Structure Theory of Graded Central Simple Algebras

by

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(Under the direction of Daniel Krashen)

Abstract

This work is focused on the structure theory of graded central simple algebras. We consider algebras graded by $\mathbb{Z}/pq\mathbb{Z}$ where p, q are distinct primes different than 2. I define a new algebra type, called a p-odd algebra, and a structure theorem for these algebras. This definition and structure theorem are a generalization of the current results in the literature. We define and discuss the graded Brauer group in this context and its relation to the structure of the algebras. Moreover, we define a group of invariants and show how to view the classical Brauer group as a subgroup of the graded Brauer group.

Index words: Non-Commutative Algebra, Central Simple Algebras, Graded Algebras, Brauer Group, Graded Brauer Group

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Chapter 1

Introduction

This thesis is focused on developing a theory for $\mathbb{Z}/n\mathbb{Z}$-graded central simple algebras, where $n = pq$ and p, q are distinct primes with $p, q \neq 2$. The cases where n is prime have been thoroughly studied in [5] (see also [15]) and the other cases remain open. In this thesis we will describe the structure of a graded Brauer group and relate it to the structure of graded algebras. This includes defining a graded Brauer group and structure theorems analogous to the results in the literature for even and odd algebras. Moreover, we will show the collection of classes of even algebras with discriminant one in the graded Brauer group is isomorphic to the (ungraded) Brauer group. The majority of the literature in this area studies graded central simple algebras up to Brauer equivalence. Koç and Kurtulmaz [5] and Vela [15] have results on the structure of $\mathbb{Z}/n\mathbb{Z}$-graded central simple algebras that only address even and odd type graded central simple algebras, which are not all inclusive. That is, there are graded algebras that are neither even nor odd, which we will call p-odd algebras. As mentioned above the goal of this thesis is to work towards filling this gap in the literature.

There is a rich theory of central simple algebras. One of the most notable theorems is the Wedderburn-Artin Theorem, which states that a central simple algebra is isomorphic to

a matrix algebra over a division ring. This theorem can also be stated in more generality for semisimple algebras. The Wedderburn-Artin Theorem essentially reduces classifying central simple algebras over a field to classifying division rings with a given center. This theorem leads to a group structure on the collection of equivalence classes of central simple algebras, called the Brauer group. Two resulting central simple algebras are in the same equivalence class, or Brauer class, if they have the same underlying division algebra (the dimensions of the matrix algebras may be different). Many aspects of this theory are not fully generalized to graded algebras.

Wall [13] introduced and studied $\mathbb{Z}/2\mathbb{Z}$-graded algebras in 1964. He developed a Brauer equivalence on $\mathbb{Z}/2\mathbb{Z}$-graded algebras which forms an abelian group, called the Brauer-Wall Group. This group is closely related to quadratic forms and Clifford algebras. For an introduction to the classical Brauer group and the Brauer-Wall group (the graded Brauer group for $n = 2$) refer to [6]. Additionally, $\mathbb{Z}/2\mathbb{Z}$-graded algebras (or superalgebras) have been studied in a number of contexts [2, 3, 8, 9, 14]. In 1969, Knus [4] generalized this idea by replacing $\mathbb{Z}/2\mathbb{Z}$ with a finite abelian group. Many built on this work and there are several other generalizations of the Brauer group in the literature, including the Brauer group of a braided monoidal category, which generalizes all other known Brauer groups [11]. A survey of the various generalizations of the Brauer group can be found in [10]. There are not many structure results for the corresponding algebras. However, results by Vela, and Koç and Kurtulmaz focus on decomposition theorems for $\mathbb{Z}/n\mathbb{Z}$–graded central simple algebras.

In the following chapters we will review the necessary background material for central simple algebras, the (ungraded) Brauer group, and graded central simple algebras, then we will discuss the structure theorems for p-odd algebras and the results about the graded Brauer group.

Chapter 2

Background on Central Simple Algebras

2.1 Preliminaries

We will begin with some basic definitions and notation.

Note: In this paper we are assuming all algebras are finite-dimensional, have an identity, and are associative. Moreover, when discussing graded algebras, we are assuming the underlying field of the algebra contains a primitive nth root of unity.

Definition 2.1.1. *An algebra, A, is a vector space over F with an additional binary operation $A \times A \to A$, denoted $\cdot$, in which the following hold for all $a, b, c \in A$ and $\alpha, \beta \in F$:*

(1) $(a + b) \cdot c = a \cdot c + b \cdot c$ *(right distributivity)*

(2) $a \cdot (b + c) = a \cdot b + a \cdot c$ *(left distributivity)*

(3) $(\alpha a) \cdot (\beta b) = (\alpha\beta)(a \cdot b)$ *(compatibility with scalars)*

Definition 2.1.2. *If A is an algebra over F, $End(A)$ is the algebra of F-linear transformations from A to itself.*

Proposition 2.1.3. *If A and B are F-algebras, then $A \otimes B$ is an associative F-algebra with multiplication induced by $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$ and identity $1_A \otimes 1_B$.*

Definition 2.1.4. *An algebra is said to be a division algebra if it has a multiplicative identity and every nonzero element has a multiplicative inverse.*

2.2 Introduction to Central Simple Algebras

In this section, F is a field, and by 'an algebra over F' we mean a finite dimensional associative algebra over the field F. The purpose of this thesis is to study graded central simple algebras, but we will use some theory of standard central simple algebras. We will briefly review standard definitions and facts about central simple algebras, as can be found in Lam [6, Section 4.1], but which are included here for convenience.

Definition 2.2.1. *For any subset, B, of A the centralizer of B is given by*

$$C_A(B) = \{a \in A \mid ab = ba,\ \forall b \in B\}.$$

Definition 2.2.2. *The center of an algebra, A, is given by*

$$Z(A) = \{a \in A \mid ab = ba,\ \forall b \in A\}.$$

We say an algebra A over F is central if $Z(A) = F$.

The above definition for the center of A can be equivalently defined as $Z(A) = C_A(A)$.

Definition 2.2.3. *An algebra over F is simple if it has no proper two-sided ideals.*

Definition 2.2.4. *An algebra is a central simple algebra (CSA) if it is both central and simple.*

The following are common examples of central simple algebras.

Example 2.2.5. *The endomorphism algebra $A = End(V) \cong \mathbb{M}_n(F)$, where V is an n-dimensional vector space over F, is a central simple algebra.*

Example 2.2.6. *The quaternion algebra $A = \left(\frac{a,b}{F}\right)$, which has two generators i, j and relations $i^2 = a$, $j^2 = b$, $ij = -ji$, is a central simple algebra over F.*

We now review some theorems from Lam [6, Section 4.1] regarding central simple algebras, including the double centralizer theorem and the Noether-Skolem Theorem.

Proposition 2.2.7 ([6, Theorem 4.1.2])**.** *(1) If A, B are F-algebras, and $A' \subset A$ and $B' \subset B$ are subalgebras, then*

$$C_{A\otimes B}(A' \otimes B') = C_A(A') \otimes C_B(B').$$

In particular if A, B are F-central, so is $A \otimes B$.

(2) If A is a central simple algebra over F and B is a simple algebra over F, then $A \otimes B$ is simple.

(3) If A and B are both central simple algebras, then $A \otimes B$ is also a central simple algebra.

Now, we review the double centralizer theorem and an important corollary.

Theorem 2.2.8 (Double Centralizer Theorem, [6, Proposition 4.1.6])**.** *Let A be a CSA over F, and B a simple subalgebra of A. Let $C = C_A(B)$. Then,*

(1) C is simple;

(2) $B = C_A(C)$;

(3) $\dim A = \dim B \cdot \dim C$.

Corollary 2.2.9 ([6, Corollary 4.1.7]). *Suppose $B \subset A$ and both are central simple algebras over F. If $C = C_A(B)$, then C is also a central simple algebra, $B = C_A(C)$, and $B \otimes C \cong A$.*

We will now state the Noether-Skolem Theorem, which is an important theorem in the theory of central simple algebras. We will later see an even type graded central simple algebra is, in fact, a (ungraded) central simple algebra. This theorem will play a large role in this theory of graded central simple algebras, as it shows the existence of an element in an even algebra that plays a vital role in defining the discriminant of an algebra.

Theorem 2.2.10 (Noether-Skolem Theorem, [6, Theorem 4.1.8]). *Let A be a central simple algebra over F and B a simple algebra. If f, g are algebra homomorphisms from B to A, then there exists an invertible element $s \in A$, such that $f(b) = s^{-1}g(b)s$ for every $b \in B$ (i.e., f and g differ by an inner automorphism of A).*

Corollary 2.2.11 ([6, Corollary 4.1.9]). *If A is a central simple algebra over F, then every automorphism of A is an inner automorphism.*

The Wedderburn-Artin theorem is another theorem which is very important in the theory of central simple algebras, as it is vital in defining the (ungraded) Brauer group. There are several versions of the Wedderburn-Artin theorem with a less restrictive hypothesis. For example, there are versions where you only need to have a semisimple ring, semisimple algebra, or an Artinian ring. However, we are only considering central simple algebras in this paper, so we will state the following version which is directly applicable to our topic.

Theorem 2.2.12 (Wedderburn-Artin). *Let A be a central simple algebra over F. Then $A \cong \mathbb{M}_r(D)$ for some r and some division algebra, D, over F. Moreover, D is uniquely determined up to isomorphism by A.*

2.3 The Brauer Group

In this section we will discuss the construction of the classical Brauer group, which we will generalize for graded algebras in Section 5.

We begin by defining an equivalence relation on central simple algebras, called the Brauer equivalence. The purpose of the Brauer group is to classify all central simple algebras using this relation.

Definition 2.3.1. *Two central simple algebras A and B over F are Brauer equivalent if there exist finite-dimensional vector spaces V and W such that $A \otimes End(V) \cong B \otimes End(W)$ as F-algebras.*

It is easy to check that this forms an equivalence relation on the set of central simple algebras over F. We will denote the equivalence class of A by $[A]$. The collection of equivalences classes form an abelian group, called the Brauer group. The group operation is given by $[A] \cdot [B] = [A \otimes B]$, which is well defined and the identity element is $[F] = [\mathbb{M}_n(F)]$. We will need the following definition in order to define inverses.

Definition 2.3.2. *The opposite algebra is defined to be $A^{op} = \{a^{op} \mid a \in A\}$ with operation given by $a^{op} \cdot b^{op} = (ba)^{op}$.*

Proposition 2.3.3 ([6, Proposition 4.1.3])**.** *If A is a central simple algebra, so is A^{op} and $A \otimes A^{op} \cong End(A)$.*

The above proposition makes it clear that $[A]^{-1}$ is given by $[A^{op}]$. It should also be noted that by the Wedderburn-Artin Theorem, if A is a central simple algebra, then $A \cong \mathbb{M}_r(D)$ for some central division algebra D over F. So, $[A] = [D]$ in the Brauer group since $\mathbb{M}_r(D) \cong \mathbb{M}_r(F) \otimes D$.

Chapter 3

Background on Central Simple Graded Algebras

3.1 Introduction to Central Simple Graded Algebras

For the remainder of this paper, let $n \geq 2$ and F be a field containing ρ, a primitive nth root of unity, and char$F \nmid n$. By an algebra, we still mean a finite dimensional associative algebra with identity over the field F. In Chapter 4, we will study the structure of the graded Brauer group and p-odd graded central simple algebras. In this chapter, we begin with some basic definitions regarding central simple graded algebras and will review the structure theory for even and odd type algebras.

Definition 3.1.1. *If A is a $\mathbb{Z}/n\mathbb{Z}$-graded algebra, then A has a decomposition of the form $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{n-1}$, such that $A_iA_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}/n\mathbb{Z}$. An element $a \in A_k$ is called homogeneous of degree k, which we denote by $\partial a = k$ or $\deg(a) = k$. We denote the set of all homogeneous elements of A by $\mathcal{H}(A)$.*

It is important to note that you can define a grading on an algebra by an arbitrary group, however this paper will focus on gradings by $\mathbb{Z}/n\mathbb{Z}$. We will now consider two important examples of gradings that arise in the structure theorems for graded central simple algebras.

Remark. *Algebras graded by $\mathbb{Z}/2\mathbb{Z}$ (i.e., the case $n = 2$) have been thoroughly studied and are called superalgebras.*

Example 3.1.2. *Given any any algebra A, it can be considered trivially graded by concentrating it in degree 0. That is, $A = A_0$ and $A_i = 0$ for $i \neq 0$. We denote this trivial grading on A by (A).*

Example 3.1.3. *A more interesting example of a $\mathbb{Z}/n\mathbb{Z}$-graded algebra that arises in the structure theorems is, $A = F[x]/(x^n - d) = F \oplus Fx \oplus \cdots \oplus Fx^{n-1}$, for $d \neq 0$. Under this grading the monomials are the homogeneous elements with ax^k being degree k. This forms a grading since the exponents are added when multiplying two monomials.*

Example 3.1.4. *We will consider two gradings on the matrix algebras $\mathbb{M}_r(A)$, where A is a graded algebra:*

1. *The first, which we denote $\widetilde{\mathbb{M}}_r(A)$ indicates that a matrix is of degree i if all the entries of the matrix are degree i elements of A.*

2. *The second grading on $\mathbb{M}_r(A)$ is called the generalized checkerboard grading and is denoted $\hat{\mathbb{M}}_r(F)$. Under this $\mathbb{Z}/n\mathbb{Z}$ grading, a matrix M is homogeneous of degree k if $M_{ij} = 0$ for $j - i \not\equiv k \pmod{n}$. Let us consider specific example, where $n = 3$:*

$$A_0 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$$

The following example will be vital in defining the graded Brauer group in Section 5.

Example 3.1.5. *$E = End(V)$ is a graded F-algebra, where $V = \oplus_{i=0}^{n-1} V_i$ is a graded F-vector space. The grading on E is obtained by defining $E_i = \{f \in End(V) \mid f(V_j) \subset V_{j+i}\}$. In fact, this graded structure makes E a graded central simple algebra. From a matrix point of view this is the checkerboard grading described in the example above.*

Definition 3.1.6. *A graded subspace (or subalgebra) B of an algebra A is a subspace (or subalgebra) that preserves the graded structure, i.e. $B = \oplus_{i=0}^{n-1}(B \cap A_i)$.*

$C_A(H)$ is a graded subalgebra for any $H \subset \mathcal{H}(A)$. The center of A, $Z(A)$, is also a graded subalgebra.

Definition 3.1.7. *We say an ideal of a $\mathbb{Z}/n\mathbb{Z}$-graded algebra is graded if it is generated by homogeneous elements, or equivalently if it can be written as $I = \oplus_{i=0}^{n-1}(I \cap A_i)$.*

The ideal $\langle H \rangle$ generated by $H \subset \mathcal{H}(A)$ is a graded ideal of A.

Definition 3.1.8. *A graded algebra is a simple graded algebra (SGA) if it has no proper graded two sided ideals.*

Definition 3.1.9. *A homogeneous element $a \in A$ is said to left graded commute with a homogenous element $b \in A$ if $ab = \rho^{\partial a \cdot \partial b} ba$. We define the left graded center to be the set generated by all homogeneous elements that left graded commute with all the elements of A, i.e. $\hat{Z}_L(A) = span\{a \in \mathcal{H}(A) \mid ah = \rho^{\partial a \cdot \partial h} ha, \forall h \in \mathcal{H}(A)\}$. Similarly we define the right graded center, $\hat{Z}_R(A)$.*

Remark. *We will use the notation $\hat{Z}(A)$ to mean the left graded center, $\hat{Z}_L(A)$.*

Notice, if we have $ab = \rho^{\partial a \cdot \partial b} ba$ (i.e. a left graded commutes with b), it does not necessarily imply that $ba = \rho^{\partial a \cdot \partial b} ab$ (i.e. a right graded commutes with b). Hence, in general there is a need to distinguish between the left and right graded center. Since the left graded

center is defined and standardly used in the literature, if there is no designation we will mean the left graded center. However, at the end of Section 4.1, in Proposition 4.2.6 we will show that for a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra we, in fact, have $\hat{Z}_L(A) = \hat{Z}_R(A)$, so for the majority of this paper the notation $\hat{Z}(A)$ is not ambiguous.

