The Fekete-Szegö Theorem with Splitting Conditions on the Projective Line of Positive Characteristic p

by

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(Under the direction of Robert S. Rumely)

Abstract

The logarithmic capacity is a measure of size for sets in $\mathbb{C}$, which has arithmetic consequences. It initially arose in classical potential theory, but occurs in several contexts (including probability and number theory), and goes under several names, including "Transfinite diameter", "Chebychev constant", and "exponential of the Robin constant". If the logarithmic capacity of a set can be computed, the Fekete-Szegö theorem gives a finiteness criterion/strong existence in arithmetic geometry [8]. Namely, whether the capacity is < 1 or > 1 determines the finiteness (or infiniteness) of the collection of algebraic integers whose conjugates lie near the set.

In this dissertation, we are concerned with capacities for adelic sets in the positive characteristic projective line. Let K be a function field in one variable over a finite field of constants, with characteristic $p > 0$, and let $\overline{K}$ be its algebraic closure. We prove a "Fekete-Szegö theorem with splitting conditions" for an adelic set $\mathbb{E}_K := \prod_{v \in P_K} E_v \subset \prod_{v \in P_K} \mathbb{P}^1(\mathbb{C}_v)$ with respect to a finite set of global points $\mathfrak{X} \subset \mathbb{P}^1(\overline{K})$. Loosely, the theorem says that if the capacity of $\mathbb{E}_K$ is > 1, and if $E_v \subseteq \mathbb{P}^1(K_v)$ for each v in a finite set S of places of K, then for any collection of neighborhoods U_v of the E_v, there are infinitely many points of $\mathbb{P}^1(\overline{K}_v)$ whose conjugates over K lie in the neighborhood U_v for all v, are bounded away from $\mathfrak{X}$ at all v, and belong to $\mathbb{P}^1(K_v)$ for each $v \in S$. The proof consists of constructing a rational function $f(z) \in K(z)$ whose poles are supported on $\mathfrak{X}$, and whose zeros belong to the neighborhood U_v for each v, and also belong to $\mathbb{P}^1(K_v)$ at $v \in S$. This is done by first constructing local functions $f_v(z) \in K_v(z)$ which have these properties for each v, and then constructing the global function $f(z) \in K(z)$ by sequentially modifying ("patching") the Laurent coefficients at the points in S to obtain a single function which locally looks like $f_v(z)$ for each v. There are three key parts in the proof: the construction of the basic local approximating functions; the local patching process, which preserves the properties of the roots of the $f_v(z)$; and the global patching process, which achieves K-rationality.

Two important difficulties arise in function fields of positive characteristic which do not occur in the case of number fields. First, in the number field case, even though it is hard to construct the basic local approximating functions at real archimedean places, there is considerable freedom in carrying out the patching precess. In positive characteristic, there are no archimedean places, so the patching process is extremely rigid. The second difficulty concerns inseparability. In the number field case, as soon as the patching functions are invariant under the Galois action, they become rational over the base field. However, in the case of positive characteristic, even though a function may be Galois-invariant, it does not need to be rational over the base field. To achieve rationality, it is necessary to consider the expansions of the $f_v(z)$ with respect to two sets of basis functions at the same time, one of which is used to check rationality and the other to determine approximation properties of the functions.

INDEX WORDS: Fekete-Szegö theorem, capacity theory, splitting conditions, projective line, positive characteristic

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Dedication

To my loving wife Okkyung Cho and son Justin Jinseo Park.

Acknowledgments

This dissertation is in large part a synthesis of what I have learned from my advisor, Robert S. Rumely. I would like to thank him for his patience, support, careful proofreading and suggestions. He encouraged me throughout my graduate study, and helped me succeed in writing this dissertation. Especially, I really thank him for his prayers when I was depressed and frustrated.

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Many results which are now considered “standard” have been presented as such. Even though I presented my own proof, I can claim no originality for any of the unlabeled theorems in the dissertation, and apologize to anyone who may feel slighted. For more details, readers may refer [22] and [25] for the chapters 2, 3 and 4, 5, respectively.

I would like to thank my (late) father, mother, parents-in-law, and my brother’s and sisters’ families. Without their sincere support, I would not be here.

These few words cannot adequately express the thanks I owe to my wife, Okkyung Cho. Even though she was also a graduate student in the department of Mathematics at the University of Georgia and was writing her own dissertation, she created a warm home for me and our son, Justin Jinseo Park, and provided a rich human context for my life through her constant support and friendship. Also, I am so proud of my son, Jinseo, who has been a great kid and has made us laugh at our lives.

Finally, I would like to thank God for His grace and abundant love. He put me in the Korean Catholic Community in Athens. They provided a beautiful environment around my family and have been faithful friends to us.

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Notes on Chapters

I have organized the chapters in the dissertation to motivate definitions of constants and choices of parameters, focusing on the flow of the proof of the main theorem (the Fekete-Szegö Theorem with Splitting Conditions).

I state the main theorem in Chapter 1, giving a brief history of prior results.

In Chapter 2, I discuss background material. The first section examines some well-known properties of power series in one variable, including the theory of Newton polygons, Weierstrass Preparation Theorem, and the Maximum Modulus Principle. The second section introduces the canonical distances related with a uniformizing parameter at a point.

Readers find general notions about Local and Global Capacity Theory in Chapter 3. I discuss properties of this capacity theory for the canonical distance with respect to a single point and related objects, including equilibrium distributions, Robin constants, Capacity, Green's functions, and Green's matrices. Part of this theory is classical, and part was developed by Cantor [6]. Here, as well one find some examples.

In Chapter 4, I discuss capacity theory with respect to a finite set $\mathfrak{X}$ and a probability vector $\vec{s}$, called $(\mathfrak{X}, \vec{s})$-capacity theory. I establish the facts about this theory which we need.

The proof of the main theorem involves constructing global rational functions $G(z) \in K(z)$ of arbitrary large degree, whose poles are supported on $\mathfrak{X}$ and whose zeros are the numbers produced by the theorem. The construction has two parts, a local part and a global part.

The local part involves first constructing "basic local approximating functions", rational functions $f_v(z) \in K_v(z)$ with poles supported on $\mathfrak{X}$ whose zeros belong to the prespecified neighborhood U_v in the main theorem. These functions are renormalized on the basis of global considerations, giving "normalized local approximating functions" $g_v(z)$, and then composed with "degree-raising" polynomials to construct "initial local patching functions" $G_v^{(0)}(z)$. Lastly, the local part gives a "patching process" which involves sequentially modifying the Laurent coefficients of the $G_v^{(0)}(z)$, from highest to lowest order, while keeping the roots in U_v. There are two cases to deal with, the case of RL-domains and the case of compact sets.

The global part concerns achieving K-rationality, melding the initial patching functions $G_v^{(0)}(z) \in K_v(z)$ into a single global function $G(z) \in K(z)$. Important aspects include modifying the leading coefficients to be global S_L-units, adjusting the high-order coefficients to be all 0, and using the Very Strong Approximation Theorem to choose target values for the low-order coefficients.

Chapter 5 introduces the notion of regular sequences in $\mathcal{O}_v$ and their properties. I construct rugular sequences for a compact subset of $\mathbb{P}^1(K_v)$. These are used in constructing the basic local approximating functions in the case of compact sets.

Chapter 6 constructs the basic local approximating functions for a compact subset of $\mathbb{P}^1(K_v)$.

In Chapter 7, I state the Gap Principle which plays a key role in the global patching process. I reduce the main theorem to tractable special case, involving only RL-domains and compact sets which are K_v-rational. I obtain the final constraints on the integer N, which is the degree of the normalized local approximating functions.

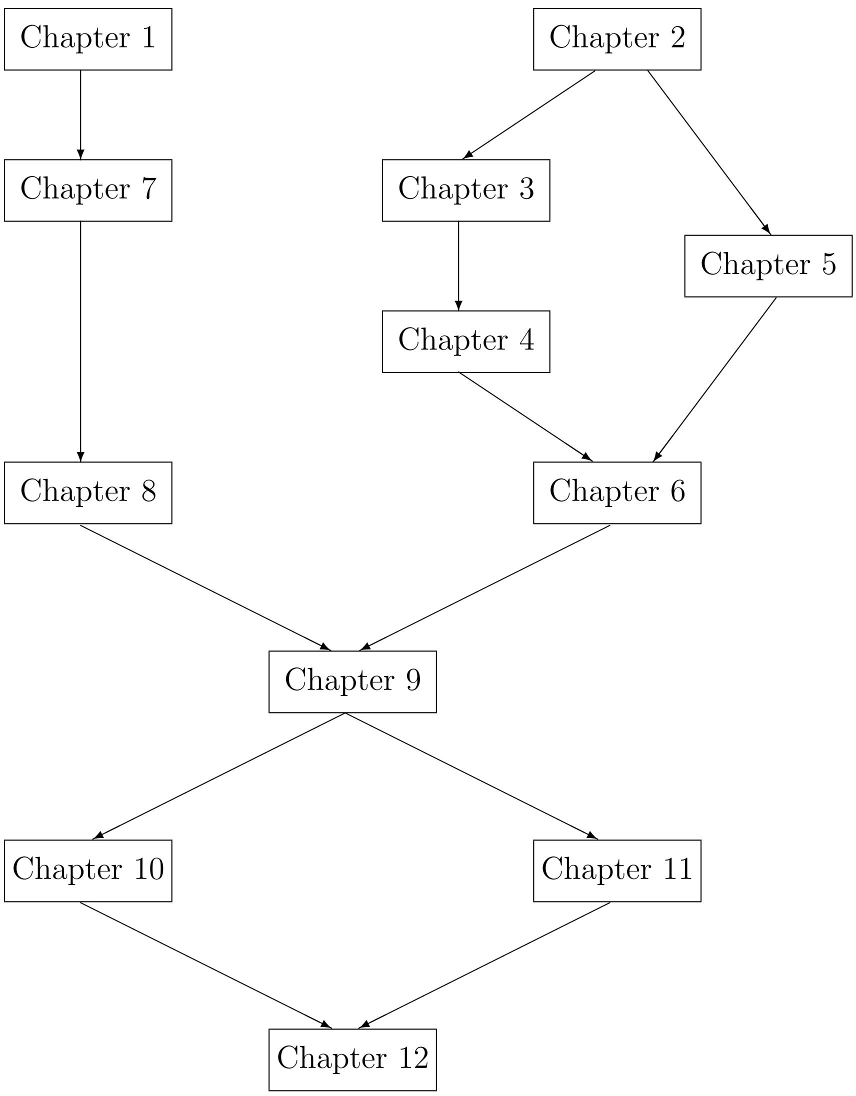
I achieve the product formula for the leading coefficients and construct the normalized local approximating functions in Chapter 8.

In Chapter 9, I prove the Very Strong Approximation Theorem and choose the global patching parameters k_0 and n. The number k_0 makes the break between coefficients deemed high-order and low-order. The integer n is used to determine the degree of each initial patching function. It must be sufficiently large and satisfy certain divisibility conditions.

In Chapters 10 and 11, I give the details of the local patching process for RL-domains and compact sets. This process will preserve the locations of roots of each initial pathcing function. In the compact case, the process for low-order coefficients is basically the same as the one in [25].

I complete the proof of the main theorem in Chapter 12 by arranging the high-order coefficients to be zero and the low-order coefficients to be global elements using the Very Strong Approximation Theorem in Chapter 9.

Here is a table of chapter interdependence:



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Chapter 1

Introduction

In 1955, Fekete and Szegö proved that if $E \subset \mathbb{C}$ is a compact set with logarithmic capacity $\gamma(E) \geq 1$, stable under complex conjugation, then every complex neighborhood of E contains infinitely many conjugate sets of algebraic integers [8]. In 1964, Raphael Robinson refined this, showing that if E is a subset of the real line, then every real neighborhood of E contains infinitely many conjugate sets of totally real algebraic integers [20]. Robinson's theorem is the prototype for the "Fekete-Szegö theorem with splitting conditions", with which we are concerned in this dissertation.

In 1980, David Cantor developed a theory of capacity for adelic sets in $\mathbb{P}^1$ over a number field K [6]. Fix an algebraic closure $\overline{K}$ of K. Let K_v be the completion of K at a place v of K and denote $\mathbb{C}_v$ the completion of a fixed algebraic closure of K_v at v. Let $\mathbb{E}_K := \prod_{v \in P_K} E_v \subset \prod_{v \in P_K} \mathbb{P}^1(\mathbb{C}_v)$ be an adelic set over K and fix a finite Galois-stable set $\mathfrak{X} \subset \mathbb{P}^1(\overline{K})$. Under appropriate hypotheses on $\mathbb{E}_K$ and $\mathfrak{X}$, Cantor defined a number $\gamma(\mathbb{E}, \mathfrak{X})$, which is called the capacity of $\mathbb{E}_K$ with respect to $\mathfrak{X}$. Let S be a finite set of places of K satisfying the following condition: for each $v \in S$, there is a finite Galois extension $\mathcal{K}_u/K_v$ such that $E_v \subset \mathbb{P}^1(\mathcal{K}_u)$. Cantor formulated a "Fekete-Szegö theorem with splitting conditions", which asserted that if $\gamma(\mathbb{E}, \mathfrak{X}) > 1$, then for any adelic neighborbhood $\mathbb{U}_K := \prod_{v \in P_K} U_v \subset \prod_{v \in P_K} \mathbb{P}^1(\mathbb{C}_v)$ of $\mathbb{E}_K$, there are infinitely many points in $\mathbb{P}^1(\overline{K})$ whose conjugates in $\mathbb{C}_v$ belong to U_v for all $v \in P_K$, and lie in $\mathbb{P}^1(\mathcal{K}_u)$ for all $v \in S$. The latter constraints are the "splitting conditions". Unfortunately, there was a gap in the proof of the theorem.

Robert Rumely extended Cantor's theory, including the Fekete-Szegö theorem without splitting conditions, to arbitrary algebraic curves over any global field [22]. In 2000, he established the Fekete-Szegö theorem with splitting conditions in the special case where $K = \mathbb{Q}$ and $\mathfrak{X} = \{\infty\}$, with $E_\infty = [-2r, 2r]$, and $E_p = \mathbb{Z}_p$ for finitely many primes p [23]. In 2002, he proved the theorem for $\mathfrak{X} = \{\infty\}$ over an arbitrary number field K, and for general sets E_v [24]. Recently, he generalized the Fekete-Szegö theorem with splitting conditions for any finite set $\mathfrak{X}$ with a general adelic set $\mathbb{E}_K$ on a smooth connected algebraic curve over a *number field* [25].

Here, we prove the Fekete-Szegö theorem with splitting conditions for any finite set $\mathfrak{X}$ with general sets E_v in $\mathbb{P}^1(\mathbb{C}_v)$ when the base field K is a function field *of characteristic* $p > 0$. Let q be a power of p and F denote the field $\mathbb{F}_q(T)$ with a transcendental element T. Fix a finite extension K of F. Let K_v be the completion of the field K at a place v of K and $\mathbb{C}_v$ the completion of a fixed algebraic closure of K_v at the place v. Our main theorem is as follows:

Theorem 1.1 (Fekete-Szegö Theorem with Splitting Conditions) *Let K be a finite algebraic extension of $F := \mathbb{F}_q(T)$, and let $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \mathbb{P}^1(\overline{K})$ be finite and stable under Gal$(\overline{K}/K)$. For each place v of K, let $E_v \subset \mathbb{P}^1(\mathbb{C}_v)$ be stable under Gal$_c(\mathbb{C}_v/K_v)$ (the group of continuous automorphisms of $\mathbb{C}_v$ over K_v), and put $\mathbb{E}_K := \prod_{v \in P_K} E_v$. Suppose that $\mathbb{E}_K$ is compatible with $\mathfrak{X}$, i.e., each E_v is bounded away from $\mathfrak{X}$, and for all but finitely many v, E_v is the $\mathfrak{X}$-trivial set (consisting of all points in $\mathbb{P}^1(\mathbb{C}_v)$ which do not specialize to the points of $\mathfrak{X} \mod v$). Let $S \subset P_K$ be a finite (possibly empty) set of places of K with the property that for each $v \in S$, there exists a finite Galois extension $\mathcal{K}_u/K_v$ such that $E_v \subset \mathbb{P}^1(\mathcal{K}_u)$. If $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$, then for any adelic neighborhood $\mathbb{U}_K := \prod_{v \in P_K} U_v$ of $\mathbb{E}_K$, there are infinitely many points $\alpha \in \mathbb{P}^1(\overline{K})$ such that all of their conjugates over K belong to U_v for all v, and lie in $\mathbb{P}^1(\mathcal{K}_u) \cap U_v$ for each $v \in S$.*

Chapter 2

Preliminaries

Throughout this dissertation, we will keep the notations in Theorem 1.1: K is a finite algebraic extension of $F := \mathbb{F}_q(T)$, where q is a power of a prime number $p \in \mathbb{Z}_{>0}$; $\overline{K}$ is a fixed algebraic closure of K; K_v is the completion of K at a place v of K, and $\mathbb{C}_v$ denotes the completion of a fixed algebraic closure $\overline{K}_v$ of K_v at v. It will be useful to regard $\overline{K}$ as embedded in $\mathbb{C}_v$; for each v, fix such an embedding. Without loss of generality, we can assume that $\mathbb{F}_q$ is the exact field of constants of K. Note that every absolute value of K is nonarchimedean. In particular, K_v is a nonarchimedean local field and $\mathbb{C}_v$ is both complete and algebraically closed.

Given a place v of K, let $\mathcal{O}_v = \{z \in K_v : |z|_v \leq 1\}$ be the ring of integers of K_v and $\mathfrak{m}_v$ its unique maximal ideal; let $\widehat{\mathcal{O}_v} = \{z \in \mathbb{C}_v : |z|_v \leq 1\}$ be the ring of integers of $\mathbb{C}_v$ and $\widehat{\mathfrak{m}_v}$ its unique maximal ideal. We will write π_v for a uniformizing element at v, which means that $\mathfrak{m}_v = \pi_v\mathcal{O}_v$. Let $q_v = \#(\mathcal{O}_v/\pi_v\mathcal{O}_v)$ be the order of the residue field of K_v; note that $q_v = q^{f_v}$ for some $f_v \in \mathbb{Z}_{>0}$, while $\widehat{\mathcal{O}_v}/\widehat{\mathfrak{m}_v}$ is isomorphic to the algebraic closure $\overline{\mathbb{F}}_q$ of the field of constants $\mathbb{F}_q$. However, $\widehat{\mathfrak{m}_v}$ is not finitely generated. For any $x \in K_v \setminus \{0\}$, there is a unique integer n such that $x = \pi_v^n \cdot u$ for some $u \in \mathcal{O}_v^\times$, where $\mathcal{O}_v^\times := \mathcal{O}_v \setminus \pi_v\mathcal{O}_v$. We denote n by $\mathrm{ord}_v(x)$ and put $\mathrm{ord}_v(0) := \infty$. We take $|\cdot|_v$ to be the normalized absolute value on K_v given by $|x|_v = q_v^{-\mathrm{ord}_v(x)}$. We will also denote by $|\cdot|_v$ the unique extension of this absolute value to $\mathbb{C}_v$. The Galois group $\mathrm{Gal}(\overline{K}_v/K_v)$ respects $|\cdot|_v$, and its action on $\overline{K}_v$ extends to a continuous action on $\mathbb{C}_v$. Conversely, each continuous automorphism of $\mathbb{C}_v/K_v$ arises from an element of $\mathrm{Gal}(\overline{K}_v/K_v)$. By $\mathrm{Gal}_c(\mathbb{C}_v/K_v) \cong \mathrm{Gal}(\overline{K}_v/K_v)$, we mean the group of continuous automorphisms of $\mathbb{C}_v$, fixing K_v.

Write $\log_v(\cdot)$ for the logarithm to the base q_v. Then $-\log_v(|x|_v) = \mathrm{ord}_v(x)$ for all $x \in K_v$.

Let P_K be the set of all places of K. For each $x \in K^\times := K \setminus \{0\}$, the product formula holds in the form

$$\prod_{v \in P_K} |x|_v = 1.$$

Let L be a finite normal extension of K. If w is a place of L lying above v, then $|\cdot|_w$ will denote the normalized absolute value on L_w and the absolute value $|\cdot|_v$ on L_w will denote the unique extension of $|\cdot|_v$ on K_v to L_w, induced by any embedding of L_w into $\mathbb{C}_v$. Since $|y|_v^{[L_w:K_v]} = |N_{L_w/K_v}(y)|_v$ for all $y \in K_v$, it follows that for any $x \in L_w$,

$$|x|_w = |x|_v^{[L_w:K_v]}. \tag{2.1}$$

To see this, let $e_{w/v}$ and $f_{w/v}$ be the ramification index and residue degree of the extension L/K. Then we have $[L_w : K_v] = e_{w/v} \cdot f_{w/v}$. For any $x \in K_v$, since $\mathrm{ord}_w(x) = e_{w/v}\mathrm{ord}_v(x)$ and $q_w = q_v^{f_{w/v}}$, it follows that regarding x as an element of L_w,

$$\begin{aligned} |x|_w &= q_w^{-\mathrm{ord}_w(x)} = (q_v^{f_{w/v}})^{-e_{w/v}\mathrm{ord}_v(x)} \\ &= (q_v^{-\mathrm{ord}_v(x)})^{e_{w/v}f_{w/v}} = |x|_v^{[L_w:K_v]}. \end{aligned}$$

That is, for any $x \in K_v$,

$$|x|_w = |x|_v^{[L_w:K_v]} = |N_{L_w/K_v}(x)|_v.$$

So, we get two extentions of $|\cdot|_v$ on K_v to L_w defined by

$$|x|_w^{1/[L_w:K_v]} \text{ or } |N_{L_w/K_v}(x)|_v^{1/[L_w:K_v]}$$

for all $x \in L_w$. But [4], Lemma 2, p.411 says that there is only one extension of $|\cdot|_v$ on K_v to L_w. Thus, for any $x \in L_w$,

$$|x|_v = |x|_w^{1/[L_w:K_v]} = |N_{L_w/K_v}(x)|_v^{1/[L_w:K_v]}, \tag{2.2}$$

which is equivalent to (2.1). Furthermore, we have $|x|_w = |N_{L_w/K_v}(x)|_v$ for all $x \in L_w$. It follows that absolute values $|x|_v$ and $|x|_w$ on L_w are invariant under $\mathrm{Gal}(L_w/K_v)$.

We will denote $|L_w^\times|$ the value group of $L_w^\times$. Note that $|L_w^\times| = q_w^{\mathbb{Z}} \cup \{0\}$ and $|\mathbb{C}_v^\times| = q_v^{\mathbb{Q}} \cup \{0\}$.

We use the following notations for the *open* and *closed balls* in $\mathbb{C}_v$: for any $a \in \mathbb{C}_v$ and any positive real number r,

$$B(a,r)^- = \{z \in \mathbb{C}_v : |z-a|_v < r\}$$
$$B(a,r) = \{z \in \mathbb{C}_v : |z-a|_v \leq r\}. \tag{2.3}$$

Let $\partial B(a,r) := B(a,r) \setminus B(a,r)^- = \{z \in \mathbb{C}_v : |z-a|_v = r\}$ be the *boundary* of $B(a,r)$ or $B(a,r)^-$.

2.1 Power Series in One Variable

The *Newton polygon* of a polynomial $g(z) = \sum_{k=0}^n c_k z^k \in \mathbb{C}_v[z]$ is defined to be the lower convex hull of the set of points $\{(k, \mathrm{ord}_v(c_k))\}$, with a vertical side above the point corresponding to the first nonzero coefficient. Let $f(z) = \sum_{k=0}^\infty c_k z^k \in \mathbb{C}_v[[z]]$ be a power series. Define $f_n(z) = \sum_{k=0}^n c_k z^k$ to be the n^{th} partial sum of $f(z)$. The Newton polygon of $f(z)$ is then the limit of the Newton polygons of the $f_n(z)$.

In the nonarchimedean case, we have the following necessary and sufficient condition for convergence of a power series, which is not true for archimedean places.

Lemma 2.1 *Let* $f(z) = \sum_{k=0}^\infty c_k z^k \in \mathbb{C}_v[[z]]$ *be a power series. Then* $f(z)$ *converges at* $x \in \mathbb{C}_v$ *if and only if* $\lim_{k\to\infty} |c_k x^k|_v = 0$.

Proof: Since we are dealing with a nonarchimedean absolute value, a sequence is Cauchy if and only if the absolute value of the difference between adjacent terms approaches zero.

Noting that $\mathbb{C}_v$ is complete,

$$\begin{aligned}\sum_{k\geq 0} c_k x^k \quad \text{converges} \quad &\Longleftrightarrow \quad \{S_n := \sum_{k=0}^{n} c_k x^k\} \quad \text{converges}\\ &\Longleftrightarrow \quad \{S_n\} \quad \text{is Cauchy}\\ &\Longleftrightarrow \quad |S_n - S_{n-1}|_v \to 0\\ &\Longleftrightarrow \quad |c_n x^n|_v \to 0\end{aligned}$$

□

Corollary 2.2 *Let $r = |b|_v$ for some $b \in \mathbb{C}_v$. Then $f(z)$ converges on $B(0, r)$ if and only if $\lim_{k\to\infty} |c_k|_v r^k = 0$, or equivalently, $\lim_{k\to\infty} \mathrm{ord}_v(c_k b^k) = \infty$.*

Proof: For any $x \in B(0, r)$, Lemma 2.1 implies that

$$f(x) \quad \text{converges} \quad \Longleftrightarrow \quad |c_n x^n|_v \to 0. \tag{2.4}$$

Since $|x|_v \leq r = |b|_v$ and $f(z)$ converges for all $x \in B(0, r)$, the RHS of (2.4) is equivalent to $|c_n|_v r^n \to 0$, which is equivalent to $\mathrm{ord}_v(c_n b^n) \to \infty$. □

Lemma 2.3 *Suppose r and M belong to $|\mathbb{C}_v^\times|$. If $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$ is a nonzero power series converging on $B(0, r)$ with $\|f\|_{B(0,r)} \leq M$, then $|c_k|_v \leq \frac{M}{r^k}$ for each $k \geq 0$.*

Proof: Choose π and b in $\mathbb{C}_v^\times$ such that $r = |\pi|_w$ and $M = |b|_w$. After replacing $f(z)$ by $\frac{f(\pi z)}{b}$, we can assume that $r = M = 1$. We must show that $|c_k|_v \leq 1$ for each $k \geq 0$, whenever $\|f\|_{B(0,r)} \leq 1$.

Suppose that $|c_\ell|_v > 1$ for some ℓ. Since $f(z)$ converges on $B(0, 1)$, we have $\lim_{k\to\infty} |c_k|_v = 0$, and so there is the largest index ℓ_0 for which $|c_{\ell_0}|_v = \max_{k\leq 0} |c_k|_v$. Putting $M_0 := |c_{\ell_0}|_v$, we get $M_0 \geq |c_\ell|_v > 1$. Rewriting

$$f(z) = \sum_{|c_k|_v < M_0} c_k z^k + \sum_{|c_k|_v = M_0} c_k z^k,$$

put

$$g(z) := \sum_{|c_k|_v = M_0} c_k z^k,$$

which is a polynomial. We claim that there is a $z_0 \in B(0,1)$ such that $|f(z_0)|_v = M_0 > 1$, which is a contradiction. Letting $\widetilde{g}(z) = \frac{1}{c_{\ell_0}} g(z)$ and $\overline{k}_v = \widehat{\mathcal{O}_v}/\widehat{\mathfrak{m}_v}$, $\widetilde{g}(z) \in \widehat{\mathcal{O}_v}[z]$ is a monic polynomial with coefficients having absolute value 1, and so $\widetilde{g}(z) \pmod{\widehat{\mathfrak{m}_v}}$ is a monic polynomial in $\overline{k}_v[z]$, which has only finitely many roots in $\overline{k}_v$. Choose $z_0 \in B(0,1)$ so that $z_0 \pmod{\widehat{\mathfrak{m}_v}}$ is not a root of $\widetilde{g}(z) \pmod{\widehat{\mathfrak{m}_v}}$, and hence $\widetilde{g}(z_0) \notin \widehat{\mathfrak{m}_v}$. Thus we have $|\widetilde{g}(z_0)|_v = 1$ and so $|g(z_0)|_v = |c_{\ell_0}|_v = M_0$. Since $|\sum_{|c_k|_v < M_0} c_k z_0^k|_v < M_0$, we get $|f(z_0)|_v = |g(z_0)|_v = M_0$. □

Proposition 2.4 *Suppose that* $h(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \in \mathbb{C}_v[[z]]$ *converges on* $B(0,r)$ *for some* $r \in |\mathbb{C}_v^\times|$, *and that* $|b_k|_v \cdot r^k < 1$ *for all* $k \geq 1$. *Then the inverse* $h(z)^{-1} = 1 + \sum_{k=1}^{\infty} b'_k z^k$ *converges on* $B(0,r)$ *as well. Furthermore,* $|b'_k|_v \cdot r^k < 1$ *for all* $k \geq 1$ *and* $|h(z)|_v = |h(z)^{-1}|_v = 1$ *for all* $z \in B(0,r)$.

Proof: Since $|b_k|_v \cdot r^k < 1$ for all $k \geq 1$, we see that $|\sum_{k=1}^{\infty} b_k z^k|_v < 1$ for all $z \in B(0,r)$. So, we can expand

$$\begin{aligned} h(z)^{-1} &:= \frac{1}{1 + \sum_{k=1}^{\infty} b_k z^k} \\ &= 1 - \sum_{k=1}^{\infty} b_k z^k + (\sum_{k=1}^{\infty} b_k z^k)^2 - \cdots \\ &= 1 + \sum_{k=1}^{\infty} b'_k z^k, \end{aligned}$$

where b'_k is a homogeneous polynomial $P_k(b_1, \cdots, b_k)$ of weight k in $b_1, \cdots, b_k$ over $\mathbb{Z}$ if b_k is assigned weight k. Clearly, we obtain that for all $k \geq 1$,

$$|b'_k|_v \cdot r^k < 1. \tag{2.5}$$

Fix $\varepsilon > 0$. Since $\lim_{k \to \infty} |b_k|_v \cdot r^k = 0$, there is $k_0 \in \mathbb{Z}_{>0}$ such that $|b_k|_v \cdot r^k < \varepsilon$ for all $k \geq k_0$. There is also a number $\delta < 1$ such that $|b_k|_v \cdot r^k < \delta^k$ for $k = 1, \cdots, k_0 - 1$ because $|b_k|_v \cdot r^k < 1$ for all $k \geq 1$. Let A be a positive integer such that $\delta^A < \varepsilon$. Fix $m \in \mathbb{Z}_{>0}$ with $m \geq A$. If a term $a_{e_1, \cdots, e_m} b_1^{e_1} \cdots b_m^{e_m}$ in $P_m(b_1, \cdots, b_m)$ contains b_k for some $k \geq k_0$, then $|a_{e_1, \cdots, e_m} b_1^{e_1} \cdots b_m^{e_m}|_v \cdot r^m < (|b_k|_v \cdot r^k)^{e_k} < \varepsilon$ since $\sum_{i=1}^{m} i \cdot e_i = m$, $|a_{e_1, \cdots, e_m}|_v \leq 1$, and $|b_i|_v \cdot r^i < 1$

for all $i \geq 1$. Otherwise, we have $|a_{e_1,\cdots,e_m} b_1^{e_1} \cdots b_m^{e_m}|_v \cdot r^m = |a_{e_1,\cdots,e_m}|_v \cdot \prod_{i=1}^m (|b_i|_v \cdot r^i)^{e_i} < \delta^{\sum_{i=1}^m i \cdot e_i} = \delta^m \leq \delta^A < \varepsilon$. Hence, $|P_m(b_1, \cdots, b_m)|_v \cdot r^m < \varepsilon$. This implies that $1 + \sum_{k=1}^\infty d_k z^k$ converges on $B(0, r)$.

The last assertion immediately follows from the hypothesis on the b_k and (2.5). □

Definition 2.5 We will call a power series $h(z) \in \mathbb{C}_v[[z]]$ with the properties in Proposition 2.4 a *unit power series.*

Proposition 2.6 (Weierstrass Preparation Theorem) *Let* $f(z) \sum_{k=0}^\infty c_k z^k \in \mathbb{C}_v[[z]]$ *be a nonzero power series converging in* $B(0, r)$ *where* r *belongs to* $|\mathbb{C}_v^\times|$. *Then* $f(z)$ *has a finite number of roots in* $B(0, r)$, *there is a largest index* m *for which* $|c_m|_v r^m$ *is maximal, and* $f(z)$ *can be factored as*

$$f(z) = c_m \, g(z) \, h(z),$$

where $g(z) \in \mathbb{C}_v[z]$ *is a monic polynomial of degree* m *whose roots are those of* $f(z)$ *in* $B(0, r)$, *and* $h(z) \in \mathbb{C}_v[[z]]$ *is a unit power series converging in* $B(0, r)$. *These properties uniquely determine the factorization of* $f(z)$.

Furthermore, if $f(z)$ *belongs to* $L_w[[z]]$ *for some finite extension* L_w/K_v, *then* $c_m \in L_w$, $g(z)$ *belongs to* $L_w[z]$ *and* $h(z)$ *belongs to* $L_w[[z]]$.

Proof: See [22], Proposition 1.2.7, p.43. □

The following property is immediate from Proposition 2.6:

Corollary 2.7 *Let* L_w *be a finite extension of* K_v. *Suppose that* $f(z) = \sum_{k=0}^\infty c_k z^k \in L_w[[z]]$ *is a power series converging on* $B(0, r)$ *for* $r \in |L_w^\times|$. *If* $f(z)$ *has exactly one root in* $B(0, r)$, *then the root must belong to* L_w.

Corollary 2.8 (Refined Maximum Modulus Principle for Power Series) *If* $f(z) = \sum_{k \geq 0} c_k z^k \in \mathbb{C}_v[[z]]$ *is a nonzero power series converging in* $B(0, r)$ *for* $r \in |\mathbb{C}_v^\times|$, *then for any finite set* $\{a_1, \cdots, a_t\} \subset B(0, r)$, $|f(z)|_v$ *achieves its maximum value for* $z \in B(0, r)$ *at a point of* $B(0, r) \setminus \cup_{i=1}^t B(a_i, r)^-$.

Proof: After scaling $f(z)$, we can assume that $r = 1$. By the Weierstrass Preparation Theorem, there is a factorization $f(z) = c \cdot g(z) \cdot h(z)$ such that $g(z) = \prod_{j=1}^{m}(z - \alpha_j)$ is a monic polynomial of degree m whose roots are the same as those of $f(z)$ in $B(0,1)$, $h(z) \in \mathbb{C}_v[[z]]$ is a unit power series in $B(0,1)$, with $h(0) = 1$, and $|h(z)|_v = 1$ for all $z \in B(0,1)$, and $c = c_m$, where $|c_m|_v = \max\{|c_k|_v : k \geq 0\}$. Noting that $\widehat{\mathcal{O}_v}/\widehat{\mathfrak{m}_v}$ is infinite, we can choose a nonzero element $z_0 \in \widehat{\mathcal{O}_v}$ such that $z_0 \not\equiv a_i$ and $z_0 \not\equiv \alpha_j$ in $\widehat{\mathcal{O}_v}/\widehat{\mathfrak{m}_v}$ for all i and j. Clearly, $|z_0|_v = 1$, $|z_0 - a_i|_v = 1$, and $|z_0 - \alpha_j|_v = 1$ for all i and j. Hence $z_0 \in B(0,1) \setminus (\cup_{i=1}^{t} B(a_i,1)^- \cup \cup_{j=1}^{m} B(\alpha_j,1)^-)$. Given an arbitrary $z \in B(0,1)$, since $|h(z)|_v = 1$ and $|z - \alpha_j|_v \leq 1 = |z_0 - \alpha_j|_v$,

$$\begin{aligned} |f(z)|_v &= |c \cdot g(z)|_v = |c|_v \cdot \prod_{j=1}^{m} |z - \alpha_j|_v \\ &\leq |c|_v \cdot \prod_{j=1}^{m} |z_0 - \alpha_j|_v = |c \cdot g(z_0)|_v = |f(z_0)|_v. \end{aligned}$$

□

Remark 2.9 Let $f(z) \in \mathbb{C}_v[[z]]$ be a nonzero power series converging in $B(0,r)$, where r belongs to $|\mathbb{C}_v^\times|$. Then $f(z)$ has only finitely many zeros $\alpha_1, \cdots, \alpha_m$ in $B(0,r)$ by the Weierstrass Preparation Theorem. Corollary 2.8 says that we can choose $z_0 \in B(0,r)$ so that $|z_0 - \alpha_j|_v = r$ for all j.

Corollary 2.10 *Let $r = |b|_v$ for some $b \in \mathbb{C}_v^\times$. If $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$ converges on $B(0,r)$, then we have*

$$\|f\|_{B(0,r)} = \|f\|_{\partial B(0,r)} = \max_{k \geq 0} |c_k b^k|_v.$$

Proof: The first equality holds by the Refined Maximum Modulus Principle.

For the second equality, replace $f(z)$ by $f(z/b)$ and assume that $r = 1$. Then $\max_{k\geq 0} |c_k|_v$ exists by Corollary 2.1.2. Let m be the largest index for which $|c_m|_v = \max_{k\geq 0} |c_k|_v$. By the Weierstrass Preparation theorem, we can rewrite $f(z) = c_m \cdot g(z) \cdot h(z)$. Find $z_0 \in \mathbb{C}_v$ so that $|g(z_0)|_v = 1 = |h(z_0)|_v$ as in the proof of the Maximum Modulus Principle. Then we have

$|f(z_0)|_v = c_m$. For any $z \in \partial B(0,1)$, we see that $|f(z)|_v \leq \max_{k\geq 0} |c_k z^k|_v = \max_{k\geq 0} |c_k|_v = |c_m|_v$, which implies the second equality. $\square$

Lemma 2.11 ([25]) *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$ be a nonzero power series converging in $B(0,1)$. Write $f(z) = c_m\, g(z)\, h(z)$ as in the Weierstrass Preparation Theorem, where $g(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m \in \mathbb{C}_v[z]$ is monic, and $h(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \in \mathbb{C}_v[[z]]$ is a unit power series converging in $B(0,1)$. Then the Newton polygon of $c_m\, g(z)$ coincides with the part of the Newton polygon of $f(z)$ on and to the left of $(m, \mathrm{ord}_v(c_m))$.*

Proof: After dividing through by c_m, we can assume that $c_m = 1$ and that $\mathrm{ord}_v(c_k) \geq 0$ for all $k \geq 0$. Writing $h(z)^{-1} = 1 + \sum_{k=1}^{\infty} b'_k z^k$, we have that $\mathrm{ord}_v(b'_k) > 0$ for all $k \geq 1$. Since $g(z) = f(z)\, h(z)^{-1}$, we see that if J is the smallest index for which $c_J \neq 0$, then $a_k = c_k = 0$ for all $k < J$, while $a_J = c_J$. Noting that $\mathrm{ord}_v(c_k) \geq 0$ for all $k \geq 0$ and $\mathrm{ord}_v(b'_k) > 0$ for all $k \geq 1$, we see that for $J < k < m$,

$$\begin{aligned} \mathrm{ord}_v(a_k) &= \mathrm{ord}_v(c_k + c_{k-1} b'_1 + \cdots + c_0 b'_k) \\ &\geq \min(\mathrm{ord}_v(c_k),\ \mathrm{ord}_v(c_{k-1}), \cdots, \mathrm{ord}_v(c_0)). \end{aligned} \tag{2.6}$$

If $(k,\, \mathrm{ord}_v(c_k))$ is a corner of the Newton polygon of $f(z)$ for some $J < k < m$, then we have that $\mathrm{ord}_v(c_k) < \mathrm{ord}_v(c_j)$ for all $j = 0, \cdots, k-1$ because $\mathrm{ord}_v(c_k) \geq 0$ for all $k \geq 0$ and $\mathrm{ord}_v(c_m) = 0$. Hence (2.6) implies that $\mathrm{ord}_v(a_k) = \mathrm{ord}_v(c_k)$. Since $\mathrm{ord}_v(c_k) \geq 0$ for all $k \geq 0$, the Newton polygon of $g(z)$ lies on or below the initial part of the Newton polygon of $f(z)$. Applying the same argument to $f(z) = g(z)\, h(z)$, we see that the initial part of the Newton polygon of $f(z)$ lies on or below the Newton polygon of $g(z)$. $\square$

Corollary 2.12 ([25]) *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$ have the radius of convergence $R > 0$.*

(a) For each $0 < r < R$, $f(z)$ has finitely many zeros in $B(0,r)$. If R belongs to $|\mathbb{C}_v^\times|$ and $f(z)$ converges in $B(0,R)$, then $f(z)$ has finitely many zeros in $B(0,R)$.

(b) The zeros of $f(z)$ correspond to the sides of the Newton polygon of $f(z)$ of finite projection length in the same way as for a polynomial: for each side with slope m and projection length s, $f(z)$ has exactly s roots α (counted with multiplicity) such that $\mathrm{ord}_v(\alpha) = -m$.

Proof: (a) Without loss of generality, we can assume that r belongs to $|\mathbb{C}_v^\times|$. Take $\beta \in \mathbb{C}_v^\times$ with $|\beta|_v = r$. After replacing $f(z)$ with $f(z/\beta)$, we are reduced to the case of Lemma 2.11.

(b) If the Newton polygon has a side of finite length and slope m, then taking r so that $\log_v(r) = m$, choosing β as in a), and replacing $f(z)$ with $f(z/\beta)$, we are again reduced to Lemma 2.11. □

Remark 2.13 Let $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$ have the radius of convergence $0 < R \in |\mathbb{C}_v^\times|$. Then its Newton polygon has a terminal ray of slope $\log_v(R)$. For any $0 < r \leq R$, there is a unique line touching the Newton polygon of $f(z)$ with slope $\log_v(r)$. If $(n, \mathrm{ord}_v(a_n))$ is the rightmost point where the line touches the Newton polygon, then $f(z)$ has exactly n roots in $B(0, r)$.

Corollary 2.14 *Let R be the radius of convergence of $f(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$, and let $0 < r < R$ satisfy $r = |b|_v$ for some $b \in \mathbb{C}_v$. Let ℓ be the unique line touching the Newton polygon of $f(z)$ with slope $\log_v(r)$ at the rightmost corner $(n, \mathrm{ord}_v(a_n))$. Then the y-intercept of ℓ is $-\log_v(M)$, where $M = \|f\|_{B(0,r)}$.*

Proof: Note that the y-intercept of ℓ is $\log_v(r)\cdot(-n)+\mathrm{ord}_v(a_n)$, which is equal to $n\cdot\mathrm{ord}_v(b)+\mathrm{ord}_v(a_n) = \mathrm{ord}_v(a_n b^n)$. For each $m \neq n$, since $(m, \mathrm{ord}_v(a_m))$ is above or on the line ℓ, we have $\mathrm{ord}_v(a_m) \geq \log_v(r)\cdot(m-n)+\mathrm{ord}_v(a_n)$, i.e., $-\log_v(r)\cdot m+\mathrm{ord}_v(a_m) \geq -\log_v(r)\cdot n+\mathrm{ord}_v(a_n)$, which is $\mathrm{ord}_v(a_m) + m \cdot \mathrm{ord}_v(b) \geq \mathrm{ord}_v(a_n) + n \cdot \mathrm{ord}_v(b)$. Hence $\mathrm{ord}_v(a_m b^m) \geq \mathrm{ord}_v(a_n b^n)$ for all $m \neq n$, and so $M = |a_n b^n|_v$ by Corollary 2.10. Thus the y-intercept of ℓ is $-\log_v(M)$. □

Definition 2.15 We will call a power series $f(z) \in \mathbb{C}_v[[z]]$ a ρ/r-*isometry* between $B(0, r)$ and $B(a, \rho)$ if the map $f : B(0, r) \to B(a, \rho)$ is a $1-1$ correspondence, with $|f(x)-f(y)|_v = \rho/r \cdot |x-y|_v$ for all x, $y \in B(0, r)$.

Proposition 2.16 *Suppose that $f(z) = \sum_{k=0}^{\infty} c_k z^k \in L_w[[z]]$ is a power series converging on $B(0, r)$ for $r \in |L_w^\times|$, which gives an injective map $f : B(0, r) \to \mathbb{C}_v$. Then there are an $a \in L_w$ and $\rho \in |L_w^\times|$ such that f defines a ρ/r-isometry between $B(0, r)$ and $B(a, \rho)$ and*

$f(B(0,r)\cap L_w)=B(a,\rho)\cap L_w$. Furthermore, for any $\zeta\in B(a,\rho)\cap L_w$, $f(z)=\zeta$ has exactly one root in $B(0,r)$ belonging to L_w.

Proof: Clearly, $a:=f(0)=c_0\in L_w$. For any $z,t\in B(0,r)$, we see that

$$\begin{aligned}|f(z)-f(t)|_v &= |\sum_{k=0}^{\infty} c_k(z^k-t^k)|_v = |\sum_{k=1}^{\infty} c_k(z^k-t^k)|_v \\ &= |z-t|_v\cdot|c_1+c_2(z+t)+c_3(z^2+zt+t^2)+\cdots|_v.\end{aligned}$$

Since f is injective on $B(0,r)$, $f(z)=a$ has only one solution in $B(0,r)$ at $z=0$. That is, $F(z):=f(z)-a=\sum_{k=1}^{\infty}c_kz^k$ has only one zero in $B(0,r)$. This means that for all $k\geq 2$, all vertices $(k,\mathrm{ord}_v(c_k))$ of the Newton polygon of $F(z)$ lie above the line $\ell: y=\log_v(r)(x-1)+\mathrm{ord}_v(c_1)$, and so $\mathrm{ord}_v(c_k)>\log_v(r)\cdot(k-1)+\mathrm{ord}_v(c_1)$, which implies that $|c_k|_vr^{k-1}<|c_1|_v$ for all $k\geq 2$. It follows that for all $z,t\in B(0,r)$ and all $k\geq 2$, we have $|c_k(z^{k-1}+z^{k-1}t+\cdots+t^{k-1})|_v<|c_1|_v$. Hence we get $|f(z)-f(t)|_v=|c_1|_v\cdot|z-t|_v$.

Now, taking $t=0$ and letting z range over $B(0,r)$, we have $|f(z)-a|_v=|c_1|_v\cdot|z|_v$. Putting $\rho:=|c_1|_v\cdot r\in|L_w^\times|$, it follows that $f(B(0,r))\subset B(a,\rho)$.

Conversely, fix $\zeta\in B(a,\rho)$. Then $|\zeta-c_0|_v\leq|c_1|_v\cdot r$, or equivalently, $\mathrm{ord}_v(c_0-\zeta)\geq -\log_v(r)+\mathrm{ord}_v(c_1)$. Hence the point $(0,\mathrm{ord}_v(c_0-\zeta))$ lies on or above the line ℓ. Putting $F_\zeta(z):=f(z)-\zeta$, its constant term is $c_0-\zeta$ and other coefficients of $F_\zeta(z)$ are the same as those of $F(z)$. It follows that the Newton polygon of $F_\zeta(z)$ lies on or above the line ℓ and that $(1,\mathrm{ord}_v(c_1))$ is the last vertex on the line because $F(z)$ has only one zero in $B(0,r)$. Hence there is a unque solution to $f(z)=\zeta$ in $B(0,r)$. Therefore, we have $f(B(0,r))=B(a,\rho)$. Furthermore, since $|f(z)-f(t)|_v=\rho/r\cdot|z-t|_v$ for all $z,t\in B(0,r)$, $f(z)$ gives a ρ/r-isometry between $B(0,r)$ and $B(a,\rho)$.

If $x\in B(0,r)\cap L_w$, then since $f(z)$ is defined over L_w, we have $f(x)\in B(a,\rho)\cap L_w$. Conversely, if $\zeta\in B(a,\rho)\cap L_w$, then since f is bijective, Corollary 2.7 implies that the unique root of $f(z)=\zeta$ in $B(0,r)$ must lie in L_w. Thus, we have $f(B(0,r)\cap L_w)=B(a,\rho)\cap L_w$. □

Corollary 2.17 *Suppose that* $g(z) = \sum_{k=1}^{\infty} c_k z^k \in \mathbb{C}_v[[z]]$ *is a power series converging on* $B(0,r)$ *for* $r \in |\mathbb{C}_v^\times|$, *which has exactly one zero in* $B(0,r)$ *at* $z = 0$. *If* $\rho = |c_1|_v \cdot r$, *then* g *induces a* ρ/r*-isometry from* $B(0,r)$ *onto* $B(a,\rho)$.

Proof: In the proof of Proposition 2.16, we only used the fact that $f(z) = a$ has only one solution in $B(0,r)$ at $z = 0$. In our case, $a = 0$ and $g(z)$ has exactly one zero at $z = 0$ in $B(0,r)$. Hence the proof follows. □

Definition 2.18 Let $B(0,r)$ and $B(a,\rho)$ be balls in $\mathbb{C}_v$. We will say that a power series $\lambda(z) \in \mathbb{C}_v[[z]]$ gives a *parametrization* of $B(a,\rho)$ by $B(0,r)$ if $\lambda(z)$ converges on $B(0,r)$ and defines a $1-1$ correspondence between $B(0,r)$ and $B(a,\rho)$. We will call λ an *isometric parametrization* if $r = \rho$ and $|\lambda(z) - \lambda(x)|_v = |z - x|_v$ for all $z,\ x \in B(0,r)$.

Remark 2.19 If $\lambda : B(0,r) \to B(a,\rho)$ is a parametrization, it follows from Proposition 2.16 that for all $z,\ x \in B(0,r)$, $|\lambda(z) - \lambda(x)|_v = \rho/r \cdot |z - x|_v$. Furthermore, if $a \in L_w$, $r \in |L_w^\times|$ and $\lambda(z) \in L_w[[z]]$, then we have $\lambda(B(0,r) \cap L_w) = B(a,\rho) \cap L_w$.

We give some examples of isometric parametrizations.

Example 2.20 (i) For any $\varepsilon \in \widehat{\mathcal{O}_v}^\times$, the map $\lambda : B(0,r) \to B(a,r)$ given by $\lambda(z) = a + \varepsilon\, z$ is an isometric parametrization of $B(a,r)$ by $B(0,r)$. Thus an arbitrary ball $B(a,r) \subset \mathbb{C}_v$ is isometrically parametrizable.

(ii) With the same notations as in Proposition 2.16, if $|c|_v = r/\rho$ for some $c \in \mathbb{C}_v$, then the map $\lambda : B(0,\rho) \to B(a,\rho)$ given by $\lambda(z) = f(c\, z)$ defines an isometric parametrization of $B(a,\rho)$ by $B(0,\rho)$.

Proposition 2.21 (Expansion of algebraic functions) *Let* $B(a,\rho)$ *be a parametrizable ball with a parametrization* $\lambda : B(0,r) \to B(a,\rho)$. *Given* $f(z) \in \mathbb{C}_v(z)$, *let* $\alpha_1, \cdots, \alpha_m$ *be the zeros and poles of* $f(z)$ *in* $B(a,\rho)$, *and let* n_i *be the multiplicity of* α_i *in div*(f) *for each* i. *Let* $a_1, \cdots, a_m$ *be the points in* $B(0,r)$ *corresponding to* $\alpha_1, \cdots, \alpha_m$ *under* λ. *Then there* *exist a constant* $C \in \mathbb{C}_v$*, a rational function* $g(x) = \prod_{i=1}^{m}(x - a_i)^{n_i}$ *and a unit power series* $h(x) \in \mathbb{C}_v[[x]]$ *converging in* $B(0, r)$*, with*

$$f(\lambda(x)) = C \cdot g(x) \cdot h(x)$$

for all $x \in B(0, r)$*. These properties uniquely determine* C*,* $g(x)$ *and* $h(x)$*.*

It follows that there is a constant $C_0 \in |\mathbb{C}_v^\times|$ *such that*

$$|f(z)|_v = C_0 \cdot \prod_{i=1}^{m} |z - \alpha_i|_v^{n_i}$$

for all $z \in B(a, \rho)$*. In addition, if* $\lambda(z) \in L_w[[z]]$*,* $a \in L_w$*, and* $f(z)$ *belongs to* $L_w(z)$ *for some finite extension* L_w/K_v*, then* $C \in L_w$*,* $g(x)$ *belongs to* $L_w(x)$ *and* $h(x)$ *belongs to* $L_w[[x]]$*.*

Proof: Writing $f(z) = f_1(z)/f_2(z)$ for some $f_1(z),\ f_2(z) \in \mathbb{C}_v[z]$, the $f_i(\lambda(x))$ become power series over $\mathbb{C}_v$. Without loss of generality, we can assume that $f_1(x)$ and $f_2(x)$ have no common factors. By Proposition 2.6, there are a constant $C_i \in \mathbb{C}_v$, a monic polynomial $g_i(x) \in \mathbb{C}_v[x]$ and a unit power series $h_i(x) \in \mathbb{C}_v[[x]]$ in $B(0, r)$ such that $f_i(\lambda(x)) = C_i \cdot g_i(x) \cdot h_i(x)$, and $f_i(\lambda(x))$ and $g_i(x)$ have the same zeros in $B(0, r)$. Letting $C = C_1/C_2$, $g(x) = g_1(x)/g_2(x)$, and $h(x) = h_1(x)/h_2(x)$, we have $f(\lambda(x)) = C \cdot g(x) \cdot h(x)$. Clearly, $h(x)$ is a unit power series in $B(0, r)$. [22], Proposition 1.2.5, p.40 implies that $g(x)$ also has the form in the proposition. That is, $g(x) = \prod_{i=1}^{m}(x - a_i)^{n_i}$. The rationality assertion follows from Proposition 2.6. For the uniqueness, it is clear that C is unique because the $g_i(x)$ are monic polynomials and the $h_i(x)$ are unit power series. Now suppose that $C \cdot g(x) \cdot h(x) = C \cdot g'(x) \cdot h'(x)$ were two factorizations of $f(\lambda(x))$ of the form in the proposition. Multiplying through by $g_2(x) \cdot g_2'(x)$, we obtain two Weierstrass factorizations of the same power series. By the uniqueness assertion in Proposition 2.6, it follows that $h(x) = h'(x)$ and hence $g(x) = g'(x)$.

Finally, for all $z \in B(a,\rho)$, or equivalently, for the unique $x \in B(0,r)$ with $z = \lambda(x)$, since $|z-\alpha_i|_v = |\lambda(x)-\lambda(a_i)|_v = \rho/r \cdot |x-a_i|_v$,

$$\begin{aligned} |f(z)|_v &= |f(\lambda(x))|_v = |C|_v \cdot |g(x)|_v \cdot |h(x)|_v \\ &= |C|_v \cdot \prod_{i=1}^{m} |x-a_i|_v^{n_i} = |C|_v \cdot \prod_{i=1}^{m} \big(r/\rho \cdot |z-\alpha_i|_v\big)^{n_i} \\ &= |C|_v \cdot (r/\rho)^{\sum_{i=1}^m n_i} \cdot \prod_{i=1}^{m} |z-\alpha_i|_v^{n_i}, \end{aligned}$$

so we can take $C_0 = |C|_v \cdot (r/\rho)^{\sum_{i=1}^m n_i} \in |\mathbb{C}_v^\times|$. □

2.2 The Canonical Distance

The *spherical metric* on $\mathbb{P}^n(\mathbb{C}_v)$ is defined by

$$\|z,x\|_v = \frac{\max_{0\le i<j\le n} |z_i x_j - z_j x_i|_v}{(\max_{0\le i\le n} |z_i|_v)\cdot(\max_{0\le i\le n}|x_i|_v)} \tag{2.7}$$

if we write $z = (z_0 : \cdots : z_n)$ and $x = (x_0 : \cdots : x_n)$.

Let $\mathrm{GL}(n+1, \widehat{\mathcal{O}_v})$ be the set of all $(n+1)\times(n+1)$ matrices with entries in $\widehat{\mathcal{O}_v}$, and having determinant of absolute value 1. The spherical metric on $\mathbb{P}^n(\mathbb{C}_v)$ is invariant under $\mathrm{GL}(n+1,\widehat{\mathcal{O}_v})$. To see this, fix z, x in $\mathbb{P}^n(\mathbb{C}_v)$. After scaling the coefficients in their homogeneous representations, we can assume that $\max_{0\le i\le n}|z_i|_v = \max_{0\le i\le n}|x_i|_v = 1$. For any $A = (a_{ij})_{0\le i,j\le n} \in \mathrm{GL}(n+1,\widehat{\mathcal{O}_v})$, clearly $\max_{0\le i\le n}|(Az)_i|_v \le 1$. Suppose that $\max_{0\le i\le n}|(Az)_i|_v < 1$. Then we have $\max_{0\le i\le n}|(A^{-1}(Az))_i|_v < 1$. But $1 = \max_{0\le i\le n}|z_i|_v = \max_{0\le i\le n}|(A^{-1}(Az))_i|_v < 1$, which is a contradiction. Hence we must have $\max_{0\le i\le n}|(Az)_i|_v = \max_{0\le i\le n}|(Ax)_i|_v = 1$. We then have

$$\begin{aligned} \|Az,Ax\|_v &= \max_{0\le i<j\le n} |(Az)_i(Ax)_j - (Az)_j(Ax)_i|_v \\ &= \max_{0\le i<j\le n} \left| \Big(\sum_{k=0}^n a_{ik}z_k\Big)\cdot\Big(\sum_{m=0}^n a_{jm}x_m\Big) - \Big(\sum_{m=0}^n a_{jm}z_m\Big)\cdot\Big(\sum_{k=0}^n a_{ik}x_k\Big)\right|_v \\ &= \max_{0\le i<j\le n} \left| \sum_{0\le k<m\le n} (a_{ik}\cdot a_{jm})(z_k x_m - z_m x_k)\right|_v \\ &\le \max_{0\le k<m\le n} |z_k x_m - z_m x_k|_v = \|z,x\|_v. \end{aligned}$$

The symmetric argument gives the assertion. Therefore, we have

$$\|Az, Ax\|_v = \|z, x\|_v \tag{2.8}$$

for all $A \in \mathrm{GL}(n+1, \widehat{\mathcal{O}_v})$.

In this dissertation, we will be only working with $\mathbb{P}^1(\mathbb{C}_v)$. Henceforth, to simplify notations, we will identify $\mathbb{P}^1(\mathbb{C}_v)$ with $\mathbb{C}_v \cup \{\infty\}$. Letting z, $x \in \mathbb{C}_v$ correspond to $(z:1)$, $(x:1)$ in $\mathbb{P}^1(\mathbb{C}_v)$, we obtain that for all $z, x \in \mathbb{C}_v$,

$$\|z, x\|_v = \frac{|z-x|_v}{\max\{1, |z|_v\} \cdot \max\{1, |x|_v\}}, \tag{2.9}$$

while $\|z, \infty\|_v = 1/\max\{1, |z|_v\}$ if $z \neq \infty$. Clearly, for any $z \in \mathbb{P}^1(\mathbb{C}_v)$, $\|z, z\|_v = 0$.

We use the following notations for the *open* and *closed balls* in $\mathbb{P}^1(\mathbb{C}_v)$: for any $a \in \mathbb{P}^1(\mathbb{C}_v)$ and any positive real number r,

$$\mathfrak{B}(a, r)^- = \{z \in \mathbb{P}^1(\mathbb{C}_v) : \|z, a\|_v < r\}$$
$$\mathfrak{B}(a, r) = \{z \in \mathbb{P}^1(\mathbb{C}_v) : \|z, a\|_v \leq r\}.$$

Let $\partial\mathfrak{B}(a, r) := \mathfrak{B}(a, r) \setminus \mathfrak{B}(a, r)^- = \{z \in \mathbb{P}^1(\mathbb{C}_v) : \|z, a\|_v = r\}$ be the *boundary* of $\mathfrak{B}(a, r)$ or $\mathfrak{B}(a, r)^-$.

We have the following correspondence between the balls in $\mathbb{C}_v$ and those in $\mathbb{P}^1(\mathbb{C}_v)$:

Proposition 2.22 *Given $a \in \mathbb{C}_v$ and $r > 0$,*

(i) if $|a|_v \leq 1$ and $r < 1$, then $B(a, r) = \mathfrak{B}(a, r)$;

(ii) if $|a|_v \leq 1$ and $r \geq 1$, then $B(a, r) = B(0, r) = \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{B}(\infty, 1/r)^-$;

(iii) if $|a|_v > 1$ and $r < |a|_v$, then $B(a, r) = \mathfrak{B}(a, r/|a|_v^2)$;

(iv) if $|a|_v > 1$ and $r \geq |a|_v$, then $B(a, r) = B(0, r) = \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{B}(\infty, 1/r)^-$.

Proof: For the second and last assertions, we have $|z-a|_v \leq r \Leftrightarrow |z|_v \leq r \Leftrightarrow \max\{1, |z|_v\} \leq r \Leftrightarrow 1/\max\{1, |z|_v\} \geq 1/r \Leftrightarrow \|z, \infty\|_v \geq 1/r$, so we are done.

For the first assertion, suppose that $|a|_v \leq 1$ and $r < 1$. If $|z-a|_v \leq r$, then $|z|_v \leq 1$, so $\|z, a\|_v = |z-a|_v \leq r$. Conversely, assume that $\|z, a\|_v \leq r$. We claim that $|z|_v \leq 1$. If not, we see that $|z-a|_v = |z|_v > 1$ and $\max\{1, |z|_v\} = |z|_v$. So, $\|z, a\|_v = 1 > r$, which is a contradiction. Hence $|z|_v \leq 1$. This implies that $|z-a|_v = \|z, a\|_v \leq r$.

Finally, to prove the third assertion, suppose that $|a|_v > 1$ and $r < |a|_v$. If $|z-a|_v \leq r$, then $|z|_v = |a|_v$, so $\|z, a\|_v = |z-a|_v/|a|_v^2 \leq r/|a|_v^2$. Conversely, assume that $\|z, a\|_v \leq r/|a|_v^2$. We claim that $\max\{1, |z|_v\} \leq |a|_v$. If not, then $|a|_v < \max\{1, |z|_v\} = |z|_v$, so $|z-a|_v = |z|_v$. Hence $\|z, a\|_v = |z-a|_v/(\max\{1, |z|_v\} \cdot |a|_v) = 1/|a|_v$, which is a contradiction because $\|z, a\|_v \leq r/|a|_v^2 < 1/|a|_v$. We thus have $\max\{1, |z|_v\} \leq |a|_v$. Since $\|z, a\|_v \leq r/|a|_v^2$, $|z-a|_v/\max\{1, |z|_v\} \leq r/|a|_v$, and hence $|z-a|_v \leq r \cdot \max\{1, |z|_v\}/|a|_v \leq r$. □

Remark 2.23 By the same argument as Proposition 2.22, if $|a|_v \leq 1$ and $r \leq 1$, then $B(a, r)^- = \mathfrak{B}(a, r)^-$; if $|a|_v \leq 1$ and $r > 1$, then $B(a, r)^- = B(0, r)^- = \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{B}(\infty, 1/r)$; if $|a|_v > 1$ and $r \leq |a|_v$, then $B(a, r)^- = \mathfrak{B}(a, r/|a|_v^2)^-$; if $|a|_v > 1$ and $r > |a|_v$, then $B(a, r)^- = B(0, r)^- = \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{B}(\infty, 1/r)$.

Definition 2.24 A ball $\mathfrak{B}(a, \rho) \subset \mathbb{P}^1(\mathbb{C}_v)$ is said to be *parametrized* by a power series $\lambda(z) \in \mathbb{C}_v[[z]]$ if there is an affine patch $\mathbb{A}^1$ and a ball $B(A, R) \subset \mathbb{C}_v$ such that $\mathfrak{B}(a, \rho)$ corresponds to $B(A, R)$ in that patch, and the map $\lambda : B(0, r) \to B(A, R)$ is a parametrization; and λ is an isometric parametrization if $r = R$.

Remark 2.25 Proposition 2.22 shows that in appropriate affine coordinates, any ball in $\mathbb{P}^1(\mathbb{C}_v)$ corresponds to a ball in $\mathbb{C}_v$. It follows from Example 2.20 that any ball $\mathfrak{B}(a, \rho) \subset \mathbb{P}^1(\mathbb{C}_v)$ is isometrically parametrizable.

In fact, for any ball $\mathfrak{B}(a, \rho) \subset \mathbb{P}^1(\mathbb{C}_v)$, we necessarily have $\rho < 1$ and it is always possible to choose affine coordinates in which there is an isometric parametrization $\lambda : B(0, \rho) \to \mathfrak{B}(a, \rho)$ defined by $\lambda(z) = a + z$ (see [22], Corollary 1.2.4, p.39).

The following proposition is an analogue of Proposition 2.21.

Proposition 2.26 (Expansion of algebraic functions) *Let $\mathfrak{B}(a, \rho)$ be a parametrizable ball with a parametrization $\lambda : B(0, r) \to \mathfrak{B}(a, \rho)$. Given $f(z) \in \mathbb{C}_v(z)$, let $\alpha_1, \cdots, \alpha_m$ be* *the zeros and poles of $f(z)$ in $\mathfrak{B}(a,\rho)$, and let n_i be the multiplicity of α_i in div(f) for each i. Let $a_1, \cdots, a_m$ be the points in $B(0,r)$ corresponding to $\alpha_1, \cdots, \alpha_m$ under λ. Then there exist a constant $C \in \mathbb{C}_v$, a rational function $g(x) = \prod_{i=1}^m (x-a_i)^{n_i}$ and a unit power series $h(x) \in \mathbb{C}_v[[x]]$ converging in $B(0,r)$, with*

$$f(\lambda(x)) = C \cdot g(x) \cdot h(x)$$

for all $x \in B(0,r)$. These properties uniquely determine C, $g(x)$ and $h(x)$.

Furthermore, it follows that there is a constant $C_0 \in |\mathbb{C}_v^\times|$ such that

$$|f(z)|_v = C_0 \cdot \prod_{i=1}^m \|z,\alpha_i\|_v^{n_i}$$

for all $z \in \mathfrak{B}(a,\rho)$. In addition, if $\lambda(z) \in L_w[[z]]$, $a \in L_w$, and $f(z)$ belongs to $L_w(z)$ for some finite extension L_w/K_v, then $C \in L_w$, $g(x)$ belongs to $L_w(x)$ and $h(x)$ belongs to $L_w[[x]]$.

