ORIGAMI-CONSTRUCTIBLE NUMBERS

by

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(Under the Direction of Daniel Krashen)

Abstract

In this thesis, I present an exposition of origami constructible objects and their associated algebraic fields. I first review the basic definitions and theorems of field theory that are relevant and discuss the more commonly known straightedge and compass constructions. Next, I introduce what origami is and discuss the basic single-fold operations of origami. Using the set of single-fold operations, I explain what it means for an object to be origami-constructible and show how to prove or disprove the constructibility of some origami objects. Finally, I present extensions of origami theory in literature and pose some additional questions for future studies.

Index words: Field Theory, Compass and Straightedge Construction, Origami Construction, Constructible Numbers

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Contents

List of Figures

Chapter 1

Introduction

1.1 Geometric Constructions

There are various tools, such as the straightedge and compass, one can use to construct geometric objects. Each construction tool has its unique rules of operating when making geometric constructions. To *construct* a geometric object means to carry out a set of operations the tool permits that collectively produce the geometrical object. Note that constructing a geometric object differs from making a freehand illustration or drawing of the object.

Take the classical straightedge and compass as an example. In straightedge and compass constructions, one can use either tool but not both simultaneously. The straightedge can only be used to draw the line through two given points but cannot be used to measure length. The compass can be used to construct a circle or arc with a given center and point on the circle or arc. Some basic straightedge and compass constructions will be discussed in Chapter 3 (see Figures 3.1 and3.2).

The system of assumptions or axioms concerning the unique rules of operating for each tool have a profound effect on what objects are constructible within a particular framework. While the classical framework of straightedge and compass constructions (described below) is better known, one can expand this framework by adding other constructions (for example, see Section 3.4 on the marked ruler and verging), or one can consider new frameworks, such as constructions by origami, which will be explored later in detail.

The straightedge and compass, the rules of construction, and the study of constructible objects go back to the ancient Greeks and Egyptians. For example, consider the famous Greek geometry problems:

Using only straightedge and compass,

1. (Doubling a cube) Is it possible to construct a cube whose volume is equivalent to twice the volume of a given cube?

2. (Trisecting an angle) Is it possible to trisect any given angle of measure θ?

3. (Squaring a circle) Is it possible to construct a square whose area is equivalent to that of a given circle?

The study of constructible objects including the famous Greek geometry problems emerged from everyday life, such as in architecture and surveying [14]. For example, according to Cox [4, page 266], there are two versions of the origin of doubling a cube: King Minos and the tomb of his son; the cubical altar of Apollo in Athens. Both involved the construction of some architectural structure. Although straightedge and compass may have lost their place in our everyday lives, geometric constructions with straightedge and compass are still introduced in high school geometry providing opportunities for students to better understand definitions and characteristics of geometric figures.

Going back to the Greek geometry problems, the Greeks were not able to solve the famous geometry problems at that time but the search for solutions led to other mathematical creations (e.g., Hippocrates of Chios and the lune; Hippias of Elis and the quadratrix, Menaechmus and parabolas, the spiral of Archimedes) [4]. In 1837 Wantzel showed that trisecting an angle and doubling a cube by straightedge and compass were not always possible; in 1882 Lindemann showed that squaring the circle was impossible with straightedge and compass when he showed that π is transcendental over $\mathbb{Q}$ [4]. The proofs will be discussed in Chapter 3.

The development of Modern algebra and coordinate geometry played a big role in solving these problems and studying geometric constructions with other various tools, such as the marked ruler, divider, Origami (paperfolding), and Mira. Specifically, the collection of lengths that various geometric tools can theoretically construct became associated with various algebraic fields. Among the various tools of geometric constructions, this thesis presents an exposition of origami constructible objects and its associated algebraic field.

1.2 Origami Constructions

Origami is the Japanese art of paper folding, in which one starts with a square-shaped sheet of paper and folds it into various three-dimensional shapes. Typically in origami, one starts with an unmarked square-shaped sheet of paper using only folding (usually cutting is not allowed) with the goal of constructing reference points that are used to define folds that produce the final object [14]. Reference points can be points on the square-shaped papers (e.g., the four corners) but also can be generated as intersections of lines formed by the edges of the paper or creases that align a combination of points, edges, and creases [14].

Although origami originally started as an activity in everyday life, origami has become a topic of research. According to Lang [14, page 42], "Starting in the 1970s, several folders began to systematically enumerate the possible combinations of folds to study what types of distances were constructible by combining them in various ways". In the 1980s, several researchers identified a fixed set of well-defined folds one can make in origami constructions (see Figure 4.1) and formalized the modern study of geometric constructions with origami.

In origami theory, starting with the fixed set of well-defined folds, researchers have investigated the objects possible or impossible to construct using origami. For example, it was shown that trisecting an angle and doubling a cube are possible with origami constructions; however, constructing the regular 11-gon or solving the general quintic equation was shown impossible [2].

There is an international conference, 'The International Meeting on Origami Science, Mathematics and Education,' at which the origami community have gathered since 1989 to discuss origami in science, mathematics, education, technology, and art [18]. With the development of the computer, computational systems of origami simulation have been developed and a more systematic study of origami has evolved.

Origami has also been applied to other fields such as science and technology. According to Lucero [15], the application of origami can be found in aerospace and automotive technology, materials science, computer science, biology, civil engineering, robotics, and acoustics. However, as folding techniques have been adapted in industry, more rigor and formalization of origami theory has been called for. For example, Kasem et al. [13] and Ghourabi et al. [7] called for more precise statements of the folding operations and introduced the extension of origami constructions with an additional tool of the compass.

1.3 Overview of Thesis Chapters

In Chapter 2, I will first review the basic definitions and theorems of field theory that will be used in subsequent chapters. In Chapter 3, I will discuss the more commonly known straightedge and compass constructions as a lead into the discussion on origami constructions. In Chapter 4, I will introduce the basic single-fold operations of origami and discuss what it means for an object to be origami-constructible. Through algebraizing geometric constructions with origami, I will show how to prove or disprove the constructibility of some objects. In Chapter 5, I end this thesis with closing remarks and some additional thoughts for future studies.

Chapter 2

Preliminaries in Field Theory

In this Chapter 2, I will review the basic definitions and theorems of field theory that will be used in subsequent chapters.

2.1 Basic Definitions

Definition 2.1.1 (Group)**.** *A **group** G is a set together with a binary operation (usually denoted by $\cdot$ and called the group operation) that satisfy the following:*

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*, for all* $a, b, c \in G$*. (Associativity)*

(ii) There exists an element $e \in G$ *(called the identity of* G*) such that* $a \cdot e = e \cdot a = a$ *for all* $a \in G$*. (Identity)*

(iii) For each $a \in G$*, there exists an element* $a^{-1} \in G$ *(called the inverse of* a*) such that* $a \cdot a^{-1} = a^{-1} \cdot a = e$*. (Inverse)*

The group G *is called **abelian** if* $a \cdot b = b \cdot a$ *for all* $a, b \in G$*.*

Definition 2.1.2 (Subgroup). *Given a group G, a subset H of G is a* ***subgroup*** *of G (denoted $H \leq G$) if H is nonempty and closed under multiplication and inverse, i.e., $a, b \in H \Rightarrow ab \in H$ and $a^{-1} \in H$.*

Whereas the theory of groups involves general properties of objects having an algebraic structure defined by a single binary operation, the theory of rings involves objects having an algebraic structure defined by two binary operations related by the distributive laws [5, p.222].

Definition 2.1.3 (Ring). *A* ***ring*** *R is a set together with two operations, $+, \times$ (addition and multiplication) that satisfy the following:*

(i) *R is an abelian group under addition (denote the additive identity 0 and additive inverse of element a as $-a$).*

(ii) $(a \times b) \times c = a \times (b \times c)$, *for all $a, b, c \in R$. (Multiplicative associativity)*

(iii) $(a + b) \times c = (a \times c) + (b \times c)$ *and* $a \times (b + c) = (a \times b) + (a \times c)$, *for all $a, b, c \in R$. (Left and right distributivity)*

(iv) *There exists an element $1 \in R$ (called the multiplicative identity) such that $1 \times a = a \times 1 = a$ for all $a \in R$ (multiplicative unit).*

Rings may also satisfy optional conditions such as:

(v) $a \times b = b \times a$ *for all $a, b \in R$.*

In this case the ring R is called ***commutative****.*

(vi) *For every* $a \in R \setminus \{0\}$*, there exists an element* $a^{-1} \in R$ *such that* $a \times a^{-1} = 1 = a^{-1} \times a$*.*

In this case the ring R is called a ***division ring****.*

Definition 2.1.4 (Subring)**.** *Given a ring R, a subset S of R is a* ***subring*** *of R if S is a subgroup of R closed under multiplication, and if* $1 \in S$*. In other words, S is a subring of R if the operations of addition and multiplication in R restricted to S give S the structure of a ring.*

A commutative division ring (a ring that satisfies all conditions (i) through (vi) in Definition 2.1.3) is called a field. More concisely:

Definition 2.1.5 (Field)**.** *A* ***field*** *is a set F together with two binary operations* $+$ *and* $\cdot$ *that satisfy the following:*

(i) *F is an abelian group under* $+$ *(with identity 0)*

(ii) $F^\times = F - \{0\}$ *is an abelian group under* $\cdot$ *(with identity 1)*

(iii) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$*, for all* $a, b, c \in F$*. (Distributivity)*

For example, any set closed under all the arithmetic operations $+, -, \times, \div$ (division by nonzero elements) is a field.

2.2 Field Theory

Of particular interest in this thesis is the idea of extending a given field into a (minimally) larger field so that the new field contains specific elements in addition to all the elements of the given field. We first define a field extension.

Definition 2.2.1. *If K is a field with subfield F, then K is an **extension field** of F, denoted K/F (read "K over F"). If K/F is a field extension, then K is a vector space over F with degree $dim_F K$, denoted $[K:F]$. The extension is finite if $[K:F]$ is finite and infinite otherwise.*

Definition 2.2.2. *If K is an extension of field F and if $\alpha, \beta, \cdots \in K$, then the smallest subfield of K containing F and $\alpha, \beta, \cdots \in K$ is called the field **generated by** $\alpha, \beta, \cdots$ **over F**, denoted $F(\alpha, \beta, \cdots)$. An extension K/F is **finitely generated** if there are finitely many elements $\alpha_1, \ldots, \alpha_k \in K$ such that $K = F(\alpha_1, \ldots, \alpha_k)$. If the field is generated by a single element α over F, then K is a **simple extension** of F with α a **primitive element** for the extension $K = F(\alpha)$.*

We are specifically interested in field extensions that contain roots of specific polynomials over a given field, with the new field extending the field. The following propositions of such field extensions are taken as given without proof[1].

Proposition 2.2.3. *Given any field F and irreducible polynomial $p(x) \in F[x]$, there exists an extension of F in which $p(x)$ has a root. Namely, the field $K = F[x]/(p(x))$ in which $F[x]/(p(x))$ is the quotient of the ring $F[x]$ by the maximal ideal $(p(x))$. Further, if $\deg(p(x)) = n$ and $\alpha \in K$ denotes the class of x modulo $p(x)$, then $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ is a basis for K as a vector space over F, with $[K:F] = n$, so*

$$K = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} | a_0, \ldots, a_{n-1} \in F\}.$$

[1]For a detailed proof of these statements, see Dummit & Foote [5, pp. 512-514].

While Proposition 2.2.3 states the existence of an extension field K of F that contains a solution to a specific equation, the next proposition indicates that any extension of F that contains a solution to that equation has a subfield isomorphic to K and that K is the smallest extension of F (up to isomorphism) that contains such a solution.

Proposition 2.2.4. *Given any field F and irreducible polynomial $p(x) \in F[x]$, suppose L is an extension field of F containing a root α of $p(x)$ and let $F(\alpha)$ denote the subfield of L generated over F by α. Then $F(\alpha) \cong F[x]/(p(x))$. If $\deg(p(x)) = n$, then $F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} | a_0, a_1, \ldots, a_{n-1} \in F\} \subseteq L$.*

Example 2.2.5. Consider $F = \mathbb{R}$ and $p(x) = x^2 + 1$, an irreducible polynomial over $\mathbb{R}$. Then we obtain the field $K = \mathbb{R}/(x^2 + 1) \cong \mathbb{R}(i) \cong \mathbb{R}(-i)$, an extension of $\mathbb{R}$ that contains the solution of $x^2 + 1 = 0$. $[K : F] = 2$ with $K = \{a + bi | a, b \in \mathbb{R}\} = \mathbb{C}$.

The elements of a field extension K of F need not always be a root of a polynomial over F.

Definition 2.2.6. *Let K be an extension of a field F. The element $\alpha \in K$ is **algebraic over F** if α is a root of some nonzero polynomial $f(x) \in F[x]$. If $\alpha \in K$ is not the root of any nonzero polynomial over F, then α is **transcendental** over F. The extension K/F is **algebraic** if every element of K is algebraic over F.*

Proposition 2.2.7. *Let K be an extension of a field F and α algebraic over F. Then there is a unique monic irreducible polynomial $m_\alpha(x) \in F[x]$ with α as a root. This polynomial $m_\alpha(x)$ is called the **minimal polynomial** for α over F and $\deg(m_\alpha)$ is the **degree of** α. So $F(\alpha) \cong F[x]/(m_\alpha(x))$ with $[F(\alpha) : F] = \deg(m_\alpha(x)) = \deg \alpha$.*

Proposition 2.2.8. *If α is algebraic over F, i.e., if α is a root to some polynomial of degree n over F, then $[F(\alpha) : F] \leq n$. On the other hand, if $[F(\alpha) : F] = n$, then α is a root of a polynomial of degree at most n over F. It follows that any finite extension K/F is algebraic.*

Example 2.2.9 (Quadratic Extensions over F with $char(F) \neq 2$)**.** Let K an extension of a field F with the $char(F) \neq 2$[2] and $[K : F] = 2$. Let $\alpha \in K$ but $\alpha \notin F$. Then α must be algebraic, i.e., a root of a nonzero polynomial over F of degree at most 2, by Proposition 2.2.8. Since $\alpha \notin F$, this polynomial cannot be of degree 1. Hence, by Proposition 2.2.7, the minimal polynomial of α is a monic quadratic $m_\alpha(x) = x^2 + bx + c$ with $b, c \in \mathbb{Q}$. $F \subset F(\alpha) \subseteq K$ and $[K : F] = 2$, so $K = F(\alpha)$ (see Proposition 2.2.7 and 2.2.8). From the quadratic formula (possible since $char(F) \neq 2$), the elements

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2} \in F(\sqrt{d})$$

are roots of $m_\alpha(x)$ with $d = b^2 - 4c$. We can, by Proposition 2.2.7, identify $F(\alpha)$ with a subfield of $F(\sqrt{d})$ by identifying α with $\frac{1}{2}(-b \pm \sqrt{d})$ (either sign choice works). We know that d is not a square in F, since $\alpha \notin F$. By construction $F(\alpha) \subset F(\sqrt{d})$. On the other hand, $\sqrt{d} = \mp(b + 2\alpha) \in F(\alpha)$, so $F(\sqrt{d}) \subseteq F(\alpha)$. Therefore, $F(\alpha) = F(\sqrt{d})$. As such, any extension K of F of degree 2 is of the form $F(\sqrt{d})$, where $d \in F$ is not a square in F. Conversely, any extension of the form $F(\sqrt{d})$, where

[2] The characteristic of a field F is the smallest integer n such that $n \cdot 1_F = 0$. If such n does not exist, then the characteristic is defined to be 0.

