THE DEVELOPMENT OF STRIP DIAGRAMS AND DOUBLE NUMBER LINES IN A CONTENT COURSE FOR PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS

by

ERIC PHILLIP SIY

(Under the Direction of Andrew G. Izsák)

ABSTRACT

The purpose of this study was to understand how prospective middle school teachers used strip diagrams and double number lines in a content course. Representations play a critical role in mathematics teaching and learning. They help students and teachers solve problems, communicate their thinking, and access mathematics. However, teachers largely privilege symbols over non-symbolic representations such as drawings and diagrams, thus restricting their own and their students' mathematical thinking and communication. I conjecture the genesis of a culture shift to legitimizing drawings in school mathematics begins in mathematics teacher education content courses, an under-researched space.

In this study, I analyzed a year-long content course for prospective middle school teachers where they learned to consistently use two drawings, strip diagrams and double number lines, to solve mathematical problems. I collected video data of classroom lessons and analyzed how the teachers created their drawings. To analyze the drawings, I constructed an explicit set of methods heavily shaped by Geoffrey Saxe's Papua New Guinea and classroom studies. I distilled the teachers' drawings down to a set of coarse forms–sets of inscriptions used to create drawings to serve certain functions. I also identified three task features shaping which coarse forms teachers' used when creating their drawings. Using these methods and results, I provided an account of a community of teachers who reasoned with drawings to understand, organize, and connect critical concepts in middle grades mathematics.

INDEX WORDS: Mathematics Teacher Education, Middle School Mathematics, Teacher Knowledge, Representations, Sociohistorical Analysis

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MATHEMATICS TEACHERS

by

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WILL: I was gonna propose a toast to family.
Family that loves you and accepts you for exactly who you are.

JACK: Boring.

KAREN: Too real.

WILL: You know what's funny. We haven't changed a bit.

GRACE: It's kinda nice, isn't it?

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This is the part where people usually thank a significant other for their patience, kindness, and cooking but here I am with an empty paragraph. And that's fine.

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Maraming salamat sa inyong lahat.

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CHAPTER 1

INTRODUCTION

In 2008, the National Mathematics Advisory Panel compiled a list of recommendations to improve mathematics education in the United States. They first recommended a "focused, coherent progression of mathematics learning, with an emphasis on proficiency with key topics." By the term coherent, the Panel refers to curricula marked by "effective, logical progressions from earlier, less sophisticated topics into later, more sophisticated ones" (National Mathematics Advisory Panel, 2008, pp. xvi-xvii). Thompson (2008) critiqued this recommendation by arguing that coherence "is a property of a body of meanings, not a property of a list of topics" (p. 583). I also argue coherence should be viewed as a property of mathematical activity (i.e., doing mathematics across topics is a similar activity producing and drawing from coherent meanings). However, school mathematics has been characterized as a set of disjoint procedures and representations without coherent meanings. Thus, teachers and students are expected to know how to accurately and quickly execute these disjoint procedures without a sense of the underpinnings or how the procedures are related.

The National Council of Teachers of Mathematics (NCTM) also advocated for a coherent vision of mathematics (NCTM, 2000). However, when discussing representations, they described representations as mathematical objects that could be understood as disjoint. They encouraged the use of multiple representations in order to “reflect on their use of representations to develop an understanding of the relative strengths and weaknesses of various representations for different purposes” (NCTM, 2000, p. 70). Although this statement may hold true when selecting features to communicate mathematical ideas, multiple representations may hinder the development of coherent meanings. For example, consider the meaning of slope. When working with a line on a coordinate plane, a student determined the slope of a line by counting on a grid to obtain the slope as “rise over run.” When working on an algebraic equation, the same student executed a series of algebraic manipulations to obtain an equation in point-slope form and determined the slope as the coefficient of *x*. Although the student engaged in multiple representations and worked on the idea of slope, the type of representation confined their meaning for slope (see Lobato and Bowers (2000) for a discussion on a coherent meaning of slope). Thus, multiple representations itself must not be a goal of teachers but rather coherent meanings across multiple representations.

Issues of representations also arise from research on teacher knowledge and beliefs. First, in teachers’ conceptions, representations are relegated to the periphery of mathematical activity—creating displays is not “real” mathematics (Stylianou, 2010). Teachers prefer to prioritize abstract, procedural rules over productive representations (Borko et al., 1992). Second, teachers’ content knowledge constrains their pedagogical purposes for using representations (Izsák, 2008), thus well-developed teachers’ content knowledge is related to using representations. However, teachers’ mathematical knowledge has been shown to be primarily procedural without a strong grasp of the mathematical underpinnings (Mewborn, 2003). One avenue to develop both teachers’ productive use of representations and content knowledge is providing them with opportunities to learn how to use representations (Jacobson & Izsák, 2015).

Although this narrative permeates most of the practice of mathematics education, some classroom communities have created and developed ways of doing mathematics to unify mathematical concepts into a coherent whole. This study is an analysis of such a community. In particular, I analyzed how a coherent practice of using representations developed in a content course for prospective middle school teachers to support their unification of concepts of multiplication.

Two Teachers Across Time

In this dissertation, I argue a practice of using representations supports prospective middle school teachers to productively engage in the mathematical content they will teach. Investigating the teacher education spaces where they learn to use representations can provide insight into how teacher educators can best support prospective teachers. To illustrate how powerful teacher preparation courses can be in the development of a mathematics teacher, I recount the stories of two prospective teachers, Ms. Daniels from the seminal pieces Borko et al. (1992) and Eisenhart, M. et al. (1993) and Elizabeth who participated in my study. In both cases, the teachers enrolled in courses to support their mathematical knowledge using representations.

Ms. Daniels. Ms. Daniels, a senior, participated in a methods course in a K-8 teacher education program. The methods course instructor designed his class around a sequence of developmental levels for teaching where each mathematical idea was presented in a sequence of concrete (manipulatives), semi-concrete (drawings, diagrams), and abstract (formal mathematical symbols) representations. He sequenced each mathematical idea this way because he believed students learn conceptually with the first two levels and the third level supports students' understanding of mathematical symbols. However, he did not always follow this sequence. During a lesson on fraction division, he drew on a "how-many-groups" definition of division e.g., $1/3 \div 2/5$ is the same as "how many 2/5's is there in 1/3?" The instructor explained the invert-and-multiply rule i.e., $1/3 \div 2/5 = 1/3 \cdot 5/2$, with symbols and formal mathematical language suggesting that "(a) there is no direct relationship between stories or concrete and semi-concrete representations of the [how-many-groups] interpretation of division of fractions and the standard algorithm; (b) representations can be used to verify a solution obtained through use of the algorithm, but not to derive the algorithm" (Borko et al., 1992, p. 214).

Even with a methods course designed to help Ms. Daniels further her mathematical knowledge, she struggled when using representations to explain the mathematics she already knew. She expressed difficulty when asked by both the researcher and her students during student teaching to explain the invert-and-multiply rule. During an interview, she provided a limited explanation for the rule by arguing, "maybe give [students] a problem, a division problem and have them come up with a story behind it or how you could use that in real life… Just that using something visual they could show me how they got that answer. That would be to support when they flip the second number" (p. 209). When she taught a lesson on fraction division, one student asked her why the invert-and-multiply rule worked. She attempted to draw a representation for the problem but realized she drew a representation for fraction multiplication instead. As the researchers noted, “she continued to be unable to draw on this knowledge to construct coherent explanations or powerful representations, even away from the pressure of the classroom. Further, she seemed to be confused about the role that applications and representations could play in developing an understanding of the invert-and-multiply algorithm” (Borko et al., 1992, p. 207).

Elizabeth. Like Ms. Daniels, Elizabeth, a junior and prospective middle school mathematics teacher, enrolled in a course designed to help her understand middle school mathematical content with an emphasis on representations. The instructor designed the course with a focus on one definition of multiplication, $N \cdot M = P$ where N is the number of units in one group, M is the number of groups, and P refers to the number of units in M groups. Additionally, the instructor focused on two representations, strip diagrams and double number lines.

During the course, Elizabeth applied the same definition of multiplication to different problem situations, created story problems from number sentences, and reasoned through problem situations with representations. For example, the instructor asked the class to both create a problem for 1/3 ÷ 2/5 = ?, solve the problem, and explain keep-change-flip (i.e., the invert-and-multiply rule). Based on the number sentence, Elizabeth created a “how-many-in-one-group” division word problem: “A third of a pound of chicken is enough for 2/5 of a bowl of chicken soup. How many pounds of chicken is in 1 whole bowl of chicken soup?” (see Figure 1). She then wrote two annotated equivalent expressions, a division expression and the equivalent multiplication expression following the class definition of multiplication. Elizabeth drew a strip diagram to solve the problem. First, she drew the strip on the left with five parts and annotated her parts as 1/5 of the bowl. She colored in two parts of the strip and indicated this was one-third of a pound. She pulled out one of the parts and described it in two ways, as one-fifth of the bowl because of the original annotation and one-sixth of a pound because half of one-third is one-sixth. She replicated this part five times to build the whole bowl of soup and kept track of the total amount with respect to the size of the bowl and the amount of chicken in the soup simultaneously. Counting up by both fifths of a bowl and sixths of a pound of chicken, she ended, “when you add them up, you get five-sixths.”

When asked to explain why keep-change-flip works, Elizabeth re-interpreted her diagram and once again used the definition of multiplication. She described the situation by shifting her “group” from the original group of one bowl to a new group where one group is two-fifths of a bowl. In other words, one part refers to one-sixth pound of chicken, one-fifth of the bowl, and one-half of two-fifths of a bowl. Considering this new group, she explained that there is one-third pounds of chicken in one group (i.e., the highlighted parts of her drawing). She described one part of the strip as one-half of two-fifths of the bowl. Just like her initial explanation, she counted up but by halves, “one-half of two-fifths, two-halves of two-fifths, three-halves of two-fifths, fourth-halves of two-fifths, and five-halves of two-fifths.” Using the new group, Elizabeth wrote the expression 1/3 · 5/2 following the definition of multiplication. She explained there is one third pound of chicken in one group (amount in one group, *N*), five-halves of the new group in the whole bowl of soup (amount of groups, *M*), and five-sixths pounds of chicken in five-halves of the new group (amount of units in *M* groups, *P*).

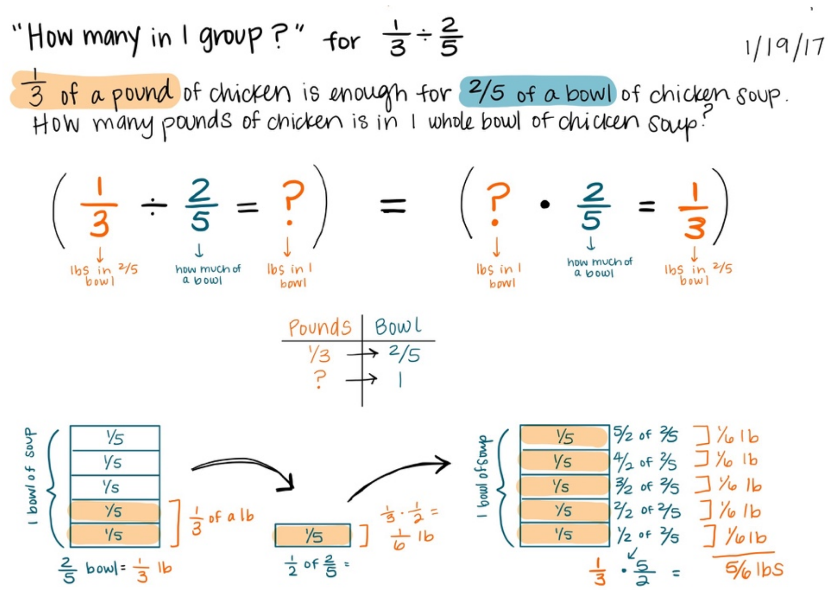


Figure 1. Elizabeth's Drawing for 1/3 ÷ 2/5.

Although there are marked contextual differences between the two teachers, both cases demonstrate the influence of teacher preparation courses on teachers' mathematical knowledge. Elizabeth and Ms. Daniels enrolled in different courses where their instructors focused on developing their mathematical understanding using representations. Elizabeth's instructor provided the class with a set of coherent mathematical tools to interpret and solve different mathematical problems and focused on using representations as a legitimate way to think about and communicate mathematics. Ms. Daniels's instructor provided a sequence of concepts from concrete to abstract, ultimately privileging mathematical notation even though he argued using concrete and semi-concrete representations develop conceptual understanding.

Because the instructors designed their courses differently, the prospective teachers engaged in mathematics differently. Although Ms. Daniels worked with concrete and semi-concrete representations, she explained a commonly taught algorithm without representations and relied on memorized procedures both in interview settings and student teaching. Elizabeth, on the other hand, explained the same algorithm drawing using the unifying definition of multiplication and a representation.

Rationale for the Project

Researchers have argued prospective teachers' prevailing mathematical knowledge is primarily a collection of disjoint, symbolic procedures, such as Ms. Daniels's. Elizabeth, however, demonstrated coherent mathematical understanding using a strip diagram. I investigated what happened in Elizabeth's teacher preparation course to support her to reason through a procedure Ms. Daniels could not. I analyzed video data collected from the course to describe how the prospective teachers used representations, such as strip diagrams, to engage in mathematical content they were expected to teach in the future.

A need to study middle school teacher preparation courses. This study addresses two under-researched areas in mathematics teacher education. First, researchers have provided limited understanding into what occurs during teacher education programs, particularly in developing mathematical content. Second, literature on middle school teacher preparation is sparse.

Experiences in mathematics teacher preparation courses generate a particular mathematical knowledge outside of everyday and school mathematics. The Association of Mathematics Teacher Educators [AMTE] (2017) and Ball, Thames, and Phelps (2008) argued that mathematics teachers should not only be knowledgeable about the content they are teaching, but "they require a kind of mathematical reasoning that most adults do not need on a regular basis" (Ball, Thames, and Phelps, 2008, p. 397) called specialized content knowledge (SCK). SCK involves "unpacking of mathematics that is not needed—or even desirable—in settings other than teaching" (p. 400) such as understanding keep-change-flip. In other words, teachers must not only be prepared to teach students "how to drive" but they must also be knowledgeable of what is "under the hood" of the mathematical content they are teaching.

I investigated a teacher preparation course intended to support prospective teachers' SCK, called "content courses" (as compared to "methods courses" intended to support prospective teachers to teach mathematics). There is limited research on mathematics content courses (Hart, Oesterle, & Swars, 2013; Li & Castro Superfine, 2016). Researchers focused on the design of the course (Li & Castro Superfine, 2016), who taught these courses (Masingila, Olanoff, & Kwaka, 2012), goals (Appova & Taylor, 2017), and perceptions of the course (Hart et al., 2013). Most of the data in these studies are limited to retrospective reflections and curricular materials, not video or audio from the courses.

Researchers know little about middle school teachers' knowledge because studies with prospective middle school teachers are not as extensive as studies with elementary and secondary prospective teachers. One reason may be the merging of elementary and middle school bands (K-6) or middle and secondary school bands (7-12) in some certification programs; exclusive middle school certification in the United States is rare (Baldi, Warner-Griffin, & Tadler, 2015). In the limited research conducted in this grade band, well-documented findings showing disjoint mathematical knowledge still hold. For example, Beswick, Callingham, and Watson (2011) found middle grades teachers with more advanced knowledge were likely to view mathematics as computation and mathematics teaching as expository in nature.

I described what happened in a content course that supported prospective middle school mathematics teachers, like Elizabeth. The results provide much needed insight into what can occur in these courses. In particular, I will discuss how she and her peers developed coherence across middle school topics. Worth noting, although Ms. Daniels was enrolled in a methods course, her instructor also focused on developing her mathematical knowledge similar to a content course.

A need to study teachers' use of representations. This study illuminates a counter narrative to how researchers have characterized teacher knowledge. Researchers found that teachers believe drawings and diagrams are not "real" mathematics (Stylianou, 2010) and teachers' mathematical knowledge has been shown to be primarily procedural without a strong grasp of the mathematical underpinnings (Mewborn, 2003). In this study, I provide a case to demonstrate that teachers, particularly prospective teachers, can sensibly engage in mathematics primarily with drawings and diagrams.

A critical aspect of both Elizabeth and Ms. Daniels's teacher preparation courses was the use of representations. Representations, such as drawings and diagrams, are an essential component to mathematics teaching and learning. In *Principles and*

Standards (2000), NCTM noted representations as one of the ways students learn mathematics in that:

> [Representations are] the act of capturing a mathematical concept or relationship in some form and to the form itself… Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. (p. 67)

Researchers have also emphasized the importance of representations in mathematical thinking (Cuoco, 2001; Janvier, 1987). Both teachers and students use representations to help them solve and make sense of problems (e.g., Lobato, Hohensee, and Diamond (2014), communicate their ideas (e.g., Roth & McGinn, 1997), and participate in mathematical activity especially if their language is not the privileged language in the classroom (Turner, Dominguez, Maldonado, & Empson, 2013). Additionally, representations are a critical feature of effective mathematics teaching. In *Principles to Action* (NCTM, 2014), the authors explicitly identified teaching with representations as a principle for high quality mathematics teaching practice. Teaching with representations "[engage] students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (p. 10) and may improve teacher knowledge (Izsák & Sherin, 2003).

Researchers, however, have produced little evidence that teacher preparation programs (both for practicing and prospective teachers) prepare them to successfully integrate representations in the classroom (Stylianou, 2010). Providing teachers with opportunities to learn with representations has been seen predict teachers' knowledge as well as their purposes and frequency of representations in instruction (Jacobson & Izsák, 2015).

I analyzed what representations prospective teachers used in the content course in order to exemplify what prospective teachers can do when provided with opportunities to learn with representations. In particular, I address the need to understand how teachers can engage productively in mathematical content where representations such as diagrams are central.

A need to study classroom data. Earlier, I discussed the limited research base on what happens during content courses even though these courses are critical in the development of teachers' mathematical knowledge, particularly SCK. Perhaps, researchers have avoided using classroom data from content courses because the data is "messy" and not as controlled as interview data. Classrooms are complex spaces permitting "the joint activity of teacher and students and amenable to being further structured by such joint activity" (Mesa & Herbst, 2011, p. 113). Several phenomena occur simultaneously in classrooms and other social spaces such as negotiating and co-constructing meaning (Bauersfeld, 1998), developing normative practices (Yackel, Cobb, & Wood, 1998), and positioning oneself and others (Bishop, J. P., 2012). In this dissertation, I address this complexity by developing methods for analyzing how representations are used in class such as the content course.

There has been little methodological direction in characterizing representation use in classrooms. Researchers who have examined representation use in class (Hall &

Rubin, 1998; Izsák, 2005; Saxe, 2002) have generally agreed to distinguish what is represented and what is "doing" the representing (cf. von Glasersfeld, 1987b) but have stopped short of developing replicable methods. I have drawn on one characterization of the represented-representing paradigm, the form-function relationship based on the work of Geoffrey Saxe. I will discuss this relationship in greater detail in Chapter 2.

Saxe and his colleagues attempted to establish methodological clarity in ways previous researchers have not (Saxe, 2015), yet these methods can be developed still further. In Saxe's work, identifying a form and function is critical for analysis. A form is something perceivable used to communicate such as words on a page or a mathematical drawing. A function is the purpose for which a form is used. Researchers in this work provided little clarity to the grain size of a form or function. In Elizabeth's work, different parts of her drawing can constitute a form but can range from simple inscriptions such as a line to something complex such as the complete set of inscriptions. Thus, the first research question (RQ) I posed is a methodological question:

RQ1. What methods and grain sizes should researchers consider when characterizing forms and functions of mathematical drawings in classroom data?

Developing an explicit way of analyzing drawings in classroom data can provide a springboard for replication studies (see Cai et al. (2018)). The methods developed in this study are not meant to remain stable over time but rather critiqued and refined by other researchers. The affordances of the methods developed to answer RQ1 lie in what the analysis yields. Thus, I applied the methods to describe the drawings the prospective teachers created in the content course and addressed the gap in the literature describing what happens in content courses.

RQ2. What forms and functions of strip diagrams and double number lines emerged from student drawings in a content course for prospective middle school teachers?

Finally, I addressed a call for characterizing how teachers' knowledge develops over time by analyzing two semesters of the content course. Mewborn (2001) argued "we need more in-depth studies of teachers in action in various contexts as learners of mathematics and as teachers of mathematics. Studies that are longitudinal in nature, that provide us with "videotapes" rather than just snapshots of teachers' knowledge, are needed to enhance our understanding" (p. 34). NRC (2001) echoed her sentiment:

> Although learning is fundamentally temporal, too little research has addressed the ways in which instruction develops over time. Many studies are restricted to isolated fragments of teaching and learning, providing little understanding of how the interactions of teachers, students, and content emerge over time, and how earlier interactions shape later ones. (p. 358)

Over time, the prospective teachers' representations changed and I investigated why such changes occurred. Changes in representations over time can be expected because of the constant emergence of new goals and conditions (Saxe, 2012). I contend that changes in representations can be traced to changes in goals as new mathematical tasks are posed. I align my view of "mathematical task" with Stein and Smith (1998) where a task "can involve several related problems or extended work, up to an entire class period, on a single complex problem" (p. 269) devoted to a particular mathematical idea. Thus, the final question I address in this study addresses how the students' drawings change over time with respect to the tasks:

RQ3. What features of the mathematical tasks shaped the use of certain forms and functions over time?

Research Questions and Overview of the Dissertation

In this study, I investigated how prospective middle school teachers used representations in a content course to develop coherent mathematical knowledge. Saxe (2012) described this kind of work using the analogy of the development of the wing. What are the different forms of the wing that emerged over time? What conditions supported the precursory wing to develop into today's form which allows certain birds to fly? In the same way, my investigation traced the development representations over time and documented the conditions which necessitated changes to the representations. In summary, I answered the following questions in this project:

1. What methods and grain sizes should researchers consider when characterizing forms and functions of mathematical drawings in classroom data?
2. What forms and functions of strip diagrams and double number lines emerged from student drawings in a content course for prospective middle school teachers?
3. What features of the mathematical tasks shaped the use of certain forms and functions over time?

In this chapter, I presented the rationale for this study and the research questions emerging from the gaps identified in research. In Chapter 2, I begin by providing reviews of relevant literature to further expand on what has been found in the research on representations and middle school mathematical content. In these reviews, I describe general areas of consensus and issues in teacher education. I also present my theoretical framing of representations and how they develop. In Chapter 3, I answer RQ1 and summarize the developed methods to investigate the development of forms and functions over time in classroom data. In Chapter 4, I outline the results of executing the methods. I first describe the vital components of the representations created in class and then I enumerate the task features that shaped the development of representations over time. Finally, I summarize the study and provide connections of my results to existing literature in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

This chapter is composed of three sections summarizing the main ideas from three lines of research that guide the dissertation. The aim of this chapter is not to provide a parade of research results, but to present common themes and tensions within these lines. In the first section, I review the literature on representations in mathematics education. I summarize how scholars defined “representation.” I outline three dimensions along which researchers have characterized representations. In the second section, I present findings on research on multiplicative reasoning. I outline how scholars have conceived of multiplicative situations, the cognitive constructs supporting multiplicative reasoning, and how these ideas are manifested in middle school content. In the final section, I outline my perspective on mathematics learning and the theoretical framing guiding my investigation, the culture-cognition framework.

What Do We Know About Representations?

Defining Representations

Mahāmati, what is meant by non-duality? It means that light and shade, long and short, black and white, are relative terms, Mahāmati, and not independent of each other; as Nirvana and Samsara are, all things are not-two. There is no Nirvana except where is Samsara; there is no Samsara except where is Nirvana; for the condition of existence is not a mutually exclusive character.

–The Buddha, The Laṅkāvatāra Sūtra

Table 1.
The Dimensions of Representation Definitions and Their Poles.

Dimension	Poles and their Description	
	Representations are internal	*Representations are external*
Locus	Representations are cognitive or mental structures of an individual that can only be inferred by the researcher. (Goldin, 2002; Izsák, 2011; Kaput, 1991; Moore, 2014; Pape & Tchoshanov, 2001; von Glasersfeld, 1987b)	Representations are objects in an environment that can be observed, talked about, and interpreted by other individuals. (Dufour-Janvier, Bednarz, & Belanger, 1987; Gellert & Steinbring, 2013; Goldin, 2002; Izsák, 2011; Janvier, 1987; Kaput, 1991; Lesh, Behr, & Post, 1987b; Moschkovich, Schoenfeld, & Arcavi, 1993; White & Pea, 2011; Zhang, 1997)
	Representations belong to the individual	*Representations belong to the community*
Ownership	Representations are dynamic constructions of an individual that are not "copies" of sensed objects in the environment. (Moore, 2014; von Glasersfeld, 1987b)	Representations are formed out of cultural practices that allow a culture to collectively make sense of ideas. (Blumer, 1986; Greeno & Hall, 1997; Hall, 1996; Medina & Suthers, 2013; Roth & McGinn, 1997)
	Representations are a process	*Representations are a product*
Function	To represent is the dynamic act of producing a representation of a mathematical concept or process that affects one's thinking. (diSessa, Hammer, Sherin, & Kolpakowski, 1991; Kaput, 1991; Larkin & Simon, 1987; Lobato et al., 2014; Meira, 1995; Moschkovich et al., 1993; National Council of Teachers of Mathematics, 2000; Pólya, 1957; Sherin, 2000; Zhang, 1997)	Representations are encoded objects that result from thinking, maybe for purposes of communicating. However, the encoded information is not necessarily preserved. (Cobb, Yackel, & Wood, 1992; Greeno & Hall, 1997; Kaput, 1987; Moschkovich et al., 1993; National Council of Teachers of Mathematics, 2000; Parnafes, 2010; Pimm, 1987)

In mathematics education literature, representations have been roughly defined as observable or inferred “things” that stand for another mathematical “thing” (Goldin, 2014; Kaput, 1989; Vergnaud, 1998b; cf. von Glasersfeld, 1986). This idea is rooted in philosophy, semiotics, and cognition. A coherent definition across research areas of mathematics education, however, has not been established and thus a definition for representation is “fraught with ambiguity that, for the most part, remains hidden” (von Glasersfeld, 1987b, p. 216). In this section, I organize the different ways representations have been defined in mathematics education research.

I identified three dimensions along which researchers and scholars have characterized representations. First, researchers have examined whether a representation is inside or outside the mind. Researchers have also conceived of representations with respect to its owner (i.e., who does the representation belong to?). Finally, representations have been framed with respect to their function—for instance, as a process or a product.

In discussing each dimension, I present two poles where each pole is a strong contrast to the other pole (see

Table 1). Each pole is a characterization of how a group of researchers have described representations with respect to the dimension. Although research can frame representations as closer to one pole or the other, I believe representations themselves do not solely exist in one pole. As the Buddha stated in the opening quote to the section, one pole cannot exist without the other. Light cannot exist without the dark; noise cannot exist without silence. Each pole and dimension are deeply intertwined with the other. My goal in using poles is to employ Cobb's (2007) metaphor of *co-existence and conflict* to the field of representations where, although there may be stark conflicts among researchers of differing views, their views co-exist to ultimately make sense of the tools, questions, and strides in mathematics education.

Locus. Scholars have conceptualized representations in two ways based on where they are "located": external and internal (e.g., Dufour-Janvier et al., 1987; Izsák, 2004a; Roth & McGinn, 1998). Goldin (2014) summarized the dichotomy. External representations are perceivable objects and internal representations are cognitive or mental structures of an individual that are not directly accessible by others.

The distinct yet related nature of external and internal representations have roots in philosophy, semiotics, and cognitive psychology. The philosopher Wittgenstein (1994) asserted a distinction between a sign—a physical, perceivable object—and symbol—a sign with a corresponding meaning. He claimed symbols could be inferred through the sign's use in context, especially when signs with the same form may carry different meanings. He provided the example "Green is green" where the two signs "green" have the same verbal form but within context one could assign different meanings for both "greens." Similarly, Pierce (1955) characterized signs with a three-fold nature. A *representamen* is a thing representing something else, called the *object*. He noted the incompleteness of representamen as standing "for that object, not in all respects, but in reference to a sort of idea" (p. 99). The third element to the sign is the *interpretant,* which refers to an equivalent sign in the mind of a person which varies with each individual. Vygotsky (1978) presented the idea an externally oriented *tool* allowing a mediation of human's actions to manipulate the physical world in the same way a hammer can provide certain actions when building. An internal *sign* serves a similar function (i.e., to mediate an activity); however, the sign mediates one's own activity to alter one's own physical action. Moreover, Vyogtsky posited process of a transposition from an external tool to an internal sign called *internalization*. In other words, an external tool or action becomes an internal psychological entity.

In mathematics education literature, researchers have also distinguished the locus of representation. In constructivist literature, internal representations are foregrounded (Goldin, 2002; Izsák, 2011). Constructivists highlight how the cognizing individual makes sense of the world by adapting one's own mental structures. Thus, the experiential world is "always and irrevocably subjective" (von Glasersfeld, 1996, p. 2) and is actively constructed, rather than passively received (von Glasersfeld, 1989, 1990). Because reality cannot be copied, constructivists do not use "external representations" rather, they talk about "re-presentations." Re-presentations provide opportunities for the individual to carry out a mental operation (Moore, 2014; von Glasersfeld, 1995, 1996). Because the individual constructs the experiential world, observed things do not have meanings that exist outside the mind[1]. Re-presentations are subject to how the individual interprets and understands, thus a student's re-presentation is inseparable from their own thinking (Moore, 2014) and can be "replayed, shelved, or discarded according to their usefulness and applicability in experiential

[1] This may be the reason for the reluctance of constructivists to use the term external representation.

contexts” (von Glasersfeld, 1987a, p. 219). Researchers adapting constructivism have greatly expanded literature on how students construct meanings for mathematical representations such as coordinate systems (Lee, 2017), angles (Hardison, 2018; Moore, 2013), rates (Thompson, 1994), and formulas (Stevens, 2018).

The other pole of research in mathematics education places more weight on external representations by examining how global and local features of a representation support mathematical reasoning. These studies have examined relationships between perceivable features of inscriptions and how one thinks. Kaput (1989) argued representations have cognitive implications for the actions and connections supported by the representation. Consider addition and subtraction of integers using two different yet common representations, chips and a number line (see Figure 2). Because two different representations are used, two different meanings for addition and subtraction are supported (Bishop, J. P. et al., 2014). The chip representation supports a magnitude way of reasoning about integers where integers have cardinality or substance, thus adding and subtracting involved gaining and losing cardinality. The number line representation could support an ordinal way of reasoning where integers have an ordered sequence and adding and subtracting are movements along a number line. The two representations are both intended to capture addition and subtraction of integers, however the ways of reasoning involve different actions.

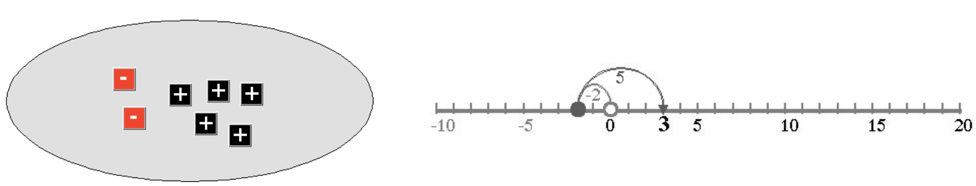


Figure 2. Two Representations for –2 + 5 = ?

As demonstrated, focusing on external representations is tied to cognitive actions supported by the representation. Kaput (1987) described three main cognitive actions associated with thinking about representations he called symbol systems: (1) reading or interpreting symbols, (2) encoding or annotating within and across symbol systems, and (3) elaboration or manipulating and identifying the referents of symbols. The first and second actions have been studied in detail.

Reading or interpreting mathematical representations places emphasis on how certain representations are externally presented. In the seminal work, Kieran, C. (1981) described ways students interpreted the equal sign. She found students interpreted the sign in two ways, as a “do this” sign or a relationship between two quantities. More recently, Knuth, Stephens, McNeil, and Alibali (2006) found a majority of the sample of their middle school students interpret the equal sign as “what the answer is” or as “the total” (p. 303). Sherin (2001) demonstrated how symbols invoke certain mathematical ideas and relationships between the meanings of the symbols and their physical configurations with undergraduate engineering students. For example, one student made sense of the equation

$$\mu = \mu_1 + C \frac{\mu_2}{m}$$

as having a term that “varies inversely as the weight” by focusing on the configuration of the second term, particularly the fraction form. Additionally, students used the problem situation to interpret the configuration of the symbols. A student, for instance, reasoned about a block hanging from a ceiling by a spring and the equation:

$$x = \frac{mg}{k}$$

where *k* refers to the spring constant and *x* to the distance between the block and the ceiling. He stated, "if you have a stiffer spring, *x* is going to decrease" using the both the problem situation and the configuration to imbue meaning to the symbols.

Encoding a system of symbols has been investigated in studies of meta-representational competence (MRC)—capabilities that individuals consider in creating and modifying scientific and mathematical representations (Danish & Saleh, 2014; diSessa et al., 1991; Izsák, 2003; Meira, 1995). Students create and modify drawings of problems contexts by using attributes such as drawing, colors, space, and line segments in order to create and modify representations (Azevdeo, 2000; Elby, 2000; Sherin, 2000). Additionally, researchers examined the cognitive processes used when creating and modifying conventional mathematical notations such as equations (Izsák, 2003; Sherin, 2001).

Finally, researchers have also described translations between symbol systems , such as those for the graphs and those for equations connected to linear relationships. Students do not exclusively use one system of representations when solving mathematical problems (Lesh, Landau, & Hamilton, 1983) but translating between systems is not trivial (Moschkovich et al., 1993). Superfine, Canty, and Marshall (2009) claimed translating between systems could consist of multiple cognitive mechanisms such as isolating mathematical conceptions from external representations and re-organizing one's knowledge in order to make sense of another representational system.

Ownership. Many researchers have ascribed representations to the individual but, as the literature has expanded to include different theoretical perspectives, researchers have begun to theorize representations as the mark of a community or a shared, cultural artifact or practice. In the seminal collection of work *Problems of Representation* (Janvier, 1987), Kaput (1987) posed a thought experiment. In an environment (e.g., a classroom), multiple observers have access to an external representation because presumably it is perceivable to everyone. Thus, is it possible to "share" a symbol system if we do not have access to an individual's conceptions? A few researchers addressed this question by attributing a representation to a culture, not just an individual. In his description of symbolic interactionism, Blumer (1986) posited that meanings arise not only from one's own view but also are derived from social interactions with others. Blumer wrote about "objects" in the world that "can be indicated, anything that is pointed or referred to" (p. 10) and that do not have inherent meanings. Rather meanings emanate from how a group of humans interact with objects. Defining, interpreting, and creating objects are a product of interaction among members of a culture, and thus meaning would "belong" or be "taken-as-shared" within a group of individuals. For example, a cup of coffee takes on different meanings when speaking to a group of food scientists and when speaking to a group of high school students at a coffee shop. Other sociocultural frameworks also emphasize the creation of meaning of an object as a social endeavor (Lave & Wenger, 1991; Wenger, 1998).

Researchers have expanded on how social groups, history, and culture influence the meanings assigned to a representation. In Vygotsky (1978) description of a tool, he acknowledged socially rooted and historically developed activities. Thus, tools are not value or culture-free nor are they frozen in time: They influence the individual's psychological functions by "both changing the user's view of the world and adopting the belief system of the culture in which they are used" (Brown, Collins, & Duguid, 1989, p. 33). In other words, the meaning of a representation is based on how both the individual and culture uses it at a certain point in time. The individual's challenge is using the tool in "a new, culturally appropriate ways" (Wertsch, 1985, p. 161). For example, if a student only holds a belief that mathematics is about using symbols to solve math problems, when they participate in a new community where they are asked to draw their thinking, the teacher would facilitate discussions in order for the student to participate in the new way the community does mathematics perhaps changing the students' beliefs. Representing is not an individual endeavor, rather "ways of symbolizing are treated as emergent phenomena interactively constituted by the class community" (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997, p. 221). By developing, negotiating, and constituting norms for representations, members of the classroom come to a taken-as-shared meaning and practice of creating drawings. Thus, examining the ways representations are developed and negotiated is a crucial site when investigating classroom cultures.

These frameworks were incorporated mathematics education research on representation. *Symbolizing and Communicating* (Cobb, Yackel, & McClain, 2000) may be viewed as the successor to *Problems of Representation* (Janvier, 1987). With social frameworks in mind, the authors in this compilation discussed representations as the result of collective activity. There is a substantial literature base in mathematics education framing reasoning with representations as participation in cultural practices (Danish & Phelps, 2010; Danish & Saleh, 2015; Granados, 2000; Greeno & Hall, 1997; Hall, 1996; Hall & Rubin, 1998; Izsák, 2004b; Medina & Suthers, 2013; Roth & McGinn, 1997, 1998; Seeger, 1998).

The work in this field shifted the focus from individual cognitive mechanisms to social mechanisms and their influence in creating and modifying representations. Such mechanisms rely on discursive practices of the community such as observing, correcting, and negotiating (Roth & McGinn, 1997). Medina and Suthers (2013) examined two representational practices of an online mathematics group. The first practice, visual decomposition, involved separating parts of a geometric figure to focus a conversation. The second practice, modulate perspective, involved transforming a figure to "see the figure in a new way" (p. 45). These practices afforded the group shared goals, activities, and resources. Izsák (2004b) found collective ways of interpreting, reasoning, and inscribing an area model for multiplication emerged from classroom activity supported by teacher actions such as labelling parts of a drawing.

Function. National Council of Teachers of Mathematics (2000) presented representations as a process standard—how students learn mathematics. They described a representation's function as a process, "the act of capturing a mathematical concept or relationship in some form" (p. 67), and a product, the result of the process. Pimm (1987) also described the idea of function based on Halliday's (1985) function of an utterance. Individuals form utterances to serve different functions. For example, the utterance “Would you like to pick up your coat?” may serve as a request to pick up a coat even if the utterance is in the form of a question. Similarly, individuals may use representations to serve different functions—for instance, as a process for solving problems or as a product, something used to help communicate and idea.

Researchers have described representations as pivotal as part of the process of thinking. Kaput’s (1989, 1991) symbol system, Goldin’s (1987) model for competency in mathematical problem solving, and Vergnaud’s (1998) expansion of the Aristotelian triangle are examples of frameworks outlining resources individuals use when reasoning with representations. For example, Goldin (1987) considered five interrelated “higher-level languages”: verbal, imagistic, formal notation, planning or heuristics, and affect as representations playing a role how one thinks mathematically. In these frameworks, the authors highlighted three fundamental components: a set of inscriptions or symbols, allowable configurations for the symbols, and connections between the symbols and the mathematical meanings they supposedly represent. Vergnaud (1998a) acknowledged connections need not be one-to-one correspondences. In fact, some of these connections may be incomplete. Both Goldin and Vergnaud also emphasized connections between and within systems of representations (e.g., formal mathematical notation, written problem situations) which Lesh et al. (1987b) called translations and transformations, respectively.

Empirically, studies of representations as a process have presented fine-grained analyses of students’ representations and how the act of creating representations supports or modifies their thinking. Lobato et al. (2014) found students with similar conceptions of speed have similar processes in which they constructed diagrams while solving tasks on speed. In a micro-genetic analysis of student inscriptions, Meira (1998) found student inscriptions were not static. Student inscriptions shifted as they considered their mathematical goals, meanings, and information over the course of solving problems. Because of the fluid nature of representations, the students' activities invoked evolving representations. Thus, representations were not simply a result of encoding one's thoughts—rather a systematic, complex process of thinking through mathematical problems.

Studies framing representation as a product treat the representation as a static object which may be used to share an interactive space to calibrate meanings (Roth & McGinn, 1997). Representations may also be used to externalize one's thoughts for the purposes of communicating (Kaput, 1989; Lesh et al., 1987b). Pimm (1987) argued symbols are used to record and retrieve knowledge and to allow others in the community to access each other's thoughts. This conceptualization places representations within the realm of social semiotics and interactionism. A social semiotic perspective recognizes representations as a semiotic resource in meaning making in the social arena (Morgan, 2006; Sáenz-Ludlow & Presmeg, 2006). Additionally, the meanings and practices emerging from social interactions are not static. Rather, they are constantly being negotiated and constituted in interaction (Bauersfeld, Krummheuer, & Voigt, 1988; Cobb & Bauersfeld, 1995) but may be based on static images used when communicating in the same way a conversation may emerge from a students' drawing presented during whole-class discussion.

The evolution of representations and their meanings may be attributed to meta-representational competence (diSessa, 2004; diSessa et al., 1991; diSessa & Sherin, 2000; Izsák, 2003; Meira, 1998; Sherin, 2000). Meta-representational competence (MRC) refers to one's capacity to create and critique external representations. Creating representations entails actions such as selection, production, and productive use of the representation. In the first study on MRC, 6th grade students created, refined, and critiqued representations for motion and speed of a motor vehicle that slows down then speeds back up (diSessa et al., 1991). Several criteria for creating representations were identified such as transparency (a representation should need little explanation), compactness (a representation does not take up a lot of space), abstractness (a representation includes relevant features of the problem), etc. Since this study, more criteria have been identified (see Table 2).

Table 2.
Some Criteria for Creating Representations in Select Publications

Criteria	Description
Color	Students use colors and coloring schemes as a means to represent quantities or to label different aspects of a representational display (Azevedo, 2000)
Temporal sequence	Sequences are characterized by a list of elements that show the story of the problem situation (Sherin, 2000)
Space = space	Spatial displacements in the representation show spatial displacements in the referent. (diSessa, 2004)
What you see is what you get	Intuitive knowledge elements map onto the interpretation of a visual representation or an aspect thereof (Elby, 2000)
Single-variable	Algebraic equations should be expressed in one variable (Izsák, Caglayan, & Olive, 2009).

The second component of MRC is the capacity to critique a representation (diSessa, 2002; diSessa et al., 1991; diSessa & Sherin, 2000). Although there are some overlaps with the capacities of students to create representations, some criteria appeared solely in the capacities to critique representations such as completeness (shows all relevant information) and conventionality (does not violate accepted conventions) (diSessa et al., 1991). When representations are framed as a product of thinking, students can create and engage in other's representations and thinking. By publicly discussing representations in class, individuals adjust their expectations of each other in an activity called negotiation (Cobb & Bauersfeld, 1995; Voigt, 1994, 1998). As students critique and modify other's representations, they reflect on their own representations and make changes based on the interactions with their peers (Danish & Phelps, 2010). Similarly, Goldin (2002) stated, "it is the internal level that largely determines the usefulness of such external representational systems, according to how the individual understands and interacts with them" (p. 211).

Summary. In this section, I identified three dimensions of research on representations. Opening this section, I quoted a section of the Buddhist text The Laṅkāvatāra Sūtra. In the quote, the Buddha argued against a dualist view of the world and claimed the world is composed of gradients of poles and the poles exist because of each other. In defining representations, scholars may place their views or operationalization of representations along the dimensions and I identified. By clearly delineating the locus, ownership, and function of representations, the researcher can focus on questions and methods relevant to the investigation at hand. My purpose of defining poles is to acknowledge the multi-faceted nature of representations research and to create a bigger picture that provides, as Cobb (2007) suggested, more light than heat. In creating a more complex and unified view of representations, strengths of one framing complement the blind spots of the other.

What Do We Know About Multiplicative Reasoning?

Those are $8 a pound, sport. – Shopkeeper

$8 a pound times, say, oh, 5 pounds, is, um, let's see… How many pounds in a gallon? –Homer Simpson

In this section, I discuss key ideas from research on multiplication and multiplicative reasoning. First, I present how researchers have investigated the structure of multiplicative situations. I then review two big ideas found to be crucial to reasoning about these situations and researchers' findings with students and teachers. Next, I discuss how the big ideas permeate three areas that are crucial in the middle grades curriculum: base-10 understanding, rational numbers, and proportional reasoning. In the discussion of proportional reasoning, I also discuss strip diagrams and double number lines two representations that can support such reasoning.

Multiplication: A Change in Structure

Several curricular documents anticipate a critical shift in the mathematical content when students enter the 3-5 grade band, more specifically, moving from addition to multiplication. In the Common Core State Standards for Mathematics (CCSS-M), students are prepared to work with multiplication concepts in the second grade by determining the parity of whole numbers by creating groups of two or two equal sized groups (National Governors Association Center for Best Practices &

Council of Chief State School Officers, 2010). By the third grade, students are expected to solve, interpret, and model multiplication and division problems and have a firm understanding of addition concepts (National Council of Teachers of Mathematics [NCTM], 2000). However, the shift from additive concepts to multiplicative concepts is not trivial (Clark & Kamii, 1996; Sowder et al., 1998). In this section, I describe the shift from addition to multiplication. By discussing the structure of multiplication, I am not claiming that these structures are “out there” for students to “find” or that these structures are another entity outside our thinking. Rather, I view these structures similar to how Stein, Remillard, and Smith (2007) viewed concepts to be learned (i.e., multiplicative situations are a backdrop to the critical ideas emerging out of students reasoning).

In my exposition of multiplicative situations and results in later chapters, I use the terms quantity and amount. I use the term quantity to describe a measurable attribute of an object (Lamon, 2007; Schwartz, 1988; Thompson, 1994). For example, when presented with water, one can select an attribute such as the volume of water without necessarily conceiving of a unit of measurement such as a gallon. Although I agree with this definition of a quantity, I will not distinguish between a unitless quantity (volume of water) and a quantity with a unit (gallons of water) because most of the problem situations posed by the instructor in the study had a stated unit of measurement. I use the word amount to refer to numerical value of a quantity (e.g., *six* gallons of water).