Proof: This follows from Proposition 2.21 and the definition of a parametrization. □

Fix a point $\zeta \in \mathbb{P}^1(\mathbb{C}_v)$. We will call any function of the form

$$[z,x]_\zeta := C_\zeta \cdot \frac{\|z,x\|_v}{\|z,\zeta\|_v \cdot \|x,\zeta\|_v}, \tag{2.10}$$

with $C_\zeta > 0$, a *canonical distance* on $\mathbb{P}^1(\mathbb{C}_v)$ with respect to ζ. This is a distance function on $\mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$ modelled on the usual distance $|z-x|_v$ on $\mathbb{C}_v = \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$. To see this, choose $A \in \mathrm{GL}(2, \widehat{\mathcal{O}_v})$ with $A\zeta = \infty$. That is, Az is of the form $C' \cdot (z-a)/(z-\zeta)$ for an appropriate nonzero constant $C' \in \mathbb{C}_v$, so that Az has a singularity at ζ. A simple computation shows that for all z, $x \neq \zeta$, ∞,

$$|Az - Ax|_v = C'' \cdot \frac{|z-x|_v}{|z-\zeta|_v \cdot |x-\zeta|_v}, \tag{2.11}$$

for an appropriate nonzero constant $C'' \in |\mathbb{C}_v^\times|$. Comparing (2.11) with (2.9), one sees that if $C''' = C'' \cdot \max\{1, |\zeta|_v\}^2 \in |\mathbb{C}_v^\times|$, then for all z, $x \neq \zeta$, ∞,

$$|Az - Ax|_v = C''' \cdot \frac{\|z,x\|_v}{\|z,\zeta\|_v \cdot \|x,\zeta\|_v}. \tag{2.12}$$

Furthermore, it follows from (2.8) that for any $B \in \mathrm{GL}(2, \widehat{\mathcal{O}_v})$,

$$[Bz, Bx]_{B\zeta} = [z, x]_\zeta. \tag{2.13}$$

Definition 2.27 Given a point $\zeta \in \mathbb{P}^1(\mathbb{C}_v)$, a function $g_\zeta(z) \in \mathbb{C}_v(z)$ is a *uniformizing parameter* at ζ if it has a simple zero at ζ.

Lemma 2.28 *Given a uniformizing parameter $g_\zeta(z) \in \mathbb{C}_v(z)$, there is a constant $C \in |\mathbb{C}_v^\times|$ such that $\|z, \zeta\|_v = C \cdot |g_\zeta(z)|_v$ for all z sufficiently near ζ.*

Proof: Put $\mathrm{div}(g_\zeta) := (\zeta) + \sum_{a_i \neq \zeta} n_i(a_i)$. Without loss of generality, we can assume that $\zeta = 0$ and the a_i are not ∞ by choosing an appropriate element $A \in \mathrm{GL}_2(\widehat{\mathcal{O}_v})$ to change coordinates and using (2.8). Then, we can write $g_0(z) = C' \cdot z \cdot \prod_{a_i \neq 0}(z - a_i)^{n_i}$ for some $C' \in \mathbb{C}_v$. Choose $r \in |\mathbb{C}_v^\times|$ so that $r < 1$ and $r < |a_i|_v$ for all i. Then $\mathfrak{B}(0, r) = B(0, r)$ by Proposition 2.22. For any $z \in B(0, r)$, $|z - a_i|_v = |a_i|_v$ by the choice of r. Letting $C = |C'|_v \cdot \prod_{a_i \neq 0} |a_i|_v^{n_i} \in |\mathbb{C}_v^\times|$, it follows that $|g_0(z)|_v = C \cdot |z|_v$. Since $|z - 0|_v = \|z, 0\|_v$, we have $|g_0(z)|_v = C \cdot \|z, 0\|_v$. $\square$

Proposition 2.29 *Let L/K be a finite extension and let $g_\zeta(z) \in L(z)$ be a uniformizing parameter at $\zeta \in \mathbb{P}^1(L)$. Then, for all but finitely many places v of K, we have $|g_\zeta(z)|_v = \|z, \zeta\|_v$ for all $z \in \mathfrak{B}(\zeta, 1)^-$.*

Proof: Put $\mathrm{div}(g_\zeta) := (\zeta) + \sum_{a_i \neq \zeta} n_i(a_i)$. Without loss of generality, we can assume that ζ and the a_i are not ∞. Otherwise, choose $A \in \mathrm{GL}_2(L)$ so that $A\zeta$ and the Aa_i are not ∞. There is a finite set S_0 of places of L such that $|\det(A)|_w \neq 1$, and all the entries of A belong to $\mathcal{O}_w$. Thus for all $w \notin S_0$, $A \in \mathrm{GL}_2(\mathcal{O}_w)$. Note that A preserves the spherical metric $\|\cdot, \cdot\|_w$ for all $w \notin S_0$ by (2.8). Write $g_\zeta(z) = C \cdot (z - \zeta) \cdot \prod_{a_i \neq \zeta}(z - a_i)^{n_i}$. Since $g_\zeta(z) \in L(z)$, it follows from Proposition 2.6 that C belongs to L. Let S' be the finite set of places w of L containing S_0 and all places w of L, where $|\zeta - a_i|_w \neq 1$, $|C|_w \neq 1$, or $|\zeta|_w > 1$. Fix $w \notin S'$. We have $\mathfrak{B}(\zeta, 1)^- = B(\zeta, 1)^-$ by Remark 2.23. For all $z \in \mathfrak{B}(\zeta, 1)^-$, since $|\zeta - a_i|_w = 1$, we see that $|z - a_i|_w = |z - \zeta + \zeta - a_i|_w = |\zeta - a_i|_w = 1$. Noting that $|C|_w = 1$, we have

$$|g_\zeta(z)|_w = |C|_w \cdot |z - \zeta|_w \cdot \prod_{a_i \neq \zeta} |z - a_i|_w^{n_i} = |z - \zeta|_w. \tag{2.14}$$

Furthermore, $|z|_w \le 1$ because $|\zeta|_w \le 1$ and $|z-\zeta|_w \le 1$. Now, let S be the set of all places of K such that for any $v \in S$, there is some $w \in S'$ lying above v. Clearly, S is a finite set. Noting that $|x|_w = |x|_v^{[L_w:K_v]}$, $|\zeta|_v \le 1$ and $|z|_v \le 1$ for all $v \notin S$. It follows from (2.14) that $|g_\zeta(z)|_v = \|z,\zeta\|_v$ for all $v \notin S$ and all $z \in \mathfrak{B}(\zeta,1)^-$. □

Lemma 2.30 *For any uniformizing parameter $g_\zeta(z) \in \mathbb{C}_v(z)$ at ζ and any choice of a canonical distance $[z,x]_\zeta$, there is a constant $C > 0$ such that*

$$\lim_{z\to\zeta} [z,x]_\zeta \cdot |g_\zeta(z)|_v = C.$$

Proof: We are given a canonical distance

$$[z,x]_\zeta = C_\zeta \cdot \frac{\|z,x\|_v}{\|z,\zeta\|_v \cdot \|x,\zeta\|_v},$$

for some constant $C_\zeta > 0$. Given a uniformizing parameter $g_\zeta(z)$, there is a constant $C' \in |\mathbb{C}_v^\times|$ such that $\|z,\zeta\|_v = C' \cdot |g_\zeta(z)|_v$ for all z sufficiently near ζ by Lemma 2.28. For any $x \ne \zeta$, letting $z \to \zeta$,

$$[z,x]_\zeta \cdot |g_\zeta(z)|_v = C_\zeta \cdot \frac{\|z,x\|_v}{\|z,\zeta\|_v \cdot \|x,\zeta\|_v} \cdot \frac{1}{C'} \cdot \|z,\zeta\|_v = \frac{C_\zeta}{C'} \cdot \frac{\|z,x\|_v}{\|x,\zeta\|_v} \to \frac{C_\zeta}{C'} =: C.$$

□

Frequently, it will be important to normalize the canonical distance, fixing a choice of C_ζ; this is most naturally done by fixing a uniformizing parameter $g_\zeta(z) \in \mathbb{C}_v(z)$ and requiring that for all $x \ne \zeta$,

$$\lim_{z\to\zeta} [z,x]_\zeta \cdot |g_\zeta(z)|_v = 1. \tag{2.15}$$

From now on, whenever we encounter the canonical distance $[z,x]_\zeta$, we will assume that it has been normalized as in (2.15), with respect to a chosen uniformizing parameter $g_\zeta(z) \in \mathbb{C}_v(z)$ at ζ. We will call this canonical distance the *(normalized) canonical distance* on $\mathbb{P}^1(\mathbb{C}_v)$ with respect to ζ. As a special case, if $\zeta = \infty$ and $g_\zeta(z) = 1/z$, then we can easily see that (2.15) yields

$$[z,x]_\infty = |z-x|_v.$$

The canonical distance has the following properties.

Proposition 2.31 *Fix* $\zeta \in \mathbb{P}^1(\mathbb{C}_v)$.

(i) (Factorization) *Suppose that* $f(z) \in \mathbb{C}_v(z)$ *has nonzero divisor* $div(f) = \sum n_i(a_i)$. *There is a constant* $C \in |\mathbb{C}_v^\times|$ *such that* $|f(z)|_v = C \cdot \prod_{a_i \neq \zeta}[z, a_i]_\zeta^{n_i}$ *for all* $z \neq \zeta$, *which are not poles of* $f(z)$.

(ii) (Galois Invariance) *If* $\zeta \in \mathbb{P}^1(K_v)$, *then for all* z, $[\sigma(z), \sigma(x)]_\zeta = [z, x]_\zeta$ *for all* $\sigma \in Gal_c(\mathbb{C}_v/K_v)$.

(iii) (Continuity) $[z, x]_\zeta$ *is jointly continuous in* z *and* x, *for all* $z, x \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$.

(iv) $[z, x]_\zeta$ *has a singularity at* ζ *and for all* $x,\ x' \neq \zeta$,

$$\lim_{z \to \zeta} \frac{[z, x]_\zeta}{[z, x']_\zeta} = 1.$$

Proof: In [22], Theorem 2.1.1, p.57, the properties (i), (ii) and (iii) are proved for the canonical distance on curves of arbitrary genus. On $\mathbb{P}^1(\mathbb{C}_v)$, (ii), (iii) and (iv) are elementary, and (i) follows from a simple computation. Indeed, write $f(z) = C' \prod_{a_i \neq \infty}(z - a_i)^{n_i}$ for some nonzero constant $C' \in \mathbb{C}_v$. First, suppose that $\zeta = \infty$. Since $[z, x]_\infty = |z - x|_v$, we are done. If $\zeta \neq \infty$, choose $A \in \mathrm{GL}(2, \widehat{\mathcal{O}_v})$ with $A\zeta = \infty$. Letting $g(z) = f(A^{-1}z)$, its divisor is $\sum_i (Aa_i)^{n_i}$, so there is a constant C such that $|g(z)|_v = C \prod_{Aa_i \neq A\zeta}[z, Aa_i]_{A\zeta}^{n_i}$ for all $z \neq A\zeta$. Since $f(z) = g(Az)$, it follows from (2.13) that for all $z \neq \zeta$, $|f(z)|_v = |g(Az)|_v = C \prod_{Aa_i \neq A\zeta}[Az, Aa_i]_{A\zeta}^{n_i} = \prod_{a_i \neq \zeta}[z, a_i]_\zeta^{n_i}$. □

Lemma 2.32 *Let* $\mathfrak{B}(a, r)$ *and* $\mathfrak{B}(a', r')$ *be disjoint isometrically parametrizable balls in* $\mathbb{P}^1(\mathbb{C}_v)$, *not containing* ζ.

(i) $[z, x]_\zeta$ *is constant on* $\mathfrak{B}(a, r) \times \mathfrak{B}(a', r')$.

(ii) If L_w/K_v *is a finite extension containing* a *and* ζ, *then there is a constant* $C_a \in |L_w^\times|$, *depending only on the point* a, *such that* $[z, x]_\zeta = C_a \|z, x\|_v$ *for all* $z,\ x \in \mathfrak{B}(a, r)$.

Proof: (i) Fix second coordinate x. If an algebraic function $f(z) \in \mathbb{C}_v(z)$ has no zeros and poles in $\mathfrak{B}(a, r)$, Proposition 2.26 says that $|f(z)|_v$ is constant for all $z \in \mathfrak{B}(a, r)$.

Considering $f(z) = \frac{z-x}{(z-\zeta)(x-\zeta)}$, $f(z)$ has no zeros and poles in $\mathfrak{B}(a,r)$, and so $|f(z)|_v = [z,x]_\zeta \cdot \max\{1, |\zeta|_v\}^{-2}$ is constant for all $z \in \mathfrak{B}(a,r)$. Thus, $[z,x]_\zeta$ is constant for all $z \in \mathfrak{B}(a,r)$.

With the symmetric argument, we have the assertion.

(ii) Fix $x \in \mathfrak{B}(a,r)$. Letting $f_x(z) = \frac{z-x}{(z-\zeta)(x-\zeta)}$, $f_x(z)$ has only one zero at x and no poles in $\mathfrak{B}(a,r)$. By Proposition 2.26, there is a constant B_x such that $|f(z)|_v = B_x \cdot \|z,x\|_v$ for all $z \in \mathfrak{B}(a,r)$. It follows that $[z,x]_\zeta = \max\{1,|\zeta|_v\}^2 \cdot |f(z)|_v = C_x \cdot \|z,x\|_v$, where $C_x := B_x \cdot \max\{1,|\zeta|_v\}^2$. We claim that $C_x = C_a$. Indeed, we see that by symmetry, $C_x \cdot \|a,x\|_v = [a,x]_\zeta = [x,a]_\zeta = C_a \cdot \|x,a\|_v = C_a \cdot \|a,x\|_v$.

Finally, since $f_a(z) = \frac{z-a}{(z-\zeta)(a-\zeta)}$ is defined over L_w, we have $C_a \in |L_w^\times|$ by Proposition 2.26. □

Remark 2.33 We can express Lemma 2.32 in terms of the absolute value and the balls in $\mathbb{C}_v$. Let $B(a,r)$ and $B(a',r')$ be disjoint isometrically parametrizable balls in $\mathbb{C}_v$, not containing ζ. We then have the following:

(i) $[z,x]_\zeta$ is constant on $B(a,r) \times B(a',r')$.

(ii) If L_w/K_v is a finite extension containing a and ζ, then there is a constant $C_a \in |L_w^\times|$, depending only on the point a, such that $[z,x]_\zeta = C_a \cdot |z-x|_v$ for all $z,\ x \in B(a,r)$.

Chapter 3

Capacity Theory

3.1 Local Capacity Theory

Let $\zeta \in \mathbb{P}^1(\mathbb{C}_v)$ be given and fix a uniformizing parameter $g_\zeta(z)$ at ζ, which determines the normalization of the canonical distance as in (2.15).

First, let E_v be a compact set in $\mathbb{P}^1(\mathbb{C}_v)$, not containing ζ. Following [22], the *Robin constant* and the *(local) capacity* are defined by

$$V_\zeta(E_v) = \inf_\nu \int\int_{E_v \times E_v} -\log_v([z,x]_\zeta) d\nu(z) d\nu(x) \tag{3.1}$$

$$\gamma_\zeta(E_v) = q_v^{-V_\zeta(E_v)}. \tag{3.2}$$

Here, ν runs over all probability measures supported on E_v. If $\gamma_\zeta(E_v) > 0$, [22], Theorem 4.1.22, p.211 says that there are a unique measure μ_ζ, called the *equilibrium distribution* of E_v with respect to ζ, and the *potential function* of E_v such that

$$V_\zeta(E_v) = \int\int_{E_v \times E_v} -\log_v([z,x]_\zeta) d\mu_\zeta(z) d\mu_\zeta(x), \tag{3.3}$$

$$u_\zeta(z; E_v) = \int_{E_v} -\log_v([z,x]_\zeta) d\mu_\zeta(x). \tag{3.4}$$

It follows from [22], Lemma 4.1.9, p.l93 that $u_\zeta(z; E_v)$ is continuous for $z \notin E_v$ and lower semicontinuous for all $z \neq \zeta$. Moreover, we have the following properties ([22], Theorem 4.1.6 (Maria's Theorem), p.191 and Theorem 4.1.11 (Frostman's Theorem), p.195):

(i) $u_\zeta(z; E_v) \leq V_\zeta(E_v)$ for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$,

(ii) the equality holds on E_v, except possibly a set of capacity 0, which is a union of compact sets, and

(iii) $<$ holds on $\mathbb{P}^1(\mathbb{C}_v) \setminus E_v$.

Remark 3.1 If E_v is a compact set, then E_v is contained in a finite union of balls $\mathfrak{B}(a_i, r_i)$ and these balls can be chosen with arbitrary small $r_i \in |\mathbb{C}_v^\times|$ for each i so that they are disjoint from each other. Fix two distinct points $\zeta,\ \zeta'$, not contained in E_v. Lemma 2.32 says that $[z,x]_\zeta/[z,x]_{\zeta'}$ is constant for all $z \in \mathfrak{B}(a_i, r_i)$ and $x \in \mathfrak{B}(a_j, r_j)$. Hence $-\log_v([z,x]_\zeta)$ and $-\log_v([z,x]_{\zeta'})$ differ by a constant on $\mathfrak{B}(a_i, r_i) \times \mathfrak{B}(a_j, r_j)$. It follows that $V_\zeta(E_v) = \infty \Leftrightarrow V_{\zeta'}(E_v) = \infty$. Thus, the property that $\gamma_\zeta(E_v) = 0$ or $\gamma_\zeta(E_v) > 0$ is independent of $\zeta \notin E_v$, and we can speak without ambiguity of "sets of capacity 0" or "sets of positive capacity".

For an arbitrary set $H_v \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$, the capacity of H_v with respect to ζ will be defined by

$$\gamma_\zeta(H_v) = \sup_{\substack{E_v \subset H_v \\ E_v \text{ compact}}} \gamma_\zeta(E_v), \tag{3.5}$$

equivalently,

$$V_\zeta(H_v) = \inf_{\substack{E_v \subset H_v \\ E_v \text{ compact}}} V_\zeta(E_v). \tag{3.6}$$

Note that for any two sets $H_v \subset H_v'$, since any probability measure on H_v is also a probability measure on H_v', it follows that $V_\zeta(H_v) \geq V_\zeta(H_v')$, or equivalently, $\gamma_\zeta(H_v) \leq \gamma_\zeta(H_v')$ for all $\zeta \notin H_v'$.

We will now define the Green's functions $G(z, \zeta; H_v)$ for arbitrary sets $H_v \subset \mathbb{P}^1(\mathbb{C}_v)$, and for $\zeta \notin H_v$, $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$. First, assume that $H_v = E_v$ is compact not containing ζ. If E_v has positive capacity, then the *Green's function* of E_v is defined by

$$G(z, \zeta; E_v) = \begin{cases} V_\zeta(E_v) - u_\zeta(z; E_v) & \text{if } z \neq \zeta \\ \infty & \text{if } z = \zeta. \end{cases} \tag{3.7}$$

If E_v has capacity 0, we put

$$G(z,\zeta;E_v) = \begin{cases} \infty & \text{if } z \notin E_v \\ 0 & \text{if } z \in E_v. \end{cases}$$

If ζ belongs to E_v, we put $G(z,\zeta;E_v) = 0$ for all z.

For any subset H_v in $\mathbb{P}^1(\mathbb{C}_v)$ not containing ζ, define the Green's function by

$$G(z,\zeta;H_v) = \inf_{\substack{E_v \subset H_v \\ E_v \text{ compact}}} G(z,\zeta;E_v). \tag{3.8}$$

If there is a compact subset in H_v with positive capacity, we say that H_v has positive capacity. In this case, $G(z,\zeta;H_v)$ is finite for all $z,\ \zeta \notin H_v$ with $z \neq \zeta$. Otherwise, H_v has capacity 0. In this case, $G(z,\zeta;H_v) \equiv \infty$ for all $z,\ \zeta \notin H_v$. Note that for any two sets $H_v \subset H_v'$, it follows from [22], Theorem 4.4.1, p.278 and Theorem 4.4.14, p.299 that $G(z,\zeta;H_v) \geq G(z,\zeta;H_v')$ and $G(z,\zeta;H_v) = G(\zeta,z;H_v)$ for all $z,\ \zeta$.

Suppose that H_v is a set in $\mathbb{P}^1(\mathbb{C}_v)$, having positive capacity. For any $\zeta \notin \overline{H}_v$, there is a ball $\mathfrak{B}(\zeta,r)$ with $\mathfrak{B}(\zeta,r) \cap H_v = \emptyset$. Let $g_\zeta(z) \in \mathbb{C}_v(z)$ be a uniformizing parameter at ζ. For any compact subset E_v of H_v with positive capacity, if follows from (3.7) that $G(z,\zeta;E_v) + u_\zeta(z;E_v) = V_\zeta(E_v)$ for all $z \neq \zeta$. [22], Proposition 4.1.5, p.188 implies that if r is sufficiently small, then $u_\zeta(z;E_v) = \log_v(|g_\zeta(z)|_v)$ for all $z \in \mathfrak{B}(\zeta,r) \setminus \{\zeta\}$. Hence, $G(z,\zeta;E_v) + \log_v(|g_\zeta(z)|_v) = V_\zeta(E_v)$ for all $z \in \mathfrak{B}(\zeta,r) \setminus \{\zeta\}$. By (3.8) and (3.6), we see that for all $z \in \mathfrak{B}(\zeta,r) \setminus \{\zeta\}$,

$$\begin{aligned} G(z,\zeta;H_v) + \log_v(|g_\zeta(z)|_v) &= \Big(\inf_{\substack{E_v \subset H_v \\ E_v \text{ compact}}} G(z,\zeta;E_v)\Big) + \log_v(|g_\zeta(z)|_v) \\ &= \inf_{\substack{E_v \subset H_v \\ E_v \text{ compact}}} \big(G(z,\zeta;E_v) + \log_v(|g_\zeta(z)|_v)\big) \\ &= \inf_{\substack{E_v \subset H_v \\ E_v \text{ compact}}} V_\zeta(E_v) = V_\zeta(H_v). \end{aligned}$$

Therefore, the Robin constant satisfies

$$V_\zeta(H_v) = \lim_{z \to \zeta}(G(z,\zeta;H_v) + \log_v(|g_\zeta(z)|_v)). \tag{3.9}$$

Remark 3.2 For any set $H_v \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$, the capacity $\gamma_\zeta(H_v)$ and the Green's function $G(z,\zeta;H_v)$ of H_v with respect to ζ as defined in (3.5) and (3.8) are called the *inner capacity* $\underline{\gamma}_\zeta(H_v)$ and the *upper Green's function* $\overline{G}(z,\zeta;H_v)$ in [22], respectively.

Definition 3.3 An *RL-domain* is a subset D_v of $\mathbb{P}^1(\mathbb{C}_v)$ of the form

$$D_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le R\},$$

for some $f(z) \in \mathbb{C}_v(z)$ and some $R \in |\mathbb{C}_v^\times|$. If $f(z)$ has poles at only one point, then we call D_v a *PL-domain.* In particular, if $f(z)$ has poles only at ζ, then we call D_v a *PL*$_\zeta$*-domain.*

Similarly, we can define the the *outer capacity* $\overline{\gamma}_\zeta(H_v)$ and the *lower Green's function* $\underline{G}(z,\zeta;H_v)$ given by

$$\begin{aligned} \overline{\gamma}_\zeta(H_v) &= \inf_{\substack{D_v \supset H_v \\ D_v \text{ PL}_\zeta\text{-domain}}} \gamma_\zeta(D_v), \\ \underline{G}(z,\zeta;H_v) &= \sup_{\substack{D_v \supset H_v \\ D_v \text{ PL}_\zeta\text{-domain}}} G(z,\zeta;D_v). \end{aligned}$$

Remark 3.4 It can be shown that $\underline{\gamma}_\zeta(H_v) \le \overline{\gamma}_\zeta(H_v)$ and $\underline{G}(z,\zeta;H_v) \le \overline{G}(z,\zeta;H_v)$ (see [22], Proposition 4.4.1, p.278). By [22] Theorems 4.3.3, p.260 and 4.3.4, p.262, if H_v is either a compact set or an RL-domain, then $\underline{\gamma}_\zeta(H_v) = \overline{\gamma}_\zeta(H_v) = \gamma_\zeta(H_v)$. It follows from [22] Theorem 4.4.4, p.283 that if E_v is a compact set, then $\underline{G}(z,\zeta;E_v) = \overline{G}(z,\zeta;E_v) = G(z,\zeta;E_v)$ for all $z \notin E_v$; if D_v is an RL-domain, then $\underline{G}(z,\zeta;D_v) = \overline{G}(z,\zeta;D_v) = G(z,\zeta;D_v)$ for all z. In particular, if D_v is a nonempty PL-domain not containing ζ defined by $f(z)$ of degree n, the *Green's function* of D_v is

$$G(z,\zeta;D_v) = \begin{cases} \frac{1}{n}\log_v(|f(z)|_v) & \text{if } z \notin D_v \text{ and } z \ne \zeta \\ \infty & \text{if } z = \zeta \\ 0 & \text{if } z \in D_v \end{cases} \tag{3.10}$$

If ζ belongs to D_v, we put $G(z,\zeta;D_v) = 0$ for all z. We define the *potential function* of D_v given by

$$u_\zeta(z;D_v) = V_\zeta(D_v) - G(z,\zeta;D_v). \tag{3.11}$$

Remark 3.5 Let $f(z) \in \mathbb{C}_v(z)$ be a function of degree N, having poles only at ζ and normalized so that $\lim_{z\to\zeta} |f(z)|_v \cdot |g_\zeta(z)|_v = 1$, where $g_\zeta(z) \in \mathbb{C}_v(z)$ is a uniformizing parameter of ζ which determines the choice of $[z,x]_\zeta$. If $D_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \leq R\}$ for some $R \in |\mathbb{C}_v^\times|$, then D_v is a PL-domain and hence $\gamma_\zeta(D_v) = R^{1/N}$ (see [22], Proposition 4.3.1, p.258). Equivalently, $V_\zeta(D_v) = -\frac{1}{N}\log_v(R)$. Since $|\mathbb{C}_v^\times| = q_v^{\mathbb{Q}} \cup \{0\}$, $V_\zeta(D_v)$ belongs to $\mathbb{Q}$. Moreover, by (3.10), $G(z,\zeta;D_v)$ has values in $\mathbb{Q}$ for all $z \neq \zeta$.

We will now give some examples of Robin constants and Green's functions for some simple sets.

Definition 3.6 Let E_v be a compact set in $\mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$. For any probability measure μ on E_v, the *logarithmic potential function* with respect to μ is defined by

$$u_\zeta(z,\mu) = \int_{E_v} -\log_v([z,x]_\zeta)d\mu(x).$$

Example 3.7 Put $E_v := \mathcal{O}_v \subset K_v$. Since $\mathcal{O}_v$ is translation invariant under $\mathcal{O}_v$, its equilibrium distribution μ must be the normalized additive Haar measure on $\mathcal{O}_v$. To see this, note that $\mu(\pi_v^n) = 1/q_v^n$ and $\pi_v^n\mathcal{O}_v^\times = \pi_v^n\mathcal{O}_v \setminus \pi_v^{n+1}\mathcal{O}_v$. Since $[z,x]_\infty = |z-x|_v$, we see that for any $z \in \mathcal{O}_v$,

$$\begin{aligned} u_\infty(z,\mu) &= \int_{\mathcal{O}_v} -\log_v(|z-x|_v)d\mu(x) = \int_{\mathcal{O}_v} -\log_v(|y|_v)d\mu(y) \\ &= \sum_{n=0}^{\infty}\int_{\pi_v^n\mathcal{O}_v^\times} -\log_v(|\pi_v^n|_v)d\mu(y) = \sum_{n=0}^{\infty} n\cdot(\frac{1}{q_v^n} - \frac{1}{q_v^{n+1}}) \\ &= \sum_{n=0}^{\infty}\frac{n}{q_v^n}\cdot(1-\frac{1}{q_v}) = \frac{1}{q_v-1}. \end{aligned}$$

Here, we used the fact that $x/(1-x)^2 = \sum_{n=0}^\infty n\cdot x^n$ if $|x|<1$ because $1/(1-x) = \sum_{n=0}^\infty x^n$. It follows by [22], Proposition 4.1.23, p.211 that μ is the equilibrium distribution of E_v with respect to ∞, $u_\infty(z;\mathcal{O}_v) = u_\infty(z,\mu)$ and

$$V_\infty(\mathcal{O}_v) = 1/(q_v-1).$$

To complete the formula for $u_\infty(z;\mathcal{O}_v)$, suppose that $|z|_v > 1$. For each $x \in \mathcal{O}_v$, we see that $|z-x|_v = |z|_v$. Hence $u_\infty(z;\mathcal{O}_v) = -\log_v(|z|_v)$. Therefore, we obtain that

$$u_\infty(z;\mathcal{O}_v) = \begin{cases} V_\infty(\mathcal{O}_v) = 1/(q_v - 1) & \text{if } z \in \mathcal{O}_v \\ -\log_v(|z|_v) & \text{if } |z|_v > 1. \end{cases}$$

More generally, if L_w/K_v is a finite extension with ramification index $e_{w/v}$ and residue degree $f_{w/v}$, then the equilibrium distribution of $\mathcal{O}_w$ with respect to ∞ is the additive Haar measure ν on L_w normalized so that $\nu(\mathcal{O}_w) = 1$ and the potential function of $\mathcal{O}_w$ is

$$u_\infty(z;\mathcal{O}_w) = \begin{cases} \left(e_{w/v} \cdot (q_v^{f_{w/v}} - 1)\right)^{-1} & \text{for } z \in \mathcal{O}_w \\ -\log_v(|z|_v) & \text{for } z \notin B(0,1). \end{cases}$$

We can generalize Example 3.7 by using [22], Proposition 4.1.25, p.214.

Example 3.8 Let $E_v = a + b\mathcal{O}_v = B(a,r) \cap K_v$ be a compact set with $a \in K_v$, where $r = |b|_v \in |K_v^\times|$. Then the equilibrium distribution of E_v with respect to ∞ is the additive Haar measure μ on K_v normalized so that $\mu(E_v) = 1$ and the potential function of E_v is

$$u_\infty(z;E_v) = \begin{cases} \frac{1}{q_v - 1} - \log_v(r) & \text{for } z \in E_v \\ -\log_v(|z-a|_v) & \text{for } z \notin B(a,r). \end{cases}$$

More generally, if L_w/K_v is a finite extension with ramification index $e_{w/v}$ and residue degree $f_{w/v}$, and if $E_w = B(a,r) \cap L_w$, then the equilibrium distribution of E_w with respect to ∞ is the additive Haar measure ν on L_w normalized so that $\nu(E_w) = 1$ and the potential function of E_w is

$$u_\infty(z;E_w) = \begin{cases} \left(e_{w/v} \cdot (q_v^{f_{w/v}} - 1)\right)^{-1} - \log_v(r) & \text{for } z \in E_w \\ -\log_v(|z-a|_v) & \text{for } z \notin B(a,r). \end{cases}$$

We have given examples of potential functions of compact sets in K_v with respect to ∞. Next, we want to extend these results to compact sets in $\mathbb{P}^1(\mathbb{C}_v)$ or $\mathbb{P}^1(K_v)$ with respect to an arbitrary point ζ.

Example 3.9 Let $E_v = \mathfrak{B}(a, r)$ be a ball in $\mathbb{P}^1(\mathbb{C}_v)$ and fix $\zeta \notin E_v$. For any $z \in E_v$, there is a constant C_a such that $[z, a]_\zeta = C_a \cdot \|z, a\|_v$ by Lemma 2.32. It follows that $E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : [z, a]_\zeta \leq C_a \cdot r\}$ (see the proof of [22], Theorem 4.2.16, p.252). If the canonical distance is normalized so that $\lim_{z \to \zeta}[z, a]_\zeta \cdot |g_\zeta(z)|_v = 1$, then $[z, a]_\zeta = C_\zeta \cdot \frac{\|z,a\|_v}{\|z,\zeta\|_v \|a,\zeta\|_v}$ for some constant $C_\zeta \in |\mathbb{C}_v^\times|$. Put $h(z) := C \cdot \frac{z-a}{z-\zeta}$, where the constant $C \in \mathbb{C}_v$ satisfies $|C|_v = C_\zeta \cdot \frac{\max\{1, |\zeta|_v\}^2}{|a-\zeta|_v}$. Clearly, $[z, a]_\zeta = |h(z)|_v$. If $g_\zeta(z)$ was the uniformizing parameter determining the normalization of $[z, a]_\zeta$, then $\lim_{z \to \zeta} |h(z)|_v \cdot |g_\zeta(z)|_v = 1$. Since $E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z)|_v \leq C_a \cdot r\}$, it follows from Remark 3.5 that $V_\zeta(E_v) = -\log_v(C_a \cdot r)$ and that for all $z \notin E_v$,

$$G(z, \zeta; E_v) = \log_v \Big(\frac{|h(z)|_v}{C_a \cdot r} \Big) = \log_v([z, a]_\zeta) - \log_v(C_a \cdot r).$$

It follows from the definition of $u_\zeta(z; E_v)$ for a PL-domain that

$$u_\zeta(z; E_v) = \begin{cases} -\log_v(C_a \cdot r) & \text{for } z \in E_v \\ -\log_v([z, a]_\zeta) & \text{for } z \notin E_v. \end{cases}$$

Example 3.10 Let $E_v = \mathfrak{B}(a, r) \cap \mathbb{P}^1(K_v)$ be a compact set with $a \in K_v$, where $r = |b|_v \in |K_v^\times|$. Fix $\zeta \notin \mathfrak{B}(a, r)$. Since $\mathfrak{B}(a, r)$ is isometrically parametrizable and $a \in K_v$, we can fix an isometric parametrization $\lambda : B(0, r) \to \mathfrak{B}(a, r)$ such that $\lambda \in K_v[[z]]$ and $\lambda(0) = a$. As in Example 3.9, $\mathfrak{B}(a, r) = \{z \in \mathbb{P}^1(\mathbb{C}_v) : [z, a]_\zeta \leq C_a \cdot r\}$ and $[z, x]_\zeta = C_a \cdot \|z, x\|_v$ for all $z,\ x \in \mathfrak{B}(a, r)$. Note that if $\lambda(\alpha) = z$ and $\lambda(\beta) = x$ for $z,\ x \in \mathfrak{B}(a, r)$, then $\|z, x\|_v = |\alpha - \beta|_v$. Since $\lambda(b\mathcal{O}_v) = E_v$, we claim that the pushforward ν of the additive Haar measure μ on K_v, normalized to give $b\mathcal{O}_v$ total mass 1, is the equilibrium distribution of E_v with respect to ζ. It suffices to verify that the logarithmic potential function $u_\zeta(z, \nu)$ takes the same value for all $z \in E_v$ by [22], Proposition 4.1.23, p.211. For any $z \in E_v$,

$$\begin{aligned} u_\zeta(z,\nu) &= \int_{E_v} -\log_v([z,x]_\zeta)\ d\nu(x) = \int_{E_v} -\log_v(C_a \cdot \|z,x\|_v)\ d\nu(x) \\ &= -\log_v(C_a) + \int_{E_v} -\log_v(\|z,x\|_v)\ d\nu(x) \\ &= -\log_v(C_a) + \int_{b\mathcal{O}_v} -\log_v(|\alpha-\beta|_v)\ d\mu(x) \\ &= -\log_v(C_a) - \log_v(|b|_v) + \frac{1}{q_v-1} \\ &= -\log_v(C_a \cdot r) + \frac{1}{q_v-1}, \end{aligned}$$

which is a constant. Hence ν is the equilibrium distribution of E_v with respect to ζ and $V_\zeta(E_v) = -\log_v(C_a \cdot r) + \frac{1}{q_v-1}$. For $z \notin \mathfrak{B}(a,r)$, since $[z,x]_\zeta$ is constant for $x \in E_v$, one sees that $u_\zeta(z,\nu) = -\log_v([z,x]_\zeta)$. We thus have

$$u_\zeta(z;E_v) = \begin{cases} \frac{1}{q_v-1} - \log_v(C_a \cdot r) & \text{for } z \in E_v \\ -\log_v([z,a]_\zeta) & \text{for } z \notin \mathfrak{B}(a,r). \end{cases}$$

Remark 3.11 Examples 3.9 and 3.10 remain true for any ball $B(a,r) \subset \mathbb{C}_v \setminus \{\zeta\}$ with $0 < r < 1$.

We are going to deal with a compact set E_v, which is a finite union of compact sets of the form $\mathfrak{B}(a_i,r_i) \cap \mathbb{P}^1(K_v)$ for $a_i \in K_v$ and $r_i = |b_i|_v \in |K_v^\times|$, where the balls $\mathfrak{B}(a_i,r_i)$ are disjoint from each other and do not contain ζ. To compute the equilibrium distribution and the potential function of E_v with respect to ζ, we need this:

Proposition 3.12 *Let $E_{v,1}, \cdots, E_{v,t}$ be pairwise disjoint compact sets in $\mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$, with positive capacity, and put $E_v := \cup_{i=1}^t E_{v,i}$. Let μ and μ_i, $i = 1, \cdots, t$ be the equilibrium distribution of E_v and $E_{v,i}$ with respect to ζ, respectively, and write $\varepsilon_i = \mu(E_{v,i})$ for $i = 1, \cdots, t$. Suppose that $[z,x]_\zeta$ is constant for $z \in E_{v,i}$ and $x \in E_{v,j}$, whenever $i \neq j$. Then $\varepsilon_i > 0$ for each i, and*

$$\mu = \sum_{i=1}^t \varepsilon_i\ \mu_i, \quad u_\zeta(z;E_v) = \sum_{i=1}^t \varepsilon_i\ u_\zeta(z;E_{v,i}).$$

Proof: See [22], Proposition 4.1.27, p.216. □

Proposition 3.13 *Let the balls $\mathfrak{B}(a_i, r_i) \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$, $i = 1, \cdots, t$, be disjoint from each other, where $a_i \in K_v$ and $r_i \in |K_v^\times|$, and put $E_{v,i} := \mathfrak{B}(a_i, r_i) \cap \mathbb{P}^1(K_v)$. Let C_i be the constant such that $[z, x]_\zeta = C_i \cdot \|z, x\|_v$ for all $z,\ x \in \mathfrak{B}(a_i, r_i)$ given by Lemma 2.32. If $E_v := \cup_{i=1}^t E_{v,i}$, then the Robin constant $V := V_\zeta(E_v)$ belongs to $\mathbb{Q}$; the weights $\varepsilon_i > 0$ belong to $\mathbb{Q}$; and V and the ε_i are uniquely determined by the $t + 1$ linear equations*

$$0 \cdot V + \sum_{i=1}^{t} \varepsilon_i = 1, \quad \textit{and}$$

$$V + \varepsilon_j \cdot \left(\log_v(C_j \cdot r_j) - \frac{1}{q_v - 1}\right) + \sum_{\substack{i=1 \\ i \neq j}}^{t} \varepsilon_i \cdot \log_v([a_j, a_i]_\zeta) = 0$$

for each $j = 1, \cdots, t$.

Proof: For the first equation, since $\mu = \sum_{i=1}^t \varepsilon_i \mu_i$, one sees that $1 = \mu(E_v) = \sum_{i=1}^t \varepsilon_i \mu_i(E_v) = \sum_{i=1}^t \varepsilon_i \mu_i(E_{v,i}) = \sum_{i=1}^t \varepsilon_i$. The rest of the equations follow from Example 3.10, upon evaluating $u_\zeta(z; E_v)$ at $a_1, \cdots, a_t$. Indeed, for each $j = 1, \cdots, t$, if $i \neq j$, then $[z, x]_\zeta = [a_j, a_i]_\zeta$ for all $z \in \mathfrak{B}(a_j, r_j),\ x \in \mathfrak{B}(a_i, r_i)$ by Lemma 2.32. Since $a_j \in E_v$,

$$\begin{aligned} V &= u_\zeta(a_j; E_v) = \int_{E_v} - \log_v([a_j, x]_\zeta) d\mu(x) \\ &= \sum_{i=1}^{t} \int_{E_{v,i}} - \log_v([a_j, x]_\zeta) d\mu(x) \\ &= \sum_{i=1}^{t} \varepsilon_i \int_{E_{v,i}} - \log_v([a_j, x]_\zeta) d\mu_i(x) \\ &= \sum_{i \neq j} \varepsilon_i \big(- \log_v([a_j, a_i]_\zeta)\big) + \varepsilon_j \Big(- \log_v(C_j \cdot r_j) + \frac{1}{q_v - 1}\Big). \end{aligned}$$

The equilibrium distribution of E_v with respect to ζ is uniquely characterized by its potential function, which takes a constant value on E_v. Hence, the system of the linear equations in V and the ε_i, with positive ε_i, has a unique solution; in particular, the matrix of the system is nonsingular. Noting that the C_j, r_j and $[a_j, a_i]_\zeta$ belong to $|\mathbb{C}_v^\times|$, the coefficients of the equations are rational, and hence V and the ε_i must be rational as well. □

Proposition 3.14 *Let E_v be the compact set in Proposition 3.13. Then $V_\zeta(E_v) \in \mathbb{Q}$ and for each $z \notin \cup_{i=1}^t \mathfrak{B}(a_i, r_i)$, $G(z, \zeta; E_v) \in \mathbb{Q}$.*

Proof: By Proposition 3.13, $V_\zeta(E_v) \in \mathbb{Q}$. It then follows from Example 3.10, Proposition 3.12 and Proposition 3.13 that for each $z \notin \cup_{i=1}^t \mathfrak{B}(a_i, r_i)$, the value

$$\begin{aligned} G(z, \zeta; E_v) &= V_\zeta(E_v) - u_\zeta(z; E_v) \\ &= V_\zeta(E_v) - \sum_{i=1}^t \varepsilon_i u_\zeta(z; E_{v,i}) \\ &= V_\zeta(E_v) + \sum_{i=1}^t \varepsilon_i \log_v([z, a_i]_\zeta), \end{aligned}$$

belongs to $\mathbb{Q}$. □

The conclusions of Proposition 3.14 also hold for RL-domains.

Proposition 3.15 *Let $D_v \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \{\zeta\}$ be an RL-domain. Then $V_\zeta(D_v) \in \mathbb{Q}$ and for all $z \neq \zeta$, $G(z, \zeta; D_v) \in \mathbb{Q}$.*

Proof: By [22], Theorem 4.2.15, p.252, D_v can be uniquely written in the form $D_v = \cap_{i=1}^n D_{v,i}$, where each $D_{v,i}$ is a PL$_\xi$-domains with respect to every point ξ in its complement, and the $D_{v,i}$ have pairwise disjoint complements. Hence there is only one j such that $\zeta \notin D_{v,j}$. Suppose that D is a PL$_\zeta$-domain containing D_v. Since $\zeta \notin D_v$ and $\zeta \notin D_{v,j}$, $D \cap D_{v,i}$ is a PL-domain by [22], Corollary 4.2.13, p.251. Clearly, $(D \cap D_{v,i}) \cap \cap_{i \neq j}^n D_{v,i} = D_v$. It follows from the uniqueness of the expression $D_v = \cap_{i=1}^n D_{v,i}$ that $D \cap D_{v,i} = D_{v,i}$, so $D_{v,i} \subset D$. By Remark 3.4 and the monotonicities of capacities and Green's functions,

$$\begin{aligned} \gamma_\zeta(D_v) &= \inf_{\substack{D \supset D_v \\ D \text{ PL}_\zeta\text{-domain}}} \gamma_\zeta(D) = \gamma_\zeta(D_{v,j}), \\ G(z, \zeta; D_v) &= \sup_{\substack{D \supset D_v \\ D \text{ PL}_\zeta\text{-domain}}} G(z, \zeta; D) = G(z, \zeta; D_{v,j}). \end{aligned}$$

This completes the proof by Remark 3.5.

3.2 Global Capacity Theory

Definition 3.16 A set $\mathfrak{X} \subset \mathbb{P}^1(\overline{K})$ is called *K-symmetric* if it is stable under $\mathrm{Gal}(\overline{K}/K)$. Suppose that $\mathfrak{X} = \{x_1, \cdots, x_m\}$ is a finite K-symmetric set. An m-dimensional real vector $\vec{s} = (s_1, \cdots, s_m)$ is a *probability vector* if $s_1, \cdots, s_m \geq 0$ and $\sum_{i=1}^m s_i = 1$. We define an action of $\mathrm{Gal}(\overline{K}/K)$ on $\vec{s}$ so that each $\sigma \in \mathrm{Gal}(\overline{K}/K)$ permutes the coordinates of $\vec{s}$ in the same way that it permutes the elements of $\mathfrak{X}$, and write $\sigma(\vec{s})$ for this action; $\vec{s}$ is called *K-symmetric* if $\sigma(\vec{s}) = \vec{s}$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

Let $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \mathbb{P}^1(\overline{K})$ be a finite K-symmetric set and $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{R}^m$ be a K-symmetric probability vector. Suppose that we are given a set $E_v \subset \mathbb{P}^1(\mathbb{C}_v)$ for each place v of K, with E_v stable under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$ and bounded away from $\mathfrak{X}$ under the spherical metric $\|z, x\|_v$. We want to define the *global capacity* $\gamma(\mathbb{E}_K, \mathfrak{X})$ of the adelic set $\mathbb{E}_K = \prod_{v \in P_K} E_v$ with respect to $\mathfrak{X}$. Let L be the normal closure of K containing $\mathfrak{X}$. For each place $w|v$ of L, fix an embedding of L_w in $\mathbb{C}_v$ and choose an topological isomorphism of $\mathbb{C}_w$ with $\mathbb{C}_v$. Pulling back E_v to a set $E_w \subset \mathbb{P}^1(\mathbb{C}_w)$, E_w is independent of the isomorphism chosen since E_v is Galois-stable. Put $\mathbb{E}_L := \prod_{w \in P_L} E_w$. We can identify $\mathbb{P}^1(\mathbb{C}_v)$ with $\mathbb{P}^1(\mathbb{C}_w)$ by the chosen isomorphism. Since $|x|_w = |x|_v^{[L_w:K_v]}$ for any $x \in L_w$, we see that $\log_w(|x|_w) = e_{w/v} \cdot \log_v(|x|_v)$, where $e_{w/v}$ is the ramification index. It follows that $V_\zeta(E_w) = e_{w/v} V_\zeta(E_v)$ and $G(z, \zeta; E_w) = e_{w/v} G(z, \zeta; E_v)$ for all $z, \zeta \notin E_v$. We then obtain that for all $z, \zeta \notin E_v$,

$$\begin{aligned} V_\zeta(E_w)\log(q_w) &= [L_w : K_v] V_\zeta(E_v)\log(q_v), \\ G(z, \zeta; E_w)\log(q_w) &= [L_w : K_v] G(z, \zeta; E_v)\log(q_v). \end{aligned} \tag{3.12}$$

The *local Green's matrix* at w is defined by

$$\Gamma(E_w, \mathfrak{X}) = \begin{pmatrix} V_{x_1}(E_w) & G(x_1, x_2; E_w) & \cdots & G(x_1, x_m; E_w) \\ G(x_2, x_1; E_w) & V_{x_2}(E_w) & \cdots & G(x_2, x_m; E_w) \\ \vdots & \vdots & \vdots & \vdots \\ G(x_m, x_1; E_w) & G(x_m, x_1; E_w) & \cdots & V_{x_m}(E_w) \end{pmatrix} \tag{3.13}$$

and, provided that the sum makes sense, the *global Green's matrix* over L is

$$\Gamma(\mathbb{E}_L, \mathfrak{X}) = \sum_{w \in P_L} \Gamma(E_w, \mathfrak{X}) \log(q_w). \tag{3.14}$$

Then, the local and global Green's matrices over K are defined by

$$\Gamma(E_v, \mathfrak{X}) \log(q_v) = \frac{1}{[L:K]} \sum_{w|v} \Gamma(E_w, \mathfrak{X}) \log(q_w),$$
$$\Gamma(\mathbb{E}_K, \mathfrak{X}) = \frac{1}{[L:K]} \Gamma(\mathbb{E}_L, \mathfrak{X}). \tag{3.15}$$

We will now give a condition on the set $\mathbb{E}_K$ which makes the sum (3.14) finite, in which case $\Gamma(\mathbb{E}_L, \mathfrak{X})$ and $\Gamma(\mathbb{E}_K, \mathfrak{X})$ are certainly well-defined. Fix K-rational system of homogeneous coordinates on $\mathbb{P}^1/K$, thus determining a model of $\mathbb{P}^1/Spec(\mathcal{O}_v)$, for each place v of K.

Definition 3.17 A set $E_v \subset \mathbb{P}^1(\mathbb{C}_v)$ is said to be $\mathfrak{X}$-*trivial* if the points of $\mathfrak{X}$ specialize to distinct points (mod v) and E_v is of the form $\mathbb{P}^1(\mathbb{C}_v) \setminus \cup_{x_i \in \mathfrak{X}} \mathfrak{B}(x_i, 1)^-$.

Remark 3.18 Fix a uniformizing parameter $g_{x_i}(z) \in L(z)$ for each x_i. Then, for all but finitely many v, we have $\|z, x_i\|_v = |g_{x_i}(z)|_v$ for all z sufficiently near x_i by Proposition 2.29. For such v, $\Gamma(E_v, \mathfrak{X})$ is the zero matrix if E_v is $\mathfrak{X}$-trivial ([22], Proposition 5.1.2, p.324).

Definition 3.19 An adelic set $\mathbb{E}_K = \prod_{v \in P_K} E_v$ is said to be *compatible with* $\mathfrak{X}$ if each E_v is bounded away from $\mathfrak{X}$, and for all but finitely many v, E_v is the $\mathfrak{X}$-trivial set. Note that the property of compatibility is independent of the choice of homogeneous coordinates.

If the adelic set $\mathbb{E}_K = \prod_{v \in P_K} E_v$ is compatible with $\mathfrak{X}$, then the definition of the global matrices $\Gamma(\mathbb{E}_K, \mathfrak{X})$ and $\Gamma(\mathbb{E}_L, \mathfrak{X})$ makes sense. We define the *(global) Robin constant* $V(\mathbb{E}_K, \mathfrak{X})$ to be the value of $\Gamma(\mathbb{E}_K, \mathfrak{X})$ given by

$$\mathrm{val}(\Gamma(\mathbb{E}_K, \mathfrak{X})) = \max_{\vec{s}} \min_{j} \sum_{i} s_i \cdot \Gamma(\mathbb{E}_K, \mathfrak{X})_{ij} = \max_{\vec{s}} \min_{\vec{t}} \sum_{i,j} s_i \cdot \Gamma(\mathbb{E}_K, \mathfrak{X})_{ij} \cdot t_j, \tag{3.16}$$

where $\vec{s}$ and $\vec{t}$ range over all m-dimensional real probability vectors, and define the *global capacity* of $\mathbb{E}_K$ with respect to $\mathfrak{X}$ by

$$\gamma(\mathbb{E}_K, \mathfrak{X}) = \exp(-V(\mathbb{E}_K, \mathfrak{X})). \tag{3.17}$$

Remark 3.20 If $f(z) \in K(z)$ has degree 1, then $\gamma(f^{-1}(\mathbb{E}_K), f^{-1}(\mathfrak{X})) = \gamma(\mathbb{E}_K, \mathfrak{X})$ by [22], Theorem 5.1.14, p.333. Hence we can assume that $\infty \notin \mathfrak{X}$ by taking $f(z) = Az$ for an appropriate $A \in \mathrm{GL}_2(\mathcal{O}_K)$.

Sometimes, we are not interested in the exact value of $\gamma(\mathbb{E}, \mathfrak{X})$, but only whether it is greater than or less than 1. For this, we have:

Proposition 3.21 *$\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$ if and only if $\Gamma(\mathbb{E}_K, \mathfrak{X})$ is negative-definite, or equivalently, $(-1)^k \cdot \det(\Gamma^{(k)}) > 0$ for $k = 1, \cdots, m$, where $\Gamma^{(k)}$ is the $k \times k$ submatrix in the upper left corner of $\Gamma(\mathbb{E}_K, \mathfrak{X})$.*

Proof: See [22], Proposition 5.1.8, p. 331. □

Proposition 3.22 *If $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$, then there is a unique K-symmetric real probability vector $\vec{s}$ with all its entries $s_i > 0$, such that $\Gamma(\mathbb{E}_K, \mathfrak{X}) \cdot \vec{s} < 0$ and $\Gamma(\mathbb{E}_K, \mathfrak{X}) \cdot \vec{s} = V(\mathbb{E}_K, \mathfrak{X}) \cdot \mathbf{1}_m$, where $\mathbf{1}_m$ is the vector of size m whose entries are all 1's.*

Proof: See [22], Theorem 5.1.6, p. 328. □

Corollary 3.23 *With the notations in Proposition 3.22, if there is a real number $\beta \neq 0$ such that each entry of the matrix $\Gamma(\mathbb{E}_K, \mathfrak{X})$ belongs to $\beta \cdot \mathbb{Q}$, then $V(\mathbb{E}_K, \mathfrak{X}) \in \beta \cdot \mathbb{Q}$ and $\vec{s} \in \mathbb{Q}^m$.*

Proof: Proposition 3.21 implies that the matrix $\Gamma(\mathbb{E}_K, \mathfrak{X})$ is nonsingular. It follows that $\vec{s} = V(\mathbb{E}_K, \mathfrak{X}) \cdot \Gamma(\mathbb{E}_K, \mathfrak{X})^{-1} \cdot \mathbf{1}_m$. Since each entry of the matrix $\Gamma(\mathbb{E}_K, \mathfrak{X})$ belongs to $\beta \cdot \mathbb{Q}$, $\Gamma(\mathbb{E}_K, \mathfrak{X})^{-1} \cdot \mathbf{1}_m$ belongs to $\beta^{-1} \cdot \mathbb{Q}^m$. Let $\Gamma(\mathbb{E}_K, \mathfrak{X})^{-1} \cdot \mathbf{1}_m = \beta^{-1} \cdot {}^t(\alpha_1, \cdots, \alpha_m) = \beta^{-1} \cdot \vec{\alpha} \in \beta^{-1} \cdot \mathbb{Q}^m$. By Proposition 3.22, $\beta^{-1} \cdot \vec{\alpha} = V(\mathbb{E}_K, \mathfrak{X})\vec{s}$. Since $\vec{s}$ is a probability vector with all entries positive and $V(\mathbb{E}_K, \mathfrak{X}) < 0$, each $\beta^{-1}\alpha_i < 0$, so $\beta^{-1} \cdot \sum_{i=1}^m \alpha_i < 0$, and in particular, $\sum_{i=1}^m \alpha_i \neq 0$. It follows that $1 = \sum_{i=1}^m s_i = V(\mathbb{E}_K, \mathfrak{X}) \cdot \beta^{-1} \cdot \sum_{i=1}^m \alpha_i$, so $V(\mathbb{E}_K, \mathfrak{X}) = \beta / \sum_{i=1}^m \alpha_i$, which belongs to $\beta \cdot \mathbb{Q}$. Moreover, $\vec{s} = V(\mathbb{E}_K, \mathfrak{X}) \cdot \beta^{-1} \cdot \vec{\alpha}$ belongs to $\mathbb{Q}^m$. □

Remark 3.24 Let $\mathbb{E}_K = \prod_{v \in P_K} E_v$ be an adelic set compatible with $\mathfrak{X}$. Suppose further that if E_v is not $\mathfrak{X}$-trivial, then E_v is either an RL-domain or a compact set of the form $\cup_{i=1}^{t} \mathfrak{B}(a_i, r_i) \cap \mathbb{P}^1(K_v)$ for $a_i \in K_v$ and $r_i \in |K_v^\times|$, where the balls $\mathfrak{B}(a_i, r_i)$ are disjoint from each other and from $\mathfrak{X}$. By Propositions 3.14 and 3.15, if E_v is not $\mathfrak{X}$-trivial, then each entry of the matrix $\Gamma(E_v, \mathfrak{X})$ belongs to $\mathbb{Q}$. Now, assume that E_v is $\mathfrak{X}$-trivial; then $E_v = \mathbb{P}^1(\mathbb{C}_v) \setminus \cup_{x_i \in \mathfrak{X}} \mathfrak{B}(x_i, 1)^-$. It follows from the proof of [22], Proposition 5.1.2, p.324 that each $\mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{B}(x_i, 1)^-$ is a PL_{x_i}-domain. Since $E_v = \cap_{x_i \in \mathfrak{X}} \big(\mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{B}(x_i, 1)^-\big)$, E_v is an RL-domain by [22], Corollary 4.2.14, p.251. Remark 3.18 and Proposition 3.15 imply that the matrix $\Gamma(E_v, \mathfrak{X})$ either is the zero matrix or has entries in $\mathbb{Q}$. Thus, each entry of the matrix $\Gamma(\mathbb{E}_K, \mathfrak{X})$ belongs to $\log(p) \cdot \mathbb{Q}$ by the definition of $\Gamma(\mathbb{E}_K, \mathfrak{X})$. It follows from Corollary 3.23 that if $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$, then $V(\mathbb{E}_K, \mathfrak{X}) \in \log(p) \cdot \mathbb{Q}$ and the probability vector $\vec{s}$ in Proposition 3.22, satisfying $\Gamma(\mathbb{E}_K, \mathfrak{X}) \cdot \vec{s} = V(\mathbb{E}_K, \mathfrak{X}) \cdot \mathbf{1}_m$, belongs to $\mathbb{Q}_{>0}^m$.

Chapter 4

$(\mathfrak{X}, \vec{s})$-Capacity Theory

In the previous chapter, we examined the classical capacity theory as developed in [22] for the canonical distance $[z, x]_\zeta$ with respect to a single point ζ, considering equilibrium distributions, Robin constants, potential functions, Green's functions, and basic facts such as Maria's Theorem and Frostman's Theorem. For the proof of the Main Theorem, we will need to extend the classical capacity theory to a theory of capacity with respect to a finite set $\mathfrak{X}$ and a probability vector $\vec{s}$. The resulting theory is completely analogous to the classical theory. Furthermore, most of the objects in the $(\mathfrak{X}, \vec{s})$-capacity theory can be expressed in terms of objects in the classical theory. We will establish the relations which we need; for more details, see the manuscript [25].

Fix a place v of K, and a finite K-symmetric set $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \mathbb{P}^1(\overline{K})$. For each $x_i \in \mathfrak{X}$, fix a uniformizing parameter $g_i(z) \in \mathbb{C}_v(z)$ at x_i, and let the canonical distance $[z, x]_{x_i}$ be normalized so that for each $x \neq x_i$,

$$\lim_{z \to x_i} [z, x]_{x_i} \cdot |g_i(z)|_v = 1. \tag{4.1}$$

Let $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{Q}^m_{\geq 0}$ be a probability vector. Define the $(\mathfrak{X}, \vec{s})$-*canonical distance*

$$[z, x]_{\mathfrak{X}, \vec{s}} = \prod_{i=1}^{m} [z, x]_{x_i}^{s_i}.$$

Let $E_v \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{X}$ be a compact set. We define the $(\mathfrak{X}, \vec{s})$-*Robin constant* $V_{\mathfrak{X},\vec{s}}(E_v)$ and $(\mathfrak{X}, \vec{s})$-*capacity* $\gamma_{\mathfrak{X},\vec{s}}(E_v)$ by

$$V_{\mathfrak{X},\vec{s}}(E_v) = \inf_\nu \int\int_{E_v \times E_v} -\log_v([z, x]_{\mathfrak{X},\vec{s}}) d\nu(z) d\nu(x),$$
$$\gamma_{\mathfrak{X},\vec{s}}(E_v) = q_v^{-V_{\mathfrak{X},\vec{s}}(E_v)},$$

where ν runs over all probability measures on E_v.

Remark 4.1 Since $[z,x]_{\mathfrak{X},\vec{s}}$ is a weighted product of the $[z,x]_{x_i}$, Lemma 2.32 remains true with respect to $[z,x]_{\mathfrak{X},\vec{s}}$, on balls in $\mathbb{P}^1(\mathbb{C}_v)$ disjoint from $\mathfrak{X}$.

Proposition 4.2 *If $E_v \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{X}$ be a compact set of positive capacity, then there is a unique probability measure $\mu_0 = \mu_{\mathfrak{X},\vec{s}}$ on E_v, called the $(\mathfrak{X},\vec{s})$-equilibrium distribution of E_v, such that*

$$V_{\mathfrak{X},\vec{s}}(E_v) = \int\int_{E_v \times E_v} -\log_v([z,x]_{\mathfrak{X},\vec{s}})d\mu_0(z)d\mu_0(x).$$

Furthermore, the $(\mathfrak{X},\vec{s})$-potential function of E_v

$$u_{\mathfrak{X},\vec{s}}(z;E_v) = \int_{E_v} -\log_v([z,x]_{\mathfrak{X},\vec{s}})d\mu_0(x)$$

satisfies $u_{\mathfrak{X},\vec{s}}(z;E_v) \le V_{\mathfrak{X},\vec{s}}(E_v)$ for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{X}$, with $u_{\mathfrak{X},\vec{s}}(z;E_v) < V_{\mathfrak{X},\vec{s}}(E_v)$ for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus E_v$ and $u_{\mathfrak{X},\vec{s}}(z;E_v) = V_{\mathfrak{X},\vec{s}}(E_v)$ for all $z \in E_v$ except possibly a set of capacity 0.

Proof: The proofs are the basically same as those for $[z,x]_{\mathfrak{X},\vec{s}}$ in [22], Theorems 4.1.11, p.195, and 4.1.22, p.211, with minor modifications, using Remark 4.1. □

In general, for any probability measure ν on E_v, put

$$u_{\mathfrak{X},\vec{s}}(z,\nu) := \int_{E_v} -\log_v([z,x]_{\mathfrak{X},\vec{s}})d\nu(x).$$

For each $x_i \in \mathfrak{X}$, let μ_i be the equilibrium distribution of E_v with respect to x_i. We defined the Green's function of E_v with respect to x_i by

$$G(z,x_i;E_v) = V_{x_i}(E_v) - u_{x_i}(z;E_v).$$

Likewise, we define the *Green's function* $G_{\mathfrak{X},\vec{s}}(z;E_v)$ of E_v with respect to $(\mathfrak{X},\vec{s})$ given by

$$G_{\mathfrak{X},\vec{s}}(z;E_v) = V_{\mathfrak{X},\vec{s}}(E_v) - u_{\mathfrak{X},\vec{s}}(z;E_v).$$

Proposition 4.3 *If $E_v \subset \mathbb{P}^1(\mathbb{C}_v) \setminus \mathfrak{X}$ is a compact set of positive capacity, then*

$$\mu_{\mathfrak{X},\vec{s}} = \sum_{i=1}^{m} s_i \mu_i, \tag{4.2}$$

$$G_{\mathfrak{X},\vec{s}}(z; E_v) = \sum_{i=1}^{m} s_i G(z, x_i; E_v), \tag{4.3}$$

$$V_{\mathfrak{X},\vec{s}}(E_v) = \sum_{i,j,k=1}^{m} s_i s_j s_k \int\int_{E_v \times E_v} -\log_v([z,x]_{x_i}) d\mu_j(z) d\mu_k(x). \tag{4.4}$$

Proof: (See [25]) Choose decreasing sequences $\{r_n\}$ and $\{\varepsilon_n\}$ of positive numbers such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} \varepsilon_n = 0$. For each n, put $E_n := \{z \in \mathbb{P}^1(\mathbb{C}_v) : \|z,x\|_v \leq r_n$ for some $x \in E_v\}$. Then the E_n form a decreasing sequence of neighborhoods of E_v with $\cap_{n=1}^{\infty} E_n = E_v$. Let $g_i(z) \in \mathbb{C}_v(z)$ be the uniformizing parameter at each x_i which determines the normalization of $[z,x]_{x_i}$. For each i and n, [22], Proposition 4.1.5, p.188 provides a function $f_i^{(n)}(z) \in \mathbb{C}_v(z)$, with poles only at x_i and all zeros in E_n, normalized so that

$$\lim_{z\to x_i} |f_i^{(n)}(z)|_v \cdot |g_i(z)^{N_{in}}|_v = 1 \tag{4.5}$$

such that for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus (E_n \cup \{x_i\})$,

$$\Big| u_{x_i}(z; E_v) - \Big(-\frac{1}{N_{in}} \log_v \left(|f_i^{(n)}(z)|_v \right) \Big) \Big|_v < \varepsilon_n, \tag{4.6}$$

where $N_{in} = \deg(f_i^{(n)}(z))$. By raising the $f_i^{(n)}(z)$ to appropriate powers, we can assume without loss of generality that for each n, they have the same degree N_n. Let the zeros of $f_i^{(n)}(z)$ be $\alpha_{i1}^{(n)}, \cdots, \alpha_{iN_n}^{(n)}$, listed with multiplicities. It follows from Proposition 2.31 (i) that for all z, $|f_i^{(n)}(z)|_v = C_{in} \cdot \prod_{j=1}^{N_n} [z, \alpha_{tj}^{(n)}]_{x_i}$, so by (4.5) and the normalization of $[z,x]_{x_i}$, $1 = C_{in} \cdot \lim_{z\to x_i} \prod_{j=1}^{N_n} \left([z, \alpha_{tj}^{(n)}]_{x_i} \cdot |g_i(z)|_v \right) = C_{in}$. Hence, for each i and n, $|f_i^{(n)}(z)|_v = \prod_{j=1}^{N_n} [z, \alpha_{tj}^{(n)}]_{x_i}$. Let $\mu_i^{(n)}$ be the probability measure which gives weight $1/N_n$ to each $\alpha_{tj}^{(n)}$. Then we have

$$\begin{aligned} u_{x_i}(z, \mu_i^{(n)}) &= \int -\log_v([z,x]_{x_i}) d\mu_i^{(n)}(x) \\ &= \sum_{j=1}^{N_n} (-\log_v([z, \alpha_{tj}^{(n)}]_{x_i}) \cdot \frac{1}{N_n} \\ &= -\frac{1}{N_n} \cdot \log_v(|f_i^{(n)}(z)|_v). \end{aligned} \tag{4.7}$$

In particular, it follows from (4.6) that for each $z \notin E_v \cup \{x_i\}$,

$$\lim_{n\to\infty} u_{x_i}(z,\mu_i^{(n)}) = \lim_{n\to\infty} -\frac{1}{N_n}\log_v(|f_i^{(n)}(z)|_v = u_{x_i}(z;E_v). \tag{4.8}$$

Now, we claim that the $\mu_i^{(n)}$ converge weakly to μ_i. Let μ be any weak limit of a subsequence $\{\mu_i^{(n_k)}\}$ of the $\mu_i^{(n)}$. Then for each $z \notin E_v \cup \{x_i\}$, $u_{x_i}(z,\mu) = \lim_{k\to\infty} u_{x_i}(z,\mu_i^{(n_k)}) = \lim_{n\to\infty} u_{x_i}(z,\mu_i^{(n)})$. It follows from (4.8) that for all $z \notin E_v \cup \{x_i\}$, $u_{x_i}(z,\mu) = u_{x_i}(z;E_v) = u_{x_i}(z,\mu_i)$. By the proof of [22], Lemma 4.1.3, p.187 and Corollary 4.1.4, p.187, the measure μ can be recovered from the restriction of $u_{x_i}(z,\mu)$ to $\mathbb{P}^1(\mathbb{C}_v) \setminus (E_v \cup \{x_i\})$. Hence, $\mu = \mu_i$.

