Equivalence of Mirror Families Constructed by Toric Degenerations of Flag Varieties

by

Joe Rusinko

(Under the direction of Valery Alexeev)

Abstract

Batyrev (et. al.) constructed a family of Calabi-Yau varieties using small toric degenerations of the full flag variety G/B. They conjecture this family to be mirror to generic anticanonical hypersurfaces in G/B. Recently Alexeev and Brion, as a part of their work on toric degenerations of spherical varieties, have constructed many degenerations of G/B. For any such degeneration we construct a family of varieties, which we prove coincides with Batyrev's in the small case. We prove that any two such families are birational, thus proving that mirror families are independent of the choice of degeneration. The birational maps involved are closely related to Berenstein and Zelevinsky's geometric lifting of tropical maps to maps between totally positive varieties.

Index words: Toric Degenerations, Calabi-Yau Varieties, Mirror Symmetry

Equivalence of Mirror Families Constructed by Toric Degenerations of Flag Varieties

by

Joe Rusinko

B.S., Davidson College, 2001

A Dissertation Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

Doctor of Philosophy

Athens, Georgia

2007

Equivalence of Mirror Families Constructed by Toric Degenerations of Flag Varieties

by

Joe Rusinko

Approved:

Major Professor: Valery Alexeev

Committee: Bill Graham
Mitch Rothstein
Robert Varley
Miljenko Zabcic

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2007

Acknowledgments

I would like to thank all of the professors, graduate students and departmental staff at the University of Georgia for their role in helping me progress over the past five years. I would especially to thank my committee members for the time they spent shaping me as a mathematician.

Valery Alexeev's inspiring lectures got me interested in algebraic geometry. He suggested my thesis problem one day in my office and patiently helped me understand the what the question was asking for the next year. It was his insight and perseverance that helped guide me through these problems. Most importantly, he has never settled for anything less than my best.

I will always be indebted to Robert Varley for the countless hours spent in his office trying to clarify and develop any mathematical ideas I happened to be wondering about at the time. These conversations helped me truly understand the subtleties of our discipline. Thanks to Sybilla Beckman for introducing me to the world of mathematics education and for her help in developing me as a teacher. I appreciate the time Dr. Galewski and Dr. Shifrin spent developing both my mathematical and teaching abilities.

Michael Guy and Val Cormani have always been there to share their ideas and advice. They have been both admired colleagues and coveted friends. I want to thank Deana, Craig, Paul and Tremaine for helping me through the nonacademic challenges of graduate school.

Finally, I would like to thank my parents who have always been there for me throughout the past 27 years. My mom has always encouraged and supported my dreams. Her faith in me has been a buoy when times were tough. My dad has shown me how hard work and compassion can lead to great success. I would not have made it through graduate school without their support.

Table of Contents

List of Figures

Chapter 1

Introduction

In the late 1980's, string theorists were studying a type of algebraic variety known as a Calabi-Yau. They discovered a surprising relationship between pairs of Calabi-Yau varieties now known as Mirror Symmetry.

To any three (complex) dimensional Calabi-Yau, physicists associate two distinct physical theories of the Universe, the A and B models. Two Calabi-Yau manifolds are considered mirror if the A-model for one is physically equivalent to the B-model for the other and vice versa. Often, if a calculation is difficult on a particular model it may be replaced by a more tractable problem on its mirror.

Using this relationship, physicists were able to predict solutions to previously unsolved mathematical problems. For example, they were able to predict the number of holomorphic rational curves of a given degree on a quintic hypersurface in $\mathbb{P}^4$. Mathematical confirmation of these predictions sparked great interest among mathematicians in string theory. During the past 20 years, string theory has been put into a more rigorous setting and mirror symmetry has become a power mathematical tool.

Classical examples of Calabi-Yau varieties include generic anticanonical hypersurfaces in flag varieties. One would like to find a family of varieties which are mirror to these hypersurfaces. Givental constructed such a family for full flag varieties. This construction was extended to partial flags in a more combinatorial setting by Batyrev, Ciocan-Fontanine, Kim, and van Straten.

Their construction relied on a particular degeneration of the flag variety to a toric variety corresponding to a reflexive polytope Δ. Every reflexive polytope has a dual polytope Δ^*, which in turn corresponds to a toric variety X_{Δ^*} along with its anticanonical class $-K_{X_{\Delta^*}}$. The mirror family of Calabi-Yau varieties were conjectured to be a special subfamily of the linear system $|-K_{X_{\Delta^*}}|$ whose coefficients satisfy a set of relations called "box equations". In later work, Batyrev showed that this construction could be applied to any Fano variety by using a nice class of toric degenerations. He called them *small toric degenerations*.

Recently, Caldero-Alexeev-Brion constructed a set of toric degenerations of the flag variety which we call *string degenerations*. We cannot use the aforementioned mirror constructions since not all string degenerations are small. The primary aim of this thesis is to construct an appropriate mirror family using these degenerations. The biggest challenge is identifying the appropriate subfamily of mirror candidates in $|-K_{X_{\Delta^*}}|$. Using combinatorial description of the degenerations, we are able to define a set of relations called "combinatorial box equations". The subfamily of $|-K_{X_{\Delta^*}}|$ whose coefficients satisfy these relations are the appropriate mirror candidates. This family coincides with the Batyrev's when the string degenerations are small.

We give an explicit birational map between any two such families. This is especially nice for two reasons. Since most mathematical definitions of "mirror properties" are birational invariants, this shows that the mirror families defined are independent of the choice of string degeneration. Finally, the toric limits of the G/B vary greatly depending on the choice of string degeneration. Our results show that these mirror families are a way of tying all of these degenerations together.

Chapter 2

Algebraic Geometry Background

2.1 Calabi-Yau Varieties

In this chapter we fix notation and review the standard results in algebraic geometry that we will need in our mirror construction.

Fix the ground field $\mathbb{C}$. Let X be a smooth irreducible projective variety of dimension n. We can define $\omega_X := \wedge^n \Omega_X$ the nth exterior power of the sheaf of differentials on X. To this sheaf we can associate a divisor class K_X called the canonical class.

Definition 2.1.1. *A smooth projective variety X is called a smooth Calabi-Yau if $K_X \sim 0$ (i.e. K_X is the divisor of a rational function on X) with $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$.*

Example 2.1.2. *Elliptic curves and generic degree $(n+1)$ hypersurfaces in $\mathbb{P}^n$ are examples of smooth Calabi-Yau varieties.*

Although the study of mirror symmetry generally involves smooth Calabi-Yau manifolds, it is sometimes necessary to extend the notion to singular varieties. In order to do this, we review some basic facts and definitions from singularity theory.

2.1.1 Singularities

Definition 2.1.3. *We say that a variety X has ($\mathbb{Q}$)-factorial singularities if every Weil divisor D on X is ($\mathbb{Q}$)-Cartier. (i.e. there exists a positive integer d such that dD is a Cartier divisor).*

Definition 2.1.4. *For any normal variety X, we can define a Weil divisor K_X by extending the anticanonical line bundle on the nonsingular locus of X to a line bundle on X, and then taking the associated divisor. (see [M02, Rem4-1-2] for details).*

Definition 2.1.5. *A normal variety X has ($\mathbb{Q}$)-Gorenstein singularities if K_X is ($\mathbb{Q}$)-Cartier.*

A proper birational map $f : Y \to X$ from a smooth projective variety Y is called a resolution of singularities of X. If X and Y are normal, and X has only $\mathbb{Q}$-Gorenstein singularities then

$$K_Y = f^* K_X + \Sigma a_i E_i$$

where E_i are the exceptional divisors of the resolution. The collection of coefficients $a := (a_1, a_2 \cdots a_n)$ is called the *discrepancy* of the resolution.

Definition 2.1.6. *If for a normal variety X there exists a resolution of singularities such that for every $a_i \in a$:*

1. $a_i > 0$ *We say that X has terminal singularities*
2. $a_i \geq 0$ *We say that X has canonical singularities*
3. $a_i > -1$ *We say that X has log-terminal singularities*
4. $a_i \geq -1$ *We say that X has log-canonical singularities*

For more details on these definitions and singularity types see [M02, Chpt. 4]. Some classical mirror constructions require the use of singular varieties. Since we are working over $\mathbb{C}$ there exists a resolution of singularities for any of our varieties. It is not always possible to both resolve the singularities of our variety and maintain a trivial canonical class. Instead, we use a partial resolution which doesn't affect the canonical class.

Definition 2.1.7. *A birational map between normal varieties*

$$f : Y \to X$$

is called crepant if $K_Y = f^*K_X$*.*

Definition 2.1.8. *[Bat94, def.2.2.13] Let* $\phi : W' \to W$ *be a projective birational morphism of normal* $\mathbb{Q}$*-Gorenstein algebraic varieties. Then* ϕ *is called a maximal projective crepant partial desingularization (MPCP-desingularization) of* W *if* ϕ *is crepant and* W' *has only* $\mathbb{Q}$*-factorial terminal singularities.*

Definition 2.1.9. *Let* X *be an normal irreducible projective variety of dimension* n*. We say that* X *is a Calabi-Yau variety if*

1. *X has only Gorenstein canonical singularities*
2. $K_X \sim 0$
3. $H^i(X, \mathcal{O}_X) = 0$ *for* $0 < i < n$*, where* $\mathcal{O}_X$ *is the sheaf of regular functions on* X

If a singular variety has $K_X \sim 0$ and an MPCP-desingularization exists, then its MPCP-desingularization will be a (possibly singular) Calabi-Yau.

2.2 Fano Varieties

We would like to start our study with a source of smooth Calabi-Yau varieties. One such source is the anticanonical class of a Fano Variety.

Definition 2.2.1. *A smooth variety* X *is called Fano if its anticanonical class* $-K_X$ *is an ample divisor.*

Lemma 2.2.2. *Let* X *be a Fano variety. Smooth elements of the linear system* $|-K_X|$ *are smooth Calabi-Yau varieties.*

Proof. Let D be a smooth element in $|-K_X|$. By the adjuction formula we have $K_D = (D+K_X)|_D = (-K_X+K_X)|_D = 0|_D = 0$. We have $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$ by Kodiara vanishing. □

2.3 Flag Varieties

Flag Varieties provide a large class of Fano Varieties. We rely on Michel Brion's lectures on flag varieties for complete proofs about the basic facts of flag varieties which we review here [Br05]. Another standard reference is Fulton's book on Young Tableaux [Ful97].

A flag of vector spaces of type $(d_1, d_2, \cdots, d_r)$ is collection of vectors spaces

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_r \subsetneq \mathbb{C}^{n+1}$$

such that $\dim V_i/V_{i-1} = d_i$.

Definition 2.3.1. *Let $\{e_1, \cdots, e_{n+1}\}$ be an ordered basis for $\mathbb{C}^{n+1}$, the* standard flag *of type $(d_1, d_2, \cdots, d_r)$ is the collection of coordinate subspaces*

$$0 \subseteq \langle e_1, \cdots, e_{d_1} \rangle \subseteq \cdots \subseteq \langle e_1, \cdots, e_{\Sigma_{i=1}^r d_r} \rangle \subseteq \mathbb{C}^{n+1}$$

Let $F(d_1, d_2, \cdots, d_r)$ be the set of flags of type $(d_1, d_2, \cdots, d_r)$. The group $G := SL_{n+1}(\mathbb{C})$ acts transitively on F. Let $P := P(d_1, d_2, \cdots, d_r)$ be the isotropy group of the standard flag in SL_{n+1}. Then P is the set of block upper triangular matrices in SL_{n+1} with diagonal blocks of size d_i. Therefore $F(d_1, d_2, \cdots, d_r) \cong G/P(d_1, d_2, \cdots, d_r)$ has the structure of a smooth homogeneous algebraic variety of dimension $\Sigma_{1 \leq i < j \leq r} d_i d_j$. When the set $(d_1, d_2, \cdots, d_r)$ is clear we refer to these *partial flag varieties* as $F = G/P$.

Examples 2.3.2.

1. *The partial flag variety $G/P(1) = \mathbb{P}^n$.*
2. *The partial flag $G/P(k)$ is the Grassmanian of k-planes in $(n+1)$-space.*
3. *$G/P(\overbrace{1, 1, \cdots, 1}^{n})$ is called the complete flag variety. Here, P is actually a minimal parabolic (or Borel) subgroup so we refer to the complete flag as G/B.*

Remark 2.3.3. *These constructions can also be carried out for other classical Lie groups G. The case $G = SL_{n+1}$ is often referred to as the type A_n flag variety.*

2.3.1 Divisors on G/B

We review a brief description of Divisors on G/B in terms of pullbacks of hyperplane sections on Grassmanians. Note that we have a forgetful morphism $f : G/B \to G/P$ for any P. In particular we have maps $f_k : G/B \to G/P(k)$, where $G/P(k)$ is the Grassmanian of k-planes in $\mathbb{C}^{n+1}$.

Definition 2.3.4. *Let H_k be the hyperplane section on $G/P(k)$ under the Plücker embedding. We define divisors D_k on G/B by $D_k := f_k^*(H_k)$.*

The following are standard facts about G/B (see [Br05])

Theorem 2.3.5.

1. *The Picard group of G/B is a free abelian group of rank n generated by D_k for $1 \leq k \leq n$.*
2. *The ample cone of divisors in G/B consists of divisors in the classes corresponding to $\mathbb{Z}^n_{>0}$ in $\mathrm{Pic}(G/B)$. Every ample divisor on G/B is very ample.*
3. $K_{G/B} \cong -2(\Sigma_{k=1}^n D_k)$

From this we see that the anticanonical class of G/B is an ample divisor, thus the full flag variety is Fano.

Remark 2.3.6. *In fact all of the partial flag varieties are Fano. Their canonical classes are described in [Br05, Prop 2.2.8]*

Lemma 2.3.7. *Generic anticanonical hypersurfaces in G/B of type A_n are smooth Calabi-Yau varieties for $n \geq 2$.*

Proof. This follows from Bertini's theorem and the adjunction formula. □

Given this collection of smooth Calabi Yau varieties, we would like to find a family whose fibers are their mirrors. To do this we first need to sketch a mathematical formulation of mirror symmetry.

2.3.2 Mirror Varieties

Physically, two smooth Calabi-Yau varieties X and X' are mirror if the A-model conformal field theory on X is the same as the B-model conformal field theory on X' and vice versa. In this section we give some of the mathematical descriptions of this property.

The most classical mathematical formulation of mirrors involves the Hodge diamonds of the varieties.

Definition 2.3.8. *For an n-dimensional smooth projective variety X, define $h^{p,q}(X) := \dim H^p(X, \Omega^q)$. We call this collection of numbers the Hodge numbers of X.*

We note some standard properties of Hodge numbers.

1. $h^{p,q}(X) = h^{q,p}(X)$ (Hodge Decomposition Theorem [GH78, p117])

2. $h^{p,q}(X) = 0$ if either p or q is greater than the dimension of X (Grothendieck's Vanishing Theorem)

3. $h^{p,q}(X) = h^{2n-p,2n-q}(X)$ (Poincaré Duality)

Definition 2.3.9. *The* Hodge diamond *is the set of Hodge numbers $h^{p,q}$ where p and q range from 0 to the dimension of X.*

Mirror Property 1. *If two smooth Calabi-Yau varieties X and X' are mirror to one another, then $h^{p,q}(X) = h^{n-p,q}(X')$ for all values of p and q.*

Remark 2.3.10. *If X and X' satisfy these conditions then their Hodge diamonds will be mirror images of one another (hence the name "mirror" symmetry).*

Remark 2.3.11. *There is a definition of homological mirror symmetry as an equivalence of the derived Fukaya category of X with the bounded derived category of coherent sheaves on X' and vice versa. This idea is described by Kontsevich in [K94]. A slightly less abstract definition is that X and X' have the same associated GKZ-hypergeometric series as described in [CK99].*

In some of the classical mirror constructions we are forced to go outside the category of smooth Calabi-Yau varieties and consider varieties which are nonsmooth and noncompact. By the work of Deligne [Del71A, Del71B], there exists a set of invariants called Hodge Deligne numbers which are defined even when the variety is singular and/or not projective. Hodge Deligne numbers as a consequence of a mixed Hodge structure, which is a more subtle version of a classical Hodge structure.

Definition 2.3.12. *A* pure Hodge structure of weight r *on a* $\mathbb{Q}$*-vector space* H *is a decomposition of its complexification*

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=r} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$.

Given a pure Hodge structure of weight r on H we can define a descending filtration F on $H_{\mathbb{C}}$ by $F^p := \bigoplus_{s \geq p} H^{s,r-s}$. We call this filtration a *Hodge filtration*. Any Hodge filtration gives a pure Hodge structure on H.

Definition 2.3.13. *[DK86, 1.2] Let* H *be a a vector space over* $\mathbb{Q}$. *A* mixed Hodge structure *on* H *consists of an ascending weight filtration* W *on* H*, and a descending Hodge filtration* F *on* $H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C}$. *The filtration* F *must induce a pure Hodge structure of wieght* r *on the complexification of* $Gr^W H = W_r/W_{r-1}$. *In particular*

$$(Gr^W H)_{\mathbb{C}} = \bigoplus_{p+q=r} H^{p,q}.$$

We define the Hodge Deligne numbers $h^{p,q}(H) := \dim H^{p,q}$.

Since the varieties involved need not be compact, it is often easier to use cohomology with compact support denoted $H^*_c(X)$. Here are some basic facts about cohomology with compact support, and it's associated mixed Hodge structure $h^{p,q}(H^k_c(X))$ [DK86, 1.4].

