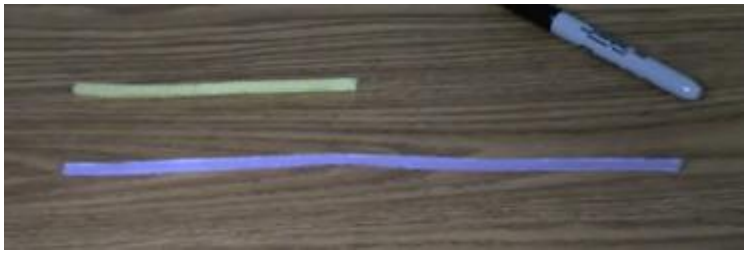


Figure 4.13. Two pipe cleaner strings on the table.

After clarifying what the measures were, Craig said that 14 and a half centimeters was needed. Craig explained,

> "I was thinking fourteen, I saw two lines and I assumed that you wanted to equal them. So I thought okay, fourteen, we double the amount, twenty-eight. And you said twenty-nine, so I realized I'll have to add another half of a number. So fourteen and a half times two would be twenty-nine."

Craig's reasoning of doubling the length of fourteen units of 1 cm [the shorter string] suggested that Craig has constructed iterable composite units. Because it seemed as though Craig did not understand the question I asked him, I presented the question again and asked him to explain what he envisioned when looking at the string. Craig explained that he envisioned two railroad tracks and wanted to make them even. After repeating the question and telling him that the purple "track" could not be adjusted and that we wanted to know how *much more* yellow "track" we needed, Craig sat looking down onto the table for approximately 25 seconds. He then replied "fifteen centimeters long string." I asked Craig how he found 15 centimeters and he explained, "Because fourteen plus fifteen is twenty-nine, so that's how much more centimeters of string you would need."

I wanted to further explore how he found the additional amount of string needed especially because of the relatively long duration of thinking that seemed to be involved in coming to his answer. However, because we have been talking about the two strings for a while, and Craig seemed to have become a little impatient with talking about this question for a while, I moved on to the next question first. I was aware from the planning of the interview that I would be able to visit a similar situation with smaller numbers involved.

So, I cut off some amount of string off of each string as shown in Figure 4.14 and told Craig that we no longer knew the lengths of each string. Then, I asked Craig how many times the yellow string was needed to measure the purple string. Looking at the pipe cleaners, Craig estimated that the purple string was two and a half or three times as long as the yellow string. When I asked him to explain why, Craig said that he pictured moving the yellow string along the purple string and demonstrated his activity, which indicated that he has iterated the yellow string along the purple string mentally, which turned out to be close to be three times.

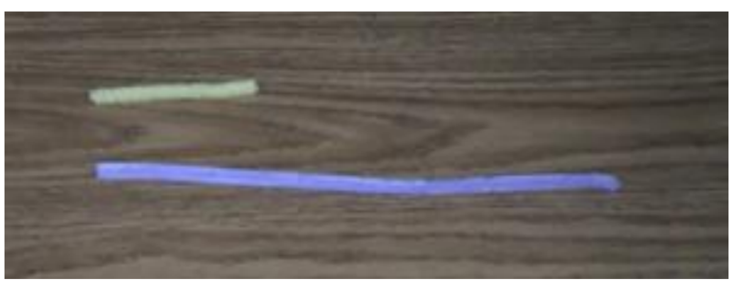


Figure 4.14. The two pipe cleaner string after I cut some amount of each of them.

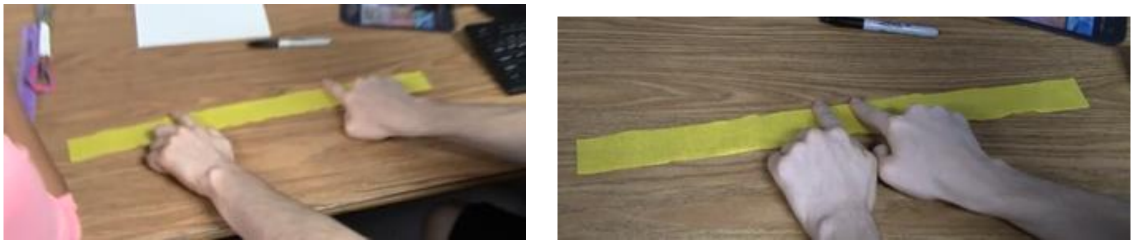
Then I suggested that we assume the yellow string is 7cm long and the purple string is 24 cm long, and asked Craig how much more string I would need in order to make the yellow string as long as the purple one. Craig recognized this situation to be similar to the previous question with the 29 cm and 14 cm long strings and replied "seven plus seventeen…No, seventeen. I subtracted twenty-four with seven." When I asked him why he subtracted four from twenty-four, Craig said that he guessed seventeen first and then he subtracted seventeen from twenty-four to check. This indicated that the 7 and 17 were each unitized units within the 24 and indicated his enactment of disembedding operations. The seventeen and seven units were each posited within the twenty-four, producing two levels of units. From this task, I observed Craig's activities that indicated the construction of the disembedding and iterating operations and iterable composite units. These constructions suggested that Craig could reason with two levels of units as given, at the least.

Recursive Partitioning Task

Based on the way Craig engaged in the equipartitoining and units coordinating tasks, I was confident that Craig has constructed two levels of units as given. The next step was to test whether Craig could reason with three levels of units as given. So, I next presented the recursive partitioning task (Appendix A, Part II, Item 5). With the paper strip representing the candy covered, I asked Craig to imagine making the first cut for one person's share. Without any prompting, Craig said "cut it into a third and then a third, and you'll have three pieces" as if he was thinking out loud. Keeping the paper strip covered, I asked Craig to imagine making the share for one of the five people, to which he replied "I still have a third, I'm thinking. So, it's a fifth of a third." When I asked Craig what amount the mini-piece would be out of all the candy, he looked straight into space for approximately four seconds, and replied "one-fifteenth?" His verbalization of what he was envisioning and his composition of the sharing as "a fifth of a third" and evaluation of the size “one-fifteenth” suggested that Craig was aware of the fractional amount of each part produced by each partitioning and enacted his unit fraction composition scheme. Craig’s engagement in the task so far seemed very similar to Kaylee’s especially in that he could engage in the task mentally, as the candy was covered the entire time.

To further explore how Craig obtained one-fifteenth, I asked Craig to explain what he was thinking when he was staring straight ahead. Craig said he was picturing the third and then the third into fifths and then said “Its’ a third and a fifth so I tried to do one over three and then times one over five and getting the even denominators, I got fifteen, and one times one is one, so I got [one-fifteenth].” Although Craig evaluated the fractional amount of each part produced by each partitioning and said he pictured the third into fifths, Craig’s explanation only revealed the calculations he used to find one-fifteenth. Therefore, I wanted to see if his one-fifteenth was constructed as a reversible multiplicative relationship between the length of the mini-part and the entire strip of paper. More specifically, I wanted to know if Craig was aware of how many of the mini-pieces would fit in the entire strip of candy as a result of distributing his partitioning. So, I uncovered the strip of candy and asked Craig to demonstrate his understanding of the situation.

First, Craig placed two of his index fingers on the paper strip as shown in Figure 4.15 (a), saying “cut into thirds.” Then moving his index fingers further into the middle of the paper strip as shown in Figure 4.15 (b), he said “and then fifths” suggesting that he has partitioned the middle one-third part into fifths.



(a) (b)

Figure 4.15. Craig cutting the paper strip into thirds.

Then he asked me to clarify what I wanted him to show. So, I repeated his explanation of how he calculated one third of one fifth earlier and asked him how he knew that the mini-part was actually one-fifteenth out of the entire strip. In rewording my question, I asked “Do you think you can explain why you think this piece would be one-fifteenth out of the whole thing without using those calculations?” My intention in asking this question was to see if Craig could justify why the mini-part had to be one-fifteenth of the entire strip by coordinating the units multiplicatively. Craig replied that it was one-fifteenth because it was a “very small piece.” So, I questioned him what if I said the piece was one-twentieth. This was to push Craig further to be more specific in explaining why the piece had to be one-fifteenth rather than relying on its “very small” size. Craig said that because he divided it into fifths from the thirds and a fifth of a third was very small, but that he did not know if he could explain it “without the math.” In retrospect, asking Craig not to use calculations may have confused him and prevented him from explicitly demonstrating his units-coordinating operations.

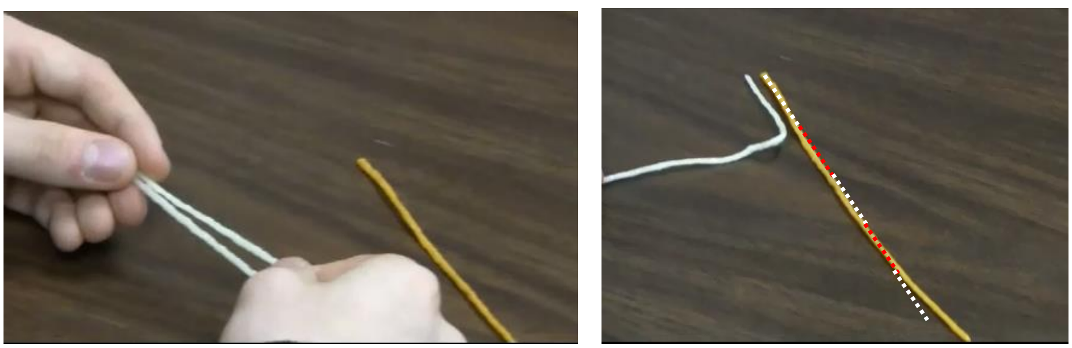
So far, I inferred that Craig has established the goal of finding the size of the mini-part in relation to the size of the whole strip through positing the partial results of the two partitions. Also, Craig demonstrated the ability to take one of the thirds and partition it further into a fifth of a third mentally; thus, partitioning the partitions.

However, in explaining how he found one-fifteenth as his answer, Craig seemed to rely on the numerical calculations of one-fifth of one-third as he learned in school, conflating some of the language such as common denominator. Craig also demonstrated gauging the size of the pieces spatially in relation to the entire strip (e.g., “very small”) rather than coordinating the units multiplicatively. It was apparent from his earlier thinking out loud that Craig was aware of “three pieces” in total when making the first-level share, as he said “cut it into a third and then a third, and you’ll have three pieces.” However, Craig did not explicitly demonstrate the distribution of the partitioning that produced the one-fifth across each one-third part. Craig’s answer of one-fifteenth was more likely derived from taking the fifth of a third relationship and applying an arithmetic procedure he learned in school. As such, I took it to be that Craig did not take the partitions of three units as input to further partition each piece into five parts to produce fifteen pieces in total. Further, his one-fifteenth was not established as a multiplicative relationship (i.e., if the mini-piece is one-fifteenth of the entire candy, then fifteen copies of the piece should make a strip equal in length to the original candy), indicating that his units-coordinating scheme did not serve as an assimilating scheme for constructing a recursive partitioning scheme.

Because the time given for the interview on the first day ended, we re-visited the recursive partitioning task on the second day of his initial interview, after the splitting task. I will discuss the second part of the recursive partitioning task after the splitting task to maintain the chronological order of Craig’s engagement in the initial interview.

Splitting Task

I started the second session of Craig's initial interview with the splitting task. After placing a piece of wax string (his string) on the table and explaining that his string was five times as long as my string, I asked Craig to tell me how long my string was. Approximately seven seconds later, Craig said that my string was "a fifth as long as mine" indicating that he established the inverse multiplicative relationship of the length of my string with the length of his string prior to enacting partitioning activities. Then I asked Craig to make my candy, placing a pile of wax string on the table. Craig picked up a new piece of wax string, bent it into half, put it against his wax string (Figure 4.16 (a)), and made it a little shorter (Figure 4.16 (b)).

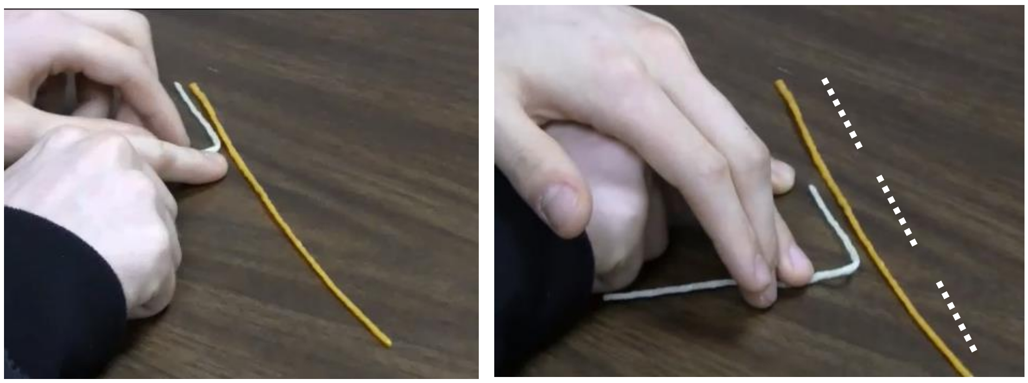


(a) (b)

Figure 4.16. Craig making my string.

Then, Craig placed the piece next to his string and said that my string "would be this long." As demonstrated by the dashed line segments, which are iterations of the length of his estimate in Figure 4.16 (b), his estimation was pretty close to a fifth of his string. These observations suggested that the partitioning of his string was likely carried out mentally prior to the sensory-motor activity in conjunction with the iteration of the hypothetical segment (Steffe & Olive, 2010). When I asked Craig how he would check if that was really my string, he marked the end of my string with his right index finger

(Figure 4.17 (a)), then keeping his index finger fixed, Craig moved my assumed string on the other side of his index finger (Figure 4.17 (b)). He repeated this iteration once more (demonstrated by the white dashed line segments in Figure 4.17 (b)), which brought him to the end of his string, at which point, he said “Three. So, it should be a bit smaller then.”



(a) (b)

Figure 4.17. Craig checking his estimate.

Craig adjusted the length of my string to make it shorter and repeated what he did earlier, which resulted in four iterations. Then, Craig said “maybe a bit more smaller,” readjusted the string to be shorter, and repeated the same activity. After iterating the new estimated hypothetical piece five times, he said “just matching up that way, this is how long I would say your chocolate licorice [interviewer’s wax string] is.” Although Craig did not explicitly count five, the way he immediately knew when to adjust or to conclude his estimation led me to infer that he was finding the length of the hypothetical piece of my string that would fit five times in his string. As demonstrated by the white dashed line segments in Figure 4.17 (b), Craig’s estimation was a little off in that the width of his index finger made a gap in between the iterations in contrast to placing the iterations with the ends adjoined. As such, the physical execution of the activity was misleading but it was clear that Craig intended to make the iterations to find if the hypothetical string would fit five times into his string. Further, he established that my string was a fifth as long as his string as a result of his splitting operation. For this reason, along with his very close estimation of my string on his first attempt, I inferred that Craig operated as if he had constructed a splitting scheme available in reasoning.

Recursive Partitioning Task Revisited

With the candy covered the entire time, Craig and I walked through the sharing situation again until Craig had the mini-part [second-level share] from the candy in his mind. When I asked Craig what fraction that mini-part was from the entire candy, Craig verbally expressed his thought process: “the strip, and then three pieces, two cuts into the first strip so there’s three equal pieces, then three cuts into the second strip, which creates five pieces, I take one of those pieces.” His explanation of his image of the sharing was consistent with that on the first day working on this task, with the exception of conflating the number of cuts—three—to produce the five pieces.

After looking down at the cover for approximately 10 seconds, he asked if I was asking for a fraction and again looked down at the cover for approximately 10 seconds. Because I thought his staring at the cover indicated that he might need the material to further his operating, I uncovered the strip of paper and asked Craig if the situation was clear to him. He explained that “I come back to this impasse of a fifth of a third. Because that seems to be the correct answer to me.” I asked Craig how much a fifth of a third was and Craig said “a very small amount. Maybe this big,” making a mark on the strip using his two index fingers, as shown in Figure 4.18.

Figure 4.18. Craig marks one mini-part in the paper strip.

I repeated again, asking Craig what fraction the part he marked off was of all the candy. After staring at the strip with his index fingers still fixed for approximately seven seconds, Craig murmured “I can try this again definitely.” Next, after staring at his index fingers still fixed for approximately ten seconds, Craig swiftly moved them to his left along the strip as shown in Figure 4.19. By the way he mentioned that he could use the same strategy, and from his behavior in staring at his fingers for a relatively long time, I inferred that Craig was mentally copying the length between his index fingers and moved it along the strip. This behavior of copying and iterating the part was consistent with his observable activities in the equipartitioning and splitting tasks. Craig repeated the activity of iterating the part while counting the number of times he iterated it, to which he reached eleven and a little bit of the whole strip left. After counting, Craig said “Maybe a twelfth” suggesting that the mini-part was a twelfth of the entire strip.

Figure 4.19. Craig iterating one mini-part along the paper strip using his index fingers.

Inferring that Craig has made copies of the mini-part along the strip but not necessarily disembedded a part, inserted five units across the three units, I asked him if he was suggesting that a fifth of a third was equal to a twelfth. Craig replied that it was "close to a twelfth." Expecting the physical activity of cutting the strip might engender a more explicit distribution of partitioning and units coordinating operations, I asked Craig to show one person's share out of three people. Craig placed his finger and the scissor on the strip as to gauge the three shares (Figure 4.20) and cut off one piece.

Figure 4.20. Craig marks one-third of the paper strip.

Then without my prompting, Craig placed the cut-off piece below the left-over piece (Figure 4.21 (a)) and slid it along the left-over piece as if to count how many times it fit into the other (Figure 4.21 (b)). After doing so, he said "three" as if he was confirming that he made a fair share. As such, Craig used his equi-partitioning scheme to partition the candy into three equal parts.

(a) (b)

Figure 4.21. Craig checks if the cut-off piece is one-third of the entire strip.

Next, I asked Craig to make one out of the five people's shares. Craig placed his two index fingers on the cut-off piece as if to gauge where to make the cut. After Craig made the cut, I rearranged the pieces as shown in Figure 4.22 and asked him how many of the mini-pieces would fit into the whole candy.

Figure 4.22. Me rearranging the pieces Craig produced from the paper strip.

Craig picked up the mini-part and moved it along the remaining pieces, in a total of 13 iterations (not including the mini-part). However, he did not count out loud so it was not clear whether he kept track of his counting or not. After that, he said he was going to do it one more time and was about to repeat his activity. I intervened and asked Craig if he could use the numbers that he knew to find out how many times the mini-part would fit in the entire strip without repeating his activity. He recalled that he found the common denominator of a third and a fifth, and got one over fifteen. I asked him if one over fifteen made sense to him, with the mini-part he cut off. The following excerpt is Craig explaining the mini-part as one-twelfth of the entire paper strip.

Excerpt 4.5. Craig explains why the mini-part is one-twelfth of the entire paper strip.

I: How many of the mini-parts would there be in the one out of three people's share?

C: [*Iterated the mini-part along the cut-off piece three times*.] Three, not including this one.

I: So, when I first asked you to make this cut, what were you intending to make?

C: A fifth.

I: A fifth, right? So, if this is a fifth, how many of this pieces would fit into this whole thing [*pointing to both the mini-part and left-over from the one-third part*]?

C: Four.

I: Including this, in the whole thing?

C: [*Repeats the iterating motion to check and once again*.] A fourth.

I: Okay, now, let's say this is a fourth. This is a fourth of this thing. And we knew that this piece was…How many of this piece would fit into the whole thing?

C: Three.

I: Okay, then how many of these pieces would fit into the whole thing?

C: [*Sits staring at the table for approximately 40 seconds*.] Since this is a fourth of this and there are three of this, that is four plus four plus four is twelve.

I: Ah, so the fraction of this would be? So what do you mean by twelve?

C: You can fit twelve of these pieces in the whole thing.

I: So, if I ask you what fraction this is out of all the candy, you would say?

C: A twelfth.

As demonstrated in Excerpt 4.5, I asked Craig explicitly how many of the mini-parts would fit in the one-third share and then how many of the one-third shares would fit into the entire strip. After the distribution of the partitioning was evoked by my explicit prompting, Craig sat in deep thought for a relatively long time and finally concluded that there had to be a total of four plus four plus four pieces in total.

There were two things that seemed important to note. First of all, Craig conflated the number of parts when discussing the second-level share. Craig seemed to be aware of the size—one-fifth—of the second-level share but conflated the number of cuts to make—four—with the number of pieces that would result from those cuts—five. This seemingly loose coordination of the number of cuts and number of pieces suggests that Craig's structuration of a unit of units was difficult for him to keep track of mentally. Perhaps, once he unitized a unit of units, the individual units and the cuts that produced them became blurry. Second, the way he described the way he found twelve was similar to the way Morgan described her answer of one-fifteenth in that both students added the composite unit of four [or five] one at a time, suggesting that the composite units were progressively integrated and inserted into each unit of three sequentially.

The discussion in Excerpt 4.5 went beyond the scope of an initial interview in that the interviewer was heavily involved in Craig's engagement in the task by providing specific prompts and questions. It seemed as though the demonstration of carrying out the sharing one by one guided by my questions evoked recursive partitioning operations. Although Craig showed a local advancement in utilizing recursive partitioning, it was difficult to impute an independent enactment of the operations. This corroborated my hypothesis that Craig was yet to construct a recursive partitioning scheme.

Although the revisit of the recursive partitioning task did not provide any significant observations of a change in Craig's reasoning, it convinced me that me asking him to find the fractional amount of one mini-part without using calculations was not the reason why he did not explicitly coordinate units multiplicatively. To the contrary, it provided more evidence that he was yet to independently distribute the partitions across all pieces recursively and coordinate the units produced by those partitions multiplicatively. However, to my surprise, the interaction with the interviewer in Excerpt 4.5 influenced the way Craig engaged in the subsequent task, which I discuss next.

Splitting/Units-Coordinating Hybrid Task

I placed the three pre-cut strips of paper on the table as shown in Figure 4.23. Then, I explained that my candy (middle length paper strip) is four times as long as Hamilton's candy (shortest paper strip), and that Craig's candy (longest paper strip) was seven times as long as mine.

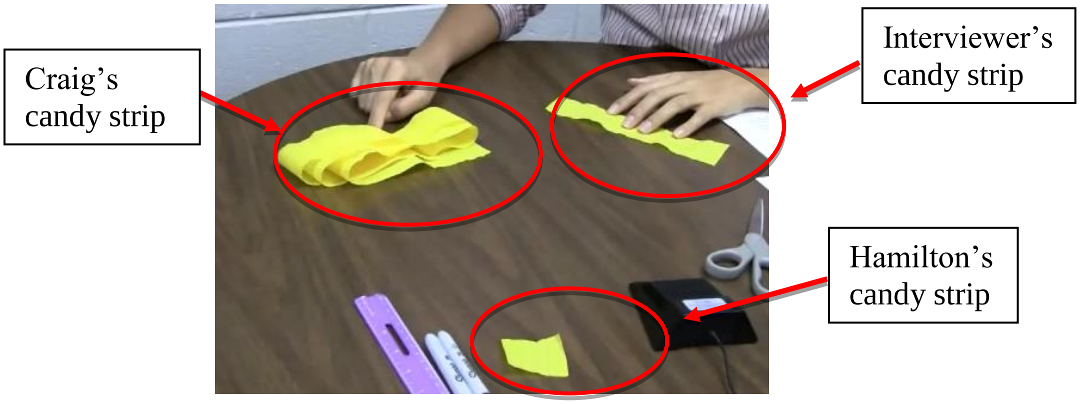


Figure 4.23. The three strips of paper representing candy of different lengths.

When I asked Craig how many times longer his candy was compared to Hamilton's[1] candy, Craig said "four…Would a fourth be correct, and then a seventh? And…I've lost the number." As such, it seemed as though Craig enacted his splitting scheme to establish the relative sizes of the pieces, in relation to the number of iterations needed to make the other piece, but lost track of the numbers. So, I repeated the prompt I presented to him earlier. Once the relationship between my candy and his, and my candy and Hamilton's was clarified, Craig solved for how long his candy was in comparison to Hamilton's candy as shown in the following excerpt.

Excerpt 4.6. Craig finds how many times longer his candy is than Hamilton's candy.

C: [Sits silently for approximately 18 seconds while looking at the table.] I'm thinking that you can put seven of these [pointing to my candy] into this [pointing to his candy]. [Then, sat for another 13 seconds, then looked straight ahead.] Four..Times seven, or… twenty-eight? Twenty… What was the question?

---

[1] Hamilton was the witness/cameraperson of this interview session.

I: So, how many times longer is yours than his?
C: Twenty-eight times longer.
I: Can you explain how?
C: I found a thing that works from figuring out that four of these fit into this third. [*Swept Hamilton's candy along my candy*.] So, four times four times four or four plus four plus four, which is four times three. Got me twelve. So, I did the same method, I fit all of those into there, and then since I know this is a fourth, since seven of those can fit into this, and then I got seven times four. Which is twenty-eight.

Craig's establishment of the one-fourth and one-seventh inverse multiplicative relations to the originating segments were immediate, likely enacting his splitting scheme. However, how to relate the two numbers did not come as immediately, suggested by the relatively long pauses at the beginning of Excerpt 4.6. Craig's comment at the beginning of Excerpt 4.6 of putting seven of my candy into his candy resembled the way I asked him to think about how many of the mini-pieces would fit into the middle-sized piece in the recursive partitioning task. The assimilation of the "method" I guided Craig into using in the recursive partitioning task became more apparent in his comment towards the end of Excerpt 4.6 when he said he "found a thing that works" and recalled what he did previously to find one-twelfth of the entire candy in the recursive partitioning task. Moreover, he explained, "I did the same method, I fit all of those into there." The particular context of the new task became one where Craig could use the same "method" using his splitting scheme to enact the partitioning and iterating of each strip.

Although Craig inserted the four units into each of the seven units, the multiplicative coordination of units seemed to have originated from a reflection on the previous activity, which was implicated by the interviewer's explicit prompts. Therefore, although Craig had produced three levels of units in activity, it was difficult to impute an independent construction of a multiplicative units coordinating scheme to Craig.

Summary of Craig's Initial Interview

In the equipartitioning task, from Craig's observable activities, I inferred that he had constructed an equipartitioning scheme. Craig later demonstrated an enactment of this scheme in finding one of three people's share in the recursive partitioning task. In the units coordinating task, Craig demonstrated the ability to iterate composite units and disembed composite units within another composite unit. Therefore, Craig's engagement in the equi-partitioning and units coordination tasks suggested that he has interiorized at least two levels of units. Craig's engagement in the splitting task indicated that he had a splitting scheme available in reasoning, suggesting that Craig could at least produce three levels of units in activity (Steffe & Olive, 2010).

In solving the recursive partitioning task, Craig established the fractional amount of the sizes of each piece in relation to the whole segment and demonstrated the ability to take one of the thirds and partition it further into a fifth of a third mentally. However, Craig did not distribute the partitioning that produced the one-third and one-fifth parts across the whole strip. His answer of one-fifteenth was more likely derived from taking the fifth of a third relationship and applying an arithmetic procedure he learned in school. Further, his one-fifteenth was not established as a multiplicative relationship between the size of the piece and the number of iterations needed to reconstruct the whole, indicating that his splitting scheme did not serve as an assimilating scheme for constructing a recursive partitioning scheme. In the second time solving the recursive partitioning task, once the partitioning was distributed to each parts, engendered by the interviewer's explicit prompts, Craig seemingly progressively integrated the composite units of four. Therefore, it was difficult to impute an independent enactment of the operations.

Although Craig could produce a three levels of units structure as demonstrated in his splitting task, he could not take that structure as given in further operating. In the hybrid splitting and units-coordinating task, Craig demonstrated an assimilation of the units-coordinating activity that he carried out in the revisit of the recursive partitioning task; however, it was difficult to impute an independent units-coordinating scheme to Craig.

In the recursive partitioning, and hybrid splitting-units coordinating tasks, Craig immediately abstracted the fractional amounts of the partitions of various segments as an inverse relation to how many times the parts could be iterated to obtain the whole, using his splitting scheme. However, the operations that would allow him to relate those results were not immediately enacted. That is, the distribution of partitioning or recursive partitioning of the segments had to be carried out explicitly to further his reasoning. Once the partitions were held across all parts, Craig could produce three levels of units. Therefore, I inferred Craig a student who operated as if he could produce three levels of units in activity.

Dan's Initial Interview

I conducted Dan's initial interview over two sessions, one on September 8 and the second on September 12, 2014. Similar to Morgan and Craig's initial interviews, Dan's initial interview often involved him carrying out sensory-motor activities in solving the tasks. Due to a miscommunication with the front office of the school, Dan's initial interview on the first day was limited in time so I did not have enough time to present the hybrid splitting-units coordinating task. In the following sections, I will discuss Dan's engagement in each task and present my analysis of his partitioning schemes and levels of units coordination.

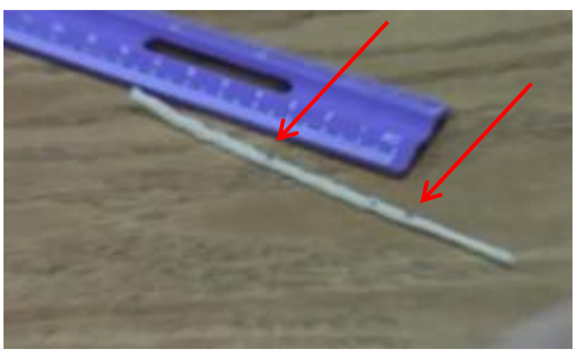
Equi-partitioning Task

When I asked Dan to mark one person's share when sharing a given wax string equally among five people, Dan stared at the wax string and made an estimate by making a mark on the string as shown in Figure 4.24. When I asked Dan what he was thinking when he was staring at the string, he explained "I was trying to figure out how big the pieces will it take" making chopping hand motions above the wax string. Although it is possible that Dan projected his concept, five, into the wax string, he did not explicitly mention how many pieces he wanted to make. Moreover, in the case that he was aware of the number of pieces he wanted to produce, Dan's demonstration of sequential chopping motions could suggest that this projection was not simultaneous as it was for the other students. That is, he could have been making sequential cuts in the wax string, until he made five pieces.

Figure 4.24. Dan's mark of one person's share of the wax string.

Next, pointing to the part he marked off, I asked Dan how he would check if it was an equal share. In retrospective analysis of the video, I realized that Dan had replied "I don't know," which in the interview, I interpreted as "cut it out." Because Dan murmured his utterance, I asked him what he said, but he looked stumped. In response, I asked "I think you said 'cut it out'?" to which Dan immediately agreed and smiled. In retrospect, my involvement in trying to better understand what Dan said unintentionally provided a way of operating that may not have been available to Dan.

After Dan agreed that he would cut it out, I asked him what he would do with the cut out piece. Dan said he would see how much it was. Because his reply was somewhat ambiguous, I asked Dan to show me what he meant. Dan cut the part that he marked off and used the ruler to measure the piece (4cm), and explained “I would start cutting the pieces [points to the leftover piece] into pieces of four centimeters and see if fits and makes an equal share.” As such, Dan did not explicitly state how many times the piece had to fit in the entire string, but focused on making copies of the cut off piece. Dan made two marks on the leftover piece in 4cm intervals as shown in Figure 4.25 (a). Next, Dan marked off the last 4 cm, which was close to the end of the wax string as shown in Figure 4.25 (b).



(a)

(b)

Figure 4.25. Dan marked the leftover string in 4cm intervals.

When I asked Dan what he had, he placed all the wax string in front of him, and counted the number of pieces he produced with his marks one by one, including the cut off piece. The way he checked that he counted the cut off piece indicated that he was aware of the cut off piece being a part of the whole. But he may not have had a strong counting pattern for four he could use mentally in recognition of the four pieces. This also suggested that his partitioning of the wax string was sequential and that he progressively integrated the pieces together one by one, as I hypothesized earlier.

When I asked Dan if the piece was a fair share Dan said it was not, so I asked him to try again. This time I asked him to try it again without using the ruler and gave him a new piece of wax string from a bundle of wax string of the same length and color. I assumed that Dan was considering the new wax string to be the same length as the first one because they were from the same bundle, but Dan and I did not explicitly talk about the length of the new string. When I asked if he would make the piece shorter or longer, Dan immediately chose to make it shorter. Then after looking at the string for approximately five seconds, he said "It was four, so it's probably going to be three," meaning that since the first mark he made was 4cm long, the new mark should probably be 3cm long.

When using the ruler, Dan did not measure the length of the whole string and divide the length into five equal sections. Neither did Dan use the 4 cm he measured out and reason that if the wax string contained four 4 cm parts, then the entire string would be 16 cm long and use that to find out the length of one out of the five shares. Instead, Dan estimated the length of one part using whole numbers of centimeters and chose one smaller than four, that is, 3cm, since he knew it had to be shorter than the last piece. Dan's attention seemed to be more focused on making precise copies of the length of the estimate rather than the number of units and the relation between the size of the part and the number of units comprising the whole. Therefore, I hypothesized that Dan's reasoning using the unit lengths relied heavily on his whole number reasoning and were progressively integrated additively rather than instantiated simultaneously.

As I explained earlier, the idea of cutting off the part he marked did not seem to be initiated by Dan. Further, even after Dan cut off his estimate, the cutting-off activity seemed unnecessary in that it seemed to be used merely as a piece to measure its length within the whole. Based on these observable activities, Dan seemed to reason as if he engaged in equi-segmenting the string.

Instead of marking off 3cm using the ruler, I asked Dan to make an estimate without using the ruler. Dan sat and stared at the wax string for approximately seven seconds and then made a mark on the string. At this point, his earlier estimate of 4cm and the remaining string after he cut off the 4cm was also lying on the table. Dan made a smaller mark and was about to mark another one by using his fingers as if to carry the same length along the string but I stopped him. Using the one mark, I asked Dan how he would check if his new mark marked off a fair share. Dan again sat looking at the string for approximately seven seconds and said that he would "make all the rest of that thing." Again, he did not explicitly indicate the number of times he needed to make in order to make the whole to check if the share was a fair share. When I asked Dan how he would "make the rest of that thing" without using the ruler, Dan sat in thought for a total of approximately 14 seconds and said "using this," pointing to the part he marked. Then, he cut off the piece he marked and slid it along the string (Figure 4.26 (a)) and marked the end of each copy, making three more copies and some remaining in the leftover string as shown in Figure 4.26 (b). The relatively long pause of 14 seconds before deciding to cut off the piece corroborated my hypothesis that his activity of cutting out the marked piece was not necessarily spontaneous.

(a) (b)

Figure 4.26. Dan iterates and marks the cut off piece along the remaining string.

After making three marks, Dan then looked up and said, “that’s five.” Dan did not align the cut-off piece along the last remaining part of the string. However, because the remaining part of the string looked almost equivalent in length with the cut-off piece, I took it for granted that Dan assumed that the last part was equivalent in length to the cut-off piece when he said that that made five pieces. I did not ask him if his cut-off piece was a fair share, but in the interview, I interpreted his comment “that’s five” to imply that he thought it was a fair share.

In any event, Dan saying that there were five pieces in total and stopping his sensory-motor activity after three iterations of the cut-off piece could have indicated that he was aware of using all of the candy up and partitioning it into five equal parts and that he disembedded and iterated the part. Moreoever, when Dan made the last estimate without the ruler, his estimate of one share was very close to a fifth of the wax string. It is possible that Dan’s reflection on his equi-segmenting operation along with my gesture of cutting out the string and limiting him to make an estimate without using the ruler altogether engendered a construction of a simultaneous partitioning scheme. However, my unintentional suggestion of cutting out the part that he marked in the beginning of the task, and his lack of utterance of how many times the part should fit in the whole string when checking if his share is indeed a fair share provide arguments against imputing an equipartitioning scheme to Dan. At best, I could only infer that Dan has constructed a simultaneous partitioning scheme in activity.

Units Coordinating Task

After giving Dan a pipe cleaner string [candy model] and holding one myself as shown in Figure 4.27, I explained that his candy was 15cm long and mine was 24 cm long.

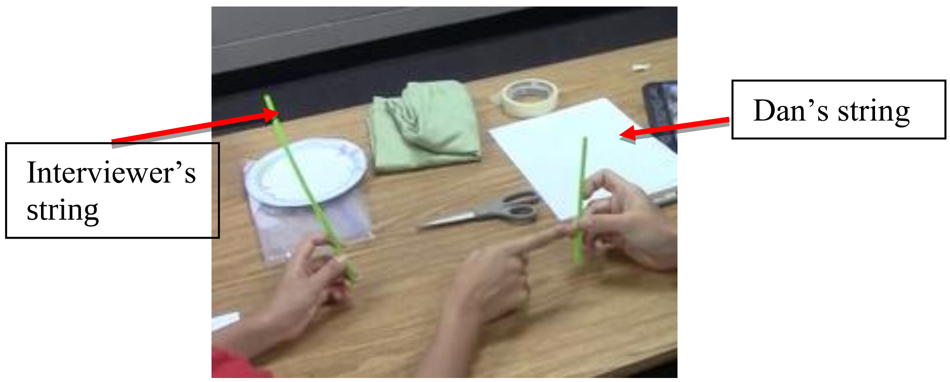


Figure 4.27. My candy and Dan's candy each represented by pipe cleaner strings.

Next, I asked Dan how much more candy he would need to make his candy as long as mine. Dan sat staring at his candy model for approximately ten seconds. During the last three seconds, he picked his hand up and stared at his fingers holding his pipe cleaner candy model. It seemed as though Dan was counting using his fingers. Then he replied "It's nine" which I interpreted to mean that my candy was 9 cm longer. When I asked Dan to explain how he got that, Dan said that he "took fifteen and added up to [*tapping finger on the desk five times,*] twenty-four." This time when he was tapping the desk with his finger, I didn't think he was keeping track of the number of taps, but that he was demonstrating what he did. His tapping on the desk corroborated my hypothesis that Dan was using his fingers in counting up to 24, one by one. I asked Dan to further elaborate and demonstrate how he found nine. Starting with his left hand, Dan tapped on the desk with each finger as he counted, as shown in Figure 4.28. While tapping his finger one by one, Dan first counted from one to nine, saying "One, two, three, four, five, six, seven, eight, nine, well, if I count up to twenty-four, that would equal up to twenty-four. [So] it's fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four." He tapped on the table again in the same order as he counted on from sixteen. The numbers in red in Figure 4.28 demonstrate the order of his counting from one to nine and the numbers in blue in Figure 4.28 demonstrate the order of his counting from 16 to 24.

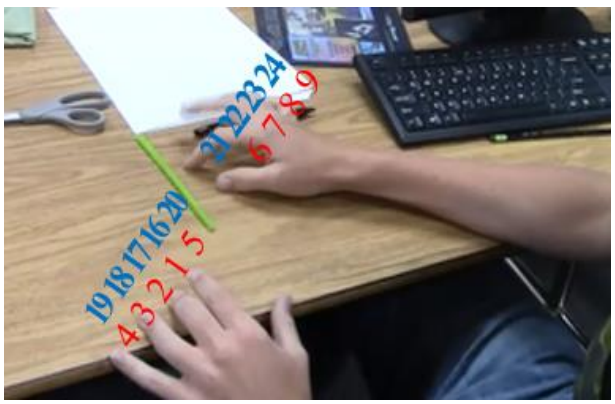
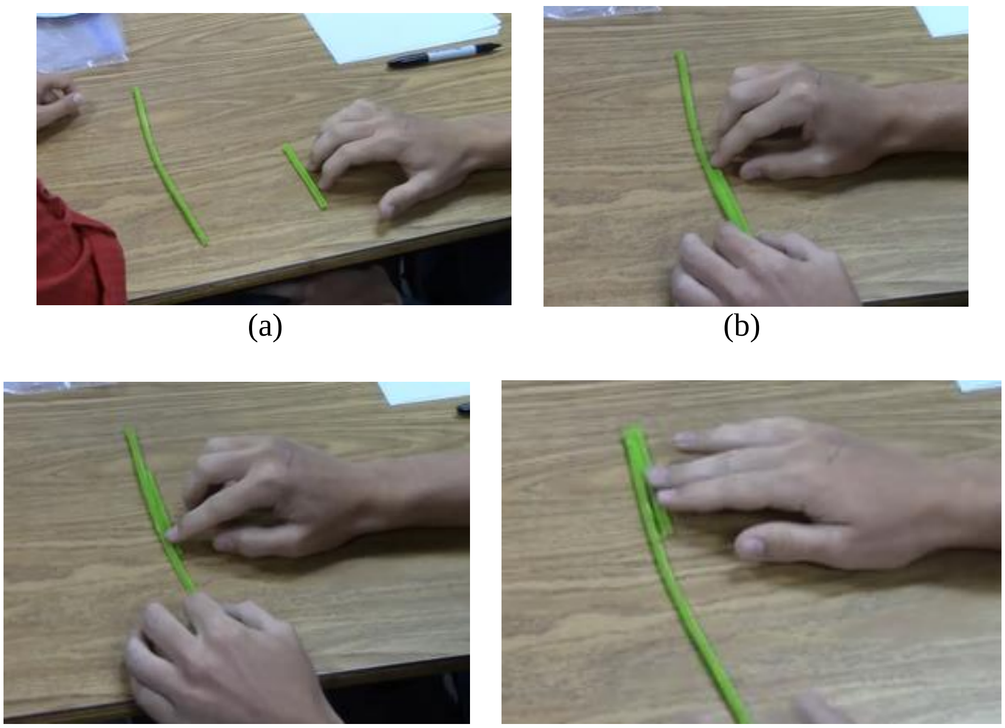


Figure 4.28. Dan counting from one to nine and then using the same fingers to count from 16 to 24.

Although Dan went back to counting starting from 15 after he counted from one to nine, by the way he said he took 15 and counted up to 24 it seemed like his demonstration was not exactly the way he initially solved the problem, but rather a demonstration to explain to me *why* 24 was nine more than 15. From an overall analysis of his behaviors in solving for nine, Dan seemed capable of keeping track of counting up to 24 starting from 15, which suggested that Dan was at least a tacitly nested number sequence (TNS) counter, which indicated the ability to coordinate two levels of units in activity (Ulrich, 2015). I did not gain any observations of certain behavior indicating that he could use 15 or some other composite unit and iterate that to compare with the 29, which would be an indication of an explicitly nested number sequence (ENS) counter and an ability to coordinate two levels of units as given. This observation corroborates my analysis of Dan's equi-segmenting scheme in that it requires an ENS for a student to construct an equipartitioning scheme.



(a) (b)

(c) (d)

Figure 4.29. Dan moving the shorter string along the longer string to find out how many times longer my string is than his.

In continuing the task, I cut off some amount of candy from each of our pipe cleaner strings (Figure 4.29 (a)), and asked how many times my candy (longer string) was longer than his (shorter string). Dan said he didn't know how long the strings were after my cutting. When I asked him how he might figure it out using what he had in front of him, Dan stared at the strings laid on the table for approximately three seconds and then put his candy next to mine as shown in Figure 4.29 (b). Then, after sitting and staring at the strings laid together for approximately three seconds, Dan slid his string along my string (Figure 4.29 (c)) and then after a second, moved his string so the end of his string met with the end of my string as shown in Figure 4.29 (d).

As demonstrated by the pauses in between his actions, Dan's sliding activity was carried out with much concentration and attention given to the string materials in front of him. Dan did not use his finger or any other marks to mark where the end of his string ended. It seemed as if Dan was using his eyes to mentally mark where the shorter string ended before sliding it to the next place. As shown in between Figures 4.29 (b) and (c), the second iteration of his string did not leave enough space for his string to fit one more time into my string, so he moved his string to make the end of his string align with the end of my string as shown in Figure 4.29 (d). This behavior of sliding the shorter string along the longer string but adjusting the sliding so the end of the shorter string aligned with the end of the longer string suggested that Dan's iteration of the shorter string was constrained to being *within* the longer string. Further, after sliding his string as such, Dan said "three times bigger," which suggested that Dan was also looking for a whole number of iterations of the shorter string.

To get a better grasp of his sliding activity between Figures 4.29 (a) and (c), I asked Dan where he decided to stop when sliding the string. Dan explained that he tried to memorize where the string stopped. It was pretty clear that Dan has iterated his string by making mental marks on my string. However, in keeping track of the marks he may have lost track along the way and decided to align the ends of our strings, looking for a whole number of iterations of the shorter string that would fit *within* the longer string. This behavior indicated that he was aware of fitting the shorter string into the longer string but the execution of the iteration was not as coordinated as he may have wanted it to be. Therefore, I suggested using a marker to make the marks to see if that would help with keeping track of the iterations. After marking off his string along mine twice, he looked at the string (Figure 4.30 (a)) and said “two and a half.” When I asked Dan how he could tell the last part was a half, Dan replied “Just kind of looking at it, it’s almost…” at which point he slid the shorter string to align with the last mark he made in Figure 4.30 (a) (Figure 4.30 (b)).

(a) (b)

Figure 4.30. Dan measuring off the last segment of the longer string using the shorter string.

Once I suggested that Dan explicitly keep track of his iteration of the shorter string using a marker, he slid and copied the shorter string to find how many times it would fit into my string and concluded that the longer string was two and a half times longer than the shorter string. Because the time was up for the first interview session, I did not ask Dan the third part of the units-coordinating task I did to Craig. However, from the first two parts of the task, I inferred that Dan’s observable activities did not imply a spontaneous use of disembedding operations.

In the first part of the task, instead of disembedding composite units of the number sequence to configure 24 into 15 and 9, like Craig did, Dan counted up from 15 to 24, one by one. Neither did he demonstrate any other type of strategic reasoning of embedded composite units (Ulrich, 2016). In the second part of the task, Dan attempted to make visual estimations of the shorter string within the longer string and not until after I suggested he used a pen to keep track of his iterations did he explicitly become aware of the number of times the shorter string would need to be iterated to obtain the longer string. The lack of disembedding operations corroborated my hypothesis that Dan was yet to construct an ENS. Therefore, based on his engagement in the equipartitioning task and units coordinating tasks on the first day of the initial interview, Dan seemed to operate as if he could reason with two levels of units in activity.

On the first day of the initial interview I had the impression that Dan found the tasks unusual and was also aware of the cameras and other researchers in the room. Perhaps because of those reasons there was a hesitance in his engagement in the tasks. Occasionally, Dan's explanation of his thinking was short and sometimes he had a difficult time verbalizing his thinking. Therefore, at the beginning of the second day of his initial interview, I had a short discussion with Dan to assure that he felt comfortable working in the particular setting. The second day of Dan's initial interview started with the splitting task, which I discuss next.

Splitting Task

Laying a piece of wax string on the table I told Dan that the string was his licorice, which we knew was five times as long as my licorice. When I asked him to make my licorice, Dan sat staring at his licorice for approximately three seconds. Dan sat in thought looking up from the desk for another three seconds and then looked back down on the wax string for approximately five seconds. Finally, Dan picked up the ruler, placed it along his wax string again as shown in Figure 4.31, and stared at the ruler while picking up a marker, at which point I intervened again.

Figure 4.31. Dan placing the ruler against his wax string.

The reason I intervened was to test whether he could explain his activities before carrying them out. So I asked Dan to explain what he was planning on doing. Dan explained "I'm trying...The same thing I did last time, by finding the measurement and go with that one to see if it'll work. If it doesn't, then try to [adjust]." As such, Dan wanted to use the same steps as he did as the equipartitioning task. The splitting task I presented to him seemed to have provoked his equi-segmenting operation in that the partitioning and iterating were not carried out in conjunction. Like before, Dan did not plan to use the measurement of the entire string to find the length of one share by reasoning about the inverse multiplicative relation between the size and number of pieces.

After looking at the wax string and the ruler as shown in Figure 4.31, Dan made a mark on the wax string at the 6cm mark, saying "six". In the following excerpt, Dan explains why he picked six and finds out if his estimate works. I, D, W each refers to interviewer, Dan, and witness, respectively, in the following and subsequent excerpts in this chapter.

Excerpt 4.7. Dan explains why he guessed 6cm and checks if it works.

T: So why did you decide to pick six out of all these numbers on the ruler [*swipes finger along the ruler*]?
D: [*Pointing to the ruler*] Seven seemed too big, and five seemed too little [*moving ruler away*].
T: Okay. Alright, okay.
D: [Cuts off the part he marked with marker.] Four, I guess.
T: Okay.
W: How were you telling if it was too big or too little?
D: [Inaudible.] Compared to the size [puts hands above the longer wax string to show the length of it,] cuz the...ah...

Figure 4.32. Dan with his hands above the two wax strings to compare their sizes.

[*Continued*.]
T: Okay, so?
D: This looks more than five.
T: Alright.
D: [*Lays cut-off piece along the leftover string, makes a mark with the marker, then slides the piece to that mark and marks off the end of the piece. Then he looks down onto the wax strings in front of him and counts the number of parts in front of him including the cut-off piece*.] Four.

Figure 4.33. Dan laying the cut-off string along the remaining of his string.

[*Continued*.]
T: Hmm?
D: I got four. Yeah, so it'll probably work with five [*picks up ruler again*].
T: What do you mean by probably five.
D: Cuz the six is a little bit too big.

There were several of Dan’s activities that I observed that seem worth noting in Excerpt 4.7. The first was Dan’s tendency on focusing on whole numbers again. Similar to how he engaged in the equipartitioning task, Dan adjusted his estimations in 1cm units, which suggested that he heavily relied on his whole number reasoning. Second, as in the other tasks, Dan demonstrated a tendency in going through the mark-and-slide activity *after* he made an estimation and then decided how to adjust it. Therefore, I hypothesized that Dan’s activities were not anticipatory and required the physical execution of them, again corroborating that he has yet to construct an equipartitioning scheme. As such, Dan’s partitioning of my wax string did not seem to entail a simultaneous partitioning into five in an anticipatory way. Rather, his partitioning was sequential, partitioning one part at a time and progressively integrating them to find if the result gave him the same length as the given string. Therefore, I inferred that Dan was yet to construct a splitting scheme.

Moving forward, I wanted to give him a chance to carry on with his new estimation of 5cm. Because he had already cut off part of his string, I suggested using some other string to make my string as I put the two pieces of his string back together. Although I intended for him to use other strings as material to use when making my hypothetical string, Dan picked up a new piece of string and took the new piece of string as his new string of licorice. The new string of licorice was shorter than his original licorice. I clarified the question several times but Dan stared at the ruler he placed along the new string for approximately 15 seconds and marked off 5cm from the left end of the new string and two more 5cm intervals on the new string. In Figure 4.34, the green string placed next to the ruler was his new string and the purple string to the right of the green string was his original string. Then, looking at the leftover on the new string after marking two iterations of the 5cm part (a total of three 5cm parts were made and there was some remaining string), Dan smiled saying "four again." Realizing that Dan was now working with the new string as his candy, I emphasized that the first string was his original candy. After I explained why I suggested using a new piece of string, Dan replied, "Oh, alright."

Figure 4.34. Dan marks off 5cm intervals on the new string.

In the moment, I thought the conflation between his original string and the new string was due to miscommunication of the situation and that his "Oh, alright" was a comment on finally having a clarification of the situation. However, in retrospect, it is likely that I was imposing a way of operating that was not available to Dan in solving the task. In other words, Dan may have not been able to posit a hypothetical string that was different from his string yet having a length of a specific relationship with that of his string. Therefore, his "Oh, alright" could have been him realizing that he could use another string to solve the problem.

Dan's next actions corroborated the hypothesis because he was yet to use the new wax string as material for making my potential string. Dan took the remaining piece of his original wax string and put it along the ruler as if he was to work with the piece that was left from cutting off the 5 cm piece. I repeated multiple times pointing at what his original string was and emphasizing what his goal was in the task. After such redirecting from me, Dan finally moved the new wax string next to his original wax string and stared at them for approximately five seconds. Then, he picked up the new wax string and cut off one of the 5cm marks he made earlier. Next, Dan moved the 5cm piece cut-off from the new wax string along his original wax string, marking the end of the 5cm intervals, to see how many times the 5cm piece went into it (see Figure 4.35).

Figure 4.35. Dan slide-marks the 5cm cut-off piece along his original wax string.

After counting as he iterated and marked the 5cm piece four times along his original string, Dan concluded it fit five times and said my string would be the cut off piece of 5cm. When I asked Dan what amount of his licorice was mine, he stared at the wax string for approximately four seconds and asked me to repeat the question. I repeated the question this time using more explicit fraction language, asking him what fraction of all his licorice was my licorice. Dan said it would be “one out of five.”

As such, although in the end Dan iterated and marked off the 5cm piece along his original wax string and concluded that my string was one out of five of his string, the process of getting to that point required repeated emphasis on what each string represented, what his goal was (to find my string when his string was five times as long as mine) and several pauses to stare and look at the strings. Therefore, consistent with my earlier hypothesis, it was difficult to impute a splitting scheme to Dan that was available in reasoning. Further, his reply that my piece was one *out of* five suggested that Dan may have established a part to whole relationship between the piece and the entire string *after* cutting a 5cm piece off and sliding and marking one at a time. Again, the reversible multiplicative relationship between the length of the piece and the length of the entire string did not seem to be established in anticipation. The inability to successfully complete the splitting task indicated that Dan operated as if he could not coordinate three levels of units in reasoning.

Recursive Partitioning Task

Dan seemed to be in very deep thought and concentrated on each step of the task as I presented him with the sharing task (cf. Appendix A, Part B, Item 5). After presenting the task and while the candy model was covered, I asked Dan what fraction his share [second-level share] would be of the entire candy. Dan looked down at the cover and sat in silence for approximately 27 seconds, then looked at me smiling. Although he did not explicitly say anything, the relatively long pause and his shy smile implied to me that Dan had a difficult time operating on the paper strip in re-presentation.

To make sure the difficulty in proceeding with the task was not due to uncertainty of what the task was asking for, I decided to ask Dan to explain the context and question of the task. Dan explained “one person will cut it into basically half… Take that person’s share and make six. Cut six.” It seemed like Dan understood that we were sharing the candy in two steps but conflated the number of people we were sharing it with. So, I further asked Dan how many people we were sharing the entire candy with, to which he replied “one”. When talking about sharing *with* other people, this could mean that the sharing *includes* the sharer and it also could be interpreted as *not including* the person doing the sharing. Noticing that Dan conflated the number of people in total with the number of people in addition to share the candy with in the context of the task, I decided to explain the task once more, trying to make it more relatable to Dan. Because I thought the language I used could have confused him in this sense, I worded the situation differently.

The following excerpt starts with Dan and I talking about the context of sharing the long strip of candy.

Excerpt 4.8. Dan solves for the fractional amount of one mini-part of the entire strip of candy.

I: So, we’re sharing among us three [*points to herself, Dan, and the Witness*]. The big candy, right? Now after sharing that, think about your share. [*Waits for approximately 3 seconds*.] Okay?
D: Yeah.
I: Okay, now take your share, we’re going to share your share with five people.
D: [*Sits thinking silently for approximately 10 seconds*.]
I: And you give the rest of the four pieces to other friends and you keep that one share.
D: [*Sits silently while the interviewer continues*.]
I: What fraction is your share out of this whole candy?
D: [After approximately four seconds,] One out of nine.
I: One out of nine?
D: Yeah.
I: Okay, can you tell me how you got that?
D: You got six and the original...I have one and there’s three, so plus six.

As shown in Excerpt 4.8, Dan added the number of pieces from the first level share (three) and the number of pieces he perceived there to be produced from the second level share (six) to obtain nine and said that the mini-part was one out of nine of the entire strip of candy. As such, Dan did not seem to distribute the partitioning that produced each part across the entire strip of candy mentally and coordinate the units produced by the partitioning multiplicatively.

To test if carrying out parts of the partitioning activity might evoke a distribution of the partitioning across the entire string recursively, I asked Dan to make the shares, uncovering the paper strip representing the candy. First, I asked Dan to make his share when sharing the entire candy equally among Dan, myself, and the witness. Dan cut off his share as shown in Figure 4.36 (a). I then asked Dan to make one person's share when sharing his share equally among five people. Dan made a cut after staring at his piece for approximately nine seconds as shown in Figure 4.36 (b).


(a)


(b)


(c)

Figure 4.36. The pieces Dan produced in sharing the long strip of candy model.

After I rearranged the strips of paper that were produced by Dan's cuts as shown in Figure 4.36 (c), I asked Dan if he could explain why the smallest share was one-ninth. Dan looked at the pieces in front of him and said "I think one eight because this one's mine and I already counted mine. Yeah, one-eighth." As such, even after re-evaluating the second-level as five, Dan still maintained adding the number of pieces together when finding the fractional amount of one mini-piece in comparison to the entire strip of candy. In Dan's case, he was able to equipartition the long strip of candy into three units and he was able to equipartition the one-third unit into five units, using his equipartitioning scheme. However, when partitioning the first-level share, the unit was no longer a one-third unit but a new unit that he partitioned into five equal parts. In other words, it is likely that Dan lost track of the relation between the new unit he was partitioning into five and the entire candy. In other words, using his composite unit of units (two levels of units) as input for further operating and further coordinating the units multiplicatively seemed yet to be available in Dan's reasoning. Instead, Dan added the number of pieces that were produced from the partitions, which were the three and five.

Next, the witness asked Dan how many of the smallest pieces would fit into the whole. Dan asked if he could use the shortest piece, picked up the mini-piece, and moved it along the longest strip up to three times and then looked at the strip. Then, Dan said "ten," explaining that he estimated it. As such, when asked how many of the small pieces would fit within the original candy, instead of reasoning reversibly, using his reply of one-eighth, Dan moved the small strip of paper along the remaining paper to count how many times it would fit in the entire candy. This again exemplified Dan's lack of establishing a reversible multiplicative relationship between the number of parts [partitioning] and the whole in relation to the length of one of the parts [iterating]. Therefore, I inferred that Dan was yet to construct a recursive partitioning scheme and produce three levels of units, including in activity.

Summary of Dan's Initial Interview

As demonstrated in the equi-partitioning, units coordinating, and splitting tasks, Dan relied heavily on his whole number reasoning. In the units coordinating and recursive partitioning tasks, Dan operated as if he coordinated units additively. Using such ways of operating, Dan successfully equi-partitioned a given wax string and counted up from 15 to 24 to find how longer a 24cm long string was than a 15cm long string. Also, although the physical enactment was not as coordinated, Dan seemed to have engaged in iterating. The way Dan cut off a part and iterated it along the given string in his second attempt in the equipartitioning task indicated so. However, the operations of partitioning and iterating were not enacted simultaneously as demonstrated by the lack of awareness of the inverse relation of the size of the part and the number of times it could be iterated to produce the whole. Not having constructed an assimilatory splitting scheme suggested that Dan was yet to construct splitting and recursive partitioning schemes. From the observations I made, I inferred that Dan could operate with two levels of units in activity.

Initial Interview and the Teaching Experiment

The initial interviews guided my pairing of the students in conducting the teaching experiment. Not only was I interested in pairing students who seemed comparable in terms of levels of units coordination, I also wanted to pair students that were socially compatible. This meant that I did not want to pair a student who would likely overtake the majority of the activities or conversations in the teaching episodes.

From the pool of potential participants in the fall of 2013, Kaylee and Morgan seemed like a reasonable pair. First of all, from their initial interviews, both were able to produce three levels of units and reason with a three-levels of units structure, either as given or in activity. Second, the two students were both articulate in sharing their thinking. Finally, Kaylee and Morgan seemed to be close friends. I thought that their friendship would make the teaching experiment an enjoyable experience for them and perhaps enhance the level of communication they would have in the teaching episodes.

After working with Kaylee and Morgan, I wanted to select a pair of students who reasoned with less than three levels of units as given. From the pool of potential participants in the fall of 2014, Craig and Dan were two of the few who the research group thought to meet this criterion. From the initial interviews, Dan was the only one who reasoned with at most two levels of units. In finding a reasonable partner for Dan from the other students in the pool, I selected Craig. Although Craig was inferred to reason with three levels of units in activity, he seemed to be the most comparable in reasoning to Dan in terms of units coordination. Also, Craig and Dan both seemed to be laid back but Craig was more talkative, so I thought he would compensate for Dan’s shy and quiet personality and hopefully add more enthusiasm to the teaching episodes.

Second, the initial interviews guided me in developing more specific hypotheses for the teaching experiment. The tasks in the initial interviews required some form of spatial material to be partitioned (e.g., the area of a rectangular candy model, the length of a linear string). Whether the students engaged in the partitioning operations in sensory-motor activity or mentally and whether the students used the actual physical material or on re-presentations of the material as abstracted items, some form of structuration of space (either perceptual or representational) seemed to be involved in their partitioning activities.

For example, Kaylee and Morgan both folded the candy strips in certain ways to make equal shares, which would have required some sort of spatial coordination along with the execution of partitioning operations. In cutting the cakes, Kaylee consistently made cuts in the same orientation (perpendicular to the longer edge) as she did with the candy sharing tasks. On the other hand, Morgan shifted from making a piece sign to making a cross sign and then later to cutting like Kaylee did. Shifting the orientation of cutting the cake to the latter two seemed to allow Morgan to reorganize her partitioning operations that would allow her to share the cakes equally and also keep track of the size of the parts produced by the partitioning.

Craig's iterating activity was not as coordinated spatially, compared to Kaylee and Morgan, when he overlooked the width of his finger he used when marking the ends of the copies. However, he seemed to have planned out the activities before carrying them out, so I inferred that he has constructed an equipartitioning scheme. Also, Craig seemed to conflate the number of cuts and the number of pieces produced from the cuts. However, once he had a certain number of partitions in mind, he could take that as a unit and reason with it as an abstracted quantity. Finally, many of the activities that Dan carried out required him carrying out sensory-motor activities on the spatial objects but they seemed to be restricted by his tendency to rely on his whole number reasoning.

As such, investigation of students' partitioning schemes and levels of units coordination in the initial interviews led me to hypothesize a linkage between these mental operations and schemes and structuration of space. It was unclear whether partitioning schemes and levels of units coordination was necessary for structuration of space or vice versa, or if they were mutually dependent on each other. In any event, I hypothesized that Kaylee's organization of space would be more intricate and systematic than Dan's organization of space. Further, I hypothesized that Morgan and Craig might demonstrate similar ways of operating as Kaylee will, but often rely on having to work with physical spatial models in order to do so. Therefore, the results from the initial interviews also guided my analysis of the teaching experiment. That is, I looked for similar or different behavioral indicators of their operations and/or schemes that I inferred them to have constructed in the initial interviews to test my hypothesis.

Summary of Chapter Four

In this chapter, I discussed the background of the initial interviews and presented the findings from the initial interviews with each student. From the initial interviews, I hypothesized Kaylee to have interiorized three levels of units, meaning that she could reason mentally with three levels of units as given. Morgan and Craig seemed to be able to produce three levels of units in activity. Finally, Dan showed behavioral indicators that he could reason with two levels of units in activity. In closing the chapter, I explained how the initial interview guided my teaching experiment.

CHAPTER 5

KAYLEE AND MORGAN CONSTRUCT COORDINATE SYSTEMS

In this chapter, I present my analysis of Kaylee's and Morgan's constructive activities in the Locating Tasks (North Pole Task and Fish Tank Task) in which I asked both students to locate a point or describe the motion of one point in two- or three-dimensional perceptual space. I designed these tasks to investigate how students organized perceptual space into representational space. More specifically, through these tasks, I explored how the students construct and use coordinate systems when representing points or the motion of a point in two- or three-dimensional perceptual space.

In discussing the North Pole Task, I will describe the ways Kaylee and Morgan each located a point in an irregular shaped and a circular shaped two-dimensional map. In the Fish Tank Task, I will present Kaylee's and Morgan's activities in locating points or describing motion of one point to another in three-dimensional cubic or cylindrical fish tanks. In these tasks, Kaylee and Morgan each constructed frames of reference and coordinated systems of measurements to organize these two- or three-dimensional spaces, which I discuss and model the mechanisms of in this chapter.

North Pole Task: Kaylee and Morgan Locate a Point in Two-Dimensional Space

On November 7 of 2013, Morgan and Kaylee worked on the North Pole Task. In this task, I asked Kaylee and Morgan to provide instructions to a rescuer, Hamilton, on the ground so that he can find a missing person, Ebru, in the North Pole region. Morgan and Kaylee were asked to imagine hovering over the region in a helicopter holding a map that showed the only road of access to the North Pole, as shown in Figure 5.1. The North Pole and the location of the missing person are labeled as points P and A, respectively. Hamilton had another map, a rectangular shaped wax paper with the same point P and one road of access to the North Pole pre-constructed on it. There was a divider on the table so the two parties could not see each other's map.

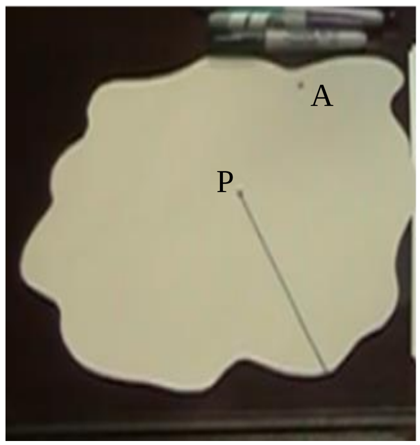


Figure 5.1. Example of North Pole Task map.

Irregular Shaped Map

Understanding the situation.

At the beginning, Morgan looked over the divider on the table to see Hamilton's map and tried to orient her map so that the two line segments representing the sole road to the North Pole were pointing the same direction. Morgan also wanted to know if they could see Hamilton walking in the snow. Morgan's actions in looking at the rescuer's map or asking if she could see the rescuer in the snow in the moment suggested that Morgan wanted to take a more temporal approach by giving in-the-moment instructions to the rescuer from the helicopter.

On the other hand, Kaylee asked if they were supposed to come up with the instructions and then give it to him at once. Kaylee's consideration of developing a complete set of instructions suggested that she was intending to take a less temporal approach by anticipating results of movements of the rescuer without having to see the rescuer in the moment. I clarified that they could not see Hamilton's map nor Hamilton and that their task was to develop the full instructions and then give them to Hamilton.

Kaylee's coordination of angle measure and distance.

Kaylee first connected the two points P and A but soon noticed that telling the rescue team to go straight from P to A was not going to help and that some sense of direction was needed for the rescuer. In order to discuss the direction, Kaylee first thought of using cardinal directions—going northeast. However, she rejected this idea because she was not sure how much northeast of a direction the rescuer would have to go. The two students sat in thought and after a twelve second pause, Kaylee suggested using the angle formed by the line segment representing the road to the North Pole and line segment connecting P and A (Figure 5.2).

Figure 5.2. Kaylee tracing an imaginary angle formed by two lines.

After estimating the measure of the angle, Kaylee then thought about how to use that angle measure. In Excerpt 5.1 Kaylee and Morgan discuss this angle measure. In the following excerpts in Chapters 5 and 6, K, M, and T each refer to Kaylee, Morgan, and the teacher-researcher, respectively.

Excerpt 5.1: Kaylee and Morgan discuss how to give directions to the rescuer to find A.

K: So, it's like a hundred and five degrees…to the right, I guess? I don't know how you would say this, instead of this [making an alternative line segment coming out of the North Pole to the opposite direction of what she drew to connect P and A (Figure 5.3)].

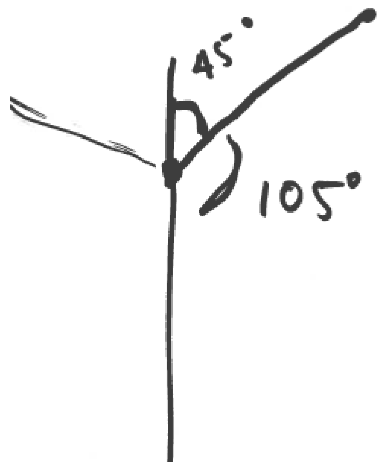


Figure 5.3. Kaylee extends the line segment and constructs a new angle[2].

[*Continued*]

M: Well, he will, he can use his protractor and draw his line and he'll know exactly where to go.

K: But what if we just told him one hundred and five degrees, what if he drew this way and started to go this way [moving her finger along the alternative path]? How would you tell him to go this way [*pointing to the path AP*]?

T: Ah, that's a good question.

M: To the…

K: The right, but I mean, that still doesn't make sense.

M: Well, you're turning to the right, not to the left.

[2] Kaylee's estimation of 45 degrees was inaccurate considering my interpretation of it being the supplement of the angle she drew earlier, which she estimated to be 105 degrees. However, because Kaylee's attention was more focused on how to construct a viable angle and to make a visual estimation of that angle measure, I did not question her estimation of 45 degrees.

K: [*Excitedly*] Or you can tell him… Hmmm. Tell him this [extends the line segment representing the road to the North Pole]. More makes sense, like, forty-five degrees [*writes 45˚ in her sketch; see Figure 5.3*]. So, it could be like, okay, when you get to the North Pole, turn forty-five degrees to the right and then go straight and you'll run into her.

As demonstrated so far, seemingly taking her imaginary perspective from in the helicopter looking down onto the ground, Kaylee first recognized that the rescuer could walk in a straight line from P to A. However, she realized that walking straight was not enough information for the rescuer to find A; Kaylee needed to define in which direction to walk straight from the rescuer's perspective. As a solution, Kaylee constructed a frame of reference through which she could gauge the amount of rotation the rescuer would have to carry out. Hence, she constructed a frame of reference consisting of an initial ray, terminal ray, and vertex where the two rays intersected, which I call an *angular frame of reference*.

Then, she anchored that frame of reference onto the map situation. However, her initial attempt in anchoring this frame of reference onto the map was unsatisfactory because the initial ray was located at the road segment to the North Pole and this did not account for Hamilton's position in the North Pole region. Therefore, Kaylee adjusted her angular frame of reference such that the initial ray was anchored onto Hamilton's imaginary line of sight, which was the extension of the road into the North Pole; the vertex was anchored at the point P, and the terminal ray, which was a result of a clockwise rotation of the initial ray would pass through the desired destination point A.

This adjustment was possible through Kaylee's *decentering* and positioning herself as if she were in the rescuer's position on the ground. Aware of both her perspective above the ground and the rescuer's perspective embedded within the space, I hypothesized that Kaylee *unitized* each perspective as two independent perspectives but also coordinated the two perspectives simultaneously in order to generate instructions for the rescuer.

Using this angular frame of reference, Kaylee measured the amount of rotation (angle measure) to define the direction in which the rescuer would have to move from point P to point A. Although Kaylee did not explicitly measure the distance between P and A, her explanation to “go straight until you find her” implied an awareness of a certain amount of distance that the rescuer would have to travel. This to me indicated that Kaylee has formulated ideas that are the basis of what I consider a *polar coordinate system* in which one coordinates radial distance and angle measure to locate points in space.

Morgan’s reaction to Kaylee’s coordination of angle measure and distance.

As demonstrated in Excerpt 5.1, Morgan did not find the 105 degrees right turn to be problematic for the rescuer; all the rescuer had to do was to draw the line forming 105 degrees with the line segment representing the road to the North Pole with his protractor and walk along that line. It was likely that Morgan imagined Hamilton looking down onto his map and drawing the line segment just like she was looking down onto her map. However, Morgan did not discuss how the rescuer should draw the line segment from his perspective. Furthermore, given the 105 degrees on the map as shown in Figure 5.3, Morgan found it sufficient to tell the rescuer to go 105 degrees to the right. Because this explanation did not account for the perspective of the rescuer, I interpreted this to mean that Morgan’s descriptions were more “ego-oriented” (Taylor & Tversky, 1996, p. 375) because it did not consider the orientation of the map the rescuer was holding, where the rescuer was on the map, which direction the rescuer was facing, or the orientation of the angle.

Prior to the discussion in Excerpt 5.1, Morgan wanted to know if they could contact the missing person and tell her how to walk to the North Pole. As such, Morgan showed a preference in developing instructions in-activity and was more focused on taking her perspective of one inside the helicopter, looking down onto the North Pole region in a figurative sense.

So far Kaylee and Morgan demonstrated differences in coordinating perspectives embedded within the space and taken from outside of the space. Kaylee first took her perspective above the ground and constructed an angular frame of reference she could anchor onto salient landmarks on the map (e.g., the road and the North Pole) to locate point A. Then, she coordinated that perspective along with the rescuer's imagined perspective in order to refine her frame of reference. On the other hand, Morgan mainly focused on her above-the-ground perspective and did not necessarily account for the rescuer's perspective.

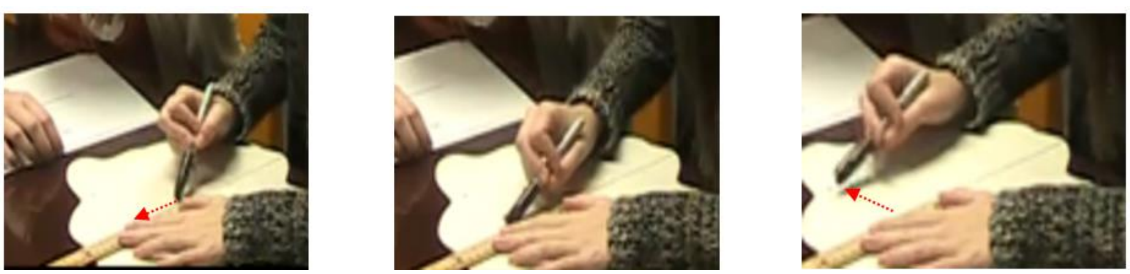
Morgan and Kaylee coordinate horizontal and vertical distances.

Kaylee mentioned earlier in the teaching episode that they did not have a way to measure distance and devised a method that would work using a protractor. Because I wanted to explore Kaylee's consideration of distance, and because I wanted to investigate whether the measuring tools they had in hand would change their approach, I asked the students if they could come up with instructions for the rescuer when they only had rulers and not a protractor.

Morgan started first, putting her ruler on their map to measure the distance between the P and A on the map (see Figure 5.4). I took this to mean that Morgan wanted to tell the rescuer how far he needed to walk straight from point P to A. It is possible that she was still thinking in terms of Kaylee's previous approach and was trying to find the distance between the two points that would supplement Kaylee's angle measurement. However, before I could ask Morgan how she would use the distance between P and A, Kaylee disagreed with Morgan's strategy saying that they would not know the angle for how much the rescuer should turn because they now did not have a protractor.

Figure 5.4. Morgan lays ruler along PA.

Instead, Kaylee aligned the ruler with the road to the North Pole as if she were to extend it (Figure 5.5 (a)), moved her pen along the ruler (see red dashed arrow in Figure 5.5 (b)), stopped at a certain point saying that he will "make an exact right angle" and turned to arrive at point A (see red dashed arrow in Figure 5.5 (c)).



(a) (b) (c)

Figure 5.5. Kaylee follows along the ruler and makes a right turn.

When Kaylee denied her approach in laying the ruler as shown in Figure 5.5, Morgan did not rebuke Kaylee's claim that they did not know the angle measure anymore. Instead, she listened to Kaylee's demonstration and seemed to agree with Kaylee's new strategy. Together, they put two rulers together (see Figure 5.6) and Kaylee traced along the rulers to draw the line segments of the trip from point P to point A as she demonstrated in Figure 5.5. Kaylee explained that they were trying to make an exact ninety degrees with the rulers so that the rescuer would be able to follow the instructions and find the missing person.

Figure 5.6. Kaylee and Morgan put two rulers together on the irregular shaped map.

From Kaylee's repeated demonstration of coordinating a horizontal and vertical distance as shown in Figures 5.5 and 5.6, I infer that Kaylee has constructed a *rectangular frame of reference* or a grid structure, consisting of horizontal and vertical lines with the intersection of one set of perpendicular lines anchored at point P. Using that rectangular frame of reference, Kaylee coordinated the horizontal distance and vertical distance from the point of intersection to define the rescuer's movement from point P to point A.

To elaborate, using the line through the road into the North Pole as a reference, Kaylee anchored a frame of horizontal and vertical lines onto the map such that one set of perpendicular lines was anchored at the North Pole and align with the road to the North Pole (see middle in Figure 5.7).

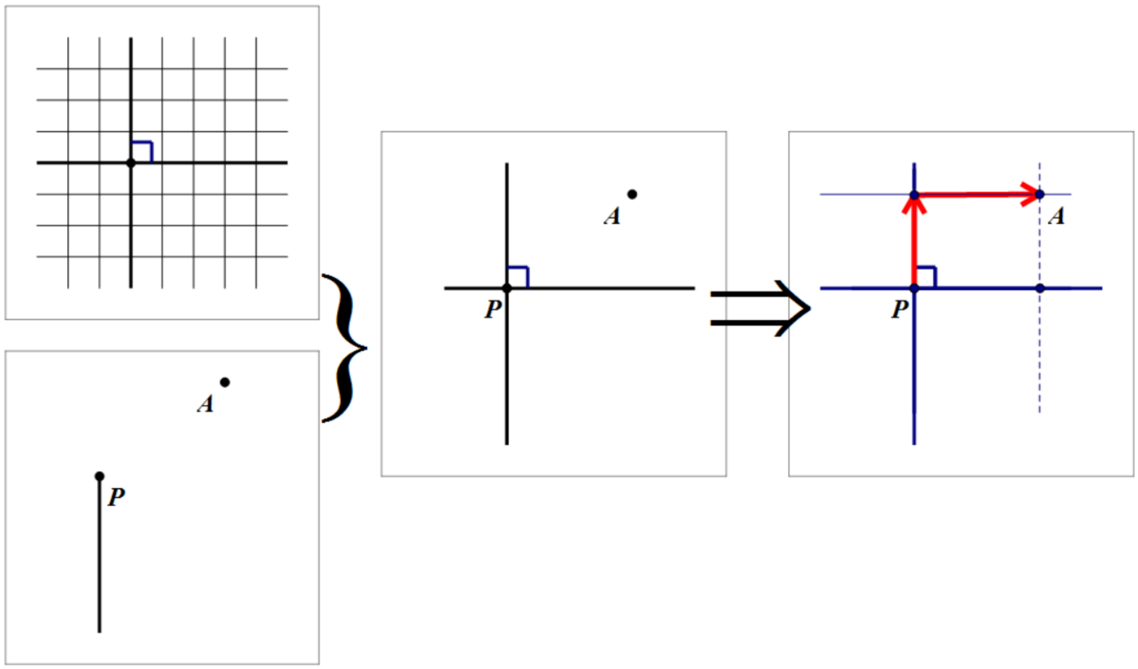


Figure 5.7. A model of Kaylee's representation of the rescuer's movement.

Then, she decomposed the movement of P to A into two spatial dimensions, which allowed her to find the horizontal and vertical distances that comprise the movement from point P to point A (as demonstrated in the right in Figure 5.7). Finally, Kaylee measured the length of each line segment that she drew in the map in inches. I recognize Kaylee's system of coordinated measurements similar to what I consider a *Cartesian-like coordinate system* in which one coordinates vertical and horizontal distances from the origin.

I hypothesize that Kaylee took an ego-oriented perspective from the helicopter when superimposing a rectangular grid onto the map. However, in her explanation, Kaylee also demonstrated awareness of the rescuer's on-the-ground perspective. For example, Kaylee said that Hamilton would need to "make an exact right angle" and that they were trying to make an exact ninety degrees with the two rulers so that he would be able to follow the instructions. As such, consistent with her previous approach, Kaylee seemed to have coordinated the rescuer's on-the-ground perspective with her helicopter above-the-ground perspective.

At the very beginning of the teaching episode, Kaylee swiftly moved her finger along the paper in a motion similar to the way she moved the pen in Figure 5.5. Further, Kaylee's reaction to Morgan's use of the ruler as shown in Figure 5.4 was immediate. I took Kaylee's behavior to indicate that her actions were operational and anticipatory in that the measurements were coordinated from the beginning without a trial-and-error process and in that she was aware that the measurements would ensure that the rescue team would find the missing person. Similar to the children in the last stage of Piaget et al.'s (1960) rectangular paper task study, Kaylee seemed to have "realized the logical necessity to take both dimensions into account and their measurements are straightaway coordinated so as to be at right angles to one another" (p. 169). By the way Kaylee laid the rulers simultaneously as shown in Figure 5.6, I hypothesized that Kaylee's vertical and horizontal distances entailed a logical multiplication of measurements (Piaget et al., 1960) in which the measurements were oriented by the rectangular frame of reference. In other words, Kaylee located point A multiplicatively as a product of a coordination of its location along one spatial dimension with the realization that the point had a specific location along the other spatial dimension.

Morgan and Kaylee discuss unit of measure.

When Kaylee finished measuring, Morgan posed a problem, which led to the following discussion between Morgan and Kaylee.

Excerpt 5.2: Morgan and Kaylee discuss using measurements in the irregular shaped map.

M: But we don't know how long that [pointing to one of the two line segments Kaylee just measured (the one that Kaylee measured to be 3 inches)] is.
K: Yes.
M: Well you can't just tell him to go three.
K: He can mark it on his map.
M: Three…
K: Oh yeah…
M: But he doesn't have… [*Looking at T*] Does he have a ruler?
K: Well, even if he does have a ruler…
T: Let's say he does. Let's give him a ruler [*M passes a ruler to Hamilton*].
K: Unless he has like a scale on his map, without it… [Inaudible].

Although brief, as shown in Excerpt 5.2, Morgan demonstrated an *awareness of a unit of measure* and Kaylee an awareness of *the scale of measurements*. At first when Morgan pointed out that they could not simply tell Hamilton to go three; her concern was of Hamilton not having a tool for measuring the three *inches*. On the other hand, Kaylee took Morgan's question in a different direction. Kaylee was attending to the possibility that the two maps could have different *scales*. Once we settled on using the same scale for both maps, Kaylee relayed the instructions to Hamilton and when the instructions were completed, they superimposed Hamilton's wax paper map onto their map and were satisfied with the result.

The irregular shaped map contained the line segment representing the road to the North Pole. The students used this line segment to orient the map or to anchor their frames of reference onto the map. To investigate the students' constructive activities when a salient spatial reference is absent from the map, I posed another map to the students, which I discuss next.

Circular Shaped Map

Kaylee folds the circular map.

Given a blank circular shaped paper as their new map, I asked the students how they would explain Ebru's location to the rescue team. Kaylee marked a point for Ebru (point A´) on the new circular shaped paper. Then, she folded the circular paper in half (Figure 5.8 (a)) and then in half again (Figure 5.8 (b)) and marked a point (point P´) on the intersection of the two diameters she made by folding the paper (Figure 5.8 (c)). Although Kaylee did not explicitly state so as she folded the paper, in the subsequent conversations between Morgan and Kaylee, both students considered point P' as the center of the circle and used it as a reference point.

(a) (b) (c)

Figure 5.8. Kaylee folds the circular paper in half twice and marks the center of the circle.

In folding the paper, Kaylee simply picked up the paper and folded it, without necessarily orienting the paper in any particular way. Although I was interested in understanding why she chose to fold the paper this way, I did not want to pose any further questions and risk interfering with her initial approach.