Let D be a common denominator for the s_i and for each n, put

$$f^{(n)}(z) := \prod_{i=1}^{m} f_i^{(n)}(z)^{Ds_i}, \tag{4.9}$$

which has degree DN_n. Let the zeros and poles of $f^{(n)}(z)$ be $\alpha_j^{(n)}$, $\beta_j^{(n)}$, $j = 1, \cdots, DN_n$, listed with multiplicities; of course, these are the $\alpha_{tj}^{(n)}$ and the x_i. Consider $\prod_{j=1}^{DN_n}[z,\alpha_j^{(n)}]_{\mathfrak{X},\vec{s}}$; there is a constant $C' \in |\mathbb{C}_v^\times|$, depending only on the choice of $[z,x]_{x_i}$, $\mathfrak{X}$, the $\alpha_j^{(n)}$ and N_n such that

$$\begin{aligned} \prod_{j=1}^{DN_n}[z,\alpha_j^{(n)}]_{\mathfrak{X},\vec{s}} &= \prod_{j=1}^{DN_n}\prod_{i=1}^{m}[z,\alpha_j^{(n)}]_{x_i}^{s_i} \\ &= C' \cdot \prod_{i=1}^{m}\frac{1}{|z-x_i|_v^{DN_n s_i}} \cdot \prod_{j=1}^{DN_n}|z-\alpha_j^{(n)}|_v. \end{aligned}$$

After looking at the zeros and poles of $f^{(n)}(z)$, one sees that there is a constant $C_n \in |\mathbb{C}_v^\times|$ such that for all z,

$$|f^{(n)}(z)|_v = C_n \cdot \prod_{j=1}^{DN_n}[z,\alpha_j^{(n)}]_{\mathfrak{X},\vec{s}}. \tag{4.10}$$

Let $\nu^{(n)}$ be the probability measure which gives weight $\frac{1}{DN_n}$ to each $\alpha_j^{(n)}$. Since there are Ds_i copies of each $\alpha_{tj}^{(n)}$ in the set $\{\alpha_j^{(n)} : j = 1, \cdots, DN_n\}$ and $\mu_i^{(n)}(\alpha_{tj}^{(n)}) = 1/N_n$, we have

$$\nu^{(n)} = \sum_{i=1}^{m} s_i \cdot \mu_i^{(n)}. \tag{4.11}$$

Combining (4.7), (4.9) and (4.10), it follows that

$$\begin{aligned}
u_{\mathfrak{X},\vec{s}}(z,\nu^{(n)}) &= \int -\log_v([z,x]_{\mathfrak{X},\vec{s}})d\nu^{(n)}(x) \\
&= \sum_{j=1}^{DN_n}(-\log_v([z,\alpha_j^{(n)}]_{\mathfrak{X},\vec{s}})\cdot\frac{1}{DN_n} \\
&= -\frac{1}{DN_n}\log_v\Big(\prod_{j=1}^{DN_n}[z,\alpha_j^{(n)}]_{\mathfrak{X},\vec{s}}\Big) \\
&= \frac{1}{DN_n}\log_v(C_n)-\frac{1}{DN_n}\log_v(|f^{(n)}(z)|_v) \\
&= \frac{1}{DN_n}\log_v(C_n)+\sum_{i=1}^{m}s_i\big(-\frac{1}{N_n}\log_v(|f_i^{(n)}(z)|_v)\big) \\
&= \frac{1}{DN_n}\log_v(C_n)+\sum_{i=1}^{m}s_i\cdot u_{x_i}(z,\mu_i^{(n)}). \qquad (4.12)
\end{aligned}$$

Put $\nu := \sum_{i=1}^{m} s_i\cdot\mu_i$. Since the $\mu_i^{(n)}$ converge weakly to the μ_i, the $\nu^{(n)}$ converge weakly to ν by (4.11). Hence, for all $z\notin E_v$,

$$\lim_{n\to\infty} u_{\mathfrak{X},\vec{s}}(z,\nu^{(n)}) = u_{\mathfrak{X},\vec{s}}(z,\nu). \qquad (4.13)$$

Combining (4.8), (4.12) and (4.13), it follows that for all $z\notin E_v$,

$$\begin{aligned}
u_{\mathfrak{X},\vec{s}}(z,\nu) &= \lim_{n\to\infty} u_{\mathfrak{X},\vec{s}}(z,\nu^{(n)}) \\
&= \lim_{n\to\infty}\frac{1}{DN_n}\log_v(C_n)+\sum_{i=1}^{m}s_i\cdot u_{x_i}(z,\mu_i^{(n)}) \\
&= C+\sum_{i=1}^{m}s_i\cdot u_{x_i}(z;E_v), \qquad (4.14)
\end{aligned}$$

where $C=\lim_{n\to\infty}\frac{\log_v(C_n)}{DN_n}$ exists because the potential functions $u_{\mathfrak{X},\vec{s}}(z,\nu)$ and $u_{x_i}(z;E_v)$ are finite for all $z\notin E_v\cup\mathfrak{X}$.

It follows from [22], Theorem 4.1.11, p.195 that for each i, there is a set $\mathfrak{e}_i\subset E_v$ of capacity 0 such that $u_{x_i}(z;E_v)=V_{x_i}(E_v)$ for all $z\in E_v\setminus\mathfrak{e}_i$. Since $u_{x_i}(z;E_v)\le V_{x_i}(E_v)$ for all z and $u_{x_i}(z;E_v)$ is lower semicontinuous for all $z\neq x_i$, $u_{x_i}(z;E_v)$ is continuous at each $z\in E_v\setminus\mathfrak{e}_i$. Put $\mathfrak{e}:=\cup_{i=1}^m\mathfrak{e}_i$. By the $(\mathfrak{X},\vec{s})$-analogue of [22], Lemma 4.1.9, p.193, it follows from (4.14) that for all $z \in E_v \setminus \mathfrak{e}$,

$$\begin{aligned} u_{\mathfrak{X},\vec{s}}(z,\nu) &= \liminf_{\substack{x\to z\\ x\notin E_v}} u_{\mathfrak{X},\vec{s}}(x,\nu) \\ &= \liminf_{\substack{x\to z\\ x\notin E_v}} (C + \sum_{i=1}^{m} s_i \cdot u_{x_i}(z;E_v)) \\ &= C + \sum_{i=1}^{m} s_i \cdot V_{x_i}(E_v), \end{aligned} \tag{4.15}$$

which is a constant. Noting that $\mathfrak{e}$ has capacity 0 by [22], Corollary 4.1.15, p.199, the $(\mathfrak{X},\vec{s})$-analogue of [22], Proposition 4.1.23, p.211 shows that ν is the $(\mathfrak{X},\vec{s})$-equilibrium distribution $\mu_{\mathfrak{X},\vec{s}}$ of E_v, so $\mu_0 := \mu_{\mathfrak{X},\vec{s}} = \sum_{i=1}^{m} s_i \cdot \mu_i$. Moreover, $u_{\mathfrak{X},\vec{s}}(z;E_v) = u_{\mathfrak{X},\vec{s}}(z,\nu)$ and $V_{\mathfrak{X},\vec{s}}(E_v) = C + \sum_{i=1}^{m} s_i \cdot V_{x_i}(E_v)$.

Finally, to complete the proof, we compute $G_{\mathfrak{X},\vec{s}}(z;E_v)$ and $V_{\mathfrak{X},\vec{s}}(E_v)$:

$$\begin{aligned} G_{\mathfrak{X},\vec{s}}(z;E_v) &= V_{\mathfrak{X},\vec{s}}(E_v) - u_{\mathfrak{X},\vec{s}}(z;E_v) \\ &= C + \sum_{i=1}^{m} s_i \cdot V_{x_i}(E_v) - (C + \sum_{i=1}^{m} s_i \cdot u_{x_i}(z;E_v)) \\ &= \sum_{i=1}^{m} s_i (V_{x_i}(E_v) - u_{x_i}(z;E_v)) \\ &= \sum_{i=1}^{m} s_i \cdot G(z,x_i;E_v); \end{aligned}$$

$$\begin{aligned} V_{\mathfrak{X},\vec{s}}(E_v) &= \int\int_{E_v\times E_v} -\log_v([z,x]_{\mathfrak{X},\vec{s}}) d\mu_0(z) d\mu_0(x) \\ &= \int\int_{E_v\times E_v} -\log_v\Big(\prod_{i=1}^{m} [z,x]_{x_i}^{s_i}\Big) d\mu_0(z) d\mu_0(x) \\ &= \sum_{i=1}^{m} s_i \int\int_{E_v\times E_v} -\log_v([z,x]_{x_i}) d\Big(\sum_{j=1}^{m} s_j \cdot \mu_j\Big)(z) d\Big(\sum_{k=1}^{m} s_k \cdot \mu_k\Big)(x) \\ &= \sum_{i,j,k=1}^{m} s_i s_j s_k \int\int_{E_v\times E_v} -\log_v([z,x]_{x_i}) d\mu_j(z) d\mu_k(x). \end{aligned}$$

□

Now let $\mathfrak{B}(a_h, r_h)$, for $h = 1, \cdots, t$, be isometrically parametrizable balls, disjoint from each other and from $\mathfrak{X}$. Fix a set $E_{v,h} \subset \mathfrak{B}(a_h, r_h)$ of positive capacity for each h and put $E_v := \cup_{h=1}^t E_{v,h}$. Let $\mu_{h,0} = \mu_{\mathfrak{X},\vec{s},h}$ be the $(\mathfrak{X}, \vec{s})$-equilibrium distribution of $E_{v,h}$ and $u_{\mathfrak{X},\vec{s}}(z; E_{v,h}) = \int_{E_{v,h}} -\log_v([z,x]_{\mathfrak{X},\vec{s}}) d\mu_{h,0}(x)$ be the $(\mathfrak{X}, \vec{s})$-potential function of $E_{v,h}$.

Proposition 4.4 *Let $E_v := \cup_{h=1}^t E_{v,h}$ be as above. For each $\vec{s}$, there are weights $\kappa_h > 0$ (depending on $\vec{s}$) with $\sum_{h=1}^t \kappa_h = 1$ such that*

$$\mu_{\mathfrak{X},\vec{s}} = \sum_{h=1}^{t} \kappa_h \ \mu_{h,0}, \tag{4.16}$$

$$u_{\mathfrak{X},\vec{s}}(z; E_v) = \sum_{h=1}^{t} \kappa_h \ u_{\mathfrak{X},\vec{s}}(z; E_{v,h}). \tag{4.17}$$

Proof: Put $\kappa_h := \mu_{\mathfrak{X},\vec{s}}(E_{v,h})$, which depends on $\vec{s}$. Since $E_{v,h}$ has positive capacity, it follows from [22], Lemma 4.1.7, p.193 that $\kappa_h > 0$. The same arguments as in Proposition 3.12 yield (4.16) and (4.17). Finally, $1 = \mu_0(E_v) = \sum_{h=1}^t \kappa_h \, \mu_{h,0}(E_v) = \sum_{h=1}^t \kappa_h \, \mu_{h,0}(E_{v,h}) = \sum_{h=1}^t \kappa_h$.
□

Remark 4.5 If μ_j is the equilibrium distribution of E_v with respect to x_j, and if $\kappa_{hj} = \mu_j(E_{v,h})$, it follows from (4.2) that $\kappa_h = \sum_{j=1}^m s_j \ \kappa_{hj}$.

We will now give an example analogous to Example 3.10. Let $\mathfrak{B}(a, r)$ be an isometrically parametrizable ball disjoint from $\mathfrak{X}$ for some $a \in \mathbb{P}^1(K_v)$ and $r = |b|_v \in |K_v^\times|$ and put $E_v := \mathfrak{B}(a, r) \cap \mathbb{P}^1(K_v)$. Suppose that $\vec{s}$ is a rational probability vector. Then there is a constant $C_{a,\vec{s}} \in |\mathbb{C}_v^\times|$ such that $[z, x]_{\mathfrak{X},\vec{s}} = C_{a,\vec{s}} \ \|z, x\|_v$ for all $z, \ x \in \mathfrak{B}(a, r)$ by Remark 4.1. Since $a \in \mathbb{P}^1(K_v)$, we can fix a K_v-rational isometric parametrization of $\mathfrak{B}(a, r)$ by $B(0, r)$ by [22]. As before, the equilibrium distribution of $B(0, r) \cap K_v = b\mathcal{O}_v$ with respect to ∞ is the additive Haar measure, normalized to give $b\mathcal{O}_v$ total mass 1.

Example 4.6 Let $E_v := \mathfrak{B}(a, r) \cap \mathbb{P}^1(K_v)$ be as above. Just as in Example 3.10, the $(\mathfrak{X}, \vec{s})$-equilibrium distribution of E_v is the pushforward of the additive Haar measure on K_v, normalized to give E_v total mass 1, and the $(\mathfrak{X},\vec{s})$-potential function of E_v satisfies

$$u_{\mathfrak{X},\vec{s}}(z;E_v)=\begin{cases}\frac{1}{q_v-1}-\log_v(C_{a,\vec{s}}\cdot r) & \text{for } z\in E_v\\ -\log_v([z,a]_{\mathfrak{X},\vec{s}}) & \text{for } z\notin \mathfrak{B}(a,r).\end{cases} \tag{4.18}$$

Until further notice, we will assume that E_v is a compact set in $\mathbb{P}^1(K_v)$ of the form $E_v=\bigcup_{h=1}^t E_{v,h}$, where $E_{v,h}=\mathfrak{B}(a_h,r_h)\cap\mathbb{P}^1(K_v)$ for each $h=1,\cdots,t$ and the $\mathfrak{B}(a_h,r_h)$ are disjoint from each other and from $\mathfrak{X}$, with $a_h\in\mathbb{P}^1(K_v)$ and $r_h=|b_h|_v\in|K_v^\times|$.

Proposition 4.7 *Let $\mu_0=\mu_{\mathfrak{X},\vec{s}}$ be the $(\mathfrak{X},\vec{s})$-equilibrium distribution of E_v and let $C_{h,\vec{s}}$ be the constant such that $[z,x]_{\mathfrak{X},\vec{s}}=C_{h,\vec{s}}\cdot\|z,x\|_v$ for all $z,\ x\in\mathfrak{B}(a_h,r_h)$. Then the $(\mathfrak{X},\vec{s})$-Robin constant $V:=V_{\mathfrak{X},\vec{s}}(E_v)$ belongs to $\mathbb{Q}$; the weights $\kappa_h:=\mu_0(E_{v,h})$ belong to $\mathbb{Q}_{>0}$, satisfying $\mu_0=\sum_{h=1}^t\kappa_h\mu_{h,0}$; and V and the κ_h are uniquely determined by the $t+1$ linear equations*

$$0\cdot V+\sum_{h=1}^t\kappa_h=1,\quad \textit{and}$$
$$V+\kappa_j\cdot\left(\log_v(C_{j,\vec{s}}\cdot r_j)-\frac{1}{q_v-1}\right)+\sum_{\substack{h=1\\h\neq j}}^t\kappa_h\cdot(\log_v([a_j,a_h]_{\mathfrak{X},\vec{s}}))=0$$

for each $j=1,\cdots,t$.

Proof: The first equation and $\mu_0=\sum_{h=1}^t\kappa_h\mu_{h,0}$ follow from Proposition 4.4. The rest of the equations follows from Example 4.6, upon evaluating $u_{\mathfrak{X},\vec{s}}(z;E_v)$ at $a_1,\cdots,a_t$. See the proof of Proposition 3.13. □

Remark 4.8 For each $h=1,\cdots,t$, since $V_\infty(E_{v,h})=V_\infty(b_h\mathcal{O}_v)=\frac{1}{q_v-1}-\log_v(r_h)$ by Example 3.8, Proposition 4.7 implies that

$$\begin{aligned}V_{\mathfrak{X},\vec{s}}(E_v)&=\kappa_h\Big(-\log_v(C_{h,\vec{s}}\cdot r_h)+\frac{1}{q_v-1}\Big)+\sum_{\substack{j=1\\j\neq h}}^t\kappa_j\cdot\big(-\log_v([a_h,a_j]_{\mathfrak{X},\vec{s}})\big)\\&=\kappa_h\big(-\log_v(C_{h,\vec{s}})+V_\infty(E_{v,h})\big)+\sum_{\substack{j=1\\j\neq h}}^t\kappa_j\cdot\big(-\log_v([a_h,a_j]_{\mathfrak{X},\vec{s}})\big).\end{aligned}$$

Chapter 5

Regular Sequences

Fix a uniformizing element π_v at v and let q_v be the order of the residue field of $\mathcal{O}_v$. Let $\beta_v(k)$ be a set of representatives for $\mathcal{O}_v/\pi_v\mathcal{O}_v$ with $\beta_v(0) = 0$ for $k = 0, \cdots, q_v - 1$. For $k \geq q_v$, writing $k = \sum_{i=0}^N d_i(k) q_v^i$, where $N = \lfloor \log_v(k) \rfloor$ and $0 \leq d_i(k) \leq q_v - 1$, put $\beta_v(k) := \sum_{i=0}^N \beta_v(d_i(k))\pi_v^i \in \mathcal{O}_v$. Then for each n, $\{\beta_v(k)\}_{0 \leq k < q_v^n}$ is a system of coset representatives for $\mathcal{O}_v/\pi_v^n\mathcal{O}_v$. We call $\{\beta_v(k)\}_{0\leq k<\infty}$ the *basic well-distributed sequence in* $\mathcal{O}_v$.

Define $\mathrm{val}_{q_v}(k)$ to be the smallest i so that $d_i(k) \neq 0$ and put $\mathrm{val}_{q_v}(0) := \infty$; then for all $k \geq 0$,

$$\mathrm{ord}_v(\beta_v(k)) = \mathrm{val}_{q_v}(k) \leq \lfloor \log_v(k) \rfloor \tag{5.1}$$

and for all $0 \leq k \neq \ell < n$,

$$\mathrm{ord}_v(\beta_v(k) - \beta_v(\ell)) = \mathrm{val}_{q_v}(|k - \ell|) \leq \lfloor \log_v(n) \rfloor. \tag{5.2}$$

Moreover, if $\lfloor \log_v(n) \rfloor < \log_v(n)$, then $\mathrm{ord}_v(\beta_v(k) - \beta_v(\ell)) < \log_v(n)$; if $\lfloor \log_v(n) \rfloor = \log_v(n)$, then $\lfloor \log_v(n-1) \rfloor < \log_v(n)$, so $\mathrm{ord}_v(\beta_v(k) - \beta_v(\ell)) = \mathrm{val}_{q_v}(|k-\ell|) \leq \mathrm{val}_{q_v}(n-1) \leq \lfloor \log_v(n-1) \rfloor < \log_v(n)$. Thus, for all $0 \leq k \neq \ell < n$,

$$\mathrm{ord}_v(\beta_v(k) - \beta_v(\ell)) < \log_v(n). \tag{5.3}$$

Definition 5.1 A *basic regular sequence of length n in $\mathcal{O}_v$* is a sequence $\{\alpha_k\}_{0\leq k<n} \subset \mathcal{O}_v$ such that for each k, $\mathrm{ord}_v(\alpha_k - \beta_v(k)) \geq \log_v(n)$.

Suppose $\{\alpha_k\}_{0\le k<n}$ is a basic regular sequence of length n in $\mathcal{O}_v$. For all $0 \le k \ne \ell < n$, it follows from (5.1), (5.2), and (5.3) that

$$\begin{aligned}
&\mathrm{ord}_v(\alpha_k) = \mathrm{ord}_v(\alpha_k - \beta_v(k) + \beta_v(k)) = \mathrm{ord}_v(\beta_v(k)) = \mathrm{val}_{q_v}(k),\\
&\mathrm{ord}_v(\alpha_k - \alpha_\ell) = \mathrm{ord}_v(\beta_v(k) - \beta_v(\ell)) \le \lfloor \log_v(n) \rfloor,\\
&\mathrm{ord}_v(\alpha_k - \alpha_\ell) < \log_v(n).
\end{aligned}$$

Remark 5.2 Let $\{\alpha_k\}_{0\le k<n}$ be a basic regular sequence of length n in $\mathcal{O}_v$. Fix $z \in \mathcal{O}_v$. Since $\{\beta_v(k)\}_{0\le k<q_v{}^{\lfloor \log_v(n)\rfloor}}$ is a system of coset representatives for $\mathcal{O}_v/\pi_v^{\lfloor \log_v(n)\rfloor}\mathcal{O}_v$, there is a $0 \le J < n$ such that $z \in \beta_v(J) + \pi_v^{\lfloor \log_v(n)\rfloor}\mathcal{O}_v$. Thus $\mathrm{ord}_v(z - \beta_v(J)) \ge \lfloor \log_v(n)\rfloor$. It follows from the definition of a basic regular sequence that $\mathrm{ord}_v(z - \alpha_J) \ge \lfloor \log_v(n)\rfloor$.

Lemma 5.3 *Let $\{\alpha_k\}_{0\le k<n}$ be a basic regular sequence of length n in $\mathcal{O}_v$. For some $0 \le J < n$, there is a point $z \in \mathcal{O}_v$ with $\mathrm{ord}_v(z-\alpha_J) = \lfloor \log_v(n)\rfloor$ such that $\mathrm{ord}_v(z - \alpha_J) = \max_{0\le k<n} \mathrm{ord}_v(z-\alpha_k)$.*

Proof: First, suppose that $\lfloor \log_v(n)\rfloor = \log_v(n)$. Fix any J, $0 \le J < q_v^{\lfloor \log_v(n)\rfloor} = n$ and choose $z \in \beta_v(J) + \pi_v^{\lfloor \log_v(n)\rfloor}\mathcal{O}_v$ with $\mathrm{ord}_v(z - \beta_v(J)) = \lfloor \log_v(n)\rfloor$. For each $k \ne J$, it follows that $\mathrm{ord}_v(z-\beta_v(k)) < \lfloor \log_v(n)\rfloor$ since $z \notin \beta_v(k) + \pi_v^{\lfloor \log_v(n)\rfloor}\mathcal{O}_v$. Hence $\mathrm{ord}_v(z - \beta_v(J)) = \lfloor \log_v(n)\rfloor = \max_{0\le k<n} \mathrm{ord}_v(z-\beta_v(k))$. By the definition of a basic regular sequence, it follows that $\mathrm{ord}_v(z-\alpha_J) \ge \mathrm{ord}_v(z-\beta_v(J)) = \lfloor \log_v(n)\rfloor$ and $\mathrm{ord}_v(z-\alpha_k) = \mathrm{ord}_v(z-\beta_v(k)) < \lfloor \log_v(n)\rfloor$. Thus $\mathrm{ord}_v(z-\alpha_J) = \max_{0\le k<n}\mathrm{ord}_v(z-\alpha_k)$.

Now, suppose that $\lfloor \log_v(n)\rfloor < \log_v(n)$, so $\lfloor \log_v(n)\rfloor < \log_v(n) < \lfloor \log_v(n)\rfloor + 1$. Since $n < q_v^{\lfloor \log_v(n)\rfloor+1}$, we can choose $z \in \mathcal{O}_v$ such that $z \notin \cup_{k=0}^{n-1}\left(\beta_v(k) + \pi_v^{\lfloor \log_v(n)\rfloor+1}\mathcal{O}_v\right)$. Then $\mathrm{ord}_v(z-\beta_v(k)) < \lfloor \log_v(n)\rfloor + 1$ for each $k = 0, \cdots, n-1$. But there is $0 \le J < q_v^{\lfloor \log_v(n)\rfloor}$ such that $z \in \beta_v(J) + \pi_v^{\lfloor \log_v(n)\rfloor}\mathcal{O}_v$. It follows that $\mathrm{ord}_v(z-\beta_v(J)) \ge \lfloor \log_v(n)\rfloor$. Hence $\mathrm{ord}_v(z-\beta_v(J)) = \lfloor \log_v(n)\rfloor = \max_{0\le k<n}\mathrm{ord}_v(z-\beta_v(k))$. By the same argument as above using $\lfloor \log_v(n)\rfloor < \log_v(n)$, we have $\mathrm{ord}_v(z-\alpha_J) = \lfloor \log_v(n)\rfloor = \max_{0\le k<n}\mathrm{ord}_v(z-\alpha_k)$. □

Now, fix $a \in K_v$ and $r = |b|_v \in |K_v^\times|$. Recall that $\{\beta_v(k)\}_{0\le k<\infty}$ is the basic well-distributed sequence in $\mathcal{O}_v$.

Definition 5.4 Let $I = \{i_0, i_0+1, \cdots, i_0+n-1\}$ be a sequence of n consecutive non-negative integers. A *regular sequence of length n in $a+b\mathcal{O}_v$ attached to I* is a sequence $\{\alpha_j\}_{j\in I} \subset a+b\mathcal{O}_v$ for which there is a r-isometry $\lambda : B(0,1) \to B(a,r)$ defined by a K_v-rational power series taking $\mathcal{O}_v$ to $a+b\mathcal{O}_v$, such that for each $j \in I$,

$$\mathrm{ord}_v(\alpha_j - \lambda(\beta_v(j))) \geq \mathrm{ord}_v(b) + \log_v(n).$$

We will often simply speak of a "regular sequence of length n in $a+b\mathcal{O}_v$" if I and λ are understood from context. Let $\{\alpha_j\}_{j\in I}$ be a regular sequence of length n in $a+b\mathcal{O}_v$. For each $j \neq k$ in I, we have

$$\begin{aligned} \mathrm{ord}_v(\alpha_j - \alpha_k) &= \mathrm{ord}_v(\alpha_j - \lambda(\beta_v(j)) + \lambda(\beta_v(j)) - (\alpha_k - \lambda(\beta_v(k)) + \lambda(\beta_v(j)))) \\ &= \mathrm{ord}_v(b) + \mathrm{val}_{q_v}(|k-\ell|) \leq \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor \end{aligned} \quad (5.4)$$

since $\mathrm{ord}_v(\lambda(\beta_v(j)) - \lambda(\beta_v(k))) = -\log_v(r) + \mathrm{ord}_v(\beta_v(j) - \beta_v(k)) = \mathrm{ord}_v(b) + \mathrm{val}_{q_v}(|k-\ell|) < \mathrm{ord}_v(b) + \log_v(n)$ by (5.3). Moreover, it follows from (5.3) that

$$\mathrm{ord}_v(\alpha_j - \alpha_k) < \mathrm{ord}_v(b) + \log_v(n). \quad (5.5)$$

Lemma 5.5 *For any regular sequence $\{\alpha_j\}_{j\in I}$ of length n in $\mathcal{O}_v$ and $z \in \mathcal{O}_v$, there exists $J \in I$ such that* $\mathrm{ord}_v(z - \alpha_J) \geq \lfloor \log_v(n) \rfloor$.

Proof: Note that since $\mathcal{O}_v = 0 + 1\cdot\mathcal{O}_v$, the r-isometry in Definition 5.4 is actually an isometry in this case. Write $z = \sum_{j\geq 0} b_j(z)\pi_v^j$, where the $b_j(z)$ belong to the set $\{\beta_v(0), \cdots, \beta_v(q_v - 1)\}$ of representatives for $\mathcal{O}_v/\pi_v\mathcal{O}_v$, and put $\ell := \lfloor \log_v(n) \rfloor$. Let

$$\widetilde{z} = \sum_{j=0}^{\ell} b_j(z)\pi_v^j \in \mathcal{O}_v.$$

Since $\lambda^{-1}(\widetilde{z}) \in \mathcal{O}_v$, there exists an integer J_0 with $0 \leq J_0 < {q_v}^{\ell}$ such that $\mathrm{ord}_v(\beta_v(J_0) - \lambda^{-1}(\widetilde{z})) \geq \ell$ by Remark 5.2. Noting that ${q_v}^{\ell} \leq n$, there exists $J \in I$ such that $J \equiv J_0 \pmod{{q_v}^{\ell}}$. If $J = J_0$, then $\mathrm{ord}_v(\beta_v(J) - \lambda^{-1}(\widetilde{z})) \geq \ell$. Since λ is an isometry, $\mathrm{ord}_v(\lambda(\beta_v(J)) - \widetilde{z}) = \mathrm{ord}_v(\lambda(\beta_v(J)) - \lambda(\lambda^{-1}(\widetilde{z}))) = \mathrm{ord}_v(\beta_v(J)) - \lambda^{-1}(\widetilde{z})) \geq \ell$. Hence, we have $\mathrm{ord}_v(\alpha_J - z) \geq \ell$

because $\mathrm{ord}_v(z-\widetilde{z}) > \ell$ and $\mathrm{ord}_v(\alpha_J - \lambda(\beta_v(J))) \geq \log_v(n) \geq \ell$. If $J \neq J_0$, then $\mathrm{ord}_v(\beta_v(J) - \beta_v(J_0)) = \mathrm{val}_{q_v}(|J-J_0|) \geq \ell$, since $J \equiv J_0 \pmod{{q_v}^\ell}$. Hence we get $\mathrm{ord}_v(\beta_v(J) - \lambda^{-1}(\widetilde{z})) \geq \ell$. With the same argument, we have $\mathrm{ord}_v(\alpha_J - z) \geq \ell$. □

Corollary 5.6 *For any regular sequence $\{\alpha_j\}_{j\in I}$ of length n in $a+b\mathcal{O}_v$ and $z \in a+b\mathcal{O}_v$, there is an index $J \in I$ such that* $\mathrm{ord}_v(z-\alpha_J) \geq \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor$.

Proof: Let J be an index for which $\mathrm{ord}_v(z-\alpha_J)$ is maximal. If $x \in \mathcal{O}_v$ corresponds to z by λ, then Lemma 5.5 implies that there is $j \in I$ so that $\mathrm{ord}_v(x - \beta_v(j)) \geq \lfloor \log_v(n) \rfloor$. Then we have $\mathrm{ord}_v(z-\alpha_J) \geq \mathrm{ord}_v(z-\alpha_j) = \mathrm{ord}_v(z - \lambda(\beta_v(j)) - (\alpha_j - \lambda(\beta_v(j)))) = \mathrm{ord}_v(\lambda(x) - \lambda(\beta_v(j)) - (\alpha_j - \lambda(\beta_v(j)))) \geq \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor$ because $\mathrm{ord}_v(\lambda(x) - \lambda(\beta_v(j))) = \mathrm{ord}_v(b) + \mathrm{ord}_v(x-\beta_v(j)) \geq \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor$ and $\mathrm{ord}_v(\alpha_j - \lambda(\beta_v(j))) \geq \mathrm{ord}_v(b) + \log_v(n)$.
□

Lemma 5.7 *For any regular sequence $\{\alpha_j\}_{j\in I}$ of length n in $a+b\mathcal{O}_v$ and $z \in a+b\mathcal{O}_v$, let $J \in I$ be an index for which* $\mathrm{ord}_v(z-\alpha_J)$ *is maximal. Then for each $j \neq J$, we have*

$$\mathrm{ord}_v(z-\alpha_j) = \mathrm{ord}_v(\alpha_J - \alpha_j).$$

Proof: If $\mathrm{ord}_v(z-\alpha_j) < \mathrm{ord}_v(z-\alpha_J)$, then $\mathrm{ord}_v(\alpha_J - \alpha_j) = \mathrm{ord}_v(z-\alpha_j-(z-\alpha_J)) = \mathrm{ord}_v(z-\alpha_j)$.

Now, suppose that $\mathrm{ord}_v(z-\alpha_j) = \mathrm{ord}_v(z-\alpha_J)$. It follows from Corollary 5.6 and the maximality of $\mathrm{ord}_v(z-\alpha_J)$ that $\mathrm{ord}_v(z-\alpha_J) \geq \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor$. By (5.4), we have

$$\begin{aligned} \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor &\leq \mathrm{ord}_v(z-\alpha_J) = \mathrm{ord}_v(z-\alpha_j) \leq \mathrm{ord}_v(z-\alpha_j-(z-\alpha_J)) \\ &= \mathrm{ord}_v(\alpha_J-\alpha_j) \leq \mathrm{ord}_v(b) + \lfloor \log_v(n) \rfloor. \end{aligned}$$

Hence $\mathrm{ord}_v(\alpha_J - \alpha_j) = \mathrm{ord}_v(z-\alpha_j)$. □

Now, for each $n \in \mathbb{Z}_{>0}$, write $n = \sum_{i\geq 0} d_i {q_v}^i = \sum_{i\geq 0} d_i(n) {q_v}^i$ with $0 \leq d_i < q_v$. Since there are exactly $\left\lfloor \frac{n}{{q_v}^k} \right\rfloor$ numbers j in the range $1 \leq j \leq n$ for which $\mathrm{val}_{q_v}(j) \geq k$, we have

$$\begin{aligned}
&\mathrm{ord}_v\left(\prod_{j=0}^{n-1}(\beta_v(n) - \beta_v(j))\right) = \sum_{j=0}^{n-1} \mathrm{ord}_v(\beta_v(n) - \beta_v(j)) \\
&= \sum_{j=0}^{n-1} \mathrm{val}_{q_v}(n-j) = \sum_{j=1}^{n} \mathrm{val}_{q_v}(j) \\
&= \sum_{k\geq 1} \left\lfloor \frac{n}{{q_v}^k} \right\rfloor = (d_1 + d_2 q_v + d_3 {q_v}^2 + \cdots) + (d_2 + d_3 q_v + \cdots) + \cdots \\
&= d_1 \frac{q_v - 1}{q_v - 1} + d_2 \frac{{q_v}^2 - 1}{q_v - 1} + d_3 \frac{{q_v}^3 - 1}{q_v - 1} + \cdots \\
&= \frac{d_0 + d_1 q_v + d_2 {q_v}^2 + \cdots}{q_v - 1} - \frac{d_0 + d_1 + d_2 + \cdots}{q_v - 1} \\
&= \frac{n}{q_v - 1} - \frac{1}{q_v - 1} \sum_{i\geq 0} d_i(n). \qquad (5.6)
\end{aligned}$$

Lemma 5.8 *Given an integer $n > 1$, fix k, $1 \leq k < n$, and write $k = \sum_{i\geq 0} d_i(k) {q_v}^i$. Then we have*

$$\sum_{i\geq 0} d_i(k) \leq (q_v - 1)\lceil \log_v(n) \rceil,$$

and the equality holds only when $n = q_v^{r+1}$ and $k = n - 1$ for some $r \in \mathbb{Z}_{\geq 0}$.

Proof: Suppose that $q_v^r \leq k < q_v^{r+1}$ for some $r \in \mathbb{Z}_{\geq 0}$ and write $k = \sum_{i=0}^{r} d_i(k) {q_v}^i$ with $d_r(k) \neq 0$. It follows that

$$1 \leq \sum_{i=0}^{r} d_i(k) \leq (q_v - 1)(r + 1). \qquad (5.7)$$

The second equality holds only when each $d_i(k) = q_v - 1$, i.e., $k = (q_v - 1)(1 + q_v + \cdots + q_v^r) = q_v^{r+1} - 1$. In this situation, we have $n \geq q_v^{r+1}$, so

$$\lfloor \log_v(n) \rfloor \geq r + 1. \qquad (5.8)$$

If we have $\sum_{i=0}^{r} d_i(k) = (q_v - 1)\lceil \log_v(n) \rceil$, it follows from (5.7) and (5.8) that $r + 1 \leq \lfloor \log_v(n) \rfloor \leq \lceil \log_v(n) \rceil \leq r + 1$, so $\lfloor \log_v(n) \rfloor = \lceil \log_v(n) \rceil = \log_v(n) = r + 1$. Hence $n = q_v^{r+1}$.

□

Proposition 5.9 *Fix the normalization* $[z,x]_\infty = |z-x|_v$, *so that* $V_\infty(a+b\mathcal{O}_v) = \frac{1}{q_v-1} + \mathrm{ord}_v(b)$ *(see Example 3.8). For each* $n>1$, *let* $\{\alpha_j\}_{j\in I}$ *be a regular sequence of length* n *in* $a+b\mathcal{O}_v$.

(i) For each $J \in I$,

$$nV_\infty(a+b\mathcal{O}_v) - 2\log_v(n) - \mathrm{ord}_v(b) - 2 < \mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(\alpha_J-\alpha_j)\Bigg) < nV_\infty(a+b\mathcal{O}_v) - \mathrm{ord}_v(b).$$

(ii) Put $f(z) := \prod_{j\in I}(z-\alpha_j)$. *For each* $z \in a+b\mathcal{O}_v$, *if* $J\in I$ *is an index for which* $\mathrm{ord}_v(z-\alpha_J)$ *is maximal, then we have*

$$nV_\infty(a+b\mathcal{O}_v) - \log_v(n) - 3 < \mathrm{ord}_v(f(z)) < nV_\infty(a+b\mathcal{O}_v) - \mathrm{ord}_v(b) + \mathrm{ord}_v(z-\alpha_J).$$

In particular, we see that $\|f\|_{a+b\mathcal{O}_v} \leq q_v{}^{-nV_\infty(a+b\mathcal{O}_v)+\log_v(n)+3}$.

Proof: (i) By (5.4) and (5.6),

$$\begin{aligned}
&\mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(\alpha_J-\alpha_j)\Bigg) = \sum_{\substack{j\in I\\ j\neq J}} \mathrm{ord}_v(\alpha_J-\alpha_j)\\
&= \sum_{\substack{j\in I\\ j\neq J}} \big(\mathrm{ord}_v(b) + \mathrm{val}_{q_v}(|J-j|)\big)\\
&= (n-1)\,\mathrm{ord}_v(b) + \sum_{j=i_0}^{J-1} \mathrm{val}_{q_v}(J-j) + \sum_{j=J+1}^{i_0+n-1} \mathrm{val}_{q_v}(j-J)\\
&= (n-1)\,\mathrm{ord}_v(b) + \sum_{j=1}^{J-i_0} \mathrm{val}_{q_v}(j) + \sum_{j=1}^{i_0+n-J-1} \mathrm{val}_{q_v}(j)\\
&= (n-1)\,\mathrm{ord}_v(b) + \frac{n}{q_v-1} - \frac{\sum_{j\geq 0} d_j(J-i_0) + \sum_{j\geq 0} d_j(i_0+n-J-1)+1}{q_v-1}.
\end{aligned}$$

We claim that $0 < A(n,J) := \sum_{j\geq 0} d_j(J-i_0) + \sum_{j\geq 0} d_j(i_0+n-J-1)+1 \leq 2(q_v-1)\lceil \log_v(n)\rceil$. Clearly, $A(n,J)>0$. If $J=i_0$ or $J=i_0+n-1$, it follows from Lemma 5.8 that $A(n,J) = \sum_{j\geq 0} d_j(n-1)+1 \leq (q_v-1)\lceil\log_v(n)\rceil + 1 \leq 2(q_v-1)\lceil\log_v(n)\rceil$ because $(q_v-1)\lceil\log_v(n)\rceil \geq 1$. Now, suppose J, $i_0 < J < i_0+n-1$. Since $J-i_0 < n-1$ and $i_0+n-J-1 < n-1$, Lemma 5.8 implies that $\sum_{j\geq 0} d_j(J-i_0) < (q_v-1)\lceil\log_v(n)\rceil$ and $\sum_{j\geq 0} d_j(i_0+n-J-1) < (q_v-1)\lceil\log_v(n)\rceil$

and hence $A(n,J) < 2(q_v-1)\lceil \log_v(n)\rceil$. This verifies the claim. Thus, we have

$$nV_\infty(a+b\mathcal{O}_v) - 2\log_v(n) - \mathrm{ord}_v(b) - 2 < \mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(\alpha_J-\alpha_j)\Bigg) < nV_\infty(a+b\mathcal{O}_v) - \mathrm{ord}_v(b).$$

(ii) For $z \in \mathcal{O}_v$, if follows from Lemma 5.7 that

$$\begin{aligned}\mathrm{ord}_v(f(z)) &= \mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(z-\alpha_j)\Bigg) + \mathrm{ord}_v(z-\alpha_J)\\ &= \mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(\alpha_J-\alpha_j)\Bigg) + \mathrm{ord}_v(z-\alpha_J).\end{aligned}$$

Since $\mathrm{ord}_v(z-\alpha_J) \geq \mathrm{ord}_v(b)+\lfloor \log_v(n)\rfloor > \mathrm{ord}_v(b)+\log_v(n)-1$ by Corollary 5.6, the assertion follows from (i). □

Corollary 5.10 *Let $f(z)$ be as above. For each $z \in a + b\widehat{\mathcal{O}_v}$, if $0 \leq J < n$ is an index for which* $\mathrm{ord}_v(z-\alpha_J)$ *is maximal, then we have*

$$\mathrm{ord}_v(f(z)) < nV_\infty(a+b\mathcal{O}_v) - \mathrm{ord}_v(b) + \mathrm{ord}_v(z-\alpha_J).$$

Proof: Since $\mathrm{ord}_v(z-\alpha_k) \leq \mathrm{ord}_v(z-\alpha_J)$ for each $k \in I$, it follows that $\mathrm{ord}_v(\alpha_J-\alpha_k) = \mathrm{ord}_v(z-\alpha_k-(z-\alpha_J)) \geq \min\{\mathrm{ord}_v(z-\alpha_k), \mathrm{ord}_v(z-\alpha_J)\} = \mathrm{ord}_v(z-\alpha_k)$. We then have

$$\begin{aligned}\mathrm{ord}_v(f(z)) &= \mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(z-\alpha_j)\Bigg) + \mathrm{ord}_v(z-\alpha_J)\\ &\leq \mathrm{ord}_v\Bigg(\prod_{\substack{j\in I\\ j\neq J}}(\alpha_J-\alpha_j)\Bigg) + \mathrm{ord}_v(z-\alpha_J)\\ &< nV_\infty(\mathcal{O}_v) - \mathrm{ord}_v(b) + \mathrm{ord}_v(z-\alpha_J).\end{aligned}$$

□

For a nonzero $b \in \mathbb{C}_v$, put $\zeta_v(j) := b\beta_v(j)$ for each $j \geq 0$. Since $\{\beta_v(j)\}_{j\geq 0}$ is the basic well-distributed sequence in $\mathcal{O}_v$, we call $\{\zeta_v(j)\}_{j\geq 0}$ the basic *well-distributed sequence in* $b\mathcal{O}_v$. For each $n \in \mathbb{Z}_{>0}$, let

$$S_{n,v,b}(z) = \prod_{j=0}^{n-1}(z-\zeta_v(j)) \tag{5.9}$$

be the Stirling polynomial of degree n for $b\mathcal{O}_v$.

Proposition 5.11 ([25]) *Suppose $f(z) \in \mathbb{C}_v[[z]]$ is a nonzero power series converging in $B(0,r)$ and having exactly one zero in $B(0,r')$, for some $0 < r' \leq r$. Assume without loss of generality that the zero is at $z = 0$, and expand $f(z) = \sum_{k=1}^{\infty} c_k z^k$. Assume also that $r' = |a|_v$ for some $a \in \mathbb{C}_v^{\times}$. Put $b = c_1\, a$, and let $\rho = |b|_v = |c_1|_v\, r'$. Then the part of the Newton polygon of $(S_{n,v,b} \circ f)(z)$ corresponding to roots in $B(0,r')$ is given by the Newton polygon of $S_{n,v,b}(z)$, translated upward by the line $y = \mathrm{ord}_v(c_1)\, x$. That is, for $0 \leq k \leq n$, if $(k, \mathrm{ord}_v(a_k))$ is a corner of the Newton polygon of $S_{n,v,b}(z)$, then $(k, \mathrm{ord}_v(a_k) + k\, \mathrm{ord}_v(c_1))$ is a corner of the Newton polygon of $(S_{n,v,b} \circ f)(z)$, and all segments of the Newton polygon of $(S_{n,v,b} \circ f)(z)$ to the right of $x = n$ have slope $> \log_v(r')$. For each $j = 0, \cdots, n-1$, if we put $\widetilde{\zeta}_v(j) := f^{-1}(\zeta_v(j))$, then the same assertion holds for the Newton polygon of $(S_{n,v,b} \circ f)(z + \widetilde{\zeta}_v(j))$ and the Newton polygon of $S_{n,v,b}(z + \zeta_v(j))$.*

Proof: Fix j with $0 \leq j < n$ and consider the expansion of $S_{n,v,b}(Z)$ about $\zeta_v(j)$ and of $f(z)$ about $\widetilde{\zeta}_v(j)$. Let these be

$$S_{n,v,b}(Z) = \sum_{i=1}^{n} a_{ij}(Z - \zeta_v(j))^i,$$

$$f(z) = \zeta_v(j) + \sum_{i=1}^{\infty} c_{ij}(z - \widetilde{\zeta}_v(j))^i.$$

By Corollary 2.17, $f(z)$ defines an ρ/r'-isometry from $B(0,r')$ onto $B(0,\rho)$. Hence $a_{1j} \neq 0$ since $S_{n,v,b}(Z)$ has distinct roots, and $c_{1j} \neq 0$ since $f(z)$ is 1-1 on $B(0,r')$. Let the expansion of $(S_{n,v,b} \circ f)(z)$ abut $\widetilde{\zeta}_v(j)$ be

$$(S_{n,v,b} \circ f)(z) = \sum_{i=1}^{\infty} d_{ij}(z - \widetilde{\zeta}_v(j))^i.$$

Note that $d_{1j} = a_{1j}c_{1j} \neq 0$. For each $z \in B(0,r')$, $|f(z) - \zeta_v(j)|_v = |f(z) - f(\widetilde{\zeta}_v(j))|_v = \rho/r' \cdot |z - \widetilde{\zeta}_v(j)|_v = |c_1|_v \cdot |z - \widetilde{\zeta}_v(j)|_v$, and so $\left| c_{1j} + \sum_{i \geq 2} c_{ij}(z - \widetilde{\zeta}_v(j))^{i-1} \right|_v = |c_1|_v$. Taking $z = \widetilde{\zeta}_v(j)$, we get $|c_{1j}|_v = |c_1|_v$. Hence $(1, \mathrm{ord}_v(a_{1j}))$ and $(1, \mathrm{ord}_v(a_{1j}) + \mathrm{ord}_v(c_1))$ are the initial vertices of the Newton polygons of $S_{n,v,b}(Z)$ and $(S_{n,v,b} \circ f)(z)$, respectively.

The zeros of $S_{n,v,b}(Z + \zeta_v(j))$ in $B(0,\rho)$ are the $\zeta_v(j) - \zeta_v(j)$ for each j, and the zeros of $(S_{n,v,b} \circ f)(z + \widetilde{\zeta}_v(j))$ in $B(0,r')$ are the $\widetilde{\zeta}_v(j) - \widetilde{\zeta}_v(j)$ for each j. The slopes of the sides of the initial part of the Newton polygon of $S_{n,v,b} \circ f$ are determined by its zeros in $B(0, r')$. For each $\ell \neq j$, $-\mathrm{ord}_v(\widetilde{\zeta}_v(\ell) - \widetilde{\zeta}_v(j)) = -\mathrm{ord}_v(\zeta_v(\ell) - \zeta_v(j)) + \mathrm{ord}_v(c_1)$, because $f(z)$ is an ρ/r'-isometry. Thus the part of the Newton polygon of $(S_{n,v,b} \circ f)(z + \widetilde{\zeta}_v(j))$ corresponding to the roots in $B(0, r')$ has the same shape as the Newton polygon of $S_{n,v,b}(Z + \zeta_v(j))$, but the slope of each segment has been increased by $\mathrm{ord}_v(c_1)$. Note that the $\widetilde{\zeta}_v(j)$ are the only zeros of $(S_{n,v,b} \circ f)(z + \widetilde{\zeta}_v(j))$ in $B(0, r')$, so all the other sides of the Newton polygon have slopes geater than $\log_v(r')$. □

Now, fix a finite K-symmetric set $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \mathbb{P}^1(\overline{K})$ and a K-symmetric probability vector $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{Q}^m$. Consider a compact set of the form $E_v = \bigcup_{h=1}^t E_{v,h}$ bounded away from $\mathfrak{X}$, where $E_{v,h} = \mathfrak{B}(a_h, r_h) \cap \mathbb{P}^1(K_v)$ for some $a_h \in \mathbb{P}^1(K_v)$ and some $r_h = |b_h|_v \in |K_v^\times|$. By the ultrametric inequality, we can assume that the balls $\mathfrak{B}(a_h, r_h)$ are disjoint from each other and from $\mathfrak{X}$. Let $\mu_{\mathfrak{X},\vec{s}}$ be the $(\mathfrak{X}, \vec{s})$-equilibrium distribution of E_v. We want to construct the basic well-distributed sequence in E_v with respect to $(\mathfrak{X}, \vec{s})$ to assign elements to each $E_{v,h}$ in proportion to its weight $\mu_{\mathfrak{X},\vec{s}}(E_{v,h})$. This can be done with the following lemma due to Balinski and Young ([3], Theorem 3, p.714).

Lemma 5.12 *Let $\kappa_1, \cdots, \kappa_t$ be positive real numbers such that $\sum_{h=1}^t \kappa_h = 1$. Then there is a $1-1$ correspondence $\Phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}^t$ defined by $\Phi(n) = (\Phi_1(n), \cdots, \Phi_t(n))$ such that*

(i) for each h, $\Phi_h(0) = 0$ and $\Phi_h(n)$ is non-decreasing with n ;

(ii) for each n, $\sum_{h=1}^t \Phi_h(n) = n$;

(iii) for each n and h, $\lfloor \kappa_h n \rfloor \leq \Phi_h(n) \leq \lceil \kappa_h n \rceil$.

In particular, $|\Phi_h(n) - \kappa_h n| < 1$ and if $\kappa_h n \in \mathbb{Z}_{\geq 0}$, then $\Phi_h(n) = \kappa_h n$.

For each $n \in \mathbb{Z}_{\geq 0}$, there is exactly one index h for which $\Phi_h(n+1) = \Phi_h(n) + 1$ and $\Phi_j(n+1) = \Phi_j(n)$ for all $j \neq h$, because for each k, $\Phi_k(n+1) \geq \Phi_k(n)$ and $\sum_{k=1}^m \Phi_k(n+1) = n + 1 = \sum_{k=1}^m \Phi_k(n) + 1$. We will write $h(n)$ for this h.

Fix the K_v-rational isometric parametrization $\lambda_h : B(0, r_h) \to \mathfrak{B}(a_h, r_h)$, which is obtained by $z \mapsto a_h + z$ in appropriate affine coordinates. Consider the pushforward of the basic well-distributed sequence $\{\beta_v(k)\}$ in $\mathcal{O}_v$ to each $E_{v,h}$ given by $\beta_{v,h}(k) := \lambda_h(b_h\beta_v(k))$. Let $\kappa_h = \mu_{\mathfrak{X},\vec{s}}(E_{v,h})$. It follows from Proposition 4.7 that $\kappa_h > 0$ and $\sum_{h=1}^{t} \kappa_h = 1$.

Definition 5.13 The *basic well-distributed sequence in* E_v *with respect to* $(\mathfrak{X}, \vec{s})$ is the sequence $\{\lambda_{\mathfrak{X},\vec{s}}(k)\}_{0 \le k < \infty}$ defined by

$$\lambda_{\mathfrak{X},\vec{s}}(k) = \beta_{v,h(k)}\big(\Phi_{h(k)}(k)\big) = \lambda_{h(k)}\big(b_{h(k)} \cdot \beta_v(\Phi_{h(k)}(k))\big).$$

Note that the kth element of the sequence is assigned to $E_{v,h(k)}$, and the elements assigned to $E_{v,h}$ fill out $E_{v,h}$ like the basic well-distributed sequence in $\mathcal{O}_v$. To simplify notation, when $\mathfrak{X}$ and $\vec{s}$ are fixed, we will write $\lambda_v(k)$ for $\lambda_{\mathfrak{X},\vec{s}}(k)$. There are exactly $\Phi_h(n)$ elements in the set $\{\lambda_v(k)\}_{0 \le k < n}$ belonging to $E_{v,h}$.

Chapter 6

Construction of the Basic Local Approximating Functions for a Compact Set in $\mathbb{P}^1(K_v)$

Let v be a place of K. Fix a finite K-symmetric set $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \overline{K}$ and a K-symmetric probability vector $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{Q}_{>0}^m$. Let E_v be a compact set in $\mathbb{P}^1(K_v)$ of the form $E_v = \cup_{h=1}^t E_{v,h}$, where each $E_{v,h} = \mathfrak{B}(a_h, r_h) \cap \mathbb{P}^1(K_v)$ for some $a_h \in K_v$ and some $r_h = |b_h|_v \in |K_v^\times|$, and the $\mathfrak{B}(a_h, r_h)$ are disjoint from each other and from $\mathfrak{X}$. Put $L := K(\mathfrak{X})$ and let M be the separable closure of K in L, i.e., $M = K^{\mathrm{sep}} \cap L$. Then M/K is a Galois extension and L/M is a purely inseparable extension of degree $[L : K]_\iota$. Note that for any $\alpha \in L$, $\alpha^{[L:K]_\iota} \in M$. If u is a place of M lying above v, then there is only one place w of L lying above u. It follows that $[L_w : M_u] = [L_w : K_v]_\iota = [L : K]_\iota$, so $\alpha^{[L:K]_\iota} \in M_u$ for any $\alpha \in L_w$.

Let $\mathfrak{X}_{v,a}$, $a = 1, \cdots, m_v'$, be the Galois orbits in $\mathfrak{X}$ under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$. For each orbit $\mathfrak{X}_{v,a}$, fix a representative $x_i \in \mathfrak{X}_{v,a}$. Fix a uniformizing parameter $g_i(z) \in K(x_i)(z)$ at x_i and let $[z, x]_{x_i} = C_i \cdot \frac{\|z,x\|_v}{\|z,x_i\|_v \ \|x,x_i\|_v}$ be the canonical distance normalized so that $\lim_{z\to x_i} [z, x]_{x_i} \cdot |g_i(z)|_v = 1$. By Lemma 2.28, there is a constant $C_i' \in |\mathbb{C}_v^\times|$ such that $\|z, x_i\|_v = C_i' \cdot |g_i(z)|_v$ for all z sufficiently near x_i. Let μ be a place of $K(x_i)$ lying above v. Considering $z \in K(x_i)$, since $\|z, x_i\|_v$ and $|g_i(z)|_v$ are in the value group of $K(x_i)$ with respect to μ, C_i' also belongs to the value group of $K(x_i)$ with respect to μ. Hence it follows from the proof of Lemma 2.30 that C_i belongs to the value group of $K(x_i)$ with respect to μ as well, say $C_i = |\zeta_{i,v}|_\mu$ for some $\zeta_{i,v} \in K(x_i)$. For other elements x_j in the orbit $\mathfrak{X}_{v,a}$, if $\sigma(x_i) = x_j$ for some $\sigma \in \mathrm{Gal}(\mathbb{C}_v/K_v)$, then put $g_j(z) := \sigma(g_i)(z)$, which is independent of the choice of σ because

$g_i(z) \in K(x_i)(z)$. Then we have $C_j = |\zeta_{j,v}|_{\sigma(\mu)}$, where $\zeta_{j,v} := \sigma(\zeta_{i,v}) \in K(x_j)$. In this way, the set $\{\zeta_{1,v}, \cdots, \zeta_{m,v}\}$ is K_v-symmetric.

Let N_s be the least common denominator for $s_1, \cdots, s_m$. Proposition 4.7 implies that $V_{\mathfrak{X},\vec{s}}(E_v) \in \mathbb{Q}$ and $\kappa_h = \mu_{\mathfrak{X},\vec{s}}(E_{v,h}) \in \mathbb{Q}_{>0}$ with $\sum_{h=1}^{t} \kappa_h = 1$. Let $N_{r,v}$ be the denominator of $V_{\mathfrak{X},\vec{s}}(E_v)$ and $N_{\kappa,v}$ be the least common denominator for $\kappa_1, \cdots, \kappa_t$. Put

$$N_v := N_s N_{r,v} N_{\kappa,v} [L : K]_\iota. \tag{6.1}$$

Recall that $\{\lambda_v(k)\}_{0 \le k \le \infty}$ is the basic well-distributed sequence in E_v as constructed in Definition 5.13 together with the fixed K_v-rational isometric parametrization $\lambda_h : B(0, r_h) \to \mathfrak{B}(a_h, r_h)$ for each $h = 1, \cdots, t$, which was defined by $z \mapsto a_h + z$ in appropriate affine coordinates. Let N be a sufficiently large integer divisible by N_v so that each $E_{v,h}$ receives points from $\{\lambda_v(k)\}_{0 \le k < N}$. Since $\kappa_h N$ is a positive integer, $\Phi_h(N) = \kappa_h N$ for each h. Put $N_i := N s_i$ for each $i = 1, \cdots, m$.

Definition 6.1 Let E be any field. An E-rational $(\mathfrak{X}, \vec{s})$-function is a function $f(z) \in E(z)$ whose poles are supported on $\mathfrak{X}$, and such that if $n = \deg(f)$, then ns_i is the order of the pole of f at x_i.

The following theorem provides the basic local approximating functions for a compact set in $\mathbb{P}^1(K_v)$.

Theorem 6.2 *Let E_v be a compact set, with the same notations as above. In particular, N is divisible by N_v. Then there is a K_v-rational $(\mathfrak{X}, \vec{s})$-function $f_v(z)$ of degree N such that the zeros of $f_v(z)$ are distinct and form a basic well-distributed sequence of length N in E_v, and $f_v(z)$ satisfies the following properties:*

(i) If $\theta_1, \cdots, \theta_N$ are the zeros of $f_v(z)$ in E_v, then there are a number R'_v with $R'^N_v \in |K_v^\times|$ and $R'^N_v \le q_v^{-\lceil \log_v(N)+1 \rceil}$, and numbers $\rho_1, \cdots, \rho_N$ belonging to $|K_v^\times|$, with $\rho_j < 1$ for each $j = 1, \cdots, N$, such that the balls $\mathfrak{B}(\theta_j, \rho_j)$ are pairwise disjoint and $f_v(z) : \mathfrak{B}(\theta_j, \rho_j) \to$

$B(0,R_v'^N)$ is an $R_v'^N/\rho_j$-isometry satisfying

$$U_v^0 := f_v^{-1}(B(0,R_v'^N)) = \bigcup_{j=1}^{N} \mathfrak{B}(\theta_j,\rho_j) \subset \bigcup_{h=1}^{t} \mathfrak{B}(a_h,r_h),$$

$$E_v^0 := f_v^{-1}(B(0,R_v'^N) \cap K_v) = \bigcup_{j=0}^{N} (\mathfrak{B}(\theta_j,\rho_j) \cap K_v) \subset E_v;$$

(ii) For all $z \notin \bigcup_{h=1}^{t} \mathfrak{B}(a_h,r_h)$,

$$\frac{1}{N}\log_v(|f_v(z)|_v) = G_{\mathfrak{X},\vec{s}}(z;E_v) = \sum_{i=1}^{m} G(z,x_i;E_v)s_i.$$

Recall that by Remark 4.1, there is a constant $C_{h,\vec{s}}$ such that $[z,x]_{\mathfrak{X},\vec{s}} = C_{h,\vec{s}} \cdot \|z,x\|_v$ for all $z,x \in \mathfrak{B}(a_h,r_h)$; and $[z,x]_{\mathfrak{X},\vec{s}} = [a_h,a_\ell]_{\mathfrak{X},\vec{s}}$ for all $z \in \mathfrak{B}(a_h,r_\ell)$ and $x \in \mathfrak{B}(a_\ell,r_\ell)$ whenever $h \neq \ell$. Consider the "pseudo-rational function"

$$f_0(z) := \prod_{k=0}^{N-1} [z,\lambda_v(k)]_{\mathfrak{X},\vec{s}}.$$

For each $z \in \mathfrak{B}(a_h,r_h)$, write $z = \lambda_h(b_h\zeta)$ for some $\zeta \in B(0,1)$: if $\lambda_v(k) \in \mathfrak{B}(a_h,r_h)$, then $[z,\lambda_v(k)]_{\mathfrak{X},\vec{s}} = C_{h,\vec{s}} \cdot \|z,\lambda_v(k)\|_v = C_{h,\vec{s}} \cdot |b_h\zeta - b_h\beta_v(\Phi_h(k))|_v = C_{h,\vec{s}} \cdot r_h \cdot |\zeta - \beta_v(\Phi_h(k))|_v$; if $\lambda_v(k) \in \mathfrak{B}(a_\ell,r_\ell)$ for $h \neq \ell$, then $[z,\lambda_v(k)]_{\mathfrak{X},\vec{s}} = [a_h,a_\ell]_{\mathfrak{X},\vec{s}}$. Since each $E_{v,h}$ contains exactly $\Phi_h(N)$ points from the set $\{\lambda_v(k)\}_{0\le k<N}$ and since $\Phi_h(N) = \kappa_h N$ for all $h = 1,\cdots,t$, it follows from Corollary 5.10 that

$$\begin{aligned}
-\log_v(f_0(z)) = & \sum_{k=0}^{\Phi_h(N)-1} -\log_v(C_{h,\vec{s}} \cdot r_h \cdot |\zeta - \beta_v(k)|_v) \\
& + \sum_{\substack{\ell=1\\ \ell\neq h}}^{t} -\Phi_\ell(N) \cdot \log_v([a_h,a_\ell]_{\mathfrak{X},\vec{s}}) \\
= & -\Phi_h(N) \cdot \log_v(C_{h,\vec{s}} \cdot r_h) + \sum_{k=0}^{\Phi_h(N)-1} \mathrm{ord}_v(\zeta - \beta_v(k)) \\
& + \sum_{\substack{\ell=1\\ \ell\neq h}}^{t} -\Phi_\ell(N) \cdot \log_v([a_h,a_\ell]_{\mathfrak{X},\vec{s}}) \qquad (6.2) \\
< & -\Phi_h(N) \cdot \log_v(C_{h,\vec{s}} \cdot r_h) + \frac{\Phi_h(N)}{q_v - 1} + \mathrm{ord}_v(\zeta - \beta_v(J_h)) \\
& + \sum_{\substack{\ell=1\\ \ell\neq h}}^{t} -\Phi_\ell(N) \cdot \log_v([a_h,a_\ell]_{\mathfrak{X},\vec{s}})
\end{aligned}$$

$$= \Phi_h(N)\left(-\log_v(C_{h,\vec{s}} \cdot r_h) + \frac{1}{q_v - 1}\right) + \mathrm{ord}_v(\zeta - \beta_v(J_h))$$
$$+ \sum_{\substack{\ell=1\\ \ell\neq h}}^{t} -\Phi_\ell(N) \cdot \log_v([a_h, a_\ell]_{\mathfrak{X},\vec{s}})$$
$$= \kappa_h N \cdot \left(-\log_v(C_{h,\vec{s}} \cdot r_h) + \frac{1}{q_v - 1}\right) + \mathrm{ord}_v(\zeta - \beta_v(J_h))$$
$$+ \sum_{\substack{\ell=1\\ \ell\neq h}}^{t} -\kappa_\ell N \cdot \log_v([a_h, a_\ell]_{\mathfrak{X},\vec{s}})$$
$$= N\left(\kappa_h\left(-\log_v(C_{h,\vec{s}} \cdot r_h) + \frac{1}{q_v - 1}\right) + \sum_{\substack{\ell=1\\ \ell\neq h}}^{t} \kappa_\ell(-\log_v([a_h, a_\ell]_{\mathfrak{X},\vec{s}}))\right)$$
$$- \log_v(|\zeta - \beta_v(J_h)|_v).$$

Here, $0 \leq J_h < \Phi_h(N)$ is an index for which $\mathrm{ord}_v(\zeta - \beta_v(J_h))$ is maximal. Since λ_h is an isometry, $\|\lambda_h(b_h\zeta), \lambda_h(b_h\beta_v(J_h))\|_v = |b_h\zeta - b_h\beta_v(J_h)|_v = |b_h|_v \cdot |\zeta - \beta_v(J_h)|_v$, so $-\log_v(|\zeta - \beta_v(J_h)|_v) = -\log_v\left(\|\lambda_h(b_h\zeta), \lambda_h(b_h\beta_v(J_h))\|_v\right) + \log_v(r_h) = -\log_v\left(\|z, \lambda_v(J_h)\|_v\right) + \log_v(r_h)$. It follows from Remark 4.8 that for all $z \in \bigcup_{h=1}^t \mathfrak{B}(a_h, r_h)$,

$$-\log_v(f_0(z)) < NV_{\mathfrak{X},\vec{s}}(E_v) - \log_v(\|z, \lambda_v(J)\|_v) + \log_v(r),$$

where $r = \max_{1\leq h\leq t} r_h$. Here, $0 \leq J < N$ is an index for which $\|z, \lambda_v(J)\|_v$ is minimal.

Now, consider $z \notin \bigcup_{h=1}^t \mathfrak{B}(a_h, r_h)$. Since $z \notin \mathfrak{B}(a_h, r_h)$ for each h, if $\lambda_v(k) \in \mathfrak{B}(a_h, r_h)$, then $[z, \lambda_v(k)]_{\mathfrak{X},\vec{s}} = [z, a_h]_{\mathfrak{X},\vec{s}}$ by Remark 4.1. It follows from Proposition 4.4 and Example 4.6 that

$$\begin{aligned}
-\log_v(f_0(z)) &= \sum_{k=0}^{N-1} -\log_v([z, \lambda_v(k)]_{\mathfrak{X},\vec{s}}) \\
&= \sum_{h=1}^{t} \sum_{k=0}^{\Phi_h(N)-1} -\log_v([z, a_h]_{\mathfrak{X},\vec{s}}) \\
&= \sum_{h=1}^{t} \Phi_h(N) u_{\mathfrak{X},\vec{s}}(z; E_{v,h}) \\
&= N \sum_{h=1}^{t} \kappa_h u_{\mathfrak{X},\vec{s}}(z; E_{v,h}) = N u_{\mathfrak{X},\vec{s}}(z; E_v).
\end{aligned}$$

Hence, we have

$$-\frac{1}{N}\log_v(f_0(z)) = u_{\mathfrak{X},\vec{s}}(z;E_v) = V_{\mathfrak{X},\vec{s}}(E_v) - G_{\mathfrak{X},\vec{s}}(z;E_v)$$

for all $z \notin \bigcup_{h=1}^t \mathfrak{B}(a_h, r_h)$.

Thus, we have the following proposition.