Theorem 2.3.14. *[Del71A, Del71B] For any algebraic variety* X *there exists a mixed Hodge structure on* $H^*_c(X, \mathbb{Q})$ *with the following properties:*

1. *If $f : X \to Y$ is a proper morphism then $f^* : H_c^*(X) \to H_c^*(Y)$ is compatible with the Hodge structure.*

2. *The Künneth isomorphism $H_c^*(X) \otimes H_c^*(Y) \to H_c^*(X \times Y)$ is compatible with Hodge structures.*

3. *If Y is a closed subvariety in X, then the following is an exact sequence of Hodge structures.*

$$\cdots \to H_c^k(X\backslash Y) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(X\backslash Y) \to \cdots$$

4. *The Hodge Deligne numbers $h^{p,q}(H_c^k(X)) = 0$ for $p + q > k$, and for p or $q < 0$.*

5. *If X is a smooth projective variety then the Hodge Deligne structure on $H_c^*(X) \cong H^*(X)$ is consistent with the classical Hodge structure, and so $h^{p,q}(H_c^k(X))) = \dim H^p(X, \Omega^q)$.*

In addition, one can define an e-polynomial $e(X)$

$$e(X; x, \overline{x}) := \sum_{p,q} (-1)^{p+q} h^{p,q}(H_c^k(X)) x^p \overline{x}^q$$

which encapsulates the data of the Hodge Deligne numbers. This e-polynomial is additive for disjoint unions, which in some cases allows us to calculate the Hodge Deligne numbers from the Hodge numbers of the closure of the variety [DK86, 1.5].

2.4 Toric Varieties, Hypersurfaces and Degenerations

In Chapter 3 we will discuss several classical constructions of mirror families involving hypersurfaces in toric varieties and degenerations of flag varieties to toric limits. We give a basic overview of these tools here.

2.4.1 Toric Varieties

Definition 2.4.1. *A normal variety X is called a* toric variety *if it contains an algebraic torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset, and an action $T \times X \to X$ that extends the natural action of T on itself.*

Example 2.4.2. *The following are all examples of toric varieties: $(\mathbb{C}^*)^n \subset \mathbb{A}^n \subset \mathbb{P}^n$.*

The geometric properties of toric varieties can be described in terms of the combinatorics of associated polytopes and cones. We review some of the basic facts here. See [Ful93, O88] for more details.

Definition 2.4.3. *Let M be a k-dimensional lattice with $N := Hom(M, \mathbb{Z})$ its dual. Then define $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.*

Definition 2.4.4. *A polytope is the the convex hull of a finite set of points in $M_{\mathbb{R}}$.*

Definition 2.4.5. *A polytope $\Delta \subset M_{\mathbb{R}}$ is* integral *if all its vertices lie in M.*

To any n-dimensional projective toric variety with a choice of $\mathbb{Q}$-Cartier divisor D we can associate a polytope $\Delta := \Delta(X, D) \subseteq M_{\mathbb{R}}$. The following are some basic facts about Δ.

1. $\Delta(X, D)$ can be translated to an integral polytope if and only if D is Cartier.

2. Facets of Δ are in one to one correspondence with T-invariant divisors on X.

3. If D is Cartier then integral points of the polytope $k\Delta$ correspond to a basis of $H^0(X, kD)$.

The association goes in both directions. To a polytope in M we can associate a toric pair (X, D) with D a $\mathbb{Q}$-Cartier divisor. Toric varieties can also be described by cones and fans.

Definition 2.4.6. *Let N be the lattice $Hom(M, \mathbb{Z})$. A strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is a cone with apex at the origin generated by a finite number of vectors which are contained in the lattice and with the additional requirement that σ doesn't contain a line.*

Unless otherwise noted a cone will refer to a strongly convex rational polyhedral cone. Viewing M as a lattice of monomials, we can associate to a cone σ an affine toric variety $X_\sigma := Spec\mathbb{C}[\sigma \cap M]$ [Ful93]. We can also associate to any affine toric variety X a cone $\sigma(X)$ (see [Ful93] for details).

Definition 2.4.7. *A normal fan Σ is a finite collection of cones $\sigma_i \subset N_\mathbb{R}$ with the following properties.*

1. *Each face of a cone σ_i is also a cone in Σ.*

2. *The intersection of two cones in Σ is a face of each.*

Definition 2.4.8. *To any cone σ in $M_\mathbb{R}$ we define a cone $\widehat{\sigma}$ in the dual lattice $N_\mathbb{R}$ as $\widehat{\sigma} := \{x \in N | \langle x, v\rangle \geq 0 \text{ for all } v \in \sigma\}$*

Note that $\widehat{\widehat{\sigma}} = \sigma$.

Lemma 2.4.9. *If σ is a strongly convex rational polyhedral cone then so is $\widehat{\sigma}$.*

Each cone σ_i corresponds to an affine toric variety $X(\sigma_i)$. To any fan Σ we can associate a toric variety X_Σ by gluing the affine varieties X_{σ_i} together along the common faces $X(\sigma_i \cap \sigma_j)$ [Ful93].

Lemma 2.4.10. *If the union of the cones in a fan Σ is $N_\mathbb{R}$, then the corresponding toric variety X_Σ is proper.*

Every cone $\sigma \in \Sigma$ corresponds to an affine toric variety which is an open subset of X_Σ. These open subsets have a unique closure denoted $V(\sigma)$ which corresponds to the union of all of the affine varieties $X_{\sigma'}$ for each σ' containing σ.

Examples 2.4.11.

1. *If $\sigma = (0)$ then $V(\sigma) = X_\Sigma$.*

2. *If σ_i is a cone generated by a single ray then $D_i := V(\sigma_i)$ is a T-invariant divisor.*

The rays σ_i are perpendicular to the facets of Δ giving the correspondence between facets of Δ and T-invariant divisors stated above. Given a projective variety, it is often helpful to use both the fan and polytope pictures when trying to understand the geometry.

Remark 2.4.12. *To any polytope Δ we can associate a normal fan Σ in a unique way, but to any fan we can associate many different polytopes depending on the choice of line bundle (see [Ful93, O88] for details).*

2.4.2 Hypersurfaces in Toric Varieties

We view M as the lattice of monomials in the group algebra $\mathbb{C}[M] := \mathbb{C}[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$. When we intersect the elements of $|\mathcal{O}_{X_\Delta}(1)|$ with the torus $T = Spec[M]$, we can describe the corresponding affine hypersurfaces $Z_{f,\Delta}$ as solutions to the equations

$$f_\Delta := \Sigma c_m T^m = 0$$

where c_m is a complex coefficient and we sum over all integral points in Δ. We refer to this collection of toric hypersurfaces as $L(\Delta)$.

Definition 2.4.13. *If Δ is an integral polytope containing the origin in its interior we define $V(\Delta)$ as the subfamily of $L(\Delta)$ where the sum is only taken over the origin and the vertices of Δ.*

Definition 2.4.14. *For any affine toric hypersurface $Z_{f,\Delta}$ we denote its closure in X_Δ by $\overline{Z_{f,\Delta}}$. We denote an MCPC-desingularization of $\overline{Z_{f,\Delta}}$ by $\hat{Z}_{f,\Delta}$.*

Theorem 2.4.15. *[Bat94, Thm 4.2.2]For any generic affine hypersurface $Z_{f,\Delta} \in L(\Delta)$ there exists an MCPC-desingularization of the closure $\overline{Z_{f,\Delta}}$ of $Z_{f,\Delta}$ inside of X_Δ. The singular locus of $\hat{Z}_{f,\Delta}$ has codimension at least 4.*

2.4.3 Reflexive Polytopes

Definition 2.4.16. *To any polytope $\Delta \subset M_\mathbb{R}$ define*

$\Delta^* := \{y \in N_\mathbb{R} | \langle x, y \rangle \geq -1,$ *for all* $x \in \Delta\}$.

Example 2.4.17. *Let $X = \mathbb{P}^2$ and $D = -K_{\mathbb{P}^2} = 3H$, where H is the class of a hyperplane. We know that $H^0(X, D)$ is generated by homogeneous monomials of degree 3 in three variables, which after a dehomogenization can be represented by the nine points in the following lattice. The corresponding polytopes are shown in Figure 2.4.17.*

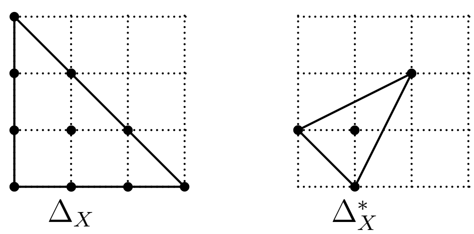


Figure 2.1: Δ and Δ^* for the pair $(\mathbb{P}^2, 3H)$

The polytopes Δ and Δ^* above are examples of what Batyrev calls reflexive polytopes. Every face H of a polytope determines a linear function h such that $h(x) = 0$ for all $x \in H$. The integral distance between a point x and a face H is defined to be $h(x)$.

Definition 2.4.18. *[Bat94, Def4.1.5] Let H be a rational hyperplane in $M_{\mathbb{R}}$, p a point in $M_{\mathbb{R}}$. Assume H is generated by integral points $H \cap M$, i.e., there exists a primitive $l \in N$ such that $H = x \in M_{\mathbb{R}} | < x, l >= c$ for some integer c. Then the integral distance between H and p is equal to $|c- < p, l > |$.*

Definition 2.4.19. *An integral polytope Δ is called* reflexive *if there exists a point p on the interior of Δ which is distance one from every facet.*

In [Bat94] Batyrev proves the following properties of reflexive polytopes.

Proposition 2.4.20. *1. A polytope Δ is reflexive if and only if Δ^* is reflexive.*

2. Δ is reflexive if and only if both Δ and Δ^* are integral polytopes whose only interior point is the origin.

3. *The pair (X, D) corresponds to a reflexive polytope if and only if $D = -K_X$ and X_Δ, i.e. a toric Fano variety with Gorenstein singularities.*

4. *If Δ is a reflexive polytope then generic elements of $|-K_X|$ are Calabi-Yau varieties (Not necessarily smooth Calabi-Yau varieties)*

2.4.4 Picard Group of Toric Varieties

We give a combinatorial description of the Picard group of the toric variety X. It is known that the T-invariant Weil divisors can be written as $D := \Sigma a_i D_i$, where D_i are the divisors corresponding to the rays of the fan Σ [Ful93]. For any ray σ_i we denote by e_i the first integral point on the ray.

Lemma 2.4.21. *[Ful93] Every divisor on a toric variety is linearly equivalent to a T-invariant divisor.*

Property 1. *A Weil divisor $D = \Sigma a_i D_i$ is Cartier if and only if for each maximal cone σ (i.e. a cone in Σ not contained in any other cone) there exists an $m \in M$ such that for every $\sigma_i \in \sigma$, $\langle m, e_i \rangle = -a_i$.*

This fact allows us to calculate the Picard group of X using the following theorem.

Theorem 2.4.22. *[Ful93] The following sequence is exact.*

$$0 \to M \to Div_T X \to \mathrm{Pic}(X) \to 0$$

Here $Div_T X$ denotes the free abelian group of T-invariant Cartier divisors on X.

2.4.5 Degenerations

As we will see in Chapter 4, Batyrev uses reflexive polytopes to construct a duality between families of Calabi-Yau hypersurfaces in toric varieties [Bat94]. This work was extended in [BCFKvS00]to find mirrors for hypersurfaces in partial flag varieties. This is done by degenerating the flag variety to a toric variety and then using a specialized version of the mirror construction for toric hypersurfaces. These constructions, and the extension to the case of small toric degenerations will be discussed in the next chapter.

Definition 2.4.23. *Let $X \subset \mathbb{P}^m$ be a smooth variety of dimension n. A toric variety $Y \subset \mathbb{P}^m$ is called a* toric degeneration *of X, if there exists a Zariski open neighborhood U of $0 \subset \mathbb{A}^1$ and an irreducible subvariety $\widetilde{X} \subset \mathbb{P}^m \times U$ such that the morphism $\pi : \widetilde{X} \mapsto U$ is flat and the following conditions hold:*

1. *the fiber $X_t := \pi^{-1}(t) \subset \mathbb{P}^m$ is isomorphic to X for all $t \in U \backslash 0$;*
2. *X_0 is isomorphic to $Y \subset \mathbb{P}^m$;*

A more refined version of toric degeneration is a "small toric degeneration". The "smallness" is a combination of conditions on singularities of the fibers and a condition on the Picard groups.

Definition 2.4.24. *[Bat04, Def 3.1] Let $X \subset \mathbb{P}^m$ be a smooth Fano variety of dimension n. A normal Gorenstein toric Fano variety $Y \subset \mathbb{P}^m$ is called a* small toric degeneration *of X, if there exists a Zariski open neighborhood U of $0 \subset \mathbb{A}^1$ and an irreducible subvariety $\widetilde{X} \subset \mathbb{P}^m \times U$ such that the morphism $\pi : \widetilde{X} \mapsto U$ is flat and the following conditions hold:*

1. *the fiber $X_t := \pi^{-1}(t) \subset \mathbb{P}^m$ is smooth for all $t \in U \backslash 0$;*
2. *the special fiber $X_0 := \pi^{-1}(0) \subset \mathbb{P}^m$ has at worst Gorenstein terminal singularities and X_0 is isomorphic to $Y \subset \mathbb{P}^m$;*
3. *the canonical homomorphism*

$$\mathrm{Pic}(\widetilde{X}/U) \mapsto \mathrm{Pic}(X_t)$$

   is an isomorphism for all $t \in U$.

An example of a small toric degeneration which was used in the classical mirror constructions of Batyrev et. al. can be found in[GL96].

Lemma 2.4.25. *[Bat04] If X is a small toric degeneration of Y then $-K_X$ is Cartier and thus $\Delta(X, -K_X)$ is a reflexive polytope.*

We are now armed with the tools needed to understand the most classical combinatorial construction of mirrors to the generic anticanonical hypersurfaces in G/B.

Chapter 3

Classical Mirror Constructions

The first example of mirror symmetry was constructed by physicists Greene and Plesser for generic quintic hypersurface in $\mathbb{P}^4$ [GrP90]. In this chapter we review several mathematical mirror constructions which include Greene and Plesser's result as a special case. Special attention is given to the construction of mirrors to the anticanonical hypersurfaces of G/B.

3.1 Calabi-Yau Hypersurfaces in Toric Varieties and their Mirrors

Conjecture 3.1.1. *[Bat94] For a reflexive polytope Δ, MPCP-desingularizations of compactifications of generic elements of $L(\Delta)$ are mirror to MPCP-desingularizations of compactifications of generic elements of $L(\Delta^*)$.*

As evidence Batyrev proves that the Hodge Deligne numbers of MPCP-desingularizations of generic hypersurfaces of $L(\Delta)$ and $L(\Delta^*)$ satisfy $h^{n-2,1}(\widehat{Z_{X_\Delta}}) = h^{1,1}(\widehat{Z_{X_{\Delta^*}}})$ and vice versa. This implies the mirror relations in the case of Calabi-Yau threefolds. Batyrev proved this result by using Danilov and Khovansky's method of calculating the Hodge Deligne numbers of generic toric hypersurfaces in terms of the number of integral points on certain faces of Δ and Δ^* [DK86].

Proposition 3.1.2. *[Bat94, Thm 4.3.7,4.4.2] For $n \geq 4$ and $Z_{X_\Delta} \in L(\Delta)$, then the Hodge number $h^{n-1,1}(\widehat{Z_{X_\Delta}}) = h^{1,1}(\widehat{Z_{X_{\Delta^*}}}) = l(\Delta) - n - 4 - \sum_{codim \theta = 1} l^*(\theta) \cdot l^*(\theta^*)$ where θ denotes a face of a reflexive polytope Δ and θ^* denotes the corresponding dual face on Δ^*.*

Example 3.1.3. *$\mathbb{P}^4$ is a toric variety which corresponds to the 4-simplex. For the 4-simplex with with vertices at distance 5 from the origin Δ is a reflexive polytope and $L(\Delta)$ consists of* *generic quintic hypersurfaces. The mirror construction coincides with Greene and Plesser's original construction of mirrors to generic quintic threefolds.*

This construction was extended to include the case of Calabi-Yau complete intersections in toric varieties in [BB96].

3.2 Mirror Construction for flag and Fano varieties

3.2.1 Example: Mirrors of Anticanonical Hypersurfaces in G/B

In this section we review the work of Batyrev, Ciocan-Fontanine, Kim, and van Straten [BCFKvS00]. In this work they used a particular small toric degeneration (namely the sagbi or equivalently Gonciulea-Lakshmibai degeneration), to construct mirrors to the anticanonical hypersurfaces in partial flag varieties. We review this example in the case of a complete flag.

Fix G/B of type A_n, and create an oriented graph Γ as follows. The vertices of Γ will be split into two groups $V = Dots \cup Stars$. The vertices lie in the first quadrant of $\mathbb{Z}^2$ (with coordinates y_1 and y_2) as follows. If $y_1 + y_2 < n$ then place a Dot on (y_1, y_2). If $y_1 + y_2 = n$ place a Star on (y_1, y_2). For the edges draw vertical and horizontal lines connecting all of the vertices. Orient the edges down and to the right. The picture for the A_2 case is shown in Figure 3.2.1. Let M be the lattice of rank $|Dots|$ with coordinates x_d corresponding to the

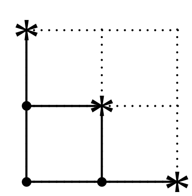


Figure 3.1: Γ for G/B of type A_2

Dots in Γ. To each oriented edge e from vertex v_{tail} to v_{head} we associate an inequality in $M_{\mathbb{R}}$

$$x_{tail} - x_{head} \geq 0$$

with the condition that if v corresponds to a Star with coordinate (y_1, y_2) then x_v is the constant y_1. We call this collection of inequalities the Γ-inequalities.