In additive situations, only one quantity is considered (e.g., apples are added to apples to get more apples). In multiplicative situations, a new quantity is introduced by composing one quantity into a new quantity. The new quantity is formed by establishing a correspondence between multiple numbers of an object and one of another object (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Harel & Confrey, 1994; Hiebert & Behr, 1988; Steffe, 1992). I provide examples of such correspondences in Table 3. As Vergnaud (1988) explained, this deviates from the addition because it changes "what counts as a number" (p. 2).

Table 3.
Sample Multiplication Problems and the Unit Correspondence

Problem	**Correspondence**
Melissa has 7 boxes of cupcakes. Each box contains 6 cupcakes. How many cupcakes does Melissa have in all?	1 box ↔ 6 cupcakes
Nancy has 48 pencils. She wants to put them into boxes with 12 pencils in each box. How many boxes does Nancy need?	1 box ↔ 12 pencils
Henry has 500 gallons of paint he wants to put into 250 equal-sized buckets. How many gallons of paint must each bucket hold?	250 buckets ↔ 500 gallons

In texts for teacher education courses, multiplication has been defined in various ways. There is no one definition that authors have employed across the literature (see Table 4); there are five prominent definitions for multiplication:

- Repeated addition: $a \times b$ refers to the value of a added to itself b times.
- Multiplier, multiplicand: $a \times b$ refers to the total value of items in a groups when there are b items in each group.
- Scaling or comparison: $a \times b$ refers to a comparison or scale a in reference to a unit b (e.g., twice as big, five times as much).

- Cartesian products: $a \times b$ refers to the total number of possible combinations of a objects and another set with b objects.
- Rectangular array: $a \times b$ refers to the total number of objects in a rectangular array with a rows and b in each row.

Table 4.
Definitions of Multiplication in Select Publications

Publication	**Repeated addition**	**Multiplier, multiplicand**	**Scaling or comparison**	**Cartesian products**	**Rectangular array**
NCTM (1964)	X				
Musser, Peterson, and Burger (2014)	X			X	X
Beckmann (2014)		X	X	X	X
Van de Walle, Karp, Bay-Williams, and Wray (2016)		X	X	X	X
Otto, Caldwell, Lubinski, and Hancock (2011)			X		
CCMS (2010)		X	X		X

In this dissertation, I primarily use the multiplier, multiplicand definition because it presents a coherent definition that can be used across multiple content areas. The class I analyzed also used this definition in lessons involving multiplication and division. In this definition, two quantities are presented, wherein one refers to the number of objects and the other, refers to the number of groups. The multiplier, multiplicand definition may be notated by the following equation (Beckmann & Izsák, 2015):

$$M \cdot N = P \tag{1}$$

Taking on the multiplier, multiplicand definition of multiplication, Beckmann and Izsák (2015) refer to the value M as the number of groups (the multiplier) and the value N as the number of units/objects in each group (the multiplicand). The value of P refers to the number of units/objects in M groups (see also Carpenter, Fennema, Franke, Levi, & Empson, 2015; Greer, 1992). This structure and definition deviate from previous conceptions of multiplication as repeated addition and division as repeated subtraction (Fischbein, Deri, Nello, & Marino, 1985; NCTM, 1964).

Vergnaud (1983, 1988, 1994) presented a similar structure in his conceptualization of the multiplicative conceptual field (MCF). He asserted a conceptual field is a set of situations wherein certain concepts are needed to master these situations. By examining the conceptual field, researchers can study connections between ideas, illuminate scenarios that can be leveraged with students, and examine sets of languages and representations within this field. Moreover, Greer (1992) echoed that problem contexts afford the researcher an analytical framework. Vergnaud hypothesized one subtype of the MCF, the isomorphism of measures (see Figure 3) where there is a direct proportion between two measure spaces, M_1 and M_2. Vergnaud did not define what a measure space is, however its use is similar to how researchers have used the word *quantity* or a measurable attribute of an object (Lamon, 2007; Schwartz, 1988; Thompson, 1994). The correspondence in the second row of Figure 2 is the defining characteristic of early multiplication and division problems where 1 in M_1 corresponds to a in M_2 or the multiplicand. Depending on what is unknown in the problem, researchers (e.g., Carpenter et al., 2015; Lamon, 2007) have given these situations different names (see the right side of Figure 3). Table 5 shows how problems in Table 3 can be schematized and classified according to problem type.

M_1	M_2	
1	a	Multiplication: c is unknown Measurement/Quotitive division: b is unknown
b	c	Partitive division: a is unknown

Figure 3. Vergnaud's Schematic for Isomorphism of Measures with Problem Types.

Table 5.
Sample Problems Classified According to Vergnaud's Schematic

Problem	Schematic	Problem type
Melissa has 7 boxes of cupcakes. Each box contains 6 cupcakes. How many cupcakes does Melissa have in all?	M_1 | M_2 1 | 6 7 | c	Multiplication
Nancy has 48 pencils. She wants to put them into boxes with 12 pencils in each box. How many boxes does Nancy need?	M_1 | M_2 1 | 12 b | 48	Measurement/Quotitive division
Henry has 500 gallons of paint he wants to put into 250 equal-sized buckets. How many gallons of paint must each bucket hold?	M_1 | M_2 1 | a 250 | 500	Partitive division

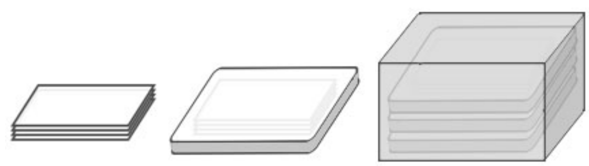
Two points about the structure of the situations are worth noting. First, the problem structure does not necessarily indicate the strategy that one might use to solve the problem (Carpenter et al., 2015; Vergnaud, 1983). However, researchers have found several considerations that contribute to the difficulty of a problem such as the location of the unknown, the unit in the problem context, and the number choices (Harel & Behr, 1989; Kaput & West, 1994; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Lesh, Behr, & Post, 1987a; Noelting, 1980b; Tourniaire & Pulos, 1985; Van Dooren, De Bock, Evers, & Verschaffel, 2009). Second, the isomorphism-of-measures construct in Table 3 is a special case of Vergnaud's rule-of-three class (1983, 1988) which will be discussed in a later section.

Big Ideas Supporting Multiplicative Reasoning

In research on multiplicative reasoning, scholars have identified key research constructs fostering productive reasoning about multiplicative situations. I have identified two ideas from the literature, which I will call big ideas: (1) a shifting unit and (2) extensive and intensive quantities. In this section, I discuss these two big ideas in the context of multiplication as presented in Figure 3. With each big idea, I also present findings from research on teacher knowledge.

The shifting unit. I have called the first big idea the shifting unit to underscore two main ideas. First, researchers have shown that the ability to coordinate multiple objects with another object (as seen in Table 3) is a key component of multiplicative reasoning (Clark & Kamii, 1996; Hackenberg & Tillema, 2009; Hiebert & Behr, 1988; Izsák, Jacobson, de Araujo, & Orrill, 2012; Kaput & West, 1994; Lamon, 2007; Steffe, 1992). For instance, consider a pack of crackers inside a box (see Figure 4). In this scenario, (a) four crackers is equivalent to one pack of crackers (b) and three packs of crackers is equivalent to one box of crackers (c). Essential to this coordination is a composite unit. Steffe (1994) described a composite unit as "a unit that itself is composed of units" (Steffe, 1994, p. 15) and that coordinating units is "mental operation of distributing a composite unit across the elements of another composite unit" (Steffe, 1992, p. 279). The big idea of a shifting unit is apparent when one can also describe the box of crackers in terms of individual crackers and packs of crackers.

The second underlying idea is viewing a quantity in multiple ways. Lamon (1994) described the construction of a quantity and the interpretation of a situation in terms of the quantity as *unitizing*. The correspondences between different groups and units shift the amount depending on the selected quantity. In Figure 4c, there are two ways to describe the amount of crackers in a box. There are three packs of crackers or 12 crackers in a box. In this example, one can describe the same “stuff” in the box of crackers by shifting the unit in two ways—with packs of crackers or crackers as units.



(a) (b) (c)

Figure 4. Structure of Individual, Pack, and a Box of Crackers

The big idea of shifting unit has been studied in research with K-12 students (e.g., Hackenberg & Tillema, 2009; Lamon, 1994). In teacher education, Sowder et al. (1998) emphasized prospective middle school teachers must have a robust understanding of this big idea in order to shape their future instruction with multiplicative concepts (e.g., discourse, task selection). Researchers have also noted that this big idea needs to be developed with American prospective teachers. In her study, Ball (1990) reported prospective elementary and secondary teachers had difficulty in reasoning through a division problem with fractions (1¾ ÷ ½) mainly because of the discrepancy of describing a fraction in relation to the different units (e.g., describing ¼ of a pizza in relation to ½ of a pizza instead of a whole pizza). In contrast, Lo and Luo (2012) found that prospective Taiwanese teachers have robust strategies for fraction division and multiplication where teachers could shift units to describe a given scenario and solve problems. For example, when teachers were asked to find a number on a number line that was 7/12 of 6/5, some teachers were able to describe 6/5 as 12/10, which entails a switch from 5ths as a unit to 10ths. Other teachers were able to consider 6/5 as a unit and simply find 7/12 of 6/5 using fraction multiplication (the report, however, does not provide the teachers meanings for multiplication of $7/12 \cdot 6/5$).

Extensive and Intensive Quantities. The concepts of extensive and intensive quantities support the development of multiplicative reasoning (Behr, Harel, Post, & Lesh, 1992; Schwartz, 1988; Thompson, 1994). A quantity has been defined as some measurable attribute of an object (Lamon, 2007; Schwartz, 1988; Thompson, 1994). Researchers described quantities in two categories, extensive and intensive. Extensive quantities are measured attributes of an object, situation, or event (Kaput & West, 1994). Intensive quantities, on the other hand, describe a fixed "quality" of *any* amount of the object, situation, or event (Schwartz, 1988) and in some cases, intensive quantities can be formed from two extensive quantities (Harel, Behr, Lesh, & Post, 1994; Hershkowitz & Schwarz, 1999; Kaput & West, 1994). For instance, one can examine tea and measure the volume of tea in cups. When one measures different volumes of tea (e.g., three cups of tea), one is identifying an extensive quantity by noting the amount of the quantity of tea. One can also measure the amount in milligrams of caffeine in tea. These quantities are tied to the amount of tea that one is measuring (i.e., extensive quantities). By combining two extensive quantities, one can form an intensive quantity, in this case, amount of caffeine per cup. To illustrate, the concentration of caffeine in green tea is 29 mg of caffeine in one cup of green tea. This would constitute an intensive quantity because it is a measure describing *any* cup of green tea, unlike volume and caffeine amount. To illustrate, 1 gallon of green tea and 1 teaspoon of green tea would have different volumes and caffeine content (extensive quantities), but would have the same caffeine concentration (intensive quantities). Moreover, the caffeine concentration is obtained by combining two extensive quantities, volume in cups and caffeine in mg.

The two quantities play a critical role in the structure of multiplication. In Equation 1, the multiplier, *M*, can be thought of as an extensive quantity because it is measuring the amount of groups while *N*, the multiplicand, can be thought of as an intensive quantity because it describes the amount of objects in one group which remains invariant no matter how many groups are given in the problem (Simon & Blume, 1994). For instance, in Melissa's cupcakes problem in Table 5, regardless of the number of boxes of cupcakes or the number of cupcakes she has, there will always be six cupcakes in a box.

The orange drink studies by Noelting (1980a, 1980b) presented the importance of these quantities in the multiplicative (and proportional) reasoning of children. Embedded in his interview questions with 321 students, aged 6 through 16, he asked students about the consistent "quality" of orange juice recipes, containing orange juice and water. Specifically, he asked, "Which [recipe] do you think the drink will have a stronger orange taste? Or do you think both drinks will have the same taste" (1980a, p. 222). In referring to the possibility of a consistent quality, Noelting investigated the idea of intensive quantities by having a student create an intensive quantity for taste from the amount of orange juice and the amount of water in the recipe. However, in a direct response to the orange drink experiments, Harel et al. (1994) demonstrated the internalized ratio (intensive quantity) of taste that have been pervasive in Noelting's studies were taken for granted. The researchers found some students considered the volume of the drink in determining the taste of the drink. For instance, one of their students, AM, said a 7 oz glass of orange juice tasted more "orangey" than a 4 oz glass of juice because it would hold more orange. Simon and Blume (1994) found prospective elementary teachers were not able to focus on an intensive quantities when examining the steepness of a ski slope. Furthermore, the teachers did not view the steepness as similar to miles per hour. These studies demonstrated that intensive quantities are a key understanding in multiplicative reasoning and that researchers should not take students' understanding of intensive quantities for granted.

Extending the Big Ideas

In the previous section, I explicated two big ideas with multiplicative reasoning but I limited my discussion and examples to early ideas of multiplication (i.e., *n* number of objects corresponding to 1 of another). In this section, I extend the ideas into two middle school concepts, rational numbers and proportional reasoning.

Rational Numbers. Rational numbers have long been mathematically defined as the set of numbers in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$ (Lamon, 2007).

However, this meaning alone does not capture differing meanings. In conceptual analyses of rational numbers, researchers have described several meanings for rational numbers such as comparisons and ratios (Behr et al., 1992; Behr, Lesh, Post, & Silver, 1983) but these meanings may have not emerged from a single, unifying system for rational numbers (Thompson & Saldanha, 2003). In this dissertation and in the classroom I analyzed, the teachers used the definition of rational numbers as written in the CCSS-M ((National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In the standards, rational numbers leverage unit fractional amounts (i.e., $\frac{1}{b}$ is the quantity formed by partitioning a whole into *b* equal parts) to define the fraction $\frac{a}{b}$ as *a* parts of size $\frac{1}{b}$ (see standard 3.NF.1). Empson and Levi (2011) recommend leveraging partitive division problems, which may be an artifact of earlier instruction, to induce fractional amounts (i.e., positive numbers less than one). In order to induce a fractional amount, the total number of objects should not be divisible by the multiplicand. In addition to the two big ideas presented earlier, some researchers found partitioning—the process of splitting a whole into smaller equal sized parts—to be crucial in learning rational number (Behr et al., 1992; Behr et al., 1983; Harel & Confrey, 1994; Lamon, 2007). All three big ideas may be leveraged in equal sharing problems to the creation of rational numbers.

For instance, modifying Henry's buckets of paint problem in Table 5 to "Henry has 5 gallons of paint he wants to put into 3 equal-sized buckets. How many gallons of paint must each bucket hold?" may induce fractional amounts. One potential strategy (see Figure 5) is to deal out one gallon for each bucket. After the first deal, two gallons are left. In order to completely deal out the remaining paint, each gallon is partitioned into three equal parts or $\frac{1}{3}$ gallon of paint and dealt out to the remaining buckets (Empson, 1999; Empson, Junk, Dominguez, & Turner, 2006). The three big ideas I have identified carry into robust understandings of rational number (Behr et al., 1992; Behr et al., 1983; Carpenter, Fennema, & Romberg, 1993). By identifying the amount in each bucket, an intensive quantity is identified by combining the two extensive quantities, gallons of paint and number of buckets. This quantity describes *any* bucket in this scenario. In this case, 1 and 2/3 gallon is in any bucket. The strategy also demonstrates a shifting unit. A gallon of paint is not only composed of the one gallon, but can also be seen as three one-third gallons.

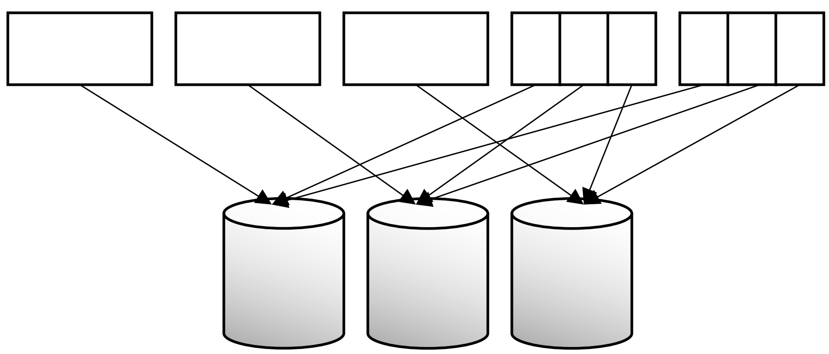


Figure 5. Inducing Fractional Amounts in a Partitive Division Problem

Proportional Reasoning. In my earlier exposition, I limited my discussion to correspondences wherein a value in one measure space is mapped to a value in the second measure space (i.e., Vergnaud's (1983) special case of isomorphism of measures). To expand this idea, consider problems and situations wherein this is not the case. Vergnaud (1983, 1988) conceptualized a general isomorphism of measures, the rule-of-three (see Figure 6a) where the numbers are not necessarily one. These problems resemble proportion problems.

M_1	M_2
x_1	$f(x_1)$
x_2	$f(x_2)$

(a)

M_1	M_2
2	3
x_2	$f(x_2)$

(b)

Figure 6. Vergnaud's Schematic for the Isomorphism of Measures.

As mentioned earlier, knowing the structure of proportion problems does not indicate knowing the strategy one would use to solve them. One would have to "discern the multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities" or reason proportionally (Lamon, 2007, p. 638). Beckmann and Izsák (2015) conceptualized two approaches to reasoning about proportional relationships in terms of Equation 1: (1) $x \cdot N = y$ (2) $M \cdot x = y$. In these ways of reasoning, the two big ideas I presented earlier can be leveraged productively. Other authors have also argued the shifting unit and intensive quantities are important in proportional reasoning (Lobato, Ellis, Charles, & Zbeik, 2010).

The first approach, wherein the number in one group is fixed, is referred to as multiple batches. This approach leverages a pair of fixed quantities to form a batch. Partitive division can be used to obtain the value of N or the amount in one group. Thus, N is an intensive quantity because it is a description of *any* group and a unit is formed by coordinating N with one unit in another measure space. The second approach leverages the ratio A to B as "if for some-sized part there are A parts of the first quantity and B parts of the second quantity" (Beckmann & Izsák, 2015, p. 21). This involves finding the amount in one part which usually entails measurement division. The second approach has been overlooked in most of the literature on proportional reasoning (Beckmann & Izsák, 2015)

To illustrate the two approaches, consider the situation "Eric makes his buttercream frosting with butter and powdered sugar in a ratio of 2 to 3." The relationships in Vergnaud's isophorphism of measures is found in Figure 6b. Using the multiple batches approach, consider the cups as the quantity for the ratio.

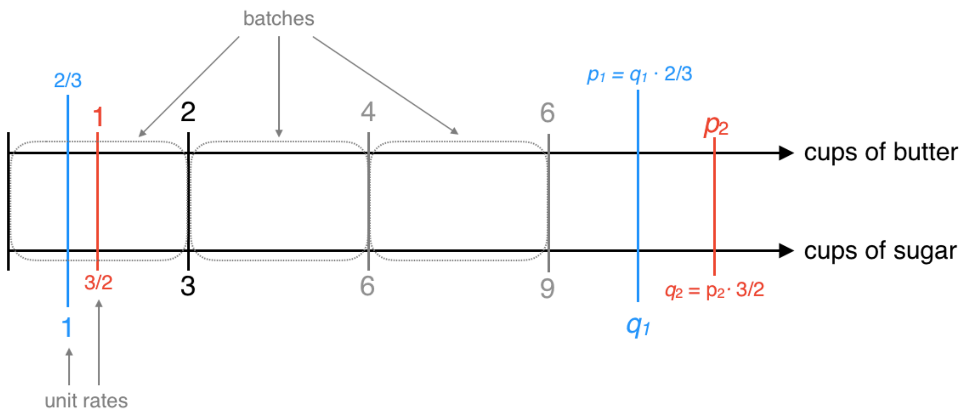


Figure 7. The Multiple Batches Approach to Solving a Proportion Problem.

Figure 7 illustrates a multiple batches approach with a double number line. On the double number line, I first aligned the ratio of two cups of butter to three cups of sugar, forming a composite unit or batch (i.e., create an intensive quantity). I can iterate the composite unit to create equivalent units such as four cups of butter to six cups of sugar or six cups of butter to nine cups of sugar. However, I can also generalize a relationship by obtaining unit rates for both measure spaces. I partition the composite unit and find the corresponding amount in one measure space to one unit in the other. For example, if I partition the original batch in two, I obtain one cup of butter and 3/2 cups of sugar; conversely, if I partition the original batch in three, I obtain one cup of sugar and 2/3 cups of butter. I shifted the unit from the original composite unit to a new batch or composite unit which I can then iterate. For instance, I can iterate the unit rate of 3/2 cups of sugar to one cup of butter to look for the corresponding amount of sugar for p_2 cups of butter. In this case, p_2 is the number of batches or groups and there are 3/2 cups of sugar in one group, thus the corresponding amount is $p_2 \cdot 3/2$ cups of butter.

The second perspective, variable parts, is mostly overlooked in literature on proportional reasoning and "fits more naturally to pairs of quantities measured in the same units" (Beckmann & Izsák, 2015, p. 23) such as the situation presented (i.e., dealing with cups). Figure 8 depicts a strip diagram for the buttercream problem.

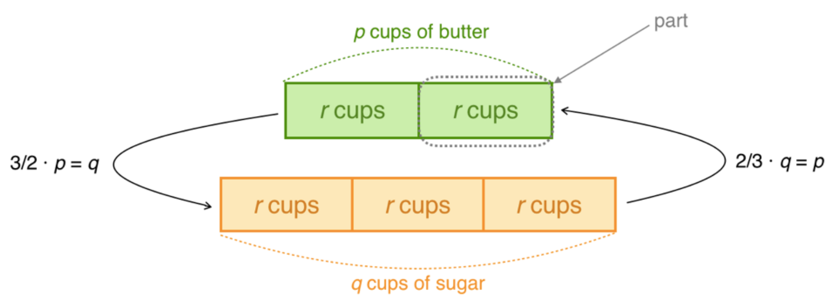


Figure 8. The Variable Parts Approach to Solving a Proportion Problem.

First, I drew the strips depicting the ratio of two parts butter to three parts sugar. If there are p cups of butter and q cups of sugar, the amount in each part can either be $1/2 \cdot p$ or $1/3 \cdot q$ respectively. To obtain the amount of butter if there are q cups of sugar is given, each part is $1/3 \cdot q$. Because there are two parts in the cups of sugar, the number of cups of sugar is $2 \cdot (1/3 \cdot q)$. Additionally, a holistic relationship between the two strips (i.e., $2/3 \cdot q = p$ or $3/2 \cdot p = q$) can be obtained. Using a holistic relationship will not necessarily induce solving for the amount in one part.

Representations Supporting Multiplicative Reasoning

Strip Diagrams. A strip diagram is usually drawn as a rectangle partitioned into different sized parts where each part may refer to a quantity (see Figure 8). The strip diagram has other names such as tape diagrams and bar models. In studies where the strip diagram is used to solve problems, the strip can be partitioned with respect to length although the diagram can be seen as an area. Researchers have used strip diagrams as inscriptions in solving different mathematical tasks (e.g., Hackenberg & Tillema, 2009) but research on strip diagrams themselves is sparse (Ng & Lee, 2009). Additionally, I cannot find a systematic analysis of knowledge resources necessitated by the strip diagram during problem solving or how teachers use the drawing in class. Researchers who have examined the strip diagram in detail situated their studies in Eastern Asian countries such as Singapore and Japan (Murata, 2008; Ng & Lee, 2009) and the Netherlands (Van Den Heuvel-Panhuizen, 2003). They examined strip diagrams to teach and learn number concepts such as the arithmetic operations (Ng & Lee, 2009) and proportions (Murata, 2008).

During problem solving, strip diagrams can support students to construct arithmetic equation and monitor their own activity (Ng & Lee, 2009). Teaching with strip diagrams can contribute to coherence across multiple topics and possibly support coherent meanings. In an analysis of the use of strip diagrams in Japanese textbooks, Murata (2009) found a consistent use of the strip diagram from grades 1 through 6 albeit their use is different depending on mathematical topic. For example, strip diagrams are used as models for addition problems where the result is unknown in the first grade. Beckmann and Izsák (2015) proposed using strip diagrams to support variable parts reasoning for proportions. Strip diagrams make relative magnitudes, not actual magnitudes, accessible from the diagram (i.e., the relationship between two measure spaces). For example, in Figure 8, the two to three ratio depicted does not necessarily refer to two cups of butter and three cups of sugar but rather the relationship between the amount of sugar and butter.

Double Number Lines (DNL). Researchers have studied how students have used number lines to solve tasks with whole numbers (Saxe, 2002), fractions (Izsák, Tillema, & Tunç-Pekkan, 2008; Litwiller & Bright, 2002), integers (Bishop, J. P. et al., 2014), and linear equations (Dickinson & Eade, 2004). In this study, the number line is used to think through multiplicative situations. The DNL is a representation where two number lines, representing two different quantities or two different perspectives on the same quantity, are drawn where each number line can be used to indicate different amounts. The vertical coordination of the amounts provides a composed unit. The DNL can support multiplicative reasoning by stretching or shrinking by the same factor on two number lines (Orrill & Brown, 2012). For example, in Figure 7, a composed unit of 12 pencils in 2 boxes may be formed or the quantities can be “shrunk” to create an equivalent composed unit of 6 pencils per box. DNLs are formally introduced in the Common Core State Standards for Mathematics in the sixth grade to help students solve rate and ratio problems.

Research on DNL is sparse in the literature (Küchemann, Hodgen, & Brown, 2011; Orrill & Brown, 2012). Based on the few studies on DNL, researchers have found

DNLs are useful in supporting students' reasoning about situations involving scaling (Küchemann et al., 2011) and teachers' reasoning about proportional situations (Orrill & Brown, 2012). Research on teaching with double number lines is more rare than controlled laboratory settings. In Hall and Rubin's (1998) study on lessons on rate, the teacher, Dr. Lampert, used a journey line. The journey line resembles previous research on DNL albeit the two lines were fused into one line (see Figure 9).

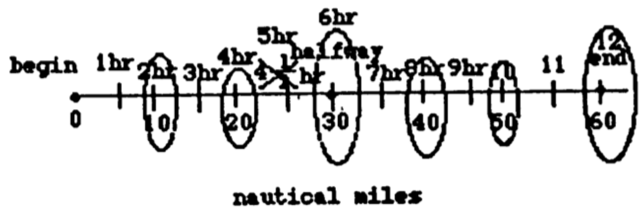


Figure 9. Lampert's Journey Line from Hall and Rubin (1998) (p. 200)

Although Hall and Rubin (1998) were primarily investigating teaching structures and mathematical practices, they present notable data on the affordances and limitations of reasoning with DNLs. First, using DNLs afforded conversations about the appropriateness of using multiplication as a mathematical operation. In public and private conversations, Karim explained why he used multiplication to find how far a car travelling at 40 mph travelled after 3.5 hours. With a journey line, Karim explained he used a repeated addition definition of multiplication because he added forty, three times. A limitation of DNLs can be seen when Ellie had to solve for how far a car, travelling at 50 mph, would travel in ten minutes. She created a journey line and worked with a partition representing both 50 miles and one hour. She initially partitioned the line segment into six partitions, each partition representing ten minutes. However, she was reached an impasse when determining the value of each partition with respect to miles since six does not divide 50 evenly. She and several members of the class settled on a trial-and-error strategy. These two episodes align with Orrill and Brown's (2012) findings about teachers' necessary knowledge components for problem solving using DNLs. In both studies, knowledge of the shifting unit is key to work with DNLs. As Karim worked with the DNL, he used the composed unit of forty miles as one hour. Additionally, partitioning is also critical in productively working with DNLs. Although Ellie chose to partition the line segment into six so she could work with segments of ten minutes, she was unable to anticipate a strategy to find the exact value of the segment with respect to miles. Beckmann and Izsák (2015) proposed using double DNLs to inscribe a multiple batches approach (see Figure 7).

Summary. In this section, I have highlighted two big ideas supporting reasoning about multiplicative situations. The two big ideas, the shifting unit and extensive and intensive quantities (in junction with a big idea from rational number), permeate middle school mathematical number topics and present a shift from earlier number concepts and should be an explicit goal of development for prospective middle school teachers (Sowder et al., 1998). By looking under the hood of the main mathematical ideas, teacher educators can tailor instruction to support future teachers in shaping their mathematical instruction. I also summarized the two approaches to solving proportion problems, a core objective of the class examined in this dissertation.

What is the Culture-Cognition Framework?

Ehara taku toa, he takitahi, he toa takitini.

My strength is not mine alone, but that of many.

–Māori proverb

In this section, I present an exposition of the key constructs of the framework I use in this project. I begin with a brief historical account of the development of sociocultural frameworks in mathematics education along with their vital commitments. Following this, I discuss the culture-cognition framework, the theoretical and analytical perspective I used to guide this study.

A Brief History of Sociocultural Frameworks in Mathematics Education

Mathematics education researchers drew heavily on the works of Piaget and von Glasersfeld to create constructivist theories of learning in mathematics education. The espousal of learning as individual activity in social contexts was de rigeuer in mathematics education research particularly in the 1980’s and 1990’s. The main tenet of constructivism is knowledge is not passively received but actively built up by the cognizing subject (von Glasersfeld, 1989, 1990, 2002). Instead of learning as the acquisition or transfer of knowledge, learning was viewed as the re-organization of one’s cognitive structures based on perturbations. Perturbations have been described as momentary upsets between the organism and the environment (Piaget, 1950/2001) or as events that produce unexpected results (von Glasersfeld, 1990). These upsets promote individuals to adapt cognitive structures via the processes of accommodation and assimilation in order to re-establish equilibrium (Jaworski, 1994). By engaging in these processes, individuals reflectively abstract their actions to form mental structures called schemes to make sense of the world.

One form of constructivism was (and perhaps still is) critical in shaping research on student learning. Radical constructivists adhere to the tenet described in the previous paragraph and an additional tenet—individuals adapt their knowledge to organize their experiences and not to discover an ontological reality. This tenet is what makes radical constructivism "radical." A reality existing outside of individuals' experience is rejected (von Glasersfeld, 2002) and knowledge is constantly re-organized and persists when it is "viable" or "fits" one's experience (Ernest, 2010; Steffe & Kieren, 1994; Thompson, 2000; von Glasersfeld, 1989, 2002). The concept of viability is also a defining characteristic of radical constructivism. For knowledge to be viable, it needs to "fit" one's abstraction of experiential constraints that one has acquired through interacting with the world. In this case, the reality constructed by the individual is "stable" (von Glasersfeld, 1989, 1990). Ernest (2010) summarized constructivism:

> What binds many of the various forms of constructivism together is the metaphor of construction from carpentry or architecture. This metaphor is about the building up of structures from pre-existing pieces, possibly specially shaped for the task. In its individualistic form the metaphor describes understanding as the building of mental structures, and the term 'restructuring', often used as a synonym for 'accommodation' or 'conceptual change' in cognitivist theory. (p. 39)

Constructivism relies on the individual's knowledge and how it changes over time. Researchers have pointed out some perceived limitations of constructivism (e.g., Kilpatrick, 2013; Otte, 1994); however, the strongest argument leveraged against radical constructivism is the absence of accounting for the social (Lerman, 1996, 2000a). Steffe and Thompson (2000) argued radical constructivism always had interaction as at its core. However, they also situate the role of the researcher as an investigator of "mathematical meanings of the *individual in interaction*" (p. 203, italics added). Such a role would still emphasize the learner where these interactions are merely a backdrop.

Some scholars became frustrated with the privileged status of this theory of learning. As Mousley (2015) recounted, she and her professional group “felt that we had to adapt paper titles and abstracts to reflect the focus on individual cognition in order to have papers accepted for presentation at the PME conference” (p. 154). Kilpatrick (1987) pushed constructivists to be humbler and to recognize the limitations of this learning theory. To address the criticism, a new strand of constructivism emerged—social constructivism. Social constructivism added considerations when thinking about learning within social contexts such as knowledge being “taken-as-shared” and communication as explained by “intersubjectivity.” This stance did not reject any of the constructivism’s theoretical commitments but acknowledged objectivity as a social construct. The emergent perspective is an example of a theory of learning surfacing from this wave. This perspective situates individuals actively constructing their own knowledge as they participate in cultural practices of the mathematics classroom (Cobb & Yackel, 1996). Thus, “learning is a process of both self-organization and a process of enculturation that occurs while participating in cultural practices” (Cobb, 1994, p. 18). By engaging in social activities, participants form an emerging culture and also reorganize their own knowledge of what it means to do mathematics by participating. However, this shift did not appease scholars who did not ascribe knowledge solely to the individual.

Theorizing learning as a social and cultural phenomenon is not a new development. Vygotsky may be seen as the prevalent scholar for theorizing learning as a product of social interaction and his de facto mantra of social learning theorists: “Every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level" (Vygotsky, 1978, p. 57). However, early non-Western philosophies such as Buddhism and Confucianism described learning as the activity of the community and the genesis of changes on a community level before Vygotsky (Merriam & Kim, 2008). Lerman (2000b) noted shifts in mathematics education research towards social perspectives began in the late 1980's. He characterized the shift as the ascent of theories characterizing social activity as the producer of mathematical thinking. The hypothesis that social activity produces mathematical thinking was different from the prevailing notion of social interaction as the catalyst for individual thought. The shift in research was not a movement from individual to social rather, it was a shift to characterizing learning as a product of both individual and social mechanisms.

The most recent shift in learning frameworks takes power and authority into account. Zevenbergen (1996) critiqued the dominance of learning theories, both individual and social, as ignoring the political aspect of meaning-making where although students can construct knowledge, these theories do not acknowledge "legitimate" knowledge. She also explained if students and teachers were to construct and demonstrate "legitimate" mathematics, they go on to reap the economic benefits which provides them with more power. Additionally, constructivist and social theories of knowledge are not well-suited for explaining differences in achievement in social groups such as women and students of color. Gutierrez (2013) proposed a shift in mathematics education to focus on power and identity to explain such differences.

The theoretical framework I espoused for this study is based on Saxe's (DATE) Papua New Guinea and classroom research. This framework falls within the constructivist, social, and political philosophies of mathematics education without following one philosophy squarely. There are four reasons why I chose the framework to guide the study. First, Saxe and his colleagues explicitly address representations as a cultural practice in the same way drawings were a cultural practice in the classroom I analyzed. Second, the framework accounts for developments across time, not just cross-sections. Third, the framework highlights the role of both an individual and the community in the process of learning. Finally, the role of power, although not heavily theorized, is acknowledged to explain shifts in the individual and community's practices.

Saxe's Culture-Cognition Framework

Geoffrey Saxe (DATEs) developed his framework on the interplay of culture and cognition based on three trips he took to Papua New Guinea. His conducted research with members of the Oksapmin tribe, a community in the Mountain-Ok region of central New Guinea. In 1978, Saxe examined how Papua New Guinean children used a culturally rooted base-27 form of counting while they solved arithmetic problems. He took a second trip in 1980 and focused on how the Oksapmin solved problems in everyday practice such as measurement and economic transactions. By the time he returned in 2001, Saxe had developed a new research trajectory based on work with other communities in Brazil, Brooklyn and Los Angeles.

Through his Papua New Guinea studies, Saxe (2002) argued culture and cognition are "reciprocally related, each participating in the constitution of the other" (p.

16) and focused his analysis on cultural forms and their cognitive functions and how the forms and functions changed across time. Cultural forms are historically rooted artifacts used in communication and the purposes for the forms are called their cognitive functions. Saxe (2012) implied forms also have a communicative function: “To be useful in communications, forms – whether they be body parts, currency tokens, or quantifiers – must be produced with an audience in mind and interpreted with attention to the speaker’s communicative intent” (p. 296). In other words, forms are both communicative and cognitive tools. For this study, I draw heavily on the identification of drawn cultural forms and their functions. I describe how I identified forms and functions in more detail in Chapter 3.

In classroom analyses, Saxe argued that, in order to characterize the role of cultural forms and their functions, the researcher must investigate the production of a “common ground” or a taken-as-shared ways of talking and doing (Saxe, de Kirby, Le, Sitabkhan, & Kang, 2015). He identified sociomathematical norms (Cobb & Yackel, 1996) as the common ground for the community (i.e., expectations for community members). He described three analytical strands describing how individuals contribute to common ground. In this dissertation, I focused on the first two strands and describe my identification of continuity over time in more detail in Chapter 3. The three strands are:

1. *Microgenesis.* This process shows how individuals contribute to a common ground, often using a form in public, by describing how forms serve certain functions. For example, a student who wants to show how 3/2 is equivalent to

6/4 may create a strip diagram partitioned into three parts and partition each part into two in order to show six sub-partitions using a different color to show the relationship between the two partitions. The student is contributing to common ground by showing a particular form to describe the fraction equivalency.

2. *Ontogenesis.* This process shows the continuity and discontinuity of forms to serve new functions. In some instances, if a new function is necessitated, some historical forms may be employed or new forms may emerge to serve the new function. The student may apply new forms (discontinuity) or use old forms (continuity) to accomplish the new goal.
3. *Sociogenesis.* This process shows how microgenetic constructions are taken up and distributed over individuals over time. If a student presents a new form, some students may take up the new form or continue using old forms.

I summarize the three stands and their relationships in Figure 10. Microgenesis occurs at a certain time where people draw on common ground to tailor public displays with available form-function relations. Individuals both draw from and contribute to the common ground. With a new situation, such as a new problem or topic, the common ground is altered to include a new set of form-function relationships individuals can use to create public displays. The ontogenetic development is schematized in the vertical arrows across the new situation to illustrate the continuity or discontinuity of previously used forms. The community may (continuity) or may not (discontinuity) use older form-function relations. Finally, the sociogenetic strand is illustrated in two ways: the use and alteration of form-function relations through time and across individuals.

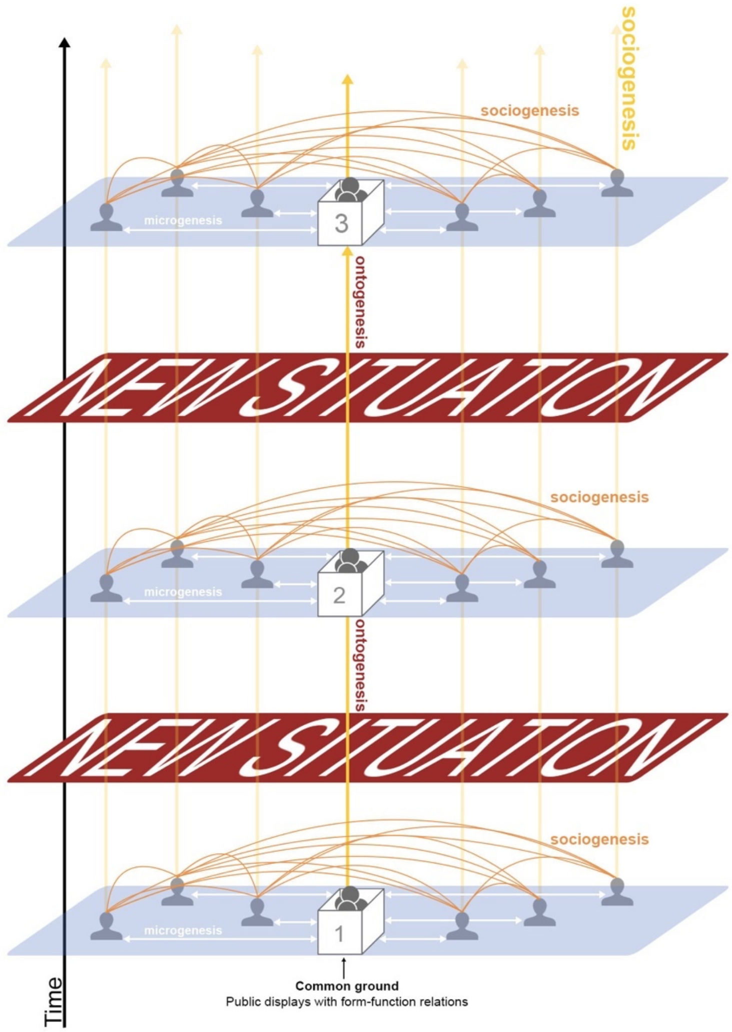


Figure 10. Three Genetic Strands of the Culture-Cognition Framework
adapted from Saxe et al. (2015, p. 73).

Papua New Guinea Studies: The Case of Haben & Fu. In the Papua New Guinea studies, Saxe and Esmonde (2012) examined the use of particular cultural forms. *Haben* and *fu* are both words from the native Oksapmin language and considered cultural word forms. Although there were other word forms and functions Saxe studied, I present haben and fu as exemplars of the form-function shift.

Saxe and Esmonde asked fifty-seven adults to identify the names of the different local currencies. The adults gave different answers to the name of the lowest denomination paper note (2-kina note). The results of the interviews pointed to several functions of one of the word forms, haben. Haben's earliest function was to refer to a leaf of a certain tree. The researchers posited haben was used to refer to any note when paper money was introduced into the community. Australian patrols introduced paper money then they made contact with the local community in the 1930's. Possibly, the function of haben shifted around this time to describe to the leaf-like nature of the notes. It is unclear when the form's function shifted to refer to the 2-kina note or why, but both schooled and unschooled adults referred to the 2-kina note as haben. Less than ten percent of the elders referred to the note as haben; the elders referred to the 2-kina note as one pound. This demonstrated the shifts in function of a specific word form throughout time based on new situations such as the introduction of paper notes.

The researchers also examined the word form *fu*. The form fu was tied to a base-27 counting system (another cultural form). The base-27 counting system starts with thumb on one hand as one. Succeeding numbers go through 26 positions on the upper body ending with the small finger on the other hand, the 27th position. Once one completes a count of 27, they raise their fists and say, "tit fu." While talking to adults of differing ages, Saxe noticed three different positions for the word fu. The elder indicated fu (in his expression, *tit fu gon a* or one complete round) as the pinky of the opposite hand or the 27th position in the count. An older adult pointed to fu as either the 27th position or the opposite elbow or 20th position while a middle-aged adult pointed to the

20th position as fu but did not refer to the 27th position as fu. For the researchers, this indicated shifts in functions for the word form fu similar to the differing functions for the word form haben. From interviews with adults in a different region than the haben study, the researchers identified a few functions for fu. Prior to contact with the Australians, fu was used to denote completeness or plenty and with the distinct expression “tit fu” referring to the complete count of 27 on the body. The number 20 acquired a more prominent status once new currencies were introduced to the community because people equated one pound and 20 shillings. The function of fu shifted from a complete set of 27 to a complete set of 20 and its corresponding body part, the opposite elbow. Papua New Guinea gained independence from the Australians and used a new type of currency, the toea and kina. In the conversion, one pound was equivalent to two kina and thus fu referred to both currencies. Fu finally took on its most recent function, doubling. This may have been rooted in counting large values of money. In both studies, the researchers used a microgenetic analysis to examine how the Oksapmin used the available cultural forms (e.g., the base-27 system, word forms), to communicate with each other. The researchers also examined ontogenetic development by identifying potential situations prompting the shift in forms such as the introduction of new currency as the impetus for different positions of fu on the body. Finally, they identified the potential loci within the community such as commercial activity inducing the new forms.

Two noteworthy occurrences were the interaction within and across forms. The functions of haben and fu intersected where both forms had functions referring to the 2-kina note. In 2001, around 70% of unschooled adults Saxe and Esmonde interviewed referred to the fu as the 2-kina note and 40% of unschooled adults in another region referred to the note as haben or leaf. Although there may have been differences based on the interview question (i.e., asking to point to fu and asking to name the 2-kina note), different functions may also have been rooted in microgenetic and sociogenetic processes such as economic transactions of the region. Additionally, a shift in the word function for fu also necessitated a shift in the body count form (i.e., the location of fu).

Summary

In this chapter, I identified three main areas of research related to this dissertation along with relevant themes and results within each area. These areas situate the rationale for the study and the results. First, mathematics education researchers have developed multiple accounts for studying representations within individual's problem-solving process; however, accounts for studying representations as a social practice have not been as robust and extensive. Second, researchers have identified critical ideas supporting engagement in the multiplicative conceptual field and some researchers have noted the need to support teachers in developing these ideas. Additionally, there is a dearth of accounts of how prospective teachers come to learn these ideas particularly within content courses. Finally, an account of the development of representation as a social practice exists within the context of a larger social structure (i.e., the Papau New Guinean population). This dissertation will contribute to the growing research that furthers how to recontextualize these studies to classroom data and will refine methods and theories gleaned from these studies.

CHAPTER 3

METHODOLOGY

In this chapter, I describe the context of the study, including the techniques for data collection and who participated in the study. I also outline the methods I used for finding forms and functions in the data, addressing the first research question posed in Chapter 1:

RQ1. What methods and grain sizes should researchers consider when characterizing forms and functions of mathematical drawings in classroom data?

I provide an account of how I developed the methods, including some analytical issues I considered when going through different iterations of the analytical techniques. Thus, this chapter does not read like a conventional methods chapter.

Context of the Study

The classes selected for this study were two mathematics content courses for prospective teachers (hereinafter called "students"). The students were enrolled in a middle grades teacher education program geared towards certification to teach 4th to 8th grade mathematics. The courses were offered in the 2016-2017 school year, one in the Fall and the other in the Spring semester. The content of the Spring course built on the content of Fall course. Each course was offered twice a week and lasted 75 minutes. The objective of both courses was to strengthen the students' mathematical understanding of middle school topics such as the base-10 number system, fractions, and ratios. The same instructor, Dr. B, taught both courses. The material of the two classes was based on Beckmann (2014) and the goals of a larger study, Investigating Proportional Relationships from Two Perspectives (InPReP2). InPReP2's goal was to investigate how prospective secondary and middle school teachers develop proportional reasoning using the two perspectives based on Beckmann and Izsák (2015).