Definition 3.1.10. *We say a graded algebra A over F is a central graded algebra (CGA) if $\hat{Z}_L(A) = F$.*

Definition 3.1.11. *An algebra A over F is a central simple graded algebra (GCSA) if it is both graded central and graded simple.*

Definition 3.1.12. *A graded homomorphism is a map $\varphi : A \to B$ which is a homomorphism (in the regular sense) and $\varphi(A_i) \subset B_i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. A graded isomorphism is a (regular) isomorphism which is also a graded homomorphism.*

Example 3.1.13. *The following are some examples of graded central simple algebras:*

(1) The algebra in Example 3.1.3, $A = F[x]/(x^n - d)$ with $d \neq 0$, is an central simple $\mathbb{Z}/n\mathbb{Z}$-graded algebra over F. However, it is not necessarily central or simple as an ungraded algebra since $Z(A) = A$ and if we take $d = 1$,then $A \cong F \times \cdots \times F$ by the Chinese remainder theorem (recall F contains a primitive root of unity), which has proper ideals.

(2) Recall the trivial grading from Example 3.1.2. If A is a central simple algebra over F, then (A) is a graded central simple algebra over F.

(3) $\mathbb{M}_n(D)$ is a graded central simple algebra over F where D is a graded division algebra over F.

We now define a graded tensor product of graded algebras, denoted $\widehat{\otimes}$, and review some of the properties of the graded tensor.

Definition 3.1.14. *The graded tensor product* $(\widehat{\otimes})$ *of two central simple graded algebras,* A *and* B*, is the same as* $A \otimes B$ *as vector spaces, but has multiplication induced by*

$$(a \mathbin{\widehat{\otimes}} b)(a' \mathbin{\widehat{\otimes}} b') = \rho^{\partial b \cdot \partial a'}(aa' \mathbin{\widehat{\otimes}} bb').$$

Proposition 3.1.15. *If* A, B, C *are* $\mathbb{Z}/n\mathbb{Z}$*-graded algebras, then* $(A \mathbin{\widehat{\otimes}} B) \mathbin{\widehat{\otimes}} C \cong A \mathbin{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}} C)$ *as graded algebras.*

Proof. We know the above are isomorphic as vector spaces since $A \mathbin{\widehat{\otimes}} B$ is the same as $A \otimes B$ as vector spaces. Since the map clearly preserves the grading, we only need to show the map induced by

$$\begin{aligned} \phi : (A \mathbin{\widehat{\otimes}} B) \mathbin{\widehat{\otimes}} C &\longrightarrow A \mathbin{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}} C) \\ (a \mathbin{\widehat{\otimes}} b) \mathbin{\widehat{\otimes}} c &\longmapsto a \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} c) \end{aligned}$$

is a homomorphism. Let a, b, c be homogeneous elements of A, B, C, respectively. Then,

$$\begin{aligned} \phi([(a \mathbin{\widehat{\otimes}} b) \mathbin{\widehat{\otimes}} c][(a' \mathbin{\widehat{\otimes}} b') \mathbin{\widehat{\otimes}} c']) &= \rho^{\partial c(\partial a' + \partial b')} \phi([(a \mathbin{\widehat{\otimes}} b)(a' \mathbin{\widehat{\otimes}} b')] \mathbin{\widehat{\otimes}} cc') \\ &= \rho^{\partial c \cdot \partial a' + \partial c \cdot \partial b' + \partial b \cdot \partial a'} \phi((aa' \mathbin{\widehat{\otimes}} bb') \mathbin{\widehat{\otimes}} cc') \\ &= \rho^{\partial c \cdot \partial a' + \partial c \cdot \partial b' + \partial b \cdot \partial a'} aa' \mathbin{\widehat{\otimes}} (bb' \mathbin{\widehat{\otimes}} cc') \\ &= \rho^{\partial a'(\partial b + \partial c) + \partial c \cdot \partial b'} aa' \mathbin{\widehat{\otimes}} (bb' \mathbin{\widehat{\otimes}} cc') \\ &= \rho^{\partial a'(\partial b + \partial c)} aa' \mathbin{\widehat{\otimes}} [(b \mathbin{\widehat{\otimes}} c)(b' \mathbin{\widehat{\otimes}} c')] \\ &= [a \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} c)][a' \mathbin{\widehat{\otimes}} b' (\widehat{\otimes} c')] \\ &= \phi((a \mathbin{\widehat{\otimes}} b) \mathbin{\widehat{\otimes}} c) \phi((a' \mathbin{\widehat{\otimes}} b') \mathbin{\widehat{\otimes}} c') \end{aligned}$$

Thus, ϕ is a homomorphism and $(A \widehat{\otimes} B) \widehat{\otimes} C \cong A \widehat{\otimes} (B \widehat{\otimes} C)$. $\square$

The following propositions from Koç and Kurtulmaz [5] explore the results when graded tensoring two graded algebras. As one would hope, the graded tensor of two central simple graded algebras results in a third graded central simple algebra.

Proposition 3.1.16. *[5, Proposition 2.3] If A and B are central graded algebras, then so is $A \widehat{\otimes} B$.*

Proposition 3.1.17. *[5, Proposition 2.4] If A is a central simple graded algebra and B is a simple graded algebra, then $A \widehat{\otimes} B$ is a simple graded algebra. In particular, if A and B are both graded central simple algebras, then $A \widehat{\otimes} B$ is also a graded central simple algebra.*

The following proposition gives criteria for when the graded tensor and regular tensor are isomorphic, which will be useful in the following sections, particularly in defining the graded Brauer group.

Proposition 3.1.18 ([5, Theorem 2.5])**.** *Let A and B be finite dimensional graded algebras. If there exists an invertible element $z \in A$ such that $z^n = 1$, $az = \rho^{\partial a} za$ for all homogeneous elements $a \in A$, then $A \widehat{\otimes} B$ and $A \otimes B$ are isomorphic. Further, if $z \in A_0$ this isomorphism is a graded isomorphism.*

It is important to note that the hypothesis that $z^n = 1$ is left out in [5], however it is a necessary hypothesis in order for the result to hold. In order to see this is necessary, we will need to use some definitions from Chapter 5. Consider the $\mathbb{Z}/15\mathbb{Z}$-graded algebras A and B. Let A be a $(1, 0)$ algebra (3-odd algebra) with discriminant 1 and B an even algebra with discriminant d (see Definition 4.1.1, 5.1.1, and 5.1.2). We will see in Chapter 5 that the discriminant is an invariant of graded algebras and two isomorphic algebras have the same discriminant. Applying Theorem 5.2.1, we see the discriminant of $A \widehat{\otimes} B$ is d^j where $j \equiv 1 - (k - i)3 \pmod{15}$, $3k \equiv 1 \pmod 5$, and $3i \equiv -1 \pmod 5$. Solving these

equivalences, we see $j \equiv 4 \pmod{15}$. So the discriminant of $A \widehat{\otimes} B$ is d^4, whereas the discriminant of $B \widehat{\otimes} A$ is d. But, if the $z^n = 1$ hypothesis were omitted, the algebras A and B would satisfy the hypotheses of Theorem 3.1.18. In the following section we will show the existence of z in an even algebra, but we do not necessarily know $z^n = 1$ in an arbitrary even algebra. So, Theorem 3.1.18 implies $A \widehat{\otimes} B \cong A \otimes B \cong B \otimes A \cong B \widehat{\otimes} A$, which is a contradiction since the graded tensor products have different discriminants.

In the following section, Propostion 3.2.6, we will see that the degree 0 element described in the above proposition always exists in even type algebra with discriminant 1. Moreover, we know $A \otimes B \cong B \otimes A$, thus this proposition also tells us $A \widehat{\otimes} B \cong A \otimes B \cong B \otimes A \cong B \widehat{\otimes} A$ if either A or B is an even algebra with discriminant 1. Thus, this proposition also provides criteria for $A \widehat{\otimes} B \cong B \widehat{\otimes} A$. The above proposition will be useful in the following sections, particularly in defining the graded Brauer group.

3.2 Structure Theorems for Even and Odd Algebras

This section will focus on the structure results of $\mathbb{Z}/n\mathbb{Z}$-graded central simple algebras, which I will generalize in the next section for $n = pq$ where p, q are distinct primes with $p, q \neq 2$. To provide context, we begin by reviewing the structure results in [5] and [15]. The following theorem in Vela [15] and corollary in Koç and Kurtulmaz [5] are key in developing theory for graded central simple algebras.

Theorem 3.2.1. *([5, Theorem 4.1],[15, Theorem 4.1]) Let A be a $\mathbb{Z}/n\mathbb{Z}$-graded central simple algebra. Then, there exists a nonzero homogeneous element $z \in Z(A)$ that is of minimal degree and generates $Z(A)$. Moreover, the degree of z divides n.*

Notice if $n = p$, a prime, there are only two options for the degree of z, 1 and p. This case has been thoroughly studied and leads itself to the following definitions of even and odd type algebras.

Definition 3.2.2. *A central simple graded algebra is of even type if $Z(A) = F$ and is of odd type if $Z(A) = F[z]$, where z is a homogeneous element of degree 1 and $z^n = d \in F^*$.*

Notice from Theorem 3.2.1 and Definition 3.2.2 that not all $\mathbb{Z}/n\mathbb{Z}$-graded algebras are either even or odd. In particular, if n is not prime, there is an algebra that is neither even nor odd. For example, $F \oplus Fz \oplus Fz^2$ graded by $\mathbb{Z}/6\mathbb{Z}$, where the degree of z is 2, is neither even nor odd. This will be the focus of Section 4.1.

Example 3.2.3. *A quaternion algebra $C = \left(\frac{a,b}{F}\right)$ with basis $\{1, i, j, k\}$ and relations $i^2 = a$, $j^2 = b$, and $ij = -ji$ is a $\mathbb{Z}/2\mathbb{Z}$-graded central simple algebra over F, with $C_0 = F \oplus Fk$ and $C_1 = Fi \oplus Fj$. Since $Z(C) = F$, this is an even type algebra.*

Example 3.2.4. *Consider $A = F[x]/(x^n - d) = F \oplus Fx \oplus \cdots \oplus Fx^{n-1}$, $d \in F^*$, from example 3.1.3. Since A is commutative, $Z(A) = A = F[x]$, and hence A is an odd algebra.*

Corollary 3.2.5 ([5, Corollary 3.4])**.** *Let A be a $\mathbb{Z}/n\mathbb{Z}$-graded central simple algebra. A is of even type if and only if A is central and simple as an ungraded algebra.*

The following proposition shows the existence of the element mentioned in Proposition 3.1.18. However it is important to observe, for an even algebra A it is not necessarily true that the element, u, has the property $u^n = 1$. We will later define u^n to be the discriminant of an even algebra A. So, if $u^n = 1$, in the following theorem (i.e. discriminant of A is 1) and either A or B is an even algebra, $A \mathbin{\widehat{\otimes}} B \cong B \mathbin{\widehat{\otimes}} A$.

Proposition 3.2.6. *Let A be a $\mathbb{Z}/n\mathbb{Z}$-graded central simple algebra of even type. Then, there exists a degree 0 element $u \in A$ such that $ua = \rho^{\partial a} au$.*

Notation: We will often refer to the element u in this Proposition as the "Noether Skolem element" in an even algebra.

Proof. Since A is an even GCSA, by Corollary 3.2.5 A is a CSA (as an ungraded algebra). Define the following F-linear map on A induced by the following,

$$\begin{aligned} A &\xrightarrow{\varphi} A \\ a \in \mathcal{H}(A) &\longmapsto \rho^{\partial a} a. \end{aligned}$$

We will show that φ is an automorphism by checking that it is an injective homomorphism. Let $a, b \in \mathcal{H}(A)$, then

$$\varphi(ab) = \rho^{\partial a + \partial b} ab = (\rho^{\partial a} a)(\rho^{\partial b} b) = \varphi(a)\varphi(b).$$

Now, we will look at the kernel of this map,

$$\ker(\varphi) = span\{a \in \mathcal{H}(A) \mid \varphi(a) = \rho^a a = 0\} = \{0\}.$$

So φ is an injective homomorphism, and thus an automorphism. Since A is a CSA, the Noether-Skolem theorem implies that this automorphism is inner. That is, there exists $u \in A$ such that $\varphi(a) = uau^{-1}$. Therefore, $uau^{-1} = \rho^{\partial a} a$ which implies the desired result $ua = \rho^{\partial a} au$.

Now, we must check $u \in A_0$, i.e. u is degree 0. We will show $C_A(u) = A_0$. Let $a \in A_0$, then $ua = au$ and so $a \in C_A(u)$. Conversely, if we take $a \in C_A(u)$, then $ua = au$. But,on the other hand we have $ua = \rho^{\partial a} au$, so $\rho^{\partial a} = 1$ and thus $\partial a = 0$. Thus, we have $C_A(u) = A_0$, and in particular, $u \in A_0$. □

In their 2012 paper [5], Koç and Kurtulmaz proved the following structure results. The structure of a graded central simple algebra depends on the type of the algebra. The structure theorem for the odd algebras gives a very concrete and explicit statement for the graded

structure. We will give an analogous result for algebras that are neither even nor odd in the next section, cf. Theorem 4.2.4.

Theorem 3.2.7. *[5, Theorem 4.4] Let A be a GCSA of odd type, graded by $\mathbb{Z}/n\mathbb{Z}$. Then*

(i) *A_0 is central simple as an ungraded algebra;*

(ii) *$A = A_0[z] = A_0 \oplus A_0 z \oplus \cdots \oplus A_0 z^{n-1}$ and $C_A(A_0) = F[z]$ for some central homogeneous element z of degree 1 such that $z^n = a \in F^*$, which is uniquely determined up to a scalar multiple with these properties;*

(iii) *There are graded isomorphisms*

$$A \cong (A_0) \widehat{\otimes} F\langle \sqrt[n]{a} \rangle \cong (A_0) \otimes F\langle \sqrt[n]{a} \rangle,$$

where $F\langle \sqrt[n]{a} \rangle$ represents the graded algebra $F[x]/(x^n - a)$.

(iv) *(a) If $x^n - a$ is irreducible over F, then A is central simple over the field $F(\sqrt[n]{a})$,*

(b) If $x^n - a$ has a root in F, then $Z(A) = \underbrace{F \times \cdots \times F}_{n\text{-copies}}$ and $A = \underbrace{A_0 \times \cdots \times A_0}_{n\text{-copies}}$.

Since an even algebra is, in fact, a central simple algebra, we know from the Wedderburn Artin Theorem that $A \cong \mathbb{M}_r(D)$ as ungraded algebras for some central division algebra, D over F. However, this does not provide any information about the graded structure of A. The following structure theorem, 3.2.9 from [5] describe the graded structure of an even algebra. However, before we state the structure theorem, we must first recall some notation and make a definition.

Recall the following grading on $\mathbb{M}_r(D)$ from Example 3.1.13 which appears in the following structure theorem: The homogeneous elements of the grading on $\mathbb{M}_r(D)$ denoted by $\widetilde{\mathbb{M}}_r(D)$, are of degree i if all the entries in the matrix are degree i elements of D.