Example 3.2.1. *For G/B of type A_2 the following inequalities define Δ.*

$$x_{0,1} \leq 2$$

$$x_{0,0} \leq x_{0,1}$$

$$x_{1,0} \leq 1$$

$$x_{1,0} \geq 0$$

$$x_{0,0} \geq x_{1,0}$$

$$x_{0,1} \geq 1$$

Theorem 3.2.2. *[BCFKvS00] The set $\Delta := \{x \in M_{\mathbb{R}}$ satisfying the Γ-inequalities $\}$ is a reflexive polytope. Moreover, this polytope corresponds to a toric variety X_Δ which appears as a small toric degeneration of G/B in [GL96].*

For our example above the polytopes Δ and Δ^* are shown in Figure 3.2.1. It would be

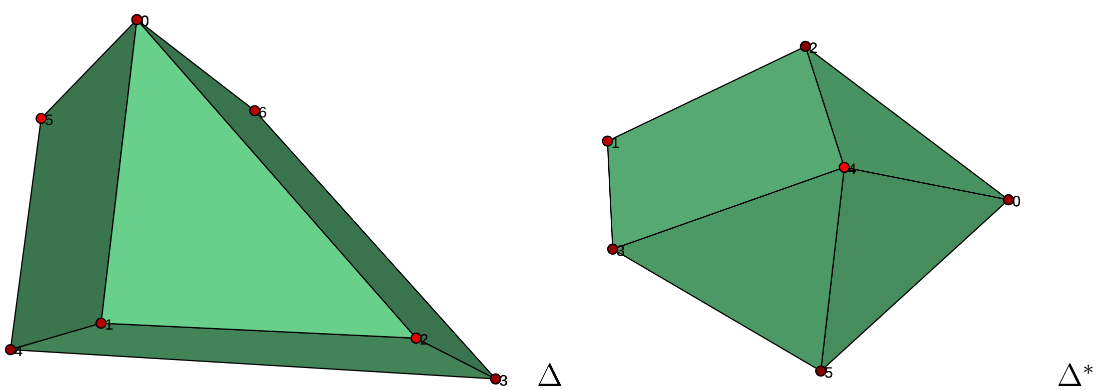


Figure 3.2: Polytopes Δ and Δ^* for G/B of type A_2.

reasonable to think of $L(\Delta)$ as a birational model of the limit of anticanonical hypersurfaces in G/B. A logical guess would be that the mirrors would thus be birational to generic elements in $L(\Delta^*)$. Unfortunately that family is too large. Instead the correct version was formulated by looking at a subfamily of $V(\Delta^*)$ whose coefficients satisfied a set of relations called *box equations.*

The box equations are given by relations among the coefficients which correspond to rays in Σ which in turn correspond to facets of Δ. These facets are defined by inequalities corresponding to edges in Γ. So we have a one to one correspondence between edges e_i and coefficients a_i. Which means the box equations can be described in terms of relations between edges of Γ.

Definition 3.2.3. *[BCFKvS00]For every square in the diagram of Γ we denote $e_{left}, e_{right}, e_{top}$ and e_{bottom} the corresponding edges. box equations are the collection of relations $a_{left}a_{bottom} = a_{top}a_{right}$ for every square in Γ.*

Example 3.2.4. *We can calculate the box equations in the A_2 case using Figure 3.2.4 The*

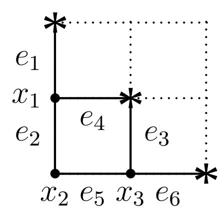


Figure 3.3: Γ with labeled edges

the equations defining $V(\Delta^*) \cap T$ are:

$$1 = a_1t_1^{-1} + a_2t_1t_2^{-1} + a_3t_3^{-1} + a_4t_1 + a_5t_2t_3^{-1} + a_6t_3$$

The corresponding box equations are

$$a_2a_5 = a_3a_4$$

.

3.2.2 Mirror Construction using Small Toric Degenerations

Let X be a small toric degeneration of a smooth Fano variety Y, with corresponding fan Σ and polytope $\Delta = \Delta(X, -K_X)$. Batyrev extends the construction from the full flag case to find mirrors to generic anticanonical hypersurfaces in Y. In particular, the mirrors will be birational to generic elements of $V(\Delta^*)$ whose coefficients relations which we will again call *box equations.*

Definition 3.2.5. *A set of coefficients $a = (a_1, \cdots, a_r)$ satisfies* box equations *if for every maximal dimensional cone in Σ there exists an $m \in M$ such that*

$$\langle m, e_i \rangle = log|a_i|$$

for every ray σ_i in the maximal cone.

Remark 3.2.6. *The set a satisfying the box equations, is is a multiplicative version of the condition for a divisor to be Cartier. Such a set a is also known as Σ-admissible.*

Conjecture 3.2.7. *[Bat04, 4] Generic elements of the subfamily of $V(\Delta^*)$ whose coefficients satisfy the box equations are birational to mirrors of generic elements of $|-K_Y|$.*

Remark 3.2.8. *The explicit construction of mirrors to anticanonical hypersurfaces in G/B described in the previous section is a specific example of this construction.*

Chapter 4

Representation Theory Background

We want to use different degenerations of G/B to construct and compare mirror families of Calabi-Yau varieties. Several such degenerations, called *string degenerations*, were constructed in [Cal02, AB04] by viewing algebraic geometry ideas from a representation theory perspective. In this section we review the background material from representation theory and Quantum Groups we need to explain and understand the string degenerations.

4.1 Definitions, Notation and Basic Facts

Working over $\mathbb{C}$, let G be the algebraic group SL_{n+1}, B the Borel subgroup of upper triangular matrices, U the subgroup of upper uni-triangular matrices, and T the maximal torus of diagonal matrices. One has $B = TU$. We let B^- and U^- denote the opposite Borel subgroup and its unipotent radical. Let Φ be the root system of (G,T), with Φ^+ the subset of positive roots. We denote $\alpha_1, ..., \alpha_n$ the corresponding simple roots.

Let W be the Weyl group of (G,T) with $s_1, ..., s_n$ the reflections associated to $\alpha_1, \ldots, \alpha_n$. Note that $W \cong S_{n+1}$ so we can view s_i as the transposition exchanging i and $i+1$. These transpositions define a length function l on W. Let ω_0 be the unique element of maximal length corresponding to the permutation sending 1 to $n+1$, 2 to n and so forth. There are $N = l(\omega_0) = \frac{n(n+1)}{2}$ positive roots, which is the dimension of the flag variety G/B.

Let Λ be the weight lattice generated by the fundamental weights $\widehat{\omega}_i$, $1 \leq i \leq n$, and let $\Lambda^+ := \bigoplus_i \mathbb{Z}_{\geq 0}\widehat{\omega}_i$ be the semigroup of integral dominant weights. If λ lies on the interior of the cone $\Lambda^+_{\mathbb{R}}$ we call it a *regular dominant weight*. We let ρ be the sum of the fundamental weights (which is equal to half the sum of the positive roots).

Definition 4.1.1. *For any weight λ let $V(\lambda) := \{v \in G | tg = \lambda(t)v$ for all $t \in T\}$. If $V(\lambda)$ is nonempty we call it a weight space.*

Note that $V(\lambda)$ is a simple G-module with highest weight λ. Set $\lambda^* := -\omega_0\lambda$, where ω_0 acts on λ by permuting the fundamental weights.

For any λ we can associate a line bundle on G/B denoted L_λ (see [FH, 23.3] for details). We have the following facts about about the line bundles L_λ [Br05, AB04].

1. L_λ is globally generated (resp. ample) if λ is dominant (resp. regular dominant).

2. If λ is a regular dominant weight then we have $H^0(G/B, L_\lambda) = V(\lambda^*)$

3. If $\lambda = 2\rho$ the $L_\lambda = -K_{G/B}$.

In constructing the string degenerations G/B is viewed as $G/B = \text{Proj}(\bigoplus_{k=1}^{\infty} H^0(G/B, L_{k\lambda}))$. It will be helpful to have a combinatorial understanding of a basis for $H^0(G/B, L_{k\lambda})$. One way of doing so is by viewing $V(\lambda^*)$ as a dequantization of quantum weight space $V_q(\lambda^*)$ since its basis elements are in one to one correspondence with integral points of a polytope. We introduce quantum groups in order to define and understand $V_q(\lambda^*)$.

4.2 Quantum Groups

Let q be a nonzero complex number which is not a root of unity. We define $\mathcal{A}_n$ as the associative algebra with unit over the field of rational functions $\mathbb{Q}(q)$ generated by the elements $x_1, \cdots, x_n$ modulo the relations:

1. $x_i x_j = x_j x_i$ for $|i - j| > 1$

2. $x_i^2 x_j - (q + q^{-1}) x_i x_j x_i + x_j x_i^2 = 0$ for $|i - j| = 1$

Remark 4.2.1. *When the rank is clear, we will write $\mathcal{A}_n$ as $\mathcal{A}$.*

We also define define an associative algebra U_+ generated by $E_1, \cdots, E_n$ satisfying the same relations. These algebras are graded by Λ^+, where the degree of x_i is $\widehat{\omega_i}$. We denote the degree γ homogeneous components of $\mathcal{A}$ by $\mathcal{A}(\gamma)$. $\mathcal{A}$ and U_+ are dual to one another by the following theorems of Berenstein and Zelevinsky.

Theorem 4.2.2. *[BZ93, Prop 1.1] There exists a unique action $(E, x) \mapsto E(x)$ of the algebra U_+ on $\mathcal{A}$ satisfying the following properties:*

1. *(Homogeneity) If $E \in U_+(\alpha), x \in \mathcal{A}(\gamma)$ then $E(x) \in \mathcal{A}(\gamma - \alpha)$.*

2. *(Leibniz formula)*

$$E_i(xy) = E_i(x)y + q^{-(\gamma,\alpha_i)} x E_i(y) \text{ for } x \in \mathcal{A}(\gamma), y \in \mathcal{A}$$

   (here (γ, α) is defined by means of the Cartan matrix).

3. *(Normalization) $E_i(x_j) = \delta_{ij}$.*

Proposition 4.2.3. *[BZ93, Prop. 1.2]*

1. *If $\gamma \in Q_+ \{0\}$ and x is a nonzero element of $\mathcal{A}(\gamma)$ then $E_i(x) \neq 0$ for some $i = 1, \cdots, n$.*

2. *For every $\gamma \in Q_+$, the mapping $(E, x) \mapsto E(x)$ defines a non-degenerate pairing*

$$U_+(\gamma) \times \mathcal{A}(\gamma) \to \mathcal{A}(o) = Q(q)$$

In addition to showing the duality of $\mathcal{A}$ and U_+, these propositions show us how the E act on x like a quantum derivation.

Example 4.2.4. *The Quantum Group $\mathcal{A}_2$ is generated by x_1 and x_2 with the relations*

$$x_1^2 x_2 + x_1 x_2^2 = (q + q^{-1}) x_1 x_2 x_1$$

$$x_2^2 x_1 + x_2 x_1^2 = (q + q^{-1}) x_2 x_1 x_2$$

The dual algebra U_+ is generated by E_1 and E_2 subject to the same relations. An example of the quantum derivative would be

$$\begin{aligned} E_1(x_1^2x_j) &= E_1((x_i^2)(x_j)) \\ &= E_1(x_1^2)x_2 + q^{-(2\alpha_1,\alpha_1)}x_1^2E_1(x_2) \\ &= E_1(x_1^2)x_2 + q^{-4}x_1^2 \cdot 0 \\ &= (E_1(x_1)x_1 + q^{-(\alpha_1,\alpha_1)}x_1E_1(x_1))x_2 \\ &= (x_1 + q^{-2}x_1)x_2 \\ &= (1+q^{-2})x_1x_2 \end{aligned}$$

4.3 String Bases

Recall, we are interested in these quantum groups because we are looking for a combinatorial structure on the basis elements of $V(\lambda^*)$. We examine a particular type of basis for $\mathcal{A}$ called a string basis. Such a basis will restrict to a basis of $V_q(\lambda^*)$. We will the use the specialization map to get a basis for $V(\lambda^*)$.

For any subset $\mathcal{B} \subset \mathcal{A}$, let $[\mathcal{B}]^+ \subset \mathcal{A}$ be the set of all linear combinations of elements of $\mathcal{B}$ with coefficients in $\mathbb{Z}[q, q^{-1}]$.

Definition 4.3.1. *A basis $\mathcal{B}$ of $\mathcal{A}$ is called a string basis if*

1. *$\mathcal{B}$ consists of homogeneous elements and contains* 1*.*

2. *For every b,$b' \in \mathcal{B}$ the product $bb' \in [\mathcal{B}]^+$.*

3. *$E_i(b) \in [\mathcal{B}]^+$ for $b \in \mathcal{B}$, $i = 1, \cdots, n$*

4. *If $b \in \mathcal{B}$, and l is the maximal integer such that $E_i^l(b) \neq 0$ then $E_i^{(l)}(b) \in \mathcal{B}$.*

Classically there were two known bases of U_+, Lusztig's canonical basis and Kashiwara's crystal basis [L90, Kash95]. Lusztig and Grojnowski later proved that these bases were equivalent [GrL93].

Proposition 4.3.2. *[BZ93, Prop 1.3] The basis in $\mathcal{A}$ dual (under the map in 4.2.3) to Lustzig's canonical basis in U_+ is a string basis. This basis is known as the* dual canonical basis.

The work of Littelmann and Berenstein-Zelevinsky [BZ93, Lit98] gives a method of associating elements of the dual canonical basis $\mathcal{B}$ to integral points of a strongly convex rational polyhedral cone. This method depends on the choice of $\overline{\omega_0}$.

4.3.1 String Cone

If $x \in \mathcal{A}$ is non zero then for each i we let

$$l_i(x) = \max \{l \in \mathbb{Z}_+ | e_i^l(x) \neq 0\}$$

and define

$$E_i^{(top)}(x) := E_i^{(l_i(x))}(x)$$

For any reduced word decomposition $\overline{\omega_0} = s_{i_1} \cdots s_{i_N}$ of ω_0 we define $a(\overline{\omega_0}, x) = (a_1, \cdots, a_n)$ the string of x in the direction $\overline{\omega_0}$ by $a_k := l_{i_k}(E_{i_{k-1}}^{(top)} E_{i_{k-2}}^{(top)} \cdots E_{i_1}^{(top)}(x)$. Note that a maps the elements of the dual canonical basis to the lattice $\mathbb{Z}^N$.

Example 4.3.3. *Fix the reduced word $\overline{\omega_0} = s_1 s_2 s_1$ for the quantum group $\mathcal{A}_2$. Assume that $x_1^2 x_2$ is an element of the dual canonical basis. Then we can calculate its string as follows. From our previous calculation we have*

$$\begin{aligned}
E_1(x_1^2 x_2) &= (1+q^{-2}) x_1 x_2 (\textit{which is not 0 so } l_1(x_1^2 x_2) \geq 1.) \\
E_1((1+q^{-2}) x_1 x_2) &= (1+q^{-2}) E_1(x_1) x_2 + q^{-(\alpha_1, \alpha_1)} x_1 E_1(x_2) \\
&= (1+q^{-2}) x_2 + 0) \\
&= (1+q^{-2}) x_2 (\textit{which is not 0 so } l_1(x_1^2 x_2) \geq 2.) \\
E_1((1+q^{-2}) x_2) &= 0 \textit{ so } l_1(x_1^2 x_2) = 2.
\end{aligned}$$

Now we need to calculate l_2:

$$E_2((1+q^{-2})x_2) = 1+q^{-2} \text{ so } l_2(x_1^2x_2) \geq 1$$

$$E_2(1+q^{-2}) = 0 \text{ so } l_2(x_1^2x_2) = 1.$$

Finally we calculate l_3:

$$E_1(1+q^{-2}) = 0$$

so $l_3(x_1^2x_2) = 0$. Therefore the element $x_1^2x_2$ would be represented by the string $(2,1,0)$.

Proposition 4.3.4. *[BZ93] Any element of the dual canonical basis $b \in \mathcal{B}$ is uniquely determined by $a(\overline{\omega_0}, b)$. In addition, under the map a, the elements of the dual canonical basis $\mathcal{B}$ are in one to one correspondence with integral points of a strongly convex rational polyhedral cone $C := C(\mathcal{B}, \overline{\omega_0}) \subset \mathbb{R}^N$.*

The inequalities describing the cone C are given in [BZ01], and a combinatorial description for them is given in [GlP00]. We will review this description later in the paper.

4.3.2 Quantum groups and a Basis for $V(\lambda^*)$

Now we want to use our understanding of quantum groups to give us some data about the basis for $V(\lambda^*) = H^0(G/B, L_\lambda)$.

Definition 4.3.5. *For λ in Λ^+, let $V_q(\lambda)$ be the simple $\mathcal{A}$ module with highest weight λ. Let v_λ be the corresponding highest weight vector.*

Definition 4.3.6. *For λ in Λ^+, $\lambda = \Sigma\lambda_i\hat{\omega}_i$ we define*
$\mathcal{B}_\lambda := \{b \in \mathcal{B}, l_i(b) \leq \lambda_i\}$.

Theorem 4.3.7. *[Kash95, 9,Prop. 8.2] For $\lambda \in \Lambda^+$ the collection $\mathcal{B}_\lambda \cdot v_\lambda$ is a basis for $V_q(\lambda)$.*

The image of the inequalities defining $\mathcal{B}_\lambda$ under the map a define a convex rational polyhedral cone in $\mathbb{Z}^N$ known as the λ-cone. This theorem shows that the basis elements of $V_q(\lambda)$ are in one to one correspondence with integral points which lie both in the string and the λ-cone. The intersection of these two cones is an N-dimensional polytope $\Delta := \Delta(\lambda, \overline{\omega_0})$.