$d \in F$ is not a square in F, is an extension of degree 2 over F. Such an extension is called a *quadratic extension* over F.

Particularly, when $F = \mathbb{Q}$, any extension K of $\mathbb{Q}$ of degree 2 is of the form $\mathbb{Q}(\sqrt{d})$, where $0 < d \in \mathbb{Q}$ is not a square in $\mathbb{Q}$. Conversely, any extension of the form $\mathbb{Q}(\sqrt{d})$, where $0 < d \in \mathbb{Q}$ is not a square in $\mathbb{Q}$, is an extension of degree 2 over $\mathbb{Q}$. Such an extension is called a *quadratic extension* over $\mathbb{Q}$.

Theorem 2.2.10 (Tower Rule)**.** *Let F a field extension of E and K a field extension of F. Then K is a field extension of E of degree $[K : E] = [K : F][F : E]$.*

Proof. Suppose $[F : E] = m$ and $[K : F] = n$ are finite. Let $\alpha_1, \ldots, \alpha_m$ a basis for F over E and $\beta_1, \ldots, \beta_n$ a basis for K over F. Let β any element of K. Since $\beta_1, \ldots, \beta_n$ are a basis for K over F, there are elements $b_1, \ldots, b_n \in F$ such that

$$\beta = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n \tag{2.1}$$

Since $\alpha_1, \ldots, \alpha_m$ are a basis for F over E, there are elements $a_{i1}, \ldots, a_{in} \in E, i = 1, 2, \ldots, m$ such that each

$$b_i = a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n. \tag{2.2}$$

Substituting (2.2) into (2.2), we obtain

$$\beta = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}(\alpha_i\beta_j), a_{ij} \in F \tag{2.3}$$

In other words, any element of K can be written as a linear combination of the nm elements $\alpha_i\beta_j$ with coefficients in F. Hence, the vectors $\alpha_i\beta_j$ span K as a vector space over E.

Suppose $\beta = 0$ in (2.3).

$$\beta = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}(\alpha_i\beta_j) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\alpha_i\right)\beta_j.$$

Since $\beta_1, \ldots, \beta_n$ are linearly independent, $\sum_{i=1}^{m} a_{ij}\alpha_i = 0$ for all $j = 1, \ldots, n$. Since $\alpha_1, \ldots, \alpha_m$ are linearly independent, $a_{ij} = 0$ for all $i = 1, 2, \ldots, m$ and $j = 1, \ldots, n$. Therefore, the nm elements $\alpha_i\beta_j$ are linearly independent over E, and form a basis for K over E. Therefore, $[K : E] = nm = [K : F][F : E]$. If either $[K : F]$ or $[F : E]$ is infinite, then there are infinitely many elements of K or F, respectively, so $[K : E]$ is also infinite. On the other hand, if $[K : E]$ is infinite, then at least one of $[K : F]$ and $[F : E]$ has to be infinite since if they are both finite, the above proof shows that $[K : E]$ is finite. $\square$

Chapter 3

Straightedge and Compass Constructions

To investigate the constructability of geometric objects using straightedge and compass, such as the problems posed by the Greeks, we translate geometric constructions into algebraic terms. In this chapter we explore geometric constructions with straightedge and compass and constructible numbers.

3.1 Initial Givens, Operations, and Basic Constructions

In straightedge and compass constructions, two points, say O and P, are initially given. Starting with these given points, we can carry out the following operations with straightedge and compass, which produce lines and circles.

Operations of straightedge and compass constructions

Given two constructed points, we can

C1. construct the line connecting the two points, or

C2. construct a circle centered at one point and passing through the other point.

We can also construct the intersection of constructed lines or circles which gives new points:

P1. We can construct the point of intersection of two distinct lines.

P2. We can construct points of intersection of a line and a circle.

P3. We can construct points of intersection of two distinct circles.

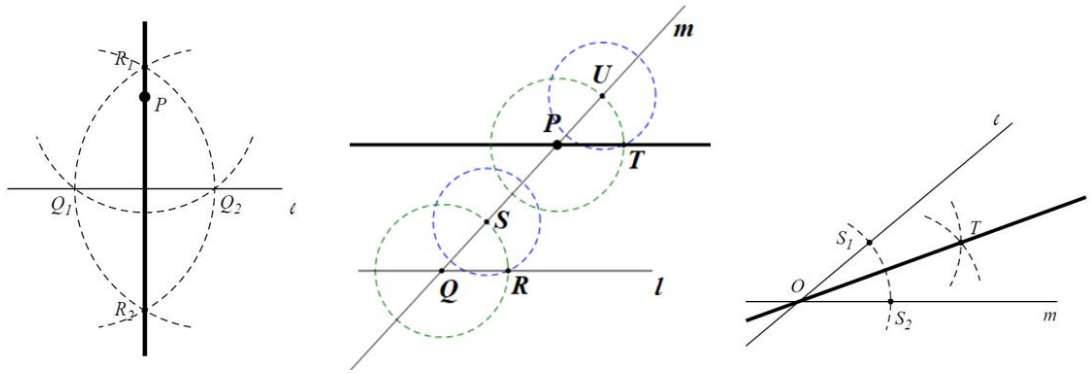
Repeating C1, C2 with P1, P2, P3 on $\{O, P\}$ produces more constructible points. A *constructible point* is any point one can construct in a finite number of steps of combinations of C1, C2, P1, P2, or P3.

Examples of basic straightedge and compass constructions

Recall the following basic constructions in Figure 3.1 from high school geometry. For example, to construct the perpendicular line through point P (see E1 in Figure 3.1), construct a circle of some radius centered at point P and let Q_1 and Q_2 be the two intersection points of the circle with ℓ. Next, construct circles centered at Q_1 and Q_2 respectively, whose radius is the distance between Q_1 and Q_2[1]. These two circles will intersect in two new points R_1 and R_2. The line through R_1 and R_2 passes through P and is perpendicular to ℓ.

Given points P, Q and line ℓ, with $P \notin \ell$ and $Q \in l$, to construct a line parallel to ℓ through P (see E2 in Figure 3.1), construct a circle of some radius centered at

[1]The radius can be any length greater than half the distance between Q_1 and Q_2 to allow the two circles to intersect.



E1. Given point P and line *l*, we can construct a line perpendicular to *l* through P.

E2. Given points P, Q and line *l*, with P$\notin$ *l* and Q$\in$ *l*, we can construct a line parallel to *l* through P.

E3. Given two non-parallel lines *l* and *m*, we can construct the bisector of the angle between them.

Figure 3.1: Three basic straightedge and compass constructions.

Q on ℓ and construct a circle centered at point P with same radius. Construct line m through P and Q and let the intersection of the circle centered at Q with m S and the intersection of the circle centered at P with m U. Let the intersection of the circle centered at Q with ℓ R and construct a circle centered at S through R. Then construct another circle with same radius centered at U. Let the intersection of the circle centered at P and the circle centered at U T. The line through P and T is parallel to ℓ through P.

To construct the angle bisector (see E3 in Figure 3.1), construct a circle of some radius centered at the point O of intersection of the two lines. Let S_1 and S_2 be the two intersection points of the circle with the two sides ℓ and m, respectively. Next, construct two congruent circles of some radius centered at S_1 and S_2 respectively.

Let point T one of the two intersections of the two circles. The line through O and T bisects the angle between lines ℓ and m. These examples will be useful later.

3.2 Constructible Numbers

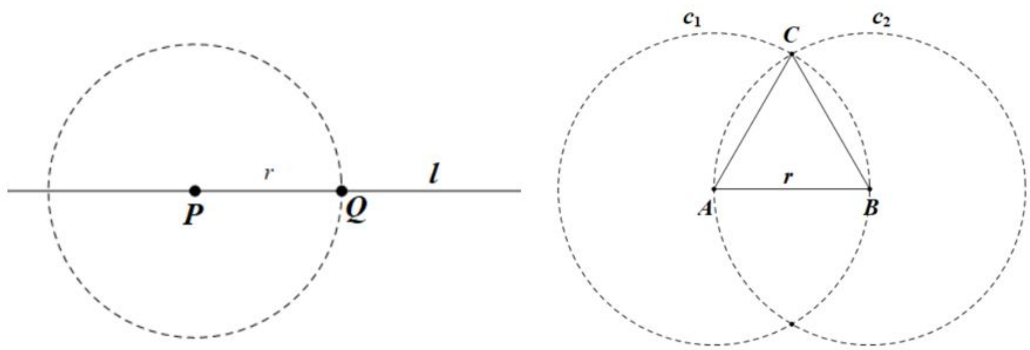
Now we translate geometric distances obtained through constructions into algebraic terms by associating lengths with elements of the real numbers. Given a fixed unit distance 1, we determine any distance by its length $1r = r1 \in \mathbb{R}$.

Definition 3.2.1. ***A real number*** r ***is constructible*** *if one can construct in a finite number of steps two points which are a distance of* $|r|$ *apart.*

The set of real numbers that are associated with lengths in $\mathbb{R}$ obtained by straightedge and compass constructions together with their negatives are constructible elements of $\mathbb{R}$. Henceforth, we do not distinguish between constructible lengths and constructible real numbers.

Given a constructible number r, we can construct various objects using straightedge and compass. Figure 3.2 shows two examples. For example, given a constructible number r, we can construct an equilateral triangle with side length r as follows. First, let A and B two points a distance of r apart. Next, construct circle c_1 centered at A with radius r and circle c_2 centered at B with radius r. Let C the intersection of circles c_1, c_2. Then, $\triangle ABC$ is an equilateral triangle.

Given the plane in which straightedge and compass constructions are made, we establish a coordinate system by taking point O as the origin and the distance between O and P as the unit length 1. Applying C1 to O and P, we can construct the



E4. Given point P and line l with P$\in l$, and constructible number r, we can construct a point Q on l such that $\overline{PQ} = r$

E5. Given a constructible number r, we can construct an equilateral triangle with side length r.

Figure 3.2: Two straightedge and compass constructions starting with given length r.

x-axis. Applying E1 to the x-axis and point O, we can construct the y-axis. Then the coordinates of the two given points O and P are $(0, 0)$ and $(1, 0)$, respectively, on the Cartesian plane. As such, using our points O and P, we can construct a coordinate system in the plane and use this to represent our points as ordered pairs, i.e., elements of $\mathbb{R}^2$[2]. We then have:

Lemma 3.2.2. ***A point*** $(a, b) \in \mathbb{R}^2$ ***is constructible*** *if and only if its coordinates* a *and* b *are constructible elements of* $\mathbb{R}$.

Proof. We can construct distances along lines (see E4 in Figure 3.2) that are perpendicular, and we can make perpendicular projections to lines (apply E1). Then, the point (a, b) can be constructed as the intersection of such perpendicular lines. On

[2]The plane can be taken as the field of complex numbers $\mathbb{C}$, with the x-axis representing the real numbers and the y-axis representing the imaginary numbers.

the other hand, if (a, b) is constructible, then we can project the constructed point to the x-axis and y-axis (apply E1) and thus the coordinates a and b are constructible as the intersection of two constructible lines. □

Lemma 3.2.3. *Every $n \in \mathbb{Z}$ is constructible.*

Proof. Construct the unit circle centered at P. Then by P2, the intersection of that unit circle and the x-axis, 2, is constructible. Next, construct the unit circle centered at 2. Then by P2, the intersection of that unit circle and the x-axis, 3, is constructible. Iterating this process of constructing the intersection of the unit circle and the x-axis (see Figure 3.3) shows that every $n \in \mathbb{Z}$ is constructible.

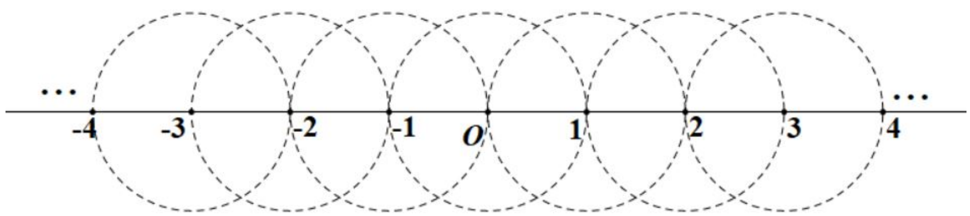


Figure 3.3: The construction of $n \in \mathbb{Z}$.

□

Lemma 3.2.4. *If two lengths a and b are given, one can construct the lengths $a \pm b$, $ab, \frac{a}{b}$, and $\sqrt{a}$.*

Proof. Use E4 in Figure 3.2 to construct $a \pm b$.

Given lengths a and b, construct parallel lines (use E2 in Figure 3.1) as shown in Figure 3.4 (a) and (b) to construct lengths ab and $\frac{a}{b}$, respectively, using similar triangles.