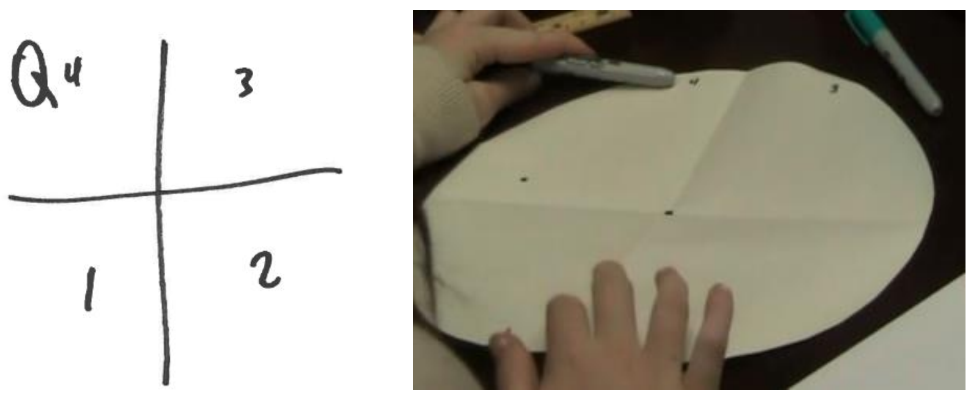
Morgan coordinates horizontal and vertical distances.

Because Kaylee had been taking over most of the episode so far and because I noticed Morgan becoming disengaged and quiet, I asked Morgan what she thought.

Morgan picked up the map. The following excerpt starts with Morgan suggesting an idea for locating Ebru in the circular map case.

Excerpt 5.3: Morgan explains how to locate Ebru on the circular map.

M: What we can do is, so we can tell him, you can fold your thing and fold it again pull it out, mark the center [*she repeats the activities Kaylee just carried out as she says them out loud*] and then…[*Pauses to think for approximately two seconds*] Okay, so, we can, make, this can act as like, um, you know how you have your graphs…[*Starts making a sketch of what she referred to as a "graph" as shown in Figure 5.9 (a)*].
K: Quadrants?
M: Your quadrants. What is this, like four, three, this is two, this is one [*marks 4, 3, 2, and 1 on each quadrant in her graph not in the conventional order as shown in Figure 5.9 (a)*].
T: Mm-hmm.
M: You can tell him to mark them… So this would be four, three, two, one [*writes 4, 3, 2, 1 in each corresponding quadrant in the circular paper] and we could say she's in quadrant four [pointing to the 4 on the circular paper (see Figure 5.12 (b))*]. And basically what you're going to have to do is… [*Picks up the ruler and looks at T*] We still have the ruler, right?



(a) (b)

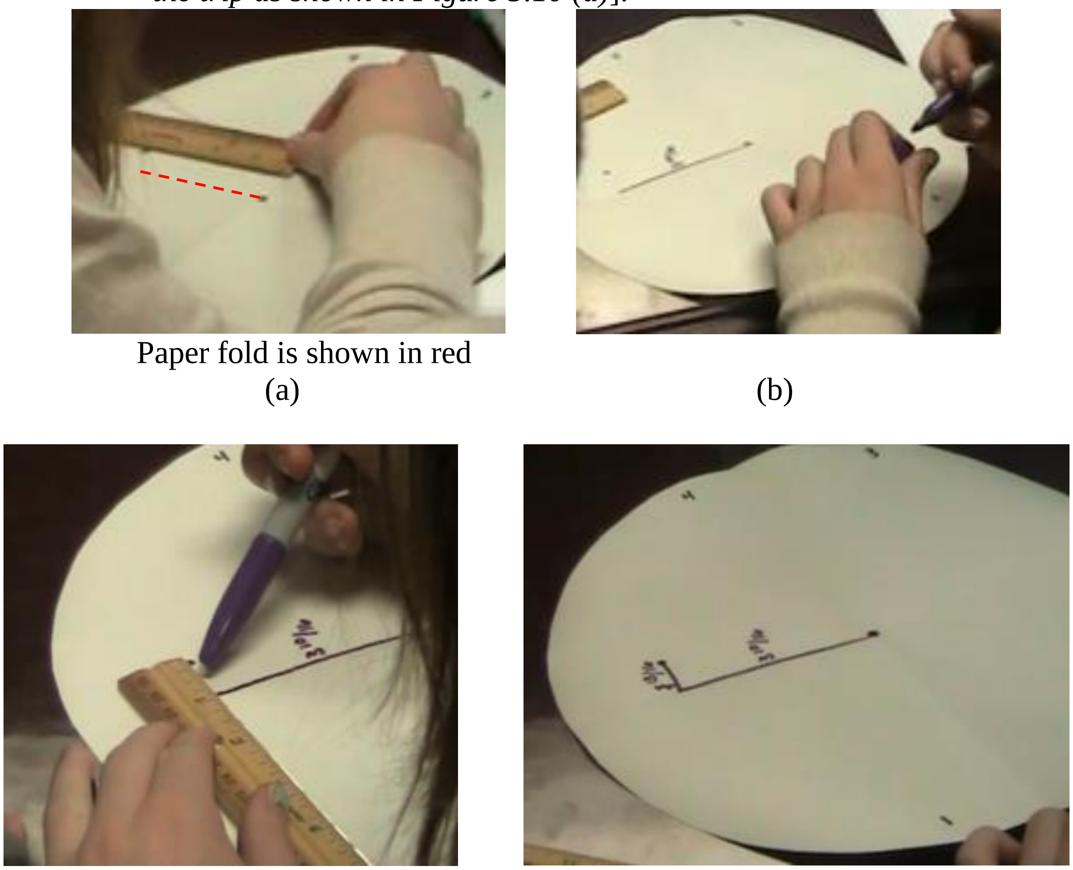
Figure 5.9. Morgan's sketch of a "graph" and her four quadrants on the circular paper.

[*Continued*.]
T: Mm-hmm.
M: Okay, so you can [*places ruler on the paper so that the edge of the ruler passes point A' parallel to the fold of the paper as shown in Figure 5.10 (a)*]. What is that, three and ten sixteenths? So… [*Draws a line segment along the fold of the paper and writes 3 10/16 as shown in Figure 5.10 (b)*]. And then… From the center…
K: To the left.

M: Yeah. And then once you reach that point, then you can step up [*starts measuring distance between the end of the line segment she drew in Figure 5.10 (b) and the point A (see Figure 5.10 (c))*].

K: Not up. If you would be walking this way [*moves her finger along the first segment of the trip*] so you'll be turning right.

M: Yeah, and you would turn… [*Finishes measuring the second segment of the trip as shown in Figure 5.10 (d)*].



Paper fold is shown in red

(a) (b)

(c) (d)

Figure 5.10. Morgan coordinating measurements on the circular map.

At the beginning of Excerpt 5.3, instead of taking Kaylee's folds as given, Morgan started out by repeating Kaylee's activities and paused to think. This to me indicated that Morgan was assimilating Kaylee's folding activities and thinking about how she could use those folds in locating point A´. Although her actions were reenactments of Kaylee's folding actions, Morgan made further progress. Instead of laying her ruler along P'A' as she did earlier in the irregular shaped map case (see Figure

5.5), Morgan decomposed the movement from P' to A' along two spatial dimensions. To elaborate, first, Morgan laid the ruler parallel to the fold of the paper through point A', as shown in Figure 5.13 (b). The place where she lined up the beginning of her ruler was the other fold of the paper, which I inferred to imply that she anchored a rectangular frame of reference onto the map using the two paper folds and the intersection of the two paper folds. Morgan then measured the distances of each line segment of the movement from P′ to A′ using this frame of reference in order to locate point A′ in reference to point P′.

There are two possible contributors to Morgan's shift in coordinating measurements. First, it is possible that the two paper folds led Morgan to enact graphing activities she learned in school. Although neither of the students mentioned the Cartesian plane or other graphical terms such as "axes" or "origin," Morgan's use of quadrants to distinguish the four sections of the circle could indicate that the paper folds of the circular map revoked her images of horizontal and vertical axes and the origin she learned in school. The second possible explanation is that Morgan assimilated Kaylee's coordination of measurements obtained through using a rectangular frame of reference, demonstrated in the irregular shaped map case. It is likely that this assimilation was enacted by Kaylee's paper folding demonstration.

In either case, whether the assimilation was one of her earlier graphing experiences in school or of Kaylee's locating activities, I claim that Morgan's construction of a rectangular frame of reference was made in activity. This is because Morgan repeated Kaylee's folding activities and paused to think how to proceed with the paper folds. Instead of having constructed the rectangular frame of reference *a priori* to her measuring activities and then anchoring it onto the map, the pre-made paper folds triggered her use of a rectangular frame of reference. Although made in activity, I hypothesize that Morgan's vertical and horizontal distances entailed a logical multiplication of measurements (Piaget et al., 1960) in which the measurements were oriented by the rectangular frame of reference. In other words, Morgan located point A multiplicatively as a product of a coordination of its location along one spatial dimension with the realization that the point had a specific location along the other spatial dimension.

The way Morgan and Kaylee had a different approach in putting the instructions into words in Excerpt 5.3 accentuated the different perspectives Kaylee and Morgan coordinated in the task. After drawing the first segment of Hamilton's trip to explain the second segment of the trip, Morgan said "you can step up" whereas Kaylee pointed out that it was not going up but that "you'll be turning right." This difference was similar to the one I observed earlier in the irregular shaped map. That is, Kaylee operated as if she combined her helicopter above-the-ground perspective and Hamilton's on-the-ground perspective in order to generate instructions for the rescuer. On the other hand, Morgan's explanation was based on her helicopter above-the-ground perspective. Consistent with her earlier way of thinking, Kaylee continuously shifted from one perspective to another and coordinated the two together while Morgan focused on one in producing instructions for Hamilton.

Morgan and Kaylee discuss the orientation of the map.

So far, neither of the students seemed to question the way they folded the paper and I anticipated an opportunity for this to occur when they relayed their instructions to Hamilton. When I asked them to give Hamilton the instructions, Morgan and Kaylee first told Hamilton to fold his circle in half and then half again to make a "pie looking shape."

After Hamilton folded his circular paper in half twice, the students told Hamilton to unfold the paper and mark the center of the circle. The following excerpt illustrates the interaction between the two students and Hamilton after Hamilton marked the center of the circle. H refers to Hamilton in the following and subsequent excerpts.

Excerpt 5.4: Morgan and Kaylee give instructions to Hamilton in the circular case.

M: Okay, mark the, like the quadrants on the graph [looks at Kaylee].
K: We can say go… [Moves finger along the first segment of the trip Morgan drew in Figure 5.10 (b).]
M: But he doesn't know because he can go…
K: Well, it doesn't matter which way he spins it [*rotates the circular paper around clockwise approximately 90 degrees.*] Oh, I guess it does.
M: Yeah.
K: [*Rotates the circular paper back to where it originally was*.] Okay, well… [*Frowns*.]
T: What's our problem?
M: That's why we tell him four, three, two, one [*pointing to each quadrant they marked on the circle*.]
K: No, but what if he has it different [*rotating the circle slightly in the clockwise direction*]. Okay, okay.
M: It doesn't matter because he still got four, three, two, and one.
K: Okay, okay. [*To Hamilton*] So, mark the top half…top two halves…
M: [*Chimes in*] the top left…
K: Or top left half…
H: Okay, how do I know which one's top left?
K: [*To Morgan*] That's what I'm saying, man! [*Laughs*.]
M: The one… Okay. You hold the circle in front of you with the line that's straight [*moves her hand along the paper fold perpendicular to her*]. Like… [*Scratches face*.]
K: See, it doesn't matter.
T: He has two lines that are straight.
M: Well, you can choose one. And…
K: Because, our map, it's like, if we hold it upside down [*rotates the circle counterclockwise 180 degrees*] she's going to be in a different place. She was up here [*pointing to where the point used to be in the circle according to the new orientation (see Figure 5.11)*] last time, so it does matter which direction it's facing and like what to tell him. So…Oh gosh.

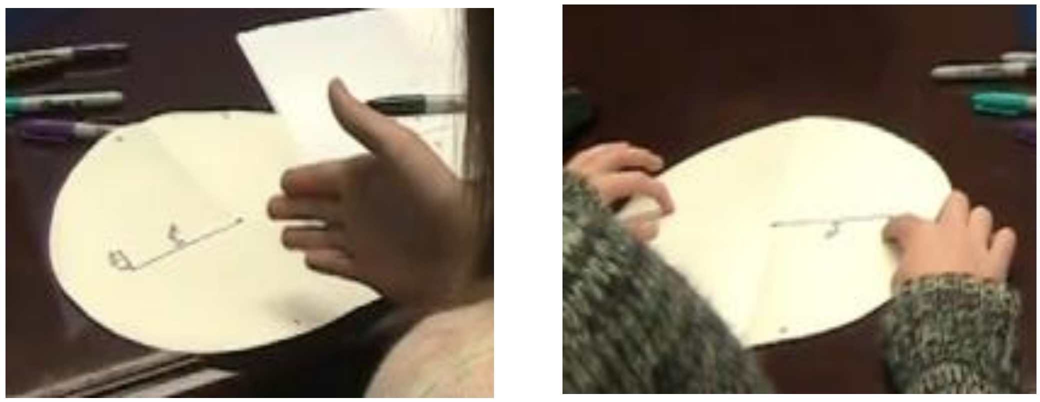


Figure 5.11. Morgan and Kaylee tell Hamilton how to find Ebru.

As shown in Excerpt 5.4, the students did not realize that there are infinitely many ways one can fold a given circle into four equal sectors. However, once they folded the paper and marked the center of the circle, the orientation of the circle with respect to the viewer of the map did have significance to Kaylee. Both Kaylee and Morgan found it important to clarify the four quadrants for Hamilton at the beginning but why it was important was different for each student.

For Morgan, the way Hamilton oriented the circular map did not matter; the quadrants were for a communicational purpose so they could tell Hamilton which quadrant Ebru is on his map. However, after rethinking her original conjecture that the orientation of the circle would not matter and after rotating the circle around a little, Kaylee seemed perturbed by the situation.

Once Hamilton asked which half of the circle was the top, Kaylee articulated why the orientation of the circular map mattered. When she rotated the circular paper 180 degrees, although the circle still had the same paper folds, she was aware that the location of Ebru in reference to those folds would be different. In other words, Kaylee anticipated what would happen if Hamilton were to pick another half of the circle to be “on top.” Figure 5.12 models how their map could be rotated 90 degrees clockwise while maintaining the same frame of reference, but admitting to different coordinate system because the direction of points induced from it differs. The different orientations of the circular map would entail different orientations of the frames of reference, which in turn would result in different definitions of directionality.

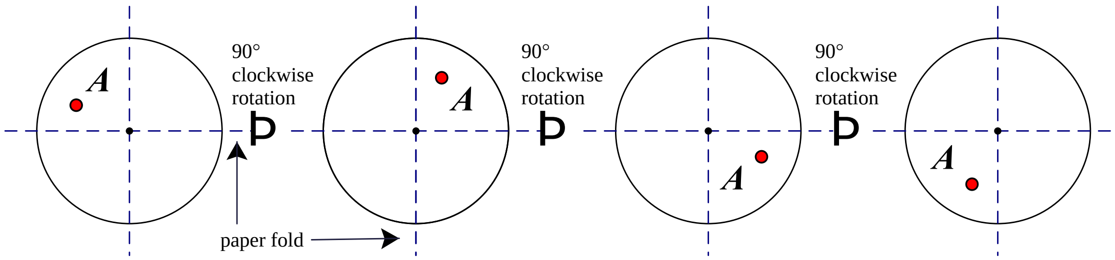


Figure 5.12. Images of various orientations of the circular map.

Different from the line segment (road) in the irregular shaped map, in the circular map, there were no spatial landmarks the students could use to define Hamilton's initial orientation of the map. In the following excerpt, Kaylee and Morgan discuss the features of the circular map that made it difficult to locate the missing person. W refers to another witness in the room that was not Hamilton.

Excerpt 5.5: Kaylee and Morgan discuss their problem with the circular map.

T: Okay. So, let's go back to what you guys were discussing about how there might be a problem.
K: Yeah. Where… Okay, let's say that our map, we're looking at it so that there's like a little forest here [*drawing a small squiggly shape on the quadrant they marked 3*] and like there's large characteristics of the place but his map could just totally be blank and we don't know where we could put, like, if his is like the same direction, I guess? I don't know…
T: Mm…
K: Like, we could be telling him to go… If he holds his map or whatever, we could tell him to go towards the forest and she's actually this way because the line was off…I don't know how…
H: Oh, there's a forest on the map?
K: I don't know [*laughing*]. I was just kind of making that up.
T: So, it seems like you need some sort of…
M: Like, characteristic of the map.
K: Feature…

T: Like the road that we had?
K&M: [*Simultaneously*] Yeah!
T: Right?
M: Yes!
T: You need to have something there to set the direction…
K&M: [*Simultaneously nodding*] Mm-hmm.
K: Because if it's like all snow it would be kind of… [*Inaudible*.]
W: You're in the helicopter, right?
K: Mm-hmm.
W: Do you know your directions?
K: Like north, south, east, west?
W: Yeah. I think there's one chance out of four he's going to miss it.
K: Yeah…
W: He may go in the wrong quadrant.
K: If we had a compass we could probably…
T: Would having a compass work?
M: It would for us but not for him.
K: Yeah, it would work but not for him.
T: Why not?
K: Because it'll still be the same like…
M: Actually, so, we're in a helicopter and we're looking down on this [*pointing to the circular map in front of them*]. You could tell him, on his paper, okay, to the top right, we have forest, to the left of that, there's going to be a river over here and then on the bottom there's like another forest or something like that. And so, he can know whenever he goes [*pointing to the center of the circle*] there's a forest, there's a river, so I know where my quadrants are going to be.
T: Oh, I see.
K: But what if it was just snow? What if there was no characteristics?
M: There's a hill right there…
K: This is all flat land and it's like just come look for her.
M: There's no…
K: That snow [*inaudible*.]
[*Both sit thinking for approximately three seconds*.]
K: That dumb circle!

As shown in Excerpt 5.5, the students identified two things to be problematic in locating point A´ in the circular shaped map. The first thing was problematic to both students: there were no spatial landmarks to use as reference in the circular paper map case. As the students pointed out, they needed some characteristic or feature of the land that they could use to communicate the directions to the rescuer.

The second problem was the circular shape of the paper, which seemed to be more problematic for Kaylee. As Kaylee exclaimed at the end of Excerpt 5.6, the map was a “dumb circle”, which was one of the reasons that made it difficult to locate A′. As Kaylee pointed out, if they were to assume that the North Pole region was flat and all covered in snow, the physical characteristic of the circle limited their ability to fix the orientation of the circle—which half of the circle should go on the top from the viewer’s perspective. Without the orientation of the map specified, Kaylee found it impossible to communicate the direction in which Hamilton should walk. As such, anchoring the rectangular frame of reference onto the spatial context and defining directionality induced from the frame of reference was challenging in the circular map case.

Summary of the North Pole Task

In the North Pole Task, I observed ways in which Kaylee and Morgan constructed frames of reference and coordinated measurements using those frames of reference to describe the location of point A in reference to point P.

In order to define the location of point A in reference to point P, the students had to conceptualize directionality in two-dimensional space. In the irregular shaped map, from her activities, I inferred that Kaylee had constructed two types of frames of reference which she used to define the direction the rescuer should walk in two different ways. First, Kaylee constructed an angular frame of reference consisting of an initial ray anchored onto the rescuer’s line of sight, a vertex at point P, and a terminal ray through point A (Figure 5.3). This frame of reference allowed Kaylee to describe the amount of rotation the rescuer would need to turn. Second, Kaylee constructed a rectangular frame of reference consisting of horizontal and vertical lines with the intersection of the lines anchored at point P. This frame of reference allowed Kaylee to break down the movement along two spatial dimensions finding horizontal and vertical distances. Using these frames of reference, Kaylee coordinated angle measure and radial distance or horizontal/vertical distances, respectively, to locate point A in reference to point P on the maps. As a result, I claimed that Kaylee had constructed what I would consider a polar-like coordinate system and a Cartesian-like coordinate system.

Morgan initially took a more temporal approach in that she wanted to give in-the-moment instructions to the rescuer or the missing person from the helicopter. Later in the teaching episode, Morgan considered connecting the two points P and A and measuring the distance between the two points in the irregular shaped map. In the circular map case, Morgan used two perpendicular paper folds Kaylee previously made through the center of the circle and their intersection (circle center P') as a references. Using the paper folds as a rectangular frame of reference, Morgan coordinated vertical and horizontal distances to locate point A' in reference to point P'. I conjectured that the paper folds of the circular map may have led to Morgan enacting graphing activities she learned in school and that Morgan has assimilated Kaylee's coordination of horizontal and vertical distances in the irregular shaped map case. In any event, I considered Morgan's construction of a rectangular frame of reference to be made in activity in that Morgan carried out the folding activities Kaylee carried out and paused to think as she engaged in the task.

Although Kaylee as given and Morgan in activity, I conjectured that both students demonstrated their coordination of vertical and horizontal distances entailed a logical multiplication of measurements (Piaget et al., 1960) in which the measurements were oriented by the rectangular frame of reference. In other words, both students eventually located point A multiplicatively as a product of a coordination of its location along one spatial dimension with the realization that the point had a specific location along the other spatial dimension.

Kaylee and Morgan demonstrated differences in their notions of directionality when developing instructions for the rescuer. I conjectured that the salient difference was in their perspective-taking. Kaylee consistently coordinated two perspectives, her above-the-ground perspective and an imaginary on-the-ground rescuer's perspective. Coordinating these two perspectives, Kaylee developed instructions using route descriptions (Taylor & Tversky, 1996) in which the descriptions are oriented from the rescuer's perspective. In this process, I inferred Kaylee transferred her ego-oriented perspective to the imaginary rescuer's perspective embedded within the perceptual space through decentering. On the other hand, in developing instructions for the rescuer, Morgan's language was based on her perspective and did not account the rescuer's line of sight. These tendencies highlighted Morgan's focus on her ego-oriented above-the-ground perspective. This perspective was not necessarily coordinated with the perspective of the rescuer. Hence, I hypothesized that Kaylee had a stronger ability to bring forth images of one perspective alongside another and coordinate them simultaneously.

In solving the task for two different shapes of the map, students thought the unit of measure and scale of measure for both maps were crucial elements to consider (Excerpt 5.2). Both students also identified it important to use salient spatial references when defining directionality (Excerpt 5.5). In the irregular shaped map, a line segment through the North Pole point P was given and it provided a reference for orienting the map. Therefore, the frames of reference the students constructed in the irregular shaped map were anchored to the space. However, in the circular map, there were no roads or other spatial landmarks the students could use to define Hamilton's initial orientation in the map. In their discussion in Excerpt 5.5, Kaylee showed an awareness of the notion of directionality depending on the orientation of the frame of reference. Although the students identified the center of the circle and a system by which they could form a set of references by using two perpendicular diameters of the circle, they did not have a way to decide where to anchor the frame of reference onto the circle nor communicate the orientation of the circle, making it difficult to locate Ebru for the rescue team.

Fish Tank Task: Kaylee and Morgan Locate a Point in Three-Dimensional Space

To explore whether the students' coordinate systems, differences in coordinating perspectives, and notions of directionality transferred to three-dimensional space, I presented the students with the Fish Tank Task. Starting on November 12, Morgan and Kaylee started working on the Fish Tank Task, which lasted for a total of three teaching episodes. In the Fish Tank Task, I asked Morgan and Kaylee to locate four fish submerged in a three-dimensional cubic or cylindrical fish tank (Figure 5.13).

Figure 5.13. Fish Tank Task Materials.

Different from the North Pole Task, the Fish Tank Task did not specifically require the students to develop instructions for a third person; they were simply asked to locate the four fish in the tank. Specifically, I asked the students to explain the location of all four fish and to give instructions for Fish 1 to swim to Fish 2, for each tank. In explaining the location of all four fish, the students produced drawings representing the fish tanks. In terms of interpreting drawings that students produced, I take Piaget and Inhelder's (1967) account for drawings:

> A drawing is a representation, which means that it implies the construction of an image, which is something altogether different from perception itself, and there is no evidence that the spatial relationships of which this image is composed are on the same plane as those revealed by the corresponding perception. A child may see the nose above the mouth, but when he tries to conjure up these elements and is no longer really perceiving them, he is liable to reverse their order, not simply from want of skill in drawing or lack of attention but also and more precisely, from the inadequacy of the instruments of spatial representation which are required to reconstruct the order along the vertical axis. (p. 47)

As such, I considered the students' sketches and the process they produced it as their representations of the perceptual fish tank space and not a trivial copying of it.

The different shapes of the tanks were selected to test if the physical shape of the tank might suggest perceptual guidance in the students' constructive activities, similar to the two different shapes of maps in the North Pole Task. Based on my findings from the North Pole Task, I anticipated the students to use the faces or edges of the cubic fish tank as spatial landmarks to anchor their rectangular or angular frames of reference onto. Also, I anticipated the circular shape of the cylindrical tank to emulate similar perturbations as did the circular map.

In the North Pole Task, there were instances in which one student took the lead of solving the task over the other. That made it difficult for me to observe activities that each student carried out independently. Therefore, in the subsequent teaching episodes, I asked the students to work individually until both were finished organizing their thoughts and then asked each student to share their solutions. In many cases, I also asked each student to compare, contrast, or critique their partner's solutions. In the following sections, I will present each student's individual activities in solving the task and the discussions they had thereafter.

For each cubic and cylindrical tank, I present on how Kaylee and Morgan each located the four fish and how they developed instructions for Fish 1 to swim to Fish 2.

Cubic tank: Locating the four fish

Kaylee locates the four fish in the cubic tank.

While looking at the tank, Kaylee first sketched a frame to represent the container of the tank and the surface of the water in the tank. To be more specific, Kaylee sketched a square that had sides with a length equal to that of the edge of the tank container and drew parallelograms to represent the top and side face of the tank. Next, she measured the distance between the surface of the water (gelatin) and the top of the tank and used that measurement to draw the surface of the water in her sketch (see Figure 5.14). After drawing the frame of the tank, she briefly sat looking at the tank. She looked from one side of the tank and then another. It seemed as though she was making a plan of action in carrying out a measurement process.

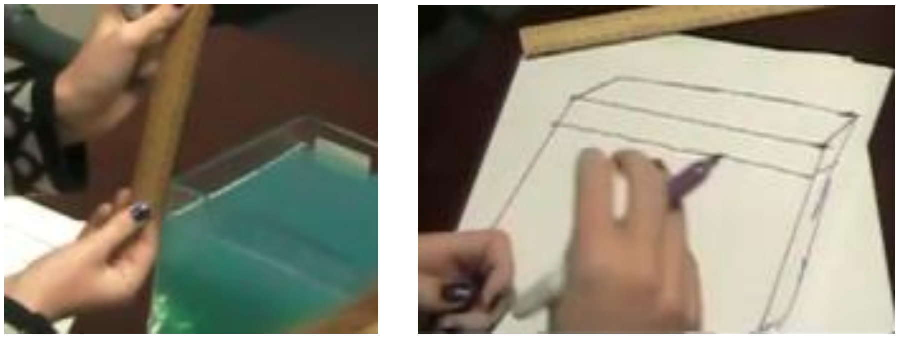


Figure 5.14. Kaylee sketching the container of the cubic tank.

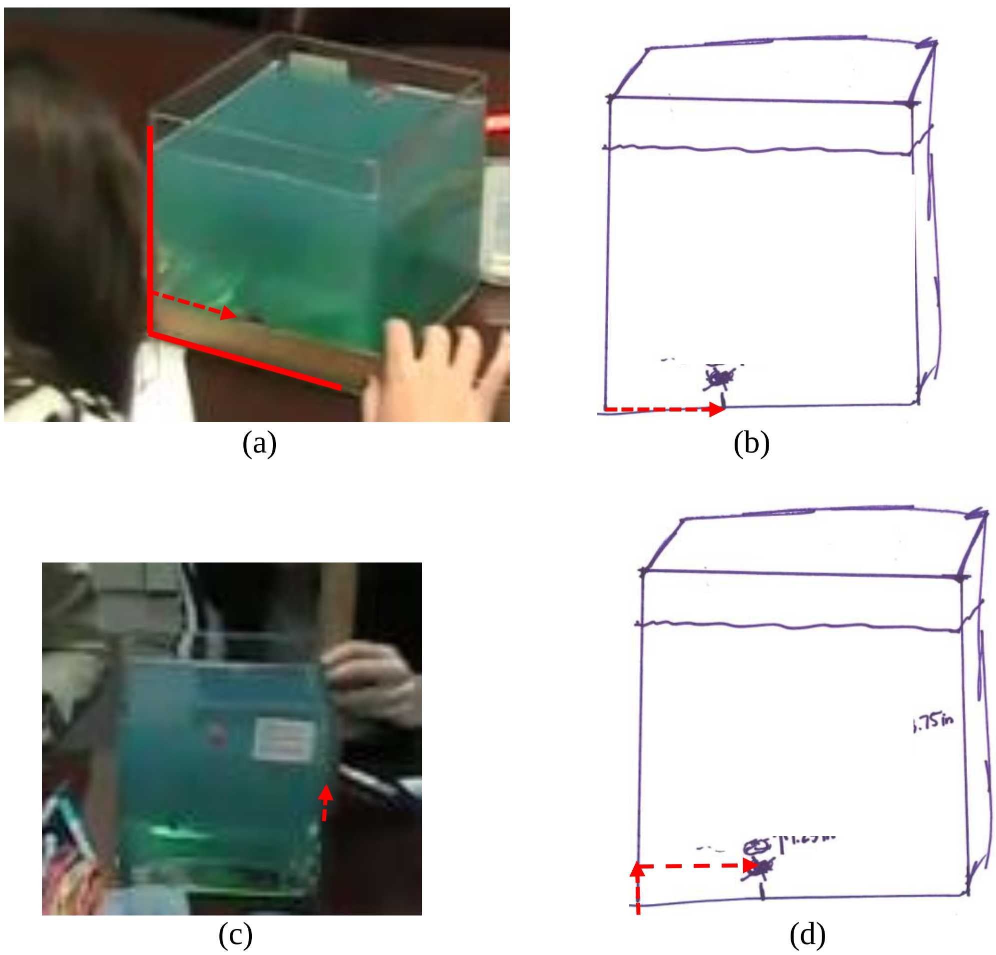
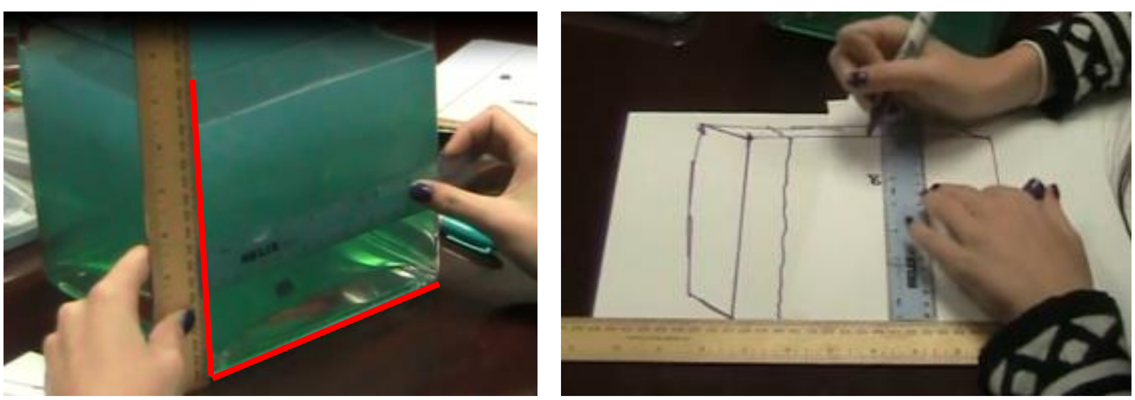


Figure 5.15. Kaylee measures the distance from the edges of the tank to the first fish.

Next, Kaylee located two fish on the face of the tank she just sketched by first measuring horizontal and vertical distances along the face. More specifically, she first measured the distance from the edge I labeled OA to the first fish, laying her ruler horizontal to the edge I labeled OB (see Figure 5.15 (a)). Then, in her sketch, she marked the same length along the bottom of the tank frame sketch (Figure 5.15 (b)).

Next, Kaylee placed the ruler along edge OA starting from OB to the position of the fish in the tank (Figure 5.15 (c)). She then copied the measure along the corresponding edge in her frame sketch and moved her pen horizontally to the right until she arrived above the mark she made earlier in Figure 5.18 (b). Finally, Kaylee marked the location of the fish (Figure 5.15 (d)).

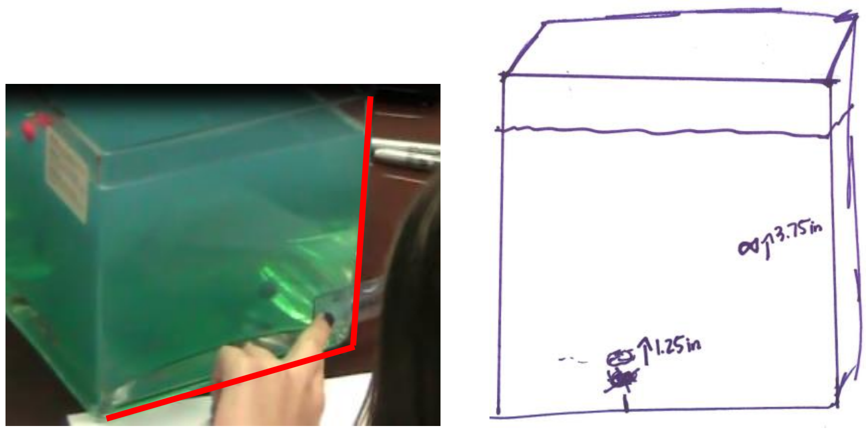
Once finished marking the location of the first fish, Kaylee moved on to the next one that she could see from the same side of the tank she was working with. This time, instead of making subsequent measurements, Kaylee picked up two rulers and placed them together perpendicularly so that one lined up along edge OA and one parallel to OB through the second fish (see Figure 5.16 (a)). Then, she moved the rulers to her sketch and placed them in the same position as they were against the tank and marked the location of the second fish (see Figure 5.16 (b)).



(a) (b)

Figure 5.16. Kaylee measures the distance from the edges of the tank to the second fish.

The way Kaylee laid the two rulers together as shown in Figure 5.16 (b) suggested she may have abstracted the measuring activities she just carried out sequentially in locating the first fish and carried them out in one sweep. In other words, it is possible that Kaylee empirically abstracted the measuring activities she carried out from one situation to another. However, the way Kaylee laid two rulers together as shown in Figure 5.16 (b) resembled her measuring process in the North Pole Task (see Figure 5.7). Therefore, I hypothesized that Kaylee had already reflectively abstracted a more general scheme for coordinating horizontal and vertical distances and enacted this general scheme across different situations. I first present more observations and then describe the general scheme.



(a) (b)

Figure 5.17. Kaylee finds more measurements in the cubic tank.

Once Kaylee located the two fish on one face of the tank in her sketch, Kaylee rotated the tank to measure one additional distance. To elaborate, she rotated the tank counterclockwise with respect to a vertical axis through the center of the tank such that vertex O was now on her bottom right. She then picked up her ruler and measured the distance from edge OA to the first fish with her ruler horizontal to edge OC (see Figure 5.17 (a)). Then in her sketch, she drew an arrow sign and wrote the length she just measured (1.25 in) next to the first fish (see Figure 5.17 (b)). Kaylee repeated the same activity for the second fish: She measured the distance from edge OA to the second fish with her ruler parallel to edge OC and marked the measurement (3.75 in) next to the fish in her sketch (see Figure 5.17 (b)).

I inferred that Kaylee's new goal in finding these measurements (1.25in, 3.75in) was to consider how far into the tank the fish were from the face including {O, OA, OB}. To do so, she superimposed a rectangular frame of reference onto the new face including {O, OA, OC}. Using this frame of reference, Kaylee coordinated the horizontal distance from edge OA parallel to OC to each fish. Here, there are two things to note. First, Kaylee only measured the horizontal distance and did not measure the vertical distance from edge OC parallel to OA to the fish. Second, she wrote the measurements she obtained (1.25in, 3.75in) in her original sketch (as shown in Figure 5.17 (b)), instead of making a new sketch for this new face she used to locate the same two fish.

For these two reasons, I hypothesized that there was more than a sequential enactment of her rectangular frames of reference. Through *disembedding*, Kaylee held her frame of reference of the first face (the face including {O, OA, OB}) as a unit structure, *inserted* her second frame of reference (the face including {O, OA, OC}) into the first frame of reference, and *translated* it along the third dimension resulting in a three-dimensional frame of reference (See Figure 5.18).

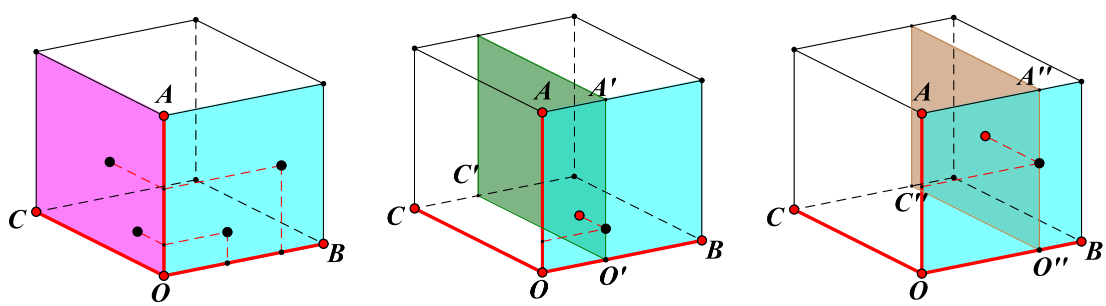
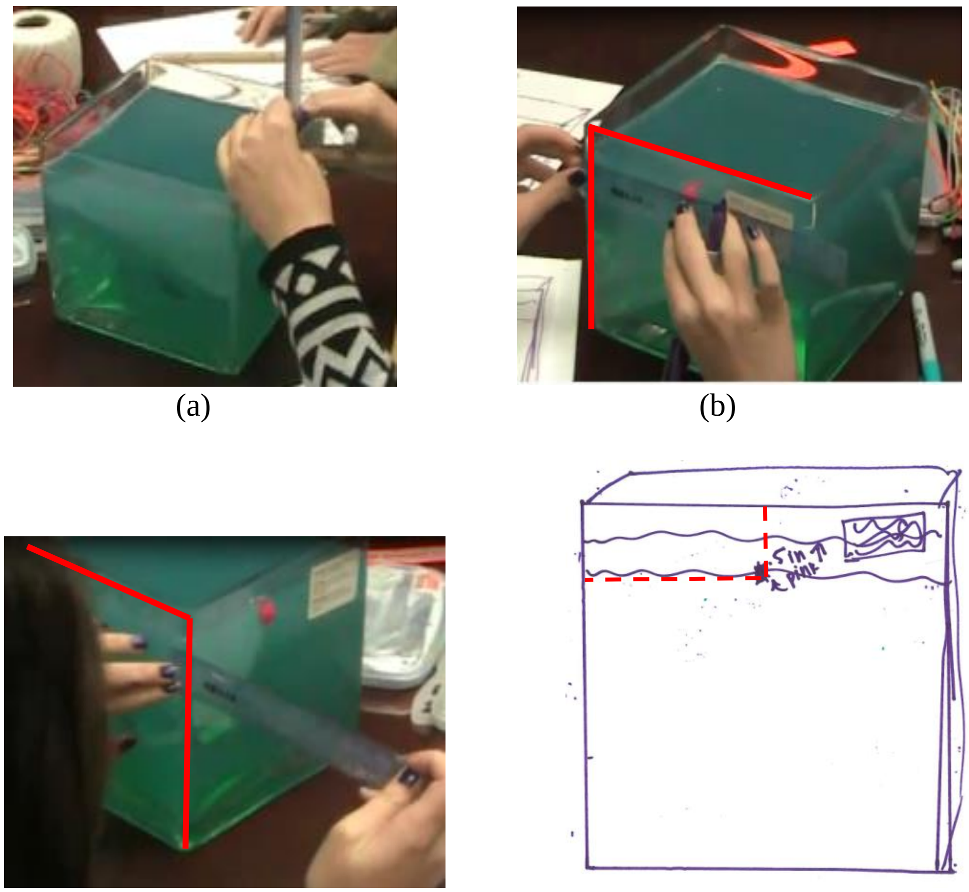


Figure 5.18. Model of Kaylee's insertion of her {O, OA, OC} frame of reference into her {O, OA, OB} frame of reference.

Simultaneously coordinating the two two-dimensional frames of reference, Kaylee gauged the location of a given point along one spatial dimension with the realization that the point had a specific location along the other two spatial dimensions. Therefore, Kaylee's locating the fish in three-dimensional space involved logical multiplication of the measurements (Piaget et al., 1960) of distances that Kaylee measured using her rectangular frames of reference. As a result, Kaylee coordinated the location of the fish along all three spatial dimensions of the space and the relevant measurements of horizontal and vertical distances, producing a three-dimensional coordinate system. Because Kaylee combined the two structures, she was aware that she did not have to measure both the horizontal and vertical distances in her second frame of reference. Her goal-directed activity in only measuring the horizontal distances of the two fish from edge OA and her adding only those distances to her first sketch corroborated this hypothesis.

After Kaylee was done locating the first two fish, she looked at the tank and rotated it counterclockwise so that the face opposite to the face she previously sketched was now facing her. Later in the teaching episode, Kaylee explained that she chose the two faces based on the visibility of the fish. That is, she selected two faces where she could best see the fish. Kaylee marked her earlier sketch as "side w/purple" and started a new sketch titled "side w/pink." The two colors referred to the color of the fish that were visible from each of the respective faces. This face that she titled "side w/pink" happened to have the label sticker attached on it, which she drew in her sketch at the end.

Similar to the first sketch, Kaylee started with a sketch of the frame of the tank and the surface of the water. Then, this time, she measured all three measurements for each fish. For the first fish, which she named “pink,” she measured the distance from the top edge PE of the tank to the pink fish, with her ruler horizontal to edge PD; measured the distance from edge PD to the pink fish, with her ruler horizontal to edge PE (see Figure 5.19 (a) and (b)); and marked the pink fish in her sketch of the cubic tank (see Figure 5.19 (d); the red dashed line segments indicate Kaylee’s placement of the ruler).



(a) (b)

(c) (d)

Figure 5.19. Kaylee measuring three measurements in locating the pink fish.

Next, she rotated the cubic tank counterclockwise with respect to a vertical axis through the center of the tank as she had done before and measured the distance from PD to the pink fish with her ruler horizontal to PF; in other words, she measured how much farther into the tank the fish was from the face containing P, PD, and PF (see Figure 5.19

(c)). Like she did before, she marked the measurement in her sketch next to the fish with an arrow, this time the measure being .5 inches (see Figure 5.19 (d)).

Kaylee repeated this activity of finding horizontal and vertical distances using {P, PD, PE} and {P, PE, PF} as references for the last fish and marked it on her sketch as shown in Figure 5.20.

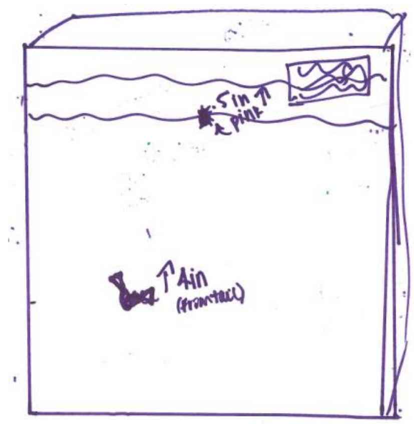


Figure 5.20. Kaylee's sketch of the two fish from the face with the sticker.

Similar to her previous activities in locating the first two fish, I infer that Kaylee inserted her second frame of reference that she superimposed onto the second face (the face including {P, PD, PF}) into her frame of reference of the first face (the one with the sticker label including {P, PD, PE}). Kaylee disembedded and held her frame of reference of the first face as a unit structure, inserted her second frame of reference into the first frame of reference, and translated it along the third dimension resulting in a three-dimensional frame of reference (see Figure 5.21). Taking each two-dimensional frame of reference as a given structure, Kaylee combined and united these two structures, which allowed her to track the location of a given point along one spatial dimension with the realization that the point had a specific location along the other two spatial dimensions. This allowed Kaylee to coordinate the location of the fish along all three spatial dimensions that span the three-dimensional space and produce a three-dimensional coordinated system of measurements.

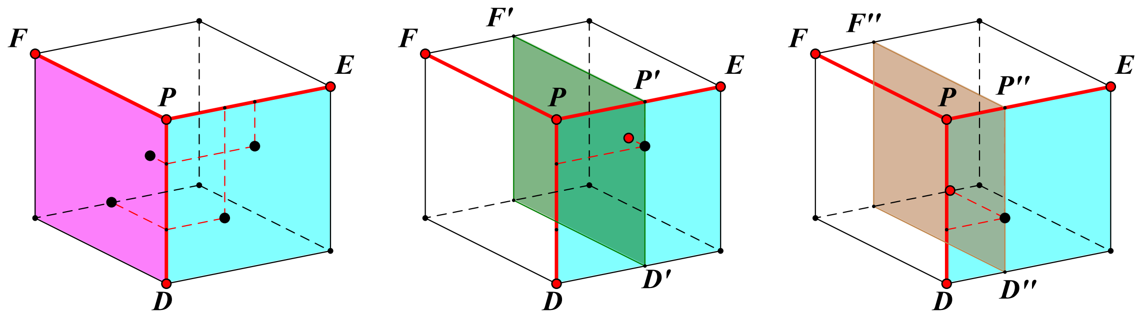


Figure 5.21. Model of Kaylee's insertion of her {O', OD, OF} frame of reference into her {O', OD, OE} frame of reference.

When I asked Kaylee to explain, in general, how she located the four fish in the tank, she explained how she located the four fish using the pink fish as an example.

Excerpt 5.6: Kaylee explains how she located the four fish in the cubic tank.

K: So, you would go to this side [*rotates the tank so that the face with the sticker label, i.e., the face she titled "w/pink" is facing her*] and you measure how far in [*moves finger horizontally from edge to fish as demonstrated in Figure 5.22 (a)*] and down [*moves finger vertically from top of tank to fish as demonstrated in Figure 5.22 (a)*] it was, according, if this [*placing hands on the face facing her*] was flat, and after I plotted that [*points to the pink fish she was referring to in her sketch*], like I drew it and I got right here [*moves finger horizontally along where she measured the "in" on her sketch (Figure 5.22 (b))*].

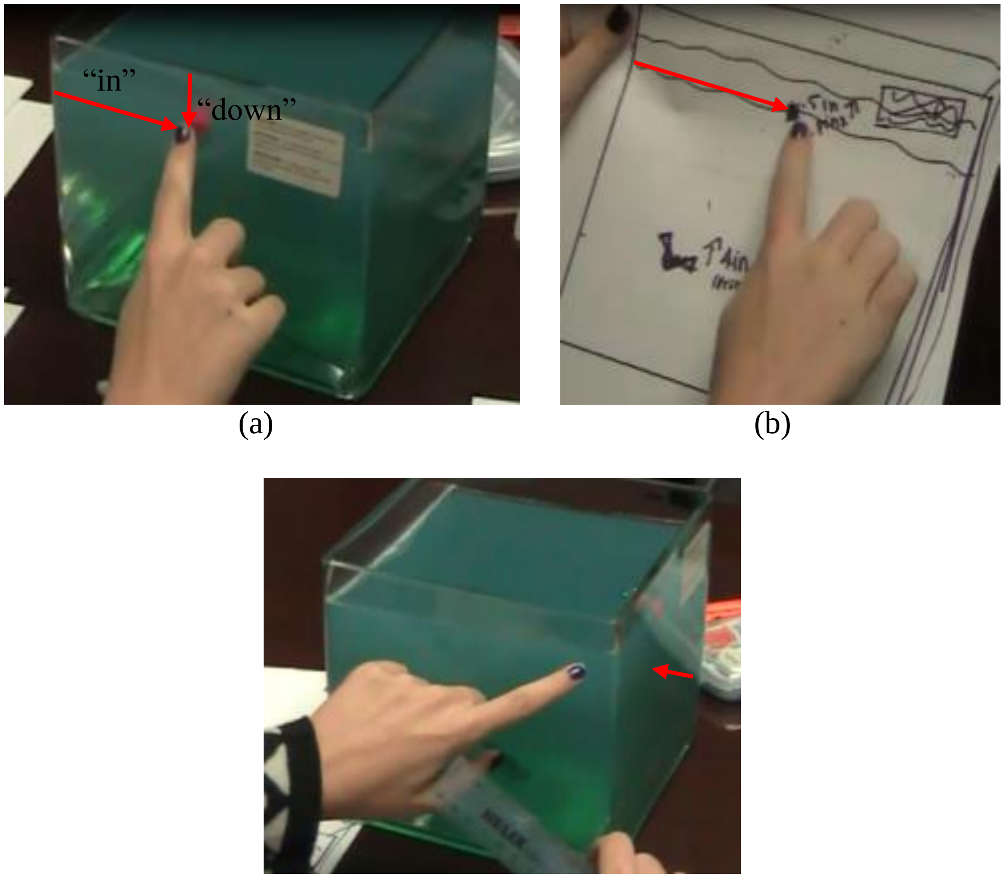
T: So, how did you decide to draw that pink thing right there? Like, what did you measure? Did you measure anything?

K: I measured [*picks up ruler*], like, because this [points to her sketch] is the same scale as this [*points to the cubic tank*]…

T: Oh, okay.

K: So I measured from here to here and from here to here [*repeating her finger motion in Figure 5.22 (a)*] and I plotted that [*pointing to the corresponding pink fish in her sketch (Figure 5.22 (b))*] and then I turned it this way [*rotates the tank counterclockwise 90 degrees so that the adjacent face is now facing her*] and measured how far in it was [*moves finger horizontally from edge of tank to the pink fish (see Figure 5.22 (c))*] and then that's what that number is [*pointing to the ".5in" she wrote in her sketch next to the pink fish*]. Point five into [*makes a hand motion referring to what she means by "in" similar to her finger motion in Figure 5.22 (c)*]… I don't know how to explain this but…

T:	Mm-hmm.
K:	And that's how I did it for every single one.



(a) (b)

(c)

Figure 5.22. Kaylee explains her sketch.

Based on Kaylee's comment, "that's how I did it for every single one," Kaylee viewed her locating activity as consistent for all four fish even though she chose different vertices (points) and edges (line segments) of the tank in locating each pair of fish.

From her consistent activities across the North Pole Task and cubic Fish Tank Task, I conjectured that Kaylee has constructed a *rectangular frame of reference scheme* to which she assimilated the different situations across the tasks. First, Kaylee recognized a situation in which she could coordinate horizontal and vertical distances to locate a point in two-dimensional perceptual space. In the North Pole Task this was a point on the map; in the Fish Tank Task this was a point projected onto the two-dimensional face of the tank. After recognizing the situation, Kaylee constructed a mental grid-like structure consisting of horizontal and vertical lines which she anchored onto spatial objects such as the North Pole point or the edges of the tank. Using this frame of reference, she gauged and represented the relative position of points in the perceptual space by coordinating the horizontal and vertical distances from the references. In the North Pole Task it was the location of point A in relation to point P and in the Fish Tank Task it was the location of one fish in relation to the tank frame that she gauged. As a result, she produced a Cartesian-like coordination of measurements along two perpendicular axes through which she represented the location of objects in the two-dimensional plane.

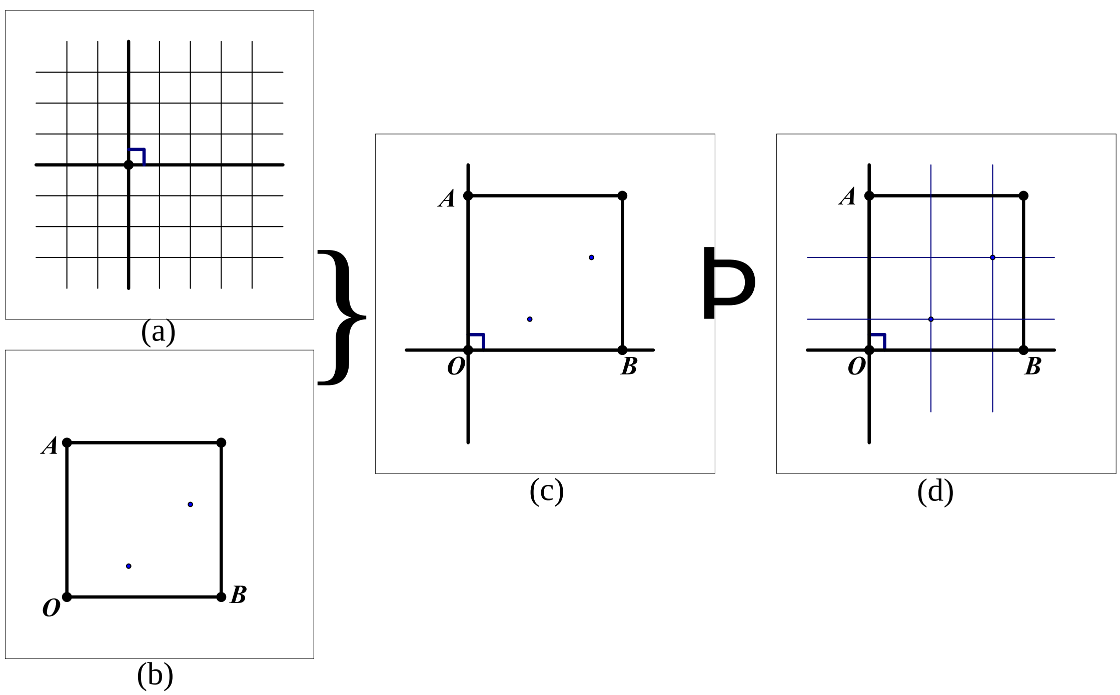


Figure 5.23. A model of Kaylee's representation of the face of the tank.

In particular to the cubic fish tank, Kaylee used a set of perpendicular lines and point of intersection (Figure 5.23 (a)) and anchored it onto {O, OA, OB} of the fish tank (Figures 5.23 (b) and (c)). Then, using this frame of reference, Kaylee constructed two sets of horizontal and vertical lines that would each pass the two fish in the tank (Figure 5.23 (d)). As a result of using this frame of reference, Kaylee was able to coordinate the respective horizontal and vertical distances consisting the spatial dimensions of the location of each fish (Figure 5.24).

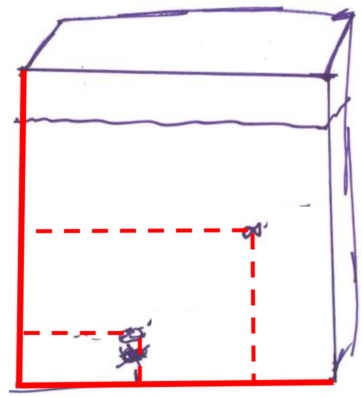


Figure 5.24. Kaylee's representation of the first face.

I inferred this to mean that Kaylee has enacted her rectangular frame of reference scheme in both faces and superimposed it onto different physical locations of the tank but was aware that she had used the same system in which coordinated three distance measures to locate each fish.

From Kaylee's consistent manner in which she coordinated two sets of rectangular frames of reference in the three-dimensional cubic fish tank case, I conjectured that Kaylee had constructed a *frames of reference coordinating scheme* (*FR coordinating scheme*). That is, she recognized of a situation in which she could posit a frame of reference as a unit and insert it into another frame of reference resulting in combined frames of reference. In the cubic fish tank, the insertion of a two-dimensional frame of reference into another and multiplicatively coordinating them allowed her to locate the fish' location along all three spatial dimensions that spanned the three-dimensional space of the tank.

In order to construct and enact a FR-coordinating scheme, I conjectured that Kaylee enacted mental actions of decentering, rotating, and brought forth images of one perspective alongside another. Then, Kaylee disembedded the frame of reference constructed for one side of the tank, taken from one perspective and inserted it into another frame of reference constructed for another side of the tank, taken from a different perspective. Uniting and multiplicatively coordinating the two sets of frames of reference resulted in a representation of the four fish along all three spatial dimensions. By multiplicative coordination I mean that the locations of a fish were gauged with the simultaneous realization that the fish had a specific location along all three dimensions. Therefore, I hypothesized that Kaylee's three levels of units coordination supported such mental actions and that the FR-coordinating scheme required mental operations that are essential for coordinating three levels of units.

In our conversation that followed Excerpt 5.6, Kaylee explained that she chose the sides where the fish were most visible. From the design of the task, the gelatin used in the tanks was semi-transparent. Therefore, the visibility of the fish from the sides of the tank was limited. For instance, some of the fish were visible from one side of the tank but barely visible from another. I attributed Kaylee's choice of two different sets of faces to the lack of transparency and visibility of the fish. Although Kaylee used different sets of spatial references for two sets of faces and the corresponding adjacent faces, Kaylee identified her method as the same and summarized her approach as finding "How far down and in [the fish] was" and "see how far into, towards the middle it was." From this, I conjectured that Kaylee has constructed a coordinated a single three-dimensional system of measurements in locating the four fish in the cubic tank.

Morgan locates the four fish in the cubic tank.

Morgan located all four fish taking the perspective of looking down onto the tank. After measuring two adjacent edges on the top of the tank, Morgan sketched a square and wrote “TOP” on top of her square figure (Figure 5.25).

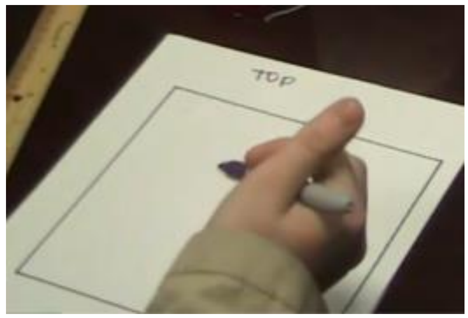
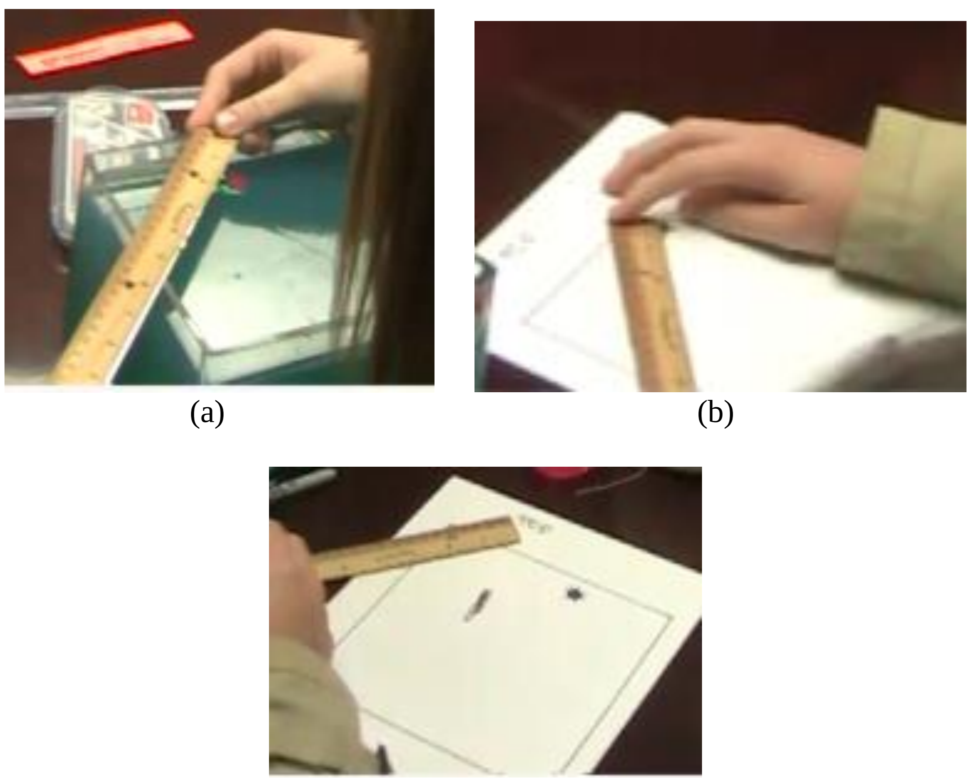


Figure 5.25. Morgan’s sketch of the top of the cubic tank.



(a) (b)

(c)

Figure 5.26. Morgan using a ruler to locate one fish from another.

Once Morgan sketched the frame of the top of the tank, she located the fish in the square. She looked at the tank from the top and plotted one fish in the square without carrying out any observable measuring activities. Next, she laid the ruler on the top of the tank so that it passed above the first fish and the second fish she was going to sketch (Figure 5.26 (a)). Then, she moved the ruler to her sketch (Figure 5.26 (b)) and marked the second fish (Figure 5.26 (c)). As such, it seemed as though Morgan gauged the location of the second fish using the first fish as a reference.

In transporting the ruler, it was unclear whether the inclination of the ruler was preserved. However, because I did not want to interfere her thought process, I did not ask Morgan how she transported the ruler in that moment. Later when I asked, Morgan did not remember how she transported the ruler. As such, I do not have confirmation of whether or not Morgan was aware of maintaining the inclination of the ruler. Even if she was aware, the inclination of the ruler was preserved based on the perceptual imagery of the ruler, not based on any specific measurements.

After sketching the first two fish, Morgan again looked down onto the tank, moved her finger vertically from top to bottom of her square (Figure 5.27 (a)) and marked the third fish in her sketch (marked in green in Figure 5.27 (b)). It seemed like Morgan guaged the location of her third fish based on the second one, continuing to use visual estimations. Finally, she looked down onto the tank and sketched the last fish. It is likely that she used the edges of the top of the tank or other fish as references in making estimations of the locations of the fish; however, because there were no measuring activities, I could not make any further inferences.

(a)

(b)

Figure 5.27. Morgan locating the other two remaining fish in her sketch.

Here I note that Morgan’s visual estimations were not mindless since the fish were not randomly marked on the paper. I acknowledge that there must have been some consideration of topological features such as order and proximity of the fish in the tank and perhaps some considerations of gross quantities (e.g., the distance between one fish and another is greater than the distance between a third fish and the fourth). However, when I claim that there was no observable measuring activity, I mean that I was not able to observe a physical measuring activity or verbal explanation involving commitment to a unit of measure (e.g., using a ruler or using fingers to mark a certain distance and iterating it along other line segments).

Once she plotted all four fish in her square as shown in Figure 5.27 (b), Morgan added details to her sketch resulting in an illustration as shown in Figure 5.28, which I describe next. First, starting with her sketch in Figure 5.27 (b), Morgan added a cardinal direction sign on the bottom of her square. Second, she measured the dimensions of the top of the tank, again, and wrote “7in” on the left edge and top edge of her square. Then using her ruler, she made 1 inch marks along the left edge and used those marks to draw horizontal lines one inch apart; she repeated the same activity in the other direction resulting in vertical lines one inch apart and formed a grid of horizontal and vertical lines.

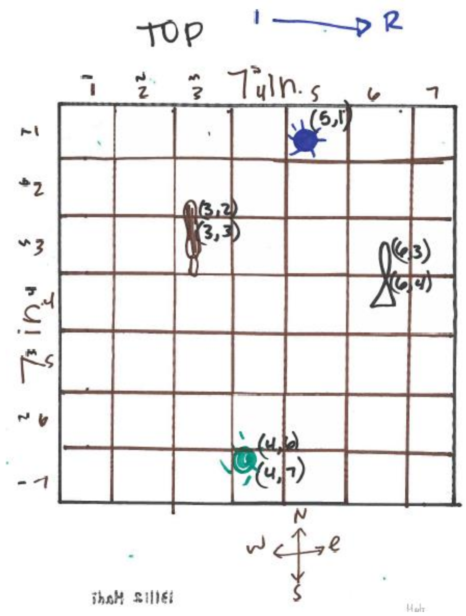


Figure 5.28. Morgan's diagram to explain the location of the four fish in the cubic tank.

Third, she labeled each one-inch interval, 1, 2, 3, 4, 5, 6, 7 along the top and side of her square. Fourth, she coordinated those numbers into pairs and next to each fish she wrote pairs of numbers in parentheses. For example, she swept her finger horizontally and then vertically and wrote (3, 2) and (3, 3) next to the brown fish in her sketch in Figure 5.28. The 3 in both coordinates referred to the unit square along the horizontal direction the fish was located in; the 2 and 3 each referred to the unit squares along the vertical direction the fish was located in. She used the same notational system for the remaining fish, accounting for the unit squares that the fish was located in. Because the fish had some amount of volume, most of them were positioned in more than one unit square and thus required two sets of coordinates pairs (with the exception of the first fish). In other words, the fish were located in terms of the section in which they were located in and not by the grid lines, similar to what Sarama et al. (2003) observed from fourth-graders using grid structures.

As such, given her initial sketch as shown in Figure 5.27 (b), the cardinal directions and the grid were added *after* she located the fish in the tank and not used to draw the fish in the first place. Instead of using the grids to gauge the locations of the fish, I conjectured that Morgan used the grids for communicational purposes so she could tell another person where each fish was located.

The following excerpt illustrates Morgan's explanation of her sketch. Note that Kaylee had already explained her sketch (cf., Excerpt 5.6).

Excerpt 5.7: Morgan explains how she located the four fish in the cubic tank.

M: Okay, so I wasn't thinking from the side, I went from the top. And um, this is basically looking at it when it's like that [*rotates tank again to obtain orientation of tank in Figure 5.29 (a)*]. And um, so, this [*pointing to her sketch; Figure 5.28*] is the same scale as this [*points to the top of the tank*] right here.

T: Okay.

M: And um, from here to here [*moves finger along QG to the pink fish (Figure 5.29 (a))*] this is five inches. And then down a unit [*moves finger vertically down one inch from QH to pink fish (Figure 5.29 (a))*] would be one inch and so practically, it's in that location. I don't know how to… Apparently, I just created like a graph where I can locate them and…

T: So, I was looking at how you were doing this because I was sitting right next to you. And it seemed like you plotted this one first [*pointing to the first (pink) fish in her sketch; Figure 5.28*].

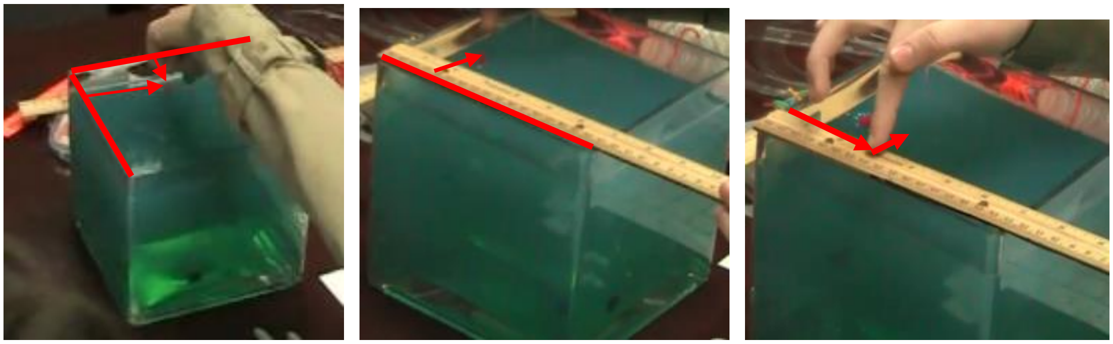
M: Mm-hmm.

T: And somehow you did something with the ruler, like, you put it like there [*hand motions laying a ruler on the top of the tank*] and you drew this one [*pointing to the brown fish she drew second in her sketch; Figure 5.28*]. Can you remember how you got this one [*pointing to the brown fish*]?

M: Oh, I mean, all I was doing… I was…[*Sits in thought and looks at top of tank again for approximately seven seconds*] Okay, so what I did was I eye-graphed it [*lays a ruler along QG*] and I found the location [*moves finger horizontally from ruler to the pink fish (see Figure 5.29 (b)*] and then I went down [*moves finger down along the ruler*] to find that one and I saw [*moves finger horizontally from ruler to the second fish (see Figure 5.29 (c))*].

T: I see. Where it was?

M: Yeah. Because that one was the closest to it [*pointing to the two fish she sketched first*]. If I were going down from top to bottom then just [*moves fingers vertically across her sketch and then horizontally across her sketch as if tracing the grid she constructed*]…



(a) (b) (c)

Figure 5.29. Morgan demonstrates how she located the second fish from the first.

I considered Morgan's explanation in Excerpt 5.7 to be different from her earlier activity in generating her sketch in Figure 5.28. Recall that when sketching the second fish, Morgan laid the ruler slant on the tank and moved the ruler onto the paper as shown in Figure 5.26. For the first and two last fish, she visually estimated the locations without laying the ruler on the tank. When I asked her to explain her earlier activity with the ruler, Morgan recalled that she had "eye-graphed" the fish by moving from one fish to another, using the fish as reference to locate other fish. In contrast, in Excerpt 5.7, as shown in Figure 5.29, Morgan used the edges of the tank as reference. Morgan's explanation in Excerpt 5.7 entailed a more explicit reference to a measurement process: she demonstrated an attendance to unit of measure (inches), scale of measure, and used her ruler.

Considering that Kaylee has first explained and demonstrated her sketch, it is likely that Morgan has assimilated Kaylee's strategy of measuring horizontal and vertical distances from the edges of the tank to the fish. Based on her behavioral indications, it is difficult to infer that Morgan has constructed a coordinated system of measurements induced from a frame of reference independently. At most, I claim that Morgan constructed a frame of reference in the activity of explaining her sketch as demonstrated in Excerpt 5.7 and Figure 5.29. However, they were not apparent enough for me to regard them as strong indications of operationalizing frames of reference in locating the fish. Therefore, I infer that Morgan's actions for locating the fish were yet in-the-moment and heavily relied on perceptual imagery.

After producing the grid and coordinates, Morgan seemed hesitant, so I asked her to explain what she was thinking about. In the following excerpt, Morgan explains the various orientations of the tank.

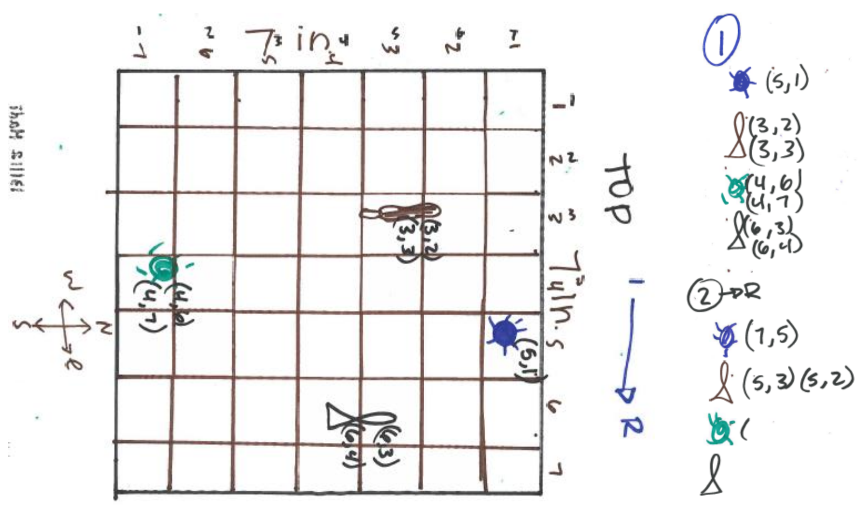
Excerpt 5.8: Morgan explains the different possible orientations of the tank.

T: Morgan, what are you thinking?
M: Well, I'm thinking about doing… Cuz, I didn't want to do it from like, the sides [*points to the side of the tank using her ruler*].
T: Mm-hmm.
M: So, if I'm doing it from the top, I'm going to have to like, um… So this [*pointing to her sketch shown in Figure 5.28*], is looking down on it seeing from like here [*rotates the tank to the orientation that corresponds to her sketch (Figure 5.30 (a))*]. This side, so I'm going to label that like, this will be one [*writes 1 on top of her sketch in Figure 5.28*] and I'm going to just say…
T: Oh, okay.
M: And so if I turn it to the right once [*rotates tank 90 degrees clockwise (Figure 5.30 (b))*] then that will be two. [*Rotating another 90 degrees*] And that will be three, [*rotating another 90 degrees*] and that could be four.

(a) (b)

Figure 5.30. Morgan rotates tank 90 degrees.

Like Morgan explained in Excerpt 5.8, because her diagram did not involve any consideration of the “sides” of the tank, she wanted to account for the positioning of the tank. After explaining her thoughts in Excerpt 5.8, Morgan made an arrow sign next to the “1” she had written on top of her sketch and wrote “R” next to the arrow indicating starting at orientation 1 and then rotating to the right. Morgan rotated her gridded square clockwise 90 degrees and wrote 1, 2, 3, 4, 5, 6, 7 on each vertical and horizontal edge as shown in Figure 5.31 (a).



(a) (b)

Figure 5.31. Morgan’s gridded square rotated 90 degrees clockwise and her new set of coordinates.

Next, Morgan located each fish under the two orientations using coordinates, as shown in Figure 5.31 (b). In sum, Morgan was aware of different orienting perspectives one can take when viewing the tank and wanted to consider these in her sketch. However, she did not seem confident about the coordinates she found for the second orientation and did not complete the list.

Later in the teaching episode, I asked Morgan about the cardinal direction sign and the two sets of {1, 2, 3, 4, 5, 6, 7} she wrote along the sides of her gridded square:

Excerpt 5.9: Morgan explains the cardinal directions and two sets of numbers.

M: Um, I noticed how, it kind of made, because I kind of demonstrated it as the north, east, south, west [*sweeping her fingers counterclockwise starting from the fish on top through all fish that seemingly lined up with the north, east, south, west direction in her sketch in Figure 5.28*].

T: Oh, okay.

M: So if I turn it this way [*turns her sketch clockwise 90 degrees as shown in Figure 5.31 (a)*] that's going to be the north [*points to the brown fish as the north (Figure 5.31 (a))*], east, south, west [*again moves her finger counterclockwise through all four fish*]… So…

T: Okay. And I noticed you changed, you turned the paper like this and you wrote one, two, three, four, five, six, seven again. Why were you doing that?

M: Just in case. Like, if you turned it a specific, another way [pointing to the tank] so, kind of like how we used the…Like when we're in the snow or whatever, if he was facing a different way, he would still know how to get to [*pointing to each fish*].

T: Ah, I see.

As shown in her comment, "when we're in the snow or whatever, if he was facing a different way, he would still know how to get to [the fish]," Morgan seemed to have recognized the situation of locating the four fish in the cubic tank similar to what she has done in the North Pole Task. She seemed to have recalled that the orientation of their map was important in communicating with the rescuer in the North Pole Task and wanted to make sure that this was accounted for in the fish tank case. As a solution, Morgan considered the cardinal directions and two different sets of numbers for each orientation of the tank sketches. As such, Morgan had an awareness of having to account for more than the top view of the tank.

Kaylee and Morgan discuss their different approaches.

Once each student had a chance to explain their way of locating the four fish, I asked the students to talk about each other's sketches. When I asked Kaylee to comment on Morgan's method, Kaylee offered a critique explaining that Morgan's sketch only accounted for the top view of the tank. Kaylee argued that all the fish could be "floating on the top," and that Morgan's sketch did not account for "how far down" the fish were in the tank.

As such, Kaylee operated as if she projected the fish into a two-dimensional plane and took that projection as an object in further operating. Consistent with her earlier behaviors, Kaylee was able to *disembed* the projection of the fish into the top layer of the water in the tank as a unit and *insert* that into any point along the third dimension. Kaylee also seemed aware that Morgan's sketch could be of any instance of those insertions of the top layer of the water in the tank into any depth of the tank. Although the top layer of the water was a different face from what Kaylee had used earlier in her sketch, she was able to flexibly transfer her frames of reference and coordinate them to the face that Morgan used in organizing the locations of the fish. Hence, corroborated my hypothesis that Kaylee constructed a rectangular FR coordinating scheme.

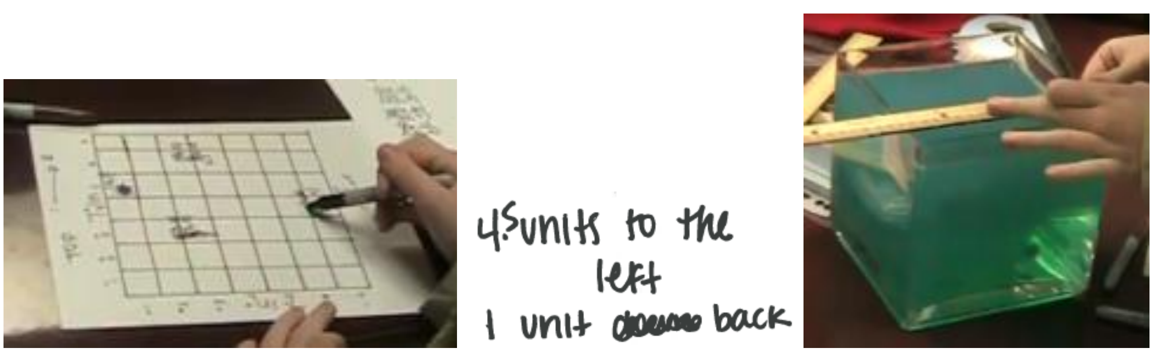
As a response to Kaylee's critique, Morgan replied that she "was going to do top and sides but didn't have enough time." Earlier in the teaching episode, Morgan did not account for the sides of the tank in her sketch and specifically stated that she "didn't want to do it from like, the sides" (Excerpt 5.8). It is possible that the social interaction between the two students led to Morgan's somewhat contradictory statements. Because Kaylee often carried out activities first and often explained her thinking before Morgan had a chance to, it is possible that Morgan wanted to find a solution different from Kaylee's. Their playful manner of competing with each other adds support to this interpretation. However, whether this was the case or not, Morgan seemed to have assimilated Kaylee's approach, which became more apparent in the next problem of the task.

Kaylee and Morgan Describe Fish 1 moving to Fish 2 in the Cubic Fish Tank

To further explore the students' coordinated systems of measurements, I asked them to work together in giving instructions to one fish ("Fish 1") to another fish ("Fish 2") in the cubic fish tank. Morgan started to coordinate her sketch with Kaylee's sketch. More specifically, Morgan suggested that they use her sketch (Figure 5.34) to tell the fish how far to travel horizontally (she moved her finger horizontally in front of the tank) and Kaylee's sketch to determine how high the fish would need to go. Her sketch has transitioned into a working frame of reference and it was being coordinated with Kaylee's sketch of one side of the tank. It seemed like Kaylee's critique of her sketch led Morgan to account for the height of each fish in the tank. Kaylee agreed with Morgan's suggestion and the two students went to work. In the following excerpt, Morgan and Kaylee formulate instructions for Fish 1 to swim to Fish 2.

Excerpt 5.10: Kaylee and Morgan discuss how Fish 1 swims to Fish 2.

M: So, this way [*rotates her sketch of the gridded square so that the locations of the fish correspond to the one in the tank*]…Okay, so, one, two, three, four [*counts the number of squares starting from Fish 1 to Fish 2 (Figure 5.32 (a)), then writes "4 units to the left," looks back at her sketch and adds "1 unit down," scratches out "down" and writes "back." (Figure 5.32 (b))*].



(a) (b) (c)

Figure 5.32. Morgan's measurement activities in Fish 1 swimming to Fish 2.

[*Continued*.]

K: [*Sketched the frame of the tank (this time only a square representing the face in front of her) and measures the horizontal and vertical distances from the fish tank and located the two fish in her square (see Figure 5.33; the red dashed arrows represent the order she measured and marked the distances using her ruler)*].

M: [*While Kaylee locates the two Fish in her sketch, Morgan used the ruler to measure the distance between the two fish from the top view and changes her earlier "4 units to the left" to "4.5 units to the left" (Figure 5.32 (c))*].

K: [*Once she has completed plotting the two fish*] So, I'm going to measure…

M: Just the length between them or the height.

K: [*Lays the ruler vertically starting from Fish 1*.]

K: [Inaudible]. [*Draws two line segments connecting the two fish, then measured the length of the first line segment and writes 2.75 next to it, as shown in Figure 5.33 (a). Then she moved the ruler to measure the length of the second line segment when Morgan interrupted her*].

M: You don't have to worry about the top one.

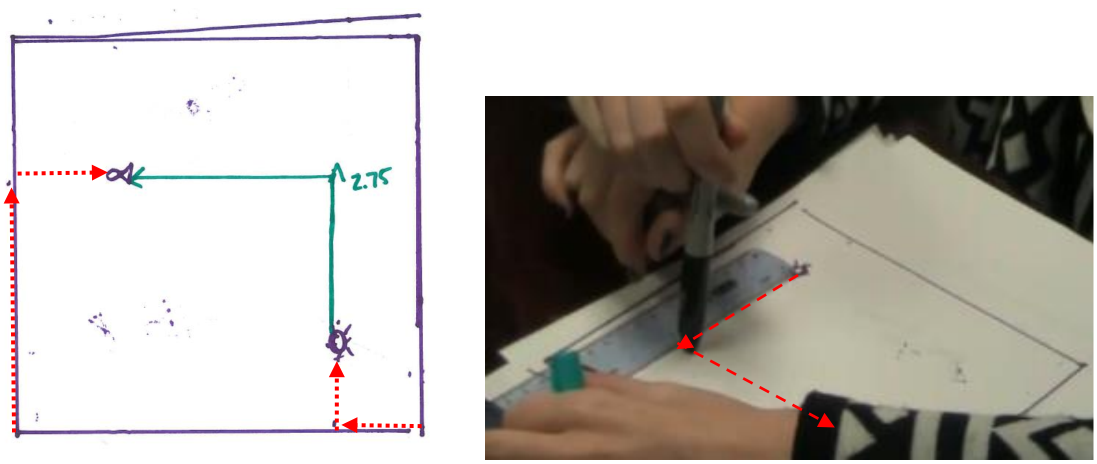
K: I mean [inaudible]… [*Moves finger along the two line segments as if following the arrows from Fish 1 to Fish 2*].

M: No, I already have it [*taps on her sketch*].

K: What? Oh, like from… [*Moves finger along first line segment in deep thought*] So, you know this [*moves finger along the second line segment*].

M: So, he'll just have to do four units to the left then he'll have to go one back, then he just go straight up something.

K: [*Smiles and nods*] Yep.



(a) (b)

Figure 5.33. Kaylee's measurement activities in Fish 1 swimming to Fish 2.

As shown in Excerpt 5.10, there was a noticeable change in Morgan's engagement in the task. In contrast to the majority of the teaching episodes so far, Morgan was leading the discussion and suggested what Kaylee should do in solving the task. For example, Morgan told Kaylee to draw a line through Fish 1, a line through Fish 2 and to find the intersection of the two lines. From her referring to the lines as “straight” and her hand motions in showing those lines, I inferred that Morgan envisioned two perpendicular lines through the two fish. As such, Morgan now seemed to use her grid to decompose the movement of Fish 1 to Fish 2 in two spatial dimensions along the top view of the fish tank. In other words, Morgan used her grid as a frame of reference defining the direction in which Fish 1 would swim to Fish 2. Interestingly, after using her sketch to count the number of inches Fish 1 would need to move to the left, Morgan measured the same distance using her ruler and changed the measure from 4 to 4.5 units. It is be possible that she wanted to double check her counting of the number of squares or that she wanted to provide more accurate measurements. I view this behavior as an indication that Morgan was transitioning from using the grids to communicate the squares in which the fish were located to using the *grids as a frame of reference* to guide her measurement activities.

Excerpt 5.10 also shows Morgan’s change in accounting for all three dimensions in the fish’s movement. Different from her earlier activities in locating the four fish in the cubic tank considering only two dimensions along the top view of the tank, Morgan now coordinated the third dimension with her representation of the two-dimensional grid. There are two elements in the teaching episode that I attribute to such progress. First, the task requirement to account for the motion of points may have pushed her to account for all three spatial dimensions. The task requirement to attend to the movements of the fish may have engendered accounts for change of position (Piaget et al., 1960) of the fish.