Example 4.3.8. *In the case* $\mathcal{A}_1$ *we must have* $\overline{\omega_0} = s_1$*. The dual canonical basis consists of elements of the form* x_1^k *for nonnegative integers* k*. Then* a *maps the basis elements to* $\mathbb{R}$ *by* $a(x_1^k) = k$*. Thus the string cone would be defined by the inequality* $x \geq 0$*. For* $\lambda = n\widehat{\omega_1}$ *we have* $\mathcal{B}_\lambda = \{x_1^m | m \leq n\}$*. From this we see the* λ*-cone is given by the inequality* $x \leq n$*. The polytope* Δ*, which is the intersection of the two cones, is an interval of length* n*.*

Lemma 4.3.9. *[Cal02, Prop 2.1.1] The basis for* $V_q(\lambda)$ *specializes to a basis for* $V(\lambda)$ *when* $q = 1$*.*

This extends our combinatorial model to a basis of $H^0(G/B, L_\lambda)$. In the following chapter we will see how this model can be used to construct different degenerations of the flag variety.

Chapter 5

String Degenerations of Flag Varieties

5.1 Constructing the Degeneration

In this Section we review the string degeneration in the case of G/B of type A_n. We follow the notation found in [AB04]. The construction is due to work of Caldero, Alexeev and Brion in [Cal02, AB04]. Let $A := \bigoplus_{\lambda \in \Lambda^+} H^0(X, \lambda)$ Recall for a fixed $\lambda \in \Lambda^+$ the associated line bundle L_λ is ample.

Now remember that $H^0(X, L_\lambda) \cong V(\lambda^*)$ which has a basis coming from the specialization of the dual canonical basis. By fixing a reduced decomposition $\overline{\omega_0}$ we can associate these dual canonical basis elements to integral points in the the polytope $\Delta(\overline{\omega_0}, \lambda)$. We denote a basis element of A as an ordered pair (λ, ϕ) where λ is the corresponding weight, and ϕ is an integral point in the polytope $\Delta(\overline{\omega_0}, \lambda)$. We will refer to the set of basis elements of A as β_A.

Lemma 5.1.1. *[Cal02, Thm2.3] Let $b_{\lambda,\phi}$ and $b_{\lambda',\phi'}$ be in β_R. The dual canonical basis has the following multiplicative property:*

$$b_{\lambda,\phi} b_{\lambda',\phi'} = b_{\lambda+\lambda',\phi+\phi'} + \Sigma_\gamma c^\gamma_{\lambda,\lambda',\phi,\phi'} b_{\lambda+\lambda',\gamma}$$

where the constant $c^\gamma_{\lambda,\lambda',\phi,\phi'} = 0$ unless $\gamma < \phi + \phi'$ in the lexicographic order of $\mathbb{N}^N$.

From this Caldero defines a filtration on A as follows.

Proposition 5.1.2. *[Cal02, Prop 3.1] $A_{\leq \phi} := \{(\gamma, \psi) \in \beta_A | \psi \leq \phi$ in the lexicographical order of of $\mathbb{Z}^N\}$ defines an $\mathbb{N}^N$ filtration on A such that the associated graded algebra $\operatorname{gr} A = \bigoplus A_{\leq \phi}/A_{<\phi}$ is naturally isomorphic to the $\mathbb{C}$-algebra of the semigroup defined by the string cone.*

This construction would yield a sequence of degenerations from G/B to an affine toric variety, but what we want is a single projective degeneration.

Using the fact that A is finitely generated, and its elements can be associated to points of a polyhedral cone. Caldero constructs an $\mathbb{N}$-filtration on A denoted $A_{\leq m}$ such that associated graded algebra $\operatorname{gr} A$ is the algebra of the string cone.

Instead of these affine degenerations, Alexeev and Brion looked at the projective case. Fix $\lambda \in \Lambda^+$.

Definition 5.1.3. $R(X, L_\lambda) := \bigoplus_{n=0}^{\infty} H^0(X, L_\lambda^n)$.

We have $G/B \cong \operatorname{Proj} R(X, L_\lambda)$. The $\mathbb{N}$-filtration on A restricts to an $\mathbb{N}$-filtration on $R(X, L_\lambda)$ denoted by $R(X, L_\lambda)_{\leq m}$.

From this Alexeev and Brion prove:

Theorem 5.1.4. *[AB04, Thm. 3.2] Fix a reduced decomposition $\overline{w_0}$ and $\lambda \in \Lambda^+$. Then there exists a family $\pi : \mathcal{X} \to \mathbb{A}^1$, where $\mathcal{X}$ is a normal variety, together with divisorial sheaves $\mathcal{O}_{\mathcal{X}}(n)$ $(n \in \mathbb{Z})$ such that*

1. *π is projective and flat.*

2. *π is trivial with fiber G/B over the complement of 0 in $\mathbb{A}^1$, and $\mathcal{O}_{\mathcal{X}}(n)|_{G/B} \cong L_{n\lambda}$ for all n.*

3. *The fiber of π at 0 is isomorphic to X_Δ where X_Δ is the projective toric variety corresponding to the polytope $\Delta(\overline{w_0}, \lambda)$.*

This family is constructed using the Rees algebra $\mathcal{R} := \bigoplus_{m=0}^{\infty} R_{\leq m} t^m$, and putting $\mathcal{X} = \operatorname{Proj} \mathcal{R}$, then, using Caldero and Alexeev-Brion's work, checking that such a variety has the above properties. We refer to these degenerations as *string degenerations.*

In the rest of the paper unless otherwise specified the string degeneration of G/B will refer to the degeneration for the pair $(G/B, -K_{G/B} = L_{2\rho})$.

5.2 Are String Degenerations Small?

The string degenerations are seen to be different, as the polytopes Δ change vastly depending on the choice of $\overline{\omega_0}$. For example the number of faces of each different dimension differ, as does the rank of the Picard group of the corresponding toric variety. Since Batyrev has a mirror construction for any small toric degeneration, we are interested in whether or not the string degenerations of G/B are small. It turns out that they are in some case but that they fail to be small in others. Recall that a key component of smallness is the Picard group of the special fiber.

Theorem 5.2.1. *For any string degeneration of* G/B *we have* $\operatorname{rank}\operatorname{Pic}(X_\Delta) \leq \operatorname{rank}\operatorname{Pic}(G/B)$.

Proof. Since $H^2(X_\Delta, \mathcal{O}_{X_\Delta}) = 0$ then there exists an open neighborhood of 0 over which $\operatorname{Pic}\mathcal{X}/S$ is smooth [FGIKNV, Prop 9.5.19]. This allows us to extend the line bundle on X_Δ to a line bundle on a nearby fiber. □

Remark 5.2.2. *The statement that the rank of the Picard group of the special fiber cannot exceed the rank of the Picard group of a generic fiber, holds for any toric degeneration.*

5.2.1 Example of a Nonsmall Degeneration

Proposition 5.2.3. *For the case of* A_3 *with* $\overline{\omega_0} = s_3s_1s_2s_1s_3s_2$ *the string degeneration is not small.*

Proof. For this choice of degeneration the polytope $\Delta = \Delta(2\rho, \overline{\omega_0})$ is a six dimensional polytope with 38 vertices and 13 facets (entire f-vector= $(38, 133, 197, 152, 63, 13)$). Recall that the set of T-Invariant divisors is generated by the corresponding rays σ_i. Using the freely available program polymake we can calculate the first integral points on each of these rays. As before we can write every T-invariant divisor on X_Δ as $D = \Sigma_{i=0}^{1} 2a_iD_i$ where D_i is the T-invariant divisor corresponding to the ray σ_i, with first integral point e_i.

We list these first integral points in a table below.

$$e_0 = (0, 1, 0, 0, 0, 0)$$
$$e_1 = (1, 0, 0, 0, 0, 0)$$
$$e_2 = (0, 0, 0, 0, 0, 1)$$
$$e_3 = (0, 0, 1, -1, 0, 0)$$
$$e_4 = (0, 0, 1, 0, -1, 0)$$
$$e_5 = (0, 0, 0, 0, 1, -1)$$
$$e_6 = (0, 0, 0, 1, 0, -1)$$
$$e_7 = (0, -1, 1, -2, 0, 1)$$
$$e_8 = (-1, 0, 1, 0, -2, 1)$$
$$e_9 = (0, 0, -1, 1, 1, -2)$$
$$e_{10} = (0, 0, 0, 0, -1, 1)$$
$$e_{11} = (0, 0, 0, -1, 0, 1)$$
$$e_{12} = (0, 0, 0, 0, 0, -1)$$

Following the methods described in Section 2.4.4, we can calculate that a T-invariant divisor is Cartier if and only if the following equations are satisfied:

$$a_0 + a_7 = a_3 + a_{11}$$
$$a_1 + a_8 = a_4 + a_{10}$$
$$a_3 + a_{10} = a_4 + a_{11}$$
$$a_4 + a_9 = a_5 + a_{12}$$
$$a_5 + a_{10} = a_6 + a_{11}$$

Using the exact sequence

$$0 \to M \to Div_T X \to \mathrm{Pic}(X) \to 0$$

we can calculate:

$$\begin{aligned}
\operatorname{rank} \operatorname{Pic} X_{Delta} &= \operatorname{rank} Div_T X - \operatorname{rank} M \\
&= \text{ Rank T-Invariant Divisors - Number of Cartier Relations} - \dim X \\
&= 13 - 4 - 6 = 3
\end{aligned}$$

Since $3 \neq 4 = \operatorname{rank} \operatorname{Pic} G/B$ we see that the degeneration isn't small. □

Chapter 6

Combinatorics of Δ

In this chapter we seek to understand the combinatorics of the polytope Δ. This will allow us to define the appropriate mirror families, and later, to prove that these families are all birational.

6.1 Gleizer and Postnikov's description of Δ

The polytope $\Delta(\lambda, \overline{\omega_0})$ is the intersection of Berenstein and Zelevinsky's string cone and a polyhedral cone called the λ-cone [AB04, Thm 1.1]. Explicit inequalities describing the λ-cone can be found in [Lit98, 1] and [AB04, Thm 1.1]. Inequalities for the string cone are given in [BZ01, 3.10] and [Lit98, 1]. A combinatorial description for them is given in [GlP00, Cor.5.8].

In this section we review Gleizer and Postnikov's combinatorial description of the string cone and add a combinatorial description of the λ-cone as well. For our description of the λ cone we fix $\lambda = 2\rho$ so that $L_\lambda = -K_{G/B}$.

Gleizer and Postnikov describe the string cone in terms of a picture called a string diagram [GlP00]. We review their construction here.

Remark 6.1.1. *In this description the order in which we write the reduced decomposition of ω_0 is reversed from their original description.*

Let $\overline{\omega_0}$=$s_{i_1} s_{i_2} \dots s_{i_N}$ be a reduced decomposition of ω_0. They represent this decomposition with a string diagram described below: Start with $n+1$ strings at the top of the diagram. Move down the diagram exchanging the string in the i_1 column with the string in the i_1+1 column. Continuing down the diagram exchange the string in the i_2 column with the string in the $i_2 + 1$ column. Continue this process until you have made N exchanges. The strings will have reversed their orders when they reach the bottom of the diagram.

Label the intersection points $t_1, t_2, \ldots, t_N$ from top to bottom. Label the strings U_1 through U_{n+1} on the top from left to right. Mark vertices u_1 through u_{n+1}(resp. b_1 through b_{n+1}) as the upper (resp. lower) ends of the strings U_1 through U_{n+1}.

Example 6.1.2. *Figure 6.1.2 shows the string diagram for $n = 3$ and reduced word decomposition $\overline{\omega_0} = s_3 s_1 s_2 s_1 s_3 s_2$:*

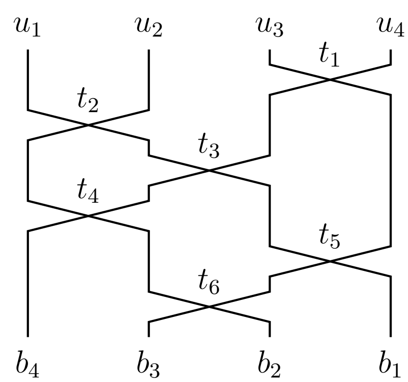


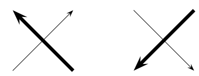
Figure 6.1: $\Gamma(s_3 s_1 s_2 s_1 s_3 s_2)$

Definition 6.1.3. *([GlP00]) Pick a number k from 1 to n. Form an oriented graph on the string diagram as follows. Orient strings U_1 through U_k upward and the rest of the strings downward. A* rigorous path *is an oriented path from b_k to b_{k+1} not meeting any vertex more than once, and avoiding the two bad fragments in Figure 6.1.3.*

Each rigorous path defines an inequality as follows:

$$\Sigma a_i x_i \geq 0 \text{ where } a_i = \begin{cases} 1 & \text{if the path switches from } U_i \text{ to } U_j \text{ and } i < j \\ -1 & \text{if the path switches from } U_i \text{ to } U_j \text{ and } i > j \end{cases}$$

We refer to these inequalities as string inequalities.



Here the thin line denotes a string, and the thick line indicates the rigorous path.

Figure 6.2: A rigorous path may not contain either of these two segments

Example 6.1.4. *For the string diagram pictured in Figure 6.1.2 we have the following list of rigorous paths and corresponding inequalities:*

$k = 1:$

$b_1 \mapsto t_5 \mapsto t_3 \mapsto t_4 \mapsto t_6 \mapsto b_2$ *corresponds to:* $x_3 \geq x_4$.

$b_1 \mapsto t_5 \mapsto t_6 \mapsto b_2$ *corresponds to:* $x_5 \geq x_6$.

$b_1 \mapsto t_5 \mapsto t_3 \mapsto t_2 \mapsto t_4 \mapsto t_6 \mapsto b_2$ *corresponds to:* $x_2 \geq 0$.

$k = 2:$

$b_2 \mapsto t_6 \mapsto b_3$ *corresponds to:* $t_g \geq 0$.

$k = 3:$

$b_3 \mapsto t_6 \mapsto t_4 \mapsto b_4$ *corresponds to:* $t_4 \geq t_6$.

$b_3 \mapsto t_6 \mapsto t_5 \mapsto t_3 \mapsto t_4 \mapsto b_4$ *corresponds to:* $t_3 \geq t_5$.

$b_3 \mapsto t_6 \mapsto t_5 \mapsto t_1 \mapsto t_3 \mapsto t_4 \mapsto t_6 \mapsto b_4$ *corresponds to:* $t_1 \geq 0$.

Note in this example all possible oriented paths from b_i *to* b_{i+1} *are rigorous. An example of a nonrigorous oriented path between two neighboring bottom vertices can found using* $\overline{\omega_0} = s_1 s_2 s_3 s_2 s_1 s_2$. *In that case for* $k = 1$ *the oriented path* $b_1 \mapsto t_3 \mapsto t_4 \mapsto t_5 \mapsto t_6 \mapsto b_2$ *is not rigorous.*

Proposition 6.1.5. *[GlP00, Cor. 5.8] The string cone is the collection of points in* $\mathbb{R}^N$ *satisfying the string inequalities.*

Lemma 6.1.6. *For $\lambda = 2\rho$, the λ-cone is the collection of points in $\mathbb{R}^N$ satisfying the following inequalities which we call λ-inequalities:*

$$\lambda_i : x_i \leq 2 + \sum_{j>i} c_j x_j$$

for i from 1 to N, where

$$c_j = \begin{cases} 1 & \textit{if the vertex } t_j \textit{ is one column to the right or left of } t_i \\ -2 & \textit{if the vertex for } t_j \textit{ is in the same column as } t_i \\ 0 & \textit{otherwise.} \end{cases}$$

This is simply a way of visualizing the Lie-algebraic definition of the λ-cone (see [AB04, Thm 1.1]), and of associating a λ-inequality to each intersection point in the string diagram.

Example 6.1.7. *For the string diagram in Figure 6.1.2 we have:*

$$\lambda_1 : x_1 \leq 2 + x_3 - 2x_5 + x_6$$

$$\lambda_2 : x_2 \leq 2 + x_3 - 2x_4 + x_6$$

$$\lambda_3 : x_3 \leq 2 + x_4 + x_5 - 2x_6$$

$$\lambda_4 : x_4 \leq 2 + x_6$$

$$\lambda_5 : x_5 \leq 2 + x_6$$

$$\lambda_6 : x_6 \leq 2$$

Definition 6.1.8. *For any weight λ we define the λ-cone as the intersection of the collection of points satisfying set of inequalities in the definition above, but with the constant 2 replaced by $< \lambda, \alpha_i >$. Similarly we define the polytope $\Delta(\lambda, \overline{\omega_0})$ to be the intersection of the string cone with the corresponding λ-cone.*

6.1.1 Boxes

In constructing mirrors using the string degenerations we will need an a set of relations that play the role of "box equations". We begin with a definition of a "box".

Definition 6.1.9. *Let* $\Gamma = \Gamma(\overline{\omega_0})$ *be the string diagram viewed in* $\mathbb{R}^2$*. Then for each bounded connected component* c *of* $\mathbb{R}^2 \backslash \Gamma$*, we call the boundary of* c *a box.*

Example 6.1.10. *For the string diagram* $\Gamma(s_3 s_1 s_2 s_1 s_3 s_2)$ *there are three boxes given by the following circuits. (See Figure 6.1.2)*

1. $t_2 \mapsto t_3 \mapsto t_4 \mapsto t_2$
2. $t_3 \mapsto t_5 \mapsto t_6 \mapsto t_4 \mapsto t_3$
3. $t_1 \mapsto t_5 \mapsto t_3 \mapsto t_1$

Definition 6.1.11. *A box bounded by a cycle* $t_i \mapsto t_{i+1} \mapsto t_{i+2} \mapsto t_i$ *is called type* $RR_{121}121$ *if* t_{i+1} *is the column to the right of* t_i*, and type* R_{212} *if* t_{i+1} *is in the column to the left of* t_i*.*

These boxes will help us describe how Δ changes for different choices of reduced word decompositions.

6.2 How points of Δ change under a Braid move

Any two reduced decompositions of ω_0 are connected by a finite sequence of the following two braid moves.