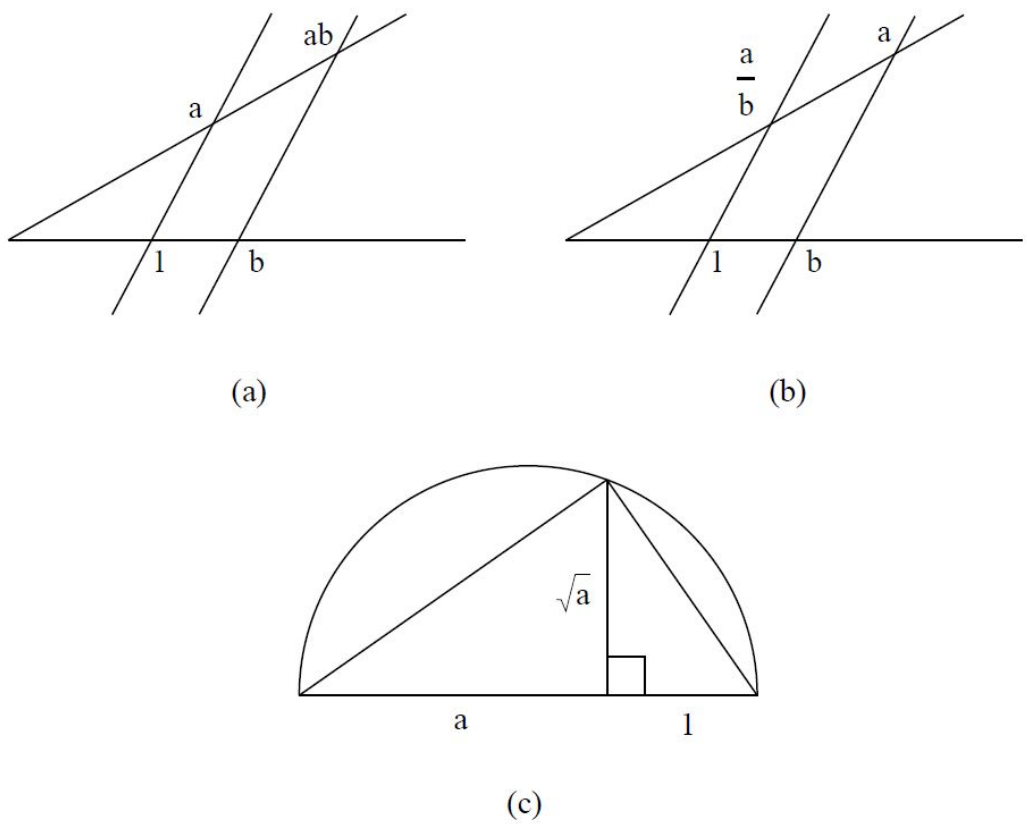


Figure 3.4: Constructing lengths ab, $\frac{a}{b}$, and $\sqrt{a}$.

Construct a semicircle with diameter of length $a + 1$ and construct the perpendicular segment to the diameter (use E1 in Figure 3.1) as shown in Figure 3.4 (c). Then, from similar triangles, the length of the perpendicular segment is $\sqrt{a}$. $\square$

Lemma 3.4 implies that straightedge and compass constructions are closed under addition, subtraction, multiplication, division, and taking square roots. The following two corollaries are immediate results.

Corollary 3.2.5. *The set of constructible numbers is a a subfield of* $\mathbb{R}$*.*

Proof. From Lemma 3.2.4, straightedge and compass constructions are closed under addition, subtraction, multiplication and division (by nonzero elements) in $\mathbb{R}$, so the set of constructible numbers is a subfield of $\mathbb{R}$. □

Corollary 3.2.6. *Every rational number is constructible.*

Proof. From Lemma 3.2.3 every integer is constructible. From Lemma 3.2.4 every quotient of a pair of integers is constructible. So, all rationals are constructible. □

From Corollary 3.2.6, we can construct all points $(a, b) \in \mathbb{R}^2$ whose coordinates are rational. We can also construct additional real numbers by taking square roots (from Lemma 3.2.4), so the the set of constructible numbers form a field strictly larger than $\mathbb{Q}$. Let $\mathscr{C}$ denote the set of constructible numbers with straightedge and compass. So far, we proved $\mathbb{Q} \subset \mathscr{C} \subseteq \mathbb{R}$. In order to determine precisely what $\mathscr{C}$ consists of, we create algebraic equations for points, lines, and circles.

First, we examine the equations of constructible lines and circles.

Lemma 3.2.7. *Let* F *be an arbitrary subfield of* $\mathbb{R}$*.*

(1) A line that contains two points whose coordinates are in F *has an equation of the form* $ax + by + c = 0$*, where* a*,* b*,* $c \in F$*.*

(2) A circle with center whose coordinates are in F and radius whose square is in F has an equation of the form $x^2 + y^2 + rx + sy + t = 0$, where $r,\ s,\ t \in F$.

Proof. (1) Suppose (x_1, y_1), (x_2, y_2) are two points on the line such that $x_1, x_2, y_1, y_2 \in F$. If $x_1 = x_2$ then the equation of the line is $x - x_1 = 0$. If $x_1 \neq x_2$, then the equation of the line is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1).$$

Rearranging both sides, we obtain

$$(\frac{y_1 - y_2}{x_1 - x_2})x - y - (\frac{y_1 - y_2}{x_1 - x_2})x_1 + y_1 = 0.$$

Since F is a field and $x_1, x_2, y_1, y_2 \in F$, each coefficient is an element of F.

(2) Suppose (x_1, y_1) is the center and k the radius such that $x_1, x_2, k \in F$. Then the equation of the circle is

$$(x - x_1)^2 + (y - y_1)^2 = k^2.$$

Rearranging the equation, we obtain

$$x^2 + y^2 - 2x_1 x - 2y_1 y + {x_1}^2 + {y_1}^2 + k^2 = 0.$$

Since F is a field and $x_1, x_2, k \in F$, each coefficient is an element of F.

$\square$

Recall that from P1, P2, P3, we can determine constructible points as intersections of two lines, a line and a circle, or of two circles.

Lemma 3.2.8. *Let F be an arbitrary subfield of $\mathbb{R}$. Let l_1, l_2 two constructible lines and c_1, c_2 two constructible circles. Then,*

(1) if l_1, l_2 intersect, the coordinates of the point of intersection are elements of F;

(2) if l_1, c_1 intersect, the coordinates of the points of intersection are elements of F or some quadratic extension field $F(\sqrt{d})$;

(3) and if c_1, c_2 intersect, the coordinates of the points of intersection are elements of F or some quadratic extension field $F(\sqrt{d})$.

Proof. (1) Let l_1, l_2 each have equations

$$l_1 : a_1x + b_1y + c_1 = 0$$

$$l_2 : a_2x + b_2y + c_2 = 0$$

where $a_i, b_i, c_i \in F$. Solving these linear equations simultaneously gives solutions also in F. Therefore, the coordinates of the point of intersection of l_1, l_2 are elements of F.

(2) Let l_1, c_1 each have equations

$$l_1 : a_1x + b_1y + c_1 = 0$$

$$c_1 : x^2 + y^2 + r_1x + s_1y + t_1 = 0$$

where $a_1, b_1, c_1, r_1, s_1, t_1 \in F$. Solving these equations simultaneously gives a quadratic equation with coefficients all in F. Hence the solutions will lie in a field of the form $F(\sqrt{d})$, where $d \in F$ (see Example 2.2.9). Therefore, the coordinates of the points of intersection of l_1, c_1 are elements of $F(\sqrt{d})$.

(3) Let c_1, c_2 each have equations

$$c_1 : x^2 + y^2 + r_1 x + s_1 y + t_1 = 0$$

$$c_2 : x^2 + y^2 + r_2 x + s_2 y + t_2 = 0$$

where $r_i, s_i, t_i \in F$. Subtracting the equation of c_2 from c_1 gives us a linear equation

$$(r_1 - r_2)x + (s_1 - s_2)y + (t_1 - t_2) = 0$$

Solving this linear equation simultaneously with the equation of c_1 gives a quadratic equation with coefficients all in F. Hence the solutions will lie in a field of the form $F(\sqrt{d})$, where $d \in F$. Therefore, the coordinates of the points of intersection of c_1, c_2 are elements of $F(\sqrt{d})$.

□

Theorem 3.2.9. *Let $r \in \mathbb{R}$. Then $r \in \mathscr{C}$ if and only if there is a finite sequence of fields $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ and for each $i = 0, 1, \ldots, n$, $[F_i : F_{i-1}] = 2$.*

Proof. ($\Rightarrow$): Suppose $r \in \mathscr{C}$. As discussed in section 3.1, $r \in \mathscr{C}$ only when it could be constructed through operations C1, C2, C3 with straightedge and compass and their intersections P1, P2, P3. From Lemma 3.2.8, such straightedge and compass constructions in F involve a field extension of degree one or two. From induction on the number of constructions required to construct r, there must be a finite sequence of fields $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ with $[F_i : F_{i-1}] = 2$.

($\Leftarrow$): Suppose $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $[F_i : F_{i-1}] = 2$. Then $F_i = F_{i-1}[\sqrt{d_i}]$ for some $d_i \in F_i$ (see Example 2.2.9). If $n = 0$, then $r \in F_0 = \mathbb{Q}$ and $\mathbb{Q} \subset \mathscr{C}$, from Corollary 3.2.6. Suppose any element of F_n is constructible for when $n = k - 1$. Then any $d_k \in F_{k-1}$ is constructible, which implies that $\sqrt{d_k}$ is also constructible, from Lemma 3.2.4. Therefore, any element of $F_k = F_{k-1}[\sqrt{d_k}]$ is also constructible. From mathematical induction, any $r \in F_n$ is constructible. □

Since straightedge and compass constructions involve a field extension of $\mathbb{Q}$ with degree at most 2, the operations can produce elements of at most a quadratic extension of $\mathbb{Q}$. So, $\mathscr{C}$ is the smallest field extension of $\mathbb{Q}$ that is closed under taking square roots.

Corollary 3.2.10. *If $r \in \mathscr{C}$ then $[\mathbb{Q}(r) : \mathbb{Q}] = 2^k$ for some integer $k \geq 0$.*

Proof. From Theorem 3.2.9, if $r \in \mathscr{C}$, there is a finite sequence of fields $\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ and for each $i = 0, 1, \ldots, n$, $[F_i : F_{i-1}] = 2$. So, by the Tower Rule (Theorem 2.2.10),

$$[F_n : \mathbb{Q}] = [F_n : F_{n-1}] \cdots [F_1 : F_0] = 2^n.$$

Since $\mathbb{Q} \subset \mathbb{Q}(r) \subset F_n$, again by the Tower Rule, $[\mathbb{Q}(r) : \mathbb{Q}] | [F_n : \mathbb{Q}] = 2^n$, so if $r \in \mathscr{C}$ then $[\mathbb{Q}(r) : \mathbb{Q}] = 2^k$ for some integer $k \geq 0$. □

It follows that every constructible number is algebraic over $\mathbb{Q}$ and the degree of its minimal polynomial over $\mathbb{Q}$ is a power of 2.

3.3 Impossibilities

Now we can revisit the three classic Greek geometry problems.

Theorem 3.3.1. *[Doubling the cube] It is impossible to construct an edge of a cube with volume 2, using straightedge and compass.*

Proof. An edge of such a cube would have length $\sqrt[3]{2}$, a root of $p(x) = x^3 - 2$. By the rational root test, $p(x)$ is irreducible, so it is the minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$. From Proposition 2.2.7, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, which is not a power of 2. So, by Corollary 3.2.10, $\sqrt[3]{2}$ is not constructible. $\square$

Before we prove the impossibility of trisecting an angle, we first define a constructible angle and prove the following Lemma.

Definition 3.3.2. *An angle θ is constructible if one can construct two lines ℓ and m with angle θ between them.*

Lemma 3.3.3. *An angle θ is constructible if and only if $\cos\theta$ and $\sin\theta$ are both constructible numbers.*

Proof. ($\Rightarrow$): Using E1, we can construct a line ℓ perpendicular to the x-axis through point P on angle θ at distance 1 from the origin. Hence, the intersection of ℓ and the x-axis, $\cos\theta$ is constructible. Similarly, using E1, we can construct $\sin\theta$.

($\Leftarrow$): From Definition 3.2.1, since $\cos\theta$ and $\sin\theta$ are both constructible, the point $(\cos\theta, \sin\theta)$ is constructible. Hence, angle θ is constructible. $\square$

Theorem 3.3.4. *[Trisecting an angle] Not all angles can be trisected using only straightedge and compass.*

Proof. We show that it is impossible to trisect a $60°$ angle. Note that $60°$ is constructible as the angle between two sides of an equilateral triangle (see construction E5 in Figure 3.2). From Lemma 3.3.3, it suffices to show that $\cos 20°$ is not constructible.

From the triple angle formulas of trigonometry, we know:

$$\cos 3\alpha = \cos^3\alpha - 3\cos\alpha\sin^2\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

Substituting $\alpha = 20°$, $\frac{1}{2} = 4\cos^3 20° - 3\cos 20°$. Thus, $\cos 20°$ is a root of the polynomial $p(x) = 8x^3 - 6x - 1$. But $p(x)$ is irreducible in $\mathbb{Q}$, by the rational root test. So by Proposition 2.2.7, $[\mathbb{Q}(\cos 20°) : \mathbb{Q}] = 3$, which is not a power of 2. Therefore, $\cos 20°$ is not constructible, by Corollary 3.2.10. $\square$

Proposition 3.3.5. *[Squaring the circle] It is impossible to construct an edge of a square with same area as the unit circle, using straightedge and compass.*

Because the proof is beyond the scope of this thesis, the last Greek problem is presented as a proposition. Proposition 3.3.5 is based on the fact that $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is not finite and hence π is not constructible, shown by Lindemann in 1882 [4].

3.4 Marked Rulers and Verging

One can add to the straightedge and compass axioms by allowing a new operation involving a straightedge marked with a distance, i.e., a marked ruler. A marked ruler is a straightedge with two marks on its edge that can be used to mark off unit distances along a line. The new operation is called *verging* [16] and with it one can make geometric constructions that were not possible with the standard straightedge and compass as previously described. Below we explore the verging operation and one example of a marked ruler and compass construction which was not possible with the standard straightedge and compass.

Given point P and two lines ℓ and m, by verging through P with respect to ℓ and m, we can construct a line through P and two points Q and R that are one unit apart with $Q \in \ell$ and $R \in m$ (Figure 3.5).

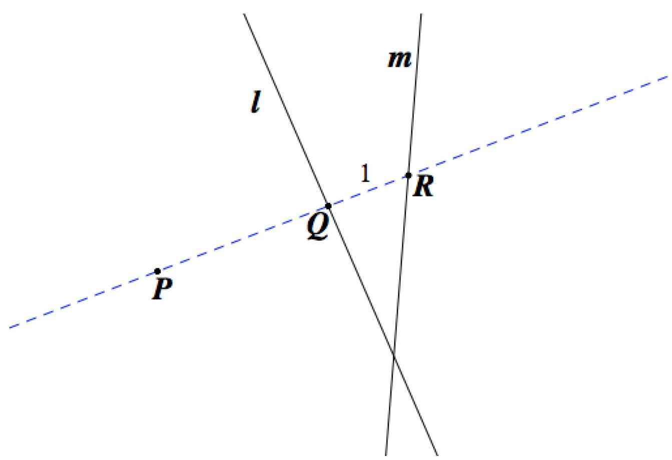


Figure 3.5: Verging through P with respect to ℓ and m.