Prior to collecting the data, I observed two iterations of both courses from 2014 to 2016 with the same instructor. Of the 13 students enrolled in Fall course, 10 students subsequently enrolled in the Spring course. Three new students joined the 10 students in the Spring. The students in both course were predominantly white and female. In addition to the instructor and students, a teaching assistant from the mathematics department, graduate students from the mathematics education doctoral program, and a post-doctoral researcher for InPReP2 attended classes. Most of the students were also enrolled in a paired methods course. A course on teaching number systems was offered in the Fall and one on teaching geometry was offered in the Spring. Notably, if the students were enrolled in the Fall methods course, they discussed similar topics such as multiplication and ratio from another instructor who approached the same content differently. It is beyond the scope of my analysis to determine the influence of the instruction of the other course, however I acknowledge instruction may have exerted some influence on the students.

The instructor, Dr. B, usually began class by orienting her students to the mathematical topics of the day such as multiplication with fractions. She gave the class a problem to solve and students worked at their table with two to five other students. As the students worked, she walked around class, asked students about their work, and pressed students to explain their thinking. Throughout the classes, students were expected to use math drawings and definitions. The instructor would redirect the student if a student violated these expectations such as when students would rely on memorized algorithms. Strip diagrams and double number lines were prioritized as the two main forms of drawings. Students were given the option of using iPads equipped with GoodNotes, a notebook app, to work on math problems. To present their strategies, students could use the whiteboards or project their iPad screen on one of four mounted screens. Some students used the iPad's camera to project their work onto a TV. The whole-class discussion focused on students' strategies and connections between different student strategies.

Because I wanted to investigate the evolution of math drawings, I purposefully selected this class as an intense sample (Patton, 2002). An intense sample displays a rich manifestation of the constructs being investigated. In this case, I was interested in the development of the math drawings used in class, specifically strip diagrams and double number lines.

Data Collection

The main data corpus for this study was video and audio-recorded lessons from class. Together with other members of InPReP2, we set up and collected data from two cameras in the classroom similar to Izsák et al. (2009). One stationary camera was set at the back of the class and captured the whole class within one frame. The other camera was also stationary during whole-class discussion and followed the instructor during small group discussions. Additionally, six microphones captured audio. Two microphones mounted on the ceiling captured audio during whole-class discussion while four flat microphones captured audio at each table. In post-production, all video and audio data were condensed into a single file. The file contained all the video and audio feeds such that I could select any microphone and listen to an isolated audio source. For example, if the instructor was at a table during small group work, I listened to the audio recorded from the flat microphone at that table.

Because I viewed the class as a culture with its own ways of doing mathematics, I acted as ethnographer. To collect additional data, I sat with one group of students and employed two methods of ethnographic data collection (Eisenhart, M. A., 1988). First, I acted as participant-observer where I became a member of the class by doing class activities and withholding my expertise with the content. I positioned myself in the group as a facilitator of conversation, not as someone to verify answers or as an extension of the instructor. As I participated with the class, I took field notes. In the Spring, I took "live" field notes on my personal device where I recreated students' work while recording the screen on my device. This allowed me to temporally note how students' public displays were made. Secondly, I informally interviewed the members of the group as they worked on class activities. I asked them how they got to certain answers and how they chose to make their drawings. Additionally, if time permitted, I asked them to compare their solutions and displays with each other or initiated help between students if someone was struggling. These conversations gave me additional data to understand displays made by certain students when they explained their solutions to the whole class.

Once the year ended, I selected days of instruction to analyze. Over the course of the year, roughly 35 hours of classroom video data were collected. I narrowed the scope of analysis to lessons with the following criteria:

- Lessons where students were requested to make a math drawing and where the math drawings were the focus of the whole-class discussion.
- Lessons where students made their own drawings or where premade drawings were provided and students annotated over the drawing.
- Lessons where multiplication and division were central such as requiring to write corresponding multiplication and division equation.

Table 6 outlines the lessons and the lessons I selected to analyze. From this base of lessons, I distilled 27 problems to analyze. A summary of all 27 problems is found in Table 7 and a full list of the problems is found in Appendix A.

Table 6.
Topics Covered and Lessons Selected for Analysis

Week	Fall Semester	Spring Semester
1	Base-10 System	Fraction Division
2	Fraction Definition	Fraction Division
3	Equivalent Fractions	Fraction Division
4	Comparing Fractions	Fraction Division
5	Percent	Appropriateness, Ratios
6	Fraction Addition and Subtraction	Ratio and Proportional Relationships
7	Multiplication	Ratio and Proportional Relationships
8	Multiplication Properties	Equations for Proportional Relationships

9	Whole-number Multiplication	Inversely Proportional Relationships
10	Fraction Multiplication	Statistics and Probability
11	Fraction Multiplication	Statistics and Probability
12	Fraction Multiplication	Statistics and Probability
13	Fraction Division	Statistics and Probability
14	Fraction Division	Number Theory
15	Division with Remainders	Number Theory
16	Division Algorithm	Review

Note. I analyzed the highlighted lessons.

Table 7.
Problems Selected for Analysis.

#	Problem Name	Problem Type	Possible Structure*
1	Playground	Multiplication	$\frac{1}{3} \cdot \frac{1}{4} = ?$
2	Jean's Casserole	Measurement Division	$? \cdot \frac{1}{2} = \frac{1}{3}$
3	Sue's Run	Partitive Division	$\frac{1}{3} \cdot ? = \frac{1}{3}$
4	Goblin Goo 1	Multiplication	$3 \cdot \frac{4}{5} = ?$
5	Pumpkin Juice	Multiplication	$\frac{1}{4} \cdot 12 = ?$
6	Bat Milk Cheese	Multiplication	$\frac{1}{4} \cdot \frac{8}{3} = ?$
7	Dragon Blood	Multiplication	$\frac{1}{3} \cdot \frac{1}{5} = ?$
8	Goblin Goo 2	Multiplication	$\frac{2}{3} \cdot \frac{4}{5} = ?$

9	Blank Multiplication	Multiplication	$\frac{2}{5} \cdot \frac{2}{3} = ?$
10	Francine's Rope	Partitive Division	$8 \cdot ? = 32$
11	Gallons of Water	Measurement Division	$? \cdot 8 = 400$
12	Drive	Partitive Division	$4 \cdot ? = 220$
13	Pizza	Partitive Division	$4 \cdot ? = 3$
14	Cookies	Measurement Division	$? \cdot 3 = 14$
15	Brownies	Partitive Division	$3 \cdot ? = 14$
16	3 ÷ 3/4	Measurement Division	$? \cdot \frac{3}{4} = 3$
17	Tonya and Chrissy	Measurement Division	$? \cdot \frac{2}{3} = 1$
18	1 1/2 ÷ 1/3	Measurement Division	$? \cdot \frac{1}{3} = 1\frac{1}{2}$
19	Noodles	Partitive Division	$\frac{2}{3} \cdot ? = 120$
20	1/3 ÷ 2/5	Partitive Division	$\frac{2}{5} \cdot ? = \frac{1}{3}$
21	Hot Chocolate	Ratio Organization	n/a

22	Hot Chocolate 2	Ratio Organization	n/a
23	Rope	Missing-Value Proportion	$x \cdot \frac{2}{3} = y$
24	Scooter	Missing-Value Proportion	$x \cdot \frac{3}{16} = y$
25	Yellow and Blue Paint	Missing-Value Proportion	$\frac{3}{2} \cdot 48 = ?;$ $\frac{2}{5} \cdot 150 = ?; \frac{3}{5} \cdot 150 = ?$
26	Punch	Missing-Value Proportion	$\frac{3}{8} \cdot x = y; \frac{5}{8} \cdot x = y$
27	Fishing	Statistics	$\frac{3}{10} \cdot 50 = ?$

*This notation of this structure is based on Beckmann and Izsak (2015).
Note: The identification of the problem type and problem structure is based on my view as an expert. Depending on the selection of a group and base unit, this structure may change.

Overview of Analysis Technique

In this section, I provide my analysis technique in detail including issues I encountered while developing final analytic techniques. I built the technique from coding specific lessons with strip diagrams. The analysis occurred in two phases. To address my first research question, I analyzed the forms and functions comprising the public displays available in the data.

Goal of Analysis

My goal for analysis was to distill the students' drawings down to a set of codes that described both form and function which could be applied to all drawings. Once all the drawings could be coded, my second goal was to look at the frequency of the codes to infer how the form and functions changed over time. For Saxe (2012), collective practices are “semidurable, socially organized activities, constituted by individuals engaged in joint actions through the use of semiotic and instrumental forms” (p. 304). Thus, once I was able to describe individual activity, I made inferences about the collective practices of the class.

Ground Zero for Analysis: Whole-Class Discussion

Inherent in the analysis of classroom interactions is the methodological decision of attributing results to the individual or the social. I determined three potential areas to begin analysis, each with its own affordances and limitations. I first considered analyzing what individual students said and did. This provides the highest levels of validity because results from analyzing an individual student’s activity can be solely attributed to the student, unless the student says otherwise (e.g., “I did what she did”). I also considered small groups of students working the same table. Although some results can be drawn from the small-group interactions, I cannot assume norms of specific groups I did not work with closely. Some students at groups who did not have graduate students or were approached by the instructor may not expect to explain their thinking to the small-group. In other words, I assumed the students tailored their drawings for whole-class discussion, not small-group discussions because the instructor developed an expectation of sharing their thinking in whole-class discussions. The final area was during whole-class discussion. This provided me with the best data set to document practices. There are two pointers from theory grounding this methodological choice. First, students construct drawings in order to communicate their thinking. As

Saxe (2012) argued, “to be useful in communications, forms… must be produced with an audience in mind and interpreted with attention to the speaker’s communicative intent” (p. 296). In other words, students created drawings with the intent to communicate their ideas to others in class. Additionally, one maxim of communication enumerated by Grice (1975) is for interlocuters to avoid obscurity and “do not say what you believe is false” (p. 27). I assumed all drawings were not constructed to confuse others or demonstrate a falsehood. If all members of class intend for others to understand their ideas, then they must necessarily rely on practices and discourses everyone in class understands, in other words, they draw on practices of the whole-class community. For instance, in order to communicate one’s ideas in class, students will use English, not Bengali or Hungarian, because they assume the practice of speaking English is the form of communication they can use to share their ideas.

In this analysis, I decided to initially code whole-class discussion and characterized the results as forms the community draws on to create drawings. I explain the criteria for these codes in a later section. Using these codes, I assume these are the forms comprising the collective practice. Then, I looked at work created and discussed during small-group work. I assumed students are creating their drawings using the same collective practices.

Techniques for the Microgenetic Analysis

The results of this study rely heavily on the identification of a form and function. Saxe and his colleagues rarely defined form and function. Because they were rarely defined or operationalized, I distinguished three levels for analysis. Before I describe the levels, I outline the different ways the constructs have been defined and used in the literature and some methodological considerations for each construct.

Analytical considerations for form. To define a form, Saxe provided examples such as “geometric forms such as elements of the number line, including the line, arrowheads, tick marks” (Saxe et al., 2015, p. 11), “counting words, written inscriptions, graphical representations, everyday speech” (Saxe, 2012, p. 1), the Oksapmin body-count system (Saxe & Esmonde, 2005), and other number systems (Saxe, 1999). However, the crux of analyzing a form is when a form “starts” and “stops.” For example, Saxe and Esmonde (2005) traced the form-function shift of the base-27 body count system, a number system form, and *fu,* a word form embedded in the number system. Saxe et al. (2015) considered multiple embedded forms such as the number line, unit intervals, and multiunit intervals. Although related, considering both a number system and a tick mark on a number line a form seemed difficult methodologically. If one was to describe a form’s development over time, a number system seems stable enough that no developments could be tracked. However, considering a smaller grain size such as a tick mark may prove useful to tracking changes over time.

First, I decided not to code a form too finely (e.g., coding each of the lines of a rectangle each as a different form). This would be too tedious and might not yield any results in characterizing form-function shifts across time. Sometimes a line is just a line. Conversely, I also did not decide to code the entire set of inscriptions as one form. This would leave me with too broad of a description of strip diagrams without capturing the complexity of how the parts fit within the larger structure of the form. This left me with form as somewhere between nothing and everything. I decided to let my data direct me towards the forms relevant to the class and used participants' utterances and gestures point at relevant inscriptions. However, participants pointed at different "sizes" of forms. Some forms were elements of larger forms such as partitions of rectangles.

Analytical considerations for function. I had initially coded for the functions as solely referents where forms were intended to show or represent other things. As I looked more through the data, I noticed some forms had other functions aside from representing other things. Students were creating drawings to serve larger functions such as using forms to measure amounts and compare quantities. I decided that functions should be verbs and not just nouns (e.g., "Show 1/2 cup" instead of "1/2 cup").

I also had to consider how other inscriptions such as tables and equations played into my analysis. Initially, I did not consider these inscriptions as part of my analysis because they are not strip diagrams or double number lines. However, students created these inscriptions to assign and construct meaning. Thus, I included these as evidence for the functions for some forms.

I considered three different grain sizes to capture my data and extant literature on form-function relations (see Figure 11). The largest grain size I identified is a system of forms. This grain size is similar to Kaput's work. Although Kaput was primarily concerned with symbol systems or the collection of inscriptions and the rules or grammar associated with the symbols (Kaput, 1987), he acknowledged a social level of symbols. He called "cultural and linguistic artifacts shared by a cultural or language community" a notational system (Kaput, 1989, p. 55). He also acknowledged that concepts and notations make sense within the boundaries of the community. Similarly, Blumer (1986) and Radford (2000) noted the social aspect of meaning creation for cultural inscriptions. However, I viewed this level as pervasive over long periods of time. It would take an act of imperialism (Bishop, A. J., 1995) or radical change in societal knowledge and practices to change the system of forms. This framed the Oksapin base-27 system as a system of forms. Because the forms developed within the community and only began to slowly accommodate the colonial base-10 system once outsiders began to change their practices, particularly in commerce. One could also count the system of variables used in K-12 American mathematics curricula as a system of forms because of its pervasive existence in our school communities. If the system of forms is colonized, then the future use of the existing system is in limbo. In some cases, vestiges of the old system may exist in the new system such as the preservation of the concept of zero or nothingness from the Mayan counting system to the colonial base-10 system. Perhaps, the system of forms disappears altogether.

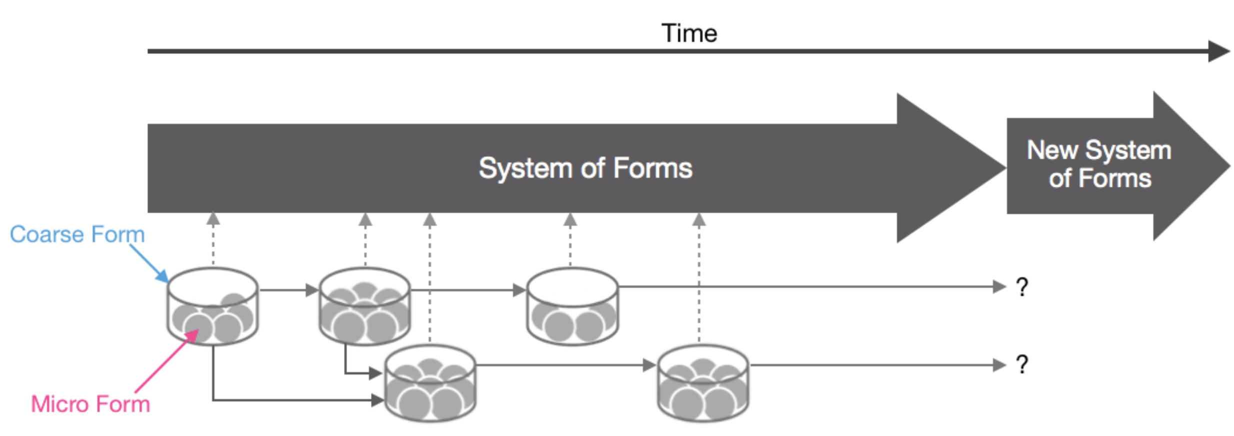


Figure 11. Three Grain Sizes for Form and Function.

The two grain sizes build the system of forms. A *micro form* is a single geometric inscription as fine as a line or rectangle and a *coarse form* was a group of microforms used to address a larger goal such as an entire strip diagram with its annotations to solve a multiplication problem. In my analysis, the bases for identifying forms rests on identifying micro forms and collapsing them into coarse forms.

An example for analytical considerations. To explain my analytical choices, I use an episode towards the end of the Fall semester. In this episode, Jack re-created his solution to the multiplication problem “You have two thirds of a serving of goblin goo. One serving of goblin goo is four fifths of a liter. How many liters of goblin goo did you have?” on a white board during whole-class discussion. Jack’s description and drawing is complex in that his drawing can be broken into different components.

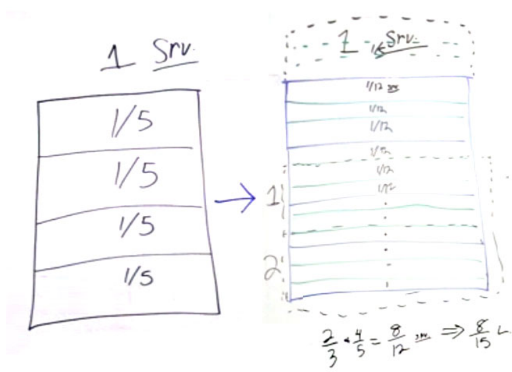


Figure 12. Jack's Drawing for the Second Goblin Goo Problem.

Following the grain sizes I illustrated, Jack used a strip diagram, a system of forms used in class. Because this data was collected in the Fall, I can conclude that Jack followed a normative or expected way of drawing his thinking and he used this form in particular because of its cultural way of solving problems. For now, and within the bounds of this community, Jack employed a system of forms to create his drawing. If I were to analyze the system of forms, I would have called this a strip diagram and stop here. This does not capture the complexity of Jack's drawing and would not differentiate this from Jack's past and future work and other students' work. Thus, I considered a smaller grain size. Moreover, I considered what parts of the drawing Jack found to be salient. By identifying the salient parts of Jack's drawing, I infered the smallest geometric elements of his drawing Jack used to assemble his drawing.

First-level Analysis: Coding for Micro Forms and Functions

The primary concern for this study is the form-function shifts that occur over time. In order to undertake a systematic investigation of the shifts, I identified both the form and function of each students' drawing whenever possible. The goal of this level of coding was to identify the microgenetic development of students' displays.

In general, I coded inscriptions as *micro forms* and how the feature was used as a *function*. More specifically, I coded all the drawings students constructed that were presented during whole-class discussion and the drawings discussed with the instructor during small-group work. I decided to include drawings made in group work because students were expected to present their drawings and solutions if they were called upon or volunteered. With this expectation, I assumed each student was creating a public display to communicate their thinking to others whether they ultimately presented their ideas during whole-class discussion or not.

Preliminary analytical considerations. I coded video to show both form and function in the first wave of data analysis. I initially captured the interaction of both form and function by coding segments of video with the structure FUNCTION as FORM (i.e., this FORM was to show FUNCTION such as PARTITION OF A STRIP to show a UNIT

FRACTION AMOUNT). However, these codes became limiting as I had not decided how I could not identify a form or function when I saw or heard it. Additionally, I limited myself to functions that “showed things.” This iteration of coding was also not sensitive to student statements appealing to memorized algorithms such as making lowest common denominators. I eventually abandoned coding video with codes that merged both the form and the function and proceeded to code them separately. I initially coded forms before determining their functions. This provided a base line of sets of forms I could determine across drawings.

To illustrate the shortcomings, consider Jack’s drawing in Figure 12. If I coded the partitions with the code structure FUNCTION as FORM, then I would code the partition as UNIT FRACTION AMOUNT as PARTITION OF A STRIP. In Jack’s description of his strategy, he drew the partitions in the first strip to show unit fractional amounts; however, in his second strip, although the partitions also represented unit fractional amounts, he drew the final partition in order to complete a liter. I also noted that the code structure did not capture the difference (e.g., COMPLETE THE UNIT as PARTITION OF A STRIP), because the function of the partition in this case was more than a representation of something–it served a particular goal.

Because of similar instances where parts of some drawings could function differently in others, I decided to focus on initially identifying the forms or inscriptions separately. Additionally, the same function could be supported by different forms. It was clear to me Jack wanted to draw an additional partition to complete the liter to determine the size of the smallest partition. Consider Sophie’s picture in Figure 13. She wanted to determine the size of one of the smaller partitions with respect to the highlighted part of the rectangle she identified as one-half. In order to determine the size of the smallest partition, she created a new strip. This indicated a difference in form but serving the same function, determine the size of a partition. Thus, I decided codes for forms and functions separately instead of coding drawings with the relationship between the forms and functions.

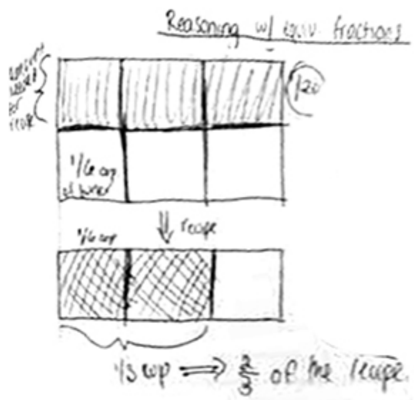


Figure 13. Sophie's Drawing for Jean's Casserole Problem.

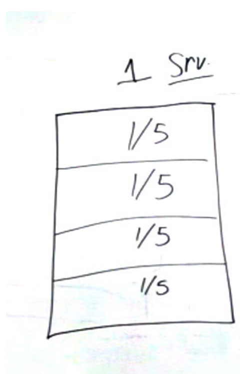
Micro Forms. In this study, I found micro forms by looking for the smallest set of perceivable inscriptions or features of an inscription as identified by a member of the class. To identify a micro form, I used the following sources of evidence:

1. Discursive evidence. I used what students said as my main source of identifying forms. Specifically, I looked for pointing language (e.g., “This is…,” “There is…”) and descriptions of actions they used to create parts of the inscriptions (e.g., “I first drew…,” “I broke…”). Often, the class would also gesture at a feature as they pointed to the drawing. In some cases, students would recreate their drawings on the board. As students recreated and explained their drawings, I would take pauses as indicators of when a form “stopped.” In most cases, students used a

pointer built into their digital notebook to refer to parts of their drawings if they presented their drawing on a screen.

2. Inscriptional resources. In some drawings, the class used resources to highlight different forms within a set of inscriptions such as using attributes by using different kinds of lines or colors to differentiate the features they want to highlight. In addition, they may use other symbols such as brackets to point to the features they highlighted.

An example of coding for forms. In Jack's drawing, I identified a micro form as he re-created his drawing. To illustrate how I coded for forms, I present the first few parts of Jack's drawing with the accompanying utterances. In the transcript, the bold font refers to utterances or gestures guiding my identification of form.

	*Jack*	[Draws the strip] So **this is one serving.** [Labels "1 srv."] This is four fifths of a liter so **we'll break this into four parts.** [Partitions strip into four pieces, labels "1/5"]
	Dr. B	Notice how nicely that uses that Common Core definition of fraction. It's four parts. **What are those four parts? They're each of size one fifth**. So, you really don't- so you just work with those four parts.

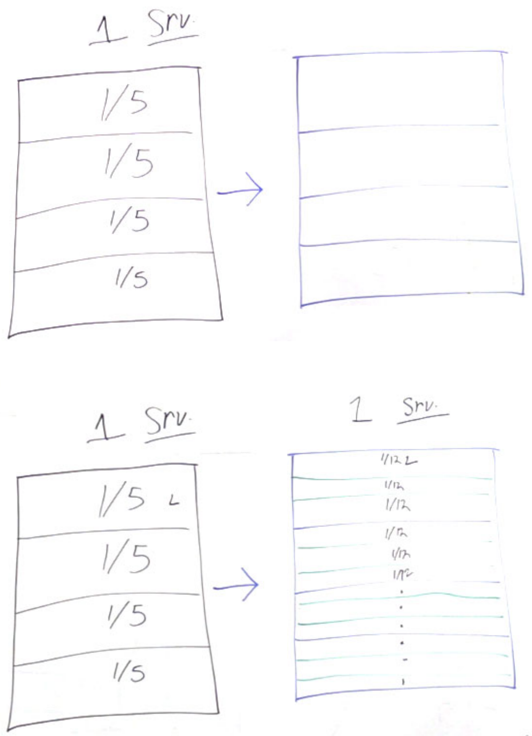
I identified coded Jack's construction of the whole rectangle without the partitions as the form. In addition to his label, Jack said the rectangle indicated a serving. Additionally, Jack identified the entire rectangle, thus I did not code this data too closely (i.e., code each line of the rectangle). Because I did not code anything "smaller" or elements of the rectangle, I took this as one of the smallest forms to code. Next, Jack indicated something to do. He broke or partitioned the rectangle into four parts. This action language helped me identify a second form, the partitions of the rectangle, indicating a shift in attention from the rectangle itself to the partitions of the rectangle. Although one can now argue that the first form, the whole rectangle, can now be broken into smaller forms, four partitions, I still considered this as one of the smallest forms used in Jack's drawing because this rectangle was one of the smallest forms at some point. Moreover, if I no longer considered the whole rectangle as a micro form, I would not be able to capture the different functions of the micro forms as I will explain in the next section.

Functions. Functions are the purposes of forms (Saxe, Dawson, Fall, & Howard, 1996). Similar to defining forms from the literature, the use of function has been varied in the literature. In one sense, functions were characterized as the referent of a form or what the form is intended to represent. For example, Saxe and Esmonde (2005) found different functions for the word *fu* such as *fu* as a complete set of an unspecified amount, *fu* as the 20th position in the base-27 counting system, and *fu* as an act of doubling. Another characterization of function is related to goals (Saxe et al., 1996). Some forms are used to achieve certain mathematical goals such as using a unit interval form to locate an ordinal number on a number line. Finally, forms may serve larger social functions such as using base-27 body system to participate in commercial transactions or using a definitional form to engage in conversation with a class. I used these characterizations to examine function as both a referent and a goal. I found functions in the data using the following sources of evidence:

1. Discursive evidence. For functions as a referent, I used pointing language to find forms and identified what the form is intended to represent (e.g., "This is one serving"). These statements, together with gestures or inscriptional resources, completes the semiotic act of the construction and interpretation of forms allowing the form to represent the object for which it is intended to stand (Werner & Kaplan, 1963). To identify functions as goals, I identified utterances that set the objective or an intent a drawing such as "I am going to show…" or "I need to…"
2. Inscriptional resources. Labels and annotations are included in the drawings to indicate the functions as referents. For example, a student may overlay the label "1/2 cup" over a partition of a rectangle to show the partition refers to 1/2 cup. Other resources could also include annotated equations and tables.

An example of coding for functions as referents. In Jack's drawing, some of his micro forms were used to refer or represent something. Based on what Jack said and his annotations, he specified the whole rectangle is intended to represent one serving. As he labeled his partitions, he wrote each partition represented one-fifth of a liter. The instructor also summarized Jack's drawing, indicating the referent of the partitions as one-fifth. Jack did not refute what the instructor said. This analysis also justifies the coding of the two different micro forms. If the micro form was solely the partitions because they are a smaller grain size than the rectangle, I would not be able to capture the referent of the strip. Jack intended for the two micro forms to refer to two different quantities, the strip as a serving and the partition as a one-fifth liter. Thus, it was pertinent for me to code every form identified with the criteria I listed in order to capture nuances in their functions.

An example of coding for functions as goals. In some cases, forms are created to serve particular mathematical goals, not simply to represent something. Consider the next part of Jack's strategy. Jack said, "So, we need to find two thirds of this four-fifths. So, like I said, four is not necessarily divisible by three so we need to turn it into something that is." "Before you do that Jack," the instructor interjected, "let's all pause and think what the issue is here. So, what is it that we're wanting at this point?" Julie suggested the goal was to divide the picture into three parts, however there are four parts at this moment." Thus Dr. B established the goal, "So everyone see that problem we solve? We have four parts but we really want to divide it into three equal parts, so how to- how to do that?" Jack continued his explanation.

	*Jack*	So we're **going to break each one into three parts** [inaudible] gonna find three parts later. [Draws a new strip with four partitions.]
	Dr. B	Do you want another color? (…) [Jack partitions each partition in the new strip into three parts] And question for everybody why does it make sense to use three parts now? Three- why divide each of the four parts into three?
	Julie	Because three time four is 12.
	Dr. B	Three times four is 12 and what does that do for you?
	Julie	**You can make 12 into three equal parts** [Jack writes “1/12” in some partitions of the new strip]
	Dr. B	Notice there was nothing about common denominators or anything like that. **It's just an issue of partitioning so that we can make three equal parts.**
	Jack	So now four- twelve total with the one serving. Still the one serving. [Writes “1 serv” above new strip]
	Dr. B	So those are twelfths of what?
	Jack	One serving [adds “L” to both one “1/5” and “1/12.”]

Because Jack indicated an action he wanted and did to the drawing, I coded a new form, the second layer of partitions. This is not dissimilar from the first set of forms I identified in the first part of his strategy. That is, Jack took a rectangle and partitioned the rectangle into three parts. Additionally, Jack also assigned the function of this partition as referring to one-twelfth of a liter (Later on in the explanation, he changes this referent to one-twelfth of a serving when suggested by other students). Additionally, the new partitions helped Jack with the initial goal set by him and the instructor in that, they need equal and different sized parts. Thus, another function of these new partitions was not only to represent one-twelfth of a liter, they were also to achieve this goal.

Heuristics for Micro Form Analysis. The goal of this analysis waqs to identify general categories of micro forms and functions students used to create their public displays. To begin analysis, I identified a form or function and created in-vivo codes to associate with each form and function. In-vivo codes are codes using names or utterances by the participants (Strauss & Corbin, 1998). As I created in-vivo codes, I noted similarities and differences with existing codes. I began to collapse the codes to describe larger categories of codes. Based on grounded theory (Strauss & Corbin, 1998), diSessa's heuristic principles (1993), and Sherin's principles to identify knowledge elements, I used the following considerations when collapsing the codes:

1. **Codes must cover all the data.** The set of generated codes must saturate all the data. Each analyzed drawing and its elements must be built from at least one identified form and function. If a drawing cannot be coded, a new code is generated or an existing, related code is modified to include the drawing under consideration.
2. **The amount of codes must be minimal.** By comparing established codes, the number of codes I use must be economical yet sufficient to cover all the data.
3. **Forms and functions must be diverse.** Different students must use the different functions and forms. In other words, if there are forms or functions unique to one student, codes must be modified to include it.

4. **Micro forms must be geometric**. In describing forms, geometric features are an essential part of their descriptions. If a form is perceived to have different geometric features from existing codes, a new code for the drawing must be generated.
5. **Codes must account for continuity and discontinuity.** Because the analysis is meant to capture the use of forms and functions over time, codes must persist throughout a substantial amount of time but because I intend to capture development, I do not expect each code to appear in every lesson. If a code appears to be persistent, I examined the code to see if it could be subsumed into different, existing codes.

Second-level Analysis: Coarse Forms

In the previous example, Jack partitioned a partition in order to address a larger goal. However, it is not just the micro forms helping Jack and the class address the goal, he considered multiple parts of his drawing. After considering multiple pieces of student work, I was able to identify a set of micro forms for both strip diagrams and double number lines. It also seemed that there were instances wherein students consistently bundled or combined certain micro forms. This pattern induced another grain size for forms—coarse forms. I noted all the combinations of the micro forms and when students used these combinations. For example, consider the following drawings (Figure 14).

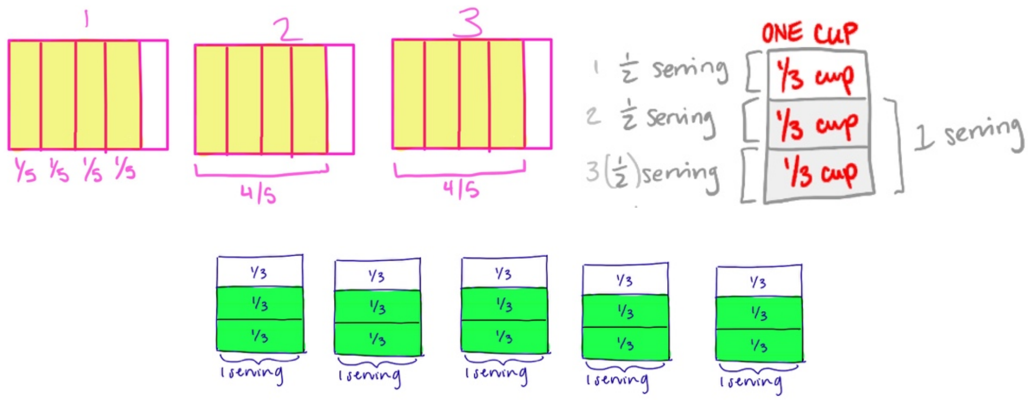


Figure 14. Samples of Coarse Forms.

Three different students created the drawings at different points in time. I identified three different micro forms in each of the: (1) a rectangle referring to one of a quantity, (2) partitions referring to a unit fraction amount of the original quantity, and (3) a set of partitions referring to a fractional amount of the original quantity. Because this set of micro forms appeared in different drawings, I created the code “common core definition” for this bundle of micro forms (i.e., coarse form). The nomenclature of this form stemmed from the common core definition of fraction introduced at the beginning of the year (i.e., *a*/*b* as the quantity formed by *a* parts of size 1/*b* of a unit amount).

Similar to how I coded functions for micro forms, sometimes a coarse form served its own function, usually for a purpose such as solving a measurement division problem. In this example, I show how a coarse form taken together serve done function. In the example where Jack partitioned his partitions further, I took all the inscriptions in that drawing as one coarse form wherein the function of this coarse form was “create equal sized partitions divisible by a number.”

However, methodologically, this presents the biggest impasse. If a coarse form is a combination of micro forms, then there is potentially a very large number of coarse forms. For example, there could potentially be one coarse form for every drawing produced. This would violate the second analytical heuristic (the number of codes must be minimal). I needed to identify a set of coarse forms that could be created from micro forms. I identified coarse forms by identifying breaks or chunks of a student's drawing based on language indicating a sequence of events (e.g., "I first… then…") or by a pause initiated by the instructor. I also found coarse forms when students recreated a drawing or parts of a drawing and added more elements or functions.

To illustrate when I identified coarse forms, consider Jack's strategy and his final drawing for the Second Goblin Goo problem (Figure 12). As seen in his first parts of his strategy, he duplicated the first strip diagram. I coded this as a coarse form because it was a part of the drawing that was recreated. Jack talked through his strategy and layered more elements onto the second drawing (Figure 15).

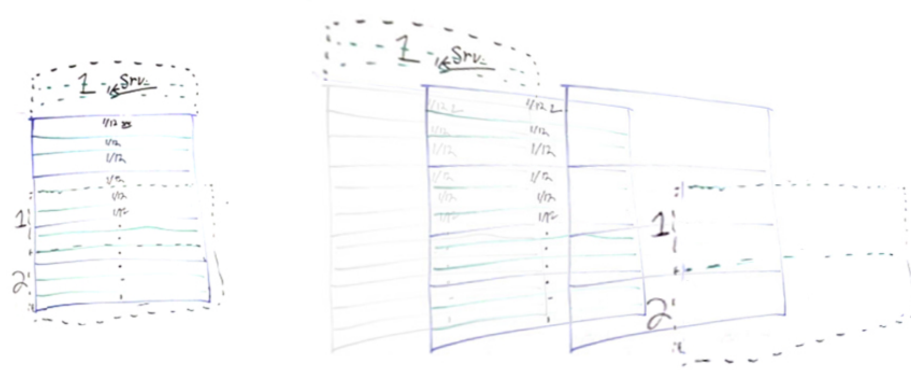


Figure 15. Coarse Forms in Jack's Drawing for Goblin Goo Problem 2.

As presented previously, the first layer of the recreation of the drawing was the partitioned partitions. Before Jack continued his explanation, the instructor interjected and talked about the goal of partitioning the partitions. He next explained that he needed two thirds of the drawing. He explained, "so now it's actually divisible by 12 or

12 can be broken up into three parts, that's four parts each. So that we just find two of the three. [draws the dashed line strip and labels the partitions] One there and one there. (…) So now we have two of our three. That’s two thirds of four fifths.” The instructor once again interjected, “So, does everybody agree the dashed part- that's two thirds of the four fifths liter?” Thus, I coded the new strip made of the dashed line as a coarse form. Finally, Jack added an additional partition and explained, “it’s eight twelfths but that’s still only eight twelfths of our four fifths but not the entire liter so I’d probably like add kind of what like Julie did- little phantom. But I’d probably put it on the same- [draws additional partition]-that’d be a fifth. And now you have your five fifths but since we partitioned it into three, we break it up again, we still have our eight, but then it's eight of fifteen. Yeah, it’d be eight of fifteen for the whole liter instead of eight twelfths of the serving.” I coded the last partition with the strip as the final coarse form in the drawing.

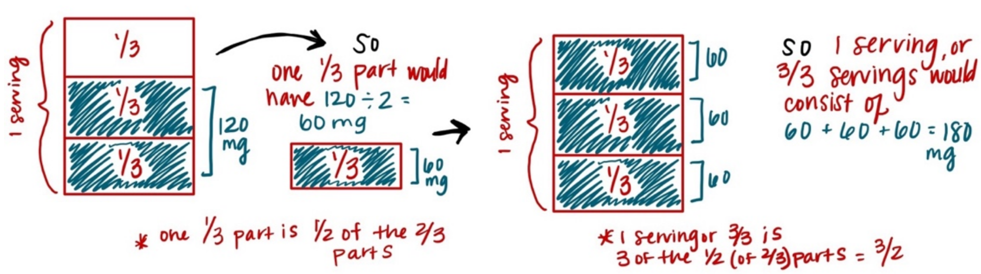


Figure 16. Elizabeth's Drawing for the Noodles Problem.

In some drawings, the coarse forms were more explicit in that students redrew some elements of their drawing to talk about in isolation. As Elizabeth’s drawing for the Noodles Problem (see Figure 16), I identified three coarse forms based on the three sub-drawings in her drawing.

Analyzing Coarse Forms. To code for the coarse forms, I followed the heuristic outlined for micro forms with the exception of (3). Instead of identifying the coarse forms as purely geometry, each coarse form was identified based on the micro forms building each coarse form. This allowed me to find similar coarse forms across students' work and across time.

Techniques for the Ontogenetic Analysis

Although the forms and functions identified from the previous analysis could be attributed to individual students in class, there are significant claims that could be made about the cultural practices of the class. In fact, Saxe described an ontogenetic analysis as the identification of continuities and discontinuities of individuals' displays and how this contributes to community practices. Similarly, Moschkovich (2013) argued in addition to mathematical practices being an individual's activity, they are also "social and cultural, because they arise from communities and mark membership in communities" (p. 295) from a Vygotskian point of view. In other words, the student's activity reflects the community's practices.

Third-level Analysis: Documenting shifts

The goal of this level is to provide a historical lens on the development of the forms and functions over the course of time. In previous analyses, proportion and frequency counts of forms and functions provided researchers with description of shifts in the activity of the class (Saxe et al., 2015; Wawro, 2014). For example, Saxe et al. (2015) documented the frequency of the appearance of certain forms (e.g., unit intervals) in classroom data and noted points in time which the forms seemed to re-emerge even after lessons where the forms were formally introduced. Similarly, I noted frequencies of the forms emerging in students' displays to analyze the emergence and re-emergence of certain forms. Saxe and colleagues also conducted after-class interviews with students who incorrectly solved opening problems. They were asked to solve the problem again to document shifts in student thinking. Saxe and his colleagues argued that these interviews are potential data to explain any shifts in the frequency of forms they documented. The researchers also used a written assessment that was administered at a few points in time to keep track of the development of individual students. Based on their description (Saxe et al., 2015), the assessment instrument did not provide students opportunities to create their own drawings. My analysis addressed some of the issues that arise in the data collection and analytical techniques I have described. First, I relied completely on what the students did and said in class. With graduate students and teaching assistants in class, I was able to probe thinking at each table similar to how Saxe's after class interviews were able to probe student's thinking on tasks posed in class. These data helped me corroborate the analysis of the frequencies of the forms. Moreover, nearly all the drawings I analyzed were created by the students.

Analytical considerations for documenting shifts. To account for shifts in the students' drawings, I needed to consider when shifts occurred. One avenue to document shifts is to identify what is normative practice and when that practice changes. Rasmussen and Stephan (2008), for example, documented shifting normative ways of reasoning in collective argumentation in a linear algebra class. Such methodologies for describing drawings in a similar manner are not common in research, based on my review. To begin the analysis, I first considered frequency counts of the coarse forms. However, comparing frequencies across problems did not provide an accurate analysis of what the students drew. The number of drawings that were produced varied per problem because for some problems, students opted to work as a pair, the camera did not pan to the student, students were absent, or students did not have enough time to complete their drawing during class.

$$\text{ratio of (coarse form)}_a = \frac{\text{number of instances of (coarse form)}_a \text{ observed}}{\text{number of drawings observed}}.$$

Streamgraphs. To describe the overall development, I chose to use Streamgraphs to show the prevalence of coarse forms over time. Using standard data representations was not sufficient to identify shifts in the frequency. A simple bar chart was too complex as each problem would have at least nine bars showing the nine SD coarse forms. A stacked bar chart would show all the coarse forms in one bar however, observing the occurrences of the coarse forms across time was difficult to track because the bar showing a coarse form is not connected across time. I input the ratios in RawGraphs, an online data visualization tool (Mauri, Elli, Caviglia, Uboldi, & Azzi, 2017) to create Streamgraphs. A Streamgraph shows multiple values of data categories across time by creating a “stream.” The Streamgraphs strength lies in analyzing continuity across time, but this is also a limitation of the visualization. A stream begins in the previous problem in order to be at the right width by the problem I want to show. For example, if at problem n there is 0% occurrence of a coarse form, and at problem $n + 1$ there is 20% occurrence, the stream emerges at problem n so that 20% can be the width at problem $n + 1$ which makes it seem as if the coarse form emerged before problem $n + 1$ was posed. This feature poses a level of inaccuracy with the Streamgraph; however, the Streamgraphs were used to track continuities and discontinuities of use of certain coarse forms. The interpretation of Streamgraph is found in Figure 17.

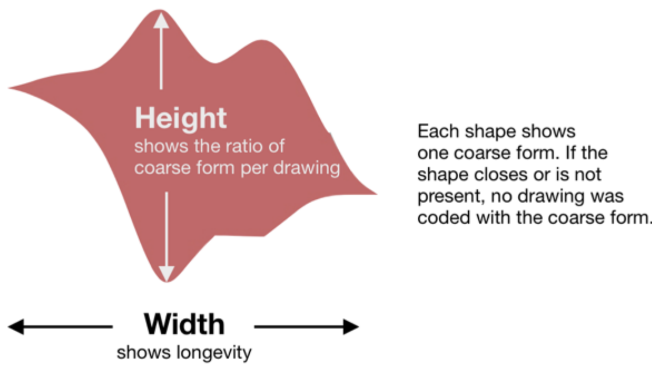


Figure 17. Interpretation of a Streamgraph.

Periods. Periods are time intervals (measured by problem) where students generally drew the same drawing. I used periods in the same manner Saxe and Esmonde (2005) identified three time periods accounting for the development of the form and function of *fu*. In their analysis, they described period where *fu* was generally similar (e.g., *fu* meant 27 or complete.) They identified specific historical events which potentially shaped the development of fu (e.g., Australian contact bringing about the 20-kina coin thus shifting *fu* from 27 to 20).

I identified periods from the Streamgraphs. I defined period as a time interval (as measured by the problems students solved) where students generally used the same coarse forms. Once a new coarse form emerged or re-emerged, I marked this as the beginning of a period and the end of a previous period. I used two heuristics:

1. **Periods are unique.** At least one new coarse form must emerge and at least one task features must be different than the previous period. I understood these differences as shifts in how students created drawings.
2. **Periods must be expansive.** Periods should cover multiple successive problems. In order to use manageable chunks of time, I created the periods spanning more than one problem. Although I could have argued that with each new problem, some shift could occur, this would not be manageable with a data set spanning multiple problems.

Once I identified the periods, I examined task features of the problems posed in the period. When a period changed, this indicated a change in the drawings, thus I assumed there was a feature of the task that prompted a change. In order for a task feature to shape the mathematical drawings of a period, the feature must not have been a feature of the previous period.

CHAPTER 4

RESULTS

In this chapter, I describe the results of the analysis I outlined in Chapter 3. I organize the results by describing the development of both strip diagrams and double number lines. I coded 190 drawings across all the students and 27 problems (Table 7). Methodologically, it was not possible to obtain one drawing for each problem for each student. I was constrained because in some cases students worked in pairs, the camera did not pan to the student, or there was no verbal explanation accompanying the drawing. Figure 18 shows counts of students' use of SD or DNL across all 27 problems.

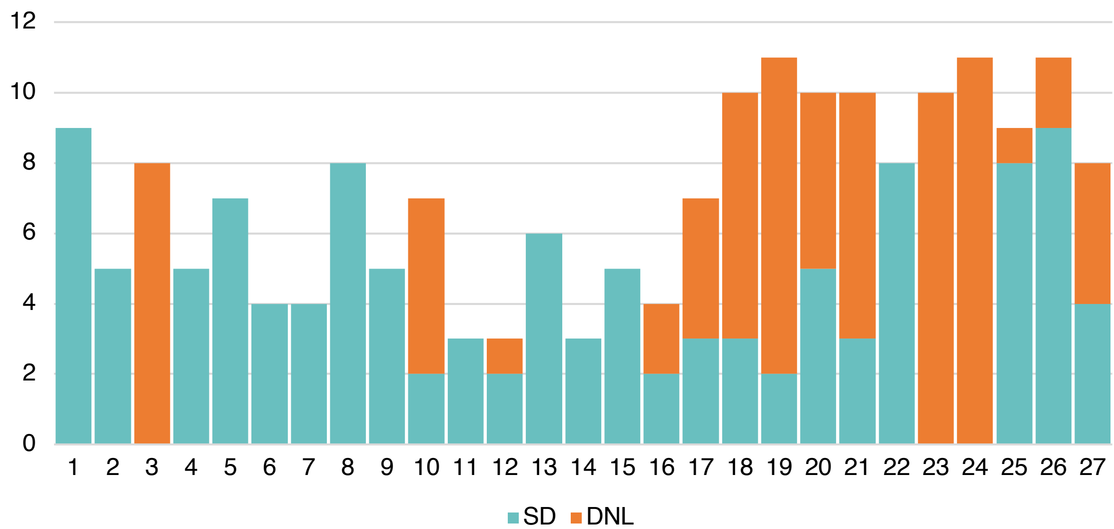


Figure 18. Number of Students who Using SDs and DNLs Across Problems.