1. 2-move: exchanges $(s_i,\, s_j)$ with $(s_j,\, s_i)$ where $|i - j| > 1$
2. 3-move: exchanges (s_i, s_j, s_i) with (s_j, s_i, s_j) where $|i - j| = 1$

Figure 6.2 shows a 2-move from $\overline{\omega_0} = \mathbf{s_3 s_1} s_2 s_1 s_3 s_2$ to $\overline{\omega_0}' = \mathbf{s_1 s_3} s_2 s_1 s_3 s_2$ Figure 6.2 shows a 3-move from $\overline{\omega_0} = s_3 \mathbf{s_1 s_2 s_1} s_3 s_2$ to $\overline{\omega_0}' = s_3 \mathbf{s_2 s_1 s_2} s_3 s_2$ A 3-move from $\overline{\omega_0}$ to $\overline{\omega_0}'$ fixes all of the string diagram except for one box of type $R_{121} or R_{212}$.

Definition 6.2.1. *Given a 3-move from* $\overline{\omega_0}$ *to* $\overline{\omega_0}'$*, label the unfixed box in the string diagram for* $\overline{\omega_0}$*,* R *and its image* R'*. Note that if* R *is of type* R_{121} *then* R' *must be of type* R_{212} *and vice versa.*

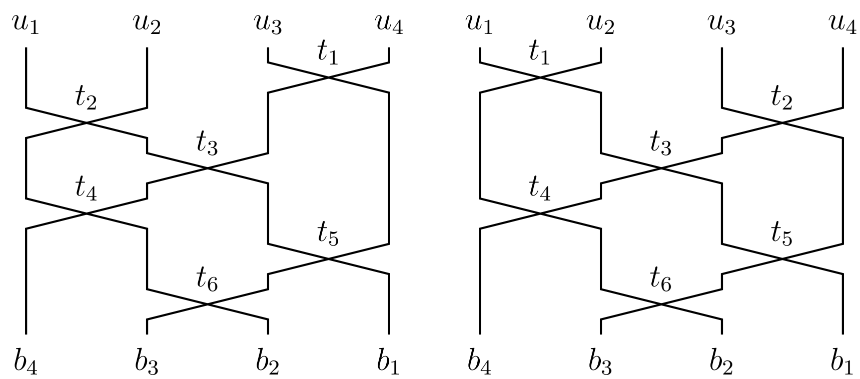


Figure 6.3: String Diagrams for $\overline{\omega_0} = \mathbf{s_3 s_1} s_2 s_1 s_3 s_2$ and $\overline{\omega_0}' = \mathbf{s_1 s_3} s_2 s_1 s_3 s_2$

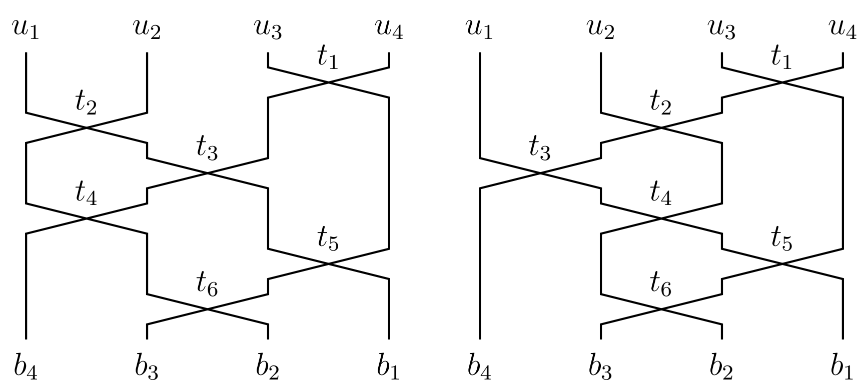


Figure 6.4: String diagrams for $\overline{\omega_0} = s_3 \mathbf{s_1 s_2 s_1} s_3 s_2$ and $\overline{\omega_0}' = s_3 \mathbf{s_2 s_1 s_2} s_3 s_2$

The map between string cones differing by a single braid move, defined by Berenstein and Zelevinsky in [BZ93, Thm 2.7], restricts to a map between polytopes $\Delta(\overline{\omega_0})$ and $\Delta(\overline{\omega_0}')$. They prove that the map fixes all of the coordinates except for the two or three corresponding to those being exchanged in the braid move. For those coordinates they have the following maps:

1. 2-move: $(x_i, x_j) \rightarrow (x_j, x_i)$
2. 3-move: $(x_i, x_j, x_k) \rightarrow (\max(x_k, x_j - x_i), x_i + x_k, \min(x_i, x_j - x_k))$. Note that i, j and k are consecutive integers.

6.3 How the Facets of Δ change under a Braid move

Recall, that we are seeking to define a mirror family in an analogous way to Batyrev's construction. Following this motivation we are interested in understanding vertices of Δ^* and how they change under the braid move. Since vertices of Δ^* correspond to facets of Δ, we examine how the facets of Δ change under Berenstein and Zelevisnky's piecewise linear map.

Notice that the piecewise linear map for a 2-move is just a relabeling of the variables so the polytopes are obviously isomorphic. In this section we classify the defining inequalities of Δ and how they change under a 3-move. From these results we prove that the dual polytope Δ^* is integral.

Definition 6.3.1. *Write the defining inequalities of $\Delta(\overline{\omega_0})$ as $\sum_i m_{i_d} x_i \leq$(0 or 2), and define $M_d = (m_{1_d}, \cdots, m_{N_d})$. Let $T^{M_d} := t_1^{m_{1_d}} t_2^{m_{2_d}} \cdots t_N^{m_{N_d}}$ and define $BF_{\overline{\omega_0}}$ as the family of hypersurfaces in the torus T=Spec$\mathbb{C}[t_1, t_1^{-1}, \ldots, t_N, t_N^{-1}]$ given by the equations:*

$$f_{\overline{\omega_0},a} := 1 - \sum_d a_d T^{M_d} = 0$$

where d runs over all defining inequalities of Δ. We will refer to individual hypersurfaces as $Z_{\overline{\omega_0},a}$, where $a = (a_1, \cdots, a_r)$ denotes the coefficient vector.

Only three coordinates change under a 3-move. In what follows we will refer to those coordinates as x_i, x_j, and x_k. We call the product of the t_i, t_j, and t_k variables inside of T^{M_d} a *monomial piece*, and refer to this product as τ^{M_d}. We will refer to the monomial pieces associated to string (resp. λ) inequalities as string (resp. λ) monomial pieces. Since each inequality corresponds to a monomial, we can classify all possible string and λ-inequalities in terms of their monomial pieces.

Example 6.3.2. *For the reduced word decomposition $\overline{\omega_0} = s_3s_1s_2s_1s_3s_2$ there is a unique possibility for performing a 3 move, namely $s_3\mathbf{s_1s_2s_1}s_3s_2 \to s_3\mathbf{s_2s_1s_2}s_3s_2$. So we can classify the monomials in terms of the three coordinates that change $t_i := t_2, t_j := t_3, t_k := t_4$. The rigorous path $b_1 \to t_5 \to t_3 \to t_4 \to t_6 \to b_2$ corresponds to the monomial $t_3^{-1}t_4$ so it will fall under the classification of $t_j^{-1}t_k$. The λ_1-inequality which corresponds to the monomial $t_1t_3^{-1}t_5^2t_6^{-1}$ would be represented by the monomial piece t_j^{-1}.*

6.3.1 Classification of λ-Monomial Pieces

Theorem 6.3.3. *The following is a complete classification of λ-monomial pieces before and after a 3-move.*

$t_it_j^{-1}t_k^2 \leftrightarrow t_it_j^{-1}t_k^2 \qquad t_jt_k^{-1} \leftrightarrow t_jt_k^{-1} \qquad t_k \leftrightarrow t_k \qquad t_i^{-1}t_k^{-1} \leftrightarrow t_j^{-1}$

$t_i^2t_j^{-1}t_k^2 \leftrightarrow t_i^{-1}t_j^2t_k^{-1}$

An arrow indicates which monomial pieces are exchanges under the braid move.

Proof. As above, we fix t_i, t_j and t_k as the coordinates on which the 3-move occurs. We classify the monomial pieces for the inequalities λ_l for all values of l. For $l > k$ the monomial for λ_l doesn't have any terms from the braid move section. This means that the monomial piece is 1 and it doesn't change after a braid move.

For λ_i, λ_j, and λ_k we have the following monomial pieces regardless if R is of type R_{121} or R_{212}: $\lambda_i : t_it_j^{-1}t_k^2 \qquad \lambda_j : t_jt_k^{-1} \qquad \lambda_k : t_k$

If $l < i$, then the λ_l monomial piece only depends on which column t_l lies (as pictured in Figure 6.3.1).

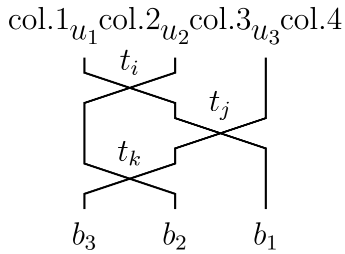


Figure 6.5: Graph of columns for λ-inequalities

1. If t_l is in column 1 we have the monomial piece $t_i^{-1}t_k^{-1}$ before, and t_j^{-1} after the braid move.
2. If t_l is in column 2 we have the monomial piece $t_i^2 t_j^{-1} t_k^2$ before, and $t_i^{-1} t_j^2 t_k^{-1}$ after the braid move.
3. If t_l is in column 3 we have the monomial piece $t_i^{-1} t_j^2 t_k^{-1}$ before, and $t_i^2 t_j^{-1} t_k^2$ after the braid move.
4. If t_l is in column 4 we have the monomial piece t_j^{-1} before, and $t_i^{-1} t_k^{-1}$ after the braid move.

Note that for R of type R_{212} we get the same collection of monomial pieces. □

Example 6.3.4. *For the reduced word* $\overline{\omega_0} = s_3 s_1 s_2 s_1 s_3 s_2$ λ_1 *corresponds to the monomial* $t_1 t_3^{-1} t_5^2 t_6^{-1}$ *represented by the monomial piece* t_j^{-1}*. Relative to the box* R_{121}*,* t_1 *is in column 4. After the braid move the* λ_1 *corresponds to the monomial* $t_1 t_2^{-1} t_4^{-1} t_5^2 t_6^{-1}$ *which corresponds to the monomial piece* $t_i^{-1} t_k^{-1}$*.*

6.3.2 Classification of String Monomial Pieces

In a similar fashion we will classify all of the string monomial pieces. We do this by examining how a rigorous path may pass through a type R_{121} region. We will then see which rigorous paths are possible after the braid move.

Theorem 6.3.5. *The following is a complete classification of the string monomial pieces, and how they are exchanged under a type B_2 braid move. The & sign indicates that a pair of monomial pieces must occur together.*

$$t_k^{-1} \leftrightarrow t_i^{-1} \ \& \ \ t_j^{-1}t_k$$

$$t_i \leftrightarrow t_k \ \& \ \ t_i^{-1}t_j$$

$$t_it_k \leftrightarrow t_j$$

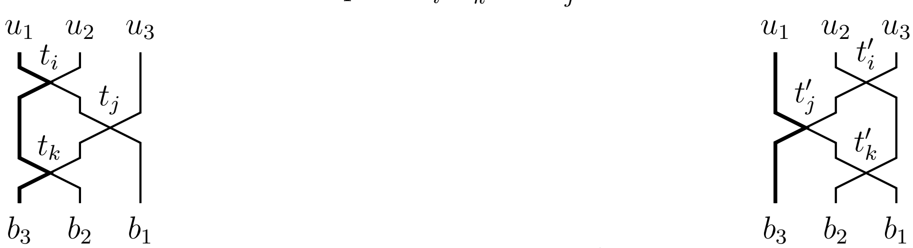
$$t_i^{-1}t_k^{-1} \leftrightarrow t_j^{-1}$$

$$t_it_j^{-1}t_k \leftrightarrow t_i^{-1}t_jt_k^{-1}$$

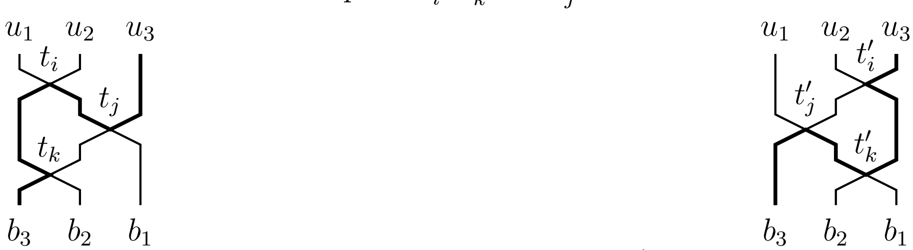
Proof. We would like to classify all of the string inequalities in terms of their monomial pieces. By symmetry we can assume that R is of type R_{121}. If a string monomial piece is anything other than 1, the corresponding rigorous path p must change strings in the region R. Any oriented path entering R more than once must intersect itself so it cannot be rigorous. Therefore, we may assume that a rigorous path only enters and exits R once. There are $6 * 5 = 30$ possible pairs of entry and exit points for p. If p enters R at the vertex b_e, then all strings U_f for $f \geq e$ must be oriented upward. Therefore p cannot exit R through b_f where $f < e$. Similarly if p enters at u_e it may not exit through u_f for $f > e$. This eliminates 6 pairs of entry and exit points. By the same reasoning no rigorous path which changes strings on R can enter at b_3 (resp. u_1) and exit at u_3 (resp. b_1). What remains are 22 entry and exit pairs.

Figure 6.3.2 correspond to cases of entry and exit points in which there is a unique rigorous path before and after the braid move. In each diagram displayed in Figure 6.3.2 there is one possible rigorous path through R before the braid move. After the braid move there are two possible rigorous paths through R'. Since the string cone is the collection of points satisfying all possible string inequalities there must be two inequalities (and hence monomial pieces) in the image. In what follows we group these monomial pieces together.

We have the monomial piece $t_i^{-1}t_k^{-1} \to {t'_j}^{-1}$



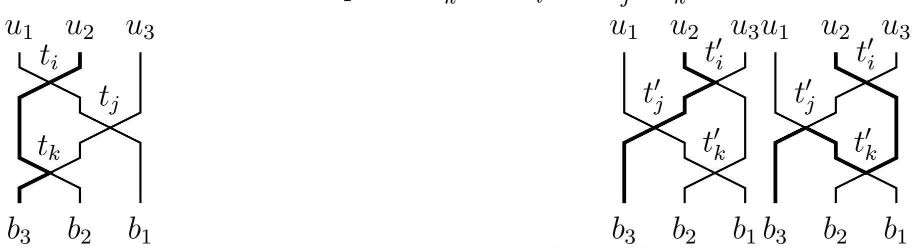
We have the monomial piece $t_i^{-1}t_k^{-1} \to {t'_j}^{-1}$



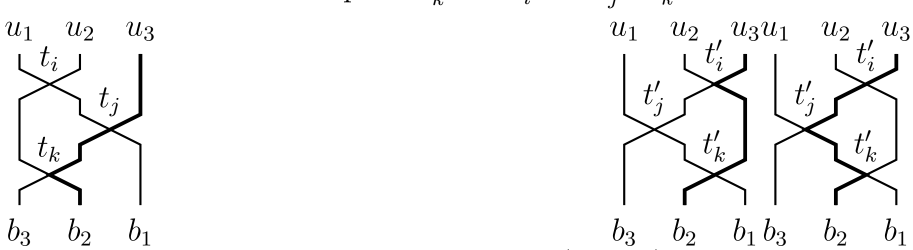
We have the monomial piece $t_i^{-1}t_jt_k^{-1} \to t'_i{t'_j}^{-1}t'_k$

Figure 6.6: String diagrams with one rigorous path before and after the 3-move

We have the monomial pieces $t_k^{-1} \to {t'_i}^{-1}$& ${t'_j}^{-1}t'_k$



We have the monomial piece $t_k^{-1} \to {t'_i}^{-1}$& ${t'_j}^{-1}t'_k$



We have the monomial pieces $t_k^{-1} \to {t'_i}^{-1}$& ${t'_j}^{-1}t'_k$.

Figure 6.7: String diagrams with one rigorous path before but two such paths after a 3-move

We can construct the other 16 cases of entry and exit points from these six using symmetry. For each rigorous path on R' listed above there is a corresponding rigorous path on R as follows. For the entry and exit points of the new path, exchange b_1 and b_3 (resp u_1 and u_3). Then construct a new path which changes strings at the same vertices as the path on R'. Unless the path on R' changes strings at every vertex, this new path will be rigorous. This is what we call horizontal path symmetry. Rigorous paths which changes strings at every vertex do not have horizontally symmetric partners. There is also a vertical symmetry. Take a path on R and exchange the "top" with "bottom" on the entry and exit points (i.e. b_3 is switched with u_1). Now create a path which changes strings at vertically flipped vertices (i.e. t_i instead of t_k). This will be a new rigorous path. The remaining 16 single entry point cases can be gotten from the 6 listed above, through a combination of horizontal and vertical symmetries. From this classification we see that the sets of monomial pieces exchanged under a 3-move. □

Example 6.3.6. *We let START (resp. FINISH) denote the piece of a rigorous path before it enters (resp. after it exits) R or R'. Assume the following is a rigorous path before the braid move: START $\mapsto b_2 \mapsto t_k \mapsto b_3 \mapsto$ FINISH. Since the 3-move only effects the braid move region, we can follow the "START" and "FINISH" pieces of the path in the exact same way. What remains to see is how many ways we can get from b_2 to b_3 in a rigorous setting. It turns out there are two such paths:*

START $\mapsto b_2 \mapsto t'_k \mapsto t'_i \mapsto t'_j \mapsto b_3 \mapsto$ FINISH: and

START $\mapsto b_2 \mapsto t'_k \mapsto t'_j \mapsto b_3 \mapsto$ FINISH.

