A Fitch-Style Derivation System for Hybrid Logic

by

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(Under the Direction of Charles Cross)

ABSTRACT

Standard modal logic is typically parsed as concerning possible worlds. However there is no way for normal modal logic to refer to those possible worlds directly, which hampers its expressiveness. Hybrid logic develops tools to talk about particular possible worlds and so avoids the curtailed expressive power modal logic possesses. There are several proof systems for hybrid logic. Here I develop one in Fitch-style. This type of proof system naturally fits the goals of hybrid logic because of its clarity and resemblance to our natural process of reasoning. I prove this newly developed system is equivalent to established systems for building proofs in hybrid logic. Finally I examine what traits we should look for in a proof system.

INDEX WORDS: Hybrid Logic, Modal Logic, Fitch-style Natural Deduction

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# Chapter 1

# Introduction

The subject matter of modal logic is the mode of statements. Typically that means a necessity or possibility claim. "Bachelors are necessarily unmarried," and "It is possible that there are green swans," are examples of modal statements. Strictly speaking though, necessity and possibility are the priority of the alethic logic branch of modal logic. Modal logic covers other, related, types of qualifiers. Doxastic logic is a branch of modal logic that deals with the logic of belief. Just as Doxastic logic covers belief, Epistemic logic deals with knowledge claims. Alethic, Doxastic, and Epistemic are just three categories of modal logic. What connects these and all the modal logics together is that the truth of a modal statement cannot be determined with only a truth-value assignment (TVA).[1] For a statement of sentential logic such as (1), we can construct a TVA to determine its truth-value.

(1) Barack and Michelle play basketball.

Let "$K$" be "Barack plays basketball." and "$M$" be "Michelle plays basketball." Then our example sentence could be formalized as "$K\&M$" where "&" means "and" as usual.

[1] A truth-value assignment assigns a value of true or false to the atomic sentences of a logical language

With just the TVA, we can prove that given a set of sentences including "$K$" and "$M$," the sentence "$K\&M$" follows. Or as it might be stated in the formal language, $\{K,M\} \vDash K\&M$.

> **Proof:** Suppose for reductio that $\{K,M\} \nvDash K\&M$.[2] Hence there is some TVA, $\mathscr{A}_0$, on which $K\&M$ is false but $K$ is true and $M$ is true. For a conjunction to be false on a TVA means at least one of the conjuncts is false. So if $K\&M$ is false on $\mathscr{A}_0$, then either $K$ is false on $\mathscr{A}_0$ or $M$ is false on $\mathscr{A}_0$. In either case, there is an immediate contradiction. Hence $\{K,M\} \vDash K\&M$.

We can apply TVAs to any sentential logic formula, but they are insufficient for modal statements. Consider what happens in (2) when we turn (1) into a modal claim.

(2) Barack and Michelle *played* basketball.

Let "$K$" = "Barack played basketball." and "$M$" = "Michelle played basketball." Then the new example sentence maintains the same form, "$K\&M$." In the case of temporal logic, a statement is true at some time, but not at others. But the TVA for our sentence is constant even though the truth-value of (2) depends on whether it is uttered before or after the President's basketball game. Necessity and possibility claims bring about the same problem. But whereas tense logic has moments of time by which to evaluate a sentence's truth or falsity, nothing apparent leaps out to judge the truth-value of necessity and possibility statements. It is possible, however, by introducing possible worlds semantics.

Possible worlds are hardly a new notion in philosophy. Leibniz secured their place in the canon with his "best of all possible worlds" argument. But the history goes back even further to the middle ages in the work of Duns Scotus and his contemporaries.[3]

[2] The proof proceeds by reductio. The idea behind a reductio proof is to show some statement is true because its negation leads to a contradiction. So here we assume that the set of sentences, $\{K,M\}$, does not entail $K\&M$. When we show that a contradiction follows from that assumption, we can conclude that $\{K,M\}$ does, in fact, entail $K\&M$.

[3] Copeland, "The Genesis of Possible Worlds Semantics", p. 99.

Before getting into detail, possible worlds semantics dictates that a necessity statement is true if and only if it is true in every possible world. Possibility statements are true if and only if there is some possible world in which the statement is true. Determining the truth-value of the basic necessity/possibility operators depends on our definition of validity which comes from the formal semantics of modal logic.[4] Possible world semantics defines validity in terms of models, which have three components: two sets and a function. The first set, *W*, is the set of points of evaluation. The second set, *R*, is the set of relations between points. The function, *V*, assigns a truth-value to a given formula relative to a possible world.

In possible world semantics, the individual elements of *W* are essential because the truth-value of a statement is determined relative to each element. We use sentence (3) to illustrate.

(3) It is raining.

In a model where points are possible worlds, (3) will be true at some worlds and false at others. Specifically, that sentence will be true if and only if it is really raining at the world of its evaluation. If instead of worlds the elements of *W* are times, and the sentence is evaluated during a downpour, then (3) is true. If, on the other hand, the sentence is evaluated on a sunny day, (3) is false.

According to Copeland, the binary relations between possible worlds in the set *R* are the key to obtaining relatively weak modal logics.[5] Some systems, Lewis's S5 for example, are too strong for many applications. Arthur Prior mentions a situation with respect to temporal logic in which another Lewis system, S4, is preferable.

> Transitiveness we surely want, i.e. a possible outcome of a possible outcome of the way things now are, is itself a possible outcome of the way things

[4]Typically, necessity and possibility are denoted with unary connectives, $\Box$ and $\Diamond$ respectively
[5]Copeland, "The Genesis of Possible Worlds Semantics", p. 99.

> now are; so we have at least S4. But have we any more? Not symmetry, surely, for it seems at least possible to do things that are irrevocable; and not connectedness, if we believe that there really are alternative possible futures, so that there may be states of affairs A and B either of which could be reached from the actual present but neither of them from the other.[6]

In fact, again according to Copeland, Prior was the first to employ the binary relations in context of modal logic.[7] Prior coined the phrase "accessibility relations" to designate the relation between moments in his temporal logic.[8],[9] One of the conditions Prior implements is transitivity. From the transitivity relation, he is able to develop a version of S4 for tense logic.[10]

Modal logic is great for formalizing sentences that mention necessity or possibility. With the normal modal operators, we can formalize both "It might rain" and "It always rains." And yet nothing in modal syntax can describe a particular point in a model.[11] The expressive power of ordinary modal logic is not enough to capture highly specific sentences; the type of sentence that is true at one and only one possible world. But it is well known that modal logic can be translated into first order logic.[12] Since first order logic does not suffer a lack of expressive power, some might wonder why we bother with modal logic at all. For one reason, modal logic edges out first-order logic on simplicity. The standard translation shows how modal operators function as a way to hide quantification of related elements of $W$.

Another reason we pursue modal logic despite its expressiveness problem is its decidability. Some systems of logic lack the ability to decide whether a statement is provable or

[6] Prior, "Possible Worlds", p. 41.
[7] Copeland, "The Genesis of Possible Worlds Semantics", p. 108.
[8] Ibid., p. 100.
[9] Arthur Prior's work, especially on temporal logic, heavily influenced the development of hybrid logic.
[10] Copeland, "The Genesis of Possible Worlds Semantics", p. 108.
[11] Blackburn, "Representation, Reasoning, and Relational Structures", p. 344.
[12] Braüner, *Hybrid Logic and Its Proof-theory*, p. 8.

$$ST_a(\Diamond\phi) = \exists b(R(a,b) \& ST_b(\phi))$$
$$ST_a(\Box\phi) = \forall b(R(a,b) \supset ST_b(phi))$$

**Figure 1.1:** Standard translation of modal operators

not because they lack a decision procedure. A decision procedure is a mechanical process that will always determine, in a finite number of steps, whether a given formula is provable. Systems that possess a decision procedure, one of which is the truth table method, are decidable. First order logic is undecidable. Modal logic on the other hand *is* decidable. Modal logic allows us to exchange the expressive power of first order logic for decidability. Hybrid logic attempts to restore that expressive power by adding nominals, a new propositional symbol. Nominals reference particular points in a model on which to evaluate a statement. This is how hybrid logic is able to represent these highly specific statements. There have been several proof systems proposed for hybrid logic. Here I present another system, in Fitch-style, for hybrid logic (FHL).

Fitch-style systems are examples of natural deduction. The numerous systems for constructing derivations fall into a few broad categories, but the different types of systems are not important here. What is important is that the dominant style in the first two decades of the twentieth century was the axiomatic, or Hilbert-style. That is the context out of which natural deduction grew. The Polish logician and historian Jan Łukasiewicz first pointed out that Russell, Hilbert, et. al. were using a system that was very different from the way we normally reason.[13] Instead of assumptions, Hilbert-style systems makes use of axioms and rules that are all theorem preserving, i.e. if the premise is a theorem, then the result must be a theorem too. But ordinarily we do not start with statements that are impossible to be false. When we reason, we suppose that some statement is true and

[13]Prawitz, *Natural Deduction*, p. 98.

see where that supposition leads. In the audience of the lecture where Łukasiewicz made these remarks sat another Polish logician, Stanisław Jaśkowski, who created a system for reasoning from assumptions instead of axioms based on Łukasiewicz's suggestion.

The first version of Jaśkowski-style lists formulas in boxes with the assumed formula at the top of each box. The other formulas follow from the first. Once a proof reaches the goal of the assumption, that formula is the first one written outside the box. The next version of Jaśkowski-style left out the boxes and used a list of numbers next to the formula to indicate which assumption preceded it.[14]

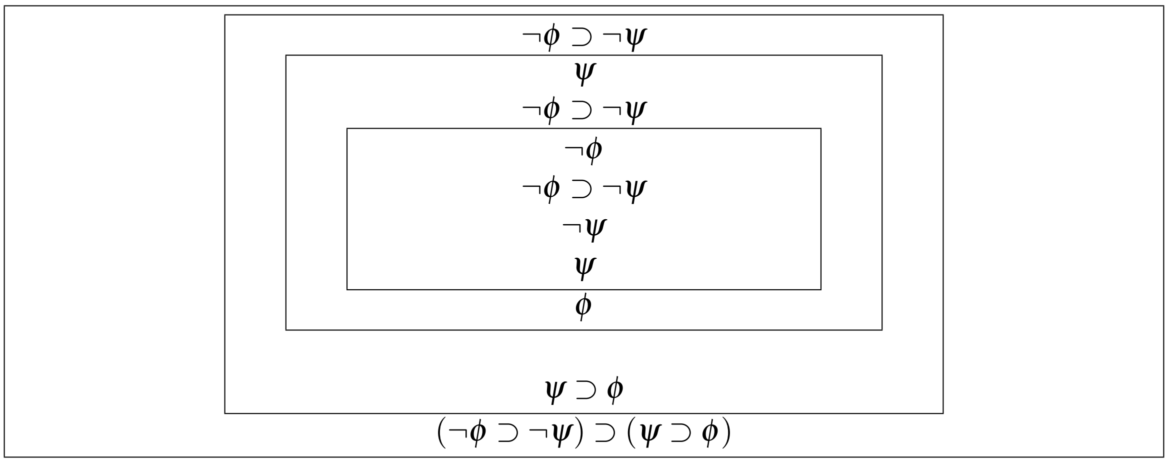

**Figure 1.2:** Jaśkowski's first style

| | |
|---|---|
| I | $\neg\phi \supset \neg\psi$ |
| I, I | $\psi$ |
| I, I, I | $\neg\phi$ |
| I, I, I | $\neg\psi$ |
| I, I | $\phi$ |
| I | $\psi \supset \phi$ |
| | $(\neg\phi \supset \neg\psi) \supset (\psi \supset \phi)$ |

**Figure 1.3:** Jaśkowski's second style

Fitch points to Jaśkowski and Gerhard Gentzen both for influencing his system.[15] Nevertheless the affinity for Jaśkowski's systems is apparent. A Fitch-style proof is a finite list

[14] Prawitz, *Natural Deduction*, pp. 98-101.
[15] Fitch, *Symbolic Logic, An Introduction*, p. vii.

| | | |
|---|---|---|
| 1 | $K$ | A |
| 2 | $M$ | A |
| 3 | $K\&M$ | &I, 1, 2 |

**Figure 1.4:** A simple Fitch-style derivation

of sentences such that each item is either an assumption or the direct result of an inference rule applied to some previous steps. These proofs are also organized into columns. On the left, the actual steps of the proof and on the right, the justification for each step. For the most part, the right-hand column contains inference rules and references to line numbers. The inference rules of a language are normally based on the connectives of that language. For each connective, there is a rule that introduces a formula based on that connective. And there is an elimination rule that separates such a formula around the connective. But before any inference rule can be applied, there has to be some step(s) to apply it to. These initial steps are often the main assumptions of a derivation. All the main assumptions are grouped together at the top of the derivation and have a horizontal line separating them from the rest of the proof.

At each step (line) there must be something in the right column to indicate the corresponding entry on the left column is the result of applying such and such rule to such and such steps or that the left side entry is an assumption. Typically parentheses, brackets, etc. are omitted unless they are necessary (or in cases when they greatly enhance readability). So say we have the set of sentences $\{K, M\}$ again. Each element in the set we are given is an assumption in the derivation of "$K\&M$". Since there are just two sentences in the set, we have just two assumptions in the derivation in Figure 1.4. To these assumptions we can apply the rule (&I) to infer their conjunction.

A vertical line to the left of each formula indicates the scope of the assumption. The entire proof occurs in the scope of the main assumptions - so the left most scope line extends from the first step to the last. But why have a scope line at all if the main assumptions are never discharged? Two reasons: uniformity with the scope lines of subderivations, and the ability to derive theorems.

The method of subordinate proof is a distinctive feature of Fitch's system and perhaps the systems greatest asset. Subproofs, or subderivations, are how Fitch-style systems let us make an assumption and see where that assumption goes. Other systems of natural deduction do that also, but not as explicitly as (or as well as) Fitch-style. Take for example the tree-style system Braüner uses. In tree-style, we might have to write our assumption several places, leading to different branches. It forces us to keep track of these branches, the contents of each, and the subscripts of the assumptions to ensure the branches convene and the assumptions are discharged at the right point.[16]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_b@_a(\phi \supset \psi)^2}{@_a(\phi \supset \psi)}\,@E \qquad \dfrac{@_b@_a\phi^1}{@_a\phi}\,@E}{@_a\psi}\,\supset\text{E}}{@_b@_a\psi}\,@E}{@_b(@_a\phi \supset @_a\psi)}\,\supset I^1}{@_b(@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi))}\,\supset I^2$$

**Figure 1.5:** Version of the K axiom for nominals derived in tree-style.

In contrast, Fitch-style uses the scope lines of subderivations instead of subscripts to keep track of assumptions. The subproof method is as straightforward as the name suggests. It consists in simply writing one proof inside another. Since scope lines can be nested, there is no need for branching either. Subproofs that are nested must be discharged (or closed) before the end of the proof and they have to be discharged one at a time beginning with the last to be introduced and moving back toward the main scope line. Subproofs

[16] Braüner, *Hybrid Logic and Its Proof-theory*, p. 27.

have to be for some particular purpose. The basic types of subproofs are for (¬I), (¬E), and (⊃I). Whatever the case, the kind of subproof is determined by the rule used to move back to the primary scope line. The (¬I), (¬E) subproofs are classic reductio arguments. Assume some proposition, derive a contradiction in the subproof and then conclude the opposite of the assumption on the main scope line. For (⊃I), make some supposition and for whatever follows from it, conclude on the main scope line that the supposition is antecedent to whatever follows from it. The types of rules for moving out of a subproof back to the main scope line are not the only restrictions however. For one thing, the subordinate proof must only have one assumption. Two or more assumptions cannot introduce a subderivation as in Figure 1.6.

| | | | |
|---|---|---|---|
| 1 | $P$ | | A |
| 2 | | $R \supset \neg P$ | A |
| 3 | | $R$ | A |
| 4 | | $\neg P$ | ⊃E, 2, 3 |
| 5 | $\neg P$ | | ¬I, 2 to 4 |

**Figure 1.6:** A malformed subderivation with two assumptions.

Besides the obvious downside that two assumptions on the same scope line make it easy to place contradictions on the first scope line, there is a conventional complication.[17] Each subproof assumption should be introduced with “A” followed by the type of subproof the assumption introduces. With two assumptions, it is unclear where to label the subproof. A convention could solve it, but with two assumptions, there is a larger problem once the subproof is completed and the result is returned to the adjacent scope line. What kind of subproof is it? Like all other steps, the result of the subproof must be labeled, but as

[17] Note that requiring only one assumption per subproof does not completely eliminate this. Conjoin steps 2 and 3 in Figure 1.6 into a single assumption. The new assumption will still lead to contradiction.

with steps 2-4 in Figure 1.6, it is not really a (⊃I) subproof. Nor is the subproof for (¬I) or (¬E). In fact, none of the types of subproof rules can really begin with more than one assumption. The reductio rules depend on supposing just one thing is the case and drawing a contradiction from it and it alone which is the reason for concluding its opposite. The implication rule depends on a single antecedent also. There can, of course, be several terms conjoined, disjointed, etc. in the antecedent. But even if they were separated into multiple rows of assumptions, they would all be required to derive the last formula in the subproof, and so must be included in the conditional formula on the main scope line.