Second, Morgan may have assimilated Kaylee's approach in her measuring activities. Morgan's illustrations, actions, and explanations gradually shifted from making visual estimations of the location of the fish to using frames of reference to coordinate measurements of distances between the two fish. This change became apparent after Morgan listened to Kaylee's explanation of her sketch and how she located the four fish. It is possible that listening to Kaylee's critique provided Morgan an opportunity for reflection as well.

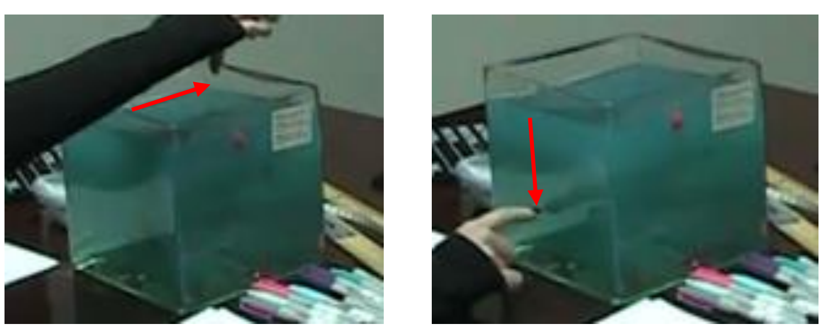
Extending the Cubic Fish Tank Task

Morgan locates the fish in the cubic tank again.

At the beginning of the next teaching episode, on November 14, I asked the two students to find a way they could combine their ideas for locating the fish. The main purpose of asking them to do so was to test whether Morgan 's aforementioned modifications were temporary or not. The following excerpt starts with Kaylee making a suggestion about how to combine their sketches in locating the four fish.

Excerpt 5.11: Kaylee and Morgan combine their ideas to locate the four fish.

K: Morgan, you can find the… With the grid, you can find how deep it is [*moves finger from the front of the tank towards the back of the tank (Figure 5.34 (a))*], in here and with mine you can find how far down it is [*moves finger from top of the tank towards the bottom of the tank (Figure 5.34 (b))*].



(a) (b)

Figure 5.34. Kaylee explaining her sketch and Morgan's sketch.

[*Continued*]

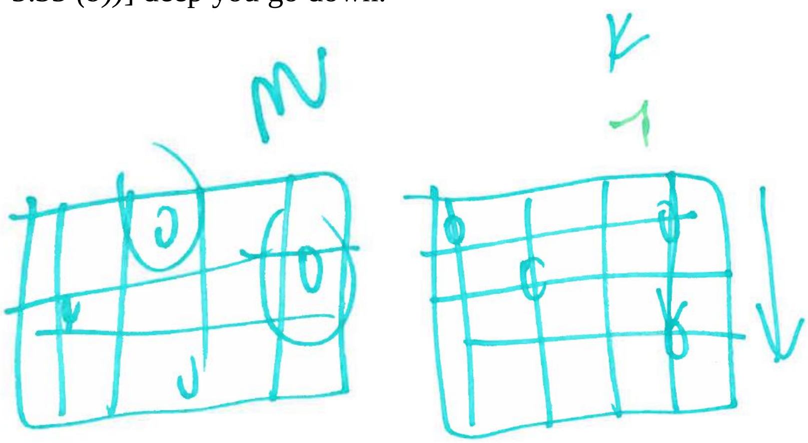
M: Yours is finding how deep it is mine is finding where they are [*points finger in various points on the top of the tank*]. Like, the location.

K: Yeah, the location and mines like the dimension, I guess.

M: What we need to do, okay, we can draw it. So, that's like the top, right? We can draw our grid again [*illustrates a rectangle and a grid inside of it*] and then draw wherever they are, whatever… [*Randomly draws four small circles in her grid*]. And then, like, this would be mine [*writes M on top of her sketch (Figure 5.35 (a))*] and then Kaylee can draw hers from the side [*writes K and then a rectangle beneath her sketch*] and showing where she sees them [*again, randomly draws four small circles in the new rectangle*]…

K: Mm-hmm. And so you can see, like, this one [*pointing to the top right circle in the M rectangle (Figure 5.35 (a))*] is like…

M: [*As Kaylee is speaking jumps in*] And so like, this is how far in [*draws an arrow from the circle on the top right to the circle underneath it (Figure 5.35 (b))*] deep you go down.



(a) (b)

Figure 5.35. Morgan's re-generation of their two sketches.

In Excerpt 5.11, although Kaylee first initiated the conversation, Morgan took a more active role in the discussion. Morgan initiated her drawing (Figure 5.35) and explanation without Kaylee's help. Different from her earlier actions in the first teaching episode of the cubic fish tank, Morgan started her sketch of the top layer with the grids and then added in the fish. From such observations, I hypothesized that Morgan's construction of a rectangular frame of reference was not temporary. By the way Morgan randomly added the four fish to her sketch without looking at the fish tank, I inferred that

Morgan no longer depended on the perceptual imagery of the cubic fish tank; now the coordination became operative and she seemed to be aware that her coordination would work for any location of the fish in the tank. Finally, by the way that Morgan was able to point to one fish on the top of the tank and relate the location of that fish in the side face of the tank, and discuss what each sketch of the tank allowed her to find in terms of the location of the fish, I inferred that she has coordinated the two perspectives multiplicatively (one taken from the side view and one taken from the top view) to locate the fish.

Fixing one origin.

Noticing that both students agreed on using the grids, I then asked them to sketch the grids onto the tank; Kaylee used a dry-erase marker to add the grids to one side face of the tank and Morgan started to sketch a gridded square. After the grids were placed on the cubic tank as shown in Figure 5.36, I asked the two students if the two grids would provide enough information to locate the fish.

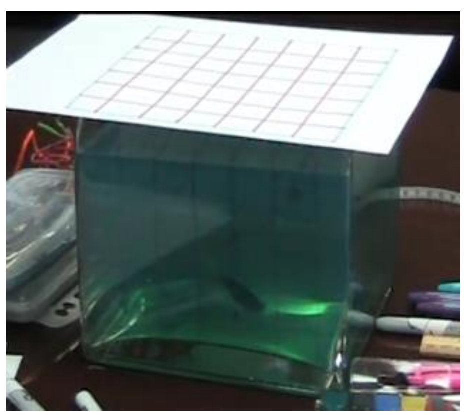


Figure 5.36. Two grids on the cubic tank.

Kaylee explained that it should, because "mine shows the height [*moving finger vertically along the face in front of her, on which she drew a grid using a board marker*] and the length [*moving finger horizontally along the face in front of her, on which she* *drew a grid using a board marker*] and Morgan's shows the width [*moving finger vertically along the grid on top of the tank*]." Kaylee generalized the three measurements that consisted of the location of each fish in relation to the frames of reference to height, length, and width. I then asked the two students to explain specifically how they would use the grids in locating the fish. The following is Kaylee describing how she would use the grids to locate one fish.

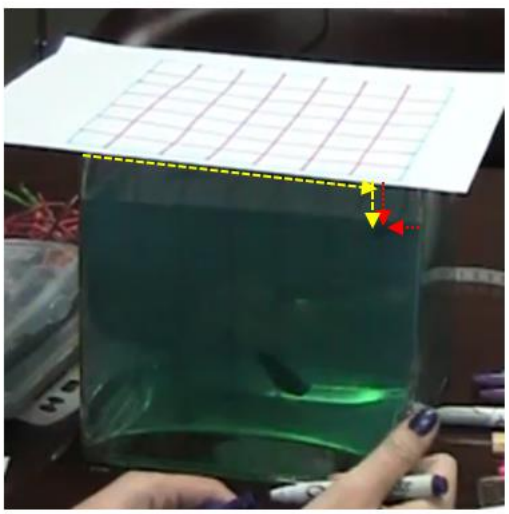
Excerpt 5.12: Kaylee explains how she would use the two grids to locate one fish.

K: So, let's say this is where the fish is [*Marks fish's location on the front face of the tank (circled in red in Figure 5.37 (a))*]. So, you could tell whoever's trying to find these fish, okay, going on the side, it's…It's at the point what's that, six one? [*Writes (6, 1) next to the point on the face.*] Or no…

M: It's close.

K: I want to look at it like a graph, so [*erases (6, 1)*] so it should be six, six [*writes (6, 6) next to the point on the face*]. And so that would tell them that's one inch down and one inch to the right [*making arrows from the top of the tank and the right edge of the tank to the point (red arrows in Figure 5.37 (b))*] or left, I mean. And then this, on this one [*pointing to the grid on top of the tank*]…

(a)



(b)

Figure 5.37. Kaylee and Morgan explain the location of one fish using two grids.

[*Continued*.]

M: Why would we do it like this though? Why wouldn't we do six and then one [*moves finger across the top of tank and then down along the right edge of the tank (yellow arrows in Figure 5.37 (b))*]?

K: Well, I'm reading this like a graph. This would be the origin [*pointing to the bottom left corner of the face in front of her; circled in white in Figure 5.37 (a)*].

M: Oh.

T: Oh, so this [*pointing to the same place Kaylee pointed to*] is your origin?

K: Yeah.

M: And that's going to be your [*moves finger along the left edge and bottom edge of the face of the tank (labeled with red arrows in Figure 5.37 (a))*].

K: Yeah.

M: Okay. So, pretty much, it just matters where you put your origin.

T: What about this grid, then? [*Pointing to the grid on top of the tank*.]

K: Then, you would do the same thing like…

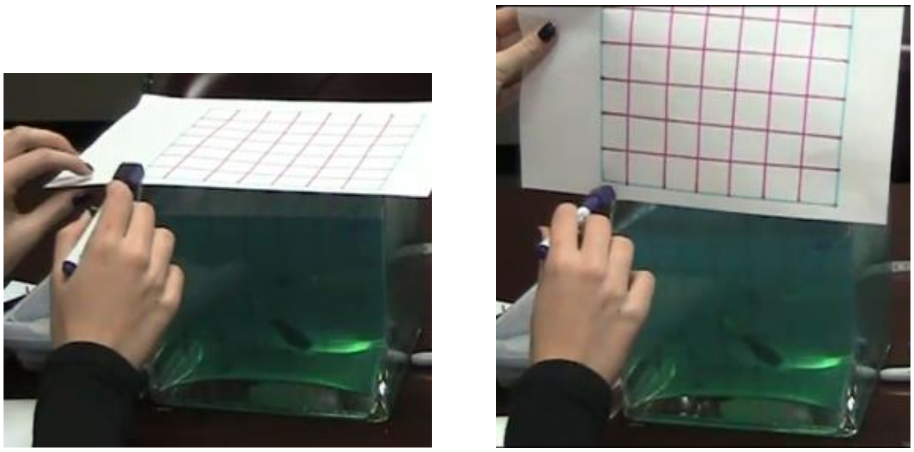
M: So, you could do the same thing right here [*pulls the grid off the tank and places it in front of the tank and then puts it back on the top*].

K: Now, it will be like right here [*points to a random point on the grid*] so you say it would be at six [moves pen along grid horizontally] three [*moves pen along grid vertically and arrives at the assumed location of the fish*] or four. And so that will tell them it's four inches in [*points towards the back of the tank*] to the water, so you would know four inches in and one inch deep.

As shown in Excerpt 5.12, looking at the grids on the cubic tank, Kaylee enacted her previous learning experience of what she referred to as a "graph." She assigned the bottom left corner of the fish tank the origin and used coordinate pairs to specify the location of one fish. Morgan soon caught on to the idea as well; although Morgan did not explicitly use the word axes, from her hand motions shown in Figure 5.37, both students superimposed horizontal and vertical axes onto the two edges of the tank and agreed on the bottom left corner as the origin. Because it seemed as though Kaylee was superimposing two two-dimensional "graphs" onto each of the two faces of the tank and coordinated them, I wanted to know if they could develop a three-dimensional "graph" that would share one origin. The following excerpt starts with Kaylee answering my question whether there would be a way to use one origin and not one for each face.

Excerpt 5.13: Kaylee explains how she would use one origin to locate one fish.

K: Yeah, like here [*points to the top left corner of the tank (Figure 5.38 (a))*], I guess. So that, it would be like this [*Picks up the grid on the top of the tank and puts it vertically on top of the tank and points to her origin as shown in Figure 5.38(b)*] is the origin [*points to the top left corner of the tank*]…



(a) (b)

Figure 5.38. Kaylee fixing one origin.

[*Continued*]

M: [*Interrupts*.] So, it's a graph…

K: [*Continues*.] So this would be negative [*pointing to the front face of the tank*] and this would be positive [*pointing to the grid above the tank*]. Negative y's [*pointing to the front face of the tank*] or no, negative y's and the positive y's [*pointing to the grid above the tank*]. So this would be like…

M: So, this is what you're doing…

K: Six, negative one [*Erases the (6, 6) inscription next to the fish on the front face and changes it to (6, -1)*].

M: This is our origin, okay? [*Starts a sketch with two perpendicular line segments*.] This is the top and this is the side that we're talking about right now. So, this would be the other side and these are like negatives and positives [*writes +, - signs in the right "side" face in her sketch*].

K: Yeah, positive, negative.

M: Yeah. And so, this would be positive, positive [*writes +, + signs in the right "top" face in her sketch*].

K: So this corner right here is your origin [*points to the top left corner of the front face of the tank*], so this would be plotted at six, negative one [*pointing to the point shown in Figure 5.37 (a)*] cuz you're going six and down one [*moves pen from origin to the six's line segment and then down one unit in the grid*].

T: Ah, I see.

M: So, it would be like negative one below sea level or whatever. [*Writes -1 next to her sketch as shown in Figure 5.39*].
K: Yeah, because you're going down.
M: [*Simultaneously,*] down.

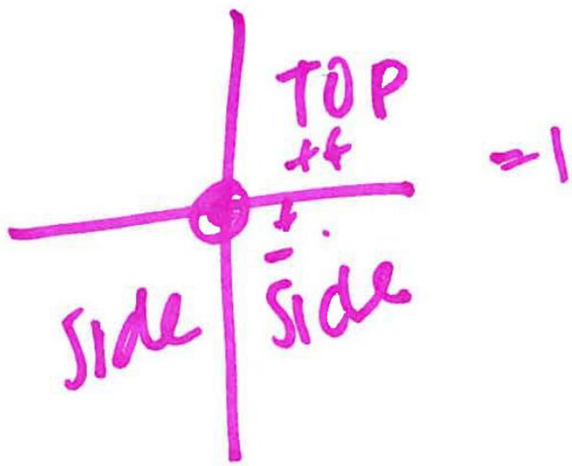


Figure 5.39. Morgan's sketch of Kaylee's explanation.

Asking Kaylee and Morgan if they could use one origin rather than two for each face provoked their consideration of direction (below sea level). However, the third axis seemed to be an extension of the second axis and not a separate one. They were aware that the first coordinate of the fish, 6, was consistent for both the side grid and the top grid. However, the second coordinate, which Kaylee referred to as the "y's" which were the -1 and 4 in each grid, respectively, were not differentiated. That is, they would need to use both (6, -1) and (6, 4) and to know which grid they are referring to when using these coordinates. The coordination of the two two-dimensional coordinate systems allowed them to locate the fish and accounted for all three dimensions in which the fish were situated; yet, the coordinates did not entail the distinction of the two measurements of -1 and 4, which would have been made if they considered (6, 4, -1) as a coordinated triple. Had I asked them to identify which measurements referred to the length, width, and height, I could have provoked the awareness of -1 and 4 as measurements of different elements (-1 the height and 4 the width). However, whether that would have led to a construction of a third axis was not further investigated.

Although Kaylee did not use the conventional coordinate triple, because of the way she was aware of the third measurement in relation to the other two measurements, i.e., she engaged in a logical multiplication of measurements, I believe that Kaylee has constructed a system of measurements for three-dimensional space. Because Kaylee predominantly took over the discussion, I do not have enough confirming evidence to claim that Morgan would have been able to develop a similar explanation independently. However, because Morgan was able to re-present Kaylee's fixation of one origin in her own picture as shown in Figure 5.39, I claim that Morgan reasoned compatibly with Kaylee's reasoning to the extent that she could coordinate the three measurements in activity.

Curious to know if their coordination of frames of reference was limited to the interior of the tank, I asked both students how they would locate a fish that was outside of the tank, as shown in Figure 5.40.

(a) (b)

Figure 5.40. The fish outside of the tank.

Both students agreed that all they needed to do was to extend their "graph" or add another "graph" and connect it. From such claims, I infer that both students were not restricted to the interior of the tank but were able to enact their FR coordinating scheme to any point in three-dimensional space either in activity (Morgan) or as given (Kaylee).

Summary of Cubic Fish Tank

Rectangular frame of reference scheme.

From her consistent activities across the North Pole Task and cubic Fish Tank Task, I conjectured that Kaylee has constructed a *rectangular frame of reference scheme* to which she assimilated the different situations across the tasks. That is, she has constructed a recognition template for a situation in which she could associate the use of a rectangular frame of reference, activated the activity of superimposing a rectangular frame of reference onto the spatial object, and resulted in a Cartesian-like coordination of measurements along two perpendicular axes through which she represented the location of objects in a two-dimensional perceptual space.

Morgan's illustrations, actions, and explanations gradually shifted from making visual estimations of the location of the fish to using spatial references to coordinate measurements. When representing the fish tank from visual estimations, Morgan added a grid onto her sketch after she located the fish to specify which section on the grid the fish were located in. After listening to Kaylee's way of locating the four fish in the tank and Kaylee's critique about her sketch, Morgan seemed to assimilate Kaylee's approach and make modifications in her spatial organizing activities. Morgan transitioned from using the grids to communicate the squares in which the fish were located to using the *grids as a frame of reference* to guide her measurement activities.

Frame of reference coordinating scheme.

From Kaylee’s consistent manner in which she coordinated two sets of rectangular frames of reference in the three-dimensional cubic fish tank case, I conjectured that Kaylee had constructed a *frames of reference coordinating scheme* (*FR coordinating scheme*). That is, she recognized of a situation in which she could posit a frame of reference as a unit and insert it into another frame of reference resulting in combined frames of reference. In the cubic fish tank, the insertion of a two-dimensional frame of reference into another and multiplicatively coordinating them allowed her to locate the fish’ location along all three spatial dimensions that spanned the three-dimensional space of the tank. Because the length, width, and height measurements for each fish were multiplicatively coordinated, when demonstrating the movement from one fish to another, she considered the change in each measurement between the two fish.

To construct and enact a FR coordinating scheme, I conjectured that Kaylee enacted operations of decentering, rotating, and bringing forth images of one perspective alongside another. Then, Kaylee disembedded the frame of reference constructed for one side of the tank, taken from one perspective and inserted it into another frame of reference constructed for another side of the tank, taken from a different perspective. Uniting and multiplicatively coordinating the two sets of frames of reference resulted in a representation of the four fish along all three spatial dimensions. By multiplicative coordination I mean that the locations of a fish were gauged with the simultaneous realization that the fish had a specific location along all three dimensions. Therefore, I hypothesized that Kaylee’s three levels of units coordination supported such mental actions and that the FR coordinating scheme required mental operations that are essential for coordinating three levels of units.

Although she started off with accounting for only two spatial dimensions in her fish tank representation, Morgan showed progress in accounting for all three dimensions in the fish's locations. There were two elements in the teaching episode that I attributed to such progress—the task requirement to account for the motion of points and her assimilation of Kaylee's measuring activities. I claimed that through attending to the movement of one object to another within the three-dimensional space and through assimilating Kaylee's measuring activities, Morgan constructed a rectangular frame of reference at least in activity and coordinated measurements that allowed her to explain the movement of one fish to another or to re-organize her locating of the four fish.

By the way Morgan randomly added the four fish to her sketch without looking at the fish tank, I inferred that Morgan no longer depended on the perceptual imagery of the cubic fish tank; now the coordination became operative and she seemed to be aware that her coordination would work for any location of the fish in the tank. Finally, by the way that Morgan was able to point to one fish on the top of the tank and relate the location of that fish in the side face of the tank, and discuss what each sketch of the tank allowed her to find in terms of the location of the fish, I inferred that she has coordinated the two perspectives multiplicatively (one taken from the side view and one taken from the top view) to locate the fish.

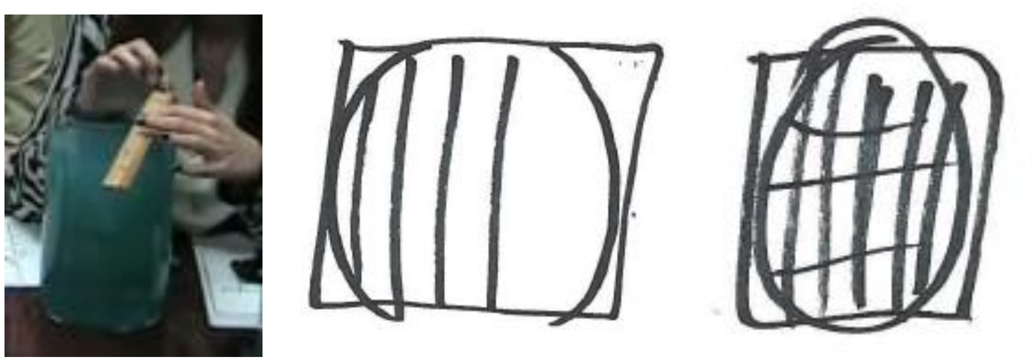
Cylindrical Tank: Locating the Four Fish

Exploratory phase.

Towards the end of our teaching episode on November 12, I asked the students to explore the cylindrical tank and think about how they might locate the fish in the tank. The round shape of the tank provided a new challenge to the students. Their initial approach to the task is illustrated in the following excerpt.

Excerpt 5.14: Kaylee and Morgan share initial thoughts about the cylindrical tank.

M: I don’t know how you would find them.
T: So, it seems like you’re thinking differently from what you just did a while ago [*pointing to Morgan’s gridded square sketch; Figure 5.34*].
M: Yeah, because it’s not a …
K: You can’t do a…
M: You can’t do a grid because it’s not…
T: Ah…
K: Unless you make a square kind of edges in there…
T: What do you mean by that?
K: Like…
M: You can draw a circle out of this [*drawing a circle in her gridded square sketch with her finger*]. Like you could draw…
K: [*Lays ruler on top of the cylindrical tank (Figure 5.41 (a) and looks at the tank. Then she looks at Morgan’s sketch in Figure 5.41 (b)*].
M: Here’s your little grid thingy and then you can draw the circle out of it [*Starts sketching the square grid and circle inside of it as shown in Figure 5.41 (b)*] and then you can still have your grid in it. So, it’s like you pretty much draw your thing, grid, and then you draw your circle [*remakes a sketch as shown in Figure 5.41 (c)*]. And then you have your grid.



(a) (b) (c)

Figure 5.41. Kaylee and Morgan share initial thoughts about the cylindrical tank.

[*Continued*.]

T: Okay, that's cool. What were you [*referring to Kaylee*] talking about?

K: Kind of like the same thing but just opposite. Put the square inside the circle but this would be kind of like outside of it [*points to the side of the cylindrical tank*].

At first, both students seemed troubled by the new shape of the tank case; different from the cubic tank, the cylindrical tank was curved and did not have any faces. Although I pointed to the grid to refer to their activity in the cubic fish tank case, which may have induced some association of the new situation with the grid they used earlier, both students decided how to use the grid independently. Morgan and Kaylee continued working on the cylindrical tank task on November 14 and November 19.

On November 14, I asked the students to work on locating the four fish in the cylindrical tank. Earlier in the teaching episode, Morgan had made a square-shaped grid. Kaylee suggested that Morgan trace the top of the tank onto the grid (see Figure 5.42).

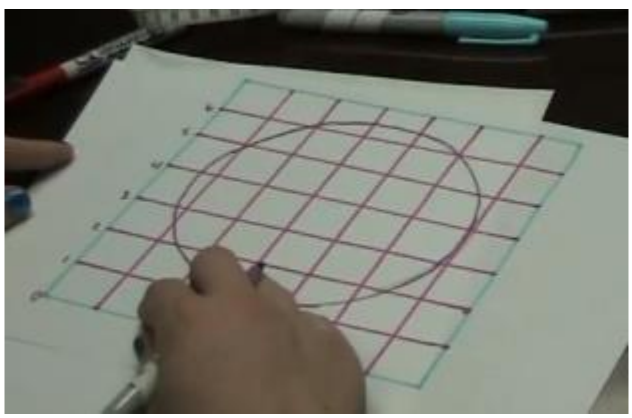


Figure 5.42. Morgan's trace of the top of the cylindrical tank.

After Morgan traced the circled onto the grid, Morgan said she was going to cut it out. However, Kaylee suggested that "you need this point to know exactly where it would be" as she was pointing to the bottom left corner of the square, perhaps referring to the origin they discussed earlier in the cubic fish tank case. Morgan agreed with Kaylee's observation and then wrote 0, 1, 2, 3, 4, 5, 6 next to each one-inch mark along the vertical axis as shown in Figure 5.42. Now her labels were next to the lines and not next to the segments between them; Morgan no longer looked for the sections after plotting the fish but used the grid lines as a guide for locating the fish.

Both students sat in thought until Morgan changed the course of the discussion by introducing another idea for locating the fish in the cylindrical tank. Morgan asked Kaylee if she measured the height of the tank. Using a ruler, Kaylee determined the height to be 10 inches. Excerpt 5.15 starts with Morgan sharing her new idea using the height of the tank.

Excerpt 5.15: Morgan partitions the height of the cylindrical tank.

M: So, here's our little cylinder [*Draws the outline of a cylinder on her paper*]. We can, um, so you said it was ten, right? [*Places ruler vertically against cylindrical tank.*]
K: Mm-hmm.
M: What we can do is, um, we can plot out like ten points [*moves ruler to her piece of paper and starts marking 1 inch intervals along the ruler up to 10 inches (Figure 5.43 (a))*].

(a) (b) (c)

Figure 5.43. Morgan and Kaylee constructing circles 1 inch apart on the side of the tank.

[*Continued*.]
T: Mm-hmm.
M: And from that point, pretty much just round it around the thing [*draws an ellipse starting from the 1-inch mark on the top (Figure 5.43 (b))*]. So you can draw cir…Things around it [*waves hand in circular motion around the cylindrical tank*]. You know what I'm saying?

K: Oh, yeah, yeah, yeah.
M: And then, you can go from there.
K: Let me do that [*picks up ruler and places it against the cylindrical tank and makes 1-inch marks along the side of the cylindrical tank using a dry-erase marker (Figure 5.43 (c))*].

As shown in Excerpt 5.15, Morgan first initiated the idea of marking 1 inch intervals horizontally along the side of the tank. Morgan partitioned the ten inches of length consisting the height of the tank into ten one inch units, similar to her earlier activity in partitioning the edges of the cubic tank. From Morgan's sketches shown in Figure 5.43 (a) and (b) and her circular hand motion, I inferred that she was thinking of circular cross sections horizontal to the top and bottom of the tank, 1 inch apart. Combining this idea with her partitioning of the height of the tank, I inferred that Morgan partitioned the three-dimensional tank into circular disk layers that were each 1 inch high. Superimposing these layers into the cylindrical tank *in order to* locate the fish was different from her earlier activities in the cubic tank where she superimposed the grids *after* she located the fish.

Figure 5.44. Kaylee and Morgan wrapped wax string around the tank 1 inch apart.

Because the dry-erase marker was not visible enough and since Morgan made a circular motion around the tank as if she was wrapping a string around the tank, I asked them if they could use the wax string on the table. Immediately, the two girls starting wrapping the wax string around the tank using the 1 inch marks Kaylee made earlier, as shown in Figure 5.44.

Because we were running out of time, I suggested that I would prepare the tank with 1 inch marks for the next teaching episode. From the students' actions I observed so far, I inferred that they imagined the grid on top of the tank. So, putting the gridded square on top of the tank, I asked them to imagine they had all the wax strings along the 1-inch marks wrapped around the tank and talk about how they would locate the fish. Although I initiated the activity of laying the grid on top of the tank, my action did not seem out of reach of the students. In the following excerpt, Morgan continues sharing her idea.

Excerpt 5.16: Morgan talks about how she would use the grid and the circles.

M: Okay, so, say there is one right there [*makes a random point in the circle on the grid*].

T: Uh-huh.

M: You would say, oh, that's in the middle…That's at two point five… [*moving her pen vertically along the left side edge of the square (where she inscribed the 0, 1, 2, 3, 4, 5, 6).*] One, two, three, two…Three point five [*moving her pen horizontally along the bottom edge of the square*]. And then, it's like one, two, three, four [*moves her pen along the side of the tank vertically, counting for each wax string*] units down. It's four units deep. So, you go here [*pointing to the point in the circle on the paper laid on the top of the tank*] and then you go four units deep [*pointing her pen downwards*].

K: Yeah. Because this one [*pointing to the side of the tank*] won't help you with how far in or… This [*sweeps her finger along the wax string that is tied onto the side of the tank*] just shows you how deep it is because you can't tell how far in it is because it's not straight.

M: Yeah.

Consistent with her organization of the cubic tank, Morgan first located the fish on the top of the tank and then coordinated the location of the fish in her drawing of the top of the tank along with the depth of the fish. Morgan did not refer to any specific fish in the cylindrical tank but used an imaginary fish to explain how she would use the gridded square and the one-inch marks on the side of the tank. In inferred this to indicate that her coordinated system of measurements has become operational and no longer relied on perceptual imagery of the fish within the tank. Thus, her system of measurements obtained through a coordination of horizontal and vertical distances in the top view of the tank along with the depth of each fish from the side view of the tank now also worked for another case—the cylindrical tank.

In the next teaching episode on November 19, five days after the previous teaching episode, I asked Kaylee and Morgan to work separately on locating the four fish in the cylindrical tank and then to share their ideas. One of my goals was to explore if Morgan would initiate the activity of laying the grid on the top of the cylindrical tank and coordinating it with her organization of the side of the tank to locate the fish in a consistent manner. Second, by asking the two students to work separately, my goal was to explore Kaylee's way of thinking, because Morgan led the majority of the discussions in the exploratory phase of the cylindrical tank. Additionally, I prepared the 1 inch marks as shown in Figure 5.45, because I promised to do so in the previous episode. I also brought the circular map they used in the North Pole Task and the gridded square Morgan constructed in the previous episode (Figure 5.42) and told the students that they could use any of the artifacts that were on the table. In the following sections I elaborate on how Kaylee and Morgan each located the fish in the cylindrical tank.

Figure 5.45. Cylindrical tank with strips of tape attached one inch apart.

Morgan locates the four fish in the cylindrical tank.

At the beginning, Morgan recalled that she had found a strategy in the last teaching episode. After a few seconds trying to remember what the strategy was, Morgan started with a sketch of the frame of the cylinder, which was not to scale with the actual length of the side of the cylindrical tank. Next, she counted the number of tape strips along the side of the tank and drew the tape strips into her cylindrical frame sketch (see pink curved lines in Figure 5.46). Next, Morgan looked at the side of the tank and counted the number of tape strips starting from the top rim of the tank down to where each fish was, resulting in a list of each fish and the layer number they were contained in (see list of layers written on top of the cylindrical figure in Figure 5.46). Then, without looking at the tank, she added “x” marks to the corresponding layers she identified in her list above the tank, as shown in Figure 5.46.

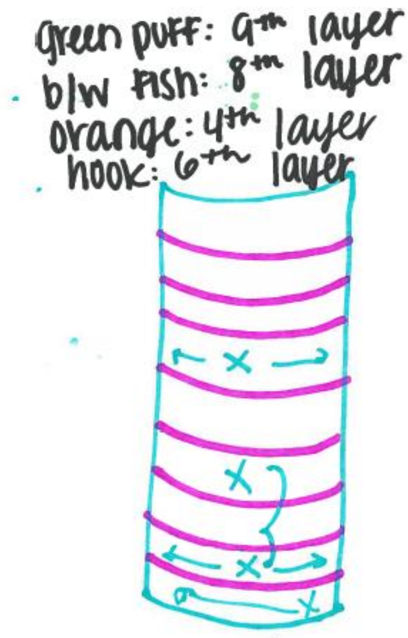


Figure 5.46. Morgan's representation of the cylindrical tank.

After she completed her drawing of the tank from the side view, Morgan stood up and looked down onto the tank. After thinking for a few seconds, she laid her ruler on the top of the tank while looking down onto her sketch of the side of the tank (Figure 5.47).

Figure 5.47. Morgan thinking about the two views (top and side) of the cylindrical tank.

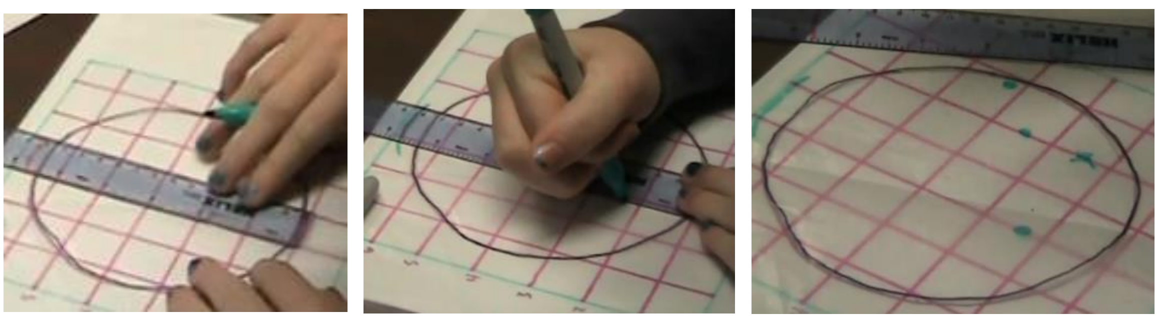
The way she intently looked at her sketch of the side of the tank as she laid the ruler on the top of the tank suggested that Morgan was coordinating the two perspectives (side view and top view of the tank) in activity. Switching back and forth from both perspectives, Morgan seemed to think about what she would need to find from the top view. After exclaiming how difficult it was to see through the tank from the top, Morgan looked at her sketch and then at Kaylee's, then she whispered that she had an idea. She rotated the tank so that the sticker label was in front of her, and laid her ruler on top of the tank so that the ruler was in line with the sticker label, as shown in Figure 5.48. At this point, Kaylee had sketched the sticker label onto her drawing of the cylindrical tank. It seemed like seeing Kaylee's idea of using the sticker label to fix the orientation of the tank triggered in her a new idea.

Figure 5.48. Morgan laying the ruler on the top of the cylindrical tank.

After laying the ruler on the top of the cylindrical tank as if measuring the diameter of the circle as shown in Figure 5.48, Morgan pointed out how her circle on the grid was not the same size as the top of the tank. Morgan looked intently at the ruler for a longer time than one would expect someone to simply measure the diameter of a circle. After looking at the ruler, Morgan moved the ruler to her gridded square as shown in Figure 5.49 (a) and marked a point on her sketch, presumably where she measured off one of the fish visually. Therefore, I inferred that Morgan also measured the distance along the ruler from one end of the rim of the tank to where she saw one of the fish from the top view. The way she placed her ruler in the middle of the sticker, I hypothesized that Morgan intended to measure the distance in line with her vertical lines of her grid.

Therefore, I conjectured that Morgan used her grid structure to guide her measuring activities.

After Morgan marked her first fish onto her sketch of the top view (Figure 5.49 (a) and (b)), Kaylee rotated the tank and wanted to work on it from that specific orientation, which required not moving the tank around; therefore, Morgan improvised. Looking from the side of the tank, Morgan added three more points to her sketch (Figure 5.49 (c)). I interpreted this action to indicate that Morgan made estimations of where the remaining fish might be, not wanting to wait until Kaylee was finished.



(a) (b) (c)

Figure 5.49. Morgan locates the four fish on her representation of the top view of the cylindrical tank.

In the following excerpt, Morgan explains her way of locating the four fish in the cylindrical tank.

Excerpt 5.17: Morgan talks about how she located the fish in the cylindrical tank.

M: Okay, so, mine was, I was looking from, I guess the side point of view [*lowers her eye level as if she's leveling her line of sight with the side of the tank*]. It doesn't matter which side, way, you're looking at. Because you can see what layer, I guess, [*pointing to the tape strips*], they're in.

K: Oh…

M: And so, I went from layers [*pointing to the layers in her sketch (Figure 5.46)*]. And then, so, you know what layer they're in [*Moves her sketch (Figure 5.46) to the side and pulls her other sketch (Figure 5.49 (c)) in front of her*]. And then from, like, one point of view [*pointing to the top of the tank*], I said, well, if you have that fish closest to you [*rotates the tank so that the orange fish closest to the top of the tank is in front of her*

(Figure 5.50)] and you look over [*again, points to the top of the tank*], and you can see where they all are. And then you look at them [*pointing to a point she sketched in the circle on the grid (Figure 5.49 (c))*] and like, oh, it's down in the sixth layer you go down six layers.

K: [*Nods vigorously, as if she understands and agrees with Morgan's explanation*].

T: So, you're using both of these [*pointing to her two sketches (Figures 5.46 & 5.49 (c)) at the same time*]…

M: Yes.

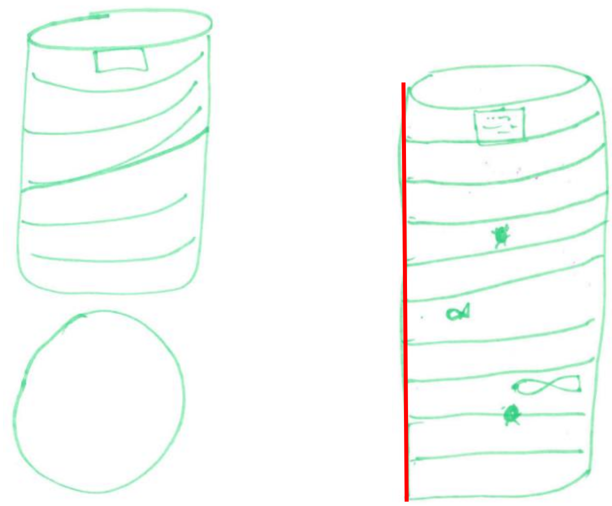
Figure 5.50. Morgan explaining her way of locating the fish in the cylindrical tank.

In Excerpt 5.17, Morgan's explanation was more focused on how she coordinated the two views of the fish tank. She indeed coordinated the top view induced sketch with the side view induced layers of the fish tank. However, from the way she intently looked at her sketch of the side of the tank as she laid the ruler on the top of the tank I inferred that Morgan was coordinating the two perspectives (side view and top view of the tank) in activity. Switching back and forth from both perspectives, I conjectured that Morgan's coordination of the two perspectives (top view and side view) was sequential. That is, first taking the side view of the tank, Morgan identified the layer in which the fish was located in. Then, putting that aside, Morgan identified where on the top view of the tank the fish was in. Finally, she put together the two locations she found from the two views. Whether her coordination of the two perspectives were brought forth in co-presence with each other or sequentially was to be further tested.

Although she previously coordinated measurements along her grid to represent the top view (Figures 5.48 and 5.49), Morgan did not mention the grid, the sticker, nor how she anchored the grid onto the top view of the tank. Instead, she relied on perceptual imagery commenting "if you have that fish closest to you" and "you look over and you can see where they all are." Therefore, related to my conjecture that her coordination of the two perspectives was sequential, I conjectured the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension. Therefore, different from Kaylee, I hypothesized that Morgan was yet to construct a FR coordinating scheme but could enact the action of coordinating frames of reference sequentially *in activity*. This hypothesis was consistent with my previous conjecture that the FR coordinating scheme required mental operations essential for coordinating three levels of units and Morgan's initial interview (reasoning as if she could operate with three levels of units *in activity*).

Kaylee locates the four fish in the cylindrical tank.

Different from the cubic tank case, the cylindrical tank required time to think and some trial and error for Kaylee. First, after looking at the tank for a few seconds, Kaylee produced a drawing as shown in Figure 5.51 (a). More specifically, Kaylee sketched the frame of the tank, added curved line segments as to depict the strips of tape marking the one inch intervals along the side of the tank. After sitting in thought again, looking at her sketch and then the tank several times for approximately 10 seconds, Kaylee added a circle below her sketch of the cylindrical tank. Then, she rotated the tank so that the sticker label on the tank was in front of her and depicted the sticker label in her drawing of the cylindrical tank (Figure 5.51 (a)).



(a) (b)

Figure 5.51. Kaylee's first and second drawings of the cylindrical tank.

However, Kaylee did not seem to be satisfied with this first approach. After looking at the tank for approximately five seconds, she turned to a new piece of paper. She started sketching a new cylindrical frame; this time she used her ruler to measure the height of the tank, which was 10 inches and made the height of her second drawing of the cylinder 10 inches long. Then, she drew in the sticker label on her cylinder sketch, marked off 1-inch marks along the left edge[3] I labeled OA in Figure 5.51 (b) and drew in the strips of tape onto her cylinder representation as shown in Figure 5.51 (b).

Finally, Kaylee started to copy the fish onto the frame of the cylinder. Moving her body away from the tank and lowering her torso, she looked straight as if she was trying

[3] Cylinders do not have edges. By *left edge* I refer to the left edge of the rectangular cross section passing the center of the cylinder, perpendicular to its base; it corresponds to OA in Figure 5.55 (b) in Kaylee's sketch of the cylindrical tank.

to align her line of sight with the side of the tank. Then, she tapped on each strip of tape from the top of the tank, moving downwards to where her first fish was, copied that tapping action along the curved lines she drew in her sketch and fixed her finger on the place corresponding to the place on the tank, in reference to the number of strips she needed to count starting from the top of the tank. Staring back and forth at the tank and her sketch for approximately ten seconds, she marked her first fish (the one I labeled “1” in Figure 5.51 (b)). It seemed like once she identified the layer, Kaylee visually estimated the location of the fish within that layer.

Next, Kaylee laid a ruler horizontally along the tank with the starting point of the ruler at the left edge of the tank going pass the next fish. She used the ruler to measure how far to the right the next fish was from the left edge of the tank. Then she moved her ruler to her sketch of the cylinder, tapped on the curved lines and marked the horizontal distance she just measured onto the corresponding layer, with reference to the tape strip. Coordinating these two measurements, Kaylee marked the point and drew a fish shape onto her sketch (the one I labeled “2” in Figure 5.51 (b)). After staring at the fish tank for a few seconds, Kaylee added fish 3 in her sketch in Figure 5.51 (b). Finally, she repeated a similar activity as the second fish by lowering her torso to align her line of sight with the location of the fish, laid the ruler horizontally starting from the left edge of the tank (Figure 5.52(a)), measured the distance of how far to the right into the tank the fish was from the left edge of the tank, counted along the tape strips starting from the top, and coordinated those measurements in her sketch to complete locating her fish 4 shown in Figure 5.51 (b).

(a) (b)

Figure 5.52. Kaylee locating the second and fourth fish in the cylindrical tank.

I identified the way Kaylee located fish 1 and 3 to be similar to the locating activities that Morgan initially demonstrated in the cubic tank case. Kaylee stared at the tank and the fish for relatively long amounts of time and made visual copies of the location of the fish, especially along the horizontal axis. Kaylee did not carry out any explicit measuring activities other than making visual estimations using the tape strips on the side of the tank. Kaylee's way of locating fish 2 and 4 seemed very similar to the locating activities that she demonstrated in the two-dimensional faces of the cubic tank. From the way Kaylee laid the ruler on the tank and lowered her torso to align her line of sight with the side of the tank, I hypothesized that Kaylee had simplified the side view of the cylindrical tank as a rectangle and has superimposed a rectangular frame of reference. This hypothesis was corroborated by Kaylee's explanations of her locating activity. In the following excerpt, Kaylee explains how she located the fish in the cylindrical tank.

Excerpt 5.18: Kaylee explains how she located the fish in the cylindrical tank.

K: So, I kind of like, looked at it like, from this view, with the sticker right in the middle [*tapping on the sticker label on the tank that was facing towards her*]. I looked at kind of like a rectangle [*makes a rectangle along the edges of the tank with both index fingers*] instead of a… Like if this were to be a, I don't know. So, I just plot how far from where I can see to the thing is [*puts two index fingers together; the left index to show the left edge of the cylindrical tank as her reference and the right index moving* *horizontally, parallel to the tape, from the fish towards her left index (Figure 5.52 (b))*], and then how far down, so… Exactly like the cube, I guess. Way less exact [*laughing nervously*].

T: So, you were making kind of like the side view of the tank.

K: Yeah.

As demonstrated in her comment, "I looked at kind of like a rectangle," I inferred that Kaylee superimposed a rectangular frame of reference onto the vertical cross section of the cylindrical tank, as modeled in Figure 5.53.

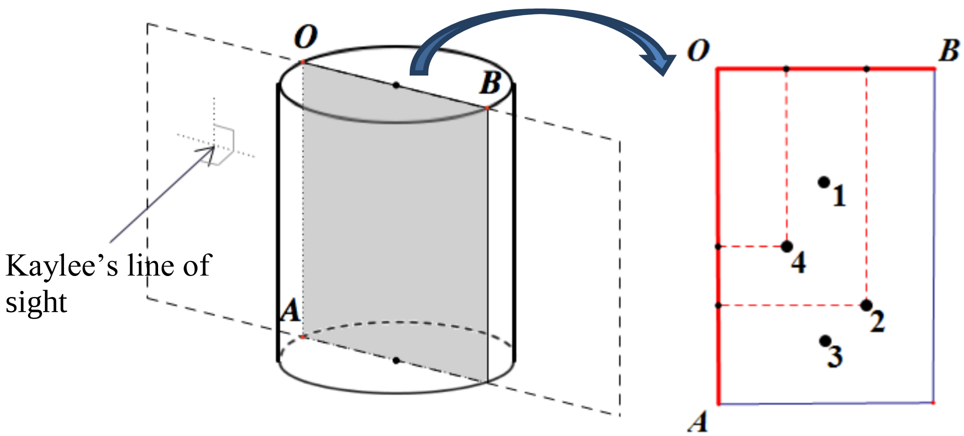


Figure 5.53. A model of Kaylee's rectangular frame of reference superimposed onto the side of the cylindrical tank.

Different from the cubic tank case, Kaylee did not explicitly address the third dimension in her explanation or in her sketch in Figure 5.51 (b). Her hesitance in carrying out her measuring activities, indicated by several pauses and frequent glances at the fish tank, along with her not considering the third dimension was different from her earlier activities in the cubic fish tank case. Although Kaylee's approach seemed similar to Morgan's first approach in the cubic fish tank case, in Kaylee's case, she appeared to be aware that her description was "[w]ay less exact" than the perceptual space she was representing.

I attributed this difference to the physical characteristic of the cylindrical tank. Earlier in the previous teaching episode on November 12, when I mentioned the origin they have identified in the cubic tank case, Kaylee was skeptical about identifying an origin in the cylindrical tank case. Kaylee pointed out how there was no "side" to the tank. Because there was no specific "side" she could anchor her system of measurements onto, this may have provided a perturbation for Kaylee. To resolve part of this perturbation, Kaylee seemed to latch onto the sticker on the fish tank to fix the orientation of the tank and instead produced a cross section of the tank and a representation of the tank using that cross section, as depicted in Figures 5.51 (b). After both Kaylee and Morgan discussed their approaches, Kaylee reflected on her way of thinking.

Kaylee's critique of her locating activity in the cylindrical tank.

Once both students had a chance to explain their re-presentations of the cylindrical tank and the four fish, I asked the two students to talk about each other's approach. The following excerpt starts with Kaylee's explanation of her way of locating the fish in the cylindrical tank, in comparison to Morgan's approach.

Excerpt 5.17: Kaylee's reflection on her locating the fish in the cylindrical tank.

K: Yours [*referring to Morgan*] is more like what we did earlier, like that, on the top and the side [*moves hand from top of the tank to side of the tank as she said top and side*].
M: Yeah.
K: Where mine doesn't show like, how far in [*moves her pen in front of her to demonstrate the direction*], it's just how, like, if it were to be flat, where they would be, I guess. Like, mine doesn't show the width, no, yeah, I guess width. Like, how far in [*again, moves her pen in front of her towards the tank*].
M: Yeah.
K: Like, I guess, mine just shows if this were to be flat and bring all this up [*moves hands along the sides of the tank as if she's pulling the volume towards the front of the tank*] where it would be like, length and height [*moves finger along the left side of the tank and then along the bottom of the tank*].

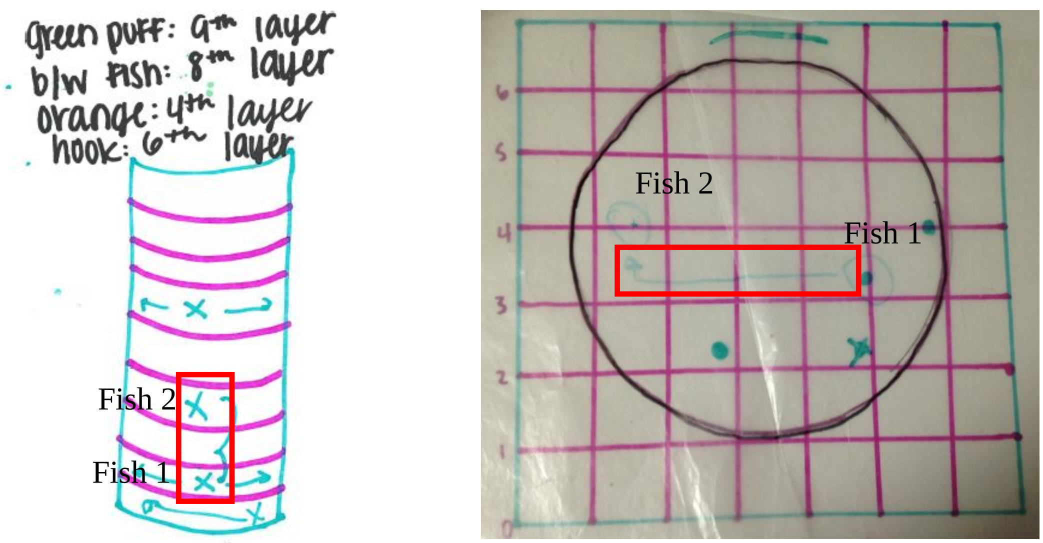
T: Mm-hmm. Mm-hmm.
K: Yeah, so yours just kind of with the layers and the, this thing [*pointing to Morgan's sketch (Figure 5.49 (c))*] shows like, everything.

Her earlier comment that her description was "way less exact," her reaction to Morgan's explanation in Excerpt 5.17 ("Oh…" while nodding vigorously), and her remarks in Excerpt 5.19 (admitting that she omitted it) suggested Kaylee was aware of the third dimension but it seemed like she did not know how to account for it in her representation. Once she listened to and saw Morgan's demonstration, she was quick to reflect on the limitations of her approach and articulated what Morgan had considered, mentioning length, width, and height.

After Kaylee expressed her critique, I wanted to test whether Kaylee would adjust her representation of the cylindrical tank to eliminate the perturbations she met in the new situation and assimilate the new situation to her existing conceptual structure of three-dimensional perceptual space, which she demonstrated in the cubic fish tank case. Therefore, I asked the students to give directions for Fish 1 to swim to Fish 2 in the cylindrical fish tank. Another reason I posed this question was to test my hypothesis that Morgan sequentially coordinated the two perspectives (top view and side view), and that her the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension.

Kaylee and Morgan Describe Fish 1 moving to Fish 2 in the Cylindrical tank

Using her two sketches, Morgan located Fish 1 in the eighth layer and Fish 2 in the sixth layer. Therefore, she concluded that "he [Fish 1] will go up two [*making inscriptions as outlined in red in Figure 5.54 (a)*] and then he'll have to go over however many this is [*drawing the green arrow outlined in a red rectangle in Figure 5.54 (b)*]."



(a) (b)

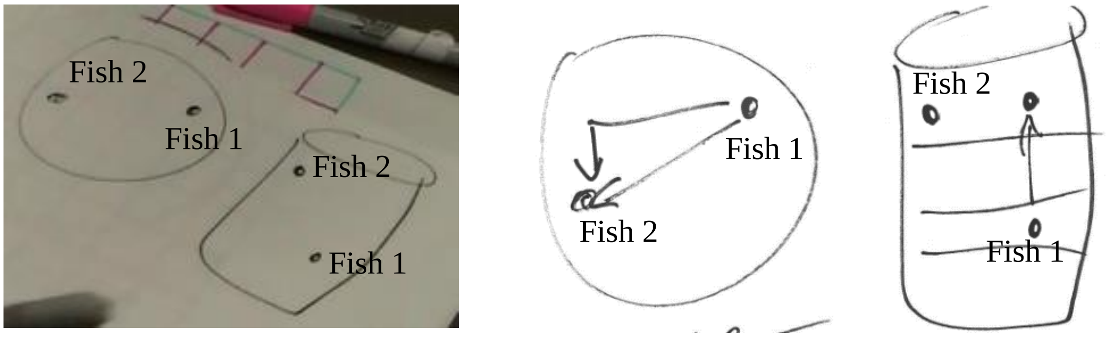
Figure 5.54. Morgan's diagram explaining the movement of Fish 1 to Fish 2 in the fish tank.

I interpreted Morgan's inscription along with her explanation of "and then he'll have to go over however many this is" to indicate that she viewed the motion on the top of the tank in a straight line connecting the two points representing Fish 1 and Fish 2. Kaylee also interpreted Morgan's explanation in a similar manner and disagreed with Morgan saying that "he [Fish 1] has to move three times." The following excerpt is the discussion that unfolded after they met that disagreement. The excerpt starts with Morgan explaining how Kaylee's approach is different from hers.

Excerpt 5.18: Morgan and Kaylee talk about Fish 1 swimming to Fish 2 in the cylindrical tank.

M: So, she's basing it off of length, width, and height. Mine is based off of well, units. If you go up two then you go over like four. So that's going to be two turns, or lines you're going to take, instead of like three.

K: Yeah, I get what you're saying. Okay, let's say here's one and here's two and then here's one and here's two [*draws diagrams shown in Figure 5.55 (a)*]. That's actually pretty good. So, you have to go up the layers. So he goes, let's say it's, that's two layers. So, he goes up two layers, or two inches, or whatever. So, now he's in line. But he's over here [*pointing to the point on the right in the circle in Figure 5.66 (a)*] but they're still in line. So, let's say they're on the top now. So, he has to go this way and then this way, that's three units [*draws the horizontal and vertical path with an arrow at the end shown in the diagram as shown Figure 5.55 (b)*], three different measurements you'd have to go.



(a) (b)

Figure 5.55. Kaylee's diagram explaining the movement of Fish 1 to Fish 2 in the cylindrical tank.

[*Continued.*]

T: Ah, so it's kind of like the…

M: But it's in the, why would you have to if it's a circle, though.

K: I mean, we're still using a square grid [*shows the grid (Figure 5.54 (b) to Morgan*].

M: You only need two straight lines though, you know what I'm trying to say? Like, they're already in line [*moves her pen along the straight arrow from Fish 1 to Fish 2 in the left of Figure 5.55 (b)*]. Then he could just go straight to him, instead of moving two times.

K: Yeah, but how would you know if you're plotting a point, it's easier to do [*points to the grid*], I guess.

M: Okay, so, that's how they are. So you want to go up the two, and they're going to be even. So why wouldn't he just go...[*draws a diagram as shown in Figure 5.56 demonstrating Fish 1 moving to Fish 2 in a somewhat circular motion*].

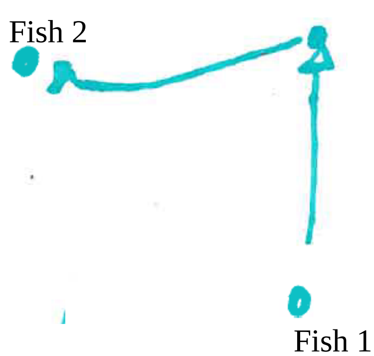


Figure 5.56. Morgan's diagram explaining the movement of Fish 1 to Fish 2 in the cylindrical tank.

[*Continued*.]

K: No, they're not even. They're on the same layer but they're in different spots [*placing two fingers one at each fish in the circle in Figure 5.55 (b)*.]

M: Yeah, so he can just swim to him [*uses her pen and makes a circular motion on the top of the tank as if the fish is swimming along the side of the tank*].

K: Well, how would you tell him to find this length [*adds the arrow connecting Fish 1 and Fish 2 in the circle in Figure 5.55 (b)*]? On the plot [*pointing to the grid*]?

M: You wouldn't be going straight, because this is, you'd be going around the circle to find him [*repeats the sweeping of her pen in a circular motion on the top of the tank*].

K: Well then how would you measure that?

Morgan's explanation at the beginning of Excerpt 5.18, "If you go up two then you go over like four. So that's going to be two turns, or lines you're going to take, instead of like three" corroborated my hypothesis that her green arrow inscription in Figure 5.54 (b) indicated a straight movement. Morgan was aware of Kaylee's length, width, height approach but disagreed with it. As such, when asked to describe the motion of Fish 1 to Fish 2, Morgan said that Fish 1 would need to take two motions; one going up and then another movement as she demonstrated with an arrow in Figure 5.54 (b). Morgan described these motions as two straight lines, arguing with Kaylee that it was not necessary to take a trip of three lines and only needed two movements.

I took Morgan's claim, "You only need two straight lines though … Like, they're already in line. Then he could *just go straight to him*, instead of moving two times" as a corroboration of my conjecture that the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension. Although she used the grid to locate the two fish on the top view representation of the tank by coordinating horizontal and vertical distances, when describing the movement of the fish once they were on the same layer, the movement no longer entailed a coordination of horizontal/vertical distances. Therefore, it is more likely that Morgan coordinated the top view and side view of the tank in succession without carrying over her measurements along the two dimensions in the top view representation to the side view representation.

Later in Excerpt 5.18, Morgan changed to using a circular motion of Fish 1 to Fish 2 once Fish 1 came to the same layer as Fish 2, as illustrated in her claim at the end of Excerpt 5.18, “You wouldn’t be going straight, because this is, you’d be going around the circle to find him.” Here, Morgan seemed to attend to the shape of the tank was circular. From her sketch in Figure 5.56 and her last comments in Excerpt 5.18, “You wouldn’t be going straight, because this is, you’d be going around the circle,” I infer that she was thinking of a circular motion of the fish. In either case, her description of “going straight” or “going around” in the tank of Fish 1 moving to Fish 2 would end in different results based on the perspective that Fish 1 is taking in the moment. Morgan would have had to make in-the-moment adjustments to her explanation based on her perceptual imagery of Fish 1 swimming to Fish 2.

On the other hand, although Kaylee did not initiate the coordination of the two drawings of the cylindrical tank from different perspectives (top and side view of the tank), she has assimilated this way of operating in describing the motion of Fish 1 to Fish 2. Similar to her actions in the North Pole Task and the cubic tank case, Kaylee engaged in decentering from her physical perspective and mentally positioned herself in two different positions—one looking down from the top of the tank and the other looking from the side of the tank. Then she brought forth images from those perspectives alongside each other, simultaneously coordinating the two representations from each perspective. This allowed Kaylee to re-enact her organizational scheme for the two-dimensional space in the top view perspective and coordinate that with the side view perspective. As a result, Kaylee decomposed the spatial movement of Fish 1 to Fish 2 into three spatial dimensional movements, which she referred to as length, width, and height.

Summary of Cylindrical Fish Tank

Summary of Morgan's and Kaylee's coordinate systems.

In locating the fish in the cylindrical tank, Morgan first initiated the idea of partitioning the cylindrical tank into circular cross sections to find the depth of each fish. Then, Morgan coordinated the location of the fish in her drawing of the top of the tank, using a gridded square, along with the depth of the fish. Based on her physical actions upon the tank (illustrated in Figures 5.47–50), sketches of the fish tank (in Figures 5.46, 5.49, 5.54, & 5.56), and explanations of her locating activities (in Excerpts 5.17 & 5.18), I conjectured that Morgan coordinated the two perspectives (side view and top view of the tank) in activity. Switching back and forth from both perspectives, I conjectured that Morgan's coordination of the two perspectives (top view and side view) was sequential.

Related to my conjecture that her coordination of the two perspectives was sequential, I conjectured the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension. Therefore, I hypothesized that Morgan was yet to construct a FR coordinating scheme but could enact the action of coordinating frames of reference sequentially *in activity*. This hypothesis was consistent with my previous conjecture that the FR coordinating scheme required mental operations essential for coordinating three levels of units and Morgan's initial interview (reasoning as if she could operate with three levels of units *in activity*).

On the other hand, Kaylee initially enacted her rectangular frame of reference scheme to locate each fish in a rectangular cross section of the tank. Kaylee explained that she simplified the tank into a flat rectangle. As a result, her initial approach lacked a coordination of measurements along the third dimension, which I attributed to the roundness of the tank, similar to a perturbation she met in the circular map in the North Pole Task. Although Kaylee did not initiate the coordination of the two drawings of the cylindrical tank from different perspectives (top and side view of the tank), she has assimilated this way of operating in describing the motion of Fish 1 to Fish 2.

Similar to her actions in the North Pole Task and the cubic tank case, Kaylee engaged in decentering from her physical perspective and mentally positioned herself in two different positions—one looking down from the top of the tank and the other looking from the side of the tank. Then she brought forth images from those perspectives alongside each other, simultaneously coordinating the two representations from each perspective. This allowed Kaylee to re-enact her organizational scheme for the two-dimensional space in the top view perspective and coordinate that with the side view perspective. Enacting her FR coordinating scheme, Kaylee constructed a three-dimensional Cartesian-like coordinate system and decomposed the spatial movement of Fish 1 to Fish 2 into three spatial dimensional movements, which she referred to as length, width, and height.

Extending the cylindrical fish tank.

Towards the end of teaching episode on November 19, I asked the two students to discuss the differences in the cylindrical tank and the cubic tank that impacted their way of thinking. Both students contrasted the physical characteristics of the two tanks. As Kaylee said, “there aren’t straight lines” in the cylinder. Morgan pointed out that the cube has multiple sides whereas the cylinder has only one side. And both students seemed to agree that the fact that that feature of the cylinder made it difficult to use elements of the tank as a spatial reference in locating the fish.

To investigate whether the students could generalize their coordinated systems of measurements to spatial situations which did not involve any “faces” or “sides,” I asked both students to imagine they were in the ocean and how they would locate somebody in the ocean. The following excerpts shows Morgan and Kaylee discussing how they would locate a person in the ocean including Morgan’s approach to the task.

Excerpt 5.19: Morgan locates a person in the ocean.

M: The features around it, I guess.
K: You can still do, I guess…
M: Because that’s kind of like here [*tapping on the cylindrical tank*]…
K: If I were like in the ocean…[*Starts to pick up paper and pen to write something*].
M: Alright, here’s the ocean or whatever [*starts sketch of Figure 5.57*] and like, oh, there’s an island thingy right there, there’s a shore right there. So, if we’re standing from the shore, you’ll go blah, blah, blah over there [*draws a line segment starting from the shore to the left towards the ocean*] and then you go blah, blah, blah down [*draws a line segment starting from the surface of the ocean water down into the ocean, ending with an arrow (Figure 5.57)*] and then you should find that thing.

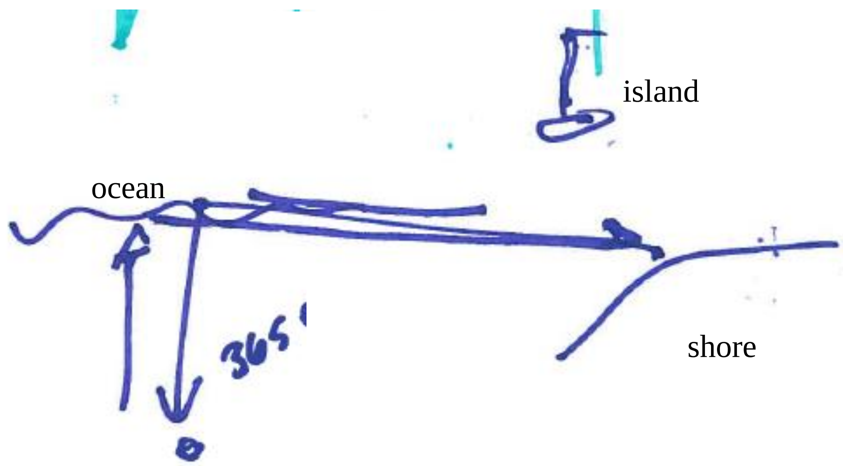


Figure 5.57. Morgan's diagram of locating a point in the ocean.

Consistent with her explanation of Fish 1 swimming to Fish 2 in the cylindrical tank, Morgan considered two movements, one going straight across from the shore to the surface of the ocean and then going down into the water. I find these two movements to be analogous to Fish 1 moving up two layers and then swimming in a straight line (or in a circular motion) to Fish 2, once Fish 1 is on the same layer as Fish 2. Morgan pointed to the cylindrical tank and said "that's kind of like here," which is an indication that she recognized the ocean situation to one like the cylindrical tank.

Morgan constructed her own "side" of the ocean, which she called the shore and used that as a spatial reference to superimpose a rectangular frame of reference. Based on her description, I inferred that Morgan has embedded her perspective within the implicit plane through the person at the shore and the person in the water when superimposing her rectangular frame of reference. Because her perspective was embedded within the plane, she did not consider a movement along the third spatial dimension.

In the following excerpt starts with Kaylee discussing how she would locate a person in the ocean and Kaylee and Morgan's conversation thereafter.

Excerpt 5.20: Kaylee locates a person in the ocean

K: It's like, I would do it like I'm in the ocean, this is me right now and this is the ocean, I'm under water [*starts diagram shown in Figure 5.58 with the surface of the ocean and the back of her head*]. And so, I would still do it like I would have to go this far [*draws a line segment starting from the diagram of her head vertically, upwards, ending with an arrow (Figure 5.58)*] and then like this far [*draws a second line segment starting from where the previous arrow ended, horizontally, to the right, ending with another arrow (Figure 5.58)*] and then like this far down [*draws a third line segment starting from where the previous arrow ended, vertically, downwards, ending with an arrow(Figure 5.58)*]. Like deep, I don't know.

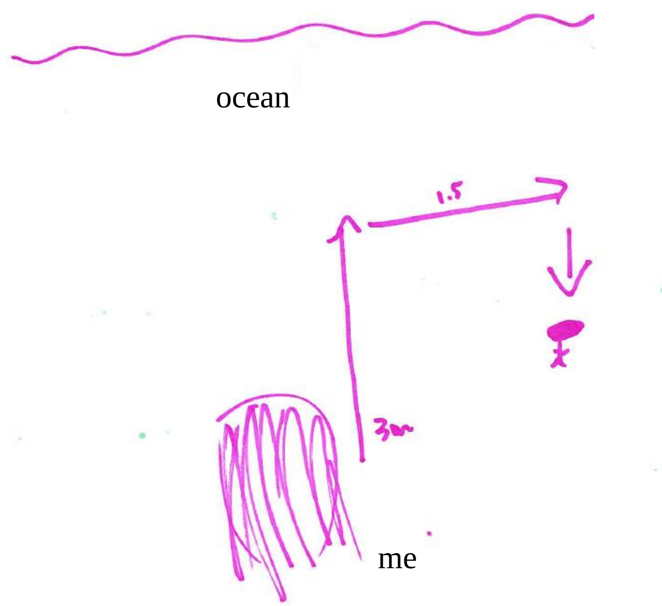


Figure 5.58. Kaylee's diagram of locating a point in the ocean.

[*Continued*]

M: Why would you go back down? [*Points to the third line segment and arrow Kaylee drew in her diagram (Figure 5.58)*.]

K: Well I mean like, okay, let's say that this [*pointing to her first line segment and arrow*,] is just straight ahead. Not going up.

T&M: Oh…

K: Because I can't really draw that. And this [*pointing to her third line segment and arrow (Figure 5.58)*] is deep.

M: Okay, that makes sense. Now I understand why you're moving by those [*pointing to an earlier diagram (Figure 5.55) made to demonstrate the two movements (horizontal and vertical) of Fish 1 to Fish 2 in the same layer*].

K: Yeah.

M: Because that [*pointing to Kaylee's three line segments with arrows in Figure 5.58*] was confusing me.

K: Yeah. It's hard to do on flat paper… So, I'd be like, oh, go like three miles straight [*writes 3m next to her first line segment with arrow (Figure 5.70)*] and then like one and a half miles on to the left [*writes 1.5 above her second line segment with arrow (Figure 5.58)*] and then go deep, like…

M: Point five…

K: Point five miles and you'll get him.

At the beginning of Excerpt 5.20, Kaylee also seemed to have assimilated this situation to the cylindrical tank case as she said "you can still do, I guess." I infer that in assimilation, Kaylee enacted her FR coordinating scheme, which resulted in a coordination of the top view and side view of the cylindrical tank (and perhaps a similar coordination of the two side views of the cubic tank), which in turn enacted her decomposition of the person's movement as length, width, and height. Therefore, consistent with her explanation of Fish 1 swimming to Fish 2 in the cubic tank and cylindrical tank, Kaylee considered three movements, a movement along each spatial dimension.

Based on Kaylee's description, I hypothesized that she embedded her perspective within the ocean and related the three movements in respect to herself in the ocean, resulting in three movements of front/back, right/left, up/down along three spatial dimensions. Inferring from the order she gave instructions (go straight, move to the side, and then go down) in Excerpt 5.20, Kaylee first coordinated the horizontal right/left and front/back movements within a plane, which was the plane in line with her line of sight. Then, disembedding this plane as a unit structure, Kaylee inserted it along the third dimension which entailed the vertical up/down movement.

The discussion in Excerpts 5.19 and 5.20 and Figures 5.57 and 5.58 highlighted the different ways of operating Morgan and Kaylee demonstrated in the North Pole Task and Fish Tank Task, which leads to a summary of Chapter Five.

Summary of Chapter Five

In this chapter, I presented my analysis of Kaylee's and Morgan's constructive activities in the Locating Tasks (North Pole Task and Fish Tank Task) in which I asked both students to locate a point or describe the motion of one point in two- or three-dimensional perceptual space. Through my observations of the students' locating activities, I analyzed how Kaylee and Morgan constructed frames of reference and coordinated measurements using those frames of reference to represent perceptual space.

Kaylee's Coordinate Systems

North Pole Task.

In the North Pole Task, Kaylee constructed two types of coordinate systems which she used to locate point A in relation to point P on the irregular shaped map. First, Kaylee constructed an angular frame of reference consisting of an initial ray anchored onto the rescuer's line of sight, a vertex at point P, and a terminal ray through point A (Figure 5.3). This frame of reference allowed Kaylee to gauge the amount of rotation the rescuer would need to turn to find the missing person. Second, Kaylee constructed a rectangular frame of reference consisting of horizontal and vertical lines with the intersection of the lines anchored at point P of the map. This frame of reference allowed Kaylee to break down the movement along two spatial dimensions finding horizontal and vertical distances. Using these frames of reference, Kaylee coordinated angle measure and distance or horizontal/vertical distances, respectively, to locate point A in reference to point A on the maps. As a result, I claimed that Kaylee had constructed what I would consider a polar-like coordinate system and a Cartesian-like coordinate system.

Fish Tank Task.

Kaylee's use of a rectangular frame of reference and coordination of horizontal/vertical distances continued in the Fish Tank Task. From her consistent activities across the North Pole Task and Fish Tank Task, I conjectured that Kaylee has constructed a *rectangular frame of reference scheme* to which she assimilated the different situations across the tasks. That is, she has constructed a recognition template for a situation in which she could associate the use of a rectangular frame of reference, activated the activity of superimposing a rectangular frame of reference onto the spatial object, and resulted in a Cartesian-like coordination of measurements along two perpendicular axes through which she represented the location of objects in a two-dimensional perceptual space.

From Kaylee's consistent manner in which she coordinated two sets of rectangular frames of reference in the three-dimensional cubic fish tank case, I conjectured that Kaylee had constructed a FR coordinating scheme. That is, she recognized of a situation in which she could posit a frame of reference as a unit and insert it into another frame of reference resulting in combined frames of reference. In the cubic fish tank, she inserted the rectangular frame of reference of the first face along the third dimension across the second face she coordinated. In the cylindrical fish tank, she inserted the rectangular frame of reference superimposed onto the top view of the tank along the third dimension across the side of the tank.

The insertion of a two-dimensional frame of reference into another and multiplicatively coordinating them allowed her to locate the fish' location along all three spatial dimensions that spanned the three-dimensional space of the tank. Because the length, width, and height measurements for each fish were multiplicatively coordinated, when demonstrating the movement from one fish to another, she considered the change in each measurement between the two fish. As a result, Kaylee coordinated all three spatial dimensions that build up the three-dimensional space and the relevant measurements of horizontal and vertical distances, producing a three-dimensional Cartesian-like coordinate system. Kaylee identified her method as the same across both tanks and summarized her approach as finding the length, width, and height of each fish. From this, I conjectured that Kaylee has constructed a coordinated a single system of measurements in locating the four fish in the fish tanks.

To construct and enact a FR coordinating scheme, I conjectured that Kaylee enacted operations of decentering, rotating, and bringing forth images of one perspective alongside another. Then, Kaylee disembedded the frame of reference constructed for one side of the tank, taken from one perspective and inserted it into another frame of reference constructed for another side of the tank, taken from a different perspective. Uniting and multiplicatively coordinating the two sets of frames of reference resulted in a representation of the four fish along all three spatial dimensions. By multiplicative coordination I mean that the locations of a fish were gauged with the simultaneous realization that the fish had a specific location along all three dimensions. Therefore, I hypothesized that Kaylee's three levels of units coordination supported such mental actions and that the FR coordinating scheme required mental operations that are essential for coordinating three levels of units.

Perspective-taking.

When developing instructions for the rescuer (North Pole Task) or for Fish 1 to swim to Fish 2 (Fish Tank Task), Kaylee consistently coordinated two perspectives. In the North Pole Task, she coordinated her above-the-ground perspective and an imaginary on-the-ground rescuer's perspective. Coordinating these two perspectives obtained through decentration, Kaylee developed instructions using route descriptions (Taylor & Tversky, 1996) in which the descriptions are oriented from the rescuer's perspective. Similarly, in the Fish Tank Task, Kaylee coordinated her outside-of-the fish tank ego-oriented perspective and an imaginary in-the-water fish's perspective. Coordinating these two perspectives, Kaylee developed instructions for the fish, breaking down its movement along all three spatial dimensions. In these instances, I hypothesize that Kaylee transferred her ego-oriented perspective to the imaginary rescuer's or fish's perspective embedded within the perceptual space through decentering. In addition to decentering, I conjecture that Kaylee unitized each perspective as two independent perspectives but also coordinated the two perspectives simultaneously. This allowed her to coordinate multiple frames of reference superimposed onto the perceptual space from different perspectives.

Morgan's Coordinate Systems

North Pole Task.

In the North Pole Task, Morgan initially took a more temporal approach in that she wanted to give in-the-moment instructions to the rescuer or the missing person from the helicopter. Later in the teaching episode, Morgan considered connecting the two points P and A and measuring the distance between the two points in the irregular shaped map. In the circular map case, Morgan used two perpendicular paper folds Kaylee previously made through the center of the circle and their intersection (circle center P') as a references. Using the paper folds as a rectangular frame of reference, Morgan coordinated vertical and horizontal distances to locate point A' in reference to point P'. I conjectured that the paper folds of the circular map may have led to Morgan enacting graphing activities she learned in school and that Morgan has assimilated Kaylee's coordination of horizontal and vertical distances in the irregular shaped map case.

In any event, I considered Morgan's construction of a rectangular frame of reference to be made in activity in that Morgan carried out the folding activities Kaylee carried out and paused to think as she engaged in the task. Although in activity, I conjectured that Morgan demonstrated their coordination of vertical and horizontal distances entailed a logical multiplication of measurements (Piaget et al., 1960) in which the measurements were oriented by the rectangular frame of reference. In other words, Morgan eventually located point A multiplicatively as a product of a coordination of its location along one spatial dimension with the realization that the point had a specific location along the other spatial dimension.

Fish Tank Task.

Morgan's illustrations, actions, and explanations gradually shifted from making visual estimations of the location of the fish to using spatial references to coordinate measurements. When representing the fish tank from visual estimations, Morgan added a grid onto her sketch after she located the fish to specify which section on the grid the fish were located in. After listening to Kaylee's way of locating the four fish in the tank and Kaylee's critique about her sketch, Morgan seemed to assimilate Kaylee's approach and make modifications in her spatial organizing activities. Morgan transitioned from using the grids to communicate the squares in which the fish were located to using the *grids as a frame of reference* to guide her measurement activities.

Although she started off with accounting for only two spatial dimensions in her fish tank representation, Morgan showed progress in accounting for all three dimensions in the fish's locations. There were two elements in the teaching episode that I attributed to such progress—the task requirement to account for the motion of points and her assimilation of Kaylee's measuring activities. I claimed that through attending to the movement of one object to another within the three-dimensional space and through assimilating Kaylee's measuring activities, Morgan constructed a rectangular frame of reference in activity and coordinated measurements that allowed her to explain the movement of one fish to another or to re-organize her locating of the four fish.

By the way Morgan was able to point to one fish on the top of the tank and relate the location of that fish in the side face of the tank, and discuss what each sketch of the tank allowed her to find in terms of the location of the fish, I inferred that she has coordinated the two perspectives multiplicatively (one taken from the side view and one taken from the top view) to locate the fish. However, I conjectured that Morgan coordinated the two perspectives (side view and top view of the tank) in activity. Switching back and forth from both perspectives, I conjectured that Morgan's coordination of the two perspectives (top view and side view) was sequential.

Related to my conjecture that her coordination of the two perspectives was sequential, I conjectured the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension. Therefore, I hypothesized that Morgan was yet to construct a FR coordinating scheme but could enact the action of coordinating frames of reference sequentially *in activity*. This hypothesis was consistent with my previous conjecture that the FR coordinating scheme required mental operations essential for coordinating three levels of units and Morgan's initial interview (reasoning as if she could operate with three levels of units *in activity*).

Perspective-taking.

When developing instructions for the rescuer (North Pole Task) or for Fish 1 to swim to Fish 2 (Fish Tank Task), Morgan's descriptions were based on her perspective looking down onto the perceptual space. This perspective was not necessarily coordinated with the rescuer's or fish's line of sight. Hence, I hypothesized that Kaylee had a stronger ability to bring forth images of one perspective alongside another and coordinate them simultaneously. It is worth noting that different from the North Pole Task, in the Fish Tank Task and task of locating another person in the ocean, Morgan started to coordinate more than one perspective. In the fish tanks, she coordinated the top view and side views of the tanks. In the open ocean task, Morgan coordinated the view looking at the situation as shown in Figure 5.57 and the imaginary view of the person at the shore. However, I conjectured that Morgan's coordination of the two perspectives was sequential.

Based on the results discussed in this chapter, I propose that the FR coordinating scheme requires mental operations essential for coordinating three levels of units. Hence, there is a parallel between the students' levels of units coordination and coordination of measurements within frames of reference in three-dimensional space. To further explore the students' coordination of perspectives and construction and enactment of the FR coordinating scheme in relation to their levels of units coordination, I investigated how they coordinated units along three spatial dimensions. This question was explored through the Cubic Block Task, which I discuss in the following chapter.

CHAPTER 6

KAYLEE AND MORGAN COORDINATE UNITS WITHIN THREE SPATIAL DIMENSIONS

In the previous chapter, I presented my analysis of Kaylee's and Morgan's constructive activities in the Locating Tasks (North Pole Task and Fish Tank Task). Through these tasks, I explored how Kaylee and Morgan constructed frames of reference and coordinate systems to organize and represent two- or there-dimensional perceptual space.

The analysis in Chapter Five emphasized the difference in perspectives the two students coordinated when engaging in these tasks and the mental operations and schemes involved in their construction of coordinate systems. Although in the end, both students demonstrated a coordination of frames of reference induced by multiple perspectives, from the analysis, I claimed that Kaylee's and Morgan's coordinations were different. Kaylee was able to bring forth her previous frame of reference as she was operating with another, so the frames of reference co-occurred in her coordination of measurements. On the other hand, Morgan's frames of reference seemed to be activated sequentially, which led to different ways of locating objects and coordinating measurements in organizing space.

Based on the results in Chapter Five, I proposed that the FR coordinating scheme requires mental operations essential for coordinating three levels of units; hence, a parallel between the students' levels of units coordination and coordination of measurements within frames of reference in three-dimensional space. To further explore the students' coordination of multiple perspectives and construction and enactment of the FR coordinating scheme in relation to their levels of units coordination, I investigated how Kaylee and Morgan coordinated units along three spatial dimensions. This question was explored through the Cubic Block Task, which I discuss in this chapter.

In the Cubic Block Task, I asked Kaylee and Morgan to reason with three cubic blocks of various sizes (see Figure 6.1). I asked them questions such as finding the total number of unit-cubes contained in each cubic block or the number of unit-cubes that are painted. In the teaching episodes, we referred to each block by the number of unit-cubes constituting one edge of the cubic block. For example, the 2×2×2 cubic block was called "the block with two cubes on each edge." I refer to the blocks by the dimensions such as the 2×2×2 cubic block for the efficiency of writing.

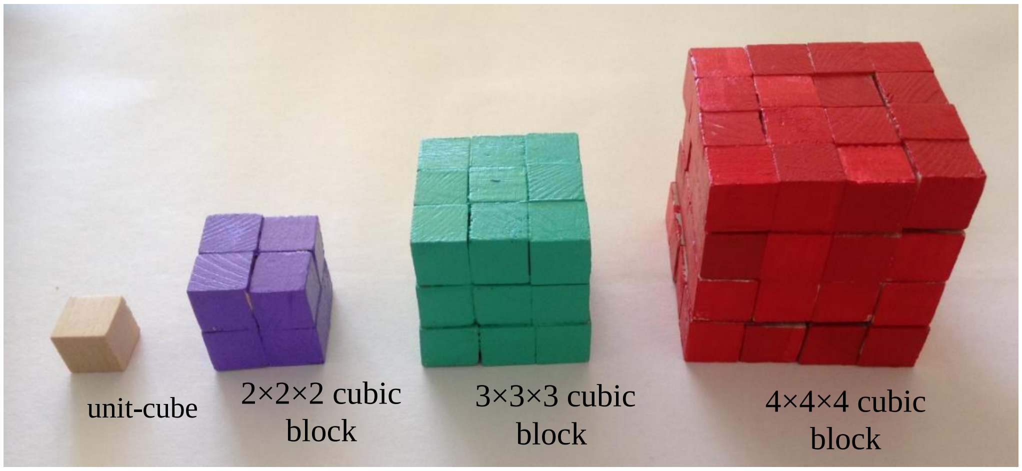


Figure 6.1. The unit-cube and cubic blocks of various dimensions painted on the exterior.

In the Locating Tasks, the students were required to locate a point or describe the motion of one point in two- or three-dimensional perceptual space. The missing person point A or the fish in these spatial contexts were visible to the students. On the other hand, in the Cubic Block Task, other than the unit-cubes on the faces of the cubic blocks, the task required students to anticipate and represent the unit-cubes that they did not have immediate perceptual access to.