Thus after the braid move there are two corresponding rigorous paths. This gives the associate of monomial pieces $t_k^{-1} \mapsto t_i'^{-1} \& t_j'^{-1} t'_k$.

6.4 Integrality of Δ^*

Given this classification we can prove that the polytope Δ^* is integral. To do this we want to show that each facet of Δ is at height 1 from an interior integral point.

Lemma 6.4.1. *For any braid region R, the line connecting the apex of the λ-cone and the origin lies on the hyperplane $x_i + x_k = x_j$.*

Proof. It suffices to show that the apex of the λ-cone lies on the hyperplane. At the apex all of the λ-inequalities are equalities. We can write the following equations, where A, B and are the contributions to the equation from x_d for $d > k$.

$$x_i = 2 + x_j - 2x_k + A$$
$$x_k = 2 + B$$

Note that since t_i and t_k are in the same column of the string diagram we have $A = B$. From this we can see that

$$x_i + x_k = (2 + x_j - 2x_k + A) + x_k = (x_j - x_k) + 2 + A = (x_j - x_k) + x_k = x_j.$$

□

Theorem 6.4.2. *$\Delta^*(\overline{\omega_0})$ is an integral polytope for any reduced decomposition $\overline{\omega_0}$.*

Proof. In [BCFKvS00], Batyrev (et. al.) show that for the reduced decomposition $\overline{\omega_0} = s_1 s_2 s_1 \dots s_n s_{n-1} \dots s_1$ the polytope Δ^* is an integral polytope with a unique interior point $P = (1, 2, 1, \cdots, n, n-1, \cdots, 2, 1)$. The facets of Δ determine linear functions L such that $L(P) = 1$. Let Ψ be the piecewise linear map between polytopes defined be Berenstein and Zelevinsky. We want to show that under a 3-move, the facets of Δ' determine L' such that $L'(P') = 1$ where $P' = \Psi(P)$.

The map Ψ is defined by two linear maps ψ_1 and ψ_2 which agree on the hyperplane $x_i + x_k = x_j$. For any linear function L associated to a facet of Δ there are two corresponding linear functions on Δ'; $L'_1 = L \circ \psi_1^{-1}$ and $L'_2 = L \circ \psi_2^{-1}$.

The point P is halfway between the origin (which is the apex of the string cone) and the apex of the λ-cone. By the preceding lemma this shows that P lies on the hyperplane $x_i + x_k = x_j$. Since ψ_1 and ψ_2 agree on that hyperplane we have $\psi_1(P) = \psi_2(P) = \Psi(P)$.

Therefore, we can make the calculations $L_1'(P') = L({\psi_1}^{-1}(P') = L(P) = 1$ and $L_2'(P') = L({\psi_1}^{-1}(P') = L(P) = 1$, which implies that the polytope Δ'^* is integral. Note that P' is still halfway between the origin and the apex of the λ-cone, which allows this argument to be repeated.

Under a 2-move the polytopes Δ and Δ' are isomorphic, so Δ^* is integral if and only if Δ'^* is integral.

By starting with $\overline{\omega_0} = s_1 s_2 s_1 \ldots s_n s_{n-1} \ldots s_1$ and the point $P = (1, 2, 1, \cdots, n, n-1, \cdots, 2, 1)$ we can compose braid moves and always have an interior point P' on which all of the defining linear functionals take value 1. This proves that Δ^* is integral for any choice of reduced decomposition. □

Corollary 6.4.3. *The monomials in $f_{\overline{\omega_0},a}$ correspond to vertices of Δ^* (i.e. $BF_{\overline{\omega_0}} = V(\Delta^*)$).*

Remark 6.4.4. *If in addition we knew this interior point was unique, and Δ was integral, this would show that Δ was in fact a reflexive polytope.*

Chapter 7

Mirror Construction

7.1 Definition of Mirror Candidates

Recall from Chapter 6 that $BF_{\overline{\omega_0}}$ is a family over the affine space $\mathbb{A}^r$ and is a subfamily of the linear system $|\mathcal{O}_{X_{\Delta^*}}(1)|$ intersected with the torus. We want to define a special subfamily of $BF_{\overline{\omega_0}}$ whose members are mirrors to the anticanonical hypersurfaces of G/B. In this chapter we prove such a family exists, and we give an explicit birational map between these families for any two choices of $\overline{\omega_0}$.

We restrict our attention to a subfamily of $BF_{\overline{\omega_0}}$ whose coefficients satisfy a set of relations we call *combinatorial box equations.*

Lemma 7.1.1. *Every box of the string diagram is obviously bounded above by a vertex* t_{top} *and directly below by a vertex* t_{bot}*.*

Definition 7.1.2. *Let* $T^{\lambda_{\text{top}}}$ *(resp.* $T^{\lambda_{\text{bot}}}$*) be the monomial in* $f_{\overline{\omega_0},a}$ *corresponding to the* λ*-inequality associated to* t_{top} *(resp.* t_{bot}*).*

Definition 7.1.3. *For every two string inequalities* p_1*,* p_2 *with corresponding monomials* T^{p_1} *and* T^{p_2} *satisfying the following* box conditions*:*

1. *there exists a box with corresponding monomials* $T^{\lambda_{\text{top}}}$ *and* $T^{\lambda_{\text{bot}}}$ *such that* $T^{p_1}\,T^{\lambda_{\text{top}}} = T^{p_2}\,T^{\lambda_{\text{bot}}}$*,*
2. *the* t_{top} *degree of* $T^{p_1} = -1$*,*
3. *the* t_{bot} *degree of* $T^{p_2} = 1$*,*

we define an equation $a_{p_1}a_{\lambda_{\text{top}}} = a_{p_2}a_{\lambda_{\text{bot}}}$. *We call the collection of all such equations the* combinatorial box equations.

Definition 7.1.4. *Let* $P_{\overline{\omega_0}}$ *be the subscheme defined by the combinatorial box equations in the torus Spec* $\mathbb{C}[a_1, a_1^{-1}, \cdots, a_r, a_r^{-1}]$.

Definition 7.1.5. *Let* $F_{\overline{\omega_0}}$ *be the subfamily of* $BF_{\overline{\omega_0}}$ *whose coefficients are nonzero and satisfy the combinatorial box equations. Note that* $F_{\overline{\omega_0}}$ *is a family over the base* $P_{\overline{\omega_0}}$.

Example 7.1.6. *In the case of* A_3 *with* $\overline{\omega_0} = s_3s_1s_2s_1s_3s_3$ *we have* $BF_{\overline{\omega_0}}$ *is the family of hypersurfaces satisfying :*

$a_0t_1^{-1} + a_1t_2^{-1} + a_2t_6^{-1} + a_3t_3^{-1}t_5 + a_4t_3^{-1}t_4 + a_5t_4^{-1}t_6 + a_6t_5^{-1}t_6 + a_7t_1t_3^{-1}t_52t_6^{-1} + a_8t_2t_3^{-1}t_4^2t_6^{-1} + a_9t_3t_4^{-1}t_5^{-1}t_6^2 + a_{10}t_4t_6^{-1} + a_{11}t_5t_6^{-1} + a_{12}t_6 = 1$

For this example the combinatorial box equations are

$$a_0a_7 = a_3a_{11}$$

$$a_1a_8 = a_4a_{10}$$

$$a_3a_{10} = a_4a_{11}$$

$$a_4a_9 = a_5a_{12}$$

Remark 7.1.7. *Attempting to extend Batyrev's definition of "box equations" to a nonsmall case, by restricting to multiplicative versions of the conditions for the corresponding divisors to be Cartier would lead to the extra relation*

$$a_5a_{10} = a_6a_{11}$$

7.2 Case of standard reduced decomposition

In their work Alexeev and Brion prove that for the reduced decomposition $\overline{\omega_0} = s_1s_2s_1 \cdots s_ns_{n-1} \cdots s_2s_1$ the anticanonical string degeneration of G/B is the same as that constructed in [GL96]. In addition they give a linear change of coordinates between the polytope Δ and the polytope used in [BCFKvS00].

In particular inequalities defined by horizontal edges on the graph Γ correspond to λ inequalities, while vertical edge inequalities correspond to string inequalities. By using this association, it is easy to check that the combinatorial box equations defined above, are precisely the box equations defined by Batyrev et. al.

As a consequence, for the reduced decomposition $\overline{\omega_0} = s_1 s_2 s_1 \cdots s_n s_{n-1} \cdots s_2 s_1$ the mirror family constructed above is precisely the mirror family defined by Batyrev et. al. We will see later that the combinatorial mirror family coincided with Batyrev's construction whenever the string degeneration is small.

7.3 Tropicalization and Geometric Lifting

We want to show that the families $F_{\overline{\omega_0}}$ are birational, which would imply that their generic elements would have the same mirror properties. In what follows we show the birationality by constructing an explicit map between the families. This map can be thought of as an example of geometric lifting which we describe in this section.

Definition 7.3.1. *A tropical semiring is a collection of elements S along with two operations $\oplus : S \times S \to S$ and $\otimes : S \times S \to S$ called tropical addition and multiplication, satisfying the following properties.*

1. *$\oplus$ and $\otimes$ are commutative, associative and obey the distributive property.*
2. *There exists an additive identity $0 \in S$*
3. *There exists a multiplicative identity $1 \in S$*
4. *Every nonzero element $s \in S$ has a multiplicative inverse s^{-1}*

Notice that not every element needs to have an additive inverse, so although division makes sense in this ring, subtraction does not.

Example 7.3.2. *The set* $\mathbb{R}_{trop} := \mathbb{R} \cup -\infty$ *with the operators* $\oplus$ *and* $\otimes$ *defines by*

$$a \oplus b = max(a, b) \text{ and } a \otimes b = a + b$$

(ordinary addition), is an example of a tropical semiring.

These semirings are the basis for the study of tropical geometry, which has some interesting connections with algebraic geometry.

Our work is in part motivated by one of these connections.

7.3.1 Tropicalization and Geometric Lifting

If f is a subtraction free rational function from $\mathbb{R}^n \to \mathbb{R}^m$ with integral coefficients, we can replace the multiplication and addition with tropical addition and multiplication to get a function

$$f_{trop} : \mathbb{R}^n_{trop} \to \mathbb{R}^m_{trop}.$$

Since the process of tropicalization is the limit of taking logs of the absolute value of the functions we lose any coefficients. This process is called *tropicalization*. Note that every such rational function has a unique tropicalization.

Example 7.3.3. *If* $f : \mathbb{R}^2 \to \mathbb{R}$ *is defined by* $f(x, y) = 4x^2 + 3xy + \frac{y}{x}$ *then* $f_{trop}(x, y) = max(2x, x + y, y - x)$*.*

The piecewise linear maps defining the map between string cones can be viewed as tropical maps as follows.

Example 7.3.4. *The map* $f : (x, y, z) \to (min(z, y - x), x + z, max(x, y - z))$ *can be written tropically as*

$$f : (x, y, z) \to (\frac{yz(z + \frac{y}{x})}{x}, xz, x + \frac{y}{z})$$

The inverse operation to tropicalization is called geometric lifting. If f_{trop} is a tropicalization of a rational map f we call f a geometric lift of f_{trop}. Note that geometric lifting is not a unique operation (since tropicalization doesn't see the coefficients).

7.4 Birationality of $F_{\overline{\omega_0}}$

We want to show that all of the families $F_{\overline{\omega_0}}$ are independent of the choice of reduced decomposition. We do this by giving an explicit birational map between any two such families.

Definition 7.4.1. *Assume that* $\overline{\omega_0}$ *and* $\overline{\omega_0}'$ *differ by a 3-move. Let* $C = \frac{a_{\lambda_k}}{a_{\lambda_i}}$ $T = Spec\mathbb{C}[t_1, t_1^{-1}, \ldots, t_N, t_N^{-1}]$ *and* $T' = Spec\mathbb{C}[t'_1, {t'_1}^{-1}, \ldots, t'_N, {t'_N}^{-1}]$*. For a fixed coefficient vector* a*, define* $h_a : T' \dashrightarrow T$ *by*
$(t_i, t_j, t_k) = (\frac{t'_i t'_k + C t'_j}{t'_i}, \frac{t'_i t'_k}{C}, \frac{C t'_i t'_j}{t'_i t'_k + C t'_j})$*, and* $t_q = t'_q$ *for* $q \notin \{i, j, k\}$*.*

Define a map $h'_a : T \dashrightarrow T'$ by exchanging t with t' in the map above. Then we have $h'_a \circ h_a = id$.

Remark 7.4.2. *Note that the map* h_a *is a geometric lift of the piecewise linear map between the corresponding polytopes.*

Proposition 7.4.3. *For a fixed coefficient vector* $a = (a_1, \cdots, a_r) \in P_{\overline{\omega_0}}$*,* $f' := h_a^*(f_{\overline{\omega_0},a})$ *defines a variety* $Z_{\overline{\omega_0}',a'}$ *for some coefficient vector* a'*. Therefore, we have an induced birational map* $h_a : Z_{\overline{\omega_0}',a'} \dashrightarrow Z_{\overline{\omega_0},a}$*.*

Remark 7.4.4. *We will see in Proposition 7.4.10 that the map* g *defined by* $g^*(a) = a'$ *is an isomorphism between the parameter spaces* $P'_{\overline{\omega_0}'}$ *and* $P_{\overline{\omega_0}}$*. The coordinates for* a' *are explicitly given as regular functions of the coordinates for* a *in Proposition 7.4.6.*

Proof of Proposition 7.4.3. We show that for a fixed coefficient vector a, h_a is birational in two steps. First, in Proposition 7.4.6, we show that $h_a^*(1 - \sum_d a_i T^{M_d}) = (1 - \sum_{d'} a'_i T^{M'_d})$ for some coefficient vector a'. Then, in Section 7.4.2 we show that the coefficient vector a' is actually in $P'_{\overline{\omega_0}'}$ by showing that g^* preserves the combinatorial box equations.

7.4.1 Action of h^* on $f_{\overline{\omega_0},a}$

Definition 7.4.5. *We refer to the box equations coming from the boxes* R *and* R' *as* local box equations.

Proposition 7.4.6. *Fix a coefficient vector a satisfying the local box equations. Then*

$$h_a^*(1 - \sum_d a_d T^{M_d}) = (1 - \sum_{d'} a'_d T^{M'_d})$$

for some coefficient vector a'.

Remark 7.4.7. *A priori it is not even clear that $h_a^*(1 - \sum_d a_d T^{M_d})$ is a sum of Laurent monomials. This theorem is an example of what Fomin and Zelevinsky call a* Laurent phenomenon *and may be related to their work on cluster algebras [FZ02].*

Proof. We examine how h_a^* acts on $1 - \sum_d a_i T^{M_d}$ monomial by monomial. Since h_a^* only depends on t_i, t_j, and t_k we classify its action on monomials based on the corresponding monomial pieces.

A priori $h_a^*(f_{\overline{\omega_0},a})$ is a rational function, but not a sum of monomials. In several cases, in order to see that $h_a^*(1 - \sum_d a_i T^{M_d})$ is a linear combination of monomials, we have to group pairs of monomials together. In some cases h_a^* only maps a particular combination of two monomials to a monomial, if the coefficients satisfy the local box equations. The map h_a^*

takes the following classes of monomials to monomials (or sum of monomials):

$$h_a^*(a_d t_i^{-1} t_k^{-1}) = a_d a_{\lambda_i} a_{\lambda_k}^{-1} t_j'^{-1}$$
$$h_a^*(a_d t_i^2 t_j^{-1} t_k^2) = a_d a_{\lambda_i}^{-3} a_{\lambda_k}^3 t_i'^{-1} t_j'^2 t_k'^{-1}$$
$$h_a^*(a_d t_i^{-1} t_j^2 t_k^{-1}) = a_d a_{\lambda_i}^3 a_{\lambda_k}^{-3} t_i'^2 t_j'^{-1} t_k'^2$$
$$h_a^*(a_d t_j^{-1}) = a_d a_{\lambda_i}^{-1} a_{\lambda_k} t_i'^{-1} t_k'^{-1}$$
$$h_a^*(a_d t_i^{-1} t_k^{-1}) = a_d a_{\lambda_i} a_{\lambda_k}^{-1} t_j'^{-1}$$
$$h_a^*(a_d t_j^{-1}) = a_d a_{\lambda_i}^{-1} a_{\lambda_k} t_i'^{-1} t_k'^{-1}$$
$$h_a^*(a_d t_i t_k) = a_d a_{\lambda_i}^{-1} a_{\lambda_k} t_j'$$
$$h_a^*(a_d t_j) = a_d a_{\lambda_i} a_{\lambda_k}^{-1} t_i' t_k'$$
$$h_a^*(a_d t_i t_j^{-1} t_k) = a_d a_{\lambda_k}^2 a_{\lambda_i}^{-2} t_i'^{-1} t_j' t_k'^{-1}$$
$$h_a^*(a_{\lambda_j} t_j t_k^{-1}) = a_{\lambda_i}^2 a_{\lambda_j} a_{\lambda_k}^{-2} t_i' t_j'^{-1} t_k'^2 + a_{\lambda_i} a_{\lambda_j} a_{\lambda_k}^{-1} t_k'$$
$$h_a^*(a_d t_i^{-1} t_j t_k^{-1}) = a_d a_{\lambda_k}^{-2} a_{\lambda_i}^2 t_i' t_j'^{-1} t_k'$$
$$h_a^*(a_d t_k^{-1}) = a_d t_i'^{-1} + a_d a_{\lambda_i} a_{\lambda_k}^{-1} t_j'^{-1} t_k'$$
$$h_a^*(a_d t_i) = a_d t_k' + a_d a_{\lambda_i}^{-1} a_{\lambda_k} t_i'^{-1} t_j'$$

The following classes of monomials must be grouped together in order for h_a^* to take them to a monomial.

$$h_a^*(a_{\lambda_i} t_i t_j^{-1} t_k^2 + a_{\lambda_k} t_k) = a_{\lambda_i}^{-1} a_{\lambda_k}^2 t_j' t_k'^{-1}$$

The following classes of monomials must be grouped together, and their coefficients must satisfy the local box equations in order for h_a^* to take them to monomials. We write the second coefficient in terms of the local box equation.

$$h_a^*(a_d t_k + a_d a_{\lambda_i}^{-1} a_{\lambda_k} t_i^{-1} t_j) = a_d t_i'$$
$$h_a^*(a_d t_i^{-1} + a_d a_{\lambda_i} a_{\lambda_k}^{-1} t_j^{-1} t_k) = a_d t_k'^{-1}$$

The monomials occurring in image($h_a{}^*$) are precisely those occurring in $f_{\overline{\omega_0},a}$. From this we see that

$h_a^*(1-\sum_d a_i T^{M_d}) = (1-\sum_{d'} a_i' T^{M_d'})$ for some coefficient vector a_i'. The inverse map is given by using the same construction for the braid move from $\overline{\omega_0}'$ to $\overline{\omega_0}$. □

7.4.2 Preservation of Box Equations.

Proposition 7.4.8. *For any 3-move, the map g^* preserves box equations.*

Proof. Let R' be the image of the box R (we can assume it is of type R_{212} with the proof of the other case following using the same methods). We classify all boxes O' of the string diagram for $\overline{\omega_0}'$ by their positions in comparison with R'.