Proposition 6.3 *Let $f_0(z) = \prod_{k=0}^{N-1}[z,\lambda_v(k)]_{\mathfrak{X},\vec{s}}$. Then for all $z \in \bigcup_{h=1}^t \mathfrak{B}(a_h, r_h)$,*

$$-\log_v(f_0(z)) < NV_{\mathfrak{X},\vec{s}}(E_v) - \log_v(\|z,\lambda_v(J)\|_v) + \log_v(r),$$

where $r = \max_{1\le h\le t} r_h$ and $0 \le J < N$ is an index for which $\|z,\lambda_v(J)\|_v$ is minimal; for all $z \notin \bigcup_{h=1}^t \mathfrak{B}(a_h, r_h)$,

$$-\frac{1}{N}\log_v(f_0(z)) = u_{\mathfrak{X},\vec{s}}(z;E_v) = V_{\mathfrak{X},\vec{s}}(E_v) - G_{\mathfrak{X},\vec{s}}(z;E_v).$$

Now, we want to construct a rational function $f_1(z)$ for this pseudo-rational function $f_0(z)$, satisfying $|f_1(z)|_v = f_0(z)$ for all z. To do so, we need to express $f_0(z)$ in terms of the absolute value $|\cdot|_v$. If $\lambda_v(k) \in E_{v,h}$, then $\|\lambda_v(k), x_i\|_v = \|a_h, x_i\|_v$ because $x_i \notin E_v$. Noting that $\sum_{i=1}^m s_i = 1$ and $\sum_{h=1}^t \Phi_h(N) = N$,

$$\begin{aligned}
f_0(z) &= \prod_{k=0}^{N-1}[z,\lambda_v(k)]_{\mathfrak{X},\vec{s}} = \prod_{k=0}^{N-1}\prod_{i=1}^{m}[z,\lambda_v(k)]_{x_i}^{s_i} \\
&= \prod_{k=0}^{N-1}\prod_{i=1}^{m} C_i^{s_i}\cdot\frac{\|z,\lambda_v(k)\|_v^{s_i}}{\|z,x_i\|_v^{s_i}\cdot\|\lambda_v(k),x_i\|_v^{s_i}} \\
&= \prod_{i=1}^{m} C_i^{Ns_i}\cdot\prod_{k=0}^{N-1}\prod_{i=1}^{m}\|z,\lambda_v(k)\|_v^{s_i} \\
&\quad\cdot\frac{1}{\prod_{i=1}^m\prod_{k=0}^{N-1}\|z,x_i\|_v^{s_i}}\cdot\frac{1}{\prod_{i=1}^m\prod_{k=0}^{N-1}\|\lambda_v(k),x_i\|_v^{s_i}}.
\end{aligned}$$

Since $N_i = Ns_i$ and $\sum_{i=1}^m s_i = 1$,

$$\begin{aligned}
f_0(z) &= \prod_{i=1}^{m} C_i^{N_i}\cdot\prod_{k=0}^{N-1}\|z,\lambda_v(k)\|_v\cdot\frac{1}{\prod_{i=1}^m\|z,x_i\|_v^{N_i}} \\
&\quad\cdot\frac{1}{\prod_{i=1}^m\prod_{h=1}^t\prod_{\lambda_v(k)\in E_{v,h}}\|\lambda_v(k),x_i\|_v^{s_i}}.
\end{aligned}$$

Next, since $\|\lambda_v(k), x_i\|_v = \|a_h, x_i\|_v$,

$$
\begin{aligned}
f_0(z) &= \prod_{i=1}^{m} C_i^{N_i} \cdot \prod_{k=0}^{N-1} \frac{|z-\lambda_v(k)|_v}{\max\{1,|z|_v\}\max\{1,|\lambda_v(k)|_v\}} \\
&\cdot \prod_{i=1}^{m} \frac{\max\{1,|z|_v\}^{N_i}\max\{1,|x_i|_v\}^{N_i}}{|z-x_i|_v^{N_i}} \cdot \frac{1}{\prod_{i=1}^{m}\prod_{h=1}^{t}\|a_h,x_i\|_v^{\Phi_h(N)s_i}} \\
&= \prod_{i=1}^{m} C_i^{N_i} \cdot \frac{\prod_{k=0}^{N-1}|z-\lambda_v(k)|_v}{\max\{1,|z|_v\}^N \cdot \prod_{k=0}^{N-1}\max\{1,|\lambda_v(k)|_v\}} \\
&\cdot \frac{\max\{1,|z|_v\}^N \cdot \prod_{i=1}^{m}\max\{1,|x_i|_v\}^{N_i}}{\prod_{i=1}^{m}|z-x_i|_v^{N_i}} \\
&\cdot \prod_{i=1}^{m}\prod_{h=1}^{t}\left(\frac{\max\{1,|a_h|_v\}\max\{1,|x_i|_v\}}{|a_h-x_i|_v}\right)^{\Phi_h(N)s_i} \\
&= \prod_{i=1}^{m} C_i^{N_i} \cdot \frac{\prod_{k=0}^{N-1}|z-\lambda_v(k)|_v}{\prod_{k=0}^{N-1}\max\{1,|\lambda_v(k)|_v\}} \cdot \frac{\prod_{i=1}^{m}\max\{1,|x_i|_v\}^{N_i}}{\prod_{i=1}^{m}|z-x_i|_v^{N_i}} \\
&\cdot \frac{\prod_{h=1}^{t}\left(\prod_{i=1}^{m}\max\{1,|a_h|_v\}^{s_i}\right)^{\Phi_h(N)} \cdot \prod_{i=1}^{m}\left(\prod_{h=1}^{t}\max\{1,|x_i|_v\}^{\Phi_h(N)}\right)^{s_i}}{\prod_{h=1}^{t}\prod_{i=1}^{m}|a_h-x_i|_v{}^{\Phi_h(N)s_i}}.
\end{aligned}
$$

Finally, since $\sum_{i=1}^{m} s_i = 1$ and $\sum_{h=1}^{t}\Phi_h(N) = N$,

$$
\begin{aligned}
f_0(z) &= \prod_{i=1}^{m} C_i^{N_i} \cdot \frac{\prod_{k=0}^{N-1}|z-\lambda_v(k)|_v}{\prod_{k=0}^{N-1}\max\{1,|\lambda_v(k)|_v\}} \cdot \frac{\prod_{i=1}^{m}\max\{1,|x_i|_v\}^{N_i}}{\prod_{i=1}^{m}|z-x_i|_v^{N_i}} \\
&\cdot \frac{\prod_{h=1}^{t}\max\{1,|a_h|_v\}^{\Phi_h(N)} \cdot \prod_{i=1}^{m}\max\{1,|x_i|_v\}^{N_i}}{\prod_{h=1}^{t}\prod_{i=1}^{m}|a_h-x_i|_v{}^{\Phi_h(N)s_i}} \\
&= \frac{\prod_{i=1}^{m} C_i^{N_i} \cdot \prod_{i=1}^{m}\max\{1,|x_i|_v\}^{2N_i} \cdot \prod_{h=1}^{t}\max\{1,|a_h|_v\}^{\Phi_h(N)}}{\prod_{k=0}^{N-1}\max\{1,|\lambda_v(k)|_v\} \cdot \prod_{h=1}^{t}\prod_{i=1}^{m}|a_h-x_i|_v^{\Phi_h(N)s_i}} \\
&\cdot \frac{\prod_{k=0}^{N-1}|z-\lambda_v(k)|_v}{\prod_{i=1}^{m}|z-x_i|_v^{N_i}}.
\end{aligned}
\tag{6.3}
$$

Put

$$
\mathfrak{C} := \frac{\prod_{i=1}^{m} C_i^{N_i} \cdot \prod_{i=1}^{m}\max\{1,|x_i|_v\}^{2N_i} \cdot \prod_{h=1}^{t}\max\{1,|a_h|_v\}^{\Phi_h(N)}}{\prod_{k=0}^{N-1}\max\{1,|\lambda_v(k)|_v\} \cdot \prod_{h=1}^{t}\prod_{i=1}^{m}|a_h-x_i|_v^{\Phi_h(N)s_i}}.
$$

We claim that $\mathfrak{C}$ belongs to $|K_v^{\times}|$. Since $[L:K]_\iota$ divides N_i, $\zeta_{i,v}^{N_i} \in M$ and $x_i^{2N_i} \in M$, so $\prod_{i=1}^{m} C_i^{N_i} \cdot \prod_{i=1}^{m}\max\{1,|x_i|_v\}^{2N_i}$ belongs to $|K_v^{\times}|$ because $\mathfrak{X}$ and the set $\{\zeta_{1,v}, \cdots, \zeta_{m,v}\}$ are K_v-symmetric. Clearly, $\prod_{h=1}^{t}\max\{1,|a_h|_v\}^{\Phi_h(N)}$ and $\prod_{k=0}^{N-1}\max\{1,|\lambda_v(k)|_v\}$ belong to $|K_v^{\times}|$ because $a_h \in K_v$ and $\lambda_v(k) \in K_v$. For each $h = 1, \cdots, t$, since $\Phi_h(N)s_i = N\kappa_h s_i$, $[L:K]_\iota$

divides $\Phi_h(N)s_i$, so $(a_h - x_i)^{\Phi_h(N)s_i}$ belongs to M_u, and hence the K_v-symmetry of $\mathfrak{X}$ implies that $\prod_{i=1}^m |a_h - x_i|_v^{\Phi_h(N)s_i}$ belongs to $|K_v^\times|$. Thus, $\mathfrak{C}$ belongs to $|K_v^\times|$. Likewise, $\prod_{i=1}^m \frac{1}{(z-x_i)^{N_i}}$ belongs to $K(z)$. Clearly, $\prod_{k=0}^{N-1}(z - \lambda_v(k))$ belongs to $K_v(z)$. Let $C \in K_v$ be such that $|C|_v = \mathfrak{C}$ and put

$$f_1(z) := C \cdot \frac{\prod_{k=0}^{N-1}(z - \lambda_v(k))}{\prod_{i=1}^m (z - x_i)^{N_i}}. \tag{6.4}$$

We have the following proposition:

Proposition 6.4 *The function $f_1(z)$ is an $(\mathfrak{X}, \vec{s})$-function of degree N and rational over K_v, having poles supported on $\mathfrak{X}$ and zeros in E_v which form a basic well-distributed sequence $\{\lambda_v(k)\}_{0\le k<N}$ of length N in E_v. Furthermore, $f_0(z) = |f_1(z)|_v$ and for all $z \notin \bigcup_{h=1}^t \mathfrak{B}(a_h, r_h)$,*

$$-\frac{1}{N}\log_v(|f_1(z)_v) = u_{\mathfrak{X},\vec{s}}(z; E_v) = V_{\mathfrak{X},\vec{s}}(E_v) - G_{\mathfrak{X},\vec{s}}(z; E_v). \tag{6.5}$$

Note that $\{\lambda_v(k)\}_{0\le k<N}$ is the set of all zeros of $f_1(z)$ and $\mathfrak{X}$ is the set of all poles of $f_1(z)$. Fix $J,\ 0 \le J < N$ and let $\lambda_v(J) \in E_{v,h} \subset \mathfrak{B}(a_h, r_h)$ for some $h = 1, \cdots, t$. Then the ultrametric inequality implies that $\mathfrak{B}(a_h, r_h) = \mathfrak{B}(\lambda_v(J), r_h)$. We had the K_v-rational isometric parametrization $\lambda_h : B(0, r_h) \to \mathfrak{B}(a_h, r_h)$ obtained by $z \mapsto a_h + z$ in appropriate affine coordinates. Note that $\lambda_v(J) = \lambda_h(b_h\beta_v(\Phi_h(J)))$. Pulling the affine coordinates back by the map $Z \to Z + b_h\beta_v(\Phi_h(J))$, we have the K_v-rational isometric parametrization $\lambda_{h,J} : B(0, r_h) \to \mathfrak{B}(a_h, r_h)$ defined by $\lambda_{h,J}(Z) = \lambda_h(Z + b_h\beta_v(\Phi_h(J)))$. Clearly, 0 corresponds to $\lambda_v(J)$ by $\lambda_{h,J}(Z)$.

Remark 6.5 For each $x \in \mathfrak{X}$, if X corresponds to x by the new affine coordinates, then we can expand $1/(Z - X)$ as a power series about $Z = 0$:

$$\begin{aligned} \frac{1}{Z - X} &= -\frac{1}{X} \cdot \frac{1}{1 - \frac{Z}{X}} \\ &= -\frac{1}{X} \cdot \left(1 + \left(\frac{Z}{X}\right) + \left(\frac{Z}{X}\right)^2 + \cdots\right), \end{aligned}$$

which converges on $\{Z \in \mathbb{C}_v : |Z/X|_v < 1\}$, or equivalently on $B(0, |X|_v)^-$. Since $X \notin B(0, r_h)$, we have $|X|_v > r_h$, so $B(0, r_h) \subset B(0, |X|_v)^-$. Thus, we can expand $f_1(Z)$ as a power series about $Z = 0$ converging on $B(0, r_h)$. Since N_i is divisible by $[L : K]_\iota$, every coefficient in the expansion of $1/(Z - X)^{N_i}$ belongs to M_u. It follows from the K-symmetry of $\mathfrak{X}$ and $\vec{s}$ that the power series $f_1(Z) = \sum_{k=0}^{\infty} c_k Z^k$ is rational over K_v. Clearly, $c_0 = 0$ because $f_1(Z)$ has a zero at $Z = 0$. Since $\{\lambda_v(k)\}_{0 \le k < N} \cap E_{v,h}$ is the basic well-distributed sequence of length $\Phi_h(N)$ in $E_{v,h}$, let $\{\tau_k\}_{0 \le k < \Phi_h(N)}$ be the corresponding regular sequence of length $\Phi_h(N)$ in $B(0, r_h) \cap K_v = b_h \mathcal{O}_v$ by $\lambda_{h,J}$. Of course, $\{\tau_k\}_{0 \le k < \Phi_h(N)}$ is the set of all zeros of $f_1(Z)$ in $B(0, r_h)$.

Now, consider the coefficients c_k of $f_1(Z)$.

Proposition 6.6 *The coefficients c_k of $f_1(Z) = \sum_{k=1}^{\infty} c_k Z^k$ satisfy the following:*

$$\mathrm{ord}_v(c_1) < N V_{\mathfrak{X},\vec{s}}(E_v) + \log_v(r_h) \tag{6.6}$$

and for each $k \ge 2$,

$$\mathrm{ord}_v(c_k) \ge \mathrm{ord}_v(c_1) - (k-1)(\log_v(N) - \log_v(r_h)). \tag{6.7}$$

Proof: It follows from the Weierstrass Preparation Theorem that there are a constant $c \in K_v$ and functions

$$G(Z) = \prod_{k=0}^{\Phi_h(N)-1} (Z - \tau_k) = \sum_{k=1}^{\Phi_h(N)} a_k Z^k \in K_v[Z],$$
$$H(Z) = 1 + \sum_{k \ge 1} d_k Z^k \in K_v[[Z]]$$

such that $f_1(Z) = c \cdot G(Z) \cdot H(Z)$, $G(Z)$ has the same zeros as $f_1(Z)$ in $B(0, r_h)$, and $H(Z)$ is a unit power series on $B(0, r_h)$. Letting $a_\ell = 0$ for all $\ell > \Phi_h(N)$ and $d_0 = 1$, it follows that $c_1 = c \cdot a_1$ and $c_k = c \cdot (a_k d_0 + a_{k-1} d_1 + \cdots + a_1 d_{k-1})$ for each $k \ge 2$. If $\tau_{k_0} = 0$ corresponds to $\lambda_v(J)$ by $\lambda_{h,J}$, then we have

$$a_1 = \pm \prod_{\substack{k=0 \\ k \ne k_0}}^{\Phi_h(N)-1} (\tau_k - \tau_{k_0})$$

and for each $2 \leq k \leq \Phi_h(N)$,

$$a_k = \pm a_1 \sum_{\substack{0 \leq j_1 < \cdots < j_{k-1} < \Phi_h(N) \\ \text{each } j_i \neq k_0}} \frac{1}{(\tau_{j_1} - \tau_{k_0}) \cdots (\tau_{j_{k-1}} - \tau_{k_0})}.$$

Now, fix an element $Z_0 = b_h\zeta_0 \in b_h\mathcal{O}_v$ and let $0 \leq J < \Phi_h(N)$ be an index for which $\mathrm{ord}_v(\zeta_0 - \beta_v(J))$ is maximal. If $A(n, J) = \sum_{j\geq 0} d_j(J) + \sum_{j\geq 0} d_j(n-J-1) + 1$ as in the proof of Proposition 5.9, then $\sum_{k=0}^{\Phi_h(N)-1} \mathrm{ord}_v(\zeta_0 - \beta_v(k)) = \Phi_h(N)V_\infty(\mathcal{O}_v) - \frac{A(\Phi_h(N),J)}{q_v-1} + \mathrm{ord}_v(\zeta_0 - \beta_v(J))$. If $z_0 \in E_{v,h}$ corresponds to $Z_0 \in b_h\mathcal{O}_v$ by $\lambda_{h,J}$, Remark 4.8 and (6.2) imply that

$$-\log_v(f_0(z_0)) = NV_{\mathfrak{X},\vec{s}}(E_v) - \frac{A(\Phi_h(N), J)}{q_v - 1} + \mathrm{ord}_v(\zeta_0 - \beta_v(J)).$$

On the other hand, Proposition 2.26 and the proof of Proposition 5.9 imply that

$$\begin{aligned} -\log_v(f_0(z_0)) &= -\log_v(|f_1(z_0)|_v) = -\log_v(|c|_v \cdot |g(Z_0)|_v \cdot |h(Z_0)|_v) \\ &= -\log_v(|c|_v) - \log_v(|g(Z_0)|_v) - \log_v(1) \\ &= -\log_v(|c|_v) + \Phi_h(N)V_\infty(b_h\mathcal{O}_v) - \mathrm{ord}_v(b_h) - \frac{A(\Phi_h(N), J)}{q_v - 1} \\ &\quad +\mathrm{ord}_v(\zeta_0 - \beta_v(J)) + \mathrm{ord}_v(b_h) \\ &= \mathrm{ord}_v(c) + \Phi_h(N)V_\infty(b_h\mathcal{O}_v) - \frac{A(\Phi_h(N), J)}{q_v - 1} + \mathrm{ord}_v(\zeta_0 - \beta_v(J)). \end{aligned}$$

Hence it follows that

$$\mathrm{ord}_v(c) = NV_{\mathfrak{X},\vec{s}}(E_v) - \Phi_h(N)V_\infty(b_h\mathcal{O}_v). \tag{6.8}$$

On the other hand, since $\Phi_h(N) = N\kappa_h$ for each $h = 1, \cdots, t$, Proposition 5.9 and (5.5) imply that

$$\mathrm{ord}_v(a_1) < \Phi_h(N)V_\infty(b_h\mathcal{O}_v) - \mathrm{ord}_v(b_h),$$

and for each $2 \leq k \leq \Phi_h(N)$,

$$\mathrm{ord}_v(a_k) > \mathrm{ord}_v(a_1) - (k-1)(\log_v(\Phi_h(N)) - \log_v(r_h)).$$

As $H(Z)$ is a unit power series in $B(0, r_h)$, it follows from the definition of a unit power seires that $|d_k|_v < 1/r_h^k$, i.e., $\mathrm{ord}_v(d_k) > k \cdot \log_v(r_h)$. For each $1 \leq j \leq \Phi_h(N)$ and each $k \geq j$, we have

$$\begin{aligned}\mathrm{ord}_v(a_j d_{k-j}) &= \mathrm{ord}_v(a_j) + \mathrm{ord}_v(d_{k-j}) \\ &\geq \mathrm{ord}_v(a_1) - (j-1)(\log_v(\Phi_h(N)) - \log_v(r_h)) + (k-j)\log_v(r_h) \\ &= \mathrm{ord}_v(a_1) + (k-1)\log_v(r_h) - (j-1)\log_v(\Phi_h(N)) \\ &\geq \mathrm{ord}_v(a_1) - (k-1)(\log_v(\Phi_h(N)) - \log_v(r_h)).\end{aligned}$$

Thus it follows that

$$\begin{aligned}\mathrm{ord}_v(c_1) &= \mathrm{ord}_v(c) + \mathrm{ord}_v(a_1) \\ &< NV_{\mathfrak{X},\vec{s}}(E_v) + \log_v(r_h)\end{aligned}$$

and for each $k \geq 2$,

$$\begin{aligned}\mathrm{ord}_v(c_k) &= \mathrm{ord}_v(c) + \mathrm{ord}_v(a_k d_0 + a_{k-1} d_1 + \cdots + a_1 d_{k-1}) \\ &\geq \mathrm{ord}_v(c) + \mathrm{ord}_v(a_1) - (k-1)(\log_v(\Phi_h(N)) - \log_v(r_h)) \\ &\geq \mathrm{ord}_v(c_1) - (k-1)(\log_v(N) - \log_v(r_h)).\end{aligned}$$

□

Lemma 6.7 *There is a number* $r_0 \in |K_v^\times|$ *such that* $q_v^{-(NV_{\mathfrak{X},\vec{s}}(E_v)+\log_v(N)+2)} < r_0 < q_v^{-(NV_{\mathfrak{X},\vec{s}}(E_v)+\log_v(N))}$ *and for each* $x \in B(0, r_0)$*, the Newton polygon of* $f_1(Z) = x$ *has a break at* $k = 1$*.*

Proof: Put $R := \lceil NV_{\mathfrak{X},\vec{s}}(E_v) + \log_v(N) + 1 \rceil$ and let $r_0 := |\pi_v^R|_v$. If $x \in B(0, r_0)$, then $\mathrm{ord}_v(x) \geq R > NV_{\mathfrak{X},\vec{s}}(E_v) + \log_v(N)$, and hence it follows from (6.6) and (6.7) that for each $k \geq 2$,

$$\begin{aligned}\mathrm{ord}_v(c_1) - \mathrm{ord}_v(x) &< \log_v(r_h) - \log_v(N) \\ &= \frac{(\mathrm{ord}_v(c_1) - (k-1)(\log_v(N) - \log_v(r_h))) - \mathrm{ord}_v(c_1)}{k-1} \\ &\leq \frac{\mathrm{ord}_v(c_k) - \mathrm{ord}_v(c_1)}{k-1}.\end{aligned}$$

Thus the Newton polygon of $f_1(Z) = x$ has a break at $k = 1$. $\square$

Proposition 6.8 *There are numbers $\rho_J \in |K_v^\times|$, $J = 0, \cdots, N-1$, such that for each J, $f_1 : \mathfrak{B}(\lambda_v(J), \rho_J) \to B(0, r_0)$ is an r_0/ρ_J-isometry, and*

$$f_1^{-1}\big(B(0,r_0)\big) = \bigcup_{J=0}^{N-1} \mathfrak{B}(\lambda_v(J), \rho_J),$$
$$f_1^{-1}\big(B(0,r_0) \cap K_v\big) = \bigcup_{J=0}^{N-1} \big(\mathfrak{B}(\lambda_v(J), \rho_J) \cap \mathbb{P}^1(K_v)\big).$$

Furthermore, for each $z \notin \bigcup_{h=1}^{t} B(a_h, r_h)$,

$$\frac{1}{N}\log_v(|f_1(z)|_v) = -V_{\mathfrak{X},\vec{s}}(E_v) + G_{\mathfrak{X},\vec{s}}(z; E_v). \tag{6.9}$$

Proof: Fix J. Lemma 6.7 implies that for each $x \in B(0, r_0)$, $f_1(Z) = x$ has a unique solution $\alpha_{J,x}$ with

$$\mathrm{ord}_v(\alpha_{J,x}) = -\frac{\mathrm{ord}_v(c_1) - \mathrm{ord}_v(x)}{1-0} = \mathrm{ord}_v(x) - \mathrm{ord}_v(c_1) \geq R - \mathrm{ord}_v(c_1). \tag{6.10}$$

Choose $a \in K_v$ so that $\mathrm{ord}_v(a) = R - \mathrm{ord}_v(c_1)$ and let $\rho_J = |a|_v$. Clearly, $\alpha_{J,x} \in B(0, \rho_J)$ by (6.10). It follows from (6.6) that

$$-\log_v(\rho_J) = R - \mathrm{ord}_v(c_1) > \log_v(N) - \log_v(r_h) = -\log_v(r_h/N),$$

so $\rho_J < r_h/N < r_h/\Phi_h(N) < 1$, which implies that the balls $B(\tau_k, \rho_k)$, $k = 0, \cdots, \Phi_h(N) - 1$, are disjoint from each other because $\mathrm{ord}_v(\tau_j - \tau_k) < -\log_v(r_h) + \log_v(\Phi_h(N)) = -\log_v(r_h/\Phi_h(N))$. Futhermore, since the balls $\mathfrak{B}(a_h, r_h)$ are disjoint from each other, the balls $\mathfrak{B}(\lambda_v(J), \rho_J)$, $J = 0, \cdots, N-1$, are disjoint from each other as well by the isometric parametrizations $\lambda_{h,J}$. Since $r_0 = |c_1|_v \rho_J$, it follows from Proposition 2.16 that $f_1 : B(0, \rho_J) \to B(0, r_0)$ is an r_0/ρ_J-isometry. Pulling this map back to $\mathfrak{B}(\lambda_v(J), \rho_J)$, we can regard $f_1(z)$ as an r_0/ρ_J-isometry from $\mathfrak{B}(\lambda_v(J), \rho_J)$ to $B(0, r_0)$ with $f_1(\mathfrak{B}(\lambda_v(J), \rho_J) \cap \mathbb{P}^1(K_v)) = B(0, r_0) \cap K_v$.

Furthermore, note that $f_1(z)$ has degree N and $f_1(z) = x$ has only one zero in each ball $\mathfrak{B}(\lambda_v(J), \rho_J)$ for each $0 \le J \le N-1$ and each $x \in B(0, r_0)$. It follows that

$$f_1^{-1}\big(B(0,r_0)\big) = \bigcup_{J=0}^{N-1} \mathfrak{B}(\lambda_v(J), \rho_J),$$

$$f_1^{-1}\big(B(0,r_0) \cap K_v\big) = \bigcup_{J=0}^{N-1} \big(\mathfrak{B}(\lambda_v(J), \rho_J) \cap \mathbb{P}^1(K_v)\big).$$

For each $z \notin \bigcup_{h=1}^{t} B(a_h, r_h)$, since $-\frac{1}{N}\log_v(f_0(z)) = u_{\mathfrak{X},\vec{s}}(z) = V_{\mathfrak{X},\vec{s}}(E_v) - G_{\mathfrak{X},\vec{s}}(z; E_v)$ by (6.5),

$$\frac{1}{N}\log_v(|f_1(z)|_v) = -V_{\mathfrak{X},\vec{s}}(E_v) + G_{\mathfrak{X},\vec{s}}(z; E_v).$$

□

Now, we are ready to prove Theorem 6.2.

Proof of Theorem 6.2: Since $N \cdot V_{\mathfrak{X},\vec{s}}(E_v)$ is an integer, put $f_v(z) = \pi_v^{-N\cdot V_{\mathfrak{X},\vec{s}}(E_v)} \cdot f_1(z)$. We can do the same argument as in Proposition 6.6 with the same a_i's, d_i's, after replacing c by $c' := \pi_v^{-N\cdot V_{\mathfrak{X},\vec{s}}(E_v)} \cdot c$ and c_k by $c_k' := \pi_v^{-N\cdot V_{\mathfrak{X},\vec{s}}(E_v)} \cdot c_k$. It follows that

$$\mathrm{ord}_v(c_1') < \log_v(r_h),$$

$$\mathrm{ord}_v(c_k') \ge \mathrm{ord}_v(c_1') - (k-1)(\log_v(N) - \log_v(r_h)).$$

Put $A_v := \lceil \log_v(N) + 1 \rceil$ and fix $R_v' \in |\mathbb{C}_v^\times|$ with $R_v'^N \in |K_v^\times|$ so that $-\log_v(R_v'^N) \ge A_v$, i.e., $R_v'^N \le q_v^{-A_v}$. Choose an element $b_v \in K_v^\times$ so that $|b_v|_v = R_v'^N$, i.e., $\mathrm{ord}_v(b_v) = -\log_v(R_v'^N) \ge A_v$. We can do the same argument as in Lemma 6.7 and Proposition 6.8. For each $y \in B(0, R_v'^N)$, the Newton polygon of $f_v(z) = y$ has a break at $k = 1$ and there is a unique solution $\alpha_{J,y}$ to $f_v(z) = y$ such that

$$\mathrm{ord}_v(\alpha_{J,y}) = -\frac{\mathrm{ord}_v(c_1') - \mathrm{ord}_v(y)}{1-0} \ge \mathrm{ord}_v(b_v) - \mathrm{ord}_v(c_1') > \log_v(N) - \log_v(r_h).$$

Choose $a' \in K_v$ so that $\mathrm{ord}_v(a') = \mathrm{ord}_v(b_v) - \mathrm{ord}_v(c_1')$ and let $\rho_J' = |a'|_v$. Clearly, $\alpha_{J,y} \in B(0, \rho_J')$. It follows that

$$-\log_v(\rho_J') > \log_v(N) - \log_v(r_h) = -\log_v(r_h/N),$$

so $\rho'_J < r_h/N < r_h/\Phi_h(N) < 1$, which implies that the balls $\mathfrak{B}(\lambda_v(J), \rho'_J)$, $\ J = 0, \cdots, N-1$, are disjoint from each other. Replacing ρ'_J with ρ_J, we can regard $f_v(z)$ as an $R_v'^N/\rho_J$-isometry from $\mathfrak{B}(\lambda_v(J), \rho_J) \to B(0, R_v'^N)$. Since each ball $\mathfrak{B}(\lambda_v(J), \rho_J)$ is contained in $\mathfrak{B}(a_h, r_h)$ for some $h = 1, \cdots, t$, we then obtain that

$$f_v^{-1}(B(0, R_v'^N)) = \bigcup_{J=0}^{N-1} \mathfrak{B}(\lambda_v(J), \rho_J) \subset \bigcup_{h=1}^{t} \mathfrak{B}(a_h, r_h),$$

$$f_v^{-1}(B(0, R_v'^N) \cap K_v) = \bigcup_{J=0}^{N-1} (\mathfrak{B}(\lambda_v(J), \rho_J) \cap K_v) \subset E_v.$$

Furthermore, for each $z \notin \bigcup_{h=1}^{t} B(a_h, r_h)$, it follows from (6.9) and Proposition 4.3 that

$$\begin{aligned} \frac{1}{N} \log_v(|f_v(z)|_v) &= \frac{1}{N} \log_v(|\pi_v^{-N \cdot V_{\mathfrak{X},\vec{s}}(E_v)} \cdot f_1(z)|_v) \\ &= \frac{1}{N}(N \cdot V_{\mathfrak{X},\vec{s}}(E_v) + \log_v(|f_1(z)|_v)) \\ &= G_{\mathfrak{X},\vec{s}}(z; E_v) = \sum_{i=1}^{m} G(z, x_i; E_v) s_i. \end{aligned}$$

$\square$

Chapter 7

Global Patching Argument I

Let F denote the field $\mathbb{F}_q(T)$ and K be a fixed finite extension of F. Let $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \mathbb{P}^1(\overline{K})$ be a finite K-symmetric set. Without loss of generality, we can assume that $\mathfrak{X}$ does not contain ∞ after changing the coordinates by an element in $\mathrm{GL}(2, \mathcal{O}_K)$. The resulting set is K-symmetric as well. Put $L := K(\mathfrak{X})$. For each $\sigma \in \mathrm{Gal}(L/K)$, we will sometimes write $\sigma(i) = j$ if $\sigma(x_i) = x_j$. Let $\mathfrak{X}_a$, $a = 1, \cdots, m'$ be the distinct orbits of $\mathfrak{X}$ under the action of $\mathrm{Gal}(L/K)$. For each $x_i \in \mathfrak{X}$, let $g_i(z) \in K(x_i)(z)$ be the uniformizing parameter at x_i as constructed in Chapter 6 and let $[z, x]_{x_i}$ be the canonical distance normalized so that $\lim_{z \to x_i} [z, x]_{x_i} \cdot |g_i(z)|_v = 1$. Let M be the separable closure of K in L, i.e., $M = L \cap K^{\mathrm{sep}}$. Then L is purely inseparable over M of degree $[L : K]_\iota$. For each $x_i \in \mathfrak{X}$, let e_i be the least nonnegative integer such that $x_i^{p^{e_i}} \in M$. Note that for each i, $K(x_i) \cap M = K(x_i^{p^{e_i}})$ and $\{x_i^\ell : \ell = 0, \cdots, p^{e_i} - 1\}$ is a basis for $K(x_i)/K(x_i^{p^{e_i}})$. If $e = \max_{1 \le i \le m} e_i$, then $[L : K]_\iota = p^e$. Let $u,\ w,\ \nu,\ \mu$ be the places of $M,\ L,\ K(x_i^{p^{e_i}}),\ K(x_i)$ lying above each place v of K, repectively.

Let ∞ be the place of F defined by $1/T$; for any $f(T) \in F$, $|f(T)|_\infty = p^{\deg(\mathrm{f})}$. Let P_K be the set of all places of K and $P_{K,\infty}$ be the set of all places of K lying above the place ∞ of F. Put $P_{K,0} := P_K \setminus P_{K,\infty}$. Let $\mathfrak{S}$ be a finite set of places of K containing $P_{K,\infty}$. Denote $K_{\mathfrak{S}} = \{x \in K : |x|_v \le 1 \text{ for all } v \notin \mathfrak{S}\}$ the set of all $\mathfrak{S}$-integers in K and $K_{\mathfrak{S}}^\times$ the set of all $\mathfrak{S}$-units in K. Let μ_K be the set of all roots of unity in K.

We will need some well-known facts related to the Dirichlet's Unit theorem, which we recall without proof. For details, see [5], p.72 and [9], p.43.

Remark 7.1 Let $\mathbb{A}_K$ and $\mathbb{J}_K$ denote the adele ring and idele group of K, respectively. For any element $\alpha = (\alpha_v)_{v \in P_K} \in \mathbb{A}_K$, define the size function

$$\|\alpha\| = \prod_{v \in P_K} |\alpha_v|_v.$$

$K^\times$ can be considered as a subset in $\mathbb{J}_K$ on the diagonal and then is a discrete subgroup of $\mathbb{J}_K$. Put $\mathbb{J}_K^1 := \{\alpha \in \mathbb{J}_K : \|\alpha\| = 1\}$. It is well known that $\mathbb{J}_K^1/K^\times$ is compact. Note that μ_K is a finite set. The Dirichlet's Unit theorem says that $K_{\mathfrak{S}}^\times$ is the direct sum of the finite cyclic group μ_K and a free abelian group of rank $\#(\mathfrak{S})-1$. Let $\mathbb{J}_{K,\mathfrak{S}} = \{\alpha \in \mathbb{J}_K : |\alpha_v|_v = 1 \ \forall v \notin \mathfrak{S}\}$ and put $\mathbb{J}_{K,\mathfrak{S}}^1 := \mathbb{J}_{K,\mathfrak{S}} \cap \mathbb{J}_K^1$. Let us remove one place from $\mathfrak{S}$ and denote the resulting subset of $\mathfrak{S}$ by $\mathfrak{S}'$. The proof of the Unit Theorem implies that the map $\Lambda : \mathbb{J}_{K,\mathfrak{S}}^1 \to \mathbb{R}^{\#(\mathfrak{S}')}$ given by $\Lambda(\alpha) = (\log(|\alpha_v|_v))_{v \in \mathfrak{S}'}$ takes $K_{\mathfrak{S}}^\times$ onto a lattice in $\mathbb{R}^{\#(\mathfrak{S}')}$. Hence there are constants $0 < c_1 < c_2$ with the following property : for any collection $\{0 < b_v \in \mathbb{R} : v \in \mathfrak{S}'\}$, there is an $\mathfrak{S}$-unit $u \in K^\times$ such that $c_1 \leq b_v \cdot |u|_v \leq c_2$ for all $v \in \mathfrak{S}'$.

We will write $\mathcal{O}_K$ for the set of all $P_{K,\infty}$-integers in K. That is, $\mathcal{O}_K = K_{P_{K,\infty}}$. In particular, $\mathcal{O}_F = \mathbb{F}_q[T]$ is the polynomial ring over $\mathbb{F}_q$.

Proposition 7.2 *Let A be a finitely generated domain over a field k. For a finite extension E of the field of fractions of A, let B be the integral closure of A in E. Then B is a finitely generated A-module and a finitely generated domain over k.*

Proof: See [4], Theorem V.3.2, p.348. □

Remark 7.3 In our case, take $A = \mathcal{O}_F$. Note that $\mathcal{O}_K$ and $\mathcal{O}_L$ are the integral closures of $\mathcal{O}_F$ in K and L, repectively, and they are Dedekind domains (see [29], Theorem V.19, p.281). The above proposition implies that $\mathcal{O}_K$ is a finitely generated $\mathcal{O}_F$-module and $\mathcal{O}_L$ is a finitely generated $\mathcal{O}_K$-module.

Recall that L is a purely inseparable extension of M with degree p^e. Let $\{\beta_j : j = 1, \cdots, p^e\}$ be a basis for L/M. For any place u of M, there is a unique place w of L lying above

u. Note that $L \otimes_M M_u = \oplus_{w|u} L_w$ (see [22], p.321), so $L \otimes_M M_u = L_w$. Since $L = \oplus_{j=1}^{p^e} M \cdot \beta_j$, it follows that $L_w = \oplus_{j=1}^{p^e} M_u \cdot \beta_j$. Hence $\{\beta_j : j = 1, \cdots, p^e\}$ is a basis for L_w/M_u.

Proposition 7.4 *For any basis $\{\beta_1, \cdots, \beta_{p^e}\}$ for L/M, we have $\mathcal{O}_w = \mathcal{O}_u[\beta_1, \cdots, \beta_{p^e}]$ for all but finitely many $u \in P_M$.*

Proof: The above remark implies that $\mathcal{O}_L = \mathcal{O}_M[y_1, \cdots, y_\ell]$ for some $y_1, \cdots, y_\ell$ in $\mathcal{O}_L$. Write $y_j = \sum_{i=1}^{p^e} a_{ij}\beta_i$ for $a_{ij} \in M$. Then there is a finite set $\mathfrak{S} \subset P_M$ such that for all $u \notin \mathfrak{S}$, $a_{ij} \in \mathcal{O}_u$ and $\beta_i \in \mathcal{O}_w$ if $w|u$. Note that if $u \notin \mathfrak{S}$ and $w|u$, then the y_j belong to $\mathcal{O}_w$. Let $\mathcal{O}_M^{\mathfrak{S}}$ be the localization of $\mathcal{O}_M$ at $\mathfrak{S}$, inverting each prime in $\mathfrak{S}$. Recall that for each $u \in P_M$, there is only one place $w \in P_L$ lying above u. It follows that $\mathcal{O}_L^{\mathfrak{S}} = \mathcal{O}_M^{\mathfrak{S}}[y_1, \cdots, y_\ell]$, $y_1, \cdots, y_\ell \in \mathcal{O}_M^{\mathfrak{S}}[\beta_1, \cdots, \beta_{p^e}]$, and $\beta_1, \cdots, \beta_{p^e} \in \mathcal{O}_L^{\mathfrak{S}}$. Hence $\mathcal{O}_M^{\mathfrak{S}}[\beta_1, \cdots, \beta_{p^e}] = \mathcal{O}_M^{\mathfrak{S}}[y_1, \cdots, y_\ell] = \mathcal{O}_L^{\mathfrak{S}}$. Thus, for any $u \notin \mathfrak{S}$, $\mathcal{O}_w = \mathcal{O}_L^{\mathfrak{S}} \otimes_{\mathcal{O}_M} \mathcal{O}_u = \mathcal{O}_M^{\mathfrak{S}}[\beta_1, \cdots, \beta_{p^e}] \otimes_{\mathcal{O}_M} \mathcal{O}_u = \mathcal{O}_u[\beta_1, \cdots, \beta_{p^e}]$. □

Proposition 7.5 *Let $\mathfrak{B} = \{\beta_j : j = 1, \cdots, p^e\}$ be a basis for L/M. If $w|u$ are places of L/M, then there exists a constant $T := T_{w,\mathfrak{B}} > 1$, depending only on w and $\mathfrak{B}$, such that for any $\Delta \in L_w$, if $\Delta = \sum_{j=1}^{p^e} \Delta_j\beta_j$ for $\Delta_j \in M_u$, then $|\Delta_j|_w \leq T \cdot |\Delta|_w$.*

Proof: Put $\mathcal{M} := \oplus_{j=1}^{p^e} \mathcal{O}_u \cdot \beta_j$. Noting that $\mathcal{O}_w$ is integral over $\mathcal{O}_u$ and is a free $\mathcal{O}_u$-module of rank p^e, there is an integral basis $\{\alpha_k : k = 1, \cdots, p^e\}$ for $\mathcal{O}_w/\mathcal{O}_u$. Writing $\alpha_k = \sum_{j=1}^{p^e} \alpha_{kj}\beta_j$ for $\alpha_{kj} \in M_u$, choose $A \in \mathcal{O}_u$ with $\mathrm{ord}_u(A) = \max\{0, \max\{-\mathrm{ord}_u(\alpha_{kj}) : k, j = 1, \cdots, p^e\}\}$. It follows that $A\alpha_{kj} \in \mathcal{O}_u$ for all k, j, so $A\alpha_k \in \mathcal{M}$ for all k. Since $\mathcal{O}_w = \oplus_{k=1}^{p^e} \mathcal{O}_u \cdot \alpha_k$, $A\mathcal{O}_w = \oplus_{k=1}^{p^e} \mathcal{O}_\nu \cdot A\alpha_k \subset \mathcal{M}$.

For any $\Delta \in L_w$, there is $n \in \mathbb{Z}$ such that $|\pi_u^{n+1}|_w < |\Delta|_w \leq |\pi_u^n|_w$. Then we get $\pi_u^{-n}\Delta \in \mathcal{O}_w$, and so $A\pi_u^{-n}\Delta \in \mathcal{M}$. Writing $A\pi_u^{-n}\Delta = \sum_{j=1}^{p^e} \delta_j\beta_j$ with each $\delta_j \in \mathcal{O}_u$, we have $\Delta_j = A^{-1}\pi_u^n\delta_j \in M_u$. Since $|\pi_u^n|_w = |\pi_u^{n+1}|_w \cdot |\pi_u^{-1}|_w \leq |\Delta|_w \cdot |\pi_u^{-1}|_w$ and $|\delta_j|_w = |\delta_j|_u^{p^e} \leq 1$, we see that

$$|\Delta_j|_w = |A^{-1}\pi_u^n\delta_j|_w \leq |A^{-1}|_w \cdot |\pi_u^{-1}|_w \cdot |\Delta|_w.$$

Take $T = |A^{-1}|_w \cdot |\pi_u^{-1}|_w > 1$. □

7.1 Basis Functions

Fix a K-symmetric probability vector $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{Q}^m_{>0}$ and let N_s be the least common denominator for $s_1, \cdots, s_m$. For each $N \in \mathbb{Z}_{>0}$ divisible by $N_s[L : K]_\iota$, put $N_i = Ns_i$. Since N_i is divisible by $[L : K]_\iota$, we have $n_i := N_i/p^{e_i} \in \mathbb{Z}_{>0}$. Consider the divisor $D = \sum_{i=1}^m N_i(x_i)$. Note that $\mathcal{L}(D) = \{f \in \overline{K}(z) : \mathrm{div}(f) \geq -D\}$ is a finite dimensional $\overline{K}$-vector space and let $\ell(D)$ denote its dimension. The Riemann-Roch Theorem says that since $\deg(D) = N > 0$, $\ell(D) = \deg(D) + 1 = N + 1$. Since $x_i^{p^{e_i}} \in M$ for each i, D is rational over M. Hence $\mathcal{L}(D)$ has a basis consisting of functions rational over M, so every function in $\mathcal{L}(D)$, rational over M, can be written as a linear combination of these basis funtions over M (see [16], Theorem 5, p.174). In fact, $\mathcal{L}(D)$ has an M-basis

$$\left\{1, \frac{z^\ell}{(z - x_i)^{j \cdot p^{e_i}}} : i = 1, \cdots, m,\ \ell = 0, \cdots, p^{e_i} - 1,\ j = 1, \cdots, n_i\right\}, \tag{7.1}$$

and an L-basis

$$\left\{1, \frac{1}{(z - x_i)^k} : i = 1, \cdots, m,\ k = 1, \cdots, N_i\right\}. \tag{7.2}$$

Note that 1 is a common element of the M-basis and L-basis, and each other M-basis function can be expressed in terms of the L-basis, having no constant term and vice versa: for each $\ell,\ k = 1, \cdots, p^{e_i} - 1$,

$$\frac{z^\ell}{(z - x_i)^{p^{e_i}}} = \frac{(z - x_i + x_i)^\ell}{(z - x_i)^{p^{e_i}}} = \sum_{k=0}^{\ell} \binom{\ell}{k} x_i^{\ell - k} \frac{1}{(z - x_i)^{p^{e_i} - k}},$$

$$\frac{1}{(z - x_i)^k} = \frac{(z - x_i)^{p^{e_i} - k}}{(z - x_i)^{p^{e_i}}} = \sum_{\ell=0}^{p^{e_i} - k} \binom{p^{e_i} - k}{\ell} (-x_i)^{p^{e_i} - k - \ell} \frac{z^\ell}{(z - x_i)^{p^{e_i}}}.$$

Hence any $(\mathfrak{X}, \vec{s})$-function $f(z) \in M(z)$ of degree N can written in terms of the M-basis and L-basis: for convenience, changing the superscripts for the L-basis from top to bottom, we can write

$$\begin{aligned} f(z) &= \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{p^{e_i} - 1} b_{ij\ell} \frac{z^\ell}{(z - x_i)^{j \cdot p^{e_i}}} + b_0 \\ &= \sum_{i=1}^{m} \sum_{k=0}^{N_i - 1} c_{ik} \frac{1}{(z - x_i)^{N_i - k}} + c_0 \end{aligned}$$

with $b_{ij\ell} \in M$, $c_{ik} \in L$, and $b_0 = c_0 \in M$.

Let v be a place of K and K' be any intermediate field of the extension L/K. We know that $L \otimes_K K_v \cong \oplus_{w|v} L_w$ as topological algebras. The algebra $K' \otimes_K K_v \cong \oplus_{u|v} K'_u$ can be embedded in $L \otimes_K K_v$ in such a way that $\oplus_{u|v} x_u \in \oplus_{u|v} K'_u$ is sent to the element $\oplus_{w|v} y_w \in \oplus_{w|v} L_w$, where $y_w = x_u$ for all $w|u$. Each $\sigma \in \mathrm{Gal}(L/K)$ acts on $L \otimes_K K_v$ through its action on L. Explicitly, $\mathrm{Gal}(L/K)$ acts on $L \otimes_K K_v$ by $\sigma(\ell \otimes k) = \sigma(\ell) \otimes k$. Thus σ induces a permutation of the places $w|v$ and gives rise to a canonical isomorphism $\tau_{\sigma,w} : L_w \to L_{\sigma(w)}$ for each $w|v$.

Now, we make explicit the action of $\mathrm{Gal}(L/K)$ on $\oplus_{w|v} L_w$. For each $\sigma \in \mathrm{Gal}(L/K)$, the action of σ on $\oplus_{w|v} L_w$ is given by applying $\tau_{\sigma,w}$ to the w-coordinate for each $w|v$, followed by the permutation induced by σ. That is, $\sigma(\oplus_{w|v} a_w) = \oplus_{\sigma(w)|v} \tau_{\sigma,w}(a_w)$.

Remark 7.6 Suppose that $f(z)$ is a K-rational $(\mathfrak{X}, \vec{s})$-function. For each $\sigma \in \mathrm{Gal}(L/K) \cong \mathrm{Gal}(M/K)$, since $\sigma(f) = f$, applying σ to the above expansion gives us $\sigma(b_{ij\ell}) = b_{\sigma(i),j\ell}$ and $\sigma(b_0) = b_0$. Hence the $b_{ij\ell}$ are K-symmetric relative to the action of $\mathrm{Gal}(L/K)$ on the x_i, and $b_0 \in K$ because $b_0 \in M$. For any element $\sigma \in \mathrm{Gal}(L/K(x_i)) \cong \mathrm{Gal}(M/K(x_i^{p^{e_i}}))$, since $\sigma(i) = i$, $\sigma(b_{ij\ell}) = b_{\sigma(i),j\ell} = b_{ij\ell}$, so each $b_{ij\ell}$ must belong to $K(x_i^{p^{e_i}})$ because $b_{ij\ell} \in M$.

For each $w|u|v$, if $g_w(z) \in L_w(z)$ is an $(\mathfrak{X}, \vec{s})$-function of degree N, then we can write

$$\begin{aligned} g_w(z) &= \sum_{i=1}^{m} \sum_{k=0}^{N_i-1} c_{w,ik} \frac{1}{(z-x_i)^{N_i-k}} + c_{w,0} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{p^{e_i}-1} b_{u,ij\ell} \frac{z^{\ell}}{(z-x_i)^{j \cdot p^{e_i}}} + b_{u,0} \end{aligned} \tag{7.3}$$

with $b_{u,ij\ell} \in L_w$, $c_{w,ik} \in L_w$, and $b_{u,0} = c_{w,0} \in L_w$. If $g_w(z) \in M_u(z)$, then the $b_{u,ij\ell}$ and $b_{u,0} = c_{w,0}$ belong to M_u. Put $b_{ij\ell} := \oplus_{w|v} b_{u,ij\ell}$ and $b_0 := \oplus_{w|v} b_{u,0}$.

Lemma 7.7 *With the notation above, if $\sigma(b_0) = b_0$ for all $\sigma \in$ Gal(L/K) $\cong$ Gal(M/K), then there is $b_{v,0} \in K_v$ such that $b_{u,0} = b_{v,0}$ for all $u|v$.*

Proof Note that M is Galois over K and there is only one place of L lying above each place of M. Fix a place u of M with $u|v$. Then the decomposition group D_u of u in M/K

is isomorphic to Gal(M_u/K_v). For each $\sigma \in D_u$, since $\sigma(u) = u$, we see that $\tau_{\sigma,u}(b_{u,0}) = b_{u,0}$ because $\sigma(b_0) = b_0$. That is, $b_{u,0}$ is fixed by D_u, so it belongs to K_v. Now, let $\sigma_1, \cdots, \sigma_g$ be coset representatives for Gal$(M/K)/D_u$. For each $h = 1, \cdots, g$ and any $\sigma \in \sigma_h D_u$, if $\sigma_h(u) = u_h$, then $\sigma \in \sigma_h D_u \Leftrightarrow \sigma(u) = u_h$. Hence $\tau_{\sigma,u}(b_{u,0}) = b_{u_h,0}$ since $\sigma(b_0) = b_0$. But, since $b_{u,0} \in K_v$ and $\tau_{\sigma,u}|_{K_v}$ is the identity, $\tau_{\sigma,u}(b_{u,0}) = b_{u,0}$. Thus, $b_{u_h,0} = b_{u,0}$ for all $h = 1, \cdots, g$, i.e., all the $b_{u,0}$ are the same and belong to K_v. □

Proposition 7.8 *For each $w|u|v$, let $g_w(z) \in L_w(z)$ be a function of degree N, having poles only in $\mathfrak{X}$, such that for each orbit, the poles of $g_w(z)$ at the points $x_i \in \mathfrak{X}_a$ have the same order $N_a = N_i$, where N_i is divisible by p^{e_i}, say, $N_i = p^{e_i} n_i$. With the notation above, the following are equivalent:*

(i) There is an $f_v(z) \in K_v(z)$ such that $g_w(z) = f_v(z)$ for all $w|v$;

(ii) Each $g_w(z) \in M_u(z)$, and for the natural action of Gal(L/K) $\cong$ Gal(M/K) on $L(z) \otimes_K K_v \cong_{w|v} L_w(z)$, $\oplus_{w|v} g_w(z)$ is invariant under Gal(L/K);

(iii) If each $g_w(z)$ is expanded as in (7.3), with the $b_{u,ij\ell}$ and $b_{u,0}$ in L_w, then in fact, the $b_{u,ij\ell}$ and $b_{u,0}$ belong to M_u, and:

(a) for each i, j, ℓ, each $w|u|v$, and each $\sigma \in$ Gal(L/K), $\tau_{\sigma,w}(b_{u,ij\ell}) = b_{\sigma(u),\sigma(i),j\ell}$ and

(b) there is $b_{v,0} \in K_v$ such that $b_{u,0} = b_{v,0}$ for all $u|v$.

Under these equivalent conditions, for each i, if μ is a place of $K(x_i)$ lying above v, then $\oplus_{w|v} c_{w,ik}$ belongs to $\oplus_{\mu|v} K(x_i)_\mu$, embedded in $L \otimes_K K_v$ for each $k = 0, \cdots, N_i - 1$.

Proof The equivalence of (i) and (ii) follows from the definition of the action of Gal(L/K) because $g_w(z) \in M_u(z)$. For the equivalence of (ii) and (iii), writing

$$\oplus_{w|v} g_w(z) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{p^{e_i}-1} b_{ij\ell} \frac{z^\ell}{(z - x_i)^{j \cdot p^{e_i}}} + b_0, \tag{7.4}$$

we have

$$\sigma(\oplus_{w|v} g_w)(z) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{p^{e_i}-1} \sigma(b_{ij\ell}) \frac{z^\ell}{(z - x_{\sigma(i)})^{j \cdot p^{e_{\sigma(i)}}}} + \sigma(b_0). \tag{7.5}$$

Since $\sigma(b_{ij\ell}) = \sigma(\oplus_{w|v} b_{u,ij\ell}) = \oplus_{\sigma(w)|v} \tau_{\sigma,w}(b_{u,ij\ell})$, $\oplus_{w|v} g_w(z)$ is invariant under $\mathrm{Gal}(L/K)$ if and only if $\sigma(b_{ij\ell}) = b_{\sigma(i),j\ell}$ and $\sigma(b_0) = b_0$ for all $\sigma \in \mathrm{Gal}(L/K)$ and all i, j, ℓ, which in turn helds if and only if $\tau_{\sigma,w}(b_{u,ij\ell}) = b_{\sigma(u),\sigma(i),j\ell}$ for all σ, i, j, ℓ and there is $b_{v,0} \in K_v$ such that $b_{u,0} = b_{v,0}$ for all $u|v$ by Lemma 7.7.

For the last assertion, Remark 7.6 implies that $b_{u,ij\ell}$ must belong to $K_v(x_i^{p^{e_i}})$. Since each $c_{w,ik}$ is a combination of the $b_{u,ij\ell}$ and the x_i^ℓ for $\ell = 0, \cdots, p^{e_i} - 1$, $c_{w,ik}$ belongs to $K_v(x_i)$. Hence the assertion follows because $K_v(x_i) \subset K(x_i)_\mu$ for all $\mu|v$. □

Lemma 7.9 (Gap Principle) *For any $(\mathfrak{X}, \vec{s})$-function $f_v(z) \in K_v(z)$ of degree N, where $p^{e_i}|N_i|N$, write*

$$f_v(z) = \sum_{i=1}^{m} \sum_{k=0}^{N_i-1} b_{v,ik} \cdot \frac{1}{(z-x_i)^{N_i-k}} + b_{v,0},$$

with $b_{v,ik} \in L_w$. If the leading coefficient $b_{v,i0}$ is 0 for some $x_i \in \mathfrak{X}_a$, then $b_{v,ik} = 0$ for all $x_i \in \mathfrak{X}_a$ and all $k = 0, \cdots, p^{e_i} - 1$, so $f_v(z)$ actually has a pole of order at most $N_i - p^{e_i}$ at each $x_i \in \mathfrak{X}_a$.

Proof: Write $f_v(z)$ in terms of the M-basis:

$$f_v(z) = \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} \sum_{\ell=0}^{p^{e_i}-1} a_{v,ij\ell} \cdot \frac{z^\ell}{(z-x_i)^{(n_i-j)p^{e_i}}} + a_{v,0},$$

with $a_{v,ij\ell} \in M_u$. After rewriting this expression using the L-basis, we get $0 = b_{v,i0} = \sum_{\ell=0}^{p^{e_i}-1} a_{v,i,0,\ell} \cdot x_i^\ell$. Since $\{x_i^\ell : \ell = 0, \cdots, p^{e_i} - 1\}$ is a basis for $K(x_i)_\mu / K(x_i^{p^{e_i}})_\nu$ and $a_{v,ij\ell} \in K(x_i^{p^{e_i}})_\nu$, we must have $a_{v,i,0,\ell} = 0$ for all $\ell = 0, \cdots, p^{e_i} - 1$. Noting that $\mathfrak{X}$ is Galois-invariant, all the $b_{v,i0} = 0$ for all $x_i \in \mathfrak{X}_a$, so $a_{v,i,0,\ell} = 0$ for all $x_i \in \mathfrak{X}_a$ and all $\ell = 0, \cdots, p^{e_i} - 1$. Hence $f_v(z)$ has a pole of order $\leq N_i - p^{e_i}$ at each $x_i \in \mathfrak{X}_a$, so $b_{v,ik} = 0$ for all $x_i \in \mathfrak{X}_a$ and all $k = 0, \cdots, p^{e_i} - 1$. □

7.2 Preliminary Reductions

Recall the main theorem:

Theorem 7.10 (Fekete-Szegö Theorem with Splitting Conditions) *Let K be a finite algebraic extension of $F := \mathbb{F}_q(T)$, and let $\mathfrak{X} = \{x_1, \cdots, x_m\} \subset \mathbb{P}^1(\overline{K})$ be a finite K-symmetric set. Let $E_v \subset \mathbb{P}^1(\mathbb{C}_v)$ be stable under $Gal_c(\mathbb{C}_v/K_v)$, and put $\mathbb{E}_K := \prod_{v \in P_K} E_v$. Suppose that $\mathbb{E}_K$ is compatible with $\mathfrak{X}$. Let $S \subset P_K$ be a finite (possibly empty) set of places of K with the property that for each $v \in S$, there exists a finite Galois extension $\mathcal{K}_u/K_v$ such that $E_v \subset \mathbb{P}^1(\mathcal{K}_u)$. If $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$, then for any adelic neighborhood $\mathbb{U}_K := \prod_{v \in P_K} U_v$ of $\mathbb{E}_K$, there are infinitely many points $\alpha \in \mathbb{P}^1(\overline{K})$ such that all of their conjugates over K belong to U_v for all v, and lie in $\mathbb{P}^1(\mathcal{K}_u) \cap U_v$ for each $v \in S$.*

Without loss of generality, we can assume that $\mathfrak{X}$ does not contain ∞ after changing the coordinates by an element in $\mathrm{GL}(2, \mathcal{O}_K)$. The resulting set is K-symmetric as well. We will make some preliminary adjustments to the sets $\mathbb{E}_K := \prod_{v \in P_K} E_v$ and $\mathbb{U}_K := \prod_{v \in P_K} U_v$. By assumption, each E_v is stable under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$ and each U_v is open. In addition, we will arrange that each E_v is closed and has a simple form; and that each U_v is stable under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$. Since the theorem concerns the existence of points $\alpha \in \mathbb{P}^1(\overline{K})$ with conjugates in $\mathbb{U}_K$, the conclusions remain valid as long as we do not enlarge the sets U_v.

Remark 7.11 For any function $f(z) \in L(z)$, there is a finite set of places of L outside of which $f(z)$ has good reduction (see [27], Proposition 24, p.107).

Let $\widehat{S}_K$ be the finite set of all places of K where any of the following conditions holds:

i) $v \in S$;

ii) the points in $\mathfrak{X}$ don't specialize to distinct points (mod v);

iii) E_v is not $\mathfrak{X}$-trivial;

iv) $\|z, x_i\|_v \neq |g_i(z)|_v$ for some z with $\|z, x_i\|_v < 1$ (see Proposition 2.29);

v) for each $x_i \in \mathfrak{X}$, $\{x_i^\ell : \ell = 0, \cdots, p^{e_i} - 1\}$ does not satisfy $\mathcal{O}_\mu = \mathcal{O}_\nu[1, x_i, \cdots, x_i^{p^{e_i}-1}]$ if $\mu|\nu|v$ of $K(x_i)/K(x_i^{p^{e_i}})/K$ (see Proposition 7.4);

vi) any of the functions $1/(z - x_i)$, $i = 1, \cdots, m$ has bad reduction (mod v) (see Remark 7.11).

Let $\widehat{S}_L$ be the set of places of L lying above $\widehat{S}_K$. For each $v \notin \widehat{S}_K$, if $w|v$, the local Green's matrices $\Gamma(E_v, \mathfrak{X})$ and $\Gamma(E_w, \mathfrak{X})$ are the zero matrix by Remark 3.18.

Reduction 1. For each $v \in S$, we claim that E_v can be reduced to a compact set. Fix a place $w|v$ of L. Recall that

$$\Gamma(\mathbb{E}_L, \mathfrak{X}) = [L : K]\Gamma(\mathbb{E}_K, \mathfrak{X}). \tag{7.6}$$

If $w \notin \widehat{S}_L$, then $\Gamma(E_w, \mathfrak{X})$ is the zero matrix. Hence $\Gamma(\mathbb{E}_L, \mathfrak{X}) = \sum_{w \in \widehat{S}_L} \Gamma(E_w, \mathfrak{X}) \log(q_w)$. Since $\Gamma(\mathbb{E}_L, \mathfrak{X})$ is negative definite and $\#(\widehat{S}_L)$ is finite, there exists $\varepsilon > 0$ such that for any collection of symmetric matrices $\Gamma_w \in \mathrm{M}_m(\mathbb{R})$ whose entries satisfy $|\Gamma_{w,ij} - \Gamma(E_w, \mathfrak{X})_{ij}| < \varepsilon$ for all $w \in \widehat{S}_L$ and all i, j, the matrix

$$\Gamma = \sum_{w \in \widehat{S}_L} \Gamma_w \log(q_w)$$

is also negative definite by the continuity of the determinants in their coefficients. The definition of Green's functions says that

$$G(z, x_i; E_v) = \inf_{\substack{E \subset E_v \\ \text{compact}}} G(z, x_i; E)$$
$$V_{x_i}(E_v) = \inf_{\substack{E \subset E_v \\ \text{compact}}} V_{x_i}(E).$$

Choose a compact set $\widehat{E}_v \subset E_v$ so that $|G(x_i, x_j; E_v) - G(x_i, x_j; \widehat{E}_v)| < \varepsilon/[L : K]$ for all $i \neq j$, and $|V_{x_i}(E_v) - V_{x_i}(\widehat{E}_v)| < \varepsilon/[L : K]$ for all i. Note that $\widehat{E}_v \subset E_v \subset \mathbb{P}^1(\mathcal{K}_u)$. Put

$$E_v' := \cup_{\sigma \in \mathrm{Gal}(\mathcal{K}_u/K_v)} \sigma(\widehat{E}_v).$$

Since E_v is Galois-stable, we obtain that $\widehat{E}_v \subseteq E_v' \subset E_v$. Clearly, E_v' is compact and stable under $\mathrm{Gal}(\mathcal{K}_u/K_v)$. The monotonicity properties of Green's functions imply that $|G(x_i, x_j; E_v) - G(x_i, x_j; E_v')| \leq |G(x_i, x_j; E_v) - G(x_i, x_j; \widehat{E}_v)| < \varepsilon/[L : K]$ for all $i \neq j$. Similarly, $|V_{x_i}(E_v) - V_{x_i}(E_v')| \leq |V_{x_i}(E_v) - V_{x_i}(\widehat{E}_v)| < \varepsilon/[L : K]$ for all i. By (7.6), we see that $|\Gamma(E_w, \mathfrak{X})_{ij} - \Gamma(E_w', \mathfrak{X})_{ij}| < \varepsilon$ for all $w \in \widehat{S}_L$ and all i, j. Replacing E_v by E_v', the new matrix $\Gamma(\mathbb{E}_L, \mathfrak{X})$ is also negative definite.

Reduction 2. For each $v \in S$, we claim that U_v can be a union of finite balls $\mathfrak{B}(a_i, r_i)$, which is stable under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$ and disjoint from $\mathfrak{X}$; furthermore, each r_i belongs to $|K_v^\times|$ and $E_v = U_v \cap \mathbb{P}^1(\mathcal{K}_u)$.

Fix $\alpha \in E_v$. Since $E_v \subset \mathbb{P}^1(\mathcal{K}_u)$ is Galois-stable and U_v is an open set containing E_v, there exists $r_\alpha > 0$ such that $\mathfrak{B}(\sigma(\alpha), r_\alpha) \subset U_v$ for all $\sigma \in \mathrm{Gal}(\mathcal{K}_u/K_v)$ and $\mathfrak{B}(\alpha, r_\alpha)$ is disjoint from $\mathfrak{X}$. After shrinking r_α, we can take r_α belonging to $|K_v^\times|$. Note that E_v is compact by Reduction 1. We can select $r \in |K_v^\times|$ so that r replaces all the r_α for all $\alpha \in E_v$. Also, we can choose a union of finite balls $\mathfrak{B}(\alpha_i, r)$, $\alpha \in E_v$, which covers E_v. Replacing U_v by the union, U_v becomes Galois-stable.

Now, enlarge E_v by replacing it with $U_v \cap \mathbb{P}^1(\mathcal{K}_u)$. Since $\mathbb{P}^1(\mathcal{K}_u)$ is compact, the new E_v remains compact. Obviously, E_v is Galois-stable because $\mathbb{P}^1(\mathcal{K}_u)$ and U_v are Galois-stable. By the same argument as on Reduction 1, $\Gamma(\mathbb{E}_L, \mathfrak{X})$ remains negative definite.

Reduction 3. For each $v \in \widehat{S}_K \setminus S$, we claim that E_v can be replaced by a union of finite balls $\mathfrak{B}(a_i, r_i) \subset U_v$, where each r_i belongs to $|K_v^\times|$. Hence E_v is a PL-domain (see [22], Theorem 4.2.16, p.252). Since such an E_v is open, we can then shrink U_v and take $U_v = E_v$.

Let ε be as in Reduction 1, and fix $v \in \widehat{S}_K \setminus S$. Put $\varepsilon' := \varepsilon/2$. There is a compact set $\widehat{E}_v \subset E_v$ such that $|G(x_i, x_j; E_v) - G(x_i, x_j; \widehat{E}_v)| < \varepsilon'/[L : K]$ for all $i \neq j$, and $|V_{x_i}(E_v) - V_{x_i}(\widehat{E}_v)| < \varepsilon'/[L : K]$ for all i. For each $x \in \widehat{E}_v$, since E_v is Galois-stable, $\sigma(x) \in E_v$ for all $\sigma \in \mathrm{Gal}_c(\mathbb{C}_v/K_v)$. Since U_v is open, there is $r > 0$ such that $\mathfrak{B}(\sigma(x), r)^- \subset U_v$. Put

$$r_\sigma := \sup_{\mathfrak{B}(\sigma(x), r) \subset U_v} r.$$

Clearly, $\mathfrak{B}(\sigma(x), r_\sigma)^- \subset U_v$. Define a map

$$\varphi_x : \mathrm{Gal}_c(\mathbb{C}_v/K_v) \to \mathbb{R}_{>0}$$

given by $\varphi_x(\sigma) = r_\sigma$. For each $\tau \in \mathrm{Gal}_c(\mathbb{C}_v/K_v)$, if $\sigma(x) \in \mathfrak{B}(\tau(x), r_\tau)^-$, then $r_\sigma = r_\tau$ because $\mathfrak{B}(\tau(x), r_\tau)^- = \mathfrak{B}(\sigma(x), r_\tau)^-$. Hence φ_x is locally constant, so it is continuous. Since $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$ is compact, there is an $r_x > 0$ such that $\mathfrak{B}(\sigma(x), r_x) \subset U_v$ for all $\sigma \in$

$\mathrm{Gal}_c(\mathbb{C}_v/K_v)$. Without loss of generality, we can take $r_x \in |K_v^\times|$ small enough that $\mathfrak{B}(x, r_x)$ is disjoint from $\mathfrak{X}$. For any fixed $y \in \mathbb{P}^1(\overline{K}_v) \cap \mathfrak{B}(x, r_x)$, since $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$ preserves $|\cdot|_v$, $\mathfrak{B}(\sigma(x), r_x) = \mathfrak{B}(\sigma(y), r_x)$ for all $\sigma \in \mathrm{Gal}_c(\mathbb{C}_v/K_v)$. Noting that y has only finitely many conjugates, there are only finitely many distinct balls $\mathfrak{B}(\sigma(x), r_x)$ and they are disjoint from $\mathfrak{X}$ because $\mathfrak{X}$ is Galois-stable and $\mathfrak{B}(x, r_x)$ is disjoint from $\mathfrak{X}$. Since $\widehat{E}_v$ is compact, we can cover it with finitely many such balls $\mathfrak{B}(x, r_x)$. Let U_v' be the union of these balls and their conjugates. Then U_v' is open and contained in U_v. Obviously, U_v' is Galois-stable and contains $\widehat{E}_v$. Replace E_v and U_v by U_v'. By the same argument as in Reduction 1, $\Gamma(\mathbb{E}_L, \mathfrak{X})$ remains negative definite. Moreover, the ultrametric inequality implies that we can write $E_v = U_v = U_v'$ as a finite union of pairwise disjoint balls.