Since a subproof is within the scope of the main scope line, items that are outside, but prior to the introduction of the subproof are accessible in the subproof (unless special considerations apply or the subproof is labeled). The reiteration rule makes this explicit in Figure 1.7.

| | | |
|---|---|---|
| 1 | $P$ | A |
| 2 | $\quad R$ | Assp ⊃I |
| 3 | $\quad P$ | R, 1 |
| 4 | $R \supset P$ | ⊃I, 2 to 3 |

**Figure 1.7:** Subproof scope and reiteration

More often than not, the formulas that occur above a subproof can be copied in that subproof through reiteration. However we cannot reiterate out of a subproof.[18] The other rules will be given in detail later. For now, this suffices to give an idea of what a Fitch-style system of logic should look like.

So why have yet another system? Some systems are better for certain purposes. For example, referring to axiom systems in meta-logical discussions makes sense because of

[18] There is an exception circumstance in FHL covered in 3.2.2, but generally this holds.

the economy with which those systems are formulated. However axiom systems are far from the way we typically reason. Axiom systems are so difficult to use, in fact, systems of “natural deduction” get their name by contrast. Natural deduction systems on the whole excel in derivation construction and evaluating arguments in natural language. But even among natural deduction systems, some are easier to grasp intuitively than others. Among the natural deduction systems, I believe Fitch-style attains the best economy for intuitive clarity. Given that, it seems like the ideal vehicle for Hybrid Logic with its goal of adding to the expressive power of modal logic.

# Chapter 2

# Hybrid Logic

We begin the discussion of Hybrid Logic with the motivation for hybridizing. Then we introduce two of Braüner's systems for hybrid logic. The first is the natural deduction system, $N_{\mathscr{H}(\mathscr{O})}$. It enjoys priority in Braüner's book, being the system used throughout, at least as a reference point. The second system we discuss is the axiom system, $A_{\mathscr{H}(\mathscr{O})}$, Braüner introduces.

## 2.1 Motivation for Hybrid Logic

First-order logic has enough expressive power to "support the key deduction steps at the *object* level."[1] But modal logic appears unable to do that, which indicates something is missing. What seems to be missing is the ability to deal with individual possible worlds in the object language. The normal modal logic vocabulary consists of the vocabulary of propositional logic plus the modal operators. With the typical modal vocabulary "$\Diamond\phi$" expresses the possibility of $\phi$. That formula, $\Diamond\phi$, is usually parsed as "there is some accessible possible world such that $\phi$." But there is no way to denote which world that is in the normal vocabulary, only that there is some such world. This inability to name points

[1] Blackburn, "Representation, Reasoning, and Relational Structures", p. 346.

in a model is the basic challenge for modal logic. The inability to refer directly to elements in the set of points is a significant hindrance to the expressiveness of the language. For example, if we consider points in a model as times, there is no way to formalize sentence (4).

(4) It is raining at 2:05 PM Wednesday, October 30, 2013.

Sentence (4) cannot be formalized in modal logic because there is only one kind of propositional symbol in normal modal logic and there is no way to restrict that symbol to one and only one point where it is true. Developing the expressive power to formalize statements like sentence (4) - sentences that are true at one and only one point - is the primary purpose of Hybrid Logic.[2] We go about this by adding nominals to the language. Syntactically, a nominal functions like any other propositional symbol. Nominals can be conjoined, negated, modified by a modal symbol, and appear in any well formed formula just as any other propositional symbol. But semantically, nominals are unique because they specify a single point in a model and at that point alone the nominal is true. The nominal, then, effectively is a label for the point at which it is true.

Referring to specific points in a model is nice, but not quite enough to achieve the expressiveness we want. With nominals, we can come close to symbolizing sentence (4) as "*R*&*a*" where "*R*" is "It is raining" and "*a*" is "2:05 PM Wednesday, October 30, 2013." Since "*a*" is a nominal, it is true just in case it is "2:05 PM Wednesday, October 30, 2013." So the formula "*R*&*a*" is true if and only if both *R* and *a* are true. But it does not exactly capture the gist of the sentence. Sentence (4) does not say "It is raining *and* it is 2:05 PM Wednesday, October 30, 2013." It says "It is raining *at* 2:05 PM Wednesday, October 30, 2013." We want to express that a statement is true *at* a particular point of evaluation, which in this case is a particular time. This is where another feature of hybrid logic, satisfaction

[2]Braüner, *Hybrid Logic and Its Proof-theory*, p. 2.

operators, come in. The satisfaction operator $@_i$ for some nominal $i$, prefixes a formula $\phi$ to create a satisfaction statement.

(5) Trivia starts at 9:00 PM Wednesday, October 30, 2013.

Suppose we wanted to formalize sentence (5). We would do so with a satisfaction statement where the sentence letter "T" stands for "trivia starts" and the nominal "a" names the point "9:00 PM Wednesday October 30 2013". So the satisfaction statement $@_a T$ means exactly what we wished to symbolize: that at the point "9:00 PM Wednesday October 30, 2013," trivia begins. More generally, for a satisfaction statement $@_a \phi$, we can read it as saying that at the point to which the nominal $a$ refers, the formula $\phi$ holds. Since nominals are syntactically the same as sentence letters, nominals can be the argument in a satisfaction statement. This amounts to, as Braüner points out, making an identity claim. For the nominals $a$ and $c$ the satisfaction statement $@_a c$ is just another way of saying $a = c$.[3] Think about it this way: let "$a$" be "2:00 PM Eastern Standard Time" and let "$c$" be "1:00 PM Central Standard Time." Then "2:00 PM Eastern Standard Time" is true at "1:00 PM Central Standard Time" and vice versa.

The last part of the hybrid language to introduce are the binders. Conceptually, binders, are comparable to quantifiers in predicate logic. In fact, one of the binders *is* a recycled quantifier from predicate logic. The function of the $\forall$ binder is to tie a formula to whatever point a nominal refers to. A formula like "$\forall a \phi$," is true if and only if $\phi$ is true for every assignment of $a$. In some sense, the binder $\forall$ has global significance whereas the other common binder, $\downarrow$, is local. The $\downarrow$ binder attaches formulas to the current point. In a way, the $\downarrow$ binder says "right here." So the formula "$\downarrow a \Diamond \phi$" becomes "right here at $a$, possibly $\phi$." Interestingly though, $\downarrow$ can be defined in terms of $\forall$ because $\downarrow a \phi \equiv \forall a (a \supset \phi)$ is valid in any frame.[4] Although Braüner presents introduction and elimination rules for $\downarrow$, since

[3] Braüner, *Hybrid Logic and Its Proof-theory*, p. 9.
[4] Ibid., p. 7.

it is defined, those rules and the binder itself are omitted here. The same reasoning applies to $\exists$, which Braüner omits too.

## 2.2 Braüner's Natural Deduction System for Hybrid Logic

Braüner's natural deduction system of propositional logic is a forward-reasoning tree-style system. To derive a formula, begin with the inference rules and try to build a derivation of that formula. The derivation takes the form of a finite tree. The end-formula of the derivation is called its root. Most other formulas are either a leaf or the result of a rule of inference.[5] Every assumption is enumerated with a superscript and each assumption is discharged at just one application of a rule. At the application of a rule discharging an assumption, the application is tagged with the same superscript. The basic rule set Braüner uses, Figure 2.1, is comprised of introduction and elimination rules for conjunction and implication and one more rule representing contradiction. [6]

[5] Braüner, *Hybrid Logic and Its Proof-theory*, p. 22.

[6] Falsum represents some arbitrary contradiction. So the statement $@_a\bot$ means "At **a** both $P$ and $\neg P$ for some arbitrary proposition $P$." The satisfaction statement above means basically that there is a contradiction at point **a**. Braüner writes, "...(recall $\neg\phi$ is an abbreviation of $\phi \supset \bot$)." ibid., p. 23

$$\frac{\phi \qquad \psi}{(\phi \& \psi)}\ \&\mathrm{I} \qquad \frac{(\phi \& \psi)}{\phi}\ \&\mathrm{E} \qquad \frac{(\phi \& \psi)}{\psi}\ \&\mathrm{E}$$

$$\begin{array}{c} [\phi] \\ \vdots \\ \dfrac{\psi}{(\phi \supset \psi)} \supset I \end{array} \qquad \frac{(\phi \supset \psi) \qquad \phi}{\psi} \supset\mathrm{E}$$

$$\begin{array}{c} [\neg\phi] \\ \vdots \\ \dfrac{\bot}{(\phi)} \bot 1^{*} \end{array}$$

* $\phi$ is a propositional symbol (ordinary or a nominal)

**Figure 2.1:** Braüner's basic system

### 2.2.1 Tree-style Rules for Hybrid Logic

Braüner's hybrid logic natural deduction system $N_{\mathscr{H}(\mathscr{O})}$ extends the basic sentential logic system. Figures 2.2 and 2.3 contain the entire set of rules.[7] The natural deduction system for hybrid logic which Braüner develops here is unique for its insistence that all formulas are satisfaction statements. Notably, the propositional rules apply to the content of the satisfaction statements, not the satisfaction statements themselves. Contrast this requirement with Seligman's natural deduction system (or FHL for that matter). Braüner dedicates a chapter to comparing his system with Seligman's.[8]

[7] cf. Figures 2.2 and 2.3 Braüner, *Hybrid Logic and Its Proof-theory*, p. 26.
[8] Ibid., Ch. 4.

$$\frac{@_a\phi \qquad @_a\psi}{@_a(\phi \& \psi)}\ \&\text{I} \qquad \frac{@_a(\phi \& \psi)}{@_a\phi}\ \&\text{E} \qquad \frac{@_a(\phi \& \psi)}{@_a\psi}\ \&\text{E}$$

$$\begin{array}{c}[@_a\phi]\\ \vdots\\ \dfrac{@_a\psi}{@_a(\phi \supset \psi)}\supset I\end{array} \qquad \frac{@_a(\phi \supset \psi) \qquad @_a\phi}{@_a\psi} \supset\text{E}$$

$$\begin{array}{c}[@_a\neg\phi]\\ \vdots\\ \dfrac{@_a\bot}{@_a(\phi)}\ \bot 1^{*}\end{array} \qquad \frac{@_a\bot}{@_c\bot}\ \bot 2$$

$$\frac{@_a\phi}{@_c@_a\phi}\ @\ \text{I} \qquad \frac{@_c@_a\phi}{@_a\phi}\ @\ \text{E}$$

$$\begin{array}{c}[@_a\Diamond c]\\ \vdots\\ \dfrac{@_c\phi}{@_a\Box\phi}\ \Box I^{\star}\end{array} \qquad \frac{@_a\Box\phi \qquad @_a\Diamond e}{@_e\phi}\ \Box E$$

$$\frac{@_a\phi[c/b]}{@_a\forall b\phi}\ \forall I^{\dagger} \qquad \frac{@_a\forall b\phi}{@_a\phi[e/b]}\ \forall E$$

---

* $\phi$ is a propositional symbol (ordinary or a nominal)
⋆ c does not occur free in $@_a\Box\phi$ or in any undischarged assumptions the specified occurrences of $@_a\Diamond c$
† c does not occur free in $@_a\forall b\phi$ or in any undischarged assumptions

**Figure 2.2:** $N_{\mathscr{H}(\mathscr{O})}$ rules for connectives

$$\frac{}{@_a a}\text{ Ref} \qquad \frac{@_a c \quad @_a \phi}{@_c \phi}\text{ Nom1*} \qquad \frac{@_a c \quad @_a \Diamond b}{@_c \Diamond b}\text{ Nom2}$$

* $\phi$ is a propositional symbol (ordinary or a nominal)

**Figure 2.3:** $N_{\mathscr{H}(\mathscr{O})}$ rules for nominals

## 2.3 Braüner's Axiom System for Hybrid Logic

Besides the natural deduction system, Braüner includes an axiom system for hybrid logic, $A_{\mathscr{H}(\mathscr{O})}$, that is equivalent to $N_{\mathscr{H}(\mathscr{O})}$. For our purposes we have modified Braüner's $A_{\mathscr{H}(\mathscr{O})}$ somewhat. The changes are intended to clarify the system and minor enough that the system as shown here is obviously equivalent to Braüner's original presentation. For instance, we explicitly include every tautology as an axiom in the system. Braüner's axiom system adopts two inference rules directly from the natural deduction system: (□I) and (∀I). And the axioms (□E) and (∀E) are closely related to the natural deduction rules of the same name. But some of the rules of $N_{\mathscr{H}(\mathscr{O})}$, and FHL for that matter, lack an axiom counterpart in $A_{\mathscr{H}(\mathscr{O})}$. A rule like (&E) for instance does not have a directly comparable rule in $A_{\mathscr{H}(\mathscr{O})}$. Including any tautology forestalls this complication.

We have also forced $A_{\mathscr{H}(\mathscr{O})}$ to share the vocabulary of $N_{\mathscr{H}(\mathscr{O})}$. As before we omit $\downarrow$ and its inference rules. For the same reasons, we also alter the (Dist) and (Scope) axioms. Braüner's presentation gives them as biconditionals. Since there is no biconditional operator in $N_{\mathscr{H}(\mathscr{O})}$ we have split them into left and right conditionals.

And finally, some rules are different simply to make the translation procedure in Chapter 4 run more smoothly. Whereas Braüner's diagram of (∀I) has a single term in the antecedent of both premise and conclusion, we use a compound antecedent.

| | |
|---|---|
| (Dist-Right) | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ |
| (Dist-Left) | $(@_a\phi \supset @_a\psi) \supset @_a(\phi \supset \psi)$ |
| ($\bot$) | $@_a\bot \supset \bot$ |
| (Scope-Right) | $@_a@_b\phi \supset @_b\phi$ |
| (Scope-Left) | $@_b\phi \supset @_a@_b\phi$ |
| (Ref) | $@_a a$ |
| (Intro) | $(a\&\phi) \supset @_a\phi$ |
| ($\Box$E) | $(\Box\phi\&\Diamond e) \supset @_e\phi$ |
| ($\forall$E) | $\forall b\phi \supset \phi[e/b]$ |
| (Taut) | $\phi$ where $\phi$ is any tautology |

**Figure 2.4:** $A_{\mathscr{H}(\mathscr{O})}$ axioms

Some of the axioms are common to other systems of hybrid logic. For instance (Scope), (Ref), and (Intro), reappear from Blackburn, et al.[9] Although $A_{\mathscr{H}(\mathscr{O})}$ contains these axioms, it differs from other systems in the literature on hybrid logic. The other systems extend modal axiom systems with new axioms.[10] Braüner's system takes nominals and satisfaction operators to be extensions of propositional logic. Hence his system extends propositional logic axiomatics with the tools of hybrid logic and the modal operator simultaneously, which is the same strategy we follow with FHL.

[9]Braüner, *Hybrid Logic and Its Proof-theory*, p. 54.
[10]Ibid., p. 57.

$$\frac{\phi \supset \psi \qquad \phi}{\psi}\ (\mathrm{MP}) \qquad \frac{\phi}{@_a\phi}\ (N_@) \qquad \frac{@_a\phi}{\phi}\ (\mathrm{Name})^*$$

$$\frac{\psi_1 \supset (\ldots(\psi_n \supset (\Diamond c \supset @_c\phi))\ldots)}{\psi_1 \supset (\ldots(\psi_n \supset \Box\phi)\ldots)}\ (\Box I)^\star \qquad \frac{\psi_1 \supset (\ldots(\psi_n \supset \phi[c/b])\ldots)}{\psi_1 \supset (\ldots(\psi_n \supset \forall b\phi)\ldots)}\ (\forall I)^\dagger$$

---

* a does not occur free in $\phi$
$\star$ c does not occur free in $\phi$ or $\psi_1,\ldots,\psi_n$
† c does not occur free in $\forall b\phi$ or $\psi_1,\ldots,\psi_n$

**Figure 2.5:** $A_{\mathscr{H}(\mathscr{O})}$ rules

# Chapter 3

# Fitch-style Hybrid Logic (FHL)

## 3.1 Foundation of FHL

To capture Braüner's hybrid logic in a Fitch-style system, there is a lot of groundwork to set out. The very first thing then is establishing the basic sentential logic in the Fitch-style system, (FSL).[1] It has a vocabulary that consists of:

1. Sentence Letters are the Roman capital letters A-Z (with or without primes, subscripts, etc)
2. The connectives are &, $\supset$, $\neg$, and $\perp$.[2]
3. '(' and ')' are punctuation marks
4. The Greek letters $\phi$, $\psi$, $\rho$, are metavariables ranging over the sentences of FSL. We will use $\Gamma$ and $\Delta$ as metavariables for sets of sentences and $\Theta$ for sets of rules.

[1]FSL is very similar to the system SD in Bergmann, Moor, and Nelson, *The Logic Book*.

[2]The common connectives $\vee$ ('or') and $\equiv$ ('if and only if') are defined in terms of the others. If all we ever did was symbolize sentences, it might be useful to have many connectives available. But when we begin reasoning about them, the more connectives there are means more inference rules and therefore a more onerous system. Some systems try to balance the two and include a moderate set of connectives, some of which are definable but are more useful than burdensome. Here we will tend to eliminate definable connectives except when we begin realizing diminishing returns on their elimination.

5. If $\phi$ and $\psi$ are well formed formulas then $\phi \& \psi$, $\phi \supset \psi$, and $\neg\phi$ are well formed formulas as well.

Like any Fitch-style system, there must be inference rules for each of the connectives. The connectives &, $\supset$, and $\neg$ have introduction rules that derive sentences where these are the main connectives. They also have elimination rules that result in a subformula of the sentence to which the elimination rule applies. The connective "$\bot$" is a nullary connective.[3] Since $\bot$ takes no arguments, there would be nothing left after eliminating it from a formula where it is the main connective. For that reason, $\bot$ only has an introduction rule. Finally, our system includes a rule for reiterating formulas in a derivation after they first appear. Figure 3.3 at the end of this chapter shows each rule of our basic system.