Entering the teaching episodes for the Cubic Block Task (November 21 and December 6), I hypothesized that the frames of reference that each student constructed, reported in Chapter 5, will guide their units-coordinating activities in three-dimensional contexts. Based on the findings from the initial interviews (Chapter 4) and the two locating tasks (Chapter 5), I hypothesized that Kaylee's units-coordinating activities (inserting composite units into other units) in three-dimensional contexts would entail a coordination of multiple three levels of units structures, accounting for all three spatial dimensions simultaneously. On the other hand, I conjectured that, Morgan's units coordinating activities in three-dimensional contexts would entail a sequential coordination of two three levels of units structures in activity, along two spatial dimensions recursively. In this chapter, I will discuss how Kaylee and Morgan each reasoned with the three-dimensional cubic blocks in relation to these hypotheses.

Cubic Block Task Part One: Kaylee and Morgan Count the Blocks of Various Sizes

After showing the cubic blocks and explaining the context of the task to Kaylee and Morgan, I covered the blocks and asked the students to find how many unit-cubes in total were contained in the cubic block, how many unit-cubes had paint on them, and how many unit-cubes did not have paint on them. We went through these questions for each cubic block, one at a time. After both students completed their work for one cubic block, they discussed their solutions and we moved on to the next cubic block. These questions composed the first part of the Cubic Block Task.

Kaylee's Cubic Block Activities

The 2×2×2 cubic block.

Figure 6.2 shows Kaylee's written responses for the questions regarding the 2×2×2 cubic block. As shown in Figure 6.2, Kaylee wrote there were a total of eight unit-cubes in the first cubic block. Kaylee explained that she obtained eight from "two by two by two, like, width times length times height. Just like how you would find the area." Although she mentioned area, from her explanation, I inferred that Kaylee recalled the formula for finding the volume of the cube (width times length times height). Therefore, her inscription of '4×2=8' seemed to showcase how she calculated 2×2×2; that is, she first calculated 2×2 to obtain 4 and then multiplied that by the remaining 2 resulting in 8 as her answer.

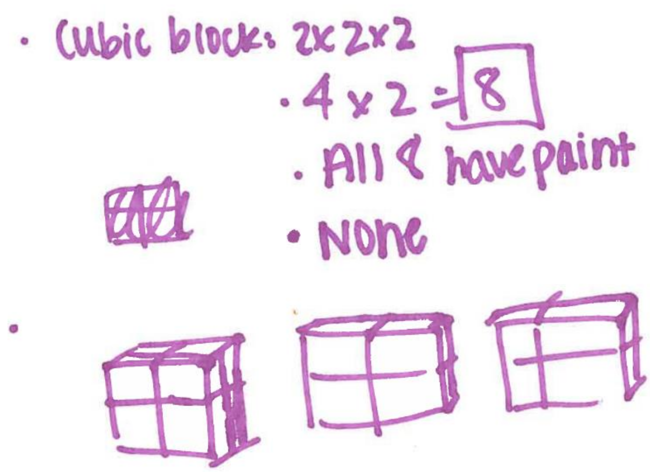


Figure 6.2. Kaylee's written responses for the 2×2×2 cubic block.

In explaining the eight painted unit-cubes, Kaylee talked about the layers of the cubic block: "it's basically like, two sections of four, like a flat… And they're next to each other [*putting her two hands together as if each hand represented each side of the cubic block*]. So, like, there's not anything in the middle." Her sketch at the bottom of

Figure 6.2 depicted the two sections of four she was referring to. Therefore, in finding the number of painted unit-cubes, Kaylee decomposed the cubic block into two sections of four. Although Kaylee did not explicitly make the connection, her inscription of ‘4×2=8’ seemed consistent with her explanation of the two layers of 2×2, 4-squares[4].

The 3×3×3 cubic block.

Figure 6.3 shows Kaylee’s written responses for the prompts regarding the 3×3×3 cubic block. As demonstrated in Figure 6.3, Kaylee labeled her cubic block 3×3×3. When asked to find the total number of unit-cubes contained in the cubic block, Kaylee immediately wrote 9×3=27 below her label of the cubic block. Her labeling and the way she found the total number of unit-cubes were consistent with those in the 2×2×2 cubic block case.

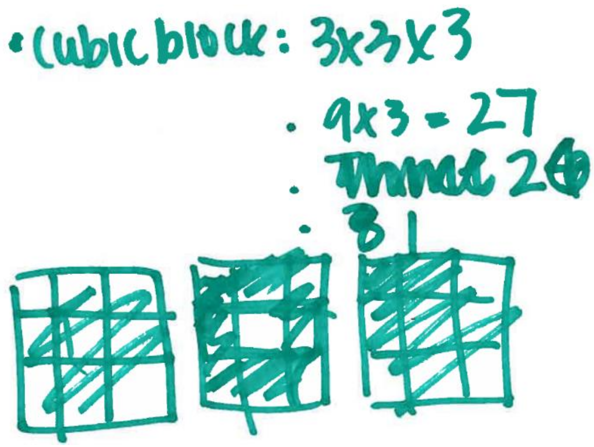
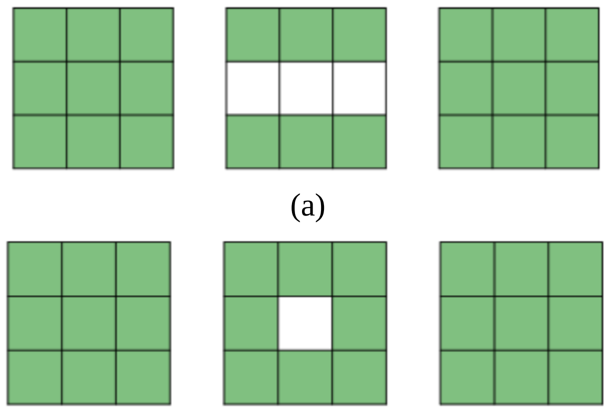


Figure 6.3. Kaylee’s written responses for the 3×3×3 cubic block.

[4] By layers of squares, I refer to the layer of the n×n×n cubic block shaped as a square that consists of n×n unit-cubes.

When I asked the students to think about the number of painted or unpainted unit-cubes, Kaylee sketched three 3×3, 9-squares in a row as shown in Figure 6.3. Originally, she colored the squares as shown in Figure 6.4 (a) and concluded that three unit-cubes were not painted. The following excerpt shows Kaylee explaining how she found three unpainted unit-cubes and then Kaylee revising her answer to one unpainted unit-cube as demonstrated in Figure 6.4 (b).



(a)

(b)

Figure 6.4. A re-generation of Kaylee's coloring of the 3×3 square layers.

Excerpt 6.1: Kaylee explains the number of unit-cubes unpainted in the 3×3×3 cubic block.

K: Uh, I did basically what I did [*pointing to her three sketches of the 2×2 squares she sketched earlier (see Figure 6.2)*] and I split them so there was, like, nine squares, there would be three layers and nine squares, that's how it's set up.

T: Okay.

K: And so one face, [*pointing to the first 3×3 square on the far left in her sketch (see Figure 6.3)*], like the two ends [*puts two hands facing each other slightly apart*] are obviously colored.

T: Mm-hmm.

K: [*Pauses for a few seconds and then smiles*.] Oh.

M: There's only one in the middle.

K: [*Nods as Morgan is talking and starts talking at the same time*.] Yeah, there's only one in the middle.

T: Wait, what was the "oh" moment. What was the "oh."

K: Alright, like, okay, so what I was thinking… is, alright, if these faces [*points to the 3×3 squares on the right and on the left]* are colored, and those take like, the front and back faces [*demonstrates the two faces on each side with her hands*]…

T: Mm-hmm.

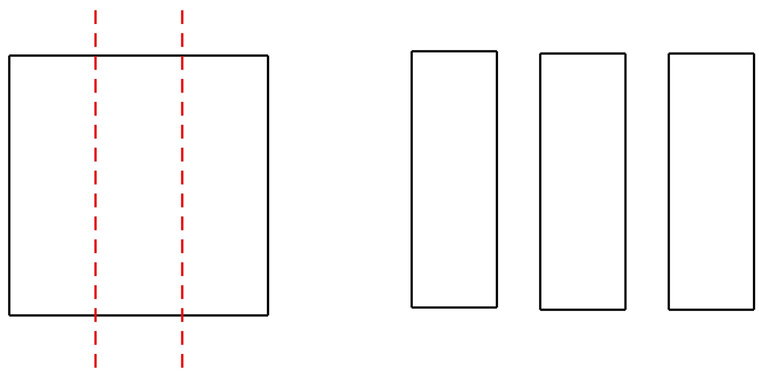
K: So, there's that one nine by nine square [*points to the 3×3 square she sketched in the middle*] in the middle. So, I knew that the top and bottom were going to be painted [*points to where she colored in earlier*] but then I forgot about the two sides over on the side.

T: Side… Okay.

K: So, I forgot to color these [*Fills in two additional unit squares as shown in Figure 6.4 (b)*]

Consistent with her explanation about the 2×2×2 cubic block, Kaylee explained that "there would be three layers and nine squares, that's how it's set up." This corroborated my hypothesis that Kaylee decomposed the cubic blocks into square-shaped layers of unit-cubes. Although Kaylee first thought there were three unit-cubes unpainted in the 3×3×3 cubic block, as she explained her sketch in Figure 6.3, Kaylee realized she forgot to color two additional unit-squares. Using the decomposition of the cubic block into 3×3 square-shaped layers, Kaylee demonstrated which unit squares in each 9-square layer corresponded to the top, bottom, or side faces of the cubic block, even though the cubic block was not in her visual field. From her consistent way of decomposing the cubic blocks, I hypothesized that Kaylee has constructed a *reversible decomposing scheme*, a systematic way of producing each unit-cube while maintaining their relative position within the cubic block in re-presentation. As a result of enacting this scheme, Kaylee was able to find the total number of unit-cubes and the unpainted unit-cubes in each cubic block.

To explain what I mean by a *reversible decomposing scheme*, consider the 3×3×3 cubic block as an example. When Kaylee formed the goal of re-presenting each unit-cube *within* the cubic block, she needed to take the cubic block apart to identify each unit-cube but then put it back together to maintain the position of each unit-cube within the cubic block. Therefore, she first chose one face of the cubic block and partitioned it vertically into three equal sections, as demonstrated in Figure 6.5 (a). This partition of one face of the cubic block resembled Kaylee's enactment of her splitting scheme of the imaginary candy strip in her initial interview. Then, holding that partition in mind, Kaylee separated the sections along those partitions, unitizing each section as a unit, as demonstrated in Figure 6.5 (b).



(a) (b)

Figure 6.5. Models of Kaylee's partitioning and segmenting of a face of the 3×3×3 cubic block.

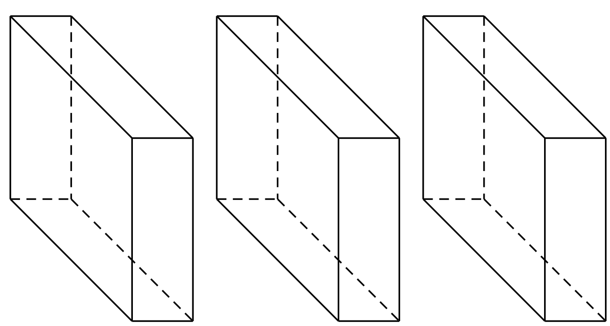
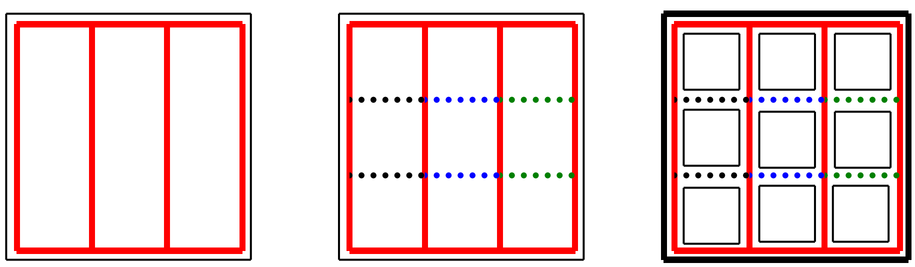


Figure 6.6. Model of Kaylee's other perspective of her segmented pieces of the 3×3×3 cubic block.

Then, shifting her perspective to the other side of the cubic block (Figure 6.6), she re-presented each square-shaped layer corresponding to each of the three sections she had just unitized. This was made possible by her ability to bring forth her images from one perspective in co-presence with her images from another perspective.

Taking the new perspective also allowed Kaylee to further partition each of the unitized square-shaped layers. From the new perspective, Kaylee enacted her recursive partitioning scheme to partition the two-dimensional square into three units (Figure 6.7 (a)) and again into three units, each distributed to the previous three units (Figure 6.7 (b)). Using the operations of her units-coordinating scheme, Kaylee inserted the composite unit of three units along one spatial dimension into the other three units along the second spatial dimension (Figure 6.7 (c)). In doing so, Kaylee not only produced each unit-cube in each square-shaped layer, but Kaylee also produced multiple three levels of units structures. As demonstrated in her explanations, this process seemed almost unnecessary, as finding the number of unit-cubes in each square-shaped layers was immediate for Kaylee, most likely induced from her units-coordinating scheme.



(a) (b) (c)

Figure 6.7. Model of Kaylee's recursive partitioning of a square layer of the 3×3×3 cubic block.

Bringing forth the results of her splitting of the cubic block into three square-shaped layers (in Figure 6.6), Kaylee inserted the nine unit-cubes in the square layer (Figure 6.7 (c)) into each of the three units along the third dimension. This insertion of two-dimensional layers into the third dimension was consistent with her insertion of unit-cubes along each spatial dimension (Figures 6.7 (a) and (b)). Further, Kaylee was aware that this insertion of each square-shaped layer into the three sections of the unit cube would re-constitute the cubic block, indicated by her comment "that's how it's set up." In other words, not only did Kaylee decompose the cubic block, but she was also able to reverse the operations she carried out to re-compose the cubic block in checking that she has indeed accounted for all the unit-cubes contained in the cubic block.

Finally, Kaylee reversibly re-composed the vertical square-shaped layers of the cubic block into a whole while holding each layer as a part of that whole, using her disembedding operation. This is why I consider the scheme to be reversible. This way of operating allowed Kaylee to individualize each unit-cube by maintaining their relative positions within the cubic block. Her awareness of which unit-squares in each square-shaped layer corresponded to which face corroborated that her decomposition of the cubic block also involved the constant re-building of the cubic block. This co-occurrence of partitioning and reversing the partitioning seemed analogous to the splitting scheme Kaylee has enacted in her initial interview. That is, these schemes both involved the co-occurrence of one activity and the reverse of the activity.

Using her reversible decomposing scheme, I claim that Kaylee re-presented the interior of the cubic block and anticipate the unpainted unit-cubes without having to physically take the cubic block apart. In conjunction with her units-coordinating scheme, Kaylee was also able to find the total number of unit-cubes in the cubic block. Although Kaylee re-enacted her previously learned volume formula to find the total number of unit-cubes for each cubic block, I claim that Kaylee's re-presentation of the cubic block entailed a *recursive coordination of three levels of units* (see Figure 6.8). To elaborate, each three levels of units structures constituted each square-shaped layer and they were inserted into each three units along the third dimension, which constituted another three levels of units structure.

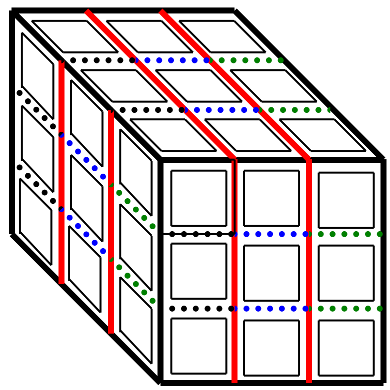


Figure 6.8. Model of Kaylee's recursive coordination of three levels of units in the 3×3×3 cubic block.

I conjecture that Kaylee's FR coordinating scheme was used as a subscheme of her reversible decomposing scheme when decomposing and inserting unit-cubes along each spatial dimension of the cubic block. The FR-coordinating scheme involved recognizing a situation in which she could posit a frame of reference as a unit and inserted it into another frame of reference. The immediate past operating of two-dimensional frames of reference are brought into the present resulting in a co-presence of multiple two-dimensional frames of reference constituting the three-dimensional space. Uniting and multiplicatively coordinating the two sets of frames of reference constructed from different perspectives, Kaylee re-presented the location of points in three-dimensional perceptual spaces in terms of their locations along all three dimensions.

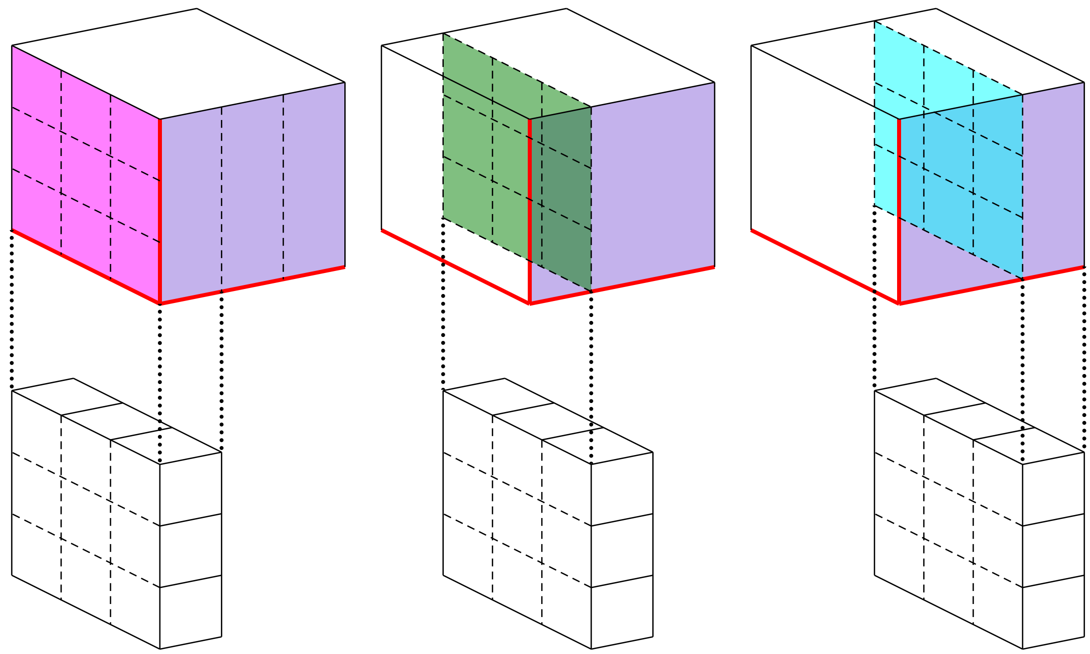


Figure 6.9. A model of Kaylee's FR coordinating scheme enacted in her reversible decomposing scheme.

Figure 6.9 is a model of Kaylee's FR-coordinating scheme enacted in her reversible decomposing scheme. To elaborate, first consider her decomposition of the cubic block into square-shaped layers. Here, I hypothesize that Kaylee superimposed a rectangular frame of reference onto one face when splitting and segmenting the cubic block into square-shaped layers (see the purple face in Figure 6.9). Shifting her perspective to the adjacent face, Kaylee superimposed another rectangular frame of reference that guided her recursive partitioning of the square-shaped layer into unit-cubes (see pink face in Figure 6.9). Finally, when Kaylee inserted the two-dimensional layers into the third dimension, this involved her insertion of her second frame of reference into the first (see the 3×3 unit-cubes underneath each layer of the cubic block in Figure 6.9). Uniting and multiplicatively coordinating the two sets of frames of reference constructed from different perspectives, Kaylee re-presented the individual unit-cubes with a realization of their relative positions along all three dimensions. That is, still aware of each square-shaped layer as unitized structures, Kaylee was able to track the location of a given point along the layer (two dimensions) with the realization that the point had a specific location along the third dimension. Therefore, the position of each unit-cube was embedded within all three dimensions multiplicatively. This awareness of the position of each unit-cube along all three dimensions is what allowed her to determine which unit-cubes in each square layer corresponded to the top, bottom, or side faces of the cubic blocks.

The 4×4×4 cubic block.

When prompted to solve for the 4×4×4 cubic block, Kaylee immediately started writing in the same format as she did for the previous cubic blocks, as shown in Figure 6.10.

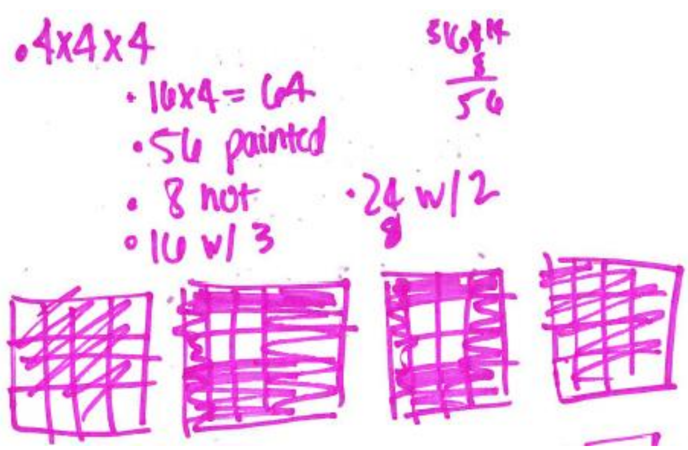


Figure 6.10. Kaylee's written responses for the 4×4×4 cubic block.

Because both students repeated a similar pattern for finding the total number of unit-cubes in this case but found a different number of unpainted unit-cubes, I focused on asking them to explain the number of unpainted unit-cubes. In the following excerpt, Kaylee explains how she found eight unpainted unit-cubes in total.

Excerpt 6.2: Kaylee explains the number of unpainted unit-cubes in the 4×4×4 cubic block.

K: So, I did the same thing. I did these two [*pointing to the far left and far right 4×4 squares that she colored in in her sketch*] are painted because they're faces [*demonstrating the faces with her hands (Figure 6.11 (a))*]. And then…

T: So, can you first explain what these [*points to the four squares*] are?

K: Yeah, like, the four...

M: The four sides.

K: [*Continuing,*] by four, just squares [*making a vertical chopping motion with hand to demonstrate the four vertical layers as modeled in Figure 6.12.*] and there's four in each little thing.

T: Okay.

K: So, these two are painted [*again, points to the two squares on the far left and far right in her illustration in Figure 6.10*] cuz the whole thing is painted as they're faces.

T: Mm-hmm.

K: And so, I took the top. From the middle [*sets hands like parentheses around the two squares in the middle*], the top four are painted [*runs pen across the top rows of the two squares in the middle*] because they're, that's like for the top [*hand motions the top layer (Figure 6.11 (b))*] face and the bottoms are painted [*points to the bottom rows across the two squares in the middle*], or, yeah, because it's the bottom face and I took the sides [*again puts hands vertically as to demonstrate the two side faces*], and so four are painted on one side and four on the other and for both of them, so there's eight. So, I subtracted eight from sixty-four.

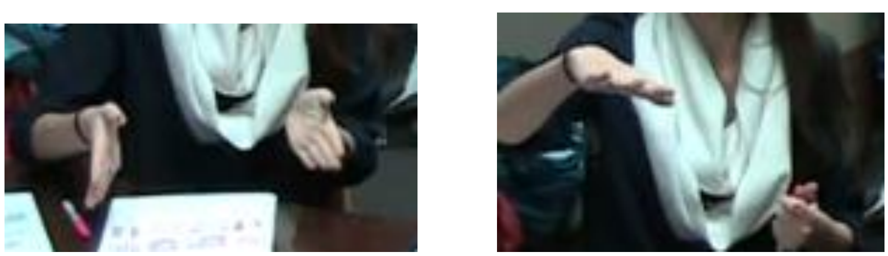
T: Mm-hmm.

M: You got eight cubes in there?

K: Yeah, there's one, two, three, four, five, six, seven, eight [*tapping on each white (uncolored) unit square in her sketch in Figure 6.10*].

M: In the middle?

K: Yeah.



(a) (b)

Figure 6.11. Kaylee explaining her illustration and the painted unit-cubes in the 4×4×4 cubic block.

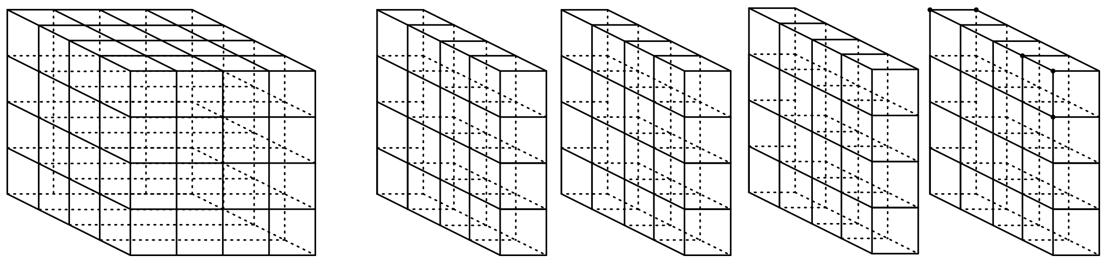


Figure 6.12. A model of Kaylee’s four layers made of four-by-four unit-cubes.

As Kaylee explained at the beginning of Excerpt 6.2, she illustrated “the four by four squares” and was aware that there were four of those in the cubic block. From her actions, I hypothesized that Kaylee coordinated multiple perspectives, decomposed the cubic block into 4×4 square layers. Doing so, she constructed a re-presentation of the cubic block to determine which unit-cubes in each square layer corresponded to the top, bottom, and side faces of that re-presentation of the cubic block. So, Kaylee established an image of the interior of the cubic block (unpainted unit-cubes) without having to physically take the cubic block apart. From the way that Kaylee consistently, independently, and confidently drew sketches of the square-shaped layers (Figures 6.2, 6.3, and 6.10) for all three cubic blocks, I impute the aforementioned reversible decomposition scheme to Kaylee.

Morgan’s Cubic Block Activities

The 2×2×2 cubic block

Figure 6.13 shows Morgan’s written responses to the 2×2×2 cubic block case. When making sketches to solve the problems, Morgan said “I’m not good at drawing three-D stuff.”

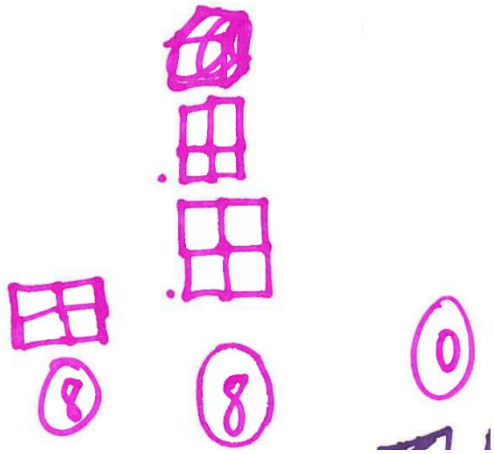


Figure 6.13. Morgan’s written responses for the 2×2×2 cubic block.

When asked how she found eight unit-cubes in total, Morgan explained, “in my head, you just, there’s four on this side [*pointing to the 2×2 square pattern she drew on the far left (see Figure 6.13)*] and then there’s four on the other side. So like, I’d do four times two or like four plus four…” As Morgan later summarized, “I was just counting from one side and then the other side.” Although her sketches only showed two-dimensional figures, from her comments, I inferred that Morgan also decomposed the cubic block in her head into two vertical layers, which she referred to as one side and the other of the cubic block. Then partitioning each side into four unit-cubes, she added the two fours to obtain eight unit-cubes in total. As such, it was possible that Morgan has also constructed a reversible decomposing scheme. In explaining the eight painted and zero unpainted unit-cubes, Kaylee explained her solution first and Morgan said that Kaylee’s approach was the same as how she thought about the problem.

The 3×3×3 cubic block

Morgan’s written responses regarding the 3×3×3 cubic block are shown in Figure 6.14. Morgan started with a sketch of a 3×3 square as shown on the left of Figure 6.14, consistent with her way of sketching the 2×2×2 cubic block.

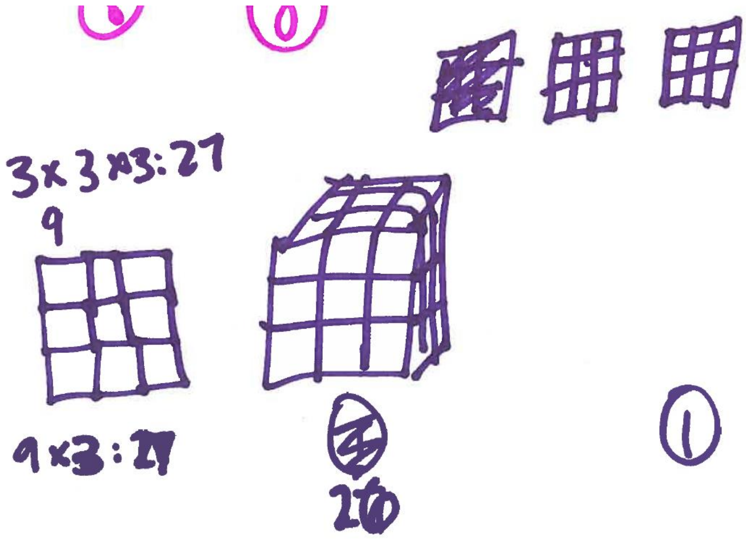


Figure 6.14. Morgan's written responses for the 3×3×3 cubic block.

Although Morgan has written 9×3=27 as the total number of unit-cubes contained in the cubic block in Figure 6.14, her original response was different. At first, Morgan ran her index finger through the small unit squares she drew in the 3×3 square in Figure 6.14 to count one by one the total number of unit-cubes in the square. Then, Morgan wrote 9×2=18 below the square as shown in Figure 6.14. This suggested that Morgan used the same counting scheme as she did in the previous cubic block and decomposed the 3×3×3 cubic block into two sides of 3×3 squares put together instead of three. This indicated that Morgan's counting scheme for the cubic blocks entailed decomposing but not necessarily re-composing the cubic block to check that the parts of the block constituted the whole.

When I asked her to find the number of painted/unpainted unit-cubes, Morgan sketched the cubic block as shown at the center of Figure 6.14 and seemed to be confused. She claimed "I hope I do this right" and expressed that this was "hard." She sat in thought for approximately five seconds and then changed her answer to 9×3=27. Thinking about the number of painted or unpainted unit-cubes and having sketched a figurative model of the cubic block together may have pushed her to re-organize her counting scheme *in* *activity* for the total number of unit-cubes in the cubic block, realizing that there were more than two sides constituting the 3×3×3 cubic block.

Morgan's comment that this task was difficult could have meant one of two things or both. First, it is possible that she meant re-presenting the 3×3×3 cubic block mentally was more complicated than the 2×2×2 cubic block. Second, it is possible that Morgan was referring to the difficulty of drawing a three-dimensional figure on a two-dimensional paper, as she mentioned earlier that she had a difficulty in drawing three-dimensional figures. In either case, once she sketched the cubic block as in the center of Figure 6.14, it helped her keep track of how many 3×3 square-shaped layers consisted the cubic block.

Because I wanted to wait until all prompts were completed by both students before we discussed any of their responses, I did not ask Morgan why she changed her answer from 18 to 27 right after she did so. However, later in the teaching episode, I asked Morgan how she obtained a total of 27 unit-cubes and the following excerpt shows Morgan's explanation.

Excerpt 6.3: Morgan explains how she found 27 unit-cubes in the 3×3×3 cubic block.

M: Okay, so I had these [*pointing to her 3×3 square in Figure 6.14*], three rows on this side [*tracing each column on the front face of the 3×3×3 cubic block she sketched when solving for the number of unit-cubes painted, demonstrated by the red dashed arrows in Figure 6.15*] and so I had my three [*writes "3" down above the 3×3 square*] and then I have times [*writes "×"*] my other side right here, three [*points to the right side face of the 3×3×3 cubic block she sketched and traced each column on that face, demonstrated by the blue dashed arrows in Figure 6.15*], which equals nine [*writes "9" above the 3×3 square*] times my other three and that equals twenty-seven [*completes writing 3×3×3=27 as shown in Figure 6.14*].

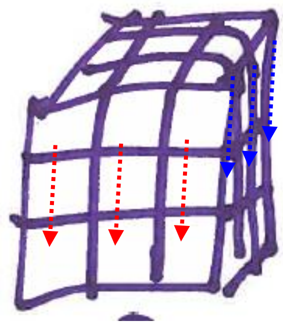


Figure 6.15. Morgan’s counting activity of the three rows in each face.

As demonstrated in Excerpt 6.3, when explaining why she multiplied 3 by two, Morgan referred to the three “rows” on two adjacent faces of the cubic block that was visible in her sketch of it (see arrows in Figure 6.15). Although she did not carry out the tracing activity in the third face based on her explanation, “my other three” referred to the three columns on the top face of the cubic block in Figure 6.15. As such, I inferred that Morgan multiplied the number of columns (which she called rows) in each face. It is possible that Morgan recalled the volume formula that Kaylee mentioned earlier (length times width times height) in the 2×2×2 cubic block case and applied it to the 3×3×3 cubic block. However, the units that Morgan multiplied did not span the entire cube. That is, multiplying those units as she described in Excerpt 6.3 was insufficient to produce all the unit-cubes comprising the cubic block.

Her explanation of how she found 27 unit-cubes in total did not support the inference I made earlier of her counting the nine unit-cubes in one layer and then multiplying that by the total number of layers consisting the cubic block. These rather contradicting actions could have rooted from a discrepancy between knowing and seeing (Parzysz, 1988). That is, what she said she has done using her drawing (seeing) could have been different from what she knew about the objects. In any event, I did not have enough evidence to impute to Morgan the operations that Kaylee's reversible decomposing scheme entailed. Although Morgan consistently started with a sketch of a 3×3 square, her counting of the nine unit squares one by one and her counting of the rows in each face of her sketch of the cubic block seemed to contra-indicate the hypothesis that Morgan has decomposed the cubic block into layers of squares using recursive partitioning and units-coordinating. That is, her recursive partitioning and units-coordinating schemes were not assimilatory in this situation of counting the total number of unit-cubes in the cubic blocks.

When I asked the students to think about the number of painted/unpainted unit-cubes, Morgan added two more faces to her sketch of the 3×3 square, transforming it into a 3×3×3 cubic block in the center of Figure 6.14. Then, after running her pen above the unit-cubes around one of the corners, she wrote '1,' referring to the number of unit-cubes that would be unpainted. Morgan was certain that there should be one unpainted unit-cube as shown in her response to Kaylee in Excerpt 6.1. Although she was certain about the one unpainted unit-cube, as shown in the following excerpt, Morgan had difficulty in decomposing and re-presenting the cubic block mentally to explain why there is only one unpainted unit-cube.

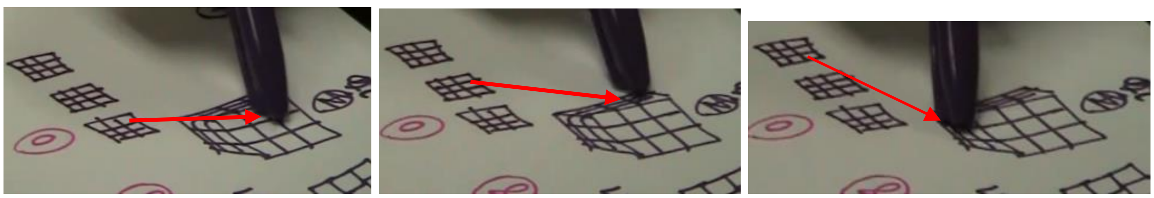
Excerpt 6.4: Morgan explains the number of unit-cubes unpainted in the 3×3×3 cubic block.

M: Well, I kind of did what Kaylee did. But I pictured it in my head because I was looking at this [*pointing to the 3×3×3 cubic block she sketched (see center of Figure 6.14)*] and well, on each of these sides [*points to the front face of the cubic block*], obviously all of these [*points to the top face of the cubic block*], but there's one in the middle that's not painted at all if you look at the rows [*runs pen along the vertical layers in the 3×3×3 cubic block (along the red arrows in Figure 6.15)*]. I'm not sure how to explain that.

T: That makes sense... [*Encouraging her to keep talking*.]

M: So, I mean, kind of like Katie was saying, [*starts sketching a 3×3 square*]

> you have your rows and everything [*sketches another 3×3 square*]… That's… [*Pauses to check if she drew a 3×3 square*] Yeah… [*Sketches a third 3×3 square; the three 3×3 squares are illustrated in Figure 6.14*]. So, this right here is that face [*first points to the first 3×3 square and then circles around the front face of the cubic block (Figure 6.16 (a))*] that one right there [*points to the second 3×3 square*] is that face [*points to the face on the side of the cubic block (Figure 6.16 (b))*] and that one right there is that face [*points to the top face of the cubic block (Figure 6.16 (c))*]. And so, if this one right here [*pointing to the first 3×3 square*] all of these are going to be painted [*colors the 3×3 square*] and like, all of these will be painted except for that one that's in the middle [*points to the unit square in the middle of the 3×3 square in the middle*], basically, I guess.



(a) (b) (c)

Figure 6.16. The three layers Morgan was referring to in her sketch.

As shown in Figure 6.16, Morgan associated each 3×3 square with one of each of the three adjacent faces that were visible in her sketch, which corroborates that her counting actions shown in Figure 6.15 were based on the faces of the cubic block in her sketch. In Excerpt 6.4, it appeared Morgan somehow identified the one unit-cube in the middle of the cubic block from looking at her sketch. Perhaps, Morgan could somehow imagine one unit-cube in the middle wrapped around by each face. However, she had a difficult time explaining why there was only one unit-cube there. I observed similar counting activities in the 4×4×4 cubic block, which I discuss next.

The 4×4×4 cubic block.

Morgan's written responses for the 4×4×4 cubic block are shown in Figure 6.17. Although in the center of Figure 6.17 her illustration shows a 4×4×4 cubic block, Morgan started with a sketch of a 4×4 square. Above the square she wrote "4·4·4" and then "16·4" and calculated the total number of unit-cubes in the cubic block.

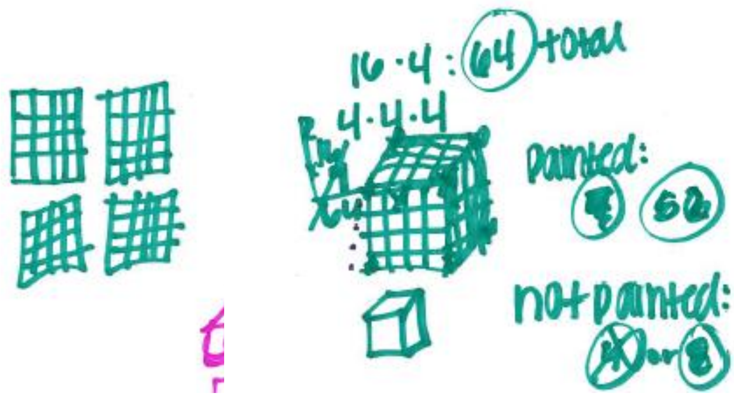
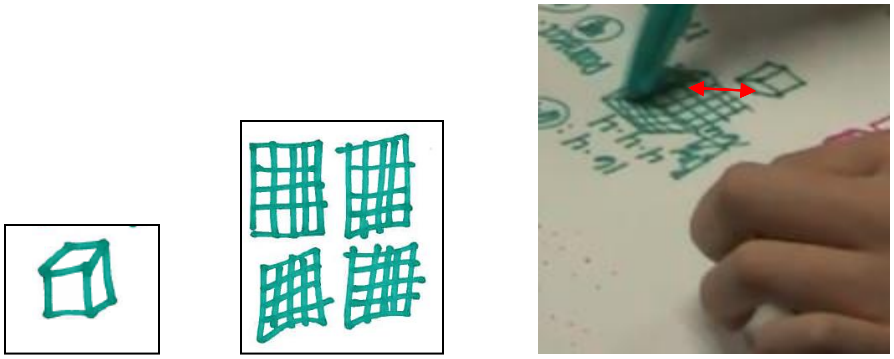


Figure 6.17. Morgan's written responses for the 4×4×4 cubic block.

Regarding the number of unpainted unit-cubes, Morgan questioned Kaylee's response of eight unit-cubes (see Excerpt 6.2), so I asked Morgan to explain how she was thinking about the problem. The following excerpt shows Morgan's explanation of the unpainted unit-cubes in the 4×4×4 cubic block.

Excerpt 6.5. Morgan explains the number of unpainted unit-cubes in the 4×4×4 cubic block.

M: Well I was thinking of four or two cubes because… Like, okay, so. I [don't] know. I kind of had to look at it just as if this was [*starts drawing a small unit-cube (see Figure 6.18 (a))*]… I remember that that was one cube, not just a little square.
T: Mm-hmm.
M: And so, if there was… I kind of pictured it like how Kaylee was doing it earlier. [*Starts sketching four 4×4 squares as shown in Figure 6.18 (b)*.]
T: Mm-hmm.
M: So, like, there is, the three of the four sides, then you would um… Wait, so this one side [points to the 4×4 square on the top left in her sketch] is all painted, this one side [points to the 4×4 square on the top right in her sketch] is all painted because you're looking at those two [*points to two adjacent faces of the 4×4×4 cubic block she had sketched earlier (Figure 6.18 (c))*]… But, ah [*puts her hands on her head as if she's having a headache*], I don't know how to say it…
T: It's okay. Maybe, would this [*uncovers the red 4×4×4 cubic block model*] help us to….
M: I don't think there's eight that aren't painted.



(a) (b) (c)

Figure 6.18. Morgan explaining her illustration and the painted unit-cubes in the 4×4×4 cubic block.

In excerpts 6.2 and 6.5, Morgan consistently refers to the four 4×4 square-shaped layers as the four “sides.” This tendency was also demonstrated in the 2×2×2 cubic block case when she explained “there’s four on this side and then there’s four on the other side.” In the 2×2×2 cubic block case, from her illustration in Figure 6.13 and her explanation, it seemed as though Morgan referred to the face in the front as “this side” and the congruent one “on the other side.” Here, her sides corresponded to what Kaylee referred to as layers. However, once the cubic blocks no longer had only two sides, this way of organizing the block did not seem to work for her , as demonstrated in her solving the 3×3×3 cubic block case. Thereafter, her “sides” shifted from layers to the visible exterior layers (consisting of each adjacent face) in her sketch of the cubic blocks. The way she demonstrated how the squares related to the cubic block in her sketch in Figures 6.15 and 6.16 corroborates such an inference.

From this consistent association of the $n \times n$ squares with the *visible exterior layers* (consisting of each adjacent *face)* of the $n \times n \times n$ cubic block and her continuous struggle to explain how she found the unpainted unit-cubes, I consider Morgan’s use of her illustrations to be limited to the perceptual elements of the cubic blocks—the unit-cubes or faces that she could see on each two-dimensional face. Often this meant that she could re-generate faces of the cubic block in her sketch but had difficulty coordinating that with the third dimension in re-presentation.

From her comment “I remember that that was one cube, not just a little square,” I inferred that Morgan had an awareness of the third dimension; positing the third dimension in conjunction with the two-dimensional faces was difficult for her. In other words, Morgan sequentially coordinated two dimensions instead of bringing forth the immediate past operating of a two-dimensional face in co-presence with another two-dimensional face. As a result, she was limited in being able to coordinate all three dimensions multiplicatively and in re-presenting the interior of the cubic block. So, Morgan guessed that there would be two or four unit-cubes that were unpainted, not being sure what was inside of the cubic block.

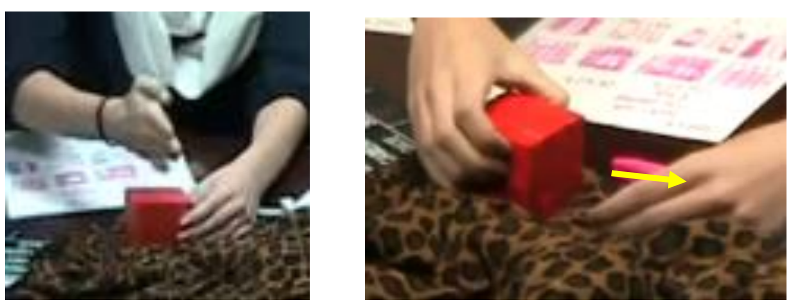
Despite her ability to partition the block along each visible face, guided by her rectangular frames of reference, her constraint in coordinating the third dimension along with the two-dimensional layers was analogous to that in her activities discussed in Chapter Five. That is, when locating the four fish in the fish tanks for describing the motion of one fish to another, Morgan sequentially coordinated the location of one fish in two-dimensional frames of reference. From her locating activities, I hypothesized that Morgan constructed a FR coordinating scheme in activity. In contrast to Kaylee—who I inferred to have used the results of her FR-coordinating scheme as input to use in structuring the spatial object and construct a reversible decomposing scheme—because Morgan was yet to construct a FR coordinating scheme for those two-dimensional frames of reference, I hypothesize that Morgan was yet to construct a reversible decomposing scheme. In other words, Morgan did not recursively coordinate the results of her FR-coordinating scheme in such a way that would allow her to decompose and re-compose the cubic block mentally as Kaylee did. Therefore, taking the cubic block apart and anticipating the interior of the cubic block in re-presentation was challenging for Morgan.

Kaylee and Morgan discuss the cubic blocks.

Because Morgan seemed to struggle with demonstrating her number of unpainted unit-cubes in re-presentation, I suggested that they demonstrate their thinking using the actual cubic block. So, I uncovered the 4×4×4 cubic block and Kaylee first demonstrated her way of thinking using the cubic block, as shown in the following excerpt.

Excerpt 6.6. Kaylee demonstrates her four layers and the painted unit-cubes.

K: Here's, you see where my four things [are]. One, two, three, four [*places hand on the cubic block as if she is chopping it into vertical layers (Figure 6.19 (a))*] and they all look like this [*points to one of the faces of the cubic block*]. Like, you see that? [*Runs her finger along the four layers once more*.] If I were to split them apart [*makes a hand motion as if she's tearing one layer out of the block at a time indicated by the yellow arrow (Figure 6.19 (b))*]…



(a) (b)

Figure 6.19. Kaylee demonstrates taking apart the four layers in the 4×4×4 cubic block.

[*Continued*.]

M: [*Interrupts K*,] that's four that aren't painted.

K: No, there's eight [*laughs*].

M: One, two…

T: Okay, this is a fun debate. Let's go. [*Encouraging the students to continue their discussion*.]

K: Okay, we take these two sides off [*puts one hand each on the outer faces of her four layers (Figure 6.20 (a))*] and then there's only these two middle ones [*points to the two vertical layers in the middle*]. And so, obviously these [*pointing to the outer two layers*] are all painted, because they're painted all over. So we have these two middle ones [*again, points to the two vertical layers in the middle*]. The four on the top [*runs index finger along the four unit-cubes on the top of the left middle vertical layer (Figure 6.20 (b))*] that are painted, right here [*again, runs index finger along the four unit-cubes on the top of the left middle vertical layer (Figure 6.19 (b))*] and…

(a) (b)

Figure 6.20. Kaylee demonstrates taking the cubic block apart in four layers.

M: [*After listening carefully to K's explanation, she interrupts K as if she realized something*.] Oh, I forget that there's [*points to the four unit-cubes in the middle of one face*] four levels and there's going to be four…

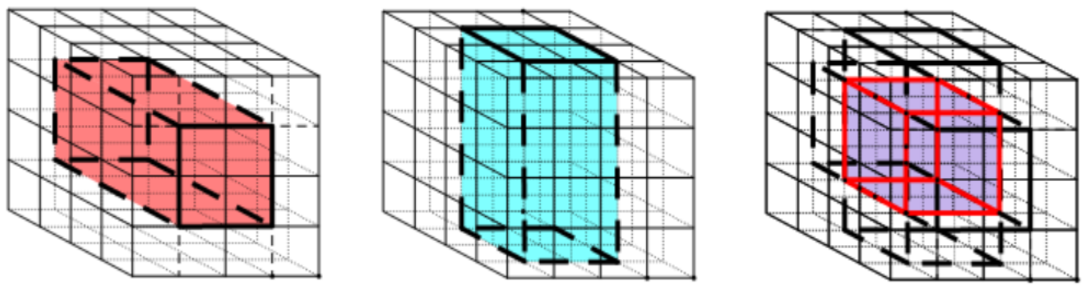
K: Yeah, there's going to be four in the middle [*runs her fingers along those four unit-cubes M was pointing at*] that are not painted, like [*points her index finger on the cubic blocks on the adjacent face and coordinates which ones are the ones that would not be painted (Figure 6.21)*)] right? Can you imagine it? [*Laughs as she is having trouble explain it through words*.] These four are painted [*runs finger along the second vertical layer on the face facing her*] in the middle two, and this top painted [*runs her finger along the second vertical layer on the top face*] and there's no other space for paint for these four in the middle [*again, touches the four unit-cubes in the middle of the face as if she's talking about the four in the second layer, then points at the top of the two vertical layers in the middle (similar to what she was doing in Figure 6.21)*.]

Figure 6.21. Kaylee demonstrates her four layers and the painted unit-cubes.

In Excerpt 6.6, Kaylee physically enacted the operations that she had used in making her sketch 6.5, in finding the painted or unpainted unit-cubes. First, she demonstrated how she decomposed the cubic block by partitioning and segmenting of the cubic block into vertical layers. From the way she showed the four layers to Morgan, it is likely that Kaylee noticed that Morgan was referring to different layers than she was—to Morgan they were the faces; to Kaylee they were the vertical square-shaped layers.

Second, as shown in Figure 6.20 (b), Kaylee pointed to the faces of the block that corresponded to each layer to determine which unit-cubes of the vertical layers should be painted. In other words, Kaylee demonstrated how she reversibly recomposed the vertical layers of the cubic block into a whole while holding each vertical as a part of that whole. Doing so, Kaylee determined which of the 16 unit-cubes in her vertical layer would be painted or unpainted and re-present the four unit-cubes in the middle, even though they were not in her direct perceptual field.

Lastly, as shown in Figure 6.21, Kaylee demonstrated her operating on the cubic block by mentally rotating her perspective and projecting her unit of four unit-cubes into the cubic block from each perspective. Figure 6.22 illustrates how I unpack her action demonstrated in Figure 6.21. As shown in Figure 6.22 (a), taking the perspective of viewing the cubic block from the side, Kaylee projected the four unit-cubes in the middle of the side face into the cubic block and inserted it into all four vertical layers. Similarly, as shown in Figure 6.22 (b), taking the perspective of viewing the cubic block from the top, Kaylee projected the four unit-cubes in the middle of the top face into the cubic block and inserted it into all four horizontal layers. Then, coordinating her two perspectives together, she mentally held the eight unit-cubes in the middle together, which is the intersection of the two projections from the two different perspectives (Figure 6.22 (c)). I conjecture that Kaylee enacted her FR coordinating scheme when inserting the four unit-cubes across the four layers from different perspectives.



(a) (b) (c)

Figure 6.22. Kaylee's projection of the four unit-cubes in the middle from both perspectives.

In Excerpt 6.6, after Kaylee explained her process of taking apart the cubic block into four layers, Morgan said "Oh, I forget that there's four levels and there's going to be four…" This indicated that using the cubic block model, Morgan was able to keep track of the four units along the third dimension that she had a difficult time coordinating in re-presentation. This corroborated my hypothesis that Morgan's coordination of her perspectives and organization of each face of the cubic block were sequential.

After carefully listening to Kaylee's explanation in Excerpt 6.6, Morgan sat in thought. I asked Kaylee to demonstrate her four layers again, and she asked us to "imagine these [layers] were like Velcro or something, so you can just pull them apart." To help Kaylee demonstrate her four layers for Morgan more visually, this time I showed them how the cubic blocks were made so we could take the layers apart, as shown in Figures 6.23 (a) and (b). As soon as the block was stripped off of the outer two vertical layers and left with the figure in Figure 6.23 (c), Morgan agreed that there were eight unit-cubes that were unpainted.

(a) (b) (c)

Figure 6.23. The teacher-researcher tearing off the vertical layers Kaylee was referring to.

When I asked Morgan what made her think that it could be four unit-cubes unpainted, she replied, “I don’t know. I guess I didn’t think it was as big as it was.” As such, Morgan demonstrated having difficulty projecting the two-dimensional 2×2 square into the cubic block and coordinating the third dimension with the middle of one of the two-dimensional faces. However, Morgan confirmed the eight unpainted unit-cubes after she heard Kaylee’s explanation and once again when she saw the cubic block taken apart. Different from Kaylee, Morgan’s operating on the cubic block seemed to rely on the perceptual material in order to find the number of unpainted unit-cubes. Therefore, I claim that Morgan’s operating on the cubic block was *in activity*, relying on carrying out sensori-motor activity on the perceptual material in order to find the number of unpainted unit-cubes.

Summary of Cubic Block Task Part One

To summarize, in reasoning about the cubic blocks of various sizes, I conclude that Kaylee constructed a reversible decomposing scheme to re-present further operate on each cubic block. When I asked her to find the total number of unit-cubes and the number of painted/unpainted unit-cubes that were, this evoked the goal of re-presenting each unit-cube *within* the cubic block, meaning that Kaylee needed to break the cubic block apart to produce each unit-cube as abstracted units but also put them back together to individualize each unit-cube by maintaining their relative positions within the cubic block. In order to achieve this goal, Kaylee partitioned each block into vertical square-shaped layers and again into unit-cubes along each spatial dimension, using her splitting and recursive partitioning schemes, guided by her FR-coordinating scheme. I conjectured that Kaylee's FR-coordinating scheme was used as a subscheme of her reversible decomposing scheme when decomposing and inserting unit-cubes along each spatial dimension of the cubic block. Uniting and multiplicatively coordinating the two sets of frames of reference constructed from different perspectives, Kaylee re-presented the individual unit-cubes with a realization of their relative positions along all three dimensions.

As a result, Kaylee was able to find the total number of unit cubes using her units-coordinating scheme and identify the position of each unit-cube embedded within all three dimensions multiplicatively. That is, still aware of each layer as unitized structures, Kaylee was able to track the location of a given unit-cube along the layer (two dimensions) with the realization that the unit-cube had a specific location along the third dimension. This awareness of the position of each unit-cube along all three dimensions was demonstrated in her ability to determine which unit-cubes in each square layer corresponded to the top, bottom, or side faces of the cubic blocks. Kaylee established an image of the interior of the cubic block (unpainted unit-cubes) without having to physically take the cubic block apart. Further, I argued that Kaylee's re-presentation of the cubic block entailed a *recursive coordination of three levels of units*.

On the other hand, Morgan seemed to have difficulties in coordinating the third dimension in the cubic blocks with the two-dimensional faces (or exterior layers) of the cubic blocks in re-presentation. Most of the time, Morgan's reasoning seemed limited to the unit-cubes or faces that she could see or visualize in her diagrams. This limitation in coordinating the third dimension along with the two-dimensional layers was analogous to that in her activities in constructing frames of reference to locate points or describe one point to another in three-dimensional space. Although Morgan had a two-dimensional FR scheme that was operational, as discussed in Chapter 5, the two-dimensional frames of reference were not recursively coordinated together to account for the third dimension. However, once she had the physical model of the cubic block in her visual field, using the cubic block model, Morgan was able to keep track of the four units along the third dimension that she had a difficult time coordinating in re-presentation. Therefore, I claimed that Morgan's operating on the cubic block was *in activity*, relying on carrying out sensori-motor activity on the perceptual material in order to find the number of unpainted unit-cubes.

Cubic Block Task Part Two: Kaylee and Morgan Extend or Reduce the Cubic Blocks

Part One of the Cubic Block Task consisted of questions regarding the subsets of each cubic block. In order to further explore the students' reasoning about the cubic blocks, I asked them to reason about extensions of the cubic blocks, towards the end of the teaching episode on November 21 and in the teaching episode conducted on December 6. The extension questions were analogous to the questions I asked in the Fish Tank Task, to locate a fish *outside* of the fish tank. The first question entailed what I considered to be extensions of the cubic blocks in units of *blocks*. That is, I asked the students to find how many unit-cubes would be contained in a cubic block that has eight unit-cubes on each edge and how many of the 4×4×4 cubic blocks would fit into it. The second type of questions were intended for the students to extend the cubic blocks in units of *unit-cubes*. In this case, I asked the students to find how many unit-cubes they would need to add to make a 4×4×4 cubic block into a 5×5×5 cubic block or a 4×4×4 cubic block into a 3×3×3 cubic block. I expected these situations in which students will need to extend their organization of the spatial objects (in this case the cubic blocks) would allow more opportunities for me to test the hypotheses I generated based on their activities in Part One of the Cubic Block Task.

The 8×8×8 Cubic Block

When I asked the students to find how many unit-cubes there would be in a cubic block with eight unit-cubes along each edge, Morgan drew an 8×8 square and counted the unit squares to check if her sketch was accurate. Morgan then pulled out the paper with her work from previous problems in Part One of the task. Then, Morgan wrote “8×8×8” and calculated the total number of unit-cubes. Meanwhile, Kaylee wrote “8×8×8” and calculated the total number of unit-cubes. Next, I asked them to find how many of the four-edged cubic blocks would fit into the eight-edged cubic block. Morgan wrote “4 blocks” on her paper while Kaylee wrote “8, 4 edges” implying that there are eight four-edged cubic blocks that would fit into the eight-edged cubic block.

When both students were finished writing on the paper, I asked them how they found the total number of unit-cubes. Simultaneously, Kaylee and Morgan replied that they multiplied eight by eight by eight. Noticing that they have come to a different conclusion for the number of 4×4×4 cubic block contained in the 8×8×8 cubic block, I asked the students to explain their different responses. The following excerpt starts with Morgan explaining how she found four four-edged cubic blocks contained in the eight-edged cubic block.

Excerpt 6.7. Morgan explains how she found four four-edged cubic blocks contained in the eight-edged cubic block.

M: Um, well, basically there's one, two, three, four, right? So, here's the four right there [*makes a bracket inscription to group the four unit-cubes she just counted in her sketch of the 8×8 square (see Figure 6.24)*]. So, that's your four-edge. Here's another four-edge [*marks another bracket of four*]. Here's another one [*marks the third group*], and then there's your last one. So, the total will be four [*writes "1" beside each bracket she just drew*], you know?

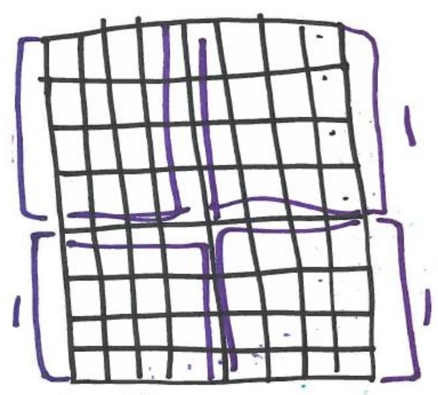


Figure 6.24. Morgan's sketch illustrating the 8×8×8 cubic block.

[*Continued*]

T: So, what is [*points to the 8×8 square*]?

M: So like, if I took this apart, like four blocks [*enacts a pinching motion to the 8×8 square as if she's taking four 4×4 squares out of it*], like you would have four… Wait. Yeah.

T: So, my question is, what is this block [*referring to the 8×8 square she sketched*], what are you representing with this?

M: Actually, you're going to need to, okay, so. You're going to need to double it because this is just one page, so it would be eight [*corrects her answer to "8"*]. So, if I had another page of this [*pointing to her 8×8 square sketch*] then you'd have one for each… Okay, so, if I had the big block of the eight, take it apart, [*motions her hand as if there's a cubic block in front of her*] there would be, you take one apart [*motions taking one chunk out of the cubic block and setting the chunk aside*], there's your one, then you take the other side [*repeats taking chunk out motion*] so like, this right here, I'll take this chunk out, take that chunk out, that chunk out, and I take that chunk out [*sections the 8×8 square into four sections of 4×4 squares*] but…

K: But it would still be four.

M: And then there would be another side of this [*sweeping hand over the 8×8 square*].

K: Cuz, there's, like it only goes four back [*sweeps her index finger along as indicated by the arrow in the Figure 6.25 on M's sketch*]. And so there's another [*slides her finger again continuing from where she left off previously*].

M: Yeah, and it would be eight back, so that's what you'll have to do.

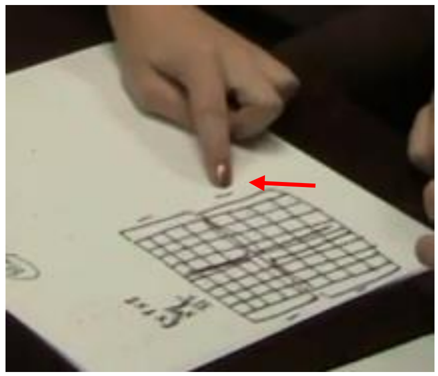


Figure 6.25. Kaylee demonstrates the eight unit-cubes "going back" in the 8×8×8 cubic block.

Initially, as shown in the beginning of Excerpt 6.7, Morgan focused on her sketch of the face of the 8×8×8 cubic block and segmented it into four sections, each of which corresponded to a face of the 4×4×4 cubic block. This attention to the face was consistent with her earlier actions in focusing on the two-dimensional faces of the three-dimensional spatial objects. Lacking was the coordination of those faces with the third dimension to find how many of the 4×4×4 cubic blocks would fit along the third dimension. However, in the moment of explaining her counting and when questioned what her sketch represented, Morgan realized that "you're going to need to double it because this is just one page, so it would be eight." Her way of referring to the additional part of the block as a “page” was similar to her explanations in the 2×2×2 cubic block case, which she said, “there’s four on this side and then there’s four on the other side.”

It is possible that in the course of her explanation, Morgan has assimilated the situation similar to the 2×2×2 cubic block, unitized each 4 unit-cubes along each edge of the 4×4×4 cubic block as a unit, and then partitioning the 8×8×8 cubic block into two “sides” each consisting of four unit-cubes put together. This way of thinking was also demonstrated in her initial engagement in solving the 3×3×3 cubic block case in that she multiplied nine by two because she thought of two “sides” each consisting of nine unit-cubes put together. From the way Morgan counted the four faces of the 4×4×4 cubic block on the first “page” and then doubled that entire “page” I inferred that Morgan was again focusing on the faces.

In addition to Morgan’s activation of her counting activity, my question about what her figure represented, and her reflection on the previous problems in Part One of the Cubic Block Task, it is possible that Kaylee’s interjection triggered Morgan’s adjustment as well. After Kaylee interjected and pointed out how many unit-cubes they needed to “go back,” Morgan was able to visualize and coordinate the third dimension consisting of eight unit-cubes with the faces within the plane formed by the other two dimensions. Therefore, I considered Morgan to have coordinated the third dimension *in activity*.

On the other hand, Kaylee used a different approach in solving the problem. Kaylee recalled that the 4×4×4 cubic block consisted of sixty-four unit-cubes in total, from her previous calculations. When solving for the total number of unit-cubes in the 8×8×8 cubic block, she explained she noticed eight multiplied by eight was sixty-four, the same number of unit-cubes contained in the 4×4×4 cubic block. So, she concluded that there must be eight 4×4×4 cubic blocks contained in the 8×8×8 cubic block. This indicated that Kaylee may have substituted the numerosity of 8×8 by the numerosity of 4×4×4. Although she seemed to operate on the numerosity of the unit-cubes in each cubic block, I hypothesize that Kaylee visualized a composite unit of the 4×4×4 cubic block symbolized in the substitution that could be iterated. This way of reasoning about multiplication was consistent with the way she calculated the total number of unit-cubes in each cubic block using her units-coordinating scheme, through which she produced multiple three levels of units structures.

Although Kaylee relied on the aforementioned numerical calculations, she immediately noticed what Morgan was trying to explain and pointed out what was not taken into consideration; that is, how many unit-cubes were "going back." Even though the four unit-cubes "going back" were not represented in Morgan's two-dimensional sketch, Kaylee was able to re-present the unit-cubes along the third dimension (see Figure 6.25). This was an indicative behavior that Kaylee has coordinated the third dimension with the two-dimensional sketch simultaneously, accounting multiple perspectives.

Making a 4×4×4 Cubic Block into a 5×5×5 Cubic Block

Towards the end of the teaching episode on November 21, I asked Kaylee and Morgan how many unit-cubes they would need to add to a four-edged cubic block to make a five-edged cubic block. After posing the question, I covered the 4×4×4 cubic block that was on the table so the students did not have a cubic block model in their visual field.

Morgan's solution.

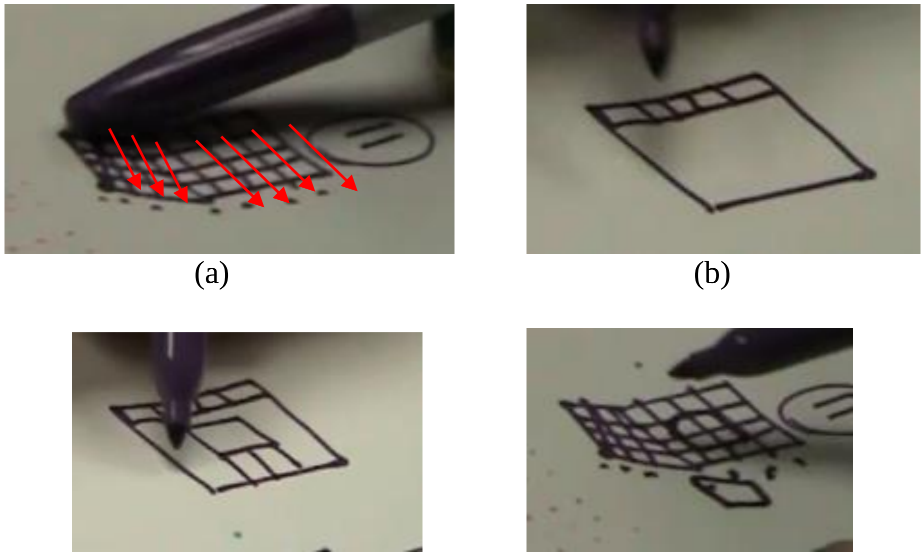
Morgan made an illustration as shown in Figure 6.26, counted along the array of squares, as demonstrated by the dots next to them and concluded 11 to be her answer.

Figure 6.26. Morgan's demonstration of finding how many unit-cubes will need to be added to the 4×4×4 cubic block.

In the following excerpt, Morgan explains her illustration and how she obtained 11 unit-cubes as her answer.

Excerpt 6.8. Morgan explains how she found eleven unit-cubes in total using her illustration.

M: Yeah, it's like, pretty much drawing that [*points to her previous 4×4×4 cubic block sketch (Figure 6.26)*] again, but um, so, to get to the five, you should add one to each side [*sweeps pen across each row of the front face (demonstrated by the red arrows in Figure 6.27 (a))*], each little layer, I guess. And then, um, so, once you, so this is pretty much what we have [*draws a square*]. So I added it to every other side [*adds five small squares inside the square (Figure 6.27 (b))*]. But I didn't add it to the middle yet [*draws a square in the middle of Figure 6.27 (b)*]. And so, I figured, well [*draws in a few line segments connecting the square in the middle with an edge of the biggest square (Figure 6.27 (c))*], once I have… Because I have [*taps with pen again along the rows in Figure 6.27 (a)*] one, two, three, four, five, six, seven right here, and I have to fill in those on that [*draws a square around the middle four (Figure 6.27 (d))*] side [*draws another square looking shape next to her figure (Figure 6.27 (d))*], like, that's facing right there, so I counted like this and eight, nine, ten, eleven [*taps on the four unit squares that she put a box around in Figure 6.27 (d)*].



(a) (b) (c) (d)

Figure 6.27. Morgan explaining her eleven unit-cubes using her illustration of the cubic block.

To get a better grasp of Morgan's counting activity, I asked her to explain again using the physical cubic block model. Excerpt 6.9 shows Morgan's explanation of the 11 unit-cubes using the cubic block model.

Excerpt 6.9: Morgan explains how she found eleven unit-cubes in total using the cubic block model.

M: Yeah. So, I could add the one, two, the five right there [*sweeps her finger along the vertical edge facing her (see black arrow in Figure 6.28 (a))*], and then I could add the other ones right there [*sweeps her finger along the horizontal edge that's adjacent to the edge she just traced (see black arrow in Figure 6.28 (b))*].

K: Oh, I see what you're saying.

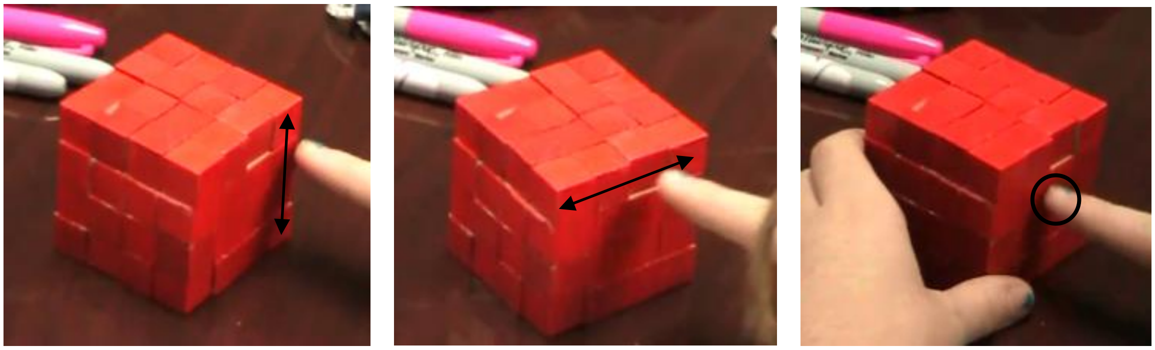
M: But I didn't add them right there [*points to the four middle blocks in the face she was referring to (see black circle in Figure 6.28 (c))*].

K: So, like, these [*pointing to the imaginary piece in the corner that would have been added on*] would be poking out but a gap [*pointing to the circle area in the figure above*] right here.

M: Yeah. Because I didn't add them yet.

K: Because you didn't fill that in yet [*makes hand motion as if she's filling in four cubes in the gap*].

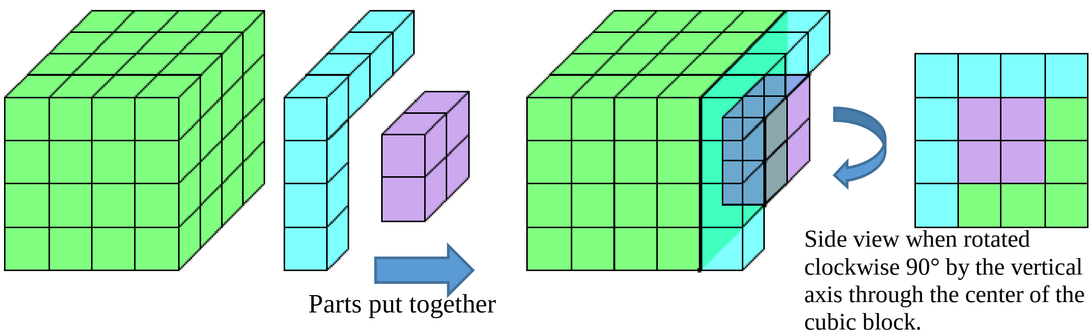
M: Yeah, that's exactly what I did.



(a) (b) (c)

Figure 6.28. Morgan explaining her eleven unit-cubes using the cubic block model.

Figure 6.29 is a model re-generating what it would have looked like if Morgan had actually added unit-cubes as she demonstrated, based on Morgan's explanations in Excerpts 6.8 and 6.9.



(a) (b) (c)

Figure 6.29. A model of Morgan's adding unit-cubes activity to obtain eleven unit-cubes.

Figure 6.29 (a) demonstrates the three parts she wanted to put together in achieving her goal of making a 5×5×5 cubic block from the 4×4×4 cubic block illustrated in green. As shown in Figure 6.29 (a) and (b), Morgan added unit-cubes—the seven unit-cubes in blue—along two adjacent edges in one face. Then, realizing that she missed some in the middle, she wanted to add four more unit-cubes depicted in purple. Figure 6.29 (b) illustrates the parts put together. Figure 6.29 (c) demonstrates the side view when Figure 6.29 (b) is rotated clockwise 90° by the vertical axis through the center of the

4×4×4 cubic block. The blue and purple unit squares would be "poking out" while the green part remains with a gap of one square short of the purple and blue sections.

From her counting activities demonstrated in Excerpts 6.8 and 6.9, I inferred that Morgan did not coordinate multiple perspectives through mental rotation, which would have assisted her to anticipate the results of her actions of adding unit-cubes, as I re-generated in Figure 6.29. Because she focused on her view of the cubic block from one perspective and the two adjacent faces visible from that perspective (Figure 6.26), Morgan did not count the total number of unit-cubes along the front face (sixteen unit-cubes) and transfer it to the other side face. Hence, she first added seven and then four more unit-cubes subsequently. Furthermore, she did not anticipate that what she would have obtained (assuming that she completed covering the entire side face) was a 5×4×4 cubic block.

However, after explaining how she found eleven unit-cubes and looking at the cubic block in front of her, Morgan said that they (at this point, Kaylee also found 11 as her answer) had "the wrong number." Then, Morgan started to count the number of unit-cubes contained in one of the faces of the cubic block. After counting them one by one, starting from one up to sixteen, she exclaimed that they had to add sixteen unit-cubes, explaining "you have to add five more to each side [*pointing to the face facing her in the red cubic block*], each thing." From this, I considered her making in-activity accommodations to her counting activity by rotating the cubic block and transferring one face to another. Adding one cubic block to each "row" along the direction as shown in Figure 6.27 (a) now became covering the entire face on the side. Still, Morgan did not anticipate that this addition would lead to a 5×4×4 cubic block and not a 5×5×5 cubic block.

Kaylee’s solution.

Kaylee calculated the total number of unit-cubes contained in the 5×5×5 cubic block and subtracted 64, the total number of unit-cubes contained in the 4×4×4 cubic block, as shown in Figure 6.30. She made an error in her calculation and calculated 5×5×5 to be equal to 75 instead of 125, resulting in 11 as her answer. Even after Morgan explained her solution in Excerpts 6.8 and 6.9, pointed out how they had the “wrong number,” and suggested 16 as the new answer, Kaylee remained confused. Based on her calculations, the answer had to be eleven unit-cubes.

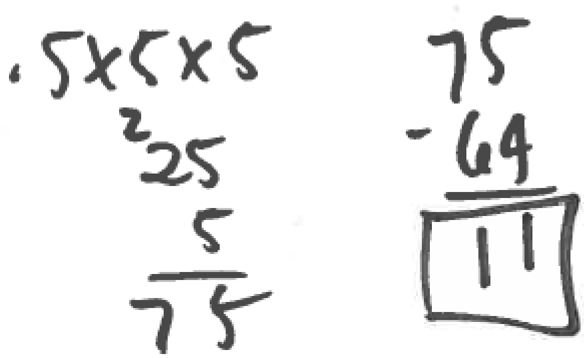


Figure 6.30. Kaylee’s calculation of finding how many unit-cubes will need to be added to the 4×4×4 cubic block.

Because both students were perturbed by the situation and their answer, I suggested they make the 5×5×5 cubic block from the 4×4×4 cubic block, providing a pile of loose unit-cubes. Morgan started to stack unit-cubes along the 4×4×4 cubic block, resulting in a figure shown in Figure 6.31.

Figure 6.31. Morgan's building of the 5×5×5 cubic block starting from the 4×4×4 cubic block.

After Morgan added the four unit-cubes along the top of the 4×4×4 cubic block, she explained, "then you do the rest of these going up [adding more cubes along the face in the front (see Figure 6.31)]. No, then we'll have to do the entire thing." While Morgan kept stacking the unit-cubes, Kaylee explained what she was thinking about the situation. The following excerpt starts with Kaylee's explanation of how she wanted to add the unit-cubes to the 4×4×4 cubic block.

Excerpt 6.10: Kaylee adds faces to the 4×4×4 cubic block.

K: I was thinking, like, you only have to add one on here [*puts her hand in front of the face M is stacking the blocks on (Figure 6.32 (a))*] and one on here [*moves her hand to the face adjacent to that she was referring to in Figure 6.32 (a) (see Figure 6.27 (b))*], like, one more face [*repeats the hand motions Figure 6.32 (a) and (b)*], I guess.

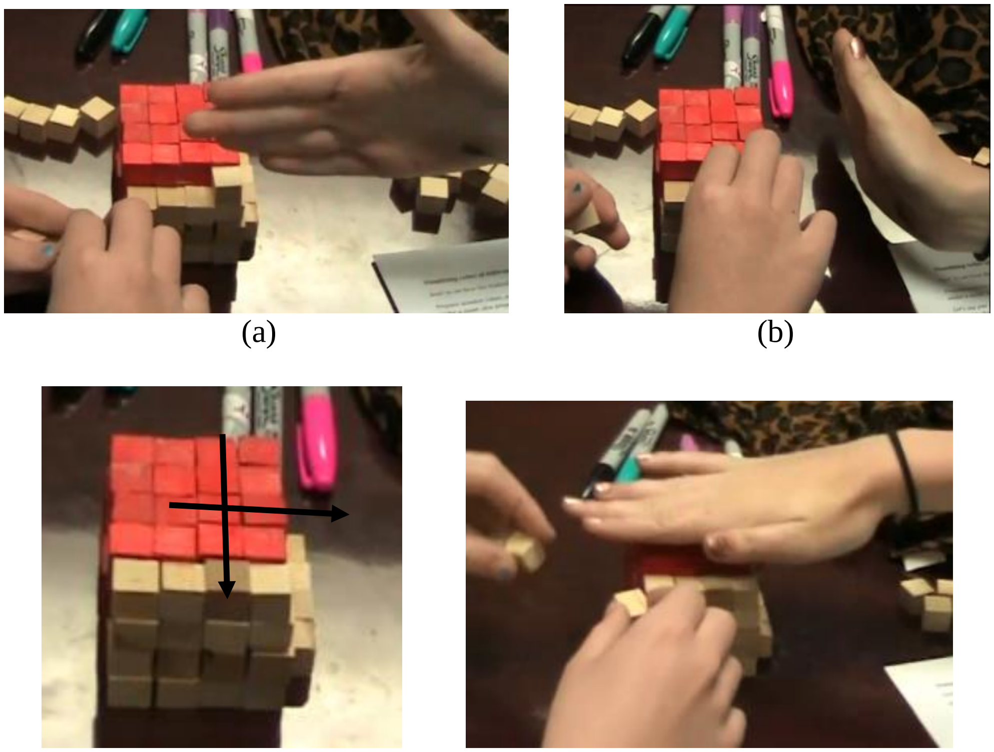
M: So it'd be thirty-two.

K: It's [going to] be this long and this long [*sweeps hand along the black arrows depicted in Figure 6.32 (c)*]

T: So, how tall is this? [*Sweeps index finger along the height of the cubic block in front of them*].

M&K: Four.

K: So, you'd have to add a layer on top [*puts hand on top of the cubic block as shown in Figure 6.32 (d)*].



(a) (b) (c) (d)

Figure 6.32. Kaylee demonstrates adding faces to the 4×4×4 cubic block.

[*Continued*.]
M: [*Simultaneously*,] on top [*starts stacking blocks on the top face*].
K: Oh my goodness.
[*Bell rings*.]
M: Well, it'd be thirty-two, I think.
T: Thirty-two?
M: Well, thirty-three, thirty-four… [*Starts counting the additional new layer on top*].
K: [*Still looks confused and looks at her paper*]
M: Thirty-six. I think it'd be thirty-six.

As demonstrated in Figures 6.31 (a) and (b), Kaylee thought about adding faces to the 4×4×4 cubic block to make it five unit-cubes long along two of the dimensions, as shown in Figure 6.32 (a). This would have resulted in a 5×5×4 cubic block. When I asked how tall the cubic block would be after adding the two faces, both Kaylee and Morgan noticed the cubic block would only be four units high, which led to Kaylee's decision in adding another face on the top, as shown in Figure 6.32 (b). However, even after my directing question, Kaylee still did not seem confident about their answer. Because the bell rang and we ran out of time, we decided to re-visit this problem in our next teaching episode.

Making a 3×3×3 Cubic Block out of a 4×4×4 Cubic Block

After Thanksgiving break, on December 6, I suggested starting with a different problem first. Instead of continuing the previous problem, I changed the problem to finding how many unit-cubes they would need to take off of the 4×4×4 cubic block in order to make a 3×3×3 cubic block. The reason I posed the new question was because I hypothesized that the cognitive demand of building up to a bigger cubic block in re-presentation was heavier than taking off unit-cubes from a cubic block that was already constructed. Therefore, I anticipated that the taking off question would still allow me opportunities to observe the students' reasoning with the cubic blocks but also provide a cognitive entry point for the students to re-visit the building up problem.

Kaylee's solution.

Figure 6.33 is Kaylee's written response for finding how many unit-cubes should be taken off from the 4×4×4 cubic block to make a 3×3×3 cubic block. Figure 6.33 shows Kaylee's initial answer of 25, obtained from adding 16 and 9 together.

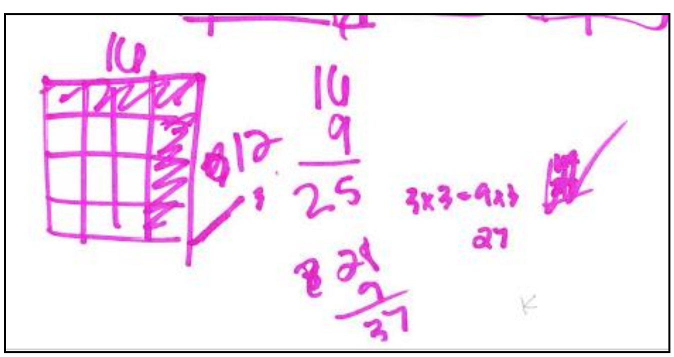


Figure 6.33. Kaylee's written response for making a 3×3×3 cubic block from a 4×4×4 cubic block.

I asked Kaylee to explain her answer of twenty-five unit-cubes. Kaylee started explaining how she found twenty-five unit-cubes but during her explanation, she changed her mind to thirty-seven unit-cubes in total, as observed in Excerpt 6.11.

Excerpt 6.11: Kaylee explains how she found the access number of unit-cubes.

K: Okay, so, I just looked at like from like the one side. And so, I did my four by four, and I knew that you have to take off the top [*points to the top four blocks in her sketch that are shaded (Figure 6.33)*] to get the length of the top, like the top square [*covers the top face of the 4×4×4 cubic block*] like the whole four by four square on the top and you just take it off. And so, now you'll have a three by four, and so if you had it like this [*covering the four squares she just talked about taking off, making it look like a 3×4 rectangle*] it'd be like a [*pauses as if she just realized something in her sketch*], oh. Be a three by four [*starts adding a line segment to the bottom right corner of the square (Figure 6.33) that illustrates the third dimension*]. I just set it to be a three by three, but there's like a three by four thing so it would be twelve [*crosses out her "9" and writes "12" next to the square*] so that would be twenty-eight [*writes "28" on her paper*].

T: So…

K: I don't know.

T: What is this sixteen?

K: It's like, if you were to look at the cube, the top four by four square [*hovers hand over the table as she did before to demonstrate the top face of the 4×4×4 cubic block*] or whatever at the top.

T: Oh, okay, the layer.

K: So you have to take all that off.

T: So you take that all off.

K: [*Covers the first row of her 4×4 square (the top layer) with her hand in Figure 6.33 as to demonstrate it's gone*.]

