## MODELING GRADE EIGHT STUDENTS' CONSTRUCTION OF FRACTION MULTIPLYING SCHEMES AND ALGEBRAIC OPERATIONS

by

#### ZELHA TUNÇ PEKKAN

(Under the Direction of Leslie P. Steffe)

#### ABSTRACT

The purpose of this research was to investigate eighth-grade students' construction of linear equations (of the form ax = b, where a, b are fractional numbers) by using their fractional multiplying schemes and the contribution of these schemes to the construction of inverse reasoning. One pair of eighth grade students participated in a 3-month teaching experiment that consisted of 18 teaching episodes. In the episodes, students engaged in solving tasks that involved quantitative relations between known and unknown quantities. When solving the tasks, the students used a computer tool called JavaBars as well as paper and pencil. The retrospective analysis of the study suggested that the students' construction of fraction multiplying schemes involved coordinating two three-levels-of-units structures and reinterpreting quantities in terms of standard measurement units. The results also suggested that the students' construction of relations between two quantities was based on the operations of their fraction multiplying schemes (distributive partitioning and recursive distributive partitioning operations) as well as their production of a measurement of an unknown quantity.

INDEX WORDS:Mathematical Learning, Fraction Multiplication, Fractional Numbers,<br/>Algebraic Reasoning, Algebraic Schemes, Quantitative Reasoning,<br/>Inverse Reasoning, Units-Coordination, Notations, Operations, Unit

Measurement, Teaching Experiment, Models, Constructivism, Radical Constructivism

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## DEDICATION

I dedicate this work to all the *Don Quixotes* in the field of education, those who never give up, including Kerem, Ziya, and Les.

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#### **CHAPTER 1: INTRODUCTION**

#### Statement of the Problem

Algebra is mostly associated with symbol manipulation by students or adults who have experienced it during their school years (Kaput, 1999; National Council of Teachers of Mathematics [NCTM], 2000). *Principles and Standards* (NCTM, 2000) claim that algebra should not be considered a subject studied only during high school but rather approached as a topic with roots developed as early as pre-kindergarten. In this sense, research on middle school algebra teaching and learning can provide us with ways in which to view algebra as a coherent part of K-12 mathematics.

Kieran (1992), one of the most cited researchers in the literature related to algebra, differentiates students' understanding of algebra as procedural from that of algebra as structural. She states, "procedural refers to arithmetic operations carried out on numbers to yield numbers...the term structural, on the other hand, refers to a different set of operations that are carried out, not on numbers, but on algebraic expressions" (p. 392). Kieran (1992) also says that students must realize that algebraic expressions and equations can no longer be interpreted as "arithmetic operations upon some number, but rather [students] must very quickly learn to view them as objects in their own right upon which higher level processes (that is, operations) are carried out" (p. 393). A number of questions might be asked about Kieran's view on this matter: First, how do students learn to treat algebraic expressions as structural entities? Second, how are those structural entities and students' operations related (what are the roles of operations in constructions of such entities)? Third, is this the only way to conceptualize school mathematics: procedural versus structural?

Kieran (1992) mentions Sfard's research on the psychological stages of how students move from procedural to structural understanding of algebra. The first stage is interiorization, the second is condensation, and the last is reification. Kieran (1992) notes that interiorization and condensation are gradual developments, whereas reification is a sudden jump, in which the objects are viewed in a static structure.<sup>1</sup> Sfard and Linchevski (1994) theorize that "mathematical objects are an outcome of *reification* – of our mind's eye's ability to envision the result of processes as permanent entities in their own right" (p. 194). This idea of reification is compatible with Piaget's idea of abstraction, especially reflective abstraction (cf. Chapter 2). However, this view of seeing algebraic concepts as static structures in their own right limits our view of how students might produce new algebraic relationships and structures. For example, how we understand radicals and give meaning to numbers like  $\sqrt{5}$ ,  $\sqrt{7}$ ,  $\sqrt{9}$ , and even to  $\sqrt{-1}$  is closely related to the processes and the mathematical structures that help us to abstract those concepts. However, neither the processes nor the structures are static when we try to give meaning to radicals.

In the last decade, there were two well-known conferences (among others) on research on algebra, one national (Wagner & Kieran, 1989) and one international (Bednarz, Kieran, & Lee, 1996). After the international colloquium on "Research Perspectives on the Emergence and Development of Algebraic Thought" in 1993, Bednarz, Kieran and Lee (1996) classified the international research on algebra teaching

<sup>&</sup>lt;sup>1</sup> "Static structure" is a term that I inferred and used to describe Sfard's conception of objects.

and learning using the presented papers and ideas. According to the editors, there are five different models of how international researchers approached algebra in the literature: (a) historical perspectives in the development of algebra (e.g., see Rojano's chapter's in the book), (b) generalizing whole numbers and geometric patterns (e.g., see Mason's chapter), (c) problem solving (e.g., see Bednarz & Janvier's chapter), (d) modeling physical or mathematical situations (e.g., see Nemirovsky & Janvier's chapter), and (e) functions (e.g., see Heid's and Kieran, Boileau, & Garacon's chapters). In addition to the organized studies in this report, some recent American based research can be viewed within the context of those five categories. For example, the studies that take algebra as an extension of arithmetic can be thought of examples of generalizing patterns (Carraher, Schliemann, & Brizuela, 2001; Schliemann, Carraher, & Brizuela, 2007); Izsák's study about middle school students' algebraic activities when modeling the motions of a physical winch (Izsák, 2000; Izsák, 2003, 2004) can be thought of as a recent example of the modeling approach to algebra; and Confrey and Smith's study (1995) on students' exponential functions using splitting operations can be thought of as an example of the functional approach to algebra.

For the edited collection of that conference's papers, *Approaches to Algebra: Perspectives for Research and Teaching*, Wheeler (1996) wrote a concluding chapter in which he discussed this international colloquium and research on algebra. He criticized the organizers for categorizing approaches to algebra research into separate trends, arguing that this is artificial.<sup>2</sup> He believes that a (good) algebra program needs all those

 $<sup>^2</sup>$  See the last four categories (2-5) in the previous paragraph. Wheeler acknowledges the "historical approach" as a useful view for conceptualizing algebra, but he does not discuss it as a viewpoint from which to teach algebra.

different ways of perceiving algebra and the categories represented by the chapters in the book are not fully separable from one another (for example, generating patterns is important in the modeling view to algebra). However, he admits that with this categorization the participants in the colloquium "had a better idea of the nature of the challenge of finding a meaningful approach to 'beginning algebra' and an enriched perspective on the possibilities" (p. 325). In addition, when recent literature on algebraic learning is reviewed, it is evident that it has become a tradition to acknowledge the last four categorizations of approaches to algebra as a legitimate way to summarize the literature (e.g., Hackenberg, 2005; Smith & Thompson, 2007). Furthermore, Wheeler revisited the four basic questions that were given as the rationale for the colloquium and explored whether they were answered throughout the discussions. One important question was: "What are the essential characteristics of algebraic thinking?" (p. 322) Wheeler says that this is an excellent long-term research question, partly because during the colloquium "there [was] no consensus on the attempt to differentiate algebraic thinking from mathematical thinking in general, or on the attempt to reduce the essential content of algebraic thinking to a set of very elementary operations" (p. 322). Therefore, one of the aims of my study was to explore what those elementary operations might consist of, in order to conceptualize algebraic thinking.

There are other studies focused on important aspects of students' algebraic learning such as students' conceptions of algebraic notations (Chae, 2005; MacGregor & Stacey, 1997), and semantics and different representations related to algebraic thinking (Kaput, 1987, 1989, 1991). However, the question of what basic algebraic operations are, and how they can be used to characterize algebraic thinking, needs further investigation. To delve deeper into students' mathematical understanding, the concepts of scheme and operation are vital to formulate possible explanations of what we observe about *students' mathematics* from the students' point of view. We cannot rely solely on (our) knowledge of mathematics when producing possible explanations of students' mathematics; we also need to rely on psychological constructs. However, scheme theory has not been used widely in the literature to explain students' construction of unknowns nor the use of symbols when they solve algebraic problems. There is research (Blanton & Kaput, 2005) related to possible identifications of algebraic structures in problems, such as the identification of problems as missing addend problems. Hackenberg's (2005) study is an important contribution to the field in this sense that her study sought understanding of students' fractional operations and schemes for creating those algebraic structures using different types of reversible multiplication problems.

Wu (2001) argued that students who are competent with fraction computation will possibly have less difficulty in manipulating symbols in algebra since solving equations requires being competent with fractions. For example, using reciprocal fractions in fractional operations can serve in the solution of linear equations. In addition to Wu (2001), Kilpatrick and Izsák (2008) asserted that fractions contribute to construction of algebraic knowledge, yet this relationship is not well investigated in the literature. Among others, they suggested *distributive property* playing a central role in that, "Both fraction and whole-number arithmetic provide opportunities for learners to develop multiplicative structures and an understanding of the distributive property, both of which are central to working with algebraic expressions and equations" (Kilpatrick & Izsák, 2008, p. 16). To investigate and provide a trajectory of the relationships of knowledge of fractions and algebra (specifically constructions of linear equations), I started with multiplicative quantitative situations that required students to reason using their knowledge of fractions. These kinds of quantitative situations provide an opportunity for me to determine how students think algebraically (meaning, what kinds of structures they use in certain types of problems), how they symbolize their thinking, and how they give meaning to unknowns. In general, the goal of the study is to understand how students construct an important part of the middle school mathematics curriculum— that is, constructing algebraic equations with one unknown— using fractional knowledge as a basis.

#### **Research Questions**

1. What operations are involved in students' construction of a fraction multiplying scheme in quantitative situations?

2. What operations and schemes are involved in a construction of inverse reasoning that is a basis for conceptual understanding (both construction and solution) of linear equations with one unknown? What is the role of fraction multiplying scheme in the constructions of inverse reasoning?

# CHAPTER 2: THEORETICAL AND CONCEPTUAL CONSTRUCTS FOR THE STUDY

What is the use of epistemological theory? How does the theory function when we construct ourselves as researchers and try to give meaning to our work as research? Crotty (1998) says, "Each epistemological stance is an attempt to explain how we know what we know and to determine the status to be ascribed to understandings we reach" (p. 18). Although many researchers commonly view epistemology and theory as separable from research, Crotty (1998) indicates that theory is not an external way of looking at the research and informs all aspects of conducting the research. We might not be explicitly aware of the fact that epistemology affects what kinds of research problems we pose and how we investigate those problems and create new knowledge through our research, but it is actually the core of how we operate as researchers. Therefore, in this chapter, I first write about the theory of knowledge that frames my study. Second, I write about the theory of learning that informs my study—Piaget's scheme theory— and then focus on describing some specific conceptual constructs that are essential to the study and are extracted from radical constructivism.

#### Mathematical Knowledge

We can define knowledge in three ways: exogenic knowledge, endogenic knowledge, and knowledge defined by radical constructivism. These ways of viewing knowledge differ in terms of how reality is viewed, how the relationship between the individual and reality is conceptualized, and how individuals come to know something. In both exogenic and endogenic traditions of knowledge, there is an acceptance of an objective reality that can be known as a thing-in-itself, independent of the mind. In the exogenic tradition, "Knowledge is achieved from this perspective when the inner states of the individual reflect accurately the existing states of the external world, or in Rorty's (1979) terms, when the mind serves as a 'mirror of nature'" (Gergen, 1995, p. 18). The external world is taken as a given, and categories are already made. This way of looking at knowledge (exogenic) implies that a student's mathematical knowledge is judged relative to objective mathematical concepts that teachers teach, and a transmission of (the same) knowledge from a teacher to a student is possible.

Gergen (1995) says that the endogenic tradition, similar to the exogenic tradition, accepts a dualism between the external world and the individual. A difference is that the endogenic tradition has roots in nativism and emphasizes the human's intrinsic capabilities, such as "reason, logic, or conceptual processing" (Gergen, 1995, p. 18). The external world can cause an individual to reason and rationalize her thoughts. In the endogenic tradition, in addition to the help of the external world, social interactions contribute (and are needed) for the mind to be cognitively active. The transmission of knowledge model implied by the exogenic tradition is not supported by the endogenic tradition, since each individual needs to experience the rationalizing process herself. Knowledge is "a mental state—an enhanced state of representation in the exogenic case and of reasoning in the endogenic [case]" (Gergen, 1995, p. 18).

Knowledge can be also explained as solely an individual's constructions; in this model, there is no concern about the dualism of mind versus the external world, since the external world is not taken as a given. This third way of looking at the relation between

knowledge and reality, radical constructivism, was inspired by the Piagetian way of understanding that relation (von Glasersfeld, 1995). Radical constructivists are not concerned with proving or disproving the existence of an external world. The emphasis is on understanding the construction of an experiential world. Rather than being concerned about whether knowledge matches a world outside of us, the concern centers on the compatibility of our experiential worlds.

I am in agreement with this philosophy that everything in our lives, even what we take for granted, such as physical objects, can not be known independent of our minds' construction. For example, think about infants. They actively construct objects and differentiate some physical properties of the objects from others. It is not usually until their second or third month that they differentiate their hands from the background, or their fifth or sixth month that they attempt to touch an object they see without knowing if the object is reachable or not. Eventually, they learn to perceive an object's distance from their own body. Therefore, physical properties of objects (such as their colors) and their spatial positions (such as distance or closeness) are constructed, and these constructions take much time and organization on the part of the infant, even though it might seem that conceiving of physical properties is innate to most observers who do not often interact with children. While these constructions are usually compatible with other people's conceptions of physical objects, this compatibility does not mean every person's construction of physical objects is the same. These constructions are different because of the different experiences and different organizations of those experiences in each person's mind. Therefore, we can only have compatible conceptions. When we use words to refer to those objects, their meanings might often seem identical within the same

culture of people. This is because their constructions usually involve feedback from and interaction with other people, which von Glasersfeld (1995) calls "development of intersubjective reality" (p. 120). According to von Glasersfeld there are fewer clashes when the conceptions of the physical objects are commonly discussed or used in everyday communication, as opposed to conceptions of objects of interest to fewer people or that are only discussed on certain occasions, such as mathematical concepts. Referring to physical objects is what we usually do in our daily life, and we do it almost always, so the meanings of those words are more refined and are "taken-to-be-shared, which does not imply actual sameness (see Cobb, 1989)" (von Glasersfeld, 1995, p.137).

According to von Glasersfeld's (1995) interpretation of Piaget's view of knowledge; "Knowledge arises from the active subject's activity, either physical or mental, and that it is goal-directed activity that gives knowledge its organization" (p. 56). In the case of mathematical knowledge and especially algebraic knowledge, students' activities are generally mental. Through mental activities that organize experiences, students construct algebraic experiences. In addition to organizing sensory experiences, mental activities that constitute an algebraic experience might include operating with concepts in hypothetical situations, reflecting on and abstracting from sensory experiences, and re-presenting past experiences. To construct a viable experiential reality (which is also valid for mathematical experiential reality), according to von Glasersfeld (1995), *re-presentations* are important because "they become the indispensable basis for the most important conceptual activities, such as the presentation of hypothetical situations, hypothetical goals, hypothetical perturbations, and thus for the making of reflective abstractions from experiences that have not actually taken place on the sensorimotor level" (p. 60). von Glasersfeld (1995) defines Piaget's term re-presentation as follows, "[it] is always the replay, or *re*-construction from memory, of a past experience and not a picture of something else, let alone a picture of the real world" (p. 59). This way of looking at knowledge implies that students' algebraic ways of knowing may well differ one from another as well as from a more or less knowledgeable person of mathematics. To make conjectures about students' algebraic knowing, I need to investigate what kinds of mental activities students carry out and how their minds organize themselves through our interactions. I do not infer causality between my interactions and their mental organization or activities, but I need to acknowledge the role of interaction when engendering learning opportunities in students.

#### Mathematical Learning

#### What is Learning? Scheme Theory

When interpreting learning in Piaget's system of knowledge, von Glasersfeld (1995) posits, "cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or reestablishes [a new] equilibrium" (p. 68). So, to understand learning, we need to understand the operations that produce accommodation or cognitive changes that reestablish equilibrium. Equilibration is often likely to change.

Equilibrium refers to a state in which an epistemic agent's cognitive structures have yielded and continue to yield expected results, without bringing to the surface conceptual conflicts or contradictions. In neither case [biological nor conceptual] is equilibrium necessarily a static affair, like the equilibrium of a balance beam, but it can be and often is dynamic, as the equilibrium maintained by a cyclist. (von Glasersfeld, 1989, p. 126)

The idea of scheme is vital to understand learning in the Piagetian framework. According to von Glasersfeld (1995), schemes are comprised of three parts: a recognition pattern, the subject's activity, and the expected result. Recognition of a situation triggers "a specific activity associated with the situation; and the expectation that the activity produces a certain previously experienced result" (von Glasersfeld, 1995, p. 65). If the student is unable to produce the expected result, even if she recognizes the situation (assimilating the situation into her current scheme) and acts upon it, disequilibrium might occur in the student's scheme. Of course, it is not certain that the student will be able to organize her experience to take an action nor that there will be such disequilibrium. However, if there is no disequilibrium, there is no way we can infer learning as a result of accommodation from our observations. For learning, the student should make accommodations in either the recognition part of the scheme or the activity part of the scheme so that the cognitive structure will reach equilibrium. L.P. Steffe (personal communication, August 17, 2006) emphasizes that by observing a child's activities and seeking repeatable actions, we can infer the non-observable parts of a scheme, the situation and the result, and construct a *scheme* as a conceptual tool when explaining how a child thinks mathematically.

#### Assimilation and Accommodation

Assimilation and accommodation are two important constructs in Piaget's learning theory. When a scheme is used in assimilation, the recognition part of the scheme is necessarily involved, whereas accommodation might affect any of the three parts of the scheme. Regarding assimilation, von Glasersfeld (1995) says that, "cognitive assimilation comes about when a cognizing organism fits an experience into a conceptual structure it already has.... In short, assimilation always reduces new experiences to already existing sensorimotor or conceptual structures" (p. 62-63). Assimilation is not restricted to the first part of a scheme:

The 'recognition' [part of a scheme] is always the result of assimilation....The recognition of the activity's result again depends on the particular pattern the agent has formed to recognize the results obtained in the course of prior experiences. That is to say, it, too, involves acts of assimilation [especially after a new scheme is constructed] (von Glasersfeld, 1995, p. 65-66).

While assimilation functions as a necessary condition for a scheme to be used, it

is not sufficient for learning to occur. As mentioned in the previous paragraph, there

needs to be "either disappointment or surprise [a disequilibrium]" in the activity or the

result of the scheme. When the activity of a scheme does not lead to an expected result,

this may evoke a review of the recognition part of the scheme that in turn may lead to an

accommodation in the scheme. Accommodation is explained further:

If the unexpected outcome of the activity was disappointing, one or more of the newly noticed characteristics may effect a change in the recognition pattern and thus in the conditions that will trigger the activity in the future. Alternatively, if the unexpected outcome was pleasant or interesting, a new recognition pattern may be formed to include the new characteristics, and this will constitute a new scheme. In both cases there would be an act of learning and we would speak of an 'accommodation'. The same possibilities are opened, if the review reveals a difference in the performance of the activity, and this again could result in an accommodation. (von Glasersfeld, 1995, p.66)

Besides these general terms that contribute to conceptualizing learning, Steffe

exploited two other useful constructs, generalizing assimilation and functional

accommodation.

*Generalizing assimilation*. Steffe and Thompson define generalizing assimilation

as follows,

An assimilation is generalizing if the scheme involved is used in situations that contain sensory material that is novel for the scheme (from the point of view of an observer), but the scheme does not recognize it (until possibly later, as a consequence of the unrecognized difference), and if there is an adjustment in the scheme without the activity of the scheme being implemented (cf. Steffe & Wiegel, 1994). (Steffe & Thompson, 2000, p. 289).

When this quotation is read, it is possible to get a sense of *learning*. However, there are two important points thwarting conceptualizing learning as a result of generalizing assimilation. One of them is that the situation is novel from the observer's point of view. This suggests that there is no change in how a subject consciously perceives the situation nor is there an awareness of change in the scheme from the perspective of the subject even when the result is produced. Therefore, it sounds like there is no surprise or uncertainty on the part of the subject. The other point is how implementation or non-implementation of activity of the scheme plays a role in conceiving generalizing assimilation as a learning case. Steffe and Thompson (2000) make the *activity* of the scheme not being implemented a necessary condition for generalizing assimilation. This situation complicates the conceptualization of generalizing assimilation as a case of learning because observing the activity of the scheme or the result of the activity is important for deciding whether a scheme is used at all or the student is just imitating a way of thinking (or approving the result) without necessarily assimilating it to the discussed scheme.

Without discussing the unimplemented activity component of the generalizing assimilation, Hackenberg (2005) states, "This type of assimilation is a reconceptualization of the notion of transferring knowledge from one situation to another, not obviously similar situation, and it probably occurs more often than accommodations do (L. P. Steffe, personal communication, April 15, 2003)" (p. 18). I think in this reconceptualization, we need to be aware that generalizing assimilation is not just a transferring of knowledge even though activity and the result of a scheme might be observed as the same. The idea of transfer might involve change, but it does not explicitly point to a change with Hackenberg's use of the term (transfer). The term *transfer* Hackenberg used is also different than how it is discussed in the literature. In the literature, Wagner (2006) indicates that there are different views of transfer and in the traditional sense of the term, the abstractions (the identification of a generic quality of instances across instances of the principle) make learning possible by transferring knowledge structures without contextual specificity. In the traditional view of transfer, there seems no discussion of awareness; the measurement of success and the failure of transfer of knowledge is determined by the observers' criteria and it is independent of how they think the individual conceive the situations of those *transfer tasks* since the precategorized tasks are used. Wagner (2006), and Lobato and Siebert (2002) approach *transfer* of knowledge with an orientation of *actor-oriented* as opposed to the traditional approach of researcher/observer and consider *transfer* of knowledge in the framework of "personal creation of relations of similarity" (Lobato, 2003, p. 18).

Importantly, there needs to be some uncertainty in understanding the situation on the student's part whether this learning be called *generalizing assimilation* or transfer of knowledge. This uncertainty (which might be some disequilibria) is necessary for extending the schemes' situations even when acting in ways that will not cause any major changes, either in the activity or in the result parts of the scheme. For example, one of the students Steffe (2002) worked with, Jason, could interpret making a unit that is three times as long as a given unit using his numerical schemes: His [Jason's] *use of his numerical schemes were indeed blocked by the language* [italics added] "twice as long" and "three times as long" and by how to make sticks [units] that were of those sizes. He did know how to make a stick by drawing and by copying a stick [unit], and he knew how to join sticks together, but *he did not know how to use these operations to make the sticks requested by the teacher* [italics added]. Nevertheless, once he recognized what Laura was doing and why, there were no major modifications necessary in his numerical schemes for him to operate as he did. He operated powerfully and smoothly as if his operating referred to discrete items that couldn't be joined together physically. (Steffe, 2002, p. 278)

Steffe's specific analysis of Jason's activities clarifies the ambiguity and the discussion about the "unimplemented activity" and its role in generalizing assimilation. Jason interpreted his partner's activities and understood the result without acting himself "once he recognized what Laura was doing and why" (p. 278). There was also some uncertainty whether he understood the situation in a way that would imply that he had an available operation associated with the situation. The debate is, however, what Jason learned. There was definitely a change involved in that he was now able to take the posed problem situation meaningfully, which means that he could act and produce a result, or interpret his partners' actions. He could assimilate the posed problem situation into his schemes and organize the structure so that when a similar problem was posed he could have acted in a similar way. The learning might be just transferring his numerical operations to the situation that was initially novel with respect to the operations, as indicated by Hackenberg, and so he conceived of the posed problem situation in a way he could interpret meaningfully. This situation adds to the debate of how deep this learning was or how big the change was in the scheme's structure in relation to whether what the

child learned or constructed could be aligned or fit with the researcher's intentions.<sup>3</sup> What I mean by this is, I accept that generalizing assimilation is *learning*, but I am questioning what is learned, and how compatible it is with the intentions of the researcher.

*Functional accommodation.* Functional accommodation is defined as an operation that changes the scheme's activity or the recognition template in the context of using the scheme. An accommodation is functional if it occurs in the context of the scheme being used. Therefore, there is possibly some awareness on the student's part that the scheme does not work that leads to possibilities for the construction of a new scheme. Steffe and Thompson (2000) elaborated functional accommodation by differentiating it from generalizing assimilation:

The elements [perturbations induced by social interaction] might block use of the schemes, they might lead to inadequacies in the schemes' activity, or they might lead to ambiguities in the results of the schemes. The accommodations that we have in mind differ from generalizing assimilation (which also can be regarded in the context of accommodation) in that they consist of a novel composition of the operations available or changes in the activity of the scheme. They go beyond use of the scheme in a situation in which it has not been used previously, which is an essential characteristic of generalizing assimilation. (p. 290)

For example, a student in Olive's (1999) study produced a quantity for 1/5 of 1/6 of a candy bar. He could only state the result as 1/5 of 1/6 of the candy (a mini-part), or he could produce the result of 1/30 of the candy bar by iterating the mini-part 30 times to check how many of those mini-parts fit into the whole unit. Eventually in the process of solving such problems, he started to use his multiplication operation to reason that there are six of the units in the whole candy bar, and each of those six units has five mini-parts, therefore there are 30 mini-parts in the candy bar. This is an example of a functional

<sup>&</sup>lt;sup>3</sup> The intentions are framed by the researcher's own mathematical knowledge, or the knowledge that is gained from the inferences she made about students' mathematical ways and means of operating.

accommodation in which the activity of the child's fraction composition scheme has changed from iterating a mini-part physically to a multiplication operation that is carried out mentally.

#### Construction of Concepts and Different Types of Abstraction

By constructing particular properties of objects, we generalize and form cognitive structures that von Glasersfeld (1991) calls *concepts*. These concepts are results of *empirical abstraction* derived from observable things or activities with perceptual materials: "To isolate certain sensory properties of an experience and to maintain them as repeatable combinations, i.e., isolating what is needed to recognize further instantiations of, say, apples, undoubtedly constitutes an empirical abstraction" (p. 55). von Glasersfeld (1991) gives an example to explain using a template (concept) in recognizing a situation and the difference in recognition from using a template in re-presentation. For example, those who speak English (or any other language) as a second language might recognize some words when they see them in writing or hear them. However, many words of the second language are not available to them (since there are no sounds or iconic symbols which might result in some recognition) either in their use of spoken or written language, so those words are not abstracted as re-presented materials. That is an example of how recognition and empirical abstraction pave the way for re-presentation of experiences.

Because there are always vastly more sensory elements than the perceiving agent can attend to and use, recognition requires the attentional selecting, grouping, and coordinating of sensory material that fits the composition program of the item to be recognized. In re-presentation, on the other hand, some substitute for the sensory raw material must be generated. (As the example of the Volkswagen indicates [when the back part of the car is seen, it is possible to imagine the whole car], the re-generation of sensory material is much easier when parts of it are supplied by perception, a fact that was well known to the proponents of Gestalt psychology.) (von Glasersfeld, 1991, p.50)

To be able to imagine or regenerate sensory material in the mind, we need to make a different type of abstraction than empirical abstraction using our experiences, von Glasersfeld talks about symbols and their function in our thinking related to this type of abstraction, reflective abstraction. For example, imagine a 2-year old hears the word *father*. To be able to understand the word *father*, he might not need to see the father himself. If he looks for something in the house to associate it with *father* such as a father's coat, a framed picture of him, and so forth then his experiences are triggered with hearing the word *father*, but they are not generalized yet. Upon being generalized, the word *father* would stand as a symbol to trigger those experiences without a need to associate the concept of father with some sensory material. He can re-present the father with the help of some material, but to be able to use the word *father* as a symbol (being operative) to point to the generalized experiences, he needs to be more proficient. In this way, he will "no longer need to actually produce the associated conceptual structures as a completely implemented re-presentation, but can simply register the occurrence of the word as a kind of 'pointer' to be followed if needed at a later moment" (von Glasersfeld, 1991, p. 51).

The type of abstraction that is needed to assimilate a word into an operative scheme is called *reflective abstraction* in von Glasersfeld's interpretations of Piaget's terms. Reflective abstraction produces a "conceptualized understanding" involving "awareness of the characteristics inherent in the *concept* of apple or whatever one is representing to oneself, and this kind of awareness constitutes a higher level of mental functioning" (von Glasersfeld, 1991, p. 57). Therefore, there is an awareness of the represented material that the subject is operating on. This awareness precedes *reflected* 

*abstraction*, in which the subject is aware of not only the re-presented material but also the operations that produce the characteristics of re-presented material (von Glasersfeld, 1991). The most important difference between these two types of abstraction (reflective and reflected) and empirical abstraction is "finding solutions to problems in the representational mode, i.e., without having to have run into them on the level of sensorymotor experience" (p. 59). For example, a child who had some experience with wooden puzzle situations might be asked a puzzle problem without being given any material for the situation. He might be asked to describe how to place triangles into a puzzle board with a kite shape that four triangles fit into perfectly. If the child can re-present the material to himself and then operate with the material cognitively and produce a solution, this would be an example of both his mathematical operations and shape concepts being re-presented as a result of his reflective abstractions. von Glasersfeld (1991) suggests that reflective abstraction should be interpreted as "projection and adjusted organization on another operational level," and reflected abstraction should be perceived "as conscious thought" (p. 58). Throughout my study I will not make a differentiation between the use of these two types of abstraction and will call both reflective abstraction. However, when there are situations that an awareness of either (or both) the re-presented material or the operations becomes important, I will make the necessary distinctions.

#### Mathematical Concepts

Concepts are usually used in the recognition part of the scheme to assimilate situations into the schemes. To define mathematical concepts, we need to discuss operations and actions in constructivism. von Glasersfeld (1995) makes a differentiation between actions and operations in his interpretation of *figurative* and *operative* in

Piaget's work. He links the actions and the figurative concepts by writing, "'Figurative' refers to the domain of sensation and includes sensations generated by motion (kinesthesia)" and "'Acting' refers to actions on that sensorimotor level, and it is observable because it involves sensory objects and physical motion" (von Glasersfeld, 1995, p. 69). On the other hand, *operation* does not "depend on specific sensory material but is determined by what the subject does... [They] are always operations of mind and, as such, not observable" (von Glasersfeld, 1995, p. 69). Piaget indicates that operations are interiorized actions:

What is already true for the sensorimotor stage appears again in all stages of development and in scientific thought itself but at levels in which the primitive actions have been transformed into *operations*. These operations are interiorized actions (e.g., addition, which can be performed either physically or mentally) that are reversible (addition acquires an inverse in subtractions) and constitute set-theoretical structures (such as the logical additive 'grouping' or algebraic groups). (Piaget, 1970, p. 705)

Olive (2001) explains interiorization of an activity as a process of reflective abstractions.<sup>4</sup> The operations that constitute mathematical concepts are initially, at least, results of interiorized actions. In addition, mathematical concepts are always operative. von Glasersfeld (1995) states that to know a word, which is a concept such as a mathematical concept, means to associate meaning with it, and "the meaning may be figurative, ([if it is] abstracted from sensorimotor experience), operative (indicating a conceptual relation or other mental operations), or a complex conceptual structure involving both figurative and operative elements" (p. 98).

<sup>&</sup>lt;sup>4</sup> Olive (2001) states, "The activity is first internalized through mental imagery; the child can mentally represent the activity. This mental re-presentation still carries with it contextual details of the activity. The activity becomes interiorized through further abstraction of these internalized re-presentations whereby they are stripped of their contextual details" (p. 4).

#### Anticipation, Mathematical Operations, and Symbols

To shed further light on the discussion of *meaning* and how researchers can make meaning using inferences of students' activities and operations, I present three examples and related discussions. In the first example, I illustrate the importance of anticipation in sensory-motor actions that play a role in constructing mathematical operations. In the second and third examples, I discuss how a mathematical operation can be thought of as a symbol, and the role of anticipation in this.

Anticipation. Discussion context 1. Imagine a 2-year-old child who has been given a set of geometric shapes (including triangles, circles, etc.) and a puzzle board on which four triangles compose a kite shape. If the child picks a triangular shape but each time needs to experiment with how to place a triangular shape on the kite-puzzle board, then he only anticipates that a triangular shape will work. He cannot coordinate the current position of the shape at the time when he picks the triangular shape and the placement of it on the board as one of the four pieces. In this case, there is an anticipation taking place in the child's mind: he knows which kind of shape will fit, but this anticipation is not operative since he does not know how to place the shape without experimenting. He also needs to have the triangular shape and the puzzle board in front of him. Even though the child could imagine how to operate with physical material in front of him, this situation is different than interiorized actions, which are anticipatory operations. The child's preference of an object might include some anticipation, but he has to actually experiment because he cannot mentally operate with the shape.

If the operations [rotating the triangular shape] are interiorized, then the child can imagine rotating the object and placing it without actually doing it. That would decrease the amount of experimentation that is necessary. The thought experimentation could be an indication of anticipating of what is going to happen. The anticipation is made possible by the mental operations. The child who must experiment can make a visual evaluation [when making a decision for the geometric shape, triangle versus a circle]... you can see some anticipation because the child is not acting randomly at all. In terms of shape fitting, he can visualize a triangle; he can re-present the shapes in his mind. That is in a sense anticipatory but it is not operative anticipation. (L.P. Steffe, personal communication, May 31, 2007)

*Complex mathematical operations. Discussion Context 2.* Imagine a child who is solving this problem: "An 8-centimeter peppermint stick is marked to show the 8 centimeters. It is 3/4 of another peppermint stick; make the other stick and tell how long it is" (Hackenberg, 2005).<sup>5</sup>

The child, Michael, established a goal to partition the 8-centimeter bar (8 unit bar) into three parts, saying "we can add on one more" (Hackenberg, 2005, p. 91). After experimenting for a while he decided to divide each of the last two units into three parts. He then pulled out 2 units (out of the 8 units of the bar) and 2/3 of a unit, and combined them to produce his answer for a third of the 8-centimeter bar. Hackenberg indicates that Michael was not aware that the way in which he operated was a distributive pattern. On the other hand, using her observations Hackenberg (2005) inferred that Michael "split distributively" (p. 92), meaning he used a distributive operation when splitting an 8-centimeter bar.

When Michael partitioned the 8-centimeter bar into three parts, which is a mathematical operation, he experimented because he did not know how to proceed before he started. He had not constructed a program of operations that he could use to make the

<sup>&</sup>lt;sup>5</sup> The video excerpt of the child's solution process with JavaBars was part of Hackenberg's (2005) study. L.P. Steffe and I watched the video excerpt and discussed mathematical and symbolic operations on March 17, 2007. I revised the inferences of our observation using Hackenberg's study. Therefore, Steffe is not responsible for the interpretations that I made as I have changed some of our original interpretations.

desired bar. If he was aware of operating distributively, without experimenting he could have partitioned each centimeter into 3 mini-parts. He could then pull out 1 mini-part from each of the units in the 8-centimeter bar and iterate this group of mini-parts four times. Had he operated in this way, I would have inferred that his distributive operations were interiorized. As it was, Michael did not anticipate partitioning even the last two centimeters into three parts prior to making the partitioning. Given a 1-centimeter bar rather than an 8-centimeter bar, he could operate to produce a bar such that the 1centimeter bar was 3/4 of that bar. But he was yet to construct distributive partitioning operations and coordinate them with the operations he used in that case. He was definitely not aware that he was actually producing the inverse of 3/4 by dividing 8 centimeters by three and multiplying it by four. If a child anticipated operating this way and abstracted his activities for these types of mathematical situations, then it would open a possibility for him to conceive of the situation as 4/3 of 8 centimeters, which undergirds the solving of linear equations like  $3/4 \times x = 8$  centimeters. This type of a network of operations is the result of reflective abstractions, and constitutes the basis of meaningful algebraic experiences similar to what von Glasersfeld explained for the functions of symbols:

There will be awareness not only of what is being operated on but also of the operations that are being carried out... symbols can be associated with operations and, once the operations have become quite familiar, the symbols can be used to point to them without the need to produce an actual re-presentation of carrying them out. (von Glasersfeld, 1995, p. 108)

*Symbols. Discussion context 3.* If there was only a 1-centimeter bar involved in the problem situation, then Michael's operations (when producing the four fourths quantity when a fractional part of it is given) could have been called symbolic operations.

L.P. Steffe (personal communication, March 17, 2007) articulates that mathematical operations are symbolic if there is an anticipation of activity and the results of the activity prior to implementing the operations. Michael knew how to produce the other candy bar; he would partition an unmarked bar into three parts and then iterate one of those three parts four times using his reversible partitive fraction scheme. In this case, *3/4* would refer to an operative mathematical concept. L.P. Steffe (personal communication, March 17, 2007) inferred that Michael knew how to act in this way even before he started, since he had observed this student on other occasions and knew the conceptual analysis that Hackenberg made about this particular student. In this sense, I do not regard the traditional written notation as the only way of conceiving of symbols.

I make a further differentiation that 3/4 can be regarded as a symbol in the case of the 1-centimeter bar, yet the concept of 3/4 may not function as a symbol (in the sense von Glasersfeld described) in the 8-centimeter bar situation because of the complexity of operations and unawareness of the distributive pattern. This distinction is important since it is a way to judge the extent to which 3/4 is used as a symbol that functions in a network of interiorized activities.

One of the indicators of interiorized activity is to take 3/4 as material to operate with, meaning viewing 3/4 as a given for further operation and also at the same time not forgetting the operational meaning associated with it. In both Michael's ways of operating (with 8-centimeter and 1-centimeter bars,) he had an operational meaning for 3/4; for example, conceiving the 8-centimeter bar as 3/4 of a bar and producing the other bar implied that 3/4 referred to interiorized operations but his distributive partitioning was not interiorized yet. However, Michael did not take what was said to be 3/4 of
another bar as a given in the sense that he could explicitly operate to find 4/3 of the 8centimeter bar. Even though 3/4 could be thought of as a symbol in his activities in that it symbolized the operations to make three-fourths of a bar, he did not have other necessary operations to take what was said to be 3/4 of another bar as something to operate with for constructing the reciprocal of 3/4. Therefore, having constructed 3/4 as a symbol does not guarantee all the three premises Hackenberg (2005) introduced as requirements for algebraic reasoning (cf. Chapter 3): generalizing (abstraction of schemes and operations into conceptual structures), reciprocity (operation on unknowns as well as knowns), and operating on notations. However, discussing the functioning of symbols as part of operations and schemes opens the possibility to view algebra as originating from students' mathematical constructions, as opposed to viewing algebra as a given.

## CHAPTER 3: ALGEBRAIC REASONING BASED ON FRACTIONAL MEASUREMENTS OF QUANTITIES

In this chapter, I have two goals: First, using extant literature, I aim to construct rationales for the two research questions (cf. Chapter 1), and discuss why they are important to investigate and how their investigations contribute to the field. Second, I also aim to explain how I conceive the essential concepts in the research questions, and discuss similarities and differences between these concepts and those in the literature. *Rationale for Research Question 1: What operations are involved in students' construction of a fraction multiplying scheme in quantitative situations?* 

Algebraic and Quantitative Reasoning

What do we mean by algebraic reasoning and how is it different than the arithmetic reasoning that precedes algebra in school mathematics? Similar to NCTM's *Standards and Principles*, Smith and Thompson (2007) make the following claim about algebra classes:

The procedures are often introduced as the mathematical means to solve specific types of problems, but the focus quickly becomes learning how to manipulate symbolic expressions. These procedures are then practiced extensively and later applied to specific problem situations (that is 'word problems'). Teaching this content involves helping students to interpret various commands—'solve,' 'reduce,' 'factor,' 'simplify'— as calls to apply memorized procedures that have little meaning beyond the immediate context. (p. 4)

How does this view of algebra that emphasizes syntax become "algebraic," while the reasoning that we want to encourage gets lost in school mathematics, and what are the effects of conceptualizing algebra in a way that is stripped of its reasoning? There are some studies discussing the reoccurring problems in the learning and teaching of algebra when algebra is viewed as only symbol manipulation (Chazan, 2000; Phillips & Lappan, 1998). To minimize the possibility of viewing algebra as symbol manipulation, Smith and Thompson (2007) suggest that quantitative reasoning should serve as the foundation of differing approaches of algebra,<sup>6</sup> especially before formal algebra classes are taken. It is possible that such an early intervention can change the goals of formal algebra classes and students' experiences in a positive way.

Regardless of students' actions, some researchers (Nathan & Koedinger, 2000) classify solutions as arithmetical when the result is unknown and as algebraic when the unknown is used within the initial steps of the solution. Smith and Thompson (2007) clarify that quantitative reasoning is the bridge between arithmetic and algebraic approaches in students' solutions and explain quantitative reasoning as follows:

In our view, conceiving of and reasoning about quantities in situations does not require knowing their numerical value (e.g., how many there are, how long or wide they are etc.). Quantities are attributes of objects or phenomena that are measurable; it is our *capacity* to measure them –whether we have carried out those measurements or not— that makes them quantities (Thompson, 1989; 1993; 1994). (p. 10)

I am in agreement with Smith and Thompson that reasoning about quantities does not require knowing their numerical value (for example, I am faster than you); however, this reasoning *about* quantities is not enough for the quantitative nor for the algebraic thinking Smith and Thompson propose. It is possible that these researchers also conceptualize quantitative reasoning with a focus on measurement of quantities, but it is not explicit in their writing. For discussing the difference between the quantitative and

<sup>&</sup>lt;sup>6</sup> With differing views of algebra, Smith and Thompson meant "algebra as modeling, as pattern finding, as the study of structure" (p. 6) that are discussed by the NCTM research groups.

algebraic approaches, they present this problem:

I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetski, 1976, p.10). (Smith & Thompson, 2007, p. 8)

Smith and Thompson discuss the algebraic solution to this problem with a traditional point of view: "A typical algebraic solution to this problem involves assigning variables, writing algebraic expressions, and eventually stating and solving an equation" (p. 8). They propose  $(t+6) \times d/40 = t \times d/30$  as an algebraic solution for the problem, and this equation could be derived by stating variables and relationships "where t represents the number of minutes I have walked, and d is the distance from home to school; my speed will be d/30 per minute, and my brother's would be d/40 per minute. Using the general relationship that 'rate multiplied by time equals distance' " (p. 8). For the quantitative approach, Smith and Thompson emphasize imagining walking from home to school and the way in which that imagined distance shrinks between the brother and the subject as they walk. The researchers then make a conceptual jump to numerical relationships between the two subjects' speed, time, and distance without discussing the necessary operations and structures for these jumps. Using the given information of 30 minutes and 40 minutes, they assert, "Since I walk 4/3 as fast as brother, the distance between us shrinks at the rate of 1/3 of brother's speed," and conclude that "the time required for the distance between us to vanish will therefore be 3 times as long as it took brother to walk it in the first place (6 minutes). Therefore, I will overtake brother in 18 minutes" (p. 9).

For both the algebraic and quantitative approaches, it is necessary that a student should reason *with* quantities (not only *about* quantities) and make some numerical

comparisons between the quantities. Without this preciseness, it is not possible to make quantitative statements about the situations, such as "I walk 4/3 times as fast as my brother walks," nor to use symbols and write an equation for representing the relationships between the quantities. While Smith and Thompson admit that a program that contains such a discussion of quantities would be ideal, but does not yet exist as a coherent path, it seems to me their writing also does not make the quantitative operations they use explicit. These operations could be the basis for important algebraic understandings, such as rate and constructions of *intensive quantities* (Schwartz, 1988) as a result of two co-varying quantities, for example, speed. I am aware that algebraic thinking has many foci (e.g., construction of extensive/intensive quantities, syntax, etc.), but I hope my study will add to two important points of discussion of quantitative reasoning and algebraic reasoning that are not emphasized by Smith and Thompson: (1) the necessity of reasoning *with* quantities (instead of reasoning only *about* quantities); (2) the explication of some of the operations that might be the bridge that Smith and Thompson claim between quantitative and algebraic reasoning. The specific context for my elaborations is a part of algebra that focuses on operations related to fractional multiplication and inverse reasoning with one unknown.

Reasoning with quantities, making *intensive* and *extensive quantities* (Schwartz, 1988), and creating relationships between the quantities are all algebraic in nature, since all require operations and abstractions based in operating. There is no quantitative reasoning if there is no measurement of quantities. It is suggested in some research, including Smith and Thompson (2007), that comparison might be sufficient for quantitative reasoning. But a quantitative comparison is also a form of measurement that

involves reference to a unit. Quantitative reasoning requires explanations of observed phenomena with quantities. For example, think about how a comparison can be done between the speeds of two people. The person, in Smith and Thompson's example, walks 4/3 times as fast as his brother. We cannot think of one person being faster than the other without talking about the measurement of some quantities and basing our observations on a (quantitative) reference. We need a reference, such as observing them walking from the same home to the same library (in which case the distance would be a reference).<sup>7</sup> When we observe someone walking the same distance in less time (brother walks 40 minutes and the person walks for 30 minutes), we may not have a numerical value for the distance, but we can take the same distance as a reference and produce a result for comparison, such as how much less time the walking takes in minutes or in any other unit (like beats). However, making a conclusion about the speed, such as 4/3 times as fast as, is not a straightforward process that is only based on the number of minutes. On this point, I am not in agreement with Smith and Thompson, who make a leap when describing quantitative reasoning about speed by using minutes. This conclusion about speed—4/3—is not only reasoning about speed quantity<sup>8</sup> (without numbers) but also abstracting the experience of walking and being faster than the other person in such a way that *fastness* becomes numerical and functional, and contributes to the problem's solution. In this sense, the measurement of the quantities (distance and time) need to be taken as a given to be able to make another quantity with a measurement, for example,

<sup>&</sup>lt;sup>7</sup> There is no need for the numerical measurement of the distance. Even though this is the case, to be able to set an algebraic equation we need to consider the distances as having the same numerical measurement. <sup>8</sup> Even though we do not assign numbers when we claim one person being faster than another one, there is

still an implicit measurement in the form of comparison of two structures; the distance is the same for two moving objects but there is a comparison of two different times.

intensive quantity such as speed. Therefore, measurement and unitizing are two important operations in my view of quantitative reasoning *with* quantities, but they are not emphasized extensively in Smith and Thompson's framework of reasoning *about* quantities.

On the other hand, I am in agreement with the way in which Smith and Thompson view two useful functions of quantitative reasoning in constructing algebraic reasoning. The functions are: (a) "to provide content for algebraic expressions so that the power of that notation can be exploited"; (b) "to support reasoning that is flexible and general in character but does not necessarily rely on symbolic expressions" (p. 12). Furthermore, they claim that quantitative reasoning affects the development of arithmetic reasoning and "[students'] future prospects in algebra." They elaborate this claim as follows:

First, the quantitative/conceptual approach makes thinking about the quantities and their relationships a central and explicit focus of solving the problem. . . . Second, this focus on thinking about and representing general relationships between quantities supports the kind of conceptual development that will eventually make algebra a sensible tool for thinking and problem solving. . . . Third, the quantitative/conceptual approach also suggests an early route to algebraic symbols in its focus on representing the general numerical relationships, rather than specific computations. (Smith & Thompson, 2007, pp. 21-22)

There is also another vein of research that examines quantitative reasoning as a basis for constructing algebraic reasoning by analyzing students' schemes, concepts and abstractions.<sup>9</sup> In this research (Tillema, 2007), in contrast to Smith's and Thompson's view, quantitative reasoning is not a transitional phenomenon between arithmetical and algebraic thinking, but it is the overarching conceptual framework of which algebraic reasoning is a specific form or part. More precisely, Tillema (2007) conjectures that

<sup>&</sup>lt;sup>9</sup> See Hackenberg (2005) and Tillema (2007).

students' activities would be algebraic if they were to operate on the abstracted concepts, symbols (words or written notations) that are derived from quantitative schemes in quantitative situations. Tillema gives the following example to illustrate what he means by a concept such as seven.<sup>10</sup> Think of a situation in which a child has a meaning for seven as the result of counting scheme whose activity is not implemented, but the word or the numerical symbol (seven) stands for the counting scheme. So when students operate with these symbols, it is possible that these operations could be thought of as algebraic reasoning. Based on this explanation, I conjecture that we can conceptualize Tillema's algebraic reasoning by extending his example with another example: Without using counting schemes to conceptualize the two numbers, a student can either add four more onto seven (using fingers or verbally counting), or just produce 11-the result of this addition in his mind without implementing the activity of adding or counting up. While these two ways of adding are different means of operating on symbols, it seems neither of them is algebraic in the traditional sense. It might be that because of this Tillema creates further criteria to help conceptualize students' activities that are algebraic. Tillema points to three important foci of algebraic reasoning: (a) using the structure of the scheme when relating similarities or differences between different problem situations (making generalizations), (b) using the scheme recursively by taking the results of the scheme to operate in a similar situation using the same scheme, and (c) using the notations—verbal or written—to either symbolize or explain the activity of the schemes and concepts (p. 36).

Tillema (2007) emphasizes that besides the role of "students' quantitative

<sup>&</sup>lt;sup>10</sup> The example is taken from Steffe, von Glasersfeld, Richards, and Cobb (1983).

operations, schemes, and concepts" in the development of algebraic reasoning, "a continued focus" of these elements is also important for the "development of algebraic symbol systems" (p. 38). To explain this claim, he gives three reasons, which are compatible with Smith and Thompson's (2007) view as it relates to the further importance of using quantitative situations. The reasons are as follows: " [1][the quantitative reasoning] opens the possibility for students' quantitative reasoning to be reflected in notation they produce, [2] it opens the possibility to build students' mental imagery for problems situations, and [3] it relates algebraic symbol systems to students' experiential realities" (p. 39).

In his three case studies, Tillema uses quantitative situations of discrete cases to construct a model of students' mental imagery and production of symbols.<sup>11</sup> In the problem situations, the discrete cases were always known quantities and the results were interpreted as unknown quantities. I am in agreement with Tillema's view on algebraic reasoning, which springs from the need for studies that define algebraic reasoning as originating from students' schemes and operations. However, considering the traditional view of algebra,<sup>12</sup> there is a need for investigation of students' operations in situations in which they are required to operate with unknowns as their initial activities. These unknowns should be both what students operate on and the result of their activities in different problem situations, including measurable continuous quantities.

Therefore, Hackenberg's study (2005) has more potential than Tillema's to

<sup>&</sup>lt;sup>11</sup> Two examples from Tillema's study are the following: (a) *The Handshake Problem*: Suppose that there are four people in this room and each person wants to shake every other person's hand. How many different handshakes would there be? (b) *The Flag Problem*: You are the President of a new country. You need to design a flag that has two stripes. You have 15 colors to choose from. How many possible flags could you make?

<sup>&</sup>lt;sup>12</sup> The traditional view of algebra emphasizes use of written symbols for unknowns and generalizability.

contribute to the discussion because of the way in which she defines the specific theoretical constructs related to unknowns and the research questions she investigates related to mathematical learning. Hackenberg's study emphasizes quantitative reasoning as a basis for investigating the underlying construction and solution of linear equations of one unknown, that is ax = b. In this sense, our overall research goals show similarities, especially in her understanding of students' constructions of fraction composition schemes and my aim to understand fractional multiplying schemes (see my first research question). Even though the students' observable activities underlying both composition and multiplying schemes might be the same (producing fractional quantities when transferring "of" in the statement into mathematical actions), I see fraction multiplying schemes as another level of complexity, since producing measurements of the quantities using unit measurement might not be possible for students who can compose two fractions. Similar to Tillema, Hackenberg (2005) emphasizes a non-traditional view of algebraic reasoning with three characteristics:

1. If a scheme is a reflected abstraction and generalizable, then it is algebraic. For example, if a child has a way of operating for dividing any number by 3, then division is a conceptual structure for the child, and therefore this way of thinking can be called algebraic. By conceptual structure, Hackenberg (2005) means "the abstraction of a 'program of operations' from the experiences of using particular schemes that includes an awareness of how the schemes are composed (their structure) and an ability to operate with this awareness" (p. 43).

2. Operating with unknown and known quantities *simultaneously* when making a relationship between them is important if the students' actions are to be conceived of as

algebraic. Hackenberg indicates that this way of operating would indicate *reciprocity*, which she conceives as another criteria for algebraic reasoning. Simultaneous operating is different from operating on known quantities *sequentially* to produce an unknown quantity, which she would call quantitative reasoning. Hackenberg gives an example for illustrating the simultaneous operation: "Tree Problem: Three-fourths of a decameter is two-thirds of the height of a tree. How tall is the tree?" (p. 39) If a student can conceive a third of the tree's height as half of 3/4 a decameter simultaneously, which means she can conceive the unknown quantity as a thing in itself and at the same time use the known quantity to make an equivalency relationship between the parts of those two quantities, then "quantitative relationships are seen as bi-directional and a student can appropriate any quantity as the basis by which another quantity is produced" (Hackenberg, 2005, p. 44). This way of making a relationship between multiplicative quantities is called *reciprocity*. The concept of reciprocity and its function in constructing and solving linear equations are important, and I take Hackenberg's work on this issue as a substantial contribution to the field. My purpose, as I will explain in the rationale to the second research question, is to extend this research and revisit Hackenberg's conjectures related to the necessary situations and operations for the possibility of such reasoning.

3. Use of notations is another indicator of algebraic reasoning, and Hackenberg states that notations do not have to be conventional algebraic symbols, but some form of expression of the conceptual structures is necessary. However, she is not certain whether to give more value to students' actions (and so judge them as algebraic) if the notations are results of finished mathematical performing or if they are the production of current mathematical activities. Depending on the particular student and his/her mathematical activity, Hackenberg seems to be flexible about when to refer to notations as indicators of algebraic reasoning. For example, for Hackenberg, drawing two rectangles to conceptualize the known and unknown quantities in the tree problem might be indications of algebraic reasoning, since there is an awareness of known and unknown quantities and they are notated, even though the operations might not yet be conceptually structured during that particular instance of solving the problem.

I will return to Hackenberg's study and the types of reversible multiplicative reasoning problems she presents in detail when I discuss the rationale for the second research question. In order to explore the role of unit measurements in the construction of fraction multiplying schemes, which contributes to the rationality for my first (construction of fraction multiplying scheme), I elaborate on how I see Hackenberg's research contributing to my thinking. Even though Hackenberg acknowledges the role of units in the construction of fraction composition scheme by framing the issue with unitscoordinating schemes, the emphasis on the units is not central. I expected more emphasis on the investigation of the role of units since the resulting quantities (results of fraction composition scheme) need to be reinterpreted in terms of measurement units (fraction multiplying scheme), and this is how I conceptualize the difference between fraction composition and fraction multiplying schemes. For example, when a student produces a quantity for 1/2 of 3/5 of a liter by operating with JavaBars and using a fraction composition scheme (creating a 3-part bar for 3/5 of a liter, then taking half of each part [producing six mini-parts], and combining the three mini-parts for a result), she may not be aware of the measurement of the resulting quantity in terms of the standard measurement unit, such as 3/10 of a liter. This awareness requires operations of a fraction

multiplying scheme. Therefore, there is a need for research that has a similar framework to Hackenberg's in terms of algebraic reasoning but investigates the fraction multiplication scheme as an extension of fraction composition scheme. I aim to investigate these points with my first research question.

Before moving to the rationale for the second research question, I will use the literature to discuss fraction multiplication and some necessary concepts (e.g., units-coordinating schemes, recursive partitioning, etc.)