## 3.2 Modifications and Additions to the Base System

There are two options for how to proceed. The first is to modify FSL to make it suitable for predicate logic and then add modal operators before introducing the nominals and satisfaction operators for hybrid logic. This strategy reflects the typical progression of logic courses. But a large part of the material introduced by this first strategy will be unnecessary if the endgame is only creating a system for hybrid logic. One example that jumps to mind is adding modal operators to predicate logic. Without going into detail, we have to weaken some predicate logic rules to avoid difficulties with statements about what is necessarily universal and universally necessary.[4] The alternative, follow Braüner and introduce the hybrid logic tools directly to propositional logic, bypasses the adjustments the first approach prescribes. With that in mind, we can begin defining FHL as an extension of FSL by accepting the basic vocabulary and adding to it.

1. The nominals are lower case Roman letters (with or without primes, subscripts, etc.).

[3] The only point of including "$\bot$" is for the sake of the equivalence proof in Chapter 4.

[4] See the discussion of the Barcan Formula in Prior, *Time and Modality*, p. 26.

2. There are new connectives including $\Box$, $\forall$, and $@_i$ where $i$ is any nominal.

3. The definition of well formed formula must be extended for the new connectives such that if $\phi$ is well formed and $i$ is any nominal, $\Box\phi$, $\forall i\phi$, and $@_i\phi$ are well formed formulas.

### 3.2.1 Rules for binders and modal operators

The symbol $\forall$ binds formulas to points denoted by nominals. It is worth mentioning again that $\forall$ defines the other binder Braüner introduces. The only rules necessary, then, are for the $\forall$ binder.

| | | | | | | |
|---|---|---|---|---|---|---|
| $i$ | $\phi(a/c)$ | | | $i$ | $\forall c\phi$ | |
| $j$ | $\forall c\phi$ | $\forall$I, $i$ | | $j$ | $\phi(a/c)$ | $\forall$E, $i$ |

**Restriction:** **a** must not occur in $\forall c\phi$ or in any undischarged assumption or as the labeling nominal of a nominal labeled subproof in which ($\forall$I) is applied

For ($\Box$I) , at some nominal $c$, if we suppose c is possible ($\Diamond c$), and derive a formula $\phi$ satisfied at $c$ ($@_c\phi$), then we should be able to derive $\phi$ in a satisfaction statement for any nominal. In which case $\phi$ is necessary ($\Box\phi$)

$i$    |   | $\Diamond c$    Assp $\Box$I

      |   | $\vdots$

$j$    |   | $@_c\phi$

$j+1$ | $\Box\phi$    $\Box$I, $i$ to $j$

**Restriction:** **c** must not occur free in $\Box\phi$, in any undischarged assumption, in any nominal labels in the scope of which ($\Box$I) occurs or in the accessible conclusion of any import rules.

$i$ | $\Box\phi$

$j$ | $\Diamond e$

$k$ | $@_e\phi$    $\Box$E, $i$, $j$

### 3.2.2 Rules for nominals and nominal labeled subproofs

Like any Fitch-style system, FHL makes prolific use of subderivations. And like many Fitch-style systems, FHL requires special kinds of subproofs. We can call the new subderivation in FHL the nominal labeled subproof or simply a nominal subproof. The label of a subproof is a way of showing the subproof specifically concerns the nominal in the label. So for example, step $i$ below is irrelevant to the nominal subproof starting at step $j$.

$i$ | $\phi$

$j$ | $@_a$ |

   |    | $\vdots$

Were the subproof beginning at step $j$ a normal, hypothetical subproof, step $i$ might be pertinent. At the very least, we could reiterate $\phi$ into the subproof. But that is not allowed with nominal labeled subproofs. Labeled subproofs differ from normal subproofs because their very purpose is limiting which formulas enter. Dealing with specific rules for moving in and out of subproofs makes for an easy first step towards defining nominal subproofs.

The standard reiteration rule's primary purpose is to move in and out of subproofs. But with the new labeled subproof, one has to be cautious with the standard reiteration rule. Actually we have to be more than cautious. We just cannot use it. To see why not, suppose that $\phi$ is "it is raining." The situation in Figure 3.1 is plausible if we allow regular reiteration into nominal subproofs.

| | | |
|---|---|---|
| $i$ | $@_a \neg\phi$ | |
| $j$ | $\phi$ | |
| $k$ | $@_a$ \| $\phi$ | R, $j$ |
| | \| $\vdots$ | |
| $l$ | $@_a \phi$ | @I, $k$ |

**Figure 3.1:** Nominal subderivation with unrestricted reiteration

Using the familiar reiteration rules allows us to say whatever we want about what is satisfied at some point. In this case we happened to already know it is not raining at $a$ by step $i$. But with standard reiteration we derived a satisfaction statement to the effect that it was, in fact, raining. But there are statements that are true at a nominal but unprovable without reiteration rules of some sort. Resolving that problem takes three reiteration rules that augment our system: (@R-in), (@R-out), and Hybrid Reiteration (HR). The rules

(@R-in) and (@R-out) allow reiterating satisfaction statements into and out of nominal subproofs.

$$\begin{array}{l|l} i & @_a\phi \\ & @_b \,|\; @_a\phi \qquad @\text{R-in}, i \\ & \quad\;\, |\;\; \vdots \end{array}
\qquad\qquad
\begin{array}{l|l} & @_b \,| \\ i & \quad\;\, |\; @_a\phi \\ & @_a\phi \qquad @\text{R-out}, i \end{array}$$

The idea behind @R-in is simple - if some statement $\phi$ is satisfied at some nominal $a$, then at some other nominal $b$, $\phi$ is still satisfied at $a$. @R-out has the same idea. If a satisfaction statement $@_b@_a\phi$ is derivable, then $@_a\phi$ should be derivable too. Otherwise a formula could be satisfied at some nominal $a$ from the perspective of another nominal $b$, and yet *not* be satisfied at $a$ itself.

Although it is unacceptable to reiterate a formula that is not a satisfaction statement into a nominal subproof with the standard reiteration rule, it is plausible that putting such a formula into a nominal subproof would be necessary. But to avoid the problems with standard reiterating (and to avoid ad-hoc restrictions to the standard reiteration rule) the third nominal reiteration rule, Hybrid Reiteration (HR), requires that the nominal which labels the subproof appear by itself, e.g. outside of a satisfaction statement, before a formula can be reiterated into that subproof. For as complicated as that sounds, the schematic of the rule shows how simple it is.

$$\begin{array}{l|l} i & a \\ j & \phi \\ k & @_a \,|\; \phi \qquad \text{HR}, i, j \end{array}$$

Intuitively, step $i$ says that we are in world $a$ and step $j$ says that $\phi$ is true. Since we are in world $a$ and $\phi$ is true, then the subderivation at step $k$ is permitted to include $\phi$ in the list of statements that are true at $a$.

The reiteration rules assume a nominal labeled subproof occurs in a derivation. That leaves open the issue of how a nominal subproof first appears in a derivation. The rules for nominal introduction and nominal elimination introduce nominal subderivations.

$$
\begin{array}{r|l|l}
 & @_a & \vdots \\
i & & \phi \\
\hline
\end{array}
$$
$$
\begin{array}{r|ll}
j & @_a\phi & \qquad @\text{I}, i
\end{array}
$$

$$
\begin{array}{r|lll}
i & @_a\phi & & \\
j & @_a \mid & \phi & \qquad @\text{E}, i \\
 & \quad\;\, \mid & \vdots &
\end{array}
$$

Braüner's $N_{\mathscr{H}(\mathscr{O})}$ includes rules by the same name, but the FHL rules are quite different. The rules Braüner calls (@I) and (@E) are more properly called (@@I) and (@@E) respectively. The introduction and elimination rules Braüner formulates only apply to satisfaction statements. Hence, there is no application such that $\phi \vdash @_a\phi$. Only applications of the form $@_c\phi \vdash @_a@_c\phi$. The same goes for @E in $N_{\mathscr{H}(\mathscr{O})}$. The result of (@@I) and (@@E) can be obtained with the subproof method outlined above.

Together the rules @I, @E can be used to build a complete nominal subproof. Take for example Figure 3.2. The nominal subproof delivers a set of statements that are true at the labeled nominal. In this case everything between $j$ and $k$ is true at the nominal $a$.

$$
\begin{array}{l|l l}
i & @_a\phi & \\
j & @_a \mid \phi & @\text{E}, i \\
 & \quad\; \vdots & \\
k & \quad\;\; \psi & \\
l & @_a\psi & @\text{I}, k
\end{array}
$$

**Figure 3.2:** Properly formed nominal subproof

So far there has been nothing definitive about negated satisfaction statements. This is an uncomfortable limitation on any system of logic. Without negated satisfaction statements, it might look as if the system could not symbolize a sentence like (6).

(6) It is not the case that trivia starts at 9:00 Wednesday Oct 30 2013.

The obvious way to symbolize that sentence is with "$\neg @_a T$" But could we not very well negate "T" and capture the meaning of the statement in a formula like "$@_a \neg T$"? Well, yes. That is the reason for the nominal negation rules (@N1) and (@N2).

$$
\begin{array}{l|l l}
i & @_a\neg\phi & \\
j & \neg @_a\phi & @\text{N1}, i
\end{array}
\qquad\qquad
\begin{array}{l|l l}
i & \neg @_a\phi & \\
j & @_a\neg\phi & @\text{N2}, i
\end{array}
$$

The last two FHL rules really act more like axioms. The first is (Ref), an intuitive yet stubbornly persistent rule, adapted from $N_{\mathscr{H}(\mathscr{O})}$ because it cannot be derived from other rules. Its one job is to form satisfaction statements such that the nominal in the satisfaction operator is also the formula that is satisfied (i.e. at itself).

$$
\begin{array}{l|l|l l}
i & @_a & a & \text{Ref}
\end{array}
$$

The other rule is (Exists). From a syntactic perspective, the name looks misleading because the existential operator is defined out of FHL. Given the vocabulary of FHL, maybe the equivalent negated universal is not intuitive, but semantically it gives us a way to assert that some nominal exists all the same, i.e. there is an actual world, a present time, etc.

$$i \mid \neg\forall a \neg a \qquad \text{Exists}$$

## Full Description of FHL

| | | |
|---|---|---|
| $i$ | $\phi$ | |
| $j$ | $\psi$ | |
| $k$ | $\phi \& \psi$ | &I, $i$, $j$ |

| | | |
|---|---|---|
| $i$ | $\phi \& \psi$ | |
| $j$ | $\phi$ | &E1, $i$ |

| | | |
|---|---|---|
| $i$ | $\phi \& \psi$ | |
| $j$ | $\psi$ | &E2, $i$ |

| | | |
|---|---|---|
| $i$ | $\mid\ \phi$ | Assp ¬ I |
| | $\mid\ \vdots$ | |
| $j$ | $\mid\ \bot$ | |
| $k$ | $\neg\phi$ | ¬I, $i$ to $j$ |

| | | |
|---|---|---|
| $i$ | $\mid\ \neg\phi$ | Assp ¬ E |
| | $\mid\ \vdots$ | |
| $j$ | $\mid\ \bot$ | |
| $k$ | $\phi$ | ¬E, $i$ to $j$ |

| | | |
|---|---|---|
| $i$ | $\neg\phi$ | |
| $j$ | $\phi$ | |
| $k$ | $\bot$ | ⊥I, $i$, $j$ |

| | | |
|---|---|---|
| $i$ | $\phi$ | |
| $j$ | $\phi$ | R, $i$ |

| | | |
|---|---|---|
| $i$ | $\mid\ \phi$ | Assp ⊃I |
| | $\mid\ \vdots$ | |
| $j$ | $\mid\ \psi$ | |
| $k$ | $\phi \supset \psi$ | ⊃I, $i$ to $j$ |

| | | |
|---|---|---|
| $i$ | $\phi \supset \psi$ | |
| $j$ | $\phi$ | |
| $k$ | $\psi$ | ⊃E, $i$, $j$ |

**Figure 3.3:** FSL Rules for FHL

| | | | |
|---|---|---|---|
| $i$ | $\phi(a/c)$ | | |
| $j$ | $\forall c\phi$ | $\forall$I, $i$ | |

**Restriction:** **a** must not occur in $\forall c\phi$ or in any undischarged assumption or as the labeling nominal of a nominal labeled subproof in which ($\forall$I) is applied

| | | | |
|---|---|---|---|
| $i$ | $\forall c\phi$ | | |
| $j$ | $\phi(a/c)$ | $\forall$E, $i$ | |

| | | | |
|---|---|---|---|
| $i$ | | $\Diamond c$ | Assp $\Box$I |
| | | $\vdots$ | |
| $j$ | | $@_c\phi$ | |
| $j+1$ | $\Box\phi$ | | $\Box$I, $i$ to $j$ |

**Restriction:** **c** must not occur free in $\Box\phi$, in any undischarged assumption, in any nominal labels in the scope of which ($\Box$I) occurs or in the accessible conclusion of any import rules.

| | | | |
|---|---|---|---|
| $i$ | $\Box\phi$ | | |
| $j$ | $\Diamond e$ | | |
| $k$ | $@_e\phi$ | $\Box$E, $i$, $j$ | |

**Figure 3.4:** Binder and Modal Rules for FHL

$$\begin{array}{l|l|l}
 & @_a & \vdots \\
i & & \phi \\
j & @_a\phi & \qquad @\text{I}, i
\end{array}$$

$$\begin{array}{l|l|l}
i & @_a\phi & \\
j & @_a & \phi \qquad @\text{E}, i \\
 & & \vdots
\end{array}$$

$$\begin{array}{l|l|l}
i & @_a\phi & \\
j & @_b & @_a\phi \qquad @\text{R-in}, i \\
 & & \vdots
\end{array}$$

$$\begin{array}{l|l|l}
 & @_b & \\
i & & @_a\phi \\
j & @_a\phi & \qquad @\text{R-out}, i
\end{array}$$

$$\begin{array}{l|l|l}
i & a & \\
j & \phi & \\
k & @_a & \phi \qquad \text{HR}, i, j
\end{array}$$

$$\begin{array}{l|l}
i & @_a\neg\phi \\
j & \neg @_a\phi \qquad @\text{N1}, i
\end{array}$$

$$\begin{array}{l|l}
i & \neg @_a\phi \\
j & @_a\neg\phi \qquad @\text{N2}, i
\end{array}$$

$$\begin{array}{l|l|l}
i & @_a & a \qquad \text{Ref}
\end{array}$$

$$\begin{array}{l|l}
i & \neg\forall a\neg a \qquad \text{Exists}
\end{array}$$

**Figure 3.5:** Nominal Rules for FHL

# Chapter 4

# Equivalence Arguments

So far we have defined three systems: Braüner's axiomatic system, $A_{\mathscr{H}(\mathscr{O})}$, Braüner's natural deduction system, $N_{\mathscr{H}(\mathscr{O})}$, and the new Fitch-style system FHL. Since Braüner's systems are equivalent, proving FHL is equivalent to one should suffice to show FHL is equivalent to the other. And so this chapter focuses on proving the equivalence between FHL and $A_{\mathscr{H}(\mathscr{O})}$. The process of showing two systems are equivalent boils down to demonstrating that one translates into the other and vice versa. The first section shows that the axioms of $A_{\mathscr{H}(\mathscr{O})}$ are derivable in FHL and FHL rules can duplicate the effect of the axiom rules. The next section shows a procedure to translate FHL derivations into $A_{\mathscr{H}(\mathscr{O})}$ derivations. The last section demonstrates the equivalence established in the previous two sections with examples for the K axiom and Braüner's (Nom) rule.

## 4.1 From $A_{\mathscr{H}(\mathscr{O})}$ to FHL

The first equivalence principle states that if $\phi$ is derivable in $A_{\mathscr{H}(\mathscr{O})}$, then there is also an FHL derivation of $\phi$

**EP1:** If $\vdash_{A_{\mathscr{H}(\mathscr{O})}} \phi$, then $\vdash_{FHL} \phi$

But $\phi$ can be derived in a couple of ways in $A_{\mathscr{H}(\mathscr{O})}$ - $\phi$ can be either an axiom or derived from an axiom by axiom rules. FHL does not contain axioms though. Even tautologies have to be derived in FHL. Since the systems have different options for reasoning, it is convenient to introduce lemmas for proving EP1.

**Lemma 1:** If $\phi$ is an axiom of $A_{\mathscr{H}(\mathscr{O})}$, then $\vdash_{FHL} \phi$.

We can prove Lemma 1 by constructing FHL derivations for each axiom in $A_{\mathscr{H}(\mathscr{O})}$.