T: And you said because you had this left [*points to the leftover squares in her sketch (Figure 6.33)*]…

K: A three and then by four [*referring to the line segment she added to the bottom right corner of her square that demonstrates the third dimension*] cuz the length.

T: Mm-hmm, you're going down in your head, okay.

K: But then [*pauses and sighs*].

T: So, you take sixteen off, and then twelve off.

K: [*At the same time as T,*] twelve off. But then I don't know about the width. I might have to take one off from the back, I think.

T: Which back do you mean?

K: Like, the back of the cube [*hand motions the face opposite to her*] because I think it would be like three by four, like, it would be three by three on the front [*hand motions the front face in front of her*] but then by going back [*slides hand from front to back of the cubic block*] like four layers.

T: Four, okay.
K: I think I have to take off one more, three by three, which would be nine. Which gets me at thirty-seven. Is that what you got [*asking M*]?
M: Mm-hmm.
K: Okay.

Figure 6.34 is a model of Kaylee's re-presentation of the cubic block as she took off the unit-cubes as she explained in Figure 6.33 and Excerpt 6.11.

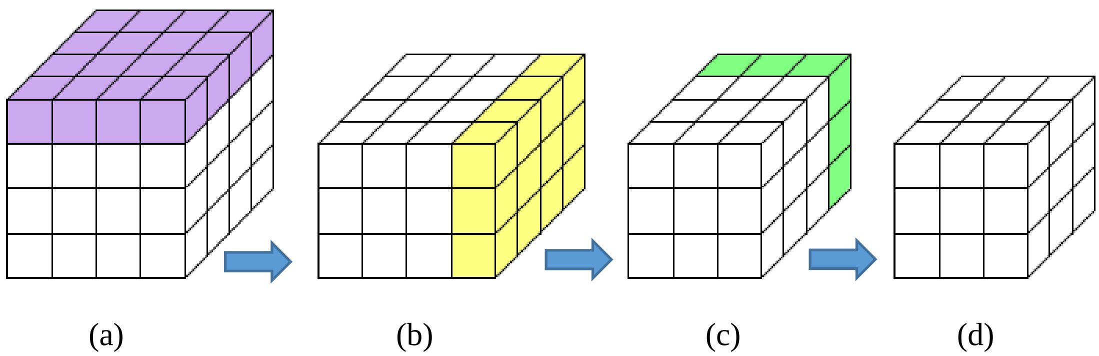


Figure 6.34. A model of Kaylee's taking off unit-cubes from a 4×4×4 cubic block to obtain a 3×3×3 cubic block.

Inferring from her inscription of 16 on top of her 4×4 square in Figure 6.33 and from her explanation in Excerpt 6.11, Kaylee considered "the whole four by four square on the top and you just take it off." That is, looking at the 4×4 square in Figure 6.33, she coordinated the four unit squares on the top and projected it through all four vertical layers of the 4×4×4 cubic block and knew that all 16 unit-cubes will need to be taken off (see Figure 6.34 (a)). Next, Kaylee explained, "I just set it to be a three by three, but there's like a three by four thing so it would be twelve." Here the critical moment is when she paused and added a line segment to the bottom right corner of the square going towards the back of the unit-cube as demonstrated in Figure 6.33.

Inferring from her illustration in Figure 6.33 and her explanation, initially Kaylee considered taking off 3×3 unit-cubes because there were three remaining unit squares that she shaded. In that process, she originally projected the three unit squares into three layers but soon realized there should be four layers in total; hence, resulting in taking off a three by four rectangular layer, consisting of 12 unit-cubes (see Figure 6.34 (b)). After repeating her explanation, Kaylee independently suggested "I might have to take one off from the back" meaning that she would take one of the four vertical layers off to complete the 3×3×3 cubic block (see Figure 6.30 (c)). Her explanation "it would be three by three on the front but then by going back like four layers" demonstrates her projection of the front face along the four vertical layers. So, Kaylee decided to take off nine additional unit-cubes, resulting in 37 unit-cubes in total.

An outstanding aspect of Kaylee's activities is that she used a two-dimensional sketch of a 4×4 square to reason about a 4×4×4 cubic block and reducing it to a 3×3×3 cubic block. This indicates Kaylee's coordination of multiple perspectives and insertion of units (units-coordinating scheme) along various spatial dimensions, supported by her coordination of multiple perspectives and her FR coordinating scheme. To elaborate, taking the four unit squares on the top row of her 4×4 square as a unit, she projected that unit of four unit squares and inserted them into the subsequent vertical layers of the cubic block. In order to anticipate how many units she would need to insert the unit of four squares into, Kaylee must have coordinated her perspective looking at the front of the cubic block with her perspective looking from the top of the cubic block.

Again, when taking the four units of three unit squares (illustrated in yellow in Figure 6.34 (b)), Kaylee had to re-orient the two perspectives she was taking. That is, it required her coordinating her perspective of looking at the cubic block from the front and then from the side. Although initially Kaylee anticipated three units of three unit squares, in reflection of her activity, she re-organized the cubic block taking the new perspectives and considered four units of three unit squares. This was demonstrated in her action in adding the line segment starting at the bottom right corner of the square going towards the back of the imaginary cubic block. Then again reflecting on her mental actions so far, she anticipated the result of her actions to be a 3×3×4 cubic block; therefore, she took an additional layer of 3×3 unit-cubes off of the cubic block. All of these actions were carried out using her illustration in Figure 6.33; Kaylee did not have the physical cubic block model in her perceptual field. Ultimately, Kaylee has generated a re-presentation of the cubic block to reason upon in finding the number of unit-cubes to be taken off of the 4×4×4 cubic block to reduce it to a 3×3×3 cubic block.

Although the goal of this activity was not to re-produce each individual unit-cube mentally and count the total number of unit-cubes or identify certain unit-cubes in the interior of the cubic block, the operations that were involved in her reversible decomposing scheme were used in solving this new problem. First, Kaylee was able to rotate her perspective and coordinate multiple perspectives by bringing forth the immediate past result of one perspective to a new perspective. Second, Kaylee operated as if she peeled off layers one at a time but at the same time she kept in mind what the whole was in order to check the resulting size of the block after taking each layer off. This involved her unitizing and disembedding operations and being able to reverse the action of taking off. Finally, her FR-coordinating scheme guided her insertion of units (units-coordinating scheme) along various spatial dimensions.

Morgan's solution.

Figure 6.35 is Morgan's written response for finding how many unit-cubes should be taken off from the 4×4×4 cubic block to make a 3×3×3 cubic block.

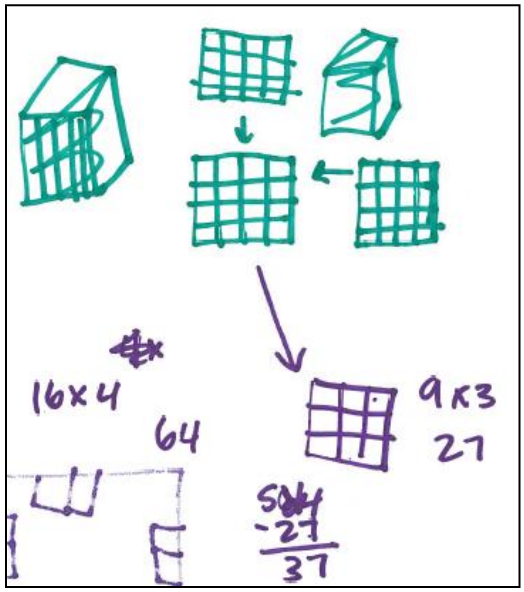


Figure 6.35. Morgan's written response for making a 3×3×3 cubic block from a 4×4×4 cubic block.

As shown in Figure 6.35, Morgan calculated the total number of unit-cubes for each cubic block and subtracted one from the other. Morgan explained her solution as shown in the following excerpt.

Excerpt 6.12: Morgan explains how she found the additional unit-cubes.

M: Okay, so, I did, how many total blocks for the entire four [*pointing to the 4×4 square she had sketched (Figure 6.31)*] so I did, I know there are sixteen on each side and there are four sides, well, there's like four layers [*makes hand motion illustrating the four layers like Katie did before*]. And so, I did sixteen times four and that's sixty-four. Then, I looked at the three [*points to the 3×3 square she had sketched (Figure 6.31)*], the three block, whatever, and there's nine on each thing, on each side and there's three layers. So, nine times three equals twenty-seven. Then sixty-four minus twenty-seven equals thirty-seven.

T: I see. So, you were subtracting the number of cubes.

M: Yes.

Although Morgan subtracted the total number of unit-cubes in the 3×3×3 cubic block from the total number of unit-cubes in the 4×4×4 cubic block using numerical calculations, there was a significant change in the way Morgan was explaining the composition of each cubic block. Previously, in Excerpts 6.3 and 6.4, Morgan referred to adjacent exterior layers or faces of each cubic block as “sides”. However, in Excerpt 6.12, she demonstrated a re-organization of the cubic blocks in which now the “sides” each containing $n \times n$ unit-cubes consisted the layers

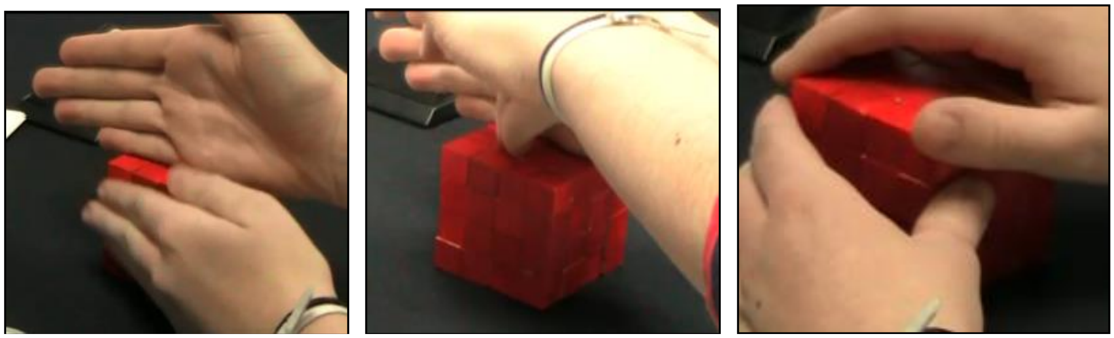
such that when iterated *n* times formed the cubic block. Morgan’s explanation “I know there are sixteen on each side and there are four sides, well, there’s like four layers” and “there’s nine on each thing, on each side and there’s three layers” indicated this re-organization. It was likely that Morgan assimilated Kaylee’s method of consistently decomposing the cubic block into square-shaped layers. Assimilating this decomposition of the cubic blocks into square-shaped layers, Morgan explained why she multiplied 4×4×4 and 3×3×3 to find the total number of unit-cubes contained in each cubic block.

To further push her to reason about the 37 unit-cubes without using her numerical calculation, I asked Morgan to identify the 37 unit-cubes using the 4×4×4 cubic block model. I suggested that the two students work together. Excerpts 6.13 and 6.14 show Morgan explaining 37 unit-cubes using the cubic block. These discussions occurred after Kaylee had explained her sketch in Excerpt 6.11.

Excerpt 6.13. Morgan explains the 37 unit-cubes using the 4×4×4 cubic block model.

M: Here’s the four thing. There’s sixteen on this little layer [*makes hand motion as to slice off one side face of the cubic block (Figure 6.36 (a))*]. There’s sixteen on this little layer [*moves hand to the next vertical layer (Figure 6.36 (b))*], there’s sixteen on this one, there’s sixteen on this one [*continues to move her hand to point at each layer*]. Then, if you… took off twenty-seven, so like sixteen right [here] [*puts hand on the top layer of* *the cubic block (Figure 6.36 (c))*]… Sixteen… How many times does sixteen go in twenty-seven? Once… I mean thirty-seven.

K: [*Mumbles as she's trying to divide 37 by 16.*]



(a) (b) (c)

Figure 6.36. Morgan demonstrates making the 4×4×4 cubic block into a 3×3×3 cubic block.

The beginning of Excerpt 6.13 suggests corroborating indications of Morgan's reorganization of the cubic block into layers. Moreover, from her hand motions, she has decomposed the cubic block into vertical layers, similar to Kaylee's decomposition of the cubic block, as illustrated in Figure 6.7. However, Morgan seemed to be confused by the number of unit-cubes she counted and what to do with them. More specifically, Morgan thought about dividing 27 (the number of unit-cubes in the 3×3×3 cubic block) by 16 (the number of unit-cubes on one of the exterior layers (face) of the 4×4×4 cubic block). Then, Morgan wanted to divide 37 by 16. Excerpt 6.14 continues with Morgan making the 4×4×4 cubic block into a 3×3×3 cubic block.

Excerpt 6.14: Morgan explains the 37 unit-cubes using the 4×4×4 cubic block model (Continued).

M: Twenty-seven or thirty-seven? You'll have to take off twenty-seven. So, sixteen, take off this sixteen [*she's pointing to the same top layer as shown in Figure 6.36 (c)*] so you only have the [*holds the cubic block at the top as if she's covering the top face and turns it around to see the leftover (Figure 6.37 (a))*] the four, like the three four layers.

T: Mm-hmm.

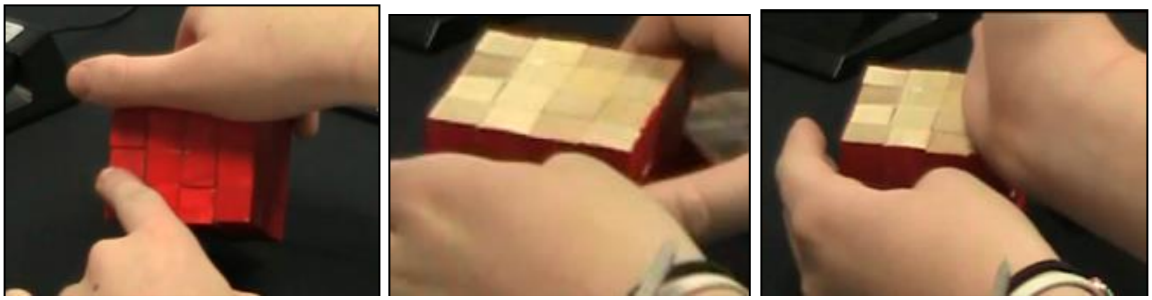
M: Like, the four by…

T: Let's… [*Takes off one layer off of the cubic block.*] Okay, so you took one layer off.

M: Okay, so then, all you have to do is just take the… [*Turns the leftover block around to a position to better fit her description (the top layer taken off) (Figure 6.37 (b))*] this side off.

T: What side are you trying to take off?

M: Like this side [*places hand on the block to demonstrate the cut*], this side right here [*Figure 6.37 (c)*].



(a) (b) (c)

Figure 6.37. Morgan continues to demonstrate her solution in making the 4×4×4 cubic block into a 3×3×3 cubic block.

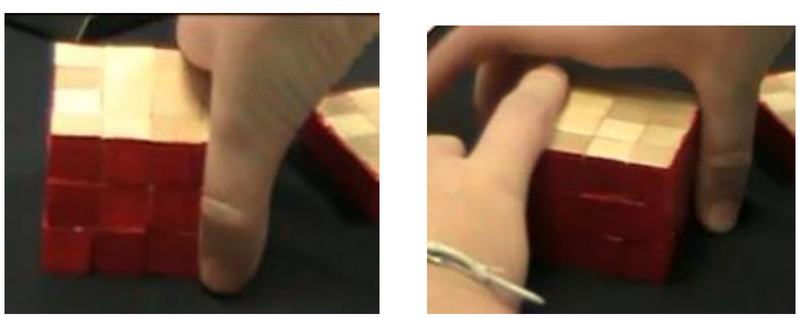
[*Continued*.]

T: Okay. I think it's too glued, stuck to it. It might not… [*Tries to take off the layer for her but fails because the layers are stuck together*.] Let's say we took it off.

M: So, like, that's not there anymore [*puts hand on the layer she wanted to take off to cover it (Figure 6.38 (a))*]. Then you have your three squares.

K: No, but this is three by four [*turns the cubic block around to show her the four blocks on one of the edges*].

M: I mean, then you would take off this one, too [*puts finger on top of the block to cover the layer she's referring to (Figure 6.38 (b))*].



(a) (b)

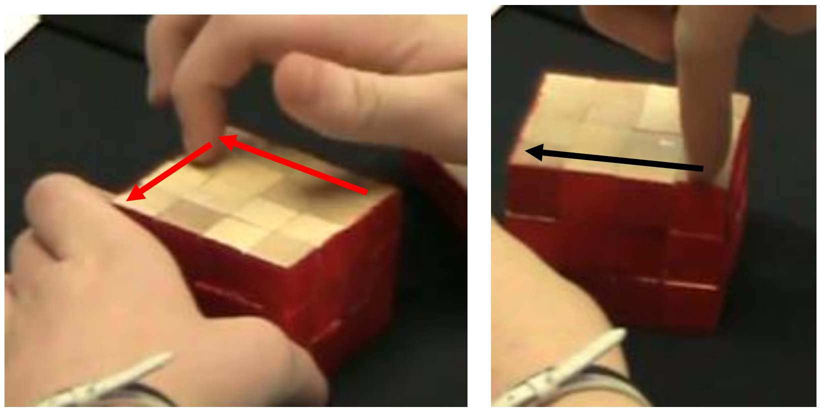
Figure 6.38. Morgan continues again to demonstrate her solution in making the 4×4×4 cubic block into a 3×3×3 cubic block.

[*Continued*.]

T: Mmm…

K: Which is nine.

M: So, sixteen [*taps on the top face that was taken off earlier*], seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three [*counts along the arrows shown in Figure 6.39 (a) then turns the cube around to the side*], umm… Twenty-four, twenty-five, twenty-six, twenty-seven… [*Counts along the blocks along the arrow shown in Figure 6.39 (b)*]. Wait, what?



(a) (b)

Figure 6.39. Morgan counting the number of unit-cubes.

Although Morgan occasionally lost track of the numerical calculations along the way and Kaylee interjected once in the process (Kaylee pointed out how the second part (Figure 6.38 (a)) was in fact a “three by four,” thus, twelve unit-cubes), using the physical cubic block model, Morgan demonstrated similar actions as Kaylee’s. That is, Morgan took off parts of the 4×4×4 cubic block to make the 3×3×3 cubic block in a similar manner as Kaylee did as modeled in Figure 6.34. Therefore, I concluded that Morgan was able to reason about the 37 unit-cubes in activity of handling the physical model of the cubic block and enacting parts of the partitioning operations. However, Morgan’s counting activity was constrained by the physical material and her actions were dependent on the actual blocks. Also, when counting the number of unit-cubes she took off, Morgan relied on counting one unit-cube at a time, instead of coordinating units of units as Kaylee did. After Excerpt 6.13, Kaylee walked Morgan through the taking off of parts of the cubic block one at a time helping her keep track of the number of unit-cubes along the way.

Making a 4×4×4 Cubic Block into a 5×5×5 Cubic Block (Re-visited)

Based on my observations from the task that required the students to reduce the 4×4×4 cubic block into a 3×3×3 cubic block, I had formed hypotheses that I wanted to test further by re-visiting the earlier quest of making a 4×4×4 cubic block into a 5×5×5 cubic block. More specifically, I wanted to test my hypothesis that what Morgan counted and what she took off the blocks was restricted to the physical blocks she could manipulate in the moment; moreover, Morgan had to carry out the activities using the cubic block model to keep track of her counting. In addition, I wanted to explore whether Kaylee could generalize her counting activity in re-presentation to a different case of the 4×4×4 cubic block using the operations she used in reducing the 4×4×4 cubic block..

Morgan's solution.

When I asked the students to re-visit the aforementioned extension question, Morgan multiplied 25 by 5 to find 125 and subtracted 64 to obtain 61. In other words, she used a similar approach as she did before by subtracting the total number of unit-cubes in the 4×4×4 cubic block from the total number of unit-cubes in the 5×5×5 cubic block. I asked Morgan if she could explain why 61 might make sense in terms of the situation, by producing an illustration of what she was thinking. Morgan generated an illustration as shown in Figure 6.40. Although Morgan's illustration in Figure 6.40 suggested yet another corroborating instance of her re-organization of decomposing the cubic blocks into layers, her illustration did not entail any of the counting activities she carried out in Excerpts 6.13 and 6.14.

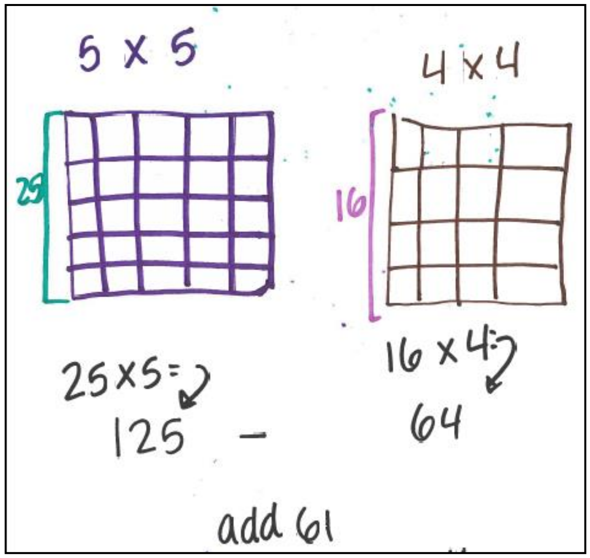


Figure 6.40. Morgan’s illustration of making a 5×5×5 cubic block from a 4×4×4 cubic block.

Kaylee’s solution.

Given the same question, Kaylee generated an illustration as shown in Figure 6.41.

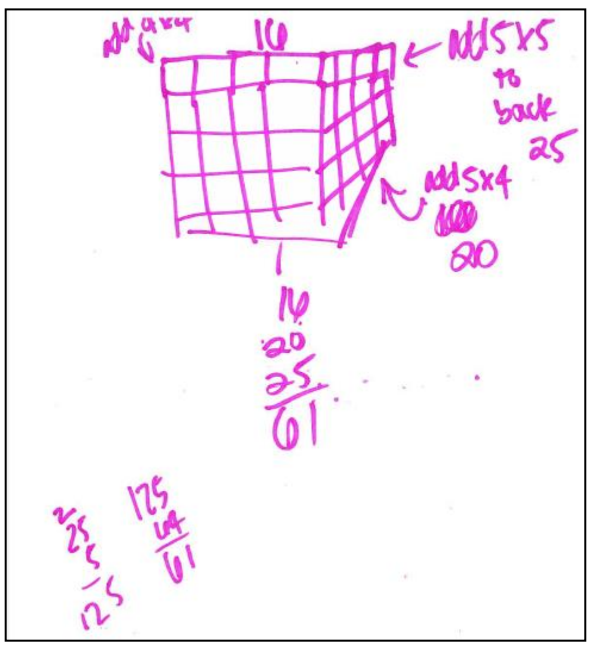


Figure 6.41. Kaylee’s illustration of making a 5×5×5 cubic block from a 4×4×4 cubic block.

Before she generated her illustration in Figure 6.41, Kaylee made hand motions of layers in the air around an imaginary block. Then, she produced the partial sketch of the block and made the inscriptions of adding onto the block as shown in Figure 6.41. As demonstrated in Figure 6.41, Kaylee used her partial illustration of the cubic block to keep track of her counting activity as she was building the cubic block in re-presentation. That is, without relying on the physical cubic block model, Kaylee was able to mentally carry out actions of building onto the block. The way she did not hesitate or stop to think in generating her illustration suggested that Kaylee had already generated the cubic block mentally. Therefore, based on her illustration, it seemed as though Kaylee assimilated the new case of building up the cubic block to her scheme of systematically taking off layers of the existing block and reversed the actions to make a cubic block larger than the given one.

Morgan and Kaylee discuss their solutions using the cubic block model.

Morgan explained how she subtracted the total unit-cubes from each other to obtain the difference in the number of unit-cubes. Although Morgan's solution was reasonable, it did not provide me an opportunity to observe her reasoning about the number of unit-cubes in the spatial context. So, I wanted to further investigate Morgan's reasoning about the extension of the cubic block. Also, I wanted to hear Kaylee's explanation of her generation of Figure 6.41 to test my hypothesis. Therefore, I provided the model of the 4×4×4 cubic block and suggested that they demonstrate their answers using the actual model. When I asked Morgan to enact the actual adding activity, she started to make sense of the problem spatially. Excerpts 6.15 and 6.16 involve Morgan and Kaylee explaining the 61 unit-cubes using the cubic block model.

Excerpt 6.15. Morgan and Kaylee explain the 61 unit-cubes using the 4×4×4 cubic block model.

M: Mmm… You add sixteen onto this side [*points to one of the faces on the side of the cubic block*].

T: Okay, I think that's what you were saying here [*pointing to the part in K's sketch where she adds 4×4=16*].

K: Yeah.

T: Okay, so let's… [*Takes out loose unit blocks so that M can make the 4×4 added on the block*.] Say we added sixteen.

M: So, we added sixteen. So it would end right there [*Figure 6.42*]. Then you would need to add sixteen [*swipes finger around the top face shown in Figure 6.42*] and add four [*points to the extra four that would be needed on the top face due to the newly added face earlier*], which equals… Twenty. [*Looking at K*] you know what I'm saying?

Figure 6.42. Morgan's demonstration of adding unit-cubes to the 4×4×4 cubic block model.

[*Continued*.]

K: Make a five by four on the top [*pats the top face of the cubic block*]? So, it would be five [*runs index finger along the five-unit-long edge (with the newly added blocks)*] by four [*runs index finger along the four-unit-long edge*]. But it would also be five up [*runs index finger up along the edge that shows the height of the cubic block*]. So, if you were to have this extended [*referring to the face that was added first (where M is holding up the blocks)*]…

M: Alright, so look…

K: And you just added to the top.

M: Okay, so you have these going all the way to the top. So you added your sixteen. And you added your sixteen [*builds the wall (Figure 6.43) and writes "16" on her paper to keep track*.] And now you have to do… I already have five on this way [*pointing to the dimension with five unit blocks*] so you need five going this way [*moving index finger vertically along the different "layers"*], so you have to add one more layer to the top…

Figure 6.43. Morgan's demonstration of adding 16 unit-cubes to the 4×4×4 cubic block model.

[Continued.]
T: And that would be?
M: Sixteen plus these four [*again, refers to the top face plus the additional four blocks*] which is twenty.
T: Okay.
M: So, twenty plus sixteen… [*Looks at K*]
K: Thirty-six.
M: Thirty-six.

Although it seemed like Kaylee and Morgan did not communicate well with each other in Excerpt 6.15 at the beginning, it seemed as though their reasoning of adding 16 unit-cubes along one face and then 20 unit-cubes along the top face of the cubic block was compatible. As shown in Excerpt 6.15, Morgan was able to reason that she would need to add 20 additional unit-cubes to the top of the cubic block by anticipating the result of her first activity—adding 16 unit-cubes to the front face. However, in moving forward, it seemed as though physically carrying out the activity of adding the unit-cubes to the cubic block was helpful. It supported her keeping track of how and what she had added so far. While Kaylee referred to each additional part by the dimensions (e.g., four by four, four by five), Morgan kept track of the number of unit-cubes along each part differently. More specifically, when finding how many unit-cubes that needed to be added to the top of the cubic block, Morgan first took the 16 unit-cubes that would cover the top face and then added four more unit-cubes.

In Excerpt 6.16, Morgan and Kaylee continue their discussion.

Excerpt 6.16. Morgan and Kaylee explain the 61 unit-cubes using the 4×4×4 cubic block model (Continued).

T: Okay. Thirty-six. What's left?
M: You need to do… [*Continues to place blocks on the top face.*]
K: Something on this side [*places hand on the face on the side completing the third dimension*.] Yeah, because that takes the bottom…
M: [*Finally completes stacking the blocks (Figure 6.44)*].

Figure 6.44. Morgan's demonstration of adding more unit-cubes to the 4×4×4 cubic block model.

[*Continued*.]
T: Okay, so is this complete?
M: No. [*Pauses*.]
K: You need to have one of these [*hand motions to either side of the block as shown in Figure 6.45 (a)*.]
M: Yes it is, it's complete.
K: No, it's not. The bottom is four. [*Points to the edge on the bottom with four units*.]
M: But…
K: Look at the front, look at the front [*moves her index finger to the face facing M and counts along the bottom edge*], one, two, three, four. You have to add one more on this side [*again hand motions another layer on the side of the block using her right hand (Figure 6.45 (b)*.]

(a) (b)

Figure 6.45. Kaylee explaining how to add onto the 4×4×4 cubic block to produce a 5×5×5 cubic block.

[*Continued*.]
M: Oh, one more layer.
K: That would be twenty-five, like a five by five [*runs finger along the five by five blocks on the side face of the block.*]
T: Okay, so that would be…
M: [*Starts stacking blocks onto the cubic block*,] okay, so we don't have to add all.
K: So, we had thirty-six. Plus twenty-five…
M&K: [*Simultaneously,*] sixty-one.
K: [*Taps on table*].
M: Boom!

Towards the end of Excerpt 6.16, an interesting interaction occurred when Morgan thought the model was complete, when she had a 5×5×4 cubic block (Figure 6.44) in front of her. Although the face facing her only had four unit-cubes along the bottom and the total number of unit-cubes they have added so far was 36 and not 61 (the answer they produced earlier), it did not seem to occur to Morgan that the cubic block model was incomplete. Kaylee then pointed out how the cubic block was incomplete and what additional action needed to be made. Once Kaylee redirected Morgan's attention to the number of unit-cubes along the third dimension, Morgan realized "one more layer" needed to be added. As such, when asked to reason through her answer of 61 unit-cubes using the cubic block model, Morgan was able to build up parts of the cubic block and follow Kaylee's guidance in counting all the necessary unit-cubes needed to extend the given cubic block. However, this activity was not explicit in her illustration and involved redirecting from Kaylee. Therefore, it is not possible to impute a scheme to Morgan that would have allowed her to independently and consistently use for reducing or extending a given cubic block. Moreover, these actions were restricted to the particular occasion of extending the 4×4×4 cubic block to the 5×5×5 cubic block.

On the other hand, Kaylee's explanations in Excerpts 6.15 and 6.16 corroborated my hypothesis that Kaylee assimilated the new case of building up the cubic block using her scheme of systematically taking off layers of the existing block and reversed the actions to make a cubic block larger than the given one. Figure 6.46 is a model generated to demonstrate what it would have looked like if Kaylee had actually added the unit-cubes as she illustrated step by step in Figure 6.41 and as she explained in Excerpts 6.15 and 6.16.

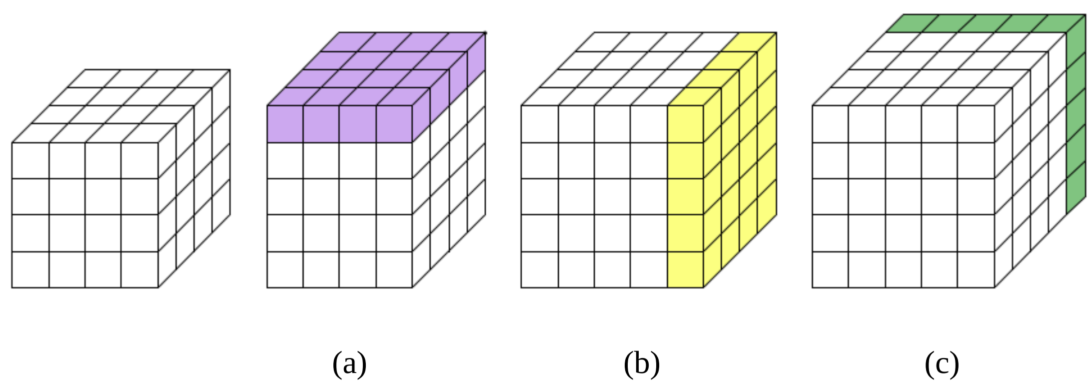


Figure 6.46. A model of Kaylee's adding unit-cubes from a 4×4×4 cubic block to obtain a 5×5×5 cubic block.

As modeled in Figure 6.46, I interpreted her sketch in Figure 6.41 and her explanations in Excerpts 6.15 and 6.16 to indicate that Kaylee added a layer of unit-cubes onto the top face (Figure 6.46 (a)), then added a layer of unit-cubes onto the side face (Figure 6.46 (b)), and finally added a layer of unit-cubes onto the face in the back (Figure

6.46 (c)). Her activity of adding onto the cubic block was analogous to her activity of taking off of unit-cubes from a 4×4×4 cubic block to obtain a 3×3×3 cubic block, as modeled in Figure 6.34.

This consistent way of adding or taking off layers of unit-cubes in re-presentation is why I conclude her counting actions to be systematic. In order to find how many unit-cubes were in each of the parts (either to be added or taken off), Kaylee anticipated what the addition or subtraction along one spatial dimension would result in along the others. For example, after adding the layer of 4×4=16 unit-cubes onto the top of the 4×4×4 cubic block, Kaylee was able to coordinate the new dimension along the height of the cubic block with the existing dimension along the width of the cubic block and reasoned that the layer of unit-cubes she would add to the side of the cubic block consisted of 5×4=20 unit-cubes. This powerful way of reasoning suggested that Kaylee's reasoning about the cubic blocks was guided by her structuration of the cubic blocks using her FR coordinating scheme. Moreover, her accounting of the position of each unit-cube along all three spatial dimensions reflected the multiplicative positioning of each fish in the Fish Tank Task and her construction of a reversible decomposing scheme when reasoning with the cubic blocks.

Summary of Cubic Block Task Part Two

When engaging in the three extension questions of the 4×4×4 cubic block—how many of them are contained in a 8×8×8 cubic block, how many unit-cubes need to be taken off to obtain a 3×3×3 cubic block, and how many unit-cubes need to be added to obtain a 5×5×5 cubic block—I observed consistent differences in Morgan's and Kaylee's reasoning.

The number of 4×4×4 cubic blocks contained in a 8×8×8 cubic block.

To find the total number of 4×4×4 cubic blocks contained in an 8×8×8 cubic block, Morgan drew a sketch of a face of the an 8×8×8 cubic block. Morgan's attention to the two-dimensional space was consistent with her earlier actions in Part One of the Cubic Block Task in that it lacked the coordination of those faces with the third dimension. However, when the teacher asked her to explain her sketch, in explaining her counting and reflecting on her sketch, Morgan realized that there needed to be two pages of her sketch to make the entire 8×8×8 cubic block. After Kaylee interjected and pointed out how many unit-cubes they needed to "go back," Morgan was able to visualize and coordinate the third dimension consisting of eight unit-cubes with the faces within the plane formed by the other two dimensions. Therefore, I considered Morgan to have coordinated the third dimension *in activity*.

Kaylee seemed to have operated numerically, by substituting the numerosity of 4×4×4 into the numerosity of 8×8. However, in that process, I hypothesized that Kaylee visualized a composite unit of the 4×4×4 cubic block symbolized in the substitution that could be iterated. This way of reasoning about multiplication was consistent with the way she calculated the total number of unit-cubes in each cubic block using her units-coordinating scheme, through which she produced multiple three levels of units structures. Although Kaylee relied on the aforementioned numerical calculations, she immediately noticed what Morgan was trying to explain and pointed out what was not taken into consideration; that is, how many unit-cubes were "going back." Even though the four unit-cubes "going back" were not represented in Morgan's two-dimensional sketch, Kaylee was able to re-present the unit-cubes along the third dimension (see Figure 6.25).

This was an indicative behavior that Kaylee has coordinated the third dimension with the two-dimensional sketch simultaneously, accounting multiple perspectives.

Extending or reducing the 4×4×4 cubic block to a 5×5×5 cubic block or a 3×3×3 cubic block.

Morgan relied on numerical calculations by subtracting the total number of unit-cubes contained in one from the other to find the number of lacking or excess unit-cubes. Although Morgan used numerical calculations, there was a significant change in the way Morgan explained the composition of each cubic block. She demonstrated a re-organization of the cubic blocks in which the "sides" each containing $n \times n$ unit-cubes consisted the layers such that when iterated *n* times formed the cubic block. It was likely that Morgan assimilated Kaylee's method of consistently decomposing the cubic block into square-shaped layers. Assimilating this decomposition of the cubic blocks into square-shaped layers, Morgan explained why she multiplied 4×4×4 and 3×3×3 to find the total number of unit-cubes contained in each cubic block.

When I asked Morgan to justify her answers in a different way, Morgan focused mainly on the faces of the cubic blocks from which I inferred that Morgan did not coordinate multiple perspectives through mental rotation, which would have assisted her to anticipate the results of her actions of adding or subtracting unit-cubes across multiple faces of the cubic block. The activity of mentally keeping track of what was taken off (or added) and what needed to further be taken off (or added) seemed to require a complexity of operations and schemes Morgan was yet to construct or coordinate together.

On the other hand, Kaylee demonstrated the ability to carry out her counting activities in re-presentation, using minimal two-dimensional sketches. This highlighted

Kaylee’s operations of coordinating multiple perspectives and FR coordinating scheme, which guided her insertion of units (units-coordinating scheme) along various spatial dimensions. Kaylee consistently added or took off layers of unit-cubes in re-presentation, allowing her to anticipate what the addition or subtraction along one spatial dimension would result in along the others. This powerful way of reasoning Kaylee demonstrated suggested that her reasoning about the cubic blocks were guided by her structuration of the cubic blocks using her frames of reference and FR coordinating scheme. Moreover, her accounting of the position of each unit-cube along all three spatial dimensions reflected the multiplicative positioning of each fish in the Fish Tank Task and her construction of a reversible decomposing scheme when reasoning with the cubic blocks.

Summary of Chapter Six

In Chapter Six, I discussed the Cubic Block Task with Kaylee and Morgan, which was a task developed to investigate the students’ units-coordinating activities (inserting composite units into other units) in a three-dimensional context. In entering the Cubic Block Task, I hypothesized that the frames of reference that each student constructed would guide their units-coordinating activities on three-dimensional objects. More specifically, I hypothesized that Kaylee’s units-coordinating activities would be a coordination of multiple three levels of units structures, accounting for all three spatial dimensions multiplicatively; whereas, Morgan’s units-coordinating activities would be a sequential coordination of three levels of units structures in activity, along two spatial dimensions recursively. In the course of the task, I was able to observe more than Kaylee and Morgan’s units-coordinating activities.

Summary of Kaylee's Cubic Block Task

In Part One of the task, Kaylee operated as if she has constructed a scheme to re-present each unit-cube within the cubic block by breaking the cubic block apart to identify each unit-cube as abstracted units but also by putting the parts together to individualize each unit-cube by maintaining their relative positions within the cubic block. I referred to this scheme as a reversible decomposing scheme to entail the decomposing and re-composing operations involved in this scheme. This scheme allowed Kaylee to find the total number of unit-cubes contained in each cubic block, in conjunction with her units-coordinating scheme. Also, this scheme resulted in Kaylee identifying which unit-cubes would have paint on them.

In Part Two of the task, Kaylee systematically counted the total number of unit-cubes to take off from a given cubic block to make a cubic block smaller than the given one. Further, she was able to reverse the operations involved in that systematic counting to find the total number of unit-cubes needed to add to make a given cubic block a bigger cubic block. An outstanding aspect of Kaylee's activities in solving this problem was that she was able to carry them out in re-presentation, using minimal two-dimensional sketches.

The operations and schemes that Kaylee has constructed in her initial interview and the tasks discussed in Chapter Five seemed to have been used as assimilatory operations and schemes in her construction of the aforementioned counting schemes involving the cubic blocks. First, unitizing and disembedding operations were key operations used in decomposing and re-composing the cubic block as well as taking off (or adding) layers of unit-cubes to reduce (or extend) cubic blocks. Second, her splitting, and recursive partitioning schemes were all involved in this decomposing process. Third, her units-coordinating scheme was also used in counting arrays of unit-cubes, either along two spatial dimensions or along three spatial dimensions.

However, before enacting all of the aforementioned operations and schemes, Kaylee must have structured the spatial object in such a way so she could use them in a meaningful way. This structuration of space was supported by her FR-coordinating scheme and operations of coordinating multiple perspectives. Especially, her ability to bring forth immediate past results of her structuration of a two-dimensional space, while structuring another two-dimensional space allowed her to account for all three spatial dimensions multiplicatively. Finally, Kaylee was able to coordinate multiple three levels of units structures recursively. So, the three levels of units that Kaylee was inferred to have interiorized in her initial interview were structures that she could take as input to coordinate further with another three levels of units structure.

Summary of Morgan's Cubic Block Task

In Part One of the task, Morgan's reasoning focused on the two-dimensional faces of the cubic blocks, especially when the cubic block model was hidden from her perceptual field. It seemed as though Morgan was yet to coordinate the third spatial dimension along with the two-dimensional faces (or exterior layers) in re-presentation. However, when I prompted her to explain her reasoning or when she had the physical model of the cubic block available to her, she reflected on her counting activities and adjusted them to account the third dimension. Her interaction with Kaylee also seemed evoke reorganizations in Morgan's counting activities as well. For example, after Kaylee demonstrated her way of decomposing the cubic blocks into square-shaped layers, Morgan assimilated this strategy and used it in organizing the 4×4×4 cubic block, as she demonstrated later in Part Two of the task.

In Part Two of the task, in extending or reducing the cubic blocks, Morgan relied on numerical calculations by subtracting the total number of unit-cubes contained in one from the other to find the number of lacking or excess unit-cubes. When I asked Morgan to justify her answers in a different way, using the cubic block, the activity of mentally keeping track of what was taken off (or added) and what needed to further be taken off (or added) seemed to require a complexity of operations and schemes Morgan was yet to construct or coordinate together, as Kaylee did.

In her initial interview, Morgan operated as if she could carry out the operations of unitizing, disembedding, equipartitioning, splitting, and recursive partitioning. However, it was difficult to impute an independent construction of the splitting and recursive partitioning schemes to Morgan. Moreover, in counting, Morgan used her whole number multiplication and the result of the units coordination seemed more additive than multiplicative in that the composite units were sequentially added through progressive integration. Therefore, it is possible that Morgan did not have the assimilatory schemes available for her to decompose and recompose the cubic blocks and count the number of unit-cubes in various situations. As discussed in her initial interview, Morgan operated as if she could produce three levels of units in activity, but the lack of interiorization of the structure could have limited her to take that structure as input to coordinate further with another three levels of units structure.

I found Morgan's constraints in coordinating the third dimension along with the two-dimensional layers analogous to that in her activities discussed in Chapter Five. That is, when locating the four fish in the three-dimensional fish tanks, Morgan sequentially coordinated the location of one fish in two-dimensional frames of reference. From her locating activities, I hypothesized that Morgan constructed a FR coordinating scheme in activity. In contrast to Kaylee—who I inferred to have used the results of her FR-coordinating scheme as input to use in structuring the spatial object and construct a reversible decomposing scheme—I hypothesized that Morgan's constraints in coordinating the third dimension along with the two-dimensional layers in the Cubic Block Task was because Morgan was yet to construct a FR coordinating scheme for three-dimensional space. In other words, Morgan did not recursively coordinate the results of her FR-coordinating scheme in such a way that would allow her to decompose and re-compose the cubic block mentally as Kaylee did.

Once she had the physical model of the cubic block in her visual field, using the cubic block model, Morgan was able to keep track of the counting activities she had a difficult time coordinating in re-presentation. Therefore, I claimed that Morgan's operating on the cubic block was *in activity*, relying on carrying out sensori-motor activity on the perceptual material.

Based on the results in Chapter Five, I proposed that the FR coordinating scheme requires mental operations essential for coordinating three levels of units; hence, a parallel between the students' levels of units coordination and coordination of measurements within frames of reference in three-dimensional space. From the findings in this chapter, I claim that coordinating multiple perspectives and the FR coordinating scheme is essential for coordinating three levels of units structures along three spatial dimensions.

CHAPTER 7

CRAIG AND DAN COORDINATE UNITS WITHIN THREE SPATIAL DIMENSIONS

In Chapters 5 and 6 I discussed Kaylee and Morgan's construction of coordinate systems and coordination of units within three spatial dimensions. In Chapter 5, I discussed the students' representations of points in two- or three-dimensional perceptual space through the construction of coordinate systems. From the findings, I conjectured that the FR coordinating scheme played an important role in representing three-dimensional perceptual space. Further, I proposed that the FR coordinating scheme requires mental operations involved in coordinating three levels of units; hence, a parallel between the students' levels of units coordination and coordination of measurements within frames of reference in three-dimensional space.

In Chapter 6, I discussed the students' units coordinating activities within a three-dimensional spatial context. From the findings, I hypothesized that the operations and schemes I imputed to students in their initial interview and the Locating Tasks seemed to have been used as assimilatory operations and schemes in their counting activities involving the cubic blocks. First, unitizing and disembedding operations and splitting and recursive partitioning schemes were used in decomposing and re-composing the cubic block as well as taking off (or adding) layers of unit-cubes to reduce (or extend) cubic blocks. The units-coordinating scheme was used in counting arrays of unit-cubes, either along two spatial dimensions or along three spatial dimensions. In terms of levels of units coordination, I hypothesized that a recursive coordination of three levels of units were produced using the aforementioned operations and schemes. On the other hand, I also hypothesized that the structuration of space was necessary because the results of such structuration guided students' enactment of the aforementioned operations and schemes. This structuration of space was constructed through coordinating multiple perspectives, the use of students' FR-coordinating schemes, and a recursive coordination of two-spatial dimensions multiplicatively.

Through the work with the second pair of students, Craig and Dan, I tested and refined these hypotheses. With Craig and Dan, I reversed the order of tasks and started with the investigation of the students' counting of units along spatial dimensions (e.g., the Cubic Block Task) and then investigated their locating activities (e.g., the North Pole Task and Fish Tank Task). The reason I started with the coordinating units tasks first was because I wanted to investigate how students might use the operations and schemes from their initial interviews as assimilatory operations and schemes in their counting activities in spatial objects, such as cubic blocks or rectangular floors. In this process, I expected to observe differences in their coordination of multiple perspectives. I also expected having Dan engage in building the spatial objects as he coordinates units along two or three spatial dimensions to enhance abstractions of the mental operations and schemes that he was yet to construct.

In this chapter, I discuss three tasks that Craig and Dan engaged in, and my analysis regarding their units coordinating activities within two- or three-dimensional contexts. The three tasks were the Cubic Block Task, the Floor Tile Task, and the Box Task. The Cubic Block Task was similar to what Kaylee and Morgan engaged in with minor variations. After the Cubic Block Task, I developed two new tasks for Craig and Dan, which I will explain in more detail in the respective sections in this chapter.

Cubic Block Task: Craig and Dan Count Blocks of Various Sizes

Craig and Dan engaged in the Cubic Block Task over two days (November 10th and 14th, 2014). On the first day, I asked Craig and Dan to build the 2×2×2 and 3×3×3 cubic blocks using unit-cubes. In this process, we established names (1-Cube, 2-Cube, and 3-Cube) for each cubic block so we could easily refer to them in our teaching episodes. After Craig and Dan were finished building the 2-Cube and 3-Cube, I covered the cubic blocks and asked them to illustrate the cubic blocks and to find the total number of unit-cubes (or 1-Cubes) contained in each cubic block. Both students worked separately at a different pace but later explained their solution to each other. On the second day, Craig was absent so I had a chance to work further with Dan in re-visiting the 3-Cube problem and extending the 2-Cube to a 3-Cube and then to a 4-Cube. In the following sections, I will present my observations and analyses of Craig's and Dan's respective activities regarding the Cubic Block Task.

Craig's Cubic Block Activities

Building the 2-Cube and 3-Cube.

Once we established that the 2-Cube referred to a cubic block that had two blocks on each edge along the length, width, and height, Craig built a 2-Cube as demonstrated in Figure 7.1. As shown in the two phases of his building process in Figure 7.1, Craig built one horizontal 2×2 square-shaped layer and then added another on top of it to form the 2-Cube.

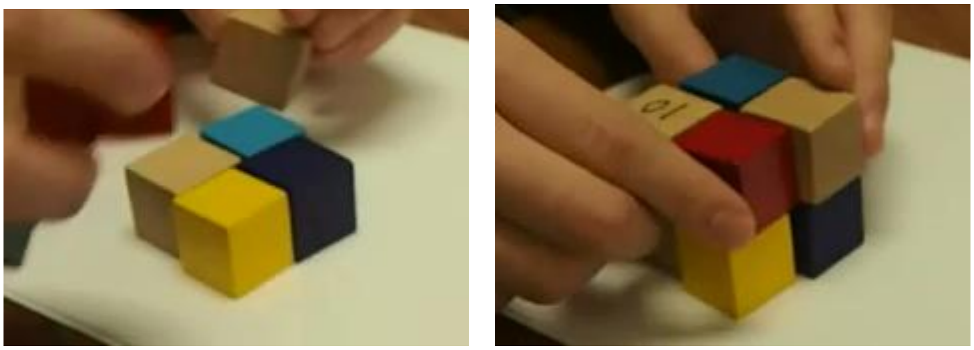


Figure 7.1. Craig's building of a 2-Cube.

Craig built the 3-Cube in successions of horizontal square-shaped layers, consistent with his way of building the 2-Cube. Figure 7.2 shows Craig's building of the 3-Cube (see figures outlined in red in Figure 7.2 that highlight the horizontal square-shaped layers).

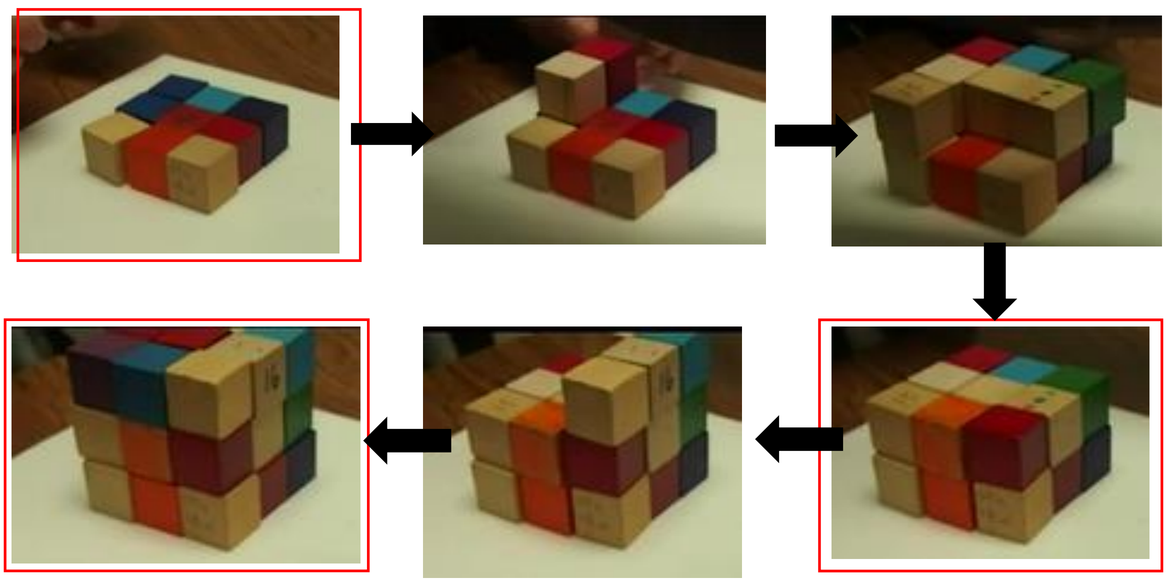
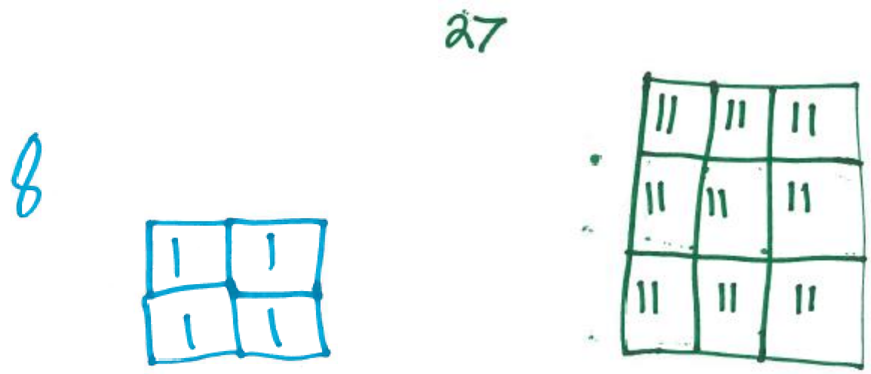


Figure 7.2. Craig's building of a 3-Cube.

Drawing and counting the 2-cube and 3-cube.

When I asked Craig to illustrate the cubic blocks, he said he forgot how to draw a cube and sat looking at his paper for a relatively long time compared to Dan. Despite his difficulty in producing illustrations of the cubic blocks, when I asked the number of 1-Cubes contained in each cubic block, Craig immediately determined that there were eight

1-Cubes and 27 1-Cubes contained in each. As such, Craig counted the cubic blocks without relying on his sketch of them. After I encouraged him to draw anything that made sense to him, Craig finally produced a sketch of the 2-Cube and 3-Cube as shown in Figure 7.3. Craig explained that he sketched the cubic blocks as if he was viewing them from one side.



(a) 2-Cube *(b)* 3-Cube

Figure 7.3. Craig's sketch of the cubic blocks.

As shown in Figure 7.3, Craig made tally marks in each unit-square that constituted the face he imagined he was viewing. Craig explained the tally marks in his illustration as the following:

> You could take it either as a roman numeral or what do you call it, those tally marks. And each one represents a block behind a block you see… It's like a puzzle to indicate how many are behind the block.

Based on Craig's building, counting, and illustrating activities, I inferred that Craig has structured the cubic blocks into units of unit-cubes. More specifically, Craig seemed to have decomposed the cubic blocks into square-shaped layers each consisting of four or nine unit-cubes. Inferring from his building activity, Craig placed each square-shaped layer on the table horizontally and stacked them on top of each other. Inferring from his illustration, Craig visualized each square-shaped layer vertically facing him and imagined two behind the one in the front. As such, not only did he decompose the cubic blocks into square-shaped layers, each consisting of a unit of unit-cubes, but he also flexibly changed the direction in which he decomposed and recomposed the cubic blocks. Therefore, I also hypothesized that such structuration was carried out in re-presentation. Further, I hypothesized that Craig's structured re-presentation of the cubic block assisted Craig's relatively quick calculation of the total number of 1-Cubes contained in each cubic block by supporting his unit-coordinating operations.

Finally, Craig seemed to be aware of his perspective in viewing the cubic blocks and was able to identify which viewpoint he was taking when re-generating the cubic blocks in his illustration. The hypotheses I developed above for Craig's reasoning with the three-dimensional objects were further tested and refined in other tasks, which I discuss later in this chapter.

Dan's Cubic Block Activities

Building the 2-Cube and 3-Cube.

Figure 7.4 shows the process of Dan building the 2-Cube.

Figure 7.4. Dan's building of a 2-Cube.

When Dan completed building his 2-Cube I asked Dan to tell me why he thought his configuration worked as a 2-Cube. Dan separated the cubic block in half vertically as shown in Figure 7.5, explaining that "there's two faces touching each other." This way of

Dan referring to each 2×2 square-shaped layer as faces put together resembled Morgan's way of describing the 2×2×2 cubic block as two "sides" put together.

Figure 7.5. Dan explains why his 2-Cube is a 2-Cube.

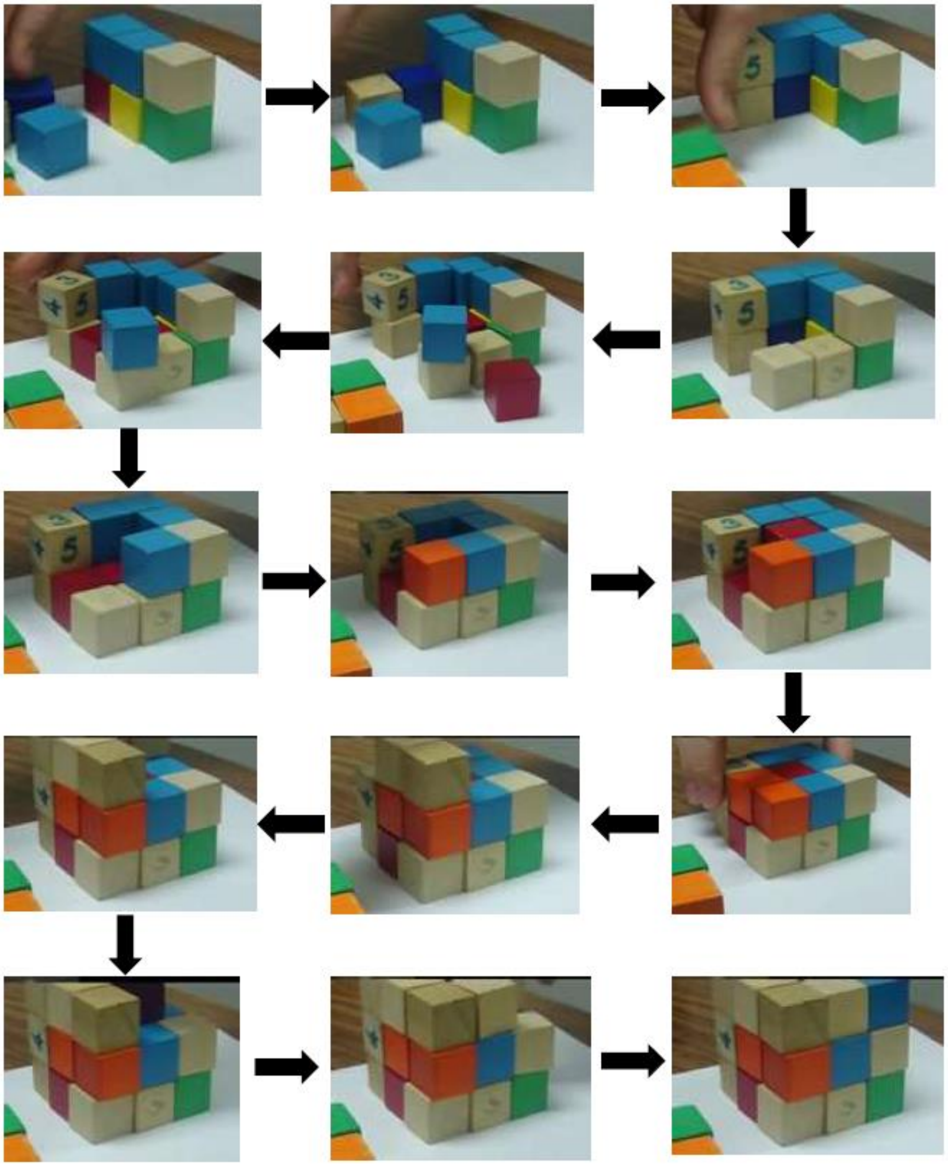


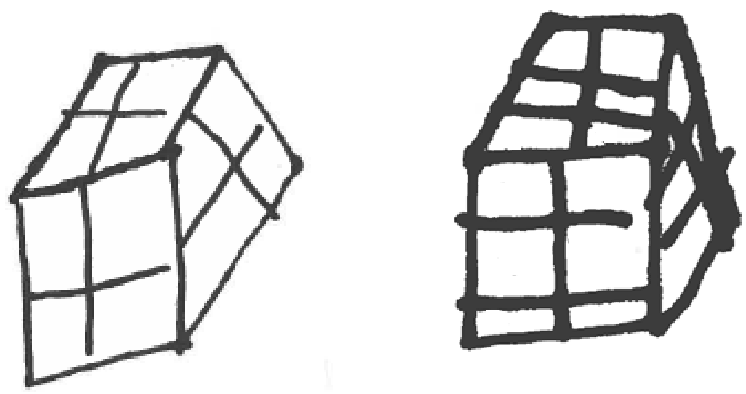
Figure 7.6. Dan's building of a 3-Cube.

Figure 7.6 shows the process of Dan building a 3-Cube. As shown in Figure 7.6, his building of the cubic block was rather haphazard, compared to Craig's in Figure 7.2. Moreover, Dan seemed to focus on building the exterior of the 3-Cube first and then filled in the interior.

Drawing and counting the 2-cube and 3-cube.

When I asked the students to draw the 2-Cube, Dan drew the outline of a cube and then partitioned each visible face into four sections by inserting cross-sign looking inscriptions, one face at a time. Eventually, Dan produced an illustration as shown in Figure 7.7 (a). When I asked him to write how many of the 1-Cubes were in the 2-Cube, Dan looked at his sketch for approximately 13 seconds and wrote "8" on his paper. The way he stared at his sketch intently for a relatively long time suggested that Dan may have been counting the 1-Cubes using his sketch.

Later in the teaching episode I asked Dan to explain how he found the eight 1-Cubes in the 2-Cube. Dan explained "At first, I just times two by four," which I interpreted to mean that he doubled four to obtain eight. When I asked Dan what was the four he was thinking of, Dan said "it took two, ah, four squares to make one side [*makes hand motion as if slicing something vertically*] and then it took another four squares on the other side." His earlier explanation that "there's two faces touching each other" (Figure 7.5) was consistent with his description of the 2-Cube as consisting of two sides, each consisting of four squares.



(a) (b)

Figure 7.7. Dan's sketch of the 2-Cube and 3-Cube.

In drawing and counting the 3-Cube, Dan used a similar approach to the one he used for the 2-Cube. In drawing the 3-Cube, after sketching an outline of a cube, Dan added a vertical line segment through the middle of the front face. Next, Dan drew two horizontal line segments perpendicular to that, partitioning the front face into 2×3 squares. Dan completed his sketch by adding the same inscription (looking like a ǂ sign) into each visible face, obtaining an illustration as shown in Figure 7.7 (b). Note that the number of horizontal partitions (two) on the top face do not correspond to the number of vertical partitions (one) on the right face.

Consistent throughout both cubic blocks, Dan's sketch started with the outline of a cube and then a partitioning of each visible face. This way of illustrating the cubic blocks was consistent with how he started with the exterior of the cubic blocks in his building activity. Also, inferring from his sketches in Figure 7.7, both cubic blocks consisted of two "sides" of some configuration put together. In other words, I inferred that Dan's counting strategy for the cubic blocks entailed decomposing the block into two "sides," focusing on the exterior faces of the cubic blocks.

When I asked Dan to write how many 1-Cubes were in the 3-Cube, Dan moved his index finger over his sketch of the 3-Cube as if he was counting the 1-Cubes along the right side of the top face (indicated by the blue dots in Figure 7.8) and then along the right side of the front face (indicated by the red dots in Figure 7.8). Next, Dan looked up as if he was thinking to himself and finally wrote “9” below his 3-Cube. This process of counting and presumably thinking intently until he came to his conclusion of nine 1-Cubes lasted for approximately 50 seconds. Putting his pen down, Dan said “I got nine,” but that he was “not sure if it’s right.”

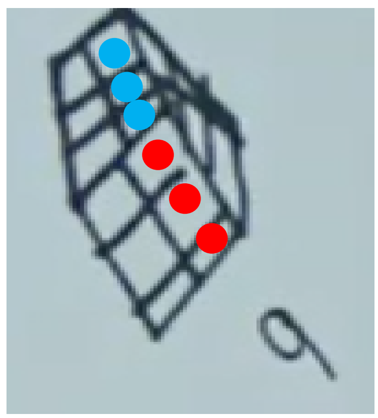


Figure 7.8. Dan’s sketch of the 3-Cube and the number of 1-Cubes he counted in the 3-Cube.

The way Dan counted along the unit-cubes indicated by the blue dots and then along the unit-cubes indicated by the red dots in Figure 7.8, and then wrote “9” after some thought suggested that he had multiplied 3 by 3 to obtain 9 unit-cubes. The relatively long hesitance to get to that conclusion and his uncertainty in his answer could have been due to the discrepancy between his answer 9 and his illustration, which only showed 6 unit-cubes along the right “side” of the cubic block. It is also possible that Dan was unsure of his answer because he tried to multiply 3 by 3 in his head. In previous sessions, Dan demonstrated difficulty in computing arithmetic in his head.

In either case, the way Dan engaged in counting the number of 1-Cubes in his drawing of the 3-Cube suggested that Dan's building of the cubic block was yet to be interiorized. That is, Dan relied on his illustration in counting the number of 1-Cubes contained in the 3-Cube. Recalling Dan's building of the 3-Cube, it is possible that Dan may have been caught up in the activity of filling in the interior of the 3-Cube and did not engage in reflective activities that he could draw upon when re-presenting the 3-Cube in his illustration.

Because Dan seemed unsure of his answer I asked him to think a little more about the 3-Cube while I was talking with Craig. After sitting in thought looking at his sketch, Dan told me that he was sure he was right. I asked Dan to compare the 2-Cube and the 3-Cube, pointing out that he said there were eight 1-Cubes in the 2-Cube and nine in the 3-Cube. I asked Dan if adding one small cube to the 2-Cube would make a 3-Cube. Dan paused to think, saying "I know if you …If you add one to all sides it will be equal." I interpreted his explanation to mean that if he added one block to each side of the 2-Cube then that will make three blocks on each side, giving him a 3-Cube. Because I was still in the middle of talking with Craig, I asked Dan to think some more about the 3-Cube.

At this point in the teaching episode, Craig had completed his work and was explaining to me that his sketch contained tally marks, which expressed how many blocks were behind the ones in the front. Dan also listened to this part of Craig's explanation. Then, Dan started to count the number of 1-Cubes in his sketch by marking them off one by one, as demonstrated by the check marks on the front face of the cubic block in Figure 7.9. This time, the way Dan marked off the 1-Cubes with the check marks suggested that his counting focused on the front vertical layer of the cubic block. After a while, Dan finished his counting of the front vertical layer and scratched the "9" out and paused to think. I did not notice that Dan was yet to finish revising his answer and asked both students to share their work.

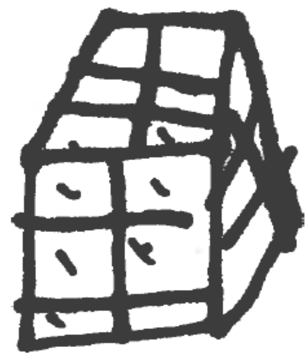


Figure 7.9. Dan counts the 3-Cube using his illustration.

After explaining how he counted eight 1-Cubes for the 2-Cube, Dan said, "I was trying to do the same thing with that one [3-Cube]. But I couldn't figure how." Corroborating my previous hypothesis, Dan wanted to use the same counting strategy he used in the 2-Cube in counting the 3-Cube but was unsure how to do so. After Craig explained his drawings, I asked Dan if Craig's sketch made sense to him. Dan slowly nodded and said, "I was thinking of mine of eighteen, too, because I was thinking six times three." Considering Dan's counting of each square on the front face in his sketch, I interpreted this to mean that Dan has thought of multiplying the number of squares in the front, 6, by the number of layers, 3.

I asked Dan what the six referred to when he said "six times three." Dan laughed shyly and there was a pause for approximately six seconds. I was about to turn to Craig when Dan replied "One row…equals six. Top to bottom [*running his index finger along the unit-cubes in his sketch indicated by the dots in Figure 7.8*]." When I observed him running his index finger along the unit-cubes in his illustration indicated by the blue and red dots in Figure 7.8, I interpreted this to mean that he was referring to the 6 unit-cubes consisting the right "side" of the 3-Cube. To my surprise, Dan pointed to a sequence of six consecutive faces of unit-cubes that were along one "side" of the cubic block. However, in retrospect, it is possible that he was referring to the row with 6 unit-cubes on the top (when sweeping his finger along the blue dots in Figure 7.8) and then the 3 unit-cubes along the top to bottom (when sweeping his finger along the red dots in Figure 7.8).

Because the bell rang, I was not able to further probe Dan for explanations. However, in the next teaching episode, I had the chance to work with Dan alone, which led to more opportunities to test and form hypotheses of Dan's reasoning about the cubic blocks, which I discuss next.

Revisiting the 3-Cube

At the start of the next teaching episode on November 14th, 2014, I asked Dan to make a 2-Cube and a 3-Cube. The purpose was to help Dan remember the context of the task in our previous teaching episode and to engage him in the physical activity which he could reflect upon. This time, Dan used the 2-Cube he made to build the 3-Cube. Figure 7.10 illustrates his new way of building the 3-Cube.

In the previous teaching episode, Dan claimed, "If you add one to all sides [of the 2-Cube] it will be equal [to the 3-Cube]." It seemed like Dan carried out this hypothetical activity in his building, as shown in Figure 7.10. However, he had to figure out how to add "sides" onto the 2-Cube in a rather trial and error manner through the actual building activity. To elaborate, after adding unit-cubes onto the side, making the configuration shown in Figure 7.10 (b), Dan paused and said "one more." This suggested that he noticed that he needed one more "side." Next, Dan added two more blocks along the bottom of the configuration, as shown in Figure 7.10 (c). However, he soon realized that it resulted in four unit-cubes on the bottom of the block so he moved the fourth block, resulting in a configuration shown in Figure 7.10 (d). Through Figures 7.10 (e) and (f), Dan completed making a 3×3×2 cubic block. Finally, noticing that the block was incomplete, Dan completed the 3-Cube by adding the last horizontal layers of 9 unit-cubes.

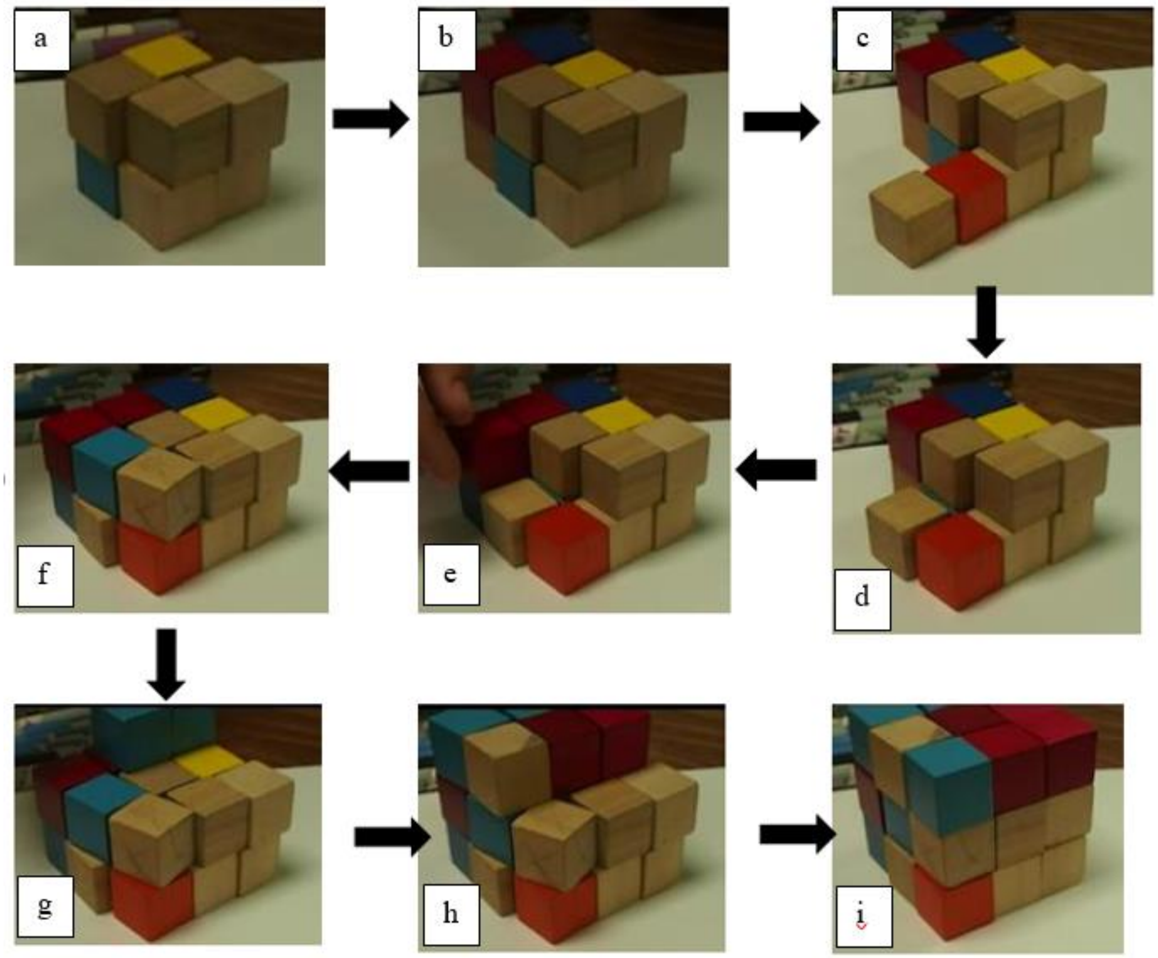


Figure 7.10. Dan's building of a 3-Cube (revisited).

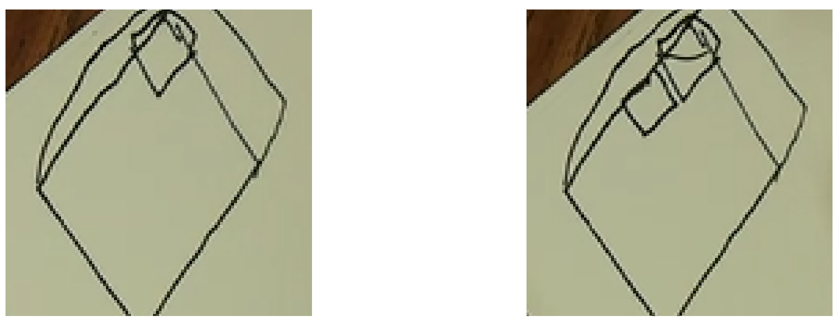
I found Dan's second time of building the 3-Cube (as demonstrated in Figure 7.10) to be quite different from his first time building it (see Figure 7.6) because he seemed to monitor his building activities. That is, Dan appeared to be more reflective on his building activities. On the first day, he first put together the exterior of the cubic block and then filled in the interior seemingly haphazardly. However, this time, Dan added "sides" onto the 2-Cube in a relatively systematic manner compared to the first time. Although Dan had to make in-the-moment realizations that the configuration was yet complete at the steps discussed above (e.g., after Figure 7.10 (b) and 7.10 (f)), I considered Dan's building activity to be more purposeful. For example, instead of adding one unit-cube at a time like he did in the first time, Dan added two unit-cubes at a time, which I interpreted to mean that elements of the cubic block were being integrated into composite units and that his building of the block entailed more anticipation than the first time.

After Dan finished building the 3-Cube, I mentioned that in the previous teaching episode, he did not seem confident about his answer for the number of 1-Cubes in the 3-Cube. Dan explained, "[c]uz I would end up counting the same one. Like, I would go like that [*pointing to one of the faces of a 1-Cube and then to the adjacent face*], end up counting the faces, not the cube." This comment explained the two tally marks he made on the same unit-cube in his sketch in Figure 7.9. It also corroborated my hypothesis that Dan primarily focused on the faces of the cubic blocks. Dan did not have his previous drawing with him when he made this comment, which suggested that Dan has reflected on his previous activities while he was building the 3-Cube for the second time. I inferred this to mean Dan started to put meaning to the squares in his illustration of the cubic block by associating it with the physical unit-cube.

Breaking the 3-Cube Dan has just built, I asked him to try to imagine the 3-Cube and count how many 1-Cubes would be contained in that 3-Cube. Dan looked up into the air as if he was in intense thought for approximately ten seconds. Then, Dan replied "eighteen," which was consistent with his last answer in the previous teaching episode. I asked Dan to explain how he was imagining the 18 cubes. The following excerpt shows Dan's explanation of his 18 unit-cubes in the 3-Cube.

Excerpt 7.1. Dan explains how he counted 18 unit-cubes in the 3-Cube.

D: [*Sketches the frame of a cube, as he did in the previous session*]. Alright, so one cube, [*sketches a 1-Cube in the corner of the frame (Figure 7.11 (a)*] I would think in my head, like, try to imagine that this is one building block so cross that one out [*makes a cross mark across the cube he just sketched in the corner*] and then there's three of them [*sketches another 1-Cube to the left of the one he just marked off (Figure 7.11 (b))*] and I would just mark those out and go down the rows [*taps twice on his illustration of the cubic block as if he is referring to two other "rows" underneath the one he sketched*].



(a) (b)

Figure 7.11. Dan's sketch explaining how he counted 18 unit-cubes in the 3-Cube.

[*Continued*.]

T: Okay.

D: Like there's three rows [*swipes finger horizontally along his sketch across the two 1-Cubes he sketched*], and I would count all of them [*again taps on his picture below the two 1-Cubes he sketched*].

T: Mmhmm. So, you're saying that you would start counting these three [*runs index finger across the first row with the two unit blocks drawn into the cube frame*]?

D: Yeah.

T: And then where do you go? After?

D: Go behind them on top. First and then…

T: Oh! So you…

D: Same on the layers, basically.

T: So are you talking about the first layer [*running finger along the first row in the frame*]?
D: Yeah.
T: How many layers are there in this [*runs index finger perpendicular to the first layer along the frame*]?
D: Three.

In the previous teaching episode, Dan first said he tried to “do the same thing” as he did for counting the 2-Cube (count each “side” and put them together) but that he “couldn’t figure it out.” Then, Dan said he thought of six times three because one row equals six, top to bottom. In these instances, there seemed to be an idea of decomposing the cubic block but the decomposition *relied* on his sketch of the 3-Cube (Figure 7.7 (b)). However, in Excerpt 7.1, Dan described each individual unit-cube and the layers *without* drawing the entire 3-Cube. As such, Dan’s decomposition of the cubic block seemed to be carried out in re-presentation. More specifically, starting with one 1-Cube, Dan counted three adjacent 1-Cubes across the top of the cubic block. When I asked him what he would count after the three in the front of the top row, Dan said he would count the cubes behind those three, forming the first layer. Further, going down the cubic block, Dan knew there were three rows altogether.

As such, Dan articulated the order in which he would count in terms of layers and rows of unit-cubes. However, the way he described his counting activity as crossing individual cubes out suggested that his decomposition of the 3-Cube was a result of a sequential re-presentation of one unit-cube at a time, which were progressively integrated. Dan’s sketch of the cubic block corroborates this hypothesis in that he sketched the individual unit-cubes and not units of several unit-cubes put together. Instead, Dan has progressively integrated the three individual unit-cubes horizontally across the cubic block into a group of unit-cubes and produced two more of those groups behind the first one. I distinguish group from composite unit in that the unit-cubes in the group are progressively integrated into a “chunk” or “group” of objects that are not necessarily unitized abstract units.

Dan’s reasoning with the cubic block and the unit-cubes constituting the cubic block seemed consistent with his reasoning with length of string and unit-lengths constituting the strings in his initial interview. In the equi-partitioning task, in which I asked him to mark one person’s share when sharing a given wax string equally among five people, Dan operated as if he engaged in equi-segmenting and at best has constructed a simultaneous partitioning scheme *in activity*. In other words, Dan’s unit-lengths were progressively integrated additively rather than instantiated simultaneously. This tendency seemed to transpire in his decomposition of the cubic block as well: the unit-cubes constituting the cubic block were re-presented sequentially and progressively integrated until it exhausted the whole.

Further, Dan’s decomposition of the 3-Cube was not a result of a reversible scheme of partitioning and re-composing the cubic block into composite units of several unit-cubes. Again, his way of reasoning about the 3-Cube resembled his reasoning in the initial interview. In the splitting task of the initial interview, I asked him to find my wax string if his given wax string was five times as long as my string. To solve the task, Dan operated as if he had enacted his equi-segmenting operation but his partitioning and iterating were not carried out in conjunction. Moreover, Dan did not use the measurement of the entire string to find the length of one share by reasoning about the inverse multiplicative relation between the size and number of pieces. Rather, Dan’s partitioning was sequential, partitioning one part at a time and progressively integrating them to find if the result of his partitioning gave him the same length as the original string. As such, I inferred that Dan was yet to construct a splitting scheme.

Both in the initial interviews and in the 3-Cube problem, Dan could progressively integrate units sequentially and could group the units into chunks. However, these units were not unitized to be used as input for further recursive reasoning. Moreover, Dan's decomposition of the cubic block or partitioning of the string were not reversible in that they did not entail a simultaneous re-composition of the cubic block or iterating of unit-strings. Therefore, I claim that Dan's ability to coordinate two levels of units in activity but not three and not yet having constructed a splitting scheme pertained throughout these tasks and explains the differences in his activities compared to the other students.

Excerpt 7.2 is a continuation of Excerpt 7.1 starting with me asking Dan to find how many unit-cubes are in the top layer.

Excerpt 7.2. Dan explains how he counted 18 unit-cubes in the 3-Cube (Continuation I).

T: Okay. Let's focus on each layer, then [*running index finger again along the first row Dan sketched*]. So, try to imagine this layer. How many cubes do you see in this layer?
D: [*After for approximately three seconds*] nine.
T: Hmm?
D: Nine.
T: Nine? Okay. So, can you explain why it should be nine?
D: Because there's three rows and three [*pause*s] and three times three equals nine.
T: Oh, okay. And so does it mean that each layer has nine blocks or…does each layer differ a little bit?
D: [*Repeats looking at his sketch and looking up from the table several times for approximately 7 seconds*.] They're all the same.
T: They're all the same?
D: Yeah.
T: Okay. So how many blocks in total do you have?
D: [*Sits quietly for approximately ten seconds*] eighteen, I think. [*Smiling shyly*,] I don't know.

In Excerpt 7.2, after I asked Dan how many unit-cubes he could imagine in the first layer, Dan seemed to be in deep thought in the three seconds he paused before answering “nine.” When I asked him why there should be nine unit-cubes, Dan explained that there were three rows and three, which I interpreted to mean that there were three cubes in each of the groups, which he referred to as rows. Considering the relatively long pause Dan took to think whether all three layers had the same number of unit-cubes, his rearrangement of the unit-cubes in the 3-Cube did not seem immediate nor did it seem to entail an abstraction of unit items. That is, each individual unit-cube and the groups were distinct objects.

Dan did not seem confident about his answer of 18 unit-cubes, so I furthered our discussion as shown in the following excerpt.

Excerpt 7.3. Dan explains how he counted 18 unit-cubes in the 3-Cube (Continuation II).

T: Okay. Can we, um, why don’t you write that down, the number of cubes for each layer. Just the numbers.
D: [*Prepares pen and poses to write*.]
T: So the first layer we have…?
D: [*Sketches in two more unit-cubes below the first unit-cube he sketched earlier (Figure 7.12) then taps on each one at a time*.] Hold on, let me first try to do that.

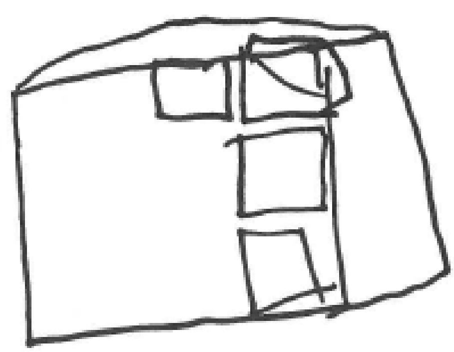


Figure 7.12. Dan adds two more unit-cubes into his sketch.

[*Continued*.]
T: Okay.

D: Alright. [*Then sits for approximately 13 seconds, looking at his sketch and then looking up from his sketch, while moving his fingers slightly*.] Twenty-four.

T: [*Smiling*,] okay, so you changed your mind.