Multiplication Operation on Fractions

What al-Khawrizmi saw as a major intellectual achievement, many students — perhaps most — see as the first significant cabalistic mystery of mathematics: operations on fractions. (Davis, 2003, p. 107)

In the last chapter of The Development of Multiplicative Reasoning in the

Learning of Mathematics (Harel & Confrey, 1994), Kieren (1994) summarizes and

critiques the chapters and related ideas in the book. He makes a point regarding a

different conceptualization of multiplication that is action-focused but not "repeated

addition." He comments:

Confrey points to actions (e.g., joining, annexing) that she claims are used to support additive approaches to multiplication and would obviously be related to a 'repeated addition' approach. As we look at other research, do we see 'actions' that could be related to counting but that point beyond 'repeated addition'? One such scheme is 'iterating.' Steffe distinguishes iterative multiplying or iteration from the usual interpretation of  $6 \times 4$  as six groups of four things. This latter interpretation can be seen as a basis for repeated addition and can support the product, twenty-four, as simply a count. In iterative multiplying, the 6 indicates six iterations of an iterable unit, 4 (as a chunk of identified quantity). (p. 392) Kieren's comment points to an important structure and scheme that is basic for

whole and fractional number multiplication: units-coordinating scheme and three levels

of units. Conceiving the result of an iteration operation (e.g., 24) as a unit structure that

has two other units nested in it is also important for partitioning operations when conceptualizing fractions. Therefore, I will first briefly review Steffe's whole number multiplicative schemes and discuss how different levels of units function in those schemes. I will then discuss how units-coordination plays a role in fractional schemes and fractional composition schemes.

## Units-coordinating scheme and two- and three- levels of units.

To explain students' ways of operating in multiplicative problem situations, Steffe (1994) explores students' activities when they were given different problem situations with blocks (or discrete objects). He first constructed *a units-coordinating scheme* when a student, Maya, produced a numerical result when in her imagination she placed two orange squares into six red rectangles. Basing her actions on coordinating two orange squares that fit into a blue rectangle and six blue rectangles that fit into a red rectangle, she used each of her six fingers as she counted 1,2; 3,4; 5,6; 7,8; 9,10; 11,12, and produced 12. Although Maya's way of acting was judged as a multiplicative scheme, it did not have an iterative structure (as it is discussed by Kieren). She was coordinating two independent units: two orange squares and six blue rectangles. Her counting implies that she inserted two units (orange squares) into each of the six units (blue rectangles), or considered each blue rectangle as composed of two orange squares. Therefore, she constructed a unit of units structure.

Using another student's activities, Steffe (1994) also constructed a related scheme that is similar to division in the traditional sense, *a reversible units-coordinating scheme*. The student, Johanna, was able to form a unit of units of units structure, such as viewing 19 as a unit composed of a unit of ten-units and a unit of nine-units. She coordinated the

two units, five and four, when finding how many blocks were in five rows of four blocks that she couldn't see. After finding how many blocks were in the first three rows, she continued, "twelve plus four is sixteen, and sixteen plus four is twenty" (Steffe, 1994, p. 26).

Steffe (1994) conceptualized a *reversible units-coordinating scheme* when Johanna modified her units-coordinating scheme to find the number of groups of three in twelve blocks by counting threes. Johanna could even find the additional number of rows of three when more blocks were added to 12 blocks and the total number of blocks was 27. Therefore, Steffe (1994) claims, "three was now an iterable unit that she could use in re-presenting a continuation of the unit containing four units of three [continuation of 12] and in counting beyond a unit containing four units of three using her units-coordinating scheme" (p. 28). Although Johanna established a three-levels of units structure in the multiplicative situations, and such situations are viewed as the roots of an *iterative multiplicative scheme*, these structures were established in the act of operating and were not structures that she could use in reflective thought. Her units-coordinating scheme was reversible when she engaged in a continuation of a given situation: e.g., forming two rows of four blocks for the extra eight blocks when she already made five rows of four blocks. Johanna could operate reversibly when called for in a given situation, but she was yet to establish reversibility as available to her in reflective thought prior to operating. When she achieved this milestone, she could take the three levels of units (e.g., a unit of 12 conceived of as three units four times) as the problem situation and the result at the same time. In this case, iterating a unit of three as many times as needed would guarantee an awareness of a unit of units of units structure, for example, twelve is four threes, and

so forth. Steffe would then call this way of operating the iterative multiplying scheme that Kieren also discussed in the previous quote incorporating "6×4."

Fractional schemes and operations for conceptualizing fraction composition scheme.

Given the difficulty of mastering the concept of unit in whole number situations, it is not surprising that changes in the nature of the unit in the middle grades bring new cognitive demands and renewed difficulties for students. (Hiebert & Behr, 1988, p. 2)

According to Steffe (2003), the units-coordinating scheme plays an important role in students' construction of unit fractional composition scheme. For example, when a student is asked to find 1/3 of 1/5 of a unit bar, three can be used to partition each of the five partitions of the unit bar in a distributive way. While this way of operating (with partitioning) provides an example of coordinating the partitions of five and three and it does not function as a uniting operation,<sup>13</sup> the scheme still involves *coordinating* units of three with each one of the five units of the unit bar. This way of embedding three units in each of the five partitions might only require using a unit of units of units structure in operating, whereas producing a result for how much 1/3 of 1/5 would be of a whole unit usually requires taking a unit of units of units structure as a given in reflective thought (which means viewing the unit bar as composed of five units, each of which is composed of three units). Hackenberg (2005) indicates that "Coordinating three levels of units prior to operating-what I will often refer to as having constructed three levels of unitsseems to be required for a good deal of fractional reasoning (cf. Steffe, 2002a, in press)" (p. 53). This type of coordination of three levels of units plays a role in students'

<sup>&</sup>lt;sup>13</sup> This was the case for the units-coordinating scheme in Maya's example in which she made a coordination for *uniting* the six groups of two units.

recursive partitioning operations as well as fraction composition schemes.

Recursive partitioning can be illustrated with two different problem situations: one example is to partition a length unit into 12 equal parts with more than one partitioning step.<sup>14</sup> In this case, the result of recursive partitioning is known (the starting unit must have 12 partitions). The other example is to produce a numerical result for the composition of two fractions, such as 3/4 of 1/4,<sup>15</sup> in which case the number of partitions in the whole unit is unknown. In both cases the student's goal is not to partition a partition, but to produce 12 equal parts from an unpartitioned bar for the first case and to reinterpret the quantity of 3/4 of 1/4 as a fractional part of the whole unit for the second case. Steffe defines recursive partitioning as follows:

Recursive partitioning is the inverse operation of first producing a composite unit, multiple copies of this composite unit, and then uniting the copies into a unit of units of units. So, producing a recursive partitioning implies that a child can engage in the operations that produce a unit of units of units, but in the reverse direction. Recursive partitioning is fundamental in the production of the unit fractional composition scheme. (Steffe, 2003, p. 240)

Steffe (2003) uses the second example (3/4 of 1/4) not only for discussing the recursive partitioning operation but also for discussing the function of the recursive operation in construction of a *fractional composition scheme*. Using a student's (Jason) activity, Steffe (2003) defines the fractional composition scheme (such as finding 1/3 of 1/4 of a whole in terms of the whole) as "embedding recursive partitioning in the reversible partitive fractional scheme in the process of achieving the goal" (p. 241). He gives further detail on the definition of a fractional composition scheme.

The goal of this scheme is to find how much a fraction is of a fractional whole,

<sup>&</sup>lt;sup>14</sup> This example is taken from Steffe (2003). See p. 204 for more details.

<sup>&</sup>lt;sup>15</sup> This example is taken from Steffe (2004). See p.137 for more details.

and the situation is the result of taking a fractional part out of a fractional part of the whole, hence the name composition. The activity of the scheme is the reverse of the operations that produced the fraction of a fraction, with the important addition of the subscheme, recursive partitioning. The result of the scheme is the fractional part of the whole constituted by the fraction of a fraction. (Steffe, 2004, p. 140)

In his two papers, Steffe (2003, 2004) analyzes Jason's and Laura's multiplication operations with fractional numbers. The situations include operations on unit fractions (such as taking one of the three shares of 1/4 of a pizza), successive halving and thirding operations (such as producing 1/2, 1/4, 1/8, etc., and 1/3, 1/9, 1/27, etc. in a sequence (cf. Steffe, 2003)), and operations with proper fractions (such as producing 3/4 of 1/4 or 3/4 of 1/2 of a 4/4-stick (cf. Steffe, 2004)). To expand our knowledge about fraction multiplying schemes in the literature, it is necessary to investigate and report students' ways and means of operating when they compose two fractions with combinations of different types of fractions (such as both proper, or both improper, or one being proper and the other being improper, etc.) For this purpose, using activities and operations of the students that I worked with, I plan to expand Steffe's definition of the fraction multiplying scheme by defining other necessary operations for a fraction multiplying scheme.

To illustrate how unit-structures function in fraction multiplication schemes for which Steffe provided details, I will introduce two studies (Mack (2001) and Olive (1999)) and use mainly fraction multiplication as a means to analyze the premises and conclusions of those studies. *Examining two studies on students' fraction multiplication using the fraction composition scheme.* 

Mack's (2001) study is one of the recent research studies often cited in relation to the learning of fraction multiplication. She believed that students come to instruction with an informal knowledge of partitioning,<sup>16</sup> which "involves a part-whole perspective that focuses on fractional quantities as counting units and does not appear to reflect the conceptual complexities which are needed for understanding multiplication of fractions (e.g., reconceptualizing different types of units and then partitioning them)" (p. 271). She investigated how students would use their knowledge of partitioning with three types of problem situations: "[1] the two terms are equal,  $a/b \times b/d$  (e.g.,  $1/4 \times 4/5$ ); [2] one term is a multiple of the other,  $a/nb \times b/d$  or  $a/b \times nb/d$  (e.g.,  $3/4 \times 2/3$  or  $2/3 \times 9/10$ , respectively); or [3] the two terms are relatively prime,  $a/b \times c/d$  (e.g.,  $3/4 \times 7/8$ )" (p. 271).

Mack taught six 5<sup>th</sup> graders over a three-month period. At the start of this period, all students had the knowledge to produce unit fractional amounts using a whole unit, e.g., one-third of one whole pizza, by partitioning the whole into a required number of pieces. However, they were not able to produce non-unit fractional amounts, such as two-thirds of a whole cookie. Mack wanted to see how their informal knowledge of partitioning would enable some of the six students to operate with different units in the problem situations and what kinds of interactions would enable such results. She started posing problem situations within an equal-sharing context, such as *10 cookies are shared* 

<sup>&</sup>lt;sup>16</sup> Mack refers to her previous work, Mack (1990), for this belief.

among four people. She later posed different types of problems, as mentioned above. In

her analysis of those problems, she described four types of mental processes.<sup>17</sup> Solving

type 3 problems (e.g., 3/4 of 7/8, where 4 and 7 are relatively prime) was the most

demanding for the students compared to the mental activities they used with problems

*type 1* and 2. The four mental phases she described are as follows:

(a) Seeing embedded fractions and not partitioning the unit. For example, finding 1/4 of 4/5 of a cake in terms of the whole cake. Four-fifths (which was interpreted as 4 parts by the students) is the embedded fraction in the 5/5, and to find 1/4 of the 4 parts (4/5) the student did not need to make further partitions and used one of the 4 parts.

(b) *Repartitioning the unit.* The problems in this category are in the form of  $a/nb \times b/d$ . For example, "Suppose you have two thirds of a bag of potato chips. You eat three fourths of what you have for a snack today. How much of the whole bag will you eat for a snack today?" (p. 285)

(c) A composite unit by grouping unit pieces. The problems in this category are in the form of  $a/b \times nb/d$ . An example is: "You have twelve fifteenths of a can of dog food. You're going to feed your dog five sixths of the amount of dog food that you have. How much of the whole can of dog food do you feed your dog?" (p. 287)

(d) Repartitioning and grouping pieces of units. The problems in this category are in the form be  $a/b \times c/d$  (where b and c are relatively prime). For example, "During your trip to the zoo, you have seven eighths of a gigantic chocolate chip cookie (after feeding the flamingos one eighth of the cookie). You go to feed the bear, and the bear is not real hungry. The bear already ate some fish. The bear eats three fourths of the remaining piece of cookie. How much of the whole cookie does he eat?" (p. 289)

When the specific discussions Mack made regarding to phases (a) and (b) are

reviewed in detail, it can be noted that Mack does not discuss units-structures when

explaining students' activities. Discussing units-structures and the units-coordinating

scheme is important to conceptualize different operations other than partitioning when

<sup>&</sup>lt;sup>17</sup> Mack's description of students' activities (mental process) in (a) refers to their activities in *type 1* problems, students activities described in mental process (b) and (c) refer to their activities in *type 2* problems, and the activities that are described in mental process (d) refers to the students' activities in *type 3* problems.

explaining students' fraction multiplying schemes. Since these two phases also function as a basis for phase (c) and (d), it is beneficial to explore students' activities and analysis of them to understand the overall contribution to the field of Mack's study. The details of the problem situations and the students' activities in those two phases are as follows:

(a) Seeing embedded fractions and not partitioning the unit (e.g., finding 1/4 of 4/5 of a cake in terms of the whole cake). After a student made a circular shape partitioned into five pieces, he marked one of the pieces and referred to the remaining parts as four pieces. He did not feel a need to further partition any of the pieces to show his answer, which was one piece out of the four pieces. Mack claims that all of the six students were able to understand the problems that were posed in this way and solve them in a similar manner. The important point for Mack when posing this type of problem was that the students should realize the different referent units. For example, a whole cake is the referent unit when conceptualizing 4/5 of a cake, and "4/4" (where each 1/4 is 1/5 of the whole cake) is the referent unit when taking 1/4 of the 4/5 of the cake. However, it is not clear from the analysis whether students constructed the equivalency relationship that 1/4 of the 4/5 of the cake is the same as a 1/5 of the whole cake. From the report of the analysis, it appears as if the two independent unit of units structures (the whole cake with five units and the 4/5 of the cake with four units) were not connected for the students. In addition, 1/4 of 4/5 of a cake might be thought of as three levels of units if there is an awareness of the whole cake from a student's point of view. The situation Mack describes is a unit of units structure in which 1/4 is a unit out of the four units that is equivalent to 4/5 of the cake, but this relationship to the whole cake is not in the awareness of the student. Therefore, the only referent for 1/4 of 4/5 of the cake is 4/4,

which refers to the quantity of four pieces. In this sense, we cannot claim that students were able to conceptualize this situation as fractional multiplication; they were only subtracting equal amounts from given equally partitioned quantities (e.g., the student said, "and I gave one to him of these four there" (Mack, 2001, p. 283).)

(b) *Repartitioning the unit (3/4 of 2/3 of a bag)*. What Mack means by this description is that the student partitioned the bar into three parts, marked a part to omit and referred to the remaining two parts as two-thirds, and then partitioned those two parts to make 4 mini-parts in total (each third was partitioned into two parts), so the student *repartitioned the unit* (repartitioned each of the two thirds). As a result of his activities, the student pointed out to 3 mini-parts and eventually said, "three sixths." The student's activity is similar to *recursive partitioning*, partitioning a partition. While Mack provides the transcription of the interaction between her and the student, Adam, it is not clear how much independence the other five students (as well as Adam) had when she claims the result in terms of the whole.

The answer that all of the students gave was *one and one half of a third of the bag* for 3/4 of 2/3, and Mack reports that she asked questions similar to the following ones: "Are each of these (one third pieces that were split in half) the same size as this (shaded one third piece on top)?" and "one and one-half thirds is the same amount as what fraction of the whole cookie [cookie context was initially introduced by the child]?" (p. 287) With these questions, the teacher/researcher's purpose was to orient the students to the number of equal partitions in the unit and to the "out of" idea in fraction conceptualization. She says, "Following this, all six students stated their answers as simple fractions [such as 3/6] for all situations they encountered involving the

multiplication of fractions" (p. 287). While Mack makes this claim for all of the six students, she simplifies the complexity of the students' thinking by minimizing or ignoring the differences in their operations. The students needed to operate with the two levels of units they constructed—a unit whole composed of three units— for conceiving the half of a third of a whole as equivalent to a sixth of the whole. To be able to do that, students needed to conceptualize a unit of units structure for 2/3 so that it was two times as much as one of the thirds and needed to further partition each third recursively to produce half of a third and coordinate this smallest unit (half of a third) as part of the fractional whole, therefore producing and operating with a three-levels-of-units structure. While Mack proposed the goal of partitioning a partition (halving the thirds) as a meaningful goal for all the students, not all six students were ready to assimilate this situation as she claimed. At least one student, Lisa, did not have this idea of a unit of units (from the descriptions that Mack gave earlier). Lisa only perceived 2/3 as two pieces out of three pieces; two-thirds was not part of a fractional unit whole, which means it was not composed of two individual one-third units. Therefore, claiming that all six students independently produced one and one-half of a third as 3/6 (or a half of the potato chips) is not a convincing argument.

Students activities in phases (c) and (d) were also grounded in using the operation of partitioning a partition (or grouping partitions), which is explored in the discussion of phase (b). Mack claims that (with her help) only half of the students could successfully operate in phase (c), and none of the students were able to solve problems in phase (d). Mack bases all of her analysis on students' informal knowledge of fractions ("out-of" and equal sharing)<sup>18</sup> and her probing questions acted as initiators for students' advancements. While Mack's study is one of the exemplary works in understanding students' fractional ways and means of operating, there are some limitations to her study. For example, one would expect in her analysis the discussions to be comparable detail to those in other studies (e.g., Olive, 1999) and to contribute to the literature by explaining how she viewed the developmental connections between the phases as well as the characteristics of students' mathematical means related to those phases. Such discussions might have included concepts compatible to; for example, unit fractions as iterable units, units as a structure (a 2-, or 3- levels of units structure), and operations that are an important part of a fraction composition scheme (such as recursive partitioning in Olive's or Steffe's terms). In addition to these reasons, Mack's study has some insufficiencies for her (as well as for the reader) to make (consistent) models of her students' knowledge of fraction multiplication: Differentiating among students' acts and differentiating between students' independent acts from those initiated by the researcher are challenging.

Earlier I suggested that Mack made interpreting the result of recursive partitioning as a simple fraction sound deceptively easy, Olive (1999, 2001) claims that similar situation was a big constraint to his two advanced students' operations with fractions. As Olive (1999) indicates, engendering ways and means of operating to eliminate such a constraint is not an easy task. While Mack (2001) was aware of Olive's study and cited it many times in her paper, she did not expand or investigate what the necessary operations or possible teacher-student interactions could be to remove such a block of students' fraction multiplication activities.

<sup>&</sup>lt;sup>18</sup> The "Out-of" idea is commonly known for causing difficulties for students when they conceptualize improper fractional quantities, which Mack seemed to overlook.

Olive (1999) analyzed two children's constructions of fraction multiplication schemes using the data which was part of a 3-year teaching experiment on 3<sup>rd</sup>. 4<sup>th</sup> and 5<sup>th</sup> grade children's constructions of rational numbers of arithmetic (RNA). Unlike the Rational Numbers Project (Behr, Harel, Post, & Lesh, 1992)—which was based on the semantics of rational numbers such as measure, quotient, ratio number, multiplicative operator, and part-whole relations—Olive's work was based on students' mathematical activities. Its aim was to investigate how students' whole-number knowledge played a role in their activities with fractions and how they could construct a scheme called rational numbers of arithmetic.<sup>19</sup> Even though the two most advanced students, Nathan and Arthur, had constructed a common partitioning fractional scheme (for example, they could partition a unit stick<sup>20</sup> and pull out an amount for both 1/5 and 1/3) and recursive partitioning operations (such as when finding 1/5 of 1/6 of a stick, they first partitioned the stick into six parts and then partitioned the disembedded sixth into five, and pulled out one mini-part), Olive (1999) found that "one stumbling block that they met was to name a fraction of a fraction as a new fraction of the original whole" (p. 292). For a while, Olive's students produced the result of 1/5 of 1/6 by iterating the mini-part 30 times and checking this against the unit stick. Olive (1999) reports that "eventually, they

<sup>&</sup>lt;sup>19</sup> Olive and Lobato (2007) explain RNA as follows:

For Steffe and Olive, the RNA are more than fractions but less than equivalence classes of fractions that belong to a quotient field. The child is aware of the operations needed, not only to reconstruct the unit whole from any one of its parts, but also to produce any fraction of the unit whole from any other fraction. For example, given a bar that is said to be 2/5 of another bar, the child would partition the given bar into two equal parts and then iterate one of those parts five times to create the other bar. If given a bar that is 3/4 of some unknown bar and asked to create 2/3 of the unknown bar, the child could partition the given bar into three equal parts and then partition one of those three parts into three smaller parts to construct 1/12 of the unknown bar. The child could think of doing this because of previous experience creating a partition of a bar from which the child could pull both a fourth and a third of the bar. The child could then use that 1/12-part to construct 2/3 by iterating the 1/12-part 8 times. (p. 12)

<sup>&</sup>lt;sup>20</sup> A unit stick is a line segment in a software program, TIMA: Sticks, which provides a stick that a child can cut, partition, iterate, etc.

developed the ability to mentally project the partition of, for example, three equal parts of 1/12 into each of the twelve 12ths in the unit to establish the value of 1/3 of 1/12 as 1/36" (p. 292), so he claims the students reversed their recursive partitioning operation to find the value of a partition of a partition in terms of the whole unit.

After the unit fractional multiplication problems, students were asked to solve multiplication problems with proper fractions. For example, a problem was given whose representation was claimed was 1/4 of 3/7 of the whole pizza in a sharing context:

A pizza (stick) is cut into seven slices (pieces). Three friends each get one slice. A fourth friend joins them, and they want to share their three slices equally among the four of them. How much of one whole pizza does each friend get? (Olive, 1999, p. 292)

After partitioning a stick into seven parts, Arthur pulled out three parts. He then

partitioned each of the three parts into four mini-parts and pulled out three mini-parts for

a share of one person. Therefore, he produced the quantity for the sharing problem.

However, Arthur only stated the result as "3/4 of 1/7 of the pizza" and attempted iterating

this group of three mini-parts to make a stick the same length as the unit stick. His

purpose was to check his result against the starting stick. Olive (1999) indicates:

The goal of finding 3/4 of 1/7 of a stick was not attainable with his [Arthur's] current operations, which were based on his strategies for finding a unit fraction of a fraction. A modification of these recursive partitioning operations was required involving units-coordinations with three different levels of units and reversal of his partitioning operations. (p. 293)

To help Arthur modify his operations, the researchers posed many problems, such as sharing 4/9 of a pizza stick among five people and finding how much of the whole pizza was one person's share. After Arthur partitioned the unit stick and produced miniparts, he said the result was 4/5 of a 9<sup>th</sup> of a pizza. The researcher oriented Arthur to focus on how much of the unit stick was a mini-part (the third pulled partition), which helped him to modify his unit fraction composition scheme. Arthur then produced 4/45 by converting "4/5(1/9-unit)-unit to 4(1/5(1/9-unit)-unit)-units" (p. 295). Using Arthur's new modification, the permanence of which he confirmed in two other tasks (e.g., "sharing 5/11 among 7 people" and "guess the stick which is 2/3 of 1/7 of the given stick"), Olive (1999) defined a *fraction composition* scheme. He claimed this scheme was "an integration of his [Arthur's] iterative fractional scheme with his reversible partitioning operations and distributive strategies" (Olive, 1999, p. 296).

While Olive investigated the fractional operations in multiplicative situations, his report does not discuss the details concerning whether the students viewed the situations as only sharing situations and the results as only what a person's share is or whether they viewed the resulting quantities as the result of the multiplication of two fractions. Since the problem context was sharing, there are possibly other operations that need to be investigated in order to understand how a sharing context is transformed to some type of algebraic context—that is, how students become aware of the mathematical operations they experience. Therefore, whether the students in the study also conceived of these sharing contexts as the multiplication of two proper fractions needs further clarification (as was also the case in Mack's study).

Based on this study, reversible partitioning, recursive partitioning, and 3-levels of units (besides other operations) play an important role in the construction of a fraction composition scheme (see Figure 3.1).



Figure 3.1. Olive's (1999, p. 297) diagram that shows the schemes and operations related to a construction of a fraction multiplication scheme.

My research also aims to investigate how these three operations play a role in extended fraction composition scheme situations, especially those which involve taking parts of improper fractional quantities without a sharing context, taking fractional parts of fractional units when the whole unit is not visible, and those which involve inverse reasoning (such as 2/3 of a liter water bottle has a capacity which is 3/5 as much as another water bottle). With the addition of new schemes and operations related to the fraction multiplying scheme, revising Olive's model (see Figure 3.1) is a strong possibility.

## Roles of (measurement) Units in Students' construction of Schemes and Operations in Fractional Quantitative Situations

A similar emphasis on quantitative reasoning, like Smith and Thompson's (2007), takes place in the Measure-Up project in the context of early-algebra. Measure-Up is a University of Hawaii based project set in a laboratory elementary school where researchers teach. The project focuses on teaching mathematics using symbols, quantities, and comparisons between the quantities of continuous situations without using numbers. Their theoretical orientation is based on Davydov's work (1975). The group (Dougherty & Slovin, 2004) reports that using measurement and quantities, students naturally use written symbols that stand for the measurement of certain quantities and represent relationships between the quantities as well as the operations. Dougherty (2002) also claims that numbers naturally develop with this kind of quantitative reasoning, when students are exposed to indirect comparison situations.

Students are given situations so that direct comparisons are not possible. When students cannot place objects next to each other, for example, to compare length, they are now forced to consider other means to do the comparison. Their suggestions on how to accomplish the task involve creating an intermediary unit, something that can be used to measure both quantities. The two measurements are then used to make inferences. For example, if students are comparing areas T and V, and they use area L as the intermediary unit, they may note that— Area T is equal to area L and area L is less than area V. Without directly measuring areas T and V, students conclude that area T must be less than area V. Their notation follows:

T = LL < VT < V

With the use of a unit, students are now ready to begin working with number. Number now represents a way that students can express the relationship between a unit and some larger quantity, both discrete and continuous. Conceptually, the introduction of number in this manner offers a more cohesive view of number systems in general. (Dougherty, 2002, p. 19)

Therefore, creating an intermediate unit in the measurement context and taking it as a reference for comparisons is crucial in the construction of numbers from the point of view of the Measure-Up project. While L.P. Steffe (personal communication, Jan 6, 2008) views the claim about the role of intermediate unit in the construction of number as open to discussion,<sup>21</sup> and at the same time, he agrees with measurement's role in the construction of discrete and continuous quantities, I think an intermediate unit for comparison is especially important when operating with fractional numbers. The fractional numbers might be viewed both as a continuous quantity and discrete quantities. Fractional numbers are continuous because we can partition a unit to produce them, and they are discrete because they are composed of unit fractions that might be viewed as discrete items. The intermediate unit that I envision is a type of abstracted unit which is produced as a result of mathematical operations on fractions; for example, if a student is given 3/5 of a liter as part of a problem statement and if the student is asked to find half of this quantity in terms of a liter, then the liter is an intermediate unit since a liter is implicit in the student's conceptualizations of the quantities. Therefore, when conceptualizing 3/5 of a liter as three out of five parts or three times one of the five parts, we refer to a unit that contains those parts (i.e., one liter). So while a liter can be thought of as a unit in itself, it is also a continuous entity that might be partitioned into smaller units.

<sup>&</sup>lt;sup>21</sup> Numbers are possibly viewed as an abstract form of both continuous and discrete quantities.

Therefore, creating a similar unit that serves as an intermediate reference is also necessary for the middle school children who are required to make conceptual units when constructing fractional operations (for example, adding or multiplying two fractions). However, to be able to advance in using units in the multiplicative fractional situations, students need to construct different and more complex mathematical operations other than the measurement and comparison mentioned in the Measure-Up project.

Extrapolating from the aforementioned research program and the ones discussed in rationale for research question 1 on quantitative reasoning and measurement, I want to investigate necessary operations for the construction of fractional multiplication in the multiplicative quantitative situations and in inverse reasoning problems as well as how *unit* plays a role in these constructions.

Moreover, when we look at the literature about fraction multiplication and students' learning, measurement unit is a common theme that is salient but does not get enough attention as one of the contributors to students' difficulties in fraction multiplication contexts.

One of the most cited studies investigating learning fractions was done by Fischbein, Deri, Nello, and Marino (1985). They investigated grade 5, 7, and 9 students' preferences of operations in word problems and were inspired by the following two studies and their findings:

[1] Bell, Swan, and Taylor (1981) have shown that when children are presented with a series of problems with the same content, they may change their minds about the operation needed to solve the problem, depending on the specific numerical data that are given. [2] Hart (1980) found that 12- to 15-year-old pupils systematically avoided multiplying by fractions when solving a problem, even though that would be the simplest way to get a solution. (Fischbein et al., 1985, p. 1)

Even today, I empathize with Hart's assertion that students avoid using fractions when solving problems. On the other hand, to understand the reasons for this avoidance, we need studies that investigate the operations by requiring students to use fractions in their activities. Extrapolating Fischbein's reasons, I hypothesize that even if students know that they need to use multiplication operations for the problems and act accordingly, they may not be able to interpret the result using the problem context; in this case, students' use of operations might not be different from symbol manipulation. As a reason for students' use of an operation other than multiplication for the multiplication word problems,<sup>22</sup> Fischbein et. al. suggest that the numbers are difficult especially when the problem statements include decimals. In connection to this finding, I further explore why the numbers are difficult for the students in the studies when they are expected to use fractions and what their image and conceptualization are for those numbers, especially parts of measurement of (whole) units. While I agree with their conclusion that as long as the operand is a whole number (p. 10) the decision to use multiplication is relatively easy, with my study I hope to gain insight into why this is the case (relative easiness), using transcripts of student-teacher interactions.

Among other researchers, Harel, Behr, Post, and Lesh (1994) further investigated preferences for using the multiplication operation in word problems after being inspired by Fischbein et al.'s study. Their extension of Fischbein et al.'s research involved controlling the text, structure, context, and syntax, and varying the number type (operand

<sup>&</sup>lt;sup>22</sup> While Fischbein et al. suggest that when conceiving multiplication, repeated addition is an intuitive primitive model for multiplication problems, I do not see how their data from multiple choice items support this claim. In addition to using multiple choice tests, some students' use of other operations (such as division) different from multiplication can not be explained by claiming that repeated addition is the intuitive model for multiplication.

being less than one, more than one, and whole number) and investigating the rule violation with 11 categories they made up for multiplication and division problems. While they found that their data do not support Fischbein et al.'s *absorption effect*, by stating "no significant difference in performance was found between multiplication problems with multipliers whose whole part is relatively large and those with multipliers whose whole part is relatively small," (Harel et al., 1994, p. 380) they produced further research questions. One important question is very much related to the role of conceptualizing a unit and to students' mental operations in multiplication problems related to this conceptualization. They asked,

If indeed this model [Fischbein et al.'s] governs subjects solution of multiplication problems, it is not at all clear why the intuitive rule derived from it-that the multiplier must be a whole number—is substantially less robust [less difficult] in the case of a non-whole-number multiplier greater than 1 than in the case of a multiplier smaller than 1. Further, it is not all clear what is the conceptual basis for the multiplier 1 being an index for the relative difficulty of multiplication problems. (Harel et al., 1994, p. 382)

I think this claim for further research emphasizes the need to understand students' fraction multiplying schemes and the role of the conceived unit in those actions. Therefore, my investigations will shed further light on why it is more difficult for students to act and produce multiplicative structures in fraction multiplication problems when one of the numbers is less than one as opposed to when one of the numbers is more than one, and why the unit of one is important in students' activities (both successful and unsuccessful).

Rationale for Research Question 2: What operations and schemes are involved in a construction of inverse reasoning that is a basis for conceptual understanding (both construction and solution) of linear equations with one unknown? What is the role of fraction multiplying scheme in the constructions of inverse reasoning?

In her dissertation, Hackenberg (2005) posed a similar question to this problem and investigated it using different types of *reversible multiplicative reasoning* problems with her four students. She categorized those problems mainly using her mathematical knowledge and the needed operations and schemes for the solution, but ultimately her purpose was to explore how the students would operate with those problems and what the necessary operations and schemes were.

[Type 5] Tree Problem: Three-fourths of a decameter is two-thirds of the height of a tree. How tall is the tree?

[Type 4] Candy Bar Problem: That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length? [Type 3] Peppermint Stick Problem: A 7-inch peppermint stick is three times

longer than another stick; how long is the other stick?

[Type 2] Money Problem: Monica has \$21, which is 3/7 of Todd's money. How much money does Todd have?

[Type 1] Juice Problem: Twenty-eight ounces of juice is four times the amount that you drank; how much did you drink? (Hackenberg, 2005, p. 60-63)

Hackenberg (2005) differentiates the problems she posed to the students

throughout the interactions. This means not all of the students were exposed to Type 5 problems –which is viewed as the most difficult type, since their observed activities did not give promising cues to Hackenberg that those type of problems would be meaningful to them. However, one of the students, Deborah, was able to solve Type 5 problems. The milestone example that Hackenberg (2005) posed towards the end of the 7-month teaching experiment, and for which she then discussed its analysis in detail, was: "The Lizards' car goes 2/3 of a meter. That's 3/4 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far it went?" (p. 193)

During Deborah's construction process the camera was not focused so Hackenberg did not report on how Deborah made this new bar, but she inferred what Deborah did after the camera captured the picture in Figure 3.2. After Deborah made a bar for the unit bar (the unit bar was always required to start with) with JavaBars, she made another bar (using this unit bar) that she partitioned into two (for 2/3 of a meter, see the second bar in Figure 3.2). She then partitioned each of the two parts (in her analysis, Hackenberg calls each part a third of a meter for communication purposes) into six miniparts. She then created a new bar with 16 mini-parts, see the last bar in Figure 3.2. Hackenberg attributes the reciprocity (use of 4/3) to Deborah using this particular explanation: "Deborah: Because that, that is—'*cause I know third is four pieces [four mini-parts, my insertion]. So four times four, because you need four thirds for this one [points to the 16-part bar, the Cobras' distance]*" (Hackenberg, 2005, p.194).



Figure 3.2. Deborah's JavaBars for Lizards' and Cobras' distance.

Deborah did not operate as Hackenberg expected—that is injecting three miniparts into each of the two one-thirds of a meter (instead Deborah injected 6 mini-parts) and then pulling out two mini-parts from one of the thirds (Hackenberg calls this distributive splitting) and repeating this 2-part bar four times to produce the traveling distance of the Cobras' car—nor did Deborah produce the measurement of 16/18 (of a meter) using her distributive partitioning and multiplicative schemes. She visually compared the 16 mini-parts bar to the original unit bar, concluding there were 18 miniparts in the unit bar. In spite of Deborah's course of action, Hackenberg asserts that a reversible iterative fraction scheme was necessary for construction of reciprocity. Hackenberg makes this assertion using this observation: "Deborah conceived of the Lizards' distance as a unit of three units, any of which could be iterated four times to produce the Cobras' distance, and simultaneously she conceived of the Cobras' distance as a unit of four units, any of which could be iterated three times to produce the Lizards' distance" (p. 293).

As Hackenberg indicates, there should be a simultaneous conception of 1/3 of the Lizards' distance and 1/4 of the Cobras' distance. With more precision, the student should simultaneously conceive of 1/3 of the Lizards' distance as being equivalent to 1/4 of the Cobras' distance. However, when the transcript is examined, whether this way of thinking could really be attributed to Deborah is questionable. One of the reasons for this is that Deborah did not explicitly state this relationship using the referent quantities, nor was there a clear-cut evidence that showed she was aware of this relationship during her operations as Hackenberg indicates. Deborah was aware that the bar she partitioned into 12 mini-parts was 3/4 and *each third* was four mini-parts. However, she did not indicate
3/4 of what, this situation shows that the 12 mini-part bar was embedded in the resulting bar—Cobras' distance—and so it is possible that the Lizards' distance was not a separate entity in Deborah's mind. So even though Deborah might have had the same quantity four mini-parts—as a referent and she might have been aware of that quantity, it seems that she was not aware of or did not pay attention to all the different names she could give to 4 mini-parts, such as a third of the Lizards' distance and a fourth of the Cobras' distance as two different names representing the same quantity. Therefore, this situation might indicate that Deborah saw the quantity that referred to the Lizards' distance in the problem situation as an identical fractional part of the Cobras' distance. Even though making this distinction by using this particular evidence might be difficult, I believe this type of discussion is important for the construction of the concept of *reciprocity* and introducing different evidences related to students' activities, which will help us to conceive of reciprocity as an algebraic operation. Reciprocity could be thought of as algebraic because equivalency that is involved in the construction of reciprocity implicitly implies two separate quantities: one of which might be a known measured quantity, e.g., 1/3 of 2/3 of a meter (Lizard's distance), and the other one might be an unknown quantity, e.g., 1/4 of unknown quantity (Cobras' distance).

In addition to this discussion about equivalency relationship, there is another complexity related to *reciprocity* and how the measurement of referent quantities contributes to this concept. In this particular problem, we need to pose the following question: How could Deborah conclude that "4/3 of 2/3 of a meter—Lizards' distance—

is 16/18 of a meter"?<sup>23</sup> I think eventually we want students to come to these types of conclusions. This "how" question did not receive enough attention in Hackenberg's study, possibly because the focus of the experiments was more to investigate the operations on quantities as opposed to the operations on the measurement units. I do not claim that I will be able to answer that particular question and inspire a ground-breaking discussion using my own data, but it seems that in order to extend the conception of *reciprocity* we need to engage in such discussions. To summarize, I see two important points that research, possibly mine, can contribute: the use of an equivalency relationship between fractional parts of known and unknown quantities, and the role of measurement of quantities in conceptualizing a general reciprocity reasoning that I call inverse reasoning.

As I mentioned earlier, Hackenberg states that a reversible iterative fraction scheme is necessary but not sufficient for the construction of reciprocity. Using the activities of another student, Michael, who could solve Type 4 problems using a distributive splitting operation, Hackenberg indicates that even though Michael "had a solid iterative fraction scheme, he did not readily use this kind of reciprocal reasoning" (p. 294). In his solution to Type 4 problems, Michael partitioned each unit (referring to an inch of the 7-inch bar) into three mini-parts, pulled out a mini-part and repeated it seven times, and throughout the discussions he made a total of five copies of this group of 7 mini-parts. His final configuration was the result of 35 mini-parts that were grouped in

 $<sup>^{23}</sup>$  After Deborah made a visual comparison between the 16 mini-parts bar and the unit bar, she suggested that the result was 16/18 of a meter.

sevens. Hackenberg calls Michael's way of operating as a general distributive splitting<sup>24</sup> that involves recursive partitioning (partitioning each unit into three). She emphasizes that distributive splitting is an important operation for the solution of reversible multiplicative reasoning problems of Type 3, 4, and 5. While Hackenberg indicates that Deborah had a blockage of using such a distributive splitting in solving the car race problem, and if she had used distributive splitting, she would have created;

One-third of 2/3-meter bar by taking one-third from each 1/3-meter part, it would have opened the possibility of embedding her recursive partitioning operation directly into her activity: One third of 2/3 is 2/9 meter, so four-thirds of the Lizards' distance would be four times 2/9 meter, or 8/9 meter. (p. 296)

Therefore, investigating this conjecture (if a student could use distributive splitting, then the possibility would be that she would produce the result of the multiplication of two fractions in terms of the measurement unit) and other operations in addition to the distributive splitting operation that play a role in producing such a result of measurement of the quantities (8/9 of a meter), is necessary for the sake of understanding the function of a fraction multiplying scheme in the construction of algebraic concepts such as reciprocity and inverse reasoning.

<sup>&</sup>lt;sup>24</sup> Hackenberg (2005) summarizes this definition as: "In distributive splitting or this more general manifestation of it, a student aims to insert units into each unit of a quantity consisting of some number of equal units. Determining the number of units to insert into each unit of the quantity comes from being able to reorganize the quantity into a different number of units of units" (p. 327).

## CHAPTER 4: METHODOLOGY- TEACHING EXPERIMENTS

#### Methodology

Building models of students' mathematical thinking through teaching is one of the basic premises of *teaching experiments*, which I use as a methodology for my research. The teaching experiment involves a sequence of teaching episodes (or interactions), in which there is a teacher, one or more students, and at least one witness-researcher who is observing the ongoing interactions. The teaching episodes need to be recorded (Cobb & Whitenack, 1996), and the on-going analysis of the records can be used for planning the following episodes and as the data for *retrospective conceptual analysis*. Retrospective conceptual analysis is an important component of this methodology in that researchers construct students' mathematical ways and means of operating as with this activity. Hackenberg (2005) indicates that the teaching time for this methodology usually varies from a period of six weeks to many years of teacher-student interactions. The analysis time, which could take several months or more, also varies since it is similar to micro genetic analysis (Schoenfeld, Smith, & Arcavi, 1993) and qualitative research analysis.

## Characteristics of Teaching Experiment Methodology

Steffe and Thompson (2000) document teaching experiments as a specialized method in mathematics education. In teaching experiments, teaching is "the scientific method of investigation" of students' ways and means of operating. The teacher's knowledge of *mathematics of students* (her interpretations of students' mathematical understandings, cf. Steffe and Thompson, 2000) depends on the teacher's experiences

with the students: mostly from a teacher's long-term observation of the students'

interactions with the teacher in mathematical contexts.

Interviews (including clinical interviews) are also a common research methodology used in mathematics education. Teaching experiments differ from interviews, in which the purpose is to learn how students operate without affecting students. For the teaching experiment, the goal is to advance students' mathematical understanding through interactions:

A goal in this is for students to make their mathematical knowledge explicit and to find the limits in their ways and means of operating. Another goal is for students to come to understand mathematics as something that belongs to them. In other words, two of the goals of the teaching experiment are to establish the zones of actual construction of the participating students and to specify the independent mathematical activity of the students in these zones. (Steffe and Thompson, 2000, p. 290)

In the remainder of this chapter, I will first write about the two components of

teaching experiment methodology-teaching and retrospective analysis-and then the

details of my teaching experiment.

Characteristic 1: Teaching

Teachers' subject matter knowledge, pedagogical content knowledge (Shulman,

1986), and beliefs about learning, mathematics, and interactions are some of the

theoretical bases for teachers' teaching actions. NCTM's Professional Standards for

Teaching Mathematics (1991) identifies the following teaching actions:

1. Creating a classroom *environment* to support teaching and learning mathematics;

2. Setting goals and selecting or creating mathematical *tasks* to help students achieve those goals;

3. Stimulating and managing classroom *discourse* so that students and teachers are clearer about what is being learned;

4. *Analyzing* student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions (NCTM, 1991, p. 4)

In my view, two important ideas derived from radical constructivism, with its theory of knowledge, enrich the teaching actions identified by NCTM: Students construct their own mathematical realities and teachers make models of students' mathematics.<sup>25</sup> While NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) base the discussion of *standards* on students' own construction of mathematics, the teaching experiment, with its laboratory-type of setting, allows us to investigate individual students' mathematical constructions and learning.

When we accept the fact that students construct their own mathematical realities, we do not emphasize that mathematics can be taught only from books and curriculum materials; rather, students should conceptualize the intended mathematical knowledge. If teachers do not stress that mathematical knowledge is independent of the students, they can shift "the focus of mathematics teaching from a process of transferring information to students to interactive mathematical communication in a consensual domain of mathematical experience" (Steffe, 1990, p. 45). Teachers can also become more independent in terms of creating or selecting *tasks* that they think will align with current ways of students' mathematical operations. These tasks might also engage students more and open up the possibilities for *productive discourses* in the classroom. *Productive discourses* refers to teacher-student or student-student interactions that advance students' mathematical operations. Those discourses establish better communication among all parties by the establishment of consensual domains of action and interaction.

 $<sup>^{25}</sup>$  NCTM (1989) states that "Our premise is that *what* a student learns depends to great degree on *how* he or she has learned it" (p. 5)

While constructivism is considered as a theory of knowing, Steffe and

D'Ambrosio (1995) claim that we can conceptualize constructivist teaching: "Similarly, if the teacher formulates a model of how she makes sense of children's mathematical knowledge, including its construction, this would be a constructivist model of teaching" (p. 146). In the next sub-section, using retrospective analysis, I will explain in detail how a model of students' mathematical knowledge can be made using this methodology, but in the remainder of this sub-section, I discuss what is necessary for making such models while teaching.

One of the responsibilities of the teacher when teaching and also making

experiential models is to determine the zone of potential construction (ZPC) of students.

The term, zone of potential construction, might seem similar to Vygotsky's term of zone

of proximal development (ZPD). The latter term is defined as:

The distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers... [The ZPD] defines those functions that have not yet matured but are in the process of maturation, functions that will mature tomorrow but are currently in an embryonic state. (Vygotsky, 1978, p. 86)

On the other hand, the ZPC could be interpreted as the students' "means of

neutralizing [the teacher provoked perturbations]" (Olive, 1994, p. 163). Olive further

elaborates on this term using a core and shell analogy:

If we think of this zone as a shell of some thickness surrounding a core of available operations, then perturbations falling outside of this shell would lead to discomfort and frustration that would be counter-productive to learning. Perturbations falling within the core of available operations can be neutralized through novel applications of these operations. This can be thought of as learning through generalizing assimilation. In this case, the perturbation is not a real problem for the child (in the sense of Polya's (1954) definition of a problem situation: a perceived goal with no immediate way of proceeding to that goal). When the perturbation falls within the shell (the zone of potential construction) the perturbation can only be neutralized through accommodations that result in new operations or applications of operations outside the scheme within which the perturbation occurred. (Olive, 1994, p. 163)

Determining students' ZPD seems highly dependent on the teacher's own mathematical knowledge, while Hackenberg (2005) indicates the constraints produced by the teacher for both the core and shell of ZPC depend on the teacher's experiences with the previous students and her knowledge of their ways of operating. Mack's (2001) study, discussed in the previous chapter, can be used to explain the teacher's role in the construction of ZPD; in this study, the students often had direct guidance manifested by the teacher's questioning. While Mack used the findings in her 1990 study to define students' core knowledge using the terms "equal partitioning" and the "out-of" idea, students' potential development in fraction multiplication problems was only determined by what they could do with direct instruction in four types of problems, which were categorized a priori. Students' activities discussed within the categories were not used to inform either the teacher's subsequent actions in the process of teaching or the decisions she made about the overall chronology of the tasks. Since the teacher/researcher only challenged the students using her mathematical knowledge, there were no inferences related to the potential of each student's mathematical constructions. Therefore, Mack's conceptualization of what students could potentially do was limited to her own mathematics and her categorization of problems.

In teaching experiments, teachers often engage in responsive and intuitive interactions with students. These kinds of interactions serve as a basis for more analytic interactions both during and after teaching (L. P. Steffe, personal communication, January 30, 2006). In constructing a ZPC, the teacher/researcher tries to think as students do and bases her decisions and interactions on that way of thinking. When our concern is making models of how students construct their mathematical knowledge, this way of teaching and researching is challenging because a teacher/researcher needs to decenter herself in interactions and to attempt to think as students do. L. P. Steffe (personal communication, January 30, 2006) says the students' ways of thinking are constraints for the teacher in two senses of the term *constraint*. First, the teacher is *constrained to* the students' actions, language, and interactions. She tries to make sense of students' acts and means of operating, and these actions are constraints for the teacher. Second, the teacher is *constrained by* the students' actions and interactions. In other words, there are limits on what kinds of actions the teacher can take in response to the actions that are imposed on the teacher by the students.

Therefore, a witness researcher, who does not feel the pressures or excitements of being an agent of teaching, can play two important roles in helping the teacher. The witness can give suggestions for how to proceed while the teacher is in action, especially when a teacher is stuck because of the constraints. The witness can also provide different interpretations (as an observer) of the past teacher-student interactions that can help the teacher to plan the subsequent episodes for dealing with the constraints.

## Characteristic 2: Retrospective Analysis of Teaching Experiments

The retrospective analysis of the sequence of recorded teaching episodes opens the possibility for the conceptual analysis of students' mathematical activities. Based on the analysis, it is possible to form a model of students as mathematical beings. Conceptual analysis is similar to making reflective abstractions. After watching particular recorded interactions in teaching episodes, researchers first engage in understanding what the students' actions are and the interactions contributing to those actions, and then justify why students acted in certain ways. This process results in the researchers' construction of the schemes that can be attributed to the students. After constructing schemes, the researcher reinvestigates the parts of the scheme more consciously by going into details of the recorded interactions and how the scheme is related to the other mathematical schemes and structures, which are constructed using analysis of a network of teaching episodes. During this process, observing an incident of learning is invaluable since hypotheses are based on those incidents that direct the flow of conceptual analysis of the data.

The process involved in looking behind what students say and do has been called conceptual analysis by von Glasersfeld (1995), and it is here one becomes explicitly aware of one's own engagement in a kind of mathematical research. For us, this awareness is essential because teaching experiment methodology is based on the necessity of providing an ontogenetic justification of mathematics; that is, a justification based on the history of its generation by individuals. (Steffe& Thompson, 2000, p. 269)

Steffe and Thompson (2000) address *data* and its role in hypothesizing in teaching experiments: "Whatever the students say or do in the context of interacting with the researchers in a medium is potential data for inferences about the students' conceptual operations and serves as confirming or disconfirming the hypothesis" (p. 296). Those hypotheses (conjectures about how the schemes and operations relate to each other) can direct both short and long term interactions of the teacher with students. In particular, the researchers use the hypotheses for conceptualizing an *epistemic* child's mathematical ways and means of operating. An epistemic child is the general concept of a child who we have observed engaging in certain mathematical activities, and we use him/her as a reference for discussing observations of particular students' mathematical operations. For

example, Hackenberg indicates that she uses an epistemic child, Jason from Steffe and Olive's study (1990), to orient her interpretation of her own students' activities and operations. In addition to this discussion, Steffe (2007b) indicates that "epistemic students are dynamic organizations of schemes of action and operation in my mental life" (p. 14).

Steffe and Thompson (2000) call this mathematical knowledge, which researchers construct after conjecturing and testing hypotheses, "second-order models of children's mathematics." The first-order model mathematics is whatever the individual students know or are aware about their mathematical operations.

Our modeling process [second-order] is only compatible because we have no access to students' mathematical realities [first-order] outside of our own ways and means of operating when bringing the students' mathematics forth. So, we cannot get outside our observations to check if our conceptual constructs are isomorphic to the student's mathematics [first-order mathematics]. But we can and do establish viable ways and means of thinking that fit within the experiential constraints that we established when interacting with the students in teaching episodes. (Steffe & Thompson, 2000, p. 293)

These models (as well as the hypothesis) are subject to revisions until the researcher's model is not countermanded by further observations. In the retrospective analysis, as with the role of witness researcher during ongoing teaching, the contributions of other researchers who were involved in the same or similar teaching experiments are vital. Those contributions are important to make compatible models of students' ways and means of operating in relation to *students' mathematical realities*.

# My Teaching Experiment and the Retrospective Analysis Selection of the Participants

The process of selecting of the participants included a 3-week classroom observation period and two interviews. During a 3-week period in the fall of 2005, I observed an eighth-grade algebra classroom at a rural middle school in Georgia. The classroom teacher was one of the participants in the Coordinating Students' and Teachers' Algebraic Reasoning (CoSTAR) project in which I was involved in various ways. The teacher was supportive of students' mathematical learning and enjoyed having researchers in her classroom. I think she also viewed our involvement as an opportunity to discuss students' learning, about which she cared most.

I made the classroom observations when the class studied a specific unit from their book, *College Preparatory Mathematics* [CPM] (Sallee, Kysh, Kasimatis, & Hoey, 2002). The unit was "Choosing a phone plan: Writing and solving equations," and its goal was to help students construct and solve linear equations using *Guess and Check* tables.<sup>26</sup> The classroom teacher and I identified eight students who contributed to the classroom discussions differently. Those eight students were of potential interest to the study since some were articulate about their thinking compared to others, and some approached constructing equations as novices, but in interesting ways that cannot be observed in traditional algebra classrooms. The students were taking the Algebra class during eighth-grade, so they were all advanced students.

 $<sup>^{26}</sup>$  For example they started the unit by making a guess and check table for the following problem: "The length of a rectangle is three centimeters more than twice the width. The perimeter is 60 centimeters. Use a Guess and Check table to find how long and how wide the rectangle is, and write an equation from the pattern developed in the table" (p.127). They made a guess for the width of the rectangle and used it to find the length and then used those two values in the general perimeter formula for rectangles to derive the perimeter. They then checked this resulting number against 60.

The student pool included Brenda, a student that I already planned to work with.<sup>27</sup> I knew Brenda's fractional knowledge extensively from the interviews we conducted in the CoSTAR project during her sixth-grade year. She was articulate about her mathematical thinking and was motivated. After the classroom observations, I conducted two interviews. The first interview, with four pairs of students, focused on students' activities related to their fractional knowledge (see Appendix A). The second interview focused on quantitative word problems, which they studied in the classroom during the unit about writing equations (see Appendix B). After the first interview, to have a manageable number of students for the teaching experiments, I eliminated three students: one whose social attitude did not suggest he would be a productive partner, and two others whose operative levels were similar to Brenda's. I then conducted the second interview with four students, but not including one of the mid-achieving students (Melanie) who I decided to work with later. After my second interview, I asked four students to be part of the study: Brenda, Dorothy, Lydia, and Melanie.

Brenda's initial partner in the two interviews had a similar fractional knowledge as Brenda had. However, her partner was more dominant in their interactions, and Brenda did not express what she thought unless I asked. I thought if I paired Brenda with her original partner, she would experience unnecessary challenges since it would be hard to establish norms in which competition between the girls was not a priority. On the other hand, Dorothy and Lydia were pairs, but they were not operating at the same level. Therefore, I made a new arrangement of the pairs basing my decision on how they would work together in a respectful and nurturing way and also whether they had similar

<sup>&</sup>lt;sup>27</sup> All student names are pseudonyms.

operational knowledge of fractions; one pair of "high-achieving" girls (Brenda and Dorothy) and another pair of "low-to-mid-achieving" girls (Lydia and Melanie).

All the classroom observations were recorded with one main video camera and field notes were taken during the classes. The two interviews were videotaped with a setting explained in Figure 4.2, and ongoing analysis of them was made during the mixing and digitizing of the videos.

## The first interview.

I will mention only briefly the first interview since the extensive analysis of students' fractional knowledge is available in the analysis chapters. In this section, I will discuss only the activities of the four students (Brenda, Dorothy, Lydia, Melanie) who became the participants in the full study. In this interview, I posed problems related to their splitting operation (e.g., This candy bar is nine times as much as yours, can you draw yours?), fractional schemes (unit, partitive, and iterative), recursive partitioning operations, reversible fractional schemes (both partitive and iterative), and inverse reasoning (see Appendix A). The high-achieving girls, Brenda and Dorothy, who were not then partners, showed that they had constructed a reversible iterative fraction scheme.<sup>28</sup> They could produce the whole unit when an improper fractional amount of the unit was given. For example, when a candy bar was given as 4/3 of another bar, Dorothy could partition the given bar into four and then separate a partition to make the other bar with three partitions. In addition, after she produced a quantity for 1/3 of 1/4 of a candy

<sup>&</sup>lt;sup>28</sup> This scheme (reversible iterative fraction scheme) developmentally follows the reversible partitive scheme, e.g., Dorothy was able to make the whole unit bar when 3/7 of it was given as a starting quantity. She partitioned the 3/7 bar into three parts and added four more parts to make the whole unit. It also follows the iterative fractional scheme in which a unit bar is given and an improper fractional quantity is asked, e.g., drawing 6/5 of a given bar.

bar in sharing context (she first shared a candy bar among four people and then shared hers with two late comers by drawing and partitioning a bar on the paper), she could reinterpret the measurement of her share as 1/12 of the candy bar. Furthermore, Dorothy could reason inversely in some situations, such as when finding how many pitchers are needed for filling the whole container if a pitcher holds 1/4 (or 2/3) as much as the container holds. However, she was not able to solve the problems when the pitcher held an improper fractional quantity (e.g., if a pitcher holds 7/4 as much as a container holds, how much of the pitcher is needed to fill the container?)