Definition 3.2.8. *The generalized Clifford algebra associated with the vector space* $V = Fe_1 \oplus Fe_2 \oplus \cdots \oplus Fe_n$ *with an ordered basis* $\{e_1, e_2, \ldots, e_n\}$ *is denoted* $C(V) = (a_1, a_2, \cdots, a_n)_{\rho}^{(n)}$. $C(V)$ *is the algebra generated by* $e_1, \cdots, e_n$ *with relations* $e_i^n = a_i$ *for* $i = 1, \ldots, n$ *and* $e_j e_i = \rho e_i e_j$ *for* $j > i$ *where* $a_1, \cdots, a_n \in F$ *and* ρ *is an* n^{th} *root of unity.*

We are now ready to state the structure theorem for even algebras.

Theorem 3.2.9. *[5, Theorem 4.6] Let* A *be a GCSA of even type, graded by* $\mathbb{Z}/n\mathbb{Z}$. *Let* D *be a central division algebra over* F *such that* $A \cong \mathbb{M}_r(D)$ *as ungraded algebras and characteristic of* F *does not divide* n. *Then,*

$$Z(A_0) = C_A(A_0) = F \oplus Fz \oplus \cdots \oplus Fz^{n-1}$$

for some $z \in Z(A_0)$ *with* $z^n = c \in F^*$ *and the following statements hold:,*

(i) If $c \in (F^*)^n$, *then there is a graded space* $V = \oplus_{i=0}^{p-1} V_i$ *such that*

  (a) $A \cong End(V) \,\widehat{\otimes}\, (D)$ *as graded algebras,*

  (b) $A_0 \cong \mathbb{M}_{r_0}(D) \times \cdots \times \mathbb{M}_{r_{n-1}}(D)$ *where* $r_i = dim(V_i)$, $i = 0, \ldots, n-1$,

  (c) $Z(A_0) \cong \underbrace{F \times \cdots \times F}_{n\text{-}copies}$.

(ii) If $x^n - c$ *is irreducible over* F *and* D *has a subfield isomorphic with* $F(\sqrt[n]{c}) = Z(A_0)$, *then there exists a grading on* D *such that*

  (a) $A \cong \widetilde{\mathbb{M}}_r(D) \cong \widetilde{\mathbb{M}}_r(F) \,\widehat{\otimes}\, D$ *as graded algebras,*

  (b) $A_0 \cong \mathbb{M}_n(D_0)$

  (c) A_0 *is central simple over* $Z(A_0)$.

(iii) if $x^n - c$ *is irreducible over* F *but* D *has no subfields isomorphic to* $F(\sqrt[n]{c}) \cong Z(A_0)$, *then*

(a) $r = nm$ *and* $a \cong (\mathbb{M}_m(D)) \widehat{\otimes} (c,1)_\rho^{(n)}$ *as graded algebras*

(b) $A_0 \cong \mathbb{M}_m(D) \otimes F(\sqrt[r]{c})$,

(c) A_0 *is central simple over* $Z(A_0)$.

Chapter 4

P-Odd Algebras and their Structure

4.1 Definitions

As mentioned previously, the current literature focuses on theory and structure theorems for GCSAs that are even or odd. There are also many generalizations of the Brauer group, however there are not many results for the corresponding algebras. We will now focus on developing structure results for algebras that are neither even nor odd. We will call these algebras p-odd type, where p is the degree of the z element that generates the center. These algebras share characteristics of both even and odd algebras. For the remainder of this thesis, we will turn our attention to $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebras, which are neither even nor odd.

In this section, we will assume p and q are distinct primes with $p > q$. However, in the next section, Section 5, we will further restrict, $p, q \neq 2$.

The main result of this section is the following structure theorem for p-odd algebras.

Theorem. *Let A be a central simple $\mathbb{Z}/pq\mathbb{Z}$ graded p-odd algebra with $z \in A$, a degree p element that generates $Z(A)$ and $z^q = d \in F^*$. Then the following statements hold:*

1. *$A_{(q)} = A_0 \oplus A_q \oplus A_{2q} \oplus \cdots \oplus A_{(p-1)q}$ is central simple as an ungraded algebra, as well as a $\mathbb{Z}/p\mathbb{Z}$-GCSA.*
2. *$A = A_{(q)}[z] = A_{(q)} \oplus A_{(q)}z \oplus A_{(q)}z^2 \oplus \cdots \oplus A_{(q)}z^{q-1}$ and $C_A(A_{(q)}) = F[z]$.*
3. *$A \cong A_{(q)} \widehat{\otimes} F[z] \cong A_{(q)} \otimes F[z]$.*

In order to prove the above theorem, we will need to make some new definitions and prove a few propositions. The three statements in the above theorem are stated and proved independently in Propositions 4.2.1, 4.2.2, and 4.2.3. The three propositions are then combined to form the above theorem, which is restated toward the end of this section as Theorem 4.2.4. Recall from Theorem 3.2.1, that in a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra there exists a special homogeneous element z of minimal degree that generates the center and whose order divides pq. This leads us to the following definition, which complements the definitions of even and odd type algebras given in the previous section.

Definition 4.1.1. *Let A be a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra. We say A is of type $(1,0)$, or p-odd, if $Z(A) = F \oplus Fz \oplus \cdots \oplus Fz^{q-1}$, where deg(z)=p and $z^q = d \in F^*$. Similarly we say A is of type $(0,1)$ or q-odd if the degree of z is q.*

Remark. *The classical even algebra is type $(0,0)$ and the classical odd algebra is type $(1,1)$.*

Example 4.1.2. *The $\mathbb{Z}/15\mathbb{Z}$-graded central simple algebra, $A = F[z]/(z^5 - d) = F \oplus Fz \oplus Fz^2 \oplus Fz^3 \oplus Fz^4$ where $\deg(z) = 3$ is a 3-odd algebra ($p = 3$ and $q = 5$).*

It should be noted that the algebra types are represented as elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whereas in the classical $\mathbb{Z}/2\mathbb{Z}$-graded case the types are represented by 0 and 1. Moreover, at the end of this section we will show that the type of a graded tensor of algebras is given

by addition in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For example, graded tensoring a $(1,0)$ algebra with a $(0,1)$ algebra will result in a $(1,1)$ algebra. This will be shown at the end of this section, as the structure result simplifies the proof in many cases.

4.2 Structure Results

We now look at the structure theorem for p-odd algebras. This theorem is analogous to the odd structure result in the previous section, Theorem 3.2.7, but in a more general setting.

Proposition 4.2.1. *Let A be a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple, p-odd algebra and let $z \in A$ be an element of degree p that generates the center of A and $z^q = d \in F^*$. Then, we can write $A = A_{(q)} \oplus A_{(q)}z \oplus A_{(q)}z^2 \oplus \cdots \oplus A_{(q)}z^{q-1}$, where $A_{(q)} = A_0 \oplus A_q \oplus A_{2q} \oplus \cdots \oplus A_{(p-1)q}$.*

Proof. We will show that $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{pq-1} = A_{(q)} \oplus A_{(q)}z \oplus A_{(q)}z^2 \oplus \cdots \oplus A_{(q)}z^{q-1}$. Since p and q are relatively prime, we can write any integer k as $k = rp+sq$ for some integers r and s. Thus we can write each of the integers $0, 1, 2, \ldots, pq-1$ as $rp + sq \pmod{pq}$, for some r, s.

The homogeneous parts of $A_{(q)}[z]$ are of the form $A_{iq}z^j$, which has degree $iq+pj \pmod{pq}$. Thus, by changing i and j, we can obtain homogeneous parts of degrees $0, \ldots, pq-1$. Notice that we are considering i modulo p and j modulo q. Hence, the Chinese Remainder Theorem ensures distinct pairs of i and j produce distinct results of $iq+pj$ modulo pq and conversely, distinct $iq + pj \pmod{pq}$ come from distinct pairs of i and j modulo p and q, respectively.

Now we check that $A_{iq+jp} = A_{iq}z^j$. Clearly, $A_{iq}z^j \subset A_{iq+jp}$ since $\deg(A_{iq}z^j) = iq + jp$. Now, let $a \in A_{iq+jp}$. We can write $a = (az^{-j})z^j$ and the degree of az^{-j} is $(iq + jp) + (-jp) = iq$. So, $az^{-j} \in A_{iq}$ and $a = (az^{-j})z^j \in A_{iq}z^j$. Thus $A_{iq+jp} \subset A_{iq}z^j$ and we have $A_{iq+jp} = A_{iq}z^j$. This, along with the observation from the Chinese Remainder Theorem above gives us $A = \oplus_{i,j} A_{iq}z^j$. Now, if we consider $A_{iq}z^j$ with a fixed j, iq will range through

all the possible multiples of q (modulo pq) and we get $A = \oplus_j A_{(q)} z^j$. We now have the desired result, $A_0 \oplus A_1 \oplus \cdots \oplus A_{pq-1} = A_{(q)} \oplus A_{(q)} z \oplus A_{(q)} z^2 \oplus \cdots \oplus A_{(q)} z^{q-1}$.

□

Thus, we can now write any $\mathbb{Z}/pq\mathbb{Z}$-graded central simple p-odd algebra as $A = A_{(q)} \oplus A_{(q)} z \oplus A_{(q)} z^2 \oplus \cdots \oplus A_{(q)} z^{q-1}$. We will now turn our attention to the subalgebra $A_{(q)}$ of A.

If we define $B = A_{(q)}$, we can view B as a $\mathbb{Z}/p\mathbb{Z}$ graded algebra with $B_k = A_{kq}$. To verify this, observe $B_i B_j = A_{iq} A_{jq} \subset A_{iq+jq} = A_{(i+j)q} = B_{i+j}$ and $B_p = A_{pq} = A_0 = B_0$.

Proposition 4.2.2. *Let A be a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple p-odd algebra and let $z \in A$ be an element of degree p that generates the center of A and $z^q = d \in F^*$. Then $A_{(q)}$ is a $\mathbb{Z}/p\mathbb{Z}$-graded central simple algebra under the grading described above, as well as a central simple algebra (as an ungraded algebra).*

Proof. We will first show that $A_{(q)}$ is graded central as a $\mathbb{Z}/p\mathbb{Z}$-graded algebra. We will show that $\hat{Z}(A_{(q)}) \subset \hat{Z}(A) = F$. Let $x \in \hat{Z}(A_{(q)})$. Then, the degree of x as an element of A is a multiple of q, say lq, and therefore in terms of the $\mathbb{Z}/p\mathbb{Z}$ grading of $A_{(q)}$, x has degree l. Moreover, x graded commutes with all elements of $A_{(q)}$ since $x \in \hat{Z}(A_{(q)})$. We want to show x graded commutes with all elements of $A = A_{(q)} \oplus A_{(q)} z \oplus \cdots \oplus A_{(q)} z^{q-1}$, so we only need to check that x graded commutes with z. Now, z is in the center of A, $Z(A)$, so $xz = zx$, but this is the same as $xz = \rho^{lqp} zx$. Note the subtle observation that since ρ is a primitive pq^{th} root of unity, $\omega = \rho^q$ is a primitive p^{th} root of unity. So we have $xz = \omega^{lp} zx$, which is the statement for graded commuting when viewing $A_{(q)}$ as a $\mathbb{Z}/p\mathbb{Z}$-graded algebra. Thus, $x \in \hat{Z}(A)$ and so $\hat{Z}(A_{(q)}) \subset \hat{Z}(A) = F$. Therefore, $\hat{Z}(A_{(q)}) = F$.

Now, we show that $A_{(q)}$ is graded simple. Let $I = I_0 \oplus I_q \oplus \cdots \oplus I_{(p-1)q}$ be a nonzero homogeneous ideal in $A_{(q)}$. Consider the ideal $J = I \oplus Iz \oplus \cdots Iz^{q-1}$ in A. J is a nonzero

graded ideal of A, but A is graded simple, so $J = A$. So we have

$$I \oplus Iz \oplus \cdots Iz^{q-1} = A_{(q)} \oplus A_{(q)}z \oplus \cdots \oplus A_{(q)}z^{q-1}.$$

As mentioned previously if r and s are integers such that $r = 0, \ldots p-1$ and $s = 0 \ldots q-1$, then the linear combinations, $rq + sp$, give distinct integers modulo pq. Since the above expressions are both decompositions of A, they have the same grading. Moreover, both the decompositions of I and $A_{(q)}$ only consist of degrees which are multiples of q. Since the decompositions give distinct integers mod pq, we do not get any repeats in the decomposition. In particular, on the right hand side the degrees that are a multiple of q only occur in $A_{(q)}$. Similarly, on the left hand side, the degrees that are a multiple of q only occur in the I term, since I is a homogeneous ideal of $A_{(q)}$. Thus, since these two decompositions have the same grading we get $A_0 = I_0$, $A_q = I_q$, $\ldots$, $A_{(p-1)q} = I_{(p-1)q}$. Therefore we have, $A_{(q)} = I$.

We will not show that $A_{(q)}$ is central simple as an ungraded algebra.

$$Z(A_{(q)}) = A_{(q)} \cap C_A(A_{(q)}) = A_{(q)} \cap Z(A).$$

Note that it is easy to see $Z(A) = C_A(A_{(q)})$ from Proposition 4.2.1 since $A = A_{(q)} \oplus A_{(q)}z \oplus \cdots \oplus A_{(q)}z^{q-1}$ and z is central. Now, $A_{(q)}$ consists of homogeneous elements with degrees that are a multiple of q, while $Z(A) = F[z]$ only consists of homogeneous elements of degrees that are multiples of p (since the degree of z is p). Since p and q are prime, $A_{(q)} \cap Z(A) = A_0 \cap F = F$. Thus, $Z(A_{(q)}) = F$, and $A_{(q)}$ is F-central.

Now, we know that $A_{(q)}$ is a $\mathbb{Z}/p\mathbb{Z}$-graded central simple algebra and $Z(A_{(q)}) = F$. Then, by Corollary 3.2.5, $A_{(q)}$ is central simple as an ungraded algebra.

□

Observe, if we let $B = A_{(q)}$, we have $A = B \oplus Bz \oplus \ldots \oplus Bz^{q-1}$, where z is degree p. Now, we just showed that B is a $\mathbb{Z}/p\mathbb{Z}$-graded central simple algebra, so B is either even or odd. But as we just showed, $Z(B) = F$, so B is even. The following Proposition shows that a p-odd algebra is the graded tensor of an even algebra with a more simple, explicit p-odd algebra. This decomposition will prove useful in the next section in regards to defining the discriminant of an algebra and various other propositions.

Proposition 4.2.3. *If A is a graded central simple algebra of p-odd type and z is a degree p element that generates the center of A and $z^p = d \in F^*$, then $A \cong A_{(q)} \widehat{\otimes} F[z] \cong A_{(q)} \otimes F[z]$.*

Proof. Recall, $A = A_{(q)}[z] = A_{(q)}F[z]$ from 4.2.1. Consider the F- linear map induced by,

$$\begin{array}{rcl} A = A_{(q)}F[z] & \xrightarrow{\varphi} & A_{(q)} \otimes F[z]. \\ az^i & \longmapsto & a \otimes z^i \end{array}$$

We first check that the above map φ is, in fact, a homomorphism. Let $a, b \in A_{(q)}$, then both a, b have degrees which are a multiple of q.

$$\begin{aligned} \varphi(az^i \cdot bz^j) &= \varphi(ab \cdot z^i z^j) \\ &= ab \otimes z^i z^j \\ &= (a \otimes z^i)(b \otimes z^j) \\ &= \varphi(az^i)\varphi(bz^j), \end{aligned}$$

since $z \in Z(A)$. It is also important to note $(a \otimes z^i)(b \otimes z^j) = \rho^{\partial z \cdot \partial b} ab \otimes zz^j = ab \otimes zz^j$ because $\partial z = p$ and $\partial b = kq$ for some k. Furthermore, this map is surjective and the dimensions of $A_{(q)}F[z]$ and $A_{(q)} \otimes F[z]$ are equal, so φ is an ungraded isomorphism.