In the first two sections of this chapter, I address my second research question: What are the forms and functions of strip diagrams (SDs) and double number lines (DNLs) emerging from a content course for prospective middle school teachers? To answer this question, I separate my analysis by the type of drawing. For each drawing, I provide a list of micro forms and coarse forms. The forms for strip diagrams are found in Table 9 (p. 99) and Table 10 (p. 103) and the forms for double number lines are found in Table 12 (p. 130) and Table 13 (p. 146). I then address the last research question by discussing the ontogenesis of both types of drawings:

RQ3. What features of the mathematical tasks shaped the use of certain forms and functions over time?

To answer this question, I describe the development of both types of drawings, describing both continuities and discontinuities. To describe continuities, I identify conditions under which the forms remained consistent. I identify points of discontinuity by identifying “periods of time” where new forms emerged or old forms re-emerged.

Table 8
Frequency of Coded Drawings by Problem and Student.

	Problem																											
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	**Total**
Alexis	S	S	S	-	S	-	W	B	-	-	S	-	-	-	-	S	W	S	S	S	S	B	B	B	S	S	-	**22**
Andrew*	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	B	S	S	S	W	-	S	S	S	S	S	**11**
Cameron*	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	W	S	S	S	W	S	S	S	S	S	S	**11**
Catherine	W	-	S	-	S	S	-	-	-	B	-	W	-	-	-	-	S	B	S	S	S	-	B	S	-	S	B	**19**
Courtney	-	-	-	-	B	-	B	-	-	S	-	-	W	B	-	-	-	S	S	S	S	-	-	S	-	-	-	**13**
Hannah†	S	S	-	B	B	W	-	B	-	S	-	-	-	-	W	-	-	-	-	-	-	-	-	-	-	-	-	**11**
Elizabeth	-	-	S	W	-	S	S	S	W	B	-	-	S	-	-	-	S	S	B	B	S	-	W	B	B	S	S	**23**
Jack	S	-	-	-	-	-	-	-	-	-	S	-	-	-	-	-	W	S	-	-	-	S	-	-	S	S	-	**7**
Julie†	S	-	W	-	S	-	S	S	-	S	-	-	S	-	S	-	-	-	-	-	-	-	-	-	-	-	-	**8**
Kelsey†	-	-	-	S	S	-	-	S	-	-	-	S	-	S	-	-	-	-	-	-	-	-	-	-	-	-	-	**5**
Lindsay	S	W	S	-	-	-	-	-	S	S	B	-	-	-	-	W	-	W	W	S	-	S	S	S	-	S	W	**16**
Molly	B	-	S	-	-	W	-	S	-	-	-	-	S	B	B	-	-	-	B	W	B	B	B	B	S	B	W	**25**
Nina	S	W	-	B	-	-	-	S	S	-	-	-	-	-	S	W	B	S	S	S	B	B	W	S	S	S	B	**23**
Sophie	B	B	B	B	S	-	B	-	W	-	B	-	S	-	-	W	-	-	B	S	B	S	B	S	S	S	B	**29**
Winnie	-	-	-	S	-	S	-	S	-	S	-	W	S	-	-	-	W	S	S	W	-	S	B	B	B	B	-	**19**
Small Grp	**8**	**3**	**6**	**5**	**7**	**3**	**4**	**8**	**2**	**7**	**4**	**1**	**5**	**3**	**3**	**1**	**4**	**9**	**10**	**9**	**7**	**8**	**8**	**11**	**9**	**11**	**6**	**162**
Whole Cla	**3**	**3**	**2**	**4**	**2**	**2**	**3**	**2**	**2**	**2**	**2**	**2**	**1**	**2**	**2**	**3**	**6**	**2**	**4**	**3**	**5**	**3**	**7**	**4**	**2**	**2**	**5**	**80**
Total	**11**	**6**	**8**	**9**	**9**	**5**	**7**	**10**	**4**	**9**	**6**	**3**	**6**	**5**	**5**	**4**	**10**	**11**	**14**	**12**	**12**	**11**	**15**	**15**	**11**	**13**	**11**	**242**

Note. S = Coded a drawing captured during small-group discussions, W = Coded a drawing captured during whole-class discussions, B = Coded a drawing during both small-group and whole-class discussions.
* indicates a student enrolled in the Spring semester but was not in enrolled in the Fall
† indicates a student enrolled in the Fall semester but was not in enrolled in the Spring

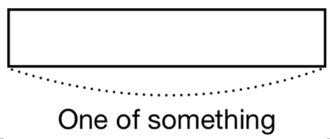
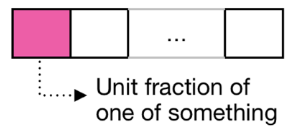
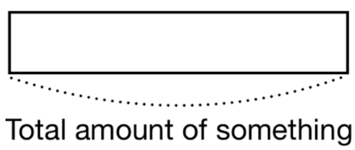
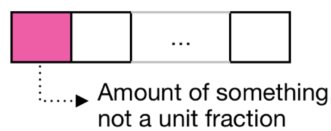
Forms and Functions of the Strip Diagram

The next two sections of this chapter address the second research question presented in Chapter 1:

> RQ2. What forms and functions of strip diagrams and double number lines emerged from student drawings in a content course for prospective middle school teachers?

In the previous chapter, I described two levels of forms that are appropriate for analysis: micro and coarse forms. I outline the micro and coarse forms for the strip diagram in this section. I begin describing the micro forms, the basic inscriptions students used to build their drawings. I describe the six micro forms for strip diagrams and offer evidence of each form. As a caveat to the reader, because micro forms are the smallest grain size I used to describe students' drawings, the examples I present may contain several micro forms. In each example, I isolated the inscriptions within the drawings. The goal of this section is to illustrate the forms and functions. I chronicle the diversity and chronology of the forms in a later section.

Table 9.
Micro Forms for Strip Diagrams.

	Strip as One (S-1)	Partition as a Unit Fraction (P-UF)
		
Form	A strip, rectangle, geometric figure/s	One partition of a rectangle partitioned into equal parts
Function	Represents “one” of a quantity	Represents a unit fractional amount of a quantity
	Strip as an Amount (S-A)	**Partition as an Amount (P-A)**
		
Form	A strip, rectangle, geometric figure/s	One partition of a rectangle partitioned into equal parts
Function	Represents an amount of a quantity but not one	Represents an amount of a quantity but not a unit fraction

Micro Forms for Strip Diagrams

Strip as One (S-1). This micro form is distinguished by a rectangle representing one of a quantity such as one serving or one gallon. I also identified amounts preceded by the word “a” or “the” as one of something such as “a serving” or “the playground.” To illustrate this, Lindsey demonstrated her strategy for Jean’s casserole problem, “Jean has a casserole recipe that calls for 1/2 cup of butter. She has 1/3 cup of butter. Assuming that Jean has enough of the other ingredients, what fraction of the casserole recipe can Jean make?” during whole class discussion. In her drawing (Figure 19), she drew a rectangle and indicated the rectangle represented one cup of butter. She drew a second adjacent rectangle with the same referent, one cup.

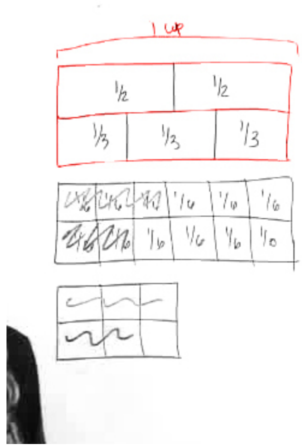


Figure 19. Lindsey's Drawing for Jean's Casserole Problem and the micro form S-1.

Partition as a Unit Fraction. This micro form is characterized by the equal partitions of a rectangle with each partition representing a unit fraction of a quantity i.e., 1/*n*. To illustrate, consider Nina's drawing for one serving for The Blank Multiplication problem, "One serving of ___ is 3/4 ___. You had 2/5 of a serving. How many ___ of ____ did you have?" (Figure 20). She partitioned her strip showing five equal parts. During whole-class discussion, she said, "because I have two-fifths of a serving, I partitioned that serving into five pieces" and highlighted two pieces until the "2/5 serv." Each partition was annotated as one-fifth of a serving at the lower right corner of each partition.

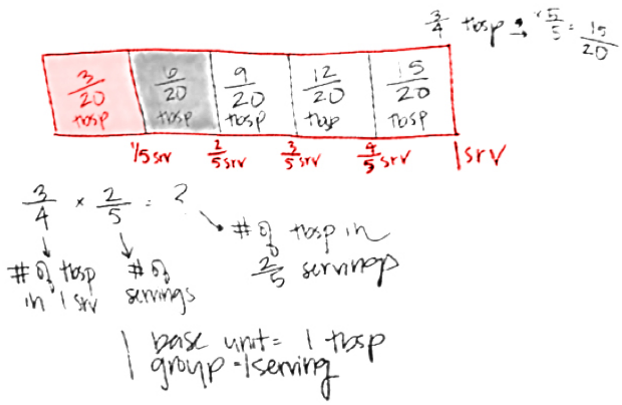


Figure 20. Nina's Drawing for the Blank Multiplication Problem and Micro Form P-UF.

Strip as an Amount (S-A). This micro form is distinguished by a rectangle representing an amount of a quantity aside from one. For example, consider Sophie's drawing for three-fourths of a cup in Figure 21. She drew her diagram with a rectangle to represent the amount three-fourths of a cup as opposed to drawing a strip to represent one cup.

Figure 21. Sophie's Drawing for 3/4 Cup and the Micro Form S-A.

Partition of a Strip as an Amount (P-A). This micro form is characterized by partitions of a rectangle representing an amount of a quantity but not a unit fractional amount. For example, in Nina's drawing in Figure 20, aside from each partition representing a unit fraction of a group, each partition also represented three-twentieths tablespoons (Figure 22).

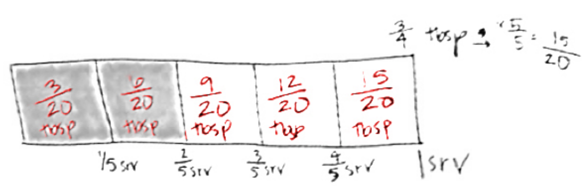


Figure 22. Nina's Drawing and the Micro Form (P-A).

Distinguishing Quantities. In the seventh week of the Fall semester, Dr. B introduced the quantitative definition of multiplication as $N \cdot M = P$ where the value of M is the number of groups, the value of N is the number of units/objects in each group and, the value of P refers to the total number of units/objects in M groups (Beckmann & Izsák, 2015). With this definition, students assigned quantities in the problem as either a group or base unit. With this identification, it was pertinent for me to identify these quantities and how it was represented in the drawings. If there was sufficient evidence, instead of coding "something," I coded the form as a "group" or "base unit." I based my identification of the quantity on tables or other annotations supporting the drawing. I also used verbal evidence when students described their strip as a group or a base unit.

An example of distinguishing quantities. Nina created her drawing (introduced in Figure 20 and Figure 22) after the lesson on the definition of multiplication. She designated a serving as a group as and a tablespoon as a base unit as seen at the bottom of her drawing. The two quantities were represented in different ways in her drawing. I coded the strip as one of a group and the partition as a unit fraction of a group (see red portions of Figure 23). I also coded the strip and the partitions as an amount of base units (see blue portions of Figure 23).

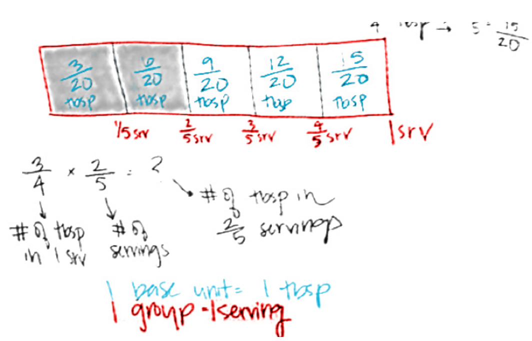
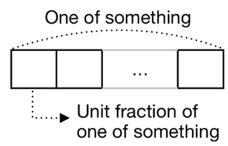
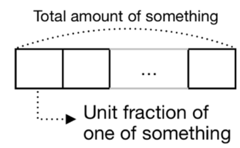
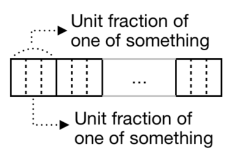
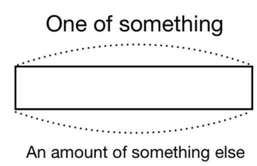
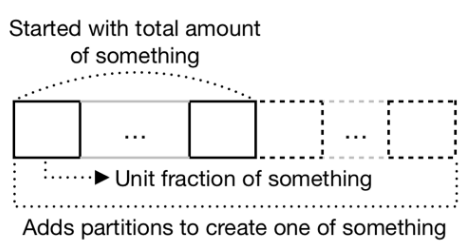


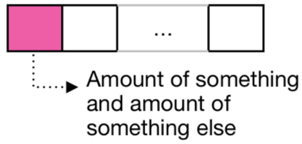
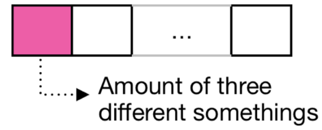
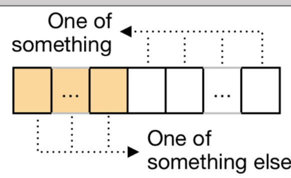
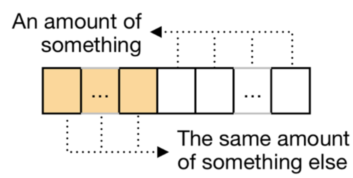
Figure 23. Nina's Drawing for the Blank Multiplication Problem with Two Quantities

Table 10.
Coarse Forms for Strip Diagrams

	Common Core Definition	Common Core Numerator
		
Description	Equi-partitioned rectangle containing a set of highlighted partitions	Equi-partitioned rectangle
Micro-forms	S-1, P-UF	S-A, P-UF
Function(s)	Represent some fractional and unit amount simultaneously	Represent some fractional amount

	Partitioned Partition	Dual function of a Strip
		
Description	Equi-partitioned rectangle with each part partitioned further	A strip, rectangle
Micro-forms	P-UF, P-UF of original P-UF	S-1, S-A
Function(s)	Display common denominator, compare partitions of different sizes, create equal parts, divisibility	Represent the relationship of two different quantities where the amount of one quantity is one

	Phantom Partitions
	
Description	Equi-partitioned rectangle then more partitions are added
Micro-forms	S-A, P-UF then S-1
Function(s)	Determine the size of a partition or partitioned partition

	Dual Function of a Partition	**Triple Function of a Partition.**
		
Description	One partition of an equi-partitioned rectangle	One partition of an equi-partitioned rectangle
Micro-forms	P-UF/A, P-UF/A (different quantities)	P-UF, P-UF, P-UF (different quantities)
Function(s)	Partition simultaneously represents some amount of a quantity and another amount of another quantity	Simultaneously represent some amount of stuff with respect to three different quantities, show keep-change-flip algorithm
	Strip as a Batch	**Variable Parts**
		
Description	Two adjacent, distinct equi-partitioned rectangles	Two adjacent, distinct equi-partitioned rectangles
Micro-forms	S-A, P-A	S-A, P-A
Function(s)	Represent a composed unit with some amount of a quantity and the corresponding amount of another quantity to be iterated	Represents a generalized relationship of some number of parts of one quantity and a number of parts of another quantity

Coarse Forms for Strip Diagrams

This section is divided into nine subsections corresponding to the nine coarse forms emerging from my analysis of the data (see Table 10). Within each subsection, I (a) provide a description of the coarse form including the micro forms comprising the coarse form (b) illustrate the first six coarse forms with student work from either The Playground Problem and The Second Goblin Goo Problem. Each problem reads:

> **The Playground Problem:** At a neighborhood park, 1/3 of the park is to be used for a new playground. Swings will be placed on 1/4 of the area of the playground. What fraction of the neighborhood will the swing area be?

> **The Second Goblin Goo Problem:** You had 2/3 of a serving of goblin goo. One serving of goblin goo is 4/5 liters. How many liters of golden goo do you have?

Finally, (c) I briefly show supporting examples from other problems to provide the reader with diverse examples from different students. The examples closing each section are intended to show that the coarse forms are not isolated to the work in the two problems.

Common Core Definition. This coarse form resembles the Common Core definition of fraction, a formal class definition was introduced early in the Fall semester. In particular, Dr. B introduced the Common Core definition of fraction as "A/B means the amount formed by A parts, each size 1/B of the unit amount" (see Figure 24). This definition is consistent with the definition of fraction offered in the Common Core State Standards for Mathematics, which first appears in the third-grade standard 3.NF.1 as "understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*." Following this definition, this form is characterized by strips or rectangles representing one of something partitioned into a number of partitions based on the denominator of the fraction. Each partition represents a unit fraction (i.e., a strip had *n* partitions with each partition each with the size 1/*n*). The micro forms comprising this coarse form are: Strip as One (S-1) and Partition as a Unit Fraction (P-UF). The function of this coarse form is to represent fractional amounts. In most cases, students would distinguish a subset of the partitions to indicate the amount they intended to represent.

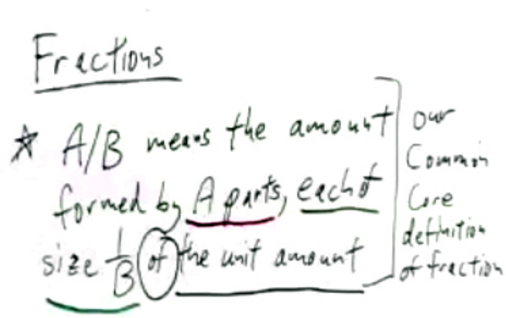
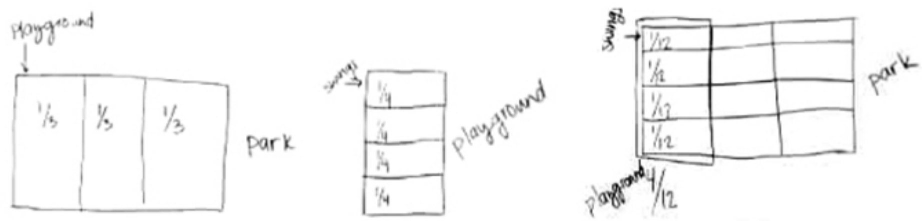


Figure 24. Dr. B's Definition of Fraction

Example. In this example, Molly leveraged this coarse form three times to solve The Playground Problem. During whole-class discussion, she explained her strategy as she recreated her work on the board as seen in Figure 25. She represented the entire park with a rectangle and then “divided the park into three equal parts, each part equal to one third of the entire park.” In this part of her drawing (Figure 25a), I identified the whole rectangle as a Strip as One (S-1), namely one park. I identified each of the three parts of the rectangle as a Partition as a Unit Fraction (P-UF), namely one-third of the park. Next, she disembedded one of the partitions and used it to create another Common Core Definition coarse form (Figure 25b). In this case, her new rectangle or a Strip as One (S-1), specifically the playground, and each of the four equal parts of the rectangle as a Partition as a Unit Fraction (P-UF), specifically one-fourth of the playground. Finally, she drew the park once again and re-embedded the “playground” rectangle back into the “park” rectangle (Figure 25c). Working with the playground rectangle (S-1), she extended her lines to show twelve partitions, each size one-twelfth of the park (P-UF).



(a) (b) (c)

Figure 25. Molly's Drawing for the Playground Problem

Examples from Other Tasks. Recall Nina's drawing for The Blank Multiplication Problem (Figure 23). Her drawing of the groups or servings also illustrated the Common Core Definition coarse form in that the entire strip represented one serving (S-1) and each partition was labelled as a unit fraction of a serving at the lower right corner of each partition (P-UF).

At the end of the year, the class was presented with the Punch Proportion Problem, "If you mix fruit and bubbly water in a ratio of 3 to 5 to make a punch, then how many liters of fruit and liters of bubbly water will you need to make 10 liters of punch?" Molly presented her drawing (Figure 26) during whole-class discussion and used the Common Core Definition coarse form in her second strategy. She said, "so when you take the one eighth of all of [the strip]… and then five of those parts are the water so, one part would be one-eighth times five so that's just looking at the water right here [points to blue partitions] and what portion of the whole container [points to the whole strip]" to describe how she used the fraction 5/8. It is worth noting that Molly called the whole rectangle a container (S-1) thus, when she was talking about one part as one-eighth of the container (P-UF). Additionally, she talked about the five blue partitions containing five parts, each one-eighth of the container (P-UF).

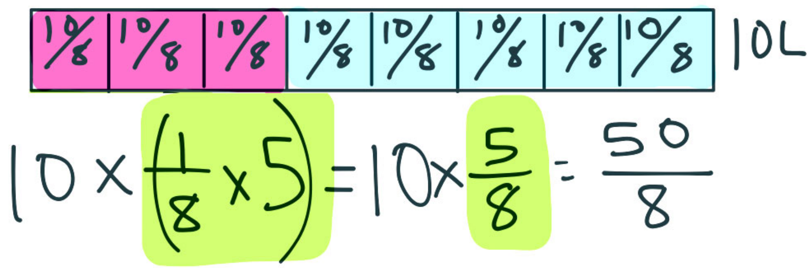


Figure 26. Molly's Drawing for the Punch Problem.

Common Core Numerator. This coarse form also resembles the Common Core definition of fraction; however, the unit amount is not drawn. This form is characterized by a rectangle representing an amount aside from one. The rectangle is also partitioned into a number of partitions based on the numerator of the fraction and each partition represents a unit fraction (i.e., a strip has *m* partitions with each partition each with the size 1/*n* and $m \neq n$ to represent the fraction *m*/*n*). The micro forms comprising this coarse form are: Strip as an Amount (S-A) and Partition as a Unit Fraction (P-UF). Similar to the previous form, the function of this coarse form is also to represent fractional quantities.

***Example*.** Jack used this coarse form at the beginning of his strategy for The Second Goblin Goo Problem (Figure 27) when he represented the amount of liters given. He first drew the rectangle and said, “So this is one serving” while writing “1 srv” above the rectangle. He continued, “This [rectangle] is four fifths of a liter so we'll break this into four parts” and proceeded to partition the rectangle into four pieces and label each piece “1/5.” I identified the rectangle representing four-fifths of a liter as Strip as an Amount (S-A). Additionally, each partition represented a unit fraction of a liter (P-UF). At this point in his strategy, Jack did not draw all the partitions to complete one liter; instead, he chose to draw the number of partitions based on the numerator.

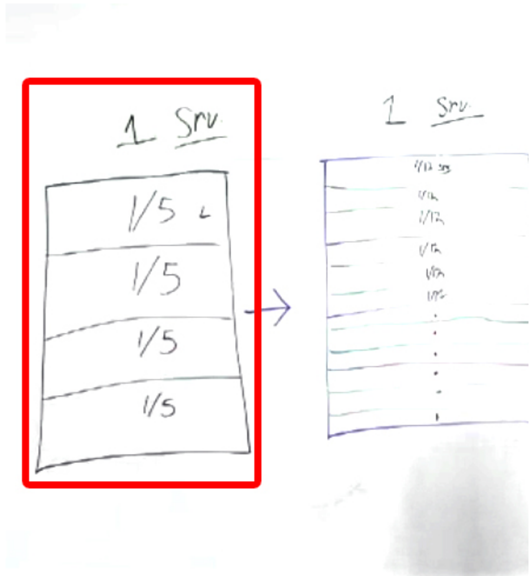


Figure 27. Jack's Drawing for 4/5 L in The Second Goblin Goo Problem.

Examples from Other Tasks. In the middle of the Fall, Sophie created a drawing for The Blank Multiplication Problem (Figure 28) for three-fourths of a cup. She drew a strip and described the entire strip as three-fourths cup (S-A) during small group discussion. She then partitioned her strip into three parts, she labelled a partition as one-fourth of a cup (P-UF). Similar to Jack, Sophie did not draw all four one-fourth partitions to show a whole cup.

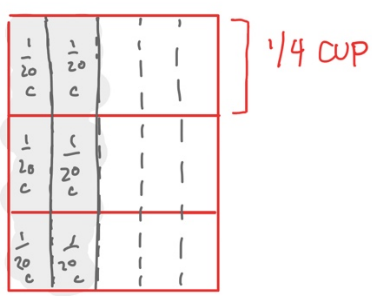


Figure 28. Part of Sophie's Drawing for the Blank Multiplication Problem.

Partitioned Partition. This coarse form resembles the two common core forms, but all partitions are partitioned into smaller, equal-sized parts where the partition's referent is the original quantity. This form is characterized by partitioned rectangles where each partition refers to a unit fraction. These partitions are partitioned further resulting in new unit fractions whose referent unit is the original quantity. The micro forms comprising this coarse form are Partitions as a Unit Fraction (P-UF) in a set of Partitions as a Unit Fraction (P-UF).

Functions and Examples of Partitioned Partitions. Unlike the previous coarse forms where each coarse form served one function, this coarse form served four functions: display common denominators, compare partitions of different sizes, create equal-sized parts, and create a number of parts divisible by a certain number. Before I describe the functions, I present an example of the form emerging during a discussion of The Playground Problem. Sophie presented her strategy, beginning with the drawing in Figure 29a. Similar to Molly's drawing in Figure 25, Sophie drew a rectangle representing the park, partitioned the rectangle into thirds of a park (P-UF), then she partitioned the third into fourths and described this new partition with the original referent, one-twelfth of the park (P-UF in a set of P-UF). Because Sophie's sets of partitions (thirds and twelfths) were both in reference to the area of the park, her drawing showed partitioned partitions.

To illustrate the four functions, I highlight certain portions of the whole-class discussion of Sophie's work. The first part of the discussion emphasizes the first two functions for the Partitioned Partition as seen in the bold text in the transcript below.

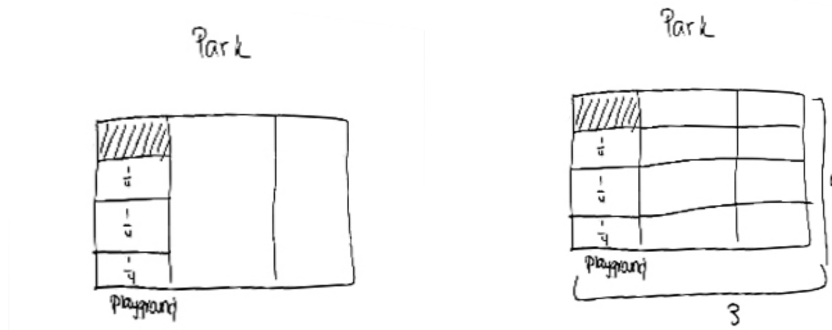


Figure 29. (a) Sophie's Initial and (b) Final Drawing for the Playground Problem.

Sophie: So we start out with the same park and then it says its divided into one thirds of the park is the playground and then the one fourth of the playground is the swings… and then instead of drawing three separate drawings thought of just making each one of the one third into fourths as well because it's hard to **compare one fourth with two thirds** so I just-

Dr. B: So let's just stop there for a second because I think this is a really crucial point… a lot of you had this drawing and then at this point you knew what the answer was gonna be and some of you were wrestling with… right this second, what is the issue? And I think Sophie said it, but someone say it again. What is the issue right this moment?

Alexis: There's not 12 parts there's just four plus the three.

Dr. B: Yeah. There were the three parts [motions at the whole rectangle], there's the four parts [motions at the left partition], and we know this [motions at the shaded part] is the stuff were interested in, right? We know- but what's the problem?

Julie: **We don't have a common denominator** so **it's hard to compare things that are unequal size or unequal measurements.**

Dr. B: Yeah and **this table was thinking least common multiple- common denominator.** We kind of know that there- these things are gonna be relevant in this- when we're working with fractions but if I look at this- put your kid eyes on- **do you see common denominators here? Do you see least common multiples here? I don't see it.**

Julie: I was thinking about like **apples to oranges or something like that**.

Display Common Denominators. The first function of this coarse form is to show the results of a memorized rule, obtaining common denominators as seen in line 17, where Julie stated that all the partitions should be partitioned further to show common denominators. Dr. B recalled some small group conversations about using common denominators in line 19-20 as a reason to show twelve partitions. In line 22-23, she pushed on this reasoning by arguing that the current drawings do not show this.

Compare Sizes. The second function emerged from the rationale that one "cannot" compare parts of different sizes. As Sophie mentioned in line 5 and Julie in line 18 and 24, fourths and the thirds are different. In line 18, Julie explained that her comparisons were based on the size of the parts and emphasized that the partitions are different things using an analogy in line 24.

Sophie explained her reasoning for Figure 29b, "If you can split of this one third of the park into fourths you can do it to the other thirds as well…[and] each of those sections is one twelfth." To extend the conversation, I asked the whole class to consider if someone said the answer to the problem was one-sixth. I asked why one could get this answer and how might they resolve it. In this conversation, a third function for partitioned partitions emerged.

Hannah Because to a kid they might think that like it's not proportional- y'know what I'm saying? Like, there's only six boxes that the swings would be one sixth because even though you've divided into one thirds then the thirds are supposed to be divided into one fourths and that's how it should all along they might not think that just because it's a whole box **even though it's not the same size**.

Jack The might not understand the whole fraction concept**. They just see the whole parts- they're just seeing there's six blocks and there's one colored of the sixth.** They might not understand that that's a fourth of a third. they just see one of six, total.

Sophie And that goes back to what **Julie was saying about you can't compare apples and oranges.** You can't say that this little box [shaded box] is comparable to those bigger boxes because **they're not the same units.**

TA So, if you were to say the definition of fraction right here, where would you run into a problem?

Sophie In this one?

TA Yeah, so say this is one part of six so just say is-

Sophie So the swings would not be one part- **they're not equal size of one sixth.** So the definition of a fraction is one part of the whole equal parts, the size one sixth and **this is not equal size to the other parts.**

Create Equal Sized Partitions. The third function of partitioned partitions is to create equal sized partitions when a strip is partitioned into different sized partitions. Jack explained that using an "out of" definition is not sufficient to understanding "the whole fraction concept." Sophie and Hannah argued that one crucial part of the drawing is knowing that prior to partitioning the partitions, the parts were not equal sizes. Thus, the park should be partitioned to show the same size partitions, in this case, partitioning the other thirds into fourths would create equal sized partitions of twelfths.

Divisibility. The final function of partitioned partitions was to create a number of partitions divisible by a certain number. I found this function in Jack's partitioned partitions for the Second Goblin Goo Problem in Figure 27 as he moved from his first strip to his second.

Jack So, we need to find two-thirds of this four-fifths. So like I said, **five is not necessarily divisible by three so we need to turn it into something that is.**

Dr. B Yeah and before you do that Jack, let's all pause and think what the issue is here. So, what is it that we're wanting at this point?

Jack So we're going to break each one into three parts [inaudible] gonna find three parts later (…)

Dr. B And question for everybody why does it make sense to use three parts now? Three- why divide each of the four parts into three?

Julie **Because three times four is 12.**

Dr. B Three times four is 12 and what does that do for you?

Julie **You can make 12 into three equal parts.**

Dr. B Notice there was nothing about common denominators or anything like that. It's just an issue of partitioning so **that we can make three equal parts.**

Jack explained his strategy by first identifying that five parts cannot be divide into three parts in line 2. He may have misspoken because he later explained he can now get three equal parts from twelve parts and not four parts. Julie expanded Jack's idea by saying partitioning the partitions further to create twelve parts is beneficial because she can identify three equal parts (lines 12 and 14).

Examples from Other Tasks. Sophie used Partitioned Partitions while solving The Blank Multiplication problem. In her drawing (Figure 30), she explained she drew her whole rectangle as a serving or three-fourths cups by partitioning the rectangle into three parts, each representing one-fourth of a cup (P-UF). She disembedded one of the partitions and partitioned this partition further. She said, "I zoomed into just that one little thing to show that one fourth of a cup and partitioned into five equal parts, each of the smaller parts is one twentieth of a cup." In this case, she described the partition of the partition in reference to the original quantity, cups (P-UF in a P-UF). She created the Partitioned Partition for divisibility. In this case, she needed five smaller parts from one part, thus partitioning the partition into five parts gave a number of parts divisible by five. She partitioned each partition separately and counted the number of purple parts to get six-twentieths cup.

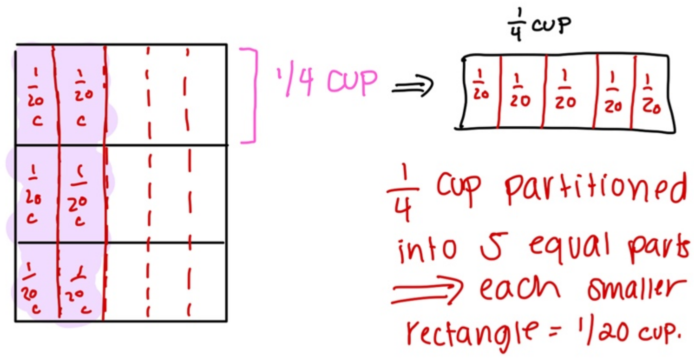


Figure 30. Sophie's Drawing for The Blank Multiplication Problem

In the Spring, Jack explained why he used Partitioned Partitions for the problem "Write a simple how many groups word problem for 1 ½ ÷ 1/3 and solve the problem with the aid of a strip diagram." During small group, he drew three-halves in Figure 31a by drawing a strip indicating one liter of apple juice, partitioning each strip into halves (P-UF), and shading three of the partitions. As he was talking with Elizabeth, he said, "You have to change the halves into sixths and thirds into sixths so you can divide 'em up… you're going to have to change it into sixths, that way- something that goes into two." This indicated the function of his partitioned partitions was to create a number of partitions divisible by both two and three. His final drawing (Figure 31b) displays partitions of size one-sixth of a liter (P-UF in P-UF). During whole-class discussion, he explained, "I was gonna use common denominators because halves and thirds don't mix perfectly." Further, he counted thirds by counting groups of two-sixths (red partitions in Figure 31b). In his final display and explanation, Jack's Partitioned Partitions was drawn to show common denominators. Similar to Julie's explanation in the previous example, Jack also indicated he wanted to compare two sized parts, halves and thirds.

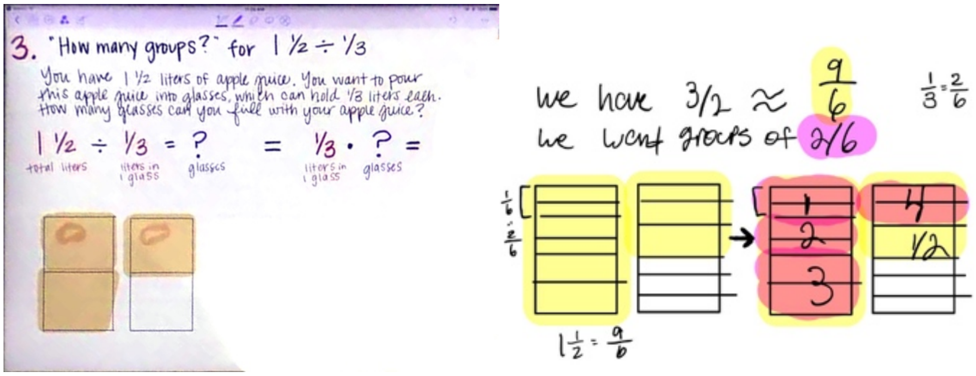


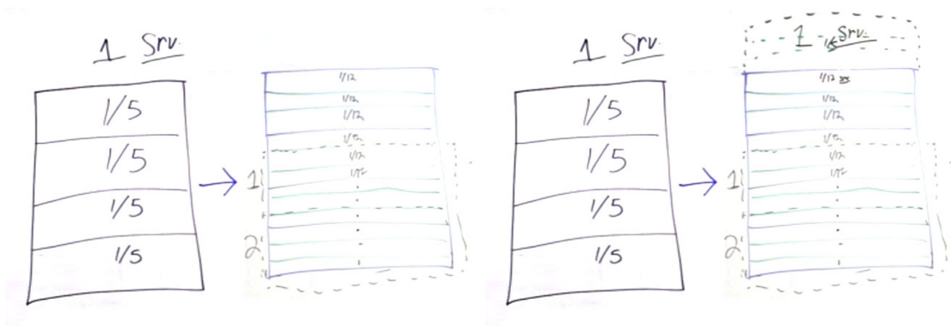
Figure 31. Jack's (a) Initial and (b) Final Drawing for 1 ½ ÷ 1/3.

Phantom Partitions. This coarse form is characterized based on a sequence starting with partitioned rectangles indicating an amount less than one. The partitions of the strip represented unit fractions. Additional partitions are drawn to create one of the original amount. The micro forms comprising this coarse form are: Strip as an Amount (S-A) then Strip as One (S-1). In most cases, the function of this coarse form was to determine the size of a partitioned partition.

Example. Recall Jack's drawing in Figure 27 representing four-fifths of a liter (S-A) with partitioned partitions to find two-thirds of four-fifths. He overlaid a strip with two partitions using dashed lines (Figure 32a). As he drew the dashed lines, he said, "The two thirds- so now it's actually divisible by 12 or 12 can be broken up into three parts, that's four parts each. So that we just find two of the three. [draws dashed lines] One there [writes "1" next to a partition] and one there [writes "2" next to the other partition]." In this dashed strip, one partition represents one-third of a serving and contains four partitions, each one-twelfth of a serving. He then explained he needed to describe the partitioned partitions in terms of liters not servings,

> I'd probably like to zoom out and focus on the- since it's eight-twelfths but that's still only eight-twelfths of our four fifths but not the entire liter so I'd probably like add kind of what like Julie did- little phantom. But I'd probably

put it on the same- [draws additional partition] -that'd be a fifth. And now you have your five fifths but since we partitioned it into three, we break it up again, we still have our eight, but then it's eight of fifteen. Yeah, it'd be eight of fifteen for the whole liter instead of eight twelfths of the serving.



(a) (b)

Figure 32. Jack's Drawing for the Second Goblin Goo Problem

Jack drew an additional partition in order to draw out the entire liter (S-1). He also noted the partition must also be partitioned into three, similar to the other partitions, thus each of the smaller partitions are one-fifteenth of a liter.

Examples from Other Tasks. Sophie drew her strategy for The Dragon Blood Problem, "One serving of dragon blood is 1/5 of a liter, but you only want 1/3 of a serving. How many liters of dragon blood is that?" (Figure 33). She first drew rectangle representing one-fifth liter (S-A). She stated, "if we had five servings, we would have one liter." She then proceeded to add more partitions (Phantom Partitions) to complete the strip for one whole liter (S-1). She then wrote, "we have 15 parts total. If we want 1/3 of 1 serving, that is 1/15 of our 1 liter."

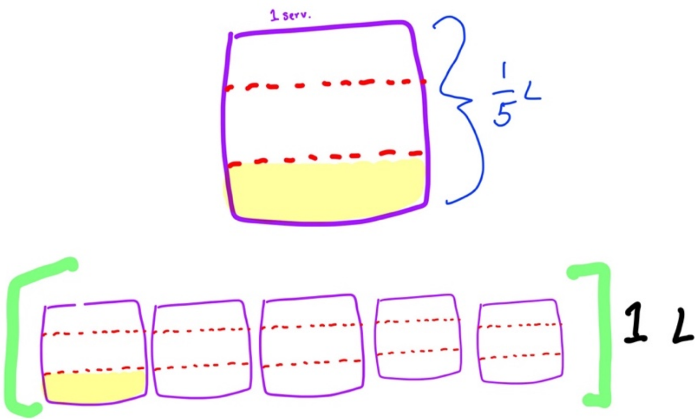


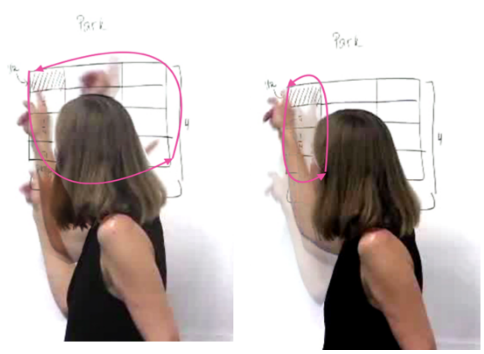
Figure 33. Sophie's Drawing for the Dragon Blood Problem.

Dual Function of a Strip. This coarse form is characterized by one rectangle representing two quantities, unlike the previous coarse forms wherein only one quantity is drawn. The micro forms comprising this coarse form are: Strip as One or an Amount (S-1/A) Strip as One or an Amount of another quantity (S-1/A). The function of this coarse form is to represent two amounts of tow different quantities simultaneously.

Example. In most examples I presented, students started their drawing using this coarse form. In Figure 27, Jack drew the rectangle to represent amounts of different quantities, four-fifths of a liter (S-A) and one serving (S-1) simultaneously. In Figure 28, Sophie drew the rectangle to represent both three-fourths of a cup (S-A) and one serving (S-1). She also drew one serving (S-1) as one-fifth of a liter (S-A).

Dual Function of a Partition. Similar to the previous form, this coarse form relates two quantities in the problem albeit the partition is considered, not the strip. This coarse form is characterized by partitions described with two different quantities. Dr. B emphasized the importance of how a partition can represent the same "stuff" but be described in two different ways, more specifically how the value of a partition is different, depending on the quantity on chooses. The micro forms comprising this coarse form are: Partition as a Unit Fraction or an Amount (P-UF/A) and the same Partition as a Unit Fraction or an Amount else (P-UF/A).

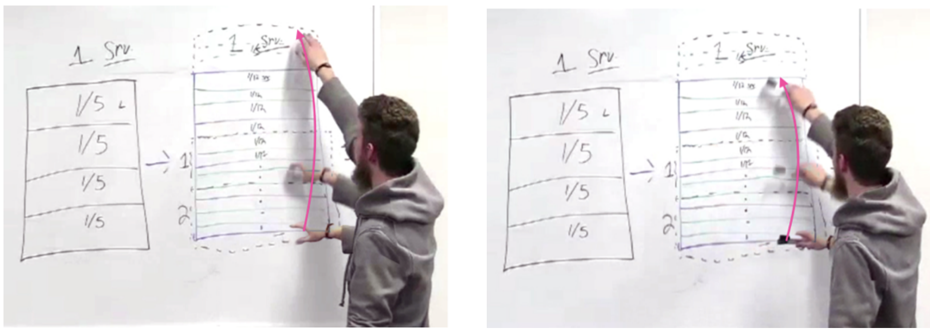
Example. Consider the discussion of the playground problem. Dr. B gestured over Sophie's picture (Figure 34) and cautioned the students, "It's the same area- it's the same region- it's the same stuff. It's a twelfth of the whole park (Figure 34a)." In this case, the partition is a unit fraction of the park (P-UF). She added, "It's only a fourth of the playground (Figure 34b)" also showing the partition as a unit fraction of the playground (P-UF) So, it's the same region. You can use the two different fractions to describe that same thing, but you have to be really clear about what is it of… you're not gonna be effective word problem solvers without this careful, careful attention to the unit amount."



(a) (b)

Figure 34. Dr. B's Gestures Over Sophie's Drawing for the Playground Problem

Similarly, as Jack ended his discussion of his drawing for the Second Goblin Goo Problem, he described partitions with respect to two quantities and gestured over his drawing. Referring to the eight partitioned partitions, he said, "we still have our eight, but then it's eight of fifteen. Yeah, it'd be eight of fifteen (Figure 35a) for the whole liter instead of eight-twelfths of the serving (Figure 35b)." Jack described the eight partitions enclosed by the dashed lines in two ways, as an amount of a serving (P-A) and as an amount of liters (P-A).



(a) (b)

Figure 35. Jack Gestures Over His Drawing for the Second Goblin Goo Problem

Examples from Other Tasks. In the Fall, Elizabeth created the drawing in Figure 36 for the Dragon Blood problem. Elizabeth's initially drew one serving of dragon's blood. After determining how many partitions she needed, she partitioned her serving into three parts, she labelled her partitions in two ways. First, she wrote 1/15 on top and 1/3 on the right of the strip on the right. In this case, she assigned two different quantities and amounts to the same partition, 1/15 of a liter (P-UF) and 1/3 of a serving (P-UF).

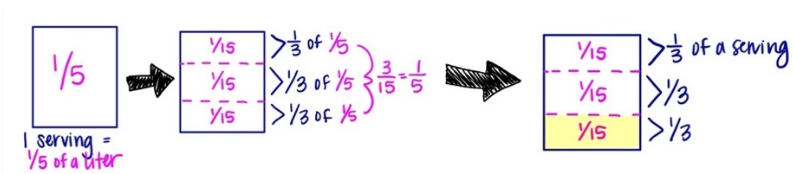


Figure 36. Elizabeth's Drawing for the Dragon Blood Problem.

In the Spring, Catherine worked on Anna's run problem, "Running at a steady pace, Anna ran 6 miles in 3/4 of an hour. At that pace, how far will Anna run in one hour?" She drew a strip showing one of a group (Figure 37). She assigned a subset of the partitions in the strip to show the size of the group in the problem, in this case, three-fourths of an hour where each partition represented one-fourth hour (P-UF). She drew her second strip with the total amount of miles in 3/4 hours. She explained she also knew this was six miles. She then drew one-fourth of an hour and made the equivalence that this partition is also two miles (P-A). She iterated this partition to complete the whole strip and counted the amount of base units in the whole strip.