Brenda, in addition to her reversible iterative fraction scheme (e.g., making a whole unit when 7/5 of it is given), had constructed recursive partitioning operations. For example, she could partition a line segment into 18 parts with more than one step, such as partitioning it first into thirds, each third into thirds again (producing ninths), and then each ninth into halves. Moreover, during her sixth grade year in one of the CoSTAR interviews (February 2, 2004), she could solve a problem requiring recursive partitioning and the unit fractional composition scheme, and produce a result in terms of the whole candy bar. She was given a fraction strip (a paper strip that is used in her sixth-grade classroom to teach fractions by folding) to pretend that it was a candy bar and was told to share the candy bar among three people by folding the strip. She claimed her share as a third of the strip. She then was asked to share her share with four late comers and determine how much her share would be in terms of the original strip (bar). She indicated that it would be really small since they had to fold a third of the strip into five. After a while she gave up folding (since she could not make it accurate) and said that it would be fifteenths and explained it as:

Brenda: If... because if we split up fifths in third of it and there's 5 pieces in that, so if you put 5 pieces in each third there's gonna be... you're gonna have to do 5 times 3, 5 times 3 is 15 and so that's 15 equal parts in each third, so it would be, like, fifteenths.

In the first interview, Brenda also solved inverse reasoning problems with proper fractions (e.g., If a pitcher holds 4/5 of what a container holds, how many pitchers would you need to fill the container?) Brenda also found the number of pitchers that are needed for filling more than one container. For example, when the pitcher held as much as 2/3 of the container, by drawing and using the pitcher as a unit, she iterated (shaded inside the container) two parts of a three-part container and found that there would be a need for 4 and a half of pitchers for filling three containers.

Therefore, in general, Brenda and Dorothy's ways of operating on fractions seemed close enough for them to be partners. As indicated with their reversible iterative fraction scheme, they were able to operate with three levels of units structure. They also constructed a unit fractional composition scheme and recursive partitioning operations, which I inferred were represented in their activities when sharing a candy bar.

The students in the other pair (low-to-mid-achieving girls) had different levels of fractional competency. One student, Lydia, demonstrated that she had a reversible partitive fraction scheme (e.g., she could produce the whole unit when 2/3 of the bar was given) and the other student in that pair demonstrated that she only had a partitive fraction scheme, where she could partition a unit into a desired number of partitions to produce a fractional quantity of the unit. The conjecture was that throughout the interactions during the teaching experiment, the two pairs would have different ways of constructing meanings for stating and solving linear equations of one unknown.

In my dissertation, I focus on the analysis of only one pair of the students—the two high-achieving girls— and the differences in those two students' activities for constructing what the necessary algebraic operations are to conceptualize and solve linear equations with one unknown. The main reason for my decision is that I became interested in students' solutions in which they constructed fractional multiplication scheme, and used this scheme and necessary operations in the inverse reasoning problems. While two-thirds of the problems posed to the pairs were similar but used at different times, the students in the low-to mid- achieving pair were not able to solve inverse reasoning problems that included fractions in a similar context to Hackenberg's Type 5 problems, and we did not discuss use of letters for representing unknowns.<sup>29</sup> Therefore, I made the decision to focus on Brenda's and Dorothy's (high-achieving pair) activities and operations. I believed this kind of in-depth analysis would give me more information on the constructions of reciprocity and how fraction multiplying schemes functioned in those constructions.

## The second interview.

This interview was allocated to the problems from the classroom observations and homework. I chose the problems according to how the research students in particular were solving them during the classroom instruction. I wanted to see whether their solutions were similar to classroom discussions or whether students had their own ways of solving the problems. In addition, I wanted to explore what kinds of difficulties they

<sup>&</sup>lt;sup>29</sup> Hackenberg (2005) developed this hierarchy among the inverse reasoning problems. The low-to-mid achieving girls in my study were mostly able to solve Type 4 problems.

<sup>[</sup>Type 4] Candy Bar Problem: That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?

<sup>[</sup>Type 5] Tree Problem: Three-fourths of a decameter is two-thirds of the height of a tree. How tall is the tree?

were experiencing (e.g., are they able to use unknowns and written symbols properly? Are they able to construct correct quantitative relationships and use those relationships when symbolizing problem situations? And lastly, were they able to solve equations for the unknown after they constructed it using an unknown (especially the ones that require fraction knowledge)?) The CPM book and the classroom discussions emphasized using Guess and Check tables and how to use symbols and unknowns in correct ways. To see what resided from those classroom experiences, I decided to conduct an interview after the students finished studying the unit. Even though I did not use this interview as one of the main contributors to how I selected the students, I report analysis of some of the problems to give information about Brenda and Dorothy's activities related to general quantitative and algebraic problem situations. Here, although this interview included Brenda, Brenda's initial partner, Dorothy, and Lydia, I will only focus on Brenda and Dorothy because they are the focus of my analysis in the dissertation study.

The first problem Brenda and her partner solved was: "Heather has twice as many dimes as nickels and two more quarters than nickels. The value of the coins is \$5.50. How many quarters does she have?" Both Brenda and Dorothy (even though they were not together) produced a similar equation: Brenda's equation was "n + 2n + n + 2 = 5.50," and Dorothy's was "n + n.2 + n + 2 = 5.50" (*n* stands for the number of nickels). They both solved the equation and produced ".8705" and realized by themselves that there was something wrong with this result. Brenda and her partner indicated that they needed to have the total number of coins, and they did not know how to include the total number of coins. This situation was such a constraint for them that Brenda and her partner could not proceed. On the other hand, Dorothy attempted to write the equation

including the monetary values of the coins, " $(n.05) + (n.01) \times 2 + (n.25) + 2 = 5.50$ " and wanted to proceed by subtracting "2" from both sides, but eventually she stopped working and did not produce a result. Both Brenda and Dorothy knew there was something missing either in how they set up the equation or in how they solved it, but they were not aware of how they were using the monetary values as a unit, and the function of the parenthesis (e.g., Dorothy wrote (n.25)+2 instead of  $(n+2)\times.25$ ). Olive and Caglayan (2007) also used a similar problem context and analyzed two groups of students. While I will not discuss the details of similarities or differences in their and my students' activities, the information about Brenda and Dorothy confirms that it is common to observe students that do not attend to different levels of units,<sup>30</sup> operate with only two levels, and do not use the parenthesis properly, which was an indication of their inadequate quantitative structural understanding in these problem situations.

In contrast to the girls, Ben and Greg [students in Olive and Caglayan's study] were not able to construct a meaningful quantitative structure using the literal symbols in these equations. They were able to work with units at the second level (number of coins and value of a coin), but did not have available a third level of units that would enable them to envision the quantitative structure required to meaningfully combine and find the value of all the coins. (Olive and Caglayan, 2007, p. 23)

Another problem (from the CPM book) I posed in the second interview was: "Chris is three years older than David. David is twice as old as Rick. The sum of Rick's age and David's age is 81. How old is Rick?" In the classroom discussions, Brenda's group presented a solution for this problem using symbols. They created mathematical

<sup>&</sup>lt;sup>30</sup> Olive and Caglayan (2007) explain the different levels of units in the coin problems: "A single coin is the first level, the value of the single coin and the number of those single coins are units established at a second level (a composite unit of units), whereas establishing the value of all the coins requires a third level of units (a composed unit of units). The ease with which Maria established the second equation, with correct parentheses, indicates that she had these three levels of units available to her prior to operating." (p. 15)

expressions that indicated relationships in each of the two sentences of the problem situation: they were C = D+3 and  $R=(C-3)\times 2$  (which was supposed to be R = (C-3)/2). However, they did not write an equation that represented or explicated the relationship of Rick and Dick's age and the given sum of two ages as 81. Neither the classroom teacher nor other students realized this situation. Therefore, I posed the problem in the interview.

Brenda thought about it for a few seconds out loud and tried to decide whose age she should be using as the central unknown; her intention was to write the other ages related to that person's age. She decided that it was David's age she would be using as the key unknown. She wrote D=D, C=D+3, and R=D/2 ("C" stands for Chris's age, D stands for David's age, and "R" stands for Rick's age) while also taking her partner's thoughts into consideration by asking her approval each time. Brenda said that "and the sum is 81 and not all of the three but just Rick and David" and asked her partner "do we even need Chris?" Her partner said, "not really," so Brenda wrote "D + D/2 = 81." At this point, when writing D/2, she asked her partner if she should write "D over 2 or D divided by 2." Her partner said since they were working with fractions they should write in fraction form. Brenda then replaced D/2 by  $\frac{D}{2}$  in her equation. However, after getting the equation, which took two minutes, neither Brenda nor her partner was able to solve the equation for David's age, which I think was a result of having a fraction in the equation. They worked on this for approximately seven minutes. I then asked Brenda's partner to continue with Guess and Check table to find the ages. Brenda was more involved in this way of finding the ages than her partner was, since her guesses were more reasonable for example, 50 and 25. Her partner started with 60 for David's age and 30 for Rick's age, then they had 56 and 30 for David's and Rick's age respectively. In their third guess, they got 54 for David's age and 27 for Rick's age, resulting in the correct sum of 81.

For this problem, Dorothy's partner stated the relationships among the three people's ages. She wrote in this order, "Sum of R+D = 81, Chris= 3 yr.+David, David is  $2 \times \text{Rick}$ , ? = Rick = x," and wanted to divide 81 by 2. She worked for almost seven minutes, but did not know how to proceed. She then asked, "How would you make an equation for this?" Dorothy offered help and started writing " $x+x \times 2+(x \times 2$ ". I then asked her to state what the terms in her expression meant. I repeated what she said (x was Rick's age, and " $x \times 2$ " was David's age) and added that the sum of Rick's and David's age was 81. She then said, "So Chris, we do not really need Chris." She then wrote the following (Figure 4.1) and produced a result of 27:



Figure 4.1. Dorothy's equation and solution for the age problem.

In this problem, Brenda and Dorothy were able to state the relationships between the quantities, and could use Rick's (or David's) age as a common unknown to conceptualize both the other person's age and the sum of the two ages. Even though having David's age as the common unknown and stating Rick's age in terms of David's age (it was half of it) resulted in difficulties for Brenda to use the equation for the solution, she was able to find their age by using guess and check and the quantitative relationship between the two ages and their sum.

From the classroom observations, I also observed students struggling when symbolizing problem situations that involved a couple of numbers related to each other in some ways. Therefore, I posed this problem, which was also from the CPM book, during the interview: "Find three consecutive numbers whose sum is 57." Students in the class still had difficulty even if they determined the unknown. They were not able to use just one unknown and the quantitative relationships properly to symbolize the other two consecutive numbers for writing the sum. For example, they wrote f (for the first number), then f+1 (for the second number) but for the third one they wrote s+1, where s stood for the second number. The usual approach to symbolize this problem could be f + (f+1) + (f+2) = 57, where f is the unknown quantity for the first number and the second and third number are written in terms of the first unknown. The other interesting classroom observation occurred when Dorothy used an unknown and approached symbolizing the given relationships differently than the usual approach. In contrast to many students, she was flexible and wrote each relationship in terms of the second unknown number: first number as, a-1; second number as, a; and the third number as, a+1. I wanted to see whether she or other interview students would be able to use these kinds of relationships flexibly in the interview situations.

Brenda labeled the first number as *n*, and the second number as n+1 and the third number as n+2. She wrote the sum as n + (n+1) + (n+2) = 57 and solved it for *n*. She got n = 18 for the first consecutive number and checked whether she got them right or not by adding those three consecutive numbers to compare against 57. After seeing Brenda

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successfully solve this problem, I asked "find three consecutive even numbers whose sum is 42." In Brenda's homework, I saw her writing n+2 for all the even numbers for a similar problem, disregarding the consecutiveness. She used *n* for the first even number, and wrote n+2 for the second number, but wrote n+3 for the third number, since she miscounted the number of circles she drew for representing the consecutiveness relationship. Her partner reminded Brenda that n+3 would be an odd number. Brenda fixed the third number and wrote the equation correctly, and solved it. Dorothy, on the other hand, did not use any symbols for either communicating about the unknowns or the relationship between the quantities in these types of problems during the interview. She divided 42 by 3 and produced 14. She then subtracted 2 from 14, and produced 12. She later started with 16 and wrote 14, 12 underneath of 16, and added all three numbers to check whether the sum was 42. This way of solving could be conceived as arithmetical reasoning since she did not start with numbers as unknowns; on the other hand, she set and used quantitative relationships properly so there was a strong emphasis on her quantitative reasoning. However, since there is an awareness on the quantitative structures (such as the sum of three consecutive numbers must be divisible and the result is the middle number), Dorothy's activities might also suggest an algebraic reasoning as Olive indicated (J. Olive, personal communication, April 13, 2008).

#### Initial Overarching Conjecture Related to the Study

When I proposed the study two years ago, my initial purpose was to understand the construction process of the concepts behind students' algebraic activities. This goal included how they viewed written and verbal symbols as meaningful mathematical activities as opposed to viewing algebra as symbol manipulation. I planned to use students' quantitative reasoning and fractional knowledge as a basis to see how they would become more competent in their algebraic activities, which included symbolic operations (cf. Chapter 2). While the general purpose has not changed, my purpose for the dissertation analysis became to see Brenda's and Dorothy's construction and solving process of the linear equations (which could be represented as ax = b)<sup>31</sup> and how their knowledge about fraction multiplying and inverse reasonings were related to each other in the process of construction of linear equations.

#### *General description.*

I taught two pairs of 8<sup>th</sup> grade students from a rural Georgia middle school. Eighteen teaching episodes took place in Spring 2006 from the beginning of March until the end of May. I met with each pair twice a week (usually Tuesday and Thursday) for between 30 minutes to one-hour per session. The meetings were in a conference room next to their classroom during their homeroom period in the mornings (during which they were free to study on their own, so they did not miss any academic instruction). All the meetings were recorded with two cameras.

There was one consistent witness researcher during the 3-month period who navigated the work-camera.<sup>32</sup> There were also times when the number of witnesses increased up to three researchers with their contributions. All the sessions were videotaped with two cameras (one recording specifically students' work [work-camera],

<sup>&</sup>lt;sup>31</sup> In the first meeting (March 3, 2008), I also posed problems whose representations could be ax + b = c, where a is a fractional number, b and c are whole numbers. However, I did not analyze those in details. <sup>32</sup> Keith Schulte was the consistent witness researcher who was pursuing his Master of Science degree in the mathematics education program. Hyung Sook Lee also contributed to the project by observing almost nine sessions and helping designing tasks along with Les Steffe and Keith Schulte. In the different times of the experiment, John Olive, Les Steffe, and Andrew Izsák added their knowledge by being a witness-researcher at least once, discussing particular recorded student-teacher interactions, as well as designing tasks.

the other one recording the whole activities [interaction-camera]), and the sound was recorded to the main camera with a sensitive microphone.

After the sessions, the videos were digitized and mixed as one file in which one can see the recorded student-teacher interactions and students' work in one screen with quality sound. As part of my on-going analysis during the experiment, when I digitized and mixed the videos, I made *lesson graphs* of all the sessions indicating what the problem was, how students approached it, and what the constraints were for the teacher as well as for the students. Using those lesson graphs, I made conjectures about what students understood or learned mathematically during the particular teaching episodes, and used my hypothesis to prepare tasks for the following session. I discussed the planned tasks with Les Steffe, and welcomed contributions from my two other witnesses (Hyung Sook Lee and Keith Schulte). While traveling by car to and from the research site for 40 minutes each way, we always discussed the tasks and their observations of the last sessions with the witness-researchers. I used their input to revise the tasks for teaching that same day or planning new tasks for the following session.

*Room configuration.* The room we met in was adjacent to the students' eighthgrade classroom. Because of this physical advantage, the communications and scheduling issues with the teacher and students were carried out smoothly. The room was rectangular and there was a big rectangular table that took up most of the space. The two students and I sat at the end of the table facing the main (interaction) camera. The microphone on the table was attached to the main camera. The other camera, which recorded students' work, was placed behind us and the witness-researcher navigated it (see Figure 4. 2). I sometimes sat in the middle of the two students (usually when they used paper and pencil), and sometimes let them sit adjacent (when they each had a computer to work with). When they sat together, we could have one camera capture the two computer screens, which was helpful when digitizing and mixing the students' work with the main camera to produce one video file.



Figure 4.2. Room configuration.

*JavaBars.* The computer program used during the experiment, *JavaBars* (Biddlecomb & Olive, 2000), lets one make variable sizes of rectangular bars using the *Bar* command. In the same toolbar there are other commands the students used frequently, such as *Erase* (erasing a bar), *Copy* (copying a bar), *Join* (joining bars which had matching dimensions either in length or width), etc. (see the toolbar in Figure 4.3) Even though the program has a *Pieces* feature (in which students can produce some number of pieces of a bar which may or may not be equal, a feature that simulates partitioning by hand), students in this study almost always used *Parts* to make equal parts

of a bar or parts of a part. For example, students can dial up to 10 (with *Parts - Bar* selected) to make a bar into a 10-part bar using Up/Down marks. They can dial *3* (with *Parts - Part* selected) to make three equal partitions of a part of the 10-part bar—with two-step partitioning where the first step is with Up/Down and the second step is with Left/Right marks (see the example bar in Figure 4.3). They can *Fill* different colors of the parts of the bar and *Pullout* parts from the bar while the part is disembedded but the bar still has the same number of partitions. In addition to those, they can also *Break*, *Combine*, and *Clear* the parts using the commands in *Parts* menu.



Figure 4.3. Using *Parts-Part* in the menu, a part in the 10-part bar is partitioned into three mini-parts and *Pullout* has disembedded the mini-part from the bar.

#### The Teaching Episodes

During the first six sessions, students used paper and pencil when solving the problems. Then I taught them how to use JavaBars, and each student usually had a laptop to work on separately. They used mostly JavaBars during the mid-eight sessions of the 18, and they were again allowed to use paper and pencil (whenever they felt they needed it) along with the JavaBars in the last six sessions. In the final sessions the students collaborated; while one of them solved the problems with the software, the other one noted the actions of her partner in mathematical symbols on the paper.

The decision to work with JavaBars was prompted by the slow progress in the fifth teaching episode on March 24, 2006, in which the students worked on the problem: "A half-inch long candy bar is cut into two parts. Find the parts if one part is thirteen thirds as much as the other part." I made the decision to work with operations on fractions with JavaBars instead of immediately moving to the problems that were similar to the second interview problems, which focused on understanding students' construction of meaning related to written symbols and construction of equations with unknowns. The reason for this decision was that I realized neither Brenda nor Dorothy could operate with quantities less than a unit measurement as fluently as they could with quantities measured with whole units. Therefore, we started using JavaBars as a means to explain and discuss their activities and operations related to fraction multiplication situations.

I posed the problems in the following table (see Table 4.1) during the 3-month teaching experiment. The problems on the left hand column are representatives of students' particular activities and operations. They are analyzed in detail in Chapter 5 and 6. The problems on the right hand column are presented to the students during the activities in those problems did not show so much differences from the activities students

engaged in solving the problems of the left-hand column.

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
<ul> <li>March 3, 2006</li> <li>Problem 5.1: If 2/3 of your sandwich is 20 inches long, how long is your original sandwich?</li> <li>Problem 5.2: If 3/4 of your sandwich is 15 inches long, how long is your original one?</li> <li>Problem 5.3: Each of us has a candy bar, and I give you some of mine and you then have sixth fifths of your original candy. You measure and see your new candy is 48 inches long. How long is your original candy?</li> <li>Problem 5.4: Each of us has a candy bar, and I give you some of mine and you then have seven fifths of your original candy. You measure and see your new candy is 49 inches long. How long is your original candy. You measure and see your new candy is 49 inches long. How long is your original candy?</li> </ul>	<ul> <li>On March 3, the students also worked on the following problems to figure out the length of the original bar (in some of the problems I used <i>x</i> to communicate the problem context. Students did not use <i>x</i> at all. When they wrote an equation, I made a discussion about it in the analysis):</li> <li>1- You have a sub-sandwich, and you measure its length with inches. Can you draw one? And after you take half of this sandwich, you see that you have a 10 inch long sandwich. So how long is your original sandwich?</li> <li>2- A fifth of the candy bar is 20 inches long. What is the length of the original bar? (e.g., 2/3 x=30, "6/4=36")</li> <li>3- You have a candy bar, and you eat half of it. You add 4 more inches to the left half and you see it is 11 inches long. How long is the original one? (They solved many problems within the same context, e.g., 1/3x+3=11, 2/3x+5=17, 8/5x+3=21, 2/5x+3=22, 8/5x-3=21. They had difficulty with 2/5x-4=11 for various reasons.)</li> </ul>
March 7, 2006	March 7, 2006
Problem 5.5: You are given two	1- If you add three consecutive
numbers, one of them is twice as much	numbers you get 36. What are those
as the other one. Find the numbers II their sum is 33	11000005?
Ului Sulli 18 33. Problem 5.6: Vou have a 52 inch	2- Alejandia cut a 40-men long board
string Vou color this string into two	nice number and the other niece
string. You color this string into two	piece purple and the other piece

Table 4.1. The 3-month teaching experiment.

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
different colors: green and white. If the	orange. The purple board is four
green part is three times as long as the	inches longer than the orange
white part, how long is each part?	board. How long is each painted
Problem 5.7: You have a 60-inch	board? (Salle et al., 2002, Unit 4, p.
string, and you have two parts. One of	132)
them is twice as much as the other part.	3- A nurse takes the temperature of a
So, how long are the parts?	patient on two different occasions.
Problem 5.8: One hundred twelve-inch	The second time, the patient's fever
string is cut into two parts. One part is	had increased by 3 degrees. If the
three times as long as the other part.	sum of the two temperatures is 203
How long are the parts?	degrees, what were the two
	temperatures? (Steffe, Saenz-
	Ludlow, & Ning, 1989, p.
	22)(variation with five degree
	increment in the second occasion)
	4- Students made up problems for
	each other similar to 1,2 and 3.
March 9, 2006	March 9, 2006
Problem 5.9: You have a string that is	1- Different variations of Problem
40 inches, and you cut it into two parts,	5.12: A 70-inch string with two
and one part is one third times as much	parts, one part is 3/4 as long as the
as the other part. So you have still two	other part. 81-inch string, one part
parts, and one part is one third times as	is $4/5$ as long as the other part.
much as the other part. How long are	2- Dorothy made up a problem for
the parts?	Brenda: You have a 144 inches
Problem 5.10: Now, Dorothy. You	long string and you cut it into two
have a string 50 inches long and again	parts, one part is 4/8 as long as the
you have two parts. But one part is	other part. How long are the parts?
two-thirds times as much as the other	3- Brenda made up a problem for
part. So how long is each part?	Dorothy: If you have 55 inches
Problem 5.11: We have still 50 inches,	long string and one part is 1/4 of
and the white part is one fourth times	the other part. How long are the
as much as the green one. So how long	parts?
are the parts?	4- A 54-inch long string has two parts.
Problem 5.12: A sixty-five inch string	One part is 2/ / times as long as the
has pink and red parts. Find the length	other part. How long are the parts?
of the parts if the pink part is 2/3 as	(Another Variation, a 65-inch long
Iong as the red part.	sum g nas two parts, one part is $2/11$ as reach as the other part )
Problem 5.15: A forty-five inch long	2/11 as much as the other part.)
string was cut into two parts. Find the	5- Dorotny created a variation on
parts 11 one part 15 three naives as long	Problem 5.13 of the previous
as the other part.	column for Brenda: A 100-inch
Froblem 5.14: Find the length of the	string has one part //3 as much as
two sub-quantities of a 45-inch string if	the other part.

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
one part is seven halves as much as the	
other part.	
March 21, 2006	<ul> <li>March 21, 2006</li> <li>1- You have a 32-inch string and cut it into two parts. If one part is 1/15 as much as the other part, how long are the parts? (Same context: A 33-inch bar is cut into two parts, and one part is 1/65 as much as the other part. How long are the parts?)</li> <li>2- A 6-inch bar is cut into two parts. Find the lengths of the parts, if one part is 1/11 of [as much as] the other part?</li> <li>3- A 2-inch bar has two parts. Find the length of the parts, if one part is 1/3 of [as much as] the other part?</li> <li>4- A 30-inch bar has two parts. One part is 4/11 as much as the other</li> </ul>
	5- A 3-inch bar has two parts One
	part is 3/4 as much as the other part.
	How long are the parts? (A witness researcher posed this problem)
March 24, 2006	March 24, 2006
<ul> <li>Problem 5.15: You have a 4-inch candy bar and you cut it into two parts. One part is three fourths as much as the other part. How long are the parts?</li> <li>Problem 5.16: A five-inch bar is cut into two parts. One part is three fourths as much as the other part. How long are the parts?</li> <li>Problem 5.17: A half-inch long candy</li> </ul>	Homework: Problem 5.18. A bar, which is 2/3 of an inch, is cut into two parts. If one part is 2/5 of the other part, find the lengths of the parts. They returned their answers written on the paper for the following meeting, and Dorothy briefly explained hers.
bar is cut into two parts. Find the parts if one part is thirteen thirds as much as	
the other part.	
March 28, 2006	<ul> <li>March 28, 2006</li> <li>1- You are given two different candy bars. Can you show 1/5 of all the bars?</li> <li>2- A 2-inch candy bar is 3/5 as much as another candy bar. How long is</li> </ul>
	the other candy bar?

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
	3- You have two different size
	candies. How can you show half of
	all the candies?
	4- You have three different type
	cakes. How can you show 1/5 of all
	the cakes?
March 30, 2006 and April 4, 2006	We started working with JavaBars.
	We worked on equivalent fractions.
April 10, 2006.	Dorothy is by herself (April 10, 2006).
	1- You have a 2-inch candy bar. Can
	you make such a bar with JavaBars,
	and show 1/3 of it? Can you show
	3/5 of it?
	2- A 2/3-inch bar is a fifth of another
	bar. Can you make the other bar
	and figure out its length?
	3- A $4/5$ -inch bar is $2/3$ of another bar.
	Can you make the other bar and
	figure out its length?
	4- A 7/5-inch bar is a third of another
	bar Can you make the other bar
	and figure out its length?
	5- This candy bar is 2 inches and it is
	5/3 as much as another bar. Can
	you make the other har and figure
	out its length? (This was a hard
	problem for Dorothy )
April 19, 2006	April 19 2006
Problem 6.1. You are given 3/5 of a	1- Finding $1/7$ of $5/6$ of a candy bar
candy bar. Can you find 1/7 of this bar	with JavaBars
and figure out how much it is of the	2- Finding $2/7$ of $3/5$ of a candy bar
whole candy har?	with IavaBars
Problem 6.2: You are given $4/5$ of a	with suvalsuis.
candy har. Can you make 1/7 of the	
given bar $(4/7 \text{ of the whole bar) and}$	
figure out how big it is (of he whole	
candy har)?	
May 2, 2006	Brenda is by herself (May 2, 2006)
	1- You are given $5/4$ of a candy bar
	Can you make 1/4 of this har and
	figure out how much it is of the
	original bar?
May 9, 2006	
Problem 6.7: For a dessert recipe, you	

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
need 4 gallons of whole milk and some	
skim milk. Four gallons of whole milk	
is five sixths as much as the skim milk	
you need. Can you make the necessary	
amount of skim milk on JavaBars and	
figure out how much it is in terms of	
gallons?	
May 11, 2006	
Problem 6.3: My water bottle holds 3/5	
of a liter and yours holds 2/3 as much	
as mine. Can you make the water	
bottles with JavaBars and figure out	
how much your bottle holds?	
Problem 6.4: If my water bottle still	
holds $3/5$ of a liter and yours holds $4/7$	
of mine, can you make your water	
bottle and figure out how much it is of	
a liter?	
Problem 6.6: My water bottle holds	
11/6 of a liter and yours holds $3/5$ as	
much as mine holds. Can you make the	
water bottles on JavaBars and figure	
out how much of a liter yours holds?	
(Please, write down your actions as you	
work with JavaBars.)	
May 12, 2006	
Problem 6.5: I have a water bottle that	
holds $4/5$ of a liter, and yours holds $7/6$	
of whatever mine holds. Can you make	
your water bottle and figure out how	
much of a liter it is?	
Problem 6.8: I have a water bottle that	
holds 3/5 of a liter. This much water is	
2/3 as much as whatever your water	
bottle holds. Can you make the water	
bottles on JavaBars and figure out how	
much your bottle holds? (The problem	
was posed twice on May 9 and May	
12. The first presentation followed	
Problem 6 7 )	
Problem 6.9. My water bottle holds 4/5	
of a liter and it is $3/7$ as much as yours	
Can you make the water bottles with	

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
liter yours holds?	
May 16, 2006	We started notating the problem situations
	along with their JavaBars activities (May
	16, 2006).
	1- Warm-up problems: Three gallons
	of milk is $3/5$ as much as chocolate
	milk. How many gallons is
	2 Estimation Droblems: Make a har
	2- Estimation Floblens. Make a bar. This har is <i>Am</i> blue ribbon and if
	this is 7/9 as much as green ribbon
	can you make a bar for an estimate
	of the green ribbon? Can you write
	down the problem situation using
	an unknown? Discussions about the
	equivalency relationships and the
	measurement of green ribbon.
May 18, 2006	1- A 5-inch bar is 3/4 of my bar. How
	long is my bar?
	They made the other bar as a separate bar
	with their estimation. We discussed the
	Idea of estimation, and I then introduced a
	rection in writing
May 10, 2008	The Same discussions held on May 16
Way 19, 2008	continue
	1- An 11-inch sub sandwich is 3/7 as
	much as mine. Can you make an
	estimate bar for my sandwich with
	JavaBars (without using the parts of
	the given bar) and state how much
	you think it is? How would you
	solve the problem with JavaBars
	and notate your actions on the
	paper as you solve?
	2- A 12-inch sandwich is 5/4 as much
	as my sandwich. Same questions.
	(with Dicha we discuss, $12 \times 4/3$ –
	individual activities from mine )
May 25, 2006	Brenda is by herself. She mainly used
	paper and pencil to set and solve the
	equation with an unknown. I checked her
	understanding by making her explain her

Analyzed problems (see Chapter 5 and	Problems presented but not analyzed in
6)	Chapter 5 or 6
	steps with JavaBars actions.
	1- We had some milk, and used 2/7 of
	it for a cake. Then we saw the
	leftover milk was 4 gallons. So how
	much milk did we have at the start?
	2- We had some milk and used 4/7 of
	it for a cake. Then we saw the
	leftover milk was 11/5 of a gallon.
	So how much milk did we have at
	the start?

## Retrospective Analysis

After the 3-month period of my teaching experiment, I used the 18 video files, my lesson graphs, and notes to make the retrospective analysis. For two weeks of intensive work, I looked at all the recorded episodes to identify the most critical points in terms of students' activities. With this work, students' activities in relation to distributive reasoning, recursive partitioning operation, 2- and 3-levels of units structures, and inverse reasoning problems promised to be an interesting theme for explaining Brenda's and Dorothy's ways and means of operating. By that time, I knew Brenda was able to solve fraction multiplication problems and interpret the results in terms of measurement, but Dorothy could not make such interpretations and I did not know why. During this process, I also made a second round of written notes similar to lesson graphs and eliminated a couple of the videos from further analysis because they were not relevant to the students' specific activities in which I became interested (e.g., I eliminated the video files about their production of equivalent fractions with JavaBars on March 30 and April 4, 2006). After this process, I went into deeper analysis of each episode and how the students schemes and actions were represented in those episodes and how they were

connected to their actions in the other episodes. This detailed analysis process took over a year.

I transcribed some of the important interactions and used them for writing two chapters of analysis. For the first chapter, I started writing about individual problems and students' activities in those problems using episodes between March 3 and 24 (see Chapter 5). Writing Chapter 5 was a learning experience for me in terms of how to write in depth analysis of the recorded observations and revise the ideas and writing. During this process, I sought consensus with another researcher, my major professor—Dr. L. P. Steffe— about the inferences I made and got help for how to construct schemes (that could be attributed to a student). After writing Chapter 5, I used the students' activities to inform me about their fractional knowledge and structural thinking that were important in my construction of second-order models of their algebraic knowing. Later using the analysis in Chapter 5, and the rest of the condensed form of the data, I made a hypothesis that I needed to specifically look at how they coordinate two 3-levels of units structures to make equivalency relationships between the parts of known and unknown quantities in inverse reasoning problems. The discussion related to this issue can be read in Chapter 6. While the data on May 16, 18, 19, and 25 is invaluable in terms of how students' connect JavaBars activities and written symbols, it only confirmed my analysis of how they constructed inverse reasoning and how fraction multiplying schemes played a role in that construction. Therefore, I did not present the analysis of these sessions in detail, but in Chapter 6 I used the other episodes in May as representative of the students' actions related to these issues.
# CHAPTER 5: ANALYSIS OF FRACTIONAL SCHEMES IN MULTIPLICATIVE QUANTITATIVE SITUATIONS

This chapter includes analysis of 18 multiplicative problems that students solved by coordinating their fractional and whole number knowledge (see the problems starting with Problem 5.1, 5.2,...5.18 in Table 4.1). The problems might be conceived as exploratory problems related to how Brenda and Dorothy operate mathematically. The analysis in this chapter gives a basis for the construction of fraction multiplying schemes and inverse reasoning that is investigated in the next chapter.

In this chapter, I start with presenting an analysis of Brenda's and Dorothy's fractional schemes for composite numbers (reversible partitive and iterative fractional schemes) and how their ways and means of operating with fractions contribute to a discussion of quantitative and symbolic (algebraic) reasoning. Then using multiplicative quantitative problems (part-part-whole and whole-part-part problems), I discuss what the two students' ways and means of operating were in those problems when they employed the two fractional schemes, what their difficulties were, and what they learned. Using Problem 5.7 and 5.8, I extensively discuss an accommodation Brenda made in her *part-part-whole reasoning scheme*. Using Problem 5.10 and 5.11, I discuss the change Dorothy made in her *whole-part-part reasoning scheme*. The analyses of the last two problems 5.17 and 5.18) are especially important because the two students had tremendous difficulty using their available fractional schemes for various reasons. The analysis of the last two problems suggested an important reason to conduct a deeper

investigation on fraction multiplying schemes, which can be read in Chapter 6.

## Fractional Problems with Quantitative Situations

Brenda and Dorothy's Reversible Partitive Fractional Schemes for Composite Numbers

Problem 5.1: If 2/3 of your sandwich is 20 inches long, how long is your original

sandwich? (March 3)

After I posed the problem, Brenda thought about it for a few seconds and made

sure she understood the relationships and numbers in the problem situation.

Protocol 5.1: Finding the length of the whole quantity using its fractional part and corresponding measurement.<sup>33</sup>

Z: Now, two thirds of that sandwich is twenty inches. Brenda, how long is the original one?Brenda: Um. You said how long? Two-thirds of it is how long?Z: Two-thirds of it is twenty inches.Brenda: Twenty, so it means one third is ten, so it will be ten, twenty, thirty.Thirty.Z: Thirty. OK.

In the previous problems (see the right column of March 3, 2006 in Table 4.1), I asked Brenda and Dorothy to draw the given candy bars as a means to discuss their solutions. In her solution to this problem, Brenda did not draw a sandwich. In addition, it took her only 15 seconds to solve the whole problem. It is possible that she neither thought about an inch quantity nor did she re-present a sandwich that is 20 times an inch long in her mind. Rather she solved this problem symbolically, perhaps operating on a minimal visualization of a (20-inch) sandwich.

Since Brenda counted by 10 three times, she knew two-thirds of the sandwich was

<sup>&</sup>lt;sup>33</sup> In the protocols, Z stands for the teacher-researcher (me). The protocols are numbered sequentially by chapter number (e.g., 5.1, 5.2, etc.). Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher-researcher's perspective. Ellipses (...) usually indicate the speaker has not finished the sentence.

two of one-third of the sandwich to be made and the whole sandwich was three-thirds, and she used 10 as a third of the whole sandwich. If a student can make the whole quantity when a part of it is given, Steffe (2002) calls these ways and means of student's operating *a reversible partitive fractional scheme*. In this case, it is obvious that Brenda used 20 inches as what she considered to be a numerical quantity for the two thirds of the sandwich. She took one half of 20 inches to produce 10 inches, and she called this a third. She disembedded 10 and then iterated it three times for the whole sandwich. When Brenda did this, I inferred that she used her reversible partitive fractional scheme to operate on composite units. In that Brenda's operations were symbolized by her quantitative language, it is perhaps more appropriate to say that her use of symbols referred to operating on composite units of 20 and 10.

To operate in this way involves establishing an identity between 1/3 of the sandwich and 10 inches. She operated very quickly, and it seemed as if she was operating with numbers without regard to their meaning as measurements of lengths of parts of a sandwich. However, she knew what the numbers 10, 20, and 30 stood in for in the problem, which was very powerful.

In the particular situation presented to her, her interpretation of "2/3 of a sandwich is 20 inches" constituted the situation of her scheme. Based on her solution, I inferred her goal was to find the length of the whole sandwich. The activity of the scheme was comprised by the operations of taking one half of 20, disembedding this quantity, and iterating that result, 10, three times. Upon her completion of the three iterations, 30 was the result of operating. For her to be able to assimilate the problem situation successfully prior to operating, I hypothesize that she needed to establish the relationship that 20 inches was identical to two-thirds of the sandwich and that 10 inches was identical to one third of the same quantity. Rather than simply producing one third of the sandwich by taking one half of two-thirds of the sandwich, Brenda took one half of the numerical length of two-thirds of the sandwich as her first action. This indicates that she intended to operate on the numerical measurements of lengths of parts of the bar. She did not start by making a drawing, but explained in words how she would come to the result if she actually operated using a drawing. These were all symbolized operations, and this way of operating was powerful because she was aware of her operations and could reflect on them. Her reversible partitive fractional scheme for composite units was well established since she knew what she was supposed to do before acting because everything took only 15 seconds.

Even though I made inferences about how Brenda solved the above problem based only on her language, there are more indicators of my inferences in the following very similar problem situation in which Dorothy initiated a solution.

Problem 5.2: If 3/4 of your sandwich is 15 inches long, how long is your original one?

Dorothy makes a drawing of the problem situation and Brenda talks about their operations:

Protocol 5.2: Dorothy's conceptualizing of the measurement of the whole quantity using a measurement of its fractional part.<sup>34</sup>

Z: Now Dorothy, three-fourths of it is fifteen.Dorothy: [Pauses for a second] Three-fourths is fifteen?Z: Ha, ha [Indicating agreement] Three-fourths of that sandwich is fifteen inches so how long is the original one?

<sup>&</sup>lt;sup>34</sup> Keith is the permanent witness-researcher who contributed in some occasions, as it is the case in this protocol.

Dorothy: Twenty [After three seconds.]

Brenda: [Nods her head to show agreement]

Z: How did you make it? If you want, you can draw. But if you only want to do so.

Dorothy: [Draws a rectangular shape and partitions it into four parts.] Three-fourths.

Z: OK. Is that the sandwich or is it three-fourths of the sandwich? [After I ask her, "Is it the whole sandwich or three-fourths of the sandwich?" she crosses out the last piece of those four pieces, see the first row of Figure 5.1.]

Dorothy: Three-fourths.

Z: OK. This is three-fourths of the sandwich [Just talking, no pointing.] Dorothy: This one? [Pointing to the drawing.]



Figure 5.1. Dorothy's drawing produced for Problem 5.2.

Explanation: Dorothy created two rows. She created the first row possibly to show the measurement of 15 inches. She created the second row possibly emphasizing the fourths and the three-fourths of the sandwich.

Z: I do not know. You tell me. I do not know what you were drawing [Dorothy extends the partitions of increments vertically to make the second row with four sections. She puts "5" in each section, see the second row of Figure 5.1] Brenda: Because each one is. You said that three fourths was fifteen so that is three divided by...like you have to think how many can fit into fifteen, and three times five is fifteen so each is five.

Z: OK.

Keith: So what is each portion of the whole when you say five, what portion of the whole is it?

Brenda: That is one fourth.

Dorothy's drawing of the problem situation and her result indicate that her

assimilation of the problem situation was very similar to Brenda's. Brenda's talk about

Dorothy's drawing indicates that their solving activity was also very similar, so I impute

a reversible partitive fractional scheme for composite units to both of them as I explain in

the following paragraphs.

Dorothy produced the answer "twenty" as a result of her mental activity. Therefore, I infer that she, too, operated symbolically. I could have insisted on her explaining her solution before I gave her the option of making a drawing, and had I done that, I think that her explanation would have been very powerful.

Based on her drawing, I infer that Dorothy took the identical relationship of three fourths of the sandwich as being 15 inches as her problem situation. I also infer that her goal was to make a bar that had four equal parts given that the length of the first three parts was 15. This was why she first drew a bar for 20 inches that had four increments of five composite units each, but said "three-fourths." After my question ("Is it the whole sandwich or three-fourths?"), she crossed out the last increment in her drawing to show the first three equal parts, which is 15 inches, as her understanding of the problem situation. This activity was very similar to how Brenda used a third as 10 inches for the previous problem and used increments of 10 three times to produce the result. Fifteen inches was a composite unit for both Brenda and Dorothy because they used it as a unit when splitting it into three equal parts. One of those equal parts was five inches long, and it was at the same time a fourth of the whole sandwich for both students. So, to communicate better that each fourth was five (inches), she revised her drawing without using increments of five composite units. Thus, Dorothy made the second row in her drawing and iterated five composite units four times to show the length of a quantity that was four times bigger than one of the equal parts (see Figure 5.1).

Both Brenda and Dorothy solved these problems symbolically because it took so little time for Brenda to solve the first problem and for Dorothy to solve the second problem. But Dorothy's drawing for the second problem and their discussion about the drawing is further indication beyond their symbolic operating that they were aware of how they operated. Operating symbolically, even though it was mainly by using number words, was algebraic in nature. Also of an algebraic nature was their use of a measure of length as identical to a fractional part of the whole sandwich. That is, both students took fifteen inches as identical to three fourths of an unmeasured sandwich.

Brenda and Dorothy's Reversible Iterative Fractional Scheme for Composite Numbers

Problem 5.3: Each of us has a candy bar, and I give you some of mine and you

then have six-fifths of your original candy bar. You measure and see your new candy is

48 inches long. How long is your original candy bar? (March 3)

This problem was a little bit different than the previous ones, mainly because

Brenda wanted to use paper and pencil to solve it.

Protocol 5.3: Conceptualizing the whole bar and its measurement when a measurement of improper quantity is given.

Z: You and I have a candy bar. They are the same size, but we do not know how big the candy bars are. I give you some of mine, and now you have six fifths of your original one, and you measure and see that is 48 inches.

Brenda: So my whole candy bar plus sixth-fifths?

Z: No. Your whole candy bar plus some more ...

Brenda: Is sixth-fifths?

Z: ... Is six-fifths of your original.

Brenda: So it is one and one sixths, [shakes her head and corrects herself] one and one fifth.

Z: Aha. [Indicating agreement] you measure that one and see that is forty-eight inches, so how long is your original one?

Brenda: Can I write it? [Asks this immediately]

Z: Yes.

Brenda: OK. You said that mine plus some of yours is how much? [She is trying to construct a meaningful problem situation for herself.]

Z: Six-fifths of your original one.

Brenda: So that is my whole candy bar together now,

Z: Um [indicating agreement]

Brenda: And it is equal to forty eight [writes 6/5 = 48],

Z: Forty-eight inches.

Brenda: So how long,

Z: How long is your original one?

Brenda: OK. So that is whole one plus [looking at the ceiling and talking] sixth, OK. Hold on. [She moves her hand on the paper as if she wants to write something. Twenty-two seconds pass.] How...

Z: Do you want to draw it? Let's draw it.

Brenda: OK. Yes. Whole one that is divided into fifths [she talks as she draws a rectangular shape partitioned into five almost equal parts], one, two, three, four, five, and you have one leftover fifth [she draws a smaller rectangle next to the previous one separately, see Figure 5.2] and this is forty-eight. Z: Forty-eight inches.

Brenda: OK. So that is one, two, three, four, five, six. [Pointing to each partition.] So six goes into forty-eight, right? Forty-eight divided by six [asking Dorothy to confirm her.] Eight times six is forty-eight? [Dorothy moves her head to show her agreement.] Oh, OK. I' ve got it. So each of these is eight [pointing to the fifths in her drawing], then eight times five [writing at the same time  $8 \times 5$ ] is forty. So it will be forty.

Z: Forty inches right? OK. Brenda: Yeah.

$$6_{15} = 48$$
  
 $6_{148}$   
 $8 \cdot 5 = 40$ 



In contrast to the first two problems, Brenda first made notations with paper and pencil (6/5=48) and then, upon my suggestion, made a drawing to enact her operations. It was unlikely that she could have operated mentally to produce 40 using her number words as in the previous problem situations. She seemed to need a drawing so she wouldn't forget the problem situation throughout her solving activity. Her notation, 6/5=48, did indicate that she established relations between the two quantities at the start

of her solving activity. It also indicates that the unknown length of her original candy bar remained implicit.

The problem situation was complicated: she had her original candy bar, some of mine taken and added to hers, her new candy bar, and the measure of the length of her new candy bar. For these components to become a meaningful situation to Brenda, she asked questions to conceptualize the actions and the quantities in the problem. At the very end, she notated 6/5=48 as a meaningful shortcut of the problem situation. Therefore, she may not have needed to re-present the problem situation, and this notation could have helped her to solve the problem: this means stating her goal, acting upon the problem situation, and getting the result.

While Brenda had a meaningful situation indicated by her notations, her 22second pause indicated that she was perturbed by the problem situation. She was not able to act. I suggested that she make a drawing. I thought if she just made a simple drawing of what she notated as 6/5 = 48, it would help her in acting purposefully. As her first action, she drew two separate bars: one whole bar with five fifths and an extra one as one fifth of the original bar. Brenda effectively used the identical relationship of 48 inches is 6/5 of the quantity to-be-made throughout her activities. This was evidenced by how she notated 6/5 = 48 at the start, how she made her drawing (with three distinct units), how she found the length of one of the fifths, and how she used the length of a fifth to produce the result. When drawing, she clearly said, "Whole one divided into fifths; one, two, three, four, five, and you have one more fifth and that is 48." She knew 6/5 as a quantity was composed of two quantities: five fifths and a fifth extra. In addition, Brenda was able to use the 6/5 quantity as six times bigger than a fifth. This is confirmed when Brenda partitioned the composite unit of 48 (inches) into six equal pieces. She looked for a number that when iterated six times produces 48. By dividing 48 into six, she generated the length of each fifth as 8 inches. For Brenda, the original bar had five fifths (see the previous quotation). That is why she multiplied eight by five to find the length of the original candy bar. Establishing and operating with the identical relationship the length of one of the six equal pieces of 48 inches is also the length of a fifth of the original bar is very important.

For this problem, this way of acting was different than how Brenda acted in the previous two problems. She did not immediately say one sixth of 48 or a fifth of the original candy bar is 8 inches (Remember she said 1/3 is 10 for the first problem). Rather than reflecting on her activities, as she did in the reversible partitive fraction scheme for composite units, she actually needed to operate on her notation and drawing of 6/5 = 48to produce the result. These are the indications that she constructed a new scheme that I call a reversible iterative fractional scheme for composite units. There is certainly reversibility involved in this new scheme because she did not have the length of the original candy bar as her situation and her goal was to find the length of the original bar. The situation and the goal were similar to the reversible partitive fractional scheme for composite units. However, she first needed to establish the identical relationship of a fifth of the original candy bar and one of the six equal pieces of 48 inches. For this, she divided the length of 6/5 quantity into six equal pieces and produced the final result by multiplying this intermediate result by five. Unlike her activities in the reversible partitive fractional scheme for composite units, this way of acting was not immediate for Brenda.

In the first teaching episode, just after Brenda solved Problem 5.3, I posed a similar problem for Dorothy.

Problem 5.4: Each of us has a candy bar, and I give you some of mine and you then have seven fifths of your original candy bar. You measure and see your new candy bar is 49 inches long. How long is your original candy bar? (March 3)

The way Dorothy conceived of the problem situation and her subsequent activities were very similar to how Brenda operated in the previous problem. In addition, Dorothy produced the result by using notation. I did not even need to ask Dorothy to make a picture of the problem situation because she could explain how she generated the result verbally. Dorothy wrote 7/5 = 49, and immediately said that the result was 35. Even though she did not write a symbol for the unknown quantity next to 7/5 to indicate that it is 7/5 of something, she was aware that 7/5 of the quantity measured 49 inches because she used the notation (7/5 = 49) when finding the five fifths of the quantity.



Figure 5.3. Dorothy's production of 35 using notation.

Dorothy said she would divide 49 by seven, "because that is how many pieces you have." I then asked Dorothy to draw the problem situation and talk about her solution using that drawing. My purpose was not to help her to solve the problem since she already solved it. I wanted her to communicate her operations by means of using a picture. Dorothy made a bar with seven almost equal partitions. She then put a long vertical line on the fifth mark with some help from Brenda to indicate the original candy bar (see Figure 5.4). She divided 49 by seven and said the result of 7 inches was the length of one of those pieces. She continued that her whole quantity was "five over five or five fifths" and multiplied 7 inches by five, so she got 5 pieces equal to 35 as her result (see Figure 5.4).



Figure 5.4. Dorothy's drawing of the problem situation.

Similar to Brenda's operations with her reversible iterative fractional scheme for composite units, Dorothy conceived of an anticipated result of her iterative fractional scheme for composite units. She used this anticipated result, 7/5 of a bar is 49 inches, as a situation to produce the original but unknown situation as a result of her reversible operations—5/5 of **the original** 7/5-bar is 35 inches.

## Multiplicative Part-Part-Whole Problems: Whole Numbers

In the next episode, March 7, we mainly worked on problems that included a whole number multiplicative relationship between two unknown whole numbers and students were asked to find those two unknown numbers. These problems were common problems in their algebra book. Other than being common problems in their book, my reason for using them was to explore how the two students operated in more complicated problem situations using their fractional and whole number knowledge when a multiplicative relationship between the two unknown numbers was stated as well as an additive relationship.

Problem 5.5: You are given two numbers, one of them is twice as much as the

other one. Find the numbers if their sum is 33. (March 7)

Protocol 5.4: Finding two multiplicatively related whole numbers.

Z: You have two numbers; one of them is twice as much as the other one. When you add them up you get thirty-three. When you find them, give me a signal. [Dorothy raises her hand immediately.] You got it [to Brenda]?Brenda: OK.Z: Brenda, you solve it and Dorothy, you will check her.

Brenda: Solve it on the paper?

Z: No.

Brenda: Or in my head? Oh. OK. Do you just want me to say it? OK. Twice. wait... it will be twelve, twenty-one. Which is...

Z: Twelve and twenty one, when you add them up you get thirty-three. That is good. But...

Brenda: But they are not, the second one is not twice as much...So it would be... Z: OK. Stop there [to Brenda]. Dorothy. How did you solve it and what did you get?

Dorothy: I divided the number by three, and I got eleven, and eleven plus twentytwo is thirty-three.

Z: So, twelve, twenty-one; eleven, twenty-two. They both add to thirty-three right? But there is a relationship, eleven and twenty two. One of them is twice as much as the other one.

Brenda started by estimating a number, which was 12, and then produced the other number, 21, based on the stated additive relationship. Twelve was a good estimate because the second number, 21, was close to twice as much as the first number, 12. I inferred that Brenda had following goals in her mind: finding two numbers whose sum is 33 and where one of the numbers is twice as much as the other. In addition to her two goals on which she based her activities, she monitored her activities. Even though she produced the sum of two numbers as 33, she was aware that the second number, 21, was not twice as much as 12, so the pair was not a satisfactory solution for her.

Dorothy, on the other hand, had an insightful strategy. She started by producing the number, three, based on the given multiplicative relationship of the two parts and on the given additive relationship. She seemed to intuitively produce a 3-part structure as a result of producing two entities, one entity being the unknown number and the other twice that number. She then joined them together to produce the 3-part structure, after which she conceptualized 33 as partitioned into three parts. Using this 3-part structure, she made an equivalency relationship between one of the equal partitions and the numerical value of the partition, which was 11. For example, she said "I divided the number by three, and got eleven," indicating that the numerical value of each of the three partitions was 11. She used this numerical value and its additive relationship to the whole quantity to find the numerical value of the two other parts—she said "eleven plus twenty two is thirty three." She used the multiplicative relationship between the parts (one of them is twice as much as the other) to find the number of parts, three, before she divided 33 by three. But when to find 22, she could have only relied on the additive relationship between the two parts and she used the complement of 11 in 33. Her ways of operating

might be conceptualized as *partitive fractional scheme for composite numbers* that she started with the whole unit and found the two fractional parts of this unit and their measurements. However, the use of this scheme may only explain Dorothy's activities after she constructed a structure of three equal parts in the whole quantity.

Brenda's and Dorothy's conceptions of the problem situation and their actions were different. It seems that by estimating, Brenda consciously made guesses for the numerosities of the two parts (which indicated that she treated them as known) and then used the additive relationship between the two numbers. She was possibly applying the Guess and Check strategy that she learned in the classroom. This approach is similar to how I initially conceived this type of problem, and because of this similarity I called them part-part-whole problems. On the other hand, Dorothy had an insightful strategy in that she operated on the known sum to figure out the multiplicatively related two parts that she treated as of unknown numerosity. Her approach suggests that she conceived the problem as a whole-part-part problem, examples of which are discussed below.

#### Multiplicative Whole-Part-Part Problems

After Problem 5.5, I posed problems with a similar structure, but those problems involved length measure of a known quantity. Further, rather than stating the multiplicative relationship among the two unknown parts and then the additive relationship of the two unknown parts to the known whole, these problems began stating the known whole as already partitioned into two unknown parts. The multiplicative relationship of the two unknown parts was then stated and the question was to find the measurement of the parts. I wanted to explore if Brenda and Dorothy would use their multiplicative reasoning in these inversely stated problems. colors: green and white. If the green part is three times as long as the white part, how

long is each part? (March 7)

Protocol 5.5: Dorothy's 4-part structure.

Brenda: Fifty-two, so three times as much as the...[She talks to herself very quietly.] I almost got it. Z: You almost got it? We are waiting for you. Fifty-two inches. One of them [one of the parts] is three times as much as the other part. The green part is three times as much as the white part. [Over a minute passed, both students were engaged in thinking quietly.] So do you want to go first, this time, Dorothy? Dorothy: Sure. I divided fifty-two by four. Since the first part and you have the other part, and the other part is three times more. So I divided by four, then I got it thirteen. And I multiplied the thirteen by three and got thirty-nine, and I added thirty-nine to thirteen and got fifty-two. Z: So, how long is the green part? Dorothy: Green part is the longer part? Z: Yes. Dorothy: Thirty-nine. Z: OK.? Brenda: Yes, I got it.

Dorothy said that she divided 52 by four. As an explanation for why she used four, she said "Since the first part, and you have the other part, and the other part is three times more" indicating that one part joined to a part that is three times more than the (initial) part is four equal parts. In this problem situation, similar to the previous one in Protocol 5.4, Dorothy used the unit of one as iterative to conceive the other part of the string as a quantity that was three times more than the first quantity. This thinking implicitly requires multiplicative reasoning, because the second quantity is three times more than the first quantity. Therefore, in total she had four equal partitions and she imagined that 52 was composed of these four parts. So, each of those partitions was 13 (inches), which she produced by dividing 52 by four.