Since the elements of $A_{(q)}$ have degrees which are a multiple of q and elements of $F[z]$ have degrees that are a multiple of p, $A_{(q)}$ and $F[z]$ commute and $A_{(q)} \widehat{\otimes} F[z] \cong A_{(q)} \otimes F[z]$. $\square$

Theorem 4.2.4. *Let A be a central simple $\mathbb{Z}/pq\mathbb{Z}$-graded p-odd algebra with $z \in A$, a element of degree p that generates the center of A and $z^q = d \in F^*$,*

1. *$A_{(q)} = A_0 \oplus A_q \oplus A_{2q} \oplus \cdots \oplus A_{(p-1)q}$ is central simple as an ungraded algebra, as well as a $\mathbb{Z}/p\mathbb{Z}$-GCSA.*

2. *$A = A_{(q)}[z] = A_{(q)} \oplus A_{(q)}z \oplus A_{(q)}z^2 \oplus \cdots \oplus A_{(q)}z^{q-1}$, and $C_A(A_{(q)}) = F[z]$.*

3. *$A \cong A_{(q)} \widehat{\otimes} F[z] \cong A_{(q)} \otimes F[z]$.*

Proof. This theorem is a combination of the previous three propositions, so all that remains is to show $C_A(A_{(q)}) = F[z]$. Since $A = A_{(q)} \oplus A_{(q)}z \oplus A_{(q)}z^2 \oplus \cdots \oplus A_{(q)}z^{q-1}$, we see $Z(A) = C_A(A_{(q)})$. But A is a p-odd algebra, so $Z(A) = F[z]$, and thus we have $C_A(A_{(q)}) = F[z]$. □

Now that we have the structure results, we will prove how the algebra types combine when graded tensored together. The following proposition will also come in useful in the next section since it shows a map from the graded Brauer group to the group of algebra types is a homomorphism.

Proposition 4.2.5. *Let $A_{(i,j)}$ represent a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra of type (i, j), where $(i, j) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $A_{(i,j)} \widehat{\otimes} A_{(k,l)} = A_{(i+k,j+l)}$, where $+$ is addition in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. We will consider each case separately.

A proof of the combinations of the classic even $(1, 1)$ and odd $(0, 0)$ algebras can be found in Vela [15, Theorem 6.4]. This includes the graded tensor of two even algebras, the graded tensor of two odd algebras, and the graded tensor of an even and an odd algebra.

Let A be a $(0, 0)$ algebra and B a $(1, 0)$ algebra. We want to show $A \widehat{\otimes} B$ is of type $(1, 0)$. Since A is even, there exists $u \in A$ of degree 0, such that $ua = \rho^{\partial a} au$ and B is p-odd, so there is an element $z \in Z(B)$ of degree p. Consider $u^{-p} \widehat{\otimes} z \in A \widehat{\otimes} B$. The degree of this

element is p and for any $a \in A$ and $b \in B$, we have

$$\begin{aligned}(u^{-p} \widehat{\otimes} z)(a \widehat{\otimes} b) &= \rho^{p\cdot\partial a}(u^{-p}a \widehat{\otimes} zb) \\ &= \rho^{p\cdot\partial a}\rho^{-p\cdot\partial a}(au^{-p} \widehat{\otimes} bz) \\ &= au^{-p} \widehat{\otimes} bz \\ &= (a \widehat{\otimes} b)(u^{-p} \widehat{\otimes} z).\end{aligned}$$

So, we have a degree p element, $u^{-p} \widehat{\otimes} z$, in $Z(A \widehat{\otimes} B)$. Recall, the element in a GCSA that generates the center is of minimal degree, so in order for $A \widehat{\otimes} B$ to be a p-odd algebra, we need to show that $Z(A \widehat{\otimes} B)$ cannot be generated by an element with degree less than p, i.e. with degree 1. Notice, we have already shown there is a degree p element in the center. If, in addition there were a degree q element in the center, we would necessarily have a degree 1 element in the center since there exist i, j such that $1 = pi + qj$. We will show there are no elements of degree q in $Z(A\widehat{\otimes}B)$. We begin by assuming there is an element $a\widehat{\otimes}b \in Z(A\widehat{\otimes}B)$ with $\partial(a+b) = \partial(a) + \partial(b) = q$. Since $B = B_{(q)} \widehat{\otimes} F[z]$, there is a degree 0 element, $v \in B_{(q)}$ such that $vb = \rho^{\partial b}bv$ for all $b \in B_{(q)}$. In particular, since $a \widehat{\otimes} b \in Z(A \widehat{\otimes} B)$, $a \widehat{\otimes} b$ commutes with $u \widehat{\otimes} 1$ and $1 \widehat{\otimes} v$,

$$(a \widehat{\otimes} b)(u \widehat{\otimes} 1) = (u \widehat{\otimes} 1)(a \widehat{\otimes} b),$$

which implies

$$au \widehat{\otimes} b = ua \widehat{\otimes} b = \rho^{\partial a}au \widehat{\otimes} b.$$

Thus, $\rho^{\partial a} = 1$ and $\partial a \equiv 0 \pmod{pq}$. Now, $q = \partial(a + b) = \partial(b)$ and from the structure theorem for p-odd algebras, we know $b \in A_{(q)}$. Now, we consider,

$$(a \widehat{\otimes} b)(1 \widehat{\otimes} v) = (1 \widehat{\otimes} v)(a \widehat{\otimes} b),$$

which impies

$$a \widehat{\otimes} bv = a \widehat{\otimes} vb = \rho^{\partial b} a \widehat{\otimes} bv,$$

since $b \in B_{(q)}$. Similar to above, we get $\rho^{\partial b} = 1$, which implies $\partial b \equiv 0 \pmod{pq}$. However, $\partial b = q$, so we have arrived at a contradiction. Therefore, there are no elements of degree q in $Z(A \widehat{\otimes} B)$ and hence no degree 1 elements. Thus, in this case, an element of degree p is the only possible degree for the element that generates the center (since it must divide pq).

By reversing the role of p and q in the above argument we see the graded tensor of a $(0,0)$ algebra and a $(1,0)$ algebra results in a $(1,0)$ algebra. A similar argument holds for $B_{(1,0)} \widehat{\otimes} A_{(0,0)}$, where $z \widehat{\otimes} u^p$ has degree p and is in the center of $B_{(1,0)} \widehat{\otimes} A_{(0,0)}$. Again by reversing the role of p and q the result holds for $B_{(0,1)} \widehat{\otimes} A_{(0,0)}$.

Now, let A and B both be p-odd algebras (i.e., type $(1,0)$) and we will show $A \widehat{\otimes} B$ is an even algebra. Using the structure theorem for p-odd algebras we can write $A = A_{(q)} \widehat{\otimes} F[x]$ and $B = B_{(q)} \widehat{\otimes} F[z]$, where both x, z are degree p elements that generate the center of A and B respectively. Now, $A \widehat{\otimes} B = A_{(q)} \widehat{\otimes} F[x] \widehat{\otimes} B_{(q)} \widehat{\otimes} F[z]$. Recall, in general $A \widehat{\otimes} B \ncong B \widehat{\otimes} A$ since you cannot, in general, define a homomorphism between them. However, since the elements of $F[x]$ have degree a multiple of p and elements of $B_{(q)}$ have degree a multiple of q and $\rho^{pq} = 1$, we can define a homomorphism and $F[x] \widehat{\otimes} B_{(q)} \cong B_{(q)} \widehat{\otimes} F[x]$. So, we have

$$A \widehat{\otimes} B \cong (A_{(q)} \widehat{\otimes} B_{(q)}) \widehat{\otimes} (F[x] \widehat{\otimes} F[z]).$$

We have already showed that the graded tensor of two even algebras is even, so we must only show that $F[x] \widehat{\otimes} F[z]$ is even. We will accomplish this by showing $Z(F[x] \widehat{\otimes} F[z]) = F$. It is clear that $F \subset Z(F[x] \widehat{\otimes} F[z])$, so we must only check the reverse inclusion. Let $x^i \widehat{\otimes} z^j \in Z(F[x] \widehat{\otimes} F[z])$ for some i, j, then

$$(x^i \widehat{\otimes} z^j)(x^k \widehat{\otimes} z^l) = (x^k \widehat{\otimes} z^l)(x^i \widehat{\otimes} z^j)$$

for all k, l. In particular, if we take $k = 1$ and $l = 0$ we get

$$(x^i \widehat{\otimes} z^j)(x \widehat{\otimes} 1) = \rho^{p^2 j}(x^i x \widehat{\otimes} z) = (x x^i \widehat{\otimes} z) = (x \widehat{\otimes} 1)(x^i \widehat{\otimes} z),$$

which forces $p^2 j \equiv 0 \pmod{pq}$, i.e. $pq | p^2 j$. This, in turn implies $q|j$, or $j \equiv 0 \pmod{q}$, and $jp \equiv 0 \pmod{pq}$. Now, $\deg(z^j) = pj \equiv 0 \pmod{pq}$, so $z^j \in F$ (the only degree 0 elements of $F[z]$ are F. Similarly, taking $k = 0$ and $l = 1$, results in $pi \equiv 0 \pmod{pq}$ and it follows that $x^i \in F$. Thus we have shown $Z(F[x] \widehat{\otimes} F[z]) = F$, and so we have an even GCSA.

Reversing the role of p and q in the above argument shows that the graded tensor of two $(0, 1)$ algebras is even, $(0, 0)$.

Let $A = A_{(1,0)}$ and $B = B_{(0,1)}$. Using the structure theorem for p-odd algebras we know $A = A_{(q)} \widehat{\otimes} F[z]$ and $B = B_{(p)} \widehat{\otimes} F[x]$, where $\deg(z) = p$ and $\deg(x) = q$ and z, x generate the centers of their respective algebras. Moreover, we know there exist degree 0 elements u and v in $A_{(q)}$ and $B_{(p)}$ (since these are even algebras), respectively, such that $ua = \rho^{\partial a} au$ and $vb = \rho^{\partial b} bv$ for all $a \in A_{(q)}$ and $b \in B_{(p)}$. Consider the element $u^k \widehat{\otimes} z^i \widehat{\otimes} v^l \widehat{\otimes} x^j \in A_{(q)} \widehat{\otimes} F[z] \widehat{\otimes} B_{(p)} \widehat{\otimes} F[x]$. Since p and q are relatively prime, there exist i, j such that $ip + jq \equiv 1 \pmod{pq}$ and so we can choose i, j such that the element has degree 1. Moreover, we choose k and l such that $k \equiv -qj \pmod{pq}$ and $l \equiv pi \pmod{pq}$. Let $a \in A_{(q)}$ and $b \in B_{(p)}$, then in the following computation $\partial a \cdot p \equiv 0 \pmod{pq}$ and $\partial b \cdot q \equiv 0 \pmod{pq}$. We will now show $u^k \widehat{\otimes} z^i \widehat{\otimes} v^l \widehat{\otimes} x^j$ is in the center of $A \widehat{\otimes} B$, however in an effort to simplify notation we will write a to mean ∂a in the exponent of ρ.

$$
\begin{aligned}
[(u^k \mathbin{\widehat{\otimes}} z^i) \mathbin{\widehat{\otimes}} (v^l \mathbin{\widehat{\otimes}} x^j)][(a \mathbin{\widehat{\otimes}} z) \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} x)] &= \rho^{qj(a+p)}(u^k \mathbin{\widehat{\otimes}} z^i)(a \mathbin{\widehat{\otimes}} z) \mathbin{\widehat{\otimes}} (v^l \mathbin{\widehat{\otimes}} x^j)(b \mathbin{\widehat{\otimes}} x) \\
&= \rho^{qja}\rho^{pia+qjb}(u^k a \mathbin{\widehat{\otimes}} z^i z) \mathbin{\widehat{\otimes}} (v^l b \mathbin{\widehat{\otimes}} x^j x) \\
&= \rho^{qja}(u^k a \mathbin{\widehat{\otimes}} z^i z) \mathbin{\widehat{\otimes}} (v^l b \mathbin{\widehat{\otimes}} x^j x) \\
&= \rho^{qja}\rho^{ka+bl}(au^k \mathbin{\widehat{\otimes}} zz^i) \mathbin{\widehat{\otimes}} (bv^l \mathbin{\widehat{\otimes}} xx^j) \\
&= \rho^{qja+ka+bl}(a \mathbin{\widehat{\otimes}} z)(u^k \mathbin{\widehat{\otimes}} z^i) \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} x)(v^l \mathbin{\widehat{\otimes}} x^j) \\
&= \rho^{bl}(a \mathbin{\widehat{\otimes}} z)(u^k \mathbin{\widehat{\otimes}} z^i) \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} x)(v^l \mathbin{\widehat{\otimes}} x^j) \\
&= \rho^{bl}\rho^{-pi(b+q)}[(a \mathbin{\widehat{\otimes}} z) \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} x)][(u^k \mathbin{\widehat{\otimes}} z^i) \mathbin{\widehat{\otimes}} (v^l \mathbin{\widehat{\otimes}} x^j)] \\
&= \rho^{bl-pib+piq}[(a \mathbin{\widehat{\otimes}} z) \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} x)][(u^k \mathbin{\widehat{\otimes}} z^i) \mathbin{\widehat{\otimes}} (v^l \mathbin{\widehat{\otimes}} x^j)] \\
&= [(a \mathbin{\widehat{\otimes}} z) \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} x)][(u^k \mathbin{\widehat{\otimes}} z^i) \mathbin{\widehat{\otimes}} (v^l \mathbin{\widehat{\otimes}} x^j)]
\end{aligned}
$$

So, we have found a degree 1 element in the center of $A \mathbin{\widehat{\otimes}} B$. We know that the center of a GCSA is generated by a central element of minimal degree by Theorem 3.2.1. Since we found a central element of degree 1, the center must be generated by an element of degree 1 and hence the resulting algebra, $A \mathbin{\widehat{\otimes}} B$, is odd.

For $A_{(0,1)} \mathbin{\widehat{\otimes}} B_{(1,0)}$, you can switch the role of p and q in the above explanation.