Reduction 4. For each $v \notin \widehat{S}_K$, E_v is the $\mathfrak{X}$-trivial set. After shrinking U_v, we can assume that $E_v = U_v$ as well.

Reduction 5. For each $v \in S$, we claim that we can assume that $\mathcal{K}_u = K_v$. We are given a finite Galois extension $\mathcal{K}_u/K_v$ with $E_v \subset \mathbb{P}^1(\mathcal{K}_u)$ for each $v \in S$. Let D be the least common multiple of the degrees $[\mathcal{K}_u : K_v]$ and put $m_v := D/[\mathcal{K}_u : K_v]$. Since $\mathcal{K}_u/K_v$ is Galois, we can choose m_v distinct non-conjugate primitive elements $\alpha_{u,j}$ for $\mathcal{K}_u$ over K_v. Let $f_{v,j}(z) \in K_v[z]$ be the minimal polynomial for $\alpha_{u,j}$ over K_v, and put

$$f_v(z) := \prod_{j=1}^{m_v} f_{v,j}(z).$$

The degree of $f_v(z)$ is D and it has distinct roots. Hence $f_v(z)$ is separable. Let ε_v be the minimal distance between the roots of $f_v(z)$ in $\mathbb{C}_v$. By the weak approximation theorem, we can find a monic separable polynomial $f(z) \in K[z]$ of degree D, which approximates each $f_v(z)$ so closely at each $v \in S$ that for each root β_ℓ of $f(z)$, there is a unique root α_ℓ of $f_v(z)$ with $|\beta_\ell - \alpha_\ell|_v < \varepsilon_v$. Moreover, we can arrange that $f(z)$ is irreducible by requiring it to be an Eisenstein polynomial at some fixed $v_0 \notin \widehat{S}_K$. Note that $\mathbb{C}_v$ is algebraically closed. By Krasner's lemma, the roots of $f_v(z)$ and $f(z)$ generate the same subfield of $\mathbb{C}_v$. Let K' be the splitting field of $f(z)$ over K. Then K'/K is a finite Galois extension satisfying $K'_{u'} \cong \mathcal{K}_u$ at each $u'|v$ for each $v \in S$ because each root of $f_v(z)$ in $\mathbb{C}_v$ is a primitive element for $\mathcal{K}_u/K_v$.

We now make a base change from K to K'. Pull back $\mathbb{E}_K$ to $\mathbb{E}_{K'}$ and $\mathbb{U}_K$ to $\mathbb{U}_{K'}$. According to the previous reductions, U_v became Galois-stable for every $v \in P_K$ and hence $\mathbb{U}_{K'}$ is well-defined. By the functorial properties of the global capacity (see [22], Theorem 5.1.13, p.333), we see that $\gamma(\mathbb{E}_{K'}, \mathfrak{X}) > 1$. We claim that if the main theorem holds for $\mathbb{E}_{K'}$ and $\mathbb{U}_{K'}$ over K', then it holds for $\mathbb{E}_K$ and $\mathbb{U}_K$ over K. Note that $\overline{K} = \overline{K'}$ (see [15], Theorem 2.8, p.233). If $\alpha \in \mathbb{P}^1(\overline{K}) = \mathbb{P}^1(\overline{K'})$ has all its K'-conjugates in $\mathbb{U}_{K'}$, then it has all its K-conjugates in $\mathbb{U}_K$ because $\mathbb{U}_{K'}$ is the pullback of $\mathbb{U}_K$. For each $v \in S$, suppose that all the K'-conjugates of α belong to $\mathbb{P}^1(K'_{u'}) = \mathbb{P}^1(\mathcal{K}_u)$ for all $u'|v$. Since $\mathrm{Gal}(K'/K)$ permutes the places $u'|v$, all the K-conjugate of α belong to $\mathbb{P}^1(\mathcal{K}_u)$. Hence we can replace K by K'.

After these reductions, we can assume the following:

(i) $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$;

(ii) Each U_v is stable under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$ and bounded away from $\mathfrak{X}$;

(iii) For each $v \in S$, $E_v = U_v \cap \mathbb{P}^1(K_v)$ is the union of a finite number of disjoint compact sets, where $U_v = \cup_{h=1}^{t} \mathfrak{B}(a_h, r_h)$ for some $a_h \in \mathbb{P}^1(K_v)$ and $|b_h|_v = r_h \in |K_v^\times|$;

(iv) For each $v \in \widehat{S}_K \setminus S$, U_v is the union of a finite number of pairwise disjoint balls in $\mathbb{P}^1(\mathbb{C}_v)$ with radii in $|K_v^\times|$; in particular, $E_v = U_v$ is a PL-domain;

(v) For each $v \notin \widehat{S}_K$, $E_v = U_v$ is the $\mathfrak{X}$-trivial set, which is an RL-domain by Remark 3.24.

7.3 Choices of $\vec{s}$, Local Patching Parameters and N

The preliminary reductions and Remark 3.24 imply that there is a K-symmetric probability vector $\vec{s} \in \mathbb{Q}_{>0}^m$ such that $\Gamma(\mathbb{E}_K, \mathfrak{X}) \cdot \vec{s} = V(\mathbb{E}_K, \mathfrak{X}) \cdot \mathbf{1}_m$. Let N_s be the least common denominator for $s_1, \cdots, s_m$. Recall that for each $v \in S$, $V_{\mathfrak{X},\vec{s}}(E_v) \in \mathbb{Q}$ and $\kappa_h = \mu_{\mathfrak{X},\vec{s}}(E_{v,h}) \in \mathbb{Q}_{>0}$ with $\sum_{i=h}^{t} \kappa_h = 1$ by Proposition 4.7. For each $v \in S$, let $N_{r,v}$ be the denominator of $V_{\mathfrak{X},\vec{s}}(E_v)$ and $N_{\kappa,v}$ be the least common denominator for $\kappa_1, \cdots, \kappa_t$. Let N_r be the least common multiple of the $N_{r,v}$, $v \in S$ and N_κ be the least common multiple of the $N_{\kappa,v}$, $v \in S$.

Now, fix an additional place $v_0 \notin \widehat{S}_K$, which will be used to obtain the product formula for leading coefficients in Section 7.5. Put

$$S_K := \widehat{S}_K \cup \{v_0\}. \tag{7.7}$$

Let S_L be the set of places of L lying above S_K. By Remark 3.24, $V(\mathbb{E}_K, \mathfrak{X})$ belongs to $\log(p) \cdot \mathbb{Q}$. Since $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$, $V(\mathbb{E}_K, \mathfrak{X}) < 0$, and we can write $V(\mathbb{E}_K, \mathfrak{X}) = -V \log(q_{v_0})$ for some $V \in \mathbb{Q}_{>0}$. The condition $\gamma(\mathbb{E}_K, \mathfrak{X}) > 1$, i.e., $V(\mathbb{E}_K, \mathfrak{X}) < 0$, will play a crucial role in obtaining the product formula for the leading coefficients in Section 7.5. Let N_g be the denominator of V and put

$$N_0 := N_s N_r N_\kappa N_g [L : K]_\iota. \tag{7.8}$$

Note that N_0 is divisible by N_v for each $v \in S$.

Let $N > 1$ be any positive integer. Since $\#(\widehat{S}_K) = \#(S_K) - 1$, Remark 7.1 implies that the map $\Lambda : \mathbb{J}^1_{K,S_K} \to \mathbb{R}^{\#(\widehat{S}_K)}$ takes $K^\times_{S_K}$ onto a lattice in $\mathbb{R}^{\#(\widehat{S}_K)}$ and hence there are constants $0 < C_1 < C_2$ with the following property : for any collection $\{0 < b_v \in \mathbb{R} : v \in \widehat{S}_K\}$, there is an S_K-unit $u \in K^\times$ such that $C_1 \le b_v \cdot |u|_v \le C_2$ for all $v \in \widehat{S}_K$. Set $P_v := \log_v(C_2) - \log_v(C_1)$ and put

$$B(N) := \#(S) + \frac{\log_{v_0}(\prod_{v \in \widehat{S}_K} q_v^{2+P_v})}{\log_{v_0}(N)}. \tag{7.9}$$

Note that $B(N) \cdot \log_{v_0}(N)/N \to 0$ as $N \to \infty$. Put

$$h_{v_0} := q_{v_0}^{V/2} > 1, \tag{7.10}$$

and choose a number h_v with $0 < h_v < 1$ for each $v \in \widehat{S}_K$ so that

$$\prod_{v \in S_K} h_v > 1. \tag{7.11}$$

If w is a place of L lying above v, then we put $h_w := h_v^{[L_w:K_v]}$.

Let N be a sufficiently large positive integer divisible by N_0 such that

$$NV \ge 2 \cdot B(N) \cdot \log_{v_0}(N), \tag{7.12}$$

and

$$h_v^N < q_v^{-2/(q_v-1)} < q_v^{-1/(q_v-1)}. \tag{7.13}$$

The conditions (7.12) and (7.13) for N will play a significant role in modifying the defining constants of the sets E_v in Chapter 8 and in acquiring appropriate conditions for the local patching parameters h_v in Chapter 9. We will prove the main theorem by constructing basic local approximating functions of degree N, and then use them to construct a rational function defined over K after raising them to appropriate powers. Since $\kappa_h N$ is a positive integer, $\Phi_h(N) = \kappa_h N$ for each h. Put $N_i := Ns_i$ for each $i = 1, \cdots, m$. Since N is divisible by $[L : K]_\iota$, we have $n_i := N_i/p^{e_i} \in \mathbb{Z}_{>0}$, where $p^{e_i} = [K(x_i) : K]_\iota$. Note that for each Galois orbit $\mathfrak{X}_a$, $a = 1, \cdots, m'$, the numbers N_i, n_i and e_i are independent of $x_i \in \mathfrak{X}_a$, so there are numbers N_a, n_a and e_a such that for each $x_i \in \mathfrak{X}_a$, we have $N_i = N_a$, $n_i = n_a$, $e_i = e_a$.

Chapter 8

Achieving the Product Formula for the Leading Coefficients and Construction of the Normalized Local Approximating Functions

In this chapter, we modify the functions $f_v(z)$ described in (8.1), (8.2), (8.4), and (8.6) so that their leading coefficients satisfies the product formula, which is used to patch the leading coefficients to common global S_L-units. Moreover, the modified functions give us freedom to select local patching parameters.

Assuming the preliminary reductions of Section 7.2, we have the finite set $\widehat{S}_K \subset P_K$ containing S. Note that $S_K = \widehat{S}_K \cup \{v_0\}$. Let $\vec{s}$ be the K-symmetric probability vector in $\mathbb{Q}^m_{>0}$ and N be a sufficiently large positive integer divisible by N_0 such that $NV \geq 2 \cdot B(N) \cdot \log_{v_0}(N)$ as in Section 7.3.

We have already chosen the constants $0 < C_1 < C_2$ with the following property (see Section 7.3): for any collection $\{0 < a_v \in \mathbb{R} : v \in \widehat{S}_K\}$, there is an S_K-unit $u \in K^\times$ such that $C_1 \leq a_v \cdot |u|_v \leq C_2$ for all $v \in \widehat{S}_K$. It follows that $\log_v(C_1) - \log_v(a_v) \leq \log_v(|u|_v) \leq \log_v(C_2) - \log_v(a_v)$. Put $P_v := \log_v(C_2) - \log_v(C_1)$.

Lemma 8.1 *There is an S_K-unit $u \in K^\times$ such that $q_v^{-\lceil \log_v(N)+1\rceil - P_v} \leq |u|_v^{-1} \leq q_v^{-\lceil \log_v(N)+1\rceil}$ for $v \in S$; $q_v^{-P_v} \leq |u|_v^{-1} \leq 1$ for $v \in \widehat{S}_K \setminus S$; and $q_{v_0}^{-B(N)\log_{v_0}(N)} < |u|_{v_0} < 1$.*

Proof: Take a_v so that $-\log_v(a_v) = \lceil \log_v(N) + 1\rceil - \log_v(C_1)$ for $v \in S$ and $-\log_v(a_v) = -\log_v(C_1)$ for $v \in \widehat{S}_K \setminus S$. It follows that there is an S_K-unit $u \in K^\times$ such that

$$\begin{aligned} q_v^{\lceil \log_v(N)+1\rceil} \leq \ & |u|_v \leq q_v^{\lceil \log_v(N)+1\rceil + P_v}, \quad \forall v \in S, \\ 1 \leq \ & |u|_v \leq q_v^{P_v}, \qquad\qquad\qquad \forall v \in \widehat{S}_K \setminus S. \end{aligned}$$

That is,

$$\begin{aligned} q_v^{-\lceil \log_v(N)+1\rceil - P_v} \le \quad & |u|_v^{-1} \quad \le q_v^{-\lceil \log_v(N)+1\rceil}, \quad \forall v \in S, \\ q_v^{-P_v} \le \quad & |u|_v^{-1} \quad \le 1, \qquad\qquad\quad \forall v \in \widehat{S}_K \setminus S. \end{aligned}$$

Note that $|u|_{v_0} = \prod_{v\in\widehat{S}_K} |u|_v^{-1}$ because u is an S_K-unit. Since $B(N)\cdot\log_{v_0}(N) = \#(S)\cdot\log_{v_0}(N) + \log_{v_0}(\prod_{v\in\widehat{S}_K} q_v^{2+P_v})$ by (7.9), it follows from the above inequalities that

$$\begin{aligned} |u|_{v_0} \quad &\ge \quad \prod_{v\in S} q_v^{-\lceil \log_v(N)+1\rceil - P_v} \cdot \prod_{v\in\widehat{S}_K\setminus S} q_v^{-P_v} \\ &> \quad \prod_{v\in S} q_v^{-(\log_v(N)+2)-P_v} \cdot \prod_{v\in\widehat{S}_K\setminus S} q_v^{-(P_v+2)} = q_{v_0}^{-B(N)\log_{v_0}(N)}, \end{aligned}$$

and

$$|u|_{v_0} \le \prod_{v\in S} q_v^{-\lceil \log_v(N)+1\rceil} < 1.$$

□

Fix the S_K-unit $u \in K^\times$ in Lemma 8.1 through this chapter. To specify basic approximating functions for $v \in \widehat{S}_K$, put $b_v := u^{-1}$ for $v \in S$ and $b_v := 1$ for $v \in \widehat{S}_K \setminus S$. Letting $R'_v = |b_v|_v^{1/N}$ for $v \in \widehat{S}_K$, we have the following basic approximationg functions:

For each $v \in S$, Theorem 6.2 implies that there is an $(\mathfrak{X}, \vec{s})$-function $f_v(z) \in K_v(z)$ of degree N such that

$$E_v^0 := \{z \in \mathbb{P}^1(\mathbb{C}_v) : f_v(z) \in b_v\mathcal{O}_v\} \subset E_v, \tag{8.1}$$

$$U_v^0 := \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f_v(z)|_v \le R_v'^N\} \subset U_v. \tag{8.2}$$

Here, U_v^0 is the union of N disjoint balls $\mathfrak{B}(\theta_j, \rho_j)$, where each $\theta_j \in K_v$ and each $\rho_j \in |K_v^\times|$. Furthermore, for each $z \notin U_v$,

$$\frac{1}{N}\log_v(|f_v(z)|_v) = \sum_{i=1}^m G(z, x_i; E_v)\cdot s_i; \tag{8.3}$$

For each $v \in \widehat{S}_K \setminus S$, [22], Theorem 4.5.4, p.316 says that there is an $(\mathfrak{X}, \vec{s})$-function $f_v(z) \in K_v(z)$ of degree N such that

$$E_v = U_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f_v(z)|_v \leq R_v'^N\}, \tag{8.4}$$

and such that for all $z \notin E_v$,

$$\frac{1}{N} \log_v(|f_v(z)|_v) = \sum_{i=1}^{m} G(z, x_i; E_v) \cdot s_i; \tag{8.5}$$

For the additional place v_0, since E_{v_0} is $\mathfrak{X}$-trivial, it follows from Remark 3.24 that E_{v_0} is an RL-domain, so [22], Theorem 4.5.4, p.316 says that there is an $(\mathfrak{X}, \vec{s})$-function $f_{v_0}(z) \in K_{v_0}(z)$ of degree N such that

$$E_{v_0} = U_{v_0} = \{z \in \mathbb{P}^1(\mathbb{C}_{v_0}) : |f_{v_0}(z)|_{v_0} \leq 1\}, \tag{8.6}$$

and such that for all $z \notin E_{v_0}$,

$$\frac{1}{N} \log_v(|f_{v_0}(z)|_{v_0}) = \sum_{i=1}^{m} G(z, x_i; E_v) \cdot s_i. \tag{8.7}$$

Definition 8.2 An $\mathfrak{S}$-*subunit* is an element $(\alpha_v)_{v \in \mathfrak{S}} \in \prod_{v \in \mathfrak{S}} \overline{K}_v^\times$ for which some power α_v^A is a global $\mathfrak{S}$-unit $\alpha \in K_{\mathfrak{S}}^\times$ for all $v \in \mathfrak{S}$. Here, A is independent of all $v \in \mathfrak{S}$.

Lemma 8.3 *Let $\mathfrak{S} = \{v_1, \cdots, v_n\}$ be a finite set of places of K. Given any $(c_i)_{1 \leq i \leq n} \in \prod_{i=1}^n K_{v_i}$ such that $\prod_{i=1}^n |c_i|_{v_i} = 1$, there is an $\mathfrak{S}$-subunit $(\alpha_i)_{1 \leq i \leq n} \in \prod_{i=1}^n \overline{K}_{v_i}^\times$ such that $|c_i|_{v_i} = |\alpha_i|_{v_i}$ for each $i = 1, \cdots, n$.*

Proof: If $\mathfrak{S}$ is empty, then there is nothing to do. If $\mathfrak{S}$ is a singleton, say $\{v_1\}$, then take $\alpha_1 = 1$.

Now, suppose that $\mathfrak{S}$ contains at least two places of K. Fix $v_1 \in \mathfrak{S}$ and for each $i = 2, \cdots, n$, choose a $\{v_1, v_i\}$-unit $\beta_i \in K^\times$ so that $r_i := \mathrm{ord}_{v_i}(\beta_i) > 0$ (such an element exists by the unit theorem). So, we have $|\beta_i|_{v_i} = |\pi_{v_i}^{r_i}|_{v_i}$. The product formula implies that $|\beta_i|_{v_1} = |\beta_i|_{v_i}^{-1}$. Let $|c_i|_{v_i} = |\pi_{v_i}^{m_i}|_{v_i}$ for some $m_i \in \mathbb{Z}$ and put $A := \mathrm{lcm}\{r_i : i = 2, \cdots, n\}$. Write $A = r_i a_i$ for some $a_i \in \mathbb{Z}_{>0}$ and put $\alpha := \prod_{i=2}^n \beta_i^{a_i m_i} \in K^\times$. Clearly, α is a global $\mathfrak{S}$-unit in

K. Let $\alpha_i = \alpha^{1/A} \in \overline{K}_{v_i}^{\times}$ be an A-th root to the equation $x^A = \alpha$ for each $i = 1, \cdots, n$. It follows that for $j = 2, \cdots, n$,

$$|\alpha_j|_{v_j} = \prod_{i=2}^{n} |\beta_i|_{v_j}^{a_i m_i/A} = |\beta_j|_{v_j}^{m_j/r_j} = |\pi_{v_j}^{r_j}|_{v_j}^{m_j/r_j} = |\pi_{v_j}|_{v_j}^{m_j} = |c_j|_{v_j}.$$

Hence, the product formula implies that

$$|\alpha_1|_{v_1} = \prod_{i=2}^{n} |\beta_i|_{v_1}^{a_i m_i/A} = \prod_{i=2}^{n} (|\beta_i|_{v_i}^{-1})^{m_i/r_i} = \prod_{i=2}^{n} |\pi_{v_i}^{r_i}|_{v_i}^{-m_i/r_i} = \prod_{i=2}^{n} |c_i|_{v_i}^{-1} = |c_1|_{v_1}.$$

Thus, we have $|\alpha_i|_{v_i} = |c_i|_{v_i}$ and $\alpha_i^A = \alpha$ for all $i = 1, \cdots, n$. $\square$

For each $x_i \in \mathfrak{X}$, the logarithmic leading coefficient of $f_v(z)$ at x_i with respect to v is

$$\Lambda(f_v, x_i) = \lim_{z \to x_i} \Big(\frac{1}{N} \log_v(|f_v(z)|_v) + s_i \log_v(|g_i(z)|_v) \Big).$$

In fact, if $c_{v,i0} := f_v(z) g_i(z)^{N s_i}|_{z=x_i}$ is the leading coefficient of $f_v(z)$ at x_i, then $\Lambda(f_v, x_i) = \frac{1}{N} \log_v(|c_{v,i0}|_v)$. Since $V_{x_i}(E_v) = \lim_{z \to x_i}(G(z, x_i; E_v) + \log_v(|g_i(z)|_v)$, it follows from (8.3), (8.5) and (8.7) that

$$\Lambda(f_v, x_i) = \sum_{j=1}^{m} \widehat{G}(x_i, x_j; E_v) \cdot s_j, \tag{8.8}$$

where

$$\widehat{G}(x_i, x_j; E_v) := \begin{cases} V_{x_i}(E_v) & \text{if } i = j \\ G(x_i, x_j; E_v) & \text{if } i \neq j. \end{cases}$$

Also, the definition of $\Gamma(E_v, \mathfrak{X})$ implies that for each $i = 1, \cdots, m$,

$$\sum_{j=1}^{m} \Gamma(E_v, \mathfrak{X})_{ij} \cdot s_j = \Lambda(f_v, x_i).$$

Thus, we obtain that

$$\frac{1}{N} \log_v(|c_{v,i0}|_v) = \sum_{j=1}^{m} \Gamma(E_v, \mathfrak{X})_{ij} \cdot s_j = \sum_{j=1}^{m} \widehat{G}(x_i, x_j; E_v) \cdot s_j,$$

and hence

$$\sum_{v \in \widehat{S}_K} \frac{1}{N} \log_v(|c_{v,i0}|_v) \cdot \log(q_v) = (\Gamma(\mathbb{E}_K, \mathfrak{X}) \cdot \vec{s})_i = V(\mathbb{E}_K, \mathfrak{X}) < 0. \tag{8.9}$$

On the other hand, since $\Gamma(E_{v_0}, \mathfrak{X}) = 0$, $\widehat{G}(x_i, x_j; E_{v_0}) = 0$, so $\frac{1}{N}\log_{v_0}(|c_{v_0,i0}|_{v_0}) = \Lambda(f_{v_0}, x_i) = 0$ for all $x_i \in \mathfrak{X}$ by (8.8). Hence $|c_{v_0,i0}|_{v_0} = 1$. Noting that $V(\mathbb{E}_K, \mathfrak{X}) = -V\log(q_{v_0})$ is negative and $NV \in \mathbb{Z}_{>0}$, it follows that $b_{v_0} := \pi_{v_0}^{-NV} \in K_{v_0}$. Put $R'_{v_0} := |b_{v_0}|_{v_0}^{1/N}$. Replacing $f_{v_0}(z)$ by $b_{v_0}f_{v_0}(z)$ and $c_{v_0,i0}$ by $b_{v_0}c_{v_0,i0}$, we have

$$E_{v_0} = U_{v_0} = \{z \in \mathbb{P}^1(\mathbb{C}_{v_0}) : |f_{v_0}(z)|_{v_0} \le R'^N_{v_0}\}. \tag{8.10}$$

Thus, we achieve the product formula for the leading coefficients of the $f_v(z)$ by (8.9):

$$\frac{1}{N}\sum_{v \in S_K} \log_v(|c_{v,i0}|_v) \cdot \log(q_v) = 0. \tag{8.11}$$

Recall (8.2), (8.4) and (8.10);

$$R'_v = \begin{cases} |b_v|_v^{1/N} & \text{for } v \in S \\ 1 & \text{for } v \in \widehat{S}_K \setminus S \\ |b_{v_0}|_{v_0}^{1/N}. \end{cases}$$

Note that $R'^N_v \in |K_v^\times|$ for all $v \in S_K$. We call them the *defining constants* of the sets E_v. We need to rescale the basic local approximating functions $f_v(z)$ for $v \in S_K$ to simplify the defining constants.

For each $v \in S_K$, put

$$g_v(z) := uf_v(z). \tag{8.12}$$

We call them the *normalized local approximating functions.* Since N was chosen so that $NV \ge 2B(N)\log_{v_0}(N)$, the leading coefficient $uc_{v_0,i0}$ of $uf_{v_0}(z)$ at x_i satisfies

$$1 < q_{v_0}^{NV/2} \le q_{v_0}^{-B(N)\log_{v_0}(N)+NV} < |uc_{v_0,i0}|_{v_0} = |ub_{v_0}|_{v_0} < |b_{v_0}|_{v_0} = R'^N_{v_0}. \tag{8.13}$$

The defining constants become

$$R_v = \begin{cases} |ub_v|_v^{1/N} = 1 & \text{for } v \in S \\ |u|_v^{1/N} \ge 1 & \text{for } v \in \widehat{S}_K \setminus S \\ |ub_{v_0}|_{v_0}^{1/N} > 1. \end{cases}$$

Furthermore, for each $x_i \in \mathfrak{X}$, the leading coefficients $c'_{v,i0}$ of the $g_v(z)$, $v \in S_K$, continue to satisfy the product formula. Indeed, it follows that

$$\frac{1}{N}\sum_{v\in S_K} \log_v(|c'_{v,i0}|_v)\cdot\log(q_v)$$
$$= \frac{1}{N}\sum_{v\in \widehat{S}_K} \log_v(|uc_{v,i0}|_v)\cdot\log(q_v) + \frac{1}{N}\log_{v_0}(|ub_{v_0}|_{v_0})\log(q_{v_0})$$
$$= \frac{1}{N}\sum_{v\in S_K} \log_v(|u|_v)\cdot\log(q_v) + \frac{1}{N}\sum_{v\in \widehat{S}_K} \log_v(|c_{v,i0}|_v)\cdot\log(q_v) + \frac{1}{N}\log_{v_0}(|b_{v_0}|_{v_0})\log(q_{v_0}),$$

by (8.9),

$$= \frac{1}{N}\sum_{v\in S_K} \log_v(|u|_v)\cdot\log(q_v) + V(\mathbb{E}_K,\mathfrak{X}) + V\log(q_{v_0}),$$

since $|u|_v = 1$ for all $v \notin S_K$ and $V(\mathbb{E}_K,\mathfrak{X}) = -V\log(q_{v_0})$,

$$= \frac{1}{N}\sum_{v\in P_K} \log_v(|u|_v)\cdot\log(q_v) + V(\mathbb{E}_K,\mathfrak{X}) - V(\mathbb{E}_K,\mathfrak{X}),$$

by the product formula,

$$= \frac{1}{N}\log\Big(\prod_{v\in P_K}|u|_v\Big) = 0.$$

For notational simplicity, replace the leading coefficient $c'_{v,i0}$ of $g_v(z)$ at x_i by $c_{v,i0}$. Note that $R_v^N \in |K_v^\times|$ for all $v \in S_K$ and $\prod_{v\in S_K} R_v > 1$. In particular, since $h_{v_0} = q_{v_0}^{V/2}$, it follows from (8.13) that that

$$1 < h_{v_0} < R_{v_0} < q_{v_0}^V. \tag{8.14}$$

We now summarize the choices of the integer N, the defining constants R_v, and the normalized local approximating functions $g_v(z)$.

Theorem 8.4 *Let N_0 be the integer as in (7.8). For any sufficiently large positive integer N divisible by N_0 and satisfying (7.12) and (7.13), there are an $(\mathfrak{X},\vec{s})$-function $g_v(z) \in K_v(z)$*

of degree N for each $v \in S_K$ and constants $R_v \in |\mathbb{C}_v^\times|$ with $R_v^N \in |K_v^\times|$ such that $R_v = 1$ for each $v \in S$, $R_v \geq 1$ for each $v \in \widehat{S}_K \setminus S$, and $1 < q_{v_0}^{V/2} < R_{v_0} < q_{v_0}^V$, having the following properties:

For each $v \in S$,

$$E_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : g_v(z) \in \mathcal{O}_v\} \subset E_v, \tag{8.15}$$

$$U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |g_v(z)|_v \leq R_v^N\} \subset U_v, \tag{8.16}$$

and U_v^0 is the union of N pairwise disjoint balls $B(\theta_t, \rho_t)$ with each $\theta_t \in K_v$ and $\rho_t \in |K_v^\times|$;

For each $v \in S_K \setminus S$,

$$U_v^0 := E_v = U_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |g_v(z)|_v \leq R_v^N\}. \tag{8.17}$$

Furthermore, for each $x_i \in \mathfrak{X}$, the leading coefficients $c_{v,i0}$ of the $g_v(z)$, $v \in S_K$, satisfy the product formula

$$\frac{1}{N} \sum_{v \in S_K} \log_v(|c_{v,i0}|_v) \cdot \log(q_v) = 0. \tag{8.18}$$

Remark 8.5 The choice of local patching parameters h_v for each $v \in S_K$ in Section 7.3 makes sense. That is, it follows from Theorem 8.4 that if $h_{v_0} = q_{v_0}^{V/2}$, then one can choose h_v's for each $v \in \widehat{S}_K$ with the following properties:

(i) $0 < h_v < 1 \leq R_v$ for each $v \in \widehat{S}_K$;

(ii) $1 < h_{v_0} < R_{v_0}$;

(iii) $\prod_{v \in S_K} h_v > 1$.

(iv) for each $v \in S$, $h_v^N < q_v^{-2/(q_v-1)} < q_v^{-1/(q_v-1)}$ by (7.13).

Remark 8.6 In the case of compact sets, Theorem 6.2 implies that the basic local approximating function $f_v(z)$ for each $v \in S$ has distinct zeros $\theta_1, \cdots, \theta_N$ in $E_v^0 \subset E_v$ with the following properties: for the number $R_v' \in |\mathbb{C}_v^\times|$ chosen above, there are numbers $\rho_1, \cdots, \rho_N$

belonging to $|K_v^\times|$, with $\rho_j < 1$ for each $j = 1, \cdots, N$, such that the balls $\mathfrak{B}(\theta_j, \rho_j)$ are pairwise disjoint and $f_v(z) : \mathfrak{B}(\theta_j, \rho_j) \to B(0, R_v'^N)$ is an $R_v'^N/\rho_j$-isometry satisfying

$$f_v^{-1}(B(0, R_v'^N)) = \bigcup_{j=1}^{N} \mathfrak{B}(\theta_j, \rho_j) \subset \bigcup_{i=1}^{t} \mathfrak{B}(a_i, r_i),$$
$$f_v^{-1}(B(0, R_v'^N) \cap K_v) = \bigcup_{j=0}^{N} (\mathfrak{B}(\theta_j, \rho_j) \cap K_v) \subset E_v.$$

Since $|u|_v = 1/R_v'^N$ and $g_v(z) = uf_v(z)$, it follows that for each $v \in S$, the normalized local approximating function $g_v(z)$ has the same zeros in $E_v^0 \subset E_v$ as $f_v(z)$, and defines a $1/\rho_j$-isometry $g_v(z) : \mathfrak{B}(\theta_j, \rho_j) \to B(0,1)$ satisfying

$$U_v^0 = g_v^{-1}(B(0,1)) = \bigcup_{j=1}^{N} \mathfrak{B}(\theta_j, \rho_j) \subset \bigcup_{i=1}^{t} \mathfrak{B}(a_i, r_i),$$
$$E_v^0 = g_v^{-1}(B(0,1) \cap K_v) = \bigcup_{j=0}^{N} (\mathfrak{B}(\theta_j, \rho_j) \cap K_v) \subset E_v.$$

Corollary 8.7 *For each $w|v$, if we put $R_w := R_v^{[L_w:K_v]}$ and $g_w(z) := f_v(z)$, then the leading coeffocients $c_{w,i0}$ of the $g_w(z)$ at each $x_i \in \mathfrak{X}$ satisfy the product formula*

$$\sum_{w \in S_L} \log_w(|c_{w,i0}|_w) \cdot \log(q_w) = 0. \tag{8.19}$$

Proof: Since $|x|_w = |x|_v^{[L_w:K_v]}$ for all $x \in L_w$ and $[L_w : K_v] = e_{w/v} f_{w/v}$, it follows that $\log_w(|c_{w,i0}|_w) = \log_v(|c_{w,i0}|_v)^{[L_w:K_v]} / \log_v(q_w) = e_{w/v} \cdot \log_v(|c_{v,i0}|_v))$. Hence we have

$$\begin{aligned}
\sum_{w \in S_L} \log_w(|c_{w,i0}|_w) \cdot \log(q_w) &= \sum_{w \in S_L} e_{w/v} \cdot \log_v(|c_{v,i0}|_v)) \cdot \log(q_v) \cdot f_{w/v} \\
&= \sum_{v \in S_K} \sum_{w|v} [L_w : K_v] \log_v(|c_{v,i0}|_v)) \cdot \log(q_v) \\
&= [L : K] \sum_{v \in S_K} \log_v(|c_{v,i0}|_v)) \cdot \log(q_v) = 0.
\end{aligned}$$

□

The following proposition provides us global S_L-units to patch the leading coefficients of $g_w(z)$ for all $w \in S_L$.

Proposition 8.8 *We are given the $(\mathfrak{X}, \vec{s})$-functions $g_w(z) \in K_v(z)$ of degree N for all $w \in S_L$ whose leading coefficients $c_{w,i0} \in L_w$ satisfy the product formula (8.19). For any collection of numbers $\delta_w > 0$ for $w \in S_L$, there is a positive integer A divisible by $[L : K]_\iota$ satisfying the following: if $n \in \mathbb{Z}_{>0}$ is divisible by A, then there is a global S_L-unit $u_i \in M^\times$ for each $x_i \in \mathfrak{X}$ such that*

$$\left| \frac{u_i}{c_{w,i0}^n} - 1 \right|_w \leq \delta_w. \tag{8.20}$$

Proof: Lemma 8.3 implies that there is an S_L-subunit $(u_{w,i})_{w \in S_L} \in \prod_{w \in S_L} \overline{L}_w^\times$ such that $|c_{w,i0}|_w = |u_{w,i}|_w$. By the definition of S_L-subunits, there exist $A' \in \mathbb{Z}_{>0}$ and a global S_L-unit $u_i' \in L^\times$ such that $u_{w,i}^{A'} = u_i'$ for all $w \in S_L$ and each i. Without loss of generality, we can assume that A' is divisible by $[L : K]_i$ and hence $u_{w,i}^{A'} = u_i'$ belongs to $M^\times$. Since $|u_{w,i}/c_{w,i0}|_w = 1$, $u_{w,i}/c_{w,i0} \in \mathcal{O}_w^\times$ for all i, i.e., $u_{w,i}/c_{w,i0}$ is not zero in $\mathcal{O}_w/\mathfrak{m}_w$, which is finite. Hence there exists $A_{w,i} \in \mathbb{Z}_{>0}$ such that $(u_{w,i}/c_{w,i0})^{A_{w,i}} = 1$ in $\mathcal{O}_w/\mathfrak{m}_w$. Letting $A'' := \operatorname{lcm}(\{A_{w,i} : w \in S_L,\ i = 1, \cdots, m\} \cup \{A'\})$, since A'' is divisible by $[L : K]_i$, $u_i'' := u_{w,i}^{A''} \in M^\times$ is a global S_L-unit, independent of all $w \in S_L$. Note that $|u_i''|_w = |u_{w,i}^{A''}|_w = |c_{w,i0}^{A''}|_w$ for all $w \in S_L$. Since $(u_{w,i}/c_{w,i0})^{A''} - 1 \in \mathfrak{m}_w$, $|(u_{w,i}/c_{w,i0})^{A''} - 1|_w < 1$. For any sufficiently small $\delta_w > 0$, there is $A_w \in \mathbb{Z}_{>0}$ so that $|(u_{w,i}/c_{w,i0})^{BA_w} - 1|_w \leq \delta_w$ for any $B \in \mathbb{Z}_{>0}$. Let A_0 be the least common multiple of the A_w for $w \in S_L$. Put $A := A''A_0$. For any $n \in \mathbb{Z}_{>0}$ divisible by A, if we put $u_i := (u_i'')^{n/A''}$, it follows that

$$\left| \frac{u_i}{c_{w,i0}^n} - 1 \right|_w = \left| \left(\frac{u_{w,i}}{c_{w,i0}} \right)^n - 1 \right|_w \leq \delta_w.$$

□

Remark 8.9 Put $\Delta_{w,t}^{(0)} := \frac{u_i}{c_{w,i0}^n} - 1$. Since n is divisible by $[L : K]_\iota$, $c_{w,i0}^n$ belongs to M_u, so $\Delta_{w,t}^{(0)} \in M_u$. Note that $\sigma(u_i) = u_{\sigma(i)}$ for any $\sigma \in \operatorname{Gal}(L/K)$.

Chapter 9

Global Patching Argument II

In Chapter 8, we have constructed the normalized local approximating functions $g_v(z) \in K_v(z)$ and defining constants R_v with $R_v^N \in |K_v^\times|$ of the sets E_v, $v \in S_K$ for any sufficiently large $N \in \mathbb{Z}_{>0}$ divisible by N_0 with the conditions (7.12) and (7.13). For each $v \in S_K$, if w is a place of L lying above v, then put $g_w(z) := f_v(z)$ and $R_w := R_v^{[L_w:K_v]}$. Denote the leading coefficients of $g_v(z)$ and $g_w(z)$ by $c_{v,i0}$ and $c_{w,i0}$, respectively. In fact, they are the same and belong to L_w. In particular, they satisfy the product formulas (8.18) and (8.19).

We will construct the local patching functions defined over K_v for each $v \in S_K$ in the local patching processes. In the final step of the global patching process, we need to meld the local patching functions into a global function defined over K, which also defines the sets E_v. To do this, we will need the following strong approximation theorem.

9.1 The Very Strong Approximation Theorem

For this section only, let K be an arbitrary function field over a finite field $\mathbb{F}_q$, i.e., K is a finite extension of $F := \mathbb{F}_q(T)$. For any finite extension L of K, let M be the separable closure of K in L. If ∞ is the place of F defined by $1/T$, then $P_{K,\infty}$ is the set of all places of L lying above ∞ and $P_{K,0} = P_K \setminus P_{K,\infty}$. Let S_K be any finite set of places of K containing $P_{K,\infty}$, and S_L be the set of all places of L lying above S_K. Write $K_{S_K} = \{x \in K : |x|_v \leq 1 \text{ for all } v \notin S_K\}$ for the set of all S_K-integers in K. Recall the finiteness of the class number of K_{S_K}:

Proposition 9.1 ([21], Theorem 14.5, p.247) *Under the hypotheses above, K_{S_K} is a Dedekind domain and there is a $1-1$ correspondence between the nonzero prime ideals of K_{S_K} and $P_K \setminus S_K$. Moreover, $C\ell(K_{S_K})$ is finite.*

Since $\mathcal{O}_K = K_{P_{K,\infty}}$, $\mathcal{O}_K$ is a Dedekind domain and the class number h_K of K is finite. Note that $\mathcal{O}_F = \mathbb{F}_q[T]$ and $\mathcal{O}_\infty = \mathbb{F}_q[[\frac{1}{T}]]$.

Lemma 9.2 *Consider the canonical embedding $\phi : K \to K \otimes_F F_\infty$. Then $(K \otimes_F F_\infty)/\phi(\mathcal{O}_K)$ is compact.*

Proof: Let $n = [K : F]$ and let $\{b_1, \cdots, b_n\}$ be a basis for $\mathcal{O}_K$ over $\mathcal{O}_F$. Then $\{b_1, \cdots, b_n\}$ is also a basis for K over F, and hence $\{\phi(b_1), \cdots, \phi(b_n)\}$ form a basis for $K \otimes_F F_\infty$ over F_∞. Note that

$$F_\infty = \mathbb{F}_q\Big(\Big(\frac{1}{T}\Big)\Big) = \mathbb{F}_q[T] + \Big\{\sum_{k\geq 1} \frac{a_k}{T^k} : a_k \in \mathbb{F}_q\Big\} = \mathcal{O}_F + D_\infty,$$

where $D_\infty = \big\{\sum_{k\geq 1} \frac{a_k}{T^k} : a_k \in \mathbb{F}_q\big\} \subseteq \mathcal{O}_\infty$. After identifying $\phi(K)$ with K, write b_i for $\phi(b_i)$. For any $\alpha \in K \otimes_F F_\infty$, write $\alpha = \sum_{i=1}^n \alpha_i b_i$ with each $\alpha_i \in F_\infty$. Then $\alpha_i = \beta_i + \gamma_i$ for some $\beta_i \in \mathcal{O}_F$ and $\gamma_i \in D_\infty$, so $\alpha = \sum_{i=1}^n \beta_i b_i + \sum_{i=1}^n \gamma_i b_i$. Clearly, $\sum_{i=1}^n \beta_i b_i \in \mathcal{O}_K$. Put $D := \{\sum_{i=1}^n \delta_i b_i : \delta_i \in D_\infty\}$. Then D is compact and $K \otimes_F F_\infty = \mathcal{O}_K + D$. Hence $(K \otimes_F F_\infty)/\mathcal{O}_K$ is compact. □

Corollary 9.3 *For any nonzero ideal $\mathfrak{A}$ of $\mathcal{O}_K$, $(K \otimes_F F_\infty)/\phi(\mathfrak{A})$ is compact.*

Proof: $\mathcal{O}_K$ is the integral closure of $\mathcal{O}_F$ and $\mathfrak{A} \subset \mathcal{O}_K$, so each $\alpha \in \mathfrak{A}$ satisfies an integral equation. Hence any $\mathcal{O}_F$-basis for $\mathfrak{A}$ is an F-basis for K. Since $\mathcal{O}_F = \mathbb{F}_q[T]$ is a PID, $\mathfrak{A}$ is a free $\mathcal{O}_F$-module of rank n. With the same argument as in the previous lemma, we are done. □

Lemma 9.4 *There is a constant $C_K > 0$ depending only on K with the following property: For any $\alpha \in \mathbb{J}_K$ with $\|\alpha\| \geq C_K$, and any $\beta \in \mathbb{A}_K$, there is an $x \in K$ such that $|x - \beta_v|_v \leq |\alpha_v|_v$ for all $v \in P_K$.*

Proof: Let $h = h_K$ be the class number of K and $\{\mathfrak{A}_1, \cdots, \mathfrak{A}_h\}$ be a set of representatives of the ideal class group of $\mathcal{O}_K$. If $\phi : K \to K \otimes_F F_\infty$ is the canonical embedding, then $(K \otimes_F F_\infty)/\mathfrak{A}_i$ is compact for each i, if we identify $\phi(\mathfrak{A}_i)$ with $\mathfrak{A}_i$. Noting that $K \otimes_F F_\infty \cong \oplus_{v \in P_{K,\infty}} K_v$, fix a constant $0 < a_1 \in \mathbb{R}$ such that

$$\{y = (y_v)_{v \in P_{K,\infty}} \in K \otimes_F F_\infty : |y_v|_v \le a_1 \ \forall v \in P_{K,\infty}\}$$

contains a fundamental domain for each $\mathfrak{A}_i$ for all $i = 1, \cdots, h$.

Fix a place $v_0 \in P_{K,\infty}$ and set $S_0 := P_{K,\infty} \setminus \{v_0\}$. Remark 7.1 implies that the map $\Lambda : \mathbb{J}^1_{K,P_{K,\infty}} \to \mathbb{R}^{\#(S_0)}$ takes $\mathcal{O}_K^\times$ onto a lattice in $\mathbb{R}^{\#(S_0)}$. Hence there are constants $0 < a_2 < 1 < a_3$ with the following property : for any collection $\{0 < a_v \in \mathbb{R} : v \in S_0\}$, there is a unit $\mu \in \mathcal{O}_K^\times$ such that $a_2 < a_v \cdot |\mu|_v < a_3$ for all $v \in S_0$. Note that for any $B > 0$, there is a unit $\mu_B \in \mathcal{O}_K^\times$ such that $|\mu_B|_v > B$ for all $v \in S_0$. Choose $B > 0$ large enough that $a_1 < Ba_2$ and put $B_0 := \max\{|\mu_B|_v : v \in S_0\}$. Replacing a_3 by $B_0 a_3$ and μ by $\mu_B \mu$, we see that there is a constant $0 < a_4$ with the following property : for any collection $\{0 < a_v \in \mathbb{R} : v \in S_0\}$, there is a unit $\mu \in \mathcal{O}_K^\times$ such that $a_1 < a_v \cdot |\mu|_v < a_4$ for all $v \in S_0$. Choose $a_0 > 1$ large enough that $a_0 \cdot a_1 \ge a_4$ and $a_0 \cdot a_1 \ge 1$. Put

$$C_K := (a_0 \cdot a_1)^{\#(P_{K,\infty})}.$$

Given $\alpha \in \mathbb{J}_K$, put

$$V(\alpha) := \{\delta \in \mathbb{A}_K : |\delta_v|_v \le |\alpha_v|_v \ \forall v \in P_K\}.$$

To prove the assertion, we need to show that $K + V(\alpha) = \mathbb{A}_K$ provided that $\|\alpha\| \ge C_K$. Let $V_\infty(\alpha)$ be the part of $V(\alpha)$ supported on $P_{K,\infty}$ and let $V_0(\alpha)$ be the part supported on the complement of $P_{K,\infty}$. Then $V(\alpha) = V_\infty(\alpha) \times V_0(\alpha)$. Let $\pi_0 : V(\alpha) \to V_0(\alpha)$ be the natural projection. Let $i(\alpha)$ be the ideal in $\mathcal{O}_K$ associated to $\pi_0(\alpha)$. For each $y \in K^\times$, we easily see that $yV(\alpha) = V(y\alpha)$, so

$$y(K + V(\alpha)) = yK + yV(\alpha) = K + V(y\alpha).$$

Hence $K + V(\alpha) = \mathbb{A}_K \Leftrightarrow K + V(y\alpha) = \mathbb{A}_K$ for each $y \in K^\times$. After replacing α by $y\alpha$ for an appropriate $y \in K^\times$, we can assume that $i(\alpha) = \mathfrak{A}_i$ for some $i = 1, \cdots, h$. We choose a unit $\mu \in \mathcal{O}_K^\times$ such that $a_1 < |\mu\alpha_v|_v < a_4$ for all $v \in S_0$. Without loss of generality, replace α by $\mu\alpha$ and assume that

$$a_1 < |\alpha_v|_v < a_4$$

for all $v \in S_0$.

We are assuming that $\|\alpha\| \geq C_K = (a_0 \cdot a_1)^{\#(P_{K,\infty})}$. Since $\|\alpha\| = \prod_{v \in P_K} |\alpha_v|_v = \frac{1}{\mathrm{N}(i(\alpha))} \cdot \prod_{v \in P_{K,\infty}} |\alpha_v|_v$ and $\mathrm{N}(i(\alpha)) \geq 1$, it follows that $\prod_{v \in P_{K,\infty}} |\alpha_v|_v \geq \prod_{v \in P_{K,\infty}} (a_0 \cdot a_1)$. Since $|\alpha_v|_v < a_4 \leq a_0 \cdot a_1$ for all $v \in S_0$, necessarily $|\alpha_{v_0}|_{v_0} > a_0 \cdot a_1 > a_1$ and hence $V_\infty(\alpha)$ contains a fundamental domain for $\mathfrak{A}_i$. Fix $\beta \in \mathbb{A}_K$. There is a finite set $S_1 \subset P_K$ such that $\beta_v \in K_v$ for each $v \in S_1$ and $\beta_v \in \mathcal{O}_v$ for each $v \notin S_1$. Since $\alpha \in \mathbb{J}_K$, there is a finite set $S_2 \subset P_K$ such that $\alpha_v \in K_v^\times$ for each $v \in S_2$ and $\alpha_v \in \mathcal{O}_v^\times$ for each $v \notin S_2$. Put $S_3 := (S_1 \cup S_2) \setminus \{v_0\}$ and put $\varepsilon := \min\{1, |\alpha_v|_v : v \in S_2\}$. The Strong Approximation Theorem ([5], Theorem, p.67) says that there exists $y \in K$ such that

$$\begin{cases} |y - \beta_v|_v < \varepsilon \ \text{ for all } v \in S_3 \\ |y|_v \leq 1, \ \text{i.e., } y \in \mathcal{O}_v \text{ for all } v \notin S_3 \text{ and } v \neq v_0. \end{cases} \tag{9.1}$$

Recall that $v_0 \in P_{K,\infty}$. For any $v \in P_{K,0}$, if $v \in S_3$, then $|y - \beta_v|_v < \varepsilon \leq |\alpha_v|_v$; if $v \notin S_3$, then $y \in \mathcal{O}_v$, $\beta_v \in \mathcal{O}_v$, and $\alpha_v \in \mathcal{O}_v^\times$ so $|y - \beta_v|_v \leq 1 = |\alpha_v|_v$. Thus $|y - \beta_v|_v \leq |\alpha_v|_v$ for all $v \in P_{K,0}$. Replacing β by $y - \beta$, we can assume that $(\beta_v)_{v \in P_{K,0}} \in V_0(\alpha)$. Since $V_\infty(\alpha)$ contains a fundamental domain for $\mathfrak{A}_i$, there exists $x \in \mathfrak{A}_i$ such that $(x - \beta_v)_{v \in P_{K,\infty}} \in V_\infty(\alpha)$. Recalling that $\mathfrak{A}_i = i(\alpha)$, we see that $|x|_v \leq |\alpha_v|_v$ for all $v \in P_{K,0}$ so $(x)_{v \in P_{K,0}} \in V_0(\alpha)$, which implies that $(x - \beta_v)_{v \in P_{K,0}} \in V_0(\alpha)$ by the ultrametric inequality. Therefore, $x - \beta$ belongs to $V(\alpha)$. □

Corollary 9.5 *Let S_K be a finite set of places of K. There is a constant $C(K, S_K) > 0$ depending only on K and S_K with the following property : For any collection $\{0 < C_v \in \mathbb{R} : v \in S_K\}$ with $\prod_{v \in S_K} C_v \geq C(K, S_K)$ and for any collection $\{\beta_v \in K_v : v \in S_K\}$, there exists*

$x \in K$ such that

$$\begin{cases} |x - \beta_v|_v \le C_v \text{ for each } v \in S_K \\ |x|_v \le 1 \text{ for all } v \notin S_K. \end{cases} \tag{9.2}$$

Proof: Let C_K be the constant from the previous lemma and put

$$C(K, S_K) := C_K \prod_{v \in S_K} q_v,$$

where q_v is the residue degree at v. Suppose $\prod_{v \in S_K} C_v \ge C(K, S_K)$ with $0 < C_v \in \mathbb{R}$. For each $v \in S_K$, there exists $\alpha_v \in K_v$ such that

$$\frac{C_v}{q_v} \le |\alpha_v|_v \le C_v.$$

Letting $\alpha_v = 1$ for each $v \notin S_K$, we have $\alpha = (\alpha_v)_{v \in P_K} \in \mathbb{J}_K$ and

$$\|\alpha\| \ge \prod_{v \in S_K} \frac{C_v}{q_v} \ge \frac{C(K, S_K)}{\prod_{v \in S_K} q_v} = C_K.$$

Given a collection $\{\beta_v \in K_v : v \in S_K\}$, let $\beta_v = 0$ for each $v \notin S_K$. By the previous lemma, there exists $x \in K$ such that $|x - \beta_v|_v \le |\alpha_v|_v$ for all $v \in P_K$. $\square$

For any place v of K, if w is a place of L lying above v, there are two natural absolute values on L_w. We will write $|\cdot|_v$ for the absolute value on L_w uniquely extending the one on K_v, and $|\cdot|_w$ for the canonically normalized absolute value on L_w. Thus, for any $x \in L_w$, $|x|_w = \#(\mathcal{O}_w/\pi_w\mathcal{O}_w)^{-\mathrm{ord}_w(x)}$ and $|x|_w = |x|_v^{[L_w:K_v]}$.

Remark 9.6 If L/K is a finite normal extension, then the separable closure M of K in L is a Galois extension and L is purely inseparable over M. Hence there is only one place w lying above each place u of M. It has the residue degree $f_{w/u} = 1$ and so the degree $[L : M]$ is equal to the ramification index $e_{w/u}$, which is the local degree $[L_w : M_u]$. It is well-known that for each place v of K and for all $u|v$ of M, the ramification indices are the same, and the residue are the same. Since all of the quantities are transitive, it follows that for all $w|v$ of L, the ramification indices are the same, and the residue degrees are the same.

Remark 9.7 By [17], Proposition X.1.9, p.343, every finite extension of F is simple, and hence there are only finitely many intermediate fields between K and L by the Primitive Element Theorem. If K/F is not separable, then there is an element $y \in K$ such that $K = F(y) = \mathbb{F}_q(T)(y)$ and K is a separable extension over $\mathbb{F}_q(y)$ by [17], Corollary X.1.10 , p.344.

Remark 9.8 In general, for field extensions of transcendence degree ≥ 2, there may be infinitely many intermediate fields for a finite extension. Let k be a field of characteristic $p > 0$ and $L = k(x, y)$ the field of rational functions in two variables. Put $K := k(x^p, y^p)$. Since $[L : k(x, y^p)] = p$ and $[k(x, y^p) : K] = p$, we have $[L : K] = p^2$. We will show that there are infinitely many intermediate fields for the finite extension L/K.

Consider a field $K_n := K(x^{np+1}+y)$ for each $n \in \mathbb{Z}_{>0}$. Clearly, every K_n is contained in L. First, we claim that $x^{np+1}+y$ does not belong to K. If $x^{np+1}+y \in K$, then $x^{np+1}+y \in K(y)$, so $x^{np+1} \in K(y)$. Since $x^p \in K \subset K(y)$, $x^{np} \in K(y)$, and hence $x \in K(y)$. This implies that $L = K(x, y) = K(y) = k(x^p, y)$, so $[L : K] = [k(x^p, y) : k(x^p, y^p)] = p$, which is a contradiction. Thus, every K_n properly contains K.

Now, we need to show that they are distinct. Suppose that $K_m = K_n$ for some $m < n$. Since $x^{np+1} + y \in K_m$, we see that $(x^{np+1} + y) - (x^{mp+1} + y) = x(x^{mp}(x^{(n-m)p} - 1)) \in K_m$. Since $x^p \in K \subset K_m$, $x^{mp}(x^{(n-m)p} - 1)$ belongs to K_m, and hence $x \in K_m$, so $y \in K_m$. Thus, $L \subset K_m$, which implies that $[K_m : K] = [L : K] = p^2$. Noting that $(x^{mp+1} + y)^p = x^{p(mp+1)}+y^p \in K$, we get $[K_m : K] = p$. We have a contradiction. Therefore, the K_n, $n \in \mathbb{Z}_{>0}$, are distinct intermediate fields between K and L.

The following theorem is an extension of Corollary 9.5 and will be used to approximate the low-order coefficients of the initial patching functions to global elements in certain ranges in the global patching process.

Theorem 9.9 (Very Strong Approximation Theorem) *Let L/K be a finite normal extension. There is a constant $C(L/K, S_K) > 0$ with the following property : Let* $\{0 <$

$D_w \in \mathbb{R} : w \in S_L\}$ *be any collection of numbers with* $\prod_{w \in S_L} D_w \geq C(L/K, S_K)$*. Fix an intermediate field* $K \subseteq K' \subseteq L$*, and suppose that for each* $u \in S_{K'}$*, the* D_w *coincide for all* $w|u$*. Then for any collection of local field elements* $\{\beta_u \in K'_u : u \in S_{K'}\}$*, there is a global field element* $x \in K'$ *such that*

$$\begin{cases} |x - \beta_u|_w \leq D_w \ \textit{for each } w \in S_L \\ |x|_w \leq 1 \ \textit{for all } w \notin S_L. \end{cases} \tag{9.3}$$

Proof: For each intermediate field $K \subseteq K' \subseteq L$, let $C(K', S_{K'})$ be the constant in Corollary 9.5. By Remark 9.7, there are only finitely many such fields K', so we can set

$$C(L/K, S_K) := \max_{K \subseteq K' \subseteq L} C(K', S_{K'})^{[L:K']}. \tag{9.4}$$

Fix an intermediate field K', and let the collection $\{0 < D_w \in \mathbb{R} : w \in S_L\}$ satisfy the conditions of the theorem. For each $u \in S_{K'}$, since L/K' is also normal, the degrees $[L_w : K'_u]$ are the same for all $w|u$ of L by Remark 9.6. Since the D_w coincide for all $w|u$, put

$$D_u := D_w^{\frac{1}{[L_w:K'_u]}}.$$

Regarding $y \in K'_u$ as an element of L_w, since $|y|_w = |y|_u^{[L_w:K'_u]}$, it follows that $|y|_u \leq D_u$ if and only if $|y|_w \leq D_w$. Noting that $[L : K'] = \sum_{w|u} [L_w : K'_u]$, we see that

$$\begin{aligned} \prod_{u \in S_{K'}} D_u &= \prod_{u \in S_{K'}} \Big(\prod_{w|u} D_w^{\frac{1}{[L_w:K'_u]}} \Big)^{\frac{1}{\#\{w \in S_L : w|u\}}} \\ &= \prod_{w \in S_L} D_w^{\frac{1}{[L:K']}} \geq C(L/K, S_K)^{\frac{1}{[L:K']}} \\ &\geq C(K', S_{K'}) \end{aligned}$$

Applying the previous corollary, there exists $x \in K'$ such that

$$\begin{cases} |x - \beta_u|_u \leq D_u \ \text{for each } u \in S_{K'} \\ |x|_u \leq 1 \ \text{for all } u \notin S_{K'}. \end{cases} \tag{9.5}$$

For each $w|u$, since $|y|_w = |y|_u^{[L_w:K'_u]}$ for all $y \in L_w$, we have

$$\begin{cases} |x - \beta_u|_w \le D_w \text{ for each } w \in S_L \\ |x|_w \le 1 \text{ for all } w \notin S_L. \end{cases} \tag{9.6}$$

□

9.2 Choices of the Local/Global Patching Parameters

In this section, we will choose the global patching parameter $k_0 \in \mathbb{Z}_{>0}$ and the local patching parameters $\delta_v > 0$, $v \in S_K$. The integer k_0 will determine the break point between the coefficients deemed "high-order" and "low-order" in the local/global patching process, and will play a significant role in the choice of $n \in \mathbb{Z}_{>0}$, which will be used to raise the degree of the normalized local approximating functions. Recall that N, the common degree of the normalized local approximating functions $g_w(z)$, is a fixed integer divisible by N_0 and subject to the conditions (7.12) and (7.13).

Fix a place $v \in S_K$. Let w be a place of L, and u be a place of M, with $w|u|v$. For each $x_i \in \mathfrak{X}$, let μ be a place of $K(x_i)$, and ν be a place of $K(x_i^{p^{e_i}})$, with $w|\mu|\nu|v$. Let $T_{\mu,i} > 1$ be the constant corresponding to the basis $\{x_i^\ell : \ell = 0, \cdots, p^{e_i} - 1\}$ for $K(x_i)/K(x_i^{p^{e_i}})$ as in Proposition 7.5. Set $T_{w,i} = T_{\mu,i}^{[L_w:K(x_i)_\mu]} > 1$ and put

$$T_w := \max\{T_{w,i} > 1 : i = 1, \cdots, m\},$$
$$T_v := \max\{T_w : w|v\}.$$

We have constructed the normalized local approximating functions $g_w(z)$ with leading coefficients $c_{w,i0}$ for each $x_i \in \mathfrak{X}$. Put

$$C_v := \max\{|c_{w,i0}|_v : i = 1, \cdots, m\},$$
$$C_w := C_v^{[L_w:K_v]}.$$

Thus it follows that $C_w = \max\{|c_{w,i0}|_w : i = 1, \cdots, m\}$.

Recall that $N_i = Ns_i = n_i p^{e_i}$ and that $D = \sum_{i=1}^m N_i(x_i)$ is an M-rational divisor. Noting that $\{1,\ z^\ell/(z-x_i)^{j\cdot p^{e_i}} : i=1,\cdots,m,\ \ell=0,\cdots,p^{e_i}-1,\ j=1,\cdots,n_i\}$ is the M-basis for $\mathcal{L}(D)$, put

$$B_v := \max\{1,\ \|z^\ell/(z-x_i)^{jp^{e_i}}\|_{E_v} : i=1,\cdots,m,\ \ell=0,\cdots,p^{e_i}-1,\ j=1,\cdots,n_i\},$$
$$B_w := B_v^{[L_w:K_v]}.$$

It follows that $B_w = \max\{1,\ |z^\ell/(z-x_i)^{jp^{e_i}}|_w$ on $U_v^0 : i=1,\cdots,m,\ \ell=0,\cdots,p^{e_i}-1,\ j=1,\cdots,n_i\}$.

We have chosen the numbers h_v so that $0 < h_v < 1 \leq R_v$ for each $v \in \widehat{S}_K$ and $1 < h_{v_0} = q_{v_0}^{V/2} < R_{v_0}$ (see Theorem 8.4).

For each $v \in S_K \setminus S$, choose $k_v \in \mathbb{Z}_{>0}$ so that for all $k \geq k_v$,

$$h_v^{kN} < \frac{R_v^{kN}}{T_v B_v}. \tag{9.7}$$

Since $T_w \leq T_v \leq T_v^{[L_v:K_v]}$, if follows that for all $w|v$ and all $k \geq k_v$,

$$h_w^{kN} < \frac{R_w^{kN}}{T_w B_w}. \tag{9.8}$$

Fix $v \in S$. We have $h_v^N < q_v^{-2/(q_v-1)} < q_v^{-1/(q_v-1)}$ by Remark 8.5. Note that for given $a,\ b$ with $0 < a < b < 1$, $a^x = b^{x+1}$ has only one zero at $x = \ln(b)/\ln(a/b)$. Let k_1 be the least positive integer such that $(h_v^N)^k < (q_v^{-2/(q_v-1)})^{k+1}$ for all $k \geq k_1$. Since $k+1$ grows much faster than $\log_v(k+1)$ does, there is a $k_2 \in \mathbb{Z}_{>0}$ such that $(k+1)/(q_v-1) \geq \log_v(k+1)$ and hence $q_v^{-(k+1)/(q_v-1)} \leq q_v^{-\log_v(k+1)}$, for all $k \geq k_2$. It follows that $(q_v^{-(k+1)/(q_v-1)})^2 \leq q_v^{-(k+1)/(q_v-1)-\log_v(k+1)}$ for all $k \geq k_2$. Letting $k_3 = \max\{k_1, k_2\}$, if $k \geq k_3$, then $(h_v^N)^k < q_v^{-(k+1)/(q_v-1)-\log_v(k+1)}$. Let $k_v \in \mathbb{Z}_{>0}$ be the smallest integer such that for all $k \geq k_v$,

$$h_v^{kN} T_v B_v < q_v^{-\frac{k+1}{q_v-1}-\log_v(k+1)} < q_v^{-\frac{k}{q_v-1}-\log_v(k)}. \tag{9.9}$$

Note that for all $w|v$ and all $k \in \mathbb{Z}_{>0}$, since $T_w^{1/[L_w:K_v]} < T_w \leq T_v$, it follows that

$$(h_w^{kN} T_w B_w)^{1/[L_w:K_v]} < h_v^{kN} T_v B_v. \tag{9.10}$$

Now, put $k_4 := \max\{k_v : v \in S_K\}$. Let $C(K, S_K)$ and $C(L/K, S_K)$ be the constants from the strong approximation theorem. Put $\chi_v := \max\{|x_i^\ell|_v : i = 1, \cdots, m, \;\; \ell = 0, \cdots, p^{e_i} - 1\}$ and set $\chi_w = \chi_v^{[L_w:K_v]}$ for all $w|v$. For our chosen $N \in \mathbb{Z}_{>0}$, choose the least positive integer k_5 so that $\prod_{v\in S_K} h_v^{Nk_5} > C(K, S_K)$.

We define $k_0 = \max\{k_4, k_5\}$ to be the global patching parameter. Thus, for any $k \geq k_0$,

$$\prod_{v\in S_K} \frac{h_v^{Nk}}{T_v\chi_v} > C(K, S_K). \tag{9.11}$$

In addition, the choice of the constant $C(L/K, S_K)$ implies that for all $w|v$ and all $k \geq k_0$,

$$\prod_{w\in S_L} \frac{h_w^{Nk}}{T_w\chi_w} > C(L/K, S_K). \tag{9.12}$$

We now summarize the properties of k_0.

Proposition 9.10 *Let $C(K, S_K)$ be the constant from the Very Strong Approximation Theorem. For any $N \in \mathbb{Z}_{>0}$ divisible by N_0 and which satisfies (7.12) and (7.13), let h_v, $v \in S_K$, be the local patching parameters constructed in Section 7.3. Then there is an positive integer k_0 such that for all $k \geq k_0$,*

$$h_v^{kN} < \frac{R_v^{kN}}{T_vB_v} \quad \text{for each } v \in S_K \setminus S,$$
$$h_v^{kN}T_vB_v < q_v^{-\frac{k+1}{q_v-1}-\log_v(k+1)} < q_v^{-\frac{k}{q_v-1}-\log_v(k)} \quad \text{for each } v \in S,$$

and

$$\prod_{v\in S_K} \frac{h_v^{Nk}}{T_v\chi_v} > C(K, S_K).$$

9.3 Choices of the Local Patching Parametes δ_v and n and Preliminaries of Local/Global Patching Process for High-order Coefficients

Fix a number $\varepsilon \in |K_v^\times|$ such that the balls $\mathfrak{B}(x_i, \varepsilon)$ are disjoint from each other and from U_v. Without loss of generality, we can assume that $\varepsilon < 1$. Since every zero of $g_w(z)$ belongs to $U_v^0 \subset U_v$, $g_w(z)$ has no zeros and only one pole at x_i in $\mathfrak{B}(x_i, \varepsilon)$. Clearly, $\varepsilon < \|x_i, x_j\|_v$ for all

$j \neq i$, because the balls $\mathfrak{B}(x_i, \varepsilon)$ are disjoint from each other. Let $\eta_i : B(0,\varepsilon) \to \mathfrak{B}(x_i,\varepsilon)$ be an isometric parametrization with $\eta_i(0) = x_i$, defined by $\eta_i(z) = x_i + z$ in appropriate affine coordinates. After pulling $\mathfrak{B}(x_i,\varepsilon)$ back to $B(0,\varepsilon)$, we can expand $1/(z - x_j)$ in a geometric series about x_i, converging on $B(x_i,\varepsilon)$ (see Remark 6.5). Let Q be the least common multiple of the q_v for all $v \in S_K$, and let $r \in \mathbb{Z}_{>0}$ be large enough that

$$p^r > k_0 N.$$

Let $\alpha \in \mathbb{Z}_{>0}$ be the least integer such that

$$\frac{1}{Q^\alpha} \leq \frac{h_w^{p^r N}\ \varepsilon^{2k_0 N}}{C_w^{p^r}\ B_w^{p^r}\ T_w^{k_0 N}\ \chi_w^{k_0 N}} \cdot \min\left\{|c_{w,i0}|_w,\ |c_{w,i0}^{k_0-1}|_w\ :\ i = 1, \cdots, m\right\}. \tag{9.13}$$

There is a unique $\alpha_v \in \mathbb{Z}_{>0}$ such that $q_v^{\alpha_v} = Q^\alpha$ for each $v \in S_K$. Put

$$p^{r_0} := Q^\alpha p^r.$$

For each $v \in S_K$, fix a number $0 < \delta_v < 1$ small enough that

$$\delta_v \leq \frac{h_v^{p^r N}\ \varepsilon^{k_0 N}}{C_v^{p^r}\ B_v^{p^r}\ T_v^{k_0 N}\ \chi_v^{k_0 N}}. \tag{9.14}$$

Putting $\delta_w := \delta_v^{[L_w:K_v]}$ for all $w|v$, it follows that $\delta_w \leq \frac{h_w^{p^r N}\varepsilon^{k_0 N}}{C_w^{p^r} B_w^{p^r} T_w^{k_0 N} \chi_w^{k_0 N}}$. This choice of the δ_v will play a role in patching the leading coefficients of the "initial local patching functions" to global units in the global patching process.