**The (Dist-Right) Axiom**

$\vdash_{FHL} @_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$

| | | |
|---|---|---|
| 1 | $@_a(\phi \supset \psi)$ | Assp $\supset$I |
| 2 | $@_a\phi$ | Assp $\supset$I |
| 3 | $@_a$ $\phi \supset \psi$ | @E, 1 |
| 4 | $\phi$ | @E, 2 |
| 5 | $\psi$ | $\supset$E, 3, 4 |
| 6 | $@_a\psi$ | @I, 5 |
| 7 | $@_a\phi \supset @_a\psi$ | $\supset$I, 2 to 6 |
| 8 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | $\supset$I, 1 to 7 |

**The (Dist-Left) Axiom**

$\vdash_{FHL} (@_a\phi \supset @_a\psi) \supset @_a(\phi \supset \psi)$

| | | |
|---|---|---|
| 1 | $@_a\phi \supset @_a\psi$ | Assp $\supset$I |
| 2 | $\neg @_a(\phi \supset \psi)$ | Assp $\neg$ E |
| 3 | $@_a\neg(\phi \supset \psi)$ | @N2, 2 |
| 4 | $@_a$ $\neg(\phi \supset \psi)$ | @E, 3 |
| 5 | $\psi$ | Assp $\neg$ I |
| 6 | $\phi$ | Assp $\supset$I |
| 7 | $\psi$ | R, 5 |
| 8 | $\phi \supset \psi$ | $\supset$I, 6 to 7 |
| 9 | $\bot$ | $\bot$I, 4, 8 |
| 10 | $\neg\psi$ | $\neg$I, 5 to 9 |
| 11 | $@_a\neg\psi$ | @I, 10 |
| 12 | $\neg @_a\psi$ | @N1, 11 |
| 13 | $@_a\phi$ | Assp $\neg$ I |
| 14 | $@_a\psi$ | $\supset$E, 1, 13 |
| 15 | $\neg @_a\psi$ | R, 12 |
| 16 | $\bot$ | $\bot$I, 14, 15 |
| 17 | $\neg @_a\phi$ | $\neg$I, 13 to 16 |

| | | |
|---|---|---|
| 18 | $@_a\neg\phi$ | @N2, 17 |
| 19 | $@_a$ $\neg\psi$ | @E, 11 |
| 20 | $\neg\phi$ | @E, 18 |
| 21 | $\neg\psi$ | Assp $\supset$I |
| 22 | $\neg\phi$ | R, 19 |
| 23 | $\neg\psi \supset \neg\phi$ | $\supset$I, 21 to 22 |
| 24 | $\phi$ | Assp $\supset$I |
| 25 | $\neg\psi$ | Assp $\neg$ E |
| 26 | $\neg\phi$ | $\supset$E, 23, 25 |
| 27 | $\phi$ | R, 24 |
| 28 | $\bot$ | $\bot$I |
| 29 | $\psi$ | $\neg$E, 25 to 28 |
| 30 | $\phi \supset \psi$ | $\supset$I, 24 to 29 |
| 31 | $@_a(\phi \supset \psi)$ | @I, 30 |
| 32 | $\bot$ | $\bot$I, 2, 31 |
| 33 | $@_a(\phi \supset \psi)$ | $\neg$E, 2 to 32 |
| 34 | $(@_a\phi \supset @_a\psi) \supset @_a(\phi \supset \psi)$ | $\supset$I, 1 to 33 |

**The (⊥) Axiom**

$\vdash_{FHL} @_a\bot \supset \bot$

| | | |
|---|---|---|
| 1 | $\quad @_a\bot$ | Assp ⊃I |
| 2 | $\quad @_a \mid \bot$ | @E, 1 |
| 3 | $\quad\quad\quad a$ | Assp ¬ E |
| 4 | $\quad\quad\quad \bot$ | R, 2 |
| 5 | $\quad\quad \neg a$ | ¬I, 3 to 4 |
| 6 | $\quad\quad a$ | Ref |
| 7 | $\quad @_a\neg a$ | @I, 5 |
| 8 | $\quad \neg @_a a$ | @N1, 7 |
| 9 | $\quad @_a a$ | @I, 6 |
| 10 | $\quad \bot$ | ⊥I, 8, 9 |
| 11 | $@_a\bot \supset \bot$ | ⊃I, 1 to 10 |

**The (Scope-Right) Axiom**

$\vdash_{FHL} @_a@_b\phi \supset @_b\phi$

| | | |
|---|---|---|
| 1 | $\quad @_a@_b\phi$ | Assp ⊃I |
| 2 | $\quad @_a \mid @_b\phi$ | @E, 1 |
| 3 | $\quad @_b\phi$ | @R-out, 2 |
| 4 | $@_a@_b\phi \supset @_b\phi$ | ⊃I, 1 to 3 |

**The (Scope-Left) Axiom**

$\vdash_{FHL} @_b\phi \supset @_a@_b\phi$

| | | |
|---|---|---|
| 1 | $@_b\phi$ | Assp $\supset$I |
| 2 | $@_a \mid @_b\phi$ | @R-in, 4 |
| 3 | $@_a@_b\phi$ | @I, 5 |
| 4 | $@_a@_b\phi \supset @_b\phi$ | $\supset$I, 1 to 3 |

**The (Ref) Axiom**

$\vdash_{FHL} @_a a$

| | | |
|---|---|---|
| 1 | $@_a \mid a$ | Ref |
| 2 | $@_a a$ | @I, 1 |

**The (Intro) Axiom**

$\vdash_{FHL} (a\&\phi) \supset @_a\phi$

| | | |
|---|---|---|
| 1 | $a\&\phi$ | Assp $\supset$I |
| 2 | $a$ | &E1, 1 |
| 3 | $\phi$ | &E2, 1 |
| 4 | $@_a \mid \phi$ | HR, 2, 3 |
| 5 | $@_a\phi$ | @I, 4 |
| 6 | $(a\&\phi) \supset @_a\phi$ | $\supset$I, 1 to 5 |

**The (□E) Axiom**

$\vdash_{FHL} (\Box\phi \& \Diamond e) \supset @_e\phi$

| | | |
|---|---|---|
| 1 | $\Box\phi \& \Diamond\psi$ | Assp $\supset$I |
| 2 | $\Box\phi$ | &E1, 1 |
| 3 | $\Diamond\psi$ | &E2, 1 |
| 4 | $@_e\phi$ | □E, 2, 3 |
| 5 | $(\Box\phi \& \Diamond\psi) \supset @_e\phi$ | $\supset$I, 1 to 4 |

**The (∀E) Axiom**

$\vdash_{FHL} \forall b\phi \supset \phi[e/b]$

| | | |
|---|---|---|
| 1 | $\forall b\phi$ | Assp $\supset$I |
| 2 | $\phi[e/b]$ | ∀E, 1 |
| 3 | $\forall b\phi \supset \phi[e/b]$ | $\supset$I, 1 to 2 |

The FHL derivations of each axiom prove Lemma 1. Next we have to address its counterpart for the rules in $A_{\mathscr{H}(\mathscr{O})}$, Lemma 2.

**Lemma 2:** For every $A_{\mathscr{H}(\mathscr{O})}$ rule, there is an equivalent application of rules from FHL.

For the axiom rules however, there are two steps to defining a procedure for translating their application to FHL derivations. The first is to reformulate the axiom rule as a Fitch-style rule. Then define the procedure for translating the rule into an application of FHL rules.

**The Rule (MP)**

Braüner's formulation of (MP) is

$$\frac{\phi \supset \psi \qquad \phi}{\psi}\ (\text{MP})$$

So if $\vdash \phi \supset \psi$ and $\vdash \phi$, then $\vdash \psi$. Suppose there are derivations such as:

$$\begin{array}{l|l} 1 & \\ & \\ i_1 & \phi \supset \psi \end{array} \qquad\qquad \begin{array}{l|l} 1 & \\ & \\ i_2 & \phi \end{array}$$

Combining the two derivations prompts us to apply Braüner's (MP) rule to infer $\psi$:

$$\begin{array}{l|ll} 1 & & \\ & & \\ i_1 & \phi & \\ i_1+1 & & \\ & & \\ i_1+i_2 & \phi \supset \psi & \\ i_1+i_2+1 & \psi & \supset\text{E}, i_1, i_1+i_2 \end{array}$$

There is little to the translation of (MP) into FHL. Since ($\supset$E) is the same rule, we can simply change the justification of step $i_1+i_2+1$. But it is a good first example of the process.

**The Rule ($N_@$)**

Braüner's formulation of ($N_@$) is:

$$\frac{\phi}{@_a\phi}\,(N_@)$$

Thus if $\vdash \phi$, then $\vdash @_a\phi$. Then suppose there is a derivation as follows:

| | |
|---|---|
| 1 | |
| $i$ | $\phi$ |

Since all the rules of $A_{\mathscr{H}(\mathscr{O})}$ are theorem preserving, there is no doubt that $\phi$ would be a theorem. Since $\phi$ is a theorem, we can derive it in a nominal labeled subproof then apply (@I) to derive "$@_a\phi$".

| | | |
|---|---|---|
| 1 | $@_a$ | |
| $i$ | $\phi$ | |
| $i+1$ | $@_a\phi$ | @I, 1 to $i$ |

**The Rule (Name)**

Braüner's formulation of (Name):

$$\frac{@_a\phi}{\phi} \text{ (Name)}$$

Thus if $\vdash @_a\phi$, then $\vdash \phi$ so long as $a$ does not occur free in $\phi$. Suppose there is a derivation as follows:

| | |
|---|---|
| 1 | |
| $i$ | $@_a\phi$ |

Remember that $a$ does not appear free here because there are *no* assumptions. Construct a new derivation as follows where $b$ does not occur in the first derivation.

| | | |
|---|---|---|
| 1 | | |
| $i$ | $@_a\phi$ | |
| $i+1$ | $\forall a @_a\phi$ | $\forall$I, $i$ |
| $i+2$ | $\neg\phi$ | Assp $\neg$ E |
| $i+3$ | $b$ | Assp $\neg$ I |
| $i+4$ | $@_b \mid \neg\phi$ | HR, $i+2$, $i+3$ |
| $i+5$ | $@_b\neg\phi$ | @I, $i+4$ |
| $i+6$ | $\neg @_b\phi$ | @N2, $i+5$ |
| $i+7$ | $@_b\phi$ | $\forall$E, $i+1$ |
| $i+8$ | $\bot$ | $\bot$I, $i+6$, $i+7$ |
| $i+9$ | $\neg b$ | $\neg$I, $i+3$ to $i+8$ |
| $i+10$ | $\forall a\neg a$ | $\forall$I, $i+9$ |
| $i+11$ | $\neg\forall a\neg a$ | Exists |
| $i+12$ | $\bot$ | $\bot$I, $i+10$, $i+11$ |
| $i+13$ | $\phi$ | $\neg$E, $i+2$ to $i+12$ |

**The Rule (□I)**

Braüner's formulation of (□I):

$$\frac{\psi_1 \supset (\ldots(\psi_n \supset (\Diamond c \supset @_c\phi))\ldots)}{\psi_1 \supset (\ldots(\psi_n \supset \Box\phi)\ldots)}\,(\Box I)$$

Assuming, of course, that $c$ is not free in $\phi$ or $\psi_1,\ldots,\psi_n$. Suppose then, there is a derivation as follows:

| | | |
|---|---|---|
| 1 | | |
| $i$ | $\psi_1 \supset (\ldots(\psi_n \supset (\Diamond c \supset @_c\phi))\ldots)$ | |

In FHL we can derive the (□I) rules conclusion by nesting the derivation above in hypothetical subproofs and repeatedly applying (⊃E)

| | | |
|---|---|---|
| 1 | $\psi_1$ | Assp ⊃I |
| | $\ddots$ | |
| $n$ | $\psi_n$ | Assp ⊃I |
| | $\vdots$ | |
| $i+n$ | $\psi_1 \supset (\ldots(\psi_n \supset (\Diamond c \supset @_c\phi))\ldots)$ | |
| | $\vdots$ | |
| $j+n$ | $\psi_n \supset (\Diamond c \supset @_c\phi)$ | |
| $j+n+1$ | $\Diamond c \supset @_c\phi$ | ⊃E, $n$, $j+n$ |
| $j+n+2$ | $\Diamond c$ | Assp □I |
| $j+n+3$ | $@_c\phi$ | ⊃E, $j+n+1$, $j+n+2$ |
| $j+n+4$ | $\Box\phi$ | □I, $j+n+2$ to $j+n+3$ |
| | $\dots$ | |
| $k+n$ | $\psi_1 \supset (\ldots(\psi_n \supset \Box\phi)\ldots)$ | |

**The Rule (∀I)**

$$\frac{\psi_1 \supset (\ldots(\psi_n \supset \phi[c/b])\ldots)}{\psi_1 \supset (\ldots(\psi_n \supset \forall b\phi)\ldots)}\ (\forall I)$$

Suppose there is a derivation as follows:

$$
\begin{array}{r|l}
1 & \\
\vdots & \\
i & \psi_1 \supset (\ldots(\psi_n \supset \forall b\phi)\ldots)
\end{array}
$$

To get the same effect as Braüner's rule, we need to create a new derivation containing the one above in $n$-number subderivations.

| | | |
|---|---|---|
| $1$ | $\psi_1$ | Assp $\supset$I |
| | $\ddots$ | |
| $n$ | $\psi_n$ | Assp $\supset$I |
| | $\vdots$ | |
| $i+n$ | $\psi_1 \supset (\ldots(\psi_n \supset \phi[c/b])\ldots)$ | |
| | $\vdots$ | |
| $j+n$ | $\psi_n \supset \phi[c/b]$ | |
| $j+n+1$ | $\phi[c/b]$ | $\supset$E, $n$, $j+n$ |
| $j+n+2$ | $\forall b\phi$ | $\forall$I, $j+n+1$ |
| | $\iddots$ | |
| $k+n$ | $\psi_1 \supset (\ldots(\psi_n \supset \forall b\phi)\ldots)$ | |

The derivations showing the effect of each $A_{\mathscr{H}(\mathscr{O})}$ rule can be obtained with FHL rules proves Lemma 2. And together Lemmas 1 and 2 prove EP1: that there is no $A_{\mathscr{H}(\mathscr{O})}$ derivation without an equivalent derivation in FHL. Showing the converse however is trickier.

## 4.2 From FHL to $A_{\mathscr{H}(\mathscr{O})}$

Continuing proving FHL and $A_{\mathscr{H}(\mathscr{O})}$ are equivalent leads to the converse of EP1. The task now is to show that any FHL derivation has a counterpart $A_{\mathscr{H}(\mathscr{O})}$ derivation.

**EP2:** If $\vdash_{FHL} \phi$, then $\vdash_{A_{\mathscr{H}(\mathscr{O})}} \phi$

Proving EP1 only required creating FHL derivations comparable to the components of $A_{\mathscr{H}(\mathscr{O})}$. But the different styles of the two systems is a concern here too. Previously the issue came down to $A_{\mathscr{H}(\mathscr{O})}$ including axioms and inference rules whereas FHL has only rules. The major challenge proving EP2 will be showing that FHL subproofs do not derive anything we cannot also derive in $A_{\mathscr{H}(\mathscr{O})}$. In that case we need a procedure to translate any FHL derivation into a derivation of $A_{\mathscr{H}(\mathscr{O})}$.[1]

Since subderivations may be nested, the transformation procedure has to be repeatable. But whether a derivation has just one subproof or a hundred, there will be at least one that contains no other subproofs. That is where the transformation procedure will begin. The process then repeats, eliminating one subproof at a time until reaching the main scope line.[2]

[1] Our process is closely related to the process in Thomason, *Symbolic Logic*, p84-90.

[2] Of course there could be a derivation with nested subproofs of the same order, or different "nests" for instance.

| | | |
|---|---|---|
| 1 | | |
| $i$ | \| $\phi$ | |
| $i'$ | \| $\psi$ | |
| $i'+1$ | $\phi \supset \psi$ | $\supset$I, $i$ to $i'$ |
| $j$ | \| $\psi$ | |
| | \| $\vdots$ | |
| $j'$ | \| $\phi$ | |
| $j'+1$ | $\psi \supset \phi$ | $\supset$I, $j$ to $j'$ |

If the innermost subproof is hypothetical, the first step is to reiterate inside the subproof any premises needed for applications in that subproof. A rule like (⊃E) can be applied to premises *outside* a subderivation to produce a formula *inside* the subproof. For all such instances, it is safe to assume the premises are reiterated into the subproof where the rule is applied, without substantially altering the proof.

In general terms, the procedure is to turn subderivations into lists of conditionals or satisfaction statements depending on the type of subproof. These new sections of the derivations might not always follow, so we might need to insert axioms in the derivation as we transition from FHL to $A_{\mathscr{H}(\mathscr{O})}$. In a sense, this creates another system that simply combines the two, FHLA.[3]

Each rule will have its own translation step, but the order for applying the translation steps for any derivation will depend on the structure of the derivation, not the order of the translation steps here. In broad terms, we eliminate rules that insert a subproof, then rules that are subproof indifferent, and finally we eliminate the subproofs themselves.

### 4.2.1 Translations that introduce a subderivation

#### TS1: (@N1) and (@N2)

The translation for the rules, (@N1) and (@N2), actually require introducing subderivations. So if there are any applications of (@N1) or (@N2) present in the innermost subderivation, these have to be the very first dealt with.

The subproofs at step *i* and step *j* have the same order; neither is more deeply nested than the other. It does not make a difference beginning with one or the other technically, but it will be less confusing eliminating to begin at the top and work down.

[3]Remember that $A_{\mathscr{H}(\mathscr{O})}$ lacks axioms comparable to some basic propositional logic rules. To work around that we decided to accept any tautology as an axiom (cf. Figure 2.4). Below when we insert a tautology, we will label it as such. If we need to include an axiom from Braüner's system, we will use the name for it in $A_{\mathscr{H}(\mathscr{O})}$, with an "A" subscript to avoid confusion where there are rules in FHL that have the same name. It is the axioms from $A_{\mathscr{H}(\mathscr{O})}$, not every tautology, that we mean to include in FHLA.