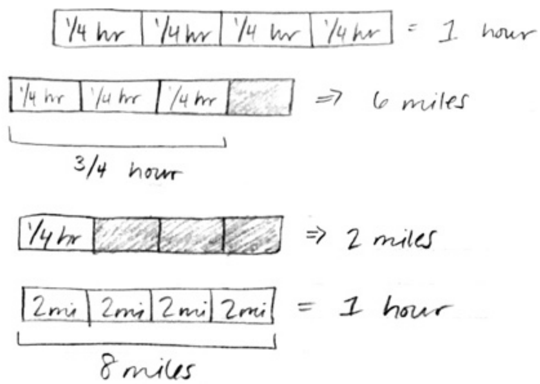


Figure 37. Catherine's Diagram for Anna's Run Problem.

Triple Function of a Partition. Similar to the Dual Function of a Partition, this coarse form also highlights how one can describe the same "stuff" or partition in multiple ways but in this case, with respect to three different quantities. The micro forms comprising this coarse form are the same as the Dual Function of a Partition with Partition as a Unit Fraction or Partition as an Amount of a third quantity (P-UF/A).

Example. Consider Elizabeth's explanation the Blank Multiplication problem and her explanation justifying 1/3 ÷ 2/5 = 1/3 · 5/2. Elizabeth drew the strip diagram in Figure 38 to show her thinking for 1/3 ÷ 2/5 = ? and explained her thinking in whole-class discussion. She created a partitive division word problem "A third of a pound of chicken is enough for 2/5 of a bowl of chicken soup. How many pounds of chicken is in 1 whole bowl of chicken soup?" Elizabeth's drew a strip on the left functioning one bowl of soup. She also partitioned the strip into five parts and annotated her parts as 1/5 of the bowl and “colored in two of the fifths and called that a third of a pound.” The set of partitions referred to the size of the group (i.e., 2/5 of the bowl), but also the corresponding quantity in base units (i.e., 1/3 pound of chicken). Using the function of the partition, she described one partition in two ways, as 1/5 of the bowl (P-UF) and 1/6 of a pound of chicken (P-UF) as seen in the middle of Figure 38. She iterated this part to build the whole bowl of soup and kept track of both quantities simultaneously to get 5/6 pounds of chicken in the whole bowl.

To explain the equivalence 1/3 ÷ 2/5 = 1/3 · 5/2, Elizabeth described the situation considering two groups—the original group of one bowl and a new group of 2/5 of a bowl. Considering this new group, she explained the partition is also one-half of two-fifths of the bowl (P-UF). This activity indicated a new function for a partition in addition to denoting a unit fractional amount of a group and base unit. She used the partition as a unit fractional amount of the size of the group of the total amount of base units in addition to one-fifth of the bowl and one-sixth of the pound. In other words, one partition refers to 1/6 pound of chicken, 1/5 of the bowl, and one-half of two-fifths of a bowl. She counted the five partitions in the whole strip and used the new function to get one bowl as five halves of two-fifths of the bowl. Using the new group, Elizabeth created the expression 1/3 · 5/2 following the definition of multiplication used in class. In the annotation, she explained there is one third pound of chicken in one of the new group, two-fifths bowl (amount in one group, *N*) and there are five-halves of the new group in the whole bowl of soup (number of groups, *M*). In summary, Elizabeth’s group changed from one bowl to two-fifths of a bowl when asked to explain keep-change-flip. Because of her new group, she added a new function to one partition. By using the class definition of multiplication, she annotated her thinking when she considered the new group to obtain the expression 1/3 · 5/2.

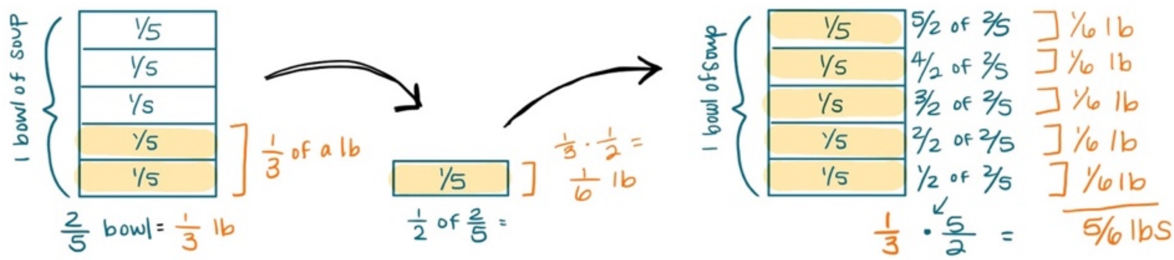


Figure 38. Elizabeth's Strip Diagram for 1/3 ÷ 2/5 = 1/3 · 5/2.

Winnie and Cameron similarly used three different quantities to describe a partition when asked to explain how they can use the reciprocal of the divisor for the Noodles problem, "2/3 of a serving of noodles contains 120 mg of sodium. How much sodium is in one bowl of noodles?" First, Winnie analyzed the problem and identified servings as a group and mg as her base units (see the table and equation in Figure 39). In their drawing, they noted three different quantities for the partition. Cameron annotated their drawing by writing three labels for the partition. He wrote 60 mg above the partition (P-A). In the partition he wrote "1/3 or 1/2 of 2/3." During whole-class discussion, he explained

> You have one-third of a serving (P-UF), two-thirds of a serving, and three-thirds of a serving and then if you know that two-thirds is the 120, that this one-half is half of two-thirds (P-UF) so that's where the reciprocal relationship comes in. So, one-third is half of two thirds, two-thirds is two halves of two-thirds and then three-thirds which is our one cup or three halves of two-thirds.

To create the whole serving, Cameron initially highlighted the first two partitions as both two-thirds of a serving and 120 mg. To build the whole serving, he iterated half of the given amount three times. He then described the resulting partition in three different ways: 60mg, one-third of a serving, and one-half of two-thirds of a serving.

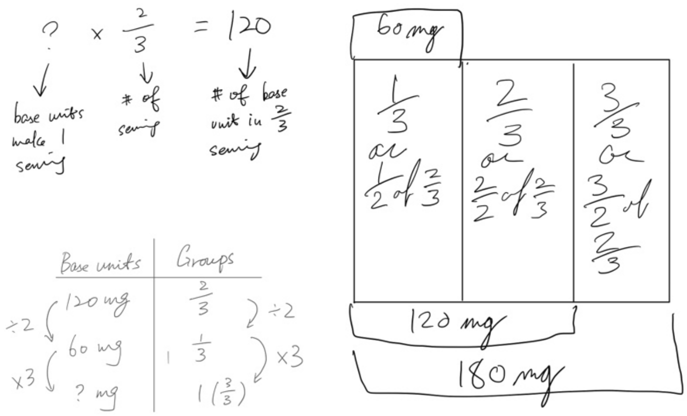


Figure 39. Winnie and Cameron's Drawing for the Noodles Problem.

Strip as a Batch. This coarse form emerged only towards the end of the lessons on proportional relationships. This form is characterized by combining two strips, each strip representing different amounts of different quantities albeit with the same unit (e.g., *x* cups of juice and *y* cups of water). In most cases, the strips were partitioned wherein each partition represented one of the quantitites. The micro forms comprising this coarse form are two Strips as an Amount (S-A) and Partitions as an Amount, in this case one (P-A). The function of this strip is to represent a composed unit which was iterated to show different amounts of the composed unit.

Examples. In the first lesson on proportional relationships, the PSMTs were given a set of black and white beads and the hot cocoa problem. They were asked to determine if a hot cocoa mixture with two drops of chocolate and three drops of milk had the same flavor as a mixture with eight drops of chocolate and 12 drops of milk (see (Noelting, 1980a)). Students initially described rearranging the set of eight black beads and 12 white beads in four groups with two black beads and three white beads to show the new mixture had the same flavor (see Figure 40).

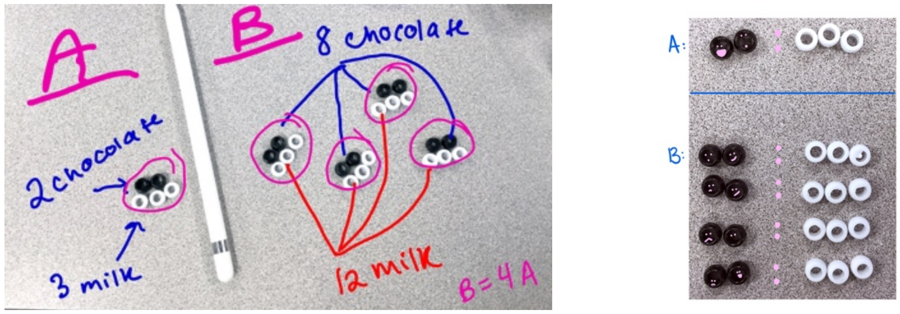


Figure 40. Students' Initial Arrangement for the Hot Cocoa Problem.

The following day, PSMTs were asked to make drawings of the hot cocoa problem that captures multiple mixtures, not just eight drops of chocolate and 12 drops of milk. Consider Cameron and Jack’s drawings (Figure 41). In both their drawings, they identified three element—a strip containing two smaller strips which are differentiated by the color (two S-A). Each partition in this case was one ounce chocolate and one ounce of milk (P-A). The strips were then iterated four times to show eight ounces of chocolate and 12 ounces of milk.

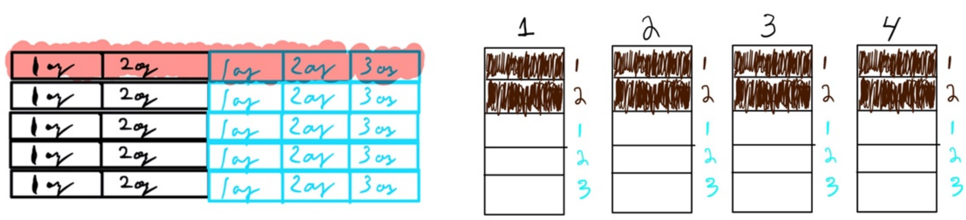


Figure 41. (a) Cameron and (b) Jack's Drawings for the Hot Cocoa Problem.

Variable Parts. Similar to the previous form, this form is characterized by combining two strips, each strip representing different amounts of different quantities albeit with the same unit. Each strip is partitioned an each partition represents any amount of a quantity. Perhaps, this indicates some result of generalizing actions in order to extend beyond the initial case of partitions representing one of something to a broader set of cases (Ellis, 2007). The micro forms comprising this coarse form are two Strips as an Amount (S-A) and Partitions as an Amount (P-A). Because the partitions do not hold a specific amount, the function of this form is to show the relationship between the quantities identified in the ratio given without necessarily considering the amounts.

Examples. In Figure 41a, Cameron drew the case where the hot cocoa recipe consists of 10 oz of chocolate and 15 oz of milk. Towards the end of the lesson on proportional reasoning, Dr. B asked the students to revisit the hot cocoa problem and "draw something to capture… a whole bunch of mixtures all at once" perhaps to promote some generalizing actions. To solve the Hot Cocoa problem, Alexis first drew a strip as a batch highlighted in pink in Figure 42. She explained that as she drew out the next set of amounts, she "just kept the number like the four remains constant all the way through… all the way, there's always two parts and three parts and the number in it remains constant." In this case, her four drawings displayed four different sets of amounts for the hot cocoa. The rectangles represented amounts for chocolate and milk (two S-A's) and each partition represented the same amount of chocolate and milk (P-A). Moreover, Alexis expressed a generalization across all four sets of strip diagrams in that the strip for chocolate contains two parts and the strip for milk contains three parts and that all the parts contained the same amount. Alexis enacted generalizing actions Ellis (2007) to describe the partition, a crucial part of her explanation not present when students reasoned with strips as batches.

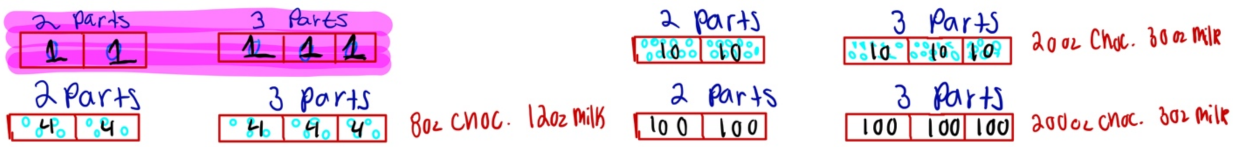


Figure 42. Alexis's Drawing for the Hot Cocoa Problem.

Elizabeth drew a "key" while working on the second Yellow and Blue Paint problem: Green paint is made in a ratio of two parts blue paint to three parts yellow paint. If you want 150 pails of green paint, how many pails of blue paint and how many pails of yellow paint will you need? She said she first drew the key (the boxed strips in Figure 43). "Up here, one green paint pail would have two parts blue paint and three parts yellow paint. It didn't really matter how many of the- in a group... so using that, you can pretty much put in whatever size you want." By stating this, Elizabeth stated a common property across the different Yellow and Blue Paint problems. Ellis (2008) called these types of statements "reflection generalizations" which are statements describing some rule or generalization. In the generalization, Elizabeth stated there are always two parts blue paint (S-A) and three parts yellow paint (S-A) and that the partition does not represent a specific amount (P-UF/A). She then applied this relationship to a specific amount of paint. First, she "knew we had 150 green paint pails" and that the green paint is "gonna have this two to three ratio so, in total you have five parts. It doesn't matter if two of them are blue and three of them are yellow. There's five parts. So, to find out how much is in each of those parts, you would divide 150 by five and you get 30." To find the amount in each part, Elizabeth's divided 150 by five to obtain 30 pails in each part. She then counted the number of blue and yellow pails using each of the different colored strips.

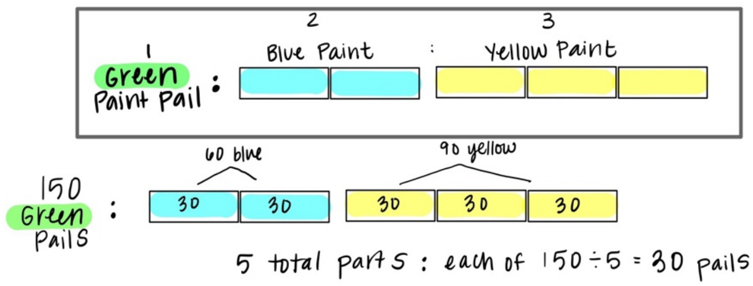
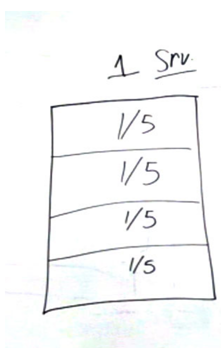
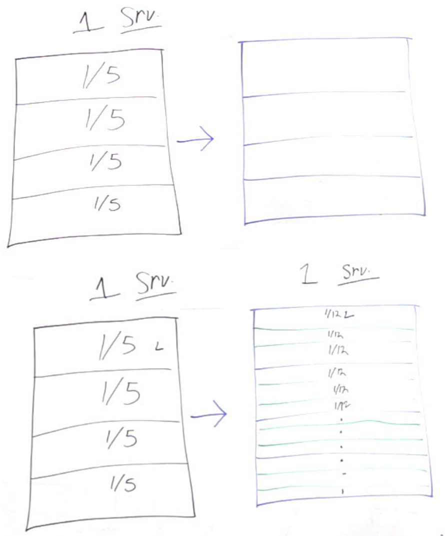
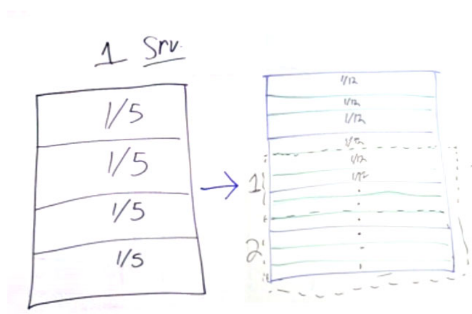
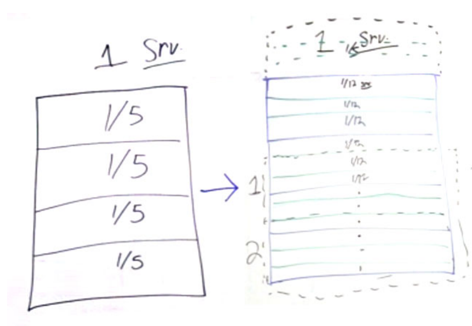


Figure 43. Elizabeth's Drawing for the Second Yellow and Blue Paint Problem.

An illustration using coarse forms. The coarse forms are intended to cover part or all of the inscriptions of a student drawing. To demonstrate how the coarse forms comprise a student's drawing, I provide a sample transcript and coding for Jack's Strategy for the Second Goblin Goo problem in Table 11.

Table 11.
Jack's Strategy for the Second Goblin Goo Problem and Corresponding Coarse Forms

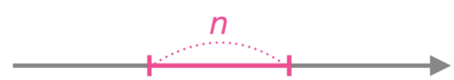
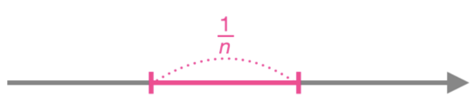
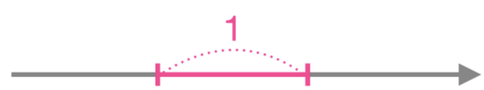
Drawing	Transcript	Codes
	[Draws the strip] So this is one serving. [Labels "1 srv."] This is four fifths of a liter so we'll break this into four parts. [Partitions strip into four pieces, labels "1/5"]	**Dual Function of a Strip:** Strip is some amount of liters and one serving. **Common Core Numerator, base units:** strip shows an amount of liters but not the unit

	So, we need to find two thirds of this four fifths. So like I said, four is not necessarily divisible by three so we need to turn it into something that is… So we're going to break each one into three parts [inaudible] gonna find three parts later. [Draws a new strip with four partitions.]	**Partitioned Partition, base units:** partitions are partitioned further with the **function** to create a number parts of parts divisible by three.
	The two thirds- so now it's actually divisible by 12 or 12 can be broken up into three parts, that's four parts each. So that we just find two of the three. One there and one there. (…) So now we have two of our three. That's two thirds of four fifths.	**Common Core Numerator, groups:** dashed line strip shows an amount of servings but not the unit
	I'd probably like to zoom out and focus on the- since it's eight twelfths but that's still only eight twelfths of our four fifths but not the entire liter so I'd probably like add kind of what like Julie did- little phantom. But I'd probably put it on the same- [draws additional partition] -that'd be a fifth. And now you have your five fifths but since we partitioned it into three, we break it up again, we still have our eight, but then it's eight of fifteen. Yeah, it'd be eight of fifteen for the whole liter instead of eight twelfths of the serving.	**Phantom Partitions, base units:** additional partitions added to determine the size of the partition

Forms and Functions of the Double Number Line

In this section, I continue to answer the first research question by outlining the micro and coarse forms for double number lines (DNLs). I structure this section similar to the previous section by first describing and illustrating the micro form and then describing and illustrating the seven coarse forms I identified for DNLs. I end the section with an example showing how the coarse forms were used to code students' work.

Table 12.
Micro Forms for Double Number Lines

	Line as a Quantity (L-Q)	**Interval as an Amount (I-A)**
		
Form	A ray	An interval enclosed by two tick marks
Function	Represent a certain quantity	Represents an amount of a quantity but not one or a unit fraction
	Interval as a Unit Fraction (I-UF)	**Interval as One (I-1)**
		
Form	An interval enclosed by two tick marks	An interval enclosed by two tick marks
Function	Represents a unit fractional amount of a quantity	Represents a one of a quantity
	Lines as Correspondence (LC)	
		
Form	A line connecting two rays	
Function	Connects two corresponding amounts of two quantities	

Micro Forms for Double Number Lines

In this section, I outline the five micro forms for DNLs I identified (see Table 12). All the examples in this section are taken from student work for The Rope Problem: If 3 yards of rope weigh 2 pounds, then how much do the following lengths of the same kind of rope weigh? 18 yards, 16 yards, 14 yards.

Line as a Quantity (L-Q). This micro form is characterized by drawing a ray (henceforth called a "number line" or "line"), usually with an annotation indicating the quantity the line represented. As illustrated in Figure 44, Elizabeth drew out two lines, her top line representing yards and the bottom representing pounds.

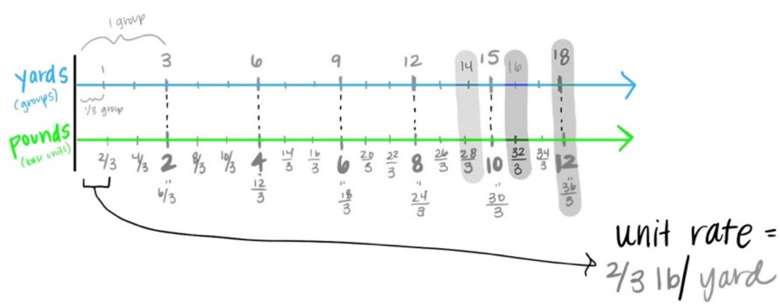
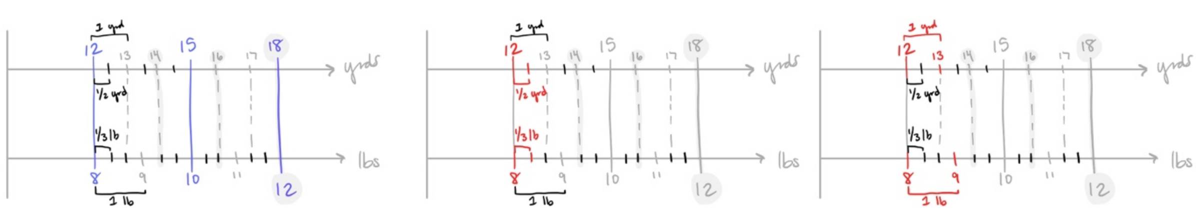


Figure 44. Elizabeth's drawing for The Rope Problem.

The next three micro forms are intervals representing different amounts. These micro forms are characterized by drawing two tick marks on a line and enclosing a length. To illustrate the three intervals, I use the same drawing from Catherine's work from The Rope Problem but highlight different parts of her drawing.



(a) (b) (c)

Figure 45. Different-Sized Intervals in Catherine's Drawing for the Rope Problem

Interval as an Amount (I-A). This micro form is characterized by an interval representing an amount but not a unit fraction or one. In Figure 45a, Catherine drew two intervals. On the top number line, the amount enclosed by each of the blue intervals is three yards. On the bottom number line, the amount enclosed by the blue intervals is two pounds. Neither of these intervals represents a unit fraction or one.

Interval as a Unit Fraction (I-UF). Similar to the previous micro form, I characterized intervals indicated by enclosing a length but this micro form's function is to represent a unit fractional amount. In Figure 45b, Catherine drew two unit fraction intervals. On the top number line, the amount enclosed by each of the red intervals is a unit fraction, specifically one-half yards and one-third pounds.

Interval as One (I-1). Finally, I characterized intervals indicated by enclosing a length representing a unit or an amount of one. In Figure 45c, I highlighted Catherine's two unit intervals. On the top number line, the amount enclosed by each of the red intervals is one yard and one pound.

Line as Correspondence (LC). When students drew two distinct lines showing two different quantities, they would often use a line to connect pairs of quantities to indicate correspondence. In Figure 46, Winnie drew lines to indicate correspondence with amounts of pounds and amounts of yards. She first drew out the pounds and corresponding yards based on the ratio given in the problem (i.e., multiples of two pounds and multiples of three yards). She then drew dashed lines to indicate correspondence. Additionally, she indicated 13, 14, 16, and 17 yards on the yards line. With each of these amounts, she drew a line from each whole number yard and marked the corresponding location on the pounds number line with a tick. During whole-class discussion, she said she found that each interval marked by the ticks was two-thirds pounds because two divided by three is two-thirds. In the second example, Catherine drew out her DNL similar to Winnie where she first drew out intervals based on the given ratio and drew a line to indicate correspondence. She also identified some correspondences with the pink line. When she drew these lines, she said she estimated where the corresponding amounts were located.

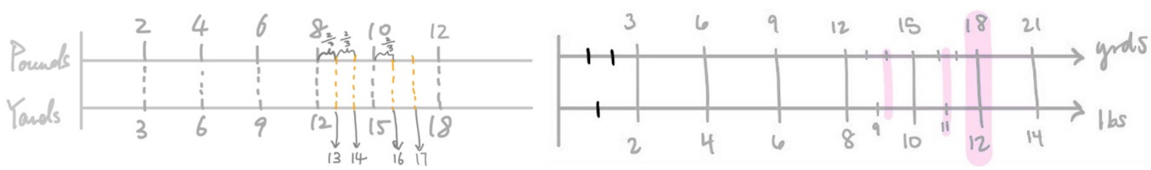
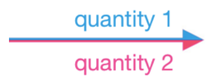
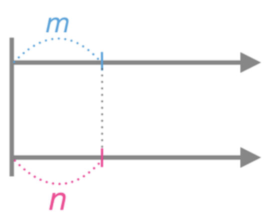
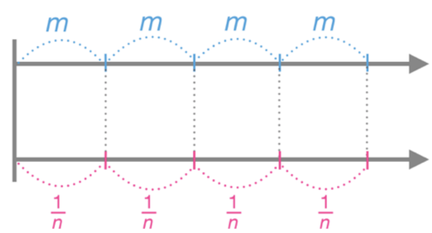


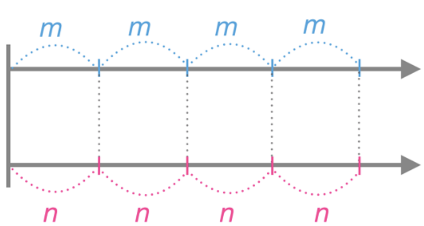
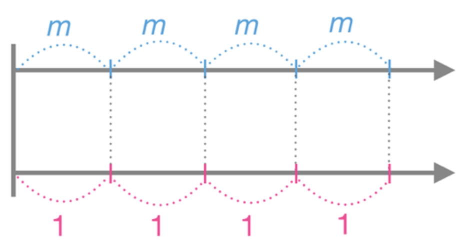
Figure 46. Winnie and Catherine's Correspondence Lines

Distinguishing Quantities. Similar to SDs, students assigned quantities in the problem as either a group or base unit after being introduced to the definition of multiplication. Thus, I also coded the form as a “group” or “base unit” when I had data to support the identification. I based my identification of the quantity on tables or other annotations supporting the drawing. For example, in Figure 44, Elizabeth explicitly identified yards as her group and pounds as her base units. I also used verbal evidence when students described their strip as a group or a base unit.

Table 13.
Coarse Forms for Double Number Lines

	Journey Line	Local Partitoned Intervals	Partitioned Intervals at Zero
			
Description	A ray	An equi-partitioned interval with neither endpoint at zero	An equi-partitioned interval with one endpoint at zero
Micro-forms	L-Q, L-Q	I-A, I-A/UF/1	I-A, I-A/UF/1
Function(s)	Represent two quantities simultaneously	Find corresponding amounts of two quantities	Create an interval to increment

	Amount Correspondence	Unit Fraction Increments
		
Description	Two corresponding amounts on separate number lines	Double number line with one ray with unit fraction increments
Micro-forms	Two I-A/UF/1 sometimes CL	I-A/UF, I-UF, sometimes CL
Function(s)	Represent the relationship between two amounts of two different quantities	Find a corresponding quantity of a certain amount

	Composed Unit Increments	Unit Increments
		
Description	Double number line with one ray with equal group increments	Double number line with one ray with increments of one
Micro-forms	I-A, I-A, sometimes CL	I-A, I-1, sometimes CL
Function(s)	Find a corresponding quantity of a certain amount	Find a corresponding quantity of a certain amount

Coarse Forms for Double Number Lines

This section is divided into seven subsections corresponding to the seven coarse forms outlined in Table 13. Within each subsection, I (a) provide a description of the coarse form including the micro forms comprising the form (b) illustrate the coarse forms with student work from the following problems:

> **Sue's Run Problem:** So far, Sue has run 1/4 of a mile but that is only 2/3 of her total running distance. Plot Sue's total running distance and determine how many miles it is.
>
> **The Hot Chocolate Problem:** Make a drawing to show a whole bunch of mixtures of chocolate in that same 2 to 3 ratio from the multiple batches and variable parts perspective.
>
> **The Scooter Problem** A scooter is going 3/4 of mile every 4 minutes at a constant speed. How far does the scooter go in the following number of minutes: 12 minutes, 17 minutes?

In some cases, I briefly show supporting examples from other problems to provide the reader with diverse examples from different students. I provide a more detailed analysis of the diversity and coverage of the coarse forms in the next section.

Journey Line. This resembled the number lines drawn in Hall and Rubin's study (see Figure 9). The Journey Line is characterized by one ray that represents two different quantities simultaneously. The micro forms comprising this coarse form are two fused Lines as a Quantity (L-Q). The function of this coarse form is to show the relationship of two amounts of two different quantities. Notably, this coarse form was perhaps prompted by the presentation of Sue's Run Problem where a number line was given for students to plot their answers.

Example. Sue's Run problem was presented to the students on a handout (see Appendix) as a line marked "miles" (L-Q) with one tick mark labelled "1/4." Students annotated the drawing to produce Journey Lines. Consider Nina's work in Figure 47. She first marked the tick mark above 1/4 as "2/3 of ~~run~~ total" on the same line (second L-Q). At her table, they partitioned the space between the given tick mark into two intervals and labelled each interval as "1/3" The group discussed the need for a fraction equivalent to 1/4 in order to determine how many miles 1/3 of the distance Sue ran. They first considered 3/12 because 12 is the least common denominator of the two fractions. Courtney then explained they could not use 3/12 because three cannot be partitioned into two. Nina proposed using the fraction 2/8 is more appropriate because it can be partitioned into two. She then wrote 2/8 underneath 1/4 and labelled each of the 1/3 intervals as 1/8.

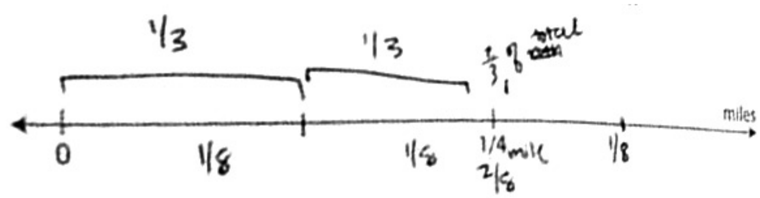


Figure 47. Nina's Drawing for Sue's Run Problem.

Examples from Other Tasks. Molly drew a Journey Line when she worked on Francine's Rope Problem which reads, "Francine has 32 yards of rope that she wants to cut into 8 equal pieces. How long will each piece be?" (Figure 48). During whole-class discussion, she explained she started with the blue number line to represent the 32 yards of rope (L-Q). She then used red tick marks to indicate her pieces of rope, each 4 yards (second L-Q) to obtain eight pieces of rope.

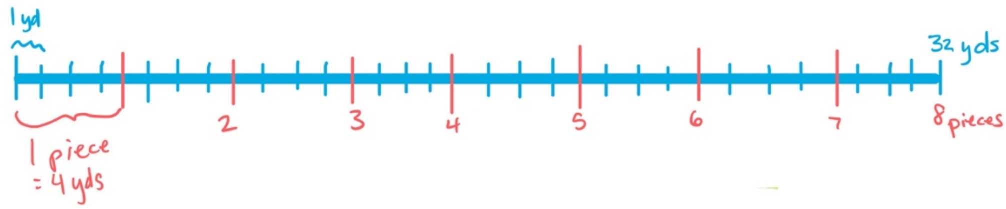


Figure 48. Molly's Drawing for Francine's Rope Problem.

Local Partitioned Intervals. coarse form is characterized by partitioned intervals however neither of the endpoints of the partitioned interval is at zero. The micro forms comprising this coarse form are a set of Interval as Amounts, Unit Fraction, or Unit embedded in one of Interval as Amounts, Unit Fraction, or Unit. The function of the coarse form is to partition one amount in order to find the corresponding amount of the other quantity.

Example. Catherine initiated a strategy and DNL for The Scooter Problem, looking for the number of miles corresponding to 17 minutes. She first "zoomed in" to where she knew 17 minutes was located, in between the interval from 16 to 20. She began by labeling the first interval from three to the first tick mark as 1/4 miles (I-UF) on the top line. She also knew that 18 was exactly in the halfway between 16 and 20 and drew a Correspondence Line from the miles line to 18 on the minutes line (Figure 49a). In Figure 49b, she then drew two more Correspondence Lines from the miles line to the 17 and 19 on the minutes line and indicated 17 was halfway between 16 and 18. Similarly, 19 was halfway between 18 and 20. She states that her goal was to find the number that corresponded to 17 minutes and highlighted the Correspondence Line at 17 minutes. She then annotated an interval from one of the red tick marks to the Correspondence Line connected to 18 as 1/8 because the Correspondence Line equally partitioned the interval in between the two red tick marks of length one-fourth in two (I-

UF in I-UF). This prompted her to also partition each interval between two red tick marks in two, each of length 1/8. At this point she expressed confusion: She could not determine the size of the smallest interval. I asked two classmates to discuss this dilemma with her. Cameron confirmed that the Correspondence Line was in between an interval of length 1/8 on the miles line. He argued that if the Correspondence Line was halfway between the 16 and 18, then the distance between the first black tick mark and the Correspondence Line must be 1/16. Catherine annotated her drawing accordingly (Figure 49c). Catherine then drew out additional tick marks, partitioning the intervals of length 1/8 into two partitions. Cameron suggested verifying that the lengths were consistent by seeing if there were four intervals of length 1/16 in the interval labelled 1/4 and two intervals of length 1/16 in the interval labelled 1/8. Catherine agreed.

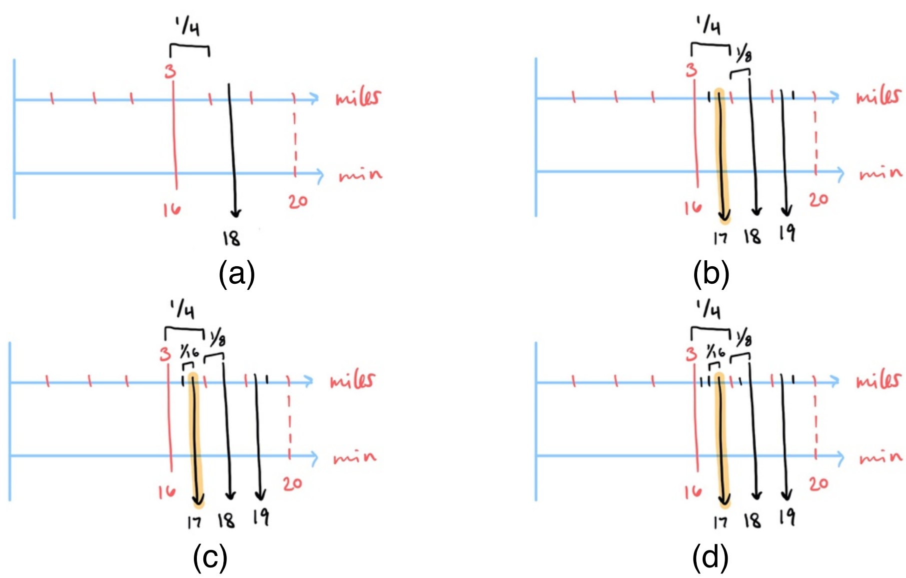


Figure 49. Catherine’s Drawing for The Scooter Problem.

Examples from Other Tasks. In creating a drawing (Figure 50) for The Rope Problem, Winnie first drew a DNL with multiples of two on the pounds line (I-A) and multiples of three on the yards line (I-A). She then drew the locations of 13, 14, 16, and

17 on the yards line saying, “I partitioned the space between 12 and 15 into three equal parts… so I want the weight of each this small part” where a small part referred to the intervals created by the Correspondence Lines. She found the size of the interval of the small parts, “because two pounds is the total pounds we have and three is the number of yards we have a 2/3–is the proud in one yard. So each of this one small part is two thirds pounds” (I-A). She then wrote two equations above the DNL to describe how she obtained two-thirds.

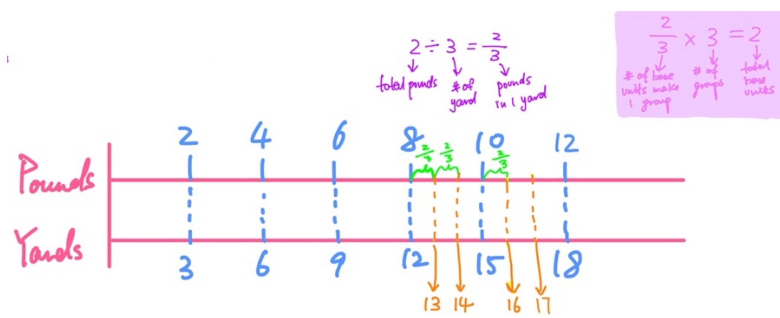


Figure 50. Winnie's Drawing for The Rope Problem.

Partitioned Intervals at Zero. This coarse form is similar to the previous coarse form; however, one endpoint of the interval is at zero. The micro forms comprising this coarse form are a set of Interval as Amounts, Unit Fraction, or Unit embedded in one of Interval as Amounts, Unit Fraction, or Unit. The function of the coarse form is also to find the corresponding amount of a quantity. The result of finding the corresponding quantities is subsequently iterated to obtain another amount wither from zero or another amount.

Example. While working on the Hot Chocolate Problem, Andrew drew out a DNL. As he explained during whole-class discussion, he started by “breaking apart the three ounces chocolate and three ounces milk” by partitioning the interval between zero and two ounces on the chocolate line resulting to two equal sized intervals: between zero and one (I-U) and between one and two (I-A). He similarly partitioned the interval from zero to three ounces on the milk line. Explaining why he partitioned both amounts, he said, “we got to one base unit of ounces of chocolate, we can kind of build that up by one and every time an ounce of chocolate goes up, you get three-halves ounces of milk to go along with it each time” indicating he could iterate these two amounts simultaneously.

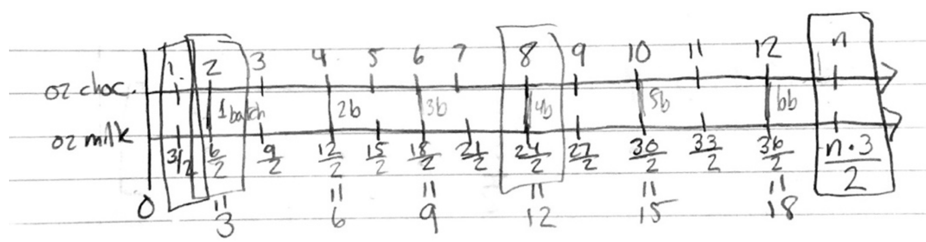


Figure 51. Andrew’s Drawing for the Hot Chocolate Problem.

Examples from Other Tasks. Lindsay created a problem for 1/3 ÷ 2/5 = ? “If there are 1/3 cup of sugar in 2/5 serving of cereal, how much sugar is there in one bowl of cereal?” She first drew a strip diagram to think through the problem. Upon further prompting, she started to draw a DNL. She first drew a tick mark labeled 2/5 on her line for the serving, and beneath it she drew a tick mark for 1/3 on her line for sugar and connected it with a Correspondence Line. She then partitioned each interval into two parts, similar to Andrew (i.e., she partitioned the interval from zero to 2/5 into an interval from zero to one-fifth (I-UF) and one-fifth to two-fifths (I-UF)). After partitioning, she knew she wanted to obtain one serving is 5/6 cup. She was confused how to obtain this. Dr. B walked over and suggested iterating the interval from zero to one-fifth servings five times, similar to how Lindsay drew the strip diagram for this problem.

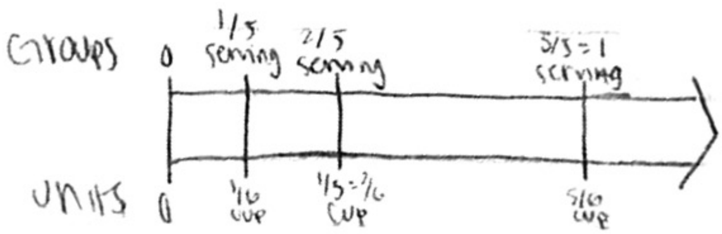


Figure 52. Lindsay's Drawing for 1/3 ÷ 2/5 = ?

Amount Correspondence. This coarse form and the rest of the coarse forms are drawn with two distinct number lines. It is characterized by tick marks on both number lines with their corresponding amounts and usually connected with a Correspondence Line. In some of the previous examples, one of the first inscriptions students drew on their DNLs was a correspondence between two quantities given. The micro forms comprising this coarse form are an Interval as Amounts, Unit Fraction, or Unit on one Line as a Quantity and Interval as Amounts, Unit Fraction, or Unit on the other Line as a Quantity. The function of the coarse form is to display the relationship between two amounts of different quantities.

Example. When explaining her drawing for Sue's Run Problem, Sophie first drew out the a DNL as seen in Figure 53. She drew out a tick mark for one-fourth of a mile (I-UF) and two-thirds of a run (I-A) and connected them with a Correspondence Line. Similarly, when Winnie drew out her DNL for the Scooter Problem, she first indicated the first correspondence in red by connecting the tick mark at three-fourths of a mile (I-A) and four minutes (I-A) with a Correspondence Line (Figure 54).

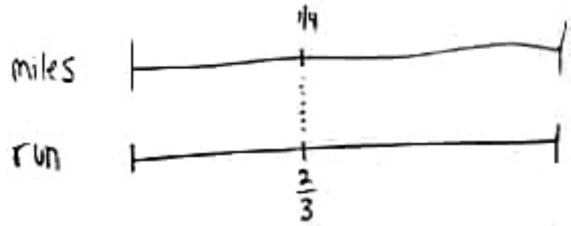


Figure 53. Sophie's Initial Drawing for Sue's Run Problem.

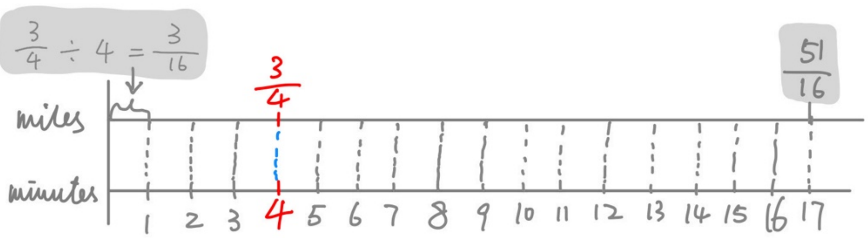


Figure 54. Winnie's Drawing for The Scooter Problem.

Unit Fraction Increments. one of the quantities increases by increments of a unit fraction amount. This coarse form begins with an Amount Correspondence. The intervals are partitioned in order to obtain the unit fraction for one quantity. Each unit fraction is paired with a corresponding amount in another quantity. The coarse form is characterized by this correspondence which is subsequently iterated either starting at zero or another amount. The micro forms comprising this coarse form are a set of Interval as a Unit Fraction, Interval as Amounts, Unit Fraction, or Unit and sometimes Correspondence Line. The function of the coarse form is to pair amounts in one quantity in order to find the corresponding amount of the other quantity.

Example. Sophie used this coarse form to complete her drawing for Sue's Run Problem. After drawing her Amount Correspondence (seen in Figure 53), she marked three-thirds on the run line and drew a Correspondence Line to an amount on the miles line, labelling the unknown amount with an asterisk. She then used Partitioned Intervals at Zero to partition the interval from zero to two-thirds into two equal partitions to create two intervals, zero to one-third (I-UF) and one-third to two-thirds. Knowing that one-third is half of two-thirds, she obtained the amount one-eighth (I-UF) as the corresponding amount on the miles line. Dr. B pressed her to explain how she knew one-eighth was half way, she argued in order to partition one-fourth into two parts, she needed the equivalent of two-eighths. She finally iterated the interval with the unit fraction and wrote three-eighths as the amount represented by the asterisk.

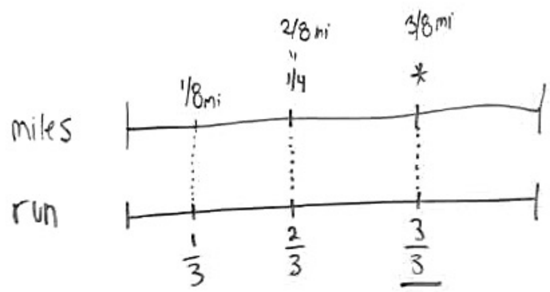


Figure 55. Sophie’s Drawing for Sue’s Run Problem.

Composed Unit Increments. In this coarse form one of the quantities increase by increments of a certain length that is not one or a unit fraction. This coarse form also begins with an Amount Correspondence. The coarse form is characterized by iterating this correspondence. The micro forms comprising this coarse form are a set of Intervals as an Amount and Correspondence Lines. The function of the coarse form is to pair amounts of one quantity in order to find the corresponding amount of the other quantity.

Example. Molly created her drawing for the Hot Chocolate Drawing by first drawing an Amount Correspondence with three ounces on the milk line (I-A) and two ounces on the chocolate line (I-A). She proceeded to iterate this correspondence, labelling each new tick mark and connecting the marks with a Correspondence Line.

She labelled each correspondence line as a batch. She finally drew tick marks in each interval to show each of the ounces on the lines.