Let $A = A_{(1,0)}$ be a p-odd algebra and $B = B_{(1,1)}$ an odd algebra. Using the structure theorems, we can write $A = A_{(q)} \mathbin{\widehat{\otimes}} F[x]$ and $B = B_0 \mathbin{\widehat{\otimes}} F[z]$, where $\partial x = p$, $\partial z = 1$, and x and z generate the center of A and B, respectively. Now, we can write

$$A \mathbin{\widehat{\otimes}} B = (A_{(q)} \mathbin{\widehat{\otimes}} F[x]) \mathbin{\widehat{\otimes}} (B_0 \mathbin{\widehat{\otimes}} F[z]) \cong (A_{(q)} \mathbin{\widehat{\otimes}} B_0) \mathbin{\widehat{\otimes}} (F[x] \mathbin{\widehat{\otimes}} F[z]),$$

since algebras concentrated in degree 0 commute with any algebra. $A_{(q)}$ and B_0 are both even, so their graded tensor is an even algebra as well. We have already shown the graded tensor of a $(0,0)$ with a $(0,1)$ algebra results in a $(0,1)$, so all that remains to show is $F[x] \mathbin{\widehat{\otimes}} F[z]$

is a $(0,1)$ algebra. The element $1 \widehat{\otimes} z^q$ is a degree q central element in $F[x] \widehat{\otimes} F[z]$. Now, we will show that there cannot be an element of degree p in $Z(F[x] \widehat{\otimes} F[z])$, which then implies there cannot be an element of degree 1 since we already know there is an element of degree q in the center. Assume there exists $x^i \widehat{\otimes} z^j \in Z(F[x] \widehat{\otimes} F[z])$ with $\deg(x^i \widehat{\otimes} z^j) = pi + j \equiv p \pmod{pq}$. Then, in particular $x^i \widehat{\otimes} z^j$ commutes with $x \widehat{\otimes} 1$,

$$(x^i \widehat{\otimes} z^j)(x \widehat{\otimes} 1) = (x \widehat{\otimes} 1)(x^i \widehat{\otimes} z^j).$$

Combining both sides, we get

$$\rho^{pj}(x^i x \widehat{\otimes} z^j) = (x x^i \widehat{\otimes} z^j) = (x^i x \widehat{\otimes} z^j),$$

which gives $pj \equiv 0 \pmod{pq}$. This congruence implies $j \equiv 0 \pmod{q}$. Similarly, we get $i \equiv 0 \pmod{q}$ by considering $(x^i \widehat{\otimes} z^j)(1 \widehat{\otimes} z) = (1 \widehat{\otimes} z)(x^i \widehat{\otimes} z^j)$. Now, rewriting the congruence $pi + j \equiv p \pmod{pq}$, we get $j \equiv p(1-i) \pmod{pq}$ or equivalently $pq | j - p(1-i)$. So we have, q divides $j - p(1-i)$ and j, so q must also divide $p(1-i)$. Since p and q are distinct, it must be the case that $q|1-i$, or $i \equiv 1 \pmod{q}$. However, we cannot have both $i \equiv 1 \pmod{q}$ and $i \equiv 0 \pmod{q}$, since neither p nor q can be equal to 2. So we have arrived at a contradiction and hence the center of $F[x] \widehat{\otimes} F[z]$ cannot contain an element of degree p, and therefore also does not contain an element of degree 1. Hence, $A \widehat{\otimes} B$ is a $(0,1)$ algebra.

A similar argument holds for $B \widehat{\otimes} A$ and the result for the graded tensor of a $(0,1)$ and a $(1,1)$ algebra follows by reversing the role of p and q in the above argument. We will briefly discuss the case $B \widehat{\otimes} A$ where B and A are as defined in the previous case. Let $A = A_{(1,0)}$ be a p-odd algebra and $B = B_{(1,1)}$ an odd algebra. Using the structure theorems, we can write $A = A_{(q)} \widehat{\otimes} F[x]$ and $B = B_0 \widehat{\otimes} F[z]$, where $\partial x = p$, $\partial z = 1$, and x and z generate the center

of A and B, respectively. Now, we can write

$$B \widehat{\otimes} A = (B_0 \widehat{\otimes} F[z]) \widehat{\otimes} (A_{(q)} \widehat{\otimes} F[x]) \cong B_0 \widehat{\otimes} (F[z] \widehat{\otimes} F[x]) \widehat{\otimes} A_{(q)}.$$

An argument similar to the previous case shows, by contradiction, $F[z] \widehat{\otimes} F[x]$ cannot contain a central element of degree p and hence cannot contain a degree 1 element.

□

Proposition 4.2.6. *If A is a $\mathbb{Z}/pq\mathbb{Z}$-graded (left) central simple algebra, then $\hat{Z}_L(A) = \hat{Z}_R(A)$.*

Proof. We will show this by considering each type of algebra (even, odd, p-odd) since this will allow us to utilize the structure of each type.

First we will show the result for an even $\mathbb{Z}/pq\mathbb{Z}$-graded (left) central simple algebra A. It is clear that $\hat{Z}_L(A) = F \subset \hat{Z}_R(A)$. Now, let $c \in \hat{Z}_R(A)$, then $ac = \rho^{\partial a \cdot \partial c} ca$ for all homogeneous elements $a \in A$. Since A is a even algebra, by Proposition 3.2.6 we know there exists a degree element $u \in A$ such that $ua = \rho^{\partial a} au$ for all homogeneous elements $a \in A$. On one hand, we have

$$uc = \rho^{\partial c} cu.$$

On the other hand, $\partial u = 0$ and c right graded commutes with u, i.e.

$$uc = \rho^{\partial c \cdot \partial u} cu = cu.$$

Thus, we must have $\rho^{\partial c} = 1$ and $\partial c = 0$. Now, the degree of c is 0 so graded commuting with c is the same as (regular) commuting with c. So, $c \in Z(A)$. Since A is an even algebra, it is an (ungraded) central simple algebra, and thus $Z(A) = F$. Therefore, $\hat{Z}_R(A) \subset F = \hat{Z}_L(A)$ and we conclude $\hat{Z}_R(A) = \hat{Z}_L(A)$.

If A is an odd algebra, then by Theorem 3.2.7 there is a graded isomorphism such that $A \cong (A_0) \otimes F\langle \sqrt[pq]{a} \rangle \cong (A_0) \widehat{\otimes} F\langle \sqrt[pq]{a} \rangle$ and $Z(A_0) = F$. For simplicity, we will use the regular tensor product in this proof. Recall the following notation, $F\langle \sqrt[pq]{a} \rangle = F[x]/(x^{pq} - a)$. It is clear that $\hat{Z}_L(A) = F \subset \hat{Z}_R(A)$. To see the reverse containment, let $c \otimes x^k \in \hat{Z}_R(A)$, where $c \in A_0$. So, $c \otimes x^k$ right graded commutes with all elements of A, in particular $c \otimes x^k$ right graded commutes with $1 \otimes x \in A$. Recall the degree of x is 1, so $\partial(c \otimes x^k) = k$ and $\partial(1 \otimes x) = 1$. Using the definition for the right graded center, we get

$$(1 \otimes x)(c \otimes x^k) = \rho^k (c \otimes x^k)(1 \otimes x),$$

which implies

$$c \otimes x x^k = \rho^k (c \otimes x^k x).$$

But on the other hand, $F\langle \sqrt[pq]{a} \rangle$ is commutative, so

$$c \otimes x x^k = c \otimes x^k x.$$

So, we now have

$$c \otimes x^k x = \rho^k (c \otimes x^k x),$$

which implies $\rho^k = 1$ and hence $k = 0$. So, our original element $c \otimes x^k = c \otimes 1 \in (A_0) \otimes F\langle \sqrt[pq]{a} \rangle$ and we know $\hat{Z}_R(A) \subset (A_0)$. So, $c \otimes 1$ must graded commute with $a \otimes 1$ for all $a \in A_0$. That is,

$$(a \otimes 1)(c \otimes 1) = \rho^0 (c \otimes 1)(a \otimes 1),$$

which gives $ac \otimes 1 = ca \otimes 1$ for all $a \in A_0$. Thus, we must have that $c \in Z(A_0) = F$. This proves $\hat{Z}_R(A) \subset F = \hat{Z}_L(A)$ and the desired conclusion follows $\hat{Z}_R(A) = F = \hat{Z}_L(A)$.

Finally, we consider a p-odd algebra, A. By Theorem 4.2.4 $A \cong A_{(q)} \widehat{\otimes} F[z] \cong A_{(q)} \otimes F[z]$, where z is a degree p element that generates the center of A and $z^{pq} = d \in F^*$. Again, it is clear that $\hat{Z}_R(A) = F \subset \hat{Z}_L(A)$ and we will show the reverse containment. Let $c \otimes z^k \in \hat{Z}_R(A)$, then $\deg(c) = iq$ for some i. Since, $c \otimes z^k$ is in the right graded center of A, it right graded commutes with all elements of A, in particular $1 \otimes z \in A$. That is,

$$\begin{aligned}(1 \otimes z)(c \otimes z^k) &= \rho^{p(iq+pk)}(c \otimes z^k)(1 \otimes z) \\ &= \rho^{p^2k}(c \otimes z^k)(1 \otimes z).\end{aligned}$$

After combining the tensors and recalling z is in the center of A we get,

$$c \otimes zz^k = \rho^{p^2k} c \otimes zz^k.$$

This implies $\rho^{p^2k} = 1$ and so we must have $p^2k \equiv 0 (\text{mod } pq)$, which implies k is a multiple of q. Thus $\deg(z^k) = p(jq) \equiv 0$ for some j. So, we have $\deg(c \otimes z^k) = \deg(c)$ and hence $\hat{Z}_R(A) \subset A_{(q)}$. Now, any element in $\hat{Z}_R(A)$ looks like $c \otimes 1$, where $c \in A_{(q)}$. Now, $c \otimes 1$ must right graded commuted with $a \otimes 1$ for any homogeneous $a \in A_{(q)}$. That is,

$$(a \otimes 1)(c \otimes 1) = \rho^{\partial a \cdot \partial c}(c \otimes 1)(a \otimes 1),$$

which implies

$$ac \otimes 1 = \rho^{\partial a \cdot \partial c} ca \otimes 1.$$

The above equation shows that c right graded commutes with any homogeneous element $a \in A_{(q)}$. Thus, $c \in \hat{Z}_R(A_{(q)}) = F$ since $A_{(q)}$ is graded central. Therefore we have shown $\hat{Z}_R(A) \subset F = \hat{Z}_L(A)$ and $\hat{Z}_R(A) = F = \hat{Z}_L(A)$. □

Chapter 5

Structure of the Graded Brauer Group

In this chapter we will review some work describing the Graded Brauer Group. We will continue to assume our algebra is graded by $\mathbb{Z}/pq\mathbb{Z}$ where p, q are distinct primes, not equal to 2. We will define the notion of a discriminant for an algebra that is neither even nor odd and show how this discriminant, together with the type, can be combined to form a non-abelian group, $Q(F)$, which keeps track of information about the algebras. In particular, one can define the graded Brauer group, the objects of which are equivalence classes of algebras. We will show that there is a natural homomorphism from the graded Brauer group to the group $Q(F)$ which provides this desired information (that is, the type and discriminant). Below we will briefly recall the definition of the discriminant in the even case, provide a new definition of the discriminant for the other algebra types and describe the structure of the group $Q(F)$ and how it relates to the graded Brauer group. Lastly, we will show how the classic Brauer group is a subgroup of the graded Brauer group.

5.1 The Graded Brauer Group

We will denote the discriminant of A using the notation, $\delta(A)$. Recall, in an even type algebra, there exists a degree 0 element u from Proposition 3.2.6, which arose from an application of the Noether-Skolem Theorem.

Definition 5.1.1. *The discriminant of an even $\mathbb{Z}/n\mathbb{Z}$ GCSA is given by $u^n = d$, where u is the element described above.*

For the following definitions, we will need to recall the structure theorems for odd and p-odd algebras. For an odd algebra A, Theorem 3.2.7 implies $A \cong (A_0) \widehat{\otimes} F[z]/(z^{pq} - d) \cong (A_0) \otimes F[z]/(z^{pq} - d)$, where z is a degree 1 element that generates the center of A. For a p-odd algebra, Theorem 4.2.4 implies $A \cong A_{(q)} \widehat{\otimes} F[z]/(z^q - d) \cong A_{(q)} \otimes F[z]/(z^q - d)$, where z is a degree p element that generates the center of A.

Definition 5.1.2. *The discriminant of an odd $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra,*

$$A = (A_0) \widehat{\otimes} F[z]/(z^{pq} - d),$$

is defined to be the discriminant of the even algebra

$$(A_0) \widehat{\otimes} F[z]/(z^{pq} - d) \widehat{\otimes} F[x]/(x^{pq} - 1),$$

where $\deg(z) = \deg(x) = 1$.

An alternate, but equivalent, definition for the discriminant of an odd $\mathbb{Z}/pq\mathbb{Z}$ GCSA is $\delta(A) = d^j$, where $j \equiv -1 \pmod{pq}$ and $d = z^{pq}$ (z is the degree 1 element that generates the center of A).

We define the discriminant of a p-odd algebra in a similar manner.

Definition 5.1.3. *The discriminant of a p-odd $\mathbb{Z}/pq\mathbb{Z}$ graded central simple algebra,*

$$A = A_{(q)} \widehat{\otimes} F[z]/(z^q - d),$$

is defined to be the discriminant of the even algebra

$$A_{(q)} \widehat{\otimes} F[z]/(z^q - d) \widehat{\otimes} F[x]/(x^q - 1),$$

where $\deg(z) = \deg(x) = p$.

Again, we have an alternate, but equivalent definition for a p-odd algebra. A p-odd $\mathbb{Z}/pq\mathbb{Z}$ GCSA has discriminant $d'd^{ip}$, where $d = z^q$ (z is the degree p element that generates the center), $ip \equiv -1 \pmod{q}$, and d' is the discriminant of the even algebra $A_{(q)}$. The definition for a q-odd algebra is analogous.

Next we will describe the structure of the graded Brauer group over F.

Definition 5.1.4. *Let A and B be two $\mathbb{Z}/pq\mathbb{Z}$ graded central simple algebras over F. We say A and B are equivalent in $GB(F)$ (or graded Brauer equivalent) if $A \widehat{\otimes} End(V) \cong B \widehat{\otimes} End(W)$, for some graded vector spaces, $V = \bigoplus_{i=0}^{-1} V_i$ and $W = \bigoplus_{i=0}^{n-1} W_i$. The grading on $End(V)$ is given by $E_i = \{f \in End(V) \mid f(V_j) \subset V_{j+i}\}$. The equivalence class of A is denoted $\langle A \rangle$.*

In this section we show the relation defined above is an equivalence relation and the collection of equivalence classes form a non-abelian group, which we call the graded Brauer Group, denoted $GB(F)$.

Definition 5.1.5. *The graded Brauer group, which we denote $GB(F)$, is the group of graded central simple algebras under the graded Brauer equivalence, defined above.*

The following proposition is necessary in order to show the above relation is an equivalence relation and the operation in the graded Brauer group is well defined.