For any application of (@N1), we have to introduce a hypothetical subderivation for negation introduction.

| | | |
|---|---|---|
| $i$ | $@_a\neg\phi$ | |
| $j$ | $\neg @_a\phi$ | @N1, $i$ |

$\Rightarrow$

| | | |
|---|---|---|
| $i$ | $@_a\neg\phi$ | |
| $i+1$ | $@_a\phi$ | A |
| $i+2$ | $@_a$ $\phi$ | @E, $i+1$ |
| $i+3$ | $\neg\phi$ | @E, $i$ |
| $i+4$ | $\bot$ | $\bot$I, $i+2$, $i+3$ |
| $i+5$ | $@_a\bot$ | @I, $i+4$ |
| $i+6$ | $@_a\bot \supset \bot$ | $\bot_A$ |
| $i+7$ | $\bot$ | $\supset$E, $i+5$, $i+6$ |
| $j+7$ | $\neg @_a\phi$ | $\neg$I, $i+1$ to $i+7$ |

For any application of (@N2), we can use the axioms through FHLA in a nominal subproof to derive the same conclusion.

| | | |
|---|---|---|
| $i$ | $\neg @_a\phi$ | |
| $j$ | $@_a\neg\phi$ | @N2, $i$ |

$\Rightarrow$

| | | |
|---|---|---|
| $i$ | $\neg @_a\phi$ | |
| $i+1$ | $\neg @_a\phi \supset (@_a\phi \supset @_a\bot)$ | Tautology |
| $i+2$ | $@_a\phi \supset @_a\bot$ | $\supset$E, $i$, $i+1$ |
| $i+3$ | $(@_a\phi \supset @_a\bot) \supset @_a(\phi \supset \bot)$ | Dist-Left |
| $i+4$ | $@_a(\phi \supset \bot)$ | $\supset$E, $i+2$, $i+3$ |
| $i+5$ | $@_a$ $\phi \supset \bot$ | @E, $i+4$ |
| $i+6$ | $(\phi \supset \bot) \supset \neg\phi$ | Tautology |
| $i+7$ | $\neg\phi$ | $\supset$E, $i+5$, $i+6$ |
| $j+7$ | $@_a\neg\phi$ | @I, $i+7$ |

**TS2: (Exists)**

To convert an application of (Exists), we will use the axiom form of (Ref) and apply (Name).

$$i \;\big|\; \neg\forall a \neg a \qquad \text{Exists, } i \qquad \Rightarrow$$

| | | |
|---|---|---|
| $i$ | $@_b b$ | Ref |
| $i+1$ | $@_b \;\big|\; b$ | @E, $i$ |
| $i+2$ | $\big|\;\big|\; \forall a \neg a$ | Assp $\neg$ I |
| $i+3$ | $\big|\;\big|\; \neg b$ | $\forall$E, $i+2$ |
| $i+4$ | $\big|\;\big|\; b$ | R, $i+1$ |
| $i+5$ | $\big|\;\big|\; \bot$ | $\bot$I, $i+3$, $i+4$ |
| $i+6$ | $\big|\; \neg\forall a \neg a$ | $\neg$I, $i+2$ to $i+5$ |
| $i+7$ | $@_b \neg\forall a \neg a$ | @I, $i+6$ |
| $i+8$ | $\neg\forall a \neg a$ | Name, $i+7$ |

**($\forall$I) and ($\Box$I)**

The strategy for these rules is to replace the FHL rule with the eponymous $A_{\mathscr{H}(\mathscr{O})}$ rule. That is to say, derive a theorem and replace ($\forall$I) with ($\forall$I$_A$) and ($\Box$I) with ($\Box$I$_A$). Deriving a suitable theorem introduces subproofs, but also presents a meta-theoretical issue. The assumptions taken into the antecedent of the theorem have to come from select positions in the derivation.

1. Let $\Gamma_k$ be the set of undischarged assumptions which are accessible at step $k$.
2. Call (@E), (@R-in), (HR), or (Ref) the import rules
3. Let $\Delta_k$ be the set of formulas accessible at $k$ that are justified by one of the import rules.
4. Where $\phi_k$ is the conclusion on step $k$, $\Gamma_k \cup \Delta_k \vdash \phi_k$.

For each $\psi_i$ in $\Gamma_k \cup \Delta_k$, introduce a new subderivation assuming $\psi_i$. That will set up the opportunity to derive the FHL rule's premise in the innermost subproof and begin discharging assumptions. Once every one of the new subproofs closes, there will be a theorem left over to which Braüner's rules apply.

**TS3: (∀I)**

Suppose that (∀I) justifies step $k$ in the example below.

| | | |
|---|---|---|
| 1 | $\psi_1$ | A |
| 2 | $\psi_2$ | A |
| | $\vdots$ | |
| $i$ | $\quad \psi_i$ | |
| | $\quad \vdots$ | |
| $i+7$ | $\quad @_a \psi_j$ | |
| | $\quad \vdots$ | |
| $j$ | $\quad @_a \quad \psi_j$ | @E, $i+7$ |
| | $\qquad \vdots$ | |
| $j'$ | $\qquad \phi(a/c)$ | |
| $k$ | $\qquad \forall a \phi$ | ∀I, $j'$ |
| | $\quad \vdots$ | |
| | $\vdots$ | |

Notice first that at step 1, $\Gamma_1 \cup \Delta_1 = \{\psi_1\}$ and at step 2, $\Gamma_2 \cup \Delta_2 = \{\psi_1, \psi_2\}$. Now really, at both of these steps, $\Delta$ is empty as these formulas are main assumptions and not justified by import rules. That is the same case at step $i$ where we introduce a hypothetical subderivation: $\Gamma_i \cup \Delta_i = \{\psi_1, \psi_2, \psi_i\}$. Step $j$ initiates a nominal subderivation making the elements of $\Gamma$ inaccessible. So at step $k$, $\Gamma_k \cup \Delta_k = \{\psi_j\}$. Though not in this example, if there were any subderivations nested in the nominal scope line, those assumptions would be in $\Gamma_k \cup \Delta_k$ also. Then, as in the derivation below, insert nested subderivations at $k$ each one assuming an element of $\Gamma_k \cup \Delta_k$. Then derive $\phi(a/c)$ again to construct the theorem and apply ($\forall I_A$). Finally we can use ($\supset$E), repeatedly depending on the elements in $\Gamma_k \cup \Delta_k$, to re-derive $\forall a\phi$.

| | | |
|---|---|---|
| 1 | $\psi_1$ | A |
| 2 | $\psi_2$ | A |
| | $\vdots$ | |
| $i$ | $\quad \psi_i$ | A |
| | $\quad \vdots$ | |
| $i+7$ | $\quad @_a\psi_j$ | |
| | $\quad \vdots$ | |
| $j$ | $\quad @_a \mid \psi_j$ | @E, $i+7$ |
| | $\qquad \vdots$ | |
| $j'$ | $\qquad \phi(a/c)$ | |
| $k$ | $\qquad\quad \psi_j$ | Assp $\supset$I |
| | $\qquad\quad \vdots$ | |
| $k'$ | $\qquad\quad \phi(a/c)$ | |
| $k'+1$ | $\qquad \psi_j \supset \phi(a/c)$ | $\supset$I, $k$ to $k'$ |
| $k'+2$ | $\qquad \psi_j \supset \forall a\phi$ | $\forall I_A$, $k'+1$ |
| $k'+3$ | $\qquad \forall a\phi$ | $\supset$E, $j$, $k'+2$ |
| | $\qquad \vdots$ | |
| | $\quad \vdots$ | |

**TS4: (□I)**

The procedure for (□I) takes a small departure from the way we dealt with (∀I) because the rule (□I) actually applies to a subderivation. But to make sure we build a theorem for Braüner's rule, we must derive a conditional from that subderivation rather than a necessity statement. See, for example, the derivation on the left.

| | | |
|---|---|---|
| 1 | $\psi_1$ | A |
| $h$ | $\psi_h$ | A |
| | $\vdots$ | |
| $i$ | $\quad \Diamond c$ | Assp □I |
| | $\quad \vdots$ | |
| $j$ | $\quad @_c\phi$ | |
| $k$ | $\Box\phi$ | □I, $i$ to $j$ |

$\Rightarrow$

| | | |
|---|---|---|
| 1 | $\psi_1$ | A |
| $h$ | $\psi_h$ | A |
| | $\vdots$ | |
| $i$ | $\Diamond c \supset \Diamond c$ | Assp □I |
| | $\vdots$ | |
| $j$ | $\Diamond c \supset @_c\phi$ | |
| $j'$ | $\quad \psi_1$ | Assp ⊃I |
| | $\quad \ddots$ | |
| | $\qquad \psi_h$ | Assp ⊃I |
| | $\qquad \vdots$ | |
| | $\qquad \Diamond c \supset @_c\phi$ | |
| $j''$ | $\quad \cdot^{\cdot^{\cdot}}$ | |
| $l$ | $\psi_1 \supset \cdots \supset (\psi_h \supset (\Diamond c \supset @_c\phi))$ | ⊃I, $j'$ to $j''$ |
| $m$ | $\psi_1 \supset \cdots \supset (\psi_h \supset \Box\phi)$ | $\Box I_A$, $l$ |
| | $\vdots$ | |
| $n$ | $\psi_h \supset \Box\phi$ | |
| $n+1$ | $\Box\phi$ | ⊃E, $h$, $n$ |

We will consider the elimination process for hypothetical subderivations in Section 4.2.3. Setting aside the details of the process, observe that if we treat the ($\Box$I) subproof like a typical hypothetical subproof, one of the resulting formulas will be the final conditional in the premise for ($\Box \text{I}_A$). In this example, that formula is $\Diamond c \supset @_c\phi$. Once we adjust the derivation, from steps 1 to $j$, the translation continues by inserting subderivations, steps $j'$ to $j''$. Eventually these derivations will disappear, but in the meantime, they build the theorem we need to apply ($\Box \text{I}_A$) by deriving $\Diamond c \supset @_c\phi$ in the innermost subderivation. By step $l$, we discharge all of the subderivations and find that a theorem appears on the scope line where we started. Now, the ($\Box \text{I}_A$) rule takes effect, justifying step $m$. Since the premise occurs on the same scope line $\Diamond c \supset @_c\phi$ on step $j$, $\Gamma_j \cup \Delta_j = \Gamma_l \cup \Delta_l$. Which means repeated applications of ($\supset$E) eventually derives $\Box\phi$.

### 4.2.2 Subderivation Indifferent Translations

The next rules to eliminate are those that do not rely on any subderivations: (&I), (&E), ($\supset$E), ($\forall$E), ($\Box$E), and axioms. Since these rules do not depend on subderivations they can be applied to any premise and derive a result. The order does not matter for applying translations to rules in this group. For instance, suppose there is a subproof with applications of ($\supset$E) and (&I). In the process of eliminating this subproof, nothing turns on whether the application of ($\supset$E) comes first, or the application of (&I). It may be more productive to work from top to bottom, but technically there is no difference.

**TS5: (&I)**

Wherever (&I) occurs, inserting a tautology and applying ($\supset$E) twice arrives at the same result.

$$
\begin{array}{l|ll}
i & \phi & \\
j & \psi & \\
k & \phi \& \psi & \&\mathrm{I},\, i,\, j
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi & \\
j & \psi & \\
j+1 & \phi \supset (\psi \supset (\phi \& \psi)) & \text{Tautology} \\
j+2 & \psi \supset (\phi \& \psi) & \supset\mathrm{E},\, i,\, j+1 \\
k+2 & \phi \& \psi & \supset\mathrm{E},\, j,\, j+2
\end{array}
$$

**TS6: (&E1) and (&E2)**

For any application of either (&E1) or (&E2), we can insert a tautology and apply (⊃E) to eliminate that occurrence.

$$
\begin{array}{l|ll}
i & \phi \& \psi & \\
j & \phi & \&\mathrm{E1},\, i
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi \& \psi & \\
j & (\phi \& \psi) \supset \phi & \text{Tautology} \\
j+1 & \phi & \supset\mathrm{E},\, i,\, j
\end{array}
$$

$$
\begin{array}{l|ll}
i & \phi \& \psi & \\
j & \psi & \&\mathrm{E2},\, i
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi \& \psi & \\
j & (\phi \& \psi) \supset \psi & \text{Tautology} \\
j+1 & \psi & \supset\mathrm{E},\, i,\, j
\end{array}
$$

**TS7: (⊃E)**

Applications of (⊃E) are not actually eliminated from the derivation. All there is to do for instances of (⊃E) is to eventually rewrite the justification.

$$
\begin{array}{l|ll}
i & \phi \supset \psi & \\
j & \phi & \\
k & \psi & \supset\mathrm{E},\, i,\, j
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi \supset \psi & \\
j & \phi & \\
k & \psi & \supset\mathrm{E},\, i,\, j
\end{array}
$$

**TS8: (∀E)**

We eliminate applications of (∀E) by writing in Braüner's axiom for ($\forall \mathrm{E}_A$) and using (⊃E).

| | | |
|---|---|---|
| $i$ | $\forall c\phi$ | |
| $j$ | $\phi(a/c)$ | $\forall$E, $i$ |

$\Rightarrow$

| | | |
|---|---|---|
| $i$ | $\forall c\phi$ | |
| $j$ | $\forall c\phi \supset \phi(a/c)$ | $\forall \mathrm{E}_A$ |
| $j+1$ | $\phi(a/c)$ | $\supset$E, $i$, $j$ |

**TS9: (□E)**

Braüner's (□E) axiom adds complication because its antecedent is a conjunction. So to eliminate any application of (□E) in FHL, we cannot just write in Braüner's axiom and apply (⊃E) the way TS8 eliminated (∀E). We have to derive a conjunction the way we did in TS5 then insert the axiom and use (⊃E).

| | | |
|---|---|---|
| $i$ | $\Box\phi$ | |
| $j$ | $\Diamond e$ | |
| $k$ | $@_e\phi$ | $\Box$E, $i$, $j$ |

$\Rightarrow$

| | | |
|---|---|---|
| $i$ | $\Box\phi$ | |
| $j$ | $\Diamond e$ | |
| $j+1$ | $\Box\phi \supset (\Diamond e \supset (\Box\phi \& \Diamond e))$ | Tautology |
| $j+2$ | $\Diamond e \supset (\Box\phi \& \Diamond e)$ | $\supset$E, $i$, $j+1$ |
| $j+3$ | $\Box\phi \& \Diamond e$ | $\supset$E, $j$, $j+2$ |
| $j+4$ | $(\Box\phi \& \Diamond e) \supset @_e\phi$ | $\Box \mathrm{E}_A$ |
| $k+4$ | $@_e\phi$ | $\supset$E, $j+3$, $j+4$ |

**TS10: (¬I) and (¬E)**

Although the example below is for (¬I), the process for (¬E) is basically the same. The only difference is the negation operator placement. We have to derive from the subderivation a conditional then use TS21-TS24 to eliminate the subproof.

| | | | |
|---|---|---|---|
| $i$ | | $\phi$ | Assp $\neg$ I |
| | | $\vdots$ | |
| $j$ | | $\bot$ | |
| $k$ | $\neg\phi$ | | $\neg$I, $i$ to $j$ |

$$\Rightarrow$$

| | | | |
|---|---|---|---|
| $i$ | | $\phi$ | Assp $\neg$ I |
| | | $\vdots$ | |
| $j$ | | $\bot$ | |
| $k$ | $\phi \supset \bot$ | | $\supset$I, $i$ to $j$ |
| $k+1$ | $(\phi \supset \bot) \supset \neg\phi$ | | Tautology |
| $k+2$ | $\neg\phi$ | | $\supset$E, $k$, $k+1$ |

**TS11: (⊥I)**

Every occurrence of (⊥I) will have to appear in a subderivation for either (¬I) or (¬E). The process for those rules will resolve instances of (⊥I).

### 4.2.3 Eliminating Subderivations

Once all the premise/result rules have been eliminated from the innermost subderivation, it is time to eliminate the subderivation itself. Because TS4 and TS10 enables us to change (□I), (¬I), and (¬E) subderivations into subderivations for (⊃I), there are effectively only two cases: either the subderivation is for (⊃I) or it is a nominal labeled subderivation.

#### Eliminating Nominal Labeled Subderivations

We will begin by considering a paradigmatic derivation in which a nominal subderivation happens to be the most deeply nested. The primary task is eliminating the subderivation, so the process begins by erasing the nominal scope line and prefixing each formula from that scope with the satisfaction operator from the subproof's label.