Dorothy established an equivalency between each one of the four equal parts and its numerical value of 13 inches when she divided 52 by four. When Dorothy multiplied 13 by three, she operated multiplicatively to accomplish her goal of finding the numerical value of the longer part. Dorothy's conceptualization of the situation and her goaldirected activities that were observed in this solution will possibly serve in solving future problems when the relationship between the two parts are unit fractions (e.g., Problem 5.11).

In my interactions with the two students, I concentrated on the details of Dorothy's strategy. Meanwhile, Brenda assimilated the problem situation by focusing on the length of 52 inches and the relationship between the two parts. However, her two long pauses and her comments, "I almost got it" before Dorothy's solution and "Yes, I got it" after Dorothy's explanations, indicate that Brenda could not solve this problem as quickly as Dorothy. Although I did not check whether Brenda produced any results for herself, her final comment indicated that she concentrated on understanding Dorothy's solution. *Brenda's Attempt to Assimilate Dorothy's Solution* 

Following Problem 5.6, Brenda attempted to solve a problem but did not use Dorothy's strategy in her independent activities. She could, however, follow the strategy with Dorothy's help and with leading questions by the teacher.

Problem 5.7: You have a 60-inch string, and you have two parts. One of them is twice as much as the other part. Let's say the white part is twice as much as the other part. So, how long are the parts? (March 7)

Protocol 5.6: Brenda's attempt to make a 3-part structure.

Brenda: [Asking Dorothy] Do you have it? [Dorothy nods her head] Z: No, we are going to wait for you [Brenda].

Brenda: Divide by two.

Z: So you divide by two, sixty divided by two?

Brenda: I think so, is that what you did? [She asks Dorothy]

Z: Because you have two parts?

Brenda: Well then you have to think, one is twice as much as the other one. So I thought, if I divided that, but I got thirty divided by two is fifteen. And fifteen plus thirty is...it probably isn't right.

Z: So, what is the relationship between the parts? One of them is twice as much as the other one.

Brenda: So, it is double.

Z: Yes. When you add them up you need to get sixty, right?

Brenda: So it would be like [Twenty seconds pass.]

Z: So, you started with thirty and fifteen,

Brenda: Yes.

Z: And because it was forty-five?

Brenda: So it would be, fifteen...

Z: Think about how Dorothy solved the other problem. What was the other problem? Fifty-two, one of them is three times as much as the other part. Right? So, what did you do Dorothy for that one?

Dorothy: Since the first one. The second one was three times as much as the first one, I divided it by four.

Brenda: So you divided by four, if it is two times as much, you would divide it by... not four [smiling]...

Dorothy: If the second one is two times as much as the first one, you have three. So the first one is one, and the second one is two times as more. So you have three.

Z: Three parts, equal parts.

Brenda: Oh. OK. So you divide it by three. So, sixty divided by three is twenty. So it would be, twenty and [Ten seconds pass.]

Z: So are you subtracting twenty to find the piece or what do you do? Brenda: Yeah, subtract it, which is forty. Oh. OK. I can't believe I did not think of that before. It would be twenty and forty. I would kept on thinking that the piece was thirty, and then. Sorry.

Brenda first checked whether Dorothy had solved the problem and hesitated

before explaining how she would solve it. She divided 60 by two and produced 30. She

then assigned 30 as the numerical value of one of the two parts of 60-inch string and split

30 into two parts to find the numerical value of the smaller part. She construed the

situation as 30 being the part that was twice as much as the smaller part. In this sense she

used the splitting operation to produce a quantity such that 30 was twice as much as the

unknown quantity. She was aware that 30 was twice as much as 15, but the sum of these two numbers was not 60, which was a constraint for her.

When I suggested to Dorothy that she repeat how she solved the 52-inch problem, Brenda had still not conceptualized the current problem in terms of 60 inches composed of three equal parts. Brenda's conceptualization of the problem was based on an identical relationship of the two parts and their numerical values: she broke 60 into two equal parts. Only then did she use the multiplicative relationship between the two parts and say that 30 was twice as much as the other part. Even though she did not say the sum of 30 and 15 is 45, she said "fifteen plus thirty is…it probably isn't right." Therefore, we can infer that she knew that 30 and 15 would not work.

Brenda's difficulty resided in her not positing a part of unknown numerosity. In Protocol 5.4, the parts Brenda used in her estimates were known and also specific numbers. She used a quite similar strategy in Protocol 5.6 where she produced the numerosity 30 of one of the two parts and then a multiplicative relationship between these two parts. To produce a 3-part structure, like Dorothy, she would need a part of unknown numerosity and then two other parts of the same unknown numerosity. In Dorothy's case, the unknown numerosities were implicit in her solution, not explicit. This was why Brenda could not make up a 3-part structure. Unlike Brenda, Dorothy used the given relationship between the two parts of unknown numerosity to find the number of equal parts that would constitute the whole 60-inch quantity. So, in this sense, for Dorothy the problem situation was to establish an equivalency between the parts of unknown numerosity and their known numerical values using the number of equal partitions that composed the whole string. This indicates that Dorothy can operate with three levels of units. For example, in Dorothy's re-explanation of how she solved the 52inch problem, she indicated awareness that one of the four equal parts was also embedded in the longer (second) part that was composed of three of those equal parts. The whole string was the containing unit that included both of the two unequal (shorter and longer) parts (which were multiplicatively related), and the three units in the longer of the two unequal parts. Therefore, the string could be conceived with a structure of a unit of units of units.

In this interaction, I also guided Brenda on how to use the result of 60 divided by three, 20, to find the numerical value for the other part as 40. She could not operate independently because from the start she did not conceptualize the structure Dorothy established as three equal parts of unknown numerosity in the whole string, so she did not understand why Dorothy proposed dividing 60 by three. When she produced 40 with my explanation, she had two numbers, 20 and 40 that added up to 60, and one of them was twice as much as the other one. She felt confident that Dorothy's method worked because the relationships between the numerical values of two sub-quantities and the sum of those values satisfied the given problem situation she conceived at the beginning.

Warrants of Brenda's Accommodation: Brenda's Independent Solution

At the end of the March 7 meeting, Brenda herself independently solved a similar problem.

Problem 5.8: One-hundred-twelve-inch string is cut into two parts. One part is three times as long as the other part. How long are the parts? (March 7)

After Brenda and Dorothy spent 90 seconds in deep concentration and Brenda thought out loud a few times, Brenda spoke first.

Protocol 5.7: Brenda's construction of a 4-part structure.

Brenda: I guess you would divide by four... Z: Why?

Brenda: Because if it is three times as much, then you have to have that one extra, so you can get this. If it is four times, if you divide it by four, you know you are gonna have one number and the three times [traces the table with right hand and holds her left hand with thumb open]. So one plus three is four. And so I was thinking one hundred twelve divided by four is forty point five. But it is not. It is, one twelve divided by four is ... that will be three. Four goes into eleven twice right? [Dorothy approves] Eight, so it is twenty-eight. So it is twenty-eight times three is eighty-four [As she talks she uses paper and pencil for division and multiplication.] Dorothy: That is how I did it. Z: The same way?

Dorothy: Yes.

From Brenda's explanations, I inferred that she operated similarly to how Dorothy operated in the previous protocols. She made a differentiation between the two parts of 112 and the relation "three times as much" between the two parts. Her comment in Protocol 5.6, "I can't believe that I did not think of that before" when coupled with her establishing four parts in Protocol 5.7 solidly indicates that this 4-part structure was within her zone of potential construction and that she made a functional accommodation to her part-part-whole reasoning scheme in Protocol 5.6. With this 4-part structure we can say that Brenda conceived this problem as a whole-part-part situation unlike her conception of Problem 5.5 as a part-part-whole problem. Because of this new conception, she constructed new ways of using operations that were available to her before Protocol 5.7. Instead of estimating the first number and then finding the other number based on the multiplicative relationship between known quantities, she used the multiplicative relationship to operate on a unit of unknown numerosity to produce a 4-part structure where each part was of unknown numerosity and then found the numbers that

corresponded to the lengths of the parts by dividing the whole length of the string into four parts.

Brenda's conception of the problem situation might be slightly different than how Dorothy used a unit of one iteratively to make the other quantity. Brenda first thought about the quantity that was three times as much as the first quantity and then she posited the extra quantity, which she called "one," using the greater part. So, I infer that Brenda used her splitting operation to conceptualize the situation. Brenda's comment "If you divide by four, you know you are gonna have one number and the three times" indicated that she conceived the singleton part as a unit and the other part as three times the singleton part. Saying that she was going to divide by four indicates that she conceived four by partitioning the greater of the two unequal parts into three equal parts and joining them to the singleton part (the unknown numerosity of the parts was implicit). Because all she was given was that the greater of the two unequal parts was three times the singleton part, the inference that she split the greater unequal part into three parts is warranted. Brenda operated with three levels of units with a three-step solution: the first step is positing the four parts, the second step is dividing the numerical value of the whole into four parts, and the third step is multiplying the result by three to find the greater part.

Unlike Dorothy, Brenda did not base her activities on only thinking about the number of equal parts of unknown numerosity. She used the number of parts for talking about the multiplicative relationship between the known numerical values of two parts similar to her actions in Protocol 5.4. Because of Brenda's activities in Protocol 5.6, I am inclined to think that using a 4-part structure enabled Brenda to be certain about her

estimated number. Thus, Brenda seemed to be working with parts that she posited as known in the sense that she treated the unknown numerosity of the part as if it was known. Dorothy, on the other hand, operated to find the unknown numerosity of the part. So, her solution could be judged as more algebraic than Brenda's solution. Dorothy's solution required that the equivalence between the unit of unknown numerosity and its numerical value be found by operating with the unit as an unknown, however implicit, to produce its numerical value. However, both Brenda's and Dorothy's activities were based on their use of three levels of units.

#### Multiplicative Whole-Part-Part Problems: Fractions

In the March 9 meeting, we started using unit fractions to characterize the relationship between two parts. I continued posing problems with length. I was wondering how Brenda and Dorothy's fractional schemes would be useful or not useful for solving problems when the known quantity (whole number) was composed of two unknown sub-quantities and those sub-quantities were multiplicatively related to each other with a unit, proper, or improper fraction. In addition, in some problems, the number of equal parts did not divide the length of whole quantity evenly, for example, "Problem 5.15: You have a 4-inch candy bar and you cut it into two parts. One part is three fourths as much as the other part. How long are the parts?" There were also some problems that I used the fractional part of a measurement unit (e.g., an inch) as the length of quantities, for example, "Problem 5.17: A half-inch long candy bar is cut into two parts. Find the parts if one part is thirteen thirds as much as the other part."

Eventually, through my analysis, I realized that Dorothy did not have undue difficulties with the whole-part-part problems posed when the length of the whole

quantity was a whole number of inches, but she had difficulties with the last two

problems, Problem 5.17 and 5.18 (see Protocol 5.14), when the whole quantity was a

fractional part of an inch.

Brenda's Extension of n-part Structure for Situations Including Unit Fractions:

Generalizing Assimilation

As in the previous problems, the length of the whole quantity and a multiplicative

relationship between the two sub-quantities were given, but this time the multiplicative

relationship was a unit fractional relationship.

Problem 5.9: You have a string that is 40 inches, and you cut it into two parts,

and one part is one third times as much as the other part. How long are the parts?

(March 9)

Protocol 5.8: Brenda's assimilation of a unit fractional relationship into an n-part structure.

Brenda: It would be ten.
Z: Wait. [Asking Dorothy] Did you get an answer?
Dorothy: Hmm [means yes].
Z: OK. Brenda what did you get?
Brenda: Ten and thirty.
Z: How did you get ten and thirty?
Brenda: Um. I divided forty by four and got ten. Then three times ten is thirty. So thirty plus ten is forty.
Z: So why did you divide by four?
Brenda: Because I knew that: if you wanted to have something three times as much as one thing, you have to have three more so three plus one is four.

Brenda interpreted the lesser of the two parts as 1/3 times as long as the greater and the greater as "something three times as much as (the other) one thing." The greater quantity is not stripped of its multiplicative relationship to the smaller quantity because Brenda defined the greater part "as three times as much as one thing." This indicates that she engaged in reciprocal reasoning. In addition, she did not use the word "pieces" when giving quantitative definitions for the cut parts. This is important because it indicates that Brenda did not conceive the sub-quantities as independent quantities of each other by only emphasizing "threeness" or "oneness" with the number of equal partitions. In addition, it is also a sign of Brenda's engagement of a different type of relationship other than equivalence, which is used as an explanation for Dorothy's activities in Protocol 5.5 and 5.6. Brenda was engaged in an identical relationship as explained in Protocol 5.7.

When Brenda talked about why she divided 40 by four, she said, "you have to have three more, so three plus one is four." So for Brenda, the problem situation became: What would be the length of one part of a 40-inch long string, if it is partitioned into four equal parts? These four equal parts did not stand alone as four individual units or pieces. Rather, Brenda operated with them to produce the multiplicative relationship between the numerical values of the two parts. Brenda's immediate results of 10 and 30 are indications that Brenda used an identical relationship efficiently, that each ten was the quantity for one of the four equal pieces. After dividing 40 by four and producing 10 symbolically, she used ten as the smaller part. She then multiplied 10 by three to find the measure of the other part. Even when Brenda divided 40 by four, she did not lose the multiplicative relationship of one quantity being three times as much as the other. Her explanation indicated that she was aware of the fact that if a quantity is three times as much as another quantity, then the measure of the first quantity will be three times as much as the measure of the second quantity. This awareness requires the identical relationship of conceptualizing 10 as being the same quantity as the small part. It also enables using that identical relationship multiplicatively for generating the quantity that

was three times as much as the smaller quantity.

Brenda even checked whether those two measures added up to 40. With this checking, I inferred that she wanted to confirm that the results satisfied the sum of the two numbers was 40, which was one of her goals in Protocol 5.4. This shows that Brenda did not abandon part-part-whole reasoning, but she could conceive the problem as a whole-part-part and operate using this situation, and then check the results as she did with her part-part-whole structure.

Dorothy's Accommodation of Whole-Part-Part Reasoning Scheme for Solving Problems with Composite Fractions as Multiplicative Relationships

I inferred from Brenda's solution and Dorothy's agreement with her solution of the previous problem that they could easily solve the problems when unit fractions were given as the multiplicative relationship between the two sub-quantities. Therefore, I presented the following problem to Dorothy that involved a proper fraction.

Problem 5.10: Now, Dorothy. You have a string 50 inches long and again you have two parts. But this time, one part is two-thirds times as much as the other part. So

how long is each part? (March 9)

Protocol 5.9: Making a 5-part structure with a proper fraction.

Z: [Dorothy sits quietly.] But you also think [to Brenda]. If you want you can draw [Handing out paper and pencil to each of them]. Anything that will be helpful to you.
Dorothy: [Quietly speaks to herself. A minute passes.]
Z: I can't hear you.
Dorothy: Oh. I divide and got sixteen and two-thirds, I think.
Z: Sixteen and two-thirds?
Dorothy: Hmm. [Indicating agreement], and I multiplied... the small one.
Brenda: [Said something but it was inaudible]
Z: [To Brenda] say it again?
Brenda: I do not know.
Dorothy: I think that is right. I divided fifty, is it sixteen and two-thirds?

Z: Rephrase the problem for me, restate the problem. What was it?

Dorothy: Whole string is fifty, and you have two pieces, one piece is two-thirds, is it two-thirds of or two-thirds bigger than the first one?

Z: Two-thirds as much as the other part. [Dorothy talks to herself quietly]... How would you interpret it Brenda?

Brenda: Two-thirds of... It is not two-thirds of the string; it is two-thirds more than one part?

Dorothy: Thirty-three one third?

Z: Thirty-three one third?

Dorothy: I think.

Z: Let's draw it. Or whatever will be helpful to you. [Dorothy divides 50 by 3 on her paper.] So why are you dividing it by three?

Dorothy: I have three pieces, not three pieces but you have one third and then you have the one; that is two-thirds.

Z: OK. One piece is two-thirds of the other piece.

Brenda: So one is two-thirds of one piece, so...

Z: So which piece is bigger?

Brenda: The one, that is "of".

Z: Let's say the white part is the two-thirds of the green part, OK.? So is the white part longer or the green part?

Brenda: The green. The white is two-thirds of the green. Green is longer. Z: Yes.

Dorothy: So you do not divide it by three?

Z: What do you think [to Dorothy]? Why do you divide by three? I am just curious about it.

Dorothy: Um...

Z: Can you draw the whole string? [Dorothy draws a line segment.] OK. that is fifty inches; we have two parts, green and white. The white part is two-thirds as much as the green one right? [Dorothy partitions the segment into two parts and puts "G" under the bigger part, "W" and "2/3" under the other part. See Figure 5.5.] Two-thirds of what?

Brenda: Of the green.

Z: Of this green [pointing to the "G" part, Dorothy wrote "2/3 of G".] Dorothy: So I divide it by two...well I divide by two and multiply by three? Z: Why would you do that?

Dorothy: I do not know.



Figure 5.5. Dorothy's drawing of the two parts for Problem 5.10.

Dorothy's dividing 50 by three indicated that she thought that the whole string was cut into three-thirds, where one part was one third of the string, and the other part was two-thirds of the string. Dorothy established an additive relationship between the two sub-quantities as one of them being two-thirds, and the other one being the complement of two thirds when the whole quantity was three thirds. Those thirds were implicit thirds of the whole string.

I asked Dorothy to restate the problem situation. She indicated that she was not certain how to interpret "two-thirds as much as the other part" whether it was "two-thirds of or two-thirds bigger than the first one." Dorothy's comment that "two-thirds bigger than the first one" can be explained that she thought how to complete the whole using 2/3. That she said, "two-thirds of" indicates that she was aware of another type of relationship, a possible multiplicative relationship, but she did not know how to establish it. So, I asked Brenda how she would have interpreted it. Not only did I want to check Brenda's conceptualization of the situation, but also if her contribution would be helpful to Dorothy. Brenda said, "It is not two-thirds of the string." However, she did not know how to interpret it as a multiplicative relationship between the parts either.

Discussing what "two-thirds of" means instead of "two-thirds as much as the other part" and adding "the white part is two thirds of the green part" to the problem situation possibly opened new paths for the students. Brenda then became aware of the (multiplicative) relationship of the two sub-quantities; she said, "The green is longer, the white is two-thirds of the green." However, Dorothy's conception of the problem situation did not change, but it was perturbed because she asked, "So you do not divide it by three?"

I asked Dorothy to make a drawing of the situation, and she drew a line segment (for 50 inches) with one mark to show the two parts. She used "G" for indicating the green part and "W" and "2/3" for the other part on the segment. When I asked " "2/3" of what?" and Brenda answered "of green," at that moment Dorothy added "of G" next to "2/3"(see Figure 5.5). This indicates, for Dorothy, 2/3 referred to the white part of the string rather than a relationship between the two parts.

As a matter of fact, after writing "2/3 of G," Dorothy did not know how to produce the number of equal parts which compose the whole quantity. This constituted a perturbation for her because up to this point, she had successfully produced the number of equal parts for every situation using her whole-part-part reasoning scheme.

Even though she did not know how to produce the three-thirds quantity for the greater part using two-thirds of it, she was at a better place than at the start when she conceptualized the situation as "I have three pieces, not three pieces but you have one third and then you have the one, that is two-thirds" because she was in a state of perturbation. So, seeing that Dorothy was unable to continue, I shifted our focus to a similar situation like Brenda's problem in Protocol 5.8 (A forty-inch string cut into two parts, and one part is one-third times as much as the other part). The following problem had a simpler multiplicative relationship (unit fractions) between the unequal parts compared to having composite fractions.

Problem 5.11: We have still 50 inches, and the white part is one-fourth times as much as the green one. So, how long are the parts?

Protocol 5.10: Simpler situation with a unit fraction.

Z: Let's go to the previous situation, we have still fifty inches, and white part is four times as much as the –no, white part is one fourth times as much as the green

one. How are you goanna draw that? You have fifty inches [Dorothy draws a line segment.] We have still two parts, white and green [puts a partition mark]. White part is one fourth times as much as the green one [Dorothy writes "1/4" under the small part, she then immediately divides 50 by five using a long division algorithm.] Um. So why do you divide by five?

Dorothy: Because this is five. This is four-fourths and that is one-fourth. Z: Good... Right? [Brenda agrees with her head.] So can you show the fourfourths here [asking Dorothy, pointing to the line segment]? And mark the each fourth? [Dorothy puts three marks for the 4/4 part. See Figure 5.6.] So how many parts do you have all together in the fifty inches? Dorothy: Five.

Z: Five. Can you do the same thing for this [pointing to her work for Figure 5.5.]? Maybe? Or no?



Figure 5.6. Dorothy's drawing for the solution of Problem 5.11.

Dorothy divided 50 by five as her solution and said, "this is four fourths and this is one fourth" as her explanation. After drawing a line segment and making two parts, she labeled the small part as "1/4" and then said the other part was four fourths. At this point, she used the operations that she had been using for generating the greater sub-quantity using the smaller quantity as a unit. She assimilated this problem situation to the whole-part-part reasoning scheme with whole number multiplication relationship between the parts. This was possible because she reasoned reciprocally for establishing the fourfourths quantity (or four parts) when one fourth of it was given as the relationship between the parts. She used one-fourth as an iterable unit to produce the four-fourths

quantity. That is why she divided 50 by five, because there were five equal partitions composing the whole. This was also the first time she independently named the two unequal parts using fractions and operated with those quantities.

Right after Problem 5.11 (subsequently referred to as "the 1/4 problem"), I asked Dorothy whether she could use her idea of finding the number of equal parts and lengths of those parts to solve the problem in Problem 5.10 (subsequently referred to as "the 2/3 problem"). Dorothy said, "Three thirds, two thirds, so I still divide by five." She notated 50 divided by five using the traditional paper and pencil algorithm on her paper. Upon my request to show thirds, Dorothy placed two marks on the "G" part of the line segment for three equal partitions and one mark for "2/3 of G" part for two even partitions for a total of five partitions (see Figure 5.7).



Figure 5.7. Dorothy's drawing of 50-inch string with green and white parts.

Dorothy then placed "10" on top of the last two partitions. She said the length of that part would be 20, and the length of the other part would be 30 inches. Producing the greater part when the smaller part of unknown numerosity was given as a unit fraction in the 1/4 problem, and accessing her meaning of fractions (two pieces out of three pieces for two thirds) enabled Dorothy to conceptualize the green part as 3/3. She definitely made an accommodation in the context of solving the problem, but it is not clear if it was

a generalizing assimilation or a functional accommodation. This accommodation might be generalizing assimilation if she only transferred her whole-part-part reasoning scheme to be used in this novel situation with proper fractional relationship. It would be a functional accommodation if she produced a new and different scheme by changing how she viewed this novel situation and operating differently than the problems when a string had two sub-parts, one part being twice, three times, or one-fourth times, etc., of the other part. Her activities or operations in the similar contexts will help us decide what this accommodation is. What is certain is that this accommodation (either generalizing assimilation or functional accommodation, cf. Chapter 2) enabled her to produce the numerical values of the two sub-quantities when the composite fractional numbers are given as the multiplicative relationship between the sub-quantities.

Before solving the 1/4 problem, Dorothy did not know that the green part was 3/3 if the white part was 2/3 of the green part. With the 1/4 problem, she became aware that four-fourths can be generated using one-fourth four times and, reciprocally, one-fourth can be produced by disembedding one of the four equal parts of the four-fourths quantity. For this disembedding operation, she needed to take four-fourths as a given or as the result of her previous activities. I did not observe Dorothy talking about her reciprocal reasoning in this way, but I do infer reciprocal reasoning in the sense that I made a possible explanation of how she produced the relationships between green and white part for the 2/3 problem (see *Explanation 1* below).

Dorothy's observed activities for producing four-fourths quantity in the 1/4 problem would not be enough for producing 3/3 as a quantity in this problem because 2/3 was not a unit fraction. She could not take the 2/3 quantity an iterable unit to make the

greater part of 3/3. There were three things she could have possibly thought of when she made 3/3.

*Explanation 1.* She started with the three-thirds quantity to make the two-thirds quantity by reasoning reversibly. In this case I would call this change a functional accommodation. Her saying "three-thirds" first and placing 3/3 under G and then talking about a 2/3 quantity might be some indication of this thinking but not a strong enough one to claim she thought reversibly. For the 1/4 problem, she knew the bigger part is four times as much as the smaller part and she could make one-fourth taking one out of four equal parts. If Dorothy abstracted the result of this relationship that one-forth is one out of four-fourths and used it in the 2/3 problem, her operations would be also different from the ones she had been using; she would start with three-thirds (greater quantity) and then give meaning to two-thirds (smaller quantity).

*Explanation 2.* She might have thought about a quantity when two out of three of it was used to make two-thirds of it: That quantity would be three thirds of the green part. This thinking would be similar to the thinking when one out of four parts is used to make one fourth of a quantity to produce the four-fourths quantity. It was not only "times" (in the case of smaller part that was always the iterable unit, e.g., 1/4) that would result in naming the greater part, but it was also the "out of" operation that would lead Dorothy to conceptualize the 3/3 quantity. The accommodation would be in conceiving the two thirds as twice as long as one third of the other piece of the string, i.e., 2/3 is two times one third of 3/3. For this conception of 1/3, she should have thought of using two out of three parts to name each partition as a third, and then using 1/3 as an iterable unit to establish 3/3 quantity. This would be a significant modification in her whole-part-part

reasoning scheme, so it would be a functional accommodation.

*Explanation 3.* After solving the 1/4 problem, Dorothy could have assimilated the 2/3 problem situation the same way. The operation that gives meaning to three in the two out of three situation might be the same as the four in one out of four situation. Therefore, it does not matter whether it is two out of three or four out of three; the important part is what the given quantity is out of. The 2/3 problem is a "novel" situation for the whole-part-part reasoning scheme since this is the first time a non-unit fraction was given as the relationship between the parts and Dorothy solved the problem easily after she solved the 1/4 problem. Therefore, there was no change in the scheme's activity part (no functional accommodation) and the change might be just a generalizing assimilation, which is defined as:

An assimilation is generalizing if the scheme involved is used in situations that contain sensory material that is novel for the scheme (from the point of view of an observer), but the scheme does not recognize it (until possibly later, as a consequence of the unrecognized difference), and if there is an adjustment in the scheme without the activity of the scheme being implemented. (Steffe & Thompson, 2000, p. 289)

Dorothy might have had a functional accommodation (as described in *Explanation 1*) to produce the green part as 3/3 because she was in a state of perturbation as indicated in Protocol 5.9 and she attempted to solve the 2/3 problem before she solved the 1/4 problem. On the other hand, she could have done both a functional accommodation and generalizing assimilation described in *Explanation 2* and *Explanation 3*. This means she could have multiplicatively thought that the two thirds as twice as long as one third of the other piece of string—i.e., 2/3 is two times one third of 3/3—and, at the same time, she could have focused on what two is out of three, so the

second part would be three thirds. Making the longer part as three thirds, because of two out of three, is kind of "inserting" the same structure she had used for one out of four when making the four-fourths quantity. Because of the nature of this accommodation, we might call this *insertive functional accommodation* (L.P. Steffe, personal communication, January, 12, 2007). *Explanation 2* in combination with *Explanation 3* is more plausible compared to *Explanation 1* in terms of how she made the accommodation; there is not a strong corroboration of reversibility. Dorothy had a partitive fractional scheme which could have enabled her using the "out of" idea for establishing the 1/3 an iterable unit as one of its corroboration for *Explanation 2*.

Regardless, Dorothy's comment in the following problem is a strong

corroboration of the change and of operating with the length of 1/3 as an iterable

measurement unit that enabled her to solve problems that included proper fractions as the

relationship between the unequal parts.

Problem 5.12: A sixty-five inch string has pink and red parts. Find the length of

the parts if the pink part is 2/3 as long as the red part. (March 9)

Protocol 5.11: Dorothy's conceptualization of one third of three-thirds as 13 inches.

[Both Brenda and Dorothy are engaged independently. Brenda talks to herself.] Brenda: Wouldn't it be a thirteen? Z: One piece? Pink? Brenda: The smaller. Or one piece is thirteen right? I think. Z: Do you want to draw it? Brenda: If you have a string, this is all equal to sixty-five [drawing a line segment and puts 65 on the right end of the segment and puts a mark on the segment] and this part was two-thirds [placing "2/3" under the small part] of this which is three thirds [placing "3/3" under the bigger part], and three plus two equals to five. So you divide sixty-five by five. And you get thirteen, which is...Um...which is like this part, isn't it [pointing to the 2/3]? or is it this one [pointing to the 3/3 part]? Dorothy: Thirteen will be one third. I mean not one third but, yeah. If you want to put this into three [pointing to "3/3" part], put that in two [pointing to "2/3" part].



It would be one [Brenda places marks on the parts. See Figure 5.8.]

Figure 5.8. Brenda's drawing (with help from Dorothy) for the solution of Problem 5.12

Brenda's division of 65 by five was based on assimilating Dorothy's operations in the previous problem (A 50-inch bar had two parts, and one part was 2/3 times as much as the other part). After placing "2/3" and "3/3" on her drawing without any hesitation, Brenda said there would be five partitions and that was why she divided 65 by five. However, when Brenda produced the result of 13 she did not know how to relate it to her problem situation. She thought for a while that it was the measure for the small part (2/3 part). She was not sure where to place 13 in her drawing either. At that time, Dorothy said that 13 would be one third, and also signified that it would be one of the partitions of three thirds and one of the partitions of two thirds. Dorothy with this contribution confirmed that she could give meaning to the result of the division by using the accommodation she made with her revisiting of Problem 5.10. She helped Brenda to place the marks for each third in those two partitions. This shows that Dorothy can operate sophisticatedly to produce the measure of those two sub-quantities when a composite fraction is given as the relationship between the sub-quantities.
### Whole-Part-Part Problems: Improper Fractional Relationships

After Brenda and Dorothy solved problems similar to Problem 5.10 and 5.12 with proper fractions as the multiplicative relationship between the parts, I posed Problem 5.13 to investigate their activities with improper fractions. As an analyst, I am particularly interested in whether they would use the same scheme or differentiate the ways they assimilated the problem situation in their scheme when improper fractions were given as

the multiplicative relationship (from the observer's point of view) between the parts.

Problem 5.13: A forty-five inch long string was cut into two parts. Find the parts

if one part is three halves as long as the other part. (March 9)

Protocol 5.12: Establishing a 5-part structure with improper fractions.

Z: Let's say you have another ribbon, string. Now this time one part is three halves as much as the other part. And your whole thing is forty-five inches. Dorothy: Three halves is three over two?
Z: What else can it be? [Laughing and Dorothy also laughs.]
Brenda: And how long?
Z: Forty-five inches.
Brenda: OK. got it.
Z: Got it? [Both nods their heads quietly.] Do you want to draw it Brenda? So we can talk?
Brenda: Sure. Whole thing is forty-five inches [draws a line segment], and this part is three over two [puts one mark on the line segment to show two parts and writes 3/2 on the shorter part], and this is three over three [writes 3/3 under the longer part. See Figure 5.9] So three OK. Three and three is six, so forty-five divided by six would be seven or... [Writes 45 divided by six using the traditional algorithm. She attempts to erase 3/3 she put under the longer part.]



Figure 5.9. Brenda's first drawing of 45-inch string with two sub-quantities of 3/3 and 3/2.

Z: So which part would be longer?

Brenda: [At the same time] This part should be two over two [pointing to 3/3 part], I did this in my head; that is why it was not coming out right...

Z: So, which part is longer?

Brenda: Um... this part [points to the part with 3/2 written in Figure 5.9], that should be different.

Z: OK.

Brenda: I'll draw right. [She erases and changes the placement of the mark she put on the line segment.] I did this way in my head. I just did not write on the paper. That was it was not making sense to me. OK. So this part is two over two and this part is three over two. And this is five.

Z: Can you show the marks for one half?

Brenda: Yah. [She partitions the part labeled "2/2" into two parts, and partitions the other part labeled "3/2" into three almost equal parts, see Figure 5.10.] So that is five equal pieces so that is forty-five divided by five. So each piece is nine, and so this "two over two" is eighteen. So this "three over two" is twenty-seven. So the whole thing...[She adds 18 and 27 and gets 45.] Yah, that is right.



Figure 5.10. Brenda's drawing for the solution of Problem 5.13.

Z: So how much is this nine inches of the whole?Brenda: One fifth.Z: And it is also, how much is it of the short part?Brenda: One half.Z: And the big part?Brenda: One third.Z: So one third of three halves is?Brenda: It is nine inches.Z: And it is also one half of two halves?Brenda: Mmm-hmm.

Both Brenda and Dorothy indicated that they solved the problem mentally. Unfortunately, I do not know how Dorothy conceptualized the problem or how she solved it because I was concentrating on Brenda's solution when teaching. I asked Brenda to draw the situation and talk about it on the paper. She drew a line segment with two parts that she labeled as "3/2" (shorter part) and "3/3" (longer part) (see Figure 5.9).

Brenda made the drawing of the problem situation (Figure 5.9) similar to how she assimilated the previous problems. Until this problem, the shorter part was given in relation to the longer part (like one part is 2/3 as much as the other part). However, in this problem the longer part was given first and the shorter part needed to be found. Different than the previous problems, the longer part was an improper fractional number so Brenda needed to take into consideration that this situation (3/2 of something) was the result of operating on a quantity of 2/2. So to make the 2/2 quantity, she needed to reverse her operations of making 3/2. Reversing her operations means that when 3/2 is viewed as three (times) one half or one half iterated three times, she needed to think about the quantity that the halves came from or to conceive what they were half of. By going through this kind of thinking process, she could generate the quantity of 2/2 that was twice as much as a one half quantity.

Instead Brenda then added the number of parts, three and three, she envisioned making using 3/2 and 3/3. An explanation for this situation might be that she inverted 3/2 and viewed it as 2/3 for the given part and consequently, the longer part was 3/3. It did not matter to her that 3/2 was three halves of something and 3/3 was also three thirds of the same quantity; she treated halves and thirds as if they were equal quantities based on the same unit of one. That was why she added the numerators of three and three from

each fractional numbers and divided 45 into six. Up to this point, from her activities, we can infer that for Brenda 3/2 was not a relationship between the two parts but it was only one of the two sub-quantities. She then quickly reviewed what she did, when the result of her division of 45 by six and her result in her mental solution did not fit. At the same time, when I asked her "which part would be longer?" she said, "this part should be two over two" for the 3/3 labeled part. She revised her drawing for the shorter part as 2/2 and longer part as 3/2. She again mentally added the number of parts she visualized for 2/2and 3/2 and said the sum would be five. Once she was confident that there were five equal parts (or five units) in the 45-inch string, and the string had two sub-parts with units of two and units of three each, she completed her solution using the whole-part-part reasoning scheme she had been using (see Figure 5.10). She constructed and used that scheme for the problems when a multiplicative relationship between the two subquantities was whole number, unit fractional numbers, or proper fractional numbers. So, she enlarged the scheme's possible situations to solve problems that had improper fractions as a multiplicative relationship between the two parts. She was aware of her activities when she assimilated this situation, because her assigning 3/3 for the unknown fractional quantity first and then assigning 2/2 made her realize that this situation was different, yet she could solve this problem the same way after she revised her conception of the situation. This was only possible because she already had her reversible iterative fractional scheme and she used it to make the 2/2 quantity mentally.

Just after Brenda solved this problem, to see how Dorothy would solve such a problem for herself, I asked Dorothy to solve Problem 5.14.

Problem 5.14: Find the length of the two sub-quantities of a 45-inch string, if one

part is seven halves as much as the other part. (March 9)

Protocol 5.13: Establishing a 9-part structure with improper fractions.

Dorothy: Well, the whole thing is forty-five and the larger part would be seven over two. And the other part will be two over two. And you divide...well, if you add the numerators, you get nine. And you divide... Z: Why do you add numerators? Dorothy: That is how many pieces are in all. Divide by nine get five and you multiply five times seven and you get thirty-five. Z: For the short part or big part? Dorothy: Big part. And for the two over two, you multiply two times five to get ten...

From Dorothy's explanations we can infer that, similar to Brenda, she also extended her whole-part-part reasoning scheme's situations to solve problems including improper fractions as a relationship between the sub-quantities. Even though Dorothy had a reversible iterative fractional scheme (see *Brenda's and Dorothy's reversible iterative fractional scheme for composite units* in this chapter) for Problem 5.14, it is not clear how Dorothy decided the shorter part would be 2/2. She might have assimilated the problem situation in a similar way to *Explanation 3* which I made in relation to her revisit of the Problem 5.10. It did not matter to her whether it was improper fraction or proper fraction, she probably focused on the denominator of the fraction and what the given proper fraction is "out of," to make the number of partitions for the second part. Because of this, there might be a qualitative difference between how Brenda and Dorothy assimilated the problem situation with improper fractions into their whole-part-part reasoning schemes.

Although Dorothy had previously used fraction names with composite fractions, she did not use fraction names when talking about her solution or conceptualization of

problems with improper fractions. For example, when talking about seven halves she read it as how it is written: "seven over two", or for two halves she read it as "two over two." In addition, in Protocol 5.12 for "three halves," she asked whether it was same as three over two. Consequently, it seems that Dorothy decided the shorter part would be two over two, since seven halves was notated as 7/2 (her naming as seven over two) and the denominator of 7/2 was coincidently the determinant of the shorter part. In any case, this way of operating was functional for Dorothy. She might not have been necessarily thinking about the multiplicative relationships between the parts (one part is seven halves as much as the other part), but she might have related those two parts using the denominator of given notation (7/2). This way of operating with the fraction notation is a generalizing assimilation and might seem like an abstraction on Dorothy's part, but in Protocol 5.14, we will see that her way of interpreting fractional notation won't be helpful when the whole known quantity is a fractional part of an inch.

### Dorothy's Struggle When the Result of a Division Activity is a Fractional Number

In the next teaching episode, March 24, Brenda came late. Meanwhile, Dorothy solved two problems with the same context as the Problem 5.12 by herself. However, this time the results were not a whole number multiple of an inch. In my analysis, I observed that this situation perturbed Dorothy since she could not decide which notation to use to represent her mathematical activity.

Problem 5.15: You have a 4-inch candy bar and you cut it into two parts. One part is three fourths as much as the other part. How long are the parts? (March 24)

This was the first time the number of equal parts she found, seven, was more than the numerical value of the whole, which was four (inches). Therefore, unlike the whole number results in the previous problems, the result was a fractional number. Even though Dorothy said that the result would be four divided by seven, she spent almost 90 seconds deciding how to notate dividing four by seven with fractions. She was unsure whether it was 4/7 or 7/4. At that point, a witness-researcher intervened and asked, "If it was four divided by seven will the result be less than or more than one?" In response, Dorothy pointed to 7/4 and the observer rephrased his question as "If you divide four into seven equal parts, will the result be more than or less than one?" At that moment, Dorothy said, "It would be less than one" and pointed to 4/7 as the right notation. Therefore, even though Dorothy knew how to act, she wanted to divide four by seven, and produce the result in this problem as "4/7," she could not independently notate the result as a fraction of an inch. J. Olive (personal communication, April 13, 2008) indicates that language of *divided by* is ambiguous for many students since they interpret 4)7 in the same word order as the numerals as 4 divided into 7, therefore 4 divided into 7 is more meaningful for them as opposed to 4 divided by 7 when interpreting long divisions.

Dorothy assimilated how to use the result of a division algorithm as a fraction without really understanding how much it was of an inch. Neither in her drawing nor in her talking, did she pay attention to the relative sizes of 1-inch and 4/7 of the 1-inch. She did not make a judgment using the "out of" structure she had been using, such as 4/7 is four partitions out of seven equal partitions of 1-inch, or realizing seven of "4/7 in [which is Dorothy's notations on the paper]" completed the whole four inches. Dorothy might have been perturbed with this result, which is not a whole number and also is notated differently. As indicated in this paragraph, she did not seem able to treat "4/7 in" as a number. "4/7 in" was a functional result: it was a place holder for a notation of one

of the seven pieces of the whole candy bar. With this way of using fractional notation, Dorothy could also produce the results for Problem 5.16.

Problem 5.16: A five-inch bar is cut into two parts. One part is three-fourths as much as the other part. How long are the parts? (March 24)

Similar to her actions in the previous problem, Dorothy made the two sub-parts in her drawing and labeled the smaller part as "3/4" with three partitions, and labeled the greater part as "4/4" with four partitions. She then wrote, "7 pieces" next to the line segment she drew for the whole quantity. She then notated that each of those seven pieces would be 5/7 ("in"), and could find the length of those two sub-parts by multiplying 5/7 by three and four respectively, producing "15/7 in" and "20/7 in" as her results. Dorothy could operate and produce correct fractional results and then label those results by writing "in" next to them. However, as it can be read in Protocol 5.14, I observed that Dorothy had tremendous difficulty because she could not extend her whole-part-part reasoning scheme to problem situations when the given quantity was a fraction of an inch.

#### Whole-Part-Part Problems with Fractional Known Quantity

The measurement of the whole quantity—1/2 inch—in Problem 5.17 was a fractional number, and the solution required operations on fractions. The structure of the problem was the same (two parts of unknown numerosity but multiplicatively related sub-quantities of a known quantity) as the previous problems, and I wanted to see whether students' possible operations would be different than those they used in Protocol 5.12 with improper fractions because in the following two problems the known quantity was part of an inch. It took a total of 15 minutes for both Brenda and Dorothy to produce

a solution for Problem 5.17 that they, the teacher, and the observers were satisfied with. During this time, Dorothy's activities were the most complicated ones and changed immensely depending on the interactions she had with me and the observers. This might be because of two reasons: First, 1/2 inch was a strong perturbation for Dorothy; it was part of an inch, and Dorothy did not know how to operate on part of a fractional whole. She did not seem to have fractional meaning for part of an inch for the results of the Problems 5.15 and 5.16. Second, as discussed in Protocol 5.13, Dorothy did not establish a structure with a multiplicative relationship of improper fractions between the sub-parts, but only operated on fractional notation without operating on the referent quantities. The problem was as follows:

Problem 5.17: A half-inch long candy bar is cut into two parts. Find the parts if one part is thirteen thirds as much as the other part. (March 24)

After Brenda found a numerical value (1/32) for one of the sixteen equal pieces of the 1/2- inch bar, she neither labeled that numerical value as part of an inch nor used it to find the lengths of the two sub-parts, yet she could contribute to the discussions we were having with Dorothy and gain some insight for herself as well.

In the following protocol, I first analyze how Brenda found the length of one of the 16 equal pieces with some help. I then partition the remaining 15-minute discussion into three sub-sections, considering how Dorothy's problem situations and activities changed. I will also include some discussion related to Brenda's understanding in those sub-sections, but I mainly focus on analyzing Dorothy's operations and understanding of this problem. Protocol 5.14: Producing length of one sixteenth of a half-inch bar.<sup>35</sup>

Brenda: I think you would divide one half by sixteen.

Z: Um. Think about why. Why would you do that?

Brenda: Um. Because there are three parts over here in one whole, and there are thirteen parts [pointing to the two labeled parts in her drawing respectively] over here of that part [pointing to three thirds part]. So, there are thirteen parts here and three parts over here, so sixteen parts all together. And then you distribute sixteen equally into one half.

Z: Equally into one half. OK.

Brenda: Which would be [thinks quietly]... eight.

Z: Eight? Try to use fractions [Brenda notated 16 divided by 0.5 in a traditional division algorithm. She erased that and wrote 16/1 and 1/2 but couldn't decide which operation notation to use for those numbers. She asked something quietly.] Brenda: Sixteen goes into...would you divide or mult... [Talks very quietly.] Z: So you said that.

Observer [to Z]: Did you hear her question?

Z: Yes. Would you divide or...

Observer: Or multiply. Right.

Z [to Brenda]: Or multiply, did you say that? You did not say multiply but you said something "or"?

Brenda: Or multiply.

Z: Or multiply. Um. I will ask you [to Brenda] the same question because you were thinking that you needed to have sixteen equal pieces in one half inch, right? Brenda: Yes. So you would divide.

Z: Which one are you gonna divide into which?

Brenda: Sixteen into one half [says it very quietly].

Z: So, if that is the case, um. The pieces will give you what?

Brenda: Um. probably one eighth [quietly].

Z: One eighth?

Brenda: Sixteen times one eighth [Talks to herself as she writes  $16/1 \times 1/8$  then erases it] I do not know.

Z: You had this one half inch long thing right? And you are trying to have 16 pieces in this, so how long will each piece be?

Brenda: Um. [Pauses for few seconds.]

Z: If it was one inch, the length of the candy bar, how long would each piece be if we had sixteen parts?

Brenda: One sixteenth. So if it is a half, it would be one eighth.

Z: You have a smaller, half of the... half of an inch.

Brenda: So, it would be one thirty two. One thirty two.

<sup>&</sup>lt;sup>35</sup> The Observer in Protocol 5.14 (Problem 5.17) is Dr. John Olive. The focus of Protocol 5.14 is what Brenda said and did. During Brenda's activities, Dorothy worked by herself and produced  $16/3 \times 13 =$ 208/3 and  $16/3 \times 3 = 48/3$  by using 16/3 as the length of one of the 16 equal parts. I will describe and discuss Dorothy's this particular work in the Sub-sections 1-3 (see Figure 5.11).

It took a while and some help for Brenda to set up the goal of dividing one half by 16, even though she said, "And then you distribute sixteen equally into one half." Brenda produced the same result of one-eighth, even though I proposed ratio reasoning hoping that she would follow this reasoning: If it was one inch and it had 16 pieces in it, how much would each piece be? If it was half inch and it had 16 pieces (in half inch), then how much would each piece be? She continued perceiving the number of parts in one inch as 16, and I told her that she had half of an inch and reminded her that there were 16 pieces in the half inch. This kind of reasoning should have come from her independently if Brenda had produced an abstract unit fraction as the result of taking one sixteenth of a half inch. I know Brenda had a recursive partitioning scheme—For example, before we started the teaching experiments, in the first interview, she could divide a candy bar into 18 pieces with 3-step partitioning. Also, in the sixth grade, she could give a fraction name in terms of the whole candy bar when 1/5 of 1/3 of a candy bar was taken in a sharing context. However, she could not access either recursive partitioning or unit fractional composition scheme in her activities for this problem. Brenda said, "And then you distribute sixteen equally into one half." I inferred that Brenda's goal was the same as that described in Steffe's fractional composition scheme:

The goal of this scheme is to find how much a fraction is of a fractional whole, and the situation is the result of taking a fractional part out of a fractional part of the whole, hence the name composition. The activity of the scheme is the reverse of the operations that produced the fraction of a fraction, with the important addition of the subscheme, recursive partitioning. The result of the scheme is the fractional part of the whole constituted by the fraction of a fraction. (Steffe, 2004, p. 140)

For this particular example, even though the situation Brenda assimilated was a fractional composition, "distributing sixteen into 1/2 equally", Steffe says "She was using

a divisional scheme, which is apparently now not coordinated with the composition of two fractions" (L.P.Steffe, personal communication, January 23, 2007). Steffe's proposed reason for this non-coordination was that there was something preventing Brenda from thinking recursively so that she would have realized there were 32 equal parts in one inch. She might be in a stage of her recursive partitioning in which she needed to see the second part of the 1/2 inch that completed the 1-inch bar. Brenda was using one of the division schemes; *equi-portioning scheme* (whole number division; she conceived 16 pieces as the total number of pieces that needed to be distributed and thought 1/2 inch was one of the two portions, so said there would be "eight" in each portion, and the result would be "one-eighth"). If she had been using the *distributive partitive scheme*, she would have conceived that there would be 16 equal parts in each of the two partitions that composed the whole 1-inch. Even though she started with the distributive partitive scheme (she said "distribute sixteen equally into one half"), she could not reverse her operations to produce the problem situation in which there would be 32 equal parts in one-inch, so there would be 16 equal parts in a half inch. Indeed, this way of reasoning would require more than having a reversible partitive fractional scheme (figuring out the whole length of 1-inch when 1/2 inch is given) and recursive partitioning operation (asking what would be the fraction name of one of the 16 pieces of 1/2 inch in terms of 1inch), because they should have been combined for the goal "to find how much a fraction is of a fractional whole." By then Brenda's activities would aim to compose two fractions (1/16 and 1/2) for finding the result in terms of the whole and she would have constructed a fractional composition scheme.

Sub-Section 1: Problem Situation for Dorothy without 1/2 inch

After seeing that Brenda found the numerical value for one of the 16 equal parts

and thinking that she could then proceed independently, I asked Dorothy to talk about her

activities. Interestingly, Dorothy produced the results for the two sub-quantities without

using the information on the length of the candy bar.

Z: Brenda, now you do whatever you think. Let's look at Dorothy. Dorothy, you have big numbers. OK. Let's see. So this two hundred eight over three [Dorothy writes "in" next to 208/3 and 48/3 on her drawing. See Figure 5.11.] Tell us what you did.

Dorothy: Um. The thirteen over three was the inches over here. And the three over three. Not the inches, pieces over here, and those are the pieces over here [pointing to 3/3 in her drawing]. Altogether there are sixteen pieces, and each little piece is one sixteenth.

Z: Of the whole thing, which is one half inch? Right?

Dorothy: I forgot the half inch.

Z: So, what did you do without using that one half inch?

Dorothy: Um [pauses for 10 seconds]. I got lost.

Z: So, how did you get this two hundred eight over three?

Dorothy: I got the sixteen over three.

Z: Sixteen over three...

Dorothy: Inches.

Z: Inches?

Observer : How long is the whole candy bar?

Dorothy: Um. Point five, or one half inch. Um. Sixteen over three is the whole thing. [inaudible] ... the pieces.

Z: Which is also? How long is that sixteen over three piece in terms of inches? Dorothy: Half of an inch. And um. I multiplied that by thirteen [pointing to  $16/3 \times 13$ ], and the other one by three. That is how I got those [pointing to 208/3 and 48/3 respectively on her drawing.] But they are wrong I believe. And I changed sixteen over three into a mixed fraction. Just because I do not know I just did, and that is what I got. But I forgot about that [pointing to 1/2 inch.]



Figure 5.11. Dorothy's drawings and notations for her solution of Problem 5.17.

Dorothy made a different problem situation (than what I intended) by forgetting the 1/2 inch. Although she stated that there were 16 pieces, and each of those equal pieces was one sixteenth of the whole thing, she could not take into consideration that the whole quantity was 1/2 inch. That she "forgot" it means that it was such a strong perturbation for her that she focused her attention on what she could deal with (number of pieces) and unintentionally suppressed the 1/2 so it was not within her awareness. She conflated the meaning of 16/3 by using it as the measure of one of those sixteen pieces by writing "in," instead of reserving 16/3 as the fraction name for the whole quantity when the shorter part is conceived as 3/3. She then multiplied 16/3 by three and 13 for finding the length of each sub-quantity, respectively. She produced 48/3 and 208/3 as her results, and inserted "in" next to those numbers. She was satisfied with her results, since she accomplished her goal of finding the length of the parts even though she forgot 1/2 inch.

It is important to note that after the observer's questioning of Dorothy "How long is the whole candy bar?" it seemed that Dorothy was aware that the measure of the whole candy bar (half inch) and 16/3 were different names for the same quantity. She said "One half inch. Sixteen over three is the whole thing".

As a summary of this sub-section, we can infer that Dorothy did not know how 16/3 was related to the 3/3, which was actually a multiplicative relationship and she had a strong perturbation after I reminded her "1/2 inch." It seemed she had 16 individual pieces, each of them was called thirds not because 16/3 is 16 times one of those 3/3, where whole 3/3 is important, but because thirteen-thirds was the given fraction name in the problem, so "third" was a name for one of those equal pieces by default as it was the case in Protocol 5.13.

Dorothy believed her results of 48/3 and 208/3 inches were wrong since she had not taken into consideration of the measure of 1/2 inch. My reminder that the whole candy bar was 1/2 inch long might have helped her to question her results. Therefore, Dorothy had a constraint: She did not know how to include this new information of 1/2 inch into her whole-part-part structure. Half of an inch was not a whole number that she was used to dividing into a number of equal pieces. For a while she did not know how to proceed, so she had neither a goal nor an activity; she was perturbed. Even though she did not know how to conceptualize the new problem situation with using 1/2 inch, she knew her results were incorrect, so she had some awareness.

Sub-Section 2: Dorothy's Conceptualizations of 16 over Three (sixteen-thirds) and 16 Pieces

Dorothy's following explanations confirms the inferences I made in the previous sub-section that for Dorothy, thirds were coming from the thirteen thirds; they were just names of the equal pieces.

Dorothy: Now we have sixteen over three [writes 16/3]. Observer: I am confused where the sixteen over three is coming from actually. Can you explain where you got the sixteen over three? Dorothy: Um. There is sixteen pieces and the denominator is three. Observer: Um. the denominator is three, that is where I am confused. Denominator of what? Dorothy: Of the fraction. Observer: Where is that fraction coming from. Dorothy: Thirteen thirds and the three thirds. Brenda: She adds them together to make the whole. Observer: So what is the sixteen thirds of? Sixteen thirds of what? Dorothy: Um. That is the whole thing.

Dorothy thought thirds were coming from thirteen thirds. In addition, she possibly

made the three thirds only using the denominator of 13/3 similar to how she

conceptualized 2/2 in Protocol 5.13. Therefore, when she named sixteen thirds, these

thirds were not a third of 3/3 quantity, on which she could have based all the

relationships. However, a third was just a name of each of those 13 equal partitions.

Z: What are thirds referring when you say thirds?
Observer: Show me with the one, that is sixteen thirds. Show me one third.
[Dorothy circles a third of 3/3 part on her drawing]
Z: Can you show me another one? another one? one more [she circles different thirds on her line segment. See Figure 5.11.]
Observer: If those are one third, can you show me three thirds?
Dorothy: This whole thing [circling the whole 3/3 part on her drawing]
Observer: OK. So what is that sixteen thirds of?
Dorothy: Of this [pointing to 3/3 part on her drawing].
Observer: Is that what we are working with?
Dorothy: I am so lost. I forgot why I did all these.

Even though Dorothy could show three-thirds on her drawing (see Figure 5.11) to make a possible connection between 16/3 quantity and three thirds (it was not independent of the observer's guidance), she actually did not make a connection. She was in a state of perturbation because she wanted to use 16 pieces as the situation of whole-part-part scheme, and the observer was asking her to acknowledge that sixteen thirds were 16/3 of 3/3 quantity.

While Dorothy admitted that she was lost a couple of times in the discussion of sixteen thirds, she made a new conceptualization of the problem situation with only using 16 pieces. On the other hand, Brenda showed flexibility to establish a multiplicative relationship of 16/3 is 16 times as much as one of the thirds of 3/3 quantity; she said "If you are looking at each part being a third of something, then, and there are sixteen of them. There would be sixteen thirds." Establishing this multiplicative relationship between the smaller part and the whole quantity, in this problem, was possible for her because she used it in her activities in *Protocol 5.4* for finding the two unknown whole numbers and in *Protocol 5.7* for finding the length of two sub-quantities of 112-inch string.

Dorothy made an equation to find the length of one of the equal pieces (she wrote "×16 pieces = 1/2 in") while Brenda made explicit the 1/32 she had already found as the length of one of those 16 pieces using an inch (see Figure 5.12).



Figure 5.12. Brenda's drawings and notations for the solution of Problem 5.17.