Let $T_0 > 0$ be the least integer such that for all $k \geq T_0$,

$$h_v^{kN} T_v B_v < q_v^{-\frac{k}{q_v - 1} - \log_v(n)}. \tag{9.15}$$

Since $h_v^N < q_v^{-1/(q_v-1)}$, i.e., $\log_v(h_v^N) < -1/(q_v - 1)$ and $h_v^{(T_0-1)N} T_v B_v \geq q_v^{-(T_0-1)/(q_v-1) - \log_v(n)}$, it follows that $T_0 \leq \frac{\log_v(B_v/h_v^N) + \log_v(n)}{-(\log_v(h_v^N) + 1/(q_v-1))}$.

Let A be the positive integer in Proposition 8.8. For the moment, let $n \in \mathbb{Z}_{>0}$ be any sufficiently large integer divisible by Ap^{r_0} such that $n > T_0 > p^r$ (later other conditions will be imposed). Write $n = p^{r_0} \cdot n_0$ for some $n_0 \in \mathbb{Z}_{>0}$. This choice of n allows the leading coefficients to be patched, and will also arrange the high-order coefficients determined by k_0

to be zero for the RL-domains and to have very small absolute values for the compact sets in the local/global patching process. In the local/global patching processes, we will patch the initial patching functions $(g_w(z))^n$ (for RL-domains) and $S_n(g_w(z))$ (for compact sets).

In the RL-domain case, since the characteristic of K is p, we can expand

$$\begin{aligned} g_w(z)^n &= (g_w(z)^{n_0})^{p^{r_0}} \\ &= \left(\sum_{i=1}^{m} \sum_{k=0}^{n_0 N_i - 1} b_{w,ik} \frac{1}{(z-x_i)^{n_0 N_i - k}} + b_{w,0} \right)^{p^{r_0}} \\ &= \sum_{i=1}^{m} \sum_{k=0}^{n_0 N_i - 1} b_{w,ik}^{p^{r_0}} \frac{1}{(z-x_i)^{nN_i - kp^{r_0}}} + b_{w,0}^{p^{r_0}}. \end{aligned}$$

In the expansion, its leading coefficient at each x_i is $b_{w,i0}^{p^{r_0}}$, which is $c_{w,i0}^n$ and the next nonzero term occurs at the order $nN_i - p^{r_0}$, which means that there are no terms of orders $nN_i - 1, \cdots, nN_i - p^{r_0} + 1$ at each x_i.

In the compact set case, consider the function $S_n(g_w(z))$. We want to arrange the high-order coefficients to be very small.

First, consider $S_n(z) := S_{n,v,1}(z)$. Note that there are $q_v^{\alpha_v}$ cosets in $\mathcal{O}_v$ modulo $\pi_v^{\alpha_v}$ and every element in each coset is the same modulo $\pi_v^{\alpha_v}$. It follows that

$$\begin{aligned} S_n(z) &= \prod_{j=0}^{n-1} (z - \beta_v(j)) \equiv \prod_{j=0}^{q_v^{\alpha_v}-1} (z - \beta_v(j))^{p^r \cdot n_0} \mod \pi_v^{\alpha_v} \\ &\equiv \prod_{j=0}^{q_v^{\alpha_v}-1} (z^{p^r} - \beta_v(j)^{p^r})^{n_0} \mod \pi_v^{\alpha_v} \\ &\equiv z^n + a_{p^r} \cdot z^{n-p^r} + \cdots \mod \pi_v^{\alpha_v}. \end{aligned}$$

Writing $S_n(z) = z^n + \sum_{\ell=1}^{n} a_\ell z^{n-\ell}$ with $a_\ell \in \mathcal{O}_v$, it follows that for each $\ell = 1, \cdots, p^r - 1$,

$$|a_\ell|_v \leq q_v^{-\alpha_v} = Q^{-\alpha}. \tag{9.16}$$

Now, we consider the high-order coefficients of $S_n(g_w(z))$. Since $g_w(z) \cdot (z - x_i)^{N_i}$ has no zeros and poles in $B(x_i, \varepsilon)$, we can write

$$g_w(z) \cdot (z - x_i)^{N_i} = c_{w,i0} + \sum_{k \geq 1} c_{w,ik} \cdot (z - x_i)^k.$$

The theory of the Newton polygons implies that the Newton polygon of $g_w(z)\cdot(z-x_i)^{N_i}$ has no breaks and the unique line touching the Newton polygon with slope $\log_v(\varepsilon)$ at $(0, \mathrm{ord}_v(c_{w,i0}))$ has the y-intercept $-\log_v\left(\|g_w(z)\cdot(z-x_i)^{N_i}\|_{B(x_i,\varepsilon)}\right)$, that is, $\|g_w(z)\cdot(z-x_i)^{N_i}\|_{B(x_i,\varepsilon)} = |c_{w,i0}|_v$. By Lemma 2.3, it follows that $|c_{w,ik}|_v < |c_{w,i0}|_v \cdot \varepsilon^{-k}$ for all $k \geq 1$. Rewrite

$$g_w(z)\cdot(z-x_i)^{N_i} = c_{w,i0}\Big(1+\sum_{k\geq 1} b_{w,ik}\cdot(z-x_i)^k\Big),$$

with $|b_{w,ik}|_v < \varepsilon^{-k}$ for each $k \geq 1$. Note that for any $d \in \mathbb{Z}_{>0}$, if we write

$$\Big(g_w(z)\cdot(z-x_i)^{N_i}\Big)^d = c_{w,i0}^d\Big(1+\sum_{\ell\geq 1} b_{w,i\ell}^{(d)}(z-x_i)^\ell\Big),$$

the coefficients $b_{w,i\ell}^{(d)}$ satisfy $|b_{w,i\ell}^{(d)}|_v < \varepsilon^{-\ell}$ for each $\ell \geq 1$ by homogeneity. Since $S_n(z) = \sum_{\ell=0}^n a_\ell z^{n-\ell}$ with $a_0 = 1$, we have

$$\begin{aligned}
S_n(g_w(z)) &= g_w(z)^n + \sum_{\ell=1}^{p^r-1} a_\ell\cdot g_w(z)^{n-\ell} + \sum_{\ell=p^r}^{n} a_\ell\cdot g_w(z)^{n-\ell} \\
&= (z-x_i)^{-nN_i}\Bigg[\Big(g_w(z)\cdot(z-x_i)^{N_i}\Big)^n + \\
&\quad \sum_{\ell=1}^{p^r-1} a_\ell\cdot\Big(g_w(z)\cdot(z-x_i)^{N_i}\Big)^{n-\ell}\cdot(z-x_i)^{\ell N_i} \\
&\quad + \text{higher terms}\Bigg] \\
&= (z-x_i)^{-nN_i}\cdot c_{w,i0}^n\Bigg[\Big(1+\sum_{j\geq 1} b_{w,ij}^{(n)}(z-x_i)^j\Big) + \\
&\quad \sum_{\ell=1}^{p^r-1} a_\ell\cdot c_{w,i0}^{-\ell}(z-x_i)^{\ell N_i}\cdot\Big(1+\sum_{h\geq 1} b_{w,ih}^{(n-\ell)}(z-x_i)^h\Big) \\
&\quad + \text{higher terms}\Bigg],
\end{aligned} \tag{9.17}$$

where "higher terms" are the terms in $(z-x_i)^h$ for $h \geq p^r N_i$. We have already arranged that $b_{w,ij}^{(n)} = 0$ in the range $1 \leq j < p^{r_0}$ by the choice of n, and $|b_{w,ih}^{(n-\ell)}|_v \leq \varepsilon^{-h}$ for each $h \geq 1$ and each $\ell = 1, \cdots, p^r - 1$. Note that $c_{w,i0}^{-\ell}$ is determined by $g_w(z)$ for each $\ell = 1, \cdots, p^r - 1$.

Since $g_w : \mathfrak{B}(\theta_t, \rho_t) \cap K_v \to \mathcal{O}_v$ is bijective, the preimage $\widetilde{\beta}_v(k) := g_w^{-1}(\beta_v(k))$ belongs to $\mathfrak{B}(\theta_t, \rho_t) \cap K_v$ for all $t = 1, \cdots, N$, so every zero of $S_n(g_w(z))$ belongs to $E_v^0 \subset K_v$. Hence

$S_n(g_w(z)) \cdot (z-x_i)^{nN_i} \cdot c_{w,i0}^{-n}$ has no zeros and poles in $\mathfrak{B}(x_i, \varepsilon)$. By the same argument as above, we can write

$$S_n(g_w(z)) = (z-x_i)^{-nN_i} \cdot c_{w,i0}^{n} \left(1 + \sum_{\ell \geq 1} B_{w,i\ell}^{(0)} (z-x_i)^{\ell}\right), \tag{9.18}$$

with $|B_{w,i\ell}^{(0)}|_v \leq \varepsilon^{-\ell}$ for all $\ell \geq 1$.

For each $e = 1, \cdots, n-1$, put

$$\widetilde{S}_e(z) := \prod_{j=e}^{n-1} (z - \beta_v(j)).$$

Similarly, we can rewrite

$$\widetilde{S}_e(g_w(z)) = (z-x_i)^{-(n-e)N_i} \cdot c_{w,i0}^{n-e} \Big(1 + \sum_{\ell \geq 1} B_{w,i\ell}^{(e)} (z-x_i)^{\ell}\Big) \tag{9.19}$$

with $|B_{w,i\ell}^{(e)}|_v \leq \varepsilon^{-\ell}$ for each $\ell \geq 1$.

But we need more delicate bound condition for $B_{w,i\ell}^{(0)}$ to carry out the global patching process, when $\ell = 1, \cdots, k_0 N_i - 1$.

Proposition 9.11 *For each $\ell = 1, \cdots, k_0 N_i - 1$,*

$$|B_{w,i\ell}^{(0)}|_w \leq \frac{h_w^{p^r N}\ \varepsilon^{k_0 N}}{C_w^{p^r} B_w^{p^r} T_w^{k_0 N} \chi_w^{k_0 N}}. \tag{9.20}$$

Proof: Comparing the coefficients in (9.17) and (9.18), it follows that for each $k = 1, \cdots, k_0$ and each $\ell = (k-1)N_i, \cdots, kN_i - 1$,

$$B_{w,i\ell}^{(0)} = b_{w,i\ell}^{(n)} + \sum_{j=1}^{k-1} \frac{a_j}{c_{w,i0}^{j}} b_{w,i,\ell - jN_i}^{(n-j)},$$

where $b_{w,i0}^{(n)} = 1$. Note that $p^{r_0} > p^r > k_0 N$, $b_{w,ij}^{(n)} = 0$ for $j = 1, \cdots, p^{r_0} - 1$ and $|b_{w,ih}^{(n-d)}|_v \leq \varepsilon^{-h}$ for each $h \geq 1$ and each $d = 1, \cdots, p^r - 1$. Since $\max\{|c_{w,i0}|_w^{-1}, \cdots, |c_{w,i0}|_w^{-(k-1)}\} \cdot \min\{|c_{w,i0}|_w,\ |c_{w,i0}^{k_0-1}|_w\} \leq 1$ for each $k = 2, \cdots, k_0$, the properties (9.13) and (9.16) imply that for each $\ell = 1, \cdots, k_0 N_i - 1$,

$$|B_{w,i\ell}^{(0)}|_w \leq \frac{h_w^{p^r N}\ \varepsilon^{k_0 N}}{C_w^{p^r} B_w^{p^r} T_w^{k_0 N} \chi_w^{k_0 N}}.$$

□

Chapter 10

Local Patching Process for RL-domains

Let N be the positive integer divisible by N_0 satisfying (7.12) and (7.13) chosen in Section 7.3.

In Theorem 8.4, we have constructed a normalized local approximating function $g_v(z) \in K_v(z)$ for each $v \in S_K \setminus S$. Recall that $g_v(z)$ is an $(\mathfrak{X}, \vec{s})$-function of degree N with the following properties: there are numbers $R_v \geq 1$, $v \neq v_0$ and $R_{v_0} > 1$ such that

$$U_v^0 = E_v = U_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f_v(z)|_v \leq R_v^N\}.$$

Recall that an RL-domain is of the form $D_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z)|_v \leq R\}$ defined by a function $h(z) \in \mathbb{C}_v(z)$, for some $R \in |\mathbb{C}_v^\times|$. The boundary of D_v, relative to $h(z)$, is defined to be $\partial D_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z)|_v = R\}$.

Proposition 10.1 (Maximum Modulus Principle) *Let $D_v \subset \mathbb{P}^1(\mathbb{C}_v)$ be an RL-domain as above. If $f(z) \in \mathbb{C}_v(z)$ has no poles in D_v, then $|f(z)|_v$ achieves its maximum value on D_v at a point of ∂D_v.*

Proof: See [22], Theorem 1.4.2, p.51. □

The following lemma will be used in the patching process described below. It says that the set U_v^0 is preserved by the patching precess.

Lemma 10.2 *Let $E = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z)|_v \leq R\}$, where $h(z) \in \mathbb{C}_v(z)$ is nonconstant and $0 < R \in |\mathbb{C}_v^\times|$. Put $E^c := \mathbb{P}^1(\mathbb{C}_v) \setminus E$. If $\varphi(z) \in \mathbb{C}_v(z)$ has the following properties:*

(i) $|\varphi(z)|_v < R$ for all $z \in E$;

(ii) $\{$*poles of* $\varphi(z)\} \subseteq \{$*poles of* $h(z)\} \subset E^c$*; and*

(iii) *for each pole* x *of* $h(z)$*, the order of* $h(z)$ *at* x *is greater than or equal to the order of* $\varphi(z)$ *at* x*,*

then $E = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z) + \varphi(z)|_v \leq R\}$.

Proof: Let $D = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z)|_v \geq R\} = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |1/h(z)|_v \leq 1/R\}$. Then D is an RL-domain and $\partial E = \partial D = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z)|_v = R\}$. Set $H(z) := \frac{\varphi(z)}{h(z)}$ and suppose that $H(z)$ has a pole in D. It follows from (iii) that $h(z)$ has a zero in D, say $h(z_0) = 0$ for some $z_0 \in D$. Since $|h(z_0)|_v = |0|_v = 0 < R$, we see that $z_0 \notin D$, which is a contradiction. Hence $H(z)$ has no poles in D. On $\partial E = \partial D$, $|H(z)|_v = \frac{|\varphi(z)|_v}{|h(z)|_v} < \frac{R}{R} = 1$. By the Maximum Modulus Principle, $|H(z)|_v < 1$ on D, i.e.,

$$|\varphi(z)|_v < |h(z)|_v \text{ on } D.$$

The ultrametric inequality implies that $|h(z) + \varphi(z)|_v = |h(z)|_v \geq R$ on D. We then obtain that $|h(z) + \varphi(z)|_v = |h(z)|_v = R$ on $\partial E = \partial D$ and $|h(z) + \varphi(z)|_v = |h(z)|_v > R$ on E^c.

Now, since $h(z)$ and $\varphi(z)$ have no poles in E by (ii), $h(z)+\varphi(z)$ has no poles in E. Again, the Maximum Modulus Principle implies that $|h(z) + \varphi(z)|_v \leq R$ on E. Thus, we have

$$E = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |h(z) + \varphi(z)|_v \leq R\}.$$

□

Fix $v \in S_K \setminus S$. For each $w|v$, let $g_w(z) = g_v(z) \in K_v(z)$ be the normalized local approximating function constructed in Theorem 8.4. Let $k_0 \in \mathbb{Z}_{>0}$ be the global patching parameter as in Proposition 9.10. Recall that for all $k \geq k_0$,

$$h_w^{kN} < \frac{R_w^{kN}}{T_w B_w}. \tag{10.1}$$

We have chosen the number $0 < \delta_w < 1$ (see (9.14)) so that

$$\delta_w \leq \frac{h_w^{p^r N} \varepsilon^{k_0 N}}{C_w^{p^r} B_w^{p^r} T_w^{k_0 N} \chi_w^{k_0 N}}. \tag{10.2}$$

For any sufficiently large integer n divisible by Ap^{r_0} (see Section 9.2), we first construct the initial local patching function $G_w^{(0)}(z)$ by composing $g_w(z)$ with z^n. Put

$$G_w^{(0)}(z) := g_w(z)^n. \tag{10.3}$$

By the choice of n, since $p^{r_0} > p^r > k_0 N_i$, $G_w^{(0)}(z)$ has no terms of orders $nN_i - 1, \cdots, (n-k_0)N_i + 1$ at each x_i. We will patch the coefficients of $G_w^{(0)}(z)$ with numbers $\Delta_{w,i\ell} \in L_w$ and $\Delta_{w,0} \in L_w$ for all $w|v$, all $i = 1, \cdots, m$, and all $\ell = 0, \cdots, nN_i - 1$, while preserving the property that its roots belong to U_v^0. The $\Delta_{w,i\ell}$ will be subject only to the following conditions:

(i) $|\Delta_{w,i0}|_w \leq \delta_w$ for all w and all i;

(ii) $\Delta_{w,i\ell} = 0$ for all w, all i, and $\ell = 1, \cdots, k_0 N_i - 1$;

(iii) $|\Delta_{w,i\ell}|_w \leq h_w^{kN}$ if $(k-1)N_i \leq \ell \leq kN_i - 1$, for each $k = k_0 + 1, \cdots, n$;

(iv) $|\Delta_{w,0}|_w \leq h_w^{nN}$;

(v) $\sum_{i=1}^m \sum_{\ell=0}^{k_0 N_i - 1} c_{w,i0}^{p^r} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{p^r N_i - \ell}} \in K_v(z)$ is independent of $w|v$;

(vi) $\sum_{i=1}^m \sum_{\ell=(k-1)N_i}^{kN_i - 1} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{kN_i - \ell}} \in K_v(z)$ is independent of $w|v$, for each $k = k_0 + 1, \cdots, n$;

(vii) $\Delta_{w,0}$ is independent of $w|v$.

The conditions (i) and (v) are used in patching the leading coefficients. By the choice of n and the condition (ii), we will do nothing to patch the coefficients in the high-order blocks. The conditions (iii) and (vi) are used in patching the coefficients in the low-order blocks, and the conditions (iv) and (vii) are used in patching including the constant term. The numbers $\Delta_{w,i\ell}$ and $\Delta_{w,0}$ satisfying these conditions will be provided by the global patching process.

10.1 Patching the Leading Coefficients

We first patch the leading coefficients, i.e., the coefficients of $1/(z-x_i)^{nN_i}$ for each x_i. Given the number $\Delta_{w,i0}$ for each i, put

$$\varphi_w^{(0)}(z) := \sum_{i=1}^m c_{w,i0}^{p^r} \Delta_{w,i0} \frac{1}{(z-x_i)^{p^r N_i}}.$$

By (ii) and (v), $\varphi_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$. In particular, it follows from (10.2) that

$$\|\varphi_w^{(0)}(z)\|_{U_v^0} \le C_w^{p^r} \cdot \delta_w \cdot B_w^{p^r} < h_w^{p^r N}. \tag{10.4}$$

Define

$$\widehat{G}_w^{(0)}(z) := G_w^{(0)}(z) + \varphi_w^{(0)}(z) \cdot g_w(z)^{n-p^r}. \tag{10.5}$$

The leading coefficient of $\widehat{G}_w^{(0)}(z)$ at each x_i is $c_{w,i0}^n + c_{w,i0}^{p^r} \cdot \Delta_{w,i0} \cdot c_{w,i0}^{n-p^r} = c_{w,i0}^n(1+\Delta_{w,i0})$ and $\widehat{G}_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$. Note that $\widehat{G}_w^{(0)}(z)$ has the same constant term as $G_w^{(0)}(z)$. It follows from (10.4) that $\|\varphi_w^{(0)}(z) \cdot g_w(z)^{n-p^r}\|_{U_v^0} < h_w^{p^r N} \cdot R_w^{(n-p^r)N} < R_w^{nN}$. We then have $U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |\widetilde{G}_w^{(0)}(z)|_w \le R_w^{nN}\}$ by Lemma 10.2.

10.2 Patching the High-order Coefficients

When $k = 1$, we patch the coefficients in the block $nN_i - 1, \cdots, nN_i - N_i + 1$ by setting $\varphi_w^{(1)}(z) := 0$. Define $G_w^{(1)}(z) = \widehat{G}_w^{(0)}(z)$.

For each $k = 2, \cdots, k_0$, we patch the coefficients in the block $nN_i - (k-1)N_i, \cdots, nN_i - kN_i + 1$ by setting $\varphi_w^{(k)}(z) := 0$. Define $G_w^{(k)}(z) = G_w^{(k-1)}(z)$.

Clearly, each $G_w^{(k)}(z)$, $k = 1, \cdots, k_0$, is rational over K_v and independent of $w|v$. Furthermore, we have $U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |G_w^{(k)}(z)|_w \le R_w^{nN}\}$ by Lemma 10.2.

10.3 Patching the Low-order Coefficients

For each $k = k_0 + 1, \cdots, n$, we inductively patch the coefficients in the block $nN_i - (k-1)N_i, \cdots, nN_i - kN_i + 1$. Suppose that we have constructed functions $G_w^{(k-1)}(z) \in K_v(z)$, independent of $w|v$, such that $U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |G_w^{(k-1)}(z)|_w \le R_w^{nN}\}$. Given the numbers $\Delta_{w,i,(k-1)N_i}, \cdots, \Delta_{w,i,kN_i-1}$ for each i, put

$$\varphi_w^{(k)}(z) := \sum_{i=1}^{m} \sum_{\ell=(k-1)N_i}^{kN_i-1} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{kN_i-\ell}}.$$

By (vi), $\varphi_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$. Since $T_w > 1$, it follows that

$$\|\varphi_w^{(k)}(z)\|_{U_v^0} \leq h_w^{kN} \cdot B_w < h_w^{kN} \cdot T_w \cdot B_w. \tag{10.6}$$

Define

$$G_w^{(k)}(z) := G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot g_w(z)^{n-k}. \tag{10.7}$$

Then $G_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$ and has the same constant term as $G_w^{(0)}(z)$. It follows from (10.1) and (10.6) that

$$\|\varphi_w^{(k)}(z) \cdot g_w(z)^{n-k}\|_{U_v^0} \leq h_w^{kN} \cdot T_w \cdot B_w \cdot R_w^{(n-k)N} < \frac{R_w^{kN}}{T_w B_w} \cdot T_w \cdot B_w \cdot R_w^{(n-k)N} = R_w^{nN}.$$

We then have $U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |G_w^{(k)}(z)|_w \leq R_w^{nN}\}$ by Lemma 10.2.

Finally, at stage $k = n+1$, we patch the constant term. Given $\Delta_{w,0} \in L_w$ with $|\Delta_{w,0}|_w \leq h_w^{nN}$ for each $w|v$, define $G_w^{(n+1)}(z) := G_w^{(n)}(z) + \Delta_{w,0}$. Then $G_w^{(n+1)}(z)$ is rational over K_v and independent of $w|v$ by (vi). Furthermore, since $|\Delta_{w,0}|_w \leq h_w^{nN}$, we have $U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |G_w^{(n+1)}(z)|_w \leq R_w^{nN}\}$ by Lemma 10.2.

Chapter 11

Local Patching Process for Compact Sets

Let N be the positive integer divisible by N_0 satisfying (7.12) and (7.13) chosen in Section 7.3.

In Theorem 8.4 and Remark 8.6, we have constructed a normalized local approximating function $g_v(z)$ for each $v \in S$. Recall that $g_v(z)$ is an $(\mathfrak{X}, \vec{s})$-function of degree N with distinct zeros $\theta_1, \cdots, \theta_N$ in $E_v^0 \subset E_v$, having the following properties: there are numbers $\rho_1, \cdots, \rho_N \in |K_v^\times|$ with $0 < \rho_t < 1$, for each $t = 1, \cdots, N$, such that the balls $\mathfrak{B}(\theta_t, \rho_t)$ are disjoint from each other and such that

$$U_v^0 = g_v^{-1}(B(0,1)) = \bigcup_{t=1}^{N} \mathfrak{B}(\theta_t, \rho_t), \tag{11.1}$$

$$E_v^0 = g_v^{-1}(\mathcal{O}_v) = U_v^0 \cap K_v = \bigcup_{t=1}^{N} (\mathfrak{B}(\theta_t, \rho_t) \cap K_v). \tag{11.2}$$

Furthermore, for each t, the map $g_v : \mathfrak{B}(\theta_t, \rho_t) \to B(0,1)$ defines a K_v-rational $1/\rho_t$-isometry, that is, g_v is a $1-1$ correspondence such that for all z, $y \in \mathfrak{B}(\theta_t, \rho_t)$,

$$|g_v(z) - g_v(y)|_v = \frac{1}{\rho_t}|z - w|_v. \tag{11.3}$$

For any sufficiently large integer n divisible by Ap^{r_0} (see Section 9.2), let $\{\beta_v(k)\}_{0 \le k < n}$ be the basic well-distributed sequence of length n in $\mathcal{O}_v$ (see Chapter 5). Let

$$S_n(z) := S_{n,v,1}(z) = \prod_{k=0}^{n-1}(z - \beta_v(k))$$

be the Stirling polynomial of degree n for the ring $\mathcal{O}_v$ and put

$$G_v^{(0)}(z) := S_n(g_v(z)) = \prod_{k=0}^{n-1}(g_v(z) - \beta_v(k)), \tag{11.4}$$

which has leading coefficient $c_{v,i0}^n$ at each $x_i \in \mathfrak{X}$.

For each zero θ_t of $g_v(z)$, since $\theta_t \in K_v$, there is a K_v-rational isometric parametrization $\widetilde{\lambda}_t : B(0,\rho_t) \to \mathfrak{B}(\theta_t,\rho_t)$ with $\widetilde{\lambda}_t(0) = \theta_t$, defined by $z \mapsto \theta_t + z$ in appropriate affine coordinates. Choose $d_t \in K_v^\times$ with $\rho_t = |d_t|_v$ and define $\lambda_t : B(0,1) \to \mathfrak{B}(\theta_t,\rho_t)$ by $\lambda_t(Z) = \widetilde{\lambda}_t(d_t Z)$. It follows that $\|\lambda_t(Z),\lambda_t(Y)\|_v = \|\widetilde{\lambda}_t(d_t Z),\widetilde{\lambda}_t(d_t Y)\|_v = |d_t(Z-Y)|_v = \rho_t \cdot |Z-Y|_v$. We then have a K_v-isometry $\xi_t : B(0,1) \to \mathfrak{B}(\theta_t,\rho_t) \to B(0,1)$ defined by $\xi_t(Z) = g_v(\lambda_t(Z))$. Since $\xi_t(0) = g_v(\lambda_t(0)) = g_v(\theta_t) = 0$, we can expand $\xi_t(Z) = \sum_{k\geq 1} c_k Z^k \in K_v[[z]]$ as a power series converging in $B(0,1)$ (see Remark 6.5). Since $\|\xi_t\|_{B(0,1)} = 1$, Lemma 2.3 shows that $|c_k|_v \leq 1$ for all $k \geq 1$. Hence $\xi_t(Z)$ belongs to $\mathcal{O}_v[[Z]]$. Clearly, $\xi_t(\mathcal{O}_v) \subset \mathcal{O}_v$. For each $x \in B(0,1)$, there is only one solution $Z_0 \in B(0,1)$ to $\xi_t(Z) = x$. If $x \in K_v$, then $Z_0 \in K_v$ by Corollary 2.7, and so $Z_0 \in \mathcal{O}_v$. Thus we obtain $\xi_t(\mathcal{O}_v) = \mathcal{O}_v$.

Moreover, since $\xi_t(Z)$ has only one root at 0 in $B(0,1)$, there is a unique line ℓ with slope $\log_v(1)$ touching the Newton polygon of $\xi_t(Z)$ at the corner $(1, \mathrm{ord}_v(c_1))$ and at no other points, that is, $\ell : Y = \log_v(1)(Z-1) + \mathrm{ord}_v(c_1)$. Noting that by Corollary 2.14, the Y-intercept of ℓ is equal to $-\log_v(\|\xi_t\|_{B(0,1)})$, we have $\mathrm{ord}_v(c_1) = -\log_v(1) = 0$ and so $|c_1|_v = 1$. For each $k \geq 2$, the points $(k, \mathrm{ord}_v(c_k))$ are above the line ℓ, and hence $\mathrm{ord}_v(c_k) > \mathrm{ord}_v(c_1)$, that is, $|c_k|_v < |c_1|_v = 1$.

Let $\tau_t(z) = \sum_{k\geq 1} b_k z^k \in \mathbb{C}_v[[z]]$ be the formal inverse of $\xi_t(Z)$. For each $z \in B(0,1)$, since $\xi_t(\tau_t(z)) = z$, if we write $\xi_t(\tau_t(z)) = \sum_{k=1}^{\infty} a_k z^k$, then a_k is a $\mathbb{Z}$-linear combination of monomials in $\{c_1,\cdots,c_k,b_1,\cdots,b_k\}$. In fact, $a_k = c_1 \cdot b_k +$ a linear combination of monomials in $\{c_2,\cdots,c_k,b_1,\cdots,b_{k-1}\}$, where $x = \xi_t(\tau_t(x)) = \sum_{k\geq 1} a_k x^k$. Clearly, $1 = c_1 b_1$. Since $c_k \in \mathcal{O}_v$ for all $k \geq 1$ and $|c_k|_v < |c_1|_v = 1$ for each $k \geq 2$, it follows by induction that $b_k \in \mathcal{O}_v$ and $|b_k|_v < |b_1|_v = 1$ for each $k \geq 2$. Hence $\tau_t(z) \in \mathcal{O}_v[[z]]$ converges on $B(0,1)$ by Lemma 2.1. As above, $\tau_t(\mathcal{O}_v) = \mathcal{O}_v$.

After multiplying d_t by a unit in $\mathcal{O}_v$ if necessary, we can assume that $c_1 = b_1 = 1$ because $|c_1|_v = 1$ and $c_1 b_1 = 1$.

There is a $1-1$ correspondence between the zeros θ_{tj} of $S_n(g_v(z))$ in $\mathfrak{B}(\theta_t, \rho_t)$ and the zeros α_{tj} of $S_n(\xi_t(Z))$ in $B(0,1)$. The zeros of $S_n(z)$ are the $\beta_v(j)$, so we can index the α_{tj} in such a way that $\xi_t(\alpha_{tj}) = \beta_v(j)$. Since $\beta_v(j) \in \mathcal{O}_v$, $\alpha_{tj} = \tau_t(\beta_v(j)) \in \mathcal{O}_v$ and $\theta_{tj} = \lambda_t(\alpha_{tj}) \in K_v$. In particular, $\{\alpha_{tj}\}_{0\leq j<n}$ is a regular sequence of length n in $\mathcal{O}_v$. For each pair $0 \leq j \neq k < n$, $\mathrm{ord}_v(\alpha_{tj} - \alpha_{tk}) = \mathrm{ord}_v(\xi_t(\alpha_{tj}) - \xi_t(\alpha_{tk})) = \mathrm{ord}_v(\beta_v(j) - \beta_v(k)) = \mathrm{val}_{q_v}(|j-k|) \leq \lfloor \log_v(n) \rfloor$ because ξ_t preserves distances.

By the discussion above, $\xi_t(Z) = Z + \sum_{k=2}^{\infty} c_k Z^k$. Put

$$\xi_{v,t}^{(0)}(Z) := S_n(\xi_t(Z)). \tag{11.5}$$

Proposition 5.11 says that the part of the Newton polygon of $\xi_{v,t}^{(0)}(Z)$ corresponding to the roots of $\xi_{v,t}^{(0)}(Z)$ in $B(0,1)$ is the same as that of $S_n(Z)$. By the Weierstrass Preparation Theorem, we can write $\xi_{v,t}^{(0)}(Z) = a\prod_{j=0}^{n-1}(Z-\alpha_{tj})H_t(Z)$ for some $a \in K_v$ and unit power series $H_t(Z) \in K_v[[Z]]$. If we expand $\xi_{v,t}^{(0)}(Z) = \sum_{k\geq 1} e_k z^k \in K_v[[Z]]$, it follows from Lemma 2.11 that the Newton polygon of $a\prod_{j=0}^{n-1}(Z-\alpha_{tj})$ coincides with the part of the Newton polygon of $\xi_{v,t}^{(0)}(Z)$ up to the corner $(n, \mathrm{ord}_v(e_n))$. Hence the Newton polygon of $a\prod_{j=0}^{n-1}(Z-\alpha_{tj})$ is the same as that of $S_n(Z)$ and so we have $(n, \mathrm{ord}_v(e_n)) = (n, \mathrm{ord}_v(1))$ after comparing their leading coefficients. Thus $\mathrm{ord}_v(a) = 1$, i.e., $|a|_v = 1$. For each $Z \in B(0,1)$, since $|\xi_t(Z)|_v \leq 1$ and $|\beta_v(j)|_v \leq 1$, $|\xi_t(Z) - \beta_v(j)|_v \leq 1$ and so $\|\xi_{v,t}^{(0)}(Z)\|_{B(0,1)} \leq 1$. By Remark 2.9, there exists an element $Z_0 \in B(0,1)$ with $|Z_0 - \alpha_{tj}|_v = 1$ for all j and hence $|\xi_{v,t}^{(0)}(Z_0)|_v = |a|_v \prod_{j=0}^{n-1} |Z-\alpha_{tj}|_v |H_t(Z_0)|_v = 1$. It follows that $\|\xi_{v,t}^{(0)}(Z)\|_{B(0,1)} = 1$.

Remark 11.1 For any subset $I' \subset I = \{0, \cdots, n-1\}$ consisting of consecutive integers, it follows by the same argument above that the roots $\{\alpha_{tj}\}_{j\in I'}$ of $\prod_{j\in I'}(\xi_t(Z) - \beta_v(j))$ form a regular sequence of length $\#(I')$ in $\mathcal{O}_v$ and $\|\prod_{j\in I'}(\xi_t(Z) - \beta_v(j))\|_{B(0,1)} = 1$.

Lemma 11.2 ([25], Basic Patching Lemma) *Let $f(z) \in K_v[[z]]$ be a power series converging on $B(0,1)$, with $\|f(z)\|_{B(0,1)} = 1$. Suppose that $f(z)$ has exactly $d \geq 1$ roots in $B(0,1)$ and that these roots form a regular sequence $\{\alpha_j\}_{j\in I}$ of length d in $\mathcal{O}_v$. Take $D \geq \log_v(d)$.*

Then for any power series $\Delta(z) \in K_v[[z]]$ converging on $B(0,1)$, with

$$\|\Delta\|_{B(0,1)} \le {q_v}^{-\frac{d}{q_v-1}-D},$$

the roots α_j^* of $f^*(z) = f(z) + \Delta(z)$ in $B(0,1)$ again form a regular sequence of length d in $\mathcal{O}_v$, and can be uniquely labeled in such a way that

$$\mathrm{ord}_v(\alpha_j^* - \alpha_j) > D$$

for each $j \in I$.

Proof: Writing $f(z) = \sum_{k\ge0} b_k z^k$, Lemma 2.3 implies that $|b_k|_v \le 1$ (that is, $\mathrm{ord}_v(b_k) \ge 0$) for all $k \ge 0$, and that $|b_k|_v = 1$ for some k. Since $f(z)$ converges on $B(0,1)$, we have $\lim_{k\to\infty} |b_k|_v = 0$, i.e., $\lim_{k\to\infty} \mathrm{ord}_v(b_k) = \infty$, and so the theory of Newton polygons shows that d is the largest index for which $\mathrm{ord}_v(b_d) = 0$. Likewise, writing $\Delta(z) = \sum_{k\ge0} \delta_k z^k$, we have $\mathrm{ord}_v(\delta_k) \ge \frac{d}{q_v-1} + D > 0$ for all $k \ge 0$. Now, writing $f^*(z) = f(z) + \Delta(z) = \sum_{k\ge0} c_k z^k$, we have $c_k = b_k + \delta_k$ and so $\mathrm{ord}_v(c_k) \ge 0$ for all $k \ge 0$. Furthermore, $\mathrm{ord}_v(c_d) = \mathrm{ord}_v(b_d) = 0$ and $\mathrm{ord}_v(c_k) > 0$ for all $k > d$. Hence the theory of Newton polygons shows that $f^*(z)$ has exactly d roots in $B(0,1)$.

On the other hand, writing $f(z) = b\cdot g(z)\cdot h(z)$ by the Weierstrass Preparation Theorem, we have $b \in K_v$ and $g(z) = \prod_{j\in I}(z-\alpha_j)$, and $h(z) \in K_v[[z]]$ is a unit power series. By the proof of the Weierstrass Preparation Theorem, since d is the largest index for which $|b_d|_v$ is maximal, we have $b = b_d$, and hence $\mathrm{ord}_v(b) = 0$, i.e., $|b|_v = 1$.

Now, fix a root α_J, and expand $f(z)$ and $\Delta(z)$ as power series about α_J:

$$f(z) = \sum_{k\ge1} b_k^{(J)}(z-\alpha_J)^k,$$
$$\Delta(z) = \sum_{k\ge0} \delta_k^{(J)}(z-\alpha_J)^k.$$

By Lemma 2.11, since $\mathrm{ord}_v(b) = 0$, the initial part of the Newton polygon of $f(z)$ expanded about α_J coincides with that of $bg(z)$ and the slopes of its other segments are greater than equal to $\log_v(1) = 0$. Rewriting

$$g(z) = \prod_{j \in I}((z - \alpha_J) - (\alpha_j - \alpha_J)) = \sum_{k=1}^{d} a_k^{(J)}(z - \alpha_J)^k,$$

we find that up to sign, the coefficients $a_k^{(J)}$ are elementary symmetric polynomials in the $\alpha_j - \alpha_J$. It then follows that

$$\begin{aligned} a_1^{(J)} &= \pm \prod_{\substack{j \in I \\ j \neq J}} (\alpha_j - \alpha_J), \\ a_k^{(J)} &= \pm a_1^{(J)} \sum_{\substack{j_1 < \cdots < j_{k-1} \text{ in } I \\ \text{each } j_i \neq J}} \frac{1}{(\alpha_{j_1} - \alpha_J) \cdots (\alpha_{j_{k-1}} - \alpha_J)} \end{aligned}$$

for each $k = 2, \cdots, d$.

Proposition 5.9 says that

$$\mathrm{ord}_v(a_1^{(J)}) < \frac{d}{q_v - 1}.$$

Since $\mathrm{ord}_v(\alpha_j - \alpha_J) = \mathrm{val}_{q_v}(|j - J|) < \log_v(d)$ for each $j \neq J$ by (5.5), we get

$$\mathrm{ord}_v(a_k^{(J)}) > \mathrm{ord}_v(a_1^{(J)}) - (k-1)\log_v(d).$$

Comparing the Newton polygons of $f(z)$ and $bg(z)$, since $\mathrm{ord}_v(b) = 0$, we find that

$$\begin{aligned} \mathrm{ord}_v(b_1^{(J)}) = \mathrm{ord}_v(a_1^{(J)}) &< \frac{d}{q_v - 1} \quad \text{and} \\ \mathrm{ord}_v(b_k^{(J)}) > \mathrm{ord}_v(a_1^{(J)}) - (k-1)\log_v(d) &= \mathrm{ord}_v(b_1^{(J)}) - (k-1)\log_v(d), \end{aligned}$$

for each $2 \leq k \leq d$. For each $k \geq d$,

$$\begin{aligned} \mathrm{ord}_v(b_k^{(J)}) &\geq \mathrm{ord}_v(b_d^{(J)}) > \mathrm{ord}_v(b_1^{(J)}) - (d-1)\log_v(d) \\ &\geq \mathrm{ord}_v(b_1^{(J)}) - (k-1)\log_v(d). \end{aligned}$$

By the hypothesis on $\Delta(z)$ and Lemma 2.3, we have $\mathrm{ord}_v(\delta_k^{(J)}) \geq \frac{d}{q_v - 1} + D$ for each $k \geq 0$. For each $k \geq 0$, since $D \geq \log_v(d) \geq 0$,

$$\begin{aligned} \mathrm{ord}_v(\delta_k^{(J)}) &\geq \frac{d}{q_v - 1} + D > \mathrm{ord}_v(b_1^{(J)}) + \log_v(d) \\ &\geq \mathrm{ord}_v(b_1^{(J)}) - (k-1)\log_v(d). \end{aligned}$$

Now, writing $f^*(z) = \sum_{k\geq 0} c_k^{(J)}(z-\alpha_J)^k$, we have $c_k^{(J)} = b_k^{(J)} + \delta_k^{(J)}$ for all $k \geq 0$. By the estimates for $b_k^{(J)}$ and $\delta_k^{(J)}$, and the fact that $b_0^{(J)} = 0$,

$$\begin{aligned} \mathrm{ord}_v(c_0^{(J)}) &= \mathrm{ord}_v(\delta_0^{(J)}) \geq \frac{d}{q_v-1} + D, \\ \mathrm{ord}_v(c_1^{(J)}) &= \mathrm{ord}_v(b_1^{(J)}) < \frac{d}{q_v-1}, \quad \text{and} \\ \mathrm{ord}_v(c_k^{(J)}) &> \mathrm{ord}_v(b_1^{(J)}) - (k-1)\log_v(d), \quad \text{for each} \quad k \geq 2. \end{aligned}$$

Since $D \geq \log_v(d)$, the Newton polygon of $f^*(z)$ has a break at $k=1$, and its initial segment has slope $\mathrm{ord}_v(c_1^{(J)}) - \mathrm{ord}_v(c_0^{(J)})$, which is less than $-D$. Hence $f^*(z)$ has a unique root α_J^* satisfying

$$\mathrm{ord}_v(\alpha_J^* - \alpha_J) > D.$$

Noting that $\mathrm{ord}_v(\alpha_J^* - \alpha_J) > D \geq \log_v(d) \geq 0$ and $\alpha_J \in B(0,1) \cap K_v = \mathcal{O}_v$, it follows that $\alpha_J^* \in B(\alpha_J, 1) = B(0,1)$. Hence we get $\alpha_J^* \in B(0,1) \cap K_v = \mathcal{O}_v$ by Corollary 2.7.

Moreover, if $\alpha_J^* = \alpha_H^*$ for some $J \neq H$ in I, then $\mathrm{ord}_v(\alpha_J - \alpha_H) = \mathrm{ord}_v(\alpha_J - \alpha_J^* + \alpha_H^* - \alpha_H) > D \geq \log_v(d)$. But $\mathrm{ord}_v(\alpha_J - \alpha_H) = \mathrm{val}_{q_v}(|J-H|) < \log_v(d)$. Thus the α_j^* are distinct. By the ultrametric inequality, since $\{\alpha_j\}_{j\in I}$ is a regular sequence of length d, $\{\alpha_j^*\}_{j\in I}$ is also a regular sequence of length d in $\mathcal{O}_v$. □

Fix $v \in S$. For each $w|v$, let $g_w(z) = g_v(z) \in K_v(z)$ be the normalized local approximating function constructed in Theorem 8.4. For each $w|v$, put $G_w^{(0)}(z) := G_v^{(0)}(z)$ and $\xi_{w,t}^{(0)}(z) := \xi_{v,t}^{(0)}(z)$. In patching the coefficients of $G_w^{(0)}(z)$, we are going to deal with the zeros of the $\xi_{w,t}^{(0)}(z) = S_n(\xi_t(z))$ because the zeros $\{\alpha_{tj}\}$ of $\xi_{w,t}^{(0)}(z)$ form a regular sequence of length n in $\mathcal{O}_v$. Let $k_0 \in \mathbb{Z}_{>0}$ be the global patching parameter as in Proposition 9.10. Recall that for all $k \geq k_0$,

$$h_v^{kN} \cdot T_v \cdot B_v < q_v^{-\frac{k+1}{q_v-1} - \log_v(k+1)} < q_v^{-\frac{k}{q_v-1} - \log_v(k)}. \tag{11.6}$$

We have choosen the number $0 < \delta_w < 1$ (see (9.14)) small enough that

$$\delta_w \leq \frac{h_w^{p^r N} \varepsilon^{k_0 N}}{C_w^{p^r} B_w^{p^r} T_w^{k_0 N} \chi_w^{k_0 N}}. \tag{11.7}$$

We will now show how to patch the coefficients of the initial local patching function $G_w^{(0)}(z)$ with numbers $\Delta_{w,i\ell} \in L_w$ and $\Delta_{w,0} \in L_w$ for all $w|v$, all $i = 1, \cdots, m$, and all $\ell = 0, \cdots, nN_i - 1$, while preserving the property that its roots belong to K_v. The $\Delta_{w,i\ell}$ will be subject only to the following conditions:

(i) $|\Delta_{w,i0}|_w \le \delta_w$ for all w and all i;

(ii) $|\Delta_{w,i\ell}|_w \le \frac{h_w^{p^r N}}{C_w^{p^r} B_w^{p^r}}$, for all w, all i, and $\ell = 1, \cdots, k_0 N_i - 1$;

(iii) $|\Delta_{w,i\ell}|_w \le h_w^{kN}$ if $(k-1)N_i \le \ell \le kN_i - 1$ for each $k = k_0 + 1, \cdots, n$;

(iv) $|\Delta_{w,0}|_w \le h_w^{nN}$;

(v) $\sum_{i=1}^m c_{w,i0}^{p^r} \Delta_{w,i0} \frac{1}{(z-x_i)^{p^r N_i}} \in K_v(z)$ is independent of $w|v$;

(vi) $\sum_{i=1}^m \sum_{\ell=(k-1)N_i}^{kN_i-1} c_{w,i0}^{p^r} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{p^r N_i - \ell}} \in K_v(z)$ is independent of $w|v$, for each $k = 1, \cdots, k_0$;

(vii) $\sum_{i=1}^m (1 + \Delta_{w,i0})^{-1} \sum_{\ell=(k-1)N_i}^{kN_i-1} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{kN_i-\ell}} \in K_v(z)$ is independent of $w|v$, for each $k = k_0 + 1, \cdots, n$;

(viii) $\Delta_{w,0}$ is independent of $w|v$.

The conditions (i), (ii), (v) and (vi) are used in patching the leading coefficients and the coefficients in the high-order blocks. The conditions (iii) and (vii) are used in patching the coefficients in the low-order blocks, and the conditions (iv) and (viii) are used in patching including the constant term. As in the RL-domain case, the numbers $\Delta_{w,i\ell}$ and $\Delta_{w,0}$ satisfying these conditions will ultimately be provided by the global patching process.

11.1 Patching the Leading Coefficients

We first patch the leading coefficients, i.e., those of order nN_i, for each x_i. Given the number $\Delta_{w,i0}$ for each i, put

$$\varphi_w^{(0)}(z) := \sum_{i=1}^m c_{w,i0}^{p^r} \Delta_{w,i0} \frac{1}{(z-x_i)^{p^r N_i}}.$$

By (v), $\varphi_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$. In particular, it follows from (11.7) that

$$\|\varphi_w^{(0)}(z)\|_{U_v^0} \le C_w^{p^r} \cdot \delta_w \cdot B_w^{p^r} < h_w^{p^r N}. \tag{11.8}$$

Put $I := \{0, \cdots, n-1\}$ and let $I_{0,p^r} = \{0, \cdots, p_r - 1\} \subset I$ be the subset consisting of the first p^r consecutive integers in I. Put

$$\widetilde{Q}_{p^r}(z) := \prod_{j \in I \setminus I_{0,p^r}} (g_w(z) - \beta_v(j)). \tag{11.9}$$

Define

$$\widehat{G}_w^{(0)}(z) := G_w^{(0)}(z) + \varphi_w^{(0)}(z) \cdot \widetilde{Q}_{p^r}(z). \tag{11.10}$$

Note that $\widehat{G}_w^{(0)}(z) = (S_{p^r}(g_w(z)) + \varphi_w^{(0)}(z)) \cdot \widetilde{Q}_{p^r}(z)$. The leading coefficient of $\widehat{G}_w^{(0)}(z)$ at each x_i is $c_{w,i0}^n + c_{w,i0}^{p^r} \cdot \Delta_{w,i0} \cdot c_{w,i0}^{n-p^r} = c_{w,i0}^n(1 + \Delta_{w,i0})$ and $\widehat{G}_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$. Put $\varepsilon_{w,i} := 1 + \Delta_{w,i0}$. Since $\delta_w < 1$, $\varepsilon_{w,i}$ is a unit. Note that $\widehat{G}_w^{(0)}(z)$ has the same constant term as $G_w^{(0)}(z)$. The magnitude of the high-order coefficients of $\widehat{G}_w^{(0)}(z)$ will be discussed in Section 11.2.

11.2 Patching the High-order Coefficients

Step 1. The Patching Process for $k = 1, \cdots, k_0$

Adjusting the coefficients causes movement of the roots. For the high-order coefficients, we absorb that movement in the subsequence I_{0,p^r} of the roots of length p^r, preserving the remaining roots.

When $k = 1$, we patch the coefficients in the block $nN_i - 1, \cdots, nN_i - N_i + 1$. Given the numbers $\Delta_{w,i1}, \cdots, \Delta_{w,i,N_i-1}$ for each i, put

$$\varphi_w^{(1)}(z) := \sum_{i=1}^{m} \sum_{\ell=1}^{N_i-1} c_{w,i0}^{p^r} \Delta_{w,i\ell} \frac{1}{(z - x_i)^{p^r N_i - \ell}}.$$

By (v) and (vi), $\varphi_w^{(1)}(z) \in K_v(z)$ is independent of $w|v$. Since $T_w > 1$, it follows that

$$\|\varphi_w^{(1)}(z)\|_{U_v^0} \le C_w^{p^r} \cdot \frac{h_w^{p^r N}}{C_w^{p^r} \cdot B_w^{p^r}} \cdot B_w^{p^r} = h_w^{p^r N} < h_w^{p^r N} \cdot T_w \cdot B_w. \tag{11.11}$$

Define

$$G_w^{(1)}(z) := \widehat{G}_w^{(0)}(z) + \varphi_w^{(1)}(z) \cdot \widetilde{Q}_{p^r}(z). \tag{11.12}$$

Then $G_w^{(1)}(z) \in K_v(z)$ is independent of $w|v$ and has the same constant term as $G_w^{(0)}(z)$.

For each $k = 2, \cdots, k_0$, we inductively patch the coefficients in the block $nN_i - (k-1)N_i, \cdots, nN_i - kN_i + 1$. We continue to absorb the movement of the roots with the same sequence of roots of length p^r. Suppose that we have constructed functions $G_w^{(k-1)}(z) \in K_v(z)$, independent of all $w|v$, having the same constant term as $G_w^{(0)}(z)$. Given the numbers $\Delta_{w,i,(k-1)N_i}, \cdots, \Delta_{w,i,kN_i-1}$ for each i, put

$$\varphi_w^{(k)}(z) := \sum_{i=1}^{m} \sum_{\ell=(k-1)N_i}^{kN_i-1} c_{w,i0}^{p^r} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{p^r N_i - \ell}}.$$

By (vi), $\varphi_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$. In particular, it follows that

$$\|\varphi_w^{(k)}(z)\|_{U_v^0} \leq C_w^{p^r} \cdot \frac{h_w^{p^r N}}{C_w^{p^r} \cdot B_w^{p^r}} \cdot B_w^{p^r} = h_w^{p^r N} < h_w^{p^r N} \cdot T_w \cdot B_w. \tag{11.13}$$

Define

$$G_w^{(k)}(z) := G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot \widetilde{Q}_{p^r}(z). \tag{11.14}$$

Then $G_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$ and has the same constant term as $G_w^{(0)}(z)$.

At the end of stage k_0, we have functions $G_w^{(k_0)}(z) \in K_v(z)$, independent of $w|v$, such that $G_w^{(k_0)}(z) = G_w^{(0)}(z) + \sum_{h=0}^{k_0} \varphi_w^{(h)}(z) \cdot \widetilde{Q}_{p^r}(z)$. The leading coefficient of $G_w^{(k_0)}(z)$ at each x_i is $c_{w,i0}^n \varepsilon_{w,i}$ and $G_w^{(k_0)}(z)$ has the same constant term as $G_w^{(0)}(z)$. Since $\|\varphi_w^{(h)}(z)\|_{U_v^0} < h_w^{p^r N} \cdot T_w \cdot B_w$ for each $h = 0, \cdots, k_0$, it follows that $\|\sum_{h=0}^{k_0} \varphi_w^{(h)}(z)\|_{U_v^0} < h_w^{p^r N} \cdot T_w \cdot B_w < 1$. Note that $G_w^{(k_0)}(z) = S_n(g_w(z)) + \sum_{h=0}^{k_0} \varphi_w^{(h)}(z) \cdot \widetilde{Q}_{p^r}(z) = (S_{p^r}(g_w(z)) + \sum_{h=0}^{k_0} \varphi_w^{(h)}(z)) \cdot \widetilde{Q}_{p^r}(z)$. Put

$$Q_w^{(k_0)}(z) := S_{p^r}(g_w(z)) + \sum_{h=0}^{k_0} \varphi_w^{(h)}(z). \tag{11.15}$$

Then $Q_w^{(k_0)}(z)$ is independent of $w|v$ and has a pole of order $p^r N_i$ at each $x_i \in \mathfrak{X}$, with leading coefficient $c_{w,i0}^{p^r}(1 + \Delta_{w,i0})$. Moreover, $\|Q_w^{(k_0)}(z)\|_{U_v^0} \leq 1$ since $\|g_w(z)\|_{U_v^0} \leq 1$ and $\beta_v(j) \in \mathcal{O}_v$ for all j. Note that $\sum_{h=0}^{k_0} \varphi_w^{(h)}(z)$ can be expanded as a power series about θ_t converging on $\mathfrak{B}(\theta_t, \rho_t)$ (see Remark 6.5).

Now, to apply the basic patching lemma to $Q_w^{(k_0)}(z)$, let $\xi_{w,t}^{(k_0)}(z), \widetilde{\xi}_{w,t}^{(k_0)}(z), \Delta_w^{(k_0)}(z)$ be the power series converging on $B(0,1)$ gotten by composing $G_w^{(k_0)}(z), \widetilde{Q}_{p^r}(z), \sum_{h=0}^{k_0} \varphi_w^{(h)}(z)$ with

λ_t, respectively. It follows that $\xi_{w,t}^{(k_0)}(z) = (S_{p^r}(\xi_t(z)) + \Delta_w^{(k_0)}(z)) \cdot \widetilde{\xi}_{w,t}^{(k_0)}(z)$ and $\|\Delta_w^{(k_0)}(z)\|_{B(0,1)} < h_v^{p^r N} T_v B_v < q_v^{-p^r/(q_v-1)-\log_v(p^r)}$. It follows from Remark 11.1 that the roots $\{\alpha_{tj}\}_{j \in I_{0,p^r}}$ of $S_{p^r}(\xi_t(z))$ form a regular sequence of length p^r in $\mathcal{O}_v$. By Remark 11.1 and Lemma 11.2, the roots $\{\alpha_{tj}^*\}_{j \in I_{0,p^r}}$ of $\mathfrak{Q}_w^{(k_0)}(z) := S_{p^r}(\xi_t(z)) + \Delta_w^{(k_0)}(z)$ form a regular sequence of length p^r in $\mathcal{O}_v$, with $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{tj}) > \log_v(p^r)$ for each $j \in I_{0,p^r}$. Put $\theta_{tj}^* := \lambda_t(\alpha_{tj}^*)$. Since λ_t is defined over K_v, we have $\theta_{tj}^* \in K_v$.

Hence the roots $\{\theta_{tj}^*, \theta_{tk} : j \in I_{0,p^r}, k \in I \setminus I_{0,p^r}\}$ of $G_w^{(k_0)}(z)$ in $\mathfrak{B}(\theta_t, \rho_t)$ belong to $\mathfrak{B}(\theta_t, \rho_t) \cap K_v \subset E_v^0$ and for each t, the roots $\{\alpha_{tj}^*, \alpha_{tk} : j \in I_{0,p^r}, k \in I \setminus I_{0,p^r}\}$ of $\xi_{w,t}^{(k_0)}(z)$ in $B(0,1)$ are a union of a regular sequence of length p^r in $\mathcal{O}_v$ and a regular sequence of length $n - p^r$ in $\mathcal{O}_v$.

Step 2. Verifying that the High-order Coefficients Remain Small

We previously described our patching method for the high-order coefficients. In order for the global patching process to succeed, it is necessary that all these coefficients remain small. In this step, we examine these coefficients to determine their size. This allows the global inductive process of determining the $\Delta_{w,i\ell}$ to continue.

Proposition 11.3 *For each $\ell = 1, \cdots, k_0 N_i - 1$, the coefficient of order $nN_i - \ell$ of $\widehat{G}_w^{(0)}(z)$ for each $x_i \in \mathfrak{X}$ has absolute value bounded by $\frac{h_w^{p^r N} \varepsilon^{k_0 N - \ell}}{C_w^{p^r} B_w^{p^r} (T_w \chi_w)^{k_0 N}} \cdot |c_{w,i0}^n|_w$. For each $k = 1, \cdots, k_0$ and each $\ell = 1, \cdots, k_0 N_i - 1$, the coefficient of order $nN_i - \ell$ of $G_w^{(k)}(z)$ has absolute value bounded by $\frac{h_w^{p^r N}}{C_w^{p^r} B_w^{p^r} \varepsilon^{\ell}} \cdot |c_{w,i0}^n|_w$.*

Proof: First, we consider the function $\widehat{G}_w^{(0)}(z)$. Since $\widehat{G}_w^{(0)}(z) = G_w^{(0)}(z) + \varphi_w^{(0)}(z) \cdot \widetilde{Q}_{p^r}(z)$, we rewrite $\widehat{G}_w^{(0)}(z)$ with the notations (9.18) and (9.19):

$$\begin{aligned} \widehat{G}_w^{(0)}(z) \quad = \quad & (z - x_i)^{-nN_i} \cdot c_{w,i0}^n \Big(1 + \sum_{\ell \geq 1} B_{w,i\ell}^{(0)} (z - x_i)^{\ell}\Big) \\ & + (z - x_i)^{-p^r N_i} \cdot c_{w,i0}^{p^r} \Big(\Delta_{w,i0} + \sum_{h \geq p^r N_i} \delta_{w,ih}^{(0)} (z - x_i)^h\Big) \\ & \cdot (z - x_i)^{-(n-p^r)N_i} \cdot c_{w,i0}^{n-p^r} \Big(1 + \sum_{j \geq 1} B_{w,ij}^{(p^r)} (z - x_i)^j\Big). \end{aligned} \tag{11.16}$$

The coefficient $\widetilde{c}^{(0)}_{w,i\ell}$ of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i}\cdot c^{-n}_{w,i0}\cdot \widehat{G}^{(0)}_w(z)$ is $B^{(0)}_{w,i\ell}+\sum_{h=0}^{\ell}\delta^{(0)}_{w,ih}B^{(p^r)}_{w,i,\ell-h}$, where $B^{(0)}_{w,i0}=B^{(p^r)}_{w,i0}=1$, $\delta^{(0)}_{w,i0}=\Delta_{w,i0}$ and $\delta^{(0)}_{w,ih}=0$ for $h=1,\cdots,p^rN_i-1$. It follows that $\widetilde{c}^{(0)}_{w,i0}=\varepsilon_{w,i}$ and $\widetilde{c}^{(0)}_{w,i\ell}=B^{(0)}_{w,i\ell}+\Delta_{w,i0}B^{(p^r)}_{w,i\ell}$ for $\ell=1,\cdots,k_0N_i-1$. Since $|\Delta_{w,i0}|_w\le\delta_w\le\frac{h_w^{p^rN}\varepsilon^{k_0N}}{C_w^{p^r}B_w^{p^r}T_w^{k_0N}\chi_w^{k_0N}}$, the property (9.20) implies that for each $\ell=1,\cdots,k_0N_i-1$,

$$|\widetilde{c}^{(0)}_{w,i\ell}|_w\le\max\left\{|B^{(0)}_{w,i\ell}|_w,\ \delta_w\frac{1}{\varepsilon^\ell}\right\}\le\frac{h_w^{p^rN}\ \varepsilon^{k_0N-\ell}}{C_w^{p^r}B_w^{p^r}T_w^{k_0N}\chi_w^{k_0N}}. \tag{11.17}$$

For the function $G^{(1)}_w(z)$, we rewrite $G^{(1)}_w(z)$ with the notations (9.18) and (9.19):

$$\begin{aligned}
G^{(1)}_w(z) &= \widehat{G}^{(0)}_w(z)+\varphi^{(1)}_w(z)\cdot\widetilde{Q}_{p^r}(z)\\
&= (z-x_i)^{-nN_i}\cdot c^n_{w,i0}\Big(\sum_{\ell\ge0}\widetilde{c}^{(0)}_{w,i\ell}(z-x_i)^\ell\Big)\\
&\quad+(z-x_i)^{-p^rN_i}\cdot c^{p^r}_{w,i0}\Big(\sum_{d=1}^{N_i-1}\Delta_{w,id}(z-x_i)^d+\sum_{h\ge p^rN_i}\delta^{(1)}_{w,ih}(z-x_i)^h\Big)\\
&\qquad\cdot(z-x_i)^{-(n-p^r)N_i}\cdot c^{n-p^r}_{w,i0}\Big(1+\sum_{j\ge1}B^{(p^r)}_{w,ij}(z-x_i)^j\Big).
\end{aligned}$$

The coefficient $\widetilde{c}^{(1)}_{w,i\ell}$ of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i}\cdot c^{-n}_{w,i0}\cdot G^{(1)}_w(z)$ is $\widetilde{c}^{(0)}_{w,i\ell}+\sum_{h=0}^{\ell}\delta^{(1)}_{w,ih}B^{(p^r)}_{w,i,\ell-h}$, where $\delta^{(1)}_{w,i0}=0$, $\delta^{(1)}_{w,id}=\Delta_{w,id}$ for $d=1,\cdots,N_i-1$, and $\delta^{(1)}_{w,ih}=0$ for $h=N_i,\cdots,p^rN_i-1$. In particular, $\widetilde{c}^{(1)}_{w,i0}=\widetilde{c}^{(0)}_{w,i0}+\delta^{(1)}_{w,i0}B^{(p^r)}_{w,i0}=\widetilde{c}^{(0)}_{w,i0}=\varepsilon_{w,i}$. Since $|\Delta_{w,id}|_w\le\frac{h_w^{p^rN}}{C_w^{p^r}B_w^{p^r}}$ for $d=1,\cdots,N_i-1$, the property (11.17) implies that for each $\ell=1,\cdots,k_0N_i-1$,

$$|\widetilde{c}^{(1)}_{w,i\ell}|_w\le\frac{h_w^{p^rN}}{C_w^{p^r}B_w^{p^r}\varepsilon^\ell}.$$

For each $k=1,\cdots,k_0-1$, suppose inductively that for each $\ell=1,\cdots,k_0N_i-1$, the coefficients $\widetilde{c}^{(k)}_{w,i\ell}$ of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i}\cdot c^{-n}_{w,i0}\cdot G^{(k)}_w(z)$ satisfy $|\widetilde{c}^{(k)}_{w,i\ell}|_w\le\frac{h_w^{p^rN}}{C_w^{p^r}B_w^{p^r}\varepsilon^\ell}$. Rewrite $G^{(k+1)}_w(z)$ with the notations (9.18) and (9.19):

$$\begin{aligned}
G^{(k+1)}_w(z) &= G^{(k)}_w(z)+\varphi^{(k+1)}_w(z)\cdot\widetilde{Q}_{p^r}(z)\\
&= (z-x_i)^{-nN_i}\cdot c^n_{w,i0}\Big(\sum_{\ell\ge0}\widetilde{c}^{(d)}_{w,i\ell}(z-x_i)^\ell\Big)\\
&\quad+(z-x_i)^{-p^rN_i}\cdot c^{p^r}_{w,i0}\Big(\sum_{d=kN_i}^{(k+1)N_i-1}\Delta_{w,id}(z-x_i)^d+\sum_{h\ge p^rN_i}\delta^{(k+1)}_{w,ih}(z-x_i)^h\Big)\\
&\qquad\cdot(z-x_i)^{-(n-p^r)N_i}\cdot c^{n-p^r}_{w,i0}\Big(1+\sum_{j\ge1}B^{(p^r)}_{w,ij}(z-x_i)^j\Big).
\end{aligned}$$

The coefficient $\widetilde{c}_{w,i\ell}^{(k+1)}$ of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i}\cdot c_{w,i0}^{-n}\cdot G_w^{(k+1)}(z)$ is $\widetilde{c}_{w,i\ell}^{(k)}+\sum_{h=0}^{\ell}\delta_{w,ih}^{(k+1)}B_{w,i,\ell-h}^{(p^r)}$, where $\delta_{w,i0}^{(k+1)}=\cdots=\delta_{w,i,kN_i-1}^{(k+1)}=0$, $\delta_{w,id}^{(k+1)}=\Delta_{w,id}$ for $d=kN_i,\cdots,(k+1)N_i-1$, and $\delta_{w,ih}^{(1)}=0$ for $h=(k+1)N_i,\cdots,p^rN_i-1$. In particular, $\widetilde{c}_{w,i0}^{(k+1)}=\widetilde{c}_{w,i0}^{(k)}+\delta_{w,i0}^{(k+1)}B_{w,i0}^{(p^r)}=\widetilde{c}_{w,i0}^{(k)}=\varepsilon_{w,i}$. Since $|\Delta_{w,id}|_w\leq\frac{h_w^{p^rN}}{C_w^{p^r}B_w^{p^r}}$ for $d=kN_i,\cdots,(k+1)N_i-1$, it follows that for each $\ell=1,\cdots,k_0N_i-1$,

$$|\widetilde{c}_{w,i\ell}^{(k+1)}|_w\leq\frac{h_w^{p^rN}}{C_w^{p^r}B_w^{p^r}\varepsilon^\ell}.$$

□

11.3 Patching the Low-order Coefficients

In the first two stages, we patched the high-order coefficients, i.e., those for powers of $1/(z-x_i)$ in the range $nN_i,\cdots,nN_i-k_0N_i+1$. In the next stage, we are patching the low-order coefficients for each $k=k_0+1,\cdots,n$. This will be done differently, in several sub-ranges. Let $T_0\in\mathbb{Z}_{>0}$ be the integer with $T_0>p^r$ (see Section 9.2) such that for all $k\geq T_0$,

$$h_v^{kN}\cdot T_v\cdot B_v<q_v^{-k/(q_v-1)-\log_v(n)}. \tag{11.18}$$

Choosing this integer T_0 guarantees that for $k>T_0$, the basic patching lemma preserves the position of roots within balls of size $q_v^{-\lceil\log_v(n)\rceil}$ so that $\mathrm{ord}_v(\alpha_{tj}^*-\alpha_{tj})>\log_v(n)$ in the subsequent patching steps. Thus we must take particular care in patching the blocks for $k=k_0+1,\cdots,T_0$. Since $h_v^N<q_v^{-1/(q_v-1)}$, i.e., $\log_v(h_v^N)<-1/(q_v-1)$ and $h_v^{(T_0-1)N}\cdot T_v\cdot B_v\geq q_v^{-(T_0-1)/(q_v-1)-\log_v(n)}$, it follows that $T_0\leq\frac{\log_v(T_vB_v)+\log_v(n)}{-(\log_v(h_v^N)+1/(q_v-1))}+1$. Hence there is a constant $A_1>0$ such that

$$T_0\leq A_1\log_v(n). \tag{11.19}$$

Without loss of generality, we can assume that n is large enough that

$$n\geq A_1\log_v(n)+2>T_0. \tag{11.20}$$

Step 1. The Patching Process for $k=k_0+1,\cdots,T_0$

We use a separate subsequence of roots to absorb the movement, for each block of coefficients in the range $k = k_0+1, \cdots, T_0$. For notational convenience, set $I_{k_0} := I_{0,p^r}$ and $\widetilde{Q}_w^{(k_0)}(z) := \widetilde{Q}_{p^r}(z)$. For each $k = k_0+1, \cdots, T_0$, put $i_k := (k_0+1)+(k_0+2)+\cdots+(k-1) = (k^2-k-k_0^2-k_0)/2$ and let $I_k = \{p^r+i_k, \cdots, p^r+i_k+(k-1)\}$ be the subset of I consisting of the k consecutive integers after I_{k-1}. This is the subsequence of roots that will be used to absorb movement for changes of the coefficients in the k^{th} block. Put

$$\widetilde{Q}_w^{(k)}(z) := \prod_{j \in I \setminus \cup_{h=k_0}^{k} I_h} (g_w(z) - \beta_v(j)). \tag{11.21}$$

We will call these factors $\widetilde{Q}_w^{(k)}(z)$ the complementary parts of each patching process for $k = k_0, \cdots, T_0$.