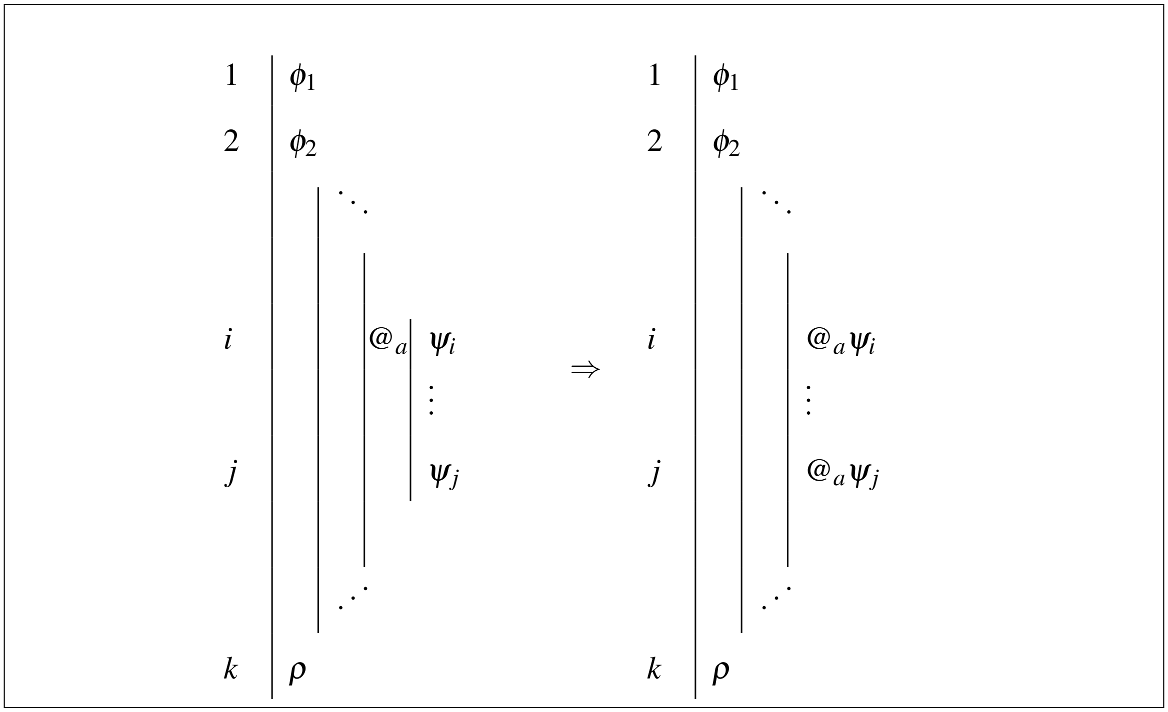

**Figure 4.1:** Eliminating nominal subderivations

That leaves a collection of satisfaction statements on the previous scope line which might not immediately follow in the new scope. Suppose there is a step $i'$ such that $i \leq i' \leq j$. Depending on how the formula entered the nominal subproof, the derivation might need additional steps to properly justify the new satisfaction statements. But only one of a few rules potentially justified $i'$ in the nominal subproof: (Ref), (@E), (⊃E), (@R-in), (HR), $A_{\mathscr{H}(\mathscr{O})}$ axioms, or $A_{\mathscr{H}(\mathscr{O})}$ rules.

**TS12: (Ref)**

Simply moving the content of the subderivation back to the previous scope line will take care of (Ref) without adding new steps. The only extra work required is adding the "A" subscript to the rule for the extent of the translation process. Eventually though, that subscript will have to be removed once the process completes and we will have no need for FHLA notation. To reach that point, though, we will likely need to translate other rules that apply to nominal subderivations.

**TS13: (@E)**

Suppose $i'$ is justified by (@E). Since eliminating the scope line produces satisfaction statements, (R) replaces (@E) as the justification for $i'$.

**TS14: (@R-in)**

Suppose $i'$ is $@_a\phi$ and is justified by (@$R$-in). After erasing, $i'$ turns into a satisfaction statement within a satisfaction statement. Inserting (Scope-Left) and applying ($\supset$E) suffices to justify the new formula.

$$\begin{array}{l|l}
h & @_a\phi \\
i' & @_b\left|\begin{array}{ll} @_a\phi & \text{@R-in}, h \\ \vdots & \end{array}\right.
\end{array}
\quad\Rightarrow\quad
\begin{array}{l|ll}
h & @_a\phi & \\
i' & @_a\phi \supset @_b@_a\phi & \text{Scope-Left} \\
i'+1 & @_b@_a\phi & \supset\text{E}, h, i'
\end{array}$$

**TS15: (HR)**

Suppose $i'$ is $\phi$ and justified by (HR). Once that formula becomes a satisfaction statement, we need to insert a tautology at step $i'$ that will yield the conjunction of steps $g$ and $h$. Apply ($\supset$E) as needed to get the conjunction by itself and write down Braüner's (Intro) axiom. We can apply ($\supset$E) once more to justify the new step $i'+4$.

$$\begin{array}{l|l}
g & a \\
h & \phi \\
i' & @_a\left|\ \phi \qquad \text{HR}, g, h\right.
\end{array}
\quad\Rightarrow\quad
\begin{array}{l|ll}
g & a & \\
h & \phi & \\
i' & a \supset (\phi \supset (a\&\phi)) & \text{Tautology} \\
i'+1 & \phi \supset (a\&\phi) & \supset\text{E}, g, i' \\
i'+2 & a\&\phi & \supset\text{E}, h, i'+1 \\
i'+3 & (a\&\phi) \supset @_a\phi & \text{Intro} \\
i'+4 & @_a\phi & \supset\text{E}, i'+2, i'+3
\end{array}$$

**TS16: $A_{\mathscr{H}(\mathscr{O})}$ Axiom on Nominal Scope Line**

Suppose $i'$ is an axiom from $A_{\mathscr{H}(\mathscr{O})}$. Probably eliminating a subproof nested within the nominal scope line we last erased required $i'$. Since we can employ $A_{\mathscr{H}(\mathscr{O})}$ rules in FHLA to justify the new $i'$ we just have to insert the original axiom and note that ($N_@$) now justifies $i' + 1$

$$
\begin{array}{l|l|ll}
i & @_a & & \\
i' & & \phi & \text{Axiom}
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & & \\
i' & \phi & \text{Axiom} \\
i'+1 & @_a\phi & N_@,\, i'
\end{array}
$$

**TS17: $A_{\mathscr{H}(\mathscr{O})}$ Rules on Nominal Scope Line**

Suppose $i'$ is justified by a rule, ($\text{Rule}_A$), from $A_{\mathscr{H}(\mathscr{O})}$. In order to justify $i'$, first we have to insert steps up to step $i$ to derive the original formula, $\phi_0$, then justify the satisfaction statement at $i+1$ with ($N_@$). Then apply to $\phi_0$ whichever $A_{\mathscr{H}(\mathscr{O})}$ rule justified $i'$ in the nominal subproof to derive $i'$ without a satisfaction operator prefix. Then we can justify the satisfaction statement left over from erasing the nominal scope line with ($N_@$).

$$
\begin{array}{l|l|ll}
i & @_a & \phi_0 & \\
i' & & \phi' & \text{Rule}_A,\, i
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi_0 & \\
i+1 & @_a\phi_0 & N_@,\, i \\
i'+1 & \phi' & \text{Rule}_A,\, i \\
i'+2 & @_a\phi' & N_@,\, i'+1
\end{array}
$$

**TS18: ($\supset$E) on Nominal Scope Line**

Suppose $i'$ is justified by ($\supset$E). Simply insert (Dist-Right) and apply ($\supset$E).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | $i$ | $@_a(\phi \supset \psi)$ | |
| $i$ | $@_a$ | $\phi \supset \psi$ | | | $i_1$ | $@_a\phi$ | |
| $i_1$ | | $\phi$ | | $\Rightarrow$ | $i'$ | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | Dist-Right |
| $i'$ | | $\psi$ | $\supset$E, $i$, $i_1$ | | $i'+1$ | $@_a\phi \supset @_a\psi$ | $\supset$E, $i$, $i'$ |
| | | | | | $i'+2$ | $@_a\psi$ | $\supset$E, $i_1$, $i'+1$ |

(In the right-hand derivation, the lines $i$ to $i'+2$ all lie inside a $@_a$ subproof.)

TS12-18 consider each possible nominal centric rule *in* the nominal subproof. But there are two rules pertaining to nominals that appear only after discharging the subderivation: (@I) and (@R-out)

**TS19: (@I)**

The last rule for nominal subproofs is perhaps the easiest to translate. In Figure 4.1, (@I) would apply at any step below $j$ to justify a satisfaction statement from the nominal subderivation. But by now that subderivation will have been replaced by satisfaction statements. Whatever is justified by (@I) below step $j$ will have to appear above step $j$ where the nominal subproof was eliminated. Hence the application of (@I) can be changed to applications of reiteration or the original justification.

**TS20: (@R-out)**

We can write down (Scope-Right) with the appropriate nominals and apply ($\supset$E).

| | | |
|---|---|---|
| | $@_b$ \| | |
| $j$ | \| $@_a\phi$ | |
| $n$ | $@_a\phi$ | @R-out, $j$ |

$\Rightarrow$

| | | |
|---|---|---|
| $j$ | $@_b@_a\phi$ | |
| $j+1$ | $@_b@_a\phi \supset @_a\phi$ | Scope-Right |
| $n+1$ | $@_a\phi$ | $\supset$E, $j$, $j+1$ |

**Eliminating Hypothetical Subderivations**

Alternately, if a hypothetical subproof is the innermost subproof, erase its scope line and place the subderivation assumption before each formula from the erased scope line to create a group of conditionals. Consider a paradigmatic derivation such as in the Figure 4.2. Before we erased the hypothetical scope line, a step, $i'$ in that subproof would have been justified by (R), ($\supset$E) or a rule from $A_{\mathscr{H}(\mathscr{O})}$ if is was not an axiom of $A_{\mathscr{H}(\mathscr{O})}$.

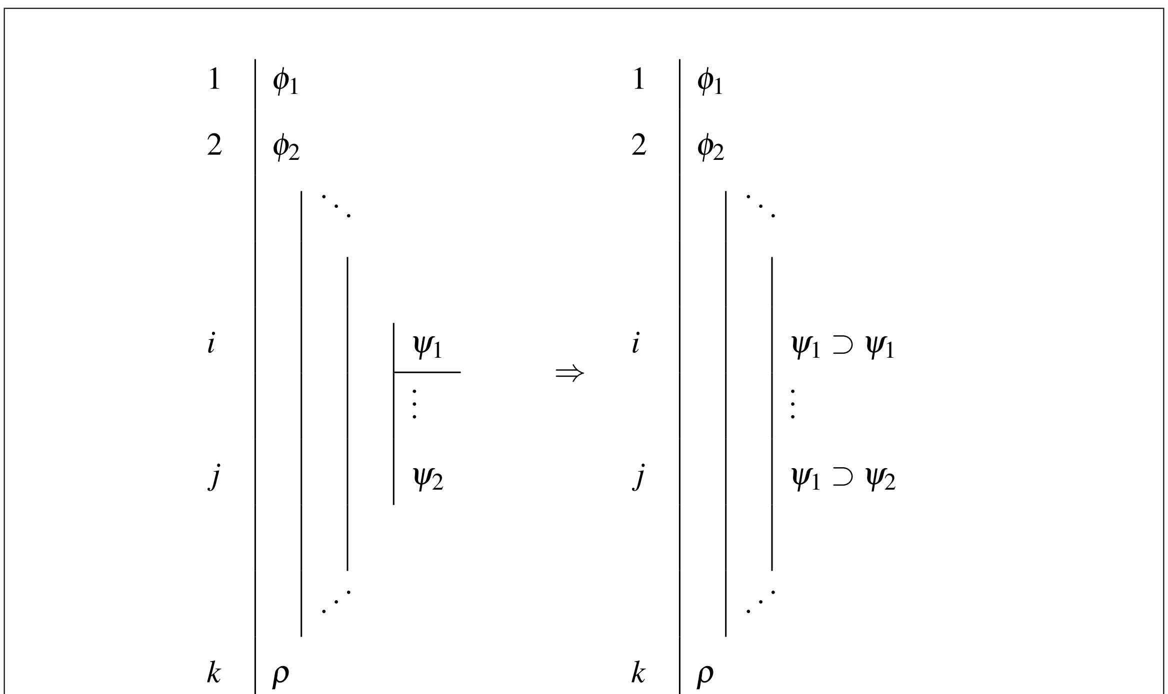


**Figure 4.2:** Eliminating hypothetical subderivations

**TS21:** $A_{\mathscr{H}(\mathscr{O})}$ **Axiom on Hypothetical Scope Line**

Suppose $i'$ was an axiom of $A_{\mathscr{H}(\mathscr{O})}$. Then write in the same axiom and another tautology. Then apply ($\supset$E) to justify its conditional statement.

$$
\begin{array}{l|l|ll}
i & & \underline{\phi} & \\
 & & \vdots & \\
i' & & \psi & \text{Axiom}
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi \supset \phi & \\
 & \vdots & \\
i' & \psi & \text{Axiom} \\
i'+1 & \psi \supset (\phi \supset \psi) & \text{Tautology} \\
i'+2 & \phi \supset \psi & \supset\text{E}, i', i'+1
\end{array}
$$

**TS22:** $A_{\mathscr{H}(\mathscr{O})}$ **Rule on Hypothetical Scope Line**

Suppose $i'$ is justified by a rule from $A_{\mathscr{H}(\mathscr{O})}$ applied to step $i$. Since ($\text{Rule}_A$) is theorem-preserving, we can derive just $\psi$ on the previous scope line at step $i'$. In which case we can insert a derivation of that theorem on the right side as well as the tautology that will ultimately justify $\phi \supset \psi$ at step $j$.

$$
\begin{array}{l|l|ll}
i & & \underline{\phi} & \\
 & & \vdots & \\
i_0 & & \psi & \\
i' & & \rho & \text{Rule}_A, i_0
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi \supset \phi & \\
 & \vdots & \\
i_0 & \phi \supset \psi & \\
 & \vdots & \\
i' & \psi & \\
i'+1 & \rho & \text{Rule}_A, i' \\
i'+2 & \rho \supset (\phi \supset \rho) & \text{Tautology} \\
i'+3 & \phi \supset \rho & \supset\text{E}, i'+1, i'+2
\end{array}
$$

**TS23: (R) on Hypothetical Scope Line**

An application of (R) is another possible justification for $i'$. Follow the translation process as defined. Then add a tautology above step $i'$ to justify using ($\supset$E).

$$
\begin{array}{l|l|ll}
h & \phi & & \\
i & & \psi & \\ \cline{3-3}
i' & & \phi & \text{R}, h
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
h & \phi & \\
i & \psi \supset \psi & \\
i' & \phi \supset (\psi \supset \phi) & \text{Tautology} \\
i'+1 & \psi \supset \phi & \supset\text{E}, h, i'
\end{array}
$$

**TS24: ($\supset$E) on Hypothetical Scope Line**

For the fourth and final possible justification, suppose $i'$ is by ($\supset$E). And for convenience, suppose both premises occur, by reiteration or otherwise, in the subderivation. The simplest solution is to write in a distribution tautology, fix the justification of $\phi \supset \rho$ if necessary and use ($\supset$E) to get $\phi \supset \psi$.

$$
\begin{array}{l|l|ll}
i & & \phi & \\ \cline{3-3}
i_0 & & \rho \supset \psi & \\
i_1 & & \rho & \\
i' & & \psi & \supset\text{E}, i_0, i_1
\end{array}
\quad \Rightarrow \quad
\begin{array}{l|ll}
i & \phi \supset \phi & \\
i_0 & \phi \supset (\rho \supset \psi) & \\
i_0+1 & (\phi \supset (\rho \supset \psi)) \supset & \\
 & \quad ((\phi \supset \rho) \supset (\phi \supset \psi)) & \text{Tautology} \\
i_0+2 & (\phi \supset \rho) \supset (\phi \supset \psi) & \supset\text{E}, i_0, i_0+1 \\
i_1+2 & \phi \supset \rho & \\
i'+2 & \phi \supset \psi & \supset\text{E}, i_0+2, i_1+2
\end{array}
$$

## 4.3 Examples of Equivalence

Philosophy, in general, could use more examples. But in logic, examples are an exceptionally rare commodity. In that spirit, this section details examples of the equivalence proofs. We begin by converting an $A_{\mathscr{H}(\mathscr{O})}$ derivation of Braüner's (Nom) rule into FHL. Then we demonstrate translating an FHL derivation of the K axiom into $A_{\mathscr{H}(\mathscr{O})}$.

### 4.3.1 Axiom derivation of (Nom)

The (Nom) rule appears only in $N_{\mathscr{H}(\mathscr{O})}$ - neither in FHL nor Braüner's other system, $A_{\mathscr{H}(\mathscr{O})}$.[4] In both of those systems (Nom) is redundant. For FHL, combining (HR) and (@R-out) in a nominal subproof duplicates the (Nom) rule's effect. In $A_{\mathscr{H}(\mathscr{O})}$, we need an axiom to reproduce (Nom). Other axiomatic systems of hybrid logic include (Nom)[5]. But in $A_{\mathscr{H}(\mathscr{O})}$ at least, it is derivable.

$$\text{(Nom)} \quad (@_c a \& @_c \phi) \supset @_a \phi$$

To convert the $A_{\mathscr{H}(\mathscr{O})}$ derivation of (Nom) into FHL, we will introduce subderivations as needed to derive formulas from Figure 4.4. Since the first line of the derivation in 4.4 is the (Intro) axiom, we can simply insert the equivalent derivation from Section 4.1.[6]

[4] cf. Figure 2.3

[5] Areces, Blackburn, and Marx, "Hybrid Logics".