Sub-Section 3: Dorothy's Confusion on how to Use 1/32 for Finding the Measures of the

Two Sub-quantities

seconds] The whole thing.

After Dorothy set up her goal of finding the numerical value for one of the 16

pieces in 1/2, she said each piece will be 1/32. I asked the same questions that I asked

Brenda: If you had one-inch candy bar and divided into 16 pieces, she said, "Actually

you get one over thirty two... I divided by two... Because, half inch is half of an inch."

Therefore, even though she produced the result of 1/32, this was not independent of my

guidance or discussions of Brenda's finding of 1/32 at the beginning of Protocol 5.14.

Z: OK. So how long will be the parts? What would you do? You can just... Dorothy: I multiply one over thirty-two by sixteen over one. You get sixteen over thirty-two. Z: What will that be? Sixteen over thirty two... Dorothy: Inches. Z: Inches, can you show it to me with your candy bar? What is it? Sixteen... Dorothy: Sixteen over three inches. Brenda: It is the same thing as one half. Z: Wait, wait. You had one thirty seconds times sixteen, can you show me what you did. One thirty seconds of an inch times sixteen, will give you what? Dorothy: Sixteen over thirty-two. Z: Which is... Brenda: One half. Z: Were you intending to find one half inch [to Dorothy] Brenda: [quietly approves] Z: What is this? Dorothy: Sixteen over three. Observer: Yes, I am not still clear on where sixteen over three came from. [Dorothy says "ooo..." with a surprise and erases 16/3 and writes 16/32. See the first line in Figure 5.13.] You are multiplying one thirty second by sixteen. Keith: And what is the one thirty seconds? [Dorothy wrote 16/32 equals 1/2 in at the same time] Observer: So what is that, what is that you found right there? Dorothy: How long one thirty two is...over one over thirty two is. [Pauses for few



Figure 5.13. Dorothy's revised notations for the solution of Problem 5.17.

When Dorothy said one of those 16 pieces would be 1/32, I expected her to use the scheme she had been using for finding the length of the sub-parts, whole-part-part reasoning scheme. If she would use that scheme, she would multiply 1/32 inch (the measure of unit of one) with three and 13, since they were the number of units of subparts. Interestingly Dorothy multiplied 16 by 1/32. I do not know whether she acted intentionally and was aware that the result would be the length measure of 1/2 inch bar. In addition, she conflated the result of 16 times 1/32 of an inch, which is 16/32 inch, with 16/3 she had used at the start (See Figure 5.13). At that point even though, Brenda might not know why Dorothy multiplied 16 by 1/32 inch, she anticipated that it would be the measure of the whole candy. This showed that Brenda was aware of her own, as well as Dorothy's, mathematical activities.

This interaction continued and I wanted to see how Dorothy would proceed:

Z: What was your intention when you multiplied one thirty second by sixteen? Did you want to find the whole thing or did you want to find something else? Dorothy: Because if I was trying to find, thirteen over three, I will multiply it by thirteen. If I was trying to get this [pointing to the small part, 3/3], I would multiply it by three. Observer: OK. So what would you get then? Dorothy: [Dorothy writes  $1/32 \times 13/1=13/32$  immediately] This [writes  $1/32 \times 3/1=3/32$ ] Observer: And what are those? Dorothy: That is this piece and this piece [not seen but probably pointing to the sub-parts of the drawing, see Figure 5.13], which you would add together to get 16 over 32.

Even though Dorothy found the length of one of the 16 pieces with some help, as explained at the beginning of Sub-section 3, she did not make her goal of producing the lengths of the sub-parts using 1/32. She was somehow still focused on the number 16, and when she did get 16/32 as the result of multiplying 1/32 by 16, she indicated that she did not intend to get the equivalent of 1/2 inch, and she wrote 16/3 as the result. Dorothy was again conflating the unmeasured quantity of 16 times one third (one third was probably still the name of one of the 13 partitions of the greater part) with the measured quantity of 16 times the length of one of the equal partitions. There was an equivalence relationship between 16/3 of the smaller part, unmeasured quantity, and the 16/32 (inch), measure of that quantity. However this equivalence, at that point, was a confusing detail since I thought Dorothy's purpose was to find the length of the two-subparts as in the case of whole-part-part reasoning scheme. The reason she multiplied 1/32 by 16 or did not multiply 1/32 by three or thirteen at first for finding the lengths of those two sub-parts can be explained because her only goal was to divide a half inch into 16 pieces to confirm that with 1/32 she could produce the whole quantity (it did not matter whether it was measured with inches or not).

Different than her actions in Problem 5.15, when Dorothy did not know how to write the result of four divided by seven as a fraction (see previous part), in this case

Dorothy did not use the written result of 1/32 as I expected. In addition, different than Problem 5.15 and 5.16, the known quantity (whole candy bar) was also a fractional number. That was too much complexity to deal with for Dorothy.

Therefore, Dorothy did not extend her scheme's situations to include fractional numbers as the length of either the known quantities or unknown sub-quantities. To further confirm this inference and to see Dorothy's independent activities, after Protocol 5.14, I posed this problem to them,

Problem 5.18: Two-thirds of an inch bar has two parts, one part is 2/5 of the other part. How long are the parts?

Dorothy drew a picture of the situation as a line segment with two parts and labeled one of those sub-parts as "2/5" underneath "2/7" and the other part as "5/5" and underneath "5/7". She also wrote "7 pieces" next to the line segment and marked each seventh. She multiplied 2/3 by 7/1 and produced 14/3 (see Figure 5.14).

4/3 4/5 Tpeices

Figure 5.14. Dorothy's drawings and solution for the Problem 5.18.

Dorothy then erased some of her writing and did not know how to proceed. She knew there were seven partitions in the "2/3 in" but was perturbed because she did not know how to find the length of one of those seven partitions. I then asked Brenda and Dorothy to work on this problem at home. When they brought their solutions to the following meeting and talked about their solutions, even though there were results of  $2/3 \div 7/1$  in her paper, Dorothy could not explain the result of  $2/3 \div 7/1$  as a quantity.

In the teaching meetings following March 24, I aimed to enhance both Brenda's and Dorothy's fraction division and multiplication operations, leaving the complexity of multiplicatively related sub-quantities of a known fractional quantity.

As a summary of the previous section, I discussed how Brenda and Dorothy engaged in "Problem 5.17: A half-inch long candy bar is cut into two parts. Find the parts if one part is thirteen thirds as much as the other part." Both students lacked some mathematical constructs. Even though Brenda said "you distribute sixteen equally into one half," she did not act as if she would use a fraction composition scheme and independently produce the result of 1/32 inches. Instead, she used a division scheme in that she perceived the situation as 16 equal pieces distributed evenly into two groups, so her result was eight. Later with my prompts, she said the result would be 1/32 inches; unfortunately, I did not ask for an explanation. Therefore, after this problem I decided to investigate whether operating successfully in fraction multiplication was in her zone of potential construction and how I could help her. Operating successfully means she would produce the result of "distribut[ing] sixteen equally into one half" and this could have happened if she had constructed a fractional composition scheme. My hypotheses for her not producing the result were as follows. First, she did not understand the necessity of completing the whole using 1/2 inch. Second, even if she thought she needed to complete the 1-inch bar, she also needed to operate on two levels of units and to distribute another 16 equal parts into the imagined 1/2-inch bar. Therefore, the whole bar would have been composed of two of the 1/2 inches, each of which had 16 equal partitions. Producing the result in terms of 1-inch is a fractional multiplying scheme and requires reversing a partitive fractional scheme along with operating on 3-levels of units, and using distributive reasoning.

During the solution process of Problem 5.17, Dorothy experienced perturbation three times. She first forgot the length of the whole candy bar because it was a fractional number. She was then confused about how to use 16/3 in her whole-part-part reasoning scheme. After she decided there were 16 pieces in the 1/2-inch bar and each piece was 1/32 in, she did not use 1/32 in to find the length of the two sub-parts. Even though I have a better formed hypothesis on what Brenda lacked (see previous paragraph) when solving this problem compared to my hypothesis about Dorothy's difficulties, I think focusing on what is necessary to be able to multiply two fractions and to use reversible fractional schemes will give an entrance on understanding both students' mathematical constructions. These foci entail investigating such questions as: how does Brenda's conceived situation of "distributing sixteen equally into one half" evolve into such a situation of finding "one sixteenth of one half?" and what are the necessary operations to produce the result for "how much of an inch is one sixteenth of one half inch"? Taking these foci as starting points and extending them with inverse reasoning problems is crucial for making my model of the two students' algebraic thinking. Therefore, in Chapter 6, I will present the analysis of problems related to 3-levels of units, inverse

fractional problems, and students' fractional operations with JavaBars along with their written notations.

### Summary of the Results of Chapter 5

## Fractional Problems with Quantitative Situations

*Reversible fraction schemes for composite numbers.* Both Brenda and Dorothy were able to solve Problems 5.1 through 5.4 by coordinating their fractional and whole number multiplication schemes. Using their activities, I have constructed and attributed to them reversible (partitive and iterative) fraction schemes. Dorothy seemed to be the more competent of the two children in using the two reversible schemes because Brenda did not immediately solve Problem 5.3 in which she demonstrated a reversible iterative fraction scheme for composite numbers.

The students solved the four problems symbolically: their words stood for the mathematical operations, and it took little time for them to produce a result since they operated with symbols. For example, in Problem 5.1—If 2/3 of a sandwich is 20 inches, how long is the whole sandwich?—Brenda said, "Twenty, so it means one third is ten, so it will be ten, twenty, thirty. Thirty." In her activities when solving Problem 5.3— If 6/5 of a candy bar is 48 inches, how long is the candy bar?—Brenda also solved the problem symbolically without explicitly using an unknown. She drew the 6/5 of the candy bar as a combination of two separate bars: the whole candy bar (5/5) and 1/5 of the whole candy bar. She then operated sequentially to find the length of one fifth as 8 inches by dividing 48 by six. Unlike her activities in Problem 5.1 where she said "ten is a third," for Brenda, conceptualizing one fifth of the candy bar as 8 inches was not immediate in Problem 5.3. Her use of notation [6/5 = 48] and drawings are important indications of her use of

symbols in the construction of an iterative fractional scheme involving whole numbers. These graphic items (drawings and notations) functioned differently from the verbal symbols (words) she used in Problem 5.1; instead of reflecting on her activities using words, as was the case for Problem 5.1, she used graphic items to construct a meaningful problem situation in Problem 5.3.

### Multiplicative Problems

*Dorothy's whole-part-part reasoning scheme and construction of an n-part structure.* The two students operated differently in the problems posed on March 7 (Problem 5.5 through 5.8). Dorothy solved all the problems using her whole-part-part reasoning scheme: for example, when solving Problem 5.5—You are given two numbers, one of them is twice as much as the other one. Find the numbers if their sum is 33.— she divided the 33 into three and used the result of this, 11, to find the two numbers. I called the structure she constructed and used in this problem an *n*-part structure, where *n* referred to the number of equal parts she used to divide the whole quantity, such as in Problems 5.6, 5.7, and 5.8.

*Brenda's part-part-whole reasoning scheme*. Brenda used a *guess and check* methodology she learned in the classroom and emphasized the multiplicative relationship between the two unequal parts when finding the two numbers in those problems. I called this a part-part-whole reasoning scheme (Problems 5.5 through 5.7). Brenda made an accommodation in her part-part-whole reasoning scheme in that she constructed and used the *n*-part structure in Problems 5.7 and 5.8. She did not totally forget her part-part-whole reasoning scheme, but modified it by constructing and using an *n*-part structure similar to Dorothy's approach. Therefore, I attributed a whole-part-part reasoning scheme to

Brenda as well.

Whole-Part-Part Problems Involving Fractional Relationships Between the Parts

Dorothy's construction of an n-part structure with proper fractions. In Problems 5.9 through 5.14, the students worked with whole-part-part problems that were similarly structured to Problems 5.5, 5.7, and 5.8. Unlike the previous problems, the two unequal parts were fractional multiples of each other. Although the students did not have difficulty conceptualizing the relation between the parts when unit fractions were given as the multiplicative relationship (they could use their whole number knowledge and could set up the *n*-part structure), they did have difficulty understanding the problem situation when one part was 2/3 as much as the other part, as discussed in Problem 5.10. Dorothy was not sure how to conceive of the relationship between the two unequal parts and how to use her whole-part-part reasoning scheme. Operating with the unit fractional relationship between the unequal parts in Problem 5.11 reoriented Dorothy and she successfully used her *n*-part structure when revisiting Problem 5.10. I discussed the change Dorothy made to her whole-part-part reasoning scheme using Problems 5.10 and 5.11 and how she operated in Problem 5.12 using this change: she now was able to establish an *n*-part structure using the given proper fractions as the relationship between the parts.

*Extending the n-part structures to improper fractional relationships between the parts.* In Problems 5.13 and 5.14, Brenda demonstrated that she could assimilate the improper fractional relationship between the two parts using her whole-part-part reasoning scheme and set up an *n*-part structure to solve the problem. While Dorothy indicated that she could establish an improper fractional relationship between the two

parts, it is not clear how she decided that the shorter of the two unequal parts would be 2/2 in Problem 5.14 (finding the lengths of two parts of a 45-inch string if one part is seven halves as much as the other one). I doubted that, unlike Brenda, Dorothy used her multiplicative ways of operating when conceptualizing the improper fractional relationships between the parts. It seemed that she was attending to the symbols as "seven over two" for seven halves and using the denominator of 7/2 to make the other quantity. Therefore, the change she made to her whole-part-part reasoning scheme in Problem 5.10 could be only a generalizing assimilation instead of a functional accommodation because she did not emphasize the fractional multiplicative relationships between the unequal parts. Rather, she emphasized the number of equal parts in the whole quantity in a way similar to how she operated with unit fractional relationships.

*Experiencing constraints when the lengths of the parts are fractional numbers.* On March 24, using whole-part-part reasoning schemes, Dorothy solved Problems 5.15 and 5.16 by herself. In Problem 5.14 (a 4-inch bar is cut into two parts. Find the lengths of the parts, if one part is 3/4 as much as the other part), after Dorothy decided there would be seven equal parts in the whole 4 inches, she could not decide what fraction to use for 4 inches (the length of the whole candy bar) divided by seven (number of equal parts). Her activities suggested that she did not treat the results of those divisions as quantities in relation to a 1-inch unit. For the result of partitioning 4 inches into seven equal parts to be constituted as a fractional quantity, I would expect the student to engage in distributive partitioning operations and partition each inch in the 4 inch bar into seven equal miniparts, and then combine four of those mini-parts (which can come from each inch) to make 4/7 of an inch.

*Experiencing constraints when the lengths of the whole and the two parts are fractional numbers.* When solving Problem 5.17 (a 1/2-inch long candy bar is cut into two parts. Find the parts, if one part is 13/3 as much as the other part), both students experienced constraints when using their whole-part-part reasoning scheme. While Brenda asserted that she would "divide one half by sixteen," she did not produce 1/32 by herself. She produced "1/32 in" after I asked, "If 1 inch has 16 equal parts, then the length of a part would be 1/16. For 1/2 inch that has 16 equal parts, what would be the length of a part?" This type of multiplication scheme, in which she needed to both reverse her fraction scheme to make the hypothetical 1-inch unit and place 16 equal parts into 1/2 inch, was not available to Brenda.

One-half inch was such a constraint to Dorothy that she appeared to suppress it as the length of all of the parts. She used her whole-part-part reasoning scheme and used 16/3 as the length of one of the 16 equal parts, instead of using it as the fractional quantity of the whole when 3/3 was taken as a reference. In a similar problem, 5.18, when the length of a bar was given as 2/3 of an inch, she attempted to produce a result including both 2/3 and seven (seven was the number of equal parts), but she first multiplied them rather than attempting to divide 2/3 by 7. She did not have anticipatory ways of operating.

Brenda and Dorothy's available operations related to fraction multiplication schemes (as demonstrated and confirmed especially with the problems 5.17 and 5.18) were not sufficient to operate successfully and independently so that a measure of one of the equal parts in the *n*-partitioned whole would be meaningful and functional. For these reasons, in the following teaching meetings, I focused on exploring and enhancing the two students' mathematical activities related to fraction multiplication by asking them to use JavaBars and/or notate their actions using paper and pencil.

# CHAPTER 6: FRACTION MULTIPLYING SCHEMES AND INVERSE REASONING

### Fraction Multiplying Problems

Throughout the analysis presented in this chapter, there were three important operations that helped me to construct the fraction multiplying schemes for Dorothy and Brenda. Those operations were related to partitioning operations (distributive partitioning, recursive distributive partitioning), and construction and coordination of three-levels-of-units structures. The problems that I posed for the investigation of students' fraction multiplying schemes were mainly finding proper and improper fractional parts of quantities. The quantities were sometimes unmeasured (unit) quantities and sometimes they were measured quantities using inches, liters, gallons, etc. In addition, the measured quantities were sometimes more than a unit measure. By using my observations on their activities in these different contexts, I conceptualized their ways and means of operating related to fraction multiplying schemes.

Brenda's Initial Distributive Partitioning Operations for Creating Fractional Parts of

### Fractional Wholes

Problem 6.1: You are given 3/5 of a candy bar. Can you find 1/7 of this bar and figure out how much it is of the whole candy bar? (April 19)

Brenda and Dorothy solved this problem individually. They used JavaBars to construct the bar and notated their steps on paper. After they made the bar, they partitioned it first into three parts. Brenda asked whether she could erase the two marks and make seven parts. I did not allow her to do so, since I did not want her to view the bar as the only fractional whole, and she then partitioned each of the parts into seven mini-parts. Brenda pulled out one of the mini-parts, and asked whether she should have one or three copies. The witness-researcher asked her to restate the problem, and this made her be confident about pulling out three mini-parts. She pulled out two more copies, thereby producing 1/7 of 3/5 of the candy bar as another bar (see Figure 6.1).



Figure 6.1. Brenda's JavaBars produced for Problem 6.1.

Dorothy agreed with Brenda's solution, so I made a decision to focus on Brenda's solution. When I asked Brenda to write down the problem and how she solved it (using JavaBars), she wrote as follows:

<sup>3</sup>15 of 5/5 1/7 of 3/5  
: each 1/5 into seven  

$$3/5 \div \frac{7}{7} = 21$$
  
 $3 \cdot 7 = 21$   
 $3_{121} = 1/7$ 

Figure 6.2. Brenda's written solution for Problem 6.1.

Brenda wrote 3/5 of 5/5 and " $\pm$ [divide] each 1/5 into seven," then multiplied three by seven, and produced 21. Looking at her notations, one would expect her to multiply five by seven and arrive at a total of 35 mini-parts, as opposed to 21. However, later she also labeled a mini-part as 1/21, and it was not by mistake. Brenda focused on the 21 mini-parts in her explanations. Since she produced three mini-parts as the result of her JavaBars activities and each mini-part was 1/21, she wrote 3/21 as 1/7 of 3/5 (see her notation 3/21 = 1/7 in Figure 6.1). It is possible that Brenda only transferred her JavaBars operations onto paper and her purpose was not to compute using the traditional algorithm. Her interest lay in utilizing JavaBars, as opposed to using the algorithm.

Brenda had some awareness that 3/5 was part of another whole, possibly a bar she called 5/5. Even though she divided each part of the 3/5 bar into seven mini-parts, she only conceived of those mini-parts as part of the bar she called 3/5; they were not part of 5/5. When she constructed a relationship based on the idea that 3/5 was part of the 5/5 quantity, she did not have the whole bar in front of her. I arrived at two possible explanations of how she formed 5/5. First, she probably reversed her partitive fraction scheme to make a visualized or symbolic 5/5 bar. If she operated this way, then a fifth in her notation would be a fifth of the 5/5 bar, and, at the same time, one of the three parts of the 3/5 bar. The second explanation is that she did not form a fractional relationship between the 3/5 and 5/5. She conceived the 5/5 in notation and used the denominator of 3/5 for making the 5/5. This means she did not form any quantitative visualization to conceptualize the 5/5. I will discuss further which explanation is more viable, but it is certain that Brenda had some awareness of a relationship between 3/5 and 5/5.

In addition to conceiving 3/5 in relation to 5/5, Brenda constructed a fractional relationship that 1/7 of one of the parts (which she called "a fifth") was one twenty-first of the bar on the computer screen (3/5 bar). Brenda did not use any reversible operations when constructing this relationship. She only coordinated the two different units using a multiplication scheme; each of the three units of the 3-part bar on the computer screen had seven mini-units (mini-parts) per unit. Therefore, the total number of mini-parts was 21. Brenda presumably constructed the 3/5 bar as a fractional relationship to the 5/5 as explained in the previous paragraph. However, she did not take this relationship, 3/5 to 5/5, as an input on which to further operate.

Brenda erected two sets of unit structures. The first set was a two-levels-of-units structure—the five-fifths composed of five of the fifths each of which was, at the same time, one of the three parts of 3/5 bar. The second set was a three-levels-of-units structure, she constructed for sure, was that the 3-part (3/5) bar was the unit containing three (smaller) units and each of those units contained seven mini-units and there were 21 mini-units. These two structures were not coordinated for Brenda since a mini-part was not a part of the 5/5 bar. What this situation means is that Brenda did not use the first structure to construct 1/21, even though she treated each of the three parts of 3/5 bar as a fifth when she wrote "divide each 1/5 into seven." Those parts were called fifths because they were one of the three parts of the bar that she called 3/5. There is no indication that at that point they were fifths of the 5/5 bar. Therefore, one seventh of a fifth was not embedded in the 5/5 bar, but was only embedded in the 3/5 bar. So the resulting quantity (a mini-part) was only part of the 3/5 bar and it was one out of all the visible 21 pieces on her computer screen (see Figure 6.1).

Brenda was not perturbed by this situation since she did not feel that one seventh of each part of 3/5 bar needed to also consistently be one seventh of each fifth of the 5/5 bar. Constructing this relationship by using the same quantity—one of the three parts of 3/5 was a fifth of 5/5— and incorporating this relationship as she operated further were essential to solving this problem. However, they were not a necessity for Brenda. I made this inference because Brenda did not realize that her labeling of a mini-part could also be one thirty-fifth of 5/5 instead of only one twenty-first of 3/5. Because of this situation I decided to ask Brenda to solve a similar problem (see Problem 6.2).

Brenda's Initial Recursive Distributive Partitioning Operations Problem 6.2: You are given 4/5 of a candy bar. Can you make 1/7 of 4/5 of the candy bar and figure out how big it is (of the whole candy bar)? (April 19)

My aim in posing Problem 6.2 was to help Brenda realize that she needed to imagine more mini-parts in order to solve this type of fraction multiplication problem. During the analysis I realized that this meant she needed to reinterpret the result in terms of the whole candy bar. For this reinterpretation, she needed to take the first unit structure as an input to further operate and coordinate it with the second one. Now I will describe what Brenda did with JavaBars and present the protocol related to our discussion of how she tried to notate her operations on the paper for Problem 6.2.

Brenda made a bar with four parts on her computer screen; she then partitioned each part into seven mini-parts. I asked her to color each fifth of the candy bar differently before proceeding further because I did not want her to lose the four parts when she further partitioned those parts. She counted the first seven mini-parts and colored them blue (in the Figure 6.3). She said this group of mini-parts was "one fifth of these four-



fifths which is four-fifths of a whole candy bar."

Figure 6.3. Brenda's JavaBars produced for 1/7 of 4/5.

I asked Brenda whether one fifth of four fifths (of the candy bar) and one fifth of

the whole candy bar were the same or different and our conversation continued as

follows:

Protocol 6.1: Making connection between Brenda's written notations  $(1/5 \div 7)$  and operations with JavaBars.<sup>36</sup>

Brenda: Yeah, it is one fifth of really five-fifths of the whole candy bar. Z: How much is that of the four-fifths?

Brenda: It is one of the fifths, so it is really kind of a fourth if you are looking at this in terms of the other candy bar.

Z: Can you write it down what you just did? You divided each fourth or each fifth of the whole thing into how many pieces?

Brenda: I divided one-fifth into um. Seven...

Z: Mathematically...

Brenda: I divided one fifth by seven pieces or seven over one which is the same thing [she wrote down  $1/5 \div 7$  and placed "1"as the denominator of "7" to make  $1/5 \div 7/1$ ]. And I got seven pieces in each fifth, and for the whole thing I got a twenty...[she puts an equal sign next to  $1/5 \div 7/1$ ]

Z: Do not think about the whole piece yet. So you have one fifth and you divided each into seven pieces, so if you do this [pointing to  $1/5 \div 7/1 =$ ] mathematically

<sup>&</sup>lt;sup>36</sup> Four periods (...) denote omitted dialogue.

what would you get as an answer?

Brenda: Just worked it out? I guess you would get, you would divide it across wouldn't you?

Z: Go ahead, do it.

Brenda: I do not know if it is right.

Z: That is okay. Dorothy, can you help us here?

Brenda: One fifth divided by seven.

Dorothy: How you do it? Five times seven, that is the numerator and... hold on. Five times seven is the denominator [Brenda also joins her and writes 1/35. See Figure 6.4.] and one times one is the numerator.

Z: So, let us stop here, can you pull an amount that shows one thirty-fifth of... can you indicate what this [pointing to 1/35] is of?

Brenda: Of one fifth. I guess because we divided the fifth into seven parts and got one thirty-fifth. I do not really see how we got one thirty-fifth.

Dorothy: Because you divided the one fifth into seven parts [she does not point to anything on the bars].



Figure 6.4. Brenda's notations produced after JavaBar activities in Problem 6.2.

Brenda: Oh yes. So, really. OK. Thirty-fifth is more or less like the whole candy bar, I was trying to think of it as the four. So, if you had the whole candy bar down here and you divided it just like you did it up here, just with one extra piece, because it is five fifths instead of four fifths. So each of these little pieces here, which is the one twenty-eight [pointing to the 1/7 of 1/5], I guess, of the four fifths, um. one over thirty-five. Z: Of?

Brenda: Of the whole candy.

Z: So can you pull out one thirty-fifth?

Brenda: [Pulls out one of the mini-parts from the bar in Figure 6.3]
Even though Brenda did not have a whole bar in front of her, after my prompts she said that one fourth of the bar with four partitions was also one fifth of the whole candy bar of which four-fifths was part. For Brenda, labeling one part of the four-part bar as a fourth or a fifth depended on the reference quantity. Since the given bar was fourfifths of a candy bar, she preferred using "a fifth" in her written notation to represent the one part out of the four parts. However, she lost the awareness that the whole candy bar was the reference quantity when she operated further on the fifth of it. When she wrote  $1/5 \div 7/1$  as representing how she operated with JavaBars, she said "twenty..." and presumably wanted to write the result of this division operation as one of the 28 pieces. Brenda probably thought that one fifth of the candy bar divided by seven and one of the 28 pieces of her bar were the same quantity. Even though the operations-dividing a fifth of a whole candy bar into seven parts or dividing a fourth of four fifths of the same candy bar into seven parts—produced the same quantity, a mini-part, stating the result as part of the candy bar or as part of the four fifths of the candy bar were different. Brenda lost the reference quantity of the whole candy bar when she took the second level of unit, a fifth, as a given and operated on it to find the third level of unit, which was one of the miniparts. For her, the result of taking a seventh of a fifth of a whole bar (which was at the same time a fourth of four fifths of the bar) was one of the 28 mini-parts of the bar that was in front of her; the result for  $1/5 \div 7/1$  was 1/28. She was not perturbed even though she said, "And I got seven pieces in each fifth, and for the whole thing I got a twenty [she puts an equal sign to  $1/5 \div 7$ ]..." I stopped her before she completed her notation of the result. I wanted her to rethink her measurement of a mini-part, which she anticipated as one twenty-eighth of the bar on her screen. Therefore, I asked her to compute the division operation. With this request, I hoped Brenda would be perturbed when she compared the result of the computation (1/35) and her production of the result with JavaBars (1/28).

Brenda was not sure how to compute the division of the two fractions. She asked, "I guess you would get, you would divide it across wouldn't you?" I thought Dorothy would have known how to compute  $1/5 \div 7/1$ . Because of that reason I asked for help from her. After they got the computational result of 1/35 together, Brenda confessed that she did not know why the result was 1/35. This was probably because she did not see all of the 35 pieces on the computer screen. Dorothy said, "You divided the one fifth into seven parts" without pointing to any of the configurations on her own or Brenda's screen. Since the camera was not focused on where Dorothy was looking, I cannot exactly tell what she focused on when she made that comment. For Dorothy, I believe the situation was straightforward: One fifth was algorithmically divided into seven and the result was 35. She did not know how Brenda decided to use "one fifth" in her writing and the process of labeling "a fifth" with the bars. Dorothy was not involved in that part of the discussion. Therefore, it is a strong possibility that Dorothy only looked at the written notations and read them as they were written. Meanwhile, Brenda was perturbed because she wanted to make a connection between the written notations and her operations with the bars on the screen.

In Problem 6.1, Brenda was not puzzled at all since the resulting quantity was one of the 21 pieces and it was also one of the seven partitions of one fifth of the candy bar. Her notations of the quantities and operations were based on her operations with the bars. In this problem, she also based her written acts on her operations with JavaBars. However, she could not observe Dorothy's imposed written result of 1/35 with the bars. It is important to note that Brenda always wanted to use both the written notations and the operations with JavaBars to explain one with the other. Unlike Brenda, Dorothy did not feel a need to combine or explain her JavaBars activities and written activities as cohesive. After Dorothy's explanation, Brenda commented as follows,

Oh, yes. So, really. OK. Thirty-fifth is more or less like the whole candy bar, I was trying to think of it as the four. So, if you had the whole candy bar down here and you divided it just like you did it up here, just with one extra piece, because it is five-fifths instead of four-fifths. So each of these little pieces here, which is the one twenty-eight [pointing to a mini-part], I guess, of the four fifths, um. one over thirty-five.

This comment shows that Brenda realized she had to complete the 4/5 of the bar with another fifth that had seven mini-parts. Dorothy's language provoked Brenda's partitioning scheme, so she partitioned each fifth into seven mini-parts. The result was not justified only because Brenda divided 1/5 by seven algorithmically (as Dorothy explained it for 1/35): Brenda imagined having five of the fifths, each of which had seven mini-parts. Brenda reinterpreted one seventh of one fifth as part of the five fifths (whole candy bar). This means she took the three-levels-of-unit structure for which 4/5 was one of the levels as an input and then operated further on it to reinterpret a mini-part.

This connection is an advancement on Brenda's part since she is aware of something new: One twenty-eighth of 4/5 of the bar is same quantity as the 1/35 of 5/5 bar and it is also the result of  $1/5 \div 7$  (see the transcription of the last two or three exchanges in Protocol 6.1). She learned how to reinterpret one of the mini-parts in terms of the whole. This way of operating is a functional accommodation that she achieved while solving the problem; it is also permanent, as I will present other cases when discussing Problems 6.4 and 6.5. The means by which she conceived one of these mini-

parts (one seventh of one fifth of the candy bar in terms of the whole candy bar) and her means of labeling the mini-part (distributing seven partitions to each fifth of the candy bar or multiplying seven by five) resulted in a new way of operating with one part of the 4/5 part bar. For this kind of labeling she also needed to construct the whole bar using only its 4/5. The fact that Brenda justified the result of 1/35 indicates that she coordinated the 2 three-levels-of-units structures and was aware of the operations that produced this coordination. She produced the figurative material of an extra fifth of the candy with seven mini-parts in her visualized imagination. She coordinated one of the mini-parts as one seventh of one fifth of the candy bar—one out of the 35 mini-parts—and one seventh of one fourth of 4/5 of the candy bar. The whole candy bar was a unit of five units (each of which contained seven mini-units) and also 35 mini-units, and the 4/5 of the candy bar was a unit of four units (each of which contained seven mini-units) and also 28 miniunits. Therefore, these two structures were composed of the three levels of units that Brenda constructed symbolically. They were symbolic because she did not need to carry out the coordination activity on the bars.

In the following problems, I present Brenda's and Dorothy's experiences with measurement units they had to imagine (e.g., liters, inches, etc). My first aim in analyzing these problems is to present how Brenda's accommodation in Problem 6.2 helped her to take three levels of units as a given, thereby producing another three levels of units and coordinating those two unit-structures in constructing a fraction multiplying scheme. My second aim is to provide details regarding why Dorothy constructed only a beginnings of a fraction multiplying scheme by emphasizing the differences in these students' operations. Problem 6.3: My water bottle holds 3/5 of a liter and yours holds 2/3 as much as mine. Can you make the water bottles with JavaBars and figure out how much your bottle holds? (May 11)

Problem 6.3 is a relatively simple fraction multiplication problem since the students did not need to produce mini-parts. With this problem, I present how Brenda and Dorothy conceived of the situation differently at the start, but later produced the same quantity and then the same result in terms of a liter. Even though Dorothy operated as if she coordinated two different unit structures, I suspect that she produced the result without referring to an imaginary whole liter. Because of this assertion, analyzing the problems following Problem 6.3 is crucial for making insightful inferences about Dorothy's (and also Brenda's) three levels of units and fraction multiplying schemes.

Each student made (drew) a bar on her own computer screen. Dorothy divided her bar into five parts and colored the top three parts blue to show, presumably, the 3/5 stated in the problem. On the other hand, Brenda did not take any actions on her bar. She attempted to conceptualize the problem verbally and our conversation continued as follows:

Protocol 6.2: Making 2/3 of a water bottle that holds 3/5 of a liter.

Brenda: It holds three-fifths or you have three-fifths of a liter in your bottle?
Z: So, it holds... What is the difference?
Brenda: If the bottle, its limit to any kind of water, is three-fifths of a liter or it has three fifths of a liter right in it [inaudible].
Z: Oh. No. It holds just that amount, it does not have any empty space. Everything here is [pointing to the unmarked bar], whole three-fifths of a liter. So, there is no empty space.
Dorothy: [Clears all four marks and puts two marks in her bar.]
Brenda: [Makes three parts in her bar subsequently referred to as 3-part bar.]
Dorothy: And ours holds what?
Z: Two-thirds of whatever mine holds.
Dorothy: [Pulls out one of the parts from her 3-part bar.]

Brenda: [Makes another copy of her bar.] Can I pull out instead? [She pulls out two parts from her bar at different times and produces a new bar by connecting those parts]

Dorothy: [REPEATS the part she pulled out from the 3-part bar one more time.] And this is three-fifths?

Z: Yes, three-fifths of a liter.

Dorothy: And ours is. . . I need to fill...[Colors the bottom two parts of the 3-part bar blue. She probably transferred the amount of the new bar to the 3-part bar, see Figure 6.5]

Z: OK? So, this is how much yours holds, right? [Pointing to Brenda and Dorothy's new bar in their screen]. So how much is that of a liter? Brenda: Two-fifths of a liter.

Dorothy: [Almost unison with Brenda] Two-fifths of a liter.



Figure 6.5. Dorothy's bars: (a) 3-part bar and (b) 2-part bar.

It took a while for Brenda to conceptualize the situation that arose in the problem. She was unsure whether the bottle's capacity was a whole liter but was not completely filled or whether the bottle was full but it was less than a liter. She was trying to combine two important features of the situation: the capacity of the water bottle and its measurement in terms of a liter.

Brenda conceived the situation in Problem 6.3 as different than the situations in Problem 6.1 and 6.2. This might be because the unit measure of one liter had to be imposed as an important part of the situation. In the previous problems, the given bar and the resulting quantity were part of a candy bar. Therefore, making the whole bar using JavaBars and recursively partitioning each part of the whole bar could be accomplished by using any parts of the given bar. However, for this problem and the following ones involving measurement units, Brenda needed to conceptualize both the capacity of the bottle and the quantity's measurement in terms of a liter. There was no full water bottle or whole candy bar that could provide a reference. A liter was both the unit measure and the reference at the same time and it was not in the students' visual field. One of the most important implications of having one liter as a reference is that the resulting quantity must be in terms of this unit measure. The result should not be in terms of a water bottle or a candy bar. To indicate this kind of awareness, I would expect a student to ask questions similar to those that Brenda asked.

Once Brenda grasped the problem situation, she operated on this problem in a way similar to the first two problems, except there was no need to make mini-parts. She pulled out two parts from the 3-part bar at different times and produced the resulting quantity of 2/3 of 3/5 of a liter. In contrast to Problem 6.2, when Brenda labeled the result in terms of a liter, she could rely less on the bar in her visual field. I think Brenda's questioning was significant and it opened up the possibility for her to imagine how to make the whole liter using 3/5 of it. Once she was satisfied with my answer that the bottle held 3/5 of a liter, she proceeded. This situation made me believe that she conceived of 3/5 of a liter and the water bottle as one entity to start with and to operate on. It is possible that an image of a liter of water was the capacity of another water bottle that held more water than her own water bottle. So the water bottle used for the measurement unit of a liter was figurative material, and she used 3/5 of it to construct figurative material for her own water bottle. My hope was that with this kind of questioning, she would become aware of some of the operations (such as reversible operations) that are

needed for constructing three levels of units symbolically, which is essential for the construction of a fraction multiplying scheme.

Dorothy, on the other hand, did not question the problem's situation. She first partitioned her bar into five parts and then colored three of them to make 3/5. She wanted to have the whole five-fifths in front of her. However, I am not sure whether the bar with five partitions constituted a liter for Dorothy or whether it was just a bar to show three parts so she could operate on the parts to solve the problem. After listening to Brenda's questioning, Dorothy changed her bar configuration to illustrate only three partitions. It is still questionable whether Dorothy's 3-part bar constituted 3/5 of a liter for her or not. Like Brenda, Dorothy also pulled out one part from the 3-part bar and copied this one more time and said the resulting bar was two-fifths of a liter. Based on this result, it would seem that the 3-part bar was 3/5 of a liter for Dorothy. However, unlike Brenda, Dorothy completed one more step before she stated the result in terms of a liter. After Dorothy made the new bar with two parts, she colored the bottom two parts of the 3-part bar in blue (see Figure 6.5). I inferred that she transposed this amount (2-part bar) to the portion of the 3-part bar. A possible explanation for why she said the result was twofifths of a liter might be that each part in the 3-part bar was called a fifth of a liter. Therefore, those two parts in the 3-part bar were two-fifths of a liter, so the resulting bar with two parts was two-fifths of a liter.

It is a possibility that Dorothy never thought of the parts of the 3-part bar as part of a liter and did not use this information when stating the result. I also investigated in detail her use of measurement units in the previous section's problems. Although I may not be able to present more evidence related to this particular problem, it was obvious

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that in the previous problems, measurement units in her written solutions did not have any function other than being a placeholder whenever Dorothy thought they were necessary to use. This detail is important since her use of language might make us think that she constructed and operated on 2 two-levels-of-units structures of which the first structure would be the 3-part bar composed of three units and of which the second structure would be the whole liter composed of five of the (smaller) units. I argue that language alone is not enough to decide whether a student operates with especially the second units-structure when finding 2/3 of 3/5 of a liter. I further discuss this issue of using measurement and its role in the construction of a fraction multiplying scheme with the following problems.

Three-levels-of-unit Structures for Fraction Multiplying Schemes

In the following problems related to three levels of units, I realized that Dorothy and Brenda were operating differently and that they constructed different fraction multiplying schemes. I used Problems 6.1, 6.2, and 6.3 to analyze the data concerning how Brenda acted when engaging basic fraction multiplying situations. The analyses of the following problems build on those actions in extended situations. In Problem 6.4 (4/7 of 3/5 of a liter), Brenda took a fractional part of a fractional part of a hypothetical whole; in Problem 6.5 (7/6 of 4/5 of a liter), she produced an improper fraction of a fractional part of a hypothetical whole; and in Problem 6.6 (3/5 of 11/6 of a liter), she took a fractional part of an improper fraction of a hypothetical whole. With the analysis of those problems, I discuss Brenda's partitioning operations, her construction of 2 three-levels-of-units structures, and her coordination of those structures. I based my construction of Brenda's fraction multiplying scheme on those discussions. The accommodation she

made in Problem 6.2 has an important role in her advancement as she solved these problems.

On the other hand, in Problem 6.1, 6.2, and 6.3, I had limited access to Dorothy's ways and means of operating, primarily because she agreed with Brenda's solutions and I inquired more into Brenda's thinking. Fortunately, Dorothy received more opportunities to comment on her solutions in Problem 6.4, 6.5, and 6.6. Dorothy's activities and operations in Problem 6.4 helped me understand how a student could conceive of fraction multiplication as a series of operations that are carried out only on visual materials. Dorothy (as well as Brenda) produced the correct quantity and constructed a three-levelsof-units structure in Problem 6.4 by taking 4/7 of a 3-part bar. However, Dorothy did not conceptualize the result in terms of a measurement unit. Not producing the result in terms of a liter could be expected because she did not construct the second three-levels-of-units structure and therefore she did not have another structure to do any coordination with the first three-levels-of-units structure. Later in Problem 6.5, Dorothy produced an improper fraction of a 4-part bar by creatively using a partitioning scheme and whole number multiplication. In this sense, she took the three-levels-of-units structure (which she constructed in Problem 6.4) as a given and operated on it so that she could produce an improper fraction. It was in this way that she extended her distributive partitioning scheme to include situations such as producing 7/6 of a 4-part bar. However, she still did not state the resulting quantity using the measurement unit. Interestingly, in Problem 6.6, Dorothy took 3/5 of 11/6 of a liter and stated the resulting quantity in terms of a liter. Her successful actions were mainly due to her conceiving of a part in her 11-part bar as also a part of a liter, and so establishing a part of the 11/6 bar as representative of a liter. That is, she could construct six of the 11 parts as the reference unit, 1 liter.

Just after the students solved Problem 6.3, I asked them to erase everything on their computer screens except the original 3-part bar. I posed Problem 6.4 thinking that they would have to operate differently than with Problem 6.3 and would need to construct a measurement unit of a liter to answer the problem.

Problem 6.4: If my water bottle still holds 3/5 of a liter and yours holds 4/7 of mine, can you make your water bottle and figure out how much it is of a liter? (May 11)

Once Dorothy had her 3-part bar on her computer screen, she erased the two marks on the bar and partitioned it into four parts, making the bar into five equal parts. When she did this, I told her that "you start with 3/5 of a liter." I wish I had not interrupted her and had waited patiently to see how she would proceed since her actions could have indicated how she conceived the measurement unit of a liter in the problem situation. After my advice, she unmarked the bar completely and repartitioned it into three parts. Dorothy then partitioned each of the three parts into seven mini-parts and colored each group of three mini-parts alternately blue and red (see the left bar in Figure (6.6). She then pulled out a mini-part and copied it two more times to make 1/7 of the 3part bar. Afterward, she copied this group two more times, joined these three groups vertically, and colored them blue and red alternately. Subsequently, she pulled out one of the mini-parts and used the *Repeat* button to make three more connected copies of it. Next, she joined this group of mini-parts to the three groups that she previously made. So, she constructed 4/7 of the 3-part bar (see the right bar in Figure 6.6). During her constructions, she did not talk.



Figure 6.6. Dorothy's bars produced during the solution of Problem 6.4.

Concurrent with Dorothy's construction process, Brenda was engaged in her own construction. Brenda began by partitioning the bottom two parts of her 3-part bar into seven mini-parts on her computer. Once she did this, I asked her to color each fifth (of a liter) differently. She then colored the first seven mini-parts black and the second seven mini-parts red. She partitioned the last third into seven mini-parts only after she colored it blue (See the left bar in Figure 6.6.1). When she completed partitioning and coloring the 3-part bar, she began talking to herself about what she wanted to do next. Unfortunately, her speech was not audible enough to follow.



Figure 6.6.1. Brenda's bars produced during her solution of Problem 6.4.

Explanation (6.6.1). On the left, her 3-part bar with 21-miniparts. On the right, her resulting bar.

After talking to herself, the first thing Brenda did was to pull out one of the mini-

parts and copy it three times. She said, "That is one seventh." and continued as follows:

Protocol 6.3: Producing 4/7 of 3/5 of a liter.

Brenda: There is another seventh [Makes another group of 3 mini-parts by copying a mini-part two more times].

Z: How many copies would you have?

Brenda: Of these three, you would have four of the three. Do you want me [to] keep copying?

Z: You can join these and copy the whole thing [Pointing to the group of 3 miniparts].

Brenda: [Brenda made a total of three more groups of three mini-parts]. Now this is my water bottle. Right here... [Pointing to the all four groups of three mini-parts that are not connected. Wants to use JOIN button but the program did not cooperate, so she stacked all the mini-parts on top of each other. See the right bar in Figure 6.6.1]. Because first we had three fifths of the water bottle, here is the one, two, three of the fifths. And mine held four sevenths of this three fifths, so I divided each piece into seven because I needed something I could pull out or it will go into seven equally. But you will still see the thirds. So then that gave me twenty-one pieces. Because seven times three is twenty-one. And so, then I pulled out three of these [pointing to the mini-parts] because seven times three is twenty-one. And that will give you one seventh of this whole thing [pointing to the left bar in Figure 6.6.1].

Z: So, Dorothy, how much is that of a liter, the one that you found? Dorothy: Four-sevenths.

Z: This was [pointing to the bar on the left in Figure 6.6] three-fifths of a liter, you remember?

Dorothy: Oh, yes. That is three [points to the bar on the left in Figure 6.6 but it is not clear what part of it she points to] that looks like one and six sevenths of... the three... fifths of a liter.

Z: Say that again.

. . .

Dorothy: One and six sevenths of three fifths of a liter. No, um, five sevenths... of a liter. That is one and I divided [it] into seven [pointing to the bar on the left in Figure 6.6]

Z: Can you color the fifth in your bar? [Dorothy colors the bottom seven miniparts of her left bar in Figure 6.6]. So, now we are trying to figure out how much this is of a liter. This black part, how much is this of a liter? Dorothy: It is one fifth of a liter.

Z: [Ten seconds pass] So one of these [pointing to the bottom mini-part of the left

bar in Figure 6.6], how much is that of a liter? Dorothy: One...[Fifteen seconds pass] one twenty first. One over twenty-one. Z: Of a liter? Dorothy: We still look at this as three-fifths of a liter? Z: Yes. Dorothy: Seven times three—yes, it is one over twenty-one. Z: Why would that be? Dorothy: Because I divided each third into seven. No. Yes, something like that. Brenda: [Eight seconds pass.] Wouldn't be a ... because you have five pieces and you divide each piece into seven. It would be thirty-five. Thirty-fifth. Z: What do you think, Dorothy? [She does not answer] Brenda: Of a liter, this would be one, two, three, four, five, six, seven, eight, nine...twelve over thirty-five, and that is how much it is of...no, this is the one liter [pointing to the left bar in Figure 6.6.1] No. no. no. Dorothy: No, that is three fifths of a liter. Brenda: So, it would be. So the one whole liter will be five fifths right? So, yes, five times seven is thirty-five and then, twelve of those are this [pointing to the bar on the right in Figure 6.6.1]. And we want to find how much this is of one liter. So, it would be twelve out of thirty five...of a liter. Z: So, Dorothy, can it be twelve over thirty-five? Dorothy: Yes, when you look at it from a liter.

Both Brenda and Dorothy acted with the same goal in mind: finding 4/7 of a 3-

part bar. They both started partitioning each part of their 3-part bar into seven mini-parts.

Afterward, Brenda kept the three parts with her color scheme by coloring each group of

seven mini-parts differently (see the left bar in Figure 6.6.1). In contrast, Dorothy colored

each group of three mini-parts in her bar alternately blue and red. So, Dorothy

transformed a 3-part bar with seven mini-parts per part to a 7-part bar with three mini-

parts per part (see the left bar in Figure 6.6). Then, both students pulled out one of the

mini-parts and made three copies. Brenda explicitly said that the group of mini-parts was

one seventh of the 3-part bar. Before operating on the bars, she said she needed to make a

total of four groups of three mini-parts for her water bottle. Dorothy acted basically the

same way and produced 4/7 of the bar by copying and joining four groups of three mini-

parts.

Both students acted as if they anticipated the partitioning, distributing, and iterating operations when producing 4/7 of the 3-part bar. Brenda was more reflective than Dorothy on her actions and how she used visual materials. After Brenda constructed 1/7 of the 3-part bar, she operated visually and iterated this quantity three times more, producing 4/7 of the bar. She was at a stage of constructing iterating operations symbolically because she anticipated producing the resulting quantity and was reflective about her anticipated actions. Imagining iterating a quantity might not be a very complex symbolic operation, but it is a fundamental operation in a fraction multiplying scheme, and Brenda became aware of and reflective about it.

Dorothy might be at a similar stage in her use of iterating operations. She was not as reflective as Brenda, so I can only infer her anticipated operations from her confidence in her actions. Dorothy anticipated partitioning the 3-part bar into seven mini-parts per part and transformed the 3-part bar into a 7-part bar with her coloring scheme. Even though Dorothy anticipated each of her actions, she always acted on the material in front of her. She did not seem to imagine the result of her anticipated actions. Imagining the result of anticipated actions might allow a student to become more cognitively aware of her actions and operations. This awareness might help students use notation or words to stand in for the actions and produce a result of the actions in visualized imagination without actually manipulating the visual materials.

While finding 4/7 of the 3-part bar as a quantity, both Brenda and Dorothy produced a structure using the bars. From their coloring scheme, it looks like they conceived the unit that contained the other levels of units differently even though it was the same quantity. For Dorothy, the quantity that was the 3-part bar was transformed to a

unit that contained seven (smaller) units. Each of these seven units contained 3 mini-units (or parts). For Brenda, the 3-part bar was the unit that contained three units and each of those units contained seven smaller units. This difference in conceiving the second level unit, whether it is seven or three units, is important for reinterpreting and coordinating any parts of the 3-part bar as part of the measure unit of a liter. If the 3-part bar is not changed to a 7-part bar when taking 4/7 of it, in my experience with these two students, it is more likely that the student will relate it to a liter; otherwise, she will be perturbed. As I observed with Brenda's activities, the general implication of this difference in the conception was to extend her fraction multiplying scheme so she could produce not only fractional parts of whole numbered quantities, such as 4/7 of a 3-part bar, but also produce measurements of fractional parts of a fractional part of a unit such as 4/7 of 3/5 of a liter.

I asked Dorothy how much of a liter her answer was and she said, "Foursevenths." It was apparent that Dorothy conceived the problem situation and operated without the measurement unit of a liter. Therefore, the only whole for Dorothy was the initial 3-part bar, which she transformed into a 7-part bar, so that the resulting bar was 4/7 of that whole. Since the bar with three partitions constituted the reference quantity and did not have any relationship to the unit liter, Dorothy was neither perturbed with my question nor with her own answer. The answer of "four-sevenths" was straightforward to her. On the other hand, not taking the external measurement unit of a liter into consideration did not deter Dorothy from using partitioning, distributing, and iterating operations and from producing three levels of units so she could take a fractional part of a whole numbered quantity (4/7 of the 3-part bar).

Those operations were as follows: Dorothy first *distributed partitioning* into seven parts across each part of the 3-part bar. She then colored every three mini-parts alternately and transformed the bar into a 7-part bar and then pulled out three mini-parts (see Figure 6.6). Since she was aware that the group of three mini-parts was a seventh of the 7-part bar, she iterated that group and produced a total of four copies to construct 4/7of the bar. Therefore, the three-levels-of-units structure (a bar composed of seven units where each unit consisted of three units [mini-parts]) she constructed can be thought as a product of Dorothy's fraction multiplying scheme (4/7 of the 3-part bar). Dorothy definitely could find a fraction of a whole number but even in this case, it is problematic whether she could have interpreted her result as 12/7 of one of the three original parts. Had I asked her to make this interpretation, I believe that she would have been able to do so with guidance, but whether the interpretation would have been a result of logical necessity is problematic. It is problematic because she transformed the 3-part bar to a 7part bar, so she changed the number of mini-parts in the parts and there was little indication that a mini-part was a seventh of one of the parts of the original 3-part bar. Still, all of her operations were anticipatory due to the fact that she was not randomly exploring the possibilities for finding the fractional parts. She acted as if she knew what she needed to do and in what order before even taking any actions.

When I asked Dorothy how much of a liter her answer was and she said "foursevenths," I did not know what Dorothy's difficulty was when she stated the result as 4/7. So, I pointed to the left bar in her configuration (see Figure 6.6) and reminded her that the bar was 3/5 of a liter. Dorothy then changed her response to "one and six sevenths of three fifths of a liter." I think Dorothy only focused on how to rename the bar with seven parts (which was at the same time 3/5 of a liter). The 7-part bar was composed of one seventh and six sevenths, so her answer was "one and six sevenths of three fifths of a liter." Just after this answer, she changed it to "one and five sevenths of a liter." I do not know why she changed her answer, but the new name emphasized "of a liter" instead of "of three fifths of a liter." A possible explanation for this situation might be that she interpreted the right bar in Figure 6.6 using the parts of her original 3-part bar.



Figure 6.6. Dorothy's bars produced during the solution of Problem 6.4.

There are 12 mini-parts in the right bar and each mini-part was made by partitioning each of three parts into seven parts, so each mini-part was possibly 1/7. Therefore, there were 12/7 in the right bar, or 1 and 5/7 of one of the original parts. Her use of "liter" is an indication that she did not conceptualize that it was 12/7 of 1/5 of a liter. Unfortunately, I did not follow up on her answer. Instead, I asked a different question. I asked Dorothy whether she could color a fifth of a liter in her 7-part bar (the left bar in Figure 6.6). I thought if Dorothy referred to the measurement unit of a liter by coloring a fifth of a liter black in the 7-part bar, she would then become perturbed. I

anticipated that the perturbation would activate an attempt to coordinate the quantities to their measurement in terms of a liter. If she had realized that the 7-part bar was not the only whole, since it was part of another whole (3/5 of a liter), she would have begun to search for an operation to coordinate the black-colored quantity and its measurement (a fifth of a liter).

Dorothy colored the bottom seven mini-parts black as a response to my question. Seeing that she could color a quantity black for a fifth of a liter, I assumed she imagined creating the whole liter using the black colored quantity. I thought she constructed an operational image of a liter by iterating the fifth quantity five times. Therefore, I asked her a question that she could possibly explore meaningfully if she had such an image: "How much is that [one of the mini-parts] of a liter?" I was quite surprised with her straight answer of "one twenty-first." When I asked her whether it was "of a liter," she responded with a question: "We still look at this as three fifths of a liter?" Even though I said "Yes," her explanation did not indicate that she considered the liter in the way I thought because she gave the same explanation: "Seven times three—yes, it is one over twenty one."

For Dorothy, a mini-part, out of all the visible ones on the computer screen, was part of a liter. She was not aware that she used "one over twenty-one" for the measurement of a mini-part that was 1/35 of a liter. I made two possible explanations for this situation: First, Dorothy did not seem to imagine producing the whole liter using the black colored quantity—a fifth of a liter— nor the whole bar that was named as 3/5 of a liter. Therefore, a fifth of a liter possibly did not have a fractional meaning in her activities, since there was no operational image of a liter in her mind. Instead, it was only an amount that constituted one of the three equal parts of the 3-part bar. Therefore, a mini-part could be only part of the 3-part bar (since there was no other conceived whole such as a quantity representing a liter), and it was one of the 21 mini-parts inside the bar.

My second possible explanation is that, since Dorothy colored the fifth of a liter in the 7-part bar when I asked, she might have some fractional meaning for the black colored quantity in terms of a liter, so a possible image of a liter container. However, she did not take this measurement—1/5 of a liter— as an input to operate on further to conceive one of the mini-parts as part of a liter. Dorothy might have lost the coordination of the measurement and the quantity at the third level: a mini-part. This situation is possible because she did not attempt to distribute more mini-parts into the imagined extra two fifths of a liter either in words or in her actions with the bars. To be able to conceive the measurement of any mini-part in the problem in terms of a liter, in addition to Dorothy's reversing her partitive fraction scheme to produce the whole liter of five fifths (using its 1/5), she needed to extend her distribution operations to partition the extra fifths for seven mini-parts per part. Therefore, Dorothy constructed only the first unit of units of units structure, when producing 4/7 of a 3-part bar, and did not coordinate a mini-part to its measurement in terms of a liter.