Using this verging characteristic of the marked ruler, we can trisect a given angle θ as shown in Figure 3.6 and known as Archimedes' construction [5]. To elaborate, start with a circle of radius 1 with central angle θ and line ℓ through the center of the circle. Then, use verging of the marked ruler to construct a line m through Q, the intersection of the terminal ray of the angle and the circle, so that the distance between A, the intersection of ℓ and m, and B, the intersection of m and circle O, is a unit distance apart. In the following we prove that the angle $\alpha = \frac{1}{3}\theta$.

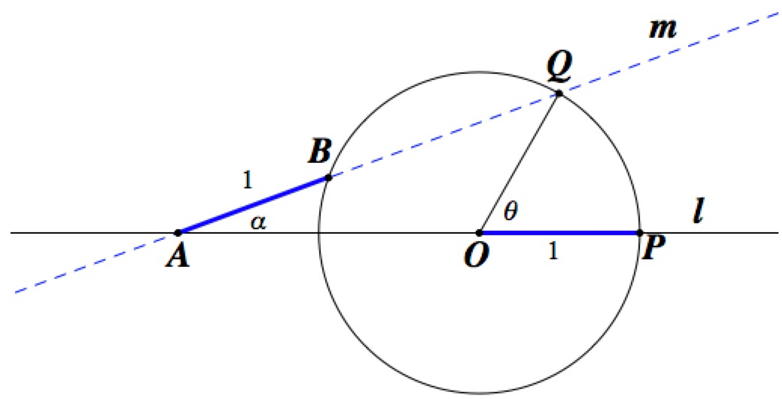


Figure 3.6: Trisecting an angle with marked ruler and compass.

Claim. $\alpha = \frac{1}{3}\theta$ in Figure 3.6.

Proof. Let $\angle OBQ = \beta$. See Figure 3.7.

Since $\overline{OB} = \overline{OQ}$, $\triangle OBQ$ is an isosceles triangle and thus $\angle OBQ = \angle OQB = \beta$. Similarly, since $\overline{BA} = \overline{BO}$, $\triangle BAO$ is an isosceles triangle and thus $\angle BAO = \angle BOA = \alpha$. Since $\angle QBO$ is the exterior angle of $\triangle BAO$, $\beta = 2\alpha$. Similarly, since $\angle QOP$ is the exterior angle of $\triangle AOQ$, $\theta = \alpha + \beta$. Therefore, $\theta = 3\alpha$, i.e., $\alpha = \frac{1}{3}\theta$. $\square$

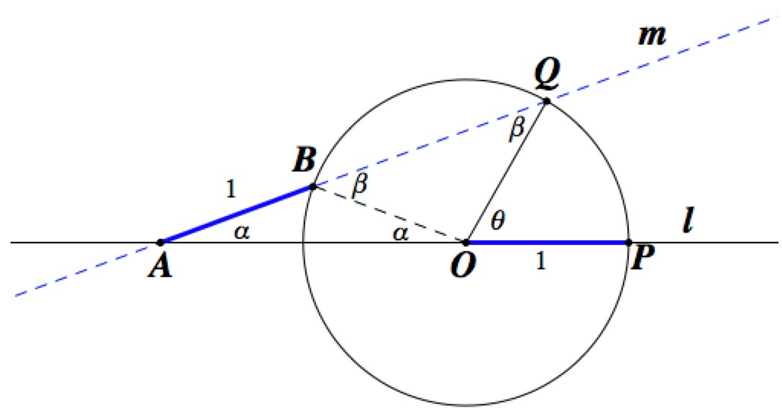


Figure 3.7: Proving angle trisection with marked ruler and compass.

Chapter 4

Origami Constructions

In this chapter we investigate the constructibility of geometric objects using origami. Similar to the discussion in Chapter 3, we translate geometric constructions into algebraic terms and explore origami constructions and origami-constructible numbers.

4.1 Initial Givens and Single-fold Operations

In origami constructions, we consider the plane on which origami occurs as a sheet of paper infinitely large, on which two points, say O and P, are initially given. Origami constructions consist of a sequence of single-fold operations that align combinations of points and lines in the plane. A single-fold refers to folding the paper once; a new fold can only be made after the paper is unfolded. Each single-fold leaves a crease which acts as a origami-constructed line. Intersections among origami-constructed lines define origami-constructed points.

The basic single-fold operations of origami are as demonstrated in Figure 4.1[1]. Carrying out O1-O4 and O7 is straightforward. O5 and O6 can be carried out using a similar process of verging, as described in section 3.4. To elaborate, one can carry out O5 by folding P_1 onto P_1' on ℓ and then sliding P_1' along ℓ until the fold line passes through P_2. Similarly, one can carry out O6 by folding P_1 onto P_1' on ℓ_1 and then sliding P_2 until it lies on ℓ_2 [1].

Starting with O and P and using operations O1-O7, we can create new lines, and the intersections of the old and new lines produce additional points. Repeating operations O1-O7 on the expanded set of points and lines produces more points and lines. Not all points on an origami-constructed line are necessarily origami-constructible points [1]. An *origami-constructible point* is any point we can construct as an intersection of two origami-constructible lines constructed in a finite number of steps of O1-O7.

The seven single-fold operations are often referred to as the "Axioms of origami" and were formulated in the late 1980s. According to Lang [14], Humiaki Huzita [10, 11] identified six basic types of single-fold operations of origami (O1-O6) in 1989 and Hatori [8] identified a seventh operation (O7) in 2003. However, it was later discovered that Jacques Justin [12] (in 1986 according to Lucero [15]; and in 1989 according to Lang [14]) has already identified all 7 axioms and that the axioms were later rediscovered by Huzita or Hatori. The seventh operation did not expand the set of origami constructible objects; however, it was not equivalent to any of the existing six operations [2]. Commonly, the 7 axioms are referred to as the Huzita-

[1] It is worth noting that not all operations involve folds that are always possible. The operations are limited to when such folds exist.

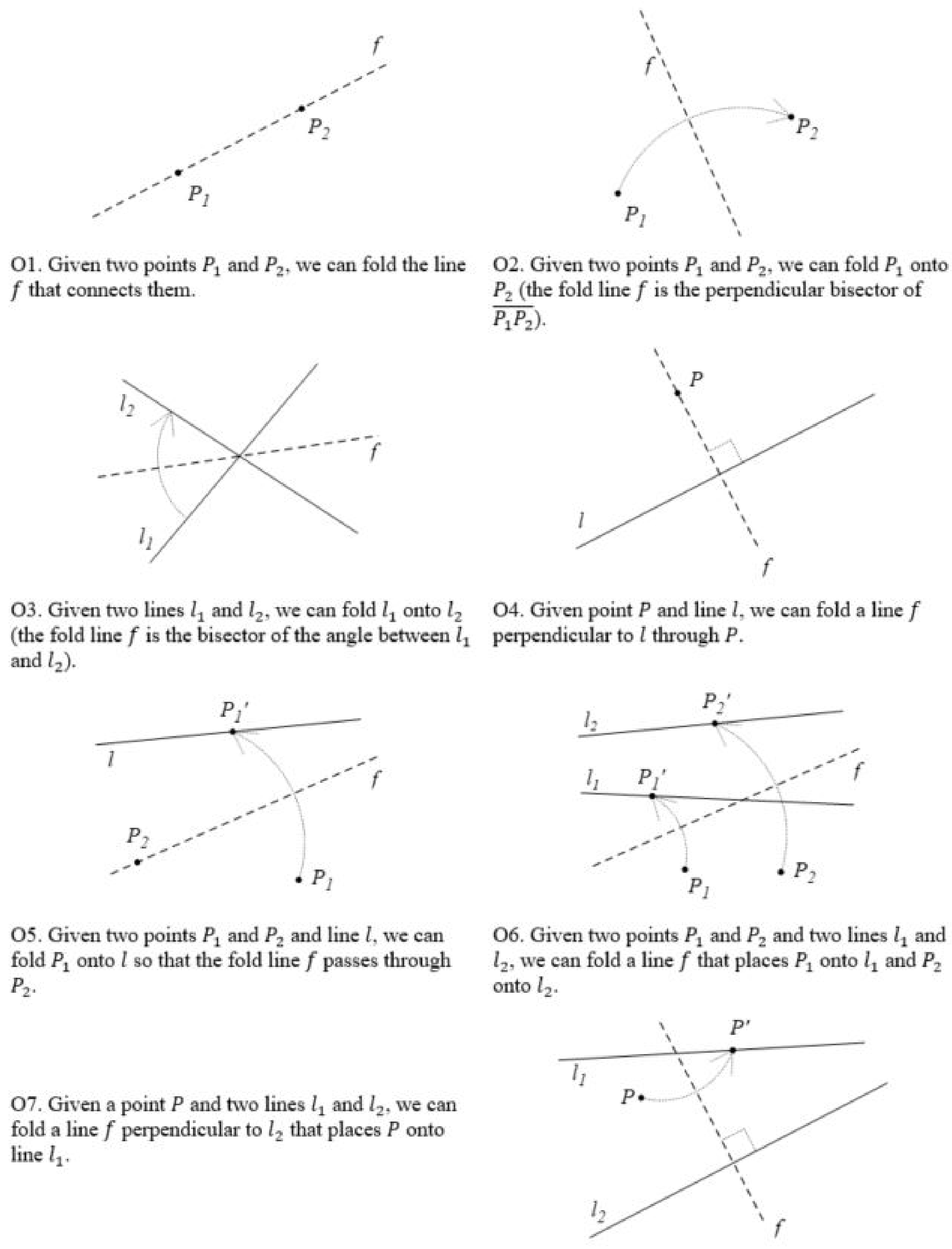


Figure 4.1: Single-fold operations of origami.

Hatori Axioms or the Huzita-Justin Axioms. These axioms are not independent: it has been shown that O1-O5 can be done using only O6 ([2], [7]).

Using exhaustive enumeration on all possible alignments of lines and points, Alperin and Lang [2] have shown that the 7 axioms are complete, meaning that the 7 operations include all possible combinations of alignments of lines and points in Origami. In other words, there are no other single-fold axioms other than the seven. However, Alperin and Lang [2] restricted the cases to those with a finite number of solutions in a finite-sized paper to exclude alignments that require infinite paper to verify. They also excluded redundant alignments that do not produce new lines or points.

More recently, there have been critiques of these axioms. For example, Kasem et al. [13] showed the impossibility of some folds, discussed the cases where infinitely many fold lines occur, and pointed out superfluous conditions in the axioms. Ghourabi et al. [7, page 146] conducted a systematic algebraic analysis of the 7 axioms and claimed, "[w]hile these statements [the 7 axioms] are suitable for practicing origami by hand, a machine needs stricter guidance. An algorithmic approach to folding requires formal definition of fold operations."

In this thesis, I assume the traditional set of 7 single-fold operations as listed in Figure 4.1 and explore the origami constructions and origami numbers derived from them. In the next chapter, I will discuss some of the more recent studies and extensions of the origami axioms.

4.2 Origami Constructions

4.2.1 Origami constructions in relation to straightedge and compass constructions

In their 1995 paper, Auckly and Cleveland [3] showed that the set of origami constructions using only operations O1-O4 is less powerful than constructions with straightedge and compass. In his 1998 book on geometric constructions, Martin [16] showed that a paper folding operation, which he termed the "fundamental folding operation," accounts for all possible cases of incidences between two given distinct points and two given lines. This fundamental folding operation is equivalent to O6 (restricted to $p_1 \neq p_2$) [2]. Further, Martin proved that this operation is sufficient for the construction of all objects constructible by O1-O6 altogether including straightedge and compass constructions. In his publication in 2000, Alperin [1] showed that operations O2, O3, and O5 together are equivalent to axioms O1-O5 and that the set of constructible numbers obtained by these sets of operations is exactly the set of constructible numbers obtained by straightedge and compass.

4.2.2 Some examples of basic origami constructions

Recall the Examples of Basic Constructions (3.1) with straightedge and compass in Chapter 3. These constructions can be made with origami as well. For example, E1 and E3 are equivalent to O4 and O3, respectively. E2 (constructing a line parallel to a given line ℓ through point P) can be done through origami as the following. Apply O4 to construct a line f_1 perpendicular to ℓ through P. Then apply O4 again to construct a line f_2 perpendicular to f_1 through P. Then f_2 is parallel to ℓ through P (see Figure 4.2). We will refer to this construction as E2′.

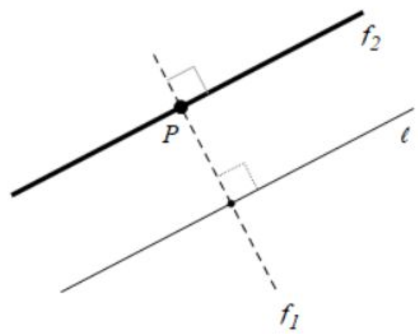


Figure 4.2: E2′: Constructing a line parallel to ℓ through P with origami.

There are constructions possible with origami but not possible with straightedge and compass, such as doubling the cube or trisecting an angle. We will explore these constructions in section 4.4.

4.3 Origami-constructible Numbers

To discuss origami-constructible numbers, we use a similar translation of geometric constructions into algebraic terms as we did in Chapter 3. That is, given a fixed unit distance 1, we determine any distance by its length $1r = r1 \in \mathbb{R}$. Using this, we can translate geometric distances into elements of the real numbers $r \in \mathbb{R}$.

Definition 4.3.1. ***A real number*** r ***is origami-constructible*** *if one can construct in a finite number of steps two points which are a distance of* $|r|$ *apart.*

The set of real numbers associated with lengths in $\mathbb{R}$ obtained by origami constructions together with their negatives are origami-constructible elements of $\mathbb{R}$.

Henceforth, we do not distinguish between origami-constructible lengths and origami-constructible real numbers.

Given the plane in which origami constructions are made, we establish a coordinate system by taking point O as the origin and the distance between O and P as the unit length 1. Then applying O1 to O and P, we can construct the x-axis. Applying O4 to the x-axis and point O, we can construct the y-axis and carry over the unit length 1 to the y-axis as demonstrated in Figure 4.3.