D: Yes, because I did, I counted those two rows [*pointing to the two unit blocks he just added to his sketch*.] and got…that was nine [*tapping on the unit block sketched on the bottom of the cube*], that was nine [*tapping on the unit block sketched above the one he just tapped*] nine times two so eighteen plus nine more…That would actually be twenty-seven. Twenty-seven.

This was the first time Dan stopped me from intervening to try something out on his own. From analysis of the video, even at the beginning of Excerpt 7.3 he did not seem to be paying attention to what I was saying. In other words, Dan seemed deeply concentrated in thought. There seemed to be several things happening in this moment. The way Dan tapped on each of the two squares he added in Figure 7.12 one at a time saying "that was nine" suggested that Dan used the two square figures as place holders for layers each consisting of 9 unit-cubes. During the 13-second pause, although the movement of his fingers did not seem systematic, I inferred that Dan was counting numbers one at a time in his head by the way he occasionally twitched his fingers. So, putting these observations together, it is likely that Dan added nine three times sequentially, keeping track of each nine by looking at the three square figures that were used as place holders to keep track of each group of nine unit-cubes.

Compared with the first day, Dan seemed to have engaged in some structuration of the cubic block and this supported his counting activities. Moreover, Dan's counting seemed to be in line with his new building process—in horizontal layers as opposed to moving from the exterior to the interior of the cubic blocks.

Based on Dan's activities throughout Excerpts 7.1, 7.2, and 7.3, I inferred that Dan coordinated two levels of units sequentially in activity. Dan repeated that grouping activity for the next two rows behind the previous one. Once he produced the three groups each consisting of three unit-cubes, Dan counted the total number of unit-cubes in the first layer, 9. Then, Dan associated the numeral 9 with the first layer and erased the mental markings of each unit-cube in the layer but rather took the result, 9, as his new unit. So, when I asked him whether each layer consisted of the same number of unit-cubes, it took him some time to think whether or not they did. Next, using the square diagrams as place holders, he inserted the 9 into the two remaining rows and sequentially added 9 three times, again, reasoning with two levels of units in activity.

Building a 3-Cube with a 2-Cube.

Next, I asked Dan how many more 1-Cubes he added when he built the 3-Cube starting with the 2-Cube. Dan sat in thought for approximately 30 seconds and said "I'm trying to figure out what two times what is equal to twenty-seven." Then, Dan changed 2 to 8, since there were eight 1-Cubes in the 2-Cube. Dan seemed perturbed, saying "But eight doesn't go into it [27] cuz the closest I will get to twenty-seven is times it by three and that would be twenty-four."

The way Dan wanted to divide the 27 by 8 again resembled some of the observations I made from Morgan's activities in the Cubic Block Task. When solving for the number of unit-cubes needed to add to the 3-Cube to make the 4-Cube, Morgan attempted to find how many times 16 (the number of unit-cubes on one of the exterior layers (face) of the 4×4×4 cubic block) would go in 27 (the number of unit-cubes in the 3×3×3 cubic block) (cf. Excerpt 6.13, Chapter 6). For Morgan and Dan, it is possible that thinking about the number of additional unit-cubes needed in extending a cubic block entailed an image of fitting the smaller cubic block into the larger one. Perhaps this image rooted from the idea that the smaller cubic block should fit inside the larger one. However, in both cases, the number that they attempted to divide was not whole number multiples of the other and hence did not work the way they expected.

Next, I asked Dan to build the 3-Cube again, giving him a 2-Cube. This time, Dan built a 3-Cube from the given 2-Cube as captured in Figure 7.13. This time, Dan added unit-cubes onto the 2-Cube in succession of layers. After completing the building activity, Dan said “twenty.” I asked Dan to identify the 2-Cube that he started with and explain his building process. Although from an observer’s perspective, Dan had added unit-cubes onto the 2-Cube to build each horizontal layer from bottom to top, Dan did not mention the layers but that “I was just adding one more to be three. And it took twenty of them to get all of the sides.” Dan admitted that he counted the number of 1-Cubes as he added them onto the 2-Cube.

Figure 7.13. Dan adds onto the 2-Cube to make a 3-Cube.

Noticing that Dan had counted the unit-cubes one by one as he added him to the 2-Cube, I decided to first establish with Dan how many additional unit-cubes there were. My goal in this task was not for Dan to find the correct number of 1-Cubes contained in the 3-Cube. My goal was to provide an opportunity for him to engage in counting unit-cubes in a more systematic way. So, I asked Dan how many 1-Cubes were in each of the 2-Cube and 3-Cube and to find how many more unit-cubes he would need from 8 to get to 27. Moving his fingers one by one as he moved his mouth, Dan counted numbers to himself and after approximately 13 seconds, guessed "Eighteen?" After I requested Dan to show me his counting, Dan explained as the following.

> "I said eight, [*putting one finger up at a time for each number starting with nine*] nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven. Eighteen. Or, nineteen. Yeah, nineteen."

Dan's counting activity resembled the counting activity he demonstrated in the initial interview when I asked how much more string he needed to add to a string of 15 cm to make the string as long as a string of 24 cm. Dan kept track of his counting from the smaller number up to the larger number, putting up one finger at a time. Consistent with his counting activity in the initial interview, Dan relied heavily on the figurative marks of his fingers to keep track of his counting. This way of counting one number at a time instead of counting in composite units seemed to be consistent with the way Dan counted the individual unit-cubes instead of counting them in composite units. In other words, Dan consistently reasoned with his tacitly nested number sequence and within two levels of units in activity, which was also consistent with his initial interview activities.

Once we established that there were 19 more unit-cubes in the 3-Cube than the 2-Cube, I asked Dan if there was a systematic way of finding the 19 blocks without subtracting the total number of unit-cubes in each cubic block or by counting one unit-cube at a time. I asked Dan to show me the 19 unit-cubes using the 3-Cube on the table. Dan sat in thought for approximately 10 seconds. I tried to intervene, by prompting him to peel off the cubes when Dan interrupted me and said, "Well, I was going to do this." He then took off the top 3×3 square-shaped layer in the 3-Cube (circled in red in Figure 7.14 (a)). Next, he took off the 2×3 rectangular-shaped layer on the side (circled in red in

Figure 7.14 (b)). Lastly, he took off the 2×2 square-shaped layer in the front to make the 2-Cube (Figure 7.14 (c)).]

(a) (b) (c)

Figure 7.14. Dan removing layers of unit-cubes off of the 3-Cube to make a 2-Cube.

The way Dan peeled off layers of unit-cubes from the 3-Cube to make the 2-Cube was different from the reverse of the way he built the 3-Cube from the 2-Cube as shown in Figure 7.13. Nevertheless, his activity was independently carried out. Interestingly, the way he took off unit-cubes was very similar to the way Kaylee took off unit-cubes from the 4×4×4 cubic block to make a 3×3×3 cubic block (cf. see Figure 6.34 in Chapter 6). Kaylee's and Dan's activities differed in that Kaylee enacted them in re-presentation while Dan relied on the physical blocks.

The crucial point in Dan's activity is that Dan grouped unit-cubes into chunks when taking off the unit-cubes. After Dan demonstrated how he would take off the unit-cubes, I suggested to him to reverse his actions to re-build the 3-Cube and keep track of the number of unit-cubes he added each time. The following excerpt starts with my prompt to go backwards.

Excerpt 7.4. Dan makes a 2-Cube from the 3-Cube.

T: Ah! That's really cool. So, can you explain, now let's go backwards, okay? So, you started with this [*pointing to the leftover two-cube*] …

D: [*Nodding*] yeah.

T: And then…? [*Pauses*.]

D: [*Looks at T*.]

T: You added this [*pointing to the 2×2 square-shaped layer he took off in the last step he described*] to here somewhere [*pointing to the 2-Cube*], right?

D: Yeah.

T: To make one part three.

D: So, I did four to get three.

T: Four to get three. And then, so, can you put it back here?

D: [*Adds the four cubes back to the 2-Cube (Figure 7.15 (a))*.]

T: So now one side is made of three [*runs pen along the dimension with three unit-cubes*.]

D: [*Nods*.]

T: Right? And in order to make that, how many were you adding?

D: Four.

T: Okay, can you keep track of those numbers for me?

D: [*Writes '4' on the paper*.]

T: Okay. And then?

D: I added six on this side [*puts the 2×3 rectangular-shaped layer onto the side of the existing configuration (Figure 7.14 (b))*.]

T: Mm-hmm. And what does that do for you?

D: It's three on that side.

T: Okay. So, and how many did you add?

D: Six. [*Writes '6' down on paper next to the 4 he wrote earlier*.]

T: Okay. And then?

D: [*Writes another ',' next to 6 as if he is ready to find another number. Picks up the remaining 3×3 unit-cubes and puts it on top of the configuration (Figure 7.15 (c))*.] Nine. [*Writes '9' down in his sequence of numbers*.] [*Once Dan wrote the three numbers on his paper, I asked him to check if the numbers gave us our earlier answer. Adding the three numbers 4, 6, and 9, one at a time, Dan concluded 19*.]

(a) (b) (c)

Figure 7.15. Dan rebuilds the 3-Cube starting with the 2-Cube.

My explicit guidance at the beginning of Excerpt 7.4 was partly due to Dan looking at me after I asked him what came next. Another reason was to engender a reflection of the actions he had just carried out and to engender an awareness of what adding each layer resulted in regarding the number of unit-cubes along each spatial dimension. This was something Kaylee seemed to be able to do independently and what allowed Kaylee to mentally re-present and count the total unit-cubes being added to the smaller cubic block to make the larger cubic block. Dan relied on the physical cubic blocks to keep track of his actions and the teacher-researcher was heavily involved as shown in Excerpt 7.4. However, Dan's reasoning with the cubic block seemed to have advanced from dividing 27 by 8 to counting the unit-cubes in layers, reversing his activity of taking off the unit-cubes.

Building a 4-Cube with a 3-Cube.

I asked Dan to find how many more 1-Cubes he would need to add to the 3-Cube to make a 4-Cube to explore to what extent this new way of operating could be generated by Dan independently. The following excerpt shows Dan's solution to the new task at hand. Dan had the 3-Cube he just made on the table in front of him.

Excerpt 7.5. Dan makes a 4-Cube with a 3-Cube.

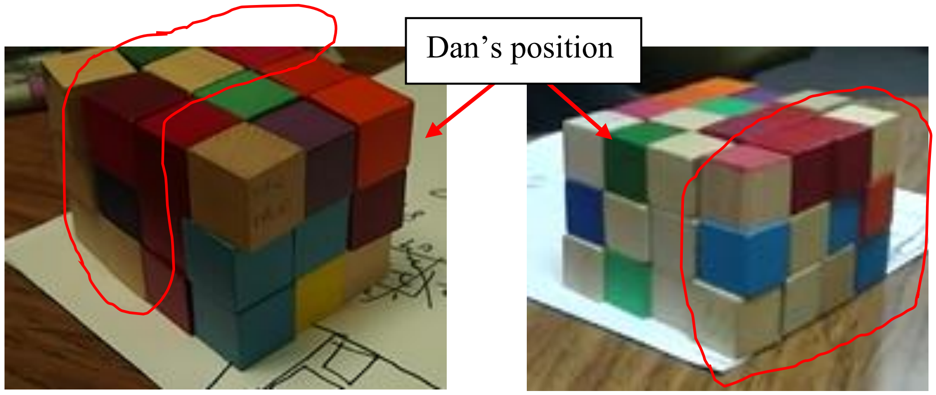
D: [*Sits looking at the 3-Cube for approximately six seconds. He shifts his body to look at the face of the cubic block to his right*.] So, place nine on this side [*pointing to the right side face that he was just looking at*].

T: Okay.

D: Let me just write what I'm thinking [*writes '9'*]. Nine… plus... I would need to add nine onto that side. [*Dan stares at the three-cube in front of him. He does not indicate which side he is referring to but his eyes are set on the side farthest from him, i.e., the face of the 3-Cube opposite to him*.] I believe. [*Writes '9' next to the 9 he wrote earlier. Then glances at T and asks*] can I do it?

T: Okay.

D: [*Adds, one unit-cube at a time, a 3×3 square-shaped layer on the right side face of the cube making a 3×3×4 configuration (see part enclosed in red in Figure 7.16 (a)). Then he adds a 4×3 rectangular-shaped layer on the face opposite of him and completes a 4×4×3 configuration (see part enclosed in red in Figure 7.16 (b))*.]



(a) (b)

Figure 7.16. Dan adds unit-cubes onto the 3-Cube to make a 4-Cube.

After Dan completed building the configuration in Figure 7.16 (b), I asked Dan to think about what he had added so far onto the 3-Cube. I pointed out how he added the nine 1-Cubes first and then asked him what he did next. Dan explained that he added twelve on the other side and changed the second 9 he wrote earlier to 12. Excerpt 7.6 starts with Dan continuing to add more 1-Cubes onto the block to complete the 4-Cube.

Excerpt 7.6. Dan makes a 4-Cube with a 3-Cube (Continuation).

D: And then I need tw… [*Pauses for approximately 15 seconds staring at the 4×4×3 configuration in front of him,*] sixteen of them.
T: Okay, how did you figure out that sixteen?
D: I know that four rows [of] four, four times four.
T: Four times four is sixteen? Okay.
D: [*Adds blocks and runs short of blocks available*.] And there's some more.
T: We have some more [*places tub of additional cubes on the table*.]
D: [*Completes a 4×4 layer on top of the 4×4×3 configuration to complete the 4-Cube and looks at T*.]
T: So how many in total?
D: [*Writes 16 next to the 12 then connects the 9 and 12 with line segments and counts for approximately 12 seconds to get 21. He moves his fingers underneath the table to keep track of his counting*.] I have to count. [*Then he adds 16 to the 21 and counts again using his fingers underneath the table for approximately 11 seconds*.] Thirty-seven would be in total.

As shown in Excerpts 7.5 and 7.6, Dan first added a rectangular-shaped layer to one side, then another, and finally added a square-shaped layer onto the top of the cubic block, all carried out independently. This to me suggested that Dan has indeed reflected on his previous activity of adding layers to the cubic block in succession to obtain the larger cubic block in the previous case and abstracted that way of organizing the cubic block. However, when re-presenting that activity in counting the additional number of unit-cubes, Dan needed the figurative material to keep track of his said activities. Hence, although Dan's systematic way of adding onto the 3-Cube resembled Kaylee's systematic way of adding onto the cubic block, Dan's was carried out in activity, using the figurative material as markers to keep track of the number of 1-Cubes that comprised each layer whereas Kaylee's operations were carried out mentally in re-presentation, using minimal sketches of the situation.

Overall, Dan was able to solve for the 3-Cube extension case, similar to how he solved for the 2-Cube extension case, with less guidance from the teacher-researcher. Dan maintained the idea of adding layers of unit-cubes onto sides of the existing cubic block to expand it into larger dimensions. He also counted the number of unit-cubes constituting each layer in re-presentation (e.g., he found the 9 and 16 unit-cubes before adding them onto the block). However, Dan seemed to have difficulty representing and anticipating the result of adding such layers and asked if he could carry out the activity. In other words, coordinating and re-presenting one instantiation of his activity in co-presence with another instantiation was challenging for Dan and limited his ability to anticipate the result of his activities.

Although Dan had to carry out the activity in order to keep track of the changes in the configuration of the block, one noticeable change was in the confidence Dan demonstrated in building the 4-Cube starting with the 3-Cube and the resemblance in his building of the 3-Cube starting with the 2-Cube. It seemed as though these extension tasks offered opportunities for Dan to reconstruct the cubic blocks into units of unit-cubes that helped him count the total number of unit-cubes in a more systematic way. When compared with his initial building and counting of the cubic blocks, Dan's activities became more systematic. Through his physical building activities Dan seemed to have abstracted a structuration of the cubic blocks into successions of layers. This reorganization of the cubic blocks also seemed to support his counting activities as evidenced by his increased engagement in counting in groups of unit-cubes. As such, I infer that Dan's structuration of the three-dimensional objects supported his two levels of units coordination.

Summary of Cubic Block Task

In the Cubic Block Task I have discussed so far, Craig and Dan engaged in building, illustrating, and counting cubic blocks of various sizes. Craig's building of the 2-Cube and 3-Cube (see Figures 7.1 and 7.2) seemed systematic in that Craig consistently built the cubic blocks in successions of horizontal square-shaped layers. In terms of illustrating the cubic blocks, Craig had difficulty recalling the conventional way of sketching a cube, but developed his own illustration (see Figure 7.3) that allowed him to demonstrate his understanding of the configuration of the cubic blocks. The way Craig made tally marks on each unit-square representing each unit-cube in the front face of the cubic block suggested that Craig had unitized several units of unit-cubes in counting. Before even sketching the cubic blocks, Craig first wrote the total number of 1-Cubes contained in each cubic block. Craig's immediate calculations of the total number of 1-

Cubes contained in each cubic block suggested that he was able to count the number of unit-cubes in units of units, in re-presentation, without having to rely on physical material.

On the other hand, Dan's initial building of the 3-Cube (see Figure 7.6) seemed somewhat haphazard with an overall approach of building the exterior of the cubic block and then filling in the interior. Dan seemed to be more reliant on his sketches in counting the number of unit-cubes contained in each cubic block. For the 2-Cube, his explanation of the configuration entailed a decomposition of the cubic block into two "sides" and he attempted to use the same idea for the 3-Cube but said he was not sure how to do so. The first sketch he made for the 3-Cube was closer to a 2×3×3 block and given the sketch he made, Dan counted nine unit-cubes. After listening to Craig's explanation of his use of tally marks, Dan started to talk about rows consisting the cubic block.

On the second day of the task, when I asked Dan to build the 3-Cube for the second time, Dan's building seemed to become more systematic in that he added chunks of unit-cubes as if he were to add horizontal layers to form the 3-Cube. In counting the 3-Cube, Dan said there were 18 and sketched an image of the 3-Cube to explain why he thought there were 18 1-Cubes in the 3-Cube. This time Dan's illustration and explanation of the 3-Cube occurred before he drew the entire 3-Cube, which is why I claimed that he was able to carry out this decomposition in re-presentation. However, from the way Dan described his counting activity as crossing individual cubes out, I claimed that his decomposition was not a result of re-presenting and counting a composite unit of several unit-cubes together. I inferred that Dan actively engaged in the putting three unit-cubes together into groups in order to keep track of his counting, as he produced one unit-cube at a time in re-presentation. I distinguished group from composite unit in that the unit-cubes in the group are progressively integrated into a “chunk” or “group” of objects that are not necessarily unitized abstract units.

Based on my observations in relation to Dan’s activities in both the initial interview tasks and Cubic Block Task, I claimed that Dan’s ability to coordinate two levels of units in activity but not three and not yet having constructed a splitting scheme pertained throughout these tasks and hypothesized that these constraints explained the differences in his activities compared to the other students.

Further, in extending the 2-Cube into a 3-Cube and a 3-Cube into a 4-Cube, with my guidance, Dan maintained the idea of adding layers of unit-cubes onto sides of the existing cubic block to expand it into larger dimensions. However, Dan seemed to have difficulty anticipating the results of adding each of those layers and asked if he could carry out the activity. Therefore, based on Dan’s activities on the second day of the task, I claimed that Dan coordinated two levels of units sequentially in activity. Moreover, I claimed that Dan’s coordination of a re-presentation of one instantiation of his activity in co-presence with another instantiation was challenging for Dan and limited his ability to anticipate the result of his activities.

From these different ways of engaging in the Cubic Block Task, I inferred that Craig operated with units of units using his unitizing operation and was able to iterate units of units in activity, guided by a systematic decomposition and structuration of the cubic blocks into layers. Therefore, I hypothesized that Craig may have the operations for constructing the reversible decomposing scheme I imputed to Kaylee. The tally marks Craig used to represent the unit-cubes behind the first layer of unit-cubes comprising one face suggested that Craig reasoned with the unit-cubes as abstracted units. As a result, the number of unit-cubes that comprised each cubic block was a result of multiplicative reasoning. In other words, Craig was able to coordinate three levels of units coordination in activity, guided by his systematic structuration of the three-dimensional objects.

On the other hand, Dan demonstrated what I called a grouping activity in order to count the individual unit-cubes efficiently. As a result, the number of unit-cubes that comprised each cubic block was a result of repeated addition of the objects within the groups. These hypotheses seemed to align with the findings from the initial interviews in that I hypothesized that Dan operated as if he could coordinate two levels of units in activity and thus coordinated units additively while Craig operated as if he could coordinate three levels of units in activity and thus coordinated units multiplicatively.

Moving forward, I wanted to test if Craig's structuration of the spatial objects were established as schemes that could independently be enacted in various situations. I also wanted to test if Dan could engage in unitizing and iterating of units of units and engender multiplicative units-coordinating operations. I also wanted to explore what his current operations allowed him to do and how to support his thinking within his zone of potential construction (Steffe & D'Ambrosio, 1995), given the operations and levels of units coordination that were available to him. Further, from Dan's engagement in the extension tasks, I noticed his difficulty in anticipating the dimensions of the additional layers prior to adding the unit-cubes in activity. Therefore, I wanted to provide similar situations in two-dimensional space to see if a reduced dimensionality would be more conducive to his current ways of operating.

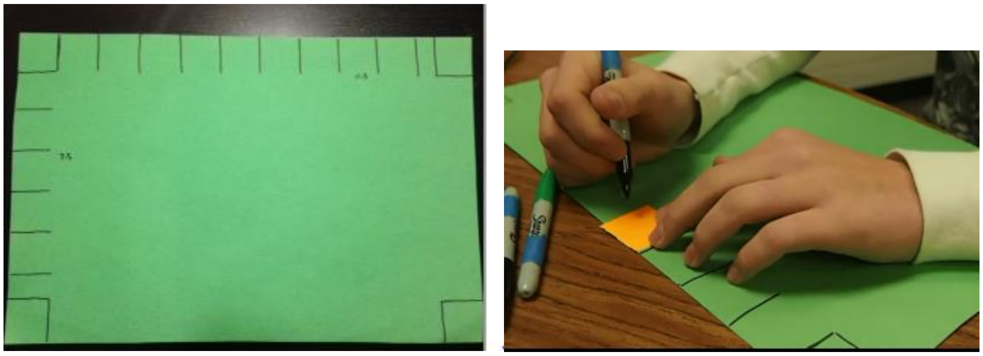
Floor Tile Task: Craig and Dan Tile Rectangular Floors of Various Dimensions

Craig and Dan Tile a Floor

On November 17, 2014, I introduced Craig and Dan to the Floor Tile Task. To introduce the task, I gave each student a rectangular sheet of paper representing a floor and a small square-shaped cutout paper modeling a tile. Then, I asked Craig and Dan to find out how many tiles they would need in covering their given floor (paper) with their tiles. The size of the rectangular paper was the same for both students but Craig's tile was smaller than Dan's to avoid them generating the same answer.

Dan finds the number of tiles to cover his floor.

Dan took his tile and started to trace it in each corner, as shown in Figure 7.17 (a). Then, he moved his tile along the length and width of the rectangular paper, marking the edges of the tile (Figure 20 (b)). As he marked the tiles, Dan counted out loud at the same time as if he were keeping track of the number of tiles. Dan's way of tracing the tiles starting from the outer corners and then moving inwards or counting one tile at a time seemed consistent with his tendency to move from exterior to interior of spatial objects or to count units one at a time, as I observed in his previous activities. Later as the number of tiles increased, Dan stopped keeping track and recounted the number of tiles from the beginning when his marking was completed. After counting the number of tile lengths marked along the length and width of the rectangular paper Dan wrote 11.5 and 7.5 on his paper along the edges.



(a) (b)

Figure 7.17. Dan measures the length and width of the paper using the side of his tile.

Craig finds the number of tiles to cover his floor.

Different from Dan who first traced the tile in each corner of the paper, Craig started by marking the length of the tiles along the length and width of the paper, as shown in Figure 7.18. When Craig remarked that the tiles did not fit exactly into the sides, I told him he could make approximations so Craig did not account for the small leftovers. After Craig completed making all tick marks, he counted the number of tiles that would fit along the width and length of the rectangular paper, respectively. Next, Craig said "eleven by sixteen."

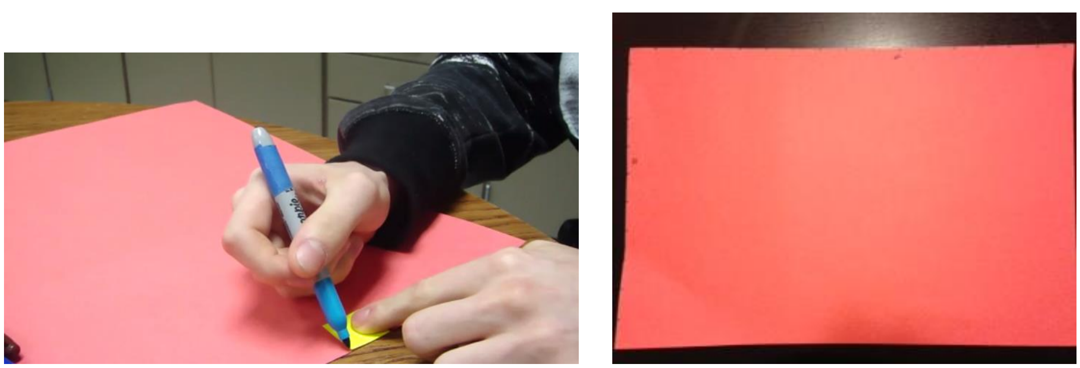


Figure 7.18. Craig measures the length and width of the paper using the side of his tile.

Craig and Dan discuss finding the number of tiles to cover the floors.

Because both students found the number of tiles that would fit into two orthogonal edges of the rectangular paper but not the number of tiles needed in order to cover the entire floor, I asked both students what they would do with the two numbers they each found. At first Dan said "Multiple… So, pre [*trying to say perimeter*]… Ah, I can't say the word, but you add them together." As such, Dan first seemed to mention multiplication but then suggested to add the two numbers. When I restated what he said, asking "so you want to add these two numbers together?" Dan replied "Yeah. Or…" and trailed off into thought. I asked Craig what he wanted to do with the numbers and Craig said he wanted to multiply his two numbers. The following excerpt starts with me asking both students to discuss which operation made more sense to use, addition or multiplication.

Excerpt 7.7. Dan and Craig discuss whether to add or multiply the measurements they obtained.

T: So, let's try to think about first what makes sense to do. If we want to multiply the two numbers or if we want to add the two numbers?
D: I'm not quite sure but I remember learning trying to find the pra… I can't say the word.
T: Perimeter?
D: Yes, perimeter around squares or somethings and you take this side and that side [*pointing to each side of the paper he marked with tile unit lengths*] and you add them together.
T: Yeah, exactly, if we wanted to find the perimeter [*sweeps fingers along the edges of the paper, where D marked off tile-lengths*] of this floor, we should add, right? But my question is, how much would we need to cover this whole floor?
C: Area.
D: Oh, the area [*sweeps the inside of the paper*]. Okay.
T: All of the tiles, right?
C: Yes.
D: So, yeah, I think it would be multiply.
T: Multiply? So you agree with Craig that we want to multiply?
D: Yeah.
T: Okay. Then, now my question is, why does multiplying make sense?
D: Because it's there's eleven rows [*runs finger along the long side with 11.5 marked*] and there are seven [*runs finger along the width side with 7.5 marked, then next to that (to correspond with the next "row")*] in each row.
T: Oh.
D: It's times by seven and a half.

T: Okay. That's how you make sense of it [*referring to D*]. What about you, Craig? Why did you think of multiplying it in the first place?

C: It's the only way I know how I can do it that way. That's the only way I know how to do it.

As shown in Excerpt 7.7, Dan recalled how to find a perimeter of a quadrilateral figure and attempted to find the perimeter of his rectangle. It is possible that the problem I posed—to find how many tiles he would need to cover the floor—meant to find the perimeter of the floor for Dan. That is, covering the space could have meant to enclose the space to Dan. When I asked how many tiles they would need in order to cover the entire floor, Craig explained that it would be the area of the floor. Then, Dan swept the interior of the paper with his hand acknowledging that covering the floor entailed finding the area of the rectangle. Thereafter, Dan agreed to multiply his two measurements.

I inferred my clarification of the situation (covering in the entire floor) and Craig's mentioning of the area to have redirected Dan's reasoning to entail both the exterior (tiles along the perimeter) and the interior of the two-dimensional spatial object (rectangle). The way that Dan first attempted to count the tiles along the peripheral of the given space was very similar to the way he originally built the cubic blocks starting with the exterior walls/faces. Dan seemed to focus mainly on the exterior of the spatial objects first than to the interior. On the other hand, although Craig immediately associated the situation with finding the area, when asked to explain why he needed to multiply the two measurements he obtained, he seemed to rely on what he learned in school. Therefore, probing Craig to articulate why area entails multiplying the two measurements and probing Dan to envision the interior of rectangular spaces in partitioning were needed. The next part of the Floor Tile Task was adequate for such explorations.

Craig and Dan Extend the Flooring

Craig and Dan familiarize a GSP simulation of the tiling activity.

For the remaining time of our teaching episode on November 17, I wanted the students to become familiar with the tiling simulation I designed on the Geometer's Sketchpad (GSP) platform. First, I showed the students the simulation, which showed a corner of a rectangular floor, and the number of tiles placed along two orthogonal edges of the floor (Figure 7.19). Note in Figure 7.19, referring to the horizontal and vertical axes are used for writing purposes. In the teaching episodes, we did not use these terms. Rather, both the students and I referred to them as "sides" or "edges" of the rectangular floor.

The horizontal edge was assigned as "Craig's side" and the vertical as "Dan's side". The students could change the number of tiles placed along their side using the +/– key. The simulation only showed the students how many tiles were on each side (length and width) of the rectangular floor. It was the students' task to imagine the rectangular section that would be made with the number of tiles placed on each edge.

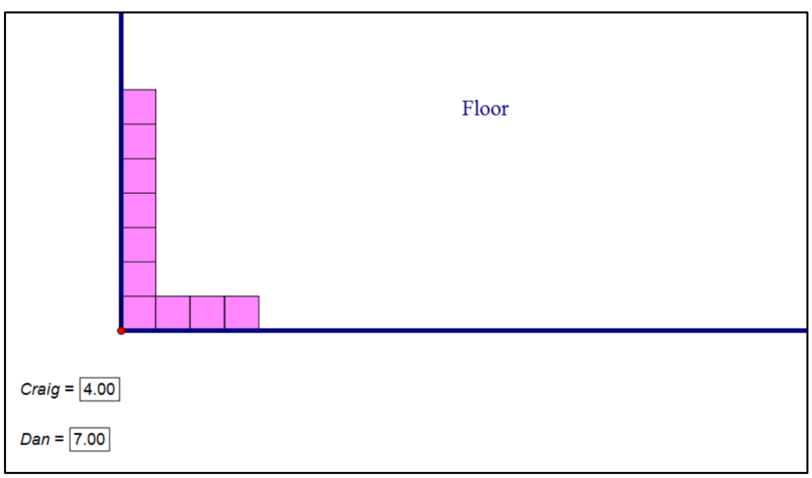


Figure 7.19. GSP sketch of the flooring with 4 tiles along the horizontal axis and 7 tiles along the vertical axis.

With the sketch set with 4 tiles along the horizontal axis and 7 tiles along the vertical axis (Figure 7.19), I asked Craig and Dan to find the number of additional tiles, other than the ones already placed on the floor, to cover the rectangular section of the floor.

Although Dan first wrote “28” as his answer, eventually both students agreed that there were 18 additional tiles. Dan explained that he multiplied, which gave him the area. It seemed like Dan had assimilated Craig’s solution of finding the area entailing multiplication. Craig elaborated, “Because you asked how many more you’d need to fill up the whole space, so I subtracted one from seven and did six times three which is eighteen.” Next, Craig leaned over to the computer screen and explained using his index finger saying:

> I did seven along the whole line [*runs finger along the vertically arranged tiles four times along each four tiles on the horizontal axis*] seven, seven, seven, and seven. But then I subtract one from each. So, since that already had seven inches, it’s six times three [*pointing to the six tiles and the three tiles (excluding the one shared by both the horizontal and vertical)*].

Although Craig did not explicitly mention subtracting 1 from the 4 tiles, I inferred from his explanation and demonstration on the screen to mean that Craig has disembedded 1 and 6 from 7, 1 and 3 from 4 and coordinated the 6 with the 3, using his units-coordinating operations. It seemed like given the GSP sketch, Craig was able to envision the 6 by 3 rectangle and the arrays of tiles comprised of three rows of six tiles. This was the first time Craig verbally explained his reasoning of multiplying two unit lengths to obtain area, other than saying that it was what he learned to do.

Craig and Dan extend the flooring of 28×5 to one of 31×7 (Day 1).

The Floor Tile Task continued on the next teaching episodes on November 21 and December 1, 2014. On November 21, the students selected numbers of tiles to place along each edge of the floor. There were 28 and 5 tiles along the horizontal and vertical axes, respectively (Figure 7.20). I asked both students to imagine that the floor they set up in Figure 7.20 was completed with tiling.

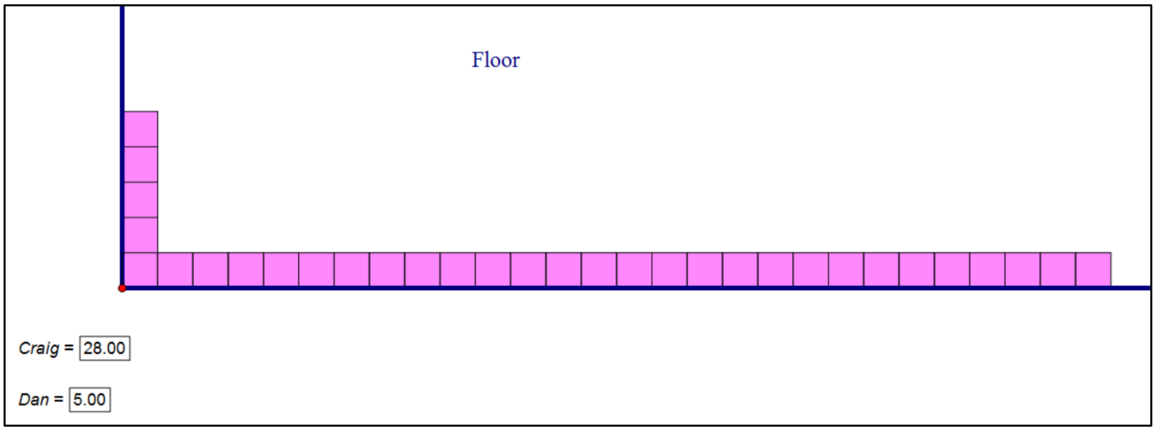


Figure 7.20. Floor configuration with 28 tiles placed on Craig's side and 5 tiles placed on Dan's side of the floor.

Both students calculated the total number of tiles to cover the floor, 28×5=140. Dan explained the configuration as twenty-eight rows of five. This explanation of the number of rows and how many were in each row was consistent with the previous teaching episodes. For example, in Excerpt 7.2, Dan said "Three rows of three and three times three equals nine." In Excerpt 7.6, Dan explained "I know four rows [of] four, four times four." Finally, in Excerpt 7.7, when discussing the number of tiles covering the floor, Dan said, "there's eleven rows and there are seven in each row." As such, Dan consistently associated multiplication of units as multiple rows with a number of units within each row. It became of significance for me to test whether his multiplication entailed a unitization of the rows, in other words, if he had constructed the area as a composite unit of units, or if it was a repeated addition of progressively integrated units.

I asked the students to imagine that we already had the 140 tiles on the floor and asked both students to imagine that Craig added 3 tiles next to the 28 tiles and Dan added

2 tiles on top of the 5 tiles. Their job was to find how many more tiles they would need in order to extend the existing floor of 140 tiles to a floor that had the 3 and 2 additional tiles added to each edge. Because the wording of the problem was complicated, I demonstrated what I was asking for using the computer screen, moving my finger along the shaded region in Figure 7.21. Note in Figure 7.21, although I traced my finger along the shaded region, there were no traces left on the screen; Craig and Dan only saw Figure 7.20 on the computer screen.

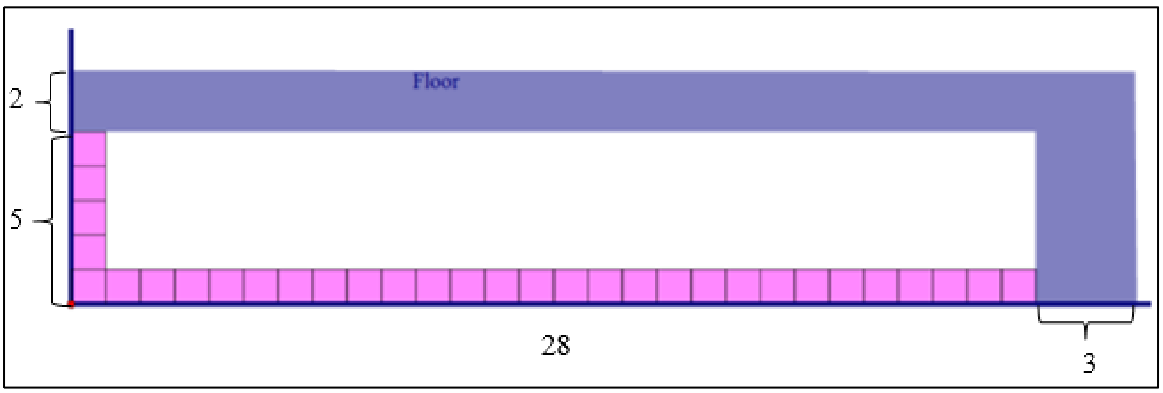


Figure 7.21. The area Craig and Dan were asked to find (region shaded in purple).

Craig's calculation. Each student worked individually on this task. Craig first calculated 31×7=217, the total number of tiles needed to cover the entire floor. After I emphasized that they already had the 140 tiles on the floor, Craig then calculated 31×2=62. This would have accounted for the amount of tiles needed to cover the top part of the additional flooring.

Dan's calculation. On the other hand, Dan calculated 32×7, supposedly in order to find the total number of tiles needed to cover the entire floor. Dan explained, "I just added it altogether and multiplied..." suggesting that Dan indeed added the three additional tiles to the 28 (which he miscalculated as 32 initially) and the 2 additional tiles to the 5 and multiplied the two numbers together. Later, Dan revised his answer after changing his 32 to 31. After solving for 217 tiles in total, Dan used long division to find 217÷140. Then, finding that 140 went into 217 once, Dan subtracted 140 from 217, getting 77 as the remainder. When I asked Dan to explain in words how he found the 77, in the middle of his explanation, he smiled and acknowledged that he meant to subtract the two numbers.

To summarize, Dan found the total number of tiles needed in the new extended floor and subtracted the number of tiles in the initial floor to find the excess amount of tiles. This approach was analogous to Morgan's approach in finding the excess number of unit-cubes needed to add onto the smaller cubic block to made a bigger cubic block. Instead of partitioning the tiles into units of units, Dan found a unit of units (28 units of 5 or 5 units of 28), found another unit of units (31 of 7 or 7 of 31) and took the resulting units (217 and 140) and subtracted one from the other.

Craig's sketch of the configuration. In order to better understand how each student visualized the situation and to support Craig's understanding of the situation, I asked both students to draw a picture to explain the numbers in their calculations. Craig produced a sketch as shown in Figure 7.22. To explain his sketch, Craig wrote "starting amount" in green referring to the green section in Figure 7.26. Then I asked how the red section was made, to which Craig replied "By adding sixty-two! By stacking two [*pointing to the additional height along the left hand side of the rectangle in red*], two by twenty-eight [*sweeping finger along the top side of the rectangle in green*], I think it was. Yeah. And then three [*pointing to the right bottom side of the rectangle in red*] by… What number would that be? Wait a minute, three by thirty-one." I asked Craig where in his picture demonstrated the "31" and he sat in silence looking at his sketch. Next, Craig wrote the number of tiles that should be laid along certain parts of his configuration, as shown in black ink in Figure 7.22 (“7”, “5”, “28”, “31, “3”).

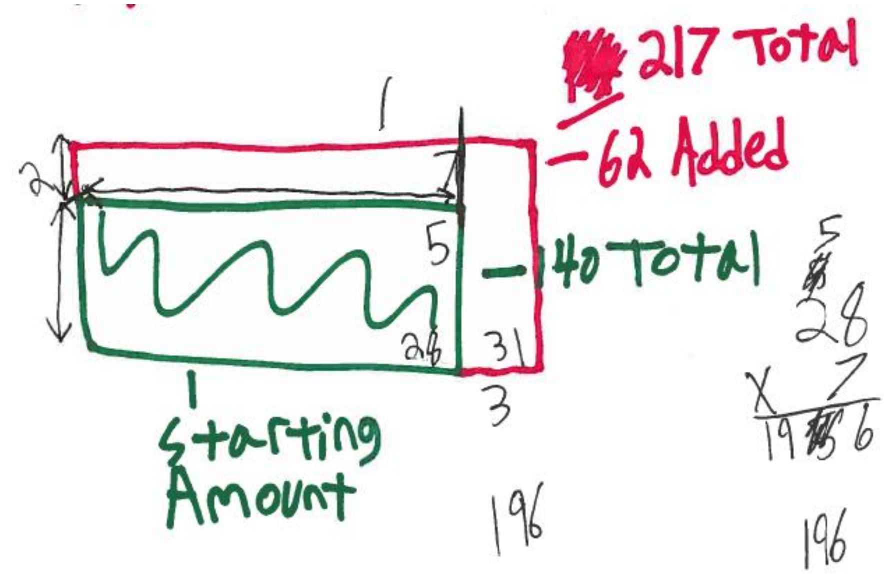


Figure 7.22. Craig’s sketch of the initial flooring (28×5) and the additional tiles to make the new flooring (31×7).

Craig said he still thought his answer 62 was correct, so I asked him where in his picture showed 31×2. After pointing to the additional section on the top in the red rectangle, he sat in thought looking at his sketch. Craig then wanted to calculate 2×28 and pointed to which section in his sketch that would cover. However, he said he could not figure out how many more he needed. As such, Craig coordinated the units along the horizontal axis with the units along the vertical axis for certain parts of his sketch. From this, I hypothesized that Craig engaged in a logical multiplication (Piaget et al., 1960) of the units along two perpendicular dimensions in order to guide his units-coordinating operations. However, he seemed unsure of how to find the total number of additional tiles he would need, using his calculations of the parts.

Dan's sketch of the configuration. When I asked Dan if he could produce a sketch to show his answer of 77, Dan first exclaimed that he did not know how to make a sketch. Finally, Dan produced a sketch, in a process captured in Figure 7.23.

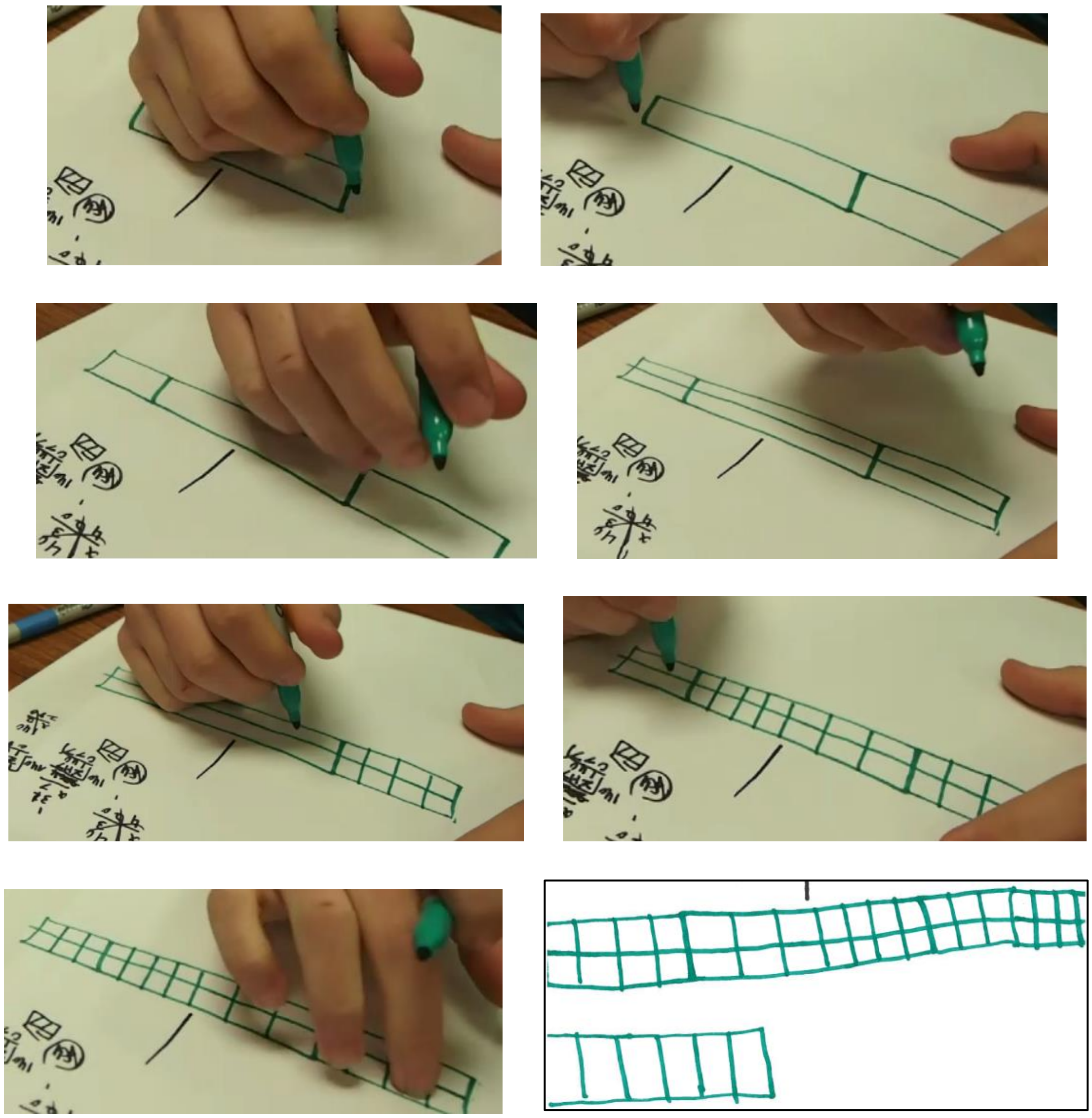


Figure 7.23. Dan's sketch of the additional tiles to make the new flooring (31×7).

As shown in Figure 7.23, analogous to his sketch of the unit-cubes in the 3-Cube, Dan's sketch suggested re-presentations of each individual tile but not necessarily a unitizing of the tiles into units of units. Instead of partitioning the long bar into 31 units, Dan haphazardly partitioned the long bar, counted the number of parts he had and added more tiles on the bottom because he did not have all 31 tiles he needed. Moreover, Dan focused on sketching the additional tiles not necessarily in relation to the initial flooring they assumed to have already covered.

In comparison with Craig, Dan seemed to lack the use of disembedding operations in solving the task. In solving the Units Coordinating Task in the initial interview, Dan did not demonstrate strategic reasoning of embedded composite units (Ulrich, 2016) nor did he demonstrate a spontaneous use of disembedding operations. For example, when I asked Dan how much more string he would need to make his string of 15cm as long as my string of 24cm, Dan counted up from 15 to 24 one by one, instead of disembedding composite units of the number sequence to configure 24 into 15 and 8, like Craig did. I claim that Dan's lack of disembedding operations, as shown in his initial interview, inhibited his systematic calculation and re-presentation of the additional tiles in the flooring situation.

Craig and Dan extend the flooring of 28×5 to one of 31×7 (Day 2).

Because we ran out of time, we continued with this task in the following teaching episode on December 1.

Craig's solution. Craig started with a new sketch showing the skeleton of the floor and the number of tiles along the length and width of the rectangular areas, as shown in Figure 7.24. Craig's sketch in Figure 7.24 suggested that he has partitioned the 31 tiles along the horizontal axis into a unit containing 28 and another unit containing 3 units. Craig disembeded the 28 units and the 3 units from the whole of 31 units.

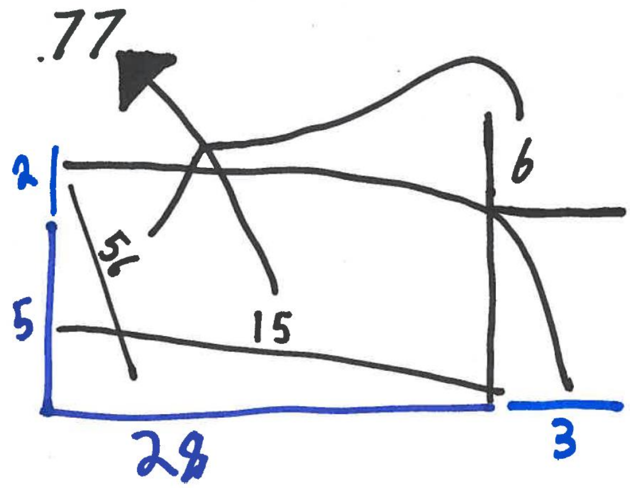


Figure 7.24. Craig's second sketch of the initial flooring (28×5) and the additional tiles to make the new flooring (31×7).

This way of operating was similar to his engagement in the second initial interview task when I asked Craig how much more string he would need to make a 14 cm long string as long as a 29 cm long string. Similarly, Craig partitioned the 7 tiles along the vertical axis into a unit containing 5 units and 2 units. He disembedded the 5 units and 2 units from the whole of 7 units. Then, Craig coordinated those with the units of 28 and 3 he disembedded from the 31 units along the horizontal axis in order to find the number of tiles that would fill in the space formed by the corresponding units. Inferring from his sketch, Craig found that he needed 28×2=56, 2×3=6, 3×5=15 tiles each to extend the initial flooring to the desired size of 31×7. It was apparent that Craig was able to re-present the tiles on the floor mentally and use his disembedding and units-coordinating operations to find the desired number of tiles.

Dan's solution. On the other hand, Dan drew a sketch on my request, with the individual tiles all drew in with an omission of some parts in the 28 because that required too many tiles to fit into the piece of paper (see Figure 7.25).

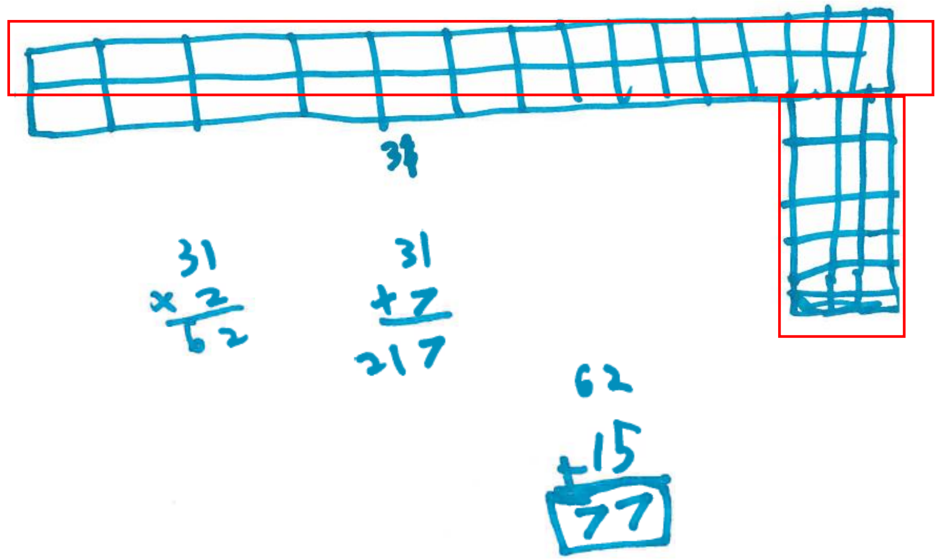


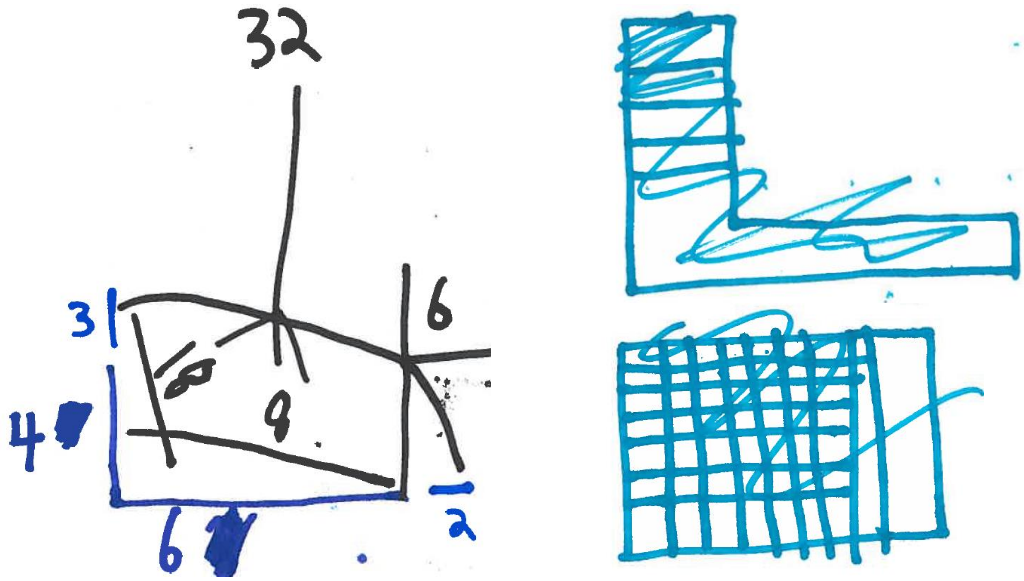
Figure 7.25. Dan's second sketch of the additional tiles to make the new flooring (31×7).

Dan's generation of Figure 7.25 was not fully independent because I was heavily involved in the process. To elaborate, in his drawing process, I asked Dan how many tiles were in total across the length of his sketch and how many there were in total along the width of his sketch. I also asked Dan how many tiles there were across the length and width of the original flooring. Then, I asked Dan to think about how he would count the total number of additional tiles using the numbers he knew. As shown in his calculations, Dan solved for 31×2+3×5=77 tiles in total. The purpose of my explicit questioning was to engender his unitizing of tiles. Although Dan produced the sketch, I could not attribute it fully to Dan's independent reasoning.

Craig and Dan extend the flooring of 6×4 to one of 8×7.

To explore whether Craig would use a consistent counting scheme with different numbers and to see whether Dan could carry out the unitizing of tiles independently, I posed a similar problem with smaller numbers. This time, starting with a floor with 6 tiles along the length and 4 tiles along the height, I asked them to imagine extending the flooring by 2 tiles and 3 tiles, respectively.

Craig looked at his previous sketch to make the sketch of this new problem, which suggested that he recognized the new situation as one similar to the previous one. Figure 7.26 (a) shows the sketch Craig produced. As shown in the figure, Craig consistently partitioned the length and the height of the rectangle into two units and disembedded each unit to coordinate with the corresponding one on the other axis.



(a) Craig's sketch (b) Dan's sketch

Figure 7.26. Craig's and Dan's sketch of the initial flooring (6×4) and the additional tiles to make the new flooring (8×7).

On the other hand, Dan produced a sketch as shown in Figure 7.26 (b). Although the given time for the session ran out and they did not finish their work, Dan demonstrated an attempt to make sketches in groups and in relation to the original floor, as opposed to individual tiles as he did in Figure 7.23.

Summary of the Floor Tile Task

One of the main differences that I observed in the two-dimensional case were in Craig and Dan's structuration and partitioning of the two-dimensional spatial object, i.e., the floor to cover with tiles, in their counting process. Craig generated sketches of the situation from which I inferred that he unitized unit-tiles into units of units of units and used his units-coordinating operations to count the total number of tiles. On the other hand, Dan seemed to produce the unit-tiles one at a time experientially or in re-presentation. In this process, the producing of the unit-tiles did not seem to entail a structuration of the spatial object; only after he produced each unit-tile did he go back to chunk them into units containing unit-tiles, often after I prompted him to think about different chunks in his sketches.

In comparison with Craig, I claimed that Dan's lack of disembedding operations, as shown in his initial interview, inhibited his systematic calculation and re-presentation of the additional tiles in the flooring situation. On the other hand, I claimed that Craig was able to re-present the tiles on the floor mentally and use his disembedding and units-coordinating operations to find the desired number of tiles. Moreover, I hypothesized that Craig engaged in a logical multiplication (Piaget et al., 1960) of the units along two perpendicular dimensions in order to guide his units-coordinating operations.

The tiling activity was helpful in observing different ways Craig and Dan enacted or used unitizing, disembedding, and units-coordinating operations. As illustrated in their different sketches when solving for the additional number of tiles needed, Craig unitized tiles into units of unit-tiles whereas Dan seemed to operate heavily with individual tiles. Although Dan demonstrated strength in inserting units into rows in his explanation of why he multiplied two numbers, I hypothesize that the rows were yet unitized into a unit containing the rows as a unit.

Based on the aforementioned observations, Craig seemed to operate with at least three levels of units in activity when Dan seemed to operate with two levels of units in activity, consistent with the findings from the initial interviews. The different sketches they produced of the situation also suggested the differences in their abstractions of the situation. Craig's sketches entailed a structuration of the given spatial situation whereas Dan's sketch entailed the individual tiles produced sequentially. Probing Dan to attend to the structure of his sketches, emphasizing the number of units in the initial flooring contained in the new unit of the extended flooring seemed to engender some awareness of the structure of the two-dimensional tiling situation.

Box Task: Craig and Dan Fill Rectangular Boxes of Various Dimensions

Filling in a Shoebox with Unit-cubes

On December 12, 2014, Craig and Dan engaged in the first part of the Box Task, in which they found the number of unit-cubes needed to fill in a shoebox. Analogous to how we first started the two-dimensional tiling task with a concrete example of tiling a floor of a fixed size, I asked Craig and Dan to find the total number of unit-cubes needed in filling a shoe box of a fixed size. Craig and Dan worked separately with boxes of different size. To put the task in context, I asked both students to imagine building brick wall posts of various sizes. The goal of this task was to explore the difference in Craig and Dan's insertion and coordination of units in counting the total number of unit-cubes comprising the volume of each shoe box.

Craig finds the number of unit-cubes needed to fill in a shoebox of fixed size.

Craig first aligned unit-cubes along two edges of the box (what he later referred to as the length and height of the box), as shown in *Figure 7.27 (a)*, counted the number of unit-cubes, and wrote "33" on his paper. Next, Craig aligned more unit-cubes along two edges of the box (which he later referred to as the width and height of the box) on the adjacent face as shown in Figure 7.27 (b) and wrote "18" on his paper. Then, he solved for "33 times 6" which he concluded was 198 unit-cubes in total.

(a) (b)

Figure 7.27. Craig's shoeboxes.

I asked Craig to explain the numbers he had written on his paper and what he did with them. Pointing to the face in Figure 7.27 (a), he explained that he multiplied three and eleven, which gave him 33 unit-cubes in total to cover the "wall." Referring to the three unit-cubes aligned along the height of the second face (Figure 7.27 (b)), and picking up the top two unit-cubes among the three, Craig said:

> "I have three here and then one, two, three, four, five, six [*moves the top two unit-cubes along each unit-cube on the bottom as he counts*] on this wall. And [it's] the same height, so I don't need those [*discards the two top unit-cubes he was holding*]. So, I have six here and it will take six of these walls [*points to the adjacent face (Figure 7.27 (a))*] to fill up this box."

Later when both students were done working on their shoe boxes, I asked Craig to explain what he did to Dan. Craig articulated his thought process as shown in Excerpt 7.8.

Excerpt 7.8. Craig explains to Dan how he counted the number of unit-cubes contained in his shoe box.

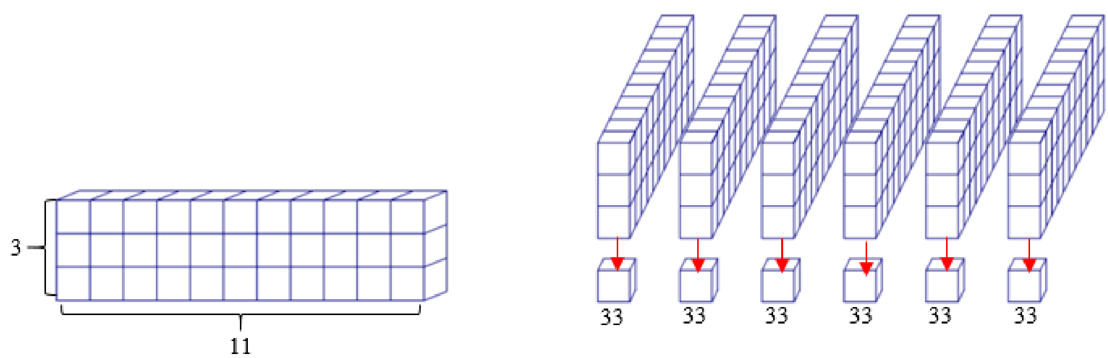
C: Well, I did, I tried to simulate what the computer program did and I did this side [*sweeping his index finger along the face shown in Figure 7.3 (a)*] and I did the height [*points to the 3 unit-cubes stacked on the left-hand side*] and the…[*hesitates*] is it width or length?

D & T: [*Together*] Length.

C: The height and the length. But then it took me a little while to figure out what to do next and what I had to do next was [*moving the 6 unit-cubes he* *placed along the bottom of the adjacent face (Figure 7.27 (b))*] line up blocks along this wall [*points to face in Figure 7.27 (b)*] over here and then [*puts the 6 unit-cubes back where they were*] just insert this wall [*pointing to the face in Figure 7.31 (a)*] into a block. And there are six blocks on this side [*again, points to the 6 unit-cubes on the bottom of the face in Figure 7.27 (b)*] and 33 blocks here [*points to the face in Figure 7.27 (a)*]. So, thirty-three times six [*moves his index finger along the top of the box as if he is simulating the wall of 33 blocks being inserted into each of the 6 unit-cubes along the third dimension*] would fill this whole box with the blocks.

D: [*Claps once and smiles*.]

Later in the teaching episode, I learned that both Craig and Dan had used a 3-D printer in their technology class to make 3-D objects, which explained what Craig meant by "the computer program" in Excerpt 7.8. Based on his explanation and motions he made with his hands, I modeled the walls and the unit-cubes Craig associated with the walls in Figure 7.28. To elaborate, I interpreted his actions and verbal descriptions to mean that Craig constructed the vertical layer, which he called a wall, of 33 unit-cubes (Figure 7.28 (a)) and inserted each of those layers into the six unit-cubes along the third dimension (Figure 7.28 (b)), using his units-coordinating operations.



(a) (b)

Figure 7.28. A model of Craig's counting of the number of unit-cubes contained in his shoebox.

Reversing his partitioning operations he used in reasoning with the 2-Cube and 3-Cube, Craig seemed to build up the box from layers of rectangles as he simultaneously partitioned the box and counted the number of "walls" he needed to fill in the entire box. So far, in the Cubic Block Task and Shoe Box Task, Craig consistently decomposed the rectangular prisms into rectangular-shaped layers. Then, Craig enacted his splitting and units-coordinating operations in counting the total number of unit-cubes in the prisms. From these observations from the Cubic Block Task and Shoe Box Task, I hypothesized that Craig has enacted the mental operations involved in a *reversible decomposing scheme*. Further, I conjectured that Craig had the operations of a FR coordinating scheme available to support his reversible decomposing activity. This was further tested through the Locating Tasks, which I discuss in Chapter 8.

Dan finds the number of unit-cubes needed to fill in a shoebox of fixed size.

Dan aligned unit-cubes along two adjacent faces. Figure 7.29 shows how Dan covered one of the two faces with unit-cubes. After placing 10 unit-cubes along the bottom of the face visible in Figure 7.29, he added 4 unit-cubes vertically on top of the first unit-cube on the bottom far right, resulting in unit-cubes aligned like a sideway L-shape. Then, he repeatedly added 4 unit-cubes vertically stacked on top of each unit-cube on the bottom until he obtained the configuration in Figure 7.29. He repeated making the sideway L-shape on the adjacent face but did not add as many unit-cubes as he did in the previous face.

Dan stopped his covering activity and counted the number of unit-cubes along the bottom of each face he covered and wrote 35 and 50 on his paper. It seemed as if he was aware that the height of each face was 5 so he only needed to count the base of each face.

After Dan wrote the two numbers 50 and 35 on his paper, I asked Dan to explain the numbers he found and what he was going to do with them. About the 50, Dan explained that he multiplied the 10 (sweeping his finger along the 10 unit-cubes on the bottom of the face shown in Figure 7.29) and 5 (sweeping his finger along the 5 unit-cubes lined along the height of that face), "but then I filled that in for some reason," referring to the blocks he added to that face. His last comment of filling the face "for some reason" suggested that he viewed the activity as unnecessary. It is possible that after repeating his activity of stacking 4 extra unit-cubes onto each unit-cube on the base, Dan abstracted the activity of inserting 5 unit-cubes in total to each unit-cube in the bottom of the face and realized he did not need to continue his activity.

Figure 7.29. Dan's shoebox.

Next, Dan said that he would double the 50 and 35 and then add them altogether. As such, Dan wanted to find the total number of unit-cubes that would cover the four side faces of the shoe box. It is possible that the way I put the task into the context of "building brick wall posts" influenced Dan's thinking of the problem because the term "wall" could have been interpreted as "face." However, his focus on the surface area was consistent with his previous activities in the Cubic Block Task, such as his building of the cubic blocks from the exterior to the interior and his focus on the "sides" of the cubic blocks when counting them.

After Dan listened to Craig explain how he counted the total number of unit-cubes in Excerpt 7.8, Dan clapped as Craig finished his explanation, which suggested that he understood Craig's explanation and potentially assimilated Craig's counting method. The next task that I had prepared served as an opportunity to test whether or not this was the case.

Craig and Dan Build a Brick Wall Post

For the next task, I prepared a big white piece of paper with two perpendicular line segments drawn on it. This paper and line segments were designed to resemble the Floor that was sketched in GSP (see Figure 7.19) with one line segment representing Craig's side and the other line segment representing Dan's side of the base. I explained that it was similar to the flooring task but that this time we were also going to build up to make a post, like the shoe box. Dan placed 13 unit-cubes and Craig placed 10 unit-cubes along the two axes on the white paper (floor). I placed 3 additional unit-cubes onto the unit-cube that was placed at the intersection of the two line segments, making the height of the configuration 4 unit-cubes long. Together, we built the frame of a rectangular prism with dimensions 13×10×4 (see Figure 7.30). Because we did not have enough time left in our session, each student worked separately on the task starting in the next teaching episode on December 15, 2014.

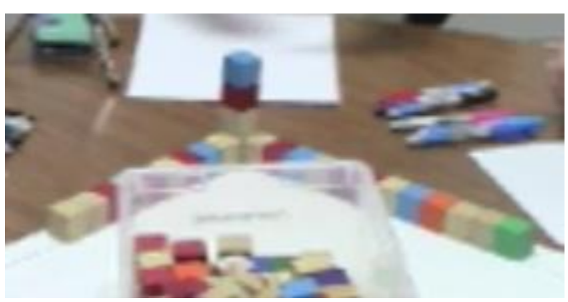


Figure 7.30. The frame of the rectangular prism (wall post) with dimensions 13×10×4.

Craig's solution.

Figure 7.31 shows Craig's written solution to the total number of unit-cubes in the configuration. Each component of his solution was written in a particular order. Craig first wrote the three numbers, 10, 13, and 4 at the top of his paper. After staring at the configuration and then his paper for approximately 30 seconds, Craig wrote the two equations $13 \cdot 10 = x$ and $x \cdot 4 = Desired\ \#$ underneath the list of three numbers. Then murmuring "I'm lazy," Craig claimed he had an answer but that it was not exact (the numbers in parentheses in Figure 7.31 were calculated later). Craig did not execute the calculations he expressed in the equations but seemed confident that he could find the number of unit-cubes using his answer. As such, Craig seemed to have abstracted the structure of the configuration and was certain that his equations would give him the desired number of unit-cubes.

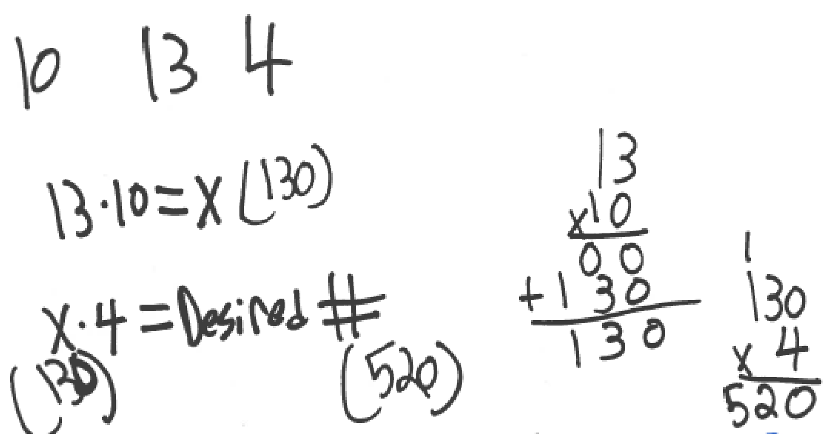


Figure 7.31. Craig's calculation of the total number of unit-cubes contained in a box with 10, 13, and 4 unit-cubes along each dimension.

Although Craig claimed he had an answer, I was turned to Dan talking with Dan about his work. So, Craig went back to work independently on his paper and finished the calculations by finding the x value and the total desired number of unit-cubes, as shown on the right and in the parentheses in Figure 7.31. When I finally turned to Craig and asked him to explain what he had on his paper, he said:

> "Okay, I got the numbers down, which is ten, thirteen, and four, were the three dimensions. And then I did thirteen times ten which equaled x, and I found out that x was one-hundred and thirty. Then I did one-hundred and thirty times four, which I knew would equal the desired number. Which is five-hundred-twenty."

Dan's solution.

Figure 7.32 shows Dan's written solution to the total number of unit-cubes in the configuration. To elaborate, Dan multiplied the 10 and 4 to obtain 40, multiplied the 13 and 4 to obtain 52, and then added the two numbers 40 and 52, which gave him 92.

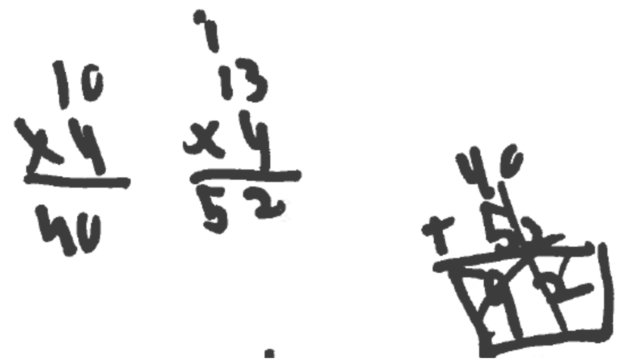


Figure 7.32. Dan's calculation of the total number of unit-cubes contained in a wall post with dimensions 13×10×4.

When I asked Dan why he added the two numbers 40 and 52, he explained that he added them because there wasn't really much he could do because he already multiplied the numbers to obtain 40 and 52. Dan said, "So, I guess there's nothing to do with that." So, I asked Dan to point to where on the configuration (Figure 7.30) the 40 were. Dan pointed to Craig's side where the 10 unit-cubes were placed on the bottom, saying "the forty is that side." Next, I asked Dan to tell me where the 52 were. Dan pointed to his side where 13 unit-cubes were placed on the paper. The following excerpt is our conversation following after Dan identified that the 40 and 52 each referred to the number of unit-cubes each consisting the two sides of the configuration.

Excerpt 7.9. Dan explains how he solved for 92 unit-cubes in total.

T: So, what you’re saying is that you want to add this side and this side, right?
D: Yeah.
T: Right? But what about the thing in between [*sweeping hand in the blank space enclosed by the frame*]?
D: [*Sits in silence, touching his mouth for approximately 5 seconds*.]
T: So, let’s imagine this [*picking up the container full of unit-cubes and placing it closer to the frame as shown in Figure 7.33*] is a box, right? What you just found was you found this side [*pointing to the side face closest to Dan*], right?
D: This side…Times those two together [*pointing subsequently to one side then the other*]. Yeah...

Figure 7.33. Teacher-researcher placed rectangular prism-like container close to the frame of 13×10×4 unit-cubes.

As shown in Excerpt 7.9, it seemed like Dan was aware that he added the number of unit-cubes contained in two sides of the configuration. But not until I asked him about the unit-cubes that would go inside of the configuration and demonstrated them with the container of unit-cubes did he change his mind to multiply the two numbers 40 and 52 to find the total number of unit-cubes contained in the post. Because I wanted Dan to have some more time to think on his own about the task, I asked Dan to think more about his answer. Dan went back to solving for the number of unit-cubes on his paper, while I turned to Craig.

Before Craig had said anything to me, Dan calculated 10 times 13 and wrote 130 on his paper. Then, he wrote another 130 on his paper and paused. He looked up from his paper towards where the configuration was on the table. At this point, Craig had started explaining his solution to the task and just as Craig had finished saying that he took the 130 and multiplied it by 4, Dan wrote ×4 underneath the second 130 he had just written, and solved for 520 unit-cubes in total.

Because I was focused on listening to Craig in the moment, I did not notice Dan's activity on the other side of the table. It was possible that the timing of the two events were coincident. That is, Dan's writing of ×4 and Craig's explanation of multiplying 130 by four may have been two independent activities that happened to occur at the same time. However, from the analysis of the video, it appeared Dan stalled after writing the second 130 on his paper but wrote ×4 immediately after Craig had just finished saying that that was what he did. So, it was also possible that Dan had mimicked Craig's calculation. To my surprise, however, as Dan finished his calculation, he murmured "that's what I was saying," which suggested that it is possible that all the way along Dan was thinking of multiplying the three numbers but his writing/drawing and execution of that idea did not portray it.

Craig and Dan discuss their answers.

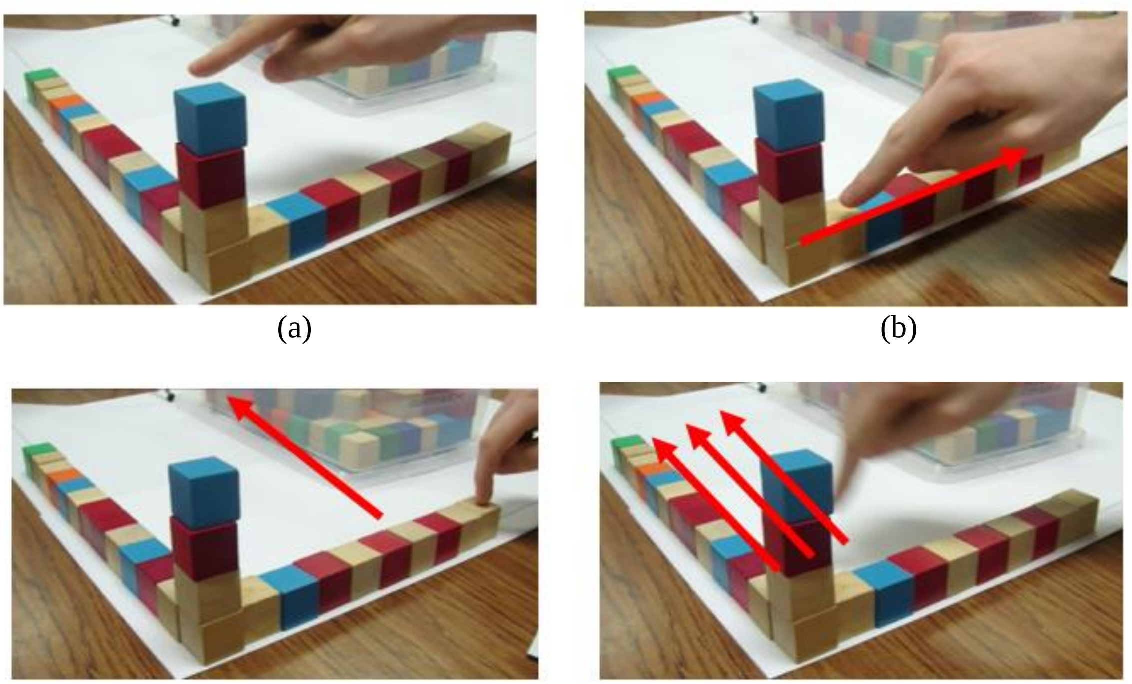
As Craig finished his explanation, Dan also acknowledged that he had gotten the same answer as Craig, so I asked both students to explain why it made sense for them to multiply the three numbers 10, 13, and 4 together, using the configuration on the table.

Dan replied “cuz it does” and shrugged as if it was obvious he did not know how to explain it. As such, Dan did not articulate why he multiplied the three numbers 10, 13, and 4.

Craig responded differently. In the next excerpt, Craig talks about how he “built” the brick post to explain why he multiplied the three numbers 10, 13, and 4.

Excerpt 7.10. Craig explains again why he multiplied the three numbers 10, 13, and 4.

C: I did thirteen [*points to the 13 unit-cubes along Dan’s side (Figure 7.34 (a))*], for each unit-cube here [*sweeps his index finger along the 10 unit-cubes along his side (Figure 7.34 (b))*]. So each unit-cube here [*once again sweeps his index finger along the 10 unit-cubes along his side (Figure 7.34 (b)*] represents a hundred, no, represents thirteen. So, thirteen [*taps on the far right unit-cube (Figure 7.34 (c))*], thirteen [*taps on the next unit-cube*], thirteen [*taps on the next unit-cube*], all the way up [*sweeps his index finger vertical to the unit-cubes aligned along his side as shown in the red arrow in Figure 7.34 (c)*].

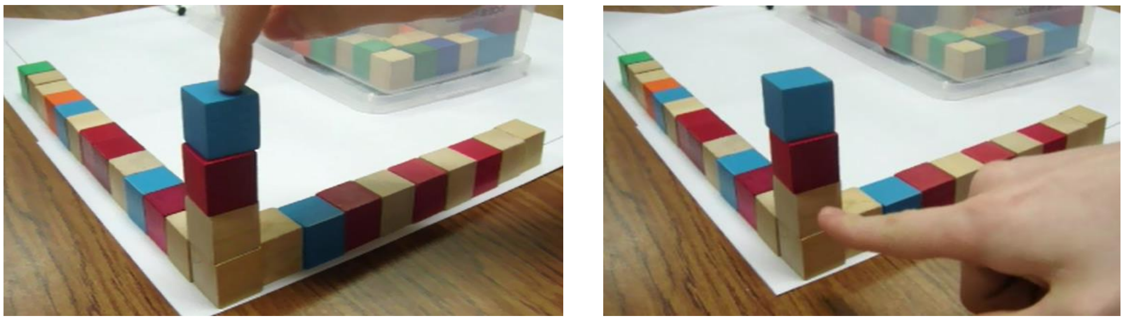


(a) (b) (c) (d)

Figure 7.34. Craig explaining his process of finding the total number of unit-cubes contained in a bottom layer with dimensions 13×10.

[*Continued*]

C: So, it's kind of like building it, thirteen, thirteen, thirteen [*moves finger along the red arrows shown in Figure 7.34 (d)*], once it gets to the end [*pointing to the unit-cube at the end, like in Figure 7.34 (c)*], it's a hundred and thirty. So, then a hundred thirty [*pointing to the unit-unit-cubes on the floor*], then each one of these [*points to the unit-cube as shown in Figure 7.35 (a)*] are a hundred-thirty. So, a hundred-thirty, a hundred-thirty, a hundred-thirty, a hundred-thirty [*tapping at each unit-cube as he moves up along the three remaining unit-cubes that I placed to represent the height (Figure 7.35 (b)*], which equals five-hundred-twenty.



(a) (b)

Figure 7.35. Craig explaining his process of finding the total number of unit-cubes contained in the wall post with dimensions 13×10×4.

As demonstrated throughout Excerpts 7.10 and 7.11, Craig consistently counted the total unit-cubes in a systematic manner by building the desired brick post, as he described. First, he mentally built the rectangular horizontal layer of 130 unit-cubes at the bottom of the post. Then, instead of rebuilding the subsequent layers over and over again, he took the first layer as given and inserted it into each unit-cube along the third dimension one by one, counting 130 four times. As such, Craig constructed a unit (rectangular layer) of (10) units of (13) units, unitized it, and took it as new input to insert into another unit of (4) units. Using his units-coordinating operations, Craig coordinated each 130 units into each of the 4 units along the height and counted the total number of unit-cubes contained in the block post. In other words, Craig seemed to have coordinated three levels of units in successions to produce the rectangular prism.

Craig's counting activities in the brick post task were consistent with his counting of the unit-cubes contained in the shoe box. That is, he found the product of the number of unit-cubes along the three dimensions. Although it seemed at first like he relied on a computational formula for finding the volume, he was articulate in explaining why he multiplied the three dimensions. Craig did this by building up the rectangular prism in representation by first constructing a layer and then inserting that layer into each unit along the third dimension. Moreover, Craig flexibly changed the layer and the third dimension into which he inserted the layer. This observation corroborated that Craig has indeed could engage in mentally decomposing and re-composing the rectangular prism into a collection of layers, each containing units of units. Hence, leading to the hypothesis that Craig has enacted the mental operations involved in a *reversible decomposing scheme*. Further, Craig's coordination of multiple perspectives and a re-presentation of one instantiation of his activity in co-presence with another instantiation of the building process seemed to support his ability to anticipate the result of his activities. Therefore, I hypothesized that Craig had the operations of a FR coordinating scheme available to support his reversible decomposing activity. This was further tested through the Locating Tasks, which I discuss in Chapter 8.

When I asked Dan if Craig's explanation made sense to him, Dan acknowledged that he understood Craig's explanation. Although Dan seemed to understand Craig's method of counting the unit-cubes, he was yet to execute them independently or explain them in his own words. Therefore, although there may have been some assimilation involved, it could have been that Dan has assimilated Craig's solution of multiplying the three dimensions but Dan was yet to independently re-generate an explanation of why it made sense to multiply them. The explanation that Dan provided involved the two-dimensional areas of the rectangular prisms but not its volume.

Craig and Dan find the number of unit-cubes needed to extend the brick wall posts.

I wanted to find out if Craig could reason recursively and take his structuration of the rectangular prism as input and further operate on it. Also, I wanted to test to what extent Dan could assimilate Craig's method of multiplying the number of units constituting each dimension. Further, I wanted to investigate what Dan would do in a situation where one could not simply multiply the measures of each dimensions. So, I posed a problem that involved extending the brick wall posts. First, I added 1, 2, and 1 unit-cubes to the dimensions of 13, 10, 4 unit-cubes, respectively. This resulted in a new configuration with dimensions 14×12×5, as shown in Figure 7.36. With that new configuration, I challenged both students to find the total number of *additional* unit-cubes in order to make such extension of the brick post that consisted of 520 unit-cubes.

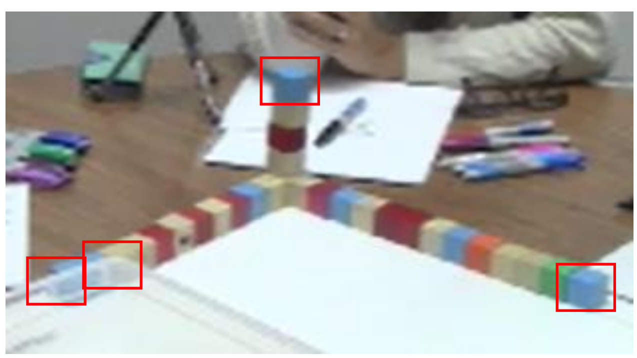


Figure 7.36. 1, 2, and 1 unit-cubes each added onto the dimensions of 13, 10, and 4, respectively.