Proposition 5.1.6. *Let V, W be graded vector spaces. Then $End(V) \widehat{\otimes} End(W) \cong End(V \widehat{\otimes} W)$.*

Proof. Recall, $A \widehat{\otimes} B \cong A \otimes B$ as vector spaces, so $End(V) \widehat{\otimes} End(W)$ and $End(V \widehat{\otimes} W)$ are isomorphic as vector spaces. Thus, to show they are graded isomorphic, we only need to define a graded algebra homomorphism. Define the following map for homogeneous elements and extend it to be F linear,

$$\begin{aligned} \varphi : End(V) \widehat{\otimes} End(W) &\longrightarrow End(V \widehat{\otimes} W), \\ f \widehat{\otimes} g &\longmapsto T_{f \widehat{\otimes} g} \end{aligned}$$

where $f : V \longrightarrow V$, $g : W \longrightarrow W$, and the map $T_{f \widehat{\otimes} g} \in End(V \widehat{\otimes} W)$ acts on homogeneous elements, $v \widehat{\otimes} w \in V \widehat{\otimes} W$ in the following way, $T_{f \widehat{\otimes} g}(v \widehat{\otimes} w) = \rho^{\partial g \cdot \partial v} f(v) \widehat{\otimes} g(w)$. We first check that φ is a homomorphism, i.e.

$$\varphi((f \widehat{\otimes} g)(f' \widehat{\otimes} g')) = \varphi(f \widehat{\otimes} g) \circ \varphi(f' \widehat{\otimes} g'), \tag{5.1.1}$$

where $\circ$ denotes composition in $End(V \widehat{\otimes} W)$. We begin by looking at the left hand side of equation 5.1.1,

$$\begin{aligned} \varphi((f \widehat{\otimes} g)(f' \widehat{\otimes} g')) &= \varphi(\rho^{\partial g \cdot \partial f'} (f \circ f') \widehat{\otimes} (g \circ g')) \\ &= \rho^{\partial g \cdot \partial f'} \, T_{f \circ f' \widehat{\otimes} g \circ g'} \end{aligned} \tag{5.1.2}$$

We now consider the right hand side of equation 5.1.1,

$$\varphi(f \widehat{\otimes} g) \circ \varphi(f' \widehat{\otimes} g') = T_{f \widehat{\otimes} g} \circ T_{f' \widehat{\otimes} g'}. \tag{5.1.3}$$

To see that both of the above describe the same map, we will apply each to an element $v \widehat{\otimes} w \in V \widehat{\otimes} W$ (v, w are homogeneous elements of V, W, respectively). Applying the map

in (5.1.2) to $v \widehat{\otimes} w$, we get,

$$\rho^{\partial g \cdot \partial f'} T_{f \circ f' \widehat{\otimes} g \circ g'}(v \widehat{\otimes} w) = \rho^{\partial g \cdot \partial f' + (\partial g + \partial g')\partial v} f \circ f'(v) \widehat{\otimes} g \circ g'(w). \tag{5.1.4}$$

Now, applying the map in 5.1.3 to $v \widehat{\otimes} w$ we get,

$$\begin{aligned} T_{f \widehat{\otimes} g} \circ T_{f' \widehat{\otimes} g'}(v \widehat{\otimes} w) &= T_{f \widehat{\otimes} g}(\rho^{\partial g' \partial v} f'(v) \widehat{\otimes} g'(w)) \\ &= \rho^{\partial g' \cdot \partial v + \partial g(\partial f' + \partial v)} f \circ f'(v) \widehat{\otimes} g \circ g'(w). \end{aligned} \tag{5.1.5}$$

It is clear that the maps in (5.1.4) and (5.1.5) are the same. Therefore, $\varphi((f \widehat{\otimes} g)(f' \widehat{\otimes} g')) = \varphi(f \widehat{\otimes} g) \circ \varphi(f' \widehat{\otimes} g')$ and φ is, indeed, a homomorphism. Moreover, it is simple to see that this homomorphism is graded. Let $f \in End(V)$ and $g \in End(W)$ with $\deg(f) = i$ and $\deg(g) = j$. Then the degree of $f \widehat{\otimes} g$ in $End(V) \widehat{\otimes} End(W)$ is $i + j$. The homomorphism is given by, $\varphi(f \widehat{\otimes} g) = T_{f \widehat{\otimes} g}$, so $\varphi(f \widehat{\otimes} g)(v \widehat{\otimes} w) = \rho^{jk} f(v) \widehat{\otimes} g(w)$, where v is a degree k element of V and w is a degree l element of W. The degree of the right hand side is $(i + k) + (j + l) = (i + j) + (k + l)$ and the degree of $v \widehat{\otimes} w$ is $k + l$ which implies that the degree of $\varphi(f \widehat{\otimes} g) = T_{f \widehat{\otimes} g}$ is $i + j$ in $End(V \widehat{\otimes} W)$. Therefore, we have defined a graded algebra homomorphism and since $End(V) \widehat{\otimes} End(W)$ and $End(V \widehat{\otimes} W)$ are isomorphic as vector spaces, they are isomorphic as graded algebras.

□

Before we check that the relation given in Definition 5.1.4 is an equivalence relation, we first make an important observation about $E = End(V)$, where V is a graded vector space. This algebra is even since $Z(E) = F$. Moreover, the discriminant of E is 1. To see this, define the endomorphism $\varphi : V \to V$ by $\varphi(v) = \rho^{\partial v} v$. This is a degree 0 homomorphism and for any $\psi \in E$ we have $\varphi \circ \psi = \rho^{\psi} \psi \circ \varphi$ since,

$$\begin{aligned}\varphi(\psi(v)) &= \rho^{\psi(v)}\psi(v)\\ &= \rho^{\partial\psi+\partial v}\psi(v)\\ &= \rho^{\psi}\psi(\rho^{v}v)\\ &= \rho^{\psi}\psi(\varphi(v))\end{aligned}$$

for any homogeneous $v \in V$. So this map φ is the Noether-Skolem element of E that exists an even algebra from Proposition 3.2.6. Now, to find the discriminant of E, by definition, we raise this element to the pqth power,

$$\varphi^{pq}(v) = \rho^{pq\cdot\partial v}v = v.$$

Hence, φ^{pq} is the identity on V, and so we have $\delta(E) = 1$.

We now check that the relation given in Definition 5.1.4 is, in fact, an equivalence relation. It is clear that the defined relation is reflexive. Recall Proposition 3.1.18, which implies that for an even algebra with discriminant 1, E, we have $A \widehat{\otimes} E \cong E \widehat{\otimes} A$. Since $End(V)$ is an even algebra with discriminant 1, the relation is symmetric. To see that the relation is transitive, assume $A \sim B$ and $B \sim C$. Now, there exist graded vector spaces V, V', W, and W' such that $A \widehat{\otimes} End(V) \cong B \widehat{\otimes} End(V')$ and $B \widehat{\otimes} End(W) \cong C \widehat{\otimes} End(W')$. If we graded tensor the first equation with $End(W)$ on the right and the second equation with $End(V')$ on the right, we get

$$A \widehat{\otimes} End(V) \widehat{\otimes} End(W) \cong B \widehat{\otimes} End(V') \widehat{\otimes} End(W)$$

and

$$B \widehat{\otimes} End(W) \widehat{\otimes} End(V') \cong C \widehat{\otimes} End(W') \widehat{\otimes} End(V').$$

Recall that if E is an even algebra with discriminant 1, then $A \widehat{\otimes} E \cong E \widehat{\otimes} A$. Finally, by 5.1.6 we see that $End(V) \widehat{\otimes} End(V') \cong End(V \widehat{\otimes} V')$, which gives

$$A \widehat{\otimes} End(V \widehat{\otimes} W) \cong B \widehat{\otimes} End(V' \widehat{\otimes} W) \cong C \widehat{\otimes} End(V' \widehat{\otimes} W'),$$

and so the relation is transitive.

We will now check that the operation $\langle A_1 \rangle \cdot \langle A_2 \rangle = \langle A_1 \widehat{\otimes} A_2 \rangle$ on equivalence classes is well defined. Let $\langle A \rangle = \langle A' \rangle$ and $\langle B \rangle = \langle B' \rangle$. Then there exist graded vector spaces V, V', W, and V' such that $A \widehat{\otimes} End(V) \cong A' \widehat{\otimes} End(V')$ and $B \widehat{\otimes} End(W) \cong B' \widehat{\otimes} End(W')$. Then we have,

$$A \widehat{\otimes} End(V) \widehat{\otimes} B \widehat{\otimes} End(W) \cong A' \widehat{\otimes} End(V') \widehat{\otimes} B' \widehat{\otimes} End(W').$$

By 3.1.18 we have $A \widehat{\otimes} End(V) \cong End(V) \widehat{\otimes} A$, which implies

$$A \widehat{\otimes} B \widehat{\otimes} End(V) \widehat{\otimes} End(W) \cong A' \widehat{\otimes} B' \widehat{\otimes} End(V') \widehat{\otimes} End(W').$$

Finally, since $End(V) \widehat{\otimes} End(V') \cong End(V \widehat{\otimes} V')$, we get

$$A \widehat{\otimes} B \widehat{\otimes} End(V \widehat{\otimes} W) \cong A' \widehat{\otimes} B' \widehat{\otimes} End(V' \widehat{\otimes} W').$$

Hence, $\langle A \widehat{\otimes} B \rangle = \langle A' \widehat{\otimes} B' \rangle$ and the operation is well defined. We now have the structure of a non-commutative monoid, where the identity is $\langle F \rangle = \langle End(V) \rangle$ and F is considered trivially graded over itself. Note that this is non-commutative since, in general, $\langle A \rangle \cdot \langle B \rangle = A \widehat{\otimes} B \ncong B \widehat{\otimes} A = \langle B \rangle \cdot \langle A \rangle$. In order to define inverses, we must first define the notion of a graded opposite algebra.

Definition 5.1.7. *The graded opposite algebra of A, denoted A^*, is defined to be $A^* = \{a^* \mid a \in A\}$ with grading given by $A_i^* = \{a^* \mid a \in A_i\}$ and operation $a^* \cdot b^* = \rho^{\partial a \cdot \partial b}(ba)^*$.*

In the following observation we need to distinguish left and right graded center, so we return to labeling the center as left or right for a moment. Let us consider $\hat{Z}_L(A^*) = \{a^* \in A \mid a^*b^* = \rho^{\partial a \cdot \partial b} b^* a^*,\ \forall b^* \in A\}$. Let $a^* \in \hat{Z}_L(A^*)$, then for any $b^* \in A$

$$\begin{aligned} a^*b^* = \rho^{\partial a \cdot \partial b} b^* a^* &\iff (ba)^* = \rho^{\partial a \cdot \partial b}(ab)^* \\ &\iff ba = \rho^{\partial a \cdot \partial b} ab. \end{aligned}$$

Thus $a^* \in \hat{Z}_L(A^*)$ if and only if $a \in \hat{Z}_R(A)$. Thus, we have $\hat{Z}_L(A^*) = \{a^* \in A \mid a \in \hat{Z}_R(A)\}$. Recall, in Proposition 4.2.6 we showed for a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra the left and right graded center are equal. So, $\hat{Z}_L(A^*) = \{a^* \in A \mid a \in \hat{Z}_R(A) = \hat{Z}_L(A)\}$. So if A is (left) graded central, then so is A^*. Additionally, if I is graded ideal in A^*, then $\{a \in A \mid a^* \in I\}$ is a graded ideal in A. Thus, if A is graded simple, A^* is also graded simple. Thus A a GCSA over F implies that A^* is a GCSA over F.

Proposition 5.1.8. *If A is a GCSA, then $A \widehat{\otimes} A^* \cong End(A)$ as graded algebras.*

Proof. Define $\theta : A \widehat{\otimes} A^* \longrightarrow End(A)$ to be the F-linear map induced by $\theta(a \widehat{\otimes} b^*)(c) = \rho^{\partial b \cdot \partial e} acb$, where a, b, c are homogeneous elements of A. θ takes an element, $a \widehat{\otimes} b^*$ of degree $\partial a + \partial b$ in $A \widehat{\otimes} A^*$ and sends it to an endomorphism, $\theta(a \widehat{\otimes} b^*)$. This endomorphism takes an element c in A of degree ∂c and send its to $\rho^{\partial b \cdot \partial c} acb$, an element of degree $\partial a + \partial b + \partial c$ in A. This implies that $\theta(a \widehat{\otimes} b)(A_i) \subset A_{\partial a + \partial b + i}$. Thus $\theta(a \widehat{\otimes} b)$ is a degree $\partial a + \partial b$ element in $End(A)$. Thus, the map θ preserves the grading since it takes an element of degree $\partial a + \partial b$ to an element of the same degree.

Now we check that θ is a graded homomorphism. Let $a, c, e \in A$ and $b^*, d^* \in A^*$ be homogeneous elements, then

$$\begin{aligned}
\theta((a \widehat{\otimes} b^*)(c \widehat{\otimes} d^*))(e) &= \theta(\rho^{\partial b \cdot \partial c} ac \widehat{\otimes} b^* d^*)(e) \\
&= \rho^{\partial b \cdot \partial c} \theta(\rho^{\partial b \cdot \partial d}(ac \widehat{\otimes} (db)^*)(e) \\
&= \rho^{\partial b(\partial c + \partial d)} \rho^{\partial e(\partial b + \partial d)} ac(e) db \\
&= \rho^{\partial b(\partial c + \partial d + \partial e)} \rho^{\partial e \cdot \partial d} a(ced) b \\
&= \theta(a \widehat{\otimes} b^*)(\rho^{\partial e \cdot \partial d} ced) \\
&= [\theta(a \widehat{\otimes} b^*)\theta(c \widehat{\otimes} d^*)](e)
\end{aligned}$$

Now, from the discussion before this proposition, we know if A is a GCSA, so is A^*. Then $A \widehat{\otimes} A^*$ is a GCSA and $\ker(\theta)$ is trivial, and so θ is injective. We see that this is, in fact, an isomorphism since both $A \widehat{\otimes} A^*$ and $End(A)$ have dimension $dim(A)^2$. □

The above proposition shows that $\langle A \rangle^{-1} = \langle A^* \rangle$ since $\langle End(A) \rangle = \langle F \rangle$ is the identity in $GB(F)$, and so $GB(F)$ is a non-abelian group, which we will call the graded Brauer group. Note that the Brauer-Wall group, which is the graded Brauer group for GCSA's graded by $\mathbb{Z}/2\mathbb{Z}$, is abelian. However, as we see here, the graded Brauer group is not necessarily abelian in general.

The two following statements will be useful when further exploring the relation between the Brauer group and the Graded Brauer group.

Lemma 5.1.9. *$F[z]/(z^q - 1) \cong (F[z]/(z^q - 1))^*$, where $\deg(z) = p$. Similarly, $F[x]/(x^{pq} - 1) \cong (F[x]/(x^{pq} - 1))^*$.*

Proof. To prove the claim, define the map, φ

$$\begin{aligned}\varphi : F[z]/(z^q-1) &\longrightarrow (F[z]/(z^q-1))^* \\ z^i &\longmapsto \rho^{p^2(1+2+\cdots+(i-1))}(z^i)^*.\end{aligned}$$

We first check this map is a well defined, graded homomorphism. Let $z^i \in F[z]$. Then,

$$\begin{aligned}[\varphi(z)]^i &= (z^*)^i \\ &= \rho^{p^2(1+2+\cdots+(i-1))}(z^i)^* \\ &= \varphi(z^i).\end{aligned}$$

Moreover,

$$\varphi(z^q-1) = \rho^{p^2(1+2+\cdots+(q-1))}(z^q)^* - 1^* = \rho^{p^2\left(\frac{(q-1)q}{2}\right)}(z^q)^* - 1^* = 1^* - 1^* = 0^*,$$

so the map is well defined. Recall the grading on the graded opposite is given by $A_i = \{a^* \mid a \in A_i\}$, so this map preserves the grading and is a graded homomorphism. To see this map is surjective, notice the preimage of an element $(z^j)^*$ is $\rho^{-p^2(1+2+\cdots+(j-1))}z^j$ since

$$\begin{aligned}\varphi(\rho^{-p^2(1+2+\cdots+(j-1))}z^j) &= \rho^{-p^2(1+2+\cdots+(j-1))}\varphi(z^j) \\ &= \rho^{-p^2(1+2+\cdots+(j-1))}\rho^{p^2(1+2+\cdots+(j-1))}(z^j)^* \\ &= (z^j)^*.\end{aligned}$$

Lastly, to see the map is injective, let $z^i \in \ker(\varphi)$. Then, $\varphi(z^i) = 0^*$, which implies $\rho^{p^2(1+2+\cdots+(i-1))}(z^i)^* = 0^*$ and hence we must have $z^i = 0$. Now, we have shown the claim holds, $F[z]/(z^q-1) \cong (F[z]/(z^q-1)^*$.

□

Corollary 5.1.10. $\langle F[z]/(z^q-1) \,\widehat{\otimes}\, F[z]/(z^q-1)\rangle = \langle F\rangle$, *where* $\deg(z) = p$. *Similarly,* $\langle F[x]/(x^{pq}-1) \,\widehat{\otimes}\, F[x]/(x^{pq}-1)\rangle = \langle F\rangle$, *where* $\deg(x) = 1$.