[6] We refer to elements in the $A_{\mathscr{H}(\mathscr{O})}$ derivation as "line" but as "step" in the FHL derivation.

| | | |
|---|---|---|
| 1 | $@_a c$ | |
| 2 | $@_a \phi$ | |
| 3 | $@_a \mid c$ | @E, 1 |
| 4 | $\quad \mid \phi$ | @E, 2 |
| 5 | $\quad \mid @_c \mid \phi$ | HR, 3, 4 |
| 6 | $\quad \mid @_c \phi$ | @I, 5 |
| 7 | $@_c \phi$ | @R-out, 6 |

**Figure 4.3:** FHL derivation that duplicates the effect of (Nom)

| | | |
|---|---|---|
| 1 | $(a \& \phi) \supset @_a \phi$ | (Intro) |
| 2 | $((a \& \phi) \supset @_a \phi) \supset (a \supset (\phi \supset @_a \phi))$ | (Taut) |
| 3 | $a \supset (\phi \supset @_a \phi)$ | (MP 2,3) |
| 4 | $@_c(a \supset (\phi \supset @_a \phi))$ | ($N_@$ 4) |
| 5 | $@_c(a \supset (\phi \supset @_a \phi)) \supset (@_c a \supset @_c(\phi \supset @_a \phi))$ | (D.R.) |
| 6 | $@_c a \supset @_c(\phi \supset @_a \phi)$ | (MP 4,5) |
| 7 | $@_c(\phi \supset @_a \phi) \supset (@_c \phi \supset @_c @_a \phi)$ | (D.R.) |
| 8 | $(@_c(\phi \supset @_a \phi) \supset (@_c \phi \supset @_c @_a \phi)) \supset$ $(@_c a \supset (@_c(\phi \supset @_a \phi) \supset (@_c \phi \supset @_c @_a \phi)))$ | (Taut) |
| 9 | $@_c a \supset (@_c(\phi \supset @_a \phi) \supset (@_c \phi \supset @_c @_a \phi))$ | (MP 7, 8) |
| 10 | $(@_c a \supset (@_c(\phi \supset @_a \phi) \supset (@_c \phi \supset @_c @_a \phi))) \supset$ $((@_c a \supset @_c(\phi \supset @_a \phi)) \supset (@_c a \supset (@_c \phi \supset @_c @_a \phi)))$ | (Taut) |
| 11 | $(@_c a \supset @_c(\phi \supset @_a \phi)) \supset (@_c a \supset (@_c \phi \supset @_c @_a \phi))$ | (MP 9,10) |
| 12 | $@_c a \supset (@_c \phi \supset @_c @_a \phi)$ | (MP 6,11) |
| 13 | $(@_c a \supset (@_c \phi \supset @_c @_a \phi)) \supset ((@_c a \& @_c \phi) \supset @_c @_a \phi)$ | (Taut) |
| 14 | $(@_c a \& @_c \phi) \supset @_c @_a \phi$ | (MP 12,13) |
| 15 | $@_c @_a \phi \supset @_a \phi$ | (S.R.) |
| 16 | $((@_c a \& @_c \phi) \supset @_c @_a \phi) \supset$ $((@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi))$ | (Taut) |
| 17 | $(@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi)$ | (MP 14,16) |
| 18 | $(@_c a \& @_c \phi) \supset @_a \phi$ | (MP 15,17) |

**Figure 4.4:** $A_{\mathscr{H}(\mathscr{O})}$ Derivation of (Nom)

| | | |
|---|---|---|
| 1 | $(a\&\phi) \supset @_a\phi$ | (Intro) |

$$\Downarrow$$

| Line | Depth | Formula | Justification |
|---|---|---|---|
| 1 | 2 | $a\&\phi$ | Assp $\supset$I |
| 2 | 2 | $a$ | &E1, 1 |
| 3 | 2 | $\phi$ | &E2, 1 |
| 4 | 2 | $@_a \mid \phi$ | HR, 2, 3 |
| 5 | 2 | $@_a\phi$ | @I, 4 |
| 6 | 1 | $(a\&\phi) \supset @_a\phi$ | $\supset$I, 1 to 5 |

Next insert subderivations as if line 2 in Figure 4.4 were a theorem we derived in FHL. Then we can use ($\supset$E) to derive the formula at line 3.

| | | |
|---|---|---|
| 2 | $((a\&\phi) \supset @_a\phi) \supset (a \supset (\phi \supset @_a\phi))$ | (Taut) |
| 3 | $a \supset (\phi \supset @_a\phi)$ | (MP 2,3) |

$$\Downarrow$$

| Line | Depth | Formula | Justification |
|---|---|---|---|
| 7 | 2 | $(a\&\phi) \supset @_a\phi$ | Assp $\supset$I |
| 8 | 3 | $a$ | Assp $\supset$I |
| 9 | 4 | $\phi$ | Assp $\supset$I |
| 10 | 4 | $a\&\phi$ | &I, 8, 9 |
| 11 | 4 | $@_a\phi$ | $\supset$E, 7, 10 |
| 12 | 3 | $\phi \supset @_a\phi$ | $\supset$I, 9 to 11 |
| 13 | 2 | $a \supset (\phi \supset @_a\phi)$ | $\supset$I, 8 to 12 |
| 14 | 1 | $((a\&\phi) \supset @_a\phi) \supset (a \supset (\phi \supset @_a\phi))$ | $\supset$I, 7 to 13 |
| 15 | 1 | $a \supset (\phi \supset @_a\phi)$ | $\supset$E, 6, 14 |

The application of ($N_@$) at line 4 of the original derivation lengthens the FHL equivalent considerably. Remember that ($N_@$) is theorem preserving. So line 3 in the original must be a theorem to apply ($N_@$) and derive line 4. To derive the equivalent in FHL, we have to derive line 3 from above in a nominal subderivation, which amounts to copying steps 1-15 inside a nominal subproof starting at step 16.

$$4 \quad @_c(a \supset (\phi \supset @_a\phi)) \quad (N_@\ 4)$$

$$\Downarrow$$

| | | |
|---|---|---|
| 16 | $@_c$ \| \| $a\&\phi$ | Assp ⊃I |
| 17 | \| \| $a$ | &E1, 16 |
| 18 | \| \| $\phi$ | &E2, 16 |
| 19 | \| \| $@_a$ \| $\phi$ | HR, 17, 18 |
| 20 | \| \| $@_a\phi$ | @I, 19 |
| 21 | \| $(a\&\phi) \supset @_a\phi$ | ⊃I, 16 to 20 |
| 22 | \| \| $(a\&\phi) \supset @_a\phi$ | Assp ⊃I |
| 23 | \| \| \| $a$ | Assp ⊃I |
| 24 | \| \| \| \| $\phi$ | Assp ⊃I |
| 25 | \| \| \| \| $a\&\phi$ | &I, 23, 24 |
| 26 | \| \| \| \| $@_a\phi$ | ⊃E, 22, 25 |
| 27 | \| \| \| $\phi \supset (@_a\phi)$ | ⊃I, 24 to 26 |
| 28 | \| \| $a \supset (\phi \supset (@_a\phi))$ | ⊃I, 23 to 27 |
| 29 | \| $((a\&\phi) \supset @_a\phi) \supset (a \supset (\phi \supset (@_a\phi)))$ | ⊃I, 22 to 28 |
| 30 | \| $a \supset (\phi \supset (@_a\phi))$ | ⊃E, 21, 29 |
| 31 | $@_c(a \supset (\phi \supset (@_a\phi)))$ | @I, 30 |

Since line 5 is an axiom, we can simply repeat the correlative derivation in Section 4.1. That will give us a subderivation at steps 32-38. Its conclusion, step 39, is the same formula as in Figure 4.4 line 5 and also the major premise to derive the formula matching line 6.

5 $@_c(a \supset (\phi \supset @_a\phi)) \supset (@_c a \supset @_c(\phi \supset @_a\phi))$ (D.R.)

6 $@_c a \supset @_c(\phi \supset @_a\phi)$ (MP 4,5)

$$\Downarrow$$

| | | |
|---|---|---|
| 32 | $@_c(a \supset (\phi \supset @_a\phi))$ | Assp $\supset$I |
| 33 | $@_c a$ | Assp $\supset$I |
| 34 | $@_c$ $a \supset (\phi \supset @_a\phi)$ | @E, 32 |
| 35 | $a$ | @E, 33 |
| 36 | $\phi \supset @_a\phi$ | $\supset$E, 34, 35 |
| 37 | $@_c(\phi \supset @_a\phi)$ | @I, 36 |
| 38 | $@_c a \supset (@_c(\phi \supset @_a\phi))$ | $\supset$I, 33 to 37 |
| 39 | $@_c(a \supset (\phi \supset @_a\phi)) \supset$ $(@_c a \supset @_c(\phi \supset @_a\phi))$ | $\supset$I, 32 to 38 |
| 40 | $@_c a \supset @_c(\phi \supset @_a\phi)$ | $\supset$E, 31, 39 |

We simply copy the derivation from (Dist-Right) in Section 4.1 to translate line 7. Line 8 proves easier still.

| | | |
|---|---|---|
| 7 | $@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi)$ | (D.R.) |
| 8 | $(@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi)) \supset$ $(@_c a \supset (@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi)))$ | (Taut) |

$$\Downarrow$$

| | | |
|---|---|---|
| 41 | $\quad\mid\ \underline{@_a(\phi \supset \psi)}$ | Assp $\supset$I |
| 42 | $\quad\mid\quad\mid\ \underline{@_a\phi}$ | Assp $\supset$I |
| 43 | $\quad\mid\quad\mid\ @_a\mid\ \phi \supset \psi$ | @E, 41 |
| 44 | $\quad\mid\quad\mid\qquad\mid\ \phi$ | @E, 42 |
| 45 | $\quad\mid\quad\mid\qquad\mid\ \psi$ | $\supset$E, 43, 44 |
| 46 | $\quad\mid\quad\mid\ @_a\psi$ | @I, 45 |
| 47 | $\quad\mid\ @_a\phi \supset @_a\psi$ | $\supset$I, 42 to 46 |
| 48 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | $\supset$I, 41 to 47 |
| 49 | $\quad\mid\ \underline{@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)}$ | Assp $\supset$I |
| 50 | $\quad\mid\quad\mid\ \underline{@_c a}$ | Assp $\supset$I |
| 51 | $\quad\mid\quad\mid\ @_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | R, 49 |
| 52 | $\quad\mid\ @_c a \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi))$ | $\supset$I, 50 to 51 |
| 53 | $(@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)) \supset$ $(@_c a \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)))$ | $\supset$I, 49 to 52 |

Lines 9 and 10, both theorems, turn into more subderivations. All of which contribute to deriving the formulas at lines 10 and 11 in the FHL derivation. Conveniently, line 12 is derivable by ($\supset$E) with steps 40 and 76.

9 $@_c a \supset (@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi))$ (MP 7, 8)

$$\Downarrow$$

| | | |
|---|---|---|
| 54 | $@_c a$ | Assp $\supset$I |
| 55 | $@_c(\phi \supset @_a\phi)$ | Assp $\supset$I |
| 56 | $@_c\phi$ | Assp $\supset$I |
| 57 | $@_c$ $\phi$ | @E, 56 |
| 58 | $\phi \supset @_a\phi$ | @E, 55 |
| 59 | $@_a\phi$ | $\supset$E, 57, 58 |
| 60 | $@_c@_a\phi$ | @I, 59 |
| 61 | $@_c\phi \supset @_c@_a\phi$ | $\supset$I, 56 to 60 |
| 62 | $@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi)$ | $\supset$I, 55 to 61 |
| 63 | $@_c a \supset (@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi))$ | $\supset$I, 54 to 62 |

| | | |
|---|---|---|
| 10 | $(@_c a \supset (@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi))) \supset$ | |
| | $((@_c a \supset @_c(\phi \supset @_a\phi)) \supset (@_c a \supset (@_c\phi \supset @_c@_a\phi)))$ | (Taut) |
| 11 | $(@_c a \supset @_c(\phi \supset @_a\phi)) \supset (@_c a \supset (@_c\phi \supset @_c@_a\phi))$ | (MP 9,10) |
| 12 | $@_c a \supset (@_c\phi \supset @_c@_a\phi)$ | (MP 6,11) |

$$\Downarrow$$

| | | |
|---|---|---|
| 64 | $@_c a \supset (@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi))$ | Assp $\supset$I |
| 65 | $@_c a \supset (@_c(\phi \supset @_a\phi))$ | Assp $\supset$I |
| 66 | $@_c a$ | Assp $\supset$I |
| 67 | $@_c\phi$ | Assp $\supset$I |
| 68 | $@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi)$ | $\supset$E, 64, 66 |
| 69 | $@_c(\phi \supset @_a\phi)$ | $\supset$E, 65, 66 |
| 70 | $@_c\phi \supset @_c@_a\phi$ | $\supset$E, 68, 69 |
| 71 | $@_c@_a\phi$ | $\supset$E, 67, 70 |
| 72 | $@_c\phi \supset @_c@_a\phi$ | $\supset$I, 67 to 71 |
| 73 | $@_c a \supset (@_c\phi \supset @_c@_a\phi)$ | $\supset$I, 66 to 72 |
| 74 | $(@_c a \supset (@_c(\phi \supset @_a\phi))) \supset$ | |
| | $(@_c a \supset (@_c\phi \supset @_c@_a\phi))$ | $\supset$I, 65 to 73 |
| 75 | $(@_c a \supset (@_c(\phi \supset @_a\phi) \supset (@_c\phi \supset @_c@_a\phi))) \supset$ | |
| | $((@_c a \supset @_c(\phi \supset @_a\phi)) \supset (@_c a \supset (@_c\phi \supset @_c@_a\phi)))$ | $\supset$I, 64 to 73 |
| 76 | $(@_c a \supset @_c(\phi \supset @_a\phi)) \supset (@_c a \supset (@_c\phi \supset @_c@_a\phi))$ | $\supset$E, 63, 74 |
| 77 | $@_c a \supset (@_c\phi \supset @_c@_a\phi)$ | $\supset$E, 40, 76 |

Lines 13 and 14 introduce, unsurprisingly, more subderivations to the FHL translation.

| | | |
|---|---|---|
| 13 | $(@_c a \supset (@_c \phi \supset @_c @_a \phi)) \supset ((@_c a \& @_c \phi) \supset @_c @_a \phi)$ | (Taut) |
| 14 | $(@_c a \& @_c \phi) \supset @_c @_a \phi$ | (MP 12,13) |

$$\Downarrow$$

| | | |
|---|---|---|
| 78 | $\quad @_c a \supset (@_c \phi \supset @_c @_a \phi)$ | Assp ⊃I |
| 79 | $\quad\quad @_c a \& @_c \phi$ | Assp ⊃I |
| 80 | $\quad\quad @_c a$ | &E1, 79 |
| 81 | $\quad\quad @_c \phi$ | &E2, 79 |
| 82 | $\quad\quad @_c \phi \supset @_c @_a \phi$ | ⊃E, 78, 80 |
| 83 | $\quad\quad @_c @_a \phi$ | ⊃E, 81, 82 |
| 84 | $\quad (@_c a \& @_c \phi) \supset @_c @_a \phi$ | ⊃I, 79 to 83 |
| 85 | $(@_c a \supset (@_c \phi \supset @_c @_a \phi)) \supset ((@_c a \& @_c \phi) \supset @_c @_a \phi)$ | ⊃I, 78 to 84 |
| 86 | $(@_c a \& @_c \phi) \supset @_c @_a \phi$ | ⊃E, 77, 85 |

Happily Section 4.1 includes an equivalent proof for (S.R.) which justifies 15. That gives the minor premise needed to derive the conclusion.

15 $\quad @_c @_a \phi \supset @_a \phi \quad$ (S.R.)

$$\Downarrow$$

| | | |
|---|---|---|
| 87 | $\quad @_a @_c \phi$ | Assp ⊃I |
| 88 | $\quad @_a \mid @_c \phi$ | @E, 87 |
| 89 | $\quad @_c \phi$ | @R-out, 88 |
| 90 | $@_a @_c \phi \supset @_c \phi$ | ⊃I, 87 to 89 |

The major premise we need to conclude the derivation comes from incorporating a derivation of line 16 and then inferring, by (⊃E), the formula from line 17.

| | | |
|---|---|---|
| 16 | $((@_c a \& @_c \phi) \supset @_c @_a \phi) \supset$ | |
| | $((@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi))$ | (Taut) |
| 17 | $(@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi)$ | (MP 14,16) |
| 18 | $(@_c a \& @_c \phi) \supset @_a \phi$ | (MP 15,17) |

$$\Downarrow$$

| | | |
|---|---|---|
| 91 | $(@_c a \& @_c \phi) \supset @_c @_a \phi$ | Assp $\supset$I |
| 92 | $@_c @_a \phi \supset @_a \phi$ | Assp $\supset$I |
| 93 | $@_c a \& @_c \phi$ | Assp $\supset$I |
| 94 | $@_c @_a \phi$ | $\supset$E, 91, 93 |
| 95 | $@_a \phi$ | $\supset$E, 92, 94 |
| 96 | $(@_c a \& @_c \phi) \supset @_a \phi$ | $\supset$I, 93 to 95 |
| 97 | $(@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi)$ | $\supset$I, 92 to 96 |
| 98 | $((@_c a \& @_c \phi) \supset @_c @_a \phi) \supset$ | |
| | $((@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi))$ | $\supset$I, 91 to 97 |
| 99 | $(@_c @_a \phi \supset @_a \phi) \supset ((@_c a \& @_c \phi) \supset @_a \phi)$ | $\supset$E, 86, 98 |
| 100 | $(@_c a \& @_c \phi) \supset @_a \phi$ | $\supset$E, 90, 99 |

Citing 90 and 99 with ($\supset$E) concludes the FHL translation. But 100 steps looks like an awful lot more than what $A_{\mathscr{H}(\mathscr{O})}$ required. Why go through the extra hassle to prove what $A_{\mathscr{H}(\mathscr{O})}$ derives in just 14 lines? In FHL, it does not actually take so many steps. This is only one derivation of (Nom) in FHL. There are other, simpler, derivations. This derivation does not make the most effective use of FHL's most useful feature - nested subproofs. Chapter 5 points out ways to make sure any given FHL derivation is as simple as it can be.

### 4.3.2 K axiom

In Section 4.2, we presented a translation procedure for turning an FHL derivation into a derivation in $A_{\mathscr{H}(\mathscr{O})}$. In this section we apply the translation process to the K axiom, beginning with an FHL derivation of the K axiom and applying the translation procedure to produce an axiomatic proof.