Dan's solution to the extension problem. Dan wrote the three numbers 2, 1, 1 on his paper. Then, he multiplied 2 and 1, took the result 2 and multiplied that by 5 to obtain 10. Finally, he added the 10 to 520, which was 530. Dan enclosed the 530 in a rectangle, indicating that it was his final answer. When I asked Dan what the number of additional unit-cubes was, he scratched out the 530 and drew a rectangle around the 10, to indicate that he needed 10 additional unit-cubes to make the extension.

Because Craig was still working on his solution, I turned to Dan to ask him some questions. First, I asked Dan where the 5 came from in his calculation, to which he replied by pointing to the 5 unit-cubes consisting the height of the configuration in Figure 7.36. Then, Dan explained that it was the same thing, pointing to his calculations of 10×13=130 and 130×4=520 unit-cubes he did earlier when solving for the total unit-cubes contained in the brick post with dimensions 13×10×4. As such, Dan recognized the new problem as the "same thing," and multiplied the numbers 2 and 1, each representing the additional unit-cubes from each side of the bottom of the configuration, and multiplied that by the new height, 5 unit-cubes.

After Craig finished working on his solution, I asked each student to take turns in explaining their solutions. The following excerpt shows Dan's explanation of how he found the 10 additional unit-cubes to make the extension.

Excerpt 7.11. Dan explains how he found 10 additional unit-cubes to make the extension.

D: I timesed [multiplied] these two [*pointing to each alignment of 12 and 14 unit-cubes on the paper*].
T: So those two mean…?
D: This [*points to the 14 unit-cubes placed on the paper in front of him*] and that [*points to the 12 unit-cubes placed on the paper in front of Craig*]. The one [*lifted the 1 blue unit-cube at the end of his side*] and two [*pointed to the 2 unit-cubes at the end of Craig's side*].
T: Mm-hmm. Okay, so one, and two [*repeats Dan's pointing activity*].
D: Which is two.
T: Two, okay.
D: And then I timesed [multiplied] two by five [*pointing to the height of the configuration*] and I got ten. [*Laughs*].
T: Okay, two times five and you got ten. Okay.
D: Yeah. Actually, I'm not even positive that's correct.

As I inferred from his calculations, based on his explanation in Excerpt 7.11, Dan had indeed multiplied the 1 and 2 additional unit-cubes and multiplied the product by the new height 5. However, as I claimed earlier, although Dan assimilated Craig's solution of multiplying the number of unit-cubes, this solution was not necessarily connected to the situation. In other words, Dan did not represent the 10 unit-cubes in relative position to the imaginary original rectangular prism of 520 unit-cubes. The coordination of a re-presentation of one instantiation of his activity in co-presence with another instantiation was challenging for Dan and limited his ability to anticipate the result of his activities. Moreover, the lack of an operationalized structuration of the three-dimensional object seemed to inhibit his calculations. Because he could not represent the 10 unit-cubes in relation to the original rectangular prism of 520 unit-cubes, Dan was unsure whether the 10 unit-cubes will make the desired extension. As shown towards the end of Excerpt 7.11, Dan claimed he was not sure if his answer was correct.

Craig's solution to the extension problem. Figure 7.37 shows Craig's written work.

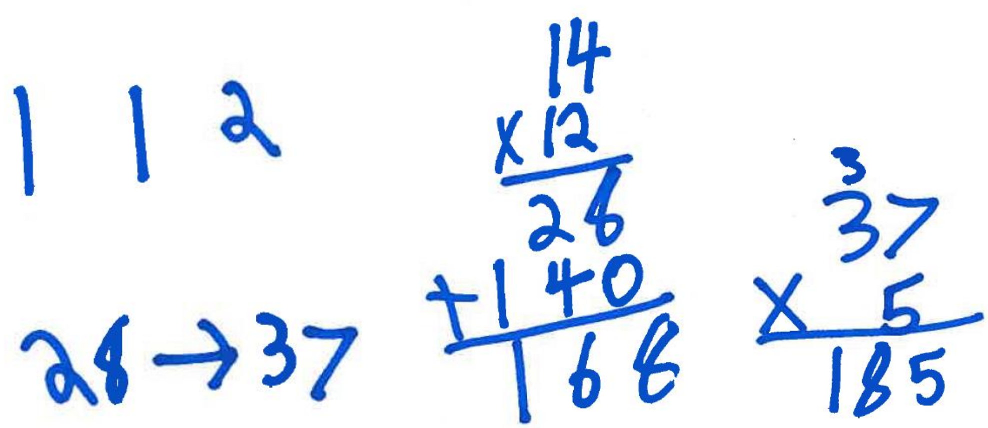


Figure 7.37. Craig's written work finding the total number of additional unit-cubes for the 1, 2, 1 extension.

Craig’s written work is better explained in his own words. The following two excerpts show Craig’s explanation of how he solved for the additional unit-cubes to make the desired extension.

Excerpt 7.12. Craig explains what he did to find the additional unit-cubes to make the extension (Part I).

C: I was like, I keep forgetting what’s supposed to be on the end, what you added, so I decided to switch out the blocks that you added with blue blocks.

T: Blue blocks. Mm-hmm.

C: And then, two [*pointing to the 2 blue unit-cubes added onto his side*] times fourteen [*pointing to the unit-cubes placed along Dan’s side*]. So this [*sweeps his finger above the table as shown with the red dashed arrow in Figure 7.38 (a)*]. I basically built it out again. So I did this [*repeats his sweeping motion illustrated in Figure 7.38 (a)*], and then I’m like what do I do next? And I had figured out I had to do this [*sweeps his finger above the table as shown with the red dashed arrow in Figure 7.38 (b) starting from the blue unit-cube at the end of Dan’s side*].

(a) (b)

Figure 7.38. Craig’s sweeping motion above the table to indicate the 28 and 10 additional unit-cubes along the bottom of the configuration.

[*Continued*]

T: So, you did two times fourteen and…

C: Yes, I did two times fourteen [*repeated sweeping motion shown in Figure 7.38 (a)*] and then one times ten [*repeated sweeping motion shown in Figure 7.38 (b)*], which is ten, obviously. So, I added ten here [*repeated sweeping motion shown in Figure 7.38 (b)*].

Based on his written work shown in Figure 7.37 and his explanation in Excerpt 7.12, I model Craig's counting of the additional unit-cubes in the extension, as the following.

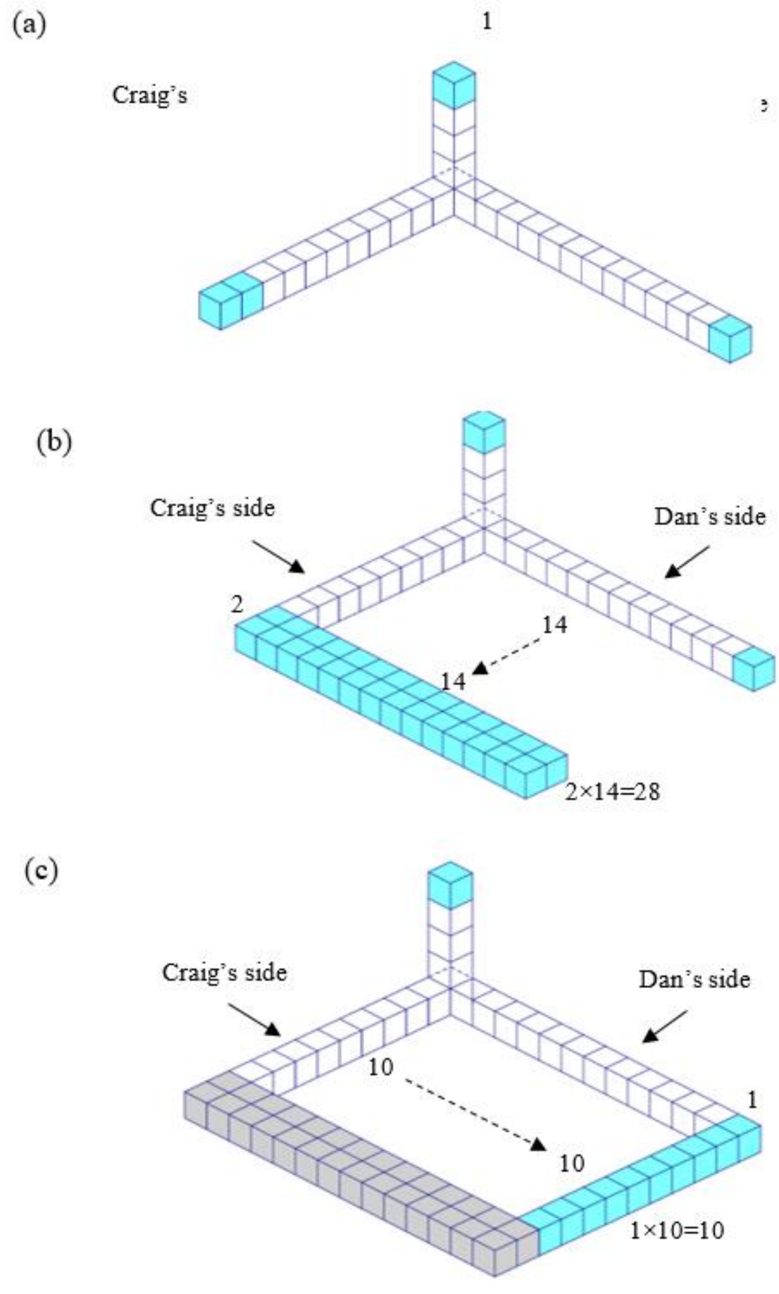


Figure 7.39. A model of Craig's representation of the extension of the brick post (Part I).

First, Craig marked the additional unit-cubes by changing their color to blue (Figure 7.39 (a)). Next, he coordinated the 2 unit-cubes added to his side of the floor with the 14 unit-cubes along Dan's side of the floor (Figure 7.39 (b)), producing 28 additional unit-cubes. Craig then coordinated the 1 unit-cube added to Dan's side of the floor with the 10 unit-cubes along his side of the floor (Figure 7.39 (c)), producing 10 additional unit-cubes.

This process explains the "28→37" he wrote on his paper in Figure 7.37. After Craig had written the 28 on his paper, his gaze on the configuration on the table shifted slowly from one side to the other, suggesting that Craig had counted up from 28 as many unit-cubes there were placed along his side. In this process, it is likely that he counted 1 unit-cube less, when meaning to add 10 unit-cubes in total. So, Craig ended up with 37 and not 38 unit-cubes in total. Craig continued his explanation in the following excerpt.

Excerpt 7.13. Craig explains what he did to find the additional unit-cubes to make the extension (Part II).

C: So, we've got this first layer added [*places each hand at each end of the two sides on the paper and sweeps both hands simultaneously, as shown in the red dashed arrows in Figure 7.40*] of how many more we would need.

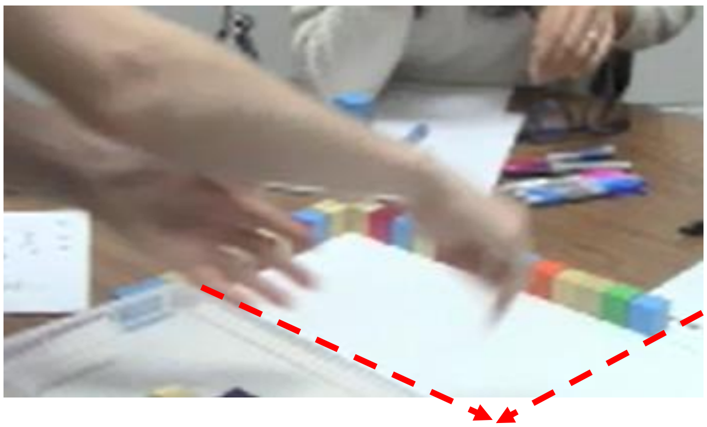


Figure 7.40. Craig makes hand motion to describe the additional unit-cubes along the bottom of the configuration.

[*Continued*.]

T: Mm-hmm…

C: And then I did the top [*pointing to the blue unit-cube placed on top of the 4 unit-cubes consisting the height of the configuration*] because I forgot something, which I'll explain in a minute. But, I did the top times… [*pauses and looks back at his paper*] It was five… Yeah, I did the bottom [*pointing to the bottom of the configuration*] times this [*pointing at the blue unit-cube placed on the top of the 4 unit-cubes consisting the height*] one, which was up to five and somehow I got a hundred-sixty-eight. But I had forgotten…[*paused and looked back at his paper*]. Yeah, I knew I wouldn't remember this. This was very complex.

Based on his written work and his explanation in Excerpt 7.13, I model Craig's continuation of counting of the additional unit-cubes in the extension as the following. First, he held the 28 and 10 unit-cubes generated along each dimension of the bottom of the configuration and unitized the units into one layer consisting of 38 (or 37) unit-cubes (Figure 7.41 (a)). Next, Craig noticed that there was 1 unit-cube added to the top of the height, which would make an additional "top" to the configuration, consisting of 14×12 unit-cubes (Figure 7.41 (b)). His calculation of 14×12 is demonstrated in Figure 7.37. Then, realizing that he left something out, Craig took the first layer as a unit (figure 7.41 (a)) and inserted it into each of the 5 unit-cubes consisting the height of the brick post (Figure 7.41 (c)). As Craig explained, he "did the bottom times this one [the height], which was up to five."

Although Craig seemed to have lost track of his thought process and later sounded less confident about his solution, it was clear to me that Craig had engaged again in a systematic structuration of the three-dimensional object in re-presentation. He was able to represent the 520 unit-cubes that consisted the imaginary brick post that was assumed to have already been built. Craig then considered the additional unit-cubes both in spatial and quantitative relation to the existing configuration. Because the existing configuration was not in sight, this required a heavy cognitive load, evidenced by Craig's engagement in deep thought, his relatively long pauses, his grabbing of his head throughout his solution of this problem, and his claim that this problem was very complex.

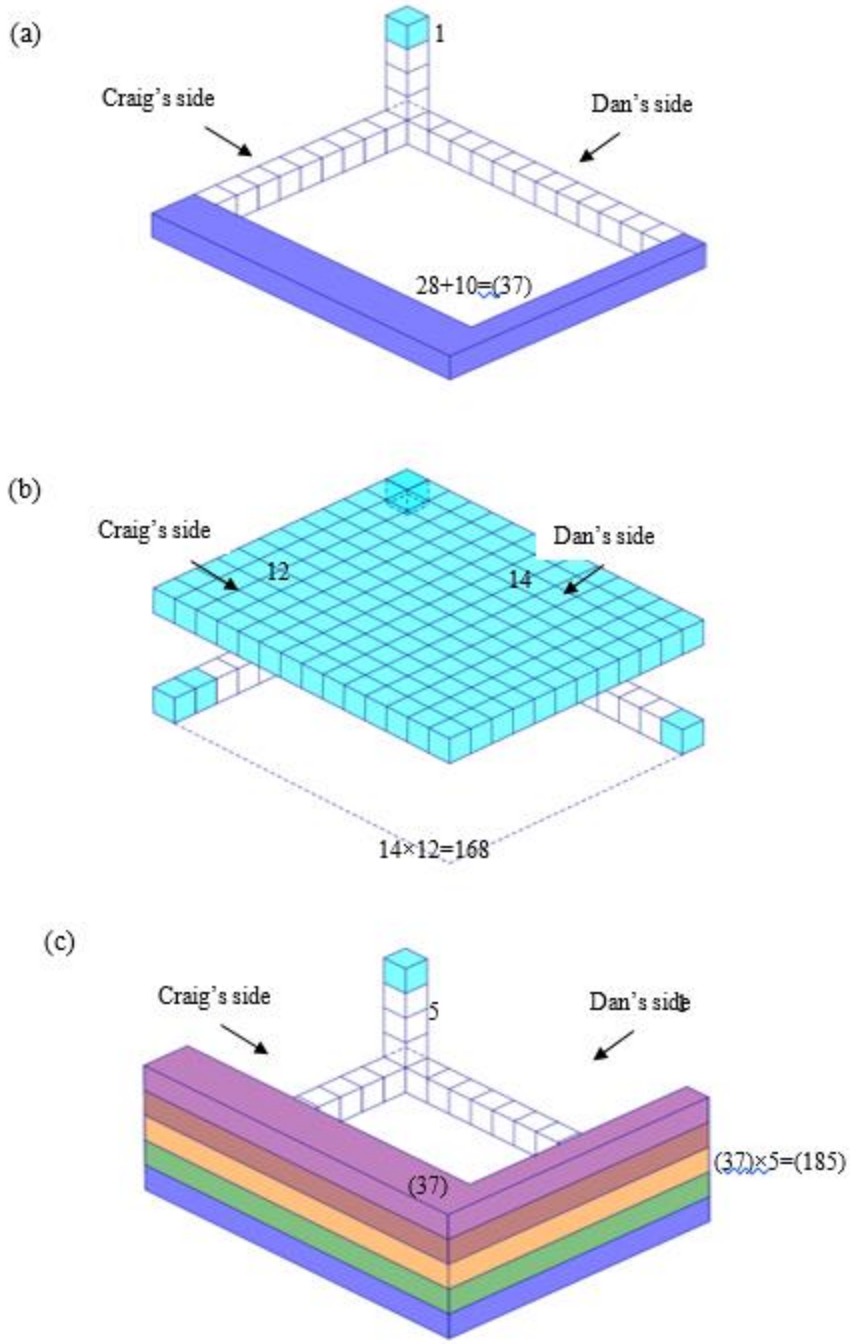


Figure 7.41. A model of Craig's representation of the extension of the brick post (Part II).

Indeed this extension problem was a novel task for Craig. Using the available operations he had, although not fully executed, Craig was able to build the extension step by step, consistent with the manner he built the existing configuration of 520 unit-cubes. That is, he constructed the first layer of the extension and inserted that layer into the total number of unit-cubes consisting the height of the new configuration. Although his additional 12×14 unit-cubes were repetitive, it demonstrated his awareness of the one additional unit-cube along the height and how that related to the extensions in the other two dimensions (14×12 unit-cubes).

Summary of Box Tasks

Craig's counting activity.

In the Box Tasks, Craig consistently found the product of the number of unit-cubes along the three dimensions to find the total number of unit-cubes needed to fill or build various sized boxes or rectangular prisms. Although it seemed at first like he relied on a computational formula for finding the volume, Craig was articulate in explaining why he multiplied the three dimensions in a consistent and systematic manner. From his demonstrations of his counting activity, I inferred that Craig mentally built units of rectangular layers consisting of units (along one spatial dimensions) of units (along a second spatial dimension) and inserted the layer into each unit-cube along the third dimension constituting the box or rectangular prisms. Moreover, Craig flexibly changed the layer and the third dimension into which he inserted the layer. As a result, Craig produced three levels of units in successions and enacted his units-coordinating operations in counting the total number of unit-cubes.

Based on the way Craig consistently decomposed and recomposed the box or rectangular prisms through enacting his splitting operations led me to hypothesize that Craig has enacted the mental operations involved in a *reversible decomposing scheme*. Further, Craig's coordination of multiple perspectives and a re-presentation of one instantiation of his activity in co-presence with another instantiation of the building process seemed to support his ability to anticipate the result of his activities. Therefore, I hypothesized that Craig had the operations of a FR-coordinating scheme available to support his reversible decomposing activity. This was further tested through the Locating Tasks, which I discuss in Chapter 8.

Dan's counting activity.

In the Box Tasks, Dan consistently showed a tendency to focus mainly on the surface areas of faces of the shoe box or rectangular prisms, consistent with his previous activities in the Cubic Block Task. For example, in the Show Box Task, Dan partially covered two adjacent faces of his shoebox with unit-cubes, counted the number of unit-cubes he will need to cover those faces, doubled each number and found the sum of those two measures. Later in the Rectangular Prisms Task, Dan seemed to assimilate to some extent Craig's method of counting the unit-cubes by multiplying the three dimensions of the prism. However, I inferred that Dan has partially assimilated Craig's solution of multiplying the three dimensions but Dan was yet to independently re-generate an explanation of why it made sense to multiply them. The explanation that Dan provided involved the two-dimensional areas of the rectangular prisms but not its volume. Although Dan seemed to have assimilated Craig's solution of multiplying the number of unit-cubes in the previous task, this solution was not necessarily connected to the situation.

Based on Dan's counting activities, I claimed that the coordination of a re-presentation of one instantiation of his activity in co-presence with another instantiation was challenging for Dan and limited his ability to anticipate the result of his activities. Moreover, the lack of an operationalized structuration of the three-dimensional object seemed to inhibit his calculations.

Summary of Chapter Seven

Summary of Craig's Counting Tasks

In building, counting, and extending the spatial objects in the three tasks Craig's activities seemed consistent. He counted the number of unit-cubes or unit-tiles along each spatial dimension and found the product of the measurements. Craig explained why he multiplied the number of units along each dimension by building up the spatial objects as successions of rows or layers. Craig did this by building up the spatial objects in representation by first constructing a row of unit-tiles or a layer of unit-cubes and inserting them into the units along another spatial dimension. From his consistent counting activities, I hypothesized the following two things.

First, I claimed that Craig produced three levels of units in successions and enacted his units-coordinating operations to count the spatial objects. In various contexts, Craig operated with units of units using his unitizing and disembedding operations and iterated units of units in activity, using his unit-coordinating operations. Therefore, Craig seemed to use their operations from his initial interviews as assimilatory operations in his counting activities of units along spatial dimensions.

Second, I conjectured that Craig's structuration of the two- or three-dimensional spatial objects were established as operations that could independently be enacted in various situations. Based on the way Craig consistently decomposed and recomposed the box or rectangular prisms through enacting his splitting operations led me to hypothesize that Craig has enacted the mental operations involved in a *reversible decomposing scheme*. Further, Craig's coordination of multiple perspectives and a re-presentation of one instantiation of his activity in co-presence with another instantiation of the building process seemed to support his ability to anticipate the result of his activities. Therefore, I hypothesized that Craig had the operations of a FR-coordinating scheme available to support his reversible decomposing activity. This was further tested through the Locating Tasks, which I discuss in Chapter 8.

Summary of Dan's Box Tasks

In building, counting, and extending the spatial objects in the three tasks Dan's activities seemed to progress throughout the three tasks. At the beginning, Dan's building and counting activities seemed to mainly focus on the exterior of the spatial objects. For example, Dan's initial building of the 3-Cube seemed somewhat haphazard with an overall approach of building the exterior of the cubic block and then filling in the interior. Another example is how Dan solved for the total number of unit-cubes needed to fill in his shoebox. Dan partially covered two adjacent faces of his shoebox with unit-cubes, counted the number of unit-cubes he will need to cover those faces, doubled each number and found the sum of those two measures.

Throughout the teaching episodes, I encouraged Dan to engage in building the spatial objects from which he could reflect upon. Also, Dan often times paid close attention to Craig's strategies and seemed to have partially assimilated them. In the later teaching episodes, Dan's building activities seemed to become more systematic in that he added chunks of unit-cubes as if he were to add horizontal layers or rows to form the three-dimensional objects. Also, Dan's explanation of the configuration of the spatial objects often occurred before he produced sketches of them, which led me to claim that he was able to produce some re-presentation of the spatial objects.

However, Dan still seemed to be reliant on his sketches or the physical models when counting and extending the spatial objects. For example, he relied on his sketch of the 3-Cube to count the total number of unit-cubes in the 3-Cube and he needed the cubic block model in order to carry out the extension of a 3-Cube into a 4-Cube. As such, I claimed that coordinating a re-presentation of one instantiation of his activity in co-presence with another instantiation was challenging for Dan and limited his ability to anticipate the result of his activities.

Furthermore, when counting units within the spatial objects, Dan demonstrated what I called a grouping activity in order to count the individual unit-cubes efficiently. As a result, the number of unit-cubes that comprised each cubic block was a result of repeated addition of the objects within the groups. These hypotheses seemed to align with the findings from the initial interviews in that Dan operated as if he could coordinate two levels of units in activity and thus coordinated units additively while Craig operated as if he could coordinate three levels of units in activity and thus coordinated units multiplicatively. Therefore, I claimed that Dan's decomposition of the spatial objects was not a result of re-presenting and counting a composite unit of several unit-cubes together. Considering Dan's activities in both the initial interview tasks and these spatial tasks, Dan's ability to coordinate two levels of units in activity but not three, not yet having constructed a splitting scheme, and not establishing a strong disembedding operation pertained throughout these tasks.

In Chapter 6, I discussed Kaylee's and Morgan's units coordinating activities within a three-dimensional spatial context. From the findings, I hypothesized that the operations and schemes I imputed to students in their initial interview and the Locating Tasks seemed to have been used as assimilatory operations and schemes in their counting activities involving the cubic blocks. In this chapter, I discussed Craig's and Dan's units coordinating activities within two- and three-dimensional spatial contexts. Consistent with the findings discussed in Chapter 6, Craig and Dan also seemed to use their operations from their initial interviews as assimilatory operations in their counting activities of units along spatial dimensions.

In Chapter 6, I also hypothesized that structuration of space was necessary because the results of such structuration guided students' enactment of the aforementioned operations and schemes. Further, I conjectured that students' structuration of space was constructed through coordinating multiple perspectives, the use of students' FR-coordinating schemes, and a recursive coordination of two-spatial dimensions multiplicatively. In this chapter, I analyzed differences in students' ability to coordinate multiple perspectives and consistent with the findings from Chapter 6, I claimed that students' structuration of spatial objects was necessary because the results of such structuration guide the individual's enactment of the aforementioned operations. Finally, I formulated conjectures about Craig's FR coordinating scheme, which I tested through the Locating Tasks and will discuss in Chapter 8.

CHAPTER 8

CRAIG AND DAN CONSTRUCT COORDINATE SYSTEMS

In Chapter 6, from the work with Kaylee and Morgan, I hypothesized that students' structuration of the spatial objects guided their units-coordinating activities. Further, I conjectured that students' structuration of the objects was formulated through coordinating multiple perspectives, the use of students' FR-coordinating schemes, and a recursive coordination of two-spatial dimensions multiplicatively. In Chapter 7, I discussed Craig and Dan's units coordinating activities within two- or three-dimensional contexts in the Counting Tasks (Cubic Block Task, the Floor Tile Task, and the Box Task). Consistent with the findings discussed in Chapter 6 about Kaylee and Morgan, Craig and Dan also seemed to use their operations from their initial interviews as assimilatory operations in their counting of units embedded within two or three spatial dimensions.

In this chapter, I present my analysis of Craig's and Dan's constructive activities in the Locating Tasks (School Map Task, North Pole Task, and Fish Tank Task) in which I asked both students to draw a map, locate a point, or describe the motion of one point in two- or three-dimensional perceptual space. I designed these tasks to investigate how students organized perceptual space into representational space. More specifically, through these tasks, I explored how the students construct and use coordinate systems when representing points or the motion of a point in two- or three-dimensional perceptual space.

In discussing the North Pole Task, I will describe the ways Craig and Dan each located a point in an irregular shaped two-dimensional map. In the Fish Tank Task, I will present Craig's and Dan's activities in locating points or describing motion of one point to another in three-dimensional cubic or cylindrical fish tanks. In these tasks, Craig and Dan each constructed frames of reference and coordinated systems of measurements to organize these two- or three-dimensional spaces, which I discuss and model the mechanisms of in this chapter.

In Chapter 5, from the work with Kaylee and Morgan, I proposed that the FR-coordinating scheme requires mental operations essential for coordinating three levels of units; hence, a parallel between the students' levels of units coordination and coordination of measurements within frames of reference in three-dimensional space. To elaborate, Morgan reasoned with three levels of units in activity in the initial interview and coordinated measurements along three spatial dimensions in activity. On the other hand, Kaylee's engagement in the initial interview tasks suggested her reasoning with three levels of units as given and she coordinated measurements within frames of reference, maintaining the logical multiplication of all three measurements along three dimensions. Therefore, entering the Locating Tasks, I anticipated similar differences in Craig and Dan's locating activities in relation to their different levels of units coordination. Finally, based on his activities in the Counting Tasks, I formulated conjectures about Craig's FR-coordinating scheme, which I tested through the Locating Tasks and will discuss in this chapter.

School Map Task: Dan and Craig Locate Five Rooms in Their School Building

In the School Map Task, I asked Dan and Craig to create a map of the first floor of their school building. To make the task approachable for the students, I asked them to include only five specific rooms in their school building on their map: the gym, teachers' lounge (the room we were in), media center, front office, and cafeteria. Detailed descriptions of the task design are in Chapter 3. Through the School Map Task, I explored the different perspectives that Craig and Dan took and the frames of reference they used when constructing a re-presentation of the first floor of their school building.

Dan's School Map

Dan started drawing his map with the Gym located at the center of his paper. Next, he drew rectangules representing the cafeteria and teacher's lounge (TL) and then the hallways connecting the gym and cafeteria. Dan then sketched in the front office, front door, and the media center, and completed the sketch with the hallway running the other direction as shown in Figure 8.1.

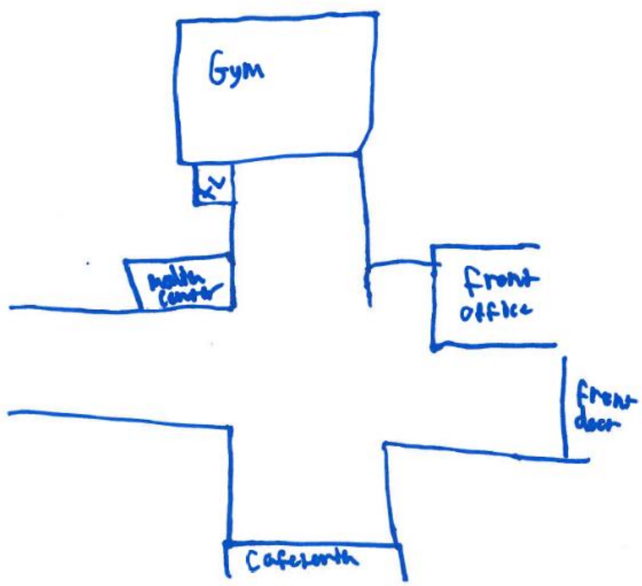


Figure 8.1. Dan's map of the first floor of school building.

I considered Dan's map to be an appropriate description of the locations of the rooms on the first floor of their school building. By appropriate I mean that the relative positions of the rooms were accurate but the distance between different rooms were not to scale.

In the following excerpt Dan explains how he made his map.

Excerpt 8.1. Dan explains how he made his map.

D: Well, the way I used to memorize my schedule is the gym is the center of the school, basically, for me, and all the hallways, almost all the hallways lead up to the gym.
T: Ah, to the gym? So, that's why you started with the gym?
D: Yeah.
T: And then, where did you go from the gym? Like, how did you [continue]?
D: I went down this hallway [*sweeps index finger along his map from the gym towards the cafeteria on his map and turns around in his chair to point towards the door of the teachers' lounge we were in*] to basically, the main area, the main hallway [*hovers finger over the intersection of the two hallways on his map*].

By the way Dan started his map with the gym at the center of his paper and explained that he viewed the gym as the center of the school, connected to most hallways in the school building, I inferred that the gym served as a spatial reference point (Sadalla et al., 1980) in Dan's organization of the first floor of his school building.

After our conversation in Excerpt 8.1, I asked Dan how he knew to put the front office on one side of the hallway and not the other. This question led Dan to rethink his map. After sitting in thought, Dan said that he drew the front office on the "wrong side." Dan either interpreted my question as a cue hinting his map was incorrect or was genuinely perturbed by my question. So, to first ensure Dan that he placed the front office on the "correct side," I asked Craig to explain where the front office was on his map. After hearing that Craig had also put the front office on the same side of the hallway as he did, Dan said that his map was correct.

Next, I asked Dan how he knew to draw the front office on one side and the media center on the other side of the hallway, and what he was thinking when he generated the map. Dan responded, “I was looking, I didn’t realize it but I was, the way I was looking at it as I was standing at the cafeteria looking towards the gym. I didn’t realize that though.” As such, Dan seemed to have mentally situated himself at the cafeteria looking towards the gym when determining the locations of the front office and media center. However, as he said, he “didn’t realize that” until after I asked him to explain what he was thinking when he generated his map. I interpreted this to mean that his locating activities shifted from unconscious to conscious thought engendered by a reflection on his mapping activities.

To summarize, in the School Map Task, Dan considered the gym as a spatial landmark and located other rooms and hallways in relation to the gym’s location. In describing how he generated his map and reflecting on his mapping activity, Dan used route descriptions (Taylor & Tversky, 1996). I inferred this to mean that he re-represented the space as he mentally traversed the area, taking an imaginary *perspective embedded within the space*. As I modeled in Figure 8.2, Dan first represented the gym, cafeteria, and the hallway connecting them by re-presenting his walking experience from the gym to cafeteria. To elaborate, Dan positioned himself at the gym looking straight across the hall and re-presented the cafeteria and anchored a line along the gym-hallway-cafeteria. Then, using that axis, he re-presented the front office/front door and media center on the right or left side to the axis, taking the viewpoint standing from the cafeteria looking toward the gym. Finally, he connected the front office/front door and media center with the hallway, completing a front door-hallway-media center axis. In Dan’s case, a line anchored onto his line of sight and one perpendicular to that constituted his frame of reference. This rectangular frame of reference was *sequentially* anchored to different places (e.g., gym and then at the cafeteria) as he mentally traversed the first floor.

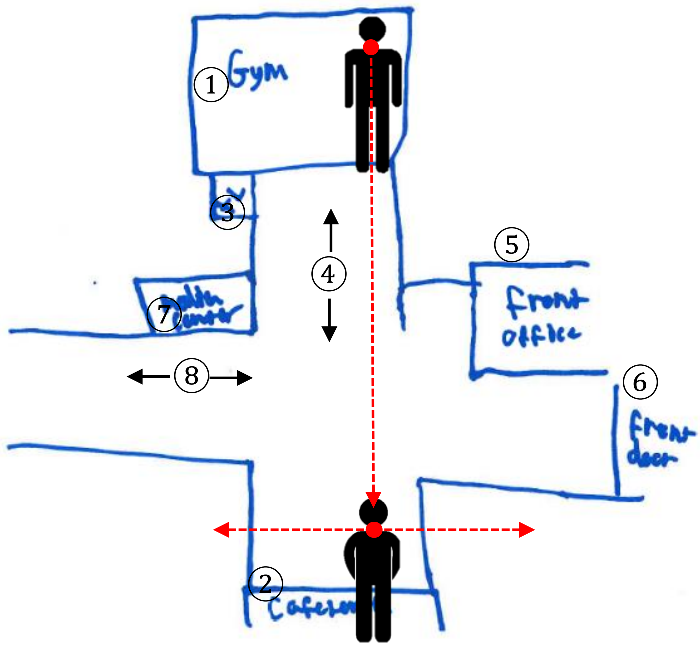
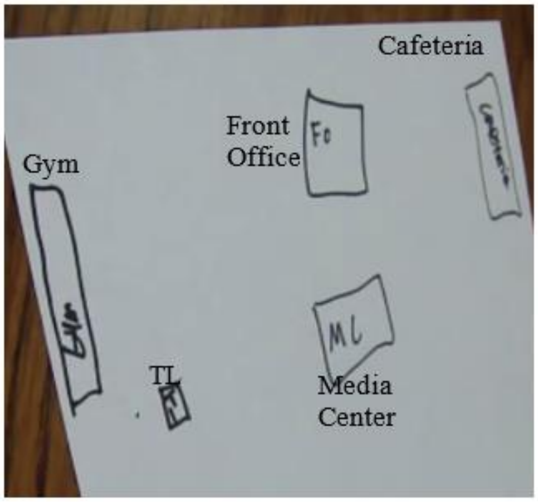
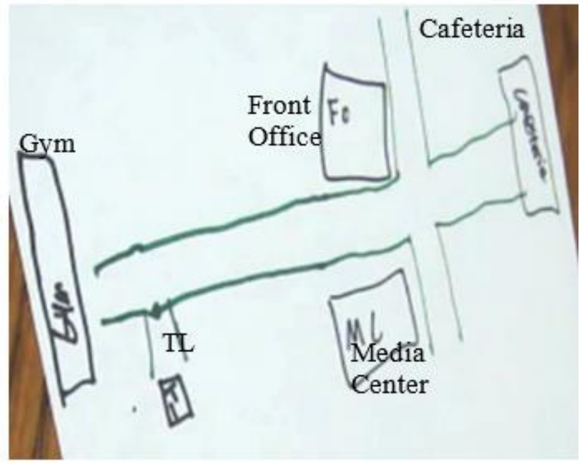


Figure 8.2. A model of Dan's mapping activity.

Craig's School Map

Contrast to Dan, Craig was hesitant to produce the map at the beginning, making comments like "my brain doesn't work like a map" or that he memorized the locations of the rooms through "muscle memory." Craig also explained that he had difficulty finding rooms in the middle of the first semester, when he had to move from one class to another. However, after sitting quietly for a few seconds, Craig started to draw each of the rooms on his paper. He sketched the front office, media center, gym, teacher's lounge, cafeteria in that order, as shown in Figure 8.3 (a).





(a) Craig's sketch of the five rooms

(b) Craig's map of the first floor

Figure 8.3. Craig's drawings of the first floor of the school building.

Different from Dan who started at the gym and sequentially added the hallways and other rooms as he mentally traversed the first floor, Craig swiftly added all the rooms without any hesitation in between drawing them. Based on his prompt sketching of the rooms, I hypothesized that Craig had an image of the rooms altogether that he re-presented on his paper in one sweep. After he produced the drawing as shown in Figure 8.3 (a), Craig looked at me and asked "Is that everything?" which to me meant that Craig did not consider the pathways connecting the rooms. I hypothesized that Craig may have taken a different perspective than Dan did in the School Map Task. After I suggested Craig to add information for a person using the map so he or she could find ways to move from one room to another, he added the hallways connecting the rooms, producing a map as shown in Figure 8.3 (b).

When I asked Craig to explain how he produced his map and where he started in his map, Craig said "I imagined myself high up in the air, removing the ceiling and centered here," pointing his index finger above the map. I asked Craig to further explain how he used this particular imagination to arrange the rooms the way he did on his map. In the following excerpt Craig describes how he made the map of the school building.

Excerpt 8.2. Craig explains how he made his map.

C: Well, I play video games and in video games, you can do things such as fly around and if you go under the map you can see, or up to high, you can see all the stuff [*sweeps index finger over his map of the first floor*] in kind of this view. So, since I'm looking this way [*points index finger frontward*] I imagined just going up [*points index finger towards the ceiling*] and looking down [*points index finger downwards*], since there would be no ceiling.

T: Mm-hmm.

C: And then seeing all this [*points again to his map*].

T: [*Turning to Dan*] Did that make sense?

D: Yeah, I can see… That's why he drew his [map] sideways [*turns his map 90 degrees to make his map correspond to the orientation Craig's map is sketched.*]

T: [*Turning back to Craig*] So, you mean you were imagining yourself coming out of this room [*pointing towards the teachers' lounge in Craig's map*] and kind of looking from above, is that what you're saying?

C: Not necessarily coming out of the room, but if I were to become a winged beast and just go up [*moves his pen starting from the teachers' lounge on his map then straight upward*] and center myself [*moves pen towards center of his map, still hovering over it*]. Because if I were to just go up, then everything will be centered around… [*Hovers finger over the teachers' lounge*.]

From his explanation in Excerpt 8.2, it was apparent that Craig had taken an imaginary perspective from above the perceptual sensorimotor space he was trying to organize. From Craig's explanation I inferred that he imagined going straight up from the teacher's lounge and translated his position to what he considered the center of the first floor. From there, he looked down onto the first floor, as I modeled in Figure 8.4. Like Dan pointed out in Excerpt 8.2, Craig's map was "sideways," demonstrating the different position Craig used in re-presenting the first floor in comparison to Dan. As I modeled in Figure 8.4, Craig re-presented the rooms in the order of top, bottom, left, then right of his imagined position. In Craig's case, a single rectangular frame of reference anchored at the center of the first floor, taken from his above-the-ground perspective, served for gauging the locations of the different rooms on the first floor.

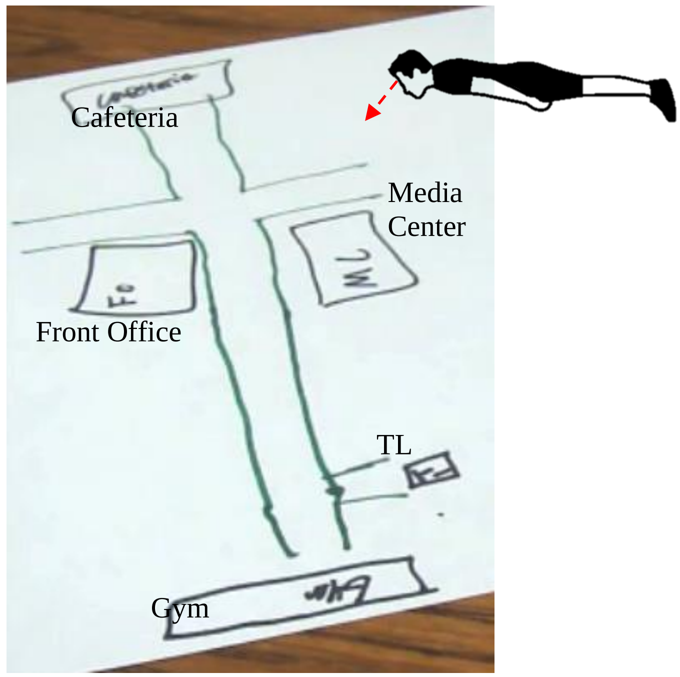


Figure 8.4. A model of Craig's mapping activity.

Earlier in the episode, Craig also demonstrated usage of a perspective embedded within the space, similar to Dan. This was demonstrated when he mentioned towards the beginning of the task that he heavily relied on his walking experience in remembering the locations of the rooms. Another instance occurred when I asked Craig to explain the locations of the front office and the media center on his map. The following is Craig's description of the location of the front office and media center:

> From how you would be walking down, if you were walking out of this room [*moves finger from teachers' lounge on his map along the hallway*], gym's there [*points to the gym on his map*] walk down the hallway, the front office is there [*pointing to the front office in his map*], the media center is there [*points to the media center on his map*] and if you keep walking forward, the cafeteria's there [*points to the cafeteria on his map*].

As such, Craig used route descriptions (Taylor & Tversky, 1996), which I inferred to mean that he also took an imaginary perspective of one embedded within the space.

To summarize, Craig demonstrated the use frames of reference induced from taking both a perspective exterior to the space and a perspective embedded within the space. When taking his perspective exterior to the space, he used a single rectangular frame of reference fixed above the ground at the center of the first floor and re-presented the rooms in co-occurrence. When taking his perspective embedded within the space, his frames of reference were sequentially anchored to different places as he mentally traversed the first floor, similar to Dan.

Summary of the School Map Task

In the School Map Task Dan and Craig each produced maps of the first floor (see Figures 8.1 and 8.3 (b)). I modeled Dan's and Craig's mapping process in Figure 8.2 and Figure 8.4, respectively. To summarize, I inferred that Dan considered the gym as a spatial reference point and located other rooms and hallways in relation to the gym's location. Based on his descriptions of how he generated his map, I hypothesized that he re-represented the space as he mentally traversed the area, taking a *perspective embedded within the space*. In Dan's case, rectangular frames of reference were *sequentially* anchored to different places (e.g., gym and then at the cafeteria) as he mentally traversed the first floor.

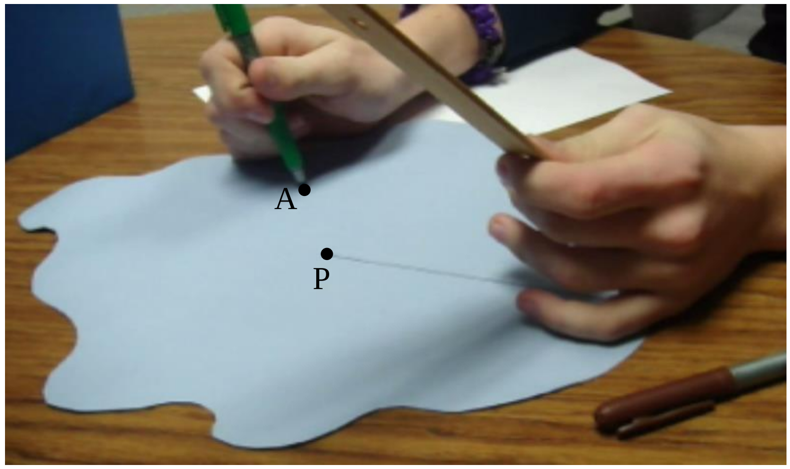
On the other hand, Craig imagined himself hovering above the school building, looking down onto the first floor of the school building, located the rooms, and *then* added the hallways connecting the rooms. It seemed as though the locations of the rooms emerged all at once, in comparison to Dan's mapping activities. From his activities in this task, I inferred that Craig had used a single rectangular frame of reference anchored at the center of the first floor, taken from his above-the-ground perspective, when re-presenting the different rooms on the first floor. Earlier in the episode, Craig also demonstrated usage of a perspective embedded within the space, similar to Dan. Therefore, I hypothesized that Craig used frames of reference induced from taking both a perspective exterior to the space and a perspective embedded within the space. When taking his perspective exterior to the space, he used a single frame of reference fixed above the ground at the center of the first floor and re-presented the rooms in co-occurrence. When taking his perspective embedded within the space, his frames of reference were sequentially anchored to different places as he mentally traversed the first floor.

Considering these findings, in the North Pole Task, I anticipated to see these differences in the perspectives they coordinate and in the coordination of frames of reference when describing a point in two-dimensional space.

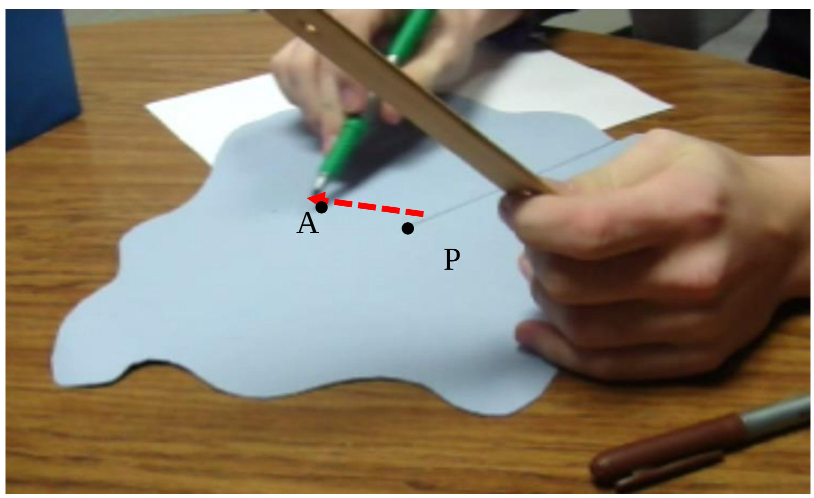
North Pole Task: Craig and Dan Locate a Point in Two-Dimensional Space

Dan Coordinates Vertical and Horizontal Distances

In our first teaching episode of the new semester held on January 23, 2015, Dan started working on the North Pole Task on his own. Because Craig was not there, I played the role of the rescuer on the ground holding the map and placed the screen on the table so I could not see Dan's map. When I asked Dan to first plot a point on his map that represented the missing person, Dan plotted a point as shown in Figure 8.5 (a). Next, without any prompting, Dan oriented his map so that the road to the North Pole was facing him, as shown in Figure 8.5 (b). After orienting his map as such, Dan moved his pen over the map starting from above North Pole (point P) straight to the above the missing person's location (point A), as demonstrated by the red dashed arrow in Figure 8.5 (b).



(a)



(b)

Figure 8.5. Dan marks a point (A) on his map, orients his map, and visualizes a straight line from point P to A.

Without knowing what Dan had done so far, I asked Dan to come up with instructions for me so that I can find the missing person on my map. Earlier in the teaching episode I showed Dan my map, which was identical in shape and size to the one he had, with the road to the North Pole and North Pole point drawn on wax paper. After looking at his map for approximately 14 seconds, Dan said “okay” as if he were ready to give me instructions. Later, from the video recording I observed Dan’s placement of the ruler and his hand (Figure 8.6) at this particular moment of the teaching episode. The way Dan placed his ruler on the map diagonally and the way he placed his right hand open as shown in Figure 8.6, along with his action demonstrated in Figure 8.5 (b) suggested that Dan initially considered measuring the distance between the two points P and A. Dan’s actions reminded me of Morgan’s activities in locating the missing person’s location in the irregular shaped map. That is, Morgan also thought of the movement of the rescuer from the North Pole to the missing person’s location in one straight movement and considered measuring the distance in between the two points (see Figure 5.4).

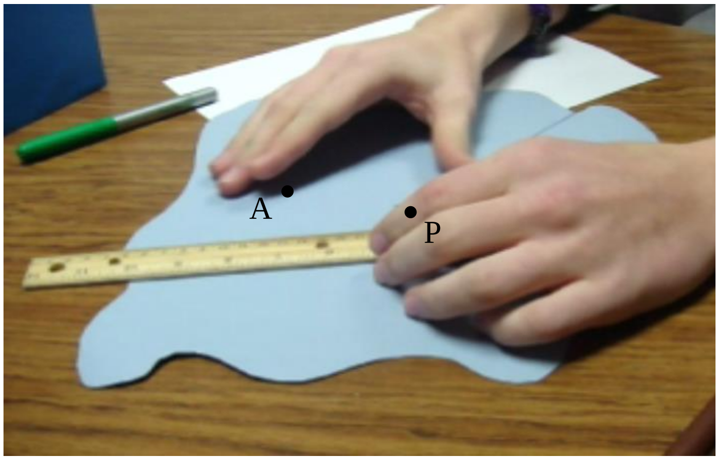
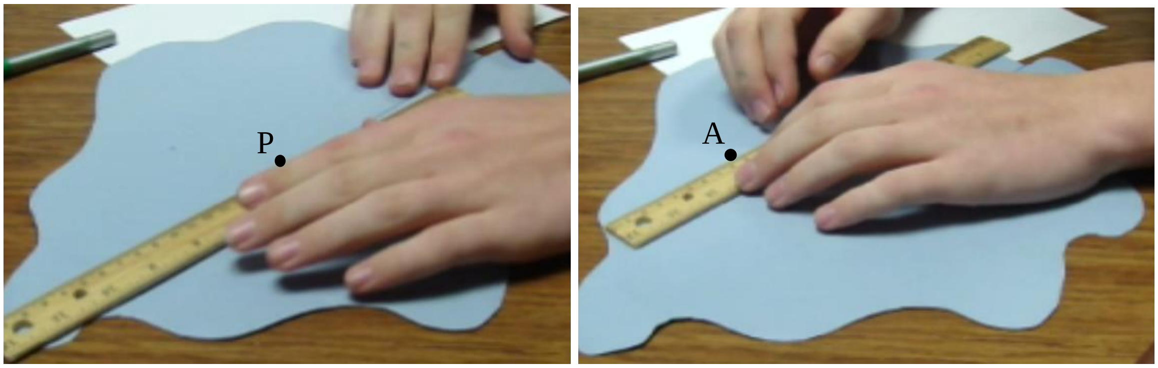


Figure 8.6. Dan places ruler and hands on his map while looking at the map.

Dan was about to start giving me instructions by saying, “Alright, at the North Pole, you’re going to … You want to go…” However, Dan paused for approximately four seconds, as if he were rethinking how to give instructions to the rescuer. Dan then moved his ruler so that it aligned with the road to the North Pole as shown in Figure 8.7 (a) and, soon after, slid his ruler keeping its orientation so that it passed through point A as shown in Figure 8.7 (b). It seemed as if Dan was translating the ruler from one point (P) to another (A). From this action, I hypothesized that Dan had visualized a horizontal movement from point P to point A.



(a) (b)

Figure 8.7. Dan places his ruler through point P and then point A on his map.

Not knowing what Dan was doing behind the screen, I said “So, I’m at the North Pole…” to prompt him to give me the instructions. Dan immediately moved his ruler back to like it was in Figure 8.7 (a). Next, holding the ruler with his left hand, he picked up a pen with his right hand and moved the pen left and right (from his perspective) as if he were connecting point A with some point on the ruler, as demonstrated by the red arrow in Figure 8.8. Dan repeated this movement of his pen once more and tapped his finger on the table. I inferred this tap on the table to mean that he was finally ready to provide instructions to the rescuer.

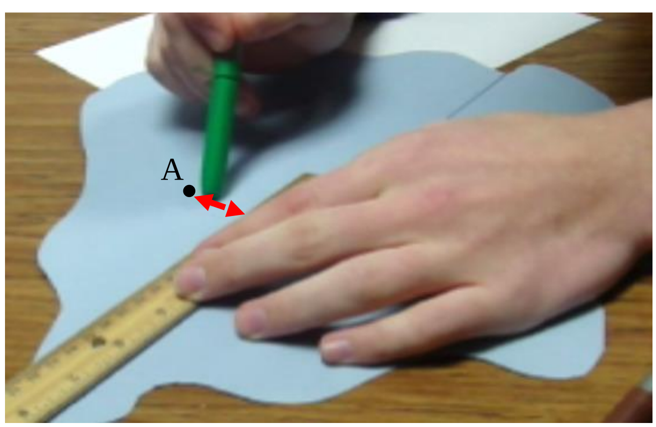


Figure 8.8. Dan moves pen from point A to a point on his ruler.

From his actions demonstrated in Figure 8.7 and 8.8, I inferred that Dan considered the horizontal (from his perspective) distance from point A to the extension of the road to the North Pole. These actions that Dan carried out were very similar to the ones I observed when Kaylee worked on the North Pole Task (c.f., Figures 5.5 and 5.6). However, compared to Kaylee's swiftness in carrying out these actions, the moment Dan moved his ruler back to the position in Figure 8.7 (a) until the moment he tapped his finger on the table took approximately 18 seconds, a relatively long time to be considered as anticipated actions like Kaylee's. In Chapter 5, I interpreted Kaylee's actions to be operational and anticipatory in that the measurements were coordinated from the beginning without a trial-and-error process and in that she was aware that this coordination along with the corresponding measurements would ensure that the rescue team would find the missing person. On the other hand, Dan's activities seemed spontaneous in that they occurred as he was verbalizing instructions and carrying out activities in the moment.

Although Dan was yet to verbalize the instructions, and although his actions took a longer time than Kaylee's, I inferred Dan's actions to indicate that Dan has constructed a frame of reference consisting of horizontal and vertical lines anchored onto landmarks such as the road to the North Pole and the North Pole point. Using this rectangular frame of reference, I hypothesized that Dan had decomposed the movement of P to A along two spatial dimensions.

After tapping his finger on the table, Dan relayed his instructions, as shown in the following excerpt.

Excerpt 8.3. Dan gives instructions to the rescuer to find the missing person.

D: You're going to want to go… one and a half inches, which is miles for me. So, one and a half inches straight from the North Pole, the dot.

T: Okay, so which… Straight from the North Pole…[*Looks down onto map*.]

D: Yeah, you just take your ruler and put it [taps on his ruler], use the inches side, it's going to be one and a half.

T: [*Places ruler on her map*] one and a half.

D: Yeah. And… [*Moves his pen along his map in the reverse direction of the red dashed arrow as shown in Figure 8.8*.] Go straight first.

T: Okay.

D: Go one and a half inches straight, and then… [*Moves his ruler into a new position as shown in Figure 8.9 (a). Next, he moves the ruler back to the first position as shown in Figure 8.9 (b), and slowly rotates the ruler back to the position in Figure 8.9 (d), as demonstrated by the red dashed arrow in Figure 8.9 (c). Dan holds the ruler down with his thumb as he rotates it*.]

D: And from there, you're going to go [*measures distance from his thumb to point A on his ruler, as demonstrated in Figure 8.9 (d)*] straight down two and a half, to the right. To the right you go two and a half inches.

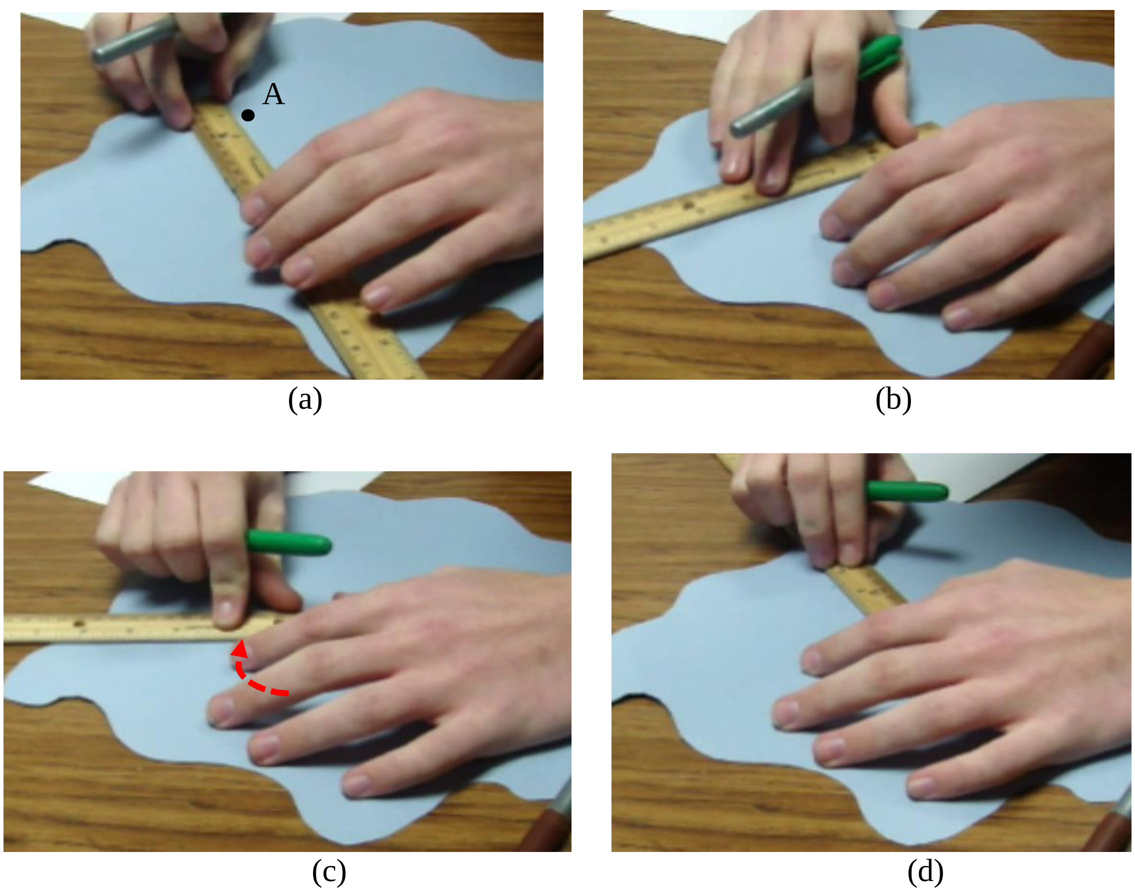


Figure 8.9. Dan uses his ruler to measure distances to provide for the rescuer.

[*Continued*.]

T: To the right…

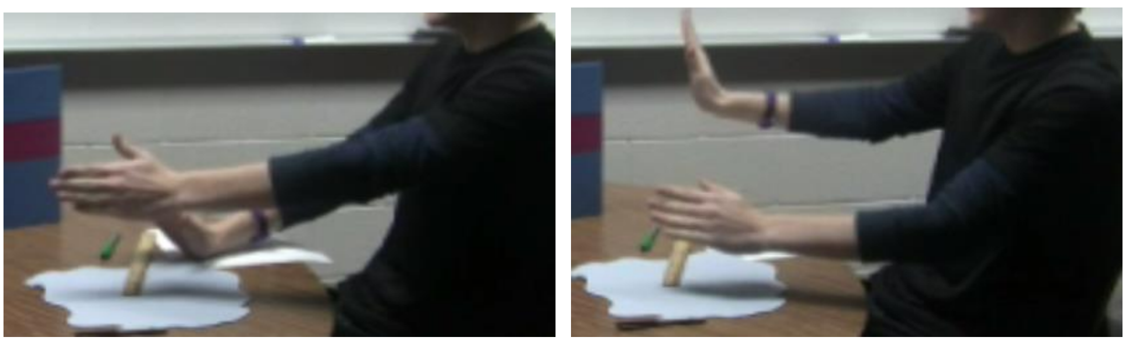
D: Yeah.

T: So, you mean…

D: So, go straight [*points his left arm in front of him as shown in Figure 8.10 (a)*] and then to the right [*places right arm next to the left arm and then moves it to the right as shown in Figure 8.10 (b)*]. And go two and a half inches.

T: Two and a half inches [*follows Dan's instructions on her map*]. Okay.

D: And that should be where you are.



(a) (b)

Figure 8.10. Dan demonstrates the movement of the rescuer using gestures.

As demonstrated by his explanation in Excerpt 8.3, Dan measured horizontal and vertical distances that constituted the rescuer's movement from point P to point A with respect to his perspective looking at the map. Later in the teaching episode, I asked Dan what was he thinking when he used the ruler to measure distances for the rescuer to walk. Dan explained that he thought about longitude and latitude lines. His comments along with his gestures in Figure 8.10 corroborated my hypothesis that Dan constructed a rectangular frame of reference to decompose the movement of the rescuer into vertical and horizontal movements. Using this frame of reference, Dan coordinated distances along the vertical/horizontal movements, constructing a Cartesian-like coordinate system. Based on the way Dan did not explicitly address the rescuer's initial orientation or perspective in Excerpt 8.3, I inferred that Dan mainly focused on his ego-oriented perspective (Taylor & Tversky, 1996) taken from above the ground and superimposed his rectangular frame of reference onto the two-dimensional plane, like Morgan did in the North Pole Task.

Craig and Dan Coordinate Angle Measure and Distance

In the following teaching episode held on January 26, 2015, Craig took the role of the rescue team in the helicopter while Dan was the rescuer on the ground. Craig plotted a point on his map as shown in Figure 8.11 but was hesitant to proceed. Craig said he did not know what he was supposed to do. Craig also had several questions about the situation such as whether his road to the North Pole was the same as the one on Dan's map and if the maps were of same size. Once we addressed these questions, Craig placed the compass with the sharp point at the North Pole (point P) and the pencil side on point A. Using the compass, he constructed an arc with radius length PA on his map, which I re-generated in Figure 8.11. Craig then said that he was thinking about an angle but that he did not know what to do with it.

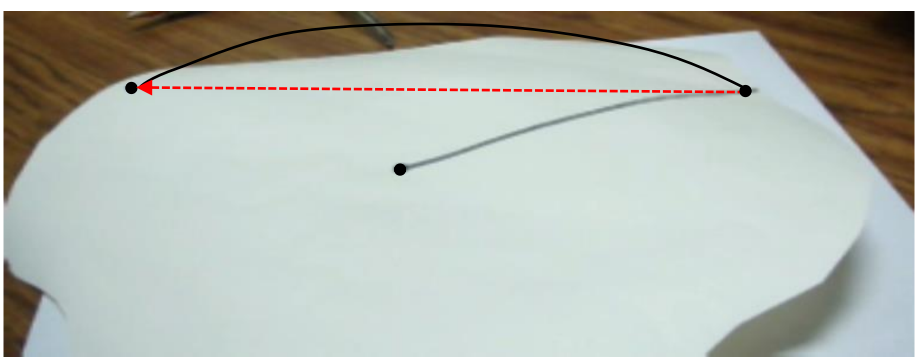


Figure 8.11. Craig's map and inscriptions describing his actions on the map.

After I asked Craig to think some more about what he could do with his idea, Craig picked up his pen, moved it above his map connecting the intersection of the arc and the road to the North Pole (point B in Figure 8.11) with point A (demonstrated by the red dashed arrow in Figure 8.11) saying, "[F]rom here to here is one-hundred and eighty degrees. From this point on the road [*pointing to Point B*], he takes a right and then something about every mile he should change 10 degrees in order to get to this location [*pointing to point A*]."

So far, Craig considered a circular movement and angle measure but he said that he did not know how to give instructions for the rescuer using what he had drawn on his map. Because Craig seemed to be stuck in moving forward, the students switched roles and Dan took a turn in leading the activity as the person in the helicopter. Finally, Craig seemed to open up again. So, I asked both students to work together to complete Craig's angle idea and placed Craig's earlier map on the table.

After I placed Craig's map (Figure 8.11) back on the table, Dan said he thought he knew what Craig was thinking and placed the protractor on the map. With the map oriented so that the road was facing him, Dan placed the protractor so that the horizontal reference on the protractor (0°) was perpendicular to the road at point P. Next, Dan explained, "go however many degrees that is" as he connected point P and A with his index finger. He then picked up the protractor and placed it on the map so that the horizontal reference of the protractor (0°) was aligned with the road at the North Pole saying, "or we can put [*the protractor*] sideways." To encourage Craig's engagement, I asked Craig if Dan's idea was similar to what he was thinking earlier. Craig said it was not but that Dan's idea "might work if you explain to put the protractor directly facing north from the North Pole," which I inferred to mean that the 90° mark aligned with the extension of the road segment.

Once both students agreed to use the protractor like in Figure 8.12, they each read off the angle measure where line segment PA passed. Note that the line segment PA in

Figure 8.12 was not yet constructed on the map at this point in the teaching episode. However, I inferred that the students imagined this line segment based on Dan's earlier movement of his index finger connecting the two points and from the angle measure they read (approximately 135°) from the compass.

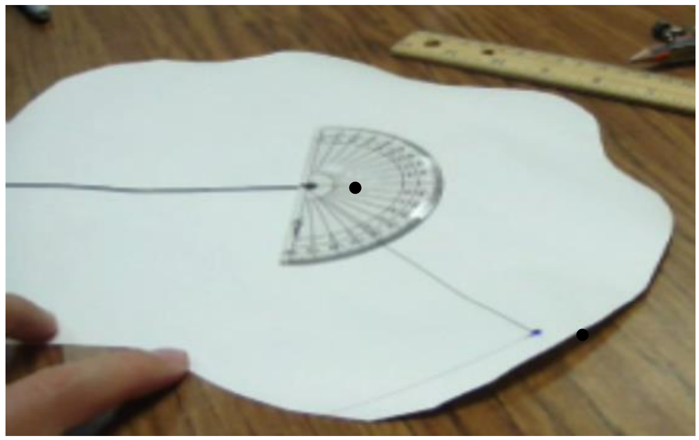


Figure 8.12. Craig's map and placement of the protractor.

I inferred their activity of measuring the angle measure to indicate that both students intended to find the inclination of PA. Because I wanted to explore whether the students were aware of the references they used in defining the angle they measured, I asked them what they meant by saying "go 135 degrees." Craig replied "I don't know. There wouldn't be a stopping point so that wouldn't work." Dan picked up the ruler saying, "Well, that's when the ruler comes in. Go a hundred and thirty degrees, five inches." In other words, Dan suggested they measure how far the person had to walk in addition to telling him the angle measure.

Although Dan made a contribution by adding another measurement of distance along with the angle measure, both students were yet to discuss what going 135 degrees meant in the situation. That is, they did not fully develop instructions for the rescuer nor did they relate the angle measure to the rescuer's orientation, as Kaylee did in the North Pole Task.

Because Craig and Dan were still trying to determine the exact angle measure using the protractor, I asked Craig to connect the two points P and A so it will help them read the angle measure off of the protractor more accurately. Craig connected the two points P and A with the protractor placed on the map as shown in Figure 8.12. Then, he explained, "Put the protractor on the North Pole facing North and then … [*reads the protractor*] so from the one hundred and forty mark the line that connects from the one forty to the center of the protractor, extend it out all the way and then explain how many inches or centimeters on the map it will take to get to the location."

As such, Craig and Dan developed instructions for the rescuer to follow in order to *plot* point A on his/her map rather than instructions to relay to the rescuer to carry out through physical movements on the ground. Craig has initiated the consideration of an angle measure and Dan contributed by adding a distance element to their locating activity. Together they developed a system of measurements of angle measure and distance similar to a polar coordinate system, with the origin of their angular frame of reference anchored at point P. However, they did not explicitly address the initial ray that constituted the angular frame of reference like Kaylee did in the North Pole Task. Eager to explore their locating activities in the three-dimensional case, I did not provide the circular map to Craig and Dan.

Summary of the North Pole Task

To summarize, in the North Pole Task, I hypothesized that Dan coordinated horizontal and vertical distances using a rectangular frame of reference with its origin anchored at point P and vertical axis aligned with the road to the North Pole. I inferred that Dan has constructed a Cartesian-like coordinated system of measurements. Based on the way Dan did not explicitly address the rescuer's initial orientation or perspective in his instructions (Excerpt 8.3), I inferred that Dan mainly focused on his ego-oriented perspective (Taylor & Tversky, 1996) taken from above the ground and superimposed his rectangular frame of reference onto the two-dimensional plane, like Morgan did in the North Pole Task.

I also compared Dan's locating activity with Kaylee's in the North Pole Task. Although both students constructed a Cartesian-like system of measurements, I interpreted Kaylee's actions to be operational and anticipatory in that the measurements were coordinated from the beginning without a trial-and-error process and in that she was aware that this coordination along with the corresponding measurements would ensure that the rescue team would find the missing person. On the other hand, Dan's activities seemed spontaneous in that they occurred as he was verbalizing instructions and carrying out activities in the moment.

I also hypothesized that Craig and Dan together coordinated angle measure and distance using an angular frame of reference with its origin anchored at point P and initial ray aligned with the line perpendicular to the road to the North Pole. I inferred that Craig and Dan has constructed a system of measurements compatible with a polar coordinate system. However, they did not explicitly address the initial ray that constituted the angular frame of reference like Kaylee did in the North Pole Task.

Finally, in the North Pole Task, similar to Kaylee and Morgan, both Craig and Dan demonstrated some commitment to a unit of measure and scale of measure. Both students explicitly used inches or miles or degrees as units of measure in their measuring activities and descriptions. In Excerpt 8.3, Dan said, “one and a half inches, which is miles for me” and Craig asked whether Dan’s map and his map were of the same size. As such, when locating points in the two-dimensional plane, both students seemed to be aware of the importance of the scale of measurements.

Fish Tank Task: Craig and Dan Locate a Point in Three-Dimensional Space

Cubic tank: Locating the four fish

Starting on January 30, I explored Craig and Dan’s ways of locating points in three-dimensional spaces through the Fish Tank Task. In the first teaching episode involving the Fish Tank Task on January 30, I asked both students to locate the four fish in the cubic fish tank on the table. Dan claimed “it’s impossible” and Craig commented, “[T]his blows my mind. I’ve never done this before.” Both students seemed perplexed by the three-dimensional situation and the task appeared to be novel to both students. It took some encouragement until the students started working on the task. In the following I will describe Dan’s and Crag’s respective locating activities in the cubic fish tank.

Dan locates the four fish in the cubic tank.

Dan started first, making diagrams of two sides of the fish tank and located the four fish in the tank based on where he saw them, as shown in Figure 8.13. I found Dan’s locating activity similar to Morgan’s in that he located the fish making visual estimations, without an observable measurement process.

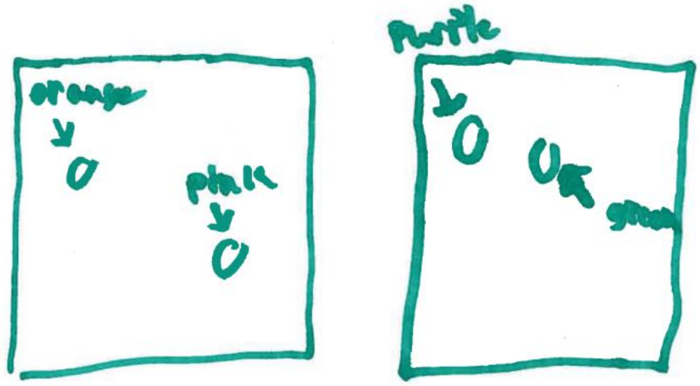


Figure 8.13. Dan’s initial diagrams of the cubic fish tank and fish.

Here I note that these visual estimations were not mindless estimations in that the locations were not randomly marked on the paper. I acknowledge that there must have been some consideration of topological features such as order and proximity of the fish in the tank and perhaps some considerations of gross quantities (e.g., the distance between the orange and pink fish is greater than the distance between the purple and green fish). However, when I claim that there was no observable measuring activity, this means that I was not able to observe a physical measuring activity or verbal explanation involving commitment to a unit of measure (e.g., using a ruler or using fingers to mark a certain distance and iterating it along other line segments).

I also found Dan’s locating activity similar to Kaylee’s in that he chose to make his diagram based on the two faces where he could best see the fish. Dan also mentioned, “there’s more than one angle you can look at it [the tank],” which suggested that he was aware of the multiple perspectives one can take in looking at the tank. However, unlike Kaylee who used the multiple perspectives to coordinate measurements in locating the fish, Dan seemed to find the multiple perspectives a confining element in solving the task.

In the next teaching episode on February 2, Dan worked alone on the cubic fish tank. To encourage more explicit measuring activities, I asked Dan to describe the locations of each fish to another person in another room, who is trying to make a replica of the fish tank. I prepared a model of a frame of the other person's tank made of wired straws. This frame was used for me or the student to enact students' instructions (see Figure 8.14 for an example).

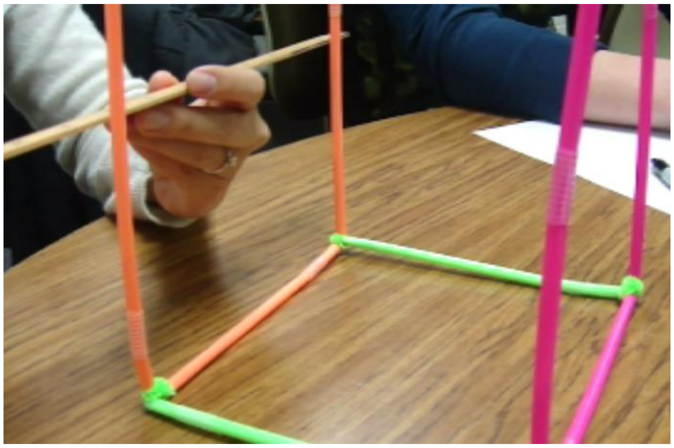


Figure 8.14. Teacher-researcher enacting Dan's instructions using the cubic straw frame.

At the beginning, after looking at the tank for a while, Dan said that he did not know what to do other than yell across the room and tell the person to "put the purple one [fish] in the corner." As such, Dan seemed to consistently rely on the perceptual imagery of the tank. Taking the other person's role, I placed my finger within the wired frame at a random corner of the fish tank hoping to demonstrate that Dan's instruction was insufficient in locating the purple fish. I also asked Dan to demonstrate what the other person would do using the wired frame if he was told to place the purple fish in the corner. After my push for more precision, Dan decided to measure how deep each fish was from the top of the water, as shown in Figure 8.15.

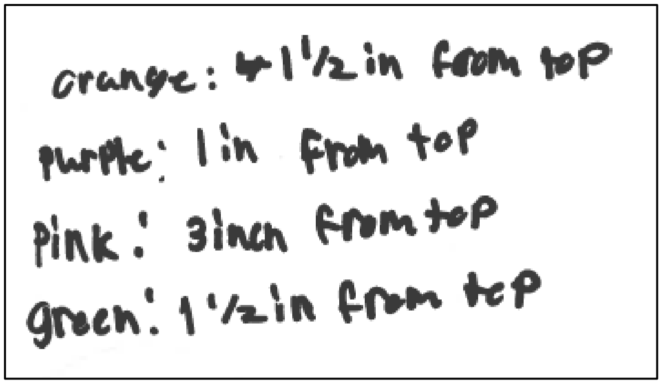


Figure 8.15. Dan measured the distance between the fish and the water surface.

After Dan measured the distance from the water surface to each fish (we called it the depth of each fish), I asked him if that was all the information the other person needed in order to make the replica of the fish tank. Dan acknowledged that it was not enough but did not say much more.

Recalling that Dan mentioned the idea of longitude/latitude in the North Pole Task I asked him whether he could use a similar idea in the cubic tank case. I asked this to see if he could assimilate the three-dimensional situation to one where he could use his coordination of horizontal and vertical distances. Dan responded, "The only way I know to use longitude and latitude is with the map. But this is three-dimensional. Latitude and longitude [inaudible]. It's like bird's eye view. Or at least in my mind for using latitude and longitude." As such, the idea of longitude/latitude lines seemed to be restricted to two-dimensional situations. From his comment, I hypothesized that Dan did not view the three-dimensional cube as a collection of infinitely many two-dimensional squares (cross sections) stacked vertically on top of each other, with each square cross section consisting of infinitely many points that share the same depth.

To test this hypothesis, I asked him to show me, in the wired straw frame, all the possible locations for a fish, given it was 2 inches from the top of the water. Dan showed me two points, one in the center of the tank and one in the corner of the tank that were each 2 inches from the imaginary water surface but he did not seem to realize that all possible points that are 2 inches below the water surface would form a square-shaped cross section plane.

After we established that the person making the replica would need to know more than just the depth of the fish, Dan explained that the person should also know how far away the fish were from each other. When I asked him to locate the purple fish using that idea, Dan claimed that he would find how far from a corner (a point on one of the side edges of the tank that seemed closest to the purple fish) the fish was. For example, Dan explained that the purple fish was 1 inch straight from the corner of the tank. As such, Dan shifted from using each fish as reference to using elements of the tank container (corner) as references to locate the fish.

Dan's comment that the purple fish was 1 inch straight from the corner of the tank reminded me of Morgan's notion of "going straight" demonstrated in her North Pole Task and cylindrical Fish Tank Task. In the North Pole Task, I hypothesized that Dan had also considered a "going straight" motion from point P to point A, as demonstrated in Figures 8.5 (b) and 8.6. However, Dan later broke down the motion into horizontal and vertical movements along his rectangular frame of reference, as demonstrated in Figures 8.8 and 8.9. Therefore, I decided to push Dan to further to explain what it means to go straight, intending it to potentially bring forth his measuring activities involving horizontal and vertical distances in the plane.

So, I asked Dan how I was supposed to go straight if I were making the replica using the wired straw frame. After laying the ruler across the top of the fish tank diagonally, Dan said I should lay my ruler from the corner to the corner diagonal from it across the top of the tank and then measure 1 inch out from the reference corner. In order to explore whether he would use a similar method for the other fish, I picked two other fish further away from the corner he used as a reference and asked how he would locate those two fish. This time, he measured how far into the tank the fish were from the face they were most visible and told me he would find how far apart the two fish were. As such, Dan started to coordinate more measurements in locating the fish but the references he used were different for each fish.

The time allowed for the teaching episode on February 2 ended so we revisited the cubic tank again in the following teaching episode on February 6. At the beginning of that episode, I asked Dan to explain to Craig what he did to locate the fish. Dan recalled his previous locating activities one by one:

> First I just looked at them to see where they were. And I said the first one, you can put it anywhere in the corner but facing this way [*sweeping his hand diagonally across the cubic tank*] and it was like two…Let me start over [*sits back in chair looking up and taps on the table three times*]. Okay, first, I measured to see how far each one was from the top. And then after I got that, I started to see how far they were from the sides of the tank… Now I get when you [*looking at Craig*] were saying. It's like the latitude and longitude lines, cuz from the top and then the sides come to one point.

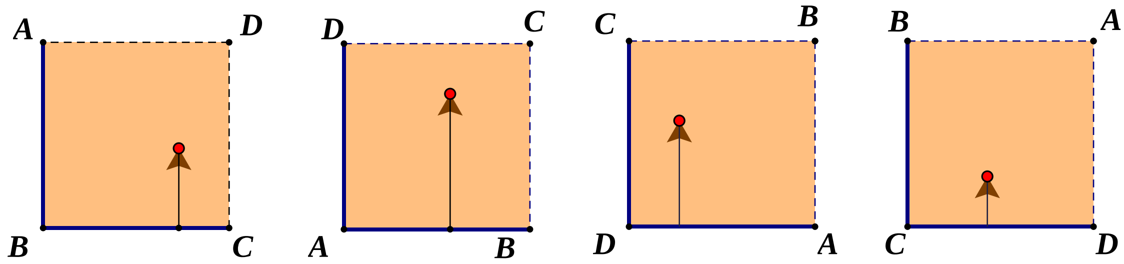
In his comment above, Dan walked through his previous locating activities, starting from making visual estimations to measuring the distances such as the depth of each fish and the distances from the sides of the tank. Dan also addressed the idea of longitude/latitude without my prompting. In his explanation of the longitude/latitude lines, Dan seemed to have become conscious of the mechanism involved in coordinating latitude and longitude lines from his comment "from the top and then the sides come to one point."

While Craig thought about how he wanted to locate the fish, I asked Dan to develop full instructions for the person making the replica. So, Dan measured the depth of each fish as he did in the previous teaching episode and then measured the distance from one fish to the closest face (2 inches) and wrote it down next to the depth of that fish. After writing those measurements on his paper, he paused and sat looking at the tank for a while. Dan looked at me to explain his method:

> I just noticed something. In my way, you will have to measure the thing from all four sides. Like, this side, this side, this side, and that side [*sequentially pointing to all four side faces of the cubic tank*]. Because, this has one measurement from this face [*points to the 2 inches he just measured*] doesn't mean anything. Because these faces aren't labeled so how would he know which face? He would need to know all of them.

To summarize, Dan recognized the location of the fish as consisting of "from the top and then the sides come to one point." So, he measured the distance from the surface of the water to each fish and then decided to measure how far it took to come to the point from the sides of the tank. At first Dan selected the face that was closest to the fish but later claimed that he would have to measure the distances from all four faces surrounding the side of the tank. The reason he chose to measure the distance from all four sides was because there did not seem to be a way to distinguish the sides from each other.

Based on his comment about longitude/latitude lines and his measuring activities, I hypothesized that Dan used rectangular frames of reference to guide his measuring of horizontal and vertical distances along the faces of the tank. However, instead of fixing one face and anchoring one set of horizontal/vertical axes onto one adjacent set of edges like Kaylee and Morgan did, I conjectured that Dan anchored his rectangular frame of reference onto all four sets of adjacent edges as I modeled in Figure 8.16. I conjectured that Dan's measurements were sequentially coordinated but not multiplicatively combined due to the lack of the logical multiplication of measurements (Piaget et al., 1960).



(a) (b) (c) (d)

Figure 8.16. A model of a top view of the fish tank and distances from each face to a fish in the water.

Nonetheless, Over the course of three teaching episodes, Dan's locating activities changed from making visual estimations and plotting them on two different side-view representations of the tank to developing a more systematic way of coordinating measurements to locate the fish. Instead of locating the fish in relation to another as he initially said he would in the previous episode, Dan located the fish using the faces of the tank as spatial references, along with the surface of the water. Also, he shifted from using different faces for each fish to using the same set of faces (all four sides of the tank) for all four fish.

Craig locates the four fish in the cubic tank.

On January 30, the first day of the Fish Tank Task, it took a while for Craig to start the task. After he sat watching Dan produce his diagrams, Craig finally drew a sketch of one side of the tank and located all four fish on that side, as shown in Figure 8.17. Similar to Morgan's approach, Craig located all fish from one perspective of the tank, in his case, his side view of the tank. Similar to both Morgan and Dan, Craig also located the fish using visual estimations, without any observable measuring activities.