Proof. By Lemma 5.1.9 $\langle F[z]/(z^q-1) \,\widehat{\otimes}\, F[z]/(z^q-1)\rangle = \langle F[z]/(z^q-1) \,\widehat{\otimes}\, (F[z]/(z^q-1))^*\rangle$ and Proposition 5.1.8 $\langle F[z]/(z^q-1) \,\widehat{\otimes}\, (F[z]/(z^q-1))^*\rangle = \langle End(F[z]/(z^q-1))\rangle = \langle F\rangle$. □

5.2 Invariants and Classification

We are interested in exploring the relationship between the classic Brauer group, $B(F)$, and the graded Brauer group, $GB(F)$. Our next goal is to define a (non abelian) group, $Q(F)$ which will provide information about two invariants: the type and discriminant of an algebra. In fact, we will define $Q(F)$ to be a semidirect product of $D = F^*/(F^*)^{pq}$, the possible discriminants and $T = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the types of a GCSA. This group, $Q(F)$ is analogous to the group $Q(F)$ defined in [6] for a $\mathbb{Z}/2\mathbb{Z}$-graded central simple algebra. However, there are some key differences between the two cases cases. For example, in the $\mathbb{Z}/2\mathbb{Z}$ case $Q(F)$ is abelian and $Q \cong T \times D$ with $(t,d)(t',d') = (t+t', (-1)^{tt'}dd')$ (in the $\mathbb{Z}/2\mathbb{Z}$ case $T = \mathbb{Z}/2\mathbb{Z}$).

The following theorem will help us to understand the relation between $Q(F)$ and the graded Brauer group.

Theorem 5.2.1. $GB(F)/GB_{E,1}(F) \cong Q(F)$ *where* $Q(F) = D \rtimes T$ *with operation* $(d,t)(d',t') = (d(d')^t, t+t')$ *and action given by*

$d^{(0,0)} = d$,

$d^{(1,1)} = d^i$, *where* $i \equiv -1 \pmod{pq}$,

$d^{(1,0)} = d^j$, *where* $j \equiv 1-(k-i)p \pmod{pq}$, $pk \equiv 1 \pmod{q}$, *and* $-pi \equiv 1 \pmod{q}$,

$d^{(0,1)} = d^l$, *where* $l \equiv 1-(r-s)q \pmod{pq}$, $qr \equiv 1 \pmod{p}$, *and* $-qs \equiv 1 \pmod{p}$.

Proof. We begin defining a well-defined map, φ, which sends an equivalence class in the graded Brauer group to its type, which is an element of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This map is well defined since all the algebras in a graded Brauer equivalence class have the same type. To see this, recall that $End(V)$ is even, i.e. type $(0,0)$, and Proposition 4.2.5 states when two algebras are graded tensored the type of the resulting algebra is the sum of the types as elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Moreover, Proposition 4.2.5 also directly implies that φ is a homomorphism. Now, the kernel of this map is the equivalence classes in $GB(F)$ with type $(0,0)$, or even algebras, which we denote $GB_E(F)$. This is represented by the following short exact sequence,

$$1 \longrightarrow GB_E(F) \longrightarrow GB(F) \xrightarrow{\varphi} T \longrightarrow 1.$$

Now, for an algebra in $GB_E(F)$ there is a well-defined discriminant map, ψ, which is part of the following short exact sequence where the kernel consists of classes of even algebras with discriminant 1, denoted $GB_{E,1}(F)$,

$$1 \longrightarrow GB_{E,1}(F) \longrightarrow GB_E(F) \xrightarrow{\psi} D \longrightarrow 1.$$

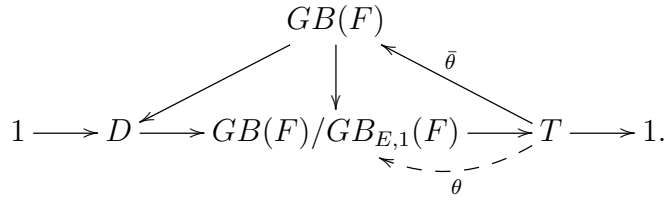
First, notice ψ is a homomorphism since we are restricted to classes of even type algebras. Let A, B be even GCSAs, and u, v the Noether Skolem elements of A, B respectively, with $u^{pq} = d$ and $v^{pq} = d'$. Then, $u \mathbin{\widehat{\otimes}} v$ is the Noether Skolem element of $A \mathbin{\widehat{\otimes}} B$. Now, $\psi(\langle A \mathbin{\widehat{\otimes}} B\rangle) = \delta(A \mathbin{\widehat{\otimes}} B) = (u \mathbin{\widehat{\otimes}} v)^{pq} = u^{pq} \mathbin{\widehat{\otimes}} v^{pq} = dd' = \delta(A)\delta(B) = \psi(\langle A\rangle)\psi(\langle B\rangle)$. The map, ψ, is well defined because $End(V)$ has discriminant 1, where V is a graded vector space. So, any two algebras in the same graded Brauer equivalence class have the same discriminant.

Since $GB_{E,1}(F) \subset GB_E(F) \subset GB(F)$, we can now combine the two previous short exact

sequences to create the following sequence, which will help us define a splitting,

$$1 \longrightarrow GB_E(F)/GB_{E,1}(F) \longrightarrow GB(F)/GB_{E,1}(F) \longrightarrow GB(F)/GB_E(F) \longrightarrow 1.$$

We use $D = GB_E(F)/GB_{E,1}(F)$ and $T = GB(F)/GB_E(F)$ obtained from the first two short exact sequences, and project $GB(F)$ to each component of the short exact sequence to obtain the diagram below,



Now, if we define a splitting, θ of the above short exact sequence, then we can recognize $GB(F)/GB_{E,1}(F)$ as a semidirect product of D and T, i.e. $GB(F)/GB_{E,1} \cong D \rtimes T$.

Remark. *We have made the restriction, $p, q \neq 2$, because there does not exists a splitting if either p or q is 2.*

Recall, we originally defined $T = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so in order to find a splitting, θ, we must only define where the generators, $(1, 0)$ and $(0, 1)$, of T map to. We will define θ by first mapping the generators to $GB(F)$ via $\bar{\theta}$ and then projecting to $GB(F)/GB_{E,1}(F)$. We will map $(1, 0)$ to the equivalence class of $F[z]$ with discriminant 1 in $GB(F)$, where the degree of z is p. Similarly, we map $(0, 1)$ to the class of $F[x]$ with discriminant 1, where the degree of x is q. Below we will show θ is a homomorphism, but let us assume this for a moment in order to describe the action of T on D. Assuming θ is a homomorphism, then $GB(F)/GB_{E,1}(F)$ is a semidirect product of D and T. The action defined from the splitting θ in the sequence,

$$1 \longrightarrow D \xrightarrow{\beta} D \rtimes T \underset{\theta}{\overset{\alpha}{\rightleftarrows}} T \longrightarrow 1,$$

is given by $\phi_t(d) = \beta^{-1}\left(\theta(t)\beta(d)\theta^{-1}(t)\right)$. Recall $D = GB_E(F)/GB_{E,1}(F)$, so the map β is an inclusion sending an algebra with discriminant d, to it's equivalence class in $GB(F)/GB_{E,1}(F) \cong D \rtimes T$. So, $\beta(d) = (d,(0,0))$. We defined θ above and t has order 2 so $\theta(t^{-1}) = \theta(t)$. Recall, $\theta(t)$ is $\langle F\rangle$, $\langle F[z]/(z^q-1)\rangle$, $\langle F[w]/(w^p-1)\rangle$, or $\langle F[x]/(x^{pq}-1)\rangle$. So, $\beta^{-1}\left(\theta(t)\beta(d)\theta^{-1}(t)\right) = \beta^{-1}\left(\theta(t)\left(d,(0,0)\right)\theta(t)\right)$. Now, the preimage of $\theta(t)\left(d,(0,0)\right)\theta(t)$ (which is an even, $(0,0)$, algebra) under β is the discriminant of $\theta(t)\left(d,(0,0)\right)\theta(t)$. Thus, the action given by the splitting is $\phi_t(d) = \delta\left(\theta(t)\left(d,(0,0)\right)\theta(t)\right)$. This can be computed explicitly in four cases by conjugating an even algebra A with discriminant d, by an algebra given by the splitting θ and finding the discriminant of the resulting even algebra (i.e finding the Noether-Skolem element and raising it to the pqth power). The resulting action is as follows:

$\phi_{(0,0)} = d,$

$\phi_{(1,1)} = d^i,$ where $i \equiv -1 \pmod{pq}$,

$\phi_{(1,0)} = d^j,$ where $j \equiv 1-(k-i)p \pmod{pq}$, $pk \equiv 1 \pmod{q}$, and $-pi \equiv 1 \pmod{q}$,

$\phi_{(0,1)} = d^l,$ where $l \equiv 1-(r-s)q \pmod{pq}$, $qr \equiv 1 \pmod{p}$, and $-qs \equiv 1 \pmod{p}$.

Notice, that the above action is the same as the action defined for $Q(F) = D \rtimes T$ in the statement of the theorem. So, $Q(F) := D \rtimes T$ and $GB(F)/GB_{E,1}(F) \cong D \rtimes_\phi T$ are semidirect products of the same group with the same action and therefore $GB(F)/GB_{E,1}(F) \cong Q(F)$.

Now, to complete the proof, we check that θ is indeed a homomorphism by checking each case. For simplicity, we will use the following notation in the below computations:

$$\begin{aligned}
&\theta(0,0) = \langle F \rangle \\
&\theta(1,0) = \langle F[z]/(z^q - 1) \rangle, \text{where } z \text{ is degree } p, \\
&\theta(0,1) = \langle F[w]/(w^p - 1) \rangle, \text{where } w \text{ is degree } q, \\
&\theta(1,1) = \langle F[x]/(x^{pq} - 1) \rangle, \text{where } x \text{ is degree } 1.
\end{aligned}$$

To see that θ is a homomorphism, we check

$$\theta(t + t') = \theta(t) \mathbin{\widehat{\otimes}} \theta(t') = \langle \theta(t) \mathbin{\widehat{\otimes}} \theta(t') \rangle \tag{5.2.1}$$

for all the combinations of the 4 algebra types. In the following proof, we will use $F[z]$ to denote $F[z]/(z^q - 1)$ in order to simplify the notation in the computations.

- $\theta((0,0) + (0,0)) = \theta(0,0) = \langle F \rangle = \langle F \mathbin{\widehat{\otimes}} F \rangle = \langle F \rangle \mathbin{\widehat{\otimes}} \langle F \rangle = \theta(0,0) \mathbin{\widehat{\otimes}} \theta(0,0)$.
- $\theta((0,0) + (1,0)) = \theta(1,0) = \langle F[z] \rangle = \langle F \mathbin{\widehat{\otimes}} F[z] \rangle = \langle F \rangle \mathbin{\widehat{\otimes}} \langle F[z] \rangle = \theta(0,0) \mathbin{\widehat{\otimes}} \theta(1,0)$.
- $\theta((0,0) + (0,1)) = \theta(0,1) = \langle F[w] \rangle = \langle F \mathbin{\widehat{\otimes}} F[w] \rangle = \langle F \rangle \mathbin{\widehat{\otimes}} \langle F[w] \rangle = \theta(0,0) \mathbin{\widehat{\otimes}} \theta(0,1)$.
- $\theta((0,0) + (1,1)) = \theta(1,1) = \langle F[x] \rangle = \langle F \mathbin{\widehat{\otimes}} F[x] \rangle = \langle F \rangle \mathbin{\widehat{\otimes}} \langle F[x] \rangle = \theta(0,0) \mathbin{\widehat{\otimes}} \theta(1,1)$.
- $\theta((1,0) + (0,0)) = \theta(1,0) = \langle F[z] \rangle = \langle F[z] \mathbin{\widehat{\otimes}} F \rangle = \langle F[z] \rangle \mathbin{\widehat{\otimes}} \langle F \rangle = \theta(1,0) \mathbin{\widehat{\otimes}} \theta(0,0)$.
- On one hand, $\theta((1,0) + (1,0)) = \theta(0,0) = \langle F \rangle$. On the other hand, $\theta(1,0) \mathbin{\widehat{\otimes}} \theta(1,0) = \langle F[z] \rangle \mathbin{\widehat{\otimes}} \langle F[z] \rangle = \langle F[z] \mathbin{\widehat{\otimes}} F[z] \rangle = \langle F \rangle$, by Corollary 5.1.10.

  Remark. *The other cases that involve only one type, $\theta(t + t) = \theta(t) \mathbin{\widehat{\otimes}} \theta(t)$ are very similar and follow the same process as this case.*

- In this case we will use a different approach than in the previous cases. We have $\theta((1,0)+(0,1)) = \theta(1,1) = \langle F[x]\rangle$, which we want to be equal to $\theta(1,0) \mathbin{\widehat{\otimes}} \theta(0,1) = \langle F[z]\rangle \mathbin{\widehat{\otimes}} \langle F[w]\rangle = \langle F[z] \mathbin{\widehat{\otimes}} F[w]\rangle$. Recall, $\deg(x) = 1$ with $x^{pq} = 1$, $\deg(z) = p$ with $z^q = 1$, and $\deg(w) = q$ with $w^p = 1$. Consider the element $z^i \mathbin{\widehat{\otimes}} w^j \in F[z] \mathbin{\widehat{\otimes}} F[w]$, where $pi + qj \equiv 1 \pmod{pq}$. This is a degree 1 central element of $F[z] \mathbin{\widehat{\otimes}} F[w]$, with $(z^i \mathbin{\widehat{\otimes}} w^j)^{pq} = 1$ and using the structure theorem for odd algebras we see $\langle F[z] \mathbin{\widehat{\otimes}} F[w]\rangle = \langle B_0 \mathbin{\widehat{\otimes}} F[z^i \mathbin{\widehat{\otimes}} w^j]\rangle$, where B_0 are the degree 0 elements of $F[z] \mathbin{\widehat{\otimes}} F[w]$.

  We now consider the elements of B_0. The degree 0 elements in $F[z] \mathbin{\widehat{\otimes}} F[w]$ are of the form $z^k \mathbin{\widehat{\otimes}} w^l$ where $pk + ql \equiv 0 \pmod{pq}$. Using the Chinese Remainder Theorem, we can consider this equivalence modulo p and q individually. So, we get $pk + ql \equiv ql \equiv 0 \pmod{p}$ and $pk + ql \equiv pk \equiv 0 \pmod{q}$ which implies $p|l$ and $q|k$. i.e. $l = pr$ and $k = qs$. Then our original element, $z^k \mathbin{\widehat{\otimes}} w^l = z^{qs} \mathbin{\widehat{\otimes}} w^{pr} = (z^q)^s \mathbin{\widehat{\otimes}} (w^p)^r = 1 \mathbin{\widehat{\otimes}} 1$. Hence, $B_0 \cong F$.