Brenda successfully carried out the operations described in the previous paragraph; therefore, they are the basis for my conceptualizing what is necessary to construct a *fraction multiplying scheme* for creating fractional parts of any fraction. Now I will present the details of the operations Brenda used in the construction of such a scheme. By emphasizing the fifths and thirds in her explanations, Brenda indicated that the 3-part bar was not only the fractional whole that was partitioned into seven equal parts but it was also related to another quantity. She said,

Because first we had three fifths of the water bottle, here is the one, two, three of the fifths [pointing to the parts of the 3-part bar]. And mine held four sevenths of this three fifths. So I divided each piece into seven because I needed something I could pull out or it will go into seven equally. But you will still see the thirds, so then that gave me twenty-one pieces.

Even after partitioning the 3-part bar into seven to make one seventh of it, Brenda wanted to have "the thirds" visible. Her persistence in maintaining the "thirds" when finding 4/7 of the 3-part bar was necessary for her to imagine how to generate a whole using 3/5 of it. It is possible that, at that time, Brenda even considered that each part of the 3-part bar constituted a fifth of a liter and she produced an operative image for a liter. In the following paragraphs, I further investigate how Brenda operated with this awareness of keeping "thirds" visible in her solution.

Approximately eight seconds later, after Dorothy claimed one of the mini-parts would be "one twenty-first," Brenda said, "Wouldn't it be because you have five pieces [five parts in a liter] and you divide each piece into seven, so thirty five [mini-parts]... this will be one, two, three, four. . . twelve over thirty-fifth [for the resulting bar]." She conceived the new problem situation as producing the whole liter. The liter consisted of five parts and each part was partitioned into seven mini-parts. Brenda imagined partitioning each of the extra two fifths of a liter into seven mini-parts without having a whole liter in front of her. Therefore, the operation Brenda used—a recursive distributive partitioning—was not only a distributive partitioning operation, but it was a more sophisticated operation. She used the operations that produced another unit of units of units structure symbolically. The containing unit of a liter had five units and 35 miniunits. In addition, Brenda also coordinated the two three-levels-of-units structures, so she could give an explanation for twelve thirty-fifths of a liter as a measurement of 4/7 of the 3-part bar.

As I explained above, even though the starting situation was to find 4/7 of a 3-part bar, Brenda was aware that the 3-part bar was not the only whole she needed to operate on when finding the measurement of the resulting quantity in terms of a liter. My claim is that this awareness is necessary for a student to search for a reversible fractional scheme and to operate recursively so she can produce a fraction multiplying scheme. In this way, the student will construct a different and a more sophisticated three-levels-of-units structure than the three-levels-of-units structure that is necessary for creating 4/7 of a 3part bar. Even though both students used similar partitioning operations in the construction of three-levels-of-units structures, I differentiate those as *recursive distributive partitioning operations* (for the construction of three levels of units with symbolic operations as Brenda did) and *distributive partitioning operations* (for the construction of three levels of units necessary for taking fractional parts of whole quantities).

In the literature, Steffe (2004) defines a recursive partitioning operation as the one the students used when the goal of their activities was a non partitioning goal. The students in his study, Jason and Laura, aimed to find 3/4 of 1/4 in terms of a 4/4 – stick (4-part bar). Steffe (2004) explains that if the student has only a reversible partitive fractional scheme, then he or she would be perturbed and would be in the "search mode induced by the perturbation with no action to perform." He conceptualized the recursive partitioning operation when Jason overcame this type of perturbation. This operation was the basis for Steffe's construction of the fraction composition scheme.

The situation Jason conceived and operated on was as follows: There was a 4/4stick (or 4-part bar), and Laura pulled out one part from the stick, and then partitioned this part into four mini-parts and pulled out three mini-parts. Jason said that the result was 3/16 and pointed to each part of the 4/4-stick and counted "4,4,4, and 4 –16. But you colored 3, so it is 3/16." Thus, to be able to describe an operation as recursive partitioning, one must observe three important conditions: students need to (1) have a reversible partitive fractional scheme, (2) be in a state of perturbation, and (3) implicitly distribute the same number of partitions into the other parts of the fractional whole.

In some sense, my use of the term "distributive partitioning operations" fits into Steffe's definition of recursive partitioning operations in the sense that in Problem 6.4, the students' goal was not to partition a part when they produced 4/7 of a 3-part bar. They decided to partition each third into seven mini-parts independently. However, the distributive partitioning operation surpasses the recursive partitioning operation because the problem situation and the product is more complex than the result of a fraction composition scheme constructed using recursive partitioning. Dorothy and Brenda not only could conceive a mini-part in terms of the 3-part bar, if asked, but they also produced the result of taking one seventh of each part of the 3-part bar as one seventh of the 3-part bar. Using those three mini-parts as a seventh of the 3-part bar, so they operated on and iterated it four times to produce the 4/7 of the 3-part bar, so they operated with a three-levels-of-units structure. Therefore, the distributive partitioning operation assumes partitioning each part of the bar recursively, and, further, it not only uses the result, a mini-part as part of the whole, as in fractional composition (e.g., 1/7 of 1/3 of 3-part bar), but it also opens up the possibility of using the mini-part for producing the second level unit in the structure, such as a seventh of the whole 3-part bar by grouping the three mini-parts as a unit, and iterating that group four times to produce 4/7 of 3-part bar. This operation was not a single operation and to describe especially Dorothy's activities, I will use *distributive partitioning scheme*. I reserve distributive partitioning operation to describe Brenda's related activities.

I extended the discussion on the distributive partitioning operations to explain Brenda's activities when she additionally produced the measurement of the quantities in terms of a liter. Brenda used a result of the distributive partitioning operation, seven miniparts is a third of the 3-part bar, as a basis when she reinterpreted a part in the 3-part bar as a fifth of a hypothetical whole; she said, "Wouldn't it be because you have five pieces [five parts in a liter] and you divide each piece into seven, so thirty five [mini-parts]..." I call the operation a recursive distributive partitioning operation to explain the activities when a student anticipates recursively distributing more mini-parts into the imaginary parts of a hypothetical whole. There is an important coordination operation taking place such that a part of the bar is also a part of this hypothetical whole and such that the given bar is conceived not only as the fractional whole but also as part of the measurement unit, such as a liter. Constructing this imaginary whole using parts and mini-parts of the given bar, and reinterpreting the quantities produced as a result of distributive partitioning operations in terms of another measurement unit are the indications of what I call the recursive distributive partitioning operation. For the construction of this operation, the student needs to be able to use a reversible fraction scheme to create the hypothetical

whole and she should have already produced the fractional quantity (such as one of the mini-parts) as a result of her fraction composition scheme.

Stating the result in terms of a liter with an explanation similar to Brenda's is the most important indication of whether or not a student used a recursive distributive partitioning operation and so constructed three levels of units symbolically. Therefore, the construction of a fraction-multiplying scheme involves using a reversible partitive fractional scheme, operating recursively on the imaginary materials, using a whole number multiplying scheme, and coordinating two three-levels-of-units structures.

In the following problem, Dorothy operated with a reversible (iterative) fractional scheme and produced an improper fractional quantity. She used a very creative partitioning scheme as an extension of her distributive partitioning scheme. However, Dorothy could not state the result in terms of a liter. This situation is interesting because she operated with sophistication, yet she did not produce the measurement of the quantity in terms of a hypothetical whole. Her reversible operations show that, she surely can operate on the three levels of units since she constructed an improper fractional quantity. In spite of this, she could not use her reversible operations to construct the hypothetical unit of a liter. Therefore, the operations available to her were not sufficient to construct the second three-levels-of-units structure that could (only) be constructed symbolically, mainly using recursive distributive partitioning operations. I stated the problem as follows:

Problem 6.5: I have a water bottle that holds 4/5 of a liter, and yours holds 7/6 of whatever mine holds. Can you make your water bottle and figure out how much of a liter it is? (May 12)

Brenda and Dorothy worked individually on their computers. They both placed a bar on their screen and partitioned it into four parts, subsequently referred to as a 4-part bar. Brenda partitioned only the bottom part of the 4-part bar into three horizontal miniparts (see Figure 6.7(a)), and Dorothy partitioned each part of the 4-part bar into three vertical mini-parts (see Figure 6.8(a)).

I asked them to color each fifth of a liter in their 4-part bar differently: Brenda colored the part with three mini-parts black and then the remaining parts red and black alternately (see Figure 6.7(a)). She then partitioned each of the remaining three parts of the 4-part bar into three mini-parts (see Figure 6.7(b)) and pulled out two mini parts from the 4-part bar at once. She used the REPEAT button to make six more vertical copies of this pair. This new bar was her result (see Figure 6.7(c)).



Figure 6.7. Brenda's bars produced during her solution of Problem 6.5.

Explanation. (a) Four-part bar colored black and red alternately; (b) Four-part bar with three mini-parts per part; (c) 7/6 of the bar in (a); (d) Black and red colored

bar showing seven pairs of mini-parts.<sup>37</sup>

In contrast, upon my request for coloring a fifth (of a liter), Dorothy colored each pair of mini-parts black, blue, red, and purple starting with the top pair (see Figure 6.8 (b)), resulting in six pairs of mini-parts. She then pulled out a mini-part and made six copies of it. Later, she arranged three of those mini-parts horizontally and created a group, then placed three of the remaining mini-parts underneath that group (see Figure 6.8(c)). She copied the group of three mini-parts two more times, and placed the groups under the bar she previously made. Effectively, she constructed a new bar with 12 mini-parts and it was the same size as the original one (see Figure 6.8(d)). Subsequently, she pulled out another mini-part from the original bar, copied it one more time and dragged those two mini-parts to the top of the bar she just created (see Figure 6.8(e)). This bar was her resulting answer.



Figure 6.8. Dorothy's bars produced during her solution of Problem 6.5.

<sup>&</sup>lt;sup>37</sup> Brenda had only (b) and (c) when she finished her solution before she started talking; the other bars are her transition work. I reference (d) later in the document.

Explanation. (a) Four-part bar with three mini-parts per part; (b) Six pairs of miniparts colored differently; (c) Two groups of three mini-parts; (d) The bar with four copies of three mini-parts; (e) Resulting bar. Dorothy had only (b) and (e) on her computer screen when she completed her solution.

Neither Brenda nor Dorothy talked during her construction. Later, I asked

Dorothy to explain what she did since the way she constructed her resulting bar seemed

advanced. She responded as follows:

Protocol 6.4: Producing 7/6 of a 4-part bar.

Dorothy: Um, I divided this into four pieces [pointing to Figure 6.8(b)]. Because it is four-fifths of a liter and mine is...seven-sixths of a liter, of yours. And I divided each fourth into three, so I have three pieces in each fourth. And then I filled in each sixths so [inaudible].

Z: So, why did not you divide it [parts in the original 4-part bar] into something else but into three pieces?

Dorothy: Because the twelve goes into six and twelve goes into...four is a factor of twelve, and so the sixths.

Z: OK. What was your purpose when you were saying sixth is a factor? Why is that important?

Dorothy: So, I know it is the least common multiple of both of those numbers. Z: So, what do you do with that?

Dorothy: I kind of used that as a denominator. So, I divided this one into six parts [pointing to the bar in Figure 6.8(b)], which meant two of these. Two-twelfths will be one sixths, and so I went over here [pointing to the bar in Figure 6.8(e)] and for each sixths.

Z: When you say "each sixths", sixths of what?

Dorothy: Sixth of a liter, and I multiplied that sixth seven times.

Z: So, how much is this [pointing to the bar in Figure 6.8(e)] of this four-fifths of a liter [pointing to the bar in Figure 6.8(b)]?

Dorothy: One six[th] plus four-fifths. Of your whole thing? Are we looking at it as four-fifths or is this one whole?

Z: First, my whole thing [water bottle] is four-fifths of a liter and then we will talk about...[Ten seconds pass] Okay. Dorothy you figure out how much is that

[pointing to Figure 6.8(e)] of mine, which is four-fifths of a liter. Brenda you will figure out how much is this of a liter [pointing to Figure 6.7(c)]?

[Twenty-five seconds pass]

Z: Do you want to write it down [handing out papers.]?

[Ten more seconds pass]

Dorothy: Fourteen-twelfths.

Z: Another fraction name?

Dorothy: Seven over six...[she writes  $4/5 \times 7/6 = 28/30 = 14/15$  on the paper]. Z: Let's see. I asked you different things, right? What was yours, and what was yours [asking Brenda and Dorothy]?

Dorothy: I think I did hers [implying her written result. Everybody laughs]. Brenda: Yes, she did mine.

Z: That is okay. I just want you realize it. So, think about yours, now. [Dorothy writes 4/5+1/6 on the paper. She continues to write 24/30, 5/30, 29/30 underneath]. What was your problem?

Dorothy: Um. [Ten seconds pass.] I forgot.

Observer: What about that you have?

Dorothy: Mine was seven-sixths of hers.

Dorothy: Seven over six were made to a mixed fraction that would be one and one sixth and they are equal...

Z: I know you can explain it that way, can you explain it with the picture how can it be seven-sixths?

Dorothy: These are six-sixths [tracing Figure 6.8(b) and part of (e) with her pencil] so that is one whole and there is one sixth left over so that right there equals to one [she probably points to a part of Figure 6.8(e)]. So it would be one whole and this will be what is left over and that is a sixth, that would be one over six.

Z: Okay. That is one and one sixth, how can you explain it as seven-sixths? Dorothy: This is seven sixths [pointing to Figure 6.8(e)], this is six-sixths [pointing to Figure 6.8(b)], and this is six-sixths and there is a sixth leftover. You have those things together. You have the six-sixths and the one sixth, you get seven over six.

Dorothy independently generated a distributive partitioning goal to partition each

part of the 4-part bar into three mini-parts per part, thereby producing 12 mini-parts. I had

expected her to distribute six mini-parts in each part of the bar and pull out or color each

group of 4 mini-parts to make a sixth of the 4-part bar, thus using distributing partitioning

multiplicatively. However, she made 1/6 of the 4-part bar in a more creative way that is

based on additively producing six-sixths for the 4-part bar.<sup>38</sup> She distributed the same

number of mini-parts across the parts, to make a sixth; she colored every pair of mini-

parts differently. She further proceeded in a sequential way which seemed as if she was

<sup>&</sup>lt;sup>38</sup> In this situation, use of "additive" does not lower the cognitive demands (of students) compared to the necessary cognitive demands of "multiplicative" distributive partitioning operation.

adding the pairs one by one. Sometimes those mini-parts were from different parts of the bar; therefore, she did not take a sixth of each part but more efficiently she took an amount for 1/6 of the 4-parts (see Figure 6.8 (b)). This is a very creative act and uses the results of the distributive partitioning scheme along with both multiplication and division operations. Her awareness of how to find a number that is divisible both by four and six was the basis of her actions. Ultimately, her goal was to make the 4-part bar into a 6-part bar. She therefore operated on the 12 mini-parts, producing one-sixth as a pair of miniparts. She used the one-sixth quantity (a pair of mini-parts) as the multiplicative unit to iterate seven times to make the 7/6 of the starting bar. Since she was not randomly exploring, she anticipated the operations for recursively partitioning the 4-part bar and then iterating a sixth of the bar seven times. Even though she anticipated these operations, she completed all of them on the visual material she had in front of her. There was no indication she would have visualized the resulting quantity before she operated on the bar. Dorothy's success at producing 7/6 of the 4-part bar showed that she extended her distributive partitioning scheme to include the situations for constructing improper fractions of whole number quantities. However, she perceived the 4-part bar as the only whole on which to operate. Whenever I asked her to reinterpret her answer using a liter, she responded as if she perceived the whole bar as a liter and the sixths as parts of a liter. When I rephrased my questions with an emphasis on the measurement of the 4-part bar (4/5 of a liter), she used four-fifths of a liter and one sixth of the bar together in her answers

After I realized Dorothy did not produce an operational image for a whole liter, I asked her to provide a fractional name that explained how much her starting bar was of

the resulting bar. She used the number of mini-parts to answer this question and her answer was 14/12. I requested another fraction name and she said "seven over six." We discussed what 7/6 meant to her and how she would relate it to the bars on her computer screen. She indicated that there was a multiplicative relationship between the bars that the 4-part bar was six-sixths and the 7/6 bar was one more sixth than the six-sixths bar.

Brenda's operations and explanations were similar to Dorothy's when finding 7/6

of the 4-part bar. She colored every other mini-part black in Figure 6.7(c) and

transformed it to Figure 6.7(d). Later I asked how she conceptualized the quantities in

relation to measurement unit of a liter. I started investigating the relationship of the one-

sixth quantity to the bar that was four-fifths of a liter and our conversation continued:

Protocol 6.4 (Continues): Figuring out 7/6 of 4/5 of a liter.

Z: So, this [is] one sixth [pointing to the bottom pair of mini parts in Figure 6.7 (d)] of what? Brenda: This is one sixth of this one [pointing to Figure 6.7(d)]. Z: Does this have a name? Brenda: The four-fifths. Z: Four-fifths of a liter. Brenda: So, then I got seven-sixths and so, that would be seven-sixths of a liter. Z: Of a liter? Brenda: Well, mine is seven-sixths of that one. Z: But how much is that of a liter? Brenda: Well, because this is four-fifths of a liter. You just divided it up. So, this here is equal to this amount down [camera is not focused on Brenda's computer screen, but she is most probably comparing Figure 6.7 (b) and (d)]. And this is four-fifths of a liter. So then you have two more here... Z: But how much is that of a liter? Brenda: It's one sixth. Z: Of a liter?

Brenda: Oh, no, it would be one, two, three...three times five is fifteen, so um. It would be like one, two, three, four, twelve and two more. It would be like. Of a liter, it would be like fourteen over fifteen. Because this is four-fifths of a liter, so [to] make this a whole liter [pointing to Figure 6.7 (b)] you have to have one more fifth, so then I divided each fifth into three parts that will make fifteen. Because three times five is fifteen and then you have like one more twelve left over here in

this water bottle [pointing to Figure 6.7 (d)], one more [of] these twelfths makes it a whole liter and you have fourteen out of the whole fifteen.Z: When you say twelfth.Brenda: Or like you have one more of these pieces, more like a fifteenth.Z: Fifteen of what and twelve of what?Brenda: Twelve will be just the four-fifths. But the fifteen and the fourteen are from the whole liter. Because there is fifteen of these [pointing to a mini-part in Figure 6.7(b)] in the whole liter and you have fourteen here [pointing to Figure 6.7 (d)], so you have fourteen out of fifteen.

Brenda: You have to make sure that you remember that this isn't a whole liter and it still has a portion missing from it so you kind of have to think of the portion missing being there too. And then you can solve like, it is easier to solve a problem from the whole than it is from a portion.

Brenda, like Dorothy, knew that the resulting bar was 7/6 of the bar that had six pairs of mini-parts. Occasionally she referred to the bar in Figure 6.7(b) as a liter, so her resulting bar was 7/6 of a liter. When I asked whether she was sure the reference quantity was "a liter," she only stated the relationship between the two bars as the resulting bar was 7/6 of the other one without really answering my question. At that point, she uttered that the reference bar was four-fifths of a liter and the resulting bar had four-fifths of a liter and an additional two mini-parts. She was aware of the different units in her answer that the two mini-parts contained in the resulting bar was "a sixth," but it was not a sixth "of a liter." Similar to her operations when finding how much of a liter one of the miniparts was in the previous problems, Brenda imagined completing the four-fifths of a liter to make the whole liter. She then recursively distributed three mini-parts into all the parts in the whole liter including the extra part, producing fifteen mini-parts. In this way she figured out the number of mini-parts in the whole liter. Her purpose was actually to find the measurement of one of the mini-parts that she conceived as the result of taking 1/12of 4/5 of a liter and reinterpreting it as the third level of unit in her second three-levels-ofunits structure. She further operated using mini-parts and said,

Twelve will be just the four-fifths. But the fifteen and the fourteen are from the whole liter. Because there is fifteen of these [pointing to a mini-part in Figure 6.7(b)] in the whole liter and you have fourteen here [pointing to Figure 6.7 (d)], so you have fourteen out of fifteen.

Therefore, while she treated all the mini-parts in her resulting bar as the same amount, she was careful to reinterpret the extra two mini-parts as fifteenths. I am not sure whether she abstracted the result of her operations as fractional multiplication as I stated here, 1/12 of 4/5 of a liter is 1/15 of a liter. However, she could take this result of one fifteenth of a liter and further operate on it to reinterpret the quantity, which she earlier gave different names, such as "7/6 of a liter, 7/6 of the starting bar, 4/5 of a liter and two more mini-parts." She produced 14/15 of a liter for the bar that was 7/6 of 4/5 of a liter. Therefore, she extended her fraction-multiplying scheme to include the situations for producing improper fractional quantities of fractional parts of a whole. This extension to a new situation was a result of an accommodation she made that I am calling the recursive distributive operation; she coordinated one part of the 4-part bar with one of the five parts of a hypothetical whole and imagined placing three mini-parts into each of those parts of the hypothetical whole.

Brenda's and Dorothy's activities in the following problem's solution provide insight into how they extended their schemes (fraction multiplying scheme and distributive partitioning scheme, respectively) for taking fractional parts of quantities that are more than a whole unit. In Problem 6.4, I asserted that Dorothy's operations depended on using visual material. Her activities in Problem 6.6 provide more evidence related to this assertion. In Problem 6.6, Dorothy had a bar that contained the whole liter. Using this relationship, a liter embedded in the given bar, she was able to construct not only the second set of units structure, but she also coordinated the 2 three-levels-of-units structures. Thus, she produced the measurement of the resulting quantity successfully as she failed to do in the previous problems.

Problem 6.6: My water bottle holds 11/6 of a liter and yours holds 3/5 as much as mine holds. Can you make the water bottles on JavaBars and figure out how much of a liter yours holds? (May 11)

Dorothy and Brenda each had a bar on her own screen and partitioned it into 11 parts. Dorothy colored the bottom six parts blue (see Figure 6.9(a)) to show, probably, the 6/6 quantity or a liter. Brenda immediately colored each part in her 11-part bar alternately red and blue. Instead of asking them to color a sixth of a liter in their bars, as I did in the previous problems, I asked Brenda and Dorothy to pull out one sixth of a liter and proceed with their actions using that part. By requiring them to operate on a separate sixth of a liter, I wanted the students, especially Dorothy, to become aware of the second level measurement unit, a sixth of a liter. If I did not require students to work with a sixth of a liter, I anticipated Dorothy would transform the 11-part bar into a 5-part bar to create 3/5 of it. Therefore, she would not be challenged to consider the parts in the 11-part bar or her resulting quantities in relation to a liter.

In response to my request, Brenda pulled out one part and wanted me to repeat the problem; I said, "Yours would be three fifths of mine." She partitioned the part (a sixth of a liter) by three, producing three mini-parts in the part. I asked her whether she could notate mathematically what she wanted to do with the JavaBars. Later she was not sure whether she wanted to divide by three or five, but she said, "I will divide it by three

pieces, so would you divide by three or times one third?<sup>39</sup> After spending some time discussing how the results of these conceptions would look with JavaBars, Brenda wanted to proceed with JavaBars instead of writing, and I supported her decision.

Despite my request to pull out a sixth of a liter and operate on it, Dorothy worked on the parts embedded in the 11-part bar. She partitioned each part into five mini-parts. She then pulled out a mini-part and repeated it 11 times and made an array. Subsequently, she repeated this array two more times underneath the first one. Dorothy waited for a while and wanted to confirm the referent quantity and asked, "Mine is three fifths of your thing or a liter?" Her question indicates that she conceived of the 11-part bar ("my thing") also as a liter, but yet as two different entities. It is possible that Dorothy was aware of "a liter" without constructing it from a sixth of a liter or a part in the 11-part bar, since she conceived of the liter as already visually embedded in the 11-part bar as indicated by coloring six of them at the start. After I responded that her bottle held three fifths of mine, Dorothy colored the first row blue and the second row black in her new bar (see Figure 6.9 (b)). At that point, possibly as a result of overhearing my interaction with Dorothy, Brenda said, "Ours is three fifths of yours? Hold on, I do not think I want to do this [she cleared all the mini-part marks on her starting bar.]"

<sup>&</sup>lt;sup>39</sup>Since her question was interesting, I asked Brenda whether "dividing by three or multiplying by one third would be the same thing or do you get the same amount (as a result)?" I required her to create the amounts that would be the result of one sixth divided by three and one sixth multiplied by one third respectively using two separate sixths pulled from the bar. She partitioned one of the parts (a sixth) into three parts and colored one mini-part blue and said, "This would be one sixth divided by three." and she did not know how to show the situation for "one-sixth times one-third" with the other part. When I asked her whether she could show one third of one sixth, Brenda quietly asked, "It would be this [pointing to the blue part of the one sixth bar], right?...So you get the same thing." The exchange shows that, for Brenda, multiplying a quantity by one third is not yet the same conception as dividing the quantity by three, but it is a reinterpretation of taking one third of the quantity.



Figure 6.9. Dorothy's bars produced during her solution of Problem 6.6. Explanation. (a) Starting bar with 11-part including 6/6 of a liter; (b) Resulting bar.

Brenda silently thought about the problem for two minutes. She did not take any action during this time. When she was ready, we started talking about both girls' solutions. The following protocol (Protocol 6.5) starts with Brenda thinking aloud about what she wanted to do with her 11-part bar (PART A). It then continues with Dorothy's explanation of her solution and the interchange between us after I requested that Dorothy combine notations with her step-by-step actions on JavaBars (PART B). The protocol ends with Brenda's explanation of her own solution (PART C) and Dorothy's and Brenda's discussions of the two solutions (PART D).

Protocol 6.5: Producing 3/5 of 11/6 of a liter.

Protocol 6.5. PART A.

Brenda: I was trying to make this [her 11-part bar] divided into a number that was divisible by three, so that you can pull out three-fifths of them, but I can't think of a number that you multiply by eleven and get a number that is divisible by three.
Dorothy: Three times eleven. Brenda: Thirty-three. Dorothy: Thirty-three divided by three is eleven. Brenda: Yes. OK. Z: So you are going to find three-fifths of this [11-part bar] right? Brenda: [She pulled out a part, partitioned it into three, and produced three miniparts] so to get one third of that or one... to get... so you will have eleven of these. Z: You get eleven of these, which one? Brenda: Of these little ones [pointing to one of the three mini-part] or no. Z: Then how much would it be in terms of the whole thing? Brenda: Um. Three times eleven is thirty-three and thirty-three divided by three is eleven, so each piece of three which is gonna be three-fifths of that...wait how many did you [asking me] say it was? Ours is how many [she turned her head to Dorothy]? Dorothy: Ours is three-fifths of eleven over six [Brenda paused for 15 seconds]. Brenda: Would you want to divide it [each sixth of a liter (or part)] by five or three?

As seen in PART A of Protocol 6.5, Brenda continued taking one-third of the 11-

part bar and using this quantity for the construction of the other bar. A reason for these

actions might be that she conceived the problem as creating a bar that was five thirds

times as much as the 11-part bar. This is only a speculation because I have not seen her

iterating the third quantity (third of the bar) five times, but I did observe her anticipating

the production of a third by partitioning each part into three mini-parts and grouping 11

of those mini-parts. My questioning about the relationship between 11 mini-parts and

11/6 of a liter might have made Brenda rethink the problem situation. Consequently, she

asked Dorothy what the problem situation was. Upon Dorothy's response, Brenda

doubted the number she needed to use to partition each of the 11 parts—whether it was

three or five. For a long time (almost 4 minutes), Brenda worked on her computer quietly

while Dorothy explained her solution.

Protocol 6.5. PART B.

Z: So what did you do, Dorothy?

Dorothy: I have divided mine into five pieces, and then I got a fifth of it, which was eleven,

Z: Can you stop there and write it down?

Dorothy: [She wrote  $11/6 \times 3/5$ ] equals to...

Z: Do not worry about the result, just write it down what you did step by step. Dorothy: I drew the bar and I divided it into eleven pieces, and just in case I had to have the information, I filled in one liter because yours was more than one liter and I divided each one into five so I can get one fifth.

Z: OK. So can we write it down, when you say each of them, each of them refers to what? How much of a liter?

Dorothy: Eleven. Um.

Z: How much of a liter is that?

Dorothy: This whole thing?

Z: No, no, no. When you say each of them, this is one [tracing a part in the 11-part bar] when you refer to each of them.

Dorothy: That is one sixth.

Z: Write it down one sixth, then.

Dorothy: I divided it into five [she writes  $1/6 \times 1/5$ ]

Z: So, which one is that, can you color it like purple?

Dorothy: [She colors a couple of mini-parts purple on the top row] Fill in sixth? Z: No, I just want you color the result of one-sixth times one-fifth, not everything. Dorothy: [She left only one mini-part colored purple (see Figure 6.9 (a)) and asked quietly] is that it? Then I took the purple one out and I repeated eleven times. I have thirty-three, yes. I divided the six into five and I pulled it out and I repeated [it] eleven times. And I multiplied by three. I pulled out one fifth and multiplied it by three. So, I have three-fifths.

Brenda: [On the other hand, Brenda already partitioned each of the bottom three parts of her 11-part bar into five on her computer. Unfortunately, the camera was not focused on the process of her work at any other time except when she pulled out the11 mini-parts].

Z: Oh. OK. So this one, this whole thing is three-fifths [tracing the Figure 6.9(b), on the right].

Dorothy: Yes.

Z: So, let's look at what you have written. One-sixth times one-fifth equals to this purple thing. Which is what?

Dorothy: One-thirtieth [she writes 1/30 after uttering it].

Z: So, how can it be one-thirtieth?

Dorothy: Because this is one sixth [pointing to a part in 11-part bar]. Well, at first, I counted how many. The way I thought about the first time was there was eleven pieces and I divided it by five and I got fifty-five and I took one of those out [pointing to a mini part in the original bar] and repeated eleven times.

Z: So that one of them, is that one fifty-fifth? Or one-thirtieth?

Dorothy: It is one-thirtieth of ele... of one liter.

Z: So, that is what we are interested in right? Dorothy: Yes. Z: OK. So can you write it down what you did after you pulled it out? We had one-thirtieth and we repeated. Dorothy: Eleven times [she writes  $1/30 \times 11/1=11/30$ ]. Z: So, we got eleven-thirtieth; is this eleven-thirtieth of a liter or is this eleventhirtieth of eleven-sixth? Dorothy: Eleven thirtieth of a liter. After the eleven-thirtieth, since it was the onefifth, I multiplied it by three so you have three-fifths that will be [she wrote  $11/30 \times 3/1$ ] thirty-three over thirty [she wrote 33/30].

In her explanations (see Protocol 6.5. PART B), Dorothy acted as if the sixth of a liter was the quantity on which she based all of her activities. To find a fifth of the 11part bar, she partitioned each sixth of the liter in the 11-part bar into five mini-parts. She then pulled out one of those mini-parts and iterated it 11 times to make a fifth of the whole bar and then constructed an array. The production of this array was an indication of Dorothy's engagement in the distributive partitioning scheme that I explained earlier in Problem 6.4. This operation was necessary for constructing the fractional part of the 11part, 3/5 of the 11-part bar, using one fifth of it. She repeated this array two more times to make the three-fifths quantity. In contrast to the previous problems, it appeared that Dorothy was aware of the measurement of the parts in the 11-part bar. The 11-part bar consisted of eleven of the sixths; therefore, it was relatively easy for Dorothy to conceive the liter both as a part of the given bar and six times as much as one of the 11 parts. She independently colored six of the parts and said, "Just in case I had to have the information, I filled in one liter because yours was more than one liter." This was a functional accommodation because Dorothy explicitly stated the measurement of each part (including mini-parts) in the given bar. In the following problems, there are instances showing this change was permanent as long as the measurement unit was embedded in

the given bar.

It is more plausible that she constructed the quantity of a liter visually. The language that is used for labeling one of the parts "a sixth" and having 11 sixths as the quantity on her computer screen could have evoked Dorothy's operations, so she colored six of those parts for a liter. In the previous problems, she could take both a part (second level unit) and a mini-part (third level unit) as if they were only quantities, say, in an 11-part bar and could operate on those quantities to produce the fractional parts of the bar. However, now she could simultaneously reinterpret a mini-part as both part of a liter and part of the 11-part bar. Having 1/6 times 1/5 written on her paper might have helped Dorothy to operate algorithmically and produce 1/30. However, I am more inclined to think that she could give meaning to a mini-part as part of a liter, 1/30, even if I did not ask her to notate the process since she had a liter with six parts and five mini-parts per part already. In addition, she was aware of these relationships from the beginning when she colored the liter in the bar differently. Writing  $1/6 \times 1/5$  only helped her reflect on the relationships she had constructed.

By creating 3/5 of the 11-part bar as another bar, Dorothy produced a unit of units of units structure; the 11-part bar was the unit, which contained five units, each of which embraced 11 mini-units. Unlike her operations in the previous problems, she created this structure without transforming the 11-part bar into a 5-part bar. This situation implies that at that point, Dorothy had a multiplicative unit structure so that the same part was both a part of a liter (1/6 of a liter) and part of the given bar (1/11 of the bar). When she was asked to color the result of dividing a sixth into five in the bar, she colored a mini-part purple (see Figure 6.9 (a)). She said the purple mini-part was 1/30 using her written

operations as a means to produce this result, but when asked, she did say that a mini-part was "one thirtieth of ele... of one liter." She preferred this labeling over "one fifty-fifth," which was a result she produced in her first explanation. At this point, I also emphasized this relationship by saying "So, that [a mini-part in terms of a liter] is what we are interested in, right?" and Dorothy said "yes." Dorothy could operate on this reinterpretation of a mini-part in terms of a liter to reinterpret 1/5 or 3/5 of the 11-part bar, as 11/30 and 33/30 of a liter respectively. Therefore, she extended a two-levels-of-units structure (a liter is a unit containing six units) to a three-levels-of-units structure by reinterpreting a mini-part as 1/30 of the liter. After working with Dorothy intensively and satisfied with her progress, I gave my attention to Brenda's activities and looked at her computer screen.

Protocol 6.5. PART C.

Z: Many little lines [l looked at Brenda's computer screen and saw Figure 6.10]. Brenda: I do not know if I am right because I am confused. I had eleven-sixths, and so I wanted to get three-fifths, so I divided each sixth by five and eleven times five is fifty-five.

Z: ...[I asked Dorothy to put her written step-by-step work next to Brenda's computer screen so that when Brenda explained her solution, Dorothy could coordinate the JavaBars and the written solution]...

Brenda: [Even though Brenda's computer screen is not very well captured (especially the resulting bar), Figure 6.10 (b) is my best guess regarding what she had on her screen.] Here, eleven of these [pointing to a mini-part] would be one-fifth of the eleven-sixth because I divided each one into five and eleven times five is fifty-five so eleven of these is one-fifth. So, I pulled out eleven. I knew I needed three-fifths so I pulled out two more so I could have three-fifths of the eleven-sixths.

Z: It looks to me you did the same thing here [Not clear where I pointed. Probably I meant their configurations on their computers]. Maybe...



Figure 6.10. Brenda's bars produced during her solution of Problem 6.6. Explanation. (a) Eleven-part bar; (b) Resulting bar.

Brenda, on the other hand, partitioned the first three parts of the 11-part bar into five mini-parts per part and pulled out a group of 11 mini-parts. She used this group as the fifth of the 11-part bar and iterated it two more times to make 3/5 of the bar. Even though she only partitioned the first three parts of the JavaBar as she said in PART C of Protocol 6.5, since she interiorized the distributive partitioning operation, she acted as if she had already partitioned the other parts of the bar resulting in 55 mini-parts in total. In all of her operations, she did not use the measurement quality either for parts or the miniparts. She produced a three-levels-of-units structure with the quantities: the 11-part bar was the unit that contained 11 units and each of those units contained five mini-units per part. When I asked Brenda how much of a liter her resulting bar was, as in the previous problems she had no difficulty reinterpreting the mini-part and one of the parts of the 11part bar as part of a liter. To conceptualize the resulting quantity in terms of a liter, Brenda went through a series of mental operations; she could indeed conceive a liter as a total of 30 mini-parts by adding five mini-parts six times. Unlike Dorothy, Brenda did not

color six parts of her bar for a liter to start with. But she did verbally construct the liter

using a mini-part and reinterpreted a mini-part in terms of a liter. Therefore, Brenda

extended her fraction multiplying scheme to include situations such as taking fractional

parts of quantities that are more than the unit measure. We then discussed how they

interpreted each other's solutions.

Protocol 6.5. PART D.

Dorothy: I think she did the way I thought about it the first time. I thought about the whole thing: It was fifty-five and I repeated it eleven, eleven, eleven [pointing to the rows in Figure 6.9(b)] and it was thirty-three, which you had to pick the denominator.

Z: So, what would be your answer for this one [pointing to Figure 6.10(b)]? How much would it be of a liter?

Brenda: It would be more than a liter wouldn't it, so it would be...

Z: How do you know it would be more than a liter?

Brenda: Because three-fifths of eleven-sixths, each fifth is eleven little pieces, so then if you are thinking there is five in each sixth, so then in six-sixth there are ten, fifteen, twenty, twenty-five, thirty [looking at her 11-part bar] and then in the three-fifths of eleven-sixths there is um, more than thirty pieces. Eleven three times so it is thirty-three pieces.

Z: Eleven three times [equals to] thirty-three pieces.

Brenda: So it is thirty-three over thirty.

Dorothy: So it would be one liter and one tenth or three over thirty.

In PART D, Dorothy commented that Brenda's thinking was similar to the

thinking she had when she solved the problem the first time. In her first approach,

Dorothy operated with the number of mini-parts (55) in the 11-part bar, took 1/5 of 55

mini-parts, and produced 3/5 of the 55 mini-parts. As Dorothy indicated, "you had to pick

the denominator" to complete the solution if this way of thinking was followed. Her

comment meant she needed to figure out how to name one of the mini-parts in terms of a

liter, so she would have a fractional name for the 33 mini-parts. Requiring Dorothy to

notate her JavaBars actions in Part B helped her to reflect on the process of producing a

mini-part as part of a liter. Therefore, in the second way, her new interpretation of a minipart as part of a liter facilitated producing a measurement for the 3/5 of the 11-part bar in terms of a liter. In this way, Dorothy could extend her distributive partitioning scheme to include situations for taking parts of quantities if those quantities are more than the standard unit measurement and they contain the standard unit measurement visually.

Since Dorothy extended her distributive partitioning scheme, similar to Brenda, to produce the measurement of a mini-part in terms of a liter, she also constructed recursive distributive partitioning operations. However, Dorothy constructed this operation as a figurative operation and so produced both a liter and a measurement of a mini-part as figurative material. This construction is different than Brenda's recursive distributive partitioning operation which could be only engaged symbolically by imagining distributing mini-parts to the parts of the imaginary whole as Brenda did in Problems 6.2, 6.4, and 6.5 and also in this problem.

## **Inverse Reasoning Problems**

The following section examines Brenda's and Dorothy's activities in the context of inverse reasoning problems. Their activities and means of operating with these problems build on their knowledge of three levels of units, their coordination of 2 threelevels-of-units structures, and their partitioning operations (distributive and recursive distributive), all of which were discussed in the earlier problems.

I used the problems in this part to explain how Brenda and Dorothy conceived of the problem situations as a series of operations. For example, when finding a quantity of which 5/6 is given as four gallons, conventionally, we set up the equation 4 gallons =  $5/6 \times \text{unknown}$ . For the solution, we take the reciprocal of 5/6 and multiply it (6/5) by 4

gallons to find the unknown quantity's measurement. While this way of solving such problems is conventional, my purpose was not to teach them this convention, but to investigate how Brenda's and Dorothy's fractional multiplying schemes helped or hindered their activities when constructing an unknown quantity and forming its measurement in terms of a liter.

Proper Fractions as Factors in Missing Factor Problems: Inverse Reasoning and Three Levels of Units

Dorothy and Brenda solved many inverse reasoning problems that I discuss in the different parts of the remaining analysis. Some of those problems were structured with using the known quantity as a multiple of a whole unit and stating an equivalency relationship between the known quantity and a fractional part of an unknown quantity: for example, four gallons of whole milk is 5/6 times as much as the skim milk (Problem 6.7). For this section, I will only present the analysis of Problem 6.7 since the ways Dorothy and Brenda operated on the other problems were similar.

Problem 6.7: For a dessert recipe, you need 4 gallons of whole milk and some skim milk. Four gallons of whole milk is five sixths as much as the skim milk you need. Can you make the needed amount of skim milk on JavaBars and figure out how much it is in terms of gallons? (May 9)

Dorothy made a bar using JavaBars and divided it into four parts to show the 4 gallons of milk. She asked, "This [the 4-part bar] is five sixths of the skim milk?" After I replied "yes," she partitioned each gallon into five mini-parts and colored alternately every four mini-parts black and red (see the top bar in Figure 6.11). She pulled out one mini-part, which was one fifth of a gallon, and repeated it three times. So she had a total

of four copies, which was equivalent to 4/5 of a gallon and to one of the five equal parts of four gallons. This way of operating and producing a result was consistent with what I called *distributive partitioning schemes* earlier. Subsequently, she copied the whole four gallons and added it to the bar of 4/5 of a gallon and this new bar was her answer (see the bottom bar in Figure 6.11).



Figure 6.11. Dorothy's bars produced during her solution of Problem 6.7.

Explanation. The top bar represents 4-gallons of milk. The bottom bar is Dorothy's solution.

On the other hand, Brenda preferred solving the problem with paper and pencil and made only a general construction of the problem situation with JavaBars. She had two independently constructed bars on her computer screen: one with 5 parts and the other one with 6 parts. The parts in the 5-part bar and the ones in the 6-part bar were different sizes. Brenda then copied these two bars onto paper and placed ".8" on each part of the two bars, after dividing 4 by 5 algorithmically. Her written solution is below (see Figure 6.12).



Figure 6.12. Brenda's written solution produced during her solution of Problem 6.7.

In our conversations, Brenda did not say what .8 meant in relation to the measurement unit of a gallon. She did take this number and multiply it by six to produce a written result for the 6-part bar as 4.8 or 4 4/5. Her JavaBars constructions did not indicate she produced the 6-part bar using a part from the 5-part bar, but she operated as if those parts were the same since she used the numerical value she assigned for one of the parts and produced a numerical value for the 6-part bar on her paper. She said the result was "four and four fifths gallons." Although Brenda used "4/5" in her written answer, neither her construction with JavaBars nor her talk about .8 suggested that she was aware of the quantitative relationship of "4/5 gallons" to a whole gallon. For these reasons, I asked Brenda to point out 4/5 of a gallon using either her JavaBars or Dorothy's JavaBars in Figure 6.11. While Brenda had difficulty understanding the problem that I posed, Dorothy used the mini-parts in her partitioned bars and easily created a gallon and four-fifths of a gallon with JavaBars. Our conversation continued as follows:

Protocol 6.6: Creating 4/5 of a gallon with JavaBars.

Z: Can you show four-fifths of a gallon?

Brenda: Of one single gallon? Or the whole four gallons? [At this point, Dorothy pulled out one mini-part from the top bar in Figure 6.11 and copied it four more times and produced a gallon, see Figure 6.13(a)]

Z: Dorothy, how much is this?

Dorothy: One gallon.

Z: One gallon, right? So can you show four-fifths of a gallon?

Dorothy: [She copied a mini part three times to make four-fifths of a gallon, see Figure 6.13(b)]



Figure 6.13. Dorothy's bars.

Explanation: Dorothy's bars for (a) one gallon, and (b) four-fifths of a gallon.

Brenda: Wouldn't you have to, like for this picture [on her computer screen], you would have to take this out like two tenths or one fifth of this piece out to get one whole gallon out and then divide that into four fifths [the camera was not focused on her actions]... to get one single gallon. Because each of these isn't a whole gallon, it is four fifths of a gallon [probably pointing to parts of the 5-part bar in Figure 6.14].

Z: OK. I was asking about four-fifths of a gallon.

Brenda: So I just pull out this? [Brenda colors the bottom part of the 5-part bar blue. See Figure 6.14.]



Figure 6.14. Brenda's JavaBars for the solution of Problem 6.7. Explanation: (a) Brenda's 5-part bar, and (b) 6-part bar.

At the start of Protocol 6.6, I explicitly asked the question, "Can you show me four-fifths of a gallon?" But Brenda did not seem to be aware that one of the parts in her 5-part bar or 6-part bar was also 4/5 of a gallon because she did not point to a part in her 5-part bar and because she asked, "Of one single gallon? Or the whole four gallons?" However, after seeing what Dorothy had made using JavaBars (Dorothy made a gallon and 4/5 of a gallon using a mini-part from her 5-part bar), Brenda pointed to the parts in her bars and said, "Each of these are not a whole gallon: it is four-fifths of a gallon." Even though her overall speech was not clear enough to follow, Brenda presumably wanted to create a gallon using her 5-part bar, but she did not produce a mini-part, the third level of unit, and did not use it for the construction of 4/5 of a gallon by using a distributive partitioning operation.<sup>40</sup> Therefore, her problem of making a gallon using her 5-part bar was a challenge to her. Instead of pursuing Brenda's challenge, I chose to focus on Dorothy's construction process and asked Dorothy to describe how she

<sup>&</sup>lt;sup>40</sup> The first level was the unit containing the four gallons, the second was one of the four gallons, and the third was the five mini-parts that Dorothy made when she partitioned each gallon into five parts.

produced the result. This situation is somewhat unfortunate because I did not encourage

Brenda to construct a means for making a connection between her written solution and

the quantities used in the problem. Protocol 6.6 continues as follows.

. . .

Protocol 6.6 (continues): Constructing a bar for an unknown quantity by using 4 gallons as 5 parts of the bar-to-be-made.

Dorothy: First, I divided this [pointing to the top bar in Figure 6.11] into four which was four gallons and I divided each gallon by five...and I took one out and repeated it four times to get...four-fifths [of a gallon]. And I added it to this original one which was this one [pointing to the bottom bar in Figure 6.11] and that is the one-gallon [pointing to Figure 6.13(a)] and that is the four-fifths [pointing to Figure 6.13(b)]. Z: And your result?

Dorothy: It is four and four-fifths of skim milk and that was four gallons and fourfifths of a gallon skim milk.

Z: What fraction name would you give for skim milk if you think of the whole milk?Brenda: Six-sixths.Dorothy: Five and one fifth.Z: Or?Dorothy: Six-fifths.Brenda: Oh. OK. Right, sorry.

Dorothy started with the known quantity of 4 gallons and created a bar for that quantity. Even though it was 4 gallons of whole milk in the problem statement, she used 4 gallons as the measurement for the part of the unknown amount of skim milk and asked, "This [the 4-part bar] is five sixths of the skim milk?" She then transformed the 4gallon bar into a 5-part bar, so she could use one of those five parts to make the bar for the skim milk. Therefore, it appears as if she conceived the problem situation as producing a whole bar with six parts by using the 5-part bar as its 5/6.

To make the 4-part bar into five parts, Dorothy partitioned each part of the 4gallon bar into five mini-parts and produced a total of 20 mini-parts (see the top bar in Figure 6.11). After she colored each group of four mini-parts alternately black and red, so producing five parts, her construction indicated that there were different levels of units in her structure. She produced a three-levels-of-units structure—the unit of 4 gallons contained five parts and each of those parts contained four mini-parts-as a result of her distributive partitioning scheme. In the first part of the Protocol 6.6, Dorothy also emphasized the production of a gallon and 4/5 of a gallon using five and four mini-parts, respectively (see Figure 6.13). Since Dorothy was explicitly aware of the measurement unit of a gallon in her statements—especially for the result of her distributive operations, 4/5 of a gallon—her activities could also be interpreted as recursive distributive partitioning operations. But while being aware of the measurement unit of a gallon is important, this awareness does not require the same cognitive demands for constructing a unit as an operative figurative image in the absence of a perceptual unit. A unit of a gallon was already visually embedded in the 4-part bar as one of the parts, and there was no need for Dorothy to imagine constructing a unit measure of a gallon for reinterpreting the results. Therefore, I determined the result of making such coordination-4/5 of a gallon is 1/5 of 4 gallons (see Figure 6.13(b))—as only an extension of her distributive partitioning scheme. This extension is based on the accommodation she made in Problem 6.6 that as long as she had a visual measurement unit embedded in the given bar, she could state the measurements for the results of her partitioning operations, such as four mini-parts, in terms of a gallon.

Using the coordination of the 5-part bar both as 4 gallons and as 5 of the 6 parts of the whole bar for skim milk along with the result of her distributive partitioning scheme, Dorothy coordinated 4/5 of a gallon as the extra sixth that she needed to add to the 5-part

bar to produce the whole bar with six parts. However, Dorothy did not operate with this coordination multiplicatively because she did not iterate 4/5 of a gallon six times to create the unknown quantity. Rather, she used it additively to produce a result for the skim milk. Her additive operations consisted of copying the 5-part bar one more time and then adding another sixth or 4/5 of a gallon to this bar to make the bar for the unknown amount of skim milk (see the bottom bar in Figure 6.11). Therefore, Dorothy took the result of her distributive partitioning scheme, which was the coordination of 4/5 of a gallon as 1/5 of the 4 gallons, and further operated with this relationship to coordinate 4/5of a gallon as the sixth of the bar representing the skim milk. However, she might not have constructed this equivalency relationship for any part of the 5-part bar and any part of the 6-part bar because the 5-part bar was embedded in the 6-part bar. They were not separate entities or quantities. Dorothy possibly did not imagine the bar for the skim milk as a separate bar before acting, but instead produced it as a result of her reversible fractional schemes. In that case, a fifth of the 4-gallon whole milk bar could not be thought of as equivalent to a sixth of the imagined 6-part bar prior to operating. This situation might suggest that Dorothy did not operate inversely since there were no two independent quantities to start with; a sixth of the bar for skim milk was constructed as a result of her reversible fractional operations. So, the partitioning and iterating operations used in the construction of the bar for skim milk were the reversible operations (not the inverse), since she produced the whole by partitioning the 4-gallon bar into five parts and used a total of six parts to make the bar for skim milk. The bar for whole milk was embedded in the bar constructed for the skim milk, so the amount of whole milk was not equivalent to a part of the skim milk but was an identical quantity.

Even though making the needed sixth for the 6-part bar (for skim milk) by distributively partitioning the 4 gallons of milk into five parts and using one of those five parts as the additional sixth for making the 6-part bar was sophisticated, this way of operating is not sufficient for claiming that iterating and partitioning were inverse operations. Instead of inverse reasoning, Dorothy might only have used a reversible partitive fraction scheme to produce the amount for skim milk using 5/6 of that amount. This production also included the distributive partitioning scheme to create the extra fifth from the 5-part bar that is needed to make the whole bar. As a result, the product cannot be said to be an independent bar constructed using the fractional relationship of 5/6. Rather, the bar is composed of five parts and one extra of these parts, where each of the parts is 4/5 of a gallon. A mathematical way of summarizing Dorothy's activities and her result would be  $6 \times 4/5$  of a gallon. On the other hand, when I asked how much the new bar would be in terms of a gallon, Dorothy said it was "4 gallons and 4/5 of a gallon skim milk." While including "skim milk" in her answer might be an indication that Dorothy conceptualized the 6-part bar as an independent bar composed of only skim milk, it is not really strong enough to claim she reasoned inversely in the construction of this bar or that she imagined a separate bar before acting.

I also asked the two students what fraction name they would give to the skim milk, if (4 gallons of) whole milk were the whole (see the continuation of Protocol 6.6). Dorothy said, "It is five and one fifth." I asked Dorothy whether she could give another fraction name and she said "six-fifths." Her answers reflect how Dorothy produced the unknown quantity: she used the parts placed in the 4-gallon bar and distributively partitioned each part. She transformed the 4-gallon bar to a bar with five parts and used one of those parts and a copy of the 5-part bar to make the bar with six parts. Therefore, when the bar with 4 gallons was the whole, the other bar, which was one fifth more than the 5-part bar, would be "five [parts] and one fifth [of the five parts]." After Dorothy used different units to reinterpret the same amount (one of the five parts of the 5- part bar) when proposing the fractional relationship between the 5-part bar and the 6-part bar, she then changed her answer to "six-fifths." This change in her answer possibly indicated that she took the same unit, a part of the 5-part bar, as a reference when interpreting the 6-part bar. However, her answer is not a strong indication that she also made the equivalency relationship between a sixth of the amount of skim milk and a fifth of the amount of whole milk, which can be only constructed as a result of inverse operations. Brenda, on the other hand, conceived the situation as if I asked what fraction name would be given to the 6-part bar if the known quantity was 5/6 of the unknown one since her answer was "six-sixths."

Even though Dorothy stated the relationship that the other bar representing 4 and 4/5 of a gallon was 6/5 of the 4-gallon bar, 6/5 was only the product of her JavaBars activities when she operated with the reversible fractional scheme. There was no indication that 6/5 was produced as a reciprocal of 5/6. This means Dorothy did not consider this result, 6/5, as something to operate with further so that she might have interpreted the problem as making a bar representing 6/5 of the 4-gallon bar and finding the measurement of this new bar in terms of a gallon. Therefore, reciprocal reasoning did not play a functional role either in her conceiving of the problem or in her activities. If the reciprocal reasoning had played a functional role in her activities, she would have used a different way of operating with the distributive partitioning scheme: had she

operated on (e.g., pulling or coloring) each gallon, attempting to make 6/5 of each gallon using distributive partitioning operations and repeating that quantity four times, she would have produced 6/5 of 4 gallons. This way of solving the problem would have indicated that an inverse relationship had been abstracted, and the inverse operations are possibly interiorized, since there would not have been any need to construct an equivalency relation between the part (4 gallons) and the whole (unknown quantity). From the start, the problem would have been conceived as finding an equivalent amount to an improper fractional quantity of each unit measurement of the known quantity, such as 6/5 of 4 gallons.

I further discuss the issue of the nature of the abstractions that produce reciprocal reasoning in the discussion chapter. However, in this section, I will elaborate on what is needed for the construction of inverse relationships and how it is related to reversible fraction schemes, partitioning operations, and three-levels-of-units structures. I have three hypothetical requirements for a construction of inverse reasoning; the first one is related to conceiving the existence of the bar-to-be-made as imaginary prior to acting, and the last two are specific to inverse operations used for the construction of an unknown quantity. Inverse reasoning is the general term for successfully acting to solve these problems, and it requires making an inverse relationship between the two quantities and performing partitioning and iterating operations as inverse operations. I base my hypotheses on my observations of the two students' activities in Problem 6.7. My hypotheses are as follows:

1. The student needs to conceptualize the bar-to-be-made as a separate, independent bar from the starting bar, whose measurement is known, even before acting. 2. The student should be explicit in her construction of the equivalency relationship that 1/5 of the 4 gallons of whole milk is 1/6 of the bar-to-be-made for the skim milk.

3. The student should disembed one of the five parts and indicate that it is a sixth of the bar-to-be-made and iterate that quantity six times to produce the 6-part bar. During this construction, the student should use a language emphasizing that one of those parts is a sixth of the skim milk.