For each $k = k_0+1, \cdots, T_0$, we inductively patch the coefficients in the block $(n-(k-1))N_i, \cdots, (n-k)N_i+1$ for each $x_i \in \mathfrak{X}$. Suppose that we have constructed

$$G_w^{(k-1)}(z) = \prod_{h=k_0}^{k-1} Q_w^{(h)}(z) \cdot \widetilde{Q}_w^{(k-1)}(z)$$

so that $G_w^{(k-1)}(z) \in K_v(z)$ is independent of all $w|v$. Here, $Q_w^{(k_0)}(z)$ has a pole of order $p^r N_i$ at each $x_i \in \mathfrak{X}$ with leading coefficient $c_{w,i0}^{p^r}\varepsilon_{w,i}$, satisfying $\|Q_w^{(k_0)}(z)\|_{U_v^0} \le 1$ and $Q_w^{(h)}(z),\ h = k_0+1, \cdots, k-1$, has a pole of order hN_i at each $x_i \in \mathfrak{X}$ with leading coefficient $c_{w,i0}^h$, satisfying $\|Q_w^{(h)}(z)\|_{U_v^0} \le 1$. Moreover, if $\mathfrak{Q}_w^{(h)}(z)$ is the power series gotten by composing $Q_w^{(h)}(z)$ with λ_t for $h = k_0, \cdots, k-1$, then the roots $\{\alpha_{tj}^*\}_{j \in I_h}$ of $\mathfrak{Q}_w^{(h)}(z)$ form a regular sequence of length h in $\mathcal{O}_v$, with $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{tj}) > \log_v(h)$ for each $j \in I_h$. Given the numbers $\Delta_{w,i,(k-1)N_i}, \cdots, \Delta_{w,i,kN_i-1}$ for each i, put

$$\varphi_w^{(k)}(z) := \sum_{i=1}^{m} \varepsilon_{w,i}^{-1} \sum_{\ell=(k-1)N_i}^{kN_i-1} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{kN_i-\ell}}.$$

By (vii), $\varphi_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$. In particular, it follows from (iii) that

$$\|\varphi_w^{(k)}(z)\|_{U_v^0} \le h_w^{kN} \cdot B_w < h_w^{kN} \cdot T_w \cdot B_w. \tag{11.22}$$

Define

$$G_w^{(k)}(z) := G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot \prod_{h=k_0}^{k-1} Q_w^{(h)}(z) \cdot \widetilde{Q}_w^{(k)}(z). \tag{11.23}$$

Then $G_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$ and has the same constant term as $G_w^{(0)}(z)$. Rewrite $G_w^{(k)}(z) = \prod_{h=k_0}^{k-1} Q_w^{(h)}(z) \cdot \left(\prod_{j\in I_k}(g_w(z)-\beta_v(j)) + \varphi_w^{(k)}(z)\right) \cdot \widetilde{Q}_w^{(k)}(z)$ and put

$$Q_w^{(k)}(z) := \prod_{j\in I_k}(g_w(z)-\beta_v(j)) + \varphi_w^{(k)}(z). \tag{11.24}$$

Then $Q_w^{(k)}(z) \in K_v(z)$ is independent of $w|v$ and has a pole of order kN_i at each $x_i \in \mathfrak{X}$, with leading coefficient $c_{w,i0}^k$. Moreover, $\|Q_w^{(k)}\|_{U_v^0} \le 1$ and $\varphi_w^{(k)}(z)$ can be expanded as a power series about θ_t converging on $\mathfrak{B}(\theta_t, \rho_t)$ (see Remark 6.5).

Now, to apply the basic patching lemma to $Q_w^{(k)}(z)$, let $\xi_{w,t}^{(k)}(z), \mathfrak{Q}_{w,t}^{(k)}(z), \widetilde{\mathfrak{Q}}_{w,t}^{(k)}(z), \Delta_{w,t}^{(k)}(z)$ be the power series gotten by composing $G_w^{(k)}(z), Q_w^{(k)}(z), \widetilde{Q}_w^{(k)}(z), \varphi_w^{(k)}(z)$ with λ_t, respectively. It follows that $\xi_{w,t}^{(k)}(z) = \prod_{h=k_0}^{k-1} \mathfrak{Q}_{w,t}^{(h)}(z) \cdot \mathfrak{Q}_{w,t}^{(k)}(z) \cdot \widetilde{\mathfrak{Q}}_{w,t}^{(k)}(z)$ and $\|\Delta_{w,t}^{(k)}(z)\|_{B(0,1)} < h_v^{kN} \cdot T_v \cdot B_v < q_v^{-k/(q_v-1)-\log_v(k)}$. Note that the roots $\{\alpha_{tj}\}_{j\in I_k}$ of $\prod_{j\in I_k}(\xi_t(z)-\beta_v(j))$ form a regular sequence of length k in $\mathcal{O}_v$. By Remark 11.1 and Lemma 11.2, the roots $\{\alpha_{tj}^*\}_{j\in I_k}$ of $\mathfrak{Q}_{w,t}^{(k)}(z)$ form a regular sequence of length k in $\mathcal{O}_v$, with $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{tj}) > \log_v(k)$ for each $j \in I_k$. Letting θ_{tj}^* correspond to α_{tj}^* for $j \in I_k$, since λ_t is defined over K_v, we have $\theta_{tj}^* \in K_v$.

Hence the roots $\{\theta_{tj}^*,\, \theta_{t\ell} : j \in \cup_{h=k_0}^k I_h,\ \ \ell \in I\setminus(\cup_{h=k_0}^k I_h)\}$ of $G_w^{(k)}(z)$ in $\mathfrak{B}(\theta_t,\rho_t)$ belong to $\mathfrak{B}(\theta_t,\rho_t)\cap K_v \subset E_v^0$ and for each i, the set of roots $\{\alpha_{tj}^*,\, \alpha_{t\ell} : j \in \cup_{h=k_0}^k I_h,\ \ \ell \in I\setminus(\cup_{h=k_0}^k I_h)\}$ of $\xi_{w,t}^{(k)}(z)$ in $B(0,1)$ is a disjoint union of a collection of regular sequences of lengths p^r, $k_0+1,\cdots,k$ in $\mathcal{O}_v$, and a regular sequence of length $n-(p^r+(k_0+1)+\cdots+k)$ in $\mathcal{O}_v$.

Step 2. Moving the patched roots apart

In Step 1, while the patching process kept the roots in $\mathcal{O}_v$, and preserved their position in the short regular sequences of length k for $k = k_0+1,\cdots,T_0$, the moved roots from different subsequences may have come very near to each other or to other non-moved roots. In order for later steps in the patching process to succeed, we need to have the roots separated from each other by a certain distance. Therefore, in this step, we show how to move the roots apart, while keeping the high-order coefficients unchanged. This is done by absorbing the movement with another subsequence of roots.

We need two lemmas in this step. They are used to recover the high-order coefficients of $G_w^{(T_0)}(z)$ after moving the patched roots apart in a certain distance. The first lemma will be used to tell how much change to the coefficients a given movement in the roots will produce. It will be applied to the Laurent expansions at the points $x_i \in \mathfrak{X}$, and relies on the fact that the balls $\mathfrak{B}(x_i, \varepsilon)$ are disjoint from U_v to control the coefficient change. The $G(z)$ will be the function $G_w^{(T_0)}(z)$. The $H(z)$ in the lemma will be a rational function which, when multiplied by $G_w^{(T_0)}(z)$, will replace the patched roots of $G_w^{(T_0)}(z)$ with new roots.

Lemma 11.4 *Let $G(z) = b_0 z^{-A} + \sum_{k=1}^{\infty} b_k z^{-A+k} \in \mathbb{C}_v((z))$ converge in $B(0,\varepsilon)\setminus\{0\}$ having no zeros there, with $b_0 \neq 0$. Suppose that $H(z) = \sum_{k=0}^{\infty} h_k z^{\ell}$ is a power series converging in $B(0,\varepsilon)$, and that there is a number $0 < \beta < 1$ such that $|H(z)-1|_v \leq \beta$ for all $z \in B(0,\varepsilon)$. If we write the product $H(z)G(z) = \sum_{j=0}^{\infty} \widetilde{b}_j z^{-A+j}$, then*

$$\left|\widetilde{b}_j - b_j\right|_v \leq \frac{\beta |b_0|_v}{\varepsilon^j} \quad \textit{for each } \ j \geq 0.$$

Proof: Multiplying through by z^A, we obtain a power series $z^A G(z)$ converging in $B(0,r)$, satisfying the same conditions as $G(z)$, and having no zeros in $B(0,r)$. By the theory of the Newton polygons, $(\mathrm{ord}_v(b_j) - \mathrm{ord}_v(b_0))/(j-0) > \log_v(\varepsilon)$ and so $|b_j|_v < |b_0|_v/\varepsilon^j$ for each $j \geq 1$. Likewise, since $|H(z)-1|_v \leq \beta$ for all $z \in B(0,\varepsilon)$, Lemma 2.3 implies that $|h_0 - 1|_v \leq \beta$ and $|h_k|_v < \beta/\varepsilon^k$ for each $k \geq 1$. In the product $H(z)G(z)$, we have $\widetilde{b}_j = \sum_{k=0}^{j} h_k b_{j-k}$ for each j. Hence it follows that

$$\begin{aligned}
|\widetilde{b}_j - b_j|_v &= |(h_0 - 1)b_j + h_1 b_{j-1} + \cdots + h_j b_0|_v \\
&\leq \max(|h_0 - 1|_v |b_j|_v, |h_1|_v |b_{j-1}|_v, \cdots, |h_j|_v |b_0|_v) \\
&\leq \max(\beta \cdot |b_0|_v/\varepsilon^j, \beta/\varepsilon \cdot |b_0|_v/\varepsilon^{j-1}, \cdots, \beta/\varepsilon^j \cdot |b_0|_v) \\
&= \beta \cdot |b_0|_v/\varepsilon^j.
\end{aligned}$$

□

The next lemma is used to tell how much we can adjust a single coefficient in the range $nN_i, \cdots, (n-T_0)N_i + 1$ for each $i = 1, \cdots, m$. The function $G(z)$ will be the function $\widetilde{Q}_{w,T_0}(z)$, which will be the complementary part used in patching $G_w^{(T_0)}(z)$.

Lemma 11.5 *Let* $G(z) = b_0 z^{-A} + \sum_{\ell=1}^{\infty} b_\ell z^{-A+\ell} \in \mathbb{C}_v((z))$ *converge in* $B(0,\varepsilon)\backslash\{0\}$ *having no zeros there. Fix* $J \geq 1$ *and put* $c_0 := b_0$. *Then there is a unique set of constants* $c_1, \cdots, c_{J-1} \in \mathbb{C}_v$, *with each* $|c_j|_v \leq 1/\varepsilon^j$ *such that* $f_J(z) = z^{-J} + \sum_{j=1}^{J-1} c_j z^{-J+j}$ *satisfies*

$$f_J(z)G(z) = c_0 z^{-(A+J)} + \sum_{j=1}^{\infty} B_j z^{-(A+1)+j}.$$

In other words, $f_J(z)G(z)$ *has no terms* $z^{-(A+j)}$ *in its expansion for* $j = J-1, \cdots, 1$. *Moreover, if there is a finite extension* L_w/K_v *such that the* $\varphi_j(z)$ *and* $G(z)$ *belong to* $L_w((z))$, *then each* c_j *belongs to* L_w *as well.*

Proof: As in the previous lemma, the theory of the Newton polygons shows that $|b_\ell|_v \leq |b_0|_v/\varepsilon^\ell$ for each ℓ. For each $j \geq 1$, we can write

$$z^{-j}G(z) = \sum_{\ell=0}^{\infty} b_\ell z^{-(A+j)+\ell}.$$

Then it follows that

$$z^{-J}G(z) - c_0 z^{-(A+J)} = \sum_{k=1}^{\infty} b_k z^{-(A+J)+k}.$$

Putting $c_1 := -b_1/b_0 \in \mathbb{C}_v$, we see that $|c_1|_v \leq \frac{1}{\varepsilon}$. Furthermore, it follows that

$$\begin{aligned}
&(z^{-J} + c_1 z^{-(J-1)})G(z) - c_0 z^{-(A+J)} \\
&= (z^{-J}G(z) - c_0 z^{-(A+J)}) + c_1 z^{-(J-1)} G(z) \\
&= \sum_{k=2}^{\infty} (b_k + c_1 b_{k-1}) z^{-(A+J)+k} \\
&=: \sum_{k=2}^{\infty} \delta_k^{(2)} z^{-(A+J)+k},
\end{aligned}$$

where $\delta_k^{(2)} = b_k + c_1 b_{k-1}$ for $k \geq 2$. Note that $|\delta_k^{(2)}|_v \leq \max\{|b_k|_v,\ |c_1|_v |b_{k-1}|_v\} \leq |b_0|_v/\varepsilon^k$ for each k.

Inductively, suppose that for some $\ell < J-1$, we have found $c_1, \cdots, c_{\ell-1} \in \mathbb{C}_v$ such that $|c_j|_v \leq 1/\varepsilon^j$ for each j and

$$\begin{aligned}
&(z^{-J} + c_1 z^{-(J-1)} + \cdots + c_{\ell-1} z^{-(J-(\ell-1))})G(z) - c_0 z^{-(A+J)} \\
&= \sum_{k=\ell}^{\infty} \delta_k^{(\ell)} z^{-(A+J)+k},
\end{aligned}$$

with $|\delta_k^{(\ell)}|_v \leq |b_0|_v/\varepsilon^k$ for each $k \geq \ell$. Put $c_\ell := -\delta_\ell^{(\ell)}/b_0 \in \mathbb{C}_v$. It follows that $|c_\ell|_v \leq 1/\varepsilon^\ell$ and

$$\begin{aligned}&(z^{-J} + c_1 z^{-(J-1)} + \cdots + c_\ell z^{-(J-\ell)})G(z) - c_0 z^{-(A+J)}\\ &= \sum_{k=\ell+1}^{\infty} (\delta_k^{(\ell)} + c_\ell b_{k-\ell}) z^{-(A+J)+k}\\ &=: \sum_{k=\ell+1}^{\infty} \delta_k^{(\ell+1)} z^{-(A+J)+k},\end{aligned}$$

where $\delta_k^{(\ell+1)} = \delta_k^{(\ell)} + c_\ell b_{k-\ell}$ for $k \geq \ell+1$. Noting that $|\delta_k^{(\ell+1)}|_v \leq \max\{|\delta_k^{(\ell)}|_v,\ |c_\ell|_v |b_{k-\ell}|_v\} \leq |b_0|_v/\varepsilon^k$ for each k, the inductive process can continue.

The final assertion concerning L_w-rationality follows from the recursive formulas for the c_j. $\square$

We now return to the main argument.

We have patched $G_w^{(0)}(z)$ for $k = 1, \cdots, T_0$ and gotten the resulting $(\mathfrak{X}, \vec{s})$-function $G_w^{(T_0)}(z) \in K_v(z)$ of degree nN, which is independent of all $w|v$. In this step, we are moving the patched roots apart in a certain distance. Recall that $T_0 \leq A_1 \log_v(n)$ and $n \geq A_1 \log_v(n) + 2$. Put

$$I[T_0] := I_{0,p^r} \cup I_{k_0+1} \cup \cdots \cup I_{T_0},$$

and

$$I^0 := I \setminus I[T_0].$$

Let B be the least positive integer satisfying condition (11.33) below, which assures that the change in the coefficients between $G_w^{(T_0)}(z)$ and $F_w^{(T_0)}(z)$ is small enough to be dealt with by Lemma 11.5, and for which $B \geq \lceil \log_v(n) \rceil$. For each $t = 1, \cdots, N$, each coset of $\mathcal{O}_v/\pi_v^B \mathcal{O}_v$ contains at most one element from each of $\{\alpha_{t\ell}\}_{\ell \in I^0}$, $\{\alpha_{tj}^*\}_{j \in I_{0,p^r}}$ and $\{\alpha_{tj}^*\}_{j \in I_k}$ for $k = k_0+1, \cdots, T_0$ because each set is a regular sequence of length less than n. Since there are $T_0 - k_0 + 2$ partitions for I, let D be the smallest positive integer satisfying the condition

$B+D \le A_3 \log_v(n)$, as will be seen as (11.34), for which $q_v^D \ge T_0 - k_0 + 3$. After dividing each coset of $\mathcal{O}_v/\pi^B\mathcal{O}_v$ into q_v^D cosets, for each t we can choose numbers $\alpha_{tj}^{\#} \in \mathcal{O}_v$ for each $j \in I[T_0]$ with the following properties:

(i) $\alpha_{tj}^{\#}$ and α_{tj}^{*} belong to the same coset of $\mathcal{O}_v/\pi^B\mathcal{O}_v$;

(ii) $\alpha_{tj}^{\#}$ and $\alpha_{t\ell}^{\#}$ belong to different cosets of $\mathcal{O}_v/\pi^{B+D}\mathcal{O}_v$ for $\ell \ne j$ in $I[T_0]$; and

(iii) $\alpha_{tj}^{\#}$ and $\alpha_{t\ell}$ belong to different cosets of $\mathcal{O}_v/\pi^{B+D}\mathcal{O}_v$ for $\ell \in I^0$.

Hence we get the following:

(a) $\mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{tj}^{*}) \ge B$;

(b) $\mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}^{\#}) \le B + D$ for each $\ell \ne j$ in $I[T_0]$; and

(c) $\mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}) \le B + D$ for each $\ell \in I^0$.

Since $\{\alpha_{tj}^{*}\}_{j \in I_{0,p^r}}$ is a regular sequence of length p^r in $\mathcal{O}_v$ and $\{\alpha_{tj}^{*}\}_{j\in I_k}$ is a regular sequence of length k in $\mathcal{O}_v$ for $k = k_0+1, \cdots, T_0$, it follows from (a) and the fact that $B \ge \lceil \log_v(n) \rceil$ that $\{\alpha_{tj}^{\#}\}_{j \in I_{0,p^r}}$ is a regular sequence of length p^r in $\mathcal{O}_v$ and $\{\alpha_{tj}^{\#}\}_{j\in I_k}$ is a regular sequence of length k in $\mathcal{O}_v$ for each $k = k_0+1, \cdots, T_0$. In particular, $\mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}^{\#}) < \log_v(p^r)$ for $j \ne \ell$ in I_{0,p^r} and $\mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}^{\#}) < \log_v(k)$ for $j \ne \ell$ in I_k for each $k = k_0+1, \cdots, T_0$. Put $\theta_{tj}^{\#} := \lambda_t(\alpha_{tj}^{\#})$. Then $\theta_{tj}^{\#} \in \mathfrak{B}(\theta_t, \rho_t) \cap K_v \subset E_v^0$ and

$$\begin{aligned} \|\theta_{tj}^{\#}, \theta_{tj}^{*}\|_v &= \|\lambda_t(\alpha_{tj}^{\#}), \lambda_t(\alpha_{tj}^{*})\|_v = \rho_t \cdot |\alpha_{tj}^{\#} - \alpha_{tj}^{*}|_v \\ &\le \rho_t \cdot {q_v}^{-B}. \end{aligned} \tag{11.25}$$

Now, put

$$h(z) := \prod_{t=1}^{N} \prod_{j \in I[T_0]} \frac{z - \theta_{tj}^{\#}}{z - \theta_{tj}^{*}} =: \prod_{t,j} \frac{z - \theta_{tj}^{\#}}{z - \theta_{tj}^{*}}.$$

Clearly, $h(z)$ is rational over K_v and independent of all $w|v$. Put

$$F_w^{(T_0)}(z) := h(z) \cdot G_w^{(T_0)}(z).$$

In passing from $G_w^{(T_0)}(z)$ to $F_w^{(T_0)}(z)$, we move the zeros $\{\theta_{tj}^{*}\}$ of $G_w^{(T_0)}(z)$ to the zeros $\{\theta_{tj}^{\#}\}$ of $F_w^{(T_0)}(z)$. Furthermore, $F_w^{(T_0)}(z)$ has the same constant term as $G_w^{(T_0)}(z)$ since $\lim_{z\to\infty} h(z) =$

1. On the other hand, the Laurent coefficients of $G_w^{(T_0)}(z)$ have changed. We now discuss how to restore the high-order Laurent coefficients to their previous values. Since $g_w(z)$ has a pole of order N_i at each $x_i \in \mathfrak{X}$, we can expand

$$G_w^{(T_0)}(z) = \sum_{i=1}^{m} \sum_{k=0}^{nN_i-1} a_{w,ik} \frac{1}{(z-x_i)^{nN_i-k}} + a_{w,0}, \tag{11.26}$$

$$F_w^{(T_0)}(z) = \sum_{i=1}^{m} \sum_{k=0}^{nN_i-1} A_{w,ik} \frac{1}{(z-x_i)^{nN_i-k}} + A_{w,0}, \tag{11.27}$$

with $a_{w,ik},\ A_{w,ik} \in K(x_i)_\mu$. Clearly, $a_{w,i0} = c_{w,i0}^n(1+\Delta_{w,i0})$ and $A_{w,i0} = a_{w,i0} \cdot g(x_i)$ for each $x_i \in \mathfrak{X}$.

We have chosen $\varepsilon \in |K_v^\times|$ so that the balls $\mathfrak{B}(x_i, \varepsilon)$ are disjoint from each other and from U_v. Let $\eta_i : B(0,\varepsilon) \to \mathfrak{B}(x_i,\varepsilon)$ be an isometric parametrization with $\eta_i(0) = x_i$, defined by $Z \mapsto x_i + Z$ in appropriate affine coordinates. If $\Theta_{tj}^{\#}$ and Θ_{tj}^* correspond to $\theta_{tj}^{\#}$ and θ_{tj}^* under the affine coordinate Z, then $\frac{Z-\Theta_{tj}^{\#}}{Z-\Theta_{tj}^*}$ can be expanded as a power series about 0, which converges when $|\frac{Z}{\Theta_{tj}^*}|_v < 1 \Leftrightarrow |Z|_v < |\Theta_{tj}^*|_v$. It follows that $h(Z)$ can be expanded as a power series about 0 converging on $\cap_{t,j} B(0, |\Theta_{tj}^*|_v)^-$. Since $\theta_{tj}^* \in E_v^0$ and $\mathfrak{B}(x_i,\varepsilon) \cap U_v = \emptyset$, $|\Theta_{tj}^*|_v > \varepsilon$ and so $\cap_{t,j} B(0, |\Theta_{tj}^*|_v)^- \supset B(0,\varepsilon)$. Thus $h(Z)$ can be expanded as a power series about 0 converging on $B(0,\varepsilon)$. Similarly, if X_h corresponds to x_h under the affine coordinate Z, we can expand $\frac{1}{(Z-X_h)^k}$ as a power series about 0 converging on $B(0,\varepsilon)$ for all $h \neq i$, because the balls are disjoint from each other. This implies that $F_w^{(T_0)}(Z)$ can be expanded as a Laurent series about 0 converging on $B(0,\varepsilon) \setminus \{0\}$.

Letting $\delta := {q_v}^{-\lceil \log_v(n) \rceil} < 1$, we have ${q_v}^{-B} < \delta$. Put $\mathfrak{B} := \cup_{t=1}^N \cup_{j \in I[T_0]} (B(\Theta_{tj}^*, \delta\rho_t)^- \cup B(\Theta_{tj}^{\#}, \delta\rho_t)^-)$ and put $\overline{\mathfrak{B}} := \cup_{t=1}^N \cup_{j \in I[T_0]} (B(\Theta_{tj}^*, \delta\rho_t) \cup B(\Theta_{tj}^{\#}, \delta\rho_t))$. Clearly, $B(\Theta_{tj}^*, \delta\rho_t) = B(\Theta_{tj}^{\#}, \delta\rho_t)$ by (11.25). *À* priori, they might have different boundaries. However, in fact $\partial B(\Theta_{tj}^*, \delta\rho_t) = \partial B(\Theta_{tj}^{\#}, \delta\rho_t)$ because $|\Theta_{tj}^{\#} - \Theta_{tj}^*|_v \le \rho_t \cdot {q_v}^{-B} < \delta\rho_t$. Now [22], Theorem 4.2.16, p.252 implies that $\overline{\mathfrak{B}}$ is an RL-domain and so $\mathfrak{B}^c$ is also an RL-domain. Clearly, $h(Z)$ has poles only at the $\Theta_{tj}^* \in \mathfrak{B}$ and it has no poles on $\mathfrak{B}^c$. The Maximum Modulus Principle implies that $|h(Z)|_v$ and $|h(Z)-1|_v$ achieve their maximum value for $Z \in \mathfrak{B}^c$ at a point of $\partial\mathfrak{B} = \partial\mathfrak{B}^c$. Fix $Z_0 \in \partial\mathfrak{B} = \partial\mathfrak{B}^c$. It follows that $|Z_0 - \Theta_{tj}^{\#}|_v \ge \delta\rho_t$ and $|Z_0 - \Theta_{tj}^*|_v \ge \delta\rho_t$ for all t, j, and so for all pairs (t,j), since $|\Theta_{tj}^{\#}-\Theta_{tj}^{*}|_v \le \rho_t \cdot q_v{}^{-B} < \delta\rho_t$ by (11.25),

$$\left|\frac{\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}}\right|_v < 1. \tag{11.28}$$

On the other hand,

$$\begin{aligned} h(Z_0) &= \prod_{t,j}\frac{Z_0-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}} = \prod_{t,j}\frac{Z_0-\Theta_{tj}^{*}+\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}} \\ &= \prod_{t,j}\left(1+\frac{\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}}\right) \\ &= 1+\sum_{t,j}\frac{\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}} + \sum_{t,j}\sum_{k,\ell}\frac{\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}}\cdot\frac{\Theta_{k\ell}^{*}-\Theta_{k\ell}^{\#}}{Z_0-\Theta_{k\ell}^{*}}+\cdots. \end{aligned}$$

From (11.28), we see that

$$\begin{aligned} &|h(Z_0)|_v = 1, \\ &|h(Z_0)-1|_v \le \max\left\{\left|\frac{\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}}\right|_v : \text{for all } t,\ j\right\}. \end{aligned}$$

Since $\left|\frac{\Theta_{tj}^{*}-\Theta_{tj}^{\#}}{Z_0-\Theta_{tj}^{*}}\right|_v \le \frac{\rho_t \cdot q_v{}^{-B}}{\delta\rho_t} = \frac{q_v{}^{-B}}{\delta}$, it follows that for all $Z \in \mathfrak{B}^c$,

$$\begin{aligned} &|h(Z)|_v \le 1, \\ &|h(Z)-1|_v \le \frac{q_v{}^{-B}}{\delta} < 1. \end{aligned} \tag{11.29}$$

Now, we want to recover the high-order coefficients in the range $nN_i, \cdots, (n-T_0)N_i+1$ of $G_w^{(T_0)}(z)$ from $F_w^{(T_0)}(z)$. After pulling $\mathfrak{B}(x_i,\varepsilon)$ back to $B(0,\varepsilon)$ by η_i, we see that each element $\frac{1}{(z-x_h)^k}$ of the L-basis converges on $B(0,\varepsilon)\setminus\{0\}$, having no zeros there. The same is true for $G_w^{(T_0)}(z)$. To apply Lemma 11.4, we rewrite

$$\begin{aligned} G_w^{(T_0)}(z) = &\sum_{j=0}^{T_0N_i-1} a_{w,ij}\frac{1}{(z-x_i)^{nN_i-j}} \\ &+ \text{higher terms in } (z-x_i)^h \text{ for } h \ge -nN_i+T_0N_i, \\ F_w^{(T_0)}(z) = &\sum_{j=0}^{T_0N_i-1} A_{w,ij}\frac{1}{(z-x_i)^{nN_i-j}} \\ &+ \text{higher terms in } (z-x_i)^h \text{ for } h \ge -nN_i+T_0N_i. \end{aligned}$$

It follows from Lemma 11.4 that for $0 \le j \le T_0 N_i - 1$,

$$|a_{w,ij} - A_{w,ij}|_v \le \frac{q_v^{-B} \cdot |a_{w,i0}|_v}{\delta \cdot \varepsilon^j}. \tag{11.30}$$

To recover the high-order coefficients of $G_w^{(T_0)}(z)$ from those of $F_w^{(T_0)}(z)$, we need a regular subsequence $\{\beta_\ell : \ell \in I \setminus I[T_0]\}$ of length T_0 for which $\mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}) < \log_v(n)$ for all t, all $j \in I[T_0]$, and all ℓ in the regular subsequence of length T_0. Put

$$I^* := \{\ell \in I^0 : \mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}) \ge \log_v(n) \text{ for some } t = 1, \cdots, N \text{ and some } j \in I[T_0]\}.$$

If $I^{\#} = I^0 \setminus I^*$, then

$$I^{\#} = \{\ell \in I^0 : \mathrm{ord}_v(\alpha_{tj}^{\#} - \alpha_{t\ell}) < \log_v(n) \text{ for all } t = 1, \cdots, N \text{ and all } j \in I[T_0]\},$$

These are the indices which are "safe" to use for the patching process. Without loss of generality, we can assume that $n > 1$. Hence there is a constant $A_2 > 0$ such that

$$\#(I[T_0]) \le A_2(\log_v(n))^2.$$

If $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{t\ell}) \ge \lceil \log_v(n) \rceil$ and $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{th}) \ge \lceil \log_v(n) \rceil$ for $\ell \ne h$ in I^0, then $\mathrm{ord}_v(\alpha_{t\ell} - \alpha_{th}) \ge \lceil \log_v(n) \rceil$, but $\mathrm{ord}_v(\alpha_{t\ell} - \alpha_{th}) < \log_v(n - \#(I[T_0])) < \log_v(n)$ because $\{\alpha_{t\ell}\}_{\ell \in I^0}$ is a regular sequence of length $n - \#(I[T_0]) < n$. Hence for each "patched" root α_{tj}^*, there is at most one "unpatched" root $\alpha_{t\ell}$ such that $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{t\ell}) \ge \lceil \log_v(n) \rceil$. It follows that $\#(I^*) \le N \cdot \#(I[T_0]) \le N \cdot A_2(\log_v(n))^2$, and in turn that $\#(I[T_0] \cup I^*) \le (N+1) \cdot A_2(\log_v(n))^2$.

Without loss of generality, we can assume that n is large enough that

$$n \ge [(N+1) \cdot A_2(\log_v(n))^2 + 1] \cdot [A_1 \log_v(n) + 1]. \tag{11.31}$$

The pigeon-hole principle implies that there is a subsequence of consecutive integers in $I^{\#}$ of length at least $A_1 \cdot \log_v(n) + 1 \ge T_0$. Let $I_0^{\#}$ be the largest sequence of consecutive integers in $I^{\#}$. Let $I^{\sharp}[T_0]$ be the subset of $I^{\#}$ consisting of the first T_0 consecutive integers in $I_0^{\#}$. Set

$$Q_w^{(0,T_0)}(z) := \prod_{j \in I^{\sharp}[T_0]} (g_w(z) - \beta_v(j)),$$
$$\widetilde{Q}_w^{(0,T_0)}(z) := \prod_{j \in I^0 \setminus I^{\sharp}[T_0]} (g_w(z) - \beta_v(j)).$$

Put

$$Q_{w,T_0}(z) := h(z)\cdot Q_w^{(k_0)}(z)\cdots Q_w^{(T_0)}(z),$$
$$\widetilde{Q}_{w,T_0}(z) = Q_{w,T_0}(z)\cdot \widetilde{Q}_w^{(0,T_0)}(z).$$

Note that $Q_{w,T_0}(z)$ has only zeros at each $\theta_{tj}^{\#}$. Since $Q_w^{(k_0)}(z),\cdots,Q_w^{(T_0)}(z)$ are rational over K_v, $\widetilde{Q}_{w,T_0}(z)$ is a K_v-rational $(\mathfrak{X},\vec{s})$-function of degree $(n-\#(I^\sharp[T_0]))N = (n-T_0)N$, independent of $w|v$. Furthermore, $\widetilde{Q}_{w,T_0}(z)$ has a pole of order $(n-T_0)N_i$ at x_i, with leading coefficient $B_{w,i0} := c_{w,i0}^{n-T_0}\ \varepsilon_{w,i}$, and a pole of order $(n-T_0)N_\ell$ at $x_\ell \neq x_i$. After pulling $\mathfrak{B}(x_i,\varepsilon)$ back to $B(0,\varepsilon)$ by η_i, we know that $\widetilde{Q}_{w,T_0}(z)$ converges on $B(0,\varepsilon)\setminus\{0\}$, having no zeros there. For each $1\le J\le T_0N_i$, taking $A=(n-T_0)N_i$, Lemma 11.5 implies that there is a unique K_v rational function

$$g_{w,iJ}(z) = \frac{1}{(z-x_i)^J} + \sum_{j=1}^{J-1} b_{w,ij}^{(J)}\frac{1}{(z-x_i)^{J-j}},$$

independent of all $w|v$, with $|b_{w,ij}^{(J)}|_v \le \frac{1}{\varepsilon^j}$ for $j=1,\cdots,J-1$ such that

$$g_{w,iJ}(z)\cdot\widetilde{Q}_{w,T_0}(z) = B_{w,i0}\cdot\frac{1}{(z-x_i)^{(n-T_0)N_i+J}} + \sum_{h\ge 1} B_{w,ih}^{(J)}\frac{1}{(z-x_i)^{(n-T_0)N_i+1-h}}.$$

That is, $g_{w,iJ}(z)\cdot\widetilde{Q}_{w,T_0}(z)\in K_v(z)$ has no terms of orders $(n-T_0)N_i+J-1,\cdots,(n-T_0)N_i+1$ in the expansion about x_i. Put $C_1 := \max\{B_v,\varepsilon^{-1}\}$, which depends only on U_v^0. Since $\|\frac{1}{(z-x_i)}\|_{U_v^0}\le B_v$, it follows that

$$\|g_{w,iJ}(z)\|_{U_v^0} \le \max_{0\le j\le J-1}\frac{B_v^{J-j}}{\varepsilon^j} \le C_1^J.$$

By (11.26) and (11.27), since $a_{w,0}=A_{w,0}$, we can write

$$\begin{aligned} G_w^{(T_0)}(z) - F_w^{(T_0)}(z) \ &= \ \sum_{i=1}^m\sum_{k=0}^{T_0N_i-1}(a_{w,ik}-A_{w,ik})\frac{1}{(z-x_i)^{nN_i-k}} \\ &\quad + \sum_{i=1}^m\sum_{\ell=T_0N_i}^{\infty}(a_{w,i\ell}-A_{w,i\ell})\frac{1}{(z-x_i)^{nN_i-\ell}}. \end{aligned}$$

Since $G_w^{(T_0)}(z)$ and $F_w^{(T_0)}(z)$ are K_v-rational, $\sum_{i=1}^m\sum_{k=0}^{T_0N_i-1}(a_{w,ik}-A_{w,ik})\frac{1}{(z-x_i)^{nN_i-k}}$ is also K_v-rational. Expand $\widetilde{Q}_{w,T_0}(z)$ in terms of the L-basis

$$\widetilde{Q}_{w,T_0}(z) = \sum_{i=1}^m\sum_{h=0}^{(n-T_0)N_i-1} B_{w,ih}\frac{1}{(z-x_i)^{(n-T_0)N_i-h}} + B_{w,0}.$$

Note that $g_{w,i,T_0N_i-j}(z)\cdot\widetilde{Q}_{w,T_0}(z)$ has no terms of orders $nN_i-j-1,\cdots,(n-T_0)N_i+1$ for each $j=0,\cdots,T_0N_i-1$. To recover the high-order coefficients, take

$$\delta_{w,ij}:=\frac{a_{w,ij}-A_{w,ij}}{B_{w,i0}}.$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{k=0}^{T_0N_i-1}(a_{w,ik}-A_{w,ik})\frac{1}{(z-x_i)^{nN_i-k}}\\&=\sum_{i=1}^{m}\sum_{j=0}^{T_0N_i-1}\delta_{w,ij}\cdot g_{w,i,T_0N_i-j}(z)\cdot\widetilde{Q}_{w,T_0}(z)\\&=\left\{\sum_{i=1}^{m}\sum_{j=0}^{T_0N_i-1}\delta_{w,ij}\cdot\left(\frac{1}{(z-x_i)^J}+\sum_{k=1}^{J-1}b_{w,ik}^{(J)}\frac{1}{(z-x_i)^{J-k}}\right)\right\}\\&\quad\cdot\left\{\sum_{i=1}^{m}\sum_{h=0}^{(n-T_0)N_i-1}B_{w,ih}\frac{1}{(z-x_i)^{(n-T_0)N_i-h}}+B_{w,0}\right\}.\end{aligned}$$

Since $a_{w,i0}=c_{w,i0}^{n}(1+\Delta_{w,i0})$ and $B_{w,i0}=c_{w,i0}^{n-T_0}(1+\Delta_{w,i0})$, it follws from (11.30) that for each $j=0,\cdots,T_0N_i-1$,

$$|\delta_{w,ij}|_v \quad\leq\quad \frac{q_v^{-B}\cdot|c_{w,i0}|_v^{T_0}}{\delta\cdot\varepsilon^j}.$$

Put

$$\Phi_w^{(T_0)}(z):=\sum_{i=1}^{m}\sum_{j=0}^{T_0N_i-1}\delta_{w,ij}\cdot g_{w,i,T_0N_i-j}(z).$$

Since $\sum_{i=1}^m\sum_{k=0}^{T_0N_i-1}(a_{w,ik}-A_{w,ik})\frac{1}{(z-x_i)^{nN_i-k}}$ and $\widetilde{Q}_{w,T_0}(z)$ are K_v-rational and independent of $w|v$, so is $\Phi_w^{(T_0)}(z)$. Put $C_2:=\max\{|c_{w,i0}|_v\cdot C_1^{N_i}:i=1,\cdots,m\}$. Since $\varepsilon\cdot C_1\geq 1$, it follows that

$$\begin{aligned}&\|\Phi_w^{(T_0)}(z)\|_{U_v^0}\\&\leq\max\left\{|\delta_{w,ij}|_v\cdot\|g_{w,i,T_0N_i-j}(z)\|_{U_v^0}\ :i=1,\cdots,m,\ j=0,\cdots,T_0N_i-1\right\}\\&\leq\max\left\{\frac{q_v^{-B}\cdot|c_{w,i0}|_v^{T_0}}{\delta\cdot\varepsilon^j}\cdot C_1^{T_0N_i-j}\ :i=1,\cdots,m,\ j=0,\cdots,T_0N_i-1\right\}\\&\leq\frac{q_v^{-B}}{\delta}\cdot C_2^{T_0}.\end{aligned}\tag{11.32}$$

To patch $F_w^{(T_0)}(z)$, put

$$\widetilde{F}_w^{(T_0)}(z) := F_w^{(T_0)}(z) + \Phi_w^{(T_0)}(z) \cdot \widetilde{Q}_{w,T_0}(z).$$

Clearly, $\widetilde{F}_w^{(T_0)}(z)$ is rational over K_v and independent of $w|v$. By the construction, $\widetilde{F}_w^{(T_0)}(z)$ has coefficients $a_{w,ik}$ of orders $nN_i, \cdots, (n - T_0)N_i + 1$ for each $i = 1, \cdots, m$, which are the same as those of $G_w^{(T_0)}(z)$.

Now, observe the movement of roots of $\widetilde{F}_w^{(T_0)}(z)$ comparing with those of $G_w^{(T_0)}(z)$. Since $F_w^{(T_0)}(z) = \widetilde{Q}_{w,T_0}(z) \cdot Q_w^{(0,T_0)}(z)$, we can rewrite

$$\widetilde{F}_w^{(T_0)}(z) = (Q_w^{(0,T_0)}(z) + \Phi_w^{(T_0)}(z)) \cdot \widetilde{Q}_{w,T_0}(z).$$

After pulling $\mathfrak{B}(\theta_t, \rho_t)$ back to $B(0,1)$ by λ_t, $Q_w^{(0,T_0)}(z)$ has exactly T_0 roots in $B(0,1)$, which form a regular sequence $\{\alpha_{t\ell}\}_{\ell \in I^\sharp[T_0]}$ of length T_0 in $\mathcal{O}_v$, and can be expanded as a power series about 0 converging on $B(0,1)$. Also, $\Phi_w^{(T_0)}(z)$ can be expanded as a power series about 0 converging on $B(0,1)$ as before. To apply Lemma 11.2 to $Q_w^{(0,T_0)}(z)$ and $\Phi_w^{(T_0)}(z)$, we need the condition

$$\|\Phi_w^{(T_0)}(z)\|_{U_v^0} \le q_v^{-\frac{T_0}{q_v-1} - \log_v(n)} < 1,$$

or equivalently, $\frac{q_v^{-B}}{\delta} \cdot C_2^{T_0} \le q_v^{-\frac{T_0}{q_v-1} - \log_v(n)}$ by (11.32), because $\lambda_t(B(0,1)) \subset U_v^0$. Recall that $T_0 \le A_1 \log_v(n)$. To satisfy this condition, we must thus have

$$B \ge \left(\log_v(C_2) + \frac{1}{q_v - 1} \right) \cdot A_1 \log_v(n) + 2\log_v(n) + 1. \tag{11.33}$$

Note that the constants appearing in (11.33) depend only on U_v^0 and ε. At the beginning of the argument, B was chosen to be the least integer satisfying (11.33). Since B and D are smallest integers satisfying the conditions described above, there is a constant A_3 such that

$$B + D \le A_3 \log_v(n), \tag{11.34}$$

for each $n \ge ((N+1) \cdot A_2(\log_v(n))^2 + 1) \cdot (A_1 \log_v(n) + 1)$. Put

$$Q_w^{(T_0\#)}(z) := Q_w^{(0,T_0)}(z) + \Phi_w^{(T_0)}(z). \tag{11.35}$$

Clearly, $\|Q_w^{(T_0\#)}(z)\|_{U_v^0} \leq 1$. Letting $\mathfrak{Q}_{w,t}^{(T_0\#)}(z)$ be the power series gotten by composing $Q_w^{(T_0\#)}(z)$ with λ_t, Lemma 11.2 implies that the roots $\{\alpha_{t\ell}^*\}_{\ell \in I^\sharp[T_0]}$ of $\mathfrak{Q}_{w,t}^{(T_0\#)}(z)$ in $B(0,1)$ form a regular sequence of length T_0 in $\mathcal{O}_v$, with

$$\mathrm{ord}_v(\alpha_{t\ell}^* - \alpha_{t\ell}) > \log_v(n).$$

Furthermore, since $I^\sharp[T_0] \subset I^\#$, it follows that for all $\ell \in I^\sharp[T_0]$,

$$\begin{aligned} \mathrm{ord}_v(\alpha_{tk}^\# - \alpha_{t\ell}^*) &= \mathrm{ord}_v(\alpha_{tk}^\# - \alpha_{tk}^* + \alpha_{tk}^* - \alpha_{t\ell} + \alpha_{t\ell} - \alpha_{t\ell}^*) \\ &= \mathrm{ord}_v(\alpha_{tk}^* - \alpha_{t\ell}) < \log_v(n). \end{aligned}$$

To summarize this step, in passing from $G_w^{(T_0)}(z)$ to $\widetilde{F}_w^{(T_0)}(z)$, we have moved the patched roots α_{tj}^* to roots $\alpha_{tj}^\# \in \mathcal{O}_v$ for each $j \in I[T_0]$, while preserving the high-order coefficients. The $\alpha_{tj}^\#$ satisfy the following properties:

(a) $\mathrm{ord}_v(\alpha_{tj}^\# - \alpha_{tj}^*) \geq B$;

(b) $\mathrm{ord}_v(\alpha_{tj}^\# - \alpha_{t\ell}^*) \leq A_3 \log_v(n)$ for each $\ell \neq j$ in $I[T_0]$;

(c) $\mathrm{ord}_v(\alpha_{tj}^\# - \alpha_{t\ell}^\#) \leq A_3 \log_v(n)$ for each $\ell \neq j$ in $I[T_0]$;

(d) $\mathrm{ord}_v(\alpha_{tj}^\# - \alpha_{t\ell}) \leq A_3 \log_v(n)$ for each $\ell \in I^0$; and

(e) $\mathrm{ord}_v(\alpha_{tj}^\# - \alpha_{t\ell}^*) \leq \log_v(n)$ for each $\ell \in I^\sharp[T_0]$.

Rename the $\alpha_{tj}^\#$ as the α_{tj}^* and replace $G_w^{(T_0)}(z)$ by $\widetilde{F}_w^{(T_0)}(z)$. For notational convenience, replace $\alpha_{t\ell}^*$ by $\alpha_{t\ell}$ for $\ell \in I^\sharp[T_0]$. Then I^* and $I^\#$ are the same sets with these new notations as before. Furthermore, $G_w^{(T_0)}(z) = Q_{w,T_0}(z) \cdot Q_w^{(T_0\#)}(z) \cdot \widetilde{Q}_w^{(0,T_0)}(z)$ has the roots $\{\lambda_t(\alpha_{tj}^*) = \theta_{tj}^*\}_{j \in I[T_0]}$ and $\{\lambda_t(\alpha_{t\ell}) = \theta_{t\ell}\}_{\ell \in I^0}$ satisfying the following properties:

(i) $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{th}^*) \leq A_3 \log_v(n)$ for all $j \neq h$ in $I[T_0]$;

(ii) $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{t\ell}) \leq A_3 \log_v(n)$ for all $j \in I[T_0]$ and $\ell \in I^*$;

(iii) $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{t\ell}) \leq \log_v(n)$ for all $j \in I[T_0]$ and $\ell \in I^\#$; and

(iv) $\mathrm{ord}_v(\alpha_{t\ell} - \alpha_{th}) \leq \log_v(n)$ for all $\ell \neq h$ in I^0, because $\{\alpha_{t\ell}^*\}_{\ell \in I^\sharp[T_0]}$ is a regular sequence of length $T_0 < n$, $\{\alpha_{t\ell}\}_{\ell \in I^0}$ is a regular sequence of length $\#(I^0) < n$, and $\mathrm{ord}_v(\alpha_{t\ell}^* - \alpha_{t\ell}) > \log_v(n)$ for $\ell \in I^\sharp[T_0]$.

Step 3. The Patching Process for the safe roots

The purpose of this step is to continue the patching process using the Basic Patching Lemma, until the dominance of h_v^{kN} over $q_v^{-\frac{k}{q_v-1}}$ is so great that further patching steps move the roots α_{tj} to α_{tj}^* with $\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{tj}) \geq A_3 \log_v(n)$.

Let T_1 be the least positive integer for which

$$h_v^{kN} \cdot T_v \cdot B_v \leq q_v^{-\frac{k}{q_v-1} - 3A_3 \log_v(n)} \tag{11.36}$$

for all $k \geq T_1$. Then there is a constant A_4 such that $T_1 \leq A_4 \log_v(n)$. Note that $T_1 > T_0$ because $3A_3 \log_v(n) > \log_v(n)$. Without loss of generality, we can assume that n is large enough that

$$n \geq ((N+1) \cdot A_2(\log_v(n))^2 + 1) \cdot (A_4 \log_v(n) + 1). \tag{11.37}$$

By the pigeon-hole principle, $\#(I_0^{\#}) \geq A_4 \log_v(n) + 1 \geq T_1$. For each $k = T_0 + 1, \cdots, T_1$, let $I^\sharp[k]$ be the subset of the first k consecutive integers in $I_0^{\#}$ and J_k be the last index in $I^\sharp[k]$. That is, $I^\sharp[k] = I^\sharp[k-1] \cup \{J_k\}$. Put

$$\widetilde{Q}_w^{(0,k)}(z) := \prod_{j \in I^0 \backslash I^\sharp[k]} (g_w(z) - \beta_v(j)). \tag{11.38}$$

We see that $\widetilde{Q}_w^{(0,k-1)}(z) = (g_w(z) - \beta_v(J_k)) \cdot \widetilde{Q}_w^{(0,k)}(z)$.

If $k = T_0 + 1$, using (11.35), rewrite

$$\begin{aligned} G_w^{(T_0)}(z) &= Q_{w,T_0}(z) \cdot Q_w^{(T_0\#)}(z) \cdot \widetilde{Q}_w^{(0,T_0)}(z) \\ &= Q_{w,T_0}(z) \cdot Q_w^{(T_0\#)}(z) \cdot (g_w(z) - \beta_v(J_{T_0+1})) \cdot \widetilde{Q}_w^{(0,T_0+1)}(z), \end{aligned}$$

and put

$$Q_w^{(0,T_0+1)}(z) := Q_w^{(T_0\#)}(z) \cdot (g_w(z) - \beta_v(J_{T_0+1})),$$

which will play the role of $f(z)$ in Lemma 11.2. We carry out the same argument as Step 1 to patch the coefficients. Given the numbers $\Delta_{w,i,T_0N_i}, \cdots, \Delta_{w,i,(T_0+1)N_i-1}$ for each i, put

$$\varphi_w^{(T_0+1)}(z) := \sum_{i=1}^{m} \varepsilon_{w,i}^{-1} \sum_{\ell=T_0N_i}^{(T_0+1)N_i-1} \Delta_{w,i\ell} \frac{1}{(z-x_i)^{(T_0+1)N_i-\ell}}.$$

By (vii), $\varphi_w^{(T_0+1)}(z) \in K_v(z)$ is indepedent of $w|v$. In particular, it follows from (iii) that

$$\|\varphi_w^{(T_0+1)}(z)\|_{U_v^0} \le h_w^{(T_0+1)N} \cdot B_w < h_w^{(T_0+1)N} \cdot T_w \cdot B_w. \tag{11.39}$$

Note that $\varphi_w^{(T_0+1)}(z)$ can be expanded as a power series about θ_t converging on $\mathfrak{B}(\theta_t, \rho_t)$ with the same argument before. Define

$$\begin{aligned} G_w^{(T_0+1)}(z) &= G_w^{(T_0)}(z) + \varphi_w^{(T_0+1)}(z) \cdot Q_{w,T_0}(z) \cdot \widetilde{Q}_w^{(0,T_0+1)}(z) \\ &= Q_{w,T_0}(z) \cdot (Q_w^{(0,T_0+1)}(z) + \varphi_w^{(T_0+1)}(z)) \cdot \widetilde{Q}_w^{(0,T_0+1)}(z). \end{aligned}$$

Then $G_w^{(T_0+1)}(z)$ is rational over K_v and independent of $w|v$. Letting $Q_w^{(T_0+1)}(z) := Q_w^{(0,T_0+1)}(z) + \varphi_w^{(T_0+1)}(z)$, the function $Q_w^{(T_0+1)}(z)$ has a pole of order $(T_0+1)N_i$ at each $x_i \in \mathfrak{X}$ with leading coefficient $c_{w,i0}^{T_0+1}$, satisfying $\|Q_w^{(T_0+1)}\|_{U_v^0} \le 1$. Now, to apply the basic patching lemma to $Q_w^{(T_0+1)}(z)$, let $\mathfrak{Q}_{w,t}^{(0,T_0+1)}(z), \mathfrak{Q}_{w,t}^{(T_0+1)}(z), \xi_{w,t}^{(T_0+1)}(z)$ be the power series gotten by composing $Q_w^{(0,T_0+1)}(z), Q_w^{(T_0+1)}(z), G_w^{(T_0+1)}(z)$ with λ_t, respectively. Consider the roots of $\mathfrak{Q}_{w,t}^{(0,T_0+1)}(z)$. They are a union of the roots $\{\alpha_{t\ell}^*\}_{\ell \in I^\sharp[T_0]}$ of $\mathfrak{Q}_{w,t}^{(T_0*)}(z)$ and the root $\alpha_{i,J_{T_0+1}}$ of $g_w(z) - \beta_v(J_{T_0+1})$. Since the roots $\{\alpha_{t\ell}\}_{\ell \in I^\sharp[T_0]}$ of $Q_w^{(0,T_0)}(z)$ and the root $\alpha_{i,J_{T_0+1}}$ of $g_w(z) - \beta_v(J_{T_0+1})$ form a regular sequence of length T_0+1 in $\mathcal{O}_v$ and $\{\alpha_{t\ell}^*\}_{\ell \in I^\sharp[T_0]}$ satisfies $\mathrm{ord}_v(\alpha_{t\ell}^* - \alpha_{t\ell}) > \log_v(n)$, the roots of $\mathfrak{Q}_{w,t}^{(0,T_0+1)}(z)$ form a regular sequence of length T_0+1 in $\mathcal{O}_v$ (via the automorphism τ_i of $B(0,1)$). Relabel $\alpha_{t\ell}^*$ as $\alpha_{t\ell}$ for $\ell \in I^\sharp[T_0]$. Then the roots $\{\alpha_{t\ell}^*\}_{\ell \in I^\sharp[T_0+1]}$ of $\mathfrak{Q}_{w,t}^{(T_0+1)}(z)$ form a regular sequence of length T_0+1 in $\mathcal{O}_v$, with $\mathrm{ord}_v(\alpha_{t\ell}^* - \alpha_{t\ell}) > \log_v(n)$.

Now, for each $k = T_0+2, \cdots, T_1$, we can carry on the same process, inductively by defining

$$\begin{aligned} G_w^{(k)}(z) &= G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot Q_{w,T_0}(z) \cdot \widetilde{Q}_w^{(0,k)}(z) \\ &= Q_{w,T_0}(z) \cdot (Q_w^{(0,k)}(z) + \varphi_w^{(k)}(z)) \cdot \widetilde{Q}_w^{(0,k)}(z), \end{aligned}$$

where $Q_w^{(0,k)}(z) := Q_w^{(k-1)}(z) \cdot (g_w(z) - \beta_v(J_k))$. Then $G_w^{(k)}(z)$ is rational over K_v and independent of $w|v$. Letting $\mathfrak{Q}_{w,t}^{(0,k)}(z), \mathfrak{Q}_{w,t}^{(k)}(z), \xi_{w,t}^{(k)}(z)$ be the power series gotten by composing $Q_w^{(0,k)}(z), Q_w^{(k)}(z), G_w^{(k)}(z)$ with λ_t, respectively, apply the basic patching lemma to $Q_w^{(k)}(z)$.

By induction, the roots of $\mathfrak{Q}_{w,t}^{(0,k)}(z)$ form a regular sequence of length k in $\mathcal{O}_v$ (via the automorphism τ_i of $B(0,1)$), so the roots $\{\alpha_{t\ell}^*\}_{\ell\in I^\sharp[k]}$ of $\mathfrak{Q}_{w,t}^{(k)}(z)$ form a regular sequence of length k in $\mathcal{O}_v$, with $\mathrm{ord}_v(\alpha_{t\ell}^*-\alpha_{t\ell})>\log_v(n)$.

To summarize this step, the roots of $G_w^{(T_1)}(z)=Q_{w,T_0}(z)\cdot Q_w^{(T_1)}(z)\cdot\widetilde{Q}_w^{(0,T_1)}(z)$ are $\{\lambda_t(\alpha_{tj}^*)=\theta_{tj}^*\}_{j\in I[T_0]}$, $\{\lambda_t(\alpha_{t\ell}^*)=\theta_{t\ell}^*\}_{\ell\in I^\sharp[T_1]}$, $\{\lambda_t(\alpha_{t\ell})=\theta_{t\ell}\}_{\ell\in I^0\backslash I^\sharp[T_1]}$. They have the following properties:

(i) the conditions for the set $\{\alpha_{tj}^*\}_{j\in I[T_0]}\cup\{\alpha_{t\ell}\}_{\ell\in I^0\backslash I^\sharp[T_1]}$ are inherited from the result of Step 2,

(ii) $\mathrm{ord}_v(\alpha_{t\ell}^*-\alpha_{th}^*)<\log_v(T_1)<\log_v(n)$ for all $\ell\neq h$ in $I^\sharp[T_1]$,

(iii) $\mathrm{ord}_v(\alpha_{t\ell}^*-\alpha_{tj}^*)=\mathrm{ord}_v(\alpha_{t\ell}^*-\alpha_{t\ell}+\alpha_{t\ell}-\alpha_{tj}^*)=\mathrm{ord}_v(\alpha_{t\ell}-\alpha_{tj}^*)<\log_v(n)$ for all $\ell\in I^\sharp[T_1]\subset I^\#$ and $j\in I[T_0]$, and

(iv) $\mathrm{ord}_v(\alpha_{t\ell}^*-\alpha_{th})=\mathrm{ord}_v(\alpha_{t\ell}^*-\alpha_{t\ell}+\alpha_{t\ell}-\alpha_{th})=\mathrm{ord}_v(\alpha_{t\ell}-\alpha_{th})<\log_v(n)$ for all $\ell\in I^\sharp[T_1]$ and $h\in I^0\setminus I^\sharp[T_1]$.

Step 4. The Patching Process for the remaining unpatched roots

Lemma 11.6 ([25] Refined Patching Lemma) *Let $f(z)\in K_v[[z]]$ be a power series converging in $B(0,1)$, with $\|f(z)\|_{B(0,1)}=1$. Suppose the roots $\{\alpha_j\}_{j\in I}$ of $f(z)$ in $B(0,1)$ can be partitioned into d disjoint regular sequences in $\mathcal{O}_v$ attached to index sets $I_1,\cdots,I_d$ of lengths $m_1,\cdots,m_d$, respectively. Put $D:=\sum_{i=1}^d m_i$. If there is a constant $R\geq\max\{\log_v(m_i):i=1,\cdots,d\}$ such that $\mathrm{ord}_v(\alpha_j-\alpha_\ell)\leq R$ for all $j\neq\ell$, then for any $H\geq R$ and any power series $\Delta(z)\in K_v[[z]]$ converging on $B(0,1)$, with*

$$\|\Delta(z)\|_{B(0,1)}\leq q_v^{-\frac{D}{q_v-1}-(d-1)R-H},$$

the roots $\{\alpha_j^*\}_{j\in I}$ of $f^*(z)=f(z)+\Delta(z)$ in $B(0,1)$ again form a union of regular sequences in $\mathcal{O}_v$ attached to $I_1,\cdots,I_d$, and can be uniquely labeled in such a way that

$$\mathrm{ord}_v(\alpha_j^*-\alpha_j)>H\quad\textit{for each } j.$$

Proof: The proof is very similar to that of the Basic Patching Lemma. See that proof for details.

Using the Weierstrass Preparation Theorem, write $f(z) = b \cdot g(z) \cdot h(z)$, where $b \in K_v$, $g(z) = \prod_{j \in I}(z - \alpha_j)$, and $h(z) \in K_v[[z]]$ is a unit power series. By the same argument as before, we see that $\mathrm{ord}_v(b) = 0$, i.e., $|b|_v = 1$ and $f^*(z)$ has the same number of roots in $B(0,1)$ as $f(z)$.

Fix a root α_J, and expand $f(z)$ and $\Delta(z)$ as power series about α_J:

$$f(z) = \sum_{k \geq 1} b_k^{(J)} (z - \alpha_J)^k,$$
$$\Delta(z) = \sum_{k \geq 0} \delta_k^{(J)} (z - \alpha_J)^k.$$

The initial part of the Newton polygon of $f(z)$ expanded about α_J coincides with that of $g(z)$ and its remaining slopes are $\geq \log_v(1) = 0$. Rewriting

$$g(z) = \prod_{j \in I}((z - \alpha_J) - (\alpha_j - \alpha_J)) = \sum_{k=1}^{D} a_k^{(J)} (z - \alpha_J)^k,$$

the coefficients $a_k^{(J)}$ are elementary symmetric polynomials in the $\alpha_j - \alpha_J$. It follows that

$$a_1^{(J)} = \pm \prod_{\substack{j \in I \\ j \neq J}} (\alpha_j - \alpha_J),$$
$$a_k^{(J)} = \pm a_1^{(J)} \sum_{\substack{j_1 < \cdots < j_{k-1} \text{ in } I \\ \text{each } j_i \neq J}} \frac{1}{(\alpha_{j_1} - \alpha_J) \cdots (\alpha_{j_{k-1}} - \alpha_J)}$$

for each $k = 2, \cdots, D$.

If $J \in I_t$ for some $1 \leq t \leq d$, then Proposition 5.9 (i) implies that

$$\mathrm{ord}_v\Big(\prod_{\substack{j \in I_t \\ j \neq J}} (\alpha_j - \alpha_J)\Big) < \frac{m_t}{q_v - 1}.$$

If $i \neq t$, then Proposition 5.9 (i) and the choice of R imply that

$$\mathrm{ord}_v\Big(\prod_{j \in I_i} (\alpha_j - \alpha_J)\Big) < \frac{m_i}{q_v - 1} + R.$$

Hence it follows that

$$\mathrm{ord}_v(a_1^{(J)}) < \frac{D}{q_v - 1} + (d-1)R.$$

For $k = 2, \cdots, D$, the choice of R implies that

$$\mathrm{ord}_v(a_k^{(J)}) > \mathrm{ord}_v(a_1^{(J)}) - (k-1)R.$$

Comparing the Newton polygons of $f(z)$ and $bg(z)$, since $\mathrm{ord}_v(b) = 0$, we find that

$$\begin{aligned}\mathrm{ord}_v(b_1^{(J)}) &= \mathrm{ord}_v(a_1^{(J)}) < \frac{D}{q_v - 1} \quad \text{and} \\ \mathrm{ord}_v(b_k^{(J)}) &> \mathrm{ord}_v(a_1^{(J)}) - (k-1)R = \mathrm{ord}_v(b_1^{(J)}) - (k-1)R\end{aligned}$$

for each $k = 2, \cdots, D$. For each $k \geq D$,

$$\begin{aligned}\mathrm{ord}_v(b_k^{(J)}) &\geq \mathrm{ord}_v(b_D^{(J)}) > \mathrm{ord}_v(b_1^{(J)}) - (D-1)R \\ &\geq \mathrm{ord}_v(b_1^{(J)}) - (k-1)R.\end{aligned}$$

By the hypothesis on $\Delta(z)$ and Lemma 2.3, we have $\mathrm{ord}_v(\delta_k^{(J)}) \geq \frac{D}{q_v-1} + (d-1)R + H$ for each $k \geq 0$. Since $H \geq R$, it follows that

$$\begin{aligned}\mathrm{ord}_v(\delta_1^{(J)}) &\geq \frac{D}{q_v - 1} + (d-1)R + R > \frac{D}{q_v - 1} + (d-1)R \\ &> \mathrm{ord}_v(b_1^{(J)}),\end{aligned}$$

$$\begin{aligned}\mathrm{ord}_v(\delta_k^{(J)}) &\geq \frac{D}{q_v - 1} + (d-1)R + R > \mathrm{ord}_v(b_1^{(J)}) + R \\ &\geq \mathrm{ord}_v(b_1^{(J)}) - (k-1)R,\end{aligned}$$

for each $k \geq 2$.

Now, writing $f^*(z) = \sum_{k \geq 0} c_k^{(J)}(z - \alpha_J)^k$, we have $c_k^{(J)} = b_k^{(J)} + \delta_k^{(J)}$ for all $k \geq 0$. By the estimates for $b_k^{(J)}$ and $\delta_k^{(J)}$,

$$\begin{aligned}\mathrm{ord}_v(c_0^{(J)}) &= \mathrm{ord}_v(\delta_0^{(J)}) \geq \frac{D}{q_v - 1} + (d-1)R + H, \\ \mathrm{ord}_v(c_1^{(J)}) &= \mathrm{ord}_v(b_1^{(J)}) < \frac{D}{q_v - 1} + (d-1)R, \quad \text{and} \\ \mathrm{ord}_v(b_k^{(J)}) &> \mathrm{ord}_v(b_1^{(J)}) - (k-1)R, \quad \text{for each } k \geq 2.\end{aligned}$$

Since $H \geq R$, the Newton polygon of $f^*(z)$ has a break at $k = 1$, and its initial segment has slope $\mathrm{ord}_v(c_1^{(J)}) - \mathrm{ord}_v(c_0^{(J)})$, which is less than $-H$. Hence $f^*(z)$ has a unique root α_J^*

satisfying

$$\mathrm{ord}_v(\alpha_J^* - \alpha_J) > H.$$

Noting that $\mathrm{ord}_v(\alpha_J^* - \alpha_J) > H \geq R \geq 0$ and $\alpha_J \in B(0,1) \cap K_v = \mathcal{O}_v$, $\alpha_J^* \in B(\alpha_J, 1) = B(0,1)$. Hence we get $\alpha_J^* \in B(0,1) \cap K_v = \mathcal{O}_v$ by Corollary 2.7.