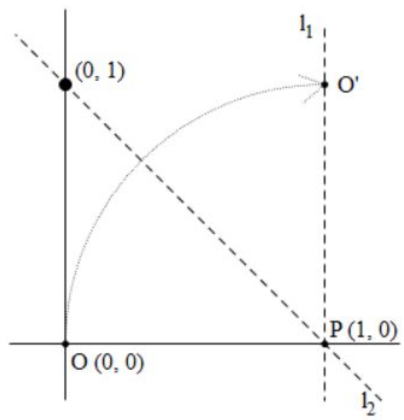


Figure 4.3: Constructing the axes and point $(0, 1)$.

To elaborate, apply O4 to P and the x-axis to obtain l_1, parallel to the y-axis. Then apply O5 to two points O, P and line l_1 to obtain l_2, which intersects the y-axis at at a distance 1 from point O. Then the coordinates of the two given points O and P and the unit on the y-axis are $(0, 0)$,$(1, 0)$, $(0, 1)$, respectively. As such, using our points O and P, we can construct a coordinate system in the plane and use it to represent our points as ordered pairs, i.e., elements of $\mathbb{R}^2$[2]. We then have:

[2]The plane can be taken as the field of complex numbers $\mathbb{C}$, with the x-axis representing the real numbers and the y-axis representing the imaginary numbers.

Lemma 4.3.2. ***A point*** *$(a, b) \in \mathbb{R}^2$* ***is origami-constructible*** *if and only if its coordinates a and b are origami-constructible elements of $\mathbb{R}$.*

Proof. We can origami-construct distances along perpendicular lines (apply E2' in Figure 4.2), and we can make perpendicular projections to lines (apply O4). Then, the point (a, b) can be origami-constructed as the intersection of such perpendicular lines. On the other hand, if (a, b) is origami-constructible, then we can project the constructed point to the x-axis and y-axis (apply O4) and thus the coordinates a and b are origami-constructible as the intersection of two origami-constructed lines. □

Since it has been shown that all straightedge and compass constructions can be made with origami, it follows that Lemmas 3.2.3, 3.2.4, and Corollaries 3.2.5, 3.2.6 hold in origami as well. In the following Lemma, we prove the closure of origami numbers under addition, subtraction, multiplication, division, and taking square roots by showing how to construct such lengths with origami.

Lemma 4.3.3. *If two lengths a and b are given, one can construct the lengths $a \pm b$, $ab, \frac{a}{b}$, and $\sqrt{a}$ in origami.*

Proof. For $a \pm b$, it suffices to show $a + b$. Given two lengths a, b, use E2$'$ (Figure 4.2) to construct lines ℓ, m parallel to $\overleftrightarrow{OP}, \overleftrightarrow{OQ}$, through points Q and P, respectively and name the intersection of ℓ, m point R (Figure 4.4).

Since $\square OPQR$ is a parallelogram, we know $QR = a$ and $RP = b$. Applying O3, we can construct line n which folds line m onto $\overleftrightarrow{OP}$. Let S the intersection of ℓ and n. Since n bisects the angle between m and $\overleftrightarrow{OP}$, and since ℓ is parallel to

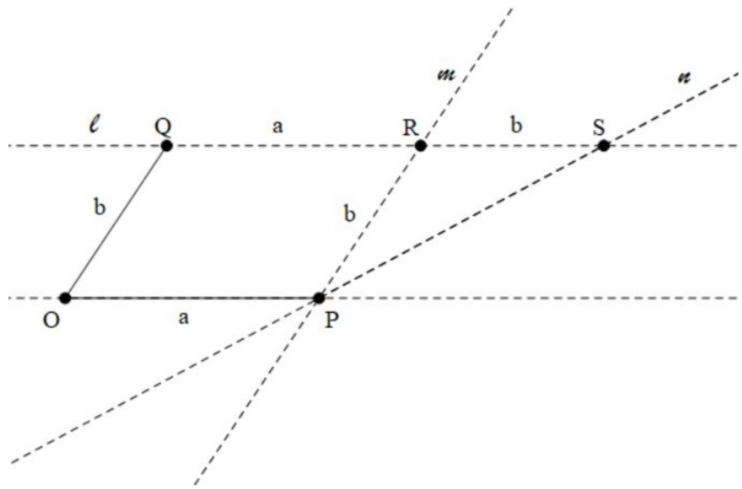


Figure 4.4: Constructing length $a + b$ using origami.

$\overleftrightarrow{OP}$, we know that $\triangle RPS$ is an isosceles triangle and thus $PQ = b$. Hence, we have constructed a line segment $\overline{QS}$ with length $a + b$.

Given lengths a and b, construct parallel lines (E2′) as shown in Figure 3.4 (a) and (b) to construct lengths ab and $\frac{a}{b}$, respectively, using similar triangles.

Given two lengths $1, a$, construct $\overline{OQ}$ with length $1 + a$, so $OP = 1$ and $PQ = a$ (Figure 4.5). Use O2 to construct the perpendicular bisector ℓ of $\overleftrightarrow{OQ}$, to construct the midpoint R of O and Q. Then $OR = \frac{1+a}{2}$. Next, apply O5 to construct line n that folds point O onto line m through point R. Finally, apply O4 to construct line t perpendicular to line n through point O. Let the intersection of lines m, t S. Then, by construction, $OR = SR = \frac{1+a}{2}$. This leads to the same construction as we made earlier in Figure 3.4(c). It follows that $SP =\sqrt{a}$.

$\square$

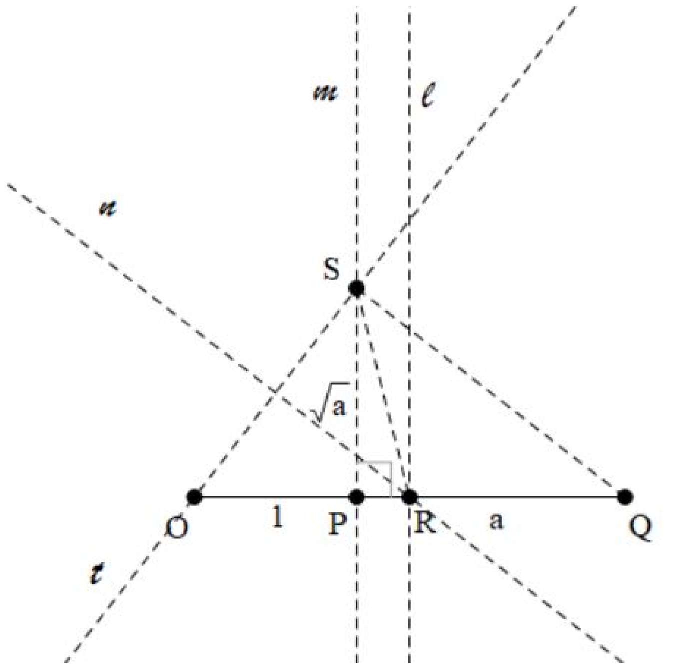


Figure 4.5: Constructing length $\sqrt{a}$ using origami.

From Lemma 4.3.3, if follows[3] that every rational number is origami-constructible and that the set of origami-constructible numbers is a subfield of $\mathbb{R}$.

Let $\mathscr{O}$ denote the set of origami-constructible numbers. So far, we have $\mathbb{Q} \subset \mathscr{O} \subseteq \mathbb{R}$. For a more precise account for origami-constructible numbers, we examine O5 and O6 algebraically (see section 4.2 for why we only examine O5 and O6).

Note that in the proof of Lemma 4.3.3, O5 was used to construct the square root of a given length. Let's examine operation O5 algebraically. Recall O5: '*Given two points P_1 and P_2 and line ℓ, we can fold P_1 onto ℓ so that the crease line passes through P_2.*' Imagine we carry out O5 with a given fixed point P_1 and a given fixed

[3]See the proof of Lemma 3.2.3, and Corollaries 3.2.5, 3.2.6

line ℓ through some auxiliary point P_2. There are infinitely many points P_1' where P_1 can fold onto ℓ, as shown in Figure 4.6.

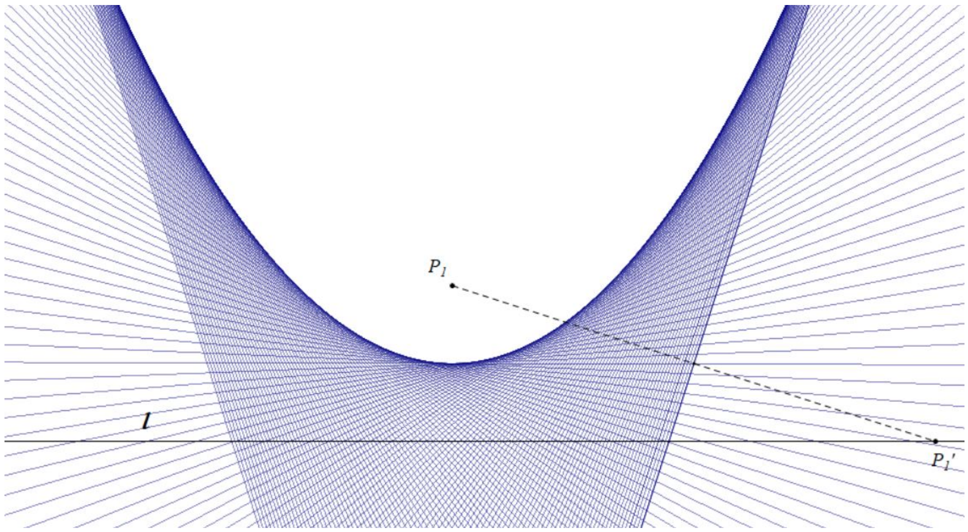


Figure 4.6: Lines folding point P onto line ℓ.

Theorem 4.3.4. *The lines that fold a given point P to a given line ℓ are tangent to the parabola with focus P and directrix ℓ.*

Proof. First, without loss of generality, we situate the point P and line ℓ in the Cartesian plane by letting $P = (0, 1)$ and ℓ the line $y = -1$ (see Figure 4.7). Then, P', the point that P folds onto line ℓ will have coordinates $(t, -1)$ for some $t \in \mathbb{R}$.

Recall from O2 that a line that folds a point P onto another point P' forms the perpendicular bisector of $\overline{PP'}$. The slope and midpoint between P and P' are $\frac{-2}{t}$ and $(\frac{t}{2}, 0)$, respectively. So, the equation of the crease line f is

$$f : y = \frac{t}{2}(x - \frac{t}{2}) \tag{4.1}$$

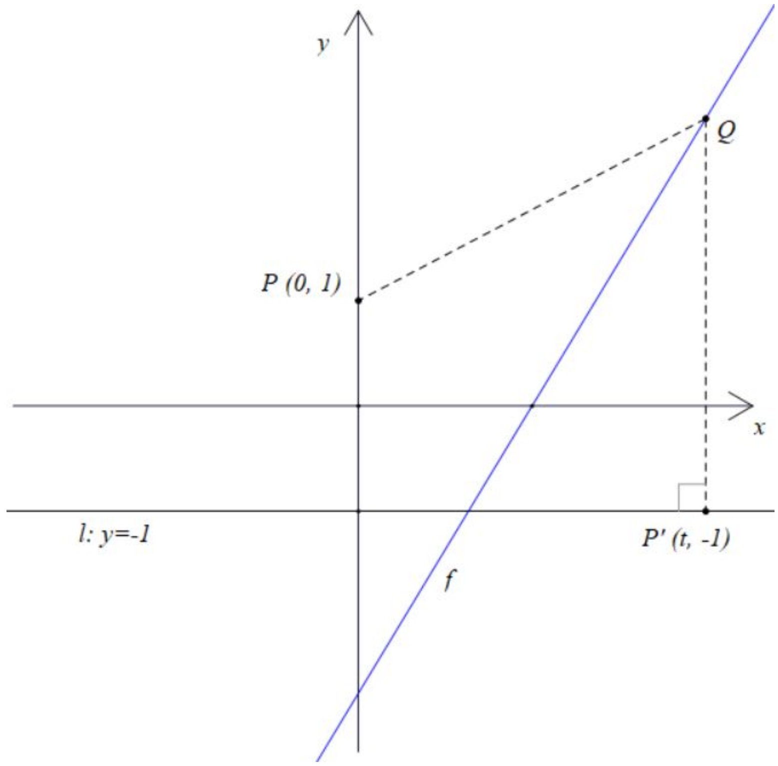


Figure 4.7: Folding point P onto a line ℓ.

Let Q the intersection of f and the line perpendicular to ℓ through P'. Since Q is a point on the crease line, we know $QP = QP'$. Note that QP' is the distance from the crease line f to line ℓ, so Q is equidistant from the given point P and given line ℓ. Therefore, the collection of points Q forms a parabola with focus P and directrix ℓ and the crease lines are tangent to the parabola. Note that evaluating $x = t$ in equation (4.1), the coordinates of $Q = (t, \frac{t^2}{4})$. Hence the curve that consists of all Q's has the quadratic equation $y = \frac{x^2}{4}$, an equation of a parabola. $\square$

Rearranging equation (4.1) and solving for t, we have

$$\frac{t^2}{4} - \frac{t}{2}x + y = 0 \Rightarrow t = \frac{\frac{x}{2} \pm \sqrt{(\frac{x}{2})^2 - y}}{\frac{1}{2}} \tag{4.2}$$

t has real values only when $(\frac{x}{2})^2 - y \geq 0$, i.e., when $y \leq \frac{x^2}{4}$. Specifically, the points on the parabola satisfy $y = \frac{x^2}{4}$ and all points P_2 in the plane that can be hit by a crease line must be $y \leq \frac{x^2}{4}$. So, O5 cannot happen when P_2 is in the inside of the parabola.

Now we can use a parabola to construct square roots with origami.

Corollary 4.3.5. *The set of origami constructible numbers $\mathscr{O}$ is closed under taking square roots*[4].