The first hypothetical requirement, conceptualizing the existence of two independent bars is fundamental; first, to making a general relationship and then to using this relationship to reconstruct partitioning and iterating operations as inverse operations (second and third requirements). The second requirement is to operate with this general relationship and to take the first observable action on this relationship, partitioning the known quantity. The purpose of the partitioning action is to make an equivalency relationship and to reconceptualize the result of distributively partitioning the known quantity as an equivalent part of the unknown quantity. This requirement assumes that the first requirement is satisfied. In the last requirement, the student operates further with the equivalency relationship she constructed (the second requirement) and uses an iteration operation to construct the unknown quantity; in this way, iterating a part of an unknown quantity and partitioning of a known quantity (for the equivalency) become inverse operations. For example, in the case of creating 6/6 of the quantity of the skim milk (result) when the measurement of 5/6 of its is given as whole milk (situation), a student will partition the quantity of whole milk into five parts and then take one of those parts as a sixth of the skim milk (equivalency) then iterate one of those parts six times to produce

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the skim milk. Actually, this kind of operating will be inverse of the situation and the result when creating 5/6 of the skim milk as an equivalent quantity for the whole milk (result) when the skim milk is given (situation). The student will partition the skim milk quantity into six parts, take one of those parts as equivalent to a part for the whole milk, and iterate one of those parts five times to produce the whole milk. Therefore, partitioning into six becomes the inverse of partitioning into five, and iterating six times becomes inverse of iterating five times,<sup>41</sup> because the situation that the student acts on and the result are inverses of each other and the operations take place on the equivalency relationships between the two quantities.

The following problems (Problem 6.8 and 6.9) are occasions for discussing Brenda's and Dorothy's activities using my hypotheses as a reference. The problems are variations of Problem 6.7 in which I changed the known quantity from whole units to a fractional part of the unit measure, but kept the proper fractional relationship between the known and unknown quantities, such as "my water bottle is 3/5 of a liter and it is 2/3 as much as yours" (Problem 6.8). This problem is also important for illustrating boundary situations for Dorothy, for example, when she created a bar for the unknown quantity and, with some help, satisfied the three requirements of inverse reasoning, but did not independently produce the measurement for the unknown quantity. For Problem 6.8, I helped Dorothy imagine making the whole liter by using parts of the known quantity. However, she did not independently act in the other situations (see Problem 6.9) to imagine making the measurement unit and did not act as if she constructed inverse operations. Dorothy conceptualized the problem situation using only her reversible

<sup>&</sup>lt;sup>41</sup> The inverse of partitioning the skim milk into six parts and iterating a part five times is partitioning the whole milk into five parts and iterating that part six times.

fraction scheme and disregarded the measurement of the known quantity (see Problem 6.9), so she was not successful at finding either the measurement of the unknown quantity or conceptualizing that quantity as a result of some inverse operations. This inability to produce measurements may be due to her not having a visual whole measurement unit and, therefore, not constructing the second set of three-levels-of-units structure that is necessary for coordinating a quantity and its measurement in terms of a liter. Therefore, Dorothy's activities suggest an important hypothesis. It is possible that constructing and using a recursive distributive partitioning operation (for the construction of a unit structure for the measurements of the quantities) and constructing inverse reasoning using inverse operations (for creating an unknown quantity by establishing and operating on an equivalency relationship) might be related psychological structures.

Proper Fractions as Factors in Missing Factor Problems: The Known Quantity is a Fractional Number

Problem 6.8: I have a water bottle that holds 3/5 of a liter. This much water is 2/3 as much as whatever your water bottle holds. Can you make the water bottles on JavaBars and figure out how much your bottle holds?<sup>42</sup>

When I presented this problem the first time (May 9<sup>th</sup>), Dorothy and Brenda understood the problem situation differently and so operated differently: Dorothy conceived her water bottle as two thirds more than my water bottle, and Brenda conceived my water bottle holding 2/3 of her water bottle. On May 9<sup>th</sup>, each student created a bar on JavaBars. Dorothy first made her bar with 5 parts and after she overheard Brenda talking about the bar being 3/5 of a liter, she erased everything and made a new

<sup>&</sup>lt;sup>42</sup> The problem was posed twice, May 9 and May 12. The first presentation followed Problem 6.7

bar that she partitioned into three parts (to show my water bottle). Dorothy said the water bottle-to-be-made was two-thirds more than the given one. Subsequently, she pulled out a part and copied it twice to show the 2/3 (see Figure 6.15). She then combined the 3-part bar and the bar for 2/3 and produced a 5-part bar (see the last bar in Figure 6.15).



Figure 6.15. Dorothy's bars produced during solution of Problem 6.8.

Explanation. (a) Dorothy's 3-part bar and 2-part bar disembedded from the original 3-part bar. (b) Dorothy's 5-part bar which is two thirds more than her original 3-part bar.

Brenda, on the other hand, created a 3-part bar, presumably as a representative of 3/5 liter. She then partitioned each part into two mini-parts and colored three mini-parts to illustrate one half of 3/5 liter (see the bar in Figure 6.16 (a)). Presumably, the three mini-parts was a representative of 1/3 of her bar. She then made a new bar with three parts for her water bottle. However, for this new bar, she did not use any of the mini-parts from the original 3-part bar she made.



(a) (b) Figure 6.16. Brenda's bars produced during solution of Problem 6.8.

Explanation: (a) The original 3-part bar whose half is colored blue for my water bottle. (b) Brenda's independently made bar for her water bottle with three parts.

We can accept Brenda's language and actions as indicators that she conceived of her water bottle (or bottle-to-be-made) as a separate and independent bar from the given bar she made as representative of 3/5 of a liter. Brenda's activities corroborate the first of the three hypotheses that I proposed as requirements for constructing an inverse relationship between the two bars. In our brief conversation, she indicated that the bar in Figure 6.16 (a) was supposed to be two-thirds of the new bar, since my water bottle was two thirds as much as hers. She was perturbed by this situation because she wanted to make a multiplicative relationship between the two bars, but did not know which part of the left bar nor which part of the new bar she needed to use to construct this relationship. While she distributively partitioned the 3-part bar, she did not use the result of this operation, three mini-parts, for either making an equivalency relationship between the parts of the bars or conceptualizing the bar for her water bottle. Therefore, Brenda did not satisfy the second and third requirements for the construction of partitioning and iterating as inverse operations, so she did not reason inversely to solve the problem. Unlike Problem 6.7, in which Brenda produced the measurement of the parts (in terms of a liter) algorithmically and made a general sketch of 5-part and 6-part bars to show one bar was 5/6 of the other one, this problem required her to operate with the result of her distributive partitioning operations and to reason with inverse operations using her two bars. At this point, we ran out of time and had to stop. I posed Problem 6.8 again in the teaching episode on May 12.

Problem 8: May 12<sup>th</sup>. The students each created a JavaBar and partitioned it into three parts. We started with a discussion of how they interpreted the problem situation and their interpretations were not different from the first time the problem was presented to them on May 9<sup>th</sup>. Dorothy said the bar-to-be-made was "two more than this one [her 3part bar]" and Brenda initially interpreted it, as the bar was "2/3 of the three fifths?" So, I repeated the problem: "We have water bottles and mine holds three fifths of a liter and this is two thirds as much as your water bottle."

Brenda said, "So this [Pointing to the only bar on her screen, the 3-part bar] is 2/3 of ours?" I said, "Yes." I asked Dorothy whether she agreed with Brenda's interpretation, but she was somewhat hesitant. Brenda was confident in her thinking and she took responsibility for explaining the situation to Dorothy. She looked at Dorothy's computer screen and made circles with one finger as she spoke. While pointing to the only bar on the screen, the 3-part bar, Brenda said, "This is 2/3 of whatever your water bottle would be [making a circular bar to indicate the bar to-be-made]" After Brenda's explanation, Dorothy set a goal of making the other bar, the process of which I discuss in the following protocol.

Protocol 6.7 has three parts: In the first part (PART A), I present the students'

construction of their resulting bars and Dorothy's explanation of her thinking. In the

second part (PART B), I present Dorothy's struggle when she was required to find the

measurement of one half of the three fifths of a liter. In the last part (PART C), I present

how I provoked Dorothy's thinking so she could imagine a liter using parts of the three-

fifths of a liter, interpret a mini-part in terms of a liter, and use this reinterpretation to find

the measurement of her resulting bar.

Protocol 6.7: Dorothy conceptualizing the water bottle and 2/3 of it without using the liter unit.

Protocol 6.7. PART A. Z: So, whose holds more? Dorothy: Ours. Z: Why would that be?

Dorothy: Because this is two-thirds of ours. I think I need a smaller part [erases her bar and makes another thinner bar. At this point, Brenda and Dorothy each had a bar with three parts and then they partitioned each part into two mini-parts, producing a total of 6 mini-parts.]

Brenda: Do you want us to fill this?

Z: Different colors? Yes.

Brenda: [Colored each pair of mini-parts black and red alternately. See Figure 6.17(a)]



Figure 6.17. Brenda's bars produced during the solution of Problem 6.8.

Explanation. On the left (a), Brenda's original 3-part bar. On the right (b), her

final bar.



Dorothy: [Colored each pair of mini-parts red, blue and black. See Figure 6.18(a)]

Figure 6.18. Dorothy's bars produced during the solution of Problem 6.8.

Explanation. On the left (a), Dorothy's 3-part bar. On the right (b), the resulting bar: Each colored column shows one-half of the 3-part bar and a third of her water bottle

Brenda: This is two thirds, you said. [Pointing to Figure 6.17(a)]

Dorothy: [Pulled out a mini-part from the bar in Figure 6.18(a) and copied it three times and joined them. She then copied this connected group of mini-parts three times and aligned them horizontally. She played with this configuration for a while; e.g., changed the colors of the mini parts etc. and ended up having the bar in Figure 6.18(b).]

Brenda: [Copied the whole bar and pulled out three mini-parts from the bar in Figure 6.17(a) and connected them to the top of the copied bar and produced Figure 6.17(b). Their construction took three minutes.]

Z: So, Dorothy, how did you do it?

Dorothy: Um. I divided it in three parts. Uhh...I forgot it was three-fifths. Um. I divided into three parts and then I divided into two, so I can figure out half of it, since it was two thirds of mine. And I took half of it that would be one third and multiplied it by three.

Z: So, half of that three-fifths, how much is that of a liter? Can you show half of three fifths?

Dorothy: Half of this [pointing to Figure 6.18(a)] or half of this in general [pointing to Figure 6.18(b)]?

Z: Half of three-fifths of a liter.

Dorothy: That will be like three-tenths [without pointing to anything on the

screen]. Z: So what is this [pointing to Figure 6.18(a)]? Dorothy: That is three-fifths and two-thirds of mine.

After Brenda rephrased the problem situation and referred to an imaginary bar on Dorothy's computer screen, I asked Dorothy "Whose [bottle] holds more?" to check whether she understood the problem situation as I intended. On her computer screen, there was only the 3-part bar, and when she spoke, she acted as if there was another bar for her water bottle. Her response of "Ours... Because this bar [her 3-part bar] is two thirds of ours" showed that Dorothy possibly did imagine the other bar as an independent bar to operate on and used it to conceptualize the given bar as 2/3 of it after she started operating on the initial 3-part bar. In this way, the first of the hypotheses relating to the requirements for the construction of inverse relationship is corroborated. She received help from Brenda in her conceptualization of the situation, so this might not be an independent act. Still, it is possible that she conceptualized a relationship between the two bars such that the 3-part bar was equivalent (as opposed to identical) to a part of the bar to-be-made since she did not construct a vertically partitioned bar in Figure 6.18(b) for a direct visual comparison between the right bar (which was given as 2/3) and the left bar (which was constructed as 3/3). While the way Dorothy constructed the bar in Figure 6.18(b)—whether she used an identity or equivalency relationship between the bars might be open to different interpretations, in this discussion, we should not forget that the other student, Brenda, conceived the bar to be-made as a separate and independent bar prior to acting. Therefore, it seems that we can make inferences about explicit equivalency relationships if we observe students explicitly indicating that general relationship between the quantities prior to acting, which otherwise could suggest that a

transformed form of the 3-part bar was embedded in the bar for the unknown quantity that is still a identity relationship.

Dorothy then partitioned each of the parts in the 3-part bar into two, producing six mini-parts. She then pulled out a mini-part, and copied the mini-part two more times. So she produced three mini-parts and conceived this group of mini-parts as "half of it [the 3-part bar]," the result of her distributive operations. When she said, "That [half of the 3-part bar] would be one third [of the bar-to-be-made]," she might have been referring to what I stated in the brackets. If so, then this would corroborate that she established an equivalency relationship between the parts of the two bars, which in turn corroborates the second hypothetical requirement. She then used this equivalency relationship in such a way that her purpose became to make a bar that was three times as much as half of the starting bar (3-part bar or "three-fifths").

Finally, Dorothy repeated (iterated) this group two more times to make the bar for her own water bottle (see Figure 6.18(b)). This equivalency of half of the 3-part bar to one third of the bar to-be-made framed her goal driven activities. Therefore, Dorothy satisfied the second hypothesis (the equivalency between the parts of the two bars) and most probably the third one<sup>43</sup> since she iterated the part three times. The satisfaction of the 2<sup>nd</sup> and 3<sup>rd</sup> hypotheses is important for asserting the construction of partitioning (of 3/5 of a liter into two) and iterating (one of those parts three times) as inverse operations. Therefore, it seemed as if she satisfied the three requirements that frame inverse reasoning. However, as she pointed out she forgot that the starting bar was "three-fifths," which caused some difficulty when she anticipated stating the measurements for the

<sup>&</sup>lt;sup>43</sup> Even though she did not explicitly say half of 3/5 liter is equivalent to a third of the bar-to-be-made at this part of the protocol, she states this relationship in PART B.

mini-parts.

Observing that Dorothy successfully produced the unknown quantity and stated the relationships between the bars (one is 2/3 of the other one) and between the parts of the bars (half of 3/5 is 1/3 of the other bar), I asked Dorothy how much of a liter half of the three-fifths of a liter would be. She was confused regarding which bar on her computer screen she needed to use as a reference for half of the three fifths of a liter. Her confusion might have been due to hearing "a liter" in the question statement. She paused for a while, and without pointing to any of the bars or parts of the bars in Figure 6.18, she said it would be "three-tenths." It was clear that the starting bar (3-part bar) did not represent 3/5 of a liter (a fractional part of a liter) to her; it was just a bar with three parts that she named three–fifths and she operated on this bar without erasing the marks. Interestingly, when she accepted the fact that she was not permitted to erase the marks on the bar, she did construct a sophisticated equivalency relation between half of the 3-part bar and a third of the bar to-be-made and proceeded accordingly. However, it seemed like I was posing a notational problem to Dorothy without any picture, and neither the problem situation (half of the 3/5 of a liter) nor the result of the problem (her saying three tenths) had any relationship to the bars in Figure 6.18. Because of this situation, I asked Dorothy how she came up with three-tenths.

Protocol 6.7. PART B.

Dorothy: Then I divided each part into two parts, and then I found half of it, which was one third of mine. Z: You said this is one third of yours, Dorothy: That is one third, that is one third, one third [pointing to the black, red, and blue colored groups of mini-parts in Figure 6.18(b)] Z: My question is, this is one third of your thing right? [Pointing to the first column of the bar in Figure 6.18(b)] Can you tell me how much is this [pointing to the same column] of a liter?

Dorothy: It is... Half of three fifths. That would be one point five over five. Z: Can you give me a fraction name? Dorothy: A half over five? Z: Say it again. Dorothy: Half over five... one and one half over five [I asked her to write down what she said since I was having difficulty understanding her response]. [She wrote "3/5" and said, "Since that one is half," and wrote "1.5/5" and said, "I do not think that is right though." She then wrote  $3/5 \times 1/2 = 3/10$  and said that it "was three tenths." She continued and said something but it was inaudible.] Z: So, three tenths, how can it be three tenths of a liter? Dorothy: Because. What was the question again? Z: That was three fifths of a liter. Dorothy: OK. Why is this three... because there is three of these, which was each color and there was five of those and then, half of the whole thing of the three fifths...why is it three tenths? Z: Yes, why is it three tenths? Dorothy: Well, first of all I multiplied the half by three fifths and I got three tenths

Because Dorothy focused on the thirds rather than my original question—the measurement of half of the three-fifths of a liter—I asked Dorothy to state the measurement of the thirds in terms of a liter. Interestingly, she said it was "half of three fifths" as if she were stating the measurement of a column in the bar in Figure 6.18(b). She then gave equivalent forms of the result of this division operation, "one point five over five" or "one and one half over five." After she notated "1.5/5," she produced another result that she was more satisfied with, 3/10, by multiplying 3/5 by 1/2. She neither realized that those two fractional notations were equivalent nor did she construct a quantitative meaning for those fractions using the bars in Figure 6.18. The productions on paper were the result of learning paper and pencil algorithms in her classroom.

Dorothy seemed to give some meaning to the three-fifths of a liter by saying "there is three of these, which was each color and there was five of those." She might have meant the three parts in the 3-part bar (Figure 6.18(a)) and five of them being a

whole liter. In this case, a part would be a fifth of a liter, and five of those parts would constitute a liter. However, she could not reinterpret half of the three-fifths in terms of a liter. It might be too much to expect Dorothy to operate further with a part that she had just conceived as a fifth of a liter and reinterpret half of it as part of a liter. Therefore, at this point, I believe Dorothy had a two-levels-of-units structure constructed; a whole liter is a unit composed of five units. But she did not operate with this structure to construct a three-levels-of-units structure in which a mini-part would be reinterpreted in terms of a liter, as the third level unit.

Dorothy was possibly aware that half of the 3-part bar was three of the mini-parts from the equivalency relationship she created between the parts of the bars in Figure 6.18. It might be the case that even though she constructed a mini-part as a quantity, she did not construct the measurement of a mini-part in terms of a liter, so she did not relate her algorithmic result of "three-tenths" to the bars or to the whole liter.

In order to explore how I could help Dorothy to conceptualize the whole liter for finding the measurement for half of three-fifths (of a liter), I asked her whether she could imagine the one-liter quantity. Our conversation follows:

Protocol 6.7. PART C.

Z: So, if you think about a liter, do you have a liter there?
Dorothy: No, I have three-fifths of a liter.
Z: So, imagine what would you have, how big will that be? If you were trying to make a liter?
Dorothy: It would be one of those multiplied by five.
Z: Which one?
Dorothy: Like each color [pointing to the parts of the bar in Figure 6.18(a)]
Z: So, multiplied by five. So, how many of this [pointing to a mini-part] will you have in a liter [without speaking I pointed to the mini-parts with my finger as if we were counting each mini part all the way from bottom of Figure 6.18(a) and continuing as if there were 2-3 more mini-parts on the top of the bar]?

Dorothy: Ten. I wasn't looking at those. Yes, it would be three of those because each of these is [a] tenth and it is same as [looking at Figure 6.18(a)]...this is sixtenths. And half of six tenths is three tenths.
Z: This is what you get, right? [Pointing to Figure 6.18(b)]. So how much is this of a liter?
Dorothy: Nine-tenths.
Z: Nine-tenths. Do you agree with her [asking Brenda]?
Brenda: Which one? Sorry.
Z: This one was three-fifths of a liter [pointing to Figure 6.18(a)] and this was two-thirds of another [pointing to the same bar and making circular motions with my finger for the other bar]...
Brenda: That is one, two, three, four, five, six, seven, eight, um. That is one, two, three, four, five, six, eight, ten [She was probably counting the mini-parts in the bar in Figure 6.17(a)].
Brenda: Yes, that is nine-tenths of a liter.

Dorothy said she did not have a whole liter but had three-fifths of it. When explicitly asked, Dorothy could imagine the whole liter by multiplying one of the three parts of the original bar five times: This way of operating is parallel to how I interpreted her conception of a liter in the PART B of Protocol 6.7 where she seemed to construct only two-levels-of-units structure. Since she imagined a whole liter in PART C, I thought she could reverse her thinking and place mini-parts in that hypothesized whole. Thus, I helped Dorothy with counting the mini-parts in Figure 6.18(a) without speaking, pointing to imagery parts with my fingers, and implied that there were a couple more mini-parts. Since she already convinced herself that there were five of those parts in the hypothesized liter, she quickly placed the implied mini-parts into the remaining (two) parts. Given that she was so quick at getting ten mini-parts as an answer, she might have reasoned that for each part there were two mini-parts and for the five parts (in the whole liter), there were ten mini-parts. Therefore, she possibly multiplied two by five. In this way, she was able to conceive the three-fifths of a liter bar in terms of tenths—six tenths—and consequently, she produced the result of taking half of three fifths of a liter as three

tenths. This advancement illustrates that she could coordinate tenths and fifths of a liter with a units-coordinating scheme and extend a two-levels-of-units structure to a threelevels-of units structure by reinterpreting a mini-part in terms of a liter. Furthermore, after she conceptualized a mini-part as a tenth, she used it to reinterpret the bar in Figure 6.18(b) as "nine-tenths." Therefore, there are indications that with help, she could operate as if she constructed a recursive distributive partitioning operation.

Brenda was also successful in this kind of interpretation, and she produced the result all by herself. We see three important changes in Brenda's activities that promise to be permanent. In her first attempt to solve this problem (Problem 6.8), I observed that Brenda conceived of the bar-to-be-made as an independent bar since she did not use any parts of the 3/5 of a liter bar to construct that bar. Therefore, she satisfied the first hypothesis for the construction of the inverse relationship between the bars. The process of her construction of Figure 6.17 indicates that she is now able to operate with this relationship meaningfully in her second interpretation of Problem 6.8. Therefore, the first important change is that she imagined and created the bar-to-be-made as independent of the first bar she made. The second important change is that, unlike her activities in Problem 6.7, for which she neither distributively partitioned the parts nor produced 4/5 of a gallon as a result of her partitioning operations, Brenda produced a half of the given bar using her distributive partitioning operations in Problem 6.8. Furthermore, in contrast to her initial activities in her first interpretation, she used the relationship (which was a result of the first change) to make the equivalency between the result of her distributive operations, half of the given bar, and a third of the bar-to-be-made. Since she did not iterate "a third" three times to make the bar-to-be-made, and did not have opportunities to

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talk about her actions in the Protocol, there is not enough evidence to make a claim about whether her partitioning and iterating operations were inverse or reversible operations. As the last change in her activities (compared to her activities in Problem 6.7), she did not have any difficulty using her recursive distributive partitioning operations to reinterpret a mini-part, once she constructed the bar-to-be-made as a result of her 1<sup>st</sup> and 2<sup>nd</sup> changes.

With the analysis of the next problem (Problem 6.9), I confirm the three changes that I inferred about Brenda's operations. I also discuss the operations that Dorothy used independently for constructing the unknown quantity. She possibly used a reversible fraction scheme to construct this unknown quantity. However, she used neither distributive partitioning operations nor the measurement of the known quantity in her operations. While Dorothy assimilated the recursive distributive partitioning operation for reinterpreting a mini-part in the context of Problem 6.8, she did not *transfer* this way of operating to Problem 6.9 and others. Dorothy did not use mini-parts in her construction of the bar-to-be-made. This situation might be due to her not receiving any outside help and her having neither a visual whole liter nor the known quantity as part of a visual liter.

Problem 6.9: My water bottle holds 4/5 of a liter and it is 3/7 as much as yours. Can you make the water bottles with JavaBars and figure out how much of a liter yours holds? (May 12)

Dorothy created a bar and partitioned it into four parts, presumably making a bottle for 4/5 of a liter. She then partitioned each of those parts into seven mini-parts, instead of three. I asked her to color each fifth (of a liter) differently and she colored each group of seven mini-parts differently. She paused for 20 seconds. She then said "Ohh...[with a surprise]" while Brenda and I were talking about Brenda's bar that the bar was 4/5 of a liter and 3/7 of the bar-to-be-made. Dorothy paused for a while because she was not sure she understood the problem. It is possible that she conceived this problem situation as if it were asking her to find three-sevenths of the original bar, unlike Problem 6.8 when Dorothy received help from Brenda and me. So she partitioned this bar into seven mini-parts and her purpose was probably to pull out four of them and, perhaps, iterate these four mini-parts three times for the bar-to-be made. Therefore, she did not conceive the problem as if the given bar was equivalent to a part of the bar-to-be-made.

Another 25 seconds passed, and Dorothy erased all the marks in her bar and partitioned it into three parts. She then erased this bar and made a smaller bar with three parts (see Figure 6.19(a)). She pulled out one of those parts and repeated it seven times, producing a new bar with seven parts. She said, "I divided [the starting bar] into three because yours is three sevenths, and I can just understand in my mind, that is four-fifths of a liter. So, yours is three-sevenths of that one [7-part bar]." Her comment indicated that she was aware of how to make the other bar using her reversible fraction scheme but without operating on the measurement of the 3-part bar. She just put aside the measurement of the known quantity by erasing all the marks (the three marks for the 4-part bar), and conceived the bar as three-sevenths of an unknown quantity. Therefore, the bar-to-be-made was seven parts or seven-sevenths (See Figure 6.19(b)).


Figure 6.19. Dorothy's bars produced during the solution of Problem 6.9. Explanation: (a) 3-part bar; (b) 7-part bar.

I asked Dorothy how much of a liter one of the three parts was if three of them were four-fifths of a liter. I told her that she could use paper and pencil if she wanted to. I was talking to Brenda so I did not see what she wrote from the beginning, but she erased something and wrote " $1/4 \times 3/7$  3/28." She was probably operating with whatever was available to her at that moment; a bar with three parts that was three-sevenths of the whole bar. Even though she was asked to figure out one-third of that bar, she used onefourth in her notations. She was told that the bar was four-fifths of a liter and she knew it had four parts, so one of the parts was one fourth and it was related to a measurement since it was part of a measured quantity of four fifths of a liter. Therefore, she included both 1/4 and 3/7 in her notations. It looks like both fractions in her notations were fractional quantities instead of one of them being an operator. Unfortunately, I did not get an explanation from Dorothy related to her writing. At the same time, Brenda gave an explanation for how she constructed the bar-to-

be-made. Her way of construction was consistent with her activities in the previous

problem and further corroborated the last two changes she made in Problem 6.8. For the

solution, Brenda first had a bar with four parts (4/5 of a liter) colored black and red

alternately, then partitioned each of those four parts into three parts. She pulled out four

of those mini-parts, producing 1/3 of 4-part bar as a result of her distributive partitioning

operation. She continued and explained her thinking in Protocol 6.8.

Protocol 6.8: Making the whole bar using 4/5 of a liter as 3/7 of it.

Brenda: I have to have three more [possibly implied a bar that was similar to Figure 6.20(a)), I mean two more because that is this, three sevenths, which I could just have copied that and then [erased the four mini-parts she already pulled out]

Z: But I want you copy whatever you pulled out.

Brenda: I need this whole thing anyway [pointing to the similar bar to Figure 6.20(a)].

Z: But you will also need parts of it?

Brenda: Yes. So [she made another copy of the bar] this is three-sevenths. So I need four more [she then erased those two bars and opened a new page to make a smaller bar. She made a 4-part bar, see Figure 6.20(a)] So this was three-sevenths. Z: Yes, three-sevenths, four fifths of a liter.

Brenda: Yes, four fifths of a liter. [Colored each part, fifths, black and red alternately upon my request. She quickly partitioned each part into three miniparts and created Figure 6.20(a). She made two copies of the whole bar and placed a group of 4-mini-parts on top of the previously made bar, producing Figure 6.20(b)]

[Meanwhile, I asked Dorothy to figure out how much is one of the parts, if three of them were four fifths of a liter.]

Brenda: Um. I did, because three sevenths. Because this whole thing is three sevenths, I copied twice. Because you need four more sevenths, to get to it, and this is three. I have added one three sevenths and then another one. And I needed, I have three sevenths. I needed four more because that is three sevenths; I pulled out one more seventh and put it on top.

Z: I will ask you the same thing, one of these sevenths [pointing to a part of the Figure 6.20(a)] which you pulled out and put it on top, how much is that of a liter?

Brenda: That would be one seventh.

Z: One seventh of which?



Figure 6.20. Brenda's bars produced during her solution of Problem 6.9. Explanation: (a) Her 4-part bar. (b) Her resulting bar.

Brenda: One seventh of this [pointing to the bar in Figure 6.20(b)], which is one liter, isn't it? Because it is a whole thing. It is not one liter?
Z: This was four-fifths of a liter [pointing to Figure 6.20(a)].
Brenda: Ok. Three. Four-fifths, so three, four so, this was four-fifths, so three more, twelve, thirteen, fourteen, fifteen, so four fifteenths of a liter.
Z: Four-fifteens of a liter? So how much is this [pointing to Figure 6.20(b)] of a liter?
Brenda: The four-fifteen times, four times seven [very quietly]
Z: Four times seven?
Brenda: Yes, it will be twenty-eight over fifteen, twenty-eight fifteenths of a liter.

Brenda's distributive operations in this solution corroborate the second change

she made in the previous problem; she could take the result of her distributive

partitioning operation, 4 mini-parts, and operate with that to produce the bar in Figure

6.20(b). Even though there are no indications that she was aware and explicitly said that a

third of the bar in Figure 6.20(a) was a seventh of the bar in Figure 6.20(b), she implicitly made that equivalency by adding the group of 4 mini-parts to the two copies of "3/7" bar (the bar in Figure 6.20(a)), so creating the resulting bar. In this sense, she did not produce an imaginary and independent water bottle for herself prior to acting, but she produced her water bottle by conceiving my water bottle as 3/7 of hers. Therefore, she started with 3/7 (situation) and produced the 7/7 (result) by using a reversible fraction scheme and operating on the result of her distributive partitioning operation.

When I asked Brenda how much of a liter "one of the sevenths [pointing to a part of Figure 6.20(a)]" would be, she said, "It would be one seventh." Even though I pointed to a part of the bar in Figure 6.20(a) whose part for "a seventh" or a third was not visually embedded, she transferred that amount to the bar in Figure 6.20(b) and made meaning for my question. Therefore, it is possible that she used an equivalency between the quantities of a third of the bar in Figure 6.20(a) and a seventh of the bar in Figure 6.20 (b). When I asked "one seventh of which?" she pointed to Figure 6.20(b), and then said, "One seventh of this, which is one liter, isn't it? Because it is a whole thing. It is not one liter?" Her comment implies that until this point, she operated and produced the resulting bar only using her reversible operations and did not feel a need to operate with the measurements of quantities in terms of unit liter. I pointed to the left bar and said, "This was four-fifths of a liter." She said, "Ohh. Okay. So this was four-fifths so three more [mini-parts], so that is twelve [mini-parts for 4/5 of a liter], thirteen, fourteen, fifteen. So it would be fourfifteenths of a liter." She imagined completing the liter not only using two levels of units, 4/5 of a liter, but also using the mini-parts. She did not complete the whole liter using the JavaBars but talked about how she would have acted and distributed three more miniparts into the last fifth of the liter. In addition, she conceived a seventh of the right bar as four fifteenths of a liter. She coordinated different levels of units and reinterpreted the quantities that she produced using the measurement of a mini-part. Her activities are quite advanced: she remembered how she produced different levels of units she worked with (e.g., a third of the 4-part bar, a mini-part, a liter, etc.) and distributed equal mini-parts into each fifth of a liter or each part of the 4-part bar. Therefore, she symbolically produced the three-levels-of-units structure that was necessary for deriving the measurements of the quantities. To state the measurement of the bar in Figure 6.20 (b), she operated further using the measurement of one seventh of the bar, which was 4 mini-parts at the same time, and said it would be "four times seven, twenty-eight over fifteen... twenty-eight fifteenths of a liter." Her way of operating in this problem also assured that the last change she made in the previous problem was permanent; she did not have any difficulty using her recursive distributive partitioning operations to reinterpret a mini-part.

I was satisfied with Brenda's explanation and I asked Dorothy whether she had her answer. While I was talking to Brenda, Dorothy independently changed the configuration for her resulting bar (see Figure 6.21(b)).



Figure 6.21. Dorothy's revised bars for Problem 6.9. Explanation: (a) Dorothy's 3-part bar. (b) Her new configuration for the 7-part bar.

Dorothy said she had six of them (referring to six parts in Figure 6.21(b)) and it was "eight-fifths." She probably doubled the four-fifths (of a liter), but she could not figure out how much of a liter one seventh of the bar-to-be-made was. This situation was partly because she did not work on the third level, a mini-part, so she did not distribute three mini-parts in each of the fourths of four fifths of a liter at the start. Most importantly, she did not feel a need to work on the unit level of a liter. Since she had an operational two-levels-of-units structure (see the discussion in Problem 6.8) and did not have a whole liter in front of her, she only used the multiple of 4/5 (of a liter) for 6/7 of the 7-part bar. She was in a state of perturbation to find how much one seventh of the 7-part bar would be in terms of a liter, and unfortunately we were out of time. Dorothy did not get a chance to explore how she could reach equilibrium on this situation. However, I believe that since she did not have an awareness of the levels of units she needed, one liter and four fifths of a liter, she would stay in this perturbed state until she constructed a three-levels-of-units structure for measurements of the quantities.

As a result, if Dorothy was given the known quantity as multiples of whole standard units (see Problem 6.7), she could transform the starting bar for the known quantity and independently produce the unknown quantity using her reversible partitive fractions scheme. This means she had to have a visual unit in front of her. If she is helped (see Problem 6.8) to imagine making a whole unit when the known quantity is given as a fractional part of the unit, she can successfully construct the unknown quantity and operate using inverse operations. However, since she did not independently produce an inverse relationship between the known and unknown quantity in Problem 6.8 nor she independently reinterpret a mini-part in terms of a unit measure, she was not able to produce a result for Problem 6.9 even though this problem was very similar to Problem 6.8. Overall, she only had a reversible fraction scheme, and this scheme was not enough to construct inverse reasoning and equivalency between the parts of the known and unknown quantities whenever the known quantity was a fractional unit and she had to produce an operative figurative image for a standard measurement unit. I believe if she had constructed recursive distributive partitioning operations, she would have independently solved the Problem 6.9 and operated very similar to how she operated in Problem 6.8. Therefore, she would have possibly constructed equivalency between the parts of quantities using inverse reasoning, and would have constructed partitioning and iterating as inverse operations.

### Summary of the Results of Chapter 6

## Fraction Multiplying Problems

Brenda's uncoordinated two units structures. In the first two problems of the set I analyzed in Chapter 6, the students mainly used JavaBars. They were occasionally asked to notate on paper the processes of their JavaBars activities and/or the relationships between their mathematical notations and their JavaBars activities. In Problem 6.1 when the students were asked to make 3/5 of a bar and then 1/7 of this amount without the whole bar being given, Brenda made a bar with three parts, and first wanted to delete the two marks and replace them with six marks.<sup>44</sup> When she was told that she was not allowed to erase the marks, Brenda partitioned each part of the 3-part bar into seven (producing seven mini-parts in each of the three parts), and with some discussion she pulled out three mini-parts to show 1/7 of 3/5 of the candy bar. While Brenda was aware of the two separate unit structures as indicated by her written notations (the 3/5 bar (3part bar) was part of a 5/5 bar and 1/7 of 1/5 in the 3/5 bar was a mini-part), she did not coordinate those two units structures and conceive the result of three mini-parts as embedded in the 5-part bar. When she labeled the three mini-parts as 3/21 it indicated that she constructed *distributive partitioning* operation, that is, to find 1/7 of the three parts together, she found 1/7 of each of the three parts, and she established three miniparts as 1/7 of the 3-part bar. But she might not have established the three-mini parts as 3/7 of one part. I think this was not a strong contributor to her lack of success of finding 1/7 of 3/5 in terms of 5/5, because she did not coordinate one part of the 3-part bar and

<sup>&</sup>lt;sup>44</sup> In this problem, Dorothy wanted to have the whole candy bar that was partitioned into five parts. With some discussion she made the bar with three parts. I did not give the details of this situation in Chapter 6 since I mainly focused on Brenda and how she used her JavaBar activities to produce notations on paper.

one part of 5/5 bar in her activities. Therefore, even though she did not construct the three mini-parts as 3/7 of one part, this type of distributive result would not have affected Brenda's construction of an initial fraction multiplying scheme when finding 1/7 of 3/5 of a bar (in contrast to what L.P. Steffe hypothesizes, personal communication, April 9, 2008). Because Brenda's operations were based on the mini-parts (the third level unit) and also on the interpretations of the mini-parts in relation to the 3/5 bar or 5/5 bar. Therefore, interpreting a mini-part as 1/7 of 1/5 and producing the result of 1/21 can be considered as fraction composition scheme and also as the beginnings of a fraction multiplying scheme.

In her activities, Brenda regarded the 3/5 bar as the unit bar and as if it was a 3part bar without a fractional connotation. Her activities with which she produced the quantity for 1/7 of the 3-part bar involved the operation of distributing the partitioning into seven parts across each of the parts of the 3-part bar, and was essentially a distributive partitioning scheme—her activities were similar to sharing three items equally among seven individuals by partitioning each of the items into seven. Each individual will then have three of the 21 smaller items, or one of seven equal parts. However, I cannot say the "sharing" fully describes and explains Brenda's operations. Since she was aware of the quantities as fractions (3/5 as 3/5 of 5/5 denoted in her writing) even if she did not coordinate one part of the 3-part bar and one part of the 5/5 bar while she operated. Therefore, her ways and means of operating constituted the beginnings of a *fraction multiplying scheme* since she considered neither having the whole candy bar nor conceptualizing the result, 3/21, as 1/7 of the 3/5 of 5/5 candy bar (not 1/7 of the whole bar). *Brenda's coordination of two units-structures.* To create a provocation for Brenda so she might consider the original bar in her activities, I posed another problem (Problem 6.2: Making 1/7 of 4/5 of a candy bar and then finding its measurement in terms of the whole candy bar). Brenda made seven mini-parts in each part of the 4-part bar and then I asked her to color each fifth differently. She did this by counting every seven mini-parts and coloring each of those groups of mini-parts alternately red and black. We then had a discussion in which she stated one fourth of the four-fifths of the candy bar is the same as one fifth of the five-fifths visually available to her, but she was able to make a coordination between two unit structures, each of which is a unit of units. This coordination explicated for her that depending on the referent quantity, while a quantity is identical to itself, it could be measured differently and it could be labeled with different fractions, such as 1/4 or 1/5.

The role of written notations in Brenda's accommodation of her fraction multiplying scheme. In the same problem (Problem 6.2), after Brenda partitioned each part of the 4-part bar into seven mini-parts, I asked her to write down what she did. She wrote  $1/5 \div 7$  and attempted to use the number of all the visible mini-parts in her JavaBars, 28 mini-parts, for the result. At that time, I intervened and asked her to produce the result using her notations. Since she did not know how to proceed, Dorothy helped using the rules that she learned in class and they produced 1/35 as a result. Brenda was constrained in giving an explanation why the result of dividing a fifth into seven pieces with her JavaBars resulted in 1/35 because she did not see 35 mini-parts in her JavaBars. Dorothy explained how she used the invert-multiply rule to write  $1/5 \times 1/7$  and to get 1/35. The language Dorothy used evoked Brenda's operations, and Brenda then made another important coordination at the mini-part level: One of the (28) mini-parts in her JavaBar could be justified as 1/35 of the five-fifths of the candy bar. She imagined distributing seven more mini-parts into the imagined extra fifth she constructed using 4/5of the candy bar. Therefore, I can conclude that Brenda made a functional accommodation in her fraction composition scheme and constructed a fraction multiplying scheme; the descriptive language that Dorothy used when computing to find  $1/5 \times 1/7$  helped Brenda to reinterpret her JavaBars distributing activity with an awareness of a mini-part as embedded in the whole candy bar.

Confirmation of the accommodation Brenda's fraction multiplying scheme with recursive distributive partitioning operation. Approximately 8 seconds after Dorothy claimed a mini-part would be 1/21 in Problem 6.4 (If my water bottle still holds 3/5 of a liter and yours holds 4/7 of mine, can you make your water bottle and figure out how much it is of a liter?), Brenda said, "Wouldn't it be because you have five pieces [five parts in a liter] and you divide each piece into seven, so thirty five [mini-parts]... this will be one, two, three, four. . . twelve over thirty-fifth [for the resulting bar]." Brenda conceived the new problem situation as producing the whole liter. The liter consisted of five parts and each part was partitioned into seven mini-parts. Brenda imagined partitioning each of the extra two fifths of a liter into seven mini-parts without having a whole liter in front of her. Therefore, with this advanced operation (compared to distributive partitioning)—that I called *recursive distributive partitioning*—Brenda produced another (second) unit-of-units-of-units structure symbolically.<sup>45</sup> The containing unit of a liter had five units and 35 mini-units. In addition, Brenda also coordinated the two three-levels-of-units structures, so she could give an explanation for twelve thirty-fifths of a liter as a measurement of 4/7 of 3/5 of a liter.

In Problem 6.6 (My water bottle holds 11/6 of a liter and yours holds 3/5 as much as mine holds), unlike Dorothy, Brenda did not color six parts of her 11-part bar for a liter to start with. But she did verbally construct the liter using a mini-part (by adding five mini-parts six times) and reinterpreted a mini-part in terms of a liter when conceptualizing the measurement of the resulting quantity. Brenda said:

Brenda: Because three-fifths of eleven-sixths, each fifth is eleven little pieces, so then if you are thinking there is five in each sixth, so then in six-sixth there are ten, fifteen, twenty, twenty-five, thirty [looking at her 11-part bar] and then in the three-fifths of eleven-sixths there is um, more than thirty pieces. Eleven three times so it is thirty-three pieces.

Therefore, she constructed 3/5 of 11/6 of a liter as 33/30 of a liter. In addition, Brenda partitioned only the first three parts (as opposed to all the parts) of the 11-part bar into five mini-parts per part and pulled out a group of 11 mini-parts. Her activities indicate that she interiorized distributive partitioning since she acted as if she had already partitioned the other parts of the bar while she only partitioned the first three parts, resulting in 55 mini-parts in total, and conceptualized 1/5 of the 11/6 of a liter as 11 miniparts.

<sup>&</sup>lt;sup>45</sup> The first three-levels of units structure was demonstrated by Brenda's starting bar that was composed of three parts and each of those parts contained seven mini-parts. Actually, another three levels of units structure can be attributed to Brenda's activities, in which the bar with three parts was also a bar with seven parts each of which contained three mini-parts.

## Dorothy's Construction of a Distributive Partitioning Scheme

In the discussion and the analysis of Problems 6.3, 6.4, and 6.5, I had opportunities to focus also on Dorothy's activities related to fraction multiplication. As in the previous problems, in Problems 6.3 and 6.4, when the given quantity was part of a whole unit measurement (such as 3/5 of a liter). Dorothy wanted to start with the whole bar with 5-parts (and to color three of the five parts) instead of an initial 3-part bar. With some discussion, she agreed to start with a 3-part bar. But it is not certain for Dorothy whether each of the parts in the 3-part bar was also a part of a liter. In the solution of Problem 6.4 (My water bottle holds 3/5 of a liter, and yours holds 4/7 as much as mine), Dorothy acted with the goal of finding 4/7 of a 3-part bar. She started partitioning each part of her 3-part bar into seven mini-parts. Afterwards, Dorothy colored each group of three mini-parts in her bar alternately blue and red.<sup>46</sup> So. Dorothy transformed her 3-part bar with seven mini-parts per part to a 7-part bar with three mini-parts per part (see the left bar in Figure 6.6). She then made a unit of three mini-parts, and repeated this unit three more times, producing a bar of total of four units of three mini-parts. Therefore, I attributed a *distributive partitioning scheme* to her. When I asked about how much of a liter the resulting bar would be (4/7 of 3/5 of a liter), she said, "Four-sevenths." Dorothy then claimed one of the mini-parts would be "one twenty-first," but I am not sure whether she reinterpreted the mini-part as the result of taking 1/3 of 1/7 of the starting quantity. Dorothy did not make a distinction between 3/5 of a liter and one liter. On the other hand, Dorothy used partitioning, distributing, and iterating operations and produced three levels of units when taking a fractional part of a whole number quantity (4/7 of the 3-part bar).

 $<sup>^{46}</sup>$  Brenda kept the three parts intact as indicated by her coloring each group of seven mini-parts differently (see the left bar in Figure 6.6.1).

The operations of her fraction multiplying scheme that she constructed a quantity for 4/7 of the 3-part bar were as follows: Dorothy first *distributed* the partitioning seven miniparts across each part of the 3-part bar. She then colored every three miniparts alternate colors and transformed the bar into a 7-part bar where each part consisted of three miniparts, and then pulled out three miniparts (see Figure 6.6). Since she was aware that a unit of three miniparts was a seventh of the 7-part bar, she iterated that unit and produced a total of four copies to construct 4/7 of the bar. Therefore, the three-levels-of-units structure (a bar composed of seven units where each unit consisted of three units [mini-parts]) she used to construct 4/7 of the starting bar can be thought of as a product of Dorothy's distributive partitioning scheme. While she did not denote the measurement of the resulting quantity either using a part or a liter as a reference, she conceptualized the result in relation to the unit of 3-part bar.

With the described (beginning) fraction multiplying scheme, Dorothy definitely could make a quantity for a fraction of a whole number, but even in this case, it is problematic whether she could have interpreted her result as 12/7 of one of the three parts. Had I asked her to make this interpretation, I believe that she would have been able to do so with guidance, but whether the interpretation would have been a result of logical necessity is problematic. It is problematic because she transformed the 3-part bar to a 7-part bar, so she changed the number of mini-parts in the parts and there was little indication that she conceptualized a mini-part as a seventh of one of the parts of the 3-part bar. In addition, there was little indication that Dorothy could coordinate a part of the 3-part bar as a fifth of a liter, and so it is questionable whether she conceptualized the 3-part bar in terms of a liter in this problem situation. In any case, she did not construct the

second three-levels-of-units structure as Brenda did using her recursive distributive operation. Still, all of Dorothy's operations and her distributive partitioning scheme were anticipatory due to the fact that she was not randomly exploring the possibilities for finding the fractional parts. She acted as if she knew what she needed to do and in what order before ever taking any action.

In Problem 6.5 (My water bottle holds 4/5 of a liter and yours holds 7/6 of whatever mine holds), Dorothy operated with a reversible (iterative) fractional scheme and produced an improper fractional quantity for 7/6 of a 4-part bar. She used a very creative partitioning scheme as an extension of her distributive partitioning operations: she transformed her 4-part bar into a 6-part bar with 12 mini-parts. However, Dorothy could not state the result in terms of a liter. This situation is interesting because she operated with sophistication, yet she did not produce the measurement of the quantity in terms of a hypothetical unit. Her reversible operations show that she certainly can operate on the three levels of units since she constructed an improper fractional quantity (7/6 of the 4-part bar). In spite of this, she could not use her reversible operations to construct the hypothetical unit of a liter. Therefore, the operations available to her were not sufficient to construct the second three-levels-of-units structure (in which a liter is the unit that contained five units (parts) each of which contained three units (mini-parts)) that could only be constructed symbolically, mainly using recursive distributive partitioning operations.

*The role of embedded measurement unit in the given quantities for Dorothy.* In Problem 6.4, I asserted that Dorothy's operations depended on using perceptual material. In Problem 6.6 (My water bottle holds 11/6 of a liter and yours holds 3/5 as much as mine holds), Dorothy made a bar that contained the whole liter. Using this relationship, a liter embedded in the given bar, she was able to construct not only the second unit structures, but she also coordinated the two three-levels-of-units structures. Thus, she successfully produced the measurement of the resulting quantity, 33/30 of a liter, as she failed to do in the previous problems. By creating 3/5 of the 11-part bar as another bar, Dorothy produced a unit of units of units structure; the 11-part bar was the unit, which contained five units, each of which included 11 mini-units. Unlike her operations in the previous problems, she created this structure without transforming the 11-part bar into a 5-part bar. This situation implies that at that point, Dorothy had a multiplicative unit structure so that the same part was both a part of a liter (1/6 of a liter) and part of the given bar (1/11 of the bar). When she was asked to color the result of dividing a sixth into five in the bar, she colored a mini-part purple (see Figure 6.9 (a)). She said the purple mini-part was 1/30, using her written operations as a means to produce this result, but when asked, she did say that a mini-part was "one thirtieth of ele... of one liter." She preferred this labeling over "one fifty-fifth," which was the result she produced in her initial explanation. Dorothy could operate on this reinterpretation of a mini-part in terms of a liter to reconceptualize 1/5 and 3/5 of the 11-part bar, as 11/30 and 33/30 of a liter respectively. Therefore, she extended a two-levels-of-units structure (a liter is a unit containing six units) to a three-levels-of-units structure by reinterpreting a mini-part as 1/30 of the liter. However, she did not have this way of operating prior to acting. The fractional results, such as 1/5 of 1/6, were consequences of her distributive partitioning operations with an addition of a measurement unit, but they were not symbolic in the sense that she did not have an anticipatory way of justifying why the result was 33/30 of

a liter.

When I stated Problems 6.3, 6.4, 6.5, and 6.6, I thought that students would conceive the other water bottle (unmeasured quantity) as a separate quantity. At the time of teaching, I did not aim to investigate whether they differentiated between the two quantities where the unknown quantity was related to the starting quantity or how I could engender this type of implicit thinking about two separate but related quantities. While this situation did not seem as significant then as it does now, conceptions of two such separate quantities became important in inverse reasoning problems and for constructing an unknown quantity. Therefore, I analyzed Problems 6.7, 6.8, and 6.9 to investigate their conceptions of two separate but related quantities and the functions of these conceptions in the construction of reciprocal fractions as an extension of their inverse reasoning. Basically, these three problems can be conceived of as stating and solving equations with one unknown in the form of ax = b, where a, b are fractional numbers and x is the unknown quantity.

### Inverse Reasoning Problems

*Brenda's initial inverse operations without distributive partitioning operations.* In Problem 6.7 (Four gallons of whole milk is 5/6 as much as the skim milk), Brenda created two independent bars (one with 5-parts and the other with 6-parts) with JavaBars and also with pencil and paper. While she labeled each part of the 5-part bar and 6-part bar as .8 after she divided 4 by 5 and stated the measurement of the 6-part bar as "4 gallons and 4/5 gallon," she had difficulty discussing how 4/5 gallon was related to her bar figures both on paper and in JavaBars. It is possible that since she did not use a distributive partitioning operation in the solution of this problem, she could not conceive of mini-parts and did not use mini-parts to conceptualize the quantity of a whole gallon.

Dorothy's reversible fraction schemes with distributive partitioning operation.

In Problem 6.7 (Four gallons of whole milk is 5/6 as much as the skim milk), Dorothy did not conceive the bar for skim milk as a separate quantity, but acted to find the whole 6/6 as if 5/6 of it was given. After she made the bar with 4-parts (for 4 gallons of whole milk), she partitioned each part of the 4-part bar into five mini-parts and colored every other four mini-parts black and red, so she transformed the 4-part bar with five mini-parts per part into a 5-part bar with four mini-parts per part. Her purpose was to make a 6-part bar for the skim milk, and she did this by copying the 5-part bar and adding a unit of four mini-parts. She constructed this unit of four mini-parts with her distributive partitioning operation when she transformed the 4-part bar into 5-part bar. Therefore, 5/6 of the bar-to-be-made was identical to the transformed form of the first bar (the 4-part bar that was transformed to a 5-part bar), which was 4 gallons.

In addition, Dorothy was able to state the measurement of the 6-part bar as "4 gallons and 4/5 of a gallon skim milk." While it seemed that she could coordinate two three-levels-of-units, this situation was only possible because she had a gallon to start with, which was embedded in the 4 gallons. Since Dorothy was explicitly aware of the measurement unit of a gallon in her statements—especially for the result of her distributive operations, 4/5 of a gallon—her activities could also be interpreted as recursive distributive partitioning operations. However, while being aware of that the measurement unit of a gallon is important, this awareness does not require the same cognitive demands for constructing a unit as an operative figurative image in the absence of a perceptual unit. A unit of a gallon was already visually embedded in the 4-part bar as

one of the parts, and there was no need for Dorothy to imagine constructing a unit measure of a gallon for reinterpreting the results. Therefore, I determined the result of making such a coordination—4/5 of a gallon is 1/5 of 4 gallons (see Figure 6.13(b))—as an extension of her distributive partitioning operations. This extension is based on the accommodation she made in Problem 6.6 that as long as she had a visual measurement unit embedded in the given bar, she could state the measurements for the results of her partitioning operations in terms of a gallon, such as four mini-parts is 4/5 of a gallon. *Three Hypotheses Related to Construction of Inverse Reasoning* 

Using Dorothy's and Brenda's activities (and what was missing in Dorothy's activities) in Problem 6.7, I made three hypotheses in terms of what kinds of operations are needed for constructing an inverse relationship between two quantities.

1. The student needs to conceptualize the bar-to-be-made (or the unmeasured/unknown quantity) as a separate, independent, and imaginary bar even before acting.

2. The student should be explicit in her construction of the equivalency relationship that 1/5 of the 4 gallons of whole milk (given/known quantity) is 1/6 of the bar-to-be-made for the skim milk (unmeasured/unknown quantity).

3. The student should disembed one of the five parts and indicate that it is a sixth of the bar-to-be-made and iterate that quantity six times to produce the 6-part bar. During this construction, the student should use language emphasizing that one of those parts is a sixth of the skim milk (unknown quantity).

The first hypothetical requirement, conceptualizing the existence of two independent bars, is fundamental: first, to making a general relationship between the two quantities and then, to using this relationship to reconstruct partitioning and iterating operations as inverse operations (the second and third requirements). The second requirement is to operate with this general relationship and to take the first observable action on this relationship, partitioning the known quantity. The purpose of the partitioning action is to make an equivalency relationship and to reconceptualize the result of distributively partitioning the known quantity as an equivalent part of the unknown quantity. This requirement assumes that the first requirement is satisfied. In the last requirement, the student operates further with the equivalency relationship she constructed (the second requirement) and uses an iteration operation to construct the unknown quantity; in this way, iterating a part of an unknown quantity and partitioning a known quantity (for the equivalency) become *inverse operations*. For example, in the case of creating 6/6 of the quantity of the skim milk (result) when the measurement of 5/6of it is given as whole milk (situation), a student will partition the quantity of whole milk into five parts, and then take one of those parts as a sixth of the skim milk (equivalency), then iterate one of those parts six times to produce the skim milk. Actually, this kind of operating will be the inverse of the situation and the result when creating 5/6 of the skim milk as an equivalent quantity for the whole milk (result) when the skim milk is given (situation). The student will partition the skim milk quantity into six parts, take one of those parts as equivalent to a part for the whole milk quantity, and iterate one of those parts five times to produce the whole milk quantity. The inverse of partitioning the skim milk into six parts and iterating a part five times is partitioning the whole milk into five parts and iterating that part six times, because the situation that the student acts on and the result are inverses of each other and the operations take place on the equivalency

relationships between the two quantities.

I further investigated these hypotheses with Problems 6.8 and 6.9. These problems are variations of Problem 6.7 in which I changed the known quantity from whole units to a fractional part of the unit measure, but kept the proper fractional relationship between the known and unknown quantities, such as "my water bottle is 3/5 of a liter and it is 2/3 as much as yours" (Problem 6.8).

Brenda's inverse reasoning with distributive partitioning operations. In Problem 6.8 (My water bottle holds 3/5 of a liter and it is 2/3 as much as yours.) Brenda had two independent bars: one with three parts for my water bottle (each of which she partitioned into two (producing six mini-parts) and colored the two units of three mini-parts red and blue (to show it was 2/3 of something)) and one with three parts for her water bottle (unknown quantity). She did not initially use any parts of the first bar to make the other bar for her water bottle. Therefore, she satisfied the first hypotheses of inverse reasoning, which was having two independent quantities prior to acting. Even though she produced each of the two bars as independent entities, she was perturbed because she did not know how to make an equivalency relationship between the parts of the bars as required in the second and third hypotheses. While Brenda used a distributive partitioning operation in her first solution to Problem 6.8 for making half of the 3-part bar (3/5 of a liter), she somehow did not use this quantity as the measurement of the third of the bar-to-be-made even though she stated that the 3-part bar (3/5 of a liter) was supposed to be 2/3 of the bar for her water bottle.