The proof of every case follows the following method: First, classify all possibilities of monomial pieces τ for which the box condition on O' could be satisfied. Since we know how the monomial pieces change under the braid move, we verify that the box conditions must be satisfied for the box $O = \text{preimage}(O')$. Since $a \in P$ we know that if the box conditions are satisfied on O, then the corresponding box equations (in a_d) are also satisfied. We use the map g to write a_d' in terms of a_d, and check that new combinatorial box equations on O' are satisfied.

Remark 7.4.9. *For some string monomials on O' there are two different corresponding string monomials in the preimage. Only one of these two string monomials is used to construct a box condition on the preimage. The first box condition may not be met in the preimage when the other choice of string monomial is used.*

We classify the boxes O' into eight groups. For each group, we state how many pair of monomial pieces could possibly correspond to monomials satisfying the box conditions on O'.

1. O' doesn't touch R' (1 pair)
2. $O' = R'$ (1 pair)

3. O' is below and left of R' (3 pair)

4. O' is above and left of R' (3 pair)

5. O' is directly above R' (3 pair)

6. O' is directly below R' (3 pair)

7. O' is to the right of R' (6 pair)

8. O' is to the left of R (6 pair)

What follows is the proof for the case when R' is directly above O'. The proof for all of the other groups follow the exact same method.

Assume R' is directly above O' as in Figure 7.4.2. Assume that p_1' and p_2' satisfy the

Figure 7.1: Relative position of O.

box conditions on O'. Let τ^{M_d} be the monomial piece associated to T^{M_d}. By comparing the λ-inequalities see the following relationship among monomial pieces.

$$\frac{\tau^{p_2'}}{\tau^{p_1'}} = \frac{\tau^{\lambda_{\text{top}'}}}{\tau^{\lambda_{\text{bot}'}}} = t_k.$$

From this we see that there are three possibilities for pairs of string monomial pieces.

1. Assume that

$$\tau^{p_1'} = t_i^{-1}t_k^{-1} \text{ and } \tau^{p_2'} = t_i^{-1}.$$

We construct a set of monomials satisfying the box conditions for the box O as follows.

$$T^{p_1'} \cdot T^{\lambda_{\mathrm{top}'}} = T^{p_2'} \cdot T^{\lambda_{\mathrm{bot}'}}$$

$$T^{p_1'} \cdot T^{\lambda_{\mathrm{top}}} \cdot t_j^{-1}t_k^2 = T^{p_2'} \cdot T^{\lambda_{\mathrm{bot}}}$$

$$T^{p_1} \cdot t_i^{-1}t_jt_k^{-1} \cdot T^{\lambda_{\mathrm{top}}} \cdot t_j^{-1}t_k^2 = T^{p_2} \cdot t_i^{-1}t_k \cdot T^{\lambda_{\mathrm{bot}}}$$

$$T^{p_1} \cdot T^{\lambda_{\mathrm{top}}} = T^{p_2} \cdot T^{\lambda_{\mathrm{bot}}}$$

Since the box condition are satisfied on O, we have the following equations in a_i.

$$a_{p_1}a_{\lambda_{\mathrm{top}}} = a_{p_2}a_{\lambda_{\mathrm{bot}}}.$$

Rewriting the a_i' in terms of a_i we see that

$$a_{p_1'}a_{\lambda_{\mathrm{top}'}} = a_{p_1}a_{\lambda_{\mathrm{top}}} \text{ and } a_{p_2'}a_{\lambda_{\mathrm{bot}'}} = a_{p_2}a_{\lambda_{\mathrm{bot}}}.$$

Therefore,

$$a_{p_1'}a_{\lambda_{\mathrm{top}'}} = a_{p_2'}a_{\lambda_{\mathrm{bot}'}}.$$

2. Assume that

$$\tau^{p_1'} = t_k^{-1} \text{ and } \tau^{p_2'} = 1.$$

We construct a set of monomials satisfying the box conditions for the box O as follows.

$$T^{p_1'} \cdot T^{\lambda_{\mathrm{top}'}} = T^{p_2'} \cdot T^{\lambda_{\mathrm{bot}'}}$$

$$T^{p_1'} \cdot T^{\lambda_{\mathrm{top}}} \cdot t_j^{-1}t_k^2 = T^{p_2'} \cdot T^{\lambda_{\mathrm{bot}}}$$

$$T^{p_1} \cdot t_jt_k^{-2} \cdot T^{\lambda_{\mathrm{top}}} \cdot t_j^{-1}t_k^2 = T^{p_2} \cdot T^{\lambda_{\mathrm{bot}}}$$

$$T^{p_1} \cdot T^{\lambda_{\mathrm{top}}} = T^{p_2} \cdot T^{\lambda_{\mathrm{bot}}}$$

Since the box condition were satisfied on O, we have the following equations in a_i.

$$a_{p_1} a_{\lambda_{\text{top}}} = a_{p_2} a_{\lambda_{\text{bot}}}.$$

Rewriting the a_i' in terms of a_i we see that

$$a_{p_1'} a_{\lambda_{\text{top}'}} = a_{p_1} a_{\lambda_{\text{top}}} \text{ and } a_{p_2'} a_{\lambda_{\text{bot}'}} = a_{p_2} a_{\lambda_{\text{bot}}}.$$

Therefore,

$$a_{p_1'} a_{\lambda_{\text{top}'}} = a_{p_2'} a_{\lambda_{\text{bot}'}}.$$

3. Assume that

$$\tau^{p_1'} = t_i^{-1} t_j t_k^{-1} \text{ and } \tau^{p_2'} = t_i^{-1} t_j.$$

We construct a set of monomials satisfying the box conditions for the box O as follows.

$$T^{p_1'} \cdot T^{\lambda_{\text{top}'}} = T^{p_2'} \cdot T^{\lambda_{\text{bot}'}}$$

$$T^{p_1'} \cdot T^{\lambda_{\text{top}}} \cdot t_j^{-1} t_k^2 = T^{p_2'} \cdot T^{\lambda_{\text{bot}}}$$

$$T^{p_1} \cdot t_i^{-2} t_j \cdot T^{\lambda_{\text{top}}} \cdot t_j^{-1} t_k^2 = T^{p_2} t_i^2 t_j^{-2} t_k^2 \cdot T^{\lambda_{\text{bot}}}$$

$$T^{p_1} \cdot T^{\lambda_{\text{top}}} = T^{p_2} \cdot T^{\lambda_{\text{bot}}}$$

Since the box condition are satisfied on O, we have the following equations in a_i.

$$a_{p_1} a_{\lambda_{\text{top}}} = a_{p_2} a_{\lambda_{\text{bot}}}.$$

Rewriting the a_i' in terms of a_i we see that

$$a_{p_1'} a_{\lambda_{\text{top}'}} = a_{p_1} \frac{a_k^2}{a_i^2} a_{\lambda_{\text{top}}} \frac{a_i}{a_k} \text{ and } a_{p_2'} a_{\lambda_{\text{bot}'}} = a_{p_2} \frac{a_k}{a_i} a_{\lambda_{\text{bot}}}.$$

Therefore,

$$a_{p_1'} a_{\lambda_{\text{top}'}} = a_{p_2'} a_{\lambda_{\text{bot}'}}.$$

The proof of the other seven cases follows the exact same argument. We have shown that the combinatorial box equations are preserved under g^*, which completes the proof of proposition 7.4.3. $\square$

7.4.3 Isomorphism of Parameter Spaces

Proposition 7.4.10. *If $\overline{\omega_0}$ and $\overline{\omega_0}'$ differ by a single braid move then $P'_{\overline{\omega_0}'} \cong P_{\overline{\omega_0}}$.*

Proof. Assume $\overline{\omega_0}$ and $\overline{\omega_0}'$ differ by a 3-move. As in remark 7.4.4, define $g : P' \to P$ by $g^*(a) = a'$. By the proof of Proposition 4.6 we see that this defines a regular map from $P_{\overline{\omega_0}}$ to the torus of coordinates of $P'_{\overline{\omega_0}'}$. Proposition 4.7 shows that the image of this map is actually in $P'_{\overline{\omega_0}'}$. Similarly we can construct a regular map $g' : P_{\overline{\omega_0}} \to P'_{\overline{\omega_0}'}$. We can use Proposition 7.4.6 to verify that these maps are inverse to one another. To complete the proof we note that $P_{\overline{\omega_0}}$ and $P'_{\overline{\omega_0}'}$ are obviously isomorphic under a 2-move. $\square$

Corollary 7.4.11. *The parameter space $P_{\overline{\omega_0}}$ is a toric variety for any choice of $\overline{\omega_0}$.*

Proof. In the case of the standard reduced decomposition $(s_1 s_2 s_1 \ldots s_n s_{n-1} \ldots s_1)$ the combinatorial box equations and Batyrev's box equations are exactly the same. By Batyrev's work [Bat04, Rm.4.2], the set of nonzero a_i satisfying the box equations form a toric variety. By the previous lemma, we see that under a braid move the parameter spaces are isomorphic. We can compose braid moves to see that the parameter spaces are isomorphic for any choice of reduced decomposition. $\square$

7.4.4 Birationality of the Families $F_{\overline{\omega_0}}$

Lemma 7.4.12. *For $\overline{\omega_0}$ and $\overline{\omega_0}'$ differing by a 2-move, the map $H : F_{\overline{\omega_0}'} \to F_{\overline{\omega_0}}$ defined by exchanging t_i with t_j and exchanging the corresponding coefficients is an isomorphism.*

Proof. This is a consequence of Berenstein and Zelevinsky's map between string cones for the 2-move case. $\square$

Theorem 7.4.13. *If $\overline{\omega_0}$ and $\overline{\omega_0}'$ differ by a 3-move, the map*

$$H : F_{\overline{\omega_0}'} \dashrightarrow F_{\overline{\omega_0}}$$

defined as $H := (g, h_a)$ is a birational map of families.

Proof. This follows from the fact that g is an isomorphism, and h_a is a birational map for any fixed coefficient vector a. □

Corollary 7.4.14. *For any two reduced decompositions $\overline{\omega_0}$ and $\overline{\omega_0}'$, the families $F_{\overline{\omega_0}}$ and $F_{\overline{\omega_0}'}$ are birational.*

Proof. Connect $\overline{\omega_0}$ and $\overline{\omega_0}'$ by a sequence of braid moves. For each braid move there is a birational map of families, so we can compose these maps to get a birational map between $F_{\overline{\omega_0}}$ and $F_{\overline{\omega_0}'}$. □

Corollary 7.4.15. *Smooth Calabi-Yau manifolds birational to the generic elements of $F_{\overline{\omega_0}}$ have the same Hodge numbers for any choice of $\overline{\omega_0}$.*

Proof. Since the generic elements are all birational, smooth Calabi-Yau manifolds birational to them have the same Hodge numbers by the work of Batyrev and Kontsevich [Bat00, K95]. □

7.5 Proof that construction coincides with Batyrev's in the small case

Recall that in [Bat04] Batyrev proposes a mirror construction for any small toric degeneration. We set up the next lemma to help show that our construction is the same as Batyrev's in the case of a small degeneration.

Lemma 7.5.1. *For any inequalities $\lambda_{\text{top}}, \lambda_{\text{bot}}, p_1,$ and p_2 whose corresponding monomials satisfy a combinatorial box condition on O, there exists a facet of Δ^* containing the vertices corresponding to $\lambda_{\text{top}}, \lambda_{\text{bot}}, p_1,$ and p_2.*

Proof. Define the linear function $L(t_1, \cdots, t_N) := t_{\text{top}} - t_{\text{bot}} + 1$. We check that $L = 0$ on the corresponding vertices $\lambda_{\text{top}}, \lambda_{\text{bot}}, p_1$, and p_2, and $L \geq 0$ on all other vertices of Δ^*.

$$L(\lambda_{\text{top}}) = 1 - 2 + 1 = 0$$
$$L(\lambda_{\text{bot}}) = 0 - 1 + 1 = 0$$
$$L(p_1) = -1 - x + 1$$
$$L(p_2) = y - 1 + 1$$

But since $\lambda_{\text{top}} + p_1 = \lambda_{\text{bot}} + p_2$ we see that $x = y = 0$ which implies that $L(p_1) = L(p_2) = 0$.

Next we want to show that for every other vertex v of Δ^*, $L(v) \geq 0$. Assume v is a vertex corresponding to a λ-inequality λ_v then

$$L(v) = \begin{cases} 0 - 0 + 1 & = 1 \text{ If } \lambda_v \text{ lies below } \lambda_{\text{bot}} \text{ or least 2 columns to the} \\ & \text{left or right of } \lambda_{\text{top}} \\ 0 - (-1) + 1 & = 2 \text{ If } \lambda_v \text{ is in the region O.} \\ 2 - 2 + 1 & = 1 \text{ If } \lambda_v \text{ lies in the column above } \lambda_{\text{top}} \\ -1 - (-1) + 1 & = 1 \text{ If } \lambda_v \text{ lies above and adjacent to } \lambda_{\text{top}}. \end{cases}$$

String vertices can only take values from the set {-1,0,1} on the coordinates t_{top} and t_{bot}. From this we see that the only way the linear function L could be negative on such a vertex v, is if t_{top} coordinate of v was -1 and the t_{bot} coordinate was 1. Any such inequality would correspond to an oriented path which intersects itself, which is a contradiction.

We see that $(L = 0)$ defines a face of Δ containing the four vertices. This completes the proof since any face is contained in a facet. □

Theorem 7.5.2. *For a small string degeneration $F_{\overline{\omega_0}}$ is same as the family constructed by Batyrev in [Bat04, Sect. 4].*

Proof. In the case of a small toric degeneration Batyrev defines a subfamily of varieties in $|\mathcal{O}_{X_{\Delta^*}}(1)|$ whose coefficients satisfy box equations. (Box equation terminology was used in [BCFKvS00]. The coefficients satisfying the box equations are referred to as Σ-admissible in [Bat04].)

Our combinatorial box equations are a subset of Batyrev's box equations if, and only if, the vertices in Δ^* corresponding to $\lambda_{\mathrm{top}}, \lambda_{\mathrm{bot}}, p_1$, and p_2 all lie in a common facet of Δ^*. Therefore, by Lemma 7.5.1 we have shown that the combinatorial box equations are a subset of Batyrev's box equations. Batyrev proves in [Bat04, Rem. 4.2] that his box equations define an irreducible variety of $\dim = \mathrm{rank}(\mathrm{Pic}(X_{\Delta^*})) + \dim(G/B)$. This is the same as the dimension of the parameter space $P_{\overline{\omega_0}}$. By Lemma 7.4.11, $P_{\overline{\omega_0}}$ is an irreducible variety. If Batyrev's family had any more independent box equations they would define a variety of smaller dimension. Therefore, Batyrev's box equations cut out exactly $P_{\overline{\omega_0}}$, and the families are the same. □

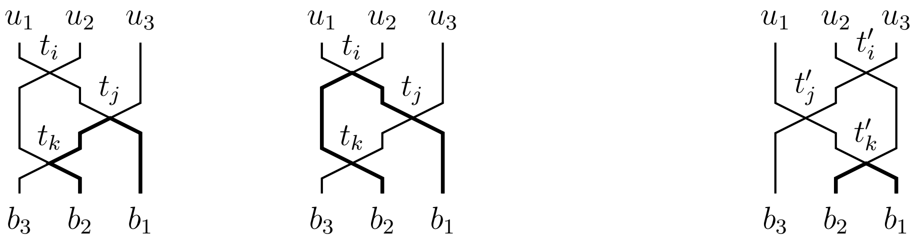
Corollary 7.5.3. *For all reduced decompositions $\overline{\omega_0}$, generic elements of $F_{\overline{\omega_0}}$ share the mirror properties of the generic elements of Batyrev's families.*

Proof. In the small case the families are the same by Theorem 7.5.2, and thus their generic elements share the same mirror properties. By Corollary 4.12 the generic elements are birational for any choice of reduced decomposition, and therefore share the same mirror properties. □

7.6 Necessity of Combinatorial Box equations

In this section we try to illuminate the role of the box conditions and equations. The following example demonstrates the way in which the local box equations are needed.