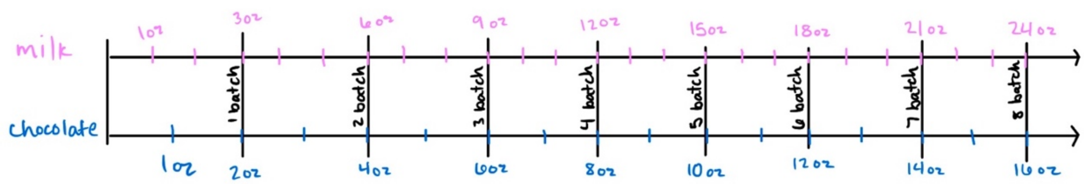


Figure 56. Molly’s Drawing for the Hot Chocolate Problem.

Winnie also iterated her Amount Correspondence when she answered the first part of The Scooter Problem. She first drew an Amount Correspondence by drawing tick marks at three-fourths miles (I-A), four minutes on the other line (I-A), and a Correspondence Line connecting the two marks. She then iterated this by drawing two sets of tick marks and a Correspondence Line labelling the miles tick marks as six-fourths and nine-fourths and the minutes marks as eight and 12. Finally, she highlighted the amounts nine-fourths miles and 12 minutes to show her answer.

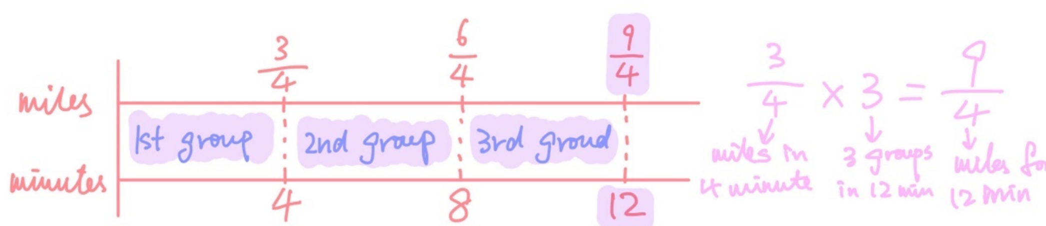


Figure 57. Winnie’s Drawing for The Scooter Problem.

Unit Increments. In this coarse form one of the quantities increases by increments of a one. This coarse form begins with an Amount Correspondence where one interval is of amount one. One interval is then partitioned in order to obtain the unit. The coarse form is characterized by this correspondence which is subsequently iterated either starting at zero or another amount. The micro forms comprising this coarse form are a set of Intervals as an Amount, Intervals as a Unit, and sometimes a Correspondence Line. The function of the coarse form is to pair amounts in one quantity in order to find the corresponding amount of the other quantity.

Example. Winnie used this coarse form to complete The Scooter Problem. After drawing the Amount Correspondence (Figure 54), she said she partitioned the interval from zero to four minutes into four intervals to obtain one minute (I-1). She then said she knew that three-fourths divided by four was three-sixteenths, thus for one minute, the scooter travelled three-sixteenths miles (I-A). She then iterated this interval seventeen times to get obtain 17 groups of three-sixteenths or 51 sixteenths.

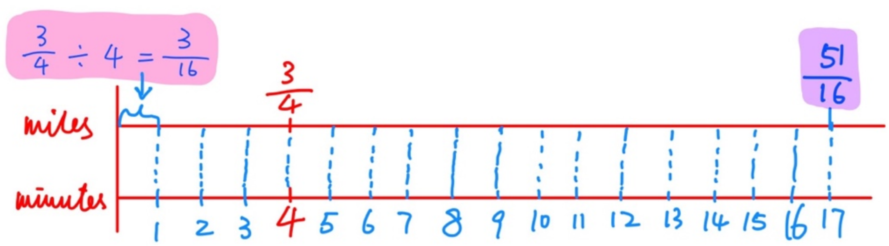
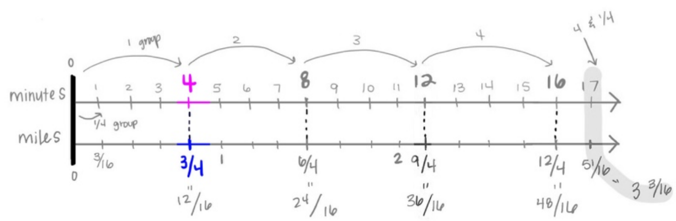
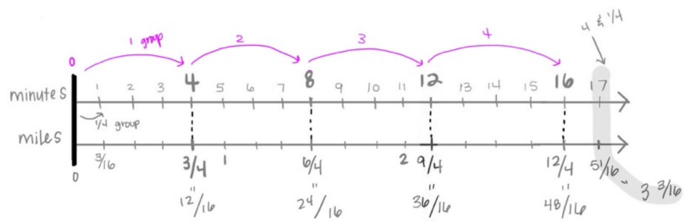
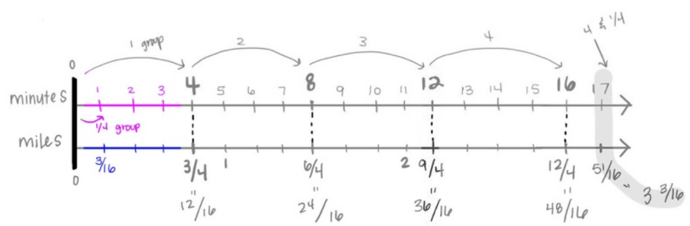
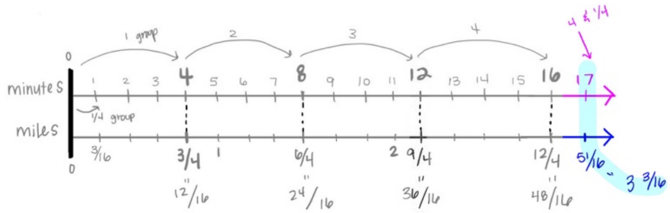


Figure 58. Winnie's Drawing for The Scooter Problem.

An illustration using coarse forms. The coarse forms are intended to cover part or all of the inscriptions of a student drawing. Similar to the section on strip diagrams, I demonstrate how the coarse forms comprise a student's drawing. I provide a sample transcript and coding for Jack's Strategy for the Scooter Problem in Table 14. In this strategy, Jack worked on looking for the number of miles after 17 minutes and pointed to parts of his drawing using a pointer which I have isolated in each figure in the table.

Table 14.
Jack's Strategy for The Scooter Problem and the Corresponding Coarse Forms

Drawing	Transcript	Codes
	First thing we did was… we realized we had three-fourths of a mile for every four minutes, so we decided to make our groups our chunks of four…	**Amount Correspondence**: two amounts on each number line are associated with each other
	For 17… we found out how many groups of our four minutes we could fit into 17 and so we can fit up four groups and got to sixteen but in order to get up to the seventeen, we needed another minute…	**Composed Unit Increments**: incremented on the number line by the established correspondence
	…which is one fourth of our group so that's when we went back and found that one minute was a fourth of our group and so we needed to take a fourth of our three fourths to give us three sixteenths…	**Partitioned Intervals at Zero**: partitioned the interval from zero to four minutes and the interval from zero to three-fourths miles into four. **Composite Unit Correspondence**: two amounts on each number line are associated with each other



…and so then we added our one minute, we added our three sixteenths so we just had to convert and add.

Unit Increments: an amount is incremented by a composite unit where one of the units is one (minutes).

Task Features Supporting the Development of Strip Diagrams

In this section, I answer the third research presented in Chapter 1:

RQ3. What features of the mathematical tasks shaped the use of certain forms and functions over time?

As explained in Chapter 3, I identified periods, time intervals where students generally drew similar drawings as indicated by the coarse forms used. When a period changed, this indicated a change in the drawings. Thus, I assumed there was a feature of the task that prompted a change. My definition of task aligns with Stein and Smith (1998) where a task "can involve several related problems or extended work, up to an entire class period, on a single complex problem" (p. 269). Thus, a task can be any prompt given by the teacher and a feature is a characteristic of the task. In order for a task feature to shape the mathematical drawings of a period, the feature must not have been a feature of the previous period. I identified three primary task features:

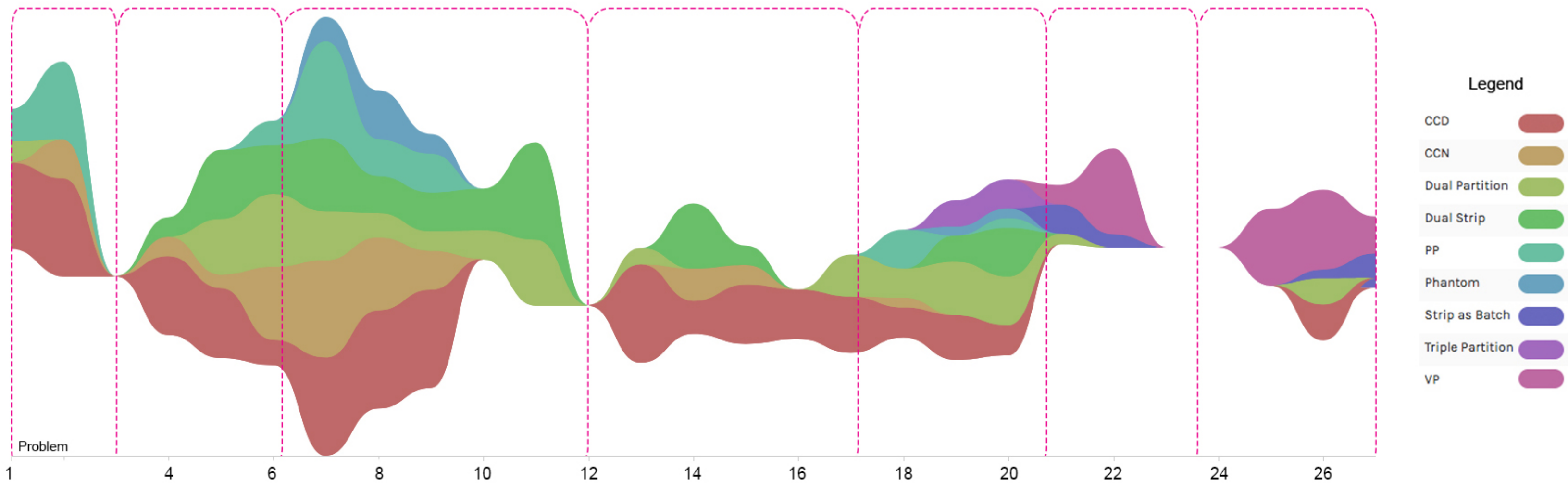
1. **Number choice (NC)**. The numbers or amounts of quantities identified in the problem and their numerical relationship (e.g., relatively prime),
2. **Problem type (PT)**. The problem structure of the situation (e.g., measurement division, missing-value proportion problem), and
3. **Teacher request (TR)**. Follow-up requests posed by Dr. B, usually not explicitly written in the task. I identified three types of teacher questions: interrogate a memorized rule, use a definition, and use a second strategy.

In this section, I first begin the discussion of each type of drawing with each overall change indicated in the Streamgraphs. The wider the stream is, the higher the ratio of the coarse form at that point in time, indicated on the horizontal axis (see Figure

17 for the interpretation of the elements of the Streamgraph). I identify the periods in the timeline and describe the drawings produced in this period. After the discussion of the general themes, I present the periods of each type of math drawing. I first describe a general overview of the drawings produced and crucial developments in the period. I then outline the task features to explain why the drawings were produced using the coarse forms of the period. To illustrate how the features, I present a subset of student work exemplifying the drawings created during each period.

A Global View of the Development of the SD

The Streamgraph for SDs is seen in Figure 59. In the first two periods, the coarse forms seen in student drawings use the same set of coarse forms, particularly the Common Core Definition, Common Core Numerator, Partitioned Partitions were dominant. Upon starting the second period, an emergence of the Dual Function of a Strip emerged and remained consistent in the periods following its emergence. There are no data in the third problem as students were asked specifically to draw number lines. Upon entering the third period, the Phantom Piece emerged as a consistent feature in some drawings. Towards the end of the period, the coarse forms reverted to two coarse forms. A re-emergence of the coarse forms occurred in the next period. In the succeeding period, the Triple Function of a Partition emerged and was unique to this period. The development ended with the emergence of the proportion coarse forms with old forms re-emerging. In the next section, I outline each period and provide illustrations of student work noting the development occurring and potential conditions in which continuity and discontinuity of the forms is seen. I summarize these periods in Table 15.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Problem	Playground	Jean's Casserole	Sue's Run	Goblin Goo 1	Pumpkin Juice	Bat Milk Cheese	Dragon Blood	Goblin Goo 2	Blank Multiplication	Francine's Rope	Gallons of Water	Drive	Pizza	Cookies	Brownies	3 ÷ 3/4	Tonya and Chrissy	1 1/2 ÷ 3/4	Noodles	1/3 ÷ 2/5	Hot Chocolate 1	Hot Chocolate 2	Scooter	Rope	Yellow and Blue	Punch	Fishing
Period	Definition of Fraction		X	Two Quantities			Phantom Piece and Refocus					X	Return to Fractions						Triple Partition		Proportional Reasoning		X	X	Return of the Definition of Fraction		

Figure 59. Streamgraph and Periods for Strip Diagrams

Table 15.
Description of the Periods in the Development of Strip Diagrams

Period	Coarse Forms	How the Features Shaped the Drawings	Task Features Shaping SD		
			PT	NC	TR
Definition of Fraction	Common Core Definition, Common Core Numerator, Partitioned Partitions	Changed the function of Partitioned Partitions			√
Two Quantities	Dual Function of Strip and Partitions, CCD, CCN, Partitioned Partitions	Distinguishing quantities in the drawings as groups and base units	√	√	√
Phantom Piece and Refocus	Same as Two Quantities period with Phantom Partition, then Dual functions of Strip and Partitions	Emergence of the Phantom Partition, return to two quantities		√	
Return to Fractions	Same as Two Quantities	Re-emergence of the Dual Functions of a Partitions and Strip	√		√
Triple Partition	Same as Return to Fractions period with Triple Partitions	Emergence of the Triple Function of a Partition	√		√
Proportional Reasoning	Emergence of Strip as a Batch and Variable Parts	Emergence of Strip as Batch, Moving to Variable Parts	√	√	√
Return of the Definition of Fraction	Same as Proportional Reasoning with Dual Function of a Partition and CCD	Re-emergence of CCD			√

Strip Diagram Periods

Definition of Fraction Period. This period is characterized by students using the definition of fraction and using strips to determine the size of partitions, particularly sizes of partitions when the Partitioned Partitions coarse form was used. The drawings began with the strip representing one of something and were subsequently partitioned to show a certain amount (i.e., using the Common Core Definition coarse form). In both tasks during this period, the students used Partitioned Partitions after drawing one amount. In both problems, the students partitioned the partitions further as schematized in Figure 60. The students then identified the amount of this partition with respect to the original quantity.

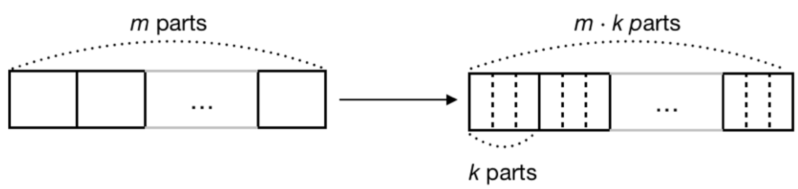


Figure 60. Schematic for Partitioned Partitions.

The teacher request task feature supported the emergence and change in the function of Partitioned Partitions, especially when she interrogated a memorized rule or encouraged using a definition. Most students justified the use of Partitioned Partitions because of the number of partitions required is the “least common denominator.” Dr. B asked to students where they could justify the use of the memorized rule in the drawing. This prompted the students to recall and use the definition of fraction (Figure 24). Dr. B reinforced the idea of using the definition of fraction to justify Partitioned Partitions.

The Playground Problem. Sophie demonstrated two predominant SDs during this period. Recall Sophie’s work for the Playground problem (Figure 29). Sophie first drew the park and partitioned it into three to obtain one-third of the park (CCD). She then used the partition as one playground and partitioned this further into four partitions. She shaded one of the partitions and called this the swings (CCD). To determine the size of the shaded partition in relation to the park, she partitioned each third of the park into fourths (Partitioned Partitions). As explained in a previous section on Partitioned Partitions, Sophie and her classmates discussed reasons for partitioning the other one-third partitions. Dr. B initiated the discussion by interrogating some students who justified the Partitioned Partitions with common denominators, “Yeah and this table was thinking least common multiple- common denominator… do you see common denominators here? Do you see least common multiples here? I don’t see it.” Sophie then drew the additional partitions in the other thirds to show the park partitioned into 12 equal-sized parts, thus the shaded part is one-twelfth.

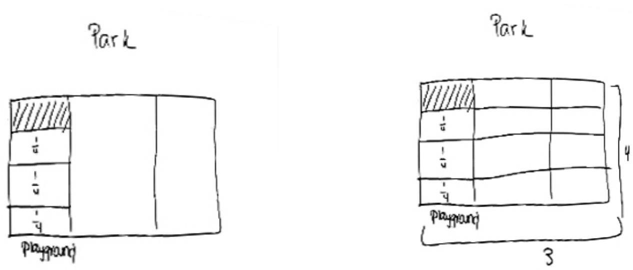


Figure 29. (a) Sophie's Initial and (b) Final Drawing for the Playground Problem.

Jean’s Casserole Problem. In Sophie’s drawing for Jean’s Casserole problem (Figure 61), she employed the CCD coarse form. In small group, she drew a rectangle representing one cup, partitioned the strip in two, then shaded one partition to show one-half of a cup (CCD). Using “the lowest common denominator for one-half and one-third”, she partitioned the strip into sixths, obtained partitions of size one-sixth of a cup, and shaded three-sixths of a cup (Partitioned Partitions). Like in the previous problem, Dr. B pressed the class for their decision to partition the partitions further. Students' initial rationales included both an appeal to the least common denominator such as Sophie. She asked the students, "Where is the least common denominator in the drawing?" Students then explained that they needed to compare one-third and one-half of a cup, thus they needed to partition the partitions further. Using her drawing of one-half or three-sixths of a cup, Sophie drew a new strip by disembedding her shaded partition to create a strip representing a recipe. Knowing one-third of a cup is equivalent to two-sixths of a cup, she shaded two of the partitions in the second strip. Finally, Sophie stated that each partition is a third of a recipe and the shaded partitions in the second strip is two-thirds of the whole recipe (i.e., a partition is both one-sixth of a cup and one-third of a recipe (CCD, Dual Function of a Partition)).

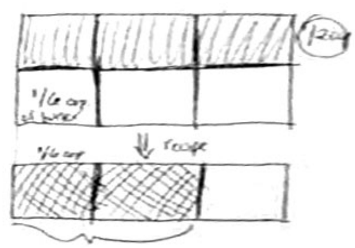


Figure 61. Sophie's Drawing for Jean's Casserole Problem.

Summary. Sophie demonstrated how the definition of fraction helped her create drawings by showing the wholes and the fractional amounts under consideration. Using Partitioned Partitions, students made sense of the different sized partitions and, when redirected by Dr. B, rationalized their partitions with a recalled procedure such as finding least common denominators.

Two Quantities Period. This period is characterized by identifying both a full strip and partitions with respect to two quantities (i.e., Dual Function of a Strip), unlike the previous period where only partitions were described with respect to two quantities. Dr. B also formally introduced the definition of multiplication, thus inducing the need to assign one quantity as a group and the other as a base unit (see Figure 62). The function of Partitioned Partitions also shifted in this period. The drawings mostly began with a strip representing one group but also representing an amount of base units (i.e., the strip was a display of the multiplicand). Based on this, the strip was either iterated or partitioned further.

Figure 62. Class Definition of Multiplication Handout

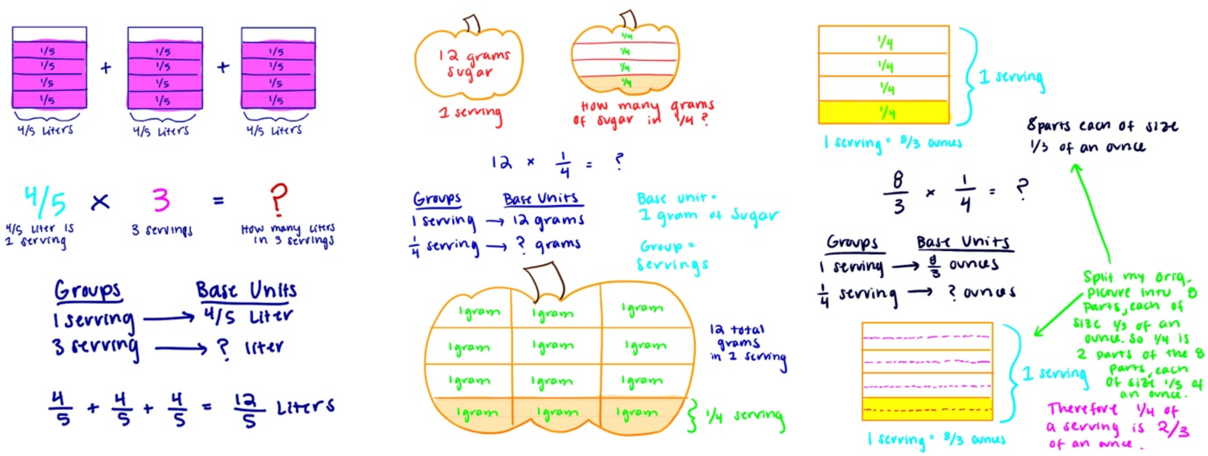
Three task features—problem type, the teacher request to use the definition of multiplication, and number choice—played a crucial role in the emergence of the coarse form Dual Function of a Strip. All problems in this period were multiplication problems, thus a multiplicand could be identified and drawn first. By drawing the multiplicand, students drew a strip showing the relationship between one of a quantity and some amount of the other quantity (i.e., Dual Function of a Strip). The teacher's request to use the definition reinforced the identification of both quantities in the drawing. Finally, the number choice played a crucial role in the function of the Partitioned Partitions. Shifting from using Partitioned Partitions to determine the size of a partition, the students justified the use of Partitioned Partitions to display a required number of partitions based on the number of partitions indicated in the multiplicand as will be demonstrated in a later example (Figure 63c).

Hannah's work for the three problems in this period showed how the multiplicand played a role in her drawing to determine the number of partitions she needed. and how she incorporated the definition of multiplication.

The First Goblin Goo Problem. In her drawing for the First Goblin Goo Problem (in Figure 63a), Hannah drew three strips, each showing one whole liter where each strip represented the multiplicand (i.e., Dual Function of a Strip). She partitioned each liter into five parts and shaded four of them to show four-fifths of a liter (CCD). She iterated the strip then counted the number of highlighted partitions, each one-fifth of a liter to obtain twelve-fifths. Notably, she assigned both groups and base units based on the definition of multiplication.

The Pumpkin Juice Problem. Hannah began her drawing by showing one-fourth of a serving whilst showing the whole and the fractional amount (CCD). Moving from the top of her drawing, she first wrote out an equation and table showing how she assigned quantities. In particular, she showed that she assigned a serving as a group. She then used the drawing in the upper right to also embed the base units in her drawing. She partitioned the four servings into three smaller partitions (Partitioned Partitions) and labelled each partition as "1 gram," because she wanted to show all 12 grams. In the partition she highlighted previously, she now had three grams displayed in the one-fourth partition. Thus, there were two functions for the partition—showing one-fourth of a serving and three grams of sugar. Similar to her response in the previous problem, Hannah organized the quantities in the problem based on the definition of multiplication.

Bat Milk Cheese Problem. In the final drawing for the period (Figure 63c), Hannah began her drawing in a similar fashion—showing one whole serving or one group and showing the size of the group by using the CCD coarse form. However, she was unsure how to proceed from this drawing. The postdoctoral researcher supported Hannah by requesting that she use the definition of fraction to think about the meaning of eight-thirds. She wrote out the definition “8 parts each size 1/3 of an ounce.” She then realized she wanted to show eight partitions in the strip and noticed she already had four partitions. She partitioned each of the partitions into two smaller partitions to show eight total partitions. She ended by saying that there are two-thirds ounces in one-fourth of a partition because there are two parts, each one-third of an ounce in the yellow partition indicating one-fourth of a serving. In this case, she used the amount indicated in the multiplicand, eight-thirds, to determine the number of partitions she needed.



(a) (b) (c)

Figure 63. Hannah's Strip Diagrams for the Two-Quantity Period

Hannah's work demonstrated how multiplication problems, number choice, and the definition of multiplication played a role in shaping the coarse forms she used. In the last two problems, she illustrated how the function of Partitioned Partitions shifted from using this coarse form to determine the size of the partitions to acquiring a certain number of partitions as indicated in the multiplicand.

Phantom Piece and Refocus Period. This period built on what they did in the previous period; however, there were two critical developments, the coarse form Phantom Piece emerged in the drawings and the function of Partitioned Partitions shifted again. Similar to the previous period, the drawings began with a strip representing one group but also representing an amount of base units as indicated by the multiplicand, all with an amount less than one base unit. The multipliers in the first set of problems were all less than one. The students used Partitioned Partitions to obtain the required size of the group. However, students noticed that the group of partitions that was obtained needed to be described in terms of the base units. They added Phantom Partitions in order to determine the size of the partitions they obtained. The period ended with a shift in problem type. However, even with the shift in problem type, the coarse forms used in the drawings reverted back to the forms used earlier.

The number choice task feature shaped the drawings in this period in two ways. First, the multiplicand given was less than one. Students needed to describe the product of a multiplication problem with respect to the base unit, so they had to draw a whole base unit in order to describe the size of the product amount in terms of base units. Second, the number of partitions they created from the multiplicand was not necessarily divisible by the number of partitions they wanted based on the multiplier. In other words (using Figure 60), the students started with a set of m partitions but they needed n partitions based on the multiplier and n was not a divisor of m. Thus, the students needed to partition further to create $m \cdot k$ partitions where $m \cdot k$ was divisible by n.

Elizabeth's created work in this period illustrated how students developed their use of a phantom partition and how she partitioned her partitions based on the number choices provided in the problem. She started the period with not using a Phantom Piece; however, she began to use the Phantom Piece in the second problem.

Dragon Blood Problem. Elizabeth did not draw a Phantom Piece and relied on a memorized number fact when she used Partitioned Partitions (Figure 64). She began by drawing the multiplicand, a strip as one-fifth of a liter (CCN) and one serving at the same time (Dual Function of a Strip). She stated she needed three parts because she needed a third of the serving and partitioned the strip into three. She determined that the size of one of the partitioned could be obtained by "multiplying by three-over-three." Dr. B redirected by saying "or take the equivalent fraction with a numerator three" probably to emphasize the necessity of focusing on the number of parts. Elizabeth explained later that she multiplied the one-fifth by three-thirds to obtain three-fifteenths as seen at the bottom of her drawing. She may not have seen the necessity of drawing out the entire liter because she determined one-third is equivalent to three-fifteenths using a memorized procedure. From this, she highlighted one of the partitions and called it one-fifteenth of a serving.

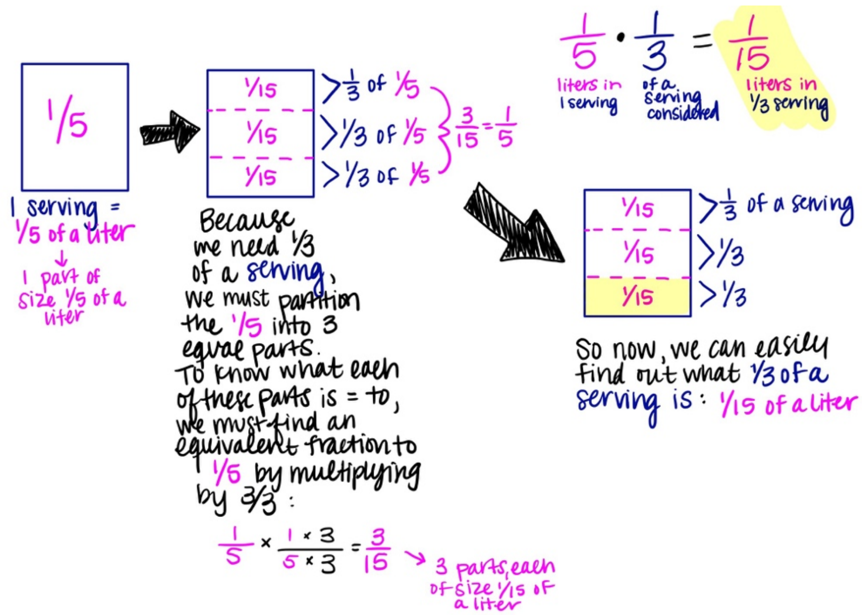
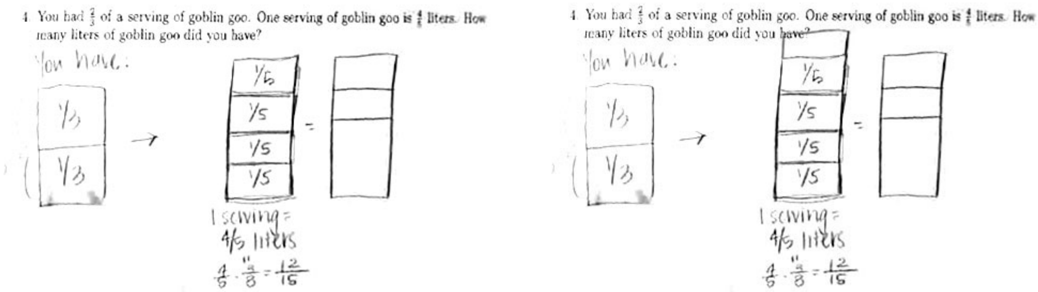


Figure 64. Elizabeth's Drawing for The Dragon Blood Problem.

Second Goblin Goo Problem. While working on this problem, Elizabeth first started by drawing the size of the group in her problem or two-thirds of a serving (CCN). She then drew a strip, similar to how she began the Dragon Blood problem, which represented both one serving and four-fifths of a liter (Figure 65a). She computed 4/5 · 3/3 = 12/15 and said she wanted each section to have three parts. During small group, when Dr. B asked why she needed to do this, she said "so that way when I'm asking for one of three parts or two of three parts, I'll be able to see it in here." Elizabeth did not have the number of partitions she needed (three parts) thus, she partitioned the partitions further. She then added an extra partition to the second strip (Figure 65b) while saying, "I might actually add the extra… just to- he can be like a ghost one fifth… he like technically is there but not really." This action of adding the additional partition to complete one of a liter is characteristic of the Phantom Partition coarse form.



(a) (b)

Figure 65. Elizabeth's Initial Work for the Second Goblin Goo Problem.

After working through the problem on paper, Elizabeth made her public display on an iPad. However, she first drew the problem by drawing a strip to represent a whole liter and highlighted four parts to show four-fifths of a liter or one serving (Figure 66). In other words, she did not subsequently add the additional partition. During whole-class discussion, she explained needed the serving in thirds, not fifths, so she needed to partition each one-fifth into three partitions. Once again, the number of partitions available to Elizabeth was not divisible by the number of partitions she wanted. Upon partitioning each fifth into three, she labelled and described each partitioned partition as one-fifteenth of a liter. She finally highlighted two-thirds of a serving by highlighting two columns of the partitioned serving to show two-thirds of a serving as eight-fifteenths.

In conversations during small-group, Elizabeth explained that seeing the whole base unit (i.e., drawing the Phantom Partition), was helpful to interpret partitions with respect to the base unit. While she did not express any reason for changing her drawing from Phantom Partition to incorporating the Phantom Partition in the initial strip, the next day while talking to one of the graduate students about this problem, she said:

> I think it helps understand how many parts there are of a liter. ‘Coz that’s why it was confusing to me was putting in in twelfths because that’s not twelfths of a liter. We went from fifth of a liter to twelfth of a serving which-

I don't know- it's too much work. You can do much less work if you just understand that there's a pretend liter… just go with liters the whole time. Don't change your wholes last minute.

Additionally, when referring to Jack's strategy (Table 11) for the Second Goblin Goo problem, she argued to keep the partitioned partitions in terms of the base unit so as to not "switch wholes."

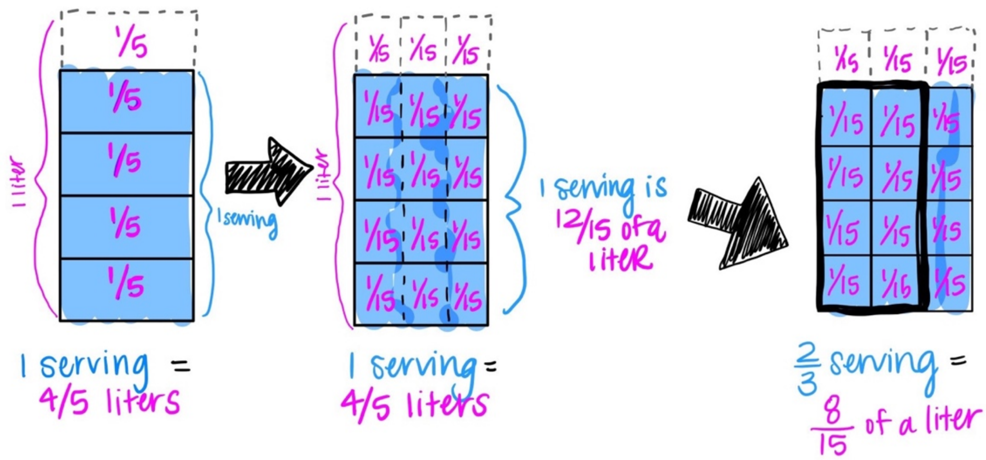


Figure 66. Elizabeth's Drawing for the Second Goblin Goo Problem.

Blank Multiplication Problem. Working with Jack through this problem, Elizabeth wrote the cheeky word problem "One serving of mascara has three-fourths of a gram of bat fecal matter. How much bat fecal matter is in two-fifths of a serving?" and drew the SD shown in Figure 67. Elizabeth explained that they started with a strip representing one gram partitioned into four and shaded three partitions representing three-fourths of a gram or one serving (CCD). She wanted to find two-fifths and partitioned each fourth into five partitions to determine the size of the partition as one-twentieth (Partitioned Partitions). When Jack suggested she should "get" two partitioned partitions from each partitions (i.e., to get two-fifths of the serving), she highlighted two-twentieths in each of the three one-fourth serving partitions. Because she highlighted six partitions, each size one-twentieth, she wrote that there are six-twentieths grams in two-fifths of a serving.

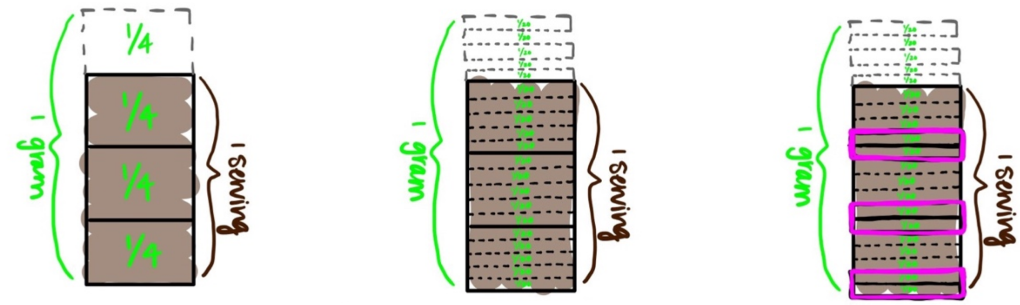


Figure 67. Elizabeth's Drawing for the Blank Multiplication Problem.

In sum, Elizabeth's drawings showed one of a group to one of a base unit by adding a phantom partition. The Phantom Partition played a role in the Second Goblin Goo problem because, although she drew partitions that were not "really there," she did not want to "change wholes." This potentially explains why she opted to create drawings starting with a CCD coarse form for the base units in the Blank Multiplication Problem (i.e., showing one whole base unit). Molly also explained why she would rather start with a base unit during whole-class discussion, "I prefer looking at the base unit because that's what the answer is in, and I like knowing the units of the answer the entire process."

This period ends with the disappearance of several course forms that were developed over the course of the previous problems. Dr. B posed measurement division problems with whole numbers, thus all the coarse forms involving fractions (i.e., CCD, CCN, partitioned partitions) were no longer present in students' drawings. The problems focused on finding the amount in one group and most of the students, when they used strip diagrams, solely used the dual function of a strip and a partition. Potentially, the number choices for the problem explain the disappearance of the other coarse forms which rely on fractions. Before posing the two problems in this period, students were given partitive division problems with whole-number quotients which I did not analyze closely. During the discussion, students drew to show a “deal out” strategy. Take Hannah’s drawing for the word problem “You have a class of ten students. You want to split them up into two groups. How many students are in each group?” for instance (Figure 68). She drew out the ten students and dealt the small blue circles out into two groups. The order in which she dealt out the students are indicated in orange.

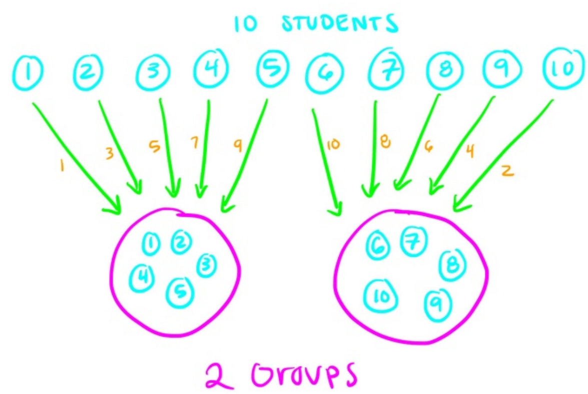


Figure 68. Hannah's Dealing Out Strategy.

Francine’s Rope Problem. Most students used DNLs to solve this problem, but Hannah used a SD to solve this problem. Hannah first began by drawing a strip showing 32 yards and eight pieces (the orange parts of her drawing in Figure 69). She then wanted to show how to obtain how many pieces were in each part. Initially, she wanted to draw another strip showing 32 parts, each part representing a yard. She explained that want to “ration out” or “place one in each and I was going to go around and place one in each” part of the eight partitions to see how many yards there were in a partition. However, she expressed hesitancy to the teaching assistant saying, “it’s going to look really messy." She attempted to draw a dealing out strategy similar to her strategy seen in Figure 68, but the teaching assistant redirected her and said that distance cannot be necessarily dealt out unlike apples into buckets. Hannah asked, "do you think it's wrong to put them in there?" He argued that it was not necessarily wrong, but it would be better to "add multiple groups" and start with a group of four and add until she got 32 yards. In other words, Hannah wanted to show how to obtain the amount in one of the pieces and the teacher assistant wanted her to show how she could iterate one piece, already knowing there are four yards in one piece. Thus, in her final drawing, she displayed a strip representing both 32 yards and eight pieces equally partitioned in eight where a partition represented both one piece and four yards.

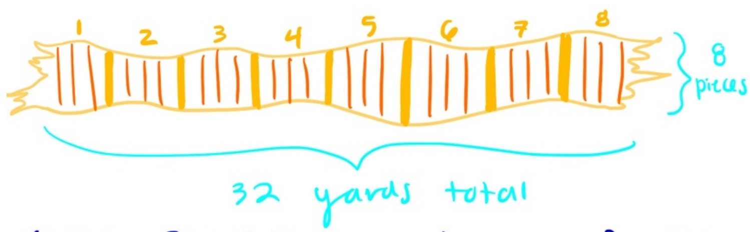


Figure 69. Hannah's Drawing for the Rope Problem.

Gallons of Water Problem. Sophie used the same coarse forms as Hannah to create her drawing. During small group, she drew one strip to show one gallon and eight pounds using the dual function of a strip (upper left of Figure 70). She then explained that she drew a larger strip to represent 400 pounds, knew that the answer was 50 gallons, and used that to "fill in" 400 pounds. She just filled the strip with 50 partitions each representing eight pounds or one gallon. In Sophie's drawing, she used both the dual function of a strip (found in the first strip) and dual function of a partition (found in the second strip).

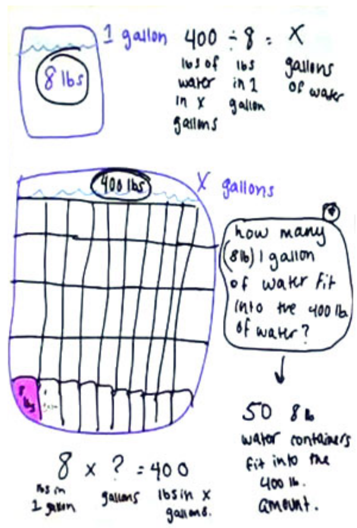


Figure 70. Sophie's Drawing for the Gallons of Water problem.

This period is marked by the emergence of the Phantom Piece for some drawings. As Elizabeth demonstrated, drawing one of a base unit proved to be useful for finding a product of a multiplication problem where the multiplicand is less than one. Additionally, some students anticipated the need for one of a base unit in the drawing as Elizabeth and Molly discussed in the Blank Multiplication Problem. This period ended where the class revisited the definition of multiplication in both types of division problems with whole numbers.

Return to Fractions Period. This period is characterized by the re-emergence of the coarse forms that were characteristic of the Two Quantities period. Similar to the Two Quantities period, the students primarily used Dual Functions of a Partition and Dual Function of a Strip to create their strip diagrams. They initially drew the product amount given and then identified groups of partitions to indicate one group based on the multiplicand. This diverged slightly from the previous period, where in the previous period the students worked on drawing the multiplicand, in this period they worked on drawing the product with the multiplicand. Notably, new coarse forms seemed to emerge and disappear with each new problem. This would have merited a new period for each new problem; however, I deemed the drawings to be similar enough to form one whole period. All the drawings drew on both definitions of multiplication and fractions and thus resembled the drawings in the Two Quantities Period.

The problem type and number choices shaped the drawings in this period. Dr. B posed division problems in this period. In the first problem, a partitive division problem, students drew drawings showing partitioned strips, following the Common Core Definition. For the measurement division problems, students first drew the product amount and then used the multiplicand to create groups of units. Ultimately, they had to describe these groups with respect the other quantity, thus necessitating the Dual Function coarse forms. The number choices influenced the use of the Dual Function of a Partition. If the number of partitions created using the product amount was not divisible by the number of partitions the students needed to draw the multiplicand, then the they used the Dual Function of a Partition and in some cases, Partitioned Partitions.

Pizza Problem and the Brownie Problem. Similar to Hannah's strategies in the previous period, Molly approached the two partitive problems posed at the beginning of the period with a dealing out strategy, but with two different interpretations of the deal. She had similar strategies for both the Pizza and Brownie Problem (Figure 71). In the Brownie Problem, she first drew three strips partitioned into four, with each partition representing a whole brownie and a strip showing one bag of four brownies. In both the Pizza Problem and the Brownie Problem, she drew out one of a base unit and a pizza or brownie, respectively. She then partitioned each strip into parts based on the number of groups and people or bags, respectively (CCD). She then assigned each partition of the base unit to each group to get three-fourths pizza for each person and four and two-thirds brownie for each bag.

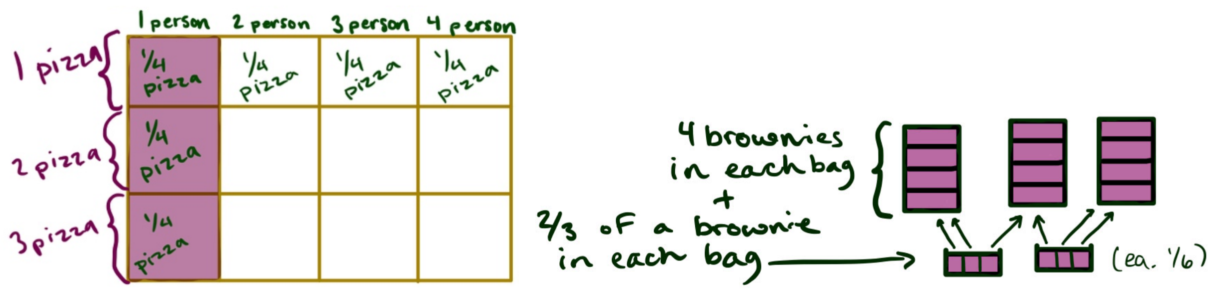


Figure 71. Molly’s Drawings for the Pizza Problem and the Brownie Problem.

For the Brownie Problem, Molly created a second strategy also using the idea of dealing out (Figure 72); however, her interpretation of two-thirds was different. As she explained in both whole-group and small-group discussions, she interpreted four and two-thirds as the number of “deal outs” made. In her drawing, she showed four on the left side with an arrow. The amount she presented represented the number of deals (i.e., the first deal resulted in one brownie in each bag, the second deal resulted in two brownies in each bag, and so on). Her last deal resulted in “2/3” deal (CCD).

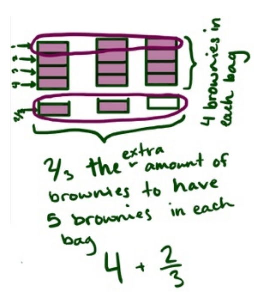


Figure 72. Molly's Second Strategy for the Brownie Problem.

The last three problems were measurement division problems with fractions and were posed at the beginning of the Spring semester. One notable development was that the DNL appeared in more drawings, thus there were fewer SDs to analyze compared previous periods either because students solely used a DNL or they started a SD as a second strategy which they were unable to finish due to time. The emergence and uptake of the DNL could be due to the prompt by Dr. B showing an image of a SD and a DNL. Because students were given agency to choose a type of drawing, I did not find a student who consistently used a SD for the next set of problems.