Moreover, if $\alpha_J^* = \alpha_T^*$ for some $J \neq T$ in I, then $\mathrm{ord}_v(\alpha_J - \alpha_T) = \mathrm{ord}_v(\alpha_J - \alpha_J^* + \alpha_T^* - \alpha_T) > H \geq R$. But $\mathrm{ord}_v(\alpha_J - \alpha_T) \leq R$ by hypothesis. Thus the α_j^* are distinct. The ultrametric inequality implies that $\{\alpha_j^*\}_{j \in I}$ form a union of regular sequences in $\mathcal{O}_v$ attached to $I_1, \cdots, I_d$. $\square$

Recalling that $I^\sharp[T_1]$ is a set of T_1 consecutive integers in I^0, $I^0 \setminus I^\sharp[T_1]$ consists of at most two sequences of consecutive integers. Let $I^\sharp[T_1] := \{J_1, \cdots, J_{T_1}\}$ and put $T_2 := \#(I^0)$. List all the elements of $I^0 \setminus I^\sharp[T_1]$ in increasing order as $\{J_{T_1+1}, J_{T_1+2}, \cdots, J_{T_2}\}$ and put $I^\sharp[k] := I^\sharp[T_1] \cup \{J_{T_1+1}, \cdots, J_k\}$, for each $k = T_1+1, \cdots, T_2$. Then we have $I^\sharp[T_2] = I^0$. We want to apply the Refined Patching Lemma using at most 3 disjoint regular sequences in $\mathcal{O}_v$ to move the roots of $\widetilde{Q}_w^{(0,T_1)}(z)$ at most the distance $q_v^{-A_3 \log_v(n)}$.

For each $k = T_1+1, \cdots, T_2 - 1$, put

$$\widetilde{Q}_w^{(0,k)}(z) := \prod_{j \in I^0 \setminus I^\sharp[k]} (g_w(z) - \beta_v(j)), \tag{11.40}$$

and set $\widetilde{Q}_w^{(0,T_2)}(z) = 1$. We see that $\widetilde{Q}_w^{(0,k-1)}(z) = (g_w(z) - \beta_v(J_k)) \cdot \widetilde{Q}_w^{(0,k)}(z)$ for each $k = T_1+1, \cdots, T_2$. The patching process in this step to construct $\varphi_w^{(k)}(z)$, $G_w^{(k)}(z)$, for each $k = T_1+1, \cdots, T_2$, is exactly the same as in the previous step. We then have the K_v rational function

$$\begin{aligned} G_w^{(k)}(z) &= G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot Q_{w,T_0}(z) \cdot \widetilde{Q}_w^{(0,k)}(z) \\ &= Q_{w,T_0}(z) \cdot (Q_w^{(0,k)}(z) + \varphi_w^{(k)}(z)) \cdot \widetilde{Q}_w^{(0,k)}(z), \end{aligned}$$

which is independent of $w|v$, where $Q_w^{(0,k)}(z) := Q_w^{(k-1)}(z) \cdot (g_w(z) - \beta_v(J_k))$. To apply the Refined Patching Lemma, take $d = 3$, $R = \log_v(n)$, and $H = A_3 \log_v(n)$. Let

$\mathfrak{Q}^{(0,k)}_{w,t}(z), \mathfrak{Q}^{(k)}_{w,t}(z), \xi^{(k)}_{w,t}(z)$ be the power series gotten by composing $Q^{(0,k)}_w(z), Q^{(k)}_w(z), G^{(k)}_w(z)$ with λ_t, respectively. By induction, the roots of $\mathfrak{Q}^{(0,k)}_{w,t}(z)$ is a union of at most 3 disjoint regular sequences in $\mathcal{O}_v$ of lengths $< n$ attached to $I^\sharp[k]$ by the automorphisim τ_i on $B(0,1)$, so the roots $\{\alpha^*_{t\ell}\}_{\ell\in I^\sharp[k]}$ of $\mathfrak{Q}^{(k)}_{w,t}(z)$ form a union of at most 3 disjoint regular sequences in $\mathcal{O}_v$ of lengths $< n$ attached to $I^\sharp[k]$ by the automorphisim τ_i on $B(0,1)$, with $\mathrm{ord}_v(\alpha^*_{t\ell} - \alpha_{t\ell}) > A_3 \log_v(n)$.

To summarize this step, the roots of $G^{(T_2)}_w(z) = Q_{w,T_0}(z)\cdot Q^{(T_2)}_w(z)\cdot \widetilde{Q}^{(0,T_2)}_w(z)$ are $\{\lambda_t(\alpha^*_{tj}) = \theta^*_{tj}\}_{j\in I[T_0]}$, $\{\lambda_t(\alpha^*_{t\ell}) = \theta^*_{t\ell}\}_{\ell\in I^\sharp[T_2]}$, $\{\lambda_t(\alpha_{t\ell}) = \theta_{t\ell}\}_{\ell\in I^0\setminus I^\sharp[T_2]}$, with the following :

(i) the conditions for the set $\{\alpha^*_{tj}\}_{j\in I[T_0]} \cup \{\alpha_{t\ell}\}_{\ell\in I^0\setminus I^\sharp[k]}$ are inherited from the result of Step 2,

(ii) $\mathrm{ord}_v(\alpha^*_{t\ell} - \alpha^*_{th}) < \log_v(T_2+1) < \log_v(n)$ for all $\ell \neq h$ in $I^\sharp[T_2]$,

(iii) $\mathrm{ord}_v(\alpha^*_{t\ell} - \alpha^*_{tj}) = \mathrm{ord}_v(\alpha^*_{t\ell} - \alpha_{t\ell} + \alpha_{t\ell} - \alpha^*_{tj}) = \mathrm{ord}_v(\alpha_{t\ell} - \alpha^*_{tj}) < A_3\log_v(n)$ for all $\ell \in I^\sharp[T_2]$ and $j \in I[T_0]$, and

(iv) $\mathrm{ord}_v(\alpha^*_{t\ell} - \alpha_{th}) = \mathrm{ord}_v(\alpha^*_{t\ell} - \alpha_{t\ell} + \alpha_{t\ell} - \alpha_{th}) = \mathrm{ord}_v(\alpha_{t\ell} - \alpha_{th}) < \log_v(n)$ for all $\ell \in I^\sharp[T_2]$ and $h \in I^0 \setminus I^\sharp[T_2] \subset I^0$.

Therefore, the roots of $G^{(T_2)}_w(z)$ are separated from each other by the distance $q_v^{-A_3\log_v(n)}$. In particular, the roots of $G^{(T_2)}_w(z) = Q_{w,T_0}(z)\cdot Q^{(T_2)}_w(z)$ are $\{\lambda_t(\alpha^*_{tj})\}_{j\in I[T_0]} \cup \{\lambda_t(\alpha^*_{t\ell})\}_{\ell\in I^\sharp[T_2]}$. For simplicity, replace α^*_{tj} by α_{tj} for $j \in I$, since $I = I[T_0] \cup I^0$ and $I^0 = I^\sharp[T_2]$. Then we observe that $\mathrm{ord}_v(\alpha_{tj} - \alpha_{t\ell}) \leq A_3 \log_v(n)$ for all $i = 1, \cdots, m$ and all $j \neq \ell$ in I.

Step 5. Completing the Patching Process

Since $I^0 \setminus I^\sharp[T_1]$ consists of at most two sequences of consecutive integers, we have separated I into at most $T_0 - k_0 + 4$ disjoint sets of consecutive integers. Recalling that $\#(I[T_0]) \leq A_2(\log_v(n))^2$, $T_2 = \#(I^0) \geq n - A_2(\log_v(n))^2$. Let k be any integer between T_2+1 and n. We can inductively patch $G^{(k-1)}_w(z)$ to $G^{(k)}_w(z)$. For each $k = T_2+1, \cdots, n$, put

$$\widetilde{Q}^{(0,k)}_w(z) := 1. \tag{11.41}$$

The patching process in this step to construct $\varphi_w^{(k)}(z)$, $G_w^{(k)}(z)$, for each $k = T_2+1, \cdots, n$, is exactly the same as in the previous step. We then have the K_v rational function

$$G_w^{(k)}(z) = G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot \widetilde{Q}_w^{(0,k)}(z),$$

which is independent of $w|v$. To apply the Refined Patching Lemma, since $h_v^{kN} \leq h_v^{T_2 N}$ for all $k \geq T_2$, we must have the condition:

$$\begin{aligned} h_v^{T_2 N} \cdot T_v \cdot B_v &\leq q_v^{-\frac{n}{q_v-1} - ((T_0-k_0+4)-1)\cdot A_3 \log_v(n) - A_3 \log_v(n)} \\ &= q_v^{-\frac{n}{q_v-1} - (T_0-k_0+4)\cdot A_3 \log_v(n)}, \quad \text{i.e.,} \end{aligned}$$

$$T_2 \log_v(h_v^N) + \log_v(B_v) \leq -\frac{n}{q_v-1} - (T_0-k_0+4)\cdot A_3 \log_v(n). \tag{11.42}$$

Since $T_2 \geq n - A_2(\log_v(n))^2$ and $T_0 \leq A_1 \log_v(n)$, we need $(n - A_2(\log_v(n))^2) \cdot \log_v(h_v^N) + \log_v(B_v) \leq -\frac{n}{q_v-1} - (A_1 \log_v(n)+3) \cdot A_3 \log_v(n)$, which is equivalent to $n \cdot (\log_v(h_v^N) + \frac{1}{q_v-1}) \leq A_2(\log_v(n))^2 \cdot \log_v(h_v^N) - \log_v(B_v) - (A_1 \log_v(n)+3) \cdot A_3 \log_v(n)$. Since $\log_v(h_v^N) + \frac{1}{q_v-1} < 0$, there is a constant $A_5 > 0$ such that (1.86) holds true for all

$$n \geq A_5(\log_v(n))^2. \tag{11.43}$$

Then $\varphi_w^{(k)}(z)$ can be expanded as a power series about θ_t converging on $\mathfrak{B}(\theta_t, \rho_t)$ with

$$\|\varphi_w^{(k)}(z)\|_{U_v^0} \leq h_w^{kN} \cdot B_w < h_w^{kN} \cdot T_w \cdot B_w. \tag{11.44}$$

Let $\xi_{w,t}^{(k)}(z)$ be the power series gotten by composing $G_w^{(k)}(z)$ with λ_t. By induction, the roots of $G_w^{(k)}(z)$ is a union of at most $T_0 - k_0 + 4$ disjoint regular sequences in $\mathcal{O}_v$ attached to $I_{0,p^r}, I_{k_0+1}, \cdots, I_{T_0}, I^\sharp[T_1], I^0 \setminus I^\sharp[T_1]$ by the automorphisim τ_i on $B(0,1)$, so the Refined Patching Lemma implies that the roots $\{\alpha_{tj}^*\}_{j\in I}$ of $\xi_{w,t}^{(k)}(z)$ in $B(0,1)$ form a union of regular sequences in $\mathcal{O}_v$ attached to $I_{0,p^r}, I_{k_0+1}, \cdots, I_{T_0}, I^\sharp[T_1], I^0 \setminus I^\sharp[T_1]$, with

$$\mathrm{ord}_v(\alpha_{tj}^* - \alpha_{tj}) > A_3 \log_v(n).$$

Furthermore, we see that

$$\begin{aligned}\operatorname{ord}_v(\alpha^*_{tj}-\alpha^*_{t\ell}) &= \operatorname{ord}_v(\alpha^*_{tj}-\alpha_{tj}+\alpha_{tj}-\alpha_{t\ell}+\alpha_{t\ell}-\alpha^*_{t\ell})\\ &= \operatorname{ord}_v(\alpha_{tj}-\alpha_{t\ell}) \le A_3\log_v(n),\end{aligned}$$

for all $j \neq \ell$ in I, i.e., $\operatorname{ord}_v(\theta^*_{tj}-\theta^*_{t\ell}) \le A_3\log_v(n)-\log_v(\rho_t)$, for all $j \neq \ell$ in I. Therefore, the roots $\{\theta_{tj}\}_{j\in I}$ in $B(\theta_t,\rho_t)$ of $G_w^{(k)}(z)$ are well-separated form each other by the distance $A_3\log_v(n)-\log_v(\rho_t)$. For simplicity, replace α_{tj} by α^*_{tj} for $j \in I$.

Finally, at stage $k = n+1$, we patch the constant term. Given $\Delta_{w,0} \in L_w$ with $|\Delta_{w,0}|_w \le h_w^{nN}$ for each $w|v$, define $G_w^{(n+1)}(z) := G_w^{(n)}(z)+\Delta_{w,0}$. Then $G_w^{(n+1)}(z)$ is rational over K_v and independent of $w|v$ by (vii). Furthermore, since $|\Delta_{w,0}|_w \le h_w^{nN}$, the refined patching lemma gives the same result as $G_w^{(n)}(z)$.

Chapter 12

Global Patching Argument III

In the local patching processes, we have constructed and used the functions $\varphi_w^{(k)}(z)$ for $k = 0, \cdots, n$ to patch the coefficients in each stage. We used the size conditions for each $\Delta_{w,i\ell}$, $\ell = 0, \cdots, nN_i - 1$, to define the functions $\varphi_w^{(k)}(z)$, and the bound conditions for each $\varphi_w^{(k)}(z)$ to carry out the local patching process. So, in this chapter we will construct the explicit functions $\varphi_w^{(k)}(z)$ with the sup norm on U_v^0 less than $h_w^{p^r N}$ for $k = 0$ or $h_w^{p^r N} T_w B_w$ for $k = 1, \cdots, k_0$ or $h_w^k T_w B_w$ for $k = k_0 + 1, \cdots, n$. Using these functions and applying the Very Strong Approximation Theorem, we will construct a K-rational function, which satisfies all the conditions in the main theorem.

Let N be any positive integer divisible by the number N_0 satisfying (7.12) and (7.13) chosen in Section 7.3. In Theorem 8.4, we have constructed a normalized local approximating $(\mathfrak{X}, \vec{s})$-function $g_v(z) \in K_v(z)$ of degree N for each $v \in S_K$ with the following properties:

For each $v \in S$,

$$E_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : g_v(z) \in \mathcal{O}_v\} \subset E_v,$$

$$U_v^0 = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |g_v(z)|_v \leq 1\} \subset U_v,$$

and U_v^0 is the union of N disjoint balls $\mathfrak{B}(\theta_t, \rho_t)$ with each $\theta_t \in K_v$ and $\rho_t \in |K_v^\times|$. Moreover, for each t, $g_v(z)$ defines a K_v-rational $1/\rho_t$-isometry between $\mathfrak{B}(\theta_t, \rho_t)$ and $B(0, 1)$;

For each $v \in S_K \setminus S$, there is the number $1 \leq R_v \in |\mathbb{C}_v^\times|$ such that $R_v^N \in |K_v^\times|$ and

$$U_v^0 := E_v = U_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |g_v(z)|_v \leq R_v^N\}.$$

In particular, the leading coefficients $c_{v,i0}$ of the $g_v(z)$ satisfy the product formula (see (8.18) and (8.19)), i.e., for each $i = 1, \cdots, m$,

$$\frac{1}{N} \sum_{v \in S_K} \log_v(|c_{v,i0}|_v) \cdot \log(q_v) = 0. \tag{12.1}$$

Let n be any sufficiently large integer divisible by Ap^{r_0} (see Section 9.2) satisfying the conditions (11.20), (11.31), (11.37) and (11.43).

For each $v \in S_K$, the local patching process has provided a collection of "degree raising" polynomials of degree n over K_v. They are z^n for $v \in S_K \setminus S$ (or RL-domains) and the Stirling polynomial $S_n(z)$ for $v \in S$ (or compact sets). The same degree-raising polynomials are used in the local patching process. Recall that $L = K(\mathfrak{X})$. For each place $w \in S_L$ with $w|v$, put $g_w(z) := g_v(z)$. We are globally patching the functions $g_w(z)$ by composing with the degree-raising polynomials. Put

$$G_w^{(0)}(z) := \begin{cases} g_w(z)^n & \text{if } w|v \text{ for some } v \in S_K \setminus S \\ S_n(g_w(z)) & \text{if } w|v \text{ for some } v \in S. \end{cases}$$

We can write $G_w^{(0)}(z)$ in terms of the L-basis and the M-basis:

$$\begin{aligned} G_w^{(0)}(z) &= \sum_{i=1}^{m} \sum_{k=0}^{nN_i-1} c_{w,ik}^{(0)} \frac{1}{(z-x_i)^{nN_i-k}} + c_{w,0}^{(0)} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{nn_i} \sum_{\ell=0}^{p^{e_i}-1} b_{w,ij\ell}^{(0)} \frac{z^\ell}{(z-x_i)^{jp^{e_i}}} + b_{w,0}^{(0)}. \end{aligned}$$

It follows from Proposition 7.8 that $c_{w,ik}^{(0)} \in K(x_i)_\mu$, $b_{w,ij\ell}^{(0)} \in K(x_i^{p^{e_i}})_\nu$ and $c_{w,0}^{(0)} = b_{w,0}^{(0)} \in K_v$.

First of all, let's consider the coefficients of the normalized local approximating function $g_w(z)$. Express $g_w(z)$ in terms of the L-basis:

$$g_w(z) = \sum_{i=1}^{m} \sum_{k=0}^{N_i-1} c_{w,ik} \frac{1}{(z-x_i)^{N_i-k}} + c_{w,0}.$$

Note that $c_{w,0}$ belongs to K_v by Proposition 7.8. For any $\sigma \in \mathrm{Gal}(L/K)$, since $g_w(z) = g_v(z) \in K_v(z)$ for all $w|v$, it follows that $\tau_{\sigma,w}(g_w(z)) = g_w(z) = g_{\sigma(w)}(z)$, so $\tau_{\sigma,w}(c_{w,ik}) = c_{w,\sigma(i),k} = c_{\sigma(w),\sigma(i),k}$ for all k.

We have chosen the global patching parameter k_0 and the local patching parameters h_w and δ_w for $w \in S_L$ (see Section 9.2) in such a way that for each $k \geq k_0$,

$$h_w^{kN} < \frac{R_w^{kN}}{T_w B_w}, \tag{12.2}$$

$$h_w^{kN} T_w B_w < q_v^{-\frac{k+1}{q_v-1} - \log_v(k+1)} < q_v^{-\frac{k}{q_v-1} - \log_v(k)}, \tag{12.3}$$

$$\prod_{w \in S_L} \frac{h_w^{Nk}}{T_w \chi_w} > C(L/K, S_K), \tag{12.4}$$

$$\delta_w \leq \frac{h_w^{p^r N} \varepsilon^{k_0 N}}{C_w^{p^r} B_w^{p^r} T_w^{k_0 N} \chi_w^{k_0 N}}. \tag{12.5}$$

12.1 Patching the Leading Coefficients

Recall that m' is the number of Galois orbits $\mathfrak{X}_a \subseteq \mathfrak{X}$. For each $a = 1, \cdots, m'$, fix $x_i \in \mathfrak{X}_a$. Given $\delta_w > 0$, since n is divisible by A, Proposition 8.8 and Remark 8.9 imply that there is a global S_L-unit $u_i \in M^\times$, independent of all $w \in S_L$, such that if $\Delta_{w,i0} = \frac{u_i}{c_{w,i0}^n} - 1 \in M_u$, then $u_i = c_{w,i0}^n + \Delta_{w,i0} \cdot c_{w,i0}^n = c_{w,i0}^n(1 + \Delta_{w,i0})$ and $|\Delta_{w,i0}|_w \leq \delta_w$, which is the same size condition as (i) in each local patching process. Put $\varepsilon_{w,i} := 1 + \Delta_{w,i0}$. Since $\delta_w < 1$, $\varepsilon_{w,i}$ is a unit. Note that $\Delta_{w,i0}$ is independent of $w|v$. Moreover, $\sigma(u_i) = u_{\sigma(i)}$ for each $\sigma \in \mathrm{Gal}(L/K)$. Define the functions

$$\varphi_w^{(0)}(z) := \sum_{i=1}^{m} c_{w,i0}^{p^r} \cdot \Delta_{w,i0} \cdot \frac{1}{(z - x_i)^{p^r N_i}},$$

$$G^{(1)}(z) := \sum_{i=1}^{m} u_i \cdot \frac{1}{(z - x_i)^{n N_i}}. \tag{12.6}$$

Let $G_a^{(1)}(z) = \sum_{x_i \in \mathfrak{X}_a} u_i \cdot \frac{1}{(z-x_i)^{nN_i}}$ be the polar part of $G^{(1)}(z)$ supported on $\mathfrak{X}_a$. Since each N_i is divisible by $[L : K]_\iota$, $\varphi_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$ and $G^{(1)}(z)$ is rational over K with leading coefficient u_i. Since $T_w > 1$, it follows that $\|\varphi_w^{(0)}(z)\|_{U_v^0} \leq C_w^{p^r} \delta_w B_w^{p^r} \leq \frac{h_w^{p^r N} \varepsilon^{k_0 N}}{T_w^{k_0 N} \chi_w^{k_0 N}} < h_w^{p^r N}$. Note that the function $\varphi_w^{(0)}(z)$ satisfies the bound conditions (10.4) and (11.8) in the local patching processes. Put

$$\widetilde{G}_w^{(1)}(z) := \begin{cases} g_w(z)^{n - p^r} & \text{if } w|v \text{ for some } v \in S_K \setminus S \\ \widetilde{Q}_{p^r}(z) & \text{if } w|v \text{ for some } v \in S. \end{cases} \tag{12.7}$$

Define

$$\widehat{G}_w^{(0)}(z) := G_w^{(0)}(z) + \varphi_w^{(0)}(z) \cdot \widetilde{G}_w^{(1)}(z).$$

Then $\widehat{G}_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$ with leading coefficient u_i at each $x_i \in \mathfrak{X}$ and $\widehat{G}_w^{(0)}(z) - G^{(1)}(z) \in K_v(z)$ is independent of $w|v$, having no terms of orders $nN_i - 1, \cdots, nN_i - p^{e_i} + 1$ at each $x_i \in \mathfrak{X}$ by the Gap Principle. Since $\widehat{G}_w^{(0)}(z)$ has the same constant term as $G_w^{(0)}(z)$, we can write

$$\widehat{G}_w^{(0)}(z) = G^{(1)}(z) + \sum_{i=1}^{m} \sum_{k=p^{e_i}}^{nN_i-1} \widehat{c}_{w,ik}^{(0)} \cdot \frac{1}{(z-x_i)^{nN_i-k}} + c_{w,0}^{(0)}.$$

Let $\widehat{G}_{w,a}^{(0)}(z) = G_a^{(1)}(z) + \sum_{x_i \in \mathfrak{X}_a} \sum_{k=p^{e_i}}^{nN_i-1} \widehat{c}_{w,ik}^{(0)} \cdot \frac{1}{(z-x_i)^{nN_i-k}}$ be the polar part of $\widehat{G}_w^{(0)}(z)$ supported on $\mathfrak{X}_a$. If $\widetilde{c}_{w,i\ell}^{(0)}$ is the coefficient of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i} \cdot c_{w,i0}^{-n} \cdot \widehat{G}_{w,a}^{(0)}(z)$, it then follows from (11.17) that for each $\ell = p^{e_i}, \cdots, k_0 N_i - 1$,

$$|\widetilde{c}_{w,i\ell}^{(0)}|_w \leq \frac{h_w^{p^r N} \, \varepsilon^{k_0 N - \ell}}{C_w^{p^r} B_w^{p^r} (T_w \chi_w)^{k_0 N}}. \tag{12.8}$$

12.2 Patching the High-order Coefficients

In this step, we patch the high-order coefficients. Because of the gap principle (Lemma 7.9) and the fact that we have patched the leading coefficients, we have in fact already patched the coefficients of orders $nN_i - 1, \cdots, nN_i - p^{e_i} + 1$ for each $x_i \in \mathfrak{X}$.

First, we will patch the coefficients of orders $nN_i - p^{e_i}, \cdots, (n-1)N_i + 1$ in $\widehat{G}_w^{(0)}(z)$ at each $x_i \in \mathfrak{X}$. If $w \in S_L$ with $w|v$ for some $v \in S_K \setminus S$, then put $\varphi_w^{(1)}(z) := 0$ since $\widehat{G}_w^{(0)}(z) - G^{(1)}(z)$ has no terms of orders $nN_i - 1, \cdots, nN_i - p^r + 1$ by the choice of n. If $w \in S_L$ with $w|v$ for some $v \in S$, then we will construct a function $\varphi_w^{(1)}(z) \in K_v(z)$ to arrange the coefficients to be zero. With the same notations as Step 2 in Section 11.2, since $\widetilde{c}_{w,i,p^{e_i}}^{(0)}$ is the coefficient of $(z-x_i)^{p^{e_i}}$ in $(z-x_i)^{nN_i} \cdot c_{w,i0}^{-n} \cdot \widehat{G}_w^{(0)}(z)$, it follows that $\widetilde{c}_{w,i,p^{e_i}}^{(0)} = \widehat{c}_{w,i,p^{e_i}}^{(0)} / c_{w,i0}^{n}$, which is independent of $w|v$. Put

$$\Delta_{w,i,p^{e_i}} := -\widetilde{c}_{w,i,p^{e_i}}^{(0)}.$$

Fix an orbit $\mathfrak{X}_a$ of $\mathfrak{X}$. Note that $N_i = N_j = N_a$ and $n_i = n_j = n_a$ when $x_i, x_j \in \mathfrak{X}_a$. Noting that $\widehat{G}_w^{(0)}(z) \in K_v(z)$ is independent of $w|v$, we see that for each $\sigma \in \mathrm{Gal}(L/K)$, if $\sigma(x_i) = x_j$, then $\sigma(\Delta_{w,i,p^{e_i}}) = \Delta_{w,j,p^{e_j}}$. It follows from (12.8) that $|\Delta_{w,i,p^{e_i}}|_w \leq \frac{h_w^{p^r N} \varepsilon^{k_0 N - p^{e_i}}}{C_w^{p^r} B_w^{p^r} (T_w \chi_w)^{k_0 N}}$. Since $\Delta_{w,i,p^{e_i}} \in K(x_i)_\mu$ by Proposition 7.8, we can write $\Delta_{w,i,p^{e_i}} = \sum_{\ell=0}^{p^{e_i}-1} \zeta_{u,i,p^{e_i},\ell} \cdot x_i^\ell$ for some $\zeta_{u,i,p^{e_i},\ell} \in K(x_i^{p^{e_i}})_\nu$. Define

$$\begin{aligned}
\varphi_{w,a}^{(1,2)}(z) &:= \sum_{x_i \in \mathfrak{X}_a} \sum_{\ell=0}^{p^{e_i}-1} c_{w,i0}^{p^r} \cdot \zeta_{u,i,p^{e_i},\ell} \cdot \frac{z^\ell}{(z-x_i)^{p^r N_i - p^{e_i}}} \\
&= (z-x_i)^{-p^r N_i} \cdot c_{w,i0}^{p^r} \Big(\sum_{\ell=0}^{p^{e_i}-1} \zeta_{u,i,p^{e_i},\ell}\, z^\ell \cdot (z-x_i)^{p^{e_i}} + \\
&\qquad \text{higher terms in } (z-x_i)^h \text{ with } h \geq p^r N_i \Big) \\
&= (z-x_i)^{-p^r N_i} \cdot c_{w,i0}^{p^r} \Big(\sum_{h=p^{e_i}}^{2p^{e_i}-1} \delta_{w,ih}^{(1,2)}\, (z-x_i)^h + \\
&\qquad \text{higher terms in } (z-x_i)^h \text{ with } h \geq p^r N_i \Big),
\end{aligned}$$

where the $\delta_{w,ih}^{(1,2)}$ are $\mathbb{Z}$-linear combinations of the $\zeta_{u,i,p^{e_i},\ell}$ and the x_i^ℓ. Clearly, $\varphi_{w,a}^{(1,2)}(z) \in M_u(z)$ is independent of $w|v$, so $\varphi_{w,a}^{(1,2)}(z)$ is rational over K_v, with leading coefficient $\sum_{\ell=0}^{p^{e_i}-1} c_{w,i0}^{p^r} \cdot \zeta_{u,i,p^{e_i},\ell} \cdot x_i^\ell = c_{w,i0}^{p^r} \cdot \Delta_{w,i,p^{e_i}}$ at each x_i. It follows from Proposition 7.5 that for each $h = p^{e_i}, \cdots, 2p^{e_i} - 1$,

$$|\delta_{w,ih}^{(1,2)}|_w \leq T_w \cdot |\Delta_{w,i,p^{e_i}}|_w \cdot \chi_w \leq \frac{h_w^{p^r N} \varepsilon^{k_0 N - p^{e_i}}}{C_w^{p^r} B_w^{p^r} (T_w \chi_w)^{k_0 N - 1}} \left(< \frac{h_w^{p^r N}}{C_w^{p^r} B_w^{p^r}} \right). \tag{12.9}$$

Moreover, Proposition 7.5 implies that $\|\varphi_{w,a}^{(1,2)}(z)\|_{U_v^0} \leq C_w^{p^r} \cdot T_w \cdot |\Delta_{w,i,p^{e_i}}|_w \cdot B_w^{p^r} < h_w^{p^r N} \cdot T_w \cdot B_w$.

We use the following modification to patch the coefficient $\widehat{c}_{w,i,p^{e_i}}^{(0)}$:

$$G_{w,a}^{(1,2)}(z) := \widehat{G}_{w,a}^{(0)}(z) + \varphi_{w,a}^{(1,2)}(z) \cdot \widetilde{G}_w^{(1)}(z).$$

Here $G_{w,a}^{(1,2)}(z)$ is rational over K_v and independent of $w|v$. Note that the leading coefficient of $\widehat{G}_{w,a}^{(0)}(z) - G_a^{(1)}(z)$ at each $x_i \in \mathfrak{X}_a$ is $\widehat{c}_{w,i,p^{e_i}}^{(0)} = -\Delta_{w,i,p^{e_i}} \cdot c_{w,i0}^n$ and $\varphi_{w,a}^{(1,2)}(z) \cdot \widetilde{G}_w^{(1)}(z)$ has the leading coefficient $\Delta_{w,i,p^{e_i}} \cdot c_{w,i0}^n$ at each x_i, with no constant term. Hence $G_{w,a}^{(1,2)}(z)$ has no terms of orders $nN_i - 1, \cdots, nN_i - 2p^{e_i} + 1$ at each $x_i \in \mathfrak{X}_a$ by Lemma 7.9. Since $\widehat{G}_{w,a}^{(0)}(z)$

has no constant term, so does $G^{(1,2)}_{w,a}(z)$. Hence we can write

$$G^{(1,2)}_{w,a}(z) = G^{(1)}_a(z) + \sum_{x_i\in\mathfrak{X}_a}\sum_{k=2p^{e_i}}^{nN_i-1} c^{(1,2)}_{w,i,k}\cdot\frac{1}{(z-x_i)^{nN_i-k}}.$$

If $\widetilde{c}^{(1,2)}_{w,i\ell}$ is the coefficient of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i}\cdot c^{-n}_{w,i0}\cdot G^{(1,2)}_{w,a}(z)$, it then follows that $\widetilde{c}^{(1,2)}_{w,i\ell} = \widetilde{c}^{(0)}_{w,i\ell} + \sum_{h=p^{e_i}}^{2p^{e_i}-1}\delta^{(1,2)}_{w,ih}B^{(p^r)}_{w,i,\ell-h}$ for each $\ell = 2p^{e_i},\cdots,k_0N_i-1$, so

$$|\widetilde{c}^{(1,2)}_{w,i\ell}|_w \le \frac{h_w^{p^rN}\ \varepsilon^{k_0N-p^{e_i}-(\ell-p^{e_i})}}{C_w^{p^r}B_w^{p^r}(T_w\chi_w)^{k_0N-1}} = \frac{h_w^{p^rN}\ \varepsilon^{k_0N-\ell}}{C_w^{p^r}B_w^{p^r}(T_w\chi_w)^{k_0N-1}}.$$

For each $d = 3,\cdots,n_i$, we inductively arrange the coefficients of orders $nN_i-(d-1)p^{e_i},\cdots,nN_i-dp^{e_i}+1$ to be zero. Suppose that we have constructed functions $G^{(1,d-1)}_{w,a}(z)\in K_v(z)$, independent of $w|v$, such that it has no terms of orders $nN_i-1,\cdots,nN_i-(d-1)p^{e_i}+1$ at each $x_i\in\mathfrak{X}_a$, with no constant term, and such that in the expansion

$$G^{(1,d-1)}_{w,a}(z) = G^{(1)}_a(z) + \sum_{x_i\in\mathfrak{X}_a}\sum_{k=(d-1)p^{e_i}}^{nN_i-1} c^{(1,d-1)}_{w,i,k}\cdot\frac{1}{(z-x_i)^{nN_i-k}},$$

if $\widetilde{c}^{(1,d-1)}_{w,i\ell}$ is the coefficient of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i}\cdot c^{-n}_{w,i0}\cdot G^{(1,d-1)}_{w,a}(z)$, then it follows that for each $\ell = (d-1)p^{e_i},\cdots,k_0N_i-1$,

$$|\widetilde{c}^{(1,d-1)}_{w,i\ell}|_w \le \frac{h_w^{p^rN}\ \varepsilon^{k_0N-\ell}}{C_w^{p^r}B_w^{p^r}(T_w\chi_w)^{k_0N-(d-2)}}.$$

Put

$$\Delta_{w,i,(d-1)p^{e_i}} := -\widetilde{c}^{(1,d-1)}_{w,i,(d-1)p^{e_i}}.$$

Since $\widetilde{c}^{(d-1)}_{w,i,(d-1)p^{e_i}} = c^{(1,d-1)}_{w,i,(d-1)p^{e_i}}/c^n_{w,i0}$, $\Delta_{w,i,(d-1)p^{e_i}}$ is independent of $w|v$. Writing $\Delta_{w,i,(d-1)p^{e_i}} = \sum_{\ell=0}^{p^{e_i}-1}\zeta_{u,i,(d-1)p^{e_i},\ell}\cdot x_i^\ell$ for some $\zeta_{u,i,(d-1)p^{e_i},\ell}\in K(x_i^{p^{e_i}})_\nu$, put

$$\varphi^{(1,d)}_{w,a}(z) := \sum_{x_i\in\mathfrak{X}_a}\sum_{\ell=0}^{p^{e_i}-1} c^{p^r}_{w,i0}\cdot\zeta_{u,i,(d-1)p^{e_i},\ell}\cdot\frac{z^\ell}{(z-x_i)^{p^rN_i-(d-1)p^{e_i}}}.$$

With the same argument as above, $\varphi^{(1,d)}_{w,a}(z)\in K_v(z)$ is independent of $w|v$ and satisfying $\|\varphi^{(1,d)}_{w,a}(z)\|_{U_v^0} < h_w^{p^rN}\cdot T_w\cdot B_w$. In particular, if $\delta^{(1,d)}_{w,ih}$ is the coefficient of $(z-x_i)^h$ in $c^{-p^r}_{w,i0}(z-x_i)^{p^rN_i}\varphi^{(1,d)}_{w,a}(z)$, then $\delta^{(1,d)}_{w,ih} = 0$ for $h = 0,\cdots,(d-1)p^{e_i}-1, dp^{e_i},\cdots,p^rN_i$ and

$$|\delta^{(1,d)}_{w,ih}|_w \le \frac{h_w^{p^rN}\ \varepsilon^{k_0N-(d-1)p^{e_i}}}{C_w^{p^r}B_w^{p^r}(T_w\chi_w)^{k_0N-(d-1)}}\left(<\frac{h_w^{p^rN}}{C_w^{p^r}B_w^{p^r}}\right) \tag{12.10}$$

for each $h = (d-1)p^{e_i}, \cdots, dp^{e_i} - 1$.

To patch the coefficient $c^{(1,d-1)}_{w,i,(d-1)p^{e_i}}$, define

$$G^{(1,d)}_{w,a}(z) := G^{(1,d-1)}_{w,a}(z) + \varphi^{(1,d)}_{w,a}(z) \cdot \widetilde{G}^{(1)}_w(z).$$

Then $G^{(1,d)}_{w,a}(z)$ is rational over K_v and independent of $w|v$. The leading coefficient of $G^{(1,d-1)}_{w,a}(z) - G^{(1)}_a(z)$ at each x_i is $c^{(1,d-1)}_{w,i,(d-1)p^{e_i}} = -\Delta_{w,i,(d-1)p^{e_i}} \cdot c^n_{w,i0}$ and $\varphi^{(1,d)}_{w,a}(z) \cdot \widetilde{G}^{(1)}_w(z)$ has the leading coefficient $(\sum_{\ell=0}^{p^{e_i}-1} c^{p^r}_{w,i0} \cdot \zeta_{u,i,(d-1)p^{e_i},\ell} \cdot x_i^{\ell}) \cdot c^{n-p^r}_{w,i0} = \Delta_{w,i,(d-1)p^{e_i}} \cdot c^n_{w,i0}$ at each x_i, with no constant term. Hence $G^{(1,d)}_{w,a}(z)$ has no terms of orders $nN_i - 1, \cdots, nN_i - dp^{e_i} + 1$ at each $x_i \in \mathfrak{X}_a$ by Lemma 7.9. Since $G^{(1,d-1)}_{w,a}(z)$ has no constant term, so does $G^{(1,d)}_{w,a}(z)$. Furthermore, if $\widetilde{c}^{(1,d)}_{w,i\ell}$ is the coefficient of $(z - x_i)^{\ell}$ in $(z - x_i)^{nN_i} \cdot c^{-n}_{w,i0} \cdot G^{(1,d)}_{w,a}(z)$, then it follows that $\widetilde{c}^{(1,d)}_{w,i\ell} = \widetilde{c}^{(1,d-1)}_{w,i\ell} + \sum_{h=p^{(d-1)e_i}}^{dp^{e_i}-1} \delta^{(1,d)}_{w,ih} B^{(p^r)}_{w,i,\ell-h}$ for each $\ell = dp^{e_i}, \cdots, k_0N_i - 1$, so

$$|\widetilde{c}^{(1,d)}_{w,i\ell}|_w \leq \frac{h_w^{p^rN} \ \varepsilon^{k_0N-\ell}}{C_w^{p^r} B_w^{p^r} (T_w\chi_w)^{k_0N-(d-1)}}.$$

Now, to complete the argument to construct $\varphi^{(1)}_w(z)$, put

$$\varphi^{(1)}_w(z) := \sum_{a=1}^{m'} \sum_{d=2}^{n_a} \varphi^{(1,d)}_{w,a}(z),$$

$$G^{(1)}_w(z) := \sum_{a=1}^{m'} G^{(1,n_a)}_{w,a}(z) + c^{(0)}_{w,0}.$$

Clearly, $G^{(1)}_w(z) = \widehat{G}^{(0)}_w(z) + \varphi^{(1)}_w(z) \cdot \widetilde{G}^{(1)}_w(z)$ and $\|\varphi^{(1)}_w(z)\|_{U^0_v} < h_w^{p^rN} \cdot T_w \cdot B_w$, which is the same bound condition as (11.11). It follows from (12.9) and (12.10) that the coefficients in the expansion of $\varphi^{(1)}_w(z)$ in terms of the L-basis satisfy the size conditions (ii) in each local patching process. Furthermore, $G^{(1)}_w(z) \in K_v(z)$ is independent of $w|v$ and has no terms of orders $nN_i - 1, \cdots, (n-1)N_i + 1$ at each $x_i \in \mathfrak{X}$, with the same constant term as $G^{(0)}_w(z)$. In particular, $G^{(1)}_w(z) - G^{(1)}(z)$ has a pole of order at most $(n-1)N_i$ at each $x_i \in \mathfrak{X}$.

Next, for each $k = 2, \cdots, k_0$, we will patch the coefficients of orders $(n-(k-1))N_i, \cdots, (n-k)N_i + 1$ for each $x_i \in \mathfrak{X}$. If $w \in S_L$ with $w|v$ for some $v \in S_K \setminus S$, then put $\varphi^{(k)}_w(z) := 0$. If $w \in S_L$ with $w|v$ for some $v \in S$, suppose that we have constructed functions $G^{(k-1)}_w(z) \in K_v(z)$, independent of $w|v$, which have no terms of orders $nN_i - 1, \cdots, (n-(k-1))N_i + 1$ at each $x_i \in \mathfrak{X}$, with the same constant term as $G_w^{(0)}(z)$. For each $d = 1, \cdots, n_i$, we inductively arrange the coefficients of orders $(n-(k-1))N_i - (d-1)p^{e_i}, \cdots, (n-(k-1))N_i - dp^{e_i} + 1$ to be zero. Put

$$G_w^{(k,0)}(z) := G_w^{(k-1)}(z).$$

We carry out the exactly same process to construct $\varphi_{w,a}^{(k,d)}(z)$ and $G_{w,a}^{(k,d)}(z)$, which are rational over K_v and independent of $w|v$. Furthermore, $\|\varphi_{w,a}^{(k,d)}(z)\|_{U_v^0} < h_w^{p^r N} \cdot T_w \cdot B_w$ and $G_{w,a}^{(k,d)}(z) = G_{w,a}^{(k,d-1)}(z) + \varphi_{w,a}^{(k,d)}(z) \cdot \widetilde{G}_w^{(1)}(z)$ has no terms of orders $nN_i - 1, \cdots, (n-(k-1))N_i - dp^{e_i} + 1$ at each $x_i \in \mathfrak{X}$, with no constant term. If $\widetilde{c}_{w,i\ell}^{(k,d)}$ is the coefficient of $(z-x_i)^\ell$ in $(z-x_i)^{nN_i} \cdot c_{w,i0}^{-n} \cdot G_{w,a}^{(k,d)}(z)$, then it follows that for each $\ell = (k-1)N_i + dp^{e_i}, \cdots, k_0 N_i - 1$,

$$|\widetilde{c}_{w,i\ell}^{(k,d)}|_w \le \frac{h_w^{p^r N}\ \varepsilon^{k_0 N - \ell}}{C_w^{p^r} B_w^{p^r} (T_w \chi_w)^{k_0 N - ((k-1)n_i + (d-1))}}.$$

At the end of this stage, put

$$\begin{aligned} \varphi_w^{(k)}(z) &:= \sum_{a=1}^{m'} \sum_{d=1}^{n_a} \varphi_{w,a}^{(k,d)}(z), \\ G_w^{(k)}(z) &:= \sum_{a=1}^{m'} G_{w,a}^{(k,n_a)}(z) + c_{w,0}^{(0)}, \\ G^{(k)}(z) &:= 0. \end{aligned}$$

Clearly, $\|\varphi_w^{(k)}(z)\|_{U_v^0} < h_w^{p^r N} \cdot T_w \cdot B_w$, which is the same bound condition as (11.13). By the inductive construction, the coefficients in the expansion of $\varphi_w^{(k)}(z)$ in terms of the L-basis satisfy the size conditions (ii) in each local patching process. It follows that $G_w^{(k)}(z) = G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot \widetilde{G}_w^{(1)}(z) \in K_v(z)$ is independent of $w|v$ and has no terms of orders $nN_i - 1, \cdots, (n-k)N_i + 1$ at each $x_i \in \mathfrak{X}$, with the same constant term as $G_w^{(0)}(z)$. Furthermore, $G_w^{(k)}(z) - (G^{(1)}(z) + \cdots + G^{(k)}(z))$ has a pole of order at most $(n-k)N_i$ at each $x_i \in \mathfrak{X}$.

Note that for each $w \in S_L$ and each $k = 1, \cdots, k_0$, the function $\varphi_w^{(k)}(z)$ satisfies the bound condition in each local patching process. Therefore, we can carry on the global patching process further.

12.3 Patching the Low-order Coefficients

We will carry out the patching process for $k = k_0 + 1, \cdots, n$, inductively. In each local patching process, we have arranged the low-order coefficients so that the resulting functions $G_w^{(k)}(z)$ are rational over K_v. So far, we have constructed the K-rational functions $G^{(1)}(z), \cdots, G^{(k_0)}(z)$ after arranging the high-order coefficients to be zero. In this section, we will arrange the low-order coefficients to generate K-rational functions $G^{(k)}(z)$ for $k = k_0 + 1, \cdots, n+1$, independent of $w|v$, so that at the final stage $G_w^{(n+1)}(z) = G^{(n+1)}(z)$ for all $w \in S_L$. Let $\mathfrak{X}_a$, $a = 1, \cdots, m'$, be the orbits of $\mathfrak{X}$ under $\mathrm{Gal}(\overline{K}/K)$. In each stage, we will patch the coefficients for each orbit $\mathfrak{X}_a$ of $\mathfrak{X}$ as a packet, maintaining the K-rationality (see Proposition 7.8) by applying the Very Strong Approximation Theorem and then put all the resulting functions together.

With the same notations as in each local patching process, put

$$\widetilde{G}_w^{(k)}(z) = \begin{cases} g_w(z)^{n-k} & \text{if } w|v \text{ for some } v \in S_K \setminus S \\ \prod_{\ell=k_0}^{k-1} Q_w^{(\ell)}(z) \cdot \widetilde{Q}_w^{(k)}(z) & \text{if } k = k_0+1, \cdots, T_0 \text{ and } w|v \text{ for some } v \in S \\ Q_{w,T_0}(z) \cdot \widetilde{Q}_w^{(0,k)}(z) & \text{if } k = T_0+1, \cdots, T_2 \text{ and } w|v \text{ for some } v \in S \\ \widetilde{Q}_w^{(0,k)}(z) & \text{if } k = T_2+1, \cdots, n \text{ and } w|v \text{ for some } v \in S \end{cases} \tag{12.11}$$

Note that $\widetilde{G}_w^{(k)}(z)$ is rational over K_v and independent of $w|v$.

We can write $G_w^{(k_0)}(z)$ in terms of the L basis;

$$G_w^{(k_0)}(z) = G^{(1)}(z) + \cdots + G^{(k)}(z) + \sum_{a=1}^{m'} \sum_{x_i \in \mathfrak{X}_a} \sum_{k=k_0 N_i}^{nN_i - 1} c_{w,ik}^{(k_0)} \cdot \frac{1}{(z-x_i)^{nN_i - k}} + c_{w,0}^{(0)},$$

with $c_{w,ik}^{(k_0)} \in K(x_i)_\mu$, independent of $w|\mu$. Letting $G_{w,a}^{(k_0)}(z) = G_a^{(1)}(z) + \sum_{x_i \in \mathfrak{X}_a} \sum_{k=k_0 N_i}^{nN_i-1} c_{w,ik}^{(k_0)} \cdot \frac{1}{(z-x_i)^{nN_i-k}}$, it follows that $G_w^{(k_0)}(z) = \sum_{a=1}^{m'} G_{w,a}^{(k_0)}(z) + c_{w,0}^{(0)}$.

We are patching each block of p^{e_i} consecutive coefficients at the same time, for each $x_i \in \mathfrak{X}$, and for each $k = k_0 + 1, \cdots, n$. Fix $a \in \{1, \cdots, m'\}$. Before giving the details of the patching process, we describe an abstract formulation of the process. Recall that $N_i = n_i p^{e_i}$.

For each $b = 1, \cdots, n_i$, let $\psi_{w,a}^{(k,b-1)}(z)$ be the function obtained in the previous step of the patching process, having poles supported only on $\mathfrak{X}_a$. In fact, $\psi_{w,a}^{(k_0+1,0)}(z) = G_{w,a}^{(k_0)}(z) - G_a^{(1)}(z)$ when $k = k_0 + 1$ and $b = 1$.

Theorem 12.1 *For each $k = k_0 + 1, \cdots, n$ and each $b = 1, \cdots, n_i$, suppose that for each $w \in S_L$, $\psi_{w,a}^{(k,b-1)}(z)$ is rational over K_v and independent of $w|v$, having poles supported on $\mathfrak{X}_a$. Let $\psi_{w,a}^{(k,b-1)}(z)$ have a pole of order $(n-(k-1))N_i - (b-1)p^{e_i}$ at each $x_i \in \mathfrak{X}_a$. We can patch the first p^{e_i} coefficients of the function $\psi_{w,a}^{(k,b-1)}(z)$ at the same time for all $x_i \in \mathfrak{X}_a$, preserving the K_v-rationality as follows:*

there are functions $\varphi_{w,a}^{(k,b)}(z) \in K_v(z)$ and $G_a^{(k,b)}(z) \in K(z)$, having poles of order $N_i - (b-1)p^{e_i}$ and $(n-(k-1))N_i - (b-1)p^{e_i}$, respectively, at $x_i \in \mathfrak{X}_a$, and no other poles, which are independent of all $w|v$. Moreover,

(i) $\|\varphi_{w,a}^{(k,b)}(z)\|_{U_v^0} < h_w^{kN} \cdot T_w \cdot B_w$;

(ii) if $\psi_{w,a}^{(k,b)}(z) = \psi_{w,a}^{(k,b-1)}(z) + \varphi_{w,a}^{(k,b)}(z) \cdot \widetilde{G}_w^{(k)}(z)$, then $\psi_{w,a}^{(k,b)}(z) \in K_v(z)$ is independent of all $w|v$, with the same constant term as $\psi_{w,a}^{(k,b-1)}(z)$; and

(iii) $\psi_{w,a}^{(k,b)}(z) - G_a^{(k,b)}(z)$ has poles supported on $\mathfrak{X}_a$ and a pole of order $(n-(k-1))N_i - bp^{e_i}$ at each $x_i \in \mathfrak{X}_a$.

Proof: First, we decompose $\psi_{w,a}^{(k,b-1)}(z)$ into its polar parts:

$$\psi_{w,a}^{(k,b-1)}(z) = \sum_{x_i \in \mathfrak{X}_a} \sum_{h=(k-1)N_i+(b-1)p^{e_i}}^{nN_i-1} c_{w,ih}^{(k,b-1)} \cdot \frac{1}{(z-x_i)^{nN_i-h}},$$

where $c_{w,ih}^{(k,b-1)} \in K(x_i)_\mu$, independent of $w|\mu$.

We need to patch the coefficient $c_{w,i,(k-1)N_i+(b-1)p^{e_i}}^{(k,b-1)}$ for all $w|v$ and each $v \in S_K$. Fix $x_i \in \mathfrak{X}_a$ and consider the set $\{c_{w,i,(k-1)N_i+(b-1)p^{e_i}}^{(k,b-1)} : w \in S_L\}$ to apply the Very Strong Approximation Theorem. Put

$$D_w := \frac{h_w^{Nk}}{T_w \chi_w} \cdot |c_{w,i0}^{n-k}|_w.$$

It follows from (12.1) and (12.4) that

$$\prod_{w \in S_L} D_w = \prod_{w \in S_L} \frac{h_w^{Nk}}{T_w \chi_w} > C(L/K, S_K),$$

and hence there exists $c_{i,(k-1)N_i+(b-1)p^{e_i}} \in K(x_i)$ such that

$$\begin{cases} |c_{i,(k-1)N_i+(b-1)p^{e_i}} - c^{(k,b-1)}_{w,i,(k-1)N_i+(b-1)p^{e_i}}|_w \leq D_w \quad \text{for all } w \in S_L, \\ |c_{i,(k-1)N_i+(b-1)p^{e_i}}|_w \leq 1 \quad \text{for all } w \notin S_L. \end{cases}$$

Put

$$\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}} := (c_{i,(k-1)N_i+(b-1)p^{e_i}} - c^{(k,b-1)}_{w,i,(k-1)N_i+(b-1)p^{e_i}})/c^{n-k}_{w,i0},$$

which belongs to $K(x_i)_\mu$, independent of all $w|\mu$. It follows that

$$\left|\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}}\right|_w \leq D_w/|c^{n-k}_{w,i0}|_w = \frac{h_w^{Nk}}{T_w\chi_w}.$$

In order to achieve Galois-invariance, for each $x_j \in \mathfrak{X}_a$ with $j \neq i$, choose $\sigma \in \mathrm{Gal}(L/K)$ so that $\sigma(x_i) = x_j$ and put $c_{j,(k-1)N_j+(b-1)p^{e_j}} := \sigma(c_{i,(k-1)N_i+(b-1)p^{e_i}})$. We define $\Delta_{w,j,(k-1)N_j+(b-1)p^{e_j}}$ for each $w|v$ as follows:
Writing $\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}} = \sum_{\ell=0}^{p^{e_i}-1} \zeta_{u,i,(k-1)N_i+(b-1)p^{e_i},\ell} \cdot x_i^\ell$ for some $\zeta_{u,i,(k-1)N_i+(b-1)p^{e_i},\ell} \in K(x_i^{p^{e_i}})_\nu$,

$$\begin{aligned} \Delta_{\sigma(w),j,(k-1)N_j+(b-1)p^{e_j}} &:= \tau_{w,\sigma}(\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}}) \\ &= \sum_{\ell=0}^{p^{e_j}-1} \tau_{w,\sigma}(\zeta_{u,i,(k-1)N_i+(b-1)p^{e_i},\ell}) \cdot x_j^\ell \\ &=: \sum_{\ell=0}^{p^{e_j}-1} \zeta_{\sigma(u),j,(k-1)N_j+(b-1)p^{e_j},\ell} \cdot x_j^\ell. \end{aligned}$$

It follows that $\zeta_{\sigma(u),j,(k-1)N_j+(b-1)p^{e_j},\ell}$ belongs to $K(x_j^{p^{e_j}})_\nu$ and

$$\begin{aligned} \oplus_{w|v}\Delta_{w,j,(k-1)N_j+(b-1)p^{e_j}} &= \oplus_{\sigma(w)|v}\Delta_{\sigma(w),j,(k-1)N_j+(b-1)p^{e_j}} \\ &= \oplus_{\sigma(w)|v}\tau_{\sigma,w}(\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}}) \\ &= \sigma(\oplus_{w|v}\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}}). \end{aligned}$$

Clearly, $\oplus_{w|v}\Delta_{w,j,(k-1)N_j+(b-1)p^{e_j}}$ belongs to $\sigma(K(x_i)) \otimes_K K_v = K(x_j) \otimes_K K_v \subset L \otimes_K K_v$ because $\oplus_{w|v}\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}}$ belongs to $K(x_i) \otimes_K K_v \subset L \otimes_K K_v$ for each $v \in S_K$. For any other $\rho \in \mathrm{Gal}(L/K)$ with $\rho(x_i) = x_j$, since $\sigma(x_i) = \rho(x_i) = x_j$ and $\sigma|_{K_v} = \rho|_{K_v}$, $\oplus_{w|v}\Delta_{w,j,(k-1)N_j+(b-1)p^{e_j}}$ is well-defined and independent of the choice of σ. Put

$$\varphi_{w,a}^{(k,b)}(z) := \sum_{x_i\in\mathfrak{X}_a}\sum_{\ell=0}^{p^{e_i}-1}\zeta_{u,i,(k-1)N_i+(b-1)p^{e_i},\ell}\cdot\frac{z^\ell}{(z-x_i)^{N_i-(b-1)p^{e_i}}}.$$

Clearly, the coefficients in the expansion of $\varphi_{w,a}^{(k,b)}(z)$ in terms of L-basis have absolute value less than equal to $T_w \cdot |\Delta_{w,i,(k-1)N_i+(b-1)p^{e_i}}|_w \cdot \chi_w \leq h_w^{kN}$, which meets the size conditions (iii) in the local patching precesses. By Proposition 7.8, $\varphi_{w,a}^{(k,b)}(z)$ is rational over K_v and independent of $w|v$, with sup norm $\|\varphi_{w,a}^{(k,b)}(z)\|_{U_v^0} \leq T_w \cdot \frac{h_w^{kN}}{T_w\chi_w} \cdot B_w < h_w^{kN} \cdot T_w \cdot B_w$.

We now consider

$$\psi_{w,a}^{(k,b)}(z) := \psi_w^{(k,b-1)}(z) + \varphi_{w,a}^{(k,b)}(z)\cdot\widetilde{G}_w^{(k)}(z).$$

Note that $\psi_{w,a}^{(k,b)}(z)$ has the same constant term as $\psi_{w,a}^{(k,b-1)}(z)$ since $\varphi_{w,a}^{(k,b)}(z)$ has no constant term. In particular, the leading coefficient of $\psi_{w,a}^{(k,b)}(z)$ at each $x_i \in \mathfrak{X}_a$ is $c_{w,i,(k-1)N_i+(b-1)p^{e_i}}^{(k,b-1)} + \left(\sum_{\ell=0}^{p^{e_i}-1}\zeta_{u,i,(k-1)N_i+(b-1)p^{e_i},\ell}\cdot x_i^\ell\right)\cdot c_{w,i0}^{n-k} = c_{i,(k-1)N_i+(b-1)p^{e_i}}$. Write $c_{i,(k-1)N_i+(b-1)p^{e_i}} = \sum_{\ell=0}^{p^{e_i}-1} d_{i,(k-1)N_i+(b-1)p^{e_i}}\cdot x_i^\ell$ for some $d_{i,(k-1)N_i+(b-1)p^{e_i}} \in K(x_i^{p^{e_i}})$. Since the leading coefficient of $\psi_{w,a}^{(k,b)}(z)$ occurs at the order $(n-(k-1))N_i - (b-1)p^{e_i}$ at each $x_i \in \mathfrak{X}_a$, if we put

$$G_a^{(k,b)}(z) := \sum_{x_i\in\mathfrak{X}_a}\sum_{\ell=0}^{p^{e_i}-1} d_{i,(k-1)N_i+(b-1)p^{e_i}}\cdot\frac{z^\ell}{(z-x_i)^{(n-(k-1))N_i-(b-1)p^{e_i}}},$$

it then follows that $G_a^{(k,b)}(z)$ is rational over M and independent of $w|v$, so $G_a^{(k,b)}(z)$ is rational over K and independent of $w|v$ by Proposition 7.8. Since $\psi_{w,a}^{(k,b)}(z)$ and $G_a^{(k,b)}(z)$ have the same leading coefficient of order $(n-(k-1))N_i-(b-1)p^{e_i}$ at each $x_i \in \mathfrak{X}_a$, Lemma 7.9 implies that $\psi_{w,a}^{(k,b)}(z) - G_a^{(k,b)}(z)$ has order $(n-(k-1))N_i - bp^{e_i}$ at each $x_i \in \mathfrak{X}_a$. Clearly, $\psi_{w,a}^{(k,b)}(z) - G_a^{(k,b)}(z)$ is rational over K_v and independent of $w|v$ for each $v \in S_K$, and hence its leading coefficient of order $(n-(k-1))N_i - bp^{e_i}$ at each $x_i \in \mathfrak{X}_a$ belongs to $K(x_i)_\mu$ by Proposition 7.8. $\square$

Now, for each $k = k_0+1, \cdots, n$ and each $b = 1, \cdots, n_i$, we inductively apply Theorem 12.1 by setting $\psi_{w,a}^{(k,0)}(z) := G_{w,a}^{(k)}(z) - (G_a^{(1)}(z) + \cdots + G_a^{(k)}(z))$, where $G_a^{(k)}(z) = 0$ for

$k = 2, \cdots, k_0$. Suppose that we have constructed $\psi_{w,a}^{(k,b-1)}(z)$, which is rational over K_v and independent of $w|v$ for each $v \in S_K$, with leading coefficient $c_{w,i,(k-1)N_i+(b-1)p^{e_i}}^{(k,b-1)} \in K(x_i)_\mu$ of order $(n-(k-1))N_i-(b-1)p^{e_i}$ at each $x_i \in \mathfrak{X}_a$. Apply Theorem 12.1 to patch the coefficient $c_{w,i,(k-1)N_i+(b-1)p^{e_i}}^{(k,b-1)}$ for all $w|v$ and each $v \in S_K$. It follows that there are functions $\varphi_{w,a}^{(k,b)}(z) \in K_v(z)$ and $G_a^{(k,b)}(z) \in K(z)$, independent of $w|v$, such that $\|\varphi_{w,a}^{(k,b)}(z)\|_{U_v^0} < h_w^{kN} \cdot T_w \cdot B_w$, $\psi_{w,a}^{(k,b)}(z) := \psi_{w,a}^{(k,b-1)}(z) + \varphi_{w,a}^{(k,b)}(z) \cdot \widetilde{G}_w^{(k)}(z)$ has poles of order $(n-(k-1))N_i-(b-1)p^{e_i}$ supported only at $x_i \in \mathfrak{X}_a$, with the same constant term as $\psi_{w,a}^{(k,b-1)}(z)$, and $\psi_{w,a}^{(k,b)}(z) - G_a^{(k,b)}(z)$ has a pole of order $(n-(k-1))N_i - bp^{e_i}$ only at each $x_i \in \mathfrak{X}_a$. Note that $\psi_{w,a}^{(k,b)}(z) - G_a^{(k,b)}(z)$ is rational over K_v and independent of $w|v$ for each $v \in S_K$.

To finish the k^{th} stage, put

$$\varphi_w^{(k)}(z) := \sum_{a=1}^{m'} \sum_{b=1}^{n_i} \varphi_{w,a}^{(k,b)}(z),$$
$$G^{(k)}(z) := \sum_{a=1}^{m'} \sum_{b=1}^{n_i} G_a^{(k,b)}(z),$$
$$G_w^{(k)}(z) := G_w^{(k-1)}(z) + \varphi_w^{(k)}(z) \cdot \widetilde{G}_w^{(k)}(z).$$

Then $\varphi_w^{(k)}(z)$ and $G_w^{(k)}(z)$ are rational over K_v and independent of $w|v$. In particular, $G^{(k)}(z)$ is rational over K and independent of $w|v$. By the proof of Theorem 12.1, the coefficients in the expansion of $\varphi_w^{(k)}(z)$ in terms of L-basis have absolute value less than equal to h_w^{kN}, which meets the size conditions (iii) in the local patching precesses. Furthermore, $\|\varphi_w^{(k)}(z)\|_{U_v^0} < h_w^{kN} \cdot T_w \cdot B_w$, which is the same bound condition as in each local patching process, and $G_w^{(k)}(z)$ has the same constant term $c_{w,0}^{(0)}$ as $G_w^{(k-1)}(z)$. Note that $G_w^{(k)}(z) - (G^{(1)}(z) + \cdots + G^{(k)}(z))$ has a pole of order $(n-k)N_i$ at each $x_i \in \mathfrak{X}$ and $G_w^{(n)}(z) - (G^{(1)}(z) + \cdots + G^{(n)}(z))$ is the constant term $c_{w,0}^{(0)}$, which belongs to K_v and is independent of $w|v$.

Finally, at stage $k = n+1$, we patch the constant terms $c_{w,0}^{(0)}$. Note that $c_{w,0}^{(0)} \in K_v$ for all $w|v$. Applying the Very Strong Approximation Theorem to the $c_{w,0}^{(0)}$ and to the $D_w := h_w^{nN} >$

$\frac{h_w^{nN}}{T_w \chi_w}$, there exists $c_0 \in K$ such that

$$\begin{cases} |c_0 - c_{w,0}^{(0)}|_w \leq D_w \quad \text{for all } w \in S_L, \\ |c_0|_w \leq 1 \quad \text{for all } w \notin S_L. \end{cases}$$

Put

$$\Delta_{w,0} := c_0 - c_{w,0}^{(0)}, \tag{12.12}$$

which belongs to K_v, independent of $w|v$. It follows that for each $w \in S_L$,

$$|\Delta_{w,0}|_w \leq D_w = h_w^{nN}.$$

Define $G_w^{(n+1)}(z) = G_w^{(n)}(z) + \Delta_{w,0}$ to patch the constant term. In particular, $G_w^{(n+1)}(z)$ is rational over K and independent of $w|v$ because $G_w^{(n+1)}(z) = G^{(1)}(z) + \sum_{h=k_0+1}^{n} G^{(h)}(z) + c_0$. Repalce $G_w^{(n+1)}(z)$ by $G^{(n+1)}(z)$.

12.4 Conclusion of the Global Patching Process

The patching process has now arranged the K-rational function $G^{(n+1)}(z)$, independent of all $w \in S_L$. Furthermore, the local patching process guarantees that all the zeros of $G^{(n+1)}(z)$ belong to U_v^0 for all $v \in S_K$. Expand $G^{(n+1)}(z)$ in terms of L-basis:

$$G^{(n+1)}(z) = \sum_{i=1}^{m} \sum_{k=0}^{nN_i-1} c_{ik} \frac{1}{(z - x_i)^{nN_i-k}} + c_0.$$

For each $v \notin S_K$, let $w|\mu|\nu|v$ be places of L, $K(x_i)$ and $K(x_i^{p^{e_i}})$ lying above v, respectively. We have defined the K-rational functions $G^{(h)}(z)$, for $h = k_0 + 1, \cdots, n$, with the numbers $c_{w,i,(h-1)N_i}, c_{w,i,(h-1)N_i+p^{e_i}}, \cdots, c_{w,i,(h-1)N_i+(n_i-1)p^{e_i}}$ in $\mathcal{O}_w \cap K(x_i) \subset \mathcal{O}_\mu$, for each $x_i \in \mathfrak{X}$. Note that by the choice of S_K, $\mathcal{O}_\mu = \mathcal{O}_\nu[1, x_i, \cdots, x^{p^{e_i}}]$ for each x_i. For each $b = 1, \cdots, n_i$, since $c_{w,i,(h-1)N_i+(b-1)p^{e_i}} = \sum_{\ell=0}^{p^{e_i}-1} d_{i,\ell}^{(h,b)} \cdot x_i^\ell$ for some $d_{i,\ell}^{(h,b)} \in K(x_i^{p^{e_i}})$, the $d_{i,\ell}^{(h,b)}$ must belong to $\mathcal{O}_\nu$. Hence if k is in the range $(n-(h-1))N_i, \cdots, (n-h)N_i + 1$, then the coefficients c_{ik} belong to $\mathcal{O}_\nu \subset \mathcal{O}_w$ because they are combinations of the $d_{i,\ell}^{(h,b)}$ and the x_i^ℓ over $\mathbb{Z}$.

On the other hand, since the $u_i \in M^\times$ are S_L-units and $c_0 \in K$ with $|c_0|_w \leq 1$ for each $w \notin S_L$, it follows that $u_i \in \mathcal{O}_w^\times$ and $c_0 \in \mathcal{O}_w$. Thus, all the coefficients in the expansion of $G^{(n+1)}(z)$ belong to $\mathcal{O}_w$ and the leading coefficients belong to $\mathcal{O}_w^\times$. Note that the functions $1/(z - x_i)$ for each $x_i \in \mathfrak{X}$ have good reduction at each $w \notin S_L$ and the points in $\mathfrak{X}$ specialize to distinct points (mod w). Hence $G^{(n+1)}(z)$ (mod w) reduces to a nonconstant function with a pole of order $nN_i > 0$ at each $x_i \in \mathfrak{X}$. This implies that for each $v \notin S_K$,

$$\{z \in \mathbb{P}^1(\mathbb{C}_v) : |G^{(n+1)}(z)|_v \leq 1\} = \mathbb{P}^1(\mathbb{C}_v) \setminus \cup_{i=1}^m \mathfrak{B}(x_i, 1)^- = U_v = E_v.$$

In particular, all the zeros of $G^{(n+1)}(z)$ belong to U_v for each $v \notin S_K$.

Finally, for each $v \in S$, all the zeros of $G^{(n+1)}(z)$ are distinct by our construction. Since n can be any sufficiently large integer divisible by Ap^{r_0} (see Section 9.2), satisfying the conditions (11.20), (11.31), (11.37) and (11.43), we obtain infinitely many numbers satisfying the conditions of the main theorem. For each $v \notin S$, consider the zeros of $G^{(n+1)}(z)^\ell - 1$ for $\ell = 1, 2, \cdots$. Noting that $\{z \in \mathbb{P}^1(\mathbb{C}_v) : |G^{(n+1)}(z)|_v \leq 1\} \subseteq U_v$ for each $v \notin S$, the zeros of $G^{(n+1)}(z)^\ell - 1$ belong to U_v. If $\gcd(\ell, p) = 1$, then there must be infinitely many distinct roots of $G^{(n+1)}(z)^\ell - 1$ as n and ℓ vary. These numbers satisfy the conditions of the main theorem.

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