Example 7.6.1. *Consider the set of rigorous paths given in Figure 7.6.1: Notice here that the box conditions are satisfied with $\lambda_{top} = \lambda$, $\lambda_{bot} = \lambda_k$, p_1 the path on the right and p_2 the path on the left. Since we would like to have the sum of the two monomials map on the left*



We have the monomial piece $t_j^{-1}t_k + t_i^{-1} \to t_k'^{-1}$

Figure 7.2: String paths used to show that the local box conditions are needed

map to the monomial on the right we would need

$$h_a^*(a_1 t_j'^{-1} t_k' + a_2 t_i'^{-1}) = A t_k^{-1}$$

for some constant A. So we calculate:

$$\begin{aligned} h_a^*(a_1 t_j'^{-1} t_k' + a_2 t_i'^{-1}) &= a_1 \frac{C}{t_i t_k} \frac{C t_i t_j}{t_i t_k + c t_j} + a_2 \frac{t_i}{t_i t_k + c t_j} \\ &= \frac{a_1 C^2 t_j t_k^{-1} + a_2 t_i}{t_i t_k + C t_j} \end{aligned}$$

So in order for

$$\frac{a_1 C^2 t_j t_k^{-1} + a_2 t_i}{t_i t_k + C t_j} = A t_k^{-1}$$

we would have to have

$$a_1 C^2 t_j t_k^{-1} + a_2 t_i = (A t_k^{-1}) \cdot (t_i t_k + C t_j) a_1 C^2 t_j t_k^{-1} + a_2 t_i \qquad = A t_i + A C t_j t_k^{-1}$$

So we get that $A = a_2$ *and* $AC = a_1 C^2$.

This forces the relation $a_2 = a_1 C$. *Substituting for* C *we see* $a_2 a_{\lambda_i} = a_1 a_{\lambda_k}$. *Which is precisely the local box equation.*

One notices that although all the box conditions held in these local cases, the only one that played a pivotal role was the first box condition. The next calculation demonstrates shows how the second and third box condition play a key role in the proof of the preservation of box equations.

Example 7.6.2. *As in our proof, assume R' is directly above O' as pictured in Figure 7.6.2. Assume that p'_1 and p'_2 satisfy the first box conditions on O' but not necessarily the second*

Figure 7.3: Picture used to show that the second and third box conditions are used in the proof of birationality

and third box condition.

By comparing the λ-inequalities see the following relationship among monomial pieces.

$$\frac{\tau^{p'_2}}{\tau^{p'_1}} = \frac{\tau^{\lambda_{\text{top}'}}}{\tau^{\lambda_{\text{bot}'}}} = t_k.$$

In addition the the possibilities that occur in the proof of the preservation of combinatorial box equations, we could have $\tau^{p'_1} = 1$ and $\tau^{p'_2} = t_k^{-1}$. But if we followed the proof above, we could not guarantee that a box condition would be satisfied on the box O, which would mean that we couldn't push the box equations forward, to see that they are preserved. It isn't clear that this situation is actually possible. It is possible that the first box condition implies the second and third.

The following result in demonstrates why all of the box equations are needed, not just the local box equations.

Proposition 7.6.3. *For any two reduced decompositions $\overline{\omega_0}$ and $\overline{\omega'_0}$, the map between the families $BF_{\overline{\omega_0}}$ and $BF_{\overline{\omega'_0}}$ defined by compositions $\Phi(a,t)$ is only a rational map when restricted to the families $F_{\overline{\omega_0}}$ and $F_{\overline{\omega'_0}}$.*

Proof. By Corollary 7.4.14, the families $F_{\overline{\omega_0}}$ are birationally mapped to one another under compositions of $\Phi(a,t)$. Let $LF_{\overline{\omega_0}}$ be the largest subfamily of $BF_{\overline{\omega_0}}$ on which restrictions of compositions of $\Phi(a,t)$ define a rational map. Assume $Z \in LF_{\overline{\omega_0}}$ and that $p_1, p_2, \lambda_{top}, \lambda_{bot}$ satisfy the box conditions for a box O. We want to show that the corresponding box equations are satisfied. This means that Z is in fact in $F_{\overline{\omega_0}}$.

The position of a box O in relation to the other boxes only changes under a braid move if O is of type R_{121} or R_{212}. Under the reduced decompositions $\overline{\omega_0} = (s_1, s_2, s_1, \cdots, s_n, \cdots, s_1)$ and $\overline{\omega_0}' = (s_n, s_{n-1}, s_n, \cdots, s_1, \cdots, s_n)$ the relative position of the boxes is completely exchanged. This shows that every box O can be transformed into one of type R_{121} by a sequence of braid moves.

Let $(b_1, b_2, \cdots, b_f)$ be a sequence of such braid moves. We label the corresponding sequence of regions $(O_1, \cdots, O_f)$ where O_f is a region of type R_{121}. In the proof of the theorem above, we see that if the box conditions are met for a box O_i then there exists $p_1', p_2', \lambda_{top'}, \lambda_{bot'}$ satisfying the box conditions on O_{i+1}. This gives us a sequence of box conditions $(BC_0, BC_1, \cdots, BC_f)$ where BC_0 is our original box condition on O, and BC_f is a local box condition on a region of type R_{121}.

By their construction the maps $\Phi(a,t)$ only take elements of $BF_{\overline{\omega_0}}$ to elements of $BF_{\overline{\omega_0}'}$ if the local box equations are met. If the local box equations aren't satisfied then the map ϕ_a^* doesn't even take the equation for $Z_{\overline{\omega_0},a}$ to a sum of monomials. Therefore if $Z \in LF_{\overline{\omega_0}}$ then the local box equations corresponding to BC_f must have been met. The proof of the preservation of box equations shows that box equations corresponding to BC_i are met if and only if those corresponding to BC_{i+1} are met. We can now repeat this process to see that the box equations corresponding to BC_0 must have been satisfied. So Z must have been an element of $F_{\overline{\omega_0}}$ which means that $LF_{\overline{\omega_0}} = F_{\overline{\omega_0}}$. □

Appendix A

Remaining Questions

A.1 Combinatorial Questions

There are several interesting questions which remain about the combinatorics of $\Delta(\lambda, \overline{\omega_0})$.

Question A.1.1. *For which combinations of* $(\lambda, \overline{\omega_0})$ *is the polytope* $\Delta(\lambda, \overline{\omega_0})$ *integral?*

Alexeev and Brion conjectured in [AB04] that that $\Delta(\lambda, \overline{\omega_0})$ is integral if and only if $< \lambda, \alpha_i >$ is integral for all i. They give examples when $\Delta(\lambda, \overline{\omega_0})$ isn't integral for the toric degenerations of G/B for G of other classical types.

For $\lambda = k\widehat{\omega_i}$ the polytope $\Delta(\lambda)$ is integral for any choice of reduced decomposition. Moreover the number of vertices doesn't depend on the reduced decomposition. [AB04, Thm 4.5]. These weights corresponds to degenerating a Grassmanian variety.

Question A.1.2. *Describe* $\Delta(\lambda)$ *for other weights* λ.

The polytope Δ can still be modeled as the intersection of the λ-cone with the string cone. The problem in using this description is that certain inequalities become irrelevant, and other inequalities become strict equalities. It would be nice to have a combinatorial rule for the defining inequalities and equalities for Δ. In the classical case the partial flags can be found by looking at a subgraph of the Dots and Stars graph and then using the same rules.

Another possible solution to this problem would be find a set of smaller base polytopes (maybe even simplices) that can be connected together in an appropriate fashion to form the $\Delta(\lambda, \overline{\omega_0})$.

A.1.1 Building Δ from Smaller Pieces

Given two polytopes Δ and Δ' we can construct a new polytope $\Delta + \Delta'$ as follows.

Definition A.1.3. *For $\Delta, \Delta' \subset M_{\mathbb{R}}$ we define the Minkowski sum of Δ and Δ' by*

$$P := \Delta + \Delta' := \{m \in M_{\mathbb{R}} \text{ such that there exist points } a \in \Delta \text{ and } b \in \Delta' \text{ with } m = a + b.$$

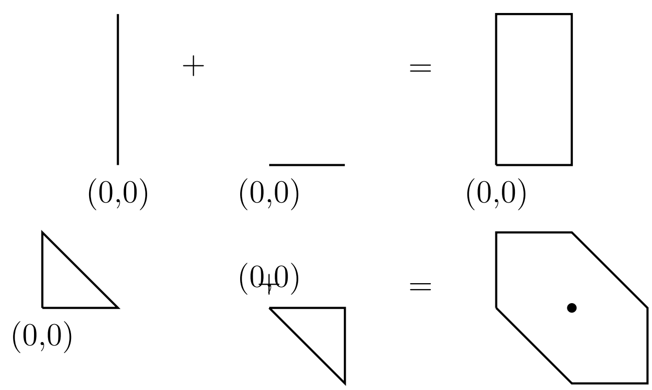


Figure A.1: Examples of Minkowski sums

Conjecture A.1.4. *For $\lambda = (k_1\widehat{\omega_1} + \cdots k_n\widehat{\omega_n})$ The string degeneration corresponding to the pair $(\lambda, \overline{\omega_0})$ is small if and only $\Delta(\lambda, \overline{\omega_0}) = \Sigma_{i=1}^{n}\Delta(k\widehat{\omega}_i, \overline{\omega_0})$.*

A.2 Algebraic Geometry of the Combinatorial Box Equations

Question A.2.1. *For other weights λ it is easy to define families $BF_{\lambda,\overline{\omega_0}}$. What are the appropriate combinatorial box equations in these cases, and how do the subfamilies defined by these equations relate?*

The answer to this question may be suggested by an appropriate path model for the other weights.

Question A.2.2. *Is there a more geometric interpretation of the combinatorial box equations?*

For example, if the degeneration is small, the combinatorial box equations are equivalent to a multiplicative version of the conditions which make a T-invariant divisor on X_Δ Cartier. This no longer hold in the nonsmall case. If we could find a geometric understanding of the combinatorial box equations it may be possible to define these families for any toric degeneration of a Fano variety.

Question A.2.3. *Danilov andKhovansky constructed a combinatorial method to for computing the HodgeDeligne numbers for "generic" toric hypersurfaces. Unfortunately, toric hypersurfaces satisfying the combinatorial box equations may fall outside of this generic locus. Can one construct a combinatorial method of computing the Hodge Deligne numbers of the toric hypersurfaces satisfying the combinatorial box equations?*

The calculation of Danilov and Khovansky applies to a open subset of toric hypersurfaces which they call "nondegenerate with respect to Δ." In the case of $\Delta(s_1s_2s_1)$ the family $F_{s_1s_2s_1}$ is precisely the closed set of hypersurfaces which are degenerate. However, a calculation in the nonsmall case shows that the combinatorial box equations don't define the degenerate locus. Is there a relationship in general between the degenerate locus and the box equations? In addition, it would be nice to have a combinatorial method for computing the Hodge Deligne numbers of the fibers of $F_{\overline{\varpi_0}}$.

A.3 The role of Tropicalization

The maps between the families can be seen as geometric lifts of tropical maps between tropical varieties associated to the families.

Question A.3.1. *The maps ϕ are only rational when restricted to the families $F_{\lambda,\overline{\varpi_0}}$, but the tropical maps don't see the corresponding coefficients. Can we understand the combinatorial box equations in the tropical setting?*

Berenstein and Zelevinsky were able to geometrically lift these piecewise linear maps to rational maps between totally positive varieties [BZ01]. Total positivity has been connected to mirror symmetry through the work of Rietsch [R06].

Our toric hypersurfaces can be viewed as tropical varieties by taking $Z_{trop} := \lim_{t \mapsto \inf} log_t(|Z_{f_\Delta}|)$. It would make sense that the piecewise linear maps defined by Berenstein and Zelevinsky would give maps between these tropical varieties. In the simplest example this is almost the case. All but one piece of the corresponding tropical variety maps as expected. Unfortunately one piece doesn't map to the anticipated image at all. Does the failure of this piecewise linear map to be a map between tropical varieties, relate to the failure of our map above to be rational unless it is restricted to $F_{\overline{\omega_0}}$? This would be very interesting since the coefficients a are lost under the tropicalization map.

Question A.3.2. *Is it possible to understand our families and maps in this larger setting? In particular is there a general theory tying mirror symmetry, total positivity, and tropical geometry together (at least in the case of flag varieties)?*

Since Berenstein and Zelevinsky's work connecting these piecewise linear maps between string cones and total positivity is completed for any classical Lie group, such an understanding may suggest how to study this problem for Flag varieties of type other than A_n. Perhaps by fitting tropical geometry into the following diagram we will be able to understand the apparent connections.

$$\begin{array}{ccc}
TPV_{\overline{\omega_0}} & \dashrightarrow & TPV_{\overline{\omega_0'}} \\
\updownarrow & & \updownarrow \\
\Delta & \longleftrightarrow & \Delta' \\
\updownarrow & & \updownarrow \\
\Delta^* & \text{no map} & \Delta'^* \\
\updownarrow & & \updownarrow \\
F_{\overline{\omega_0}} & \dashrightarrow & F_{\overline{\omega_0'}}
\end{array}$$

Bibliography

[AB04] V. Alexeev and M. Brion, *Toric Degenerations of Spherical Varieties*, Selecta Math. (N.S.) **10**(2004) no.4, 453-478.

[Bat94] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geometry., **3** (1994),493-535.

[Bat00] V.V. Batyrev,*Birational Calabi-Yau n-folds have equal Betti numbers*, New Trends in Algebraic Geometry, (Warwick 2000)1-11.

[Bat04] V.V. Batyrev,*Toric Degenerations of Fano Varieties and Constructing Mirror Manifolds*, The Fano Conference, (2004)109-122.

[BB96] V. Batyrev and L. Borisov, *On Calabi-Yau complete intersections in toric varieties.* Higher-dimensional complex varieties (Trento, 1994), 39–65, de Gruyter, Berlin, 1996.

[BCFKvS00] V. Batyrev, I. Ciocan-Fontanie, B. Kim, and D. van Straten, *Mirror symmetry and toric degenerations of partial flag manifolds*, Acta Math. **184** (2000), no. 1, 1-39.

[BZ93] A. Berenstein and A. Zelevinsky, *String bases for quantum groups of type* A_r , Adv. in Soviet Math. **16**, Part 1 (1993),51-89.

[BZ01] A. Berenstein and A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties* , Invent. Math. **143**(2001),no. 1, 77-128.

[Br05] M. Brion, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, 33-85. Trends Math., Birkhauser, Basel 2005.

[Cal02] P. Caldero, *Toric degenerations of Schubert varieties*, Transform. Groups **7** (2002), no. 1, 51-60.

[CK99] D. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical Surveys and Monographs, **68**. American Mathematical Society, Providence, RI, 1999.

[DK86] V. Danilov and A Khovanskii, *Newton Polyhedra and an algorithm for computing Hodge Deligne Numbers*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 5, 925–945.

[Del71A] P. Deligne *Thorie de Hodge. I.* Actes du Congrs International des Mathmaticiens (Nice, 1970), Tome 1, pp. 425–430. Gauthier-Villars, Paris, 1971.

[Del71B] P. Deligne *Thorie de Hodge. II.* Inst. Hautes tudes Sci. Publ. Math. No. 40 (1971), 5–57.

[FGIKNV] B. Fantechi, L. Goettsche, L. Illusie, S. Kleiman, N. Nitsure, A. Vistoli. *Fundamental algebraic geometry. Grothendieck's FGA explained. Mathematical Surveys and Monographs*, 123. American Mathematical Society, Providence, RI, 2005. x+339 pp

[FZ02] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations. (English summary)* J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[Ful93] W. Fulton, *Introduction to toric varieties*, Annals of Mathematical Studies **131**, The William H. Roever Lectures in Geometry, Princeton Univerisity Press, Princeton, NJ, 1993.

[Ful97] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts, **35**. Cambridge University Press, Cambridge, 1997.

[FH] W. Fulton, J. Harris, *Representation Theory. A first course* Graduate Texts in Mathematics, **129**. Readings in Mathematics. Springer-Verlag, New York, (1991).

[GlP00] O. Gleizer and A. Postnikov, *Littlewood-Richardson coefficients via Yang-Baxter equations*, Internat. Math. Res. Notices **14** (2000)741-774.

[GL96] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215-248.

[GrP90] B. Greene and M. Plesser, *Duality in Calabi-Yau moduli space*, Nuclear Phys. B **338** (1990) 15-37.

[GH78] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* Pure and Applied Mathematics. Wiley-Interscience , New York, 1978.

[GrL93] I. Grojnowski, G. Lusztig *A comparison of bases of quantized enveloping algebras. Linear algebraic groups and their representations* (Los Angeles, CA, 1992), 11–19. Contemp. Math., 153, Amer. Math. Soc., Providence, RI, 1993.

[Kash95] M. Kashiwara, Masaki, *Crystalizing the q-analogue of universal enveloping algebras* Comm. Math. Phys. **133** (1990), no. 2, 249–260.

[K95] M. Kontsevich, Lecture at Orsay (December 7, 1995)

[K94] M. Kontsevich, *Homological Algebra of Mirror Symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Zurich 1994), 120-139, Birkhauser, Basel 1995.

[H] R. Hartshorne, *Algebraic geometry. Graduate Texts in Mathematics*, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Lit98] P. Littelmann, *Cones, crystals, and patterns*, Transform. Groups **3** (1998), no. 2, 145-179.

[L90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[L94] G. Lusztig, *Canonical bases arising from quantized enveloping algebras. II*, Common trends in mathematics and quantum field theories (Kyoto, 1990). Progr. Theoret. Phys. Suppl. No. 102 , 175–201 (1991).

[M02] K. Matsuki, *Introduction to the Mori Program*, Universitext. Springer-Verlag, New York, 2002.

[O88] T. Oda, *Convex Bodies and Algebraic Geometry*,Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **15**. Springer-Verlag, Berlin, 1988

[R06] Rietsch, Konstanze A mirror construction for the totally nonnegative part of the Peterson variety. Nagoya Math. J. **183** (2006) 105-142