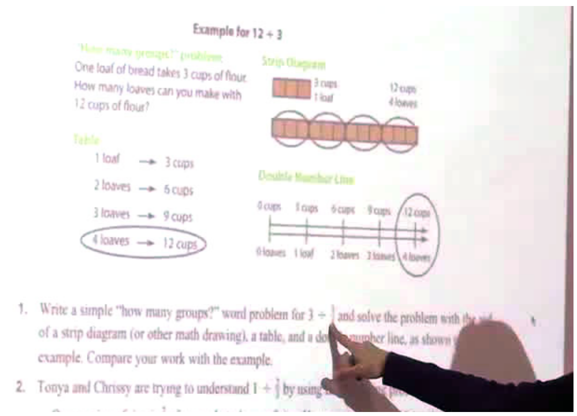


Figure 73. Dr. B Prompt Showing a SD and DNL.

3 ÷ 3/4 = ?. Dr. B asked the students to write a measurement division problem and solve the blank number sentence 3 ÷ 3/4 = ?. Sophie wrote the word problem “You have three cups of flour. Each batch required 3/4 cups of flour. How many batches can you make?” Sophie’s drawing, like Molly’s, displayed the CCD coarse form (Figure 74). She first drew three strips showing the three cups of flour. She partitioned each cup into four and called each partition one-fourth of a cup (CCD). She colored four sets of three partitions, each representing a batch or three-fourths of a cup. She counted the number of sets to get four batches. Drawings such as these did not have any other coarse forms as students counted sets of three parts, each size one-fourth.

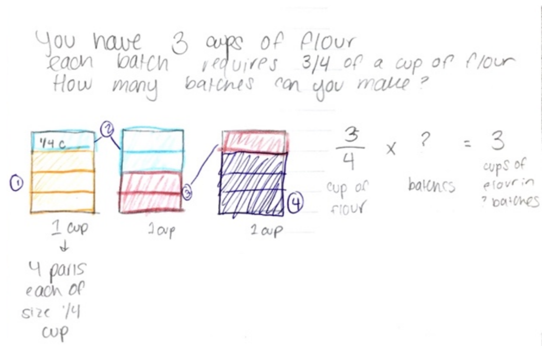


Figure 74. Sophie's Drawing for The Blank Multiplication Problem.

Tonya and Chrissy. In the Tonya and Chrissy problem, students were asked to make sense the reasoning of Tonya, a fictional student. They were provided a strip diagram, partitioned into three where two partitions were shaded. In the prompt, Tonya says, “There is one 2/3 cup of serving rice in 1 cup, and there is 1/3 cup of rice left over so the answer should be 1 1/3.” Chrissy says, “The part left over is 1/3 cup of rice but the answer is supposed to be 3/2 = 1 1/2. Did we do something wrong?” Catherine started her work by writing “Understanding 1 ÷ 2/3” and worked out the problem before engaging the fictional student’s work. Similar to Sophie’s strategy in the previous problem, Catherine started drew one of a base unit or one cup, partitioned the strip into three, and labeled each strip as one-third cup (Figure 75). She then highlighted two of them to indicate two-thirds of a cup (CCD), labelled them as one serving, and wrote one half serving next to each partition (Dual Function) to get an answer of three-halves serving in one cup. To explain the fictional student reasoning in the Tonya and Chrissy Problem, she explained that the answer given in the problem, although sensible, was incorrect. She said that Tonya’s response was in two different units, cups and servings.

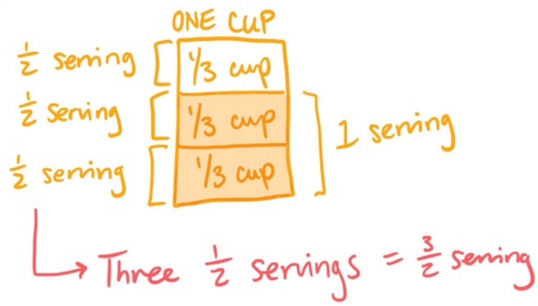
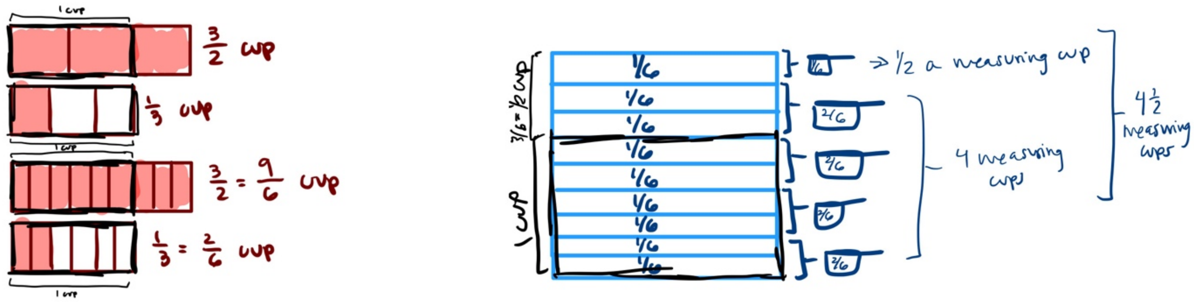


Figure 75. Catherine's Drawing for the Tonya and Chrissy Problem.

1 1/2 ÷ 1/3. Students were asked to write a measurement problem for 1 1/2 ÷ 1/3. Lindsay wrote the problem “How many times would you need to fill a 1/3 cup measuring cup with water and pour it into a container that holds 1 1/2 cup of water in order to fill that container?” She made three sets of drawings to solve the problem. In the first set of drawings (Figure 76a, top), she drew two strips showing the two amounts given in the problem using the CCD coarse form. She partitioned her strips further in her second set of drawings (Partitioned Partitions). Lindsay explained this set, “I knew my common denominator would be six and so I partitioned a cup into sixth sized parts” to create two new strips showing nine-sixths and two-sixths (Figure 76a, bottom). She finally created the last set of drawings. She re-drew the nine-sixths strips and she explained, “every time that there were two one-sixth parts that would be a measuring cup, so I saw that you could use four full measuring cups and then half of another measuring cup” (Figure 76b). In her drawing, she also noted that one of the partitions represented both one-sixth of a cup and one-half of a measuring cup and that the entire strip was both nine-sixths of a cup and four and one-half measuring cups (Dual Function of a Partition and Strip).



(a) (b)

Figure 76. Lindsay's Drawing for 1 1/2 ÷ 1/3.

In this period, the students created drawings that built on the last few problems of the previous period. Using coarse forms that were also found in the Two Quantities problem (i.e., CCD, Partitioned Partitions), thus initiating a re-emergence of using coarse forms that supported them in solving problems with fractional amounts. They also continued the coarse forms with two quantities as they did towards the end of the last period (i.e., Dual Function of the Strip and Partition).

Triple Partition Period. This period marked the last set of fraction operation problems and the emergence of the Triple Function of a Partition. In this period, the Triple Function of a Partition emerged exclusively when students were asked to explain the keep-change-flip algorithm $\left(\text{i.e., } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}\right)$ working on partitive division problems.

Students used the product to begin most of their drawings. They drew a strip showing both the given size of the group and the corresponding amount of units. They partitioned the strip based on the multiplier and then used the Dual Function of a Partition to describe one of the partitions mostly using memorized number facts. Using this partition, they determined how many base units are in a group by counting up to whole group using the partition.

The problem type and a teacher request shaped the drawings in this period. As I explained, the students began their drawings with a strip represeting the product, which is a relationship between an amount of groups and base units. Additionally, the Triple Function of a Partition emerged exclusively when the instructor requested the students to explain the keep-change-flip algorithm using the definition of multiplication. The problem type also supported the use of the Triple Function of a Partition by structuring the quantities.

Noodles Problem. In each of the three problems of this period, Catherine demonstrated the emergence of the Triple Function of a Partition, beginning with the Noodles Problem (Figure 77). She first solved the Noodles problem by drawing one strip showing a serving of noodles and partitioned it into three. She drew a new strip with one partition shaded to show two-thirds of a serving unshaded and labelled the set of two partitions as “120 mg Na” (Dual Function of a Strip). She then drew a new strip showing one-third and labelled this partition as “60 mg Na” (Dual Function of a Partition). She then drew a final strip showing the original strip and iterating the partitions labelled “60 mg” three times to represent one serving and “180 mg Na.” After some students presented similar strategies in a whole-class discussion, Dr. B asked the students to rationalize the expression $120 \cdot \frac{3}{2}$, the result of using keep-change-flip. Catherine encircled the two partitions representing 120 mg, the label 60 mg, and the 1/2 in the expression 120 · 1/2. This indicates Catherine assigned three different amounts of three different quantities to one partition, specifically, 60 mg, one-third of a serving, and one-half of two-thirds of a serving (Triple Function of a Partition). When she wrote out her equation, she annotated the equation. In her annotation, she followed the defintion of multiplication. In the multiplicand, she identified her one of a group as “2/3 serving” (cf. one group is one serving).

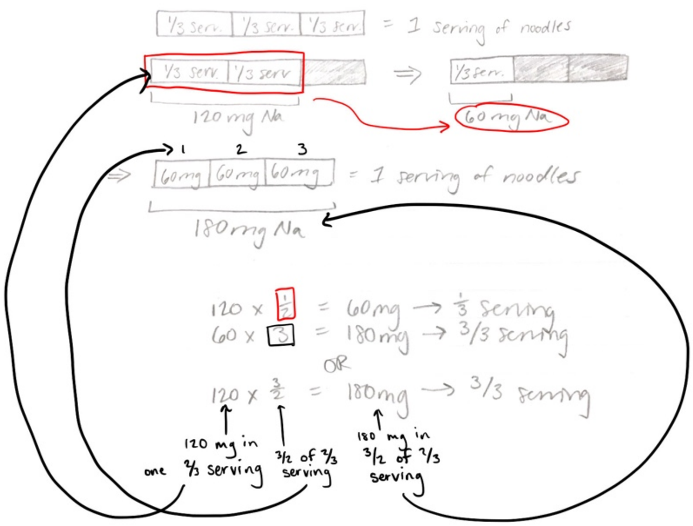


Figure 77. Catherine's Drawing for The Noodles Problem.

Scooter Problem. Catherine’s strategy for the Scooter Problem (Figure 78) resembled her Noodles problem strategy. She first drew one strip showing an hour and partitioned it into four. She drew a new strip with one partition shaded to show three-fourths of an hour (CCD), labelled the set of three partitions as “6 miles,” drew a new strip showing one-fourth of an hour, and labelled this partition as “2 miles” (Dual Function of a Strip and Partition). She then drew a final strip showing the original strip with four partitions; however, each partition was labelled “2 mi” and the whole strip was labelled “8 miles.” Upon being asked to justify the keep-change-flip algorithm, once again, Catherine assigned three different amounts of three different quantities to one partition—specifically, 2 miles, one-fourth of an hour, and one-third of three-fourths of an hour (the Triple Function of a Partition). She used the defintiion of multiplication to write the expresion 6 · 4/3.

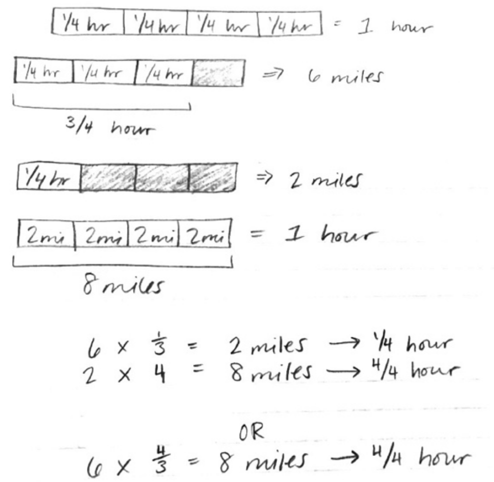


Figure 78. Catherine's Strategy for the Scooter Problem.

1/3 ÷ 2/5. In the final problem in this period, students were asked to write a partitive division problem for the expression 1/3 ÷ 2/5 and to explain the equivalence 1/3 ÷ 2/5 = 1/3 · 5/2. Catherine wrote the problem, “You have 1/3 g sugar in 2/5 of a cookie How many grams of sugar is there in one whole cookie?” Notably, Catherine drew a circle to partition as opposed to a rectangle. Based on the way I described the form of a strip, I considered the circle to be a “strip.” However, I acknowledge partitioning a circle or a 360-degree angle is more complex and potentially different than partitioning a length or area (Hardison, 2018). Catherine partitioned the strip into five parts, showing one-fifth of cookie for each partition. She colored two sets of two partitions showing the product amount and multiplier, where two partitions is two-fifths of a cookie and one-third gram of sugar. She described the last partition as both one-half of one-third gram of sugar and one-sixth grams of sugar. Like Elizabeth, she associated the partition with two amounts, one-fifth of a cookie and one-sixth of a gram. Dr. B asked the class to explain keep-change-flip similar to the previous problems. To explain 1/3 · 5/2, Catherine wrote that one partition is also one-half of two-fifths of a cookie and thus described the partition in three ways. She identified there were five partitions in the strip, and thus five-sixths gram of sugar in one cookie.

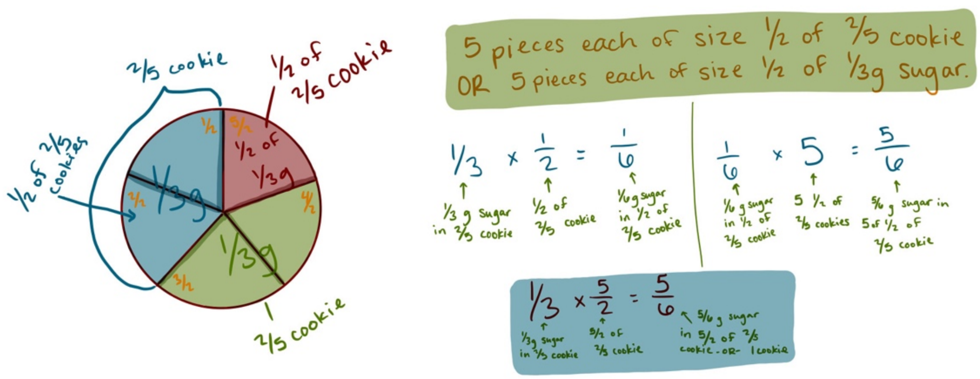


Figure 79. Catherine's Drawing for 1/3 ÷ 2/5.

Students used the Triple Function of a Partition when prompted to explain the keep-change-flip relationship $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. Students did not use this coarse form prior to the teacher request. The problem type played a role in Catherine and other students' strategy in that their drawings began by drawing product amount first as compared to measurement division problems where the multiplicand was drawn first. See Appendix B for an explanation for keep-change-flip using measurement division.

Proportional reasoning. This period marked the shift into ratio and proportion lessons. They initially worked on tasks involving concrete, discrete objects which they drew on to draw SDs. Their SDs shifted once they were pressed to think beyond discrete objects. Their initial drawings showed batches where strips represented only one quantity and the partition represented one of the quantity, indicating a suspension of the Dual Function coarse forms. When provided with more problems, the students generalized their drawings to show strips as some quantity and the partitions as any amount.

The problem type, teacher requests, and number choice shaped the SDs in this period. Proportion problems in this period could be characterized with the equation $M \cdot x = y$ (Beckmann & Izsák, 2015) which indicates that the number of groups are fixed but the amount in each group can vary. Such problem types have the ratio explictly identified. Students usually begain their drawing by creating a strip showing the ratio. In initial drawings, the students only drew strips showing a particular amount but, when requested by the instructor to generalize the amounts, shifted their drawings to show varying amounts. In the small group I was working with, number choice prompted a change in the students' strip diagrams.

The Hot Chocolate Problem. In this period, Molly demonstrated a development from using the Strip as a Batch to Variable Parts. The Strip as a Batch form shows a particular amount where a partition represents one of a quantity. These developed to Variable Parts where partitions now show multiple amounts in one partition. Using the SD to generalize a set of numbers presented some obstacles for most students. Upon changing the number choice, Molly reasoned with a SD showing multiple amounts.

Similar to other students' strategies, Molly first created a DNL for the relationship between the amount of milk and amount of chocolate (Figure 80). She then used this to create strips, partitioned into five parts with three parts representing one ounce of milk in each partition and two parts representing one ounce of chocolate in each partition (Strip as a Batch). Dr. B refocused the class to think about "a whole bunch of mixtures all at once." During small group, Molly and most her table were not sure how to draw a strip diagram that looks different from the batches as seen in Molly's drawing.

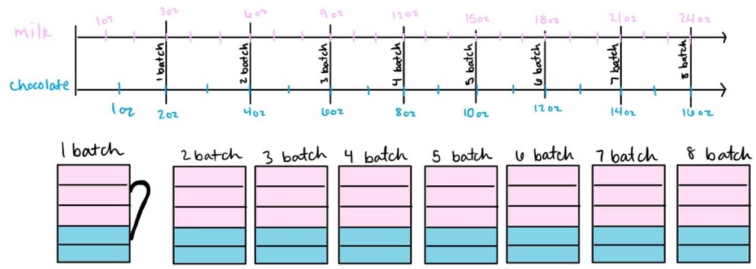


Figure 80. Molly's Two Drawings for the Hot Chocolate Problem.

Because I noticed the students were drawing out all the ounces of milk and chocolate, I asked them to think about creating a larger amount of hot chocolate and changed the number choice by asking them to draw a SD with 1000 cups of hot cocoa. Molly created a context for the larger amount—a football game where there would be 1000 people attending and everyone gets one cup of hot chocolate with the 2 to 3 ratio. As the group discussed ways to understand another way of using a 2 to 3 ratio, Molly worked independently. When she finished her drawing (Figure 81), she joined the small-group conversation and explained that one cup of hot chocolate would have "two parts, each size one-fifth of a cup of chocolate and three parts, each size one-fifth of a cup of milk…if we wanna use the same recipe for the entire football game, we need to figure out what one-fifth of the entire football game is." She obtained one-fifth of 1000 is 200 cups as seen in the equation on the right. Using the same idea that there are two parts, each size one-fifth of a cup, she said that in this larger amount of hot chocolate there were two parts, each size one-fifth of 1000. Cameron clarified what Molly meant by part and Molly wrote above each column "1 part." At this point, Dr. B refocused the class to a whole-class discussion. Molly explained the line of reasoning she discussed with her group. She explained that one cup of hot cocoa would have two one-fifth cups of chocolate and three one-fifth cups of milk as seen in the first row of her SD. She continued, explaining that 1,000 cups of hot cocoa would also have two parts chocolate and three parts milk similar to the one cup of hot cocoa. In this case, however, she stacked multiple strips to show 1,000 cups with each row representing a cup. The amount in a part was no longer one-fifth of a cup but 200 cups. She divided 1,000 cups into five parts to get 200 cups in one part. Describing the difference between one cup and 1,000 cups, she gestured towards the first column in her drawing, “so instead of having one-fifth in that one tiny box, [this column] is worth 200 cups.” Concluding, Molly suggested a generalization for her SD by considering a part with a variable amount in it (i.e., either 1/5 of a cup or 200 cups), as long as there is the same amount in each part.

Figure 81. Molly's Second Problem for the Hot Chocolate Problem.

The Hot Chocolate Problem 2. Molly drew a drawing using Variable Parts without drawing a Strip as a Batch prior to her final drawing (Figure 81). After presenting her drawings for specific amounts of chocolate and milk, she talked about a generalized drawing, “In all of the drawings, there are two-um groups of chocolate and there are three groups of milk no matter what size they are… if you put the total number of ounces of chocolate divided by two that would give you how many ounces of chocolate were in one group [points to the blue partitions] and because all the groups were the same size. So, the same thing for milk. So, no matter how many ounces of milk there are you’d do that divided by three that’d give you how many are in this one [points to the green partition].” Molly’s generalized form developed from her last strategy (Figure 81). She now explicitly expressed that any amount of base units could be represented by a partition provided there were the same number of base units in each partition (Variable Parts).

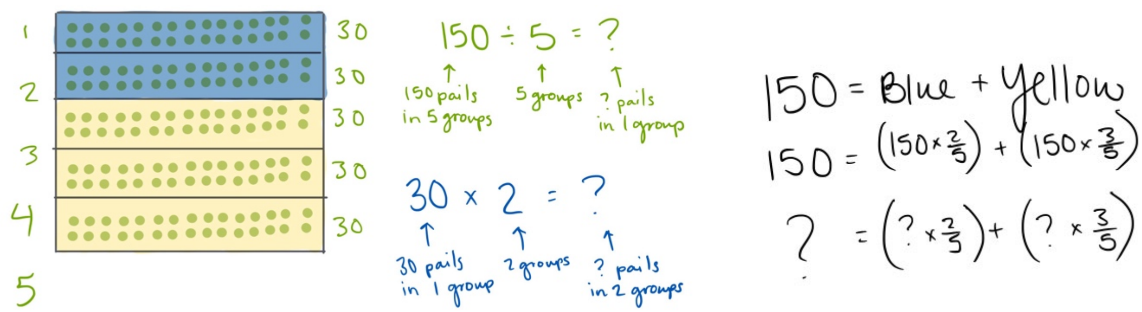
In this period, the students drew strip diagrams showing fixed amounts as determined in the given ratio. For example, the ratio two parts chocolate to three parts milk was drawn showing two ounces of chocolate and three ounces of milk . However, when pressed to generalize or draw an SD showing more quantities, the students used the strips as the number of groups with any amount in each group.

Return of the Definition of Fraction. This final period of the SD began when Dr. B requested that students return to SDs. Students drew Strips as Batches like the previous period and then shifted to drawing Variable Parts by indicating some general amount in a partition.

The teacher’s request to find another solution shaped how students interpreted their strip diagram. Initially, students used each partition as one group but upon thinking through a new strategy, they shifted their attention to using the strip as one group instead of a partition as one group. Shifting the group prompted the students to think about partitions as fractional amounts of the strip, thus the Common Core Definition of Fraction re-emerged in their drawings.

Blue and Yellow Paint. Solving the Blue and Yellow Paint problem, Molly made similar drawings to the previous problem. In particular, while solving the second prompt where 150 pails green paint is given, she initially drew a strip with five partitions of which two partitions were blue and three were yellow. She determined there would 30 pails of paint in one part, obtained by dividing 150 by five. She used this to assign 30 pails in each partition (Figure 82a). At her table, Cameron expressed confusion regarding the relationship between three amounts of yellow, blue, and green paint. He asked the table, “So does two blue and three yellow make five green or one?” Molly said there would be one because “matter cannot be created or destroyed.” When pressed by Cameron, she explained further, “it’s a ratio so it doesn’t matter… because overall you have 150 pails total. You can either look it as you have two pails of blue paint and three pails of yellow paint and then you would have five pails of yellow paint, but also you have two to three parts in one pail of paint.” Molly expressed a generalization of the amount of pails in one partition in that there is any amount of pails.

Dr. B requested a different strategy after most students created SDs similar to what Molly had already drawn. When another student at the table expressed confusion about the 150 pails of paint, Molly used a second solution that she explained as she wrote out the equation in Figure 82b, “the 150 is blue plus yellow and to find how much blue and yellow… so 150 times two-fifths plus 150 times three-fifths. because all of this green and two of those parts are blue and three parts are yellow.” The two fractions two-fifths and three-fifths emerged when she explained what fraction of the green paint is blue paint and what fraction is yellow paint (CCD). She obtained the fraction first partitioning the amount of green paint by five then multiplying the result by the number of partitions.



(a) (b)

Figure 82. Molly's Drawing for the Blue and Yellow Paint Problem.

The Punch Problem. The reemergence of the fraction of a strip using the CCD coarse form also appeared in the Punch Problem. Her first strategy resembled the previous problems in that she first divided 24 liters by eight, the number of partitions, and assigned the result of three liters to each partition. When pressed for a second strategy, she created new equations highilghting new parts of her drawing. Explaining her idea during whole-class discussion, she pointed to the strip, “so when you take the one eighth of all of [the strip]… and then five of those parts are the water so, one part would be one-eighth times five so that’s just looking at the water right here [points to blue partitions].” She then explained that the fraction of water with respect to the strip, “it doesn’t matter what size this container is, it’s just five-eighths of that container is water and you can fill in the total amount of whatever which is ten liters in this instance… you’re looking at it as what is this [blue] group right here in relation to all of it.” This also shows a reemergence of the CCD coarse form. Molly described a set of partitions using the size and number of parts with a visible whole. This strategy differed from previous strategies in that she first determined the size she considered, in this case five-eighths and then used this as the size of her group. Previously, she first determined the size of one partition and assigned this as a group and then iterated the group such as Figure 82a.

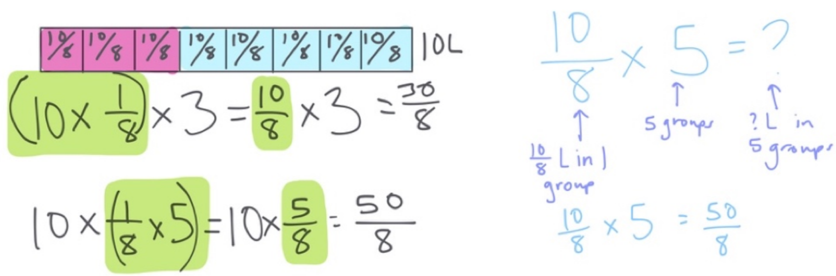
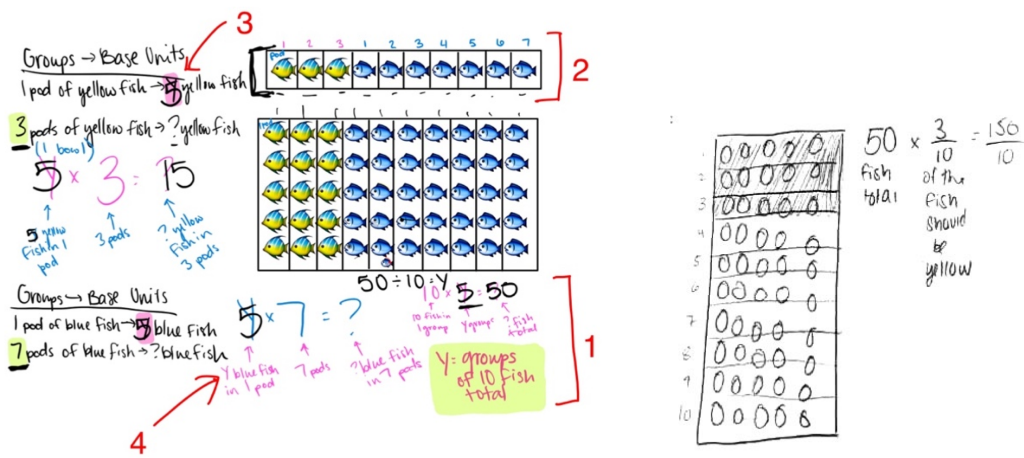


Figure 83. Molly's Drawing for the Punch Problem.

The Fish Problem. The last problem of this period was posed a month after the Punch Problem and during a statistics lesson on inferring populations from samples. Molly's drawing had both the Strip as a Batch and Variable Parts. She first explained she wanted to work it out with variables so her strategy was not limited to 50 total fish, hinting at generalization. She wrote out an equation relating the sample and population in general. When presenting her work during whole-class discussion, she wrote over the variables in her equation to talk about the specific amounts in the problem. Additionally, her drawing (Figure 84a) contained several niches of work. First, Molly identified five groups of ten in 50 because there were ten fish in the population sample and she considered 50 fish total. Second, Molly described the sample using a Strip as a Batch by explaining "there are three yellow fish and seven blue fish and they are each in a 'pod.'" Third, when talking about the population, she used her equation from the first part of her drawing and said there were five fish in one pod, thus if there were three pods, there would be fifteen yellow fish in total. Fourth, she pointed to the last part of her drawing and explained that she could do the same for the seven pods of blue fish. Molly also expressed that her drawing could convey some form of generality in that “you can do it with however many fish you have” indicating she used Variable Parts.

Notably, she did not use the CCD coarse form to explain this drawing in the same way as when she worked through the Punch Problem. Sophie, on the other hand, did (Figure 84b). She explained during whole-class discussion that, based on the sample, three-tenths of the population should be yellow fish. Thus, she created a SD showing three-tenths of the population where there were 50 fish in the population.



(a) (b)

Figure 84. Molly and Sophie’s Drawing for the Fish Problem.

In this period, students created Variable Parts SDs after drawing Strips as Batches. When drawing SDs with variable parts, the students focused on the total amount and divided the amount by the number of partitions but, when asked for a second strategy, they created strip diagrams and equations showing the strip as one group.

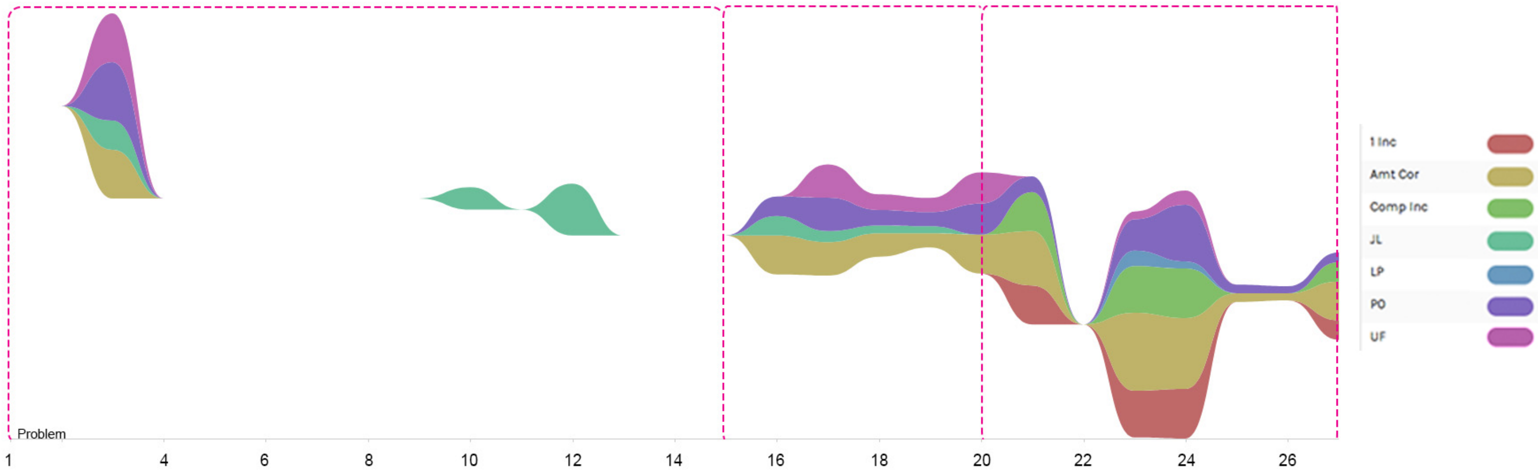
Summary of the Development of the SD. In this section, I demonstrated key time periods in the development of the SD. Beginning with the definition of fraction, the drawings contained coarse forms that supported determining the size of the partitions. Upon introduction of the definition of multiplication, students were more explicit about quantities identified in the problem and how they were related. While working with multiplication problems, the Phantom Partition coarse form helped some students determine the size of the partitions in terms of base units. By refocusing the lesson to whole-number division problems, the drawings reverted to strips and partitions describing whole-number amounts with respect to two different quantities. Upon the re-introduction of fractional amounts to the problem, the coarse forms seen in previous periods re-emerged. In addition to the old coarse forms re-emerging, the Triple Function of a Partition emerged when explaining the keep-change-flip algorithm. Finally, the introduction of SDs proportion problems led to uses of SD thatresembled uses of DNL. When asked to generalize their drawings for multiple numbers, the Variable Parts coarse form emerged. In some cases, the coarse form CCD re-emerged. The Variable Parts form re-emereged a month after the ratio and proportion lessons in a lesson on statistics, particularly inferring a population from a sample.

Task Features Supporting the Development of Double Number Lines

In this section, I outline how DNLs developed over time by describing which coarse forms emerged over time. Moreover, I discuss how task features shaped how students drew the DNLs. In each period, I provide an example of student work. The students drew DNLs at a lower frequency than SDs. Thus, I am not able to consistently provide work from one student.

A Global View of the Development of the DNL

The Streamgraph of the DNLs is found in Figure 85. There is a notable absence of DNLs in the Fall. With the exception of one instance, all the DNLs in the Fall were characterized as Journey Lines. Once Spring began, students drew more DNLs with almost all drawings containing two number lines. The first period of DNLs in the Spring was characterized by partitioning amounts in one quantity and finding the corresponding amounts in the other. Almost exclusively, students partitioned amounts to obtain unit fractional amounts which they subsequently iterated. During lessons on proportions, a new period began where students continued to partition amounts to iterate but, instead of iterating unit fractions, they would iterate sets of corresponding amounts and units of one quantity. The same set of coarse forms emerged once again in the final period when students inferred populations from a sample.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Problem	Playground	Jean's Casserole	Sue's Run	Goblin Goo 1	Pumpkin Juice	Bat Milk Cheese	Dragon Blood	Goblin Goo 2	Blank Multiplication	Francine's Rope	Gallons of Water	Drive	Pizza	Cookies	Brownies	3 ÷ 3/4	Tonya and Chrissy	1 1/2 ÷ 3/4	Noodles	1/3 ÷ 2/5	Hot Chocolate 1	Hot Chocolate 2	Scooter	Rope	Yellow and Blue	Punch	Fishing
Period	X	X	Journey Line	X	X	X	X	X	X	Journey Line	X	Journey Line	X	X	X	Unit Fraction Increments					Unit Increments	X	Composed Unit and Unit Increments				

Figure 85. Streamgraph and Periods for Double Number Lines

Table 16.
Description of the Periods in the Development of Double Number Lines

Period	Coarse Forms	How the Features Shaped the Drawings	Task Features Shaping DNL		
			PT	NC	TR
Journey Line	Predominantly Journey Line	Introduction of the Journey Line as a Drawing		√	√
Unit Fraction Increments	Amount Correspondence, Partition at Zero, Unit Fraction Increments	Partitioning amounts of two quantities simultaneously, emergence of Unit Fraction Increments		√	√
Composed Unit and Unit Increments	Amount Correspondence, Partition at Zero, Unit Increments; sometimes Unit Fraction Increments, Local Partition	Emergence of Unit Increments		√	

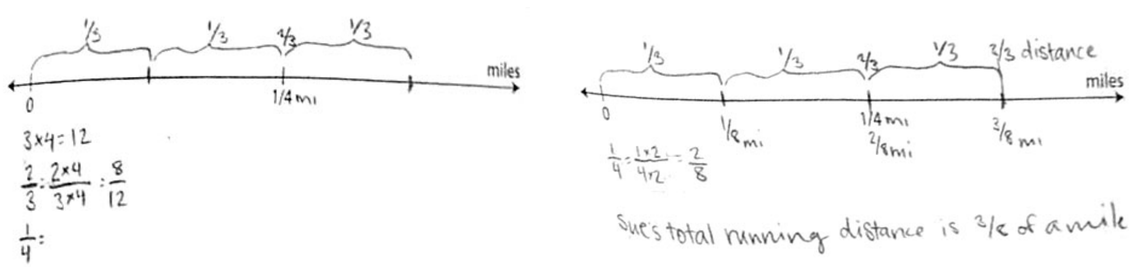
DNL Periods

Journey Line Period. This period is characterized by the dominance of the coarse form Journey Lines and is the only period spanning the Fall semester. Sophie produced the only double number line in the Fall as seen in Figure 53. Students used one number line to coordinate two different quantities to create Journey Lines.

The teacher request task feature shaped the use of a Journey Line in that the instructor requested the students to use a number line (not a double number line) that was provided on the worksheet. The divisibility of the number choices also shaped how students drew their Journey Lines, which mirrors a recurring theme in how students drew double number lines in future uses.

Sue's Run Problem. Molly and her small group initially decided they would compute for a common denominator. When Dr. B asked the group to rationalize their decision, Molly said that common denominators were needed to compare the one-fourth of a mile to two-thirds of the distance and created equally spaced marks (see computation in Figure 86a, bottom). Dr. B then asked where the total distance would be on Molly's number line. Molly drew a tick mark between 0 and the one-fourth mile mark and an additional mark after the one-fourth mile mark to indicate where the total distance would be (Figure 86a, top). She then bracketed the lengths indicating that these were each one-third of the total running distance. Dr. B asked if the common denominators would help with this and Molly quickly said, "Yes." Dr. B suggested she and her group use the common denominators. After her group worked independently, they all decided it was "easier to use eighths instead of twelfths." Molly erased her initial computation for equivalent fraction and wrote the computation for the equivalent fraction two-eighths, which she used to find the distance of the first tick mark (Figure 86b).

The number choice shaped how Molly decided to partition the Journey Line. She needed two partitions to show two-thirds of the mile, thus she needed to express one-fourth as an amount that can be drawn with two partitions.



(a) (b)

Figure 86. Molly's Drawing for Sue's Run Problem.

The Drive Problem and the Francine's Rope Problem. In both problems, Molly decided to differentiate the two quantities with two colors, an option that became available to her using the iPad (see Figure 87). To obtain the speed of 55 miles for one hour in the Drive problem, Molly explained that she did not want to draw 220 little tick marks to show the 220 miles, but rather marked 220 miles and four hours for the same tick mark. She then explained she got 110 miles for two hours and proceeded to get another half to get her answer. She subsequently marked the distance travelled at three hours. During whole class discussion, she wanted to use the number line because "both are happening at the same time," which may explain her use of the Journey Line. She created a similar drawing for Francine's Rope Problem with 32 marks to indicate the yards of rope and marked each four yards with a red mark.

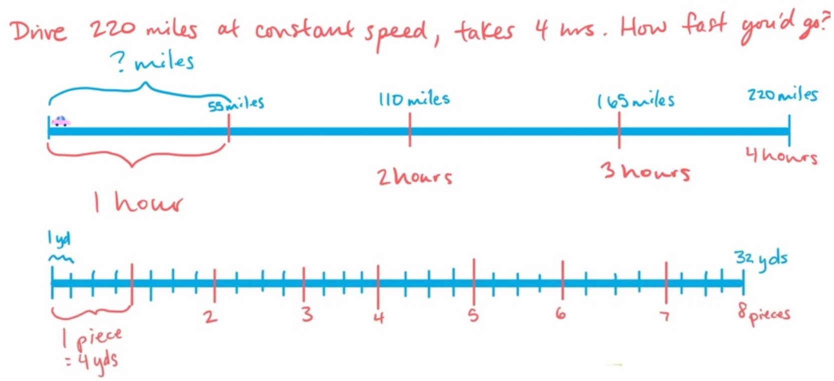


Figure 87. Molly's Drawings for the Drive and Francine's Rope problem.

In both problems, the product was divisible by the number of groups, thus in both cases, Molly did not partition her Journey Line like she did in the previous problem.

Unit Fraction Increments Period. This period is characterized by the emergence and systematic use of the Unit Fraction Increment coarse form as well as the use of double number lines as an alternative to Journey Lines. Similar to Molly's demonstration for Sue's Run Problem, students partitioned an initial correspondence into partitions showing a unit fractional amount of one quantity. They used the partitions they created to partition the other quantity.

The number choice in the tasks shaped the double number lines in that all the crucial number choices were non-whole fractional amounts. By crucial number choices, I refer to the multiplicand for measurement division problems and the product amount for partitive division problems. Students selected these amounts to begin their DNLs. There was no marked difference in how students made their DNLs between the two types of division. In both types, they partitioned the beginning correspondence to indicate a unit fractional amount similar to Molly's strategy for Sue's Run; however, this coarse form emerged more as a DNL than as a Journey Line.

A teacher request also may have shaped the use of a double number line, shifting away from a Journey Line. In her prompt, she provided an exemplar of a DNL (Figure 73). The inclusion of the drawing in the prompt and the invitation to use a DNL may explain the slow disappearance of the Journey Line coarse form and the emergence of a mix of SDs and DNLs. Because students were offered the option to use either drawing, no one student showed a consistent use of DNL in the data collected during this period.

3 ÷ 3/4. Sophie wrote the problem "You have 3 cups of flour. Each batch requires 3/4 cup of flour. How many batches can you make?" and drew a DNL as a second strategy (Figure 88). During whole-class discussion, she explained that she first marked three cups on the top number line using the three pink tick marks. Using the equivalent fraction twelve-fourths, she partitioned the top number line into twelve parts with four increments of one-fourth in each cup. She counted every three fourths cup and marked a batch on the second number line to obtain four batches (Unit Fraction Increments, Amount Correspondence).

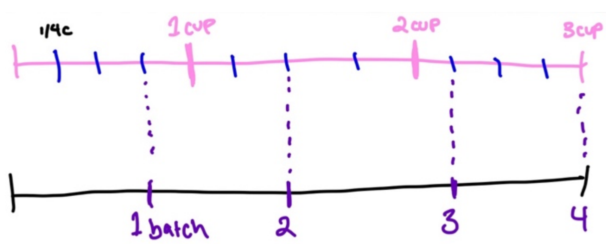


Figure 88. Sophie's Drawing for 3 ÷ 3/4.

Tonya and Chrissy. Jack also used an equivalent fraction to create a unit fraction which he subsequently iterated on his DNL (Figure 89). He marked the corresponding amounts of two-thirds of a cup and two-halves of a serving. He then "reduced" the amounts to get a correspondence between one-third of a cup and one-half serving (Partition at Zero, Amount Correspondence). Jack counted by thirds on the top number line three times to get one cup and counted up three times by half servings to get three-halves of a serving as the corresponding amount on the bottom number line (Unit Fraction Increments).

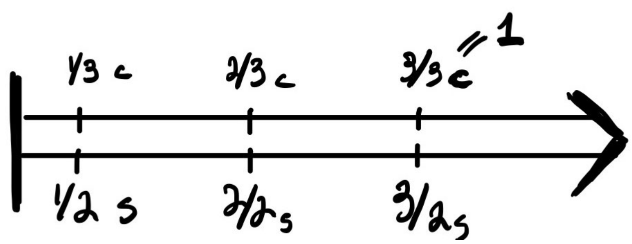


Figure 89. Jack's Drawing for the Tonya and Chrissy Problem.

1 1/2 ÷ 2/3. Elizbeth also used a unit fraction in her double number line (Figure 90). She wrote the measurement division problem “You have 1 1/2 liters of apple juice. You want to pour this apple juice into glasses which can hold 1/3 liters each. How many glasses can you fill with the apple juice?” She started her drawing with “what she knew,” one-third liter was equivalent to one glass (Amount Correspondence). Claiming she needed to find “find some kind of common denominator and make it all line up,” she knew one-third was equivalent two-sixths. Elizabeth drew two parts each size one-sixth. She also partitioned the corresponding interval on the glasses number line into two unit fraction parts. Elizabeth used this correspondence to count on the top number line. She counted up to nine-sixths on the top number line to get to one and a half liters. In the same way, she counted up nine halves on the bottom number line (Unit Fraction Increments).

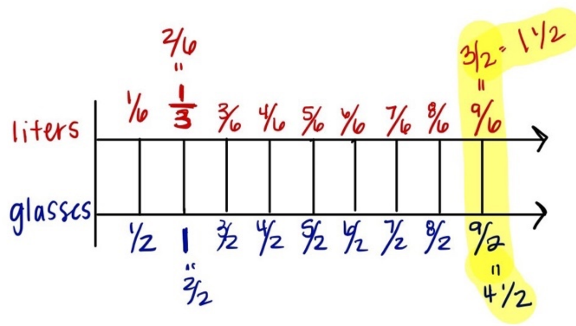


Figure 90. Elizabeth's Drawing for 1 1/2 ÷ 1/3.

Noodles Problem and 1/3 ÷ 2/5. The drawings for the last problems in this period remained consistent with the previous measurement division problems in that the Unit Fraction Increments coarse form was essential in student drawings. Courtney used unit fraction increments in both the Noodles Problem and the Blank Multiplication Problem, both partitive division problems (Figure 91). Similar to Elizabeth and Sarah's drawings, Courtney first drew the relationship between quantities for the Noodles Problem. She drew the product amount using an Amount Correspondence i.e., 120 mg connected to two-thirds servings. Similarly, she began her drawing for 1/3 ÷ 2/5 by drawing the Amount Correspondence by connecting one-third mentos in two-fifths packs. She used these intervals to iterate the interval to one of the quantity she assigned as a group. Recall that Dr. B asked students to justify the keep-change algorithm for both of problems. Courtney explained the keep-change-flip algorithm by justifying the act of partitioning an interval in c parts and iterating the new part d times is equivalent to multiplying by $\frac{d}{c}$. She annotated both DNLs by showing the first partition by an arrow with the label "÷ 2" and the iteration of this interval by an arrow. In particular, she showed $120 \cdot \frac{3}{2}$ when she iterated the unit fraction correspondence three times and $\frac{1}{3} \cdot \frac{5}{2}$ when she iterated the unit fraction correspondence five times in the two DNLs.

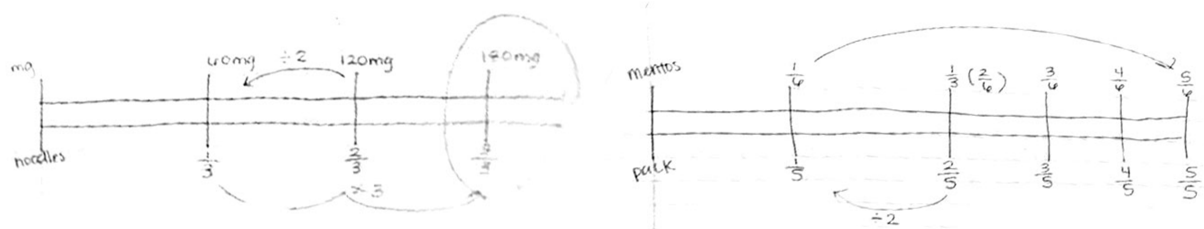


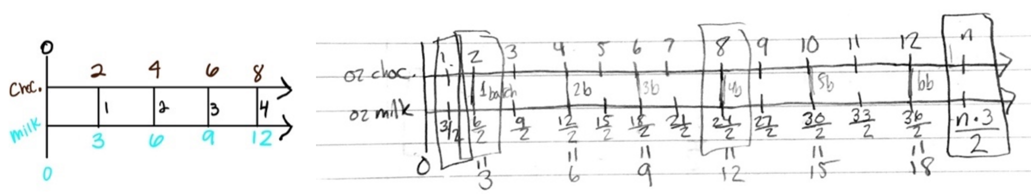
Figure 91. Courtney's Drawings for the Noodles Problem and 1/3 ÷ 2/5.