Proof. Given length r, we will show that $\sqrt{r}$ is origami-constructible. Let $P_1 = (0, 1)$ and $\ell : y = -1$. Let $P_2 = (0, -\frac{r}{4})$. Then, using O5, fold P_1 onto ℓ through P_2. We know that the equation of our crease line is $y = \frac{t}{2}x - \frac{t^2}{4}$, from equation (4.1) from the proof of Theorem 4.3.4. Since this line has to pass p_2, $-\frac{r}{4} = -\frac{t^2}{4}$, so $t = \sqrt{r}$. Therefore, the point where p_1 lands on ℓ will give us a coordinate of desired length. □

Corollary 4.3.5 implies that $\mathscr{C} \subseteq \mathscr{O}$. In fact, $\mathscr{O}$ is strictly larger than $\mathscr{C}$, from O6. Recall O6: '*Given two points p_1 and p_2 and two lines ℓ_1 and ℓ_2, we can fold a line that places p_1 onto ℓ_1 and p_2 onto ℓ_2.*'

[4] A geometric proof was made previously in Lemma 4.3.3

Theorem 4.3.6. *Given two points P_1 and P_2 and two lines ℓ_1 and ℓ_2, the line, when exists, that places P_1 onto ℓ_1 and P_2 onto ℓ_2 is the simultaneous tangent to two parabolas. Moreover, operation O6 is equivalent to solving a cubic equation.*

Proof. Fix P_1, P_2 and ℓ_1 by letting $P_1 = (0, 1), P_2 = (a, b)$ and $\ell_1 : y = -1$ (see Figure 4.8). We will see where P_2 folds onto when we fold P_1 onto ℓ_1 over and over again. In other words, we want to find the point $P_2' = (x, y)$ under each folding.

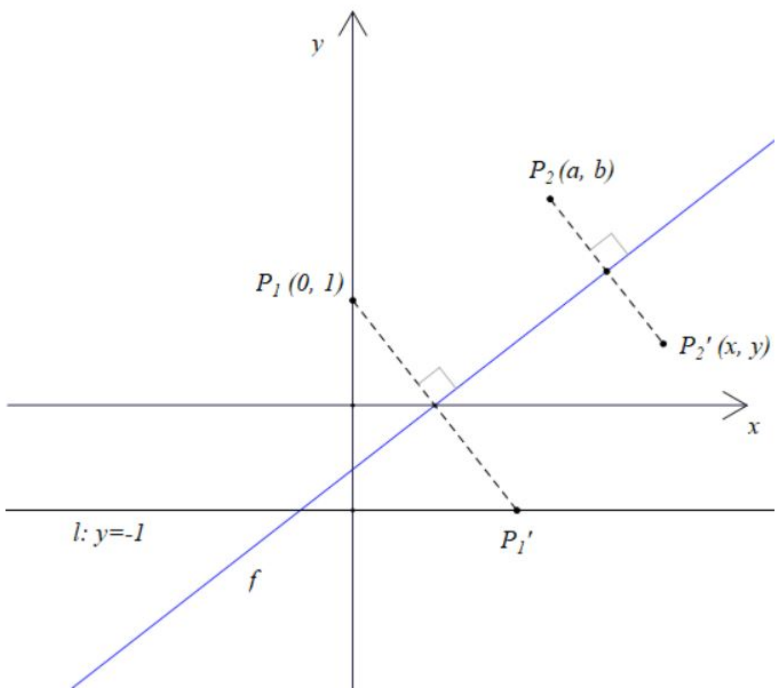


Figure 4.8: Applying O6 to $P_1 = (0, 1), P_2 = (a, b)$ and $\ell_1 : y = -1$

From Theorem 4.3.4, the crease line f folding P_1 onto ℓ_1 is tangent to the parabola with focus P_1 and directrix ℓ_1. Similarly, f is tangent to the parabola with focus P_2 and directrix ℓ_2. Therefore, f is a common tangent line to two distinct parabolas.

Recall the equation of f is $y = \frac{t}{2}x - \frac{t^2}{4}$. Since this line also folds P_2 onto P_2', it is the perpendicular bisector of $\overline{P_2P_2'}$. The slope and midpoint between P_2 onto P_2' are $\frac{y-b}{x-a}$ and $(\frac{a+x}{2}, \frac{b+y}{2})$, respectively. Since $\overline{P_2P_2'}$ is perpendicular to f, $\frac{y-b}{x-a} = -\frac{2}{t}$, so

$$\frac{t}{2} = -(\frac{x-a}{y-b}) \tag{4.3}$$

Also, since f passes the midpoint of $\overline{P_2P_2'}$,

$$\frac{y+b}{2} = \frac{t}{2}(\frac{a+x}{2}) - \frac{t^2}{4} \tag{4.4}$$

Substituting equation (4.3) into equation (4.4) we obtain

$$\frac{y+b}{2} = -(\frac{x-a}{y-b})(\frac{a+x}{2}) - (\frac{x-a}{y-b})^2$$

$$\Rightarrow (y+b)(y-b)^2 = -(x^2-a^2)(y-b) - 2(x-a)^2$$

a cubic equation. So, the possible points P_2' are points on a cubic curve. Figure 4.9 shows the curve derived from tracing point P_2' as point P_1' moves along line ℓ_1 in Figure 4.8.

If we fold P_2 onto ℓ_2, then the fold is determined by folding P_2 to where ℓ_2 intersects the cubic curve. Locating such a point is equivalent to solving the cubic equation at a specific point [9]. $\square$

According to Lang [14], this cubic curve (see Figure 4.10 for another example) has several notable features, derived from the equation. First, it usually contains

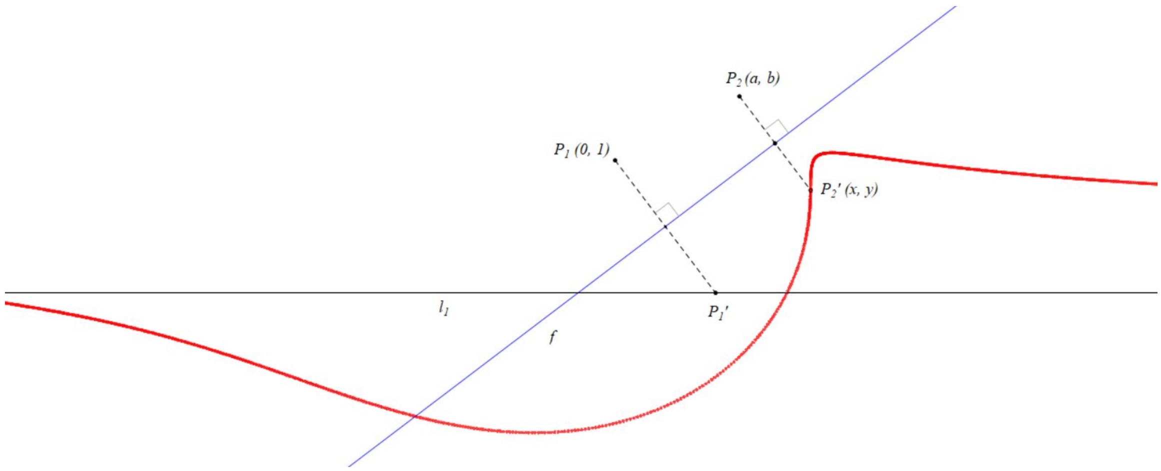


Figure 4.9: An example of a cubic curve.

a loop with the crossing of the loop at point P_2. Second, any line ℓ_2 intersects the curve at most three spots, so there are at most three possible alignments of P_2 onto ℓ_2. Note that O6 is not possible when ℓ_1 and ℓ_2 are parallel with P_1, P_2 in between the two lines.

Corollary 4.3.7. *The origami operations O1-O6 enable us to construct a real solution to any cubic equation with coefficients in $\mathscr{O}$.*

Proof. Without loss of generality, any cubic equation can be written in the form $x^3 + ax^2 + bx + c = 0$ with $a, b, c \in \mathscr{O}$. Substituting $X = x - \frac{a}{3}$, we obtain

$$X^3 + \frac{3b - a^2}{3}X - \frac{9ab - 27c - 2a^3}{27} = 0$$

Since $a, b, c \in \mathscr{O}$, the coefficients of this equation are also origami-constructible, by Lemma 4.3.3. So, we can assume the general cubic equation of the form $x^3+px+q = 0$

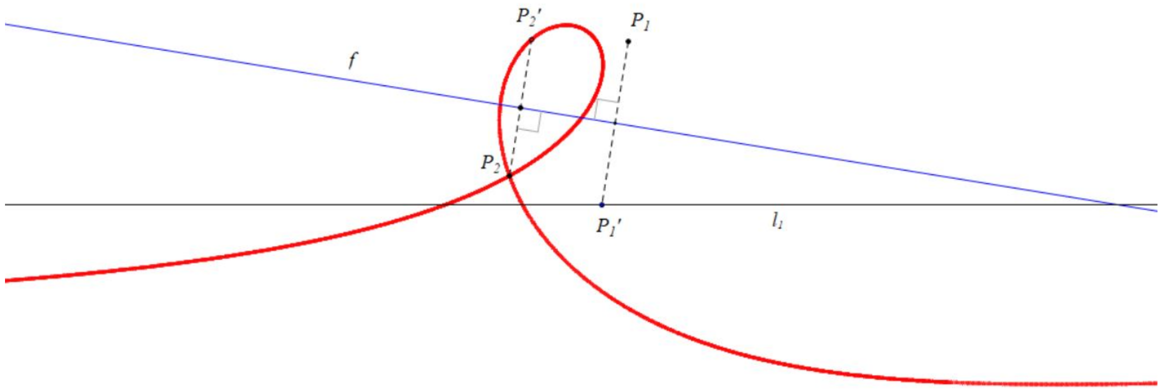


Figure 4.10: An example of a cubic curve.

where $p, q \in \mathscr{O}$. Following Alperin's [1] approach, consider two parabolas

$$(y - \frac{1}{2}p)^2 = 2qx \tag{4.5}$$

and

$$y = \frac{1}{2}x^2 \tag{4.6}$$

Parabola (4.5) has focus $(\frac{q}{2}, \frac{p}{2})$ and directrix $x = -\frac{q}{2}$; and parabola (4.6) has focus $(0, \frac{1}{2})$ and directrix $y = -\frac{1}{2}$. Since $p, q \in \mathscr{O}$, the coefficients of these equations are origami-constructible by Lemma 4.3.3, and so are the foci and directrixes.

Folding $(\frac{q}{2}, \frac{p}{2})$ onto $x = -\frac{q}{2}$ and $(0, \frac{1}{2})$ onto $y = -\frac{1}{2}$ produces a fold line f tangent to both of these parabolas. Let m be the slope of f. We claim that m is a root of $x^3 + px + q = 0$.

Let $(x_1, y_1), (x_2, y_2)$ the points of tangency of f with parabolas (4.5) and (4.6), respectively. Then from the equations of each parabola,

$$(y_1 - \frac{1}{2}p)^2 = 2qx_1 \tag{4.7}$$

and

$$y_2 = \frac{1}{2}x_1^2 \tag{4.8}$$

Taking the derivative of parabola (4.5) and evaluating it at the tangent point yields

$$2(y - \frac{1}{2}p)\frac{dy}{dx} = 2q \Rightarrow \frac{dy}{dx} = \frac{q}{y - \frac{1}{2}p}$$

So,

$$m = \frac{q}{y_1 - \frac{1}{2}p} \Rightarrow y_1 = \frac{1}{2}p + \frac{q}{m}$$

Substituting this into equation (4.7), $x_1 = \frac{q}{2m^2}$.

Similarly, taking the derivative of parabola (4.5) and evaluating it at the tangent point yields

$$\frac{dy}{dx} = x$$

So,

$$m = x_2$$

Substituting this into equation (4.8), $y_2 = \frac{1}{2}m^2$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{1}{2}m^2 - (\frac{p}{2} + \frac{q}{m})}{m - \frac{q}{2m^2}}$$

Simplifying, we have

$$m^3 + pm + q = 0 \tag{4.9}$$

So m satisfies a cubic equation with real coefficients in $\mathscr{O}$. Since the slope of f is the real root of the cubic equation, we can origami-construct a distance of m by erecting a perpendicular line at a point a unit distance from another point on the line f. Then the vertical distance between the intersection of the perpendicular line with line f is equal to m, as illustrated in Figure 4.11.

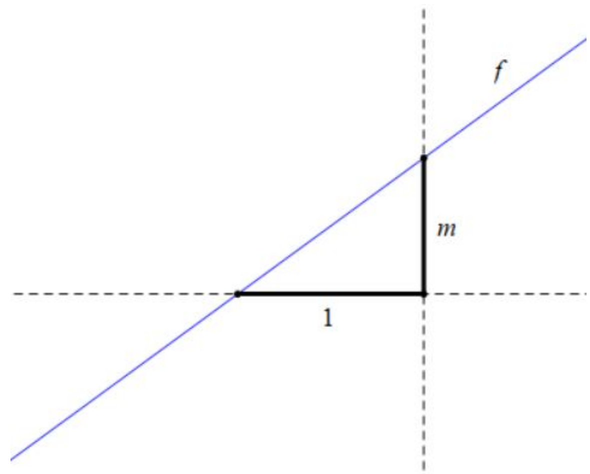


Figure 4.11: Line f with slope m.

□

Corollary 4.3.8. *The set of origami constructible numbers $\mathscr{O}$ is closed under taking cube roots.*

Proof. Given length r, we will show that $\sqrt[3]{r}$ is origami-constructible. From Theorem 4.3.6, we can apply O6 to construct a simultaneous tangent line f to the two parabolas $y^2 = -2rx$, $y = \frac{1}{2}x^2$. Note that these two parabolas are obtained by setting $p = 0, q = -r$ for the two parabolas (4.5) and (4.6) in the proof of Corollary 4.3.8. From equation 4.9, f has slope $m = \sqrt[3]{r}$, so $m = \sqrt[3]{r} \in \mathscr{O}$. $\square$

Theorem 4.3.9. *Let $r \in \mathbb{R}$. Then $r \in \mathscr{O}$ if and only if there is a finite sequence of fields $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ and $[F_i : F_{i-1}] = 2$ or 3 for each $1 \leq i \leq n$.*

Proof. $(\Rightarrow)$: Suppose $r \in \mathscr{O}$. As discussed in section 4.1, $r \in \mathscr{O}$ only when it could be constructed through operations O1-O7. So far, we have shown that origami constructions in F involve a field extension of degree 1, 2, or 3. From induction on the number of constructions required to construct r, there must be a finite sequence of fields $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ with $[F_i : F_{i-1}] = 2$ or 3.