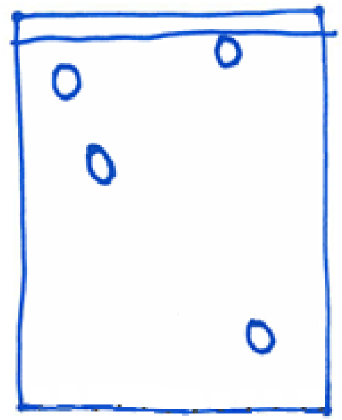


Figure 8.17. Craig's initial sketch of the fish tank.

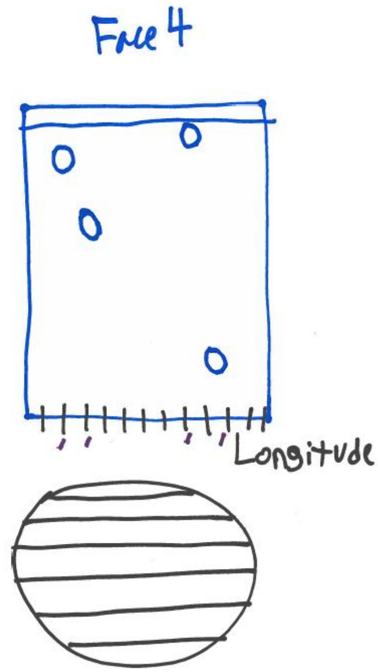


Figure 8.18. Craig added new illustrations to his Face 4 sketch.

After locating the fish in the tank like in Figure 8.17, Craig talked about the idea of longitude and latitude to describe the locations of the fish. To elaborate, Craig drew a circle below his sketch in Figure 8.17, referring to it as "the world" and drew in horizontal line segments within the circle, which he referred to as longitude (see Figure 8.18). Craig said, "there is the world and then there is longitude, which I think are this way, and then latitude, which go up and down." Although he mentioned both longitude and latitude, he did not sketch what he referred to as latitude in his circle.

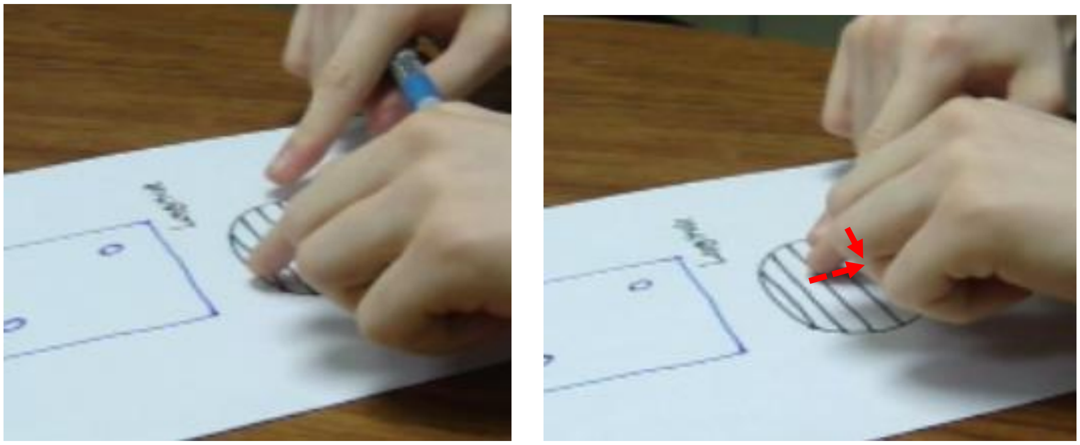


Figure 8.19. Craig demonstrates how he would use longitude and latitude lines to locate an object.

Using his diagram of the world and longitude lines, Craig explained, "The reason why you have longitude and latitude is so you can be very precise and do like this [*moves two index fingers from outside of the circle into a location within the circle, as demonstrated in Figure 8.19*] or this [*repeats same motion to another point in the circle*], and find one specific point." Craig's explanation seemed compatible with Dan's explanation, "from the top and then the sides come to one point." From Craig's explanation, I inferred that he also constructed a rectangular frame of reference consisting of horizontal and vertical lines to gauge the location of a point along two spatial dimensions. However, where these horizontal and vertical lines were anchored onto was not yet specified.

Next, Craig said, "you could be looking for an object that has actual mass" and that, "you would need a lot more lines of longitude" in this case. Craig explained "if there were enough lines of longitude, everything would be on at least one line of longitude." As such, Craig seemed to have established a goal of locating objects with mass within three-dimensional space and in order to do so, he needed an abundance of longitude lines so that the objects would be on at least one of those longitude lines. I inferred this to mean that Craig could visualize superimposing vertical and horizontal lines onto the circle representing the world, until the lines were dense enough to account for every object within that space.

Craig then took this idea of longitude/latitude to the fish tank diagram in Figure 8.17 saying, “So, to find the sea creatures, you would need something like this, labeled with many lines of longitude or in this case, it will be latitude.” Craig then added tick marks at the bottom edge of the fish tank in his sketch (see Figure 8.18), referring to them as the latitude lines. Craig explained that the tick marks were twelve latitude lines and that even though the fish tank was a three-dimensional space, if you look from a particular angle and “go to latitude line two, somewhere in that latitude, there is a thing,” with thing referring to a fish.

From his explanation, I inferred two things. First, I inferred that Craig was aware of multiple perspectives resulting in different images of the fish tank from his comment that you had to look from a particular angle. At this point in the teaching session, Dan pointed out to Craig that there were multiple angles from which one could see the fish tank. To address Dan’s comment, Craig labeled his diagram as Face 4 and said that his sketch would be made from one particular perspective, looking at Face 4. Second, I noticed that Craig only accounted for latitude lines and was aware that “somewhere in that latitude, there is a thing,” rather than specifying where in that latitude the fish was in. Craig seemed somewhat constrained by the situation being three-dimensional, inferring from his comment, “you’re looking at a three-D space, so, I don’t think I completely comprehend how longitude and latitude works so I don't think longitude will be able to calculate one thing." I interpreted these comments to suggest that Craig did not view the latitude lines as intersections of planes perpendicular to Face 4 but rather as lines superimposed onto the two-dimensional Face 4. So, his latitude/longitude lines were contrived to the two-dimensional case, similar to Dan.

By the way Craig added latitude lines after visually estimating the locations of the fish onto his sketch, I inferred that Craig superimposed horizontal/vertical lines *after* locating the fish as a way to communicate the locations like Morgan did in the cubic fish tank. In other words, the locating of the fish was made using visual estimations, not using the longitude/latitude lines. Although similar, there were some differences I observed between Morgan's and Craig's grids. In Morgan's case, she used the grid to find which square section each fish were contained within, whereas Craig described the location of each fish in terms of the intersection of some latitude or some longitude line with the fish. Another difference between Morgan and Craig was in the way Craig only considered either the latitude or longitude lines but not both simultaneously. Despite his explanation of using longitude/latitude lines as demonstrated in Figure 8.19, Craig did not mention the latitude in describing the location of each fish in the fish tank but only commented that somewhere in the latitude line there was going to be a fish.

On February 6, the next episode in which Craig was present, I asked both students to give instructions for another person in a different room who wanted to make a replica of the fish tank. Dan had explained how he located the fish in the previous episode but Craig wanted to develop his own instructions different from Dan. Craig approached the task differently than he did in his first session. To elaborate, Craig first noticed the visible layers of the gelatin in the tank[5] and considered each layer (approximately 1 inch apart) as a unit for measuring the depth of each fish. Then, Craig noticed the sticker label on one of the side faces of the tank[6]. Using the sticker face as a reference, Craig consistently measured the distance from the sticker face into the tank and the distance from the left edge towards the right edge of the sticker face.

For example, in order to locate the purple fish, Craig first identified that the fish was in the fourth layer; then, he measured that the purple fish was four inches from the sticker face into the tank (Figure 8.20 (a)) and one and one-fourths of an inch from left to right of the sticker face (Figure 8.20 (b)). He referred to these measurements as the depth, length, and width, respectively. Next, Craig wrote instructions for the other person in the other room, as shown in Figure 8.21.

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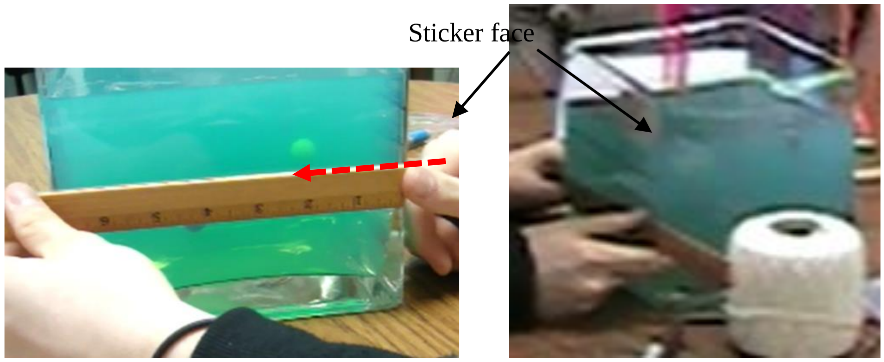
[5] When making the fish tanks, I poured gelatin into the tank in succession with a solidifying process in between layers to submerge the fish at different locations in the tank. Although I used the same mixture of food coloring, water, and gelatin for each batch, they formed layers visible from the side of the tanks. Because I made the batches of same amount, the height of each layer was constant. This was a feature that I did not intend for the fish tanks to have but one that the students realized in their activity in the task.

[6] After buying the glass tanks from an art supply store, I was not aware of the sticker labels on the tanks and did not find them to be problematic. However, this provided an interesting variation to the task that I did not anticipate.



(a) width (b) legth

Figure 8.20. Craig's measuring activities in locating the purple fish.

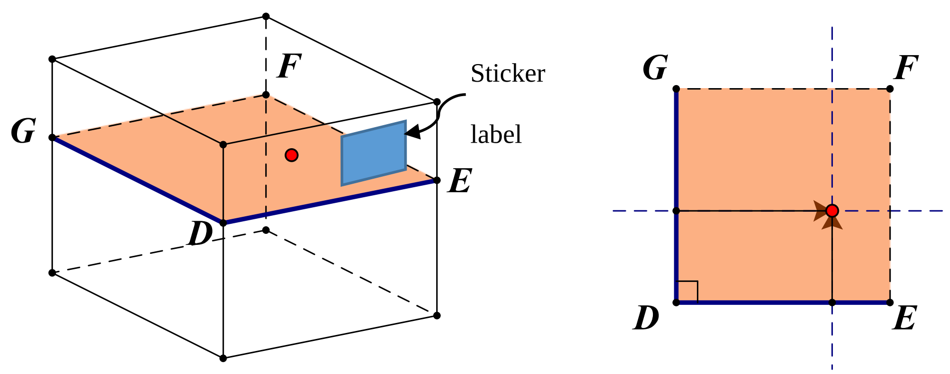
Middle of the fourth layer, 4in from the sticker face, $1\frac{1}{4}$in from left to right of the middle of the fourth layer.

Figure 8.21. Craig's written instructions for locating the purple fish in the cubic fish tank.

I inferred these new measuring activities to indicate that Craig refined his earlier locating activity using visual estimations and partial longitude/latitude lines. When finding the measurements which he referred to as the length and the width, Craig often stood up to look down onto the tank to measure the distances he wanted to measure. As such, Craig frequently shifted his perspective back and forth from the top of the tank to the side of the tank. Craig repeated this activity of first identifying the gelatin layer of the fish and then finding the distance from the sticker face into the tank and the distance from left to right along the sticker face consistently for all four fish in the tank. As such, Craig used consistent spatial references (bottom of tank and sticker face) for locating all four fish.

Although Craig had already listened to Dan's explanation of how he located the fish, Craig created a system of coordinated measurements different from Dan's. Craig's system was different from Dan's in that he fixed the sticker face as a reference and measured horizontal/vertical distances in relation to that face whereas Dan measured distances from all four sides. Craig's locating activity was similar to Morgan's in that he coordinated the top view with the side view perspective.

From Craig's locating activities, I inferred that Craig has partitioned the cubic tank horizontally into square cross-sections, cued by the layers in the gelatin, taking his side view perspective of the tank (Figure 8.22 (a)). His partitioning activity resembled Morgan's in the cylindrical fish tank. Although he relied on the visible gelatin layers, he used these layers as a way of measuring the depth of each fish. Next, Craig took each two-dimensional square layer and superimposed a set of horizontal and vertical lines onto it re-enacting his longitude and latitude concept, taking his top view perspective (Figure 8.22 (b)).



(a) (b)

Figure 8.22. Craig's rectangular frame of reference imposed onto the second face of the tank.

Further, Craig superimposed the horizontal and vertical lines such that a set of perpendicular lines lined up with the edges of the square which corresponded to that of the sticker face and the one adjacent to it on the left (modeled as DE and DG in Figure

8.22 (a)). Using the set of perpendicular lines and point of intersection as his frame of reference, Craig constructed two sets of horizontal and vertical lines that would each pass through the fish in the tank (demonstrated by the blue dashed line segments and arrows in Figure 8.22 (b)). As a result of coordinating rectangular frames of reference, Craig was able to coordinate the length and the width of the position of each fish within the particular depth of the water.

Curious to know if he would consistently use these frames of reference and the resulting system of measurements to describe movements of fish in the tank and whether his frames of reference coordinating activity was enacted sequentially or simultaneously, I asked Craig to describe the motion of one fish to another. Craig first identified the layers each fish was in, thus finding the amount of vertical movement along the "depth" dimension that was needed. Then, once the fish were in the same layer, he explained that one fish would swim straight, diagonally, to the other. I interpreted his description to entail an awareness of the parallel square sections within the tank and translating the lower one containing one fish to the higher one containing the other fish, as I modeled in Figure 8.23.

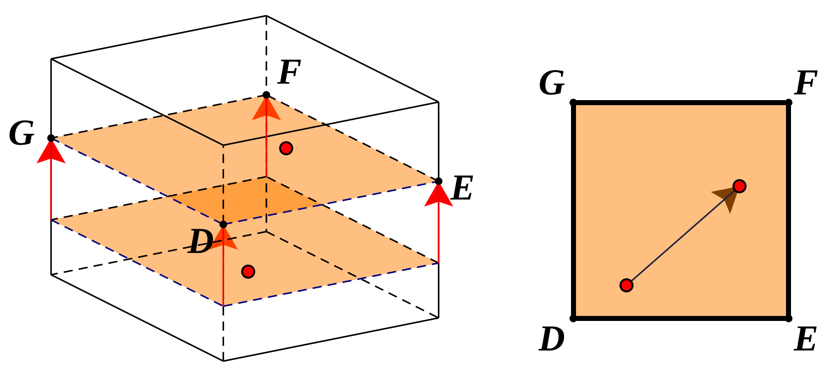


Figure 8.23. A model of Craig's explanation of one fish swimming to another in the cubic fish tank.

As demonstrated in Figure 8.23, once the fish were on the same depth, all the fish had to do was swim straight to the other fish. Craig's description of one fish swimming to another was strikingly similar to Morgan's description of Fish 1 swimming to Fish 2 in the cylindrical fish tank. Similar to Morgan, I inferred Craig's coordinate system to indicate a sequential coordination of two-dimensional rectangular frames of reference and measurements. In other words, once the depth was accounted for, it was "set aside" and not multiplicatively combined with the remaining length and width measurements.

Craig's locating activities in the cubic fish tank were similar to Kaylee's activities in that he consistently accounted for three measurements along each spatial dimension. Although similar to Kaylee's locating activities, I consider Craig's coordination of measurements different from Kaylee's. In Kaylee's case, she used her system of coordinated measurements and measured the distance between the two fish along the length, width, and height dimensions she identified earlier. Although she used different sets of faces for the fish due to the lack of visibility through the gelatin, in Kaylee's case, the length and height measurements were inserted along the width measurements. Therefore, when demonstrating the movement from one fish to another, she considered the change in each measurement between the two fish. Therefore, I consider Kaylee's coordinate system to be a multiplicative structure of three-dimensional rectangular frames of reference.

By the way Craig was able to point to one fish on a particular layer of the water and relate the location of that fish in relation to one side face of the tank, I inferred that he has coordinated the two perspectives multiplicatively (one taken from the side view and one taken from the top view) to locate the fish. However, I conjectured that Craig's coordination of the two perspectives was sequential, like in Morgan's case. I conjectured the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension. Therefore, I hypothesized that Craig was yet to construct a FR coordinating scheme but could enact the action of coordinating frames of reference sequentially *in activity*. This hypothesis was consistent with my previous conjecture that the FR coordinating scheme required mental operations essential for coordinating three levels of units and Craig's initial interview (reasoning as if he could operate with three levels of units *in activity*).

Summary of the Cubic Fish Tank

Over time in the cubic fish tank task, I observed shifts in both student's locating activities. Both started with making visual estimations of the locations of the fish in the tank but later coordinated measurements of horizontal/vertical distances to locate each fish.

Dan shifted from using fish as reference to the surface of the water and sides of the tank as spatial references. In the end, he developed a system of measurements consisting of the distance from the top of the water and the distance from all four slides to each fish. In my analysis, I conjectured that Dan did not view the three-dimensional cube as a collection of infinitely many two-dimensional squares (cross sections). Further, I hypothesized that Dan used rectangular frames of reference to guide his measuring of horizontal and vertical distances along the faces of the tank. However, instead of fixing one face and anchoring one set of horizontal/vertical axes onto one adjacent set of edges like Kaylee and Morgan did, I conjectured that Dan anchored his rectangular frame of reference onto all four sets of adjacent edges as I modeled in Figure 8.16. I conjectured that Dan's measurements were sequentially coordinated but not multiplicatively combined due to the lack of the logical multiplication of measurements (Piaget et al, 1960).

Craig shifted from using his longitude/latitude idea to developing a system of measurements consisting of the distance from the sticker face into the tank and the distance from the left edge towards the right edge of the sticker face. He referred to these measurements as the depth/height, length, and width of each fish, respectively. In finding these measurements, Craig frequently shifted his perspective back and forth from the top of the tank to the side of the tank. When it came to describing the motion of one fish to another, Craig first identified the amount of vertical movement along the "depth" dimension that was needed. Then, once the fish were in the same layer, he explained that one fish would swim straight to the other (modeled in Figure 8.23). Similar to Morgan, I inferred Craig's coordinate system to involve a sequential coordination of two-dimensional rectangular frames of reference and measurements.

By the way Craig was able to point to one fish on a particular layer of the water and relate the location of that fish in relation to one side face of the tank, I inferred that he has coordinated the two perspectives multiplicatively (one taken from the side view and one taken from the top view) to locate the fish. However, I conjectured that Craig's coordination of the two perspectives was sequential, like in Morgan's case. I conjectured the logical multiplication of measurements along the two dimensions in the first representation (top view grid) was not preserved and inserted into the third dimension. Therefore, I hypothesized that Craig was yet to construct a FR coordinating scheme but could enact the action of coordinating frames of reference sequentially *in activity*. This hypothesis was consistent with my previous conjecture that the FR coordinating scheme required mental operations essential for coordinating three levels of units and Craig's initial interview (reasoning as if he could operate with three levels of units *in activity*).

Cylindrical tank: Locating the four fish

On February 20 of 2015 I presented the cylindrical fish tank and asked both students to describe the location of the four fish in the tank to another person who is making a replica of the tank. The cylindrical feature of the tank seemed to perturb both students. Craig mentioned how he noticed two patterns in our teaching sessions. First, he commented "the pattern that I noticed with these meeting sessions is that you're taking a most understandable approach and trying to learn how we adapt, or our ability to adapt [to other situations]." The second pattern Craig commented on was how they found a way to solve the problem with faces and that I have taken away the faces. Dan said that he did not know how he would proceed because there was nothing on the tank that he could point at. I took this as an indication that Dan found this new situation challenging because there was not a salient spatial reference, like the faces of the cubic tank, he could use in locating the fish. As such, the shape of the cylindrical tank seemed to bring new challenges to the task for both students.

Craig asked me if I expected them to apply the same idea as in the cubic tank or if I wanted them to come up with an entirely different approach. I explained it was up to them and what they wanted to do. Dan responded to Craig's question, saying that they could not use the longitude and latitude lines for the cylindrical tank. Dan noted they could find the vertical layers and the distance from the glass (exterior sides of the tank) for each fish but that the distance from the glass was not sufficient because the fish could be in many different locations within the same distance from the glass.

In order to encourage both students to proceed in solving the task, I revisited Dan’s earlier comment about there being nothing they could use on the tank. In the previous cubic fish tank, there was a sticker on one of the faces that the students used as a spatial reference but the cylindrical fish tank did not have that sticker. Bringing up the sticker on the cubic fish tank, I asked them if adding an imaginary sticker on the cylindrical tank might help. Craig replied, “only if they were exactly the same because we’re dealing with, probably we’re dealing with coordinates.” So, I asked the students to draw a “sticker” on the side of the tank and to imagine that the other person making the replica had the same sticker in the same location. I was hoping that this suggestion to use something similar to what they did in the past would help them move forward. Using a dry-erase marker, Craig drew a rectangle (sticker) on the side of the tank close to the rim (see Figure 8.24).

Craig locates the four fish in the cylindrical tank.

After drawing the “sticker,” Craig started to formulate and experiment several ideas while Dan took a more observant role. First, Craig suggested using the rectangle he just drew onto the side of the tank as a viewing window and plotting the fish in that rectangle where he could see them. I took this to indicate that Craig thought of projecting the fish in the fish tank onto the two-dimensional rectangle. However, soon after, Craig explained that this method would not account for how light might bend in gelatin and he was not sure how to account for the fish very low in the tank.

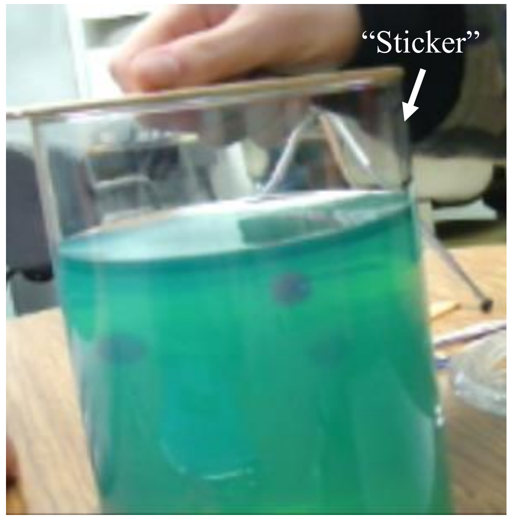


Figure 8.24. Craig measured the distance from the sticker to the fish along the top of the cylindrical fish tank.

Craig then decided to place the ruler across the top of the tank and measure the distance from the top right corner of the "sticker" to the orange fish (see Figure 8.24). He measured 2 inches but said that this was just an idea that was incomplete and that he did not account for "the angle of the ruler." To further explore Craig's reasoning about the distance of 2 inches, I asked him to show me all the possible locations of points if he gave the other person the 2 inches as instructions. After putting the ruler on the top of the tank, Craig said it was "basically an arc," making an arc with the tip of his finger as he rotated the ruler along the top of the tank.

Craig then put a protractor on the ruler going across the top of the tank so that the center of the protractor was at zero of the ruler. I asked him how he placed the protractor and Craig explained the protractor was in line with the sticker. I interpreted this to mean that the plastic bar going across the middle of the circular protractor was parallel to the horizontal side of the rectangular "sticker" (see Figure 8.25 as an example of the positioning of the protractor). I viewed the way Craig placed the protractor on the top of the cylindrical tank similar to the way he placed the protractor on the map in the North

Pole Task (see Figure 8.12). Different from the North Pole Task, in this case the endpoint of the ruler was not placed at the center of the protractor because he placed the ruler at the top right corner of the sticker whereas the center of the protractor was aligned with the sticker centered at the midpoint of the sticker. I did not consider this as a necessary error but a mindless misuse of the tools because it was apparent Craig wanted to measure "the angle of the ruler." Craig read off the angle measure where the ruler and the protractor intersected and said that if he told the other person to go 65 degrees and 2 inches, then he could locate the orange fish. As such, Craig coordinated angle measure and distance to locate the orange fish along the top view of the tank.

In the next teaching episode on February 23, I asked both students to continue their work with the cylindrical fish tank. Craig immediately said he remembered what he did in the previous session and repeated his activities. This time, he wrote instructions for the other person making the replica referring to one of the fish as the "red creature" as shown in Figure 8.25.

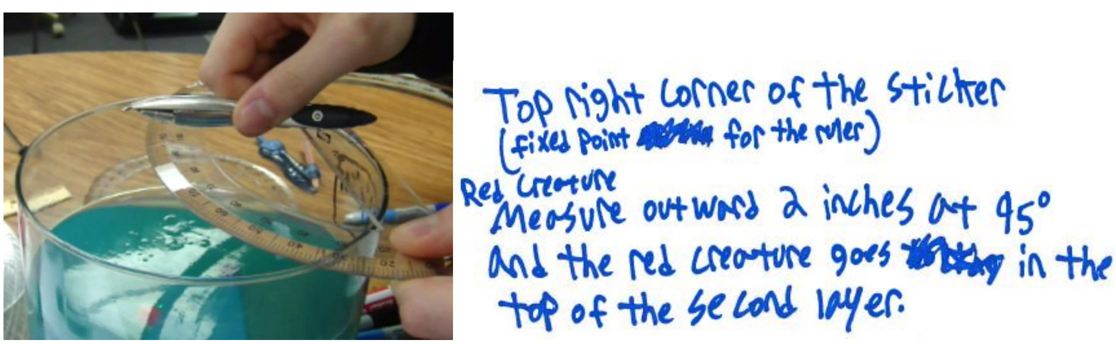


Figure 8.25. Craig describes the location of the "red creature"

As demonstrated in Figure 8.25, looking down onto the tank from the top view, Craig fixed the end of his ruler (in the figure he used a compass to demonstrate how he used his ruler) at the top right corner of the rectangular sticker so that the ruler passed above the fish he was intending to locate. After reading off the distance from the corner of the sticker, 2 inches, Craig measured the angle in which the ruler was placed. In order to do so, Craig placed the protractor such that the horizontal diameter of the circular protractor was parallel to the horizontal edge of the sticker. Then, Craig read the number on the protractor where it intersected with the ruler, 95 degrees. Finally, he identified in which vertical layer the fish was in and concluded his instructions.

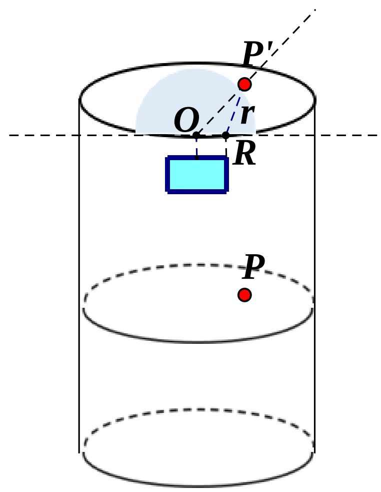


Figure 8.26. A model of Craig's frames of reference and system of measurements he coordinated in order to locate the four fish in the cylindrical tank.

As I modeled in Figure 8.26, I hypothesized that Craig located the fish in the tank by measuring a radial distance r from the top right corner of the sticker along the rim of the tank (point R), angle of rotation $\angle ROP$, and height (PP') to each fish (P) using the sticker, side of the tank, and bottom of the tank as references. From the way Craig first located the fish along his top view (circle) of the tank and then associated that with the vertical layer of the fish, I inferred that Craig viewed the cylindrical tank as a collection of infinitely many circles and each fish were contained in one of these circle layers, similar to the cubic fish tank. First taking the top view of the fish tank, he located the fish within a circle using angle measure and distance like he did in the North Pole Task. Then, Craig found the layer in which this circle was located along the side view of the tank. Once Craig identified which layer the fish was contained in, locating the fish entailed locating the fish within the two-dimensional cross section (circle) in which he utilized a polar coordinate system.

Dan locates the four fish in the cylindrical tank.

Because Dan mainly observed Craig's activities I asked Dan if he could revisit his initial idea of the layer and distance from the exterior of the tank several times. During our first session with the cylindrical tank on February 20 Dan mentioned "radius," which could have meant that Dan had imagined a radial distance from the center of the tank. However, he was not sure how to proceed with that idea. Towards the beginning of the next session on February 23, Dan drew the "sticker" using a board marker (the previous "sticker" had gotten erased during transportation) and turned the tank so the sticker was facing Craig. Dan recalled that the sticker was used like a "landmark" because there were no other features of the tank they could use. Dan also recalled that Craig measured 2 inches out into the tank from the sticker and that they found the vertical layer in which the fish was contained in. Although Dan recalled Craig's ideas we discussed in the previous episode, Dan expressed that he did not understand what Craig did the previous day. This to me indicated that Dan did not fully comprehend what Craig did with the protractor and why Craig measured an angle measure.

Craig then took the lead in producing and demonstrating his instructions (as shown in Figure 8.25) while Dan observed Craig's activities. When I asked Craig to explain to us why he used the protractor, Craig explained "to get the angle which you need to go outward." While Craig was occupied finding the 2 inches and angle measure, I asked Dan if he could help Craig. Dan measured the vertical distance between the surface of the water and the fish, which he concluded 2 inches. After Craig had finished writing the instructions as shown in Figure 8.25, I asked Craig to relay his instructions to Dan while Dan carried them out on the actual tank. As shown in Figure 8.27, Dan placed the center of the protractor at the 2-inch mark on the ruler instead of at the rim of the tank like Craig did. Later when I asked Dan how he placed the protractor on the tank, he explained that he placed it parallel to the sticker. However, Dan did not adjust the orientation of the ruler to account for the 95 degrees.

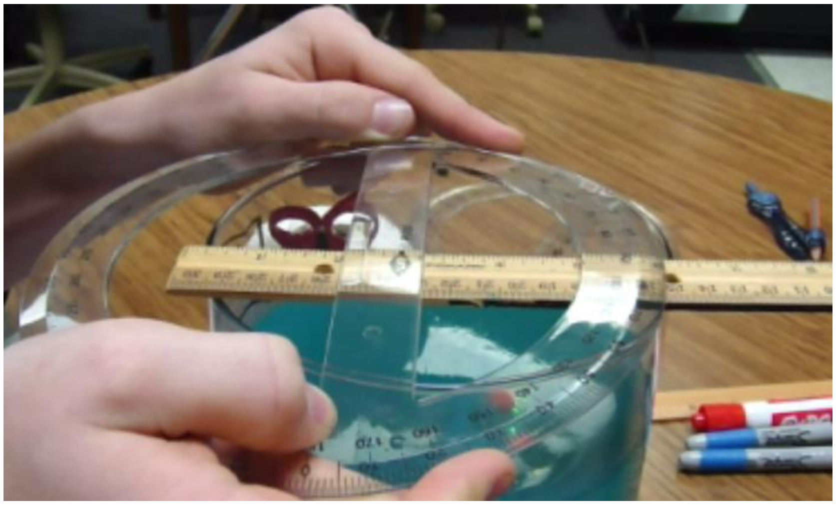


Figure 8.27. Dan enacts Craig's instructions using the ruler and protractor.

From these observations, I inferred that Dan was yet to comprehend Craig's system of measurements to locate each fish in the tank. Dan was aware that finding the vertical depth and the horizontal distance from the glass container could work but that just one distance measurement from the glass container was insufficient. Therefore, he had an awareness of the need for an additional measurement other than the vertical depth and horizontal distance in order to locate the fish. It was the third measurement that he was yet to establish.

Summary of the Cylindrical Fish Tank

In the cubic fish tank, both Craig and Dan coordinated horizontal and vertical distances along the faces and edges of the fish tank in order to locate the four fish. However, in the cylindrical tank case, other than the vertical distance, in other words, the depth of each fish, none of the students coordinated distances induced from rectangular frames of reference. In Kaylee and Morgan’s case, they both used the same rectangular frames of reference, which they superimposed onto the cylindrical tank. However, in Craig and Dan’s case, the students said that they could not use the same approach due to the cylindrical shape of the tank.

Instead of using exterior faces of the tank as spatial references, Craig and Dan used an imaginary “sticker” as a landmark to locate the fish in the tank. Craig devised a method of measuring the radial distance from the right top corner of the rectangular sticker on the tank, the angle measure with 0 placed at the top of the sticker, and the vertical layer in which each fish were embedded in. Craig used his method consistently throughout both teaching episodes. First taking the top view of the fish tank, he located the fish within a circle using angle measure and distance like he did in the North Pole Task. Then, Craig found the layer in which this circle was located along the side view of the tank. Once Craig identified which layer the fish was contained in, locating the fish entailed locating the fish within the two-dimensional cross section (circle) in which he utilized a polar coordinate system.

On the other hand, Dan explained that he could think of finding the vertical layer and the distance from the exterior of the tank but was aware that finding one length measure to express the distance from the exterior of the tank was insufficient in locating the fish.

Summary of Chapter Eight

In this chapter, I presented my analysis of Craig's and Dan's constructive activities in the Locating Tasks (School Map Task, North Pole Task, and Fish Tank Task) in which I asked both students to locate a point or describe the motion of one point in two- or three-dimensional perceptual space. Through my observations of the students' locating activities, I analyzed how Craig and Dan constructed frames of reference and coordinated measurements using those frames of reference to represent perceptual space.

Craig's Coordinate Systems

School Map Task.

In the School Map Task Craig imagined himself hovering above the school building, looking down onto the first floor of the school building, located the rooms, and *then* added the hallways connecting the rooms. It seemed as though the locations of the rooms emerged all at once, in comparison to Dan's mapping activities. From his activities in this task, I inferred that Craig had used a single frame of reference anchored at the center of the first floor, taken from his above-the-ground perspective, when re-presenting the different rooms on the first floor. Earlier in the episode, Craig also demonstrated usage of a perspective embedded within the space. Therefore, I hypothesized that Craig used frames of reference induced from taking both a perspective exterior to the space and a perspective embedded within the space. When taking his perspective exterior to the space, he used a single frame of reference fixed above the ground at the center of the first floor and re-presented the rooms in co-occurrence. When taking his perspective embedded within the space, his frames of reference were sequentially anchored to different places as he mentally traversed the first floor.

North Pole Task.

Craig initially said that he was thinking about an angle but that he did not know what to do with it. With Dan's help, Craig coordinated angle measure and distance together using an angular frame of reference with its origin anchored at point P and initial ray aligned with the line perpendicular to the road to the North Pole. From their collaborative activities in the North Pole Tasks, I inferred that Craig and Dan has constructed a system of measurements compatible to a polar coordinate system.

Fish Tank Task.

Over time in the cubic fish tank task, I observed shifts in Craig's locating activities. He started with making visual estimations of the locations of the fish in the tank but later coordinated measurements of horizontal/vertical distances to locate each fish. Craig shifted from using his longitude/latitude idea to developing a system of measurements consisting of the distance from the sticker face into the tank and the distance from the left edge towards the right edge of the sticker face. He referred to these measurements as the depth/height, length, and width of each fish, respectively. In finding these measurements, Craig frequently shifted his perspective back and forth from the top of the tank to the side of the tank. When it came to describing the motion of one fish to another, Craig first identified the amount of vertical movement along the "depth" dimension that was needed. Then, once the fish were in the same layer, he explained that one fish would swim straight to the other. In the cylindrical fish tank, Craig used an imaginary "sticker" as a landmark to locate the fish in the cylindrical tank, instead of using exterior faces of the tank as spatial references. First taking the top view of the fish tank, he located the fish within a circle using angle measure and distance like he did in the North Pole Task. Then, Craig found the layer in which this circle was located along the side view of the tank.

By the way Craig consistently identified one fish on a particular layer of the water and relate the location of that fish in relation to one side view of the tank, I inferred that he has coordinated the two perspectives multiplicatively (one taken from the side view and one taken from the top view) to locate the fish. However, I conjectured that Craig's coordination of the two perspectives was sequential, like in Morgan's case. I conjectured the logical multiplication of measurements along the two dimensions in the first representation (top view) was not preserved and inserted into the third dimension, exemplified by his description of Fish 1 swimming to Fish 2. Therefore, I hypothesized that Craig was yet to construct a FR-coordinating scheme but could enact the action of coordinating frames of reference sequentially *in activity*. This hypothesis was consistent with my previous conjecture that the FR coordinating scheme required mental operations essential for coordinating three levels of units and Craig's initial interview (reasoning as if he could operate with three levels of units *in activity*).

Dan's Coordinate Systems

School Map Task.

From his mapping activity, I conjectured that Dan considered the gym as a spatial reference point and located other rooms and hallways in relation to the gym's location. Based on his descriptions of how he generated his map, I hypothesized that he re-represented the space as he mentally traversed the area, taking a *perspective embedded* *within the space*. In Dan's case, frames of reference were *sequentially* anchored to different places (e.g., gym and then at the cafeteria) as he mentally traversed the first floor.

North Pole Task.

In the North Pole Task, I hypothesized that Dan coordinated horizontal and vertical distances using a rectangular frame of reference with its origin anchored at point P and vertical axis aligned with the road to the North Pole. I inferred that Dan has constructed a Cartesian-like coordinated system of measurements. Based on the way Dan did not explicitly address the rescuer's initial orientation or perspective in his instructions (Excerpt 8.3), I inferred that Dan mainly focused on his ego-oriented perspective (Taylor & Tversky, 1996) taken from above the ground and superimposed his rectangular frame of reference onto the two-dimensional plane, like Morgan did in the North Pole Task.

I also compared Dan's locating activity with Kaylee's in the North Pole Task. Although both students constructed a Cartesian-like system of measurements, I interpreted Kaylee's actions to be operational and anticipatory in that the measurements were coordinated from the beginning without a trial-and-error process and in that she was aware that this coordination along with the corresponding measurements would ensure that the rescue team would find the missing person. On the other hand, Dan's activities seemed spontaneous in that they occurred as he was verbalizing instructions and carrying out activities in the moment.

I also hypothesized that Craig and Dan together coordinated angle measure and distance using an angular frame of reference with its origin anchored at point P and initial ray aligned with the line perpendicular to the road to the North Pole. I inferred that Craig and Dan has constructed a system of measurements compatible to a polar coordinate system.

Fish Tank Task.

Over time in the cubic fish tank task, I observed shifts in Dan's locating activities. He started with making visual estimations of the locations of the fish in the tank but later coordinated measurements of horizontal/vertical distances to locate each fish. Dan shifted from using fish as reference to the surface of the water and sides of the tank as spatial references. In the end, he developed a system of measurements consisting of the distance from the top of the water and the distance from all four slides to each fish. In my analysis, I conjectured that Dan did not view the three-dimensional cube as a collection of infinitely many two-dimensional squares (cross sections). Further, I hypothesized that Dan used rectangular frames of reference to guide his measuring of horizontal and vertical distances along the faces of the tank. However, instead of fixing one face and anchoring one set of horizontal/vertical axes onto one adjacent set of edges like Kaylee and Morgan did, I conjectured that Dan anchored his rectangular frame of reference onto all four sets of adjacent edges as I modeled in Figure 8.16. I conjectured that Dan's measurements were sequentially coordinated but not multiplicatively combined due to the lack of the logical multiplication of measurements (Piaget et al, 1960). In the cylindrical fish tank, Dan explained that he could think of finding the vertical layer and the distance from the exterior of the tank but was aware that finding one length measure to express the distance from the exterior of the tank was insufficient in locating the fish. The round shape of the tank seemed to have constrained Dan's further operating with his rectangular frame of reference.

CHAPTER 9

CONCLUSIONS AND IMPLICATIONS

In Chapters 5 through 8, I presented findings from my retrospective analysis of the four students' constructive activities throughout the teaching experiment. More specifically, in Chapters 5 and 8, I analyzed the students' activities in constructing coordinate systems when representing points in perceptual space in the Locating Tasks (e.g., North Pole Task & Fish Tank Task). In Chapters 6 and 7, I analyzed the students' activities in coordinating units along two- or three-dimensional objects in the Counting Tasks (e.g., Cubic Block Task). In Table 3, I summarize the findings across all four students. These findings are my second-order models that account for the four students' mathematical activity and shifts in their ways of reasoning in various spatial contexts. I note here that these models are never to be interpreted as one-to-one representations of students' thinking.

In this chapter, I step back and take a wider lens to look across all four students who participated in the study. First, I revisit the research questions that guided the study and address them with a synthesis of the findings. I will zoom in and out of Table 3 as necessary, to summarize and synthesize the findings related to each research question. Second, I discuss the implications the study has for school curriculum, teaching, and research regarding students' construction and use of coordinate systems. Finally, I pose new questions and present future research directions.

Table 3. Summary of findings across all four students.

	Kaylee (3 levels of units as given)	Morgan (3 levels of units in activity)	Craig (3 levels of units in activity)	Dan (2 levels of units in activity)
North Pole Task	° Coordinated angle measure and radial distance (polar coordinate system). ° Coordination of horizontal/vertical distances (Cartesian-like coordinate system) ° *Rectangular frame of reference scheme*	° Initially wanted to give in-the-moment instructions (temporal & visual approach). ° Gradually constructed rectangular frames of reference through assimilation of Kaylee's approach; coordinated horizontal/vertical distances (Cartesian-like coordinate system)	° Together with Dan, coordinated angle measure and radial distance, but was not explicit about the initial ray (polar coordinate system).	° Constructed a Cartesian-like coordinate system in activity. ° Together with Craig, coordinated angle measure and radial distance, but not explicit about the initial ray (polar coordinate system).
School Map Task	N/A	N/A	° Used a rectangular frame of reference anchored above the center of the first floor, taken from his above-the-ground perspective, representation of the rooms in co-occurrence.	° Used gym as a reference; his frames of reference were sequentially anchored to different locations as he mentally traversed the area.
Fish Tank Task	° Coordinated length, width, height measurements for static and variable locations of fish. ° Constructed a 3-dimensional Cartesian-like coordinate system across both tanks. ° *FR-coordinating scheme*	° Initially made visual estimations of locations of fish. ° Coordinated top-view grid with depth (layer) for static locations of fish ° Described movement of fish in 2 movements (go up and straight) ° *FR-coordinating scheme in activity*	° Initially made visual estimations of locations of fish; thought of longitude/latitude but didn't know how to use it for 3-dimensions. ° In cubic tank, coordinated top-view and side-view of tank to develop a system of measurements consisting of the distance from the sticker face into the tank (width), the distance from left to right along the sticker face (length), and	° Initially made visual estimations of locations of fish. ° In cubic tank, shifted from using fish as reference to using surface of water and four sides of the tank as spatial references. In the end, he developed a system of measurements consisting of the distance from the surface of the water and the distances from all four sides to each fish. ° Dan used rectangular frames

Fish Tank Task (Continued)			the vertical layer in which the fish was embedded (depth). ° Described movement of fish in cubic tank in 2 movements (go up and straight). ° In cylindrical tank, used an imaginary sticker as reference to locate the fish. Taking the top view, coordinated angle measure and distance (polar coordinate system) then taking the side view, identified the layer in which the fish was embedded. ° *FR-coordinating scheme in activity*	of reference to guide his measuring activity but Dan's measurements were sequentially coordinated and not multiplicatively combined. ° In the cylindrical tank, the roundness of the tank constrained further operating with his rectangular frame of reference.
Counting Tasks	° Decomposed and re-composed cubic blocks into square-shaped layers. ° Produced multiple three levels of units structures recursively. ° *Reversible decomposing scheme*	° Initially focused on the exterior faces of the objects and was yet to coordinate the third dimension along with the two-dimensional faces (or exterior layers) in representation. °Using the block model, she could keep track of her counting activities she had a difficult time coordinating in representation; Morgan's operating on the block was in activity, relying on sensori-motor activity on the perceptual material.	° Decomposed the prisms into layers of unit-cubes and re-composed the prisms by building a layer of unit-cubes and inserting them into the third dimension. °Produced three levels of units in successions. Operated with units of units using his unitizing and disembedding operations and iterated units of units in activity, using his unit-coordinating operations to count the spatial objects. ° *Reversible decomposing scheme in activity*	° Initially focused on the exterior faces of the objects and was yet to coordinate the third dimension along with the two-dimensional faces (or exterior layers) in representation. °After engaging in sensori-motor activity building the blocks and partially assimilating Craig's approaches, Dan gradually engaged in representing the unit-cubes and grouping activity. However, Dan still relied heavily on his sketches or models.

Conclusions

Research Questions Revisited

The goal of this study was to investigate how four ninth-grade students construct and use coordinate systems in spatial contexts to organize perceptual space into representational space (Piaget & Inhelder, 1967). In this section, I revisit the research questions that guided the study and address them through a synthesis of the findings.

Research question 1.

> How do the students construct and use coordinate systems when representing objects in two- or three-dimensional perceptual/sensorimotor space?

As shown in Table 3 regarding the Locating Tasks, all four students constructed grid-like structures or organized a set of horizontal and vertical axes, which I referred to as rectangular frames of reference in both the North Pole and Fish Tank Task. Kaylee and Craig also constructed frames of reference consisting of initial and terminal rays joined at a vertex, which I referred to as angular frames of reference. The students used their frames of reference to gauge and represent the relative position of elements in the perceptual space they were representing. Further, they constructed coordinate systems in which they quantitatively organized the perceptual space by coordinating distances or angle measures to account for the location of objects in the space they were representing.

The functional interaction between perceptual and representational space. The students' representations of the spatial situations and their perceptual space reciprocally influenced the organization of each other and interacted functionally (Laurendeau & Pinard, 1970), consistent with what Piaget and Inhelder claimed about the relationship between conceptual and perceptual space. Borrowing Laurendeau and Pinard's (1970) words, "the information provided by perception (or the mental image) serves as raw material for the intellectual action or operation, and reciprocally, these intellectual activities exert an influence (direct or indirect) on perception" (p. 10).

For example, in the North Pole Task, Kaylee considered a structure through which she could gauge the amount of rotation the rescuer would need to make from the North Pole to find the missing person in the snow. In other words, Kaylee abstracted from the spatial context a frame of reference she could use to gauge the relative position of the missing person. On the other hand, when applying this frame of reference to specify the amount of rotation the rescuer would need to make to find the missing person, Kaylee refined her account of the spatial situation by considering the rescuer's line of sight.

Different levels of representations. The level to which the students' representations of space were operational differed across the four students. In some cases, their representations of space were constructed through trial-and-error or temporal approaches and were often dominated by the spatial characteristics of the space they were representing. For example, in the North Pole Task, Morgan initially represented the location of the missing person taking a temporal approach that relied on the perceptual imagery of the North Pole situation. Similarly, Dan moved his ruler on the map in various positions through a trial-and-error process until he accounted for horizontal and vertical distances constituting the rescuer's imagined movement. Finally, Craig used a rectangular frame of reference in the cubic fish tank whereas he used his angular frame of reference in the cylindrical fish tank because he did not find the rectangular frame of reference fitting for the latter spatial situation. In other words, his choice of frames of reference depended on the spatial characteristics of the perceptual space he was representing.

In other cases, students' representations of space were rather immediate and consistent throughout various contexts. For example, the majority of Kaylee's measuring activities were immediate without hesitation to think about how to measure or what to measure. I took this to indicate that her structuration of space was operational and anticipatory. Further, Kaylee demonstrated consistent activities across the different spatial situations. For example, although Kaylee was initially perturbed by the circular shape of the cylindrical tank, she used her rectangular frame of reference across both fish tank cases. From her consistent way of reasoning across the Locating Tasks, I imputed a *rectangular frame of reference scheme* to Kaylee. This scheme consisted of a recognition template for a situation in which she could associate the use of two perpendicular axes, a superimposition of the axes onto a spatial situation, and a result of a Cartesian-like coordination of measurements through which she represents the location of objects in perceptual space.

Frames of reference coordinating scheme. The different levels to which the students' representations of space were operational stood out the most in the three-dimensional contexts. In the three-dimensional contexts, the students used frames of reference similar to those they constructed in the two-dimensional case. For example, Kaylee, Morgan, and Craig used their rectangular frames of reference in the cubic fish tank case and Craig used his angular frame of reference in the cylindrical fish tank case. However, the way students coordinated these frames of reference to account for all three dimensions constituting the space differed.

As summarized in Table 3, in the Fish Tank Task, Kaylee swiftly and consistently coordinated of horizontal/vertical distances along three dimensions, which she referred to as length, width, and height. On the other hand, Morgan, Craig, and Dan initially relied on making visual estimations of the location of the fish in the tanks but gradually made accommodations to their representations to account for coordinated measurements.

From her consistent and flexible way of operating across both fish tanks and in the ocean context in coordinating multiple rectangular frames of reference, I imputed a *frame of reference coordinating scheme* (FR-coordinating scheme) to Kaylee. This scheme involves a recognition of a situation in which she could posit a frame of reference as a unit and insert it into another frame of reference resulting in combined frames of reference, enacting *decentering*, *rotating*, *coordinating perspectives, disembedding*, *inserting*, and *uniting* operations, resulting in multiplicatively coordinated set of measurements to represent the location of objects in a three-dimensional perceptual space.

Both Morgan and Craig demonstrated shifts in their spatial organizations, from relying on visual imagery to more sophisticated systems of measurements (i.e., coordinate systems). Both Morgan and Dan ended up coordinating vertical/horizontal distances induced from their rectangular frames of reference very much like Kaylee did in the Fish Tank Task. However, I contrasted their activities with Kaylee's by claiming that Morgan and Craig enacted *FR-coordinating operations in activity* because their rectangular frames of reference were *sequentially* coordinated. In other words, in Morgan and Craig's case, the first frame of reference was not necessarily inserted into the second frame of reference. One of the main observations that contributed to such conclusion was the way they described the motion of one fish to another in the Fish Tank Task. Instead of describing the motion of Fish 1 to Fish 2 along all three spatial dimensions, like Kaylee did, both Morgan and Craig described the fish motion as going up along the layers of the water and then going straight, once the two fish were on the same layer.

Similar to Morgan and Craig, Dan also made shifts from making visual estimations to gradually measuring distances in the cubic fish tank; he shifted from using fish as reference to using the surface of water and four sides of the tank as stationary references. In the end, he developed a system of measurements consisting of the distance from the surface of the water and the distances from all four sides to each fish. Although Dan used rectangular frames of reference to guide his measuring activity, I considered Dan's system of measurements less sophisticated because he measured distances from all four sides of the tank, some of which I considered redundant. Moreover, in the cylindrical tank, the roundness of the tank constrained further operating with his rectangular frame of reference.

To summarize, regarding the first research question, I modeled the students' frames of reference and their use of frames of reference to quantitatively represent the locations of the objects in perceptual space. I discussed the functional interaction between the students' perceptual and representational space and the different levels of sophistication of representational activity I observed across the four students. Finally, I discussed the schemes and operations to model the process through which students constructed their coordinate systems.

Research question 2.

How do the students count units within spatial objects that entail arrays of units along two or three dimensions in representation?

As shown in Table 3 regarding the Counting Tasks, students demonstrated different sophistication levels in representing units (e.g., unit-tiles or unit-cubes) in spatial objects to account for the location or number of units constituting the spatial objects.

Different levels in representations of units. The level to which the students' representations of the units were abstracted from the perceptual and/or sensorimotor activity on the spatial objects differed across the four students, which I summarize as follows. Kaylee has constructed representations of the spatial objects that she could reason upon reflectively; further, using her representations of space, she was able to coordinate units within three spatial dimensions and produced multiple three level of units structures. Her construction of powerful re-presentations of the space were supported by her immediate activities when presented with new situations and abilities to anticipate and carry out these activities without readily available perceptual material.

On the other hand, Craig and Morgan demonstrated the use of many of the operations that Kaylee used in activity. There were many instances where they sat in deep thought staring at the objects for relatively long periods of time or often needed the perceptual material in reasoning. Although, in activity, Craig and Morgan were able to sequentially coordinate two dimensions to reason within three spatial dimensions.

In contrast to the other three students, Dan reasoned primarily in two, but not three, dimensions. He did not enact the mental operations that the other three students did in decomposing and recomposing the spatial objects. His spatial objects were more perceptual than a re-presentation of the space on which he could operate on.

Reversible decomposing scheme. As summarized in Table 3, in the Cubic Block Task, Kaylee consistently decomposed and re-composed the blocks into and from square-shaped or rectangular-shaped layers. She used a similar strategy for counting in extending or reducing the cubic blocks. From her consistent counting activity, I imputed a *reversible decomposing scheme* to Kaylee, which she used to mentally decompose, re-present, and anticipate (Piaget & Inhelder, 1967) the interior of the cubic blocks in the absence of their perceptual availability.

This scheme involved a recognition of a situation in which she needed to individualize unit-cubes but also maintain their relative positions and an enactment of *unitizing*, *disembedding*, *splitting & recursive partitioning schemes* to decompose and re-compose the object in representation. The scheme also involved *coordinating multiple perspectives* by bringing forth immediate past results of her mental actions while enacting mental actions on another. I claimed that Kaylee's *FR-coordinating scheme* served as a sub-scheme to guide the decomposition/re-composition of the object. I also claimed that using her *units-coordinating scheme* as a sub-scheme, Kaylee produced multiple three levels of units recursively in re-composition of the objects. Using this scheme, Kaylee, counted the total number of unit-cubes and identified the painted/unpainted unit-cubes in representation.

Craig demonstrated similar counting strategies as Kaylee did in the Counting Tasks. Craig repeatedly explained why he multiplied the number of units along each spatial dimension to count the total number of units constituting the two- or three-dimensional figures. He decomposed rectangular prisms into layers of unit-cubes and re-composed the prisms by building a layer of unit-cubes and inserting them into the third dimension. Craig demonstrated a strong ability to coordinate a representation of one instantiation of his building activity in co-presence with another instantiation of his building activity. Therefore, I conjectured that Craig had the operations of a FR-coordinating scheme available to support his reversible decomposing activity.

However, contrast to Kaylee, Craig demonstrated having difficulty keeping track of his counting when extending the three-dimensional rectangular prisms in representation. Based on his counting activities, I conjectured that Craig produced three levels of units in successions. That is, he operated with units of units using unitizing and disembedding operations and iterated units of units in activity, using unit-coordinating operations to count the spatial objects. Further, I conjectured that Craig constructed a *reversible decomposing scheme in activity*.

Morgan and Dan initially focused on the exterior faces of the objects and was yet to coordinate the third dimension along with the two-dimensional faces (or exterior layers) in representation. Hence, there was no observable decomposing or recomposing of the units along the third dimension. Gradually, Morgan assimilated some of Kaylee's strategy in decomposing the cubic blocks into layers of unit-cubes. However, she often relied on using the block model to keep track of her counting activities. After engaging in sensori-motor activity building the blocks and partially assimilating Craig's approaches, Dan gradually engaged in representing the unit-cubes and grouping activity. However, Dan still relied heavily on his sketches or models. Coordinating a representation of one instantiation of their activity in co-presence with another instantiation of their activity seemed confining elements in their reasoning for both Morgan and Dan.

To summarize, regarding the second research question, I discussed the different levels of representations of units in the spatial objects and modeled the process through which the four students coordinated units along two- or three-dimensional objects.

Looking Across Research Questions

The models I developed in this study showcases the complexities of cognitive structures that could be involved in constructing spatial coordinate systems. In this section, I discuss three ways of reasoning in the students' constructive activities across both types of tasks I conjecture to have played as cognitive resources for some students who had them available but cognitive barriers for those who lacked such ways of reasoning.

Coordination of multiple images.

From the findings, I conjecture that the ability to coordinate multiple images is a cognitive resource for constructing spatial coordinate systems. By coordinating multiple images I mean the mental activity of *bringing forth images of one instantiation of mental activity alongside another instantiation of mental activity*. For example, in the Locating Tasks, Kaylee and Craig demonstrated the ability to bring forth images of one perspective (their ego-oriented perspective looking down onto the spatial situation) alongside another (an imaginary perspective embedded within the spatial situation). In the Counting Tasks, Kaylee and Craig demonstrated the ability to bring forth images of one instantiation of their building of blocks activity in co-presence with another instantiation of their imagined building activity. I claim their ability to engage in coordinating multiple mental images supported their coordination of multiple frames of reference and counting activities.

On the other hand, I conjecture that often Morgan and Dan's locating activity or counting units was constrained by the lack of coordination of multiple mental images. Their tendency to focus mainly on their ego-oriented perspective and to take temporal/trial-and-error approaches in the spatial situations corroborate such conjecture. For example, When I asked him to explain the location of two rooms, Dan had to imagine relocating himself at a certain room of the first floor to verify that his representation was accurate. When reasoning about the cubic blocks, Morgan had to rotate or take apart the cubic block models to reason about the number of unit-cubes or painted/unpainted unit-cubes.

Logical multiplication.

In addition to coordinating multiple images, I conjecture that logical multiplication is a cognitive resource needed for constructing spatial coordinate systems. This conclusion is consistent with Piaget et al.'s finding (1960) that locating a point in two-dimensional space "involves logical multiplication of measurements as given by rectangular coordinates" (p. 154). A logical multiplication entails *a recognition of a location of a point along one spatial dimension with the realization that the point has a specific location along the other spatial dimension*. As a result of logical multiplication, the location of a point becomes a multiplicative location in that it is a product of a simultaneous coordination of its location along two or more separate dimensions.

Based on the findings, I conclude that engaging in logical multiplication is necessary for coordinating multiple frames of reference multiplicatively so that the location of an object along one frame of reference is inserted into and maintained within another frame of reference. For example, Kaylee maintained the fish's location within her first rectangular frame of reference (e.g., one side of cubic tank) alongside that within her second rectangular frame of reference (e.g., the adjacent side of cubic tank). This allowed her to construct a coordinate system consisting of length, width, height measures induced from her rectangular frames of reference. In the Cubic Block Task, Kaylee maintained the location of an individual unit-cube in relation to other unit-cubes along all three dimensions constituting the cubic blocks.

Levels of units coordination.

In entering this hypothesized that students' levels of units coordination and the relevant operations and schemes are involved in constructing spatial coordinate systems. Therefore, I selected students who demonstrated different levels of sophistication in coordinating two or three levels of units. Kaylee reasoned as if she could operate with three levels of units as given, Morgan and Craig with three levels of units in activity, and Dan with two levels of units in activity.

From the findings, I found a parallel between the students' levels of units coordination (initial interviews) and their constructive activities in the spatial contexts across all the tasks. That is, consistent with the findings from the initial interviews, Kaylee seemed to be able to operate with a multitude of two-dimensional frames of reference in representation; Morgan and Craig were often able to enact the operations Kaylee did when carrying them out in activity; and Dan was often restricted to a single two-dimensional frame of reference.

Moreover, the students' operations and schemes involved in the partitioning and units-coordinating situations I inferred from the initial interviews were used as assimilatory operations and schemes in both the FR-coordinating scheme and reversible decomposing scheme. Therefore, I conclude that the mental operations and schemes that produce three levels of units are necessary for simultaneously coordinating multiple two-dimensional frame of reference in organizing space.

Implications and Future Research Direction

All four students reported in this study were ninth-grade students who already had experience using the Cartesian coordinate plane in school. Therefore, it seemed natural that they superimpose a Cartesian-like structure of grids in these spatial tasks. However, this was not the case and even when the students superimposed a Cartesian-like grid structure onto the spatial situations, the ways of reasoning within frames of reference and consequently the coordination of measurements differed among the students.

I believe the models I developed to account for the four students' mathematics in constructing spatial coordinate systems can inform teachers, curriculum developers, and researchers in regards to students' learning and application of coordinate systems. In this section I discuss implications of this study and future research direction.

Implications

Students' zones of potential construction.

Through the living models of the four students' mathematical activity, this study gives teachers and researchers opportunities to experience the four students' constructive activities and provides an explanatory framework (Steffe & Thompson, 2000) to describe, discuss, and think about their students' mathematical activity. The models I developed of each students' cognitive activities in this study can also be used to discuss their zones of potential construction. Here, zone of potential construction refers to "a teacher's working hypotheses of what the student can learn, given her model of the student's mathematics" (Steffe & D'Ambrosio, 1995, p. 154). In this section, I discuss the implications of my findings by discussing the zone of potential construction of each student.

Kaylee's zone of potential construction. Throughout the five months I worked with her, Kaylee demonstrated powerful ways of reasoning. Although new situations and tasks occasionally seemed to perturb her, she was quick to adjust her current ways of operating to accommodate the new situations. For example, in the cylindrical tank, Kaylee was perturbed by the circular shape but soon coordinated rectangular frames of reference consistent with her cubic tank case.

Although Kaylee mainly coordinated rectangular frames of reference as input for her FR-coordinating scheme, because she had the mental operations available, I anticipate that Kaylee's FR-coordinating scheme was not limited to a coordination of rectangular frames of reference. That is, Kaylee could have inserted different types of frames of reference like her angular frame of reference into other frames of reference. For example, I believe Kaylee would have been able to construct a cylindrical coordinate system similar to Craig's by superimposing polar coordinates along the circular cross section and coordinating that with a height axis. Further, I believe Kaylee is capable of coordinating non-perpendicular axes, such as in an oblique coordinate system, supported by her logical multiplication activity.

Supported by her FR-coordinating scheme and logical multiplication, I believe Kaylee could plot points given their horizontal and vertical coordinates and keep track of the location of the point along one axis with a realization that the point had a specific location along the other axis. Thus, for Kaylee, a point on the Cartesian plane entails its horizontal and vertical positions along each respective axis. Further, I hypothesize that Kaylee's powerful spatial organization can support her use of coordinate systems for quantitative coordination and covariational reasoning—"holding in mind a sustained image of two quantities' values simultaneously" (Saldanha & Thompson, 1998, p. 1). For example, I believe Kaylee would be able to find intervals of increase/decrease of one quantity as another quantity increases.

Supported by her FR-coordinating scheme and coordination of multiple perspectives, Kaylee conceptualized directionality as an amount of rotation or the amount of movement in horizontal/vertical segments along each spatial dimension. Especially, throughout the Fish Tank Task, Kaylee consistently described motion of one fish to another as moving along length, width, and height dimensions. Having established this conceptualization of directionality, I believe Kaylee could conceptualize the inclination of a line or distance between two points in two- or three-dimensional Cartesian coordinate systems. Further, I believe Kaylee can produce the set of all the points that are a constant distance from a combination of points and lines in two- or three-dimensional space. For example, I anticipate Kaylee would have been able to represent all the possible locations of a fish 2 inches from the surface of the water.

Finally, using her reversible decomposing scheme, I believe Kaylee could engage in other situations which require her to simultaneously decompose and re-compose spatial objects into arrays of units. Further, I believe Kaylee had the mental operations available to support engagement in differential and integral calculus.

Morgan's zone of potential construction. Throughout the five months I worked with her, Morgan showed noticeable shifts in her ways of reasoning. Although occasionally Morgan was constrained by her difficulty drawing three-dimensional figures on paper or represent the spatial situations in representation, Morgan was quick to assimilate Kaylee's ways of reasoning and showed powerful ways of reasoning when she could carry necessary mental actions out in activity.

In Morgan's case, I believe the social interaction with Kaylee was crucial in her engagement in the tasks. Given more opportunities to make the ways of reasoning she assimilated from Kaylee more permanent, I believe Morgan could engage in many of the activities I anticipate Kaylee to engage in. In addition, I hypothesize that engendering Morgan's coordination of multiple images would lead to accommodations in her FR-coordinating activity and units coordination activity along two- or three-dimensional objects.

Craig's zone of potential construction. Throughout the eight months I worked with him, Craig demonstrated powerful ways of reasoning often times similar to Kaylee but also seemed to have encountered some constraints similar to Morgan's in his ways of operating. In contrast to Morgan, Craig demonstrated a strong ability to engage in coordinating multiple images to support his representations of spatial situations. Therefore, I anticipate Craig would have been able to represent the painted/unpainted unit-cubes in the cubic blocks in the Cubic Block Task, like Kaylee did. However, because he was yet to use the FR-coordinating scheme as input for further operating, I hypothesize that Craig would have met constraints in finding the number of unit-cubes needed to extend or reduce cubic blocks in representation, especially given bigger sized numbers to coordinate. On the other hand, similar to my hypothesis of Morgan's zone of potential construction, I believe Craig could be successful in such activity when given the physical models to keep track of his units-coordinating activity.

Because Craig’s coordinate systems were more reliant on the spatial characteristics of the situation, I anticipate Craig’s constructions of coordinate systems to be more case-specific. For example, if given a spherical fish tank, I hypothesize Craig will again be perturbed by the different shape and construct a different type of coordinate system to account for that shape whereas Kaylee will consistently superimpose a Cartesian-like coordinate system onto the spherical tank.

Dan’s zone of potential construction. Over the eight months I worked with him, Dan presented the most constraints to me as the teacher-researcher. Throughout the teaching experiment, a lot of features of the tasks seemed to perturb Dan’s ways of operating and he often seemed to hit a wall due to the lack of operations and schemes the other students had available. At maximum, I inferred that Dan could produce two-dimensional coordinate systems and coordinate two spatial dimensions in activity but not three. Therefore, conceptualizing the volume of spatial objects could be out of reach of his zone of potential construction.

However, Dan made progressions as well. For example, after engaging in sensori-motor activity building the blocks and partially assimilating Craig’s approaches, Dan gradually engaged in representing the unit-cubes in grouping activity. I believe the constructive path reasonable for Dan is to allow him more opportunities to immediately reflect upon his perceptual and sensori-motor spatial activity. For example, having Dan describe a spatial configuration as he looks or touches the configuration from one perspective and then another can engender a coordination of multiple perspectives, which he seemed to lack the ability to carry out in many of the tasks. Gradually having him draw spatial configurations without having the perceptual/sensorimotor imagery available can also be helpful.

In retrospect, often Dan changed his answer when I asked him to explain his thinking or said Craig had the “correct answer” and that he was “wrong.” Sometimes, Dan waited for Craig to talk first or when Craig was not present at the teaching episodes, Dan wished Craig was there. Other than the cognitive barriers Dan experienced, I believe his learned perception of his relationship with teachers or his peers influenced such behavior. In this relationship, the teacher (i.e., an authoritative figure) or his peer (i.e., someone who he thinks is superior in mathematical ability) validate his mathematical activity. Therefore, I hypothesize that Dan’s ways of operating could be better supported by addressing such social norms in his learning environment.

Implications of zones of potential construction on teaching. Although the zones of potential construction I discussed above are those of each individual student I worked with, I believe these zones of potential construction can inform teachers and curriculum developers in constructing hypothetical learning trajectories (Steffe & D’Ambrosio, 1995) appropriate for other students at different cognitive sophistications. In other words, Kaylee’s zone of potential construction can inform teachers who might have other Kaylee’s (students who have the schemes and operations Kaylee had available) in understanding their current ways of thinking and the directions they can move forward. Alternatively, Dan’s zone of potential construction can inform teachers who might have other Dan’s (students who demonstrate similar ways of operating as Dan) why they might be struggling in certain situations such as those Dan struggled in and potential activities that could be helpful for them.

Foregrounding the background, i.e., bringing attention to the coordinate systems on which students are required to reason, can be helpful in understanding why students have difficulty constructing or interpreting graphs of geometrical or quantitative relationships represented on coordinate systems. I believe the findings of this study can provide insight for understanding students' difficulties with graphing in two- or three-dimensions, such as those identified in Dorko and Lockwood (2016).

Steffe, Moore, & Hatfield (2014) described the Epistemic Algebraic Student as "a conceptual model of what we observe as characteristic mathematical activity of students that is taken to define a level of development in the algebraic activity of the students in the context of mathematics teaching" (p. ix) and as "models consisting of dynamic organizations of schemes of action and operation in our mental life that undergo change over longish periods of time" (p. x). They also emphasized that "it is through continued interaction with students that we may conceive several epistemic algebraic students throughout the ages of schooling, each of which is distinctly different from the others" (p. x). Adopting Steffe et al.'s (2014) notion of epistemic student, I believe the models I constructed through continued interaction with students over a long time period can provide a stepping stone in building several epistemic *spatial* students. These models can characterize their mathematical activity and define levels of development in spatial organizational activities and thus can inform teaching, curriculum development, and research.

School curriculum.

Based on my conceptual analysis of coordinate systems and the findings from this study, I propose modifications in the way coordinate systems are taught in school.

Revisiting the Common Core States Standards for Mathematics, the conventional Cartesian coordinate plane is "introduced" for the first time in 5th grade geometry as follows.

> Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010, p. 38)

This rather abstract description of the Cartesian coordinate plane, stripped away from any context, only states *what* a student should know to construct one but does not necessarily address the *reasoning* that is involved in understanding *why* and *how* it might make sense to construct one. Towards the end of the teaching experiment, I had a conversation with Kaylee and Morgan about coordinate planes, as shown in Excerpt 9.1.

Excerpt 9.1. Morgan and Kaylee talking about coordinate planes.

I: Is there anything else you learned with coordinating points like this?
K: I don't know. Everything was just like (inaudible).
M: Yeah, We never really went into depth about it. You know, it's like, this is what happens to this, so just know this.
K: Mmhmm. Just graph things, make shapes.
M: Yeah, I know. We never really like, it's like this happens and that happens. They never really gave us real world examples like this.
K: Yeah.

Like Morgan said, they were told "this is what happens to this, so just know this" and like Kaylee said, they were told to "just graph things" and "make shapes." I believe the conversation I had with Kaylee and Morgan reflects and highlights how the Cartesian plane is taught in school, parallel to the presentation of it in school curriculum. Therefore, first, I propose an alternative approach to the introduction of the Cartesian plane, which encourages making sense of how coordinate systems work and what they can do for us:

> Use a pair of perpendicular number lines, called axes, to define a coordinate system through which one can quantitatively organize points in the plane, with the

> intersection of the lines located at a fixed (reference) point in the plane, by using an ordered pair of numbers, called its coordinates, which represent the direction and distance from the fixed point along each axis.

Second, I argue that school curriculum should provide students with more opportunities to construct coordinate systems of various kinds from spatial contexts. These opportunities can enhance their reasoning within multiple frames of reference and their construction of coordinate systems, which they can use for further reasoning of geometric or quantitative relationships.

Finally, I was surprised at how novel the tasks of constructing spatial coordinate systems was to the four students. I claim that attention to the distinction of the different uses of coordinate systems and support for students' balanced understanding and use of coordinate systems is needed by researchers, curriculum developers, and teachers. This study can provide insight for opportunities that afford students more cognitive access to powerful ways of reasoning of the mathematical concepts that are represented through coordinate systems.

Tasks.

Different from extant research studies in which students were provided with pre-constructed, conventional coordinate systems, the tasks in this study provided ways to observe the kinds of coordinate systems students actively construct independently in various situations. The ninth-grade students did not have any formal instruction on the polar coordinate system, but Kaylee, Craig, and Dan all constructed coordinate systems I view compatible with the polar coordinate system.

Based on my interaction with the students as they engaged in the tasks I designed for this study, I found the tasks to have provided students opportunities to construct and use coordinate systems as a means "to construct, refine, and use their own representations as tools to support learning and doing mathematics" (NCTM, 2000, p. 68). These findings can inform task development for teachers and textbook publishers to allow more meaningful opportunities for secondary students to construct, refine, and use their own coordinate systems, which can support stronger understandings of coordinate systems and other mathematical concepts such as functions and their graphs.

For example, the Locating Tasks afforded students opportunities to engage in coordinating frames of reference and constructing coordinate systems. There were four features of these tasks that I found particularly important. First, having students actively engage in describing the locations or movements of objects as opposed to making a copy of a point supported students to go beyond making visual estimations and actively engage in measuring activities. I found this requirement to have encouraged students to think about what to measure (e.g., distance or angle measure) and how to measure it. This resulted in students selecting and coordinating frames of reference in finding various measurements. Second, having students develop instructions for another person (or fish) afforded opportunities for students to consider directionality in a systematic way and coordinate multiple perspectives through decentering. Third, having various shapes and removing explicit spatial cues was helpful in investigating the type of measurements students attended to. In the North Pole Task, students coordinated what I considered horizontal and vertical distances or angle measure and radial distances. The various shape of the spaces seemed to influence students' selections of such measurements but also afforded opportunities for students to superimpose a rectangular system onto spaces not necessarily explicitly cuing rectangular coordinate systems. Finally, having students engage in locating tasks such as the North Pole and Fish Tank Tasks can enhance their constructions and enactments of operations and schemes that I found to be productive for constructions of representational space.

Future Research Direction

Replicating the study.

Working closely with a select number of students provided the opportunity for me to observe and model the students' constructions of spatial coordinate systems. These second-order models are based on my interpretations and inferences from the four students' mathematical activity. Moving forward, I would like to conduct another teaching experiment by selecting "at least three students whose language and actions indicate similar spontaneous schemes," (Steffe & Thompson, p. 299), a) to test the viability of my current models and b) to use my current models of the four students as input for building superseding models (Steffe & Thompson, 2000) of students' constructive activities in constructing coordinate systems.

Testing students' zones of potential construction and graphing activity.

In addition to replicating the study and refining my models of the students' constructive activities, I would like to test my hypotheses about the students' zones of potential construction and graphing activity. Based on the models I developed of each students' cognitive activity in this study, I discussed their zones of potential construction. These are hypotheses of what I infer the students to be capable of learning, given his or her current schemes and operations. As an extension of my teaching experiment, I would like to test these hypotheses with students whose language and actions I consider to be similar to each of the four students.

For example, I hypothesized that Kaylee's powerful spatial organization will support her use of coordinate systems for quantitative coordination and covariational reasoning. Additional research investigating students' spatial organization activity in situations that involve conceptual change, which require attending to not only static instantiations of movement but also the variability of points in space is needed. I believe such extensions of this study can inform finding connections between spatial and quantitative reasoning and identifying common mental operations invariantly involved in the construction of coordinate systems in both uses.

Levels of units coordination and spatial reasoning.

In this study, I concluded that the mental operations that produce three levels of units are necessary for simultaneously coordinating multiple two-dimensional frames of reference in organizing three-dimensional space. I also concluded that the coordination of multiple frames of reference supported students' systematic structuration of three-dimensional objects and counting activities. Regarding this finding, there are two questions that could be further investigated.

First, although I found it helpful to provide Dan with opportunities to engage in building three-dimensional objects and reflecting on his building activities, further explorations on how we can support students who are yet to construct the necessary schemes and operations for coordinating multiple frames of reference is needed. Second, further investigations on whether the development of three levels of units precedes the ability to coordinate three spatial dimensions, or the coordination of three spatial dimensions precedes the development of three levels of units, or whether the development occurs concurrently is needed.

Summary of Chapter Nine and Closing Remarks

In this chapter, I took a step back to look at the findings across all four students who participated in the study. I revisited the research questions that guided the study and addressed them with a synthesis of the findings. I also discussed the implications of the study for teaching, school curriculum, and research regarding students' construction and use of coordinate systems. Finally, I presented future research directions replicating and expanding on the findings of this study.

Although this dissertation has come to an end, I believe my conceptual models, ideas, and thoughts that emerged from this study will continue to grow through continued work with students, teachers, and researchers. This study leads me to pursue three long-term scholarship goals. First, expanding my focus beyond students' spatial coordination, I would like to investigate how students construct coordinate systems for quantitative coordination and how spatial and quantitative coordination might relate. Second, broadening my population for study beyond high school students, I would like to investigate how students at various grade levels might construct coordinate systems as powerful tools for spatial, algebraic, quantitative, and covariational reasoning. Finally, putting my research into practice, I am interested in designing curricular or professional development opportunities with an emphasis on students' meaningful use of coordinate systems in their learning of mathematical ideas in school algebra and geometry.

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APPENDIX

Initial Interview Tasks[7]

Part I: Kaylee and Morgan's Initial Interview

Introduction:
Thank you for coming to work with us this morning. We are interested in looking at how you think about the questions we are about to ask you. They might be unfamiliar questions and it might require some thinking. Take your time. We are more interested in your thinking rather than you being right or wrong. So, it's very helpful if you can tell us what you're thinking as you work on the problems we pose to you, even when you're not really sure what the answer might be. Also, we have prepared several things here on the desk for you to use (point to the various tools) and you can make use of them if you think they will help you.

- ✓ **Materials:** Wikki stix, scissors, markers, paper, ruler, thermometer (sliders)
  Part I: Partitioning
  Part II: Directed Measurements

1. Sharing a piece of string equally among five people (only ask this question if needed)

[Display a piece of wikki stix]

- *Let's pretend that this string is a piece of licorice and that you want to share this string equally to five people, can you mark off the piece of licorice that one person would get?*
- *Can you show me how you would check that this share is the right amount?*
- *What fraction of the original stick would one person receive?*

2. Sharing a piece of a piece of a string

[Prepare Fruit by the Foot. Have the student roll it out and put it in front of him/her. Then ask student to close eyes and think about the following in their head first.]

- *Cut off one person's share if we equally share this candy among 3 people.*

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[7] This interview guide was developed in work with my research group the time it was developed. I credit contributions to Dr. Les Steffe, Dave Liss, Jackie Gammaro, Eun Jung, June Chun, Ebru Ersari, and Hamilton Hardison.

- *If we share this cut off piece equally among you and 4 more of your friends, cut off your share.*
- *What amount is your piece of the whole piece of candy?*

3. Make your string such that my string is five times as long as your string

[Prepare medium wikki stix and a pile of other wikki stix and hold up medium wikki stix:

- *Suppose I have a piece of string here. I want you to make your piece of string so that my string is five times longer than yours. How would you make your string? Think about it and tell me what you're thinking.*]

If the student is unsure or struggles to answer appropriately, then have a roll of string available and ask the following question. Even if they answered the question well verbally, still ask them to make it with the piece of string and ask the following questions:

- *Here's some more string. Use this to make your string.*
- *How would you prove that my string (point to the initial piece of string you presented) is five times longer than the one you described/made?*

If they describe a string that is longer than the one given them ask:

- *What did I ask you? Whose string would be longer?*

4. Share two *same-sized, same-flavored* cakes

[Prepare two cake models (homogeneous) of SAME size.]

Suppose these are two cakes both (use different flavor) cakes of same size.

- [Cover cakes with a cover] *Let's say we're sharing these cakes equally among three people. Can you tell me how you might share all the cake?*
  When student can operate mentally, skip carrying out the cutting activity.
- *How do you know what you've an equal share? How would you check?*
- *What fraction of one cake would that be if they are identical cakes?*
  When student cannot operate mentally, show the two cakes and ask them to carry out the sharing.
  When the student puts the cake altogether, after they are finished, give a context where they can't put the two cakes together. Ask them to *find a way to find the fair share without combining the two cakes*.
- *(Point to one plate) Say this is your plate. What amount of all the cake do you have? (goes back to case a)*
- *What amount of one cake do you have?*
  Follow-up as necessary to see if they can understand and explain that they would get 2/3 of one cake. If they say 2 divided by 3 or 2/3, follow-up with asking them to show you. (We need to be sensitive to the differences between "2 divided by 3", "1/3 of 2", and "2/3 of 1".)

5. Share two *different-sized different-flavored* cakes

[Prepare two cake models of DIFFERENT size and flavor.]

- *Suppose these are two cakes both (flavor) cakes of different size*.
- [Cover cakes with a cover] *Can you tell me how you might find 1/3 of all the cake?*
  When student can operate mentally, skip carrying out the cutting activity.
- *How do you know what you've found is 1/3 of the cake? How would you check if it is 1/3 of the cake?*

When student cannot operate mentally, show the two cakes and ask them to carry out the sharing and use sharing language:

- *What would you do to share these fairly among three people?*
  When the student puts the cake altogether, after they are finished, give a context where they can't put the two cakes together. Ask them to *find a way to find 1/3 without combining the two cakes*.
- *(Point to one plate) Say this is your plate. What amount of all the cake do you have?*
  When student says "two out of six pieces" (2/6) ask
- *Are the pieces the same size?*
- [Pull out two small pieces and two big pieces] *If we think about the amount of the cake, would that be a fair share?* ***What amount*** *of all the cake do you have?*
- *If this is 2/6, then what would this (Pick one piece up) be? 1/6?*
- *How would you check if that is 1/6 of the cake? Can you use it to make all the cake? Do you get the cake you had at the beginning?*

Part II: Craig and Dan's Initial Interview

Introduction:

Thank you for coming to work with us this morning. We are interested in looking at how you think about the questions we are about to ask you. They might be unfamiliar questions and it might require some thinking. Take your time. We are more interested in your thinking rather than you being right or wrong. So, it's very helpful if you can tell us what you're thinking as you work on the problems we pose to you, even when you're not really sure what the answer might be. Also, we have prepared several things here on the desk for you to use (point to the various tools) and you can make use of them if you think they will help you.

1. Equipartitioning task

Materials: Middle sized wikki stix, scissors, markers

- ➤ Display a piece of wikki stix
- o *Let's pretend that this string is a piece of licorice and that you want to share this string equally to five people, can you mark off the piece of licorice that one person would get?*
- o *Can you show me how you would check that this share is the right amount?*
- o *What fraction of the original stick would one person receive?*

2. Additive Units Coordination task

Materials: Wikki stix, scissors, markers

- ➤ Display two pieces of wikki stix in front of student.
- o *Let's say this string is 24 centimeters long and this one is 39 centimeters long. How much more string would I need in order to make this shorter one as long as the longer one?*
- ➤ Cut off each wikki stix into shorter lengths.
- o *I cut off some length from both of these strings. We don't know the actual length of them yet. But do you think you can find how many of the shorter ones will fit in the longer one?*
- o *(Back-up) If I told you that the shorter one is 7 centimeters and the longer is 34 centimeters, how would you find that out?*

3. Splitting task

Materials: Wikki stix, scissors, markers

- Prepare medium wikki stix and a pile of other wikki stix and hold up medium wikki stix:
  - *Suppose I have a piece of string here. I want you to make your piece of string so that my string is five times longer than yours. How would you make your string? Think about it and tell me what you're thinking.*
- If the student is unsure or struggles to answer appropriately, then have a roll of string available and ask the following question. Even if they answered the question well verbally, still ask them to make it with the piece of string and ask the following questions:
  - *Here's some more string. Use this to make your string.*
  - *How would you prove that my string (point to the initial piece of string you presented) is five times longer than the one you described/made?*
- If they describe a string that is longer than the one given them, ask:
  - *What did I ask you? Whose string would be longer?*
- (Back-up) If string situation seems insufficient, change context to money:
  - *Suppose I have 24 dollars. Let's say we know that my money is 3 times more than yours. How much money would you have?*

4. Units Coordinating – Splitting hybrid task (optional)

Materials: Wikki stix, scissors, markers (if needed)

- Before displaying wikki stix, explain the situation first. Prepare stix with lengths that correspond to each situation.
  - *Let's say you, Cody, and June each have a piece of string. We know that Cody's string is 4 times longer than June's string, and your string is 7 times longer than Cody's string. Can you explain to me how long your string is in terms of June's string?*
  - *(If above is easily done) This time, let's say we know that Cody's string is 3 times longer than your string and June's string is 6 times longer than Cody's string. How long is your string in terms of Cody's string?*

5. Recursive Partitioning

Materials: Strip of ribbon, scissors, markers

- Prepare a strip of ribbon and put it in front of him/her. Then ask student to imagine this is a strip of candy and to close eyes and think about the following in their head first.
  - *Cut off one person's share if we equally share this candy among 3 people.*
  - *If we share this cut off piece equally among 5 people, cut off one share.*
  - *Let's say that share is yours. What amount is your piece of the whole strip of candy?*