  Thus far, we have shown

$$\langle F[z] \mathbin{\widehat{\otimes}} F[w]\rangle = \langle B_0 \mathbin{\widehat{\otimes}} F[z^i \mathbin{\widehat{\otimes}} w^j]\rangle = \langle F \mathbin{\widehat{\otimes}} F[z^i \mathbin{\widehat{\otimes}} w^j]\rangle = \langle F[z^i \mathbin{\widehat{\otimes}} w^j]\rangle.$$

  We will now, show $\langle F[z] \mathbin{\widehat{\otimes}} F[w]\rangle = \langle F[z^i \mathbin{\widehat{\otimes}} w^j]\rangle$, by showing $F[z] \mathbin{\widehat{\otimes}} F[w] \cong F[z^i \mathbin{\widehat{\otimes}} w^j]$ via defining an isomorphism,

$$\begin{aligned} \varphi : F[x] &\longrightarrow\!\!\!\!\!\rightarrow F[z^i \mathbin{\widehat{\otimes}} w^j]/((z^i \mathbin{\widehat{\otimes}} w^j)^{pq} - 1). \\ x &\longmapsto z^i \mathbin{\widehat{\otimes}} w^j \end{aligned}$$

  Since x and $z^i \mathbin{\widehat{\otimes}} w^j$ are both degree 1, this map clearly preserves the grading. To see

this is a homomorphism,

$$\begin{aligned}[\varphi(x)]^k &= \underbrace{(z^i \mathbin{\widehat{\otimes}} w^j) \cdots (z^i \mathbin{\widehat{\otimes}} w^j)}_{\text{k times}} \\ &= (z^i \mathbin{\widehat{\otimes}} w^j)^k \text{ since } \deg(z) = p \text{ and } \deg(w) = p \\ &= \varphi(x^k).\end{aligned}$$

So we have a graded homomorphism. Moreover, by the first isomoprhism theorem, we know $F[x]/\ker(\varphi) \cong F[z^i \mathbin{\widehat{\otimes}} w^j]/((z^i \mathbin{\widehat{\otimes}} w^j)^{pq} - 1)$ as vector spaces. It is clear, that the kernel of φ is generated by $x^{pq}-1$, and thus $F[x]/(x^{pq}-1) \cong F[z^i \mathbin{\widehat{\otimes}} w^j]/((z^i \mathbin{\widehat{\otimes}} w^j)^{pq}-1)$. We have shown the two are isomorphic as vector spaces and that the homomorphism preserves the algebraic and graded structure. Thus we have a graded isomorphism.

To summarize this case, we have

$$\begin{aligned}\theta(1,0) \mathbin{\widehat{\otimes}} \theta((0,1)) &= \langle F[z] \mathbin{\widehat{\otimes}} F[w] \rangle \\ &= \langle F[z^i \mathbin{\widehat{\otimes}} w^j] \rangle \\ &= \langle F[x] \rangle \\ &= \theta(1,1) \\ &= \theta\left((1,0) + (0,1)\right).\end{aligned}$$

- We have $\theta\left((1,0) + (1,1)\right) = \theta(0,1)$, which we want to be equal to $\theta(1,0) \mathbin{\widehat{\otimes}} \theta(1,1)$. The

following cases we have already proved will be useful in showing this case:

$$\theta(1,1) = \theta\left((1,0)+(0,1)\right) = \theta(1,0) \mathbin{\widehat{\otimes}} \theta(0,1)$$
$$\theta\left((1,0)+(1,0)\right) = \theta(1,0) \mathbin{\widehat{\otimes}} \theta(1,0)$$
$$\theta\left((0,0)+(0,1)\right) = \theta(0,0) \mathbin{\widehat{\otimes}} \theta(0,1).$$

Now, we see

$$\begin{aligned}
\theta(1,0) \mathbin{\widehat{\otimes}} \theta(1,1) &= \theta(1,0) \mathbin{\widehat{\otimes}} \theta\left((1,0)+(0,1)\right) \\
&= \theta(1,0) \mathbin{\widehat{\otimes}} \big(\theta(1,0) \mathbin{\widehat{\otimes}} \theta(0,1)\big) \\
&= \big(\theta(1,0) \mathbin{\widehat{\otimes}} \theta(1,0)\big) \mathbin{\widehat{\otimes}} \theta(0,1) \\
&= \theta\left((1,0)+(1,0)\right) \mathbin{\widehat{\otimes}} \theta(0,1) \\
&= \theta(0,0) \mathbin{\widehat{\otimes}} \theta(0,1) \\
&= \theta\left((0,0)+(0,1)\right) \\
&= \theta(0,1) \\
&= \theta\left((1,0)+(1,1)\right).
\end{aligned}$$

Remark. *The cases of the form* $\theta\left((0,1)+t\right) = \theta((0,1)) \mathbin{\widehat{\otimes}} \theta(t)$ *follow from the previous four cases by switching the role of* p *and* q*.*

- $\theta\left((1,1)+(0,0)\right) = \theta(1,1) = \langle F[x] \rangle = \langle F[x] \mathbin{\widehat{\otimes}} F \rangle = \langle F[x] \rangle \mathbin{\widehat{\otimes}} \langle F \rangle = \theta(1,1) \mathbin{\widehat{\otimes}} \theta(0,0)$.
- This is very similar to the previous case, but we include it for completion. We have $\theta\left((1,1)+(1,0)\right) = \theta(0,1)$, which we want to be equal to $\theta(1,1) \mathbin{\widehat{\otimes}} \theta(1,0)$. We will again

use some cases we have already shown,

$$\begin{aligned}
\theta(1,1) \mathbin{\widehat{\otimes}} \theta(1,0) &= \theta\left((0,1)+(1,0)\right) \mathbin{\widehat{\otimes}} \theta(1,0) \\
&= \left(\theta(0,1) \mathbin{\widehat{\otimes}} \theta(1,0)\right) \mathbin{\widehat{\otimes}} \theta(1,0) \\
&= \theta(0,1) \mathbin{\widehat{\otimes}} \left(\theta(1,0) \mathbin{\widehat{\otimes}} \theta(1,0)\right) \\
&= \theta(0,1) \mathbin{\widehat{\otimes}} \theta\left((1,0)+(1,0)\right) \\
&= \theta(0,1) \mathbin{\widehat{\otimes}} \theta(0,0) \\
&= \theta\left((0,1)+(0,0)\right) \\
&= \theta(0,1) \\
&= \theta\left((1,1)+(1,0)\right).
\end{aligned}$$

- This case is very similar to the previous two cases, but we include it for completion. We have $\theta\left((1,1)+(0,1)\right) = \theta(1,0)$, which we want to be equal to $\theta(1,1) \mathbin{\widehat{\otimes}} \theta(0,1)$. We will again use some cases we have already shown,

$$\begin{aligned}
\theta(1,1) \mathbin{\widehat{\otimes}} \theta(0,1) &= \theta\left((1,0)+(0,1)\right) \mathbin{\widehat{\otimes}} \theta(0,1) \\
&= \left(\theta(1,0) \mathbin{\widehat{\otimes}} \theta(0,1)\right) \mathbin{\widehat{\otimes}} \theta(0,1) \\
&= \theta(1,0) \mathbin{\widehat{\otimes}} \left(\theta(0,1) \mathbin{\widehat{\otimes}} \theta(0,1)\right) \\
&= \theta(1,0) \mathbin{\widehat{\otimes}} \theta\left((0,1)+(0,1)\right) \\
&= \theta(1,0) \mathbin{\widehat{\otimes}} \theta(0,0) \\
&= \theta\left((1,0)+(0,0)\right) \\
&= \theta(1,0) \\
&= \theta\left((1,1)+(0,1)\right).
\end{aligned}$$

- The final case $\theta((1,1)+(1,1)) = \theta(1,1)\widehat{\otimes}\theta(1,1)$ is very similar to the case $\theta((1,0)+(1,0)) = \theta(1,0)\widehat{\otimes}\theta(1,0)$. It can be shown that $F[x]/(x^{pq}-1) \cong (F[x]/(x^{pq}-1))^*$ (where x is degree 1) in the same manner we proved the equivalent statement for the $\theta((1,0)+(1,0))$ case. Thus we get,

$$\begin{aligned}
\theta(1,1) \,\widehat{\otimes}\, \theta(1,1) &= \langle F[x] \,\widehat{\otimes}\, F[x] \rangle \\
&= \langle F[x] \,\widehat{\otimes}\, (F[x])^* \rangle \\
&= \langle End(F[x]) \rangle \\
&= \langle F \rangle \\
&= \theta(0,0) \\
&= \theta((1,1)+(1,1)).
\end{aligned}$$

□

We will now discuss a key, but subtle, observation regarding the action of the semidirect product defined in the Theorem above. The action defined is (not obviously) trivial in certain cases. We will consider the action of $(1,0)$ on the algebra $A_{(q)}$. Recall, $A_{(q)} = A_0 \oplus A_q \oplus A_{2q} \oplus \cdots \oplus A_{(p-1)q}$. Since $A_{(q)}$ is central, it is even as a $\mathbb{Z}/pq\mathbb{Z}$-graded central simple algebra and hence contains a Noether-Skolem element u such that $u^{pq} = d$. Now, we can also view $A_{(q)} = B = B_0 \oplus B_1 \oplus \cdots \oplus B_{p-1}$ as an even $\mathbb{Z}/p\mathbb{Z}$ graded central simple algebra. So B contains a Noether-Skolem element, v, such that $v^p = d'$ and $vb = \rho^{\partial b} bv$. Now, $v^q b = \rho^{q\partial(b)} bv^q$, so we have $u = v^q$ (since the Noether-Skolem element is unique up to pqth powers). Now

$$d = u^{pq} = (v^q)^{pq} = (v^p)^{q^2} = ((d')^q)^q$$

. So, d is already a qth power. By the Chinese Remainder Theorem $F^*/(F^*)^{pq} \cong F^*/(F^*)^p \times F^*/(F^*)^q$. Under this isomorphism we have, $d \mapsto (d,0)$ (because it is a qth power) and

$d^j \mapsto (d^{1+p(i-k)}, 0)$ where $j \equiv 1 + p(i-k) \pmod{pq}$, $pk \equiv 1 \pmod{q}$, and $pi \equiv -1 \pmod{q}$. Clearly, $j \equiv 1 \pmod{p}$, so $(d^{1+p(i-k)}, 0) = (d, 0)$ in $F^*/(F^*)^p \times F^*/(F^*)^q$. So, we see that the action by $(1,0)$ on $A_{(q)}$ is trivial. There are also other cases in which the action is simplified due to the Noether-Skolem element already being a power. For example a similar argument shows that the action by a $(1,0)$ algebra on $A_{(p)}$ is $(0, d^j) = (0, d^{-1})$ in $F^*/(F^*)^p \times F^*/(F^*)^q$.

Proposition 5.2.2. *The map, $i : B(F) \longrightarrow GB(F)$, which takes a (ungraded) CSA, A over F, to the trivially graded algebra, (A), is a well-defined, injective homomorphism.*

Proof. To see this is a well-defined map, consider A and B to be CSAs over F such that $[A] = [B]$ in $B(F)$. Then there exist vector spaces V and W such that $A \otimes End(V) \cong B \otimes End(W)$. But $A \otimes End(V) \cong (A) \mathbin{\widehat{\otimes}} End((V))$, where both A and V are considered trivially graded (i.e. concentrated in degree 0) and similarly for B. This implies that $\langle (A) \rangle = \langle (B) \rangle$, and so $i([A]) = i([B])$.

Now, we check that the map i is a homomorphism. Let A and B be CSAs over F in $B(F)$, then

$$\begin{aligned} i([A] \cdot [B]) &= i([A \otimes B]) \\ &= \langle (A \otimes B) \rangle \\ &= \langle (A) \mathbin{\widehat{\otimes}} (B) \rangle \\ &= \langle (A) \rangle \cdot \langle (B) \rangle \\ &= i([A]) \cdot i([B]). \end{aligned}$$

To show injectivity, let A and B be CSAs over F. If $i([A]) = i([B])$, then there exists a graded isomorphism and graded vector spaces V and W such that,

$$(A) \mathbin{\widehat{\otimes}} End(V) \cong (B) \mathbin{\widehat{\otimes}} End(W).$$

Now, (A) and (B) are concentrated in degree 0 so $(A) \mathbin{\widehat{\otimes}} End(V) \cong A \otimes End(V)$, and similarly

$(B) \widehat{\otimes} End(W) \cong B \otimes End(W)$. Hence we get an isomorphism

$$A \otimes End(V) \cong B \otimes End(W),$$

which implies $[A] = [B]$ in $B(F)$. □

Recall, that for each graded central simple algebra, we can associate to it a type and discriminant. The group $Q(F)$ which we defined earlier in this section is a semidirect product of the algebra types ($T = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$) and discriminants ($D = F^*/F^{*pq}$). We can define a map by associating the isomorphism class of a GCSA to an element in $Q(F)$, by sending the class to it's type and discriminant.

Corollary 5.2.3. *There is a well defined group homomorphism $j : GB(F) \longrightarrow Q(F)$ via $\langle A \rangle \longmapsto (\delta(A), type(A))$ with $\ker(j) = GB_{E,1}(F)$.*

Proof. In the previous theorem we showed $Q(F) \cong GB(F)/GB_{E,1}(F)$. Then, there is a well defined homomorphism $j : GB(F) \longrightarrow GB(F)/GB_{E,1}(F) \cong Q(F)$ with $\ker(j) = GB_{E,1}(F)$.

We will now take a closer look at the group theory involved in the previous theorem, which shows $GB(F)/GB_{E,1}(F) \cong D \rtimes T = Q(F)$ in order to recognize the map j as defined in this theorem. Recall the short exact sequence defined in the proof of the previous theorem, Theorem 5.2.1

$$1 \longrightarrow D \underset{\gamma}{\overset{\beta}{\longrightarrow}} GB(F)/GB_{E,1}(F) \underset{\theta}{\overset{\alpha}{\longrightarrow}} T \longrightarrow 1.$$

We will use the above short exact sequence to define a map, $\psi : GB(F)/GB_{E,1}(F) \longrightarrow D \rtimes T$. First notice we can map from $GB(F)/GB_{E,1}(F)$ to T by the map α. The splitting γ is obtained from the splitting θ by $\gamma(\langle A \rangle) = \beta^{-1}\left(\langle A \rangle \cdot (\theta \circ \alpha)\left(\langle A \rangle\right)\right) = \beta^{-1}\left(\langle A \widehat{\otimes} (\theta \circ \alpha)(A) \rangle\right)$. Notice, that $A \widehat{\otimes} (\theta \circ \alpha)(A)$ is an even algebra (since $(\theta \circ \alpha)(A)$ is the same type as A) and so is in the kernel of α. Since this sequence is exact, $\langle A \widehat{\otimes} (\theta \circ \alpha)(A) \rangle$ is in the image of β. Now, γ is just the definition of the discriminant of A, so we can recognize the map

$\psi : GB(F)/GB_{E,1}(F) \longrightarrow D \rtimes T$ by $\langle A\rangle \mapsto (\gamma(A), \alpha(A)) = (\delta(A), type(A))$. It is important for the above explanation to recall, $\theta(\alpha(\langle A\rangle))^{-1} = \theta(\alpha(\langle A\rangle))^* = \theta(\alpha(\langle A\rangle))$ by Lemma 5.1.9.

□

Theorem 5.2.4. *Let i and j be the maps given by $B(F) \xrightarrow{i} GB(F) \xrightarrow{j} Q(F)$. The map i takes a central simple algebra A over F and sends it to the graded algebra (A). If A is a GCSA with $\langle A\rangle \in \ker(j)$, then $\langle A\rangle = i[A]$. Recall, $[A]$ denotes the equivalence class of A in $B(F)$ viewed as an ungraded CSA.*

Proof. Let A be any GCSA such that $\langle A\rangle \in \ker(j)$. Then A is an even algebra with discriminant 1. If $A_1 = A_2 = \cdots = A_{n-1} = 0$ (i.e. $A = A_0$) then $i[A] = \langle A\rangle$ since $A_0 \otimes End(V) \cong A_0 \mathbin{\widehat{\otimes}} End(V)$. If $A_i \neq 0$ for some $1 < i \leq n-1$, using the theory of descent we can assume we have case (i) ($z^{pq} = c \in (F^*)^{pq}$, where z is the degree 0 element that generates the center of A_0) of Theorem 3.2.9, i.e. there exists a graded vector space $V = \bigoplus_{i=0}^{n-1}$ such that $A \cong End(V) \mathbin{\widehat{\otimes}} (D)$, where $A \cong \mathbb{M}_r(D)$ and D is a central division algebra over F. Since $A \cong End(V) \mathbin{\