$(\Leftarrow)$: Suppose $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $[F_i : F_{i-1}] = 2$ or 3. Then $F_i = F_{i-1}[\sqrt{d_i}]$ or $F_i = F_{i-1}[\sqrt[3]{d_i}]$ for some $d_i \in F_i$. If $n = 0$, then $r \in F_0 = \mathbb{Q}$ and $\mathbb{Q} \subset \mathscr{C}$, from Lemma 4.3.3. Suppose any element of F_n is origami-constructible for when $n = k - 1$. Then any $d_k \in F_{k-1}$ is origami-constructible, which implies that $\sqrt{d_k}$ and $\sqrt[3]{d_k}$ is also constructible, from Corollary 4.3.5 and Corollary 4.3.8. Therefore, any element of $F_k = F_{k-1}[\sqrt{d_k}]$ and $F_k = F_{k-1}[\sqrt[3]{d_k}]$ is also origami-constructible. From mathematical induction, any $r \in F_n$ is origami-constructible. $\square$

Corollary 4.3.10. *If $r \in \mathscr{O}$ then $[\mathbb{Q}(r) : \mathbb{Q}] = 2^a 3^b$ for some integers $a, b \geq 0$.*

Proof. From Theorem 4.3.9, if $r \in \mathscr{O}$, there is a finite sequence of fields $\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ and for each $i = 0, 1, \ldots, n$, $[F_i : F_{i-1}] = 2$ or 3. So, by the Tower Rule (Theorem 2.2.10),

$$[F_n : \mathbb{Q}] = [F_n : F_{n-1}] \cdots [F_1 : F_0] = 2^k 3^l$$

with $k + l = n$. Since $\mathbb{Q} \subset \mathbb{Q}(r) \subset F_n$, again by the Tower Rule, $[\mathbb{Q}(r) : \mathbb{Q}] | [F_n : \mathbb{Q}] = 2^k 3^l$, so if $r \in \mathscr{O}$ then $[\mathbb{Q}(r) : \mathbb{Q}] = 2^a 3^b$ for some integers $a, b \geq 0$. $\square$

4.4 Some Origami-constructible Objects

In Chapter 3, we have seen that straightedge and compass constructions can only produce numbers that are solutions to equations with degree no greater than 2. In other words, quadratic equations are the highest order of equations straightedge and compass can solve. As a result, trisecting an angle or doubling a cube was proved impossible with straightedge and compass since they require producing lengths that are solutions to cubic equations. As shown above (Section 4.3), in addition to constructing what straightedge and compass can do, origami can also construct points that are solutions to cubic equations (from O6). Therefore, doubling a cube and trisecting an angle is possible with origami. Although we know that these constructions could be done theoretically, in the following, I elaborate on the origami operations through which one can double a cube or trisect any given angle.

Theorem 4.4.1 (Doubling the cube). *It is possible to construct an edge of a cube double the volume of a given cube, using origami.*

Proof. It suffices to show that $\sqrt[3]{2}$ is origami-constructible. According to Lang [14] and Hull [9], Peter Messer [17] developed the method of constructing $\sqrt[3]{2}$ in 1986, which is regenerated step-by-step in Figure 4.12.

First, fold the square sheet of paper into thirds[5], as shown in Figure 4.12(a). Let the left edge of the paper ℓ_1, the top one-third crease line ℓ_2, the bottom right corner of the paper P_1 and the intersection of the right edge of the paper with the bottom one-third crease line P_2.

Apply O6 to points P_1, P_2 and lines ℓ_1, ℓ_2 to make a fold line m that places P_1 onto ℓ_1 and P_2 onto ℓ_2 (Figure 4.12(b)). The point P_1' where P_1 hits the edge of the paper divides the edge by $\sqrt[3]{2}$ to 1.

To show this, consider Figure 4.12(c). Let $P_1B = x, P_1A = 1, AD = y$. Then $AB = x+1, P_1D = x+1-y, P_1P_2 = \frac{x+1}{3}, CP_1 = x - \frac{x+1}{3} = \frac{2x-1}{3}$.

From the Pythagorean Theorem on $\triangle P_1AD$,

$$(x+1-y)^2 = 1^2 + y^2$$

$$\Rightarrow y = \frac{x^2+2x}{2x+2} \tag{*}$$

Meanwhile, $m\angle P_1CP_2 = m\angle DAP_1 = 90^\circ$ and

$$m\angle CP_1P_2 + m\angle CP_2P_1 = 90^\circ,$$

$$m\angle ADP_1 + m\angle AP_1D = 90^\circ,$$

[5]For a detailed account on constructing nth folds, see Lang [14].

$$m\angle CP_1P_2 + m\angle AP_1D = 90°.$$

So, $m\angle CP_1P_2 = m\angle ADP_1$ and $m\angle CP_2P_1 = m\angle AP_1D$. Therefore, $\triangle P_1CP_2 \sim \triangle DAP_1$ and thus

$$\frac{DA}{DP_1} = \frac{P_1C}{P_1P_2}$$

$$\Rightarrow \frac{y}{x+1-y} = \frac{\frac{2x-1}{3}}{\frac{x+1}{3}} = \frac{2x-1}{x+1}$$

$$\Rightarrow \frac{x^2+2x}{x^2+2x+2} = \frac{2x-1}{x+1}, from(*)$$

$$\Rightarrow x^3 + 3x^2 + 2x = 2x^3 + 3x^2 + 2x - 2$$

$$\Rightarrow x^3 = 2$$

So, $x = \sqrt[3]{2}$. □

Theorem 4.4.2. *[Trisecting an angle] It is possible to trisect any given angle θ, using origami.*

Proof. According to Hull [9] and Alperin & Lang [2], Hisashi Abe developed the following method of angle trisection published in 1980 ([6]). This method is regenerated step-by-step in Figure 4.13.

First, place an arbitrary angle of measure θ at the bottom left corner P of a square sheet of paper formed by the bottom edge of the paper and line ℓ_1 (Figure 4.13(a)).

Next, for any constructible point Q on the left edge of the paper, apply O4 to fold a line perpendicular to the left edge of the paper through point Q. Then use O2 to fold P onto Q to produce line ℓ_2 equidistant to the parallel lines through P and Q (Figure 4.13(b)).

Finally, apply O6 to points P, Q and lines ℓ_1, ℓ_2 to make a fold line m that places P onto ℓ_2 and Q onto ℓ_1 (Figure 4.13(c)). The reflection of point P about line m, say P', can be constructed as the intersection of the line perpendicular to m through P and the line ℓ_2 (we can construct the reflection of point Q in a similar manner). The angle formed by the bottom edge of the paper and segment PP' has the measure of $\frac{\theta}{3}$.

To prove that the angle formed by the bottom edge of the paper and segment PP' indeed has the measure of $\frac{\theta}{3}$, consider Figure 4.13(d). Let R the intersection of segments PQ' and $P'Q$; S the intersection of the left edge of the paper and line ℓ_2; and T the intersection of segment PP' and line m. Also, let $m\angle RPP' = \alpha$, $m\angle PP'S = \beta$, $m\angle RP'S = \gamma$. Since P' and Q' are reflections of P and Q about line m, R lies on line m. Since $PS = QS$ and $\angle P'SQ \perp \angle P'SP$, $\triangle P'QP$ is an isosceles triangle. Therefore,

$$\beta = \gamma \tag{*}$$

Similarly, since P' is a reflection of P about m, $PT = P'T$ and $\angle RTP \perp \angle RTP'$, $\triangle RPP'$ is an isosceles triangle. Therefore,

$$\alpha = \beta + \gamma \tag{**}$$

Since ℓ_2 and the bottom edge of the paper are parallel, the angle formed by segment PP' and the bottom edge of the paper has measure β. So,

$$\theta = \alpha + \beta$$

$$= (\beta + \gamma) + \beta, \text{ from } (**)$$

$$= 3\beta, \text{ from } (*)$$

Note that the angle in Figure 4.13 has measures between 0 and $\frac{\pi}{2}$. When the angle has measure $\theta = \frac{\pi}{2}$, $Q = Q'$ and thus line m folds point P onto ℓ_2 through point Q. Since $\triangle QPP'$ is an equilateral triangle, in this case, segment PP' trisects θ. When the angle is obtuse, the angle can be divided into an acute angle α and a right angle β so $\theta = \alpha + \beta$. This can be done by using O4 to construct a line ℓ perpendicular to one side of the angle through its center O (see Figure 4.14).

$$\frac{\alpha}{3} + \frac{\beta}{3} = \frac{\alpha + \beta}{3} = \frac{\theta}{3}.$$

So, the angle between m (the trisection line of the acute angle) and n (the trisection line of the right angle) give us the trisection of the obtuse angle. Therefore, the above method of angle trisection can be applied to any arbitrary angle. $\square$

4.5 Origami-constructible Regular Polygons

In this section, we examine the regular n-gons that are origami-constructible. For the purpose of our discussion, we expand the set $\mathbb{R}^2$ in which origami constructions are made to $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$.

We first define what an origami-constructible polygon is:

Definition 4.5.1. *An origami-constructible polygon is a closed, connected plane shape formed by a finite number of origami-constructible lines. An origami-constructible regular polygon is an origami-constructible polygon in which segments of the lines form sides of equal length and angles of equal size.*

The vertices and interior angles of an origami-constructible polygon, as intersections of origami-constructible lines are also origami-constructible and the central angle $\theta = \frac{2\pi}{n}$ is constructible as the intersection of the bisector (apply O3) of two consecutive interior angles (see Figure 4.15).

Before we determine which regular polygons are origami-constructible, we define splitting fields and cyclotomic extensions:

Definition 4.5.2. *Let F a field and $f(x) \in F[x]$. Then, the extension K of F is called a splitting field for $f(x)$ if $f(x)$ factors completely into linear factors in $K[x]$ but does not factor completely into linear factors over any proper subfield of K containing F.*

Proposition 4.5.3. *For any field F, if $f(x) \in F[x]$, then there exists a splitting field K for $f(x)$ [5, p.536].*

Consider the polynomial $x^n - 1$ and its splitting field over $\mathbb{Q}$. Over $\mathbb{C}$, the polynomial $x^n - 1$ has n distinct solutions, namely the *nth roots of unity.* These solutions can be interpreted geometrically as n equally spaced points along the unit circle in the complex plane. The nth roots of unity form a cyclic group generated by a primitive nth root of unity, typically denoted ζ_n.

Definition 4.5.4. *The splitting field of $x^n - 1$ over $\mathbb{Q}$ is the field $\mathbb{Q}(\zeta_n)$ and it is called the cyclotomic field of nth roots of unity.*

Proposition 4.5.5. *The cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is generated by the nth roots of unity over $\mathbb{Q}$ with $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n)$, where ϕ denotes Euler's ϕ-function*[6]*.*

Euler's ϕ-function in Proposition 4.5.3 is defined as the number of integers a $(1 \leq a < n)$ that are relatively prime to n, which is equivalent to the order of the group $(\mathbb{Z}/n\mathbb{Z})^{\times}$. According to Cox [4, p.270], Euler's ϕ-function could be evaluated with the formula

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}) \tag{4.10}$$

Definition 4.5.6. *An isomorphism of a field K to itself is called an automorphism of K. $Aut(K)$ denotes the collection of automorphisms of K. If K/F is an extension field, then define $Aut(K/F)$ as the collection of automorphisms of K which fixes all the elements of F.*

Definition 4.5.7. *Let K/F a finite extension. If $|Aut(K/F)| = [K : F]$, then K is Galois over F and K/F is a Galois extension. If K/F is Galois $Aut(K/F)$ is called the Galois group of K/F, denoted $Gal(K/F)$.*

Proposition 4.5.8. *If K is the splitting field over F of a separable polynomial $f(x)$, then K/F is Galois [5, p. 562].*

Therefore, the cyclotomic field $\mathbb{Q}(\zeta_n)$ of nth roots of unity is a Galois extension of $\mathbb{Q}$ of degree $\phi(n)$. In fact,

[6]For a detailed proof of Proposition 4.5.3, see [5, p. 555]

Proposition 4.5.9. *The Galois group $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ (see [5, p. 596]).*

Theorem 4.5.10 (Regular n-gon)**.** *A regular n-gon is origami-constructible if and only if $n = 2^a 3^b p_1 \cdots p_r$ for some $r \in \mathbb{N}$, where each distinct p_i is of the form $p_i = 2^c 3^d + 1$ for some $a, b, c, d \geq 0$.*

Proof. Suppose a regular n-gon is origami-constructible. We can position it in our Cartesian plane such that it is centered at 0 with a vertex at 1. Then, the vertices of the n-gon are the nth roots of unity, which are origami-constructible. On the other hand, if we can origami-construct the primitive nth root of unity, $\zeta_n = e^{\frac{2\pi i}{n}}$, we can origami-construct all nth roots of unity as repeated reflection by applying O4 and the addition of lengths (Figure 4.4). Then, we can construct the regular n-gon by folding lines through consecutive nth roots of unities. Simply put, a regular n-gon is origami-constructible if and only if the primitive nth root of unity ζ_n is origami-constructible. Therefore, it suffices to show that ζ_n is origami-constructible if and only if $n = 2^a 3^b p_1 \cdots p_r$ where each p_i are distinct Fermat primes for some $r \in \mathbb{N}$.

($\Rightarrow$): Suppose ζ_n is origami-constructible. From Corollary 4.3.10, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$ is a power of 2 or 3 and from Proposition 4.5.3, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n)$. Therefore, $\phi(n) = 2^\ell 3^m$ for some $\ell, m \geq 0$. Let the prime factorization of $n = q_1^{a_1} \cdots q_s^{s_1}$, where $q_1^{a_1}, \cdots, q_s^{s_1}$ are distinct primes and $a_1, \cdots a_s \geq 1$. Then from equation (4.10),

$$\phi(n) = q_1^{a_1 - 1}(q_1 - 1) \cdots q_s^{a_s - 1}(q_s - 1)$$

and $\phi(n) = 2^\ell 3^m$ only when each q_i is either 2 or 3 or $q_i - 1 = 2^c 3^d$ for some $c, d \geq 0$.

($\Leftarrow$): Suppose $n = 2^a 3^b p_1 \cdots p_r$ for some $r \in \mathbb{N}$, where each distinct p_i is of the form $p_i = 2^c 3^d + 1$ for some $a, b, c, d \geq 0$. Then from equation (4.10),

$$\phi(n) = 2^a 3^b p_1 \cdots p_r (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_r})$$

$$= 2^a 3^{b-1} (p_1 - 1) \cdots (p_r - 1) = 2^\ell 3^m$$

for some $\ell, m \geq 0$.

If $\phi(n) = 2^\ell 3^m$, then $G = Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is an abelian group (from Proposition 4.5.9) whose order is a power of 2 or 3. Then we have a sequence of subgroups $1 = G_0 < G_1 < G_2 < \cdots < G_n = G$ with each G_i/G_{i-1} a Galois extension which is cyclic of order 2 or 3[7]. Hence, from Theorem 4.3.9, ζ_n is origami-constructible.