**Phase 1:** Construct a derivation of the K axiom in FHL.[7]

| | | |
|---|---|---|
| 1.1 | $@_b$ \| \| $@_a(\phi \supset \psi)$ | A |
| 1.2 | \| \| \| $@_a\phi$ | A |
| 1.3 | \| \| \| $@_a$ \| $\phi \supset \psi$ | @E, 1.1 |
| 1.4 | \| \| \| \| $\phi$ | @E, 1.2 |
| 1.5 | \| \| \| \| $\psi$ | ⊃E, 1.3, 1.4 |
| 1.6 | \| \| \| $@_a\psi$ | @I, 1.5 |
| 1.7 | \| \| $@_a\phi \supset @_a\psi$ | ⊃I, 1.2 to 1.6 |
| 1.8 | \| $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | ⊃I, 1.1 to 1.7 |
| 1.9 | $@_b[@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)]$ | @I, 1.8 |

**Phase 2:** Remove the nominal labeled subderivation at steps 1.3 - 1.5.

After writing the basic derivation, the translation process commences by eliminating the nominal subproof at 1.3-1.5. Steps 2.3 and 2.4 are easy to handle because by TS13 they simply become applications of standard reiteration. We have to insert (Dist-Right) and apply (⊃E) to justify 2.5. Then apply (⊃E) again to get $@_a\psi$ at step 2.7.

[7] This formulation of the K axiom comes from Braüner, *Hybrid Logic and Its Proof-theory*, p. 27.

| | | |
|---|---|---|
| 2.1 | $@_b \quad @_a(\phi \supset \psi)$ | A |
| 2.2 | $\qquad @_a\phi$ | A |
| 2.3 | $\qquad @_a(\phi \supset \psi)$ | R, 2.1 |
| 2.4 | $\qquad @_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | Dist-Right |
| 2.5 | $\qquad @_a\phi \supset @_a\psi$ | $\supset$E, 2.3, 2.4 |
| 2.6 | $\qquad @_a\phi$ | R, 2.2 |
| 2.7 | $\qquad @_a\psi$ | $\supset$E, 2.5, 2.6 |
| 2.8 | $\quad @_a\phi \supset @_a\psi$ | $\supset$I, 2.2 to 2.7 |
| 2.9 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | $\supset$I, 2.1 to 2.8 |
| 2.10 | $@_b[@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)]$ | @I, 2.9 |

**Phase 3:** Eliminate the hypothetical subderivation at steps 2.2-2.7.

Insert the tautology at 3.3 to justify the reduction of step 2.3 to 3.4. Step 3.5 reiterates the tautology at step 3.2. Step 3.6 is a tautology inserted to infer, with the axiom at step 3.7, the reduced form of 2.4 at 3.8. Then we can use ($\supset$E) from 3.4 and another tautology at 3.9 to infer 3.10. Add another distribution tautology to get 3.11 and ($\supset$E) on 3.2 and 3.11 justifies 3.12.

| | | |
|---|---|---|
| 3.1 | $@_b$ $\quad$ $@_a(\phi \supset \psi)$ | Assp $\supset$I |
| 3.2 | $@_a\phi \supset @_a\phi$ | Tautology |
| 3.3 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a(\phi \supset \psi))$ | Tautology |
| 3.4 | $@_a\phi \supset @_a(\phi \supset \psi)$ | $\supset$E, 3.1, 3.3 |
| 3.5 | $@_a\phi \supset @_a\phi$ | Tautology |
| 3.6 | $(@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)) \supset$ | |
| | $\quad (@_a\phi \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)))$ | Tautology |
| 3.7 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | Dist-Right |
| 3.8 | $@_a\phi \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi))$ | $\supset$E, 3.6, 3.7 |
| 3.9 | $(@_a\phi \supset @_a(\phi \supset \psi)) \supset (@_a\phi \supset (@_a\phi \supset @_a\psi))$ | Tautology |
| 3.10 | $@_a\phi \supset (@_a\phi \supset @_a\psi)$ | $\supset$E, 3.4, 3.9 |
| 3.11 | $@_a\phi \supset @_a\phi$ | Tautology |
| 3.12 | $(@_a\phi \supset (@_a\phi \supset @_a\psi)) \supset$ | |
| | $\quad ((@_a\phi \supset @_a\phi) \supset (@_a\phi \supset @_a\psi))$ | Tautology |
| 3.13 | $(@_a\phi \supset @_a\phi) \supset (@_a\phi \supset @_a\psi)$ | $\supset$E, 3.10, 3.12 |
| 3.14 | $@_a\phi \supset @_a\psi$ | $\supset$E, 3.11, 3.13 |
| 3.15 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi)$ | $\supset$I, 3.1 to 3.14 |
| 3.16 | $@_b(@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\psi))$ | @I, 3.15 |

**Phase 4:** Eliminate the last hypothetical subderivation from 3.1 - 3.14.

If phase 3 failed to, phases 4 and 5 would certainly reveal the tediousness inherent to the process. For that reason, those phases are truncated but still show how the translation progresses.

| | | | |
|---|---|---|---|
| 4.1 | $@_b$ | $@_a(\phi \supset \psi) \supset @_a(\phi \supset \psi)$ | Tautology |
| 4.2 | | $\phi \supset \phi$ | Tautology |
| 4.3 | | $@_a(\phi \supset \phi)$ | $N_{@}$, 4.2 |
| 4.4 | | $@_a(\phi \supset \phi) \supset (@_a\phi \supset @_a\phi)$ | Dist-Right |
| 4.5 | | $@_a\phi \supset @_a\phi$ | $\supset$E, 4.3, 4.4 |
| 4.6 | | $(@_a\phi \supset @_a\phi) \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\phi))$ | Tautology |
| 4.7 | | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\phi)$ | $\supset$E, 4.5, 4.6 |
| | | $\vdots$ | |

**Phase 5:** Eliminate the outermost nominal subderivation.

Remember that a nominal subproof is just a list of things that are true at the nominal in the label. Since most of the formulas in phase 4 turn out to be tautologies or products of $A_{\mathscr{H}(\mathscr{O})}$ rules, justifying the steps in phase 5 is easy. We can write these tautologies just as they are written in phase 4 and use the rule ($N_{@}$) to get the equivalent formula past the nominal subproof translation. Step 5.1 is obviously a tautology. Hence we can derive step 5.2 and apply the same process for every other tautology from phase 4.

| | | |
|---|---|---|
| 5.1 | $@_a(\phi \supset \psi) \supset @_a(\phi \supset \psi)$ | Tautology |
| 5.2 | $@_b(@_a(\phi \supset \psi) \supset @_a(\phi \supset \psi))$ | $N_@$, 5.1 |
| 5.3 | $\phi \supset \phi$ | Tautology |
| 5.4 | $@_a(\phi \supset \phi)$ | $N_@$, 5.3 |
| 5.5 | $@_b(@_a(\phi \supset \phi))$ | $N_@$, 5.4 |
| 5.6 | $@_a(\phi \supset \phi) \supset (@_a\phi \supset @_a\phi)$ | Dist-Right |
| 5.7 | $@_b(@_a(\phi \supset \phi) \supset (@_a\phi \supset @_a\phi))$ | $N_@$, 5.6 |
| 5.8 | $@_a\phi \supset @_a\phi$ | $\supset$E, 5.4, 5.6 |
| 5.9 | $@_b(@_a\phi \supset @_a\phi)$ | $N_@$, 5.8 |
| 5.10 | $(@_a\phi \supset @_a\phi) \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\phi))$ | Tautology |
| 5.11 | $@_b((@_a\phi \supset @_a\phi) \supset (@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\phi)))$ | $N_@$, 5.10 |
| 5.12 | $@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\phi)$ | $\supset$E, 5.8, 5.10 |
| 5.13 | $@_b(@_a(\phi \supset \psi) \supset (@_a\phi \supset @_a\phi))$ | $N_@$, 5.12 |
| | $\vdots$ | |

Once the rest of phase 5 is translated, the very last thing is to erase the main scope line and replace ($\supset$E) justifications with (MP).

# Chapter 5

# Conclusion

The primary motivation behind developing hybrid logic is increasing the expressive power of modal logic. One way to interpret that motivation is demand for reasoning about what happens at specific points in a model (states, times, possible worlds, etc.). The ability to do just that, explicitly, differentiates FHL. Braüner's $N_{\mathscr{H}(\mathscr{O})}$ requires every formula appears in a satisfaction statement. That restriction prohibits reasoning about just the content of the satisfaction statement. For instance, the $N_{\mathscr{H}(\mathscr{O})}$ version of (⊃I) concludes with a conditional in a satisfaction statement.

$$\dfrac{\begin{array}{c}[@_a\phi]\\ \vdots \\ @_a\psi\end{array}}{@_a(\phi \supset \psi)} \supset I$$

Notably, the conclusion is not the formula we would normally expect. Namely, $@_a\phi \supset @_a\psi$, which is the conclusion FHL derives from the same argument. But suppose $\phi \vdash \psi$. In FHL we can derive both $@_a(\phi \supset \psi)$ and $@_a\phi \supset @_a\psi$. Deriving the former in FHL requires only minimal adjustments to the derivation. Whereas the latter directly results from the stock rules in FHL, it is not derivable at all in $N_{\mathscr{H}(\mathscr{O})}$.

Other tree-style systems exist without the satisfaction statement restriction. These systems reason directly with what occurs at such and such a world just like FHL. But these systems tend to lack the intuitive clarity of Fitch-style systems, which seems contrary to the impetus behind hybrid logic.

The expressive power hybrid logic adds results from Arthur Prior's great insight that there can be more than one kind of propositional symbol. This paper has dealt with two, but that is not to say there are only two. Robert Bull authored a system with three: the standard predicate logic symbols, nominals, and history-propositional variables. Bull defined the third type as atomic symbols that "represent the names of courses of history..."[1] FHL is probably amenable to other propositional symbols, such as Bull's, in the same way it extends FSL with rules for binders and nominals. Whether $N_{\mathscr{H}(\mathscr{O})}$ scales up to accommodate other propositional symbols is unclear. Since the history-propositional variable is an atomic symbol, attaching it to $N_{\mathscr{H}(\mathscr{O})}$ would mean dropping the satisfaction statement requirement. But without that requirement, little distinguishes $N_{\mathscr{H}(\mathscr{O})}$ from other tree-style systems for hybrid logic. But then, Braüner appears less interested in setting $N_{\mathscr{H}(\mathscr{O})}$ apart from other hybrid logic systems than in comparing hybrid logic proof theory with modal logic. Braüner introduces his notion of well-behaved proof theories to contrast the proof theory of hybrid logic with that of regular modal logic which he argues is not well-behaved.[2] According to Braüner, there are three properties a well-behaved proof system must have:

1. Introduction and elimination rules for each connective satisfy Prawitz's inversion principle.

2. Accessibility relations can be incorporated in a uniform way by just adding appropriate rules.

3. Normalized derivations satisfy the quasi-subformula property (QSFP).

[1] Bull, "An Approach to Tense Logic", p. 291.
[2] Braüner, *Hybrid Logic and Its Proof-theory*, p. 213.

The Inversion Principle gets its name from the fact that elimination rules are to introduction rules the way a function is to its inverse.[3] This principle gives us a way to optimize derivations by removing unneeded formulas, by stating that introducing a formula to be the major premise of an elimination rule adds nothing significant to a derivation.

Given $\phi$ and $\psi$, by (&I) we can infer $\phi\&\psi$. Now if we find that later in the derivation we need $\psi$, we can conclude that from (&E) on $\phi\&\psi$. But $\psi$ is a premise of the (&I) application. So the conclusion of (&E) could be derived from the premises of (&I) by (R).

| | | |
|---|---|---|
| 1 | $\phi$ | |
| 2 | $\psi$ | |
| | $\vdots$ | |
| $i$ | $\phi\&\psi$ | &I, 1, 2 |
| | $\vdots$ | |
| $j$ | $\psi$ | &E2, $i$ |

**Figure 5.1:** Non-normal derivation with (&I)

The formula $\phi\&\psi$ at step $i$ in Figure 5.1 is an example of a *maximum formula*. According to the inversion principle, a derivation containing a maximum formula can be rewritten to omit that formula through *proper reduction rules*.[4] Braüner shows that his rules for conjunction and implication satisfy the inversion principle and he even develops the reduction rules to go along with those operators.[5] His arguments work just as well for the FHL counterparts to those rules. Although, of course, the reduction rules would have a different structure suited to Fitch-style. Proving FHL satisfies the inversion principle calls

[3]Prawitz, *Natural Deduction*, p. 33.
[4]Braüner, *Hybrid Logic and Its Proof-theory*, p. 24.
[5]Ibid., pp. 24,37-38.

for developing reduction rules for the other connectives. But since FHL is equivalent to $N_{\mathscr{H}(\mathscr{O})}$, such rules sound plausible. Satisfying the second point is more certain.

There are several accessibility relations commonly seen in the literature, reflexivity, transitivity, etc. But with the expressive power added by nominals, hybrid logic boasts some relations unavailable in typical modal logic. In normal modal logic, the relation between possible worlds is discussed in semantic terms. For example, symmetry would normally be described by stating the elements of the set of relations, $R$. So if $a$ and $c$ are nominals in a symmetric model, both $R(a,c)$ and $R(c,a)$. But with nominals, we have other options for talking about accessibility relations. Since nominals represent possible worlds in the object language, it stands to reason that the accessibility relation between those possible worlds could be formed in the object language too.

Since $R(a,c)$ simply states that $c$ is accessible from $a$, and the nominals $a$ and $c$ make it into the object language, the hybrid formula $@_a\Diamond c$ says exactly the same thing as $R(a,c)$. By the same reasoning, $@_a c$ expresses $a = c$. These satisfaction statements enable hybrid logic formulas to express accessibility relations. These accessibility formulas then lead to inference rules we can incorporate in FHL.[6]

| | | |
|---|---|---|
| 1 | Symmetry | $\forall a \forall c[@_a\Diamond c \supset @_c\Diamond a]$ |
| 2 | Antisymmetry | $\forall a \forall c[(@_a\Diamond c \& @_c\Diamond a) \supset @_a c]$ |
| 3 | Reflexivity | $\forall a(@_a\Diamond a)$ |
| 4 | Irreflexivity | $\forall a(@_a\Diamond a \supset \bot)$ |

**Figure 5.2:** Hybrid logic formulas for accessibility relations

The best way to integrate rules for accessibility relations creates a new system, FHL+Θ where Θ is the set of accessibility rules. This way, FHL remains unchanged but keeps the option of adding accessibility rule(s) to fit any occasion.

Rules in Θ, but also (¬E) and (¬I) actually, resist the normal reduction procedures that eliminate maximum formulas. Instead they introduce permutable formulas. It is not unrea-

[6] Simpson, "The Proof Theory and Semantics of Intuitionistic Modal Logic"; Braüner, *Hybrid Logic and Its Proof-theory*, p. 73.

**Symmetry**

| | | |
|---|---|---|
| $i$ | $@_a\Diamond c$ | |
| $j$ | (subproof) $@_c\Diamond a$ | |
| | $\vdots$ | |
| $j'$ | $\phi$ | |
| $k$ | $\phi$ | $R_{\theta_1}$, $i$, $j$ to $j'$ |

**Antisymmetry**

| | | |
|---|---|---|
| $i$ | $@_a\Diamond c$ | |
| $i+1$ | $@_c\Diamond a$ | |
| $j$ | (subproof) $@_a c$ | |
| | $\vdots$ | |
| $j'$ | $\phi$ | |
| $k$ | $\phi$ | $R_{\theta_2}$, $i$, $i+1$, $j$ to $j'$ |

**Reflexivity**

| | | |
|---|---|---|
| $i$ | (subproof) $@_a\Diamond a$ | |
| | $\vdots$ | |
| $i'$ | $\phi$ | |
| $j$ | $\phi$ | $R_{\theta_3}$, $i$ to $i'$ |

**Irreflexivity**

| | | |
|---|---|---|
| $i$ | $@_a\Diamond a$ | |
| | $\vdots$ | |
| $i'$ | $\bot$ | |
| $j$ | $\phi$ | $R_{\theta_3}$, $i$ to $i'$ |

**Figure 5.3:** FHL accessibility relation rules

sonable to suspect permutative reductions are available for FHL. And if it is the case that permutable reductions exist for FHL in addition to maximum reductions, it follows that repeatedly applying the reductions normalizes any FHL derivation. A normalized derivation is notable for its clarity and efficiency attributable to the absence of expendable formulas. Supposing the normalization technique applies to FHL, the next point to consider on the behavior checklist concerns the quasi-subformula property (QSFP).

The satisfaction operator keeps hybrid logic from otherwise satisfying the subformula property - that for any formula in a normal derivation, it is either a subformula of an open assumption or the conclusion. QSFP amends the subformula property making it available to hybrid logic. Approaching QSFP is untenable given this project's scope. FHL would need to be revised before beginning the proof that FHL satisfies QSFP. Braüner accounts for caveats in his system while presenting it. But the necessary considerations fell by the wayside in presenting FHL. A version satisfying QSFP would most likely include restrictions on (¬E) and (¬I) similar to how Braüner's restricts (⊥1). An alternative approach might simply make an exception for the negation rules in stating QSFP. Though making another exception to SFP, essentially creating the quasi-quasi-subformula property, looks dubious. No matter how those modifications work, either way probably requires adding to the definition of a normal derivation that every step but the conclusion is cited at some subsequent step.

These points, while interesting, would distract from the proper focus of this project: presenting a Fitch-style system for hybrid logic. Not to mention adding caveat after caveat would undermine what is perhaps the most significant benefit of FHL: its ease of use. FHL naturally bridges the span between sentential logic and higher logics and, as a bonus, FHL takes the meta-logical properties hybrid logic internalizes and formalizes those properties in a system recognizable to introductory students.

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