When we revisited the problem the second time, she asked about *as much as*, the phrase I used in the problem, and asked "so this [pointing to her 3-part bar] is 2/3 of

ours?" I agreed with her reinterpretation because that was meaningful to her. She then helped Dorothy and interpreted the problem situation using *of*. In her solution, after she distributively partitioned the 3-part bar, she pulled out three mini-parts, and placed them on top of a copy of the 3-part bar. Therefore, it seemed as if she was constructing the new bar using 2/3 of it.

Similarly in the last problem (6.9), when I rephrased the problem situation by using *of* instead of *as much as*, Brenda proceeded and made the equivalency relationship between the parts of the bars. When she proceeded using *of* for constructing a meaningful relationship between the bars, her activities can be thought of as reversible operations as opposed to inverse operations since she constructed the bar-to-be-made using the given bar as opposed to conceiving it as an independent entity prior to acting. However, she was aware of the bar-to-be-made (the unknown quantity) as a separate entity as discussed in Problem 6.7 and 6.8. Therefore, we can assume that her partitioning and iterating operations are inverse of each other as opposed to being reversible. Because this way of thinking was different than how Dorothy acted using only reversible fraction schemes in which Dorothy did not start with two independent entities but produced the unknown quantity using known quantity (or a transformed equivalent of it) with her reversible fractions.

Dorothy's dependently constructed inverse operations with distributive partitioning operations. Problem 6.8 is important for illustrating boundary situations for Dorothy's activities in inverse reasoning problems. In her first solution to Problem 6.8, Dorothy conceptualized the first bar as a 3-part bar (for 3/5 of a liter). Using two parts more (2/3 as much as) from the 3-part bar, she formed a 5-part bar for her water bottle. As similar to her initial activities in Problem 5.10, she conceived 2/3 as two parts out of the 3-part bar. Therefore, I infer that she did not conceptualize two independent but related quantities to start with.

In her second solution of Problem 6.8, with some help, while she created a bar for the unknown quantity and satisfied the last two of the three requirements of inverse reasoning, she did not independently produce the measurement for the unknown quantity. After she distributively partitioned the initial 3-part bar into half, and produced 3 miniparts as half of the 3-part bar. She then used this unit of mini-parts as a third of her water bottle and iterated it three times to make the new bar. Even though she acted as if she made an equivalency relationship between half of the 3-part bar and a third of the bar for her bottle, I cannot claim she independently operated this way. In addition, she had difficulty producing the measurement of the unit of three mini-parts. It might be the case that even though she constructed a mini-part as a quantity, she did not construct the measurement of a mini-part in terms of a liter, so she did not relate her computational result of "three-tenths" to the bars or to the whole liter. I helped Dorothy imagine making the whole liter by using parts of the known quantity. However, she did not independently act in the other situations to imagine making the measurement unit and did not act as if she constructed inverse operations (see Problem 6.9: My water bottle holds 4/5 of a liter and it is 3/7 as much as yours.)

Dorothy conceptualized the situation of Problem 6.9 using only her reversible fraction scheme and disregarded the measurement of the known quantity. For example, after she made a bar with four parts (for 4/5 of a liter) and partitioned each part into seven mini-parts, she erased the marks in the bar and made the bar with three parts and pulled out one part and repeated it seven times. She said, "I divided [the starting bar] into three because yours is three sevenths, and I can just understand in my mind, that is four-fifths of a liter. So, yours is three-sevenths of that one [7-part bar]." Since she did not conceptualize mini-parts but worked only at the second-level units (parts), this situation possibly deterred her from finding either the measurement of the unknown quantity or conceptualizing that quantity as a result of some inverse operations. This inability to produce measurements may be due to her not having a visual whole measurement unit and, therefore, not constructing the second three-levels-of-units structure that is necessary for coordinating a quantity and its measurement in terms of a liter. Therefore, Dorothy's activities suggest an important hypothesis. It is possible that constructing and using a recursive distributive partitioning operation (for the construction of a unit structure for the measurement of quantities) and constructing inverse reasoning using inverse operations (for creating an unknown quantity by establishing and operating on an equivalency relationship) might be related psychological structures.

By using Brenda's and Dorothy's activities in Problem 6.9, we can conclude that reversible fraction schemes are not sufficient to be able solve these type of inverse reasoning problems, and a student needs to construct operations such as partitioning and iterating as inverse operations in addition to constructing both distributive partitioning (to make the bar-to-be-made (unknown quantity)) and recursive distributive partitioning operations (to find the measurement of the bar-to-be-made (unknown quantity)). Both inverse operations and symbolic fraction multiplying schemes require conceiving two independent but related quantities prior to acting and making equivalency relationship between the parts of two original quantities (not the transformed ones).

I stated three components of *inverse reasoning* in simple terms as follows: conceiving separate bars prior to acting, making an equivalency relationship between the parts of the quantities, and multiplicatively constructing the unknown quantity. The reversible fraction schemes are involved in the last component of these three. The construction of the unknown quantity using the measurement of the equivalent parts of the known and unknown quantity is made possible using reversible schemes. However, since having the conception of two quantities prior to acting indicates that the unknown quantity is conceived as independent of the known—a necessary view if the statement between the known and unknown quantity is an algebraic construct—then this reversible operation is not a reversible fraction operation in the traditional sense that Steffe defined it. For example, in Problem 6.8, where the students were asked to find how much my water bottle held if 3/5 of a liter bottle held 2/3 as much as mine. I expected the student to have the awareness of working on the quantity that was *equivalent* to both, say, one-half of the starting known (before the transformation of the starting bar) and one-third of unknown quantity.

I discussed that if the student can produce results in terms of the standard measurement unit, for example, one-half of the known quantity (either the starting or the transformed bar) in relation to the standard measurement unit, that is 3/10 of a liter, and operate on this measurement as the measurement of the third of the unknown quantity, then I would claim she definitely used inverse operations. The operations are not only part of reversible fraction scheme, but they are also part of a more sophisticated scheme than the reversible fraction scheme. In this scheme, the independent quantities relate to each other with their measurements in standard units. I think this is the power of

algebraic thinking. One no longer depends on the quantities she produced as a result of operations, but one makes the relationships using what is common in all those quantities—their measurements in standard units. Therefore, that is why Brenda seems to be one step ahead of Dorothy.

Brenda did not explicitly use the measurements of standard units when she made the equivalency relationship between the parts of the known and unknown quantities, yet she was aware that the quantities have measurements in terms of standard measurement units. I hypothesize that she can use those measurements if she was asked to do so. In the problems, such as Problem 6.7, she demonstrated all of what I hypothesized; two separate bars prior to acting, the fifth of 4 gallons was equivalent to the measurements of a fifth of the starting bar and a sixth of the unknown bar, and sixth of the unknown quantity was .8 liter. However, we need to acknowledge that she did not use distributive partitioning operations in her solution when producing .8 gallon or 4/5 of a gallon, so it was not immediate for her to make the relationship of the 4/5 in her result (4 gallons and 4/5gallon) to the standard measurement unit. Therefore, I don't know how important it is to be able to interpret the results in relation to constructions that are results of distributive partitioning operations. When Beth's activities in the second attempt to the solution of Problem 6.8 and in her solution to Problem 6.9 are considered, her use of distributive partitioning operations to make the unknown quantity might prevent us from making inferences about the inverse operations because the parts of the bar Brenda constructed as the known quantity was used in the construction of the bar for unknown quantity. Brenda's operations might be viewed similar to Dorothy's activities in Problem 6.8; J. Olive (personal communication, April 22, 2008) claimed that both Beth's and Dorothy's

operations were inverse operations since Dorothy transformed the bar for the known quantity to something else other than the original bar. However, we need to acknowledge that Brenda's construction of two separate bars in Problems 6.7 and 6.8 is a strong indication of her view of the unknown quantity as a separate quantity prior to acting. Therefore, Brenda's operations are different from Dorothy's in that Brenda viewed the unknown quantity and the fractional relationship between the quantities as givens prior to operating.

# CHAPTER 7: DISCUSSION, SUGGESTIONS AND IMPLICATIONS OF THE STUDY

In this chapter, I first review the two research questions using the findings of the study derived from Dorothy's and Brenda's activities. I then provide perspectives of the findings in relation to the literature on algebraic and quantitative reasoning, and fraction multiplication and related operations. In this part, I also introduce Figure 7.1 that I created with the constructs derived from the findings of the study. In the section following the perspectives, I discuss the unresolved issues and also provide some research suggestions. Finally, I present some implications of this research for teaching and future research.

Discussion of the Research Questions in Relation to the Findings Research Question 1: What operations are involved in students' construction of a fraction multiplying scheme in quantitative situations?

Dorothy's Construction of the Beginnings of a Fraction Multiplying Scheme

Dorothy did not construct a fraction multiplying scheme that was independent of the specific numbers in problem situations. In this section, I will explain how Dorothy operated with the fraction multiplying situations, what the reasons were that prevented her from constructing a generalized fraction multiplying scheme, and how those reasons showed themselves in Dorothy's activities over the two chapters.

*Dorothy's operations*. Dorothy produced fractions of fractional quantities by distributively partitioning each part of the given quantity to produce mini-parts, grouping

a certain number of mini-parts, and then repeating that unit of mini-parts, similar to her activities in Problem 6.4 when finding 4/7 of 3/5 of a liter. Her activities constituted only the beginnings of a fraction multiplying scheme because she did not reconstruct the resulting quantity as neither 12/7 in the case of 4/7 of the 3-part bar (which would have indicated that she took each part of the three parts of the initial liter as a measurement unit) nor as 12/35 in the case of 4/7 of 3/5 of a liter (which would have indicated that she was aware of each part of the 3-part bar as a fifth of a liter). Even though with help she could state that one of the mini-parts was 1/21 of the 3-part bar, it is not certain whether she was aware of this result as 1/7 of 1/3 and abstracted the relation that 1/7 of 1/3 is 1/21. As I wrote in Chapter 6 and its summary, as long as the measurement unit was embedded in the starting quantity (such as 11/6 of a liter, Problem 6.6) she successfully found the result of making fractional parts of the given bar and further operated on it to find its measurement. In her solution to Problem 6.6, her distributive partitioning operations might seem as if they were recursive because there was an awareness of the third level (mini-part). However, her awareness was made possible by the embeddedness of the liter in the 11/6 liter and her distributive partitioning operations were still not recursive—she did not need to construct a measurement unit independently from what was given in the problem situation.

Therefore, even though she could operate in all of the problems and produce quantities, if the given quantity in the problem was a fractional part of a standard measurement unit, the quantity she produced was not the result of a fraction multiplying scheme. One of the indicators of a fraction multiplying scheme is to reinterpret the quantities Dorothy produced in relation to the given measurement unit or other units discussed in the problem situation.

*Constraining factors in Dorothy's construction of a fraction multiplying scheme.* As I explained earlier, when Dorothy used JavaBars, I observed her using distributive partitioning in both problems of inverse reasoning (cf. Problem 6.7) and problems of finding fractional parts of quantities (cf. Problem 6.4), but she did not take the results of distributive reasoning (e.g., 4/7 of the 3-part bar in Problem 6.4) as material of further operating to produce the measurement of that quantity in terms of a liter. I think there are two important factors for this situation: (a) she viewed fractions as a series of operations, and (b) she used identity relationships in these problems unlike her use of equivalency relationships in the problems of Chapter 5. These two factors functioned together and I discuss and elaborate on them in the following paragraphs.

*Inverse reasoning problems.* In the inverse reasoning problems, I can say that even though she had constructed reversible fraction schemes, Dorothy could not take the results of the schemes as givens prior to fractional operating. If she had abstracted the results of fractional operating and could take the results as given prior to operating, she would have produced two separate and independent bars to start with as Brenda did. Dorothy did not have a means of conceiving of the unknown quantity as a result of fractional operating prior to operating, so she actually had to operate to produce it, such as when she used the 5-part bar to consider the unknown quantity of 6-part bar as derived from the 5-part bar—which was actually a transformed equivalent of 4-part bar.

If Dorothy had constructed recursive distributive partitioning operations (which implies a fraction multiplying scheme), she would have been successful in finding the measurements of the unknown quantities she produced using her distributive partitioning operations. For example, in Problem 6.8 (3/5 of a liter container holds 2/3 as much as another container), even though she produced half of the 3-part bar using distributive partitioning, she only produced 3/10 of a liter as her result by using computation. Her lack of using the three mini-parts she made in JavaBars to explanation the computation is compatible with the observation that she couldn't take the results of distributive partitioning as input for operating further.

Furthermore, I hypothesize that if she conceptually knew that the three mini-parts were equivalent to 3/10 of a liter, she would not have had difficulty in conceiving the equivalency relationship between the parts of known and unknown quantities, which implicitly implies the equivalency between the measurement of those quantities. Had she made a conceptual explanation for the computational result, that would indicate that she could view 1/2 of the 3-part bar not only as partitioning the quantity into two halves as a series of operations, but also as conceiving of the result as the combination of 1/2 of each part of the 3-part bar, where each part played the role of a measurement unit. Had she engaged in such distributive reasoning<sup>47</sup>, it could have led to conceiving of the whole liter and to a construction of recursive distributive partitioning operations.

Dorothy's lack of distributive reasoning has a connection to how she could operate in inverse reasoning problems; for example, when she conceives the unknown quantity. What this means is that when asked to produce one half of a 3-part bar, she can partition all three parts of the 3-part bar into half and pull out three mini-parts. So,"1/2" refers to an operation. Yet, she did not establish the relationship that the measurement of

<sup>&</sup>lt;sup>47</sup> I distinguish distributive reasoning and distributive partitioning operations. The construction of the latter operations is based on making, say, five equal shares of three separate items by partitioning each item into five parts and taking one mini-part from each item and conceptualizing this result as one-fifth of the three items together, and the former on the insight that one of the five shares is three-fifths of one item.

1/2 of the 3-part bar was equivalent to the measurement of 1/3 of another separate bar. The reason was that she could not find the measurement of a mini-part by constructing an equivalency relationship between the part of a whole liter and the mini-part (distributive reasoning). When making the unknown quantity in the inverse reasoning problem, Dorothy conceived the three mini-parts she produced by finding 1/2 of the 3-part bar as identical to one of the three parts of a three-thirds bar that was implied by "2/3 of another bar." The unknown quantity was only implicit in her goal to make a three-thirds bar. For the lack of a better term, I used *identical relationship* to refer to the implicit relationship between the three mini-parts that she made when finding 1/2 of the 3-part bar and one of the three parts of a three-thirds bar. She produced the three-thirds bar by making a bar that was three times as much as half of the 3-part bar without using the measurements of the quantities. Because she did not construct the *equivalency* between the parts of two independent quantities, she did not operate with the measurements of the quantities she operated with (the 3-part bar was actually 3/5 of a liter and one-half of the 3-part bar was actually 3/10 of a liter). This situation shows that she can produce quantities if they are results of her fractional operations but she cannot take the results recursively as an input for operating further.

Whole-part-part problems. At the start of the whole-part-part problems, Dorothy reconceptualized the quantity expressed as a known numerosity as *equivalent* to an *n*-part structure. But she did not need to construct the measurement of the quantity because it was already given as a multiple of standard measurement units, such as 50 inches. As a consequence, she could operate on the quantity when finding the length of one of the *n* units she constructed with her *n*-part structure. When the quantity was given as a

fractional part of a measurement unit as in the problems of Chapter 6 (such as 3/5 of a liter), Dorothy conceived of those fractional quantities as multiple of whole units, such as 3/5 of a liter was a bar with three parts, and produced fractional quantities of those units, such as 4/7 of the 3-part bar. In these problems, Dorothy viewed the fractions as operations to produce identical parts of given units. While this view related to units being substituted for their measurements was salient in Dorothy's means of operating in the problems of Chapter 5 (e.g., one part is 10 since it is one of the five parts of the 50 inch quantity in Problem 5.10), this view did not *transfer* into her activities in Chapter 6, and this situation produced setbacks for her when the results (which were results of her view of fractions as operations) needed to be reconstructed in relation to the standard measurement units since Dorothy operated with the conception of the composed units as the only fractional whole without their standard measurements (e.g., 3-part bar was not 3/5 of a liter).

Identity relationships are not sufficient to construct standard measurement units as independent quantities. Recursive distributive partitioning operations (or a fraction multiplying scheme) and inverse reasoning problems are all based on the operations that use this distinction of whether the relationship between the quantities and their measurement units is equivalent or identity relationship. Therefore, Dorothy's operations with the identity relationships at least partially explains why she was unsuccessful in constructing a fraction multiplying scheme and constructing partitioning and iterating as inverse operations; She considered fractions as only operations on the known quantity to produce a quantity (which is the unknown quantity) using identical parts of the known quantity. I can summarize the discussion related to Dorothy's activities as follows: as long as the quantities in the whole-part-part problems were measured with whole numbers, Dorothy could operate and produce an equivalency relationship between the two parts of the given quantity and their unknown numerosities (using *n*-part structure). However, when there were fractional parts of the unit measurements in the problems of Chapter 6, for example, 3/5 of a liter, fractions were operations for Dorothy and she could produce quantities that were identical fractional parts of the given quantities (for example 4/7 of a 3-part bar). But she could not produce their measurements, measurements that would indicate being aware of the results of fraction multiplication. This situation is important in that in the construction of meaningful linear equations and solutions of them, quantities that are equivalent to fractional parts of unit measurements need to be taken as a given, which requires an awareness on the student's part of how a standard measurement unit can be independently produced and coordinated with an equivalent quantity, if necessary. *Brenda's Construction of a Fraction Multiplying Scheme* 

I can conclude that Brenda constructed a fraction multiplying scheme that did not depend on the specific numbers. She used her initial operation, the distributive partitioning operation, in a way that was similar to how Dorothy operated. . However, contrary to Dorothy, Brenda was able to make a connection between of computationally derived results (such as 1/7 of 1/5 is 1/35 in Problem 6.2) and results produced in JavaBars. I called the distributive partitioning operations recursive that enabled Brenda to make connections between their results and measurement units that were involved in the problem statements. I concluded that Brenda constructed a fraction multiplying scheme using recursive distributive partitioning. She could use her fraction multiplying scheme to conceptually explain why the algorithm for computing the product of two fractions using notation works.

Research Question 2: What operations and schemes are involved in a construction of inverse reasoning that is a basis for conceptual understanding (both construction and solution) of linear equations with one unknown? What is the role of the fraction multiplying scheme in the constructions of inverse reasoning?

In inverse reasoning problems in Chapter 6, Brenda conceptualized the unknown quantity as a separate and independent quantity and she viewed that quantity as a result of her fractional operations prior to operating. Her activities were similar to her operations in Chapter 5 in that when she solved the part-part-whole and whole-part-part problems, she viewed the two unknown quantities (which she treated as known numerosities) as two separate but multiplicatively related quantities prior to operating and producing them. Therefore, in this sub-section, I elaborate on the importance of conceptualizations of equivalency and identity relationships between the known and unknown quantities and the related operations students used to construct those relationships.

For example, in an inverse reasoning problem (Problem 6.7), if a water bottle holds 5/6 of another water bottle, Brenda knew her water bottle had five equal parts and the other one had six equal parts prior to acting to conceptualize the measurement of the other water bottle. While the use of distributive partitioning operations does not depend on such an understanding of the independence of the two quantities, it does enhance the construction of partitioning and iterating as inverse operations if the student has such an understanding. There are two reasons for this claim. First, Brenda constructed inverse operations and used both distributive and recursive partitioning operations with an

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awareness of equivalency between the parts of the two separate quantities. Second, while Dorothy demonstrated using distributive partitioning operation when finding fractional parts of fractional quantities (such as 4/7 of 3/5 of a liter in Problem 6.4), she did not independently use that operation in inverse reasoning problems. So to construct inverse operations, I hypothesize that conceiving two independent quantities prior to acting and establishing an equivalency relationship between the parts of those quantities are necessary. In addition, if the student also uses distributive partitioning along with this equivalency relationship, then it is possible to construct the unknown quantity using inverse operations. Related to this claim, when Dorothy's solution to Problem 6.9 is reviewed, it is seen that she could create a 7/7 of the starting quantity (which is 3/7) only using an identity relationship between the two (known and unknown) quantities and producing it as a result of her reversible fraction schemes. However, unlike Brenda, Dorothy was perturbed on how to find the measurement of one seventh of the 7/7 bar or one third of her 3/5 bar, since she dropped using distributive partitioning operation. Therefore, she neither conceptualized how the two quantities were related to the standard measurement unit of a liter nor was she able to produce the 7/7 bar as results of inverse operations (it was results of her reversible fraction scheme). That was why Dorothy was perturbed; she was not able to produce the measurement of the bar for 7/7, which could be only constructed with use of inverse operations and equivalency relationships of the quantities and standard measurement units as Brenda demonstrated with her operations.
#### Perspectives on the Results of the Study

# Algebraic and Quantitative Reasoning

In Chapter 3, I stated that I am in agreement with the way in which Smith and Thompson (2007) view two useful roles of quantitative reasoning for algebraic reasoning. These roles are: (1) "to provide content for algebraic expressions so that the power of that notation can be exploited." (2) "to support reasoning that is flexible and general in character but does not necessarily rely on symbolic expressions" (p.12). Furthermore, they claimed that quantitative reasoning affects the development of arithmetic reasoning and "[students'] future prospects in algebra." They elaborated this claim as follows:

First, the quantitative/conceptual approach makes thinking about the quantities and their relationships a central and explicit focus of solving the problem. . . . Second, this focus on thinking about and representing general relationships between quantities support the kind of conceptual development that will eventually make algebra a sensible tool for thinking and problem solving. . . . Third, the quantitative/conceptual approach also suggests an early route to algebraic symbols in its focus on representing the general numerical relationships, rather than specific computations. (Smith & Thompson, 2007, pp. 21-22)

The three roles of quantitative reasoning and its contribution to conceptualization of arithmetic reasoning helped me to make an important observation about Brenda's activities and operations. In Problems 5.1 through 5.19, Brenda was challenged to find the numbers or lengths of two multiplicatively related parts (or numbers) when the sum or the total length was given. In her reasoning, she was making educated guesses for the two numbers or lengths by using the multiplicative relationships between the numbers or the lengths and then checking against the total. The central focus in her solutions was quantities and how those were related to each other conceptually. Therefore, she seemed to use a "quantitative/conceptual approach," as explained in Smith and Thompson's first benefit of quantitative reasoning. Her approaches in the first set of problems (cf. Chapter

5) might have opened the possibilities for her to make more advancements (compared to Dorothy) in the second part of the teaching experiment (cf. Chapter 6), which focused on algebraic operations that included inverse reasoning with fractional multiplication problems. On the other hand, Dorothy's activities for the first set of data analysis (cf. Chapter 5) were different than Brenda's, and she only reasoned arithmetically with the numbers without focusing on the relationships between the numbers once she set up the *n*-part structures. It is interesting that even though Dorothy seemed more fluent in her solutions that emphasized arithmetic (numbers stripped of their qualities) as basis, she was not able to construct a fraction multiplication scheme when finding the measurements of quantities (which I elaborated in Chapter 6). This situation might be thought of as confirming Smith and Thompson's claim that quantitative reasoning falls between arithmetic reasoning and algebraic reasoning; more specifically, if a student focuses on quantities in her solutions (Brenda as opposed to Dorothy), she may conceptualize algebraic structures or use algebra as a sensible tool more easily (see Smith and Thompson's second and third claims). In the context of my research, Dorothy's situation might suggest that if a student does not reason with quantities (meaning attending to the units and measurements of the quantities), it is more likely that she will operate with written notations but cannot conceptually use those notations to justify her results as is discussed in Problems 5.15 through 5.19. Therefore, trying to give possible explanations for why some students who can reason arithmetically have difficulties with algebraic situations, such as fraction multiplication, strengthens Smith and Thompson's three claims about quantitative reasoning as the basis for algebraic reasoning.

### Fraction Multiplication as Algebraic Scheme

In this sub-section I discuss how a fraction multiplication scheme can be viewed as algebraic by using Hackenberg's (2005) three requirements of conceiving a structure as algebraic: generalizing (abstraction of schemes and operations into conceptual structures), reciprocity (operation on unknown as well as known), and operating on notations (usually unconventional algebraic notations, such as drawings or language etc.).

Hackenberg indicates that if a scheme is generalizable and is a result of reflected abstraction, then it is algebraic. She gives an example of dividing any number by any number and explains:

If this student [who partitions each unit of the 7-inch bar into three and combines seven mini-parts to show 1/3 of 7-inch quantity] abstracted the structure of her scheme as dividing each unit of a length into three parts in order to divide the entire length into three equal parts, and then used that structure to divide any number of units into any number of parts, I would likely attribute a conceptual structure of "dividing a composite unit by another composite unit" to her. I would call her way of operating algebraic reasoning because of her awareness and use of the (multiplicative) structure of her scheme. (Hackenberg, 2007, p. 44-45)

Brenda's part-part-whole and Dorothy's whole-part-part reasoning schemes that are explained in Chapter 5 could be thought of as generalized schemes with an important condition that the students were able to operate as long as their conceived problem situations did not include fractional numbers as measurements of the *whole* or the *parts*. However, there should also be some discussions related to "generalizability" introduced by Hackenberg. It seems that Hackenberg's definition does not consider the different operations needed to act in a situation so the scheme would be generalizable. In her example of dividing any number by any number, the activities or operations are changing to employ the dividing scheme in making fractional parts of composite numbers. For example, when producing 1/3 of a 7-inch bar it is not enough just to partition the whole unit into three equal parts. In this case, the student needs to have additional operations, such as partitioning each unit distributively and then combining three mini-parts and forming a unit that would be called a third of the 7-inch bar and measured as 3/7 of an inch. We might say that the situation of the dividing scheme also changes, since dividing a unit into a composite unit (e.g., 1 divided by 7) is not the same as dividing a composite unit into a composite unit (e.g., 3 divided by 7). Therefore, when attributing generalizability, I think we need to differentiate whether students make generalizations about a scheme's situations or its operations. Hackenberg's generalizability idea seems not to emphasize the different operations that lead to different reconstructions of the situations. For example, if Dorothy had been successful, the whole-part-part reasoning scheme, employed when dividing 4 inches by five in Problem 5.15 or using 1/2 inch as the length of whole quantity partitioned into 16 in Problem 5.17, could have resulted in different conceptions of the situation and different activities from those Dorothy had used in the previous problems. Therefore, if Dorothy had been successful in using and operating on fractional numbers in those problems, there would have been an accommodation to her whole-part-part reasoning scheme structure with additions of new operations, and discussing the generalizability of her particular scheme across those problems would not be possible.

Therefore, we might conceptualize the scheme as generalizable as long as the situations and the operations do not change for the student and the student is aware of the components of that scheme. For example, with problems when the length of the whole quantity and the parts were whole numbers, Dorothy was aware of how to employ the

whole-part-part reasoning scheme and constructed an n-part structure for using it to find the lengths of unequal parts. While her scheme is generalizable in those situations, I would say that her whole-part-part reasoning scheme for whole numbers can be viewed as algebraic because it is anticipatory and independent of use of particular numbers as the relationship between the parts, and she operated with symbols.

To explain how I view using schemes and operations in conceptualizing algebraic operations, I will again discuss how Brenda constructed a quantity for 1/7 of 3/5 of a candy bar in Problem 6.1; her activities were not immediate in the sense that she did not know what to do prior to acting with JavaBars. In Problem 6.2, she knew how to act with the JavaBars but she was perturbed when trying to connect the written result of  $1/5 \div 7$ , 1/35, using JavaBars. The language Dorothy used for describing the computations of fraction multiplication evoked Brenda's operations to recursively distribute seven more mini-parts into the missing fifth of the candy bar. By justifying how 1/35 could be viewed as a result of JavaBars activities, she constructed a fraction multiplication scheme. After this problem, Brenda used this scheme in different problems and did not hesitate when producing measurements of quantities in terms of the measurement unit even if the measurement unit was absent in her visual field. Therefore, her fraction multiplication scheme had become a symbolic scheme in that her verbal language as well as her written notations could stand for her operations (*symbols* as defined by von Glasersfeld, 1995), such as reversing her fraction scheme to make 5/5 of the bar using 4/5 of the bar and distributing seven mini-parts into that imagined fifth.

There will be awareness not only of what is being operated on but also of the operations that are being carried out... symbols can be associated with operations and, once the operations have become quite familiar, the symbols can be used to point to them without the need to produce an actual re-presentation of carrying

them out. (von Glasersfeld, 1995, p. 108).

In addition to notations denoted with words and drawings of situations including unknown quantities, the use of which Hackenberg (2007) viewed as algebraic, in my study we observe students operating with some conventional algebraic notations. Students sometimes used those notations with an awareness of unknown quantities (such as students' written notations on Problem 5.3 (6/5 = 48) and Problem 6.7 (where Brenda constructed two independent bars for two quantities and solved the problem using paper)) and used them to talk about their mental operations as well as to modify their mental activities. Therefore, as long as the use of notations helps students to use their anticipatory schemes when finding measurements of unknown quantities, they are indicators of at least the beginnings of algebraic reasoning. However, sometimes the concern is not whether we can judge their activities as algebraic because they used notations, but the concern is whether they can purposefully act and compose a structure that is symbolic in nature. For example, in Problem 6.7 Brenda solved the problem using written notations and drawn bars that explicated the relationship between the known and unknown quantity. Yet she had difficulty when reinterpreting "4/5 gallon" (which was part of her answer for the measurement of the drawn 6-part bar) in relation to the notations of the quantities in her written work. Therefore, even though her activities related to the production of the result of "four and four fifths gallons" were algebraic in nature (since the operations were symbolic and she operated on the equivalency of parts of known and unknown quantity), she did not complete the network of the symbolic relationships by acting purposefully to explain the relationship of 4/5 to a whole gallon.

On the other hand, while we might consider Dorothy's activities as also part of

anticipatory schemes (such as when producing parts of given bars in problems of Chapter 6), her hesitations when making the given bar in the cases where no measurement unit (bar) was given makes me think that her activities had to be evoked by the bars as she operated on them. In addition, her inability to construct a unit-of-units-of-units structure when finding the measurements of the quantities suggests that her anticipatory scheme, which I called a distributive partitioning scheme (cf. Problem 6.4) is not sufficient. It is not sufficient for her either to make the necessary connections between the written notations (or computations) and operations with JavaBars or to use either of those as symbols which could stand for a recursive distributive partitioning operation, especially when justifying the measurements of quantities in the given standard units such as liter, inch, etc.

Can we assume that the students in this study established and solved linear equations with one unknown? And what kinds of operations were needed to establish such an equation where a, and b were both fractional numbers, including proper and improper fractions?

When we look at Brenda's and Dorothy's activities in the first few problems related to reversible iterative schemes, we definitely observe that they were able to set up equations such as 6/5 = 48 or 7/5 = 49 without using an explicit symbol for an unknown quantity (in Problems 5.3 and 5.4 respectively). Even though they did not use *x* to indicate 6/5 of *x* is 48 inches or 7/5 of *x* is 49 inches, their verbal cues and activities on paper indicated that they were treating 6/5 and 7/5 in relation to an unknown quantity. In addition, they were able to solve those equations and produce a result for the length of the unknown. On the other hand, when problem situations involved fractions in inverse

reasoning problems with one unknown quantity (such as Problem 6.7), the students' operations revealed differences in that Dorothy was able to produce a result of a measurement of the unknown quantity as long as she had a measurement unit visually available to her. However, her operations were reversible operations since they are similar to the ones she used in Problem 5.4, in which she did not conceive the unknown quantity as an independent entity. In addition, she did not seem to be setting up such a structure between the parts of the known and unknown quantities similar to the one she demonstrated in Problems 5.3 and 5.4. I think the structure that Dorothy did not construct, which concerns the relationship of a known quantity to an unknown quantity, requires constructing fraction multiplication as a symbolic scheme. What I mean by fraction multiplication as a symbolic scheme is that the students need to have an anticipatory scheme, not only for constructing quantities as a result of their operations on known quantities (such as 2/3 of 3/5 of a liter, Problem 6.3), as Dorothy demonstrated, but also for constructing the measurement of those quantities using recursive distributive partitioning operations. Constructing measurements using recursive distributive partitioning operations is necessary for the fraction multiplying scheme to be symbolic, and I observed the absence of this operation in Dorothy's activities and its presence in Brenda's activities. If the fraction multiplying scheme is symbolic, then the students' activities would not depend on the contextual elements, such as operating on whole numbers or fractional numbers, or the type of problems in which the fraction multiplying scheme is used, such as being more successful in finding parts of given quantities versus constructing an unknown quantity using fraction multiplication scheme.

In our meetings of May 16, 18, 19, and 25, I posed other inverse reasoning problems for investigating constructions of reciprocity of fractions (as discussed by Hackenberg, cf. Chapter 3) and its use in the solution of linear equations of one unknown.<sup>48</sup> I asked the students to make a separate bar for the unknown quantity without using any parts of the given (known) quantity, to state the relationships between the known quantity and unknown quantity using a label for the unknown quantity, and to notate their JavaBars actions on paper as they solved the problem. For example, on May 19, the students solved "a 12-inch bar is 5/4 as much as my sandwich. Can you make my sandwich with JavaBars without using the parts of the 12-inch bar and state how much it is?" I asked students to make a bar for 12 inches and then make another bar as an estimate of the bar for my sandwich and give a numerical estimate verbally. We then discussed the relationships between the two bars using c for the length of the unknown bar. With help while Dorothy set up an equation for  $5/4 \times c = 12$  (where c is the length for my sandwich), Brenda was able to justify why  $5/4 \times c = 12$  is equivalent to  $12 \times 4/5$ = c by using her JavaBars with some help. Therefore, we can assume that while setting up an equation was in Brenda's as well as Dorothy's zone of potential construction, solving linear equations using reciprocal fractions and notations were only in Brenda's zone of potential construction.

### **Operations of Fraction Multiplication Schemes**

When I discussed Steffe's literature about the fraction composition scheme in Chapter 3, there were three important schemes contributing to the structure of the

<sup>&</sup>lt;sup>48</sup> I did not give a detailed analysis of these problems in Chapter 6 because the students independent activities did not change the hypotheses I stated related to inverse reasoning, and they confirmed my analysis of the students' ways and means of operating in inverse reasoning problems.

composition of two fractions: units-coordinating schemes, recursive partitioning schemes, and reversible fraction schemes. Steffe defined the fraction composition scheme as follows (for example, when his students produced a result for 3/4 of 1/4 using their JavaBars activities):

The goal of this scheme is to find how much a fraction is of a fractional whole, and the situation is the result of taking a fractional part out of a fractional part of the whole, hence the name composition. The activity of the scheme is the reverse of the operations that produced the fraction of a fraction, with the important addition of the subscheme, recursive partitioning. The result of the scheme is the fractional part of the whole constituted by the fraction of a fraction. (Steffe, 2004, p. 140)

The findings of my study contribute to the discussion of explaining students' activities in more complex fraction multiplication situations. Those situations include (but are not limited to) (a) finding parts of fractional wholes which are also parts of hypothetical units, for example, finding the length of one of the 16 equal parts in a 1/2 inch strip (Problem 5.17), 1/7 of 3/5 of a candy bar (Problem 6.1), or 2/3 of 3/5 of a liter (Problem 6.3); (b) finding improper fractional parts of wholes which can be fractional parts of or improper fractional parts of hypothetical units, for example, 7/6 of 4/5 of a liter (Problem 6.5) or 3/5 of 11/6 of a liter (Problem 6.6); (c) using the fraction composition scheme in inverse reasoning problems when the known and unknown quantities are parts of or improper fractional parts of hypothetical units, for example, 3/5 of a liter water bottle holds 2/3 as much as another bottle (Problem 6.8).

Steffe's fraction composition scheme was discussed (and defined) usually when the fractional whole was present: for example, in Jason's activities, when he found 3/4 of 1/4 of the whole. My definition of the fraction multiplication scheme assumes the student also operates on the parts of the hypothetical unit when the unit is not in the student's perceptual field. While the units-coordinating scheme, recursive partitioning operation, and reversible fraction scheme might be sufficient to explain the result of 3/4 of 1/4 of a unit as a quantity, they are insufficient to explain why Dorothy can produce the quantity for 4/7 of a 3-part bar in Problem 6.4 but cannot produce the measurement of that quantity which is the result of 4/7 of 3/5 of a liter. In addition, the problem situations that Steffe used for defining a fraction composition scheme (3/4 of 1/4 of a 4/4-stick or 3/4 of 1/2 of a 4/4-stick, cf. Steffe, 2004) do not take into account some operations such as distributive partitioning.

With regard to this concern, more recently Hackenberg (2005) constructed distributive splitting operations: for example, she discussed when students split distributively each unit of a 8-cm bar into three parts and combined eight of those miniunits to make a third as much as the 8-cm bar (the length of 1/3 of 8 cm). Hackenberg's definition of distributive splitting can be attributed to Dorothy's activities in Chapter 6 since Dorothy could use a distributive partitioning operation to produce fractional parts of 3/5 of a liter as long as she conceived 3/5 of a liter as a 3-part bar. Therefore, the result she produced by partitioning each part of the 3-part bar and then combining the needed number of mini-parts is similar to Hackenberg's distributive splitting operation. However, this type of operating constitutes only a fraction multiplying scheme in which the student is only aware of the quantity produced as the result of operations. While the distributive partitioning operation is different than Steffe's fraction composition scheme because of the distributivity involved in the recursive partitioning, it needs to be accommodated so the quantity can be reinterpreted in relation to a unit measurement. Brenda's operations indicated that in addition to a units-coordinating scheme and

distributive partitioning operations, there needs to be a different operation, which I call recursive distributive partitioning, to produce the result of multiplying two fractions both as a quantity in relation to the starting quantity (4/7 of 3-part bar) and as a measurement in the absence of a hypothetical unit which 3/5 is a part of it. With this new operation, Brenda coordinated the two three-levels-of-units structures especially using the mini-part quantity (the third level unit) as embedded both in the hypothetical unit and in the starting quantity.

## Fraction Multiplication and Role of Measurement Unit

In Chapter 3, I examined two studies related to students' understanding of fraction multiplication. In the first study, I presented (among others) how Mack (2001) discussed students' solutions related to fraction multiplication where two terms were equal, for example, finding 1/4 of 4/5 of a cake in terms of the whole cake. I pointed out that she did not discuss the units-coordinating schemes in students' activities or differentiate the students' activities using those schemes. In addition, she did not explain how she viewed students' subtraction activity (the student said, "I gave one to him of these four there") as a fraction multiplication represented by 1/4 of 4/5. In similar fraction multiplication contexts, when Brenda solved two problems (Problems 6.1 and 6.2) I indicated that it was not immediate for Brenda to coordinate the two units structures using fifths (as opposed to what Mack claimed about her students). While Brenda stated that 3/5 was 3/5 of 5/5 of the bar (or a liter) and operated on one of the fifths in Problem 6.1, she treated the fifths as if they were only one of the three parts of 3/5 quantity. On the other hand, in Problem 6.2, Brenda also stated that one fourth of four-fifths of the bar and one fifth of the fivefifths of the bar were the same quantity and she continued operating using one-fifth. With

this awareness, and using notations, Brenda then was able to coordinate a mini-part of the 4/5 of the bar as part of the 5/5 of the bar. Therefore, the result of 1/4 of 4/5, if it is to be conceived as the result of fraction multiplication, has to be differentiated from the result of one part of the 4-part quantity (which is 4/5 of the 5-part quantity) unlike what Mack claimed using her students' answers. The result of finding 1/4 of 4/5 (as produced by Brenda) needs to be constructed as 1/5 of the whole candy bar with the proviso that the student maintains an awareness that the same quantity could be interpreted differently depending on the referent unit whole (whether it is 4/5 of a quantity or 5/5 of a quantity).

I also highlighted Mack's discussions (cf. Chapter 3) regarding how students transformed the result of 3/4 of 2/3 of a whole bag indicated as "one and one half of the bag," to a simple fraction such as three-sixths of the bag. Since the students in Mack's study always started with a whole unit and then operated using that unit, operating with the second-level unit (thirds) and using mini-parts as parts of both thirds and also the whole unit might not have been so problematic. On the other hand, Brenda's activities in Problem 6.1 indicate that if the whole unit is not in students' perceptual field, it is not immediate for students to use the second-level unit (each part of 3/5) to reinterpret miniparts as embedded in the whole unit (5/5). The success of the activity requires some accommodation in students' fraction multiplication scheme, as I explained using distributive partitioning and recursive distributive partitioning operations in the analysis of Problem 6.4.

In the same vein, in his study, Olive (1999) indicated that "one stumbling block that they [his advanced fourth graders] met was to name a fraction of a fraction as a new fraction of the original whole" (p. 292). The students in Olive's study used their unitscoordinating schemes, reversible partitive fraction schemes, and recursive partitioning operations to accommodate their activities so they could produce a name for the quantity without iterating it and checking the iterations against the whole unit. Olive's schemes and operations do not help to explain how Dorothy could have named a quantity which is a result of a distributive partitioning scheme in relation to a hypothetical unit measurement in the absence of the unit in her perceptual field. I cannot explain either why Dorothy was not able to construct recursive distributive partitioning operations (like Brenda) or what kind of additional operations and structures were needed to advance her distributive partitioning operation.

I constructed Figure 7.1 which is an expansion of Olive's (1999) diagram that I referred to in Figure 3.1 (cf. Chapter 3). The operations that I placed between *Reversible fraction schemes* and *Fraction multiplying scheme* in the diagram are related to Olive's recursive and reversible partitioning operations in Figure 3.1, however the operations in Figure 7.1 are more detailed in terms of specifying students' certain mathematical actions. In the diagram (Figure 7.1), I also situated *Fraction multiplying scheme* (and its necessary operations) in a model that concerns a relationship to *Inverse reasoning* and *Reciprocal fractions*, which are essential for a construction (*stating*) and *solving of linear equations* with one unknown.



**Figure 7.1**. A diagram that shows the schemes and operations related to the construction of a fraction multiplying scheme and stating and solving linear equations.

The last point that I want to make in relation to the literature I reviewed is related to Fischbein et al.'s (1985) and Harel et al.'s (1994) studies and their findings. Fischbein et al. stated that students avoided using or choosing a multiplication operation in word problems. In this study, while the two eighth graders were aware of the need to use a multiplication operation (so they consciously chose this operation), as demonstrated in Problems 5.14 through 5.18 they were not able to interpret the results of such operations in the problem context. In addition, Dorothy's computational activities suggest that she did not have much difficulty choosing and notating the multiplication operation in the problems of Chapter 6, but she had difficulty interpreting those computational results

with her JavaBars activities. The students and I used the JavaBars to provide a context for our discussions related to their mathematical concepts and constructions.

In relation to Fischbein et al.'s study, Harel et al. (1994) stated that it was not clear the "conceptual basis for the multiplier 1 [was] an index for relative difficulty of multiplication problems" (p. 382). Using Dorothy's activities, I can expand on this issue: when she operated on quantities that were more than one unit of measure or multiples of one unit measure (whole number quantities), she was successful in producing both the quantity (the result of distributive partitioning operation) and also the measurement of that quantity (the result of fraction multiplication—see Problem 6.6, finding 3/5 of 11/6of a liter or Problem 6.7, finding the amount of skim milk when 4 gallons of whole milk is 5/6 as much as the amount of skim milk). For all the other problems in Chapter 6 (except some inverse reasoning problems), Dorothy was able to produce the result of the fraction multiplication operation as a quantity, but she was not able to produce their measurements. This situation in which she did not construct a measurement unit of one as an operative figurative image explains why "one" can be viewed as an index of relative difficulty when producing the results of fraction multiplications in relation to measurement units.

Unresolved Issues and Suggestions for Further Research

There are two unresolved issues that should be discussed. The first one is that even though I have sufficient warrants using Dorothy's activities and operations, especially in problems of Chapter 6, to claim that both the perceptual measurement units and operative figurative images of the measurement units are crucial for students' successful activities related to fraction multiplying schemes, I cannot explain how students (whose operations depend on the perceptual unit to produce the measurement result of fraction multiplying schemes) can construct operative figurative units. I know what operation needs to be constructed, the recursive distributive partitioning operation; however, how to engender such operation needs further investigation.

The second issue, which can be further investigated, is what kinds of relationships can be theorized between inverse reasoning (meaning the three hypotheses suggested in Chapter 6: the unknown quantity as an independent entity, an equivalency relationship between the known and unknown parts of the quantities, and using the part of unknown quantity to construct it) and recursive distributive partitioning operations. Even though I indicated that there are relationships between those constructs within the *hypothetical learning trajectory* that I constructed in Figure 7.1 and discussed some possibilities of such relationships at the beginnings of this chapter, this issue needs follow-up research. Chronologically, in the study, I first investigated students' construction (or lack of construction) of recursive distributive partitioning operations and concluded that an operative figurative image of the measurement unit is necessary and this unit can be also viewed as the result of such an operation. Since constructing such an image in Brenda's operations assured that the measurement unit could be a separate and independent quantity from the given quantity (whose measurement in relation to the unit is stated, such as 3/5 of a liter) in the problems situation, this situation suggested that Brenda could also conceive the unknown quantity in inverse reasoning problems as an independent entity to start with. In reality, I observed Brenda operating with this conceptualization as demonstrated in Problems 6.7 and 6.8 where she consider the bar-to-be-made as a separate bar. On the other hand, in Problem 6.8, after she made the two bars (one for 3/5

of a liter and the other one for the 3/3 of the unknown quantity) and distributively partitioned the bar for 3/5 of a liter to show half of it, Brenda was perturbed and used the equivalency of half of 3/5 of a liter bar and a third of the bar for the unknown quantity and operated with this equivalency to produce the measurement of the unknown quantity. When Brenda restated the problem situation using "of" instead of "as much as" for conceptualizing the relationship between the two quantities, she could produce the measurement of the unknown quantity. Therefore, in this sense I am not sure whether the first hypothesis of inverse reasoning structure related to conceiving the unknown quantity as a separate and imaginary bar (quantity) prior to operating, function simultaneously with the other two required hypotheses: establishing the equivalency between the parts of the quantities, and using measurement of the part of the unknown quantity to create the unknown quantity. Therefore, how recursive distributive partitioning operations (with which one produces an imaginary measurement unit) and inverse operations (partitioning the known quantity, stating the equivalency between the parts of quantities, and iterating the part of unknown quantity to conceive the unknown quantity) functions together needs further investigation.

In addition, in this study, I did not investigate all of the relationships presented in Figure 7.1 in detail. For example, the constructs related to linear equations are not well investigated yet, such as what kinds of operations are needed for solving equations even after the linear equations are correctly stated using quantitative relationships and unknowns. The roles of reciprocal fractions and related operations and schemes to construction of reciprocal fractions can be also investigated in this context by investigating how they contribute to stating equations and solving those equations. Implications of the Study for Teaching and further Research

While teachers should encourage students to think structurally, as demonstrated with Dorothy's whole-part-part reasoning scheme in Problems of Chapter 5, teachers should also emphasize quantities as important parts of the problem situations. This emphasis might lead to students operating with an awareness of the measurement units of the quantities. Otherwise, it is possible that students will view mathematics as a symbol manipulation without any quantitative contexts that provides "reasoning" to algebra. While students might be successful at computing, they may lack the conceptual structures that give justification to what those results mean. For example, even though Brenda and Dorothy were able to produce the result of  $1/5 \div 7$  as 1/35 in Problem 6.2 with computations, Dorothy could not conceive of this result as an important part of the problem context to explain its relation to her mathematical operations with JavaBars. Therefore, teachers need to be cautious and need to make sure that students also value different types of explanations for their computational results with which they can possibly avoid viewing algebra as symbol manipulation.

Another important implication of this study, which also confirms NCTM (2000) *Principles and Standards* general recommendations, is that we cannot assume students' construction and understanding of linear equations with one unknown as a standard and clear-cut understanding which will be valid for all students. These constructions are not as easy as they are assumed when introduced with most middle school curriculum materials (e.g., Bellman, Bragg, and Charles (2002)). In addition, the type of problems and exercises in the most curriculum materials related to finding the unknown in the equations of ax = b, where a, and b are fractional numbers, do not give enough attention

to how students' fractional knowledge, different levels of multiplicative unit structures, and their conceptions of standard measurement units could play roles in those solutions. Therefore, mathematics teachers need to use those materials considering these issues that the presentations of what is in the curriculum material should not be more important than how students conceive and construct the teacher intended knowledge. If teachers practice without paying attention to the differences in students' mathematical activities, and do not conceptualize and organize those activities using different models of how students think, and use curriculum materials as the only source, then their instruction likely will be far removed from occasioning students to construct recursive partitioning operations and their implications in the inverse reasoning that is implied by simple algebraic equations. In that case, algebra will not be different than manipulation of written symbols as criticized by many researchers (NCTM, 2000; Thompson & Smith, 2007), and the quantitative relationships, whose use can contribute to the solution of the criticized situation, will not receive enough attention even though quantities are salient in the situations of linear equations as I discussed in this research.

For designing pedagogical lessons, Simon, Tzur, Heinz, and Kinzel (2004) listed four steps: specifying students' current knowledge, specifying the pedagogical goal, identifying an activity sequence, and selecting a task. While they said that the activityeffect relationship is the underlying principle for the last two steps of designing a lesson, they suggested that a lesson designer (teacher) should also be concerned about specifying learning goals for students in the second step (specifying the pedagogical goal); the focus of the learning goal should not be "on *the mathematics as seen by the one* [teacher or the student] *who understands it*" but "on *distinctions in the learner's understanding of the*  *mathematics*" (Simon, 2002, p. 996). However, specifying learning goals is not an easy task since the teacher needs to know "at least two states of student understanding, a current state and a goal state, and the differences between them" (Simon et al., 2004, p. 322). Related to how we can conceptualize the two different states of children's understandings we can use the analytical tools that Steffe (2007) provided. Steffe's tools are the analysis of first-order models, which helps us to make the goals for the students using our own mathematical knowledge and analysis of it, and the analysis of secondorder models, which helps us to understand students' possible learning trajectories using the analytical model we made and this provides us understandings of our current students' *current states* and *goal states*. Steffe also explains how these two models can be used in the construction of a school mathematics that takes into accounts of both teachers and students mathematical activities. For example, Thompson and Saldanha's (2007) idea of *reciprocal relationships of relative sizes*,<sup>49</sup> which they created using their first order fractional knowledge, and Steffe's (2002) idea of the *splitting operation*, which he created after making models of students' fractional knowledge (second-order model), can be combined and used for designing important aspects of a fraction curriculum in school mathematics. Similarly, the conclusions derived from my study, in which I used students' fractional knowledge (their operations and schemes) to make models of their construction processes of linear equations with one unknown, can be used in providing conceptual tools for teachers to use in their instruction and in conceptualizing important aspects of algebra for developing curriculum materials.

<sup>&</sup>lt;sup>49</sup> Thompson and Saldanha (2003) explain this concept as: "Amount A is 1/n the size of amount B means that amount B is n times as large as amount A. Amount A being n times as large as amount B means that amount B is 1/n as large as amount A" (p. 107).

There is a link in how the reciprocal relationship of relative sizes (derived mainly from first-order models), the splitting operation (derived mainly from second-order models), and inverse reasoning hypotheses that I derived in my study, function in the construction of linear equations and we need to discuss how this link can contribute to a view of "school algebra." Steffe (2007a) made the link for the first two operations as: "A qualitative distinction in the two operations is that, in splitting, the child seems unaware of a reciprocal relationship between the two sticks prior to actually carrying out splitting activity" (pp. 286-287). We can expand on this issue of making a model on how a child becomes aware of the reciprocal relationship between the two quantities by using the three hypotheses related to inverse reasoning that I proposed in this study and Brenda's and Dorothy's activities related to their construction of fraction multiplying schemes. With this expansion, my research provides possibilities for making a second-order model of the construction of reciprocal fractions, and also provides an important step to make first-order analysis of solving linear equations with one unknown. Therefore, what I am suggesting here is "school mathematics" (meaning both curriculum materials and implementations of those in the classroom) should not undermine the importance of the constructions of reciprocal fractions and its role in engendering meaningful mathematical students' activities for the construction and solutions of algebraic equations.

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Appendix A: Interview Questions Related to Fractions

Students used paper and pencil to draw the candy bars and solve the problems. This interview guide was flexibly used with the students. For example, I sometimes changed the numbers in the problems, or skipped some questions, or made-up questions at the spot.

1. Here is a candy bar. We need to share it among five people, can you show the share of one person? Three person? Fraction names for those shares?

2. Here is another candy bar, four people are sharing. Can you share one of those shares with three late comers? How much will it be in terms of the whole candy bar?

3. My candy bar is twice as much as your candy bar, if this is my candy bar, can you draw yours?

4. My candy bar is six times as much as yours. If this is my candy bar can you make yours?

5. Here is a fourth of a candy bar, can you make the whole candy bar?

6. Here is a fourth of a candy bar, can you make three fourths of that candy bar?
Can you make five fourths of that candy bar?

7. Let's pretend that this is 3/7 of a candy bar, can you make the whole candy bar?

8. Here is a candy bar. Can you draw a bar picture of 6/5 of this candy bar?

9. If this is five fourths of a candy bar, can you make the candy bar?

10. Maria always receives twice as much money as her younger sister receives

from their parents. If her sister got \$5 last week, how much did Maria get from her parents?

11. Maria always receives 3/4 as much money as her older sister gets from their parents. If her sister got \$5 last week, how much did Maria get from her parents?

12. Maria always gets twice as much money as her younger sister gets from her parents. If Maria got \$5 last week, how much did her sister get?

13. A pitcher holds water as much as 1/3 of a water container holds, how many pitchers are needed to fill the empty water container?

14. Variations of Problem 13:

(a) Composite Fraction. If a pitcher holds 2/3 of a water container, how many pitchers are needed to fill the empty water container?

(b) If a pitcher holds 2/3 of a container, how many pitchers are needed to fill 2 containers, 3 containers, etc.?

When the number of containers to be filled are not whole numbers:

(c) If a pitcher holds 2/3 of a container, how many pitchers are needed to fill half of the container, to fill 3/4 of the container or to fill 1 and 1/2 of the container etc.?

Appendix B: Interview Questions Related to CPM Unit 4

1. Heather has twice as many dimes as nickels and two more quarters than nickels. The value of the coins is \$5.50. How many quarters does she have?

One number is five more than a second number. The product of the numbers is
 3300. What are the numbers?

3. Chris is three years older than David. David is twice as old as Rick. The sum of Rick's age and David's age is 81. How old is Rick?

4. Find three consecutive numbers whose sum is 57.

5. Latisha and Maisha are twins. They have a brother who is eleven years younger than them and an older sister who is four years older. The sum of the ages of all four siblings is 69. Find Latisha's age.

6. Mary sold 105 tickets for the basketball game. Each adult ticket costs \$2.50 and each student ticket costs \$1.10. Mary collected \$221.90. How many each kind of ticket did he sell? First using guess and check table, and then writing an equation, may be solving the equation.

7. Ms.Speedi keeps coins for paying the toll crossings on her commute to and from work. She presently has three more dimes than nickels and two fewer quarters than nickels. The total value is \$5.40. Find the number of each type of coins she has.