In this period, students created DNLs, as requested by the instructor. They began with an Amount Correspondence, partitioned the intervals into unit fractions, and then iterated the resulting intervals. Most of the problems in this period were division problems. Unlike in the development of the SD, the prompt of the explaining keep-change-flip did not necessitate a new coarse form.

Composed Unit and Unit Increments Period. This period is characterized by the emergence of the Unit Rate and Composed Unit Increments forms. The students drew DNLs similar to the previous period but with a different sized interval after partitioning the Amount Correspondence. This period occurred exclusively during ratio and proportion lessons. To begin their DNLs, students drew the given ratio as an Amount Correspondence. Using the given amounts in the problem, they determined which interval to iterate.

The number choices determined which coarse form or interval the students used. After students drew an amount correspondence, they iterated a Composed Unit if one of the given quantities was divisible by the matching amount in the given ratio. If not, they partitioned the amount correspondence to unit intervals or intervals of size one.

Hot Chocolate Problem. The instructor asked students to create drawings that would capture multiple mixtures of the hot chocolate problem given amounts of two ounces chocolate and three ounces milk. Most students incremented the given ratio as seen in Jack's drawing in Figure 92a. However, Andrew showed a Unit Increment coarse form where he selected an amount to partition to obtain an amount of one. In Andrew's drawing, he chose to partition two ounces of chocolate in two and the corresponding amount of three ounces of milk to obtain three-halves ounces of milk (Partition at Zero, Amount Correspondence). During small group, he explained partitioning something into two parts was "easier" than partitioning something into three. Once he obtained three-halves ounces of milk, he explained, "we can kind of build that up by one, every time an ounce of chocolate builds up, we get three-halves ounces of milk to go along with it each time" (Unit Increment). Notably, he incremented one ounce of chocolate to a generalized amount *n*. Correspondingly, he iterated the three-half ounces of milk to a generalized amount $\frac{n \cdot 3}{2}$ (Figure 92b).



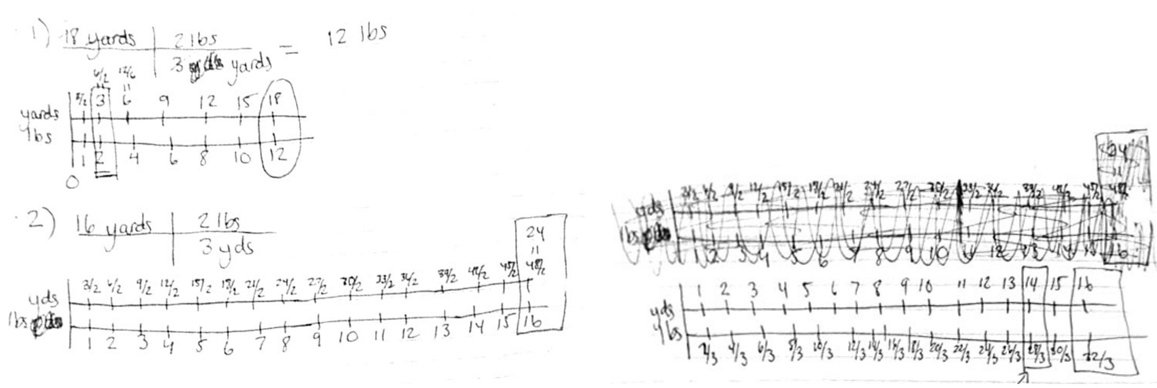
(a) (b)

Figure 92. Jack and Andrew's Drawings for the Hot Chocolate Problem.

The Rope Problem. Andrew used two coarse forms similar to how he answered the Hot Chocolate problem. Recall in the Rope problem, students needed to find the amount of pounds for 18 yards, 16 yards, and 14 yards with three yards of rope weighing two pounds. He drew a DNL (Figure 93a, top) similar to Jack's drawing in the

Hot Chocolate problem to obtain the amount of pounds for 18 yards by iterating the Amount Correspondence of three yards and two pounds to get 18 yards and 12 pounds (Composed Unit Increment). Notably, 18 yards is divisible by three yards.

When looking for amount of pounds corresponding to 16 yards, he told a graduate student sitting at the table that he used the relationship between three yards and two pounds and partitioned the interval of two pounds to get one pound. When asked why he partitioned the amount of pounds he said that he wanted to get “the pounds down to a unit” so he could increment up (Figure 93a, bottom). Upon reflecting on his strategy with the graduate student, he determined he needed to partition the yards instead of pounds because “this is what we’re given.” Additionally, he claimed that the reason for using a Unit Increment is because the amount given, 16 yards, was a whole number. He then erased his drawing and created a new drawing (Figure 93b) where he partitioned the pounds first (Partition at Zero) and then incremented the corresponding amounts one yard and two-thirds pounds (Unit Increments).

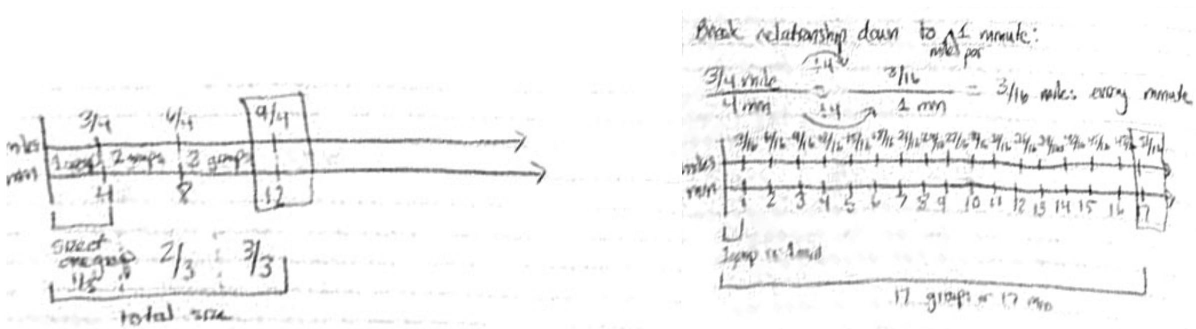


(a) (b)

Figure 93. Andrew's Drawing for the Rope Problem.

The Scooter Problem. Although Andrew’s work was seen on camera (Figure 94), there were not many opportunities to gather data on how he created his drawing.

However, there were consistencies with his previous drawings. First, he incremented by the amounts in the given ratio for his first drawing (i.e., incrementing by four minutes and three-fourths miles (Figure 94a)), when looking for the amount of miles travelled after 12 minutes (Composed Unit Increments). The amount under consideration (12 minutes) was divisible by the four minutes. In the second drawing where he needed to find the amount of miles for 17 minutes (Figure 94b), Andrew wrote a memorized algorithm to obtain how many miles correspond to one minute instead of partitioning the interval as he had done in past problems. He found three-sixteenths of a mile was the amount corresponding to one minute. He drew this relationship on the DNL and iterated the interval seventeen times (Unit Increments). He obtained fifty-one-sixteenths by counting up be three-sixteenths.



(a) (b)

Figure 94. Andrew's Drawing for the Scooter Problem.

Most students also used some computation or algorithm to find the amount of miles corresponding to one minute in this problem like Andrew. However, Nina was able to reason with a DNL using a Unit Fraction increment to find the correspondence. Based on her first drawing (Figure 95, top), she “zoomed in” to the interval from zero to three-fourths mile and four minutes. Considering the three-fourths, she “went back down” to the unit fraction amount one-fourths mile and, correspondingly, she “went down” to four-thirds minutes. She finally considered the correspondence of four-thirds minutes and one-fourth mile. She partitioned the four-thirds minutes interval into four parts to get the amount one-third minutes, another unit fraction amount. She divided one-fourth by four to get one-sixteenth using a memorized number fact and then counted up three spaces to get three-thirds minutes and three-sixteenths mile. When asked to re-explain her thinking, she realized that she “over complicated things” and she could have partitioned four minutes into four also to obtain one minute.

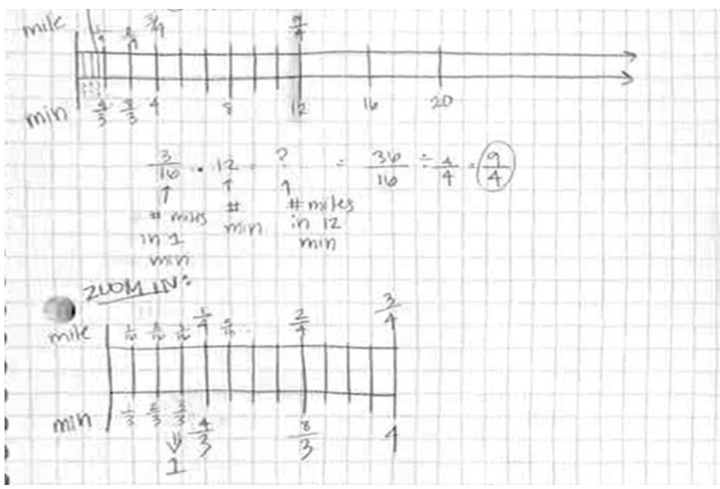


Figure 95. Nina's Drawing for the Scooter Problem.

Up to this point in this period, students used the Partitions at Zero to get an amount to one of a quantity. Once they found the corresponding amount of the other quantity, they iterated this established Amount Correspondence to the amount they wanted to examine. Notably in the Streamgraph for DNLs (Figure 85), the Local Partition coarse form emerged; however, not all students took this form up. In fact, the only instances of the Local Partition were seen in Catherine and Winnie’s work in Figure 49 and Figure 50. Additionally, the period spans the Blue and Yellow Paint and Punch problem. Although students created DNLs that I subsequently coded, these were not discussed in detail in class. Dr. B redirected students who drew DNLs to create SDs when they completed their DNLs. I did not note any significant developments of the DNL in the student work for these problems.

This period ended with the re-emergence of the Unit Rate coarse form after students primarily used SDs in the Yellow and Blue Paint problems. Using my own heuristics for a period, I decided to include this with the previous period because drawings remained similar even with a shift of content areas from Number and Operations to Statistics. The task features, as described in the previous problems, played a similar role in shaping the DNLs.

The Fishing Problem. Students created DNLs using the Composed Unit Increments similar to Nina's work in Figure 96. Given the correspondence between three yellow fish and ten total fish, Nina created an Amount Correspondence with these amounts and iterated this correspondence to get to 50 total fish and correspondingly, 15 yellow fish. Worth noting, 50 was divisible by one of the amounts in the given ratio.

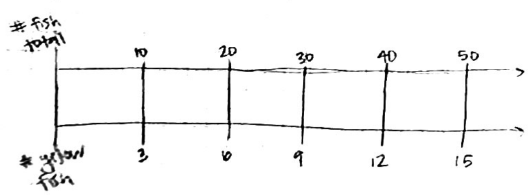


Figure 96. Nina's First Drawing for the Fishing Problem.

Nina created a second drawing by adding to her first (Figure 97). Dr. B prompted the students to another strategy. Nina decided to determine the amount of yellow fish for one fish in total. She obtained the correspondence for one fish, there "would theoretically be three-tenths yellow fish" for one fish in total by partitioning the first interval of her DNL (Partition at Zero). She used this correspondence 50 times to get to

50 total fish and correspondingly iterated fifty three-tenths yellow fish (Unit Increment). Subsequently in whole class, the class discussed the similarities and differences with Nina's drawing where three-tenths is the amount in one group and Sophie's Drawing (Figure 84b) where three-tenths is the size of the group.

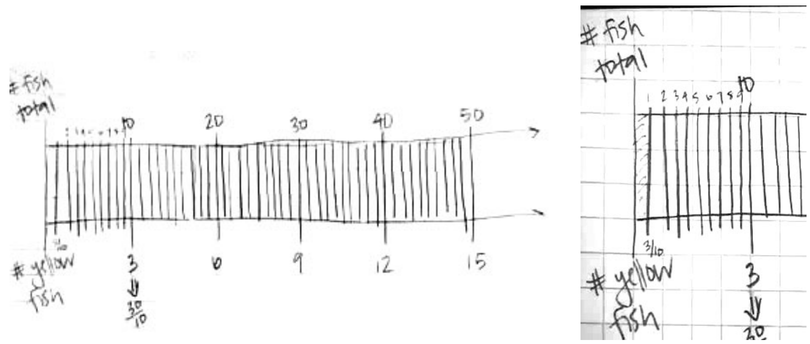


Figure 97. Nina's Second Drawing for the Fishing Problem.

Summary of the Development of the DNL. In this section, I demonstrated key time periods and task features shaping how students drew their DNL. In the Fall, DNLs were sparse and if they emerged, they were mostly Journey Lines. In the Spring, the Journey Line began to disappear and drawings with two distinct lines emerged when the instructor provided an exemplar of the DNL. The development of the DNL relied on the coarse form Partition at Zero. For all the subsequent periods, students decided the size of the interval they want to iterate. They created different sized intervals which were subsequently iterated depending on the number choice.

Summary. In the final section of this chapter, I discussed how certain task features shaped how students drew SDs and DNLs. I narrowed the task features to three broad features: problem type, number choice, and teacher request. The problem type shaped the SD more than the DNL in that depending what was given in the problem, students drew SDs encoding a relationship between two quantities; and, whatever action (e.g., partitioning) they did with one quantity, they did on the same drawing. On the other hand, students worked on number lines (i.e., each quantity), separately to determine what action was needed on the other quantity. The number choice played a role in determining how many partitions students wanted to draw on their SDs and the size of the partitions on the DNLs. Finally, teacher requests such as a request for a second strategy or a request to use a definition played a role in how they students created and reasoned with their drawings. I only concentrated on task features as a condition that shaped the drawings, thus shifting my focus away from other potential aspects of the classroom interaction such as positionality.

CHAPTER 5

CONCLUSION

I remember when my friends and I would decorate our hands with henna for special occasions. Instead of drawing flowers and patterns we would paint our hands with mathematical formulas and equations. We had a thirst for education because our future was right there in that classroom.

– Malala Yousefzai, Nobel Prize Acceptance Speech

And I promise you, the teachers here and the teachers around the country, they're not doing it for the pay — because teachers, unfortunately, still aren't paid as much as they should be. They're not doing it for the glory. They're doing it because they love you, and they believe in you, and they want to help you succeed. So, teachers deserve more than just our gratitude — they deserve our full support.

– Former President Barack Hussein Obama, Speech to Benjamin Banneker Academic High School on October 17, 2016

I was motivated to conduct this study based on my history as a mathematics student and teacher. As a student of mathematics, I struggled to recall algorithms and rules that I was taught to execute but not understand. I used different ways of doing mathematics, including drawing, to help me make sense of particular concepts. When I talked about mathematics with my classmates, I used multiple ways of communicating and notating beyond what we did in class. However, as a beginning teacher, I taught the same algorithms and rules without giving my students opportunities to make sense of what I was teaching. I created the same culture of memorized rules that made me struggle as a student. I never received support in my own teacher education program to help new teachers like me who preferred to use drawings and teach beyond algorithms and symbol pushing. A decade later, I observed a course for prospective teachers where they were doing what I wanted to do as a beginning teacher—they used drawings to solve mathematical problems. I crafted this study to understand what happened in the course. I investigated what they were drawing, how these drawings helped them think about mathematics, and what features of the problems influenced their drawings.

The main data corpus for this study was video and audio-recorded lessons from a year-long content course for prospective middle school mathematics teachers. The course was designed to help them make sense of middle school mathematics using strip diagrams and double number lines. I drew heavily on two lines of analysis from the Papua New Guinea and subsequent classroom studies lead by Geoffrey Saxe to understand how inscriptions such as drawings play a role in learning and communicating. I conducted a microgenetic analysis where I examined how individuals created drawings (i.e., forms) for specific purposes (i.e., functions). I also conducted an ontogenetic analysis to examine how the drawings changed over time and determined reasons why the drawings changed. In the following sections, I discuss the findings for my three original research questions.

RQ1. What methods and grain sizes should researchers consider when characterizing forms and functions of mathematical drawings in classroom data?

I used three grain sizes of forms and functions to reduce my data and to organize previous research (see Figure 11). The largest grain size comprises systems of forms used by larger communities and that remain stable over long periods of time such as the modern base-10 system. I used the next two levels in my research which could be refined by researchers in the future. I called the smallest grain size “micro forms” which were used to represent amounts of quantities in the problem. I use the term quantity to describe a measurable attribute of an object (Lamon, 2007; Schwartz, 1988; Thompson, 1994) and the word amount to refer to numerical measure of a quantity in terms of some unit. I also describe this form as “indestructible” in that a micro form cannot be broken down further to serve a function. For example, if a rectangle is drawn to represent a serving and the student does not indicate any significance for the line segments comprising the sides of the rectangle, then the rectangle is a micro form. I used pointing language (e.g., “This is…”), gestures, and supplementary inscriptions to identify micro forms and their functions. I also conceptualized a “coarse form.” Coarse forms are the combination of micro forms that represent quantities simultaneously or serve broader problem-solving functions. I identified these forms using temporal language (e.g., “First, I drew [coarse form 1], and then I drew [coarse form 2]…”).

When considering methods, I outlined heuristics for analysis in Chapter 3. I constructed these heuristics to ensure all drawings were coded and general themes could emerge. I also identified whole-class discussions as “ground-zero” for analysis because these discussions were the central context for use of the forms and functions I identified in the study. Theoretically, students used collective forms and functions when creating public displays. By collective, I assumed that students used acceptable and understandable practices within the community. Thus, forms and functions from whole-class discussions were assumed to be part of the collective practice. Additionally, the forms and functions permeated small-group work. Students were expected to present their ideas to the whole class and thus employed practices of the whole class, not just of the small group.

RQ2. What forms and functions of strip diagrams and double number lines emerged from student drawings in a content course for prospective middle school teachers?

Using the methodology outlined in Chapter 3 and summarized in the previous section, I found four micro forms and nine coarse forms for strip diagrams (Table 9 and Table 10, respectively). I also found five micro forms and seven coarse forms for double number lines (Table 12 and Table 13, respectively). These forms provided analytical tools to distill key features of student drawings. I outline the general findings of the research questions along with connections to literature.

Strip Diagrams. Two coarse forms (i.e., Common Core Denominator, Common Core Numerator) provided students with different approaches to representing fractional amounts with an explicit connection to the Common Core Definition of fraction. Dr. B focused on the number of partitions and the size of the partitions to make the definition explicit. Students created partitions by creating smaller parts from a physical or mental strip, a mental operation described by Kieran, T. E. (1983).

Partitioned Partitions is a form where partitions are partitioned further to serve different functions such as displaying the value of the least common denominator. Olive (1999) argued a similar action, recursive partitioning, is critical in constructing fraction schemes. This study offers more insight into why future teachers partitioned partitions. Although most reasons for partitioning were grounded in using a strip diagram, future teachers also recalled memorized rules for fractions (e.g., least common denominators) to determine how to partition partitions further.

The Dual Function of a Partition and Dual Function of a Strip are coarse forms where students described strips and partitions with respect to two different quantities, mostly based on the multiplicand or product in the given problem. In Chapter 2, I outlined how the movement from addition situations to multiplication situations increases the measure space from one to two. A measure space is “different sets of objects, different types of quantities, or different units of measure” (Lamon, 2009, p. 634). In multiplication situations, the two measure spaces are linked by an intensive quantity such as the multiplicand in multiplication problems or the multiplier in proportion problems. These two coarse forms supported students to consider two measure spaces and keep track of how the amounts in each measure space are related (i.e., establishing an isomorphism of measures (Vergnaud, 1988)).

The Phantom Partition is a coarse form where students added more partitions to determine the size of a part of a strip diagram whose whole is not drawn. This coarse form resembles of conception of Steffe and Olive’s (2010) partitive unit fraction scheme where a student considers “a unit fractional part of the fractional whole, disembed the part, and iterate it to produce another partitioned continuous unit” (p. 324).

The final two coarse forms were present in displays for proportion problems. The coarse form Strip as a Batch is where students drew a specific amount of a quantity, such as drawing a hot cocoa mixture with four parts chocolate and six parts milk where all four and six parts are explicitly drawn. This coarse form resembles Kaput and West’s

(1994) description of the building-up process where students “construct the quantities involved at a gross level” (p. 246) or an extensive quantity by displaying the measured attribute of an object, situation, or event. Variable Parts is a coarse form emerging from, perhaps, a generalizing action (Ellis, 2007) in that students focused their drawing on similarities across different cases in one drawing (i.e., the ratio). Thus, Variable Parts highlights an intensive quantity (Schwartz, 1988) by displaying the ratio as a feature of any amount rather than the specific amount of the problem.

Double Number Lines. The coarse form I identified exclusively in the Fall data is the Journey Line, a line where students showed how two amounts of two quantities were related on a single number line. In the Spring, students separated the quantities and displayed each quantity on a separate number line. These displays began with an Amount Correspondence, a coarse form showing how two amounts of two quantities are related on two separate number lines. Once the correspondence was drawn, the students used the last three coarse forms by either iterating the correspondence (Composite Unit Increments) or partitioning the correspondence to either a unit fraction (Unit Fraction Increments) or one of a quantity (Unit Increments) which was iterated. Orrill and Brown (2012) found similar results. They found coordinating units and partitioning are critical pieces of knowledge for teachers when working with DNLs.

RQ3. What features of the mathematical tasks shaped the use of certain forms and functions over time?

I found three task features critical to how the students’ displays changed over time: problem type, number choice, and teacher requests. I conceptualized the idea of a “period” to determine the task features. When a new coarse form emerged in students’ drawings, I understood this as an accommodation in response to a new task feature.

The problem type played a role mostly in determining how strip diagrams were initially created. For multiplication problems, students generally began their drawing by creating a Dual Function of a Strip to represent the multiplicand (the amount in one group). In division problems, the students began their drawings with the product amount (amount in M groups).

The number choice feature influenced how students partitioned partitions further or if they needed to add more partitions to the drawing to determine the size of the partitions. For example, if students drew the product amount for a measurement division problem with each partition representing the amount $1/m$ but needed partitions of size $1/n$ based on the multiplicand, they would partition the strip further. Also, consider a multiplicand less than one in measurement division problems. Students initially drew the multiplicand showing the amount of base units in one group (i.e., not showing one whole base unit). In most cases, partitions needed to be described in terms of the base unit. Because a whole base unit is not drawn, the students drew Phantom Partitions to determine the size of the partitions with respect to the base unit. When drawing DNLs, the number choice, particularly if the given numbers were divisible, played a role in determining which amounts the students iterated. For example, consider a student who marked the number five on a number line and wanted to iterate an interval to sixteen. Because sixteen is not divisible by five, the student would partition the interval of five to create an interval of one and iterate the new interval to sixteen.

Finally, teacher requests to use a definition, explain an algorithm, or find a different strategy changed the drawings. The teacher established a culture of using the definitions for fraction and multiplication when explaining a strategy. For example, when a student did not know how to partition aside from invoking “common denominators,” she redirected the student to use the definition of fraction to help the student think of the number of partitions required. Additionally, teacher requests to explain commonly used algorithms such as keep-change-flip invoked the emergence of some coarse forms. Finally, when the teacher requested a new strategy, new coarse forms were used in the drawings—for instance, when she asked for a second strategy in proportion problems, students shifted their attention from partitions of a strip to the whole strip.

Interpretation of Findings

This study provides some grounding for the reconceptualization of the canon in mathematics education based on the needs for the study outlined in Chapter 1. I argued for the need to study teacher preparation courses, understand teachers’ use of representations, and develop new methodologies for classroom research. In this section, I revert to using “prospective teachers” (PSTs) to refer to the students enrolled in the content course.

Revisiting the need to study teacher preparation courses. This study provides a near day-to-day analysis of what occurred in one sequence of content courses for prospective teachers. There is a limited amount of research describing what occurs in content courses. Researchers have investigated these courses by relying on retrospective descriptions. In contrast, I gathered data directly from these courses and analyzes the development and the circumstances engendering the development in those content courses. In particular, I described how strip diagrams and double number lines developed from simple drawings representing fractions to more complex diagrams showing fraction operations involving ratio and proportions. Such descriptions illuminate the intricacy of developing specialized content knowledge for teaching.

Teacher preparation programs are charged with developing content-specific knowledge of the subject matter prospective teachers will teach; however, researchers have framed mathematical knowledge as something to be learned, not questioned. Researchers have also focused on interventions that emphasize challenging future teachers' existing beliefs about mathematics teaching (Conner & Gomez, 2018; Feinman-Nemser, 2008; Mewborn & Tyminski, 2006) but not their mathematical knowledge. AMTE (2017) described an effective mathematics teacher preparation program as one providing "opportunities for candidates to learn, with understanding and depth, the school mathematics and statistics content they will teach" (indicator P.2.1); but, researchers and teacher educators often have not questioned what it means to "learn with understanding and depth" within spaces of teacher preparation. For the most part, researchers (including myself) have tended to overgeneralize results learned from K-12 research when describing teacher mathematical knowledge. Content courses are thus taught using these results and disregard the potentially different mathematical knowledge base emerging from their K-12 education.

We need to reconceptualize how we view prospective teachers and the function of content courses. When prospective teachers enter content courses, they may feel the need to leave their K-12 histories at the door because they are now asked to do certain things differently. Teacher educators need instead to acknowledge these histories. This need became evident in the present study when the PSTs demonstrated strong propensity to use memorized rules at the beginning of the year. Dr. B did not outlaw memorized rules; rather, she challenged the rules and asked the teachers to make sense of them. She also challenged the the PSTs' conceptions of what it means to do mathematics. Her expectation of making math drawings became normative mathematical practice. Dr. B provided structures, such as definitions and drawings to help them develop this practice and challenge what they knew. Teacher educators must view content courses as grounds where the PSTs actively negotiate what they experienced in K-12 and what they experience in university content courses. Such negotiations are not exclusively the jurisdiction of methods courses.

Researchers must also consider the knowledge teachers bring to teacher preparation content courses by drawing from relevant literature and adjusting theoretical frameworks. Although we rightfully draw from research on K-12 students, we have yet to identify mathematical knowledge and processes unique to the PSTs. One way we can move forward is by focusing on the interaction between knowledge learned in K-12 classrooms and knowledge learned in teacher preparation coursework by investigating if this interaction is one of conflict or resolution. During a casual conversation with Molly and Catherine in the Spring, they mused about a 6th grade classroom observation involving multiplication that they had recently conducted for another course. Reflecting on the observation, Molly told me she could not see multiplication any other way because of the course with Dr. B. Although this shift in Molly's thinking may be fleeting, for a time her knowledge shifted. Because of this study, we can understand the unique conditions supporting such shifts in knowledge in teacher preparation.

Theoretical frameworks describing teacher knowledge downplay the role of previous knowledge, especially K-12 experiences. Notably, the role of K-12 experiences is absent in most theories on teacher knowledge. Ball et al. (2008) 's characterization of teacher knowledge stopped short of identifying the genesis of the components of mathematical knowledge for teaching. Hammerness, Darling-Hammond, Grossman, Rust, and Shulman (2005) acknowledged prior experiences as relevant in learning to teach as "both problems and possibilities that derive from the apprenticeship of observation they have all experienced" (p. 400) but do not explicitly identify prior experiences as their content knowledge. Alternatively, Silverman and Thompson (2008) recognized the role of prior mathematical knowledge in developing "personally powerful understanding of particular mathematical concepts" (p. 502). However, they described development by "simply augmenting" prior knowledge with new knowledge as less desirable than knowledge developed from "reflective abstraction." Their description of prior knowledge is limited to their experience of solving a problem and that augmenting prior knowledge is "simpler" than transforming (abstracting) knowledge. The PSTs in this study often recalled ideas from K-12 experiences such as common denominators and how to set up an equation for proportion problems throughout the year even after a prolonged practice of drawing. Although they were able to do mathematics differently from their K-12 experiences, Dr. B did not whitewash these experiences. Such experiences should be allowed to exist alongside the knowledge developed in content courses, even if researchers view the two as irreconcilable. We should include K-12 experiences in results and theoretical frameworks to recognize their impact on teachers' knowledge and development.

Revisiting the need to study teachers' use of representations. Researchers have called for "opportunities to learn" that support teachers' use of drawings in their instruction but have stopped short of characterizing such opportunities. For example, the NRC (2001) defined opportunities to learn as "circumstances that allow students to engage in and spend time on academic tasks" (p. 333) and Jacobson and Izsák (2015) defined them in "a broad, polymorphous sense as the minimal educational condition by which teachers might begin to use [drawn] models in their instruction" (p. 470). At the beginning of the year, several PSTs commented they did not know what a math drawing was or they would assert, "I don't know how to make a math drawing." Over time, the teachers were able to draw their thinking for different problem types. The results of this study provide insight into specific conditions and circumstances supporting the PSTs use of drawings.

The instructor of the course privileged the use of math drawings in conjunction with other representations such as equations. She achieved this by setting explicit expectations for the PSTs to use math drawings and equations in all their strategies. The instructor promoted a sustained use of the same representations over time. Because the PSTs used the same representation across lessons and topics, they were able to modify their drawings to accommodate for changes in task features, thus creating more complex drawings in response to changes in problem type and number choice. Dr. B also prompted pro to confront previous mathematical knowledge using these representations. This supported the PSTs in making sense of what they already knew, such as why common denominators are needed for some problems or why keep-change-flip works. This demonstrates that Dr. B did not attempt to “replace” what the PSTs already knew but rather pushed them to make sense of why certain mathematical algorithms work. The drawings became a vital component in negotiating these differences between what they learned in K-12 and teacher preparation coursework.

She also expected the PSTs to use coherent meanings across representations. In particular, she supported a quantitative definition for multiplication that became crucial in creating drawings. The PSTs used quantities given in the problem to create their drawings—for instance, by using the multiplicand to draw a strip representing both one of a group and the corresponding amount of base units. The quantitative definition also supported the PSTs to identify a goal of the drawing)e.g., determine the amount of base units in a certain amount of groups), and corresponding actions to achieve the goal (e.g., partition an interval then iterate). The prospective teachers’ drawings and explanations leveraged the coherent meanings by identifying components of the definitions in their drawings and using definitions to create new forms such as partitioned partitions.

Using drawings and meanings over time supports view mathematics as connected whole. Certain forms emerged when teachers made their displays for different problem types (as seen from an expert point of view) and seemed stable for a long period of time such as the Dual Function of a Strip or the Common Core coarse forms. As indicated in the Streamgraphs (Figure 59 and Figure 85), there are forms that persisted across time. Specifically, these forms are based on the definitions of multiplication and fraction introduced in class. The stability of some forms based on coherent meanings may be indicative of the teachers' understanding of the multiplicative conceptual field, the mathematics they will teach, as a coherent whole.

Revisiting the need to study classroom data. Finally, this study contributes to methodologies for research in classrooms. Schoenfeld (2008) argued methods are lenses through which phenomena are viewed instead of being "pulled off the shelf." Perhaps because of limited word counts, researchers have opted to obscure their methods in academic publications by hand-waving over complex processes such as "multiple rounds of coding" and "creating second-order models." Thus, terse descriptions of methods come across as if the researcher selected their methods from the shelf even if this is not the case. Bikner-Ahsbahs, Knipping, and Presmeg (2015) also acknowledged the sparsity of explicit methodologies:

> Detailed descriptions on how methodologies are substantiated in a specific project, how they are implemented to investigate a research question, and how they are used to capture the research objects are normally missing… Scholars of mathematics education also should communicate their new developments in research methodologies and make them accessible to other in order to sustain a critical debate. (p. v).

Missing and incomplete reports of methods present difficulties for future researchers who are interested in studying classrooms. The classroom has complex, multiple phenomena occurring simultaneously; and, if Schoenfeld was correct to describe methods as lenses, knowing how the lenses are made and refined is critical for making sense of the analysis of classroom data. The study provides some methodological clarity in research in classrooms, particularly investigating how drawings are created. By delineating grain sizes, I support a more coherent research program for analyzing drawings created in mathematics classrooms by identifying what is analyzed. The methods I outlined are also an account of strengths of my methods but also address the limitations of previous iterations.

Specifically, the methodology I developed provides researchers with language to identify what they are analyzing. Although the nomenclature “micro forms” and “coarse forms” are unique to this study, I found similar approaches in other research. For example, Meira (1995) analyzed a pair of students’ work by segmenting their representation on a moment-to-moment level resulting in 16 sets of inscriptions. Meira’s characterization of the moment-to-moment inscriptions had a similar “texture” to identifying micro forms in that he considered the smallest possible set of inscriptions carrying meaning. Lobato et al. (2014) created diagram sequences of inscriptions and gestures to infer students’ conceptions of speed. Similar to coarse forms, they broke up diagrams based on sets of inscriptions to infer conceptions of speed such as drawing a “composed unit of elapsed distance and the corresponding elapsed time.” The sets were also composed of smaller elements such as tick marks. In both studies, the drawings were observed in interview settings, which facilitated the analysis of how the drawings were created.

Classroom data, by its nature, makes similar analytical processes difficult. Unless the researcher sets up a technologically-savvy data collection process, drawings are most likely static in classroom discussions especially in whole-class discussions. The techniques I outlined foreground what is important for the student, not necessarily the researcher. By relying on student utterances and gestures (e.g., pointing to parts of their drawings, sequencing their own drawings), researchers could segment static drawings in accordance to the student's own analysis of their drawings.

Future Research

There are three areas for future research. The first area is to continually improve the methods outlined in Chapter 3. The methods need to be tested in other environments such as K-12 classrooms and methods courses, other content areas such as geometry, and with different technologies (the teachers used the capabilities of an iPad to create drawings). A more fine-grained conception of time is also needed to refine the analysis describing development across time. In this study, I conceptualized time as measured by the problems the PSTs solved. The heuristics for defining a period may have changed if the unit of time was a day instead of a problem.

A second area for future research is working on a third strand of analysis in Saxe's work that is absent in this study. The third strand, a sociogenetic analysis, provides an explanation for how the forms and functions spread throughout a community. In the Papua New Guinea studies, he identified areas of commerce as crucial sites where the function of *fu* shifted. When he recontextualized his analysis to the classroom, his analysis seemed limited. The researchers asked students on worksheets and interviews a variant of the question "Who would you get math help from?" This provided data with which to create sociograms, visual representations of who had more “power” in class (i.e., the more nominations a student had, the more power). I found similar data in this content course where the PSTs would discursively attribute their work to a certain individual, such as prefacing their whole-class discussion with “I worked with [student] on this problem” or “I did what [student] did.” This utterance would position another student. With recent shifts in mathematics education, I found more analytical techniques, beyond a single question, to begin a more robust description of social positions in class and how they came to be. These newer methods provide a moment-to-moment view of who is positioned in class (Bishop, 2012, tracked the positioning moves of a pair of students). The data from the course provides a chance to analyze how utterances position one another and how these positions affect the propagation of coarse forms. Such an analysis needs more data, an extended timeline, and a new theoretical frame for analysis.

This study describes, in clearer terms, what opportunities to learn with drawings looks like; however, more work is needed. It is beyond the scope of my study to investigate how the PSTs taught in their field experiences. In Chapter 1, I juxtaposed two teachers, Ms. Daniels and Elizabeth. It was clear to me that, unlike Ms. Daniels, Elizabeth demonstrated robust mathematical understanding for using drawings but perhaps, during student teaching, she might struggle to integrate what she learned when teaching. In this case, her story would converge with Ms. Daniels’ story. More work is needed in teacher education and research to understand the opportunities to learn to teach with drawings. We need to identify similar conditions in coursework that promote PSTs’ productive use of drawings in instruction, especially in induction years.

Final Thoughts

To pull back towards the bigger picture, I decided to end the dissertation with two quotes found at the beginning of this chapter. I wanted to highlight how this dissertation is not just about representations, mathematics, or teacher education. As Schoenfeld (2008) asserted, researchers must "guard against the dangers of compartmentalization… This can be costly, given the systemic and deeply connected nature of educational phenomena" (p. 475). As an early researcher who has the resources to continue my education, I feel the need to continually check my privilege. In 2019, there are still large swaths of students who fight for the right to attend school. Students like Malala do not view education the way most American researchers view education. For a lot of students, education is liberation, a way out of an oppressive regime–a view of education I did not have to consider because of my place in society. Researchers in education need to be cognizant of the human rights component to their work even if they do not see it. We may never meet the future students of our prospective teachers who are fighting for a high school diploma to find better opportunities for themselves and their families. We should actively work for access to quality education for all teachers and students. We cannot analyze Brenda's mathematics, publish the analysis for decades, and not once support Brenda's teacher.

Former President Obama reminded us that teachers exhibit genuine care for their students. As researchers, we need to re-frame our work to honor this care. Superfine (2019) reminded researchers to engage in research *with* teachers rather than *on* teachers. I want to expand this notion to include research *for* teachers. We are not simply documenting opportunities to learn, creating second-order models, or describing equitable discursive mechanisms. Our research should not be about "fixing" teachers to think more like us. We are tasked with creating data-driven directions supporting teacher educators and teachers in mathematics education to ensure the right to quality education for all future generations of teachers and students.

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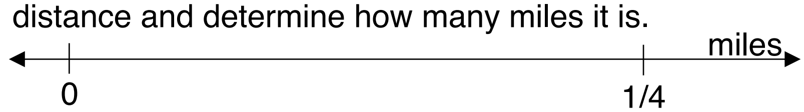
APPENDIX

Appendix A

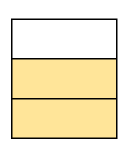
List of Class Problems

Fall 2016

August 18, 2016	
Playground	At a neighborhood park, 1/3 of the park is to be used for a new playground. Swings will be placed on 1/4 of the area of the playground. What fraction of the neighborhood will the swing area be?
August 30, 2016t	
Jean's casserole	Jean has a casserole recipe that calls for 1/2 cup of butter. She has 1/3 cup of butter. Assuming that Jean has enough of the other ingredients, what fraction of the casserole recipe can Jean make?
Sue's run	So far, Sue has run 1/4 of a mile but that is only 2/3 of her total running distance. Plot Sue's total running distance and determine how many miles it is. 
October 13, 2016 (Multiplication)	
First goblin goo	If you have 3 containers of goblin goo, each which is 4/5 liters, then how many liters of goblin goo do you have?
October 20, 2016 (Multiplication)	
Pumpkin juice	if 1 serving of pumpkin juice has 12 grams of sugar in it, then how many grams of sugar are in 1/4 serving of pumpkin juice?
Bat milk cheese	You had 1/4 of a serving of bat milk cheese. One serving of bat milk cheese is 8/3 ounces. How many ounces of cheese do you have?

October 25, 2016 (Multiplication)	
Dragon blood	One serving of dragon blood is 1/5 of a liter, but you only want 1/3 of a serving. How many liters of dragon blood is that?
Second Goblin Goo	You had 2/3 of a serving of goblin goo. One serving of goblin goo is 4/5 liters. How many liters of golden goo do you have?
October 27, 2016 (Multiplication)	
Blank multiplication	One serving of ___ is 3/4 ___. You had 2/5 of a serving. How many ___ of ____ did you have?
November 8, 2016 (Division)	
Francine's rope	Francine has 32 years of rope that she wants to cut into 8 equal pieces. How long will each piece be?
Gallons of water	If 1 gallon of water weighs 8 pounds, how many gallons will there be in 400 pounds of water?
Drive Problem	If you drive 220 miles at a constant speed, it takes you 4 hours. How fast did you go?
November 10, 2016 (Partitive Division)	
Pizza	There are 3 pizzas that will be divided equally among 4 people. How much pizza will each person get?
November 15, 2016 (Division)	
Cookies	A batch of cookies requires 3 cups of flour. How many batches of cookies can you make if you have 14 cups of flour?
Brownies	You have 14 brownies which you will divide equally among 3 bags. How many brownies should you put in each bag?

Spring 2017

January 10, 2017 (Measurement Division)	
3 ÷ ¾ = ?	Write a simple how many groups word problem for 3 ÷ 3/4 and solve the problem with the aid of a strip diagram.
Tonya and Chrissy	Tonya and Chrissy are trying to understand 1 ÷ 2/3 by using the following problem: One serving of rice is 2/3 of a cup. I ate 1 cup of rice. How many servings of rice did I eat? To solve the problem, Tonya and Chrissy drew a square divided into three equal pieces, and they shade two of those pieces. 

	Tonya says, "There is one 2/3 cup of serving rice in 1 cup, and there is 1/3 cup of rice left over so the answer should be 1 1/3." Chrissy says, "The part left over is 1/3 cup of rice but the answer is supposed to be 3/2 = 1 1/2. Did we do something wrong?" Help Tonya and Chrissy.
January 12, 2017 (Measurement Division)	
1 ½ ÷ 1/3 = ?	Write a simple how many groups word problem for 1 ½ ÷ 1/3 and solve the problem with the aid of a strip diagram.
January 17, 2017 (Partitive Division)	
Noodles	2/3 of a serving of noodles contains 120 mg of sodium. How much sodium is in one bowl of noodles?
January 19, 2017 (Partitive Division)	
1/3 ÷ 2/5 = ?	Write a how many units in one group problem for 1/3 ÷ 2/5=? Reason about a math drawing to explain why you can solve it by 1/3 · 5/2.
January 26, 2017 (Ratio and Proportions)	
Hot chocolate	Make a drawing to show a whole bunch of mixtures of chocolate and like in that same 2 to 3 ratio from multiple batches and variable parts perspective.
Hot chocolate 2	Show these amounts organized from a variable parts perspective: 8 oz chocolate, 12 oz milk; 20, 30; 200, 300 Show all the amounts in the fixed 2 to 3 ratio.
January 31, 2017 (Ratio and Proportions)	
Rope	If 3 yards of rope weigh 2 pounds, then how much do the following lengths of the same kind of rope weigh? 18 yards, 16 yards, 14 yards. Write multiplication expressions.
February 2, 2017 (Ratio and Proportions)	
Scooter	A scooter is going 3/4 of mile every 4 minutes at a constant speed. How far does the scooter go in the following number of minutes: 12 minutes, 17 minutes.
February 7, 2017 (Ratio and Proportions)	
Yellow and blue paint	A shade of green paint is made in a ration of two parts blue paint and three parts yellow paint. If you have 48 pails of blue paint, how many pails of yellow paint do you need? If you want 150 pails of green paint, how many pails of blue and yellow paint do you need?
February 17, 2017 (Ratio and Proportions)	
Punch	If you mix juice and bubbly water in a three to five ratio to make punch, how many liters of juice and how many liters of water would you need to make 24 liters of punch? 10 liters of punch?

March 14, 2017 (Samples and Populations)	
Fishing	From a tank of 50 dish, we picked random samples of 10 fish. Let's say we think that a random sample of 3 yellow fish and 7 other fish is representative of the whole population of fish in the tank. Make a drawing to help you determine approximately how many yellow fish you would expect to be in the whole tank.

Appendix B

Explaining Keep-Change-Flip with Measurement Division

I described how students Elizabeth and Catherine explained the algorithm keep-change-flip in Chapter 1 and 4, respectively. In both cases, they used a partitive division interpretation of division. The algorithm can also be explained using a measurement division interpretation. In this section, I explain how I rationalized keep-change-flip using a measurement division meaning of division for 1/3 ÷ 2/5 = ? Using Elizabeth's context of chicken soup, I wrote the following measurement division problem:

> One serving of chicken soup has two-fifths pounds of chicken. How many servings can you make with one-third pounds of chicken?

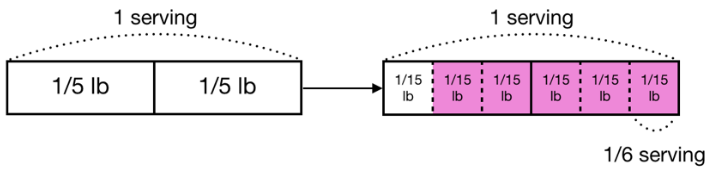
Using the definition of multiplication (Beckmann & Izsak, 2016), I interpreted the problem with the following table and equation:

Groups	Base Units
1 serving	2/5 lbs
? servings	1/3 lbs

$$? \cdot \frac{2}{5} = \frac{1}{3}$$

? servings
2/5 lbs in 1 serving
1/3 lbs in ? servings

Using a strip diagram, I first drew a strip representing one serving and two-fifths of a pound. I needed to show one-third of a serving. I knew that two-fifths is equivalent to six-fifteenths so I partitioned each fifth into three smaller parts, each one-fifteenth of a pound. Based on the strip, I also knew each partition is one-sixth of a serving because I partitioned the strip into six equal parts. Because one-third is equivalent to five-fifteenths of a pound, I highlighted five parts which is both five-fifteenths of a pound and five-sixths of a serving, thus there are one-third pounds of chicken in five-sixths of a serving.



Like the students, I re-interpreted my strip diagram to justify why 1/3 ÷ 2/5 = 1/3 · 5/2. In other words, I needed to explain 1/3 · 5/2 = 5/6 with my current diagram. I first drew the whole pound of chicken by adding three more one-fifth pound partitions. I also interpreted the strip with respect to servings. Each partition is one-sixth serving and there are fifteen partitions, thus the entire strip is fifteen-sixths of a serving or five-halves of a serving. To interpret 1/3 · 5/2, I assigned servings as the base unit and a pound as a group. There are five-halves servings in one pound (i.e. the multiplicand) and I had one-third pounds (i.e., the multiplier) This yielded the highlighted portion, five-sixths servings in one-third pounds. I annotated the resulting equation to show this structure.

Groups	**Base Units**
1 lb	5/2 serving
1/3 lb	? servings