$\square$

[7] From the Fundamental Theorem of Abelian Groups and the Fundamental Theorem of Galois Theory [5, p.158;p.574]

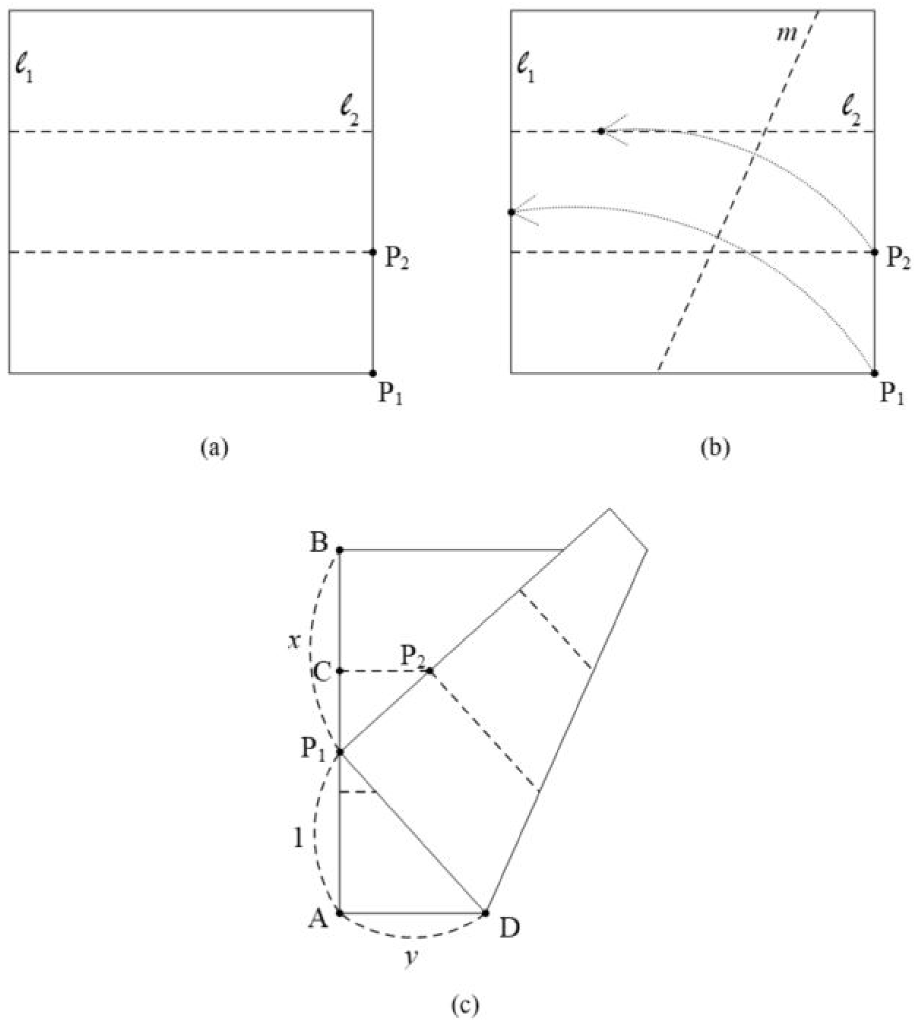


Figure 4.12: Constructing length $\sqrt[3]{2}$ using origami.

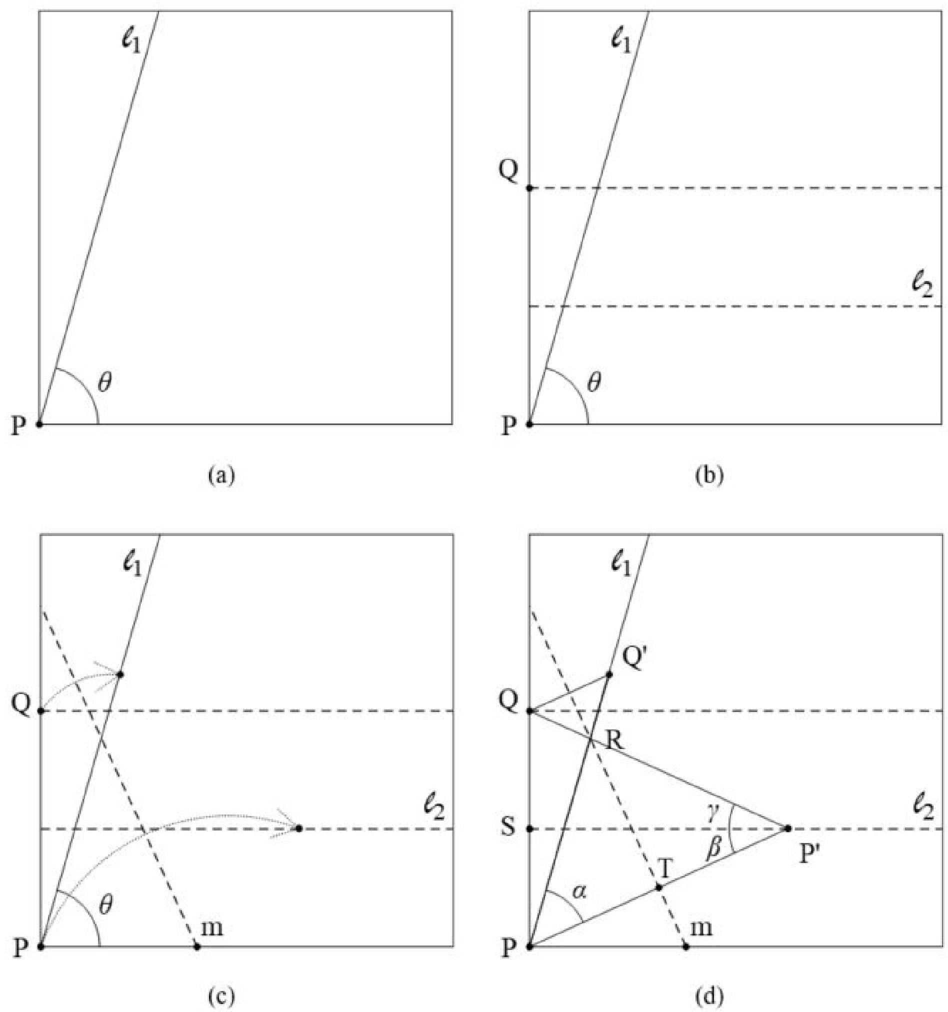


Figure 4.13: Trisecting an angle θ using origami.

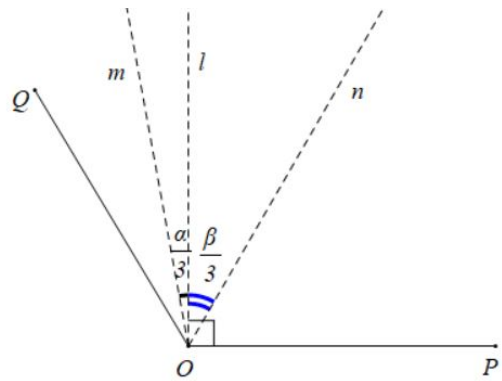


Figure 4.14: Trisecting an obtuse angle using origami.

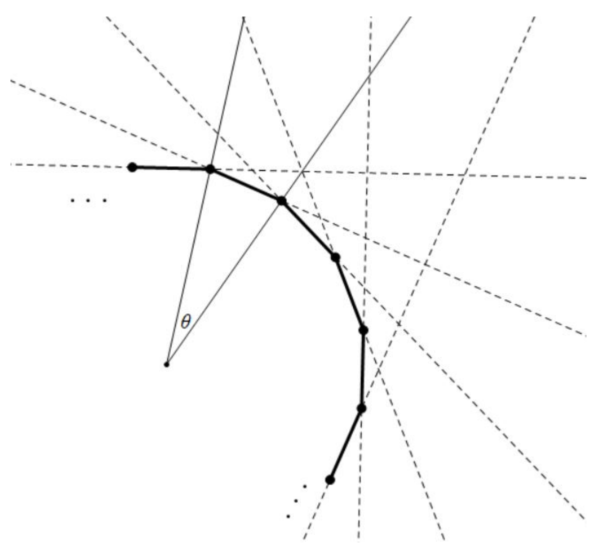


Figure 4.15: A regular polygon and its central angle.

Chapter 5

Extensions of origami theory and closing remarks

5.1 Extensions of origami theory

5.1.1 Additional tools

Several origami theorists have extended the traditional axiom set of single-fold operations by allowing additional geometrical tools for construction.

For example, in [13], Kasem et al. (2011) studied possible extensions of origami using the compass. Here, the compass allows one to construct circles centered at an origami point with radius a distance between two origami points. Kasem et al. presented three new operations, which they termed ”Origami-and-Compass Axioms” (p. 1109) and showed that the expanded set of axioms allows one to construct common tangent lines to ellipses or hyperbolas. In addition, they demonstrated a new method for trisecting a given angle and explained that they provide an interesting way of solving equations of degree 4. However, according to Kasem et al., combining the use of the compass with origami does not increase the construction power of origami beyond what is constructible by the traditional axiom set.

In [7] Ghourabi et al. (2013) took a similar approach by introducing fold operations that produce conic sections into origami. Specifically, Ghourabi et al. presented one additional fold operation which superimposes one point onto a line and another point onto a conic section. Ghourabi et al. proved that the new extended set of fold operations generate polynomials of degree up to six. They left the question of whether these fold operations solve all 5th and 6th degree equations for future research.

5.1.2 An eighth axiom

In his paper [15], Lucero revisited the single-fold operations by viewing folding as the geometry of reflections. After enumerating all possible incidences between constructible points and lines on a plane induced by a reflection, Lucero derived 8 possible fold operations that satisfy all combinations of incidences. The eight axioms included seven of the traditional axioms and one new operation, which is to fold a constructible line such that it folds onto itself. As a practical solution to allow one to fold one layer along a line marked on the other layer, Lucero introduced the additional single-fold operation as needed for the completeness of the axioms and application in practical origami.

5.1.3 Beyond the single fold

In [2], Alperin and Lang (2009) extended the traditional axiom set of single-fold operations by allowing alignments of points and lines along multiple folds. For example, a two-fold alignment is an alignment of points or lines with two simultaneous fold creases. As an example of a two-fold alignment, Alperin and Lang refer to Abe's solution (in 4.4.2) as a two-fold alignment combining O2 and O6. Since this alignment can be partitioned into two sets each of which is a single-fold, this two-fold alignment is called separable. Enumerating on non-separable alignments, Alperin and Lang identified a unique set of 17 two-fold alignments, which produce 489 distinct two-fold operations. Here, Alperin and Lang restricted to two-fold axioms to those that are non-separable and consist of fold lines"on a finite region of the Euclidean plane with a finite number of solutions" (p. 9). With the extended set of axioms, Alperin and Lang explained, "Each two-fold alignment yields a system of four equations in four variables with each equation of degree at most 4" (p. 14) but that not all degree 4 polynomials can be produced. Similarly, Alperin and Lang explained that some quintic polynomials can be solved by two-fold axioms but not in general.

Alperin and Lang acknowledged the limitation in physically carrying out the two-fold operations. For one to create a two-fold alignment requires smooth variation of the position of both folds until all alignments are satisfied, which could be practically difficult. Putting such limitations aside, Alperin and Lang investigated the construction power of origami when multi-fold operations are allowed. For example, they showed that a 3-fold alignment of 3 simultaneous folds allows solving the general quintic equation and more generally, the general nth degree equation can be solved with n-2 simultaneous folds.

As such, there are many extensions that have been studied. Propose some additional extensions to think about here and think about the implications.

5.2 Future research questions

In this section, I pose questions about other possible extensions of origami theory for future research.

5.2.1 Allowing tracing

In [15], Lucero (2017) claimed, "In origami mathematics, it is assumed that all lines and points marked on one layer are also defined on the layers above and below, as if the paper were 'transparent'" (p.12). However, such claim does not seem compatible with the definition of a single-fold (A single-fold refers to folding the paper once; a new fold can only be made after the paper is unfolded) nor does it seem feasible when considering actual paper folding of non-transparent paper. Related to such extension, one question I pose is: What implications can tracing have on origami constructions. In other words, if one is allowed to construct a line or point by tracing another line or point onto the paper when the paper is folded, would that change the set of origami-constructible numbers?

5.2.2 Allowing cutting

Typically in origami, cutting is not allowed. As another possible extension of origami theory, I pose the question of what happens if cutting is allowed. In order to explore this idea, there will need to be some specification of what cutting refers to and what type of folds are allowed after the paper is cut. For example, we can consider a case when one is allowed to cut the paper along lines or line segments connecting two origami-constructible points. In the case the paper is cut along a line into two separable parts, we can think of placing one part on top of the other and superimposing constructible points and lines from one layer onto the other (similar to tracing). In the case the paper is cut along a line segment but not fully separated into two parts, then we can think of various ways one can fold the paper about the cut. For instance, what can we do if the paper is cut along one of its edges? What can we do if there is a slit inside of the sheet of paper? In such cases, what would the set of origami-constructible numbers look like?

5.2.3 Starting with additional givens other than the unit distance

Finally, future research can investigate different types of field extensions derived from origami constructions with different initial givens. For example, in origami constructions, we start with two points and a unit distance as initials. What if we start with three points instead of two, with the collection of distances and/or angles given by these points not obtainable by only two points some unit distance apart?

5.3 Closing remarks

In this thesis, I presented an exposition of origami constructible objects and its associated algebraic field. I introduced what origami is; reviewed the basic definitions and theorems of field theory that were used in the thesis; discussed the more commonly known straightedge and compass constructions; introduced the basic single-fold operations of origami and discussed what it means for an object to be origami-constructible; showed how to prove or disprove the constructibility of some origami objects; and finally, presented some additional thoughts for future studies.

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Appendix: List of Geometric Symbols

$\overleftrightarrow{AB}$: line through two points A and B

$\overrightarrow{AB}$: ray with endpoint A that passes through B.

$\overline{AB}$: segment connecting two points A and B.

AB : distance between A and B

$\angle ABC$: angle with vertex B with rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$

$m\angle ABC$: measure of $\angle ABC$

$\triangle ABC$: triangle with vertices A, B, C

ABC : area of $\triangle ABC$

$\square ABCD$: quadrilateral with edges $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$

$p \cong q$: p congruent to q

$p \sim q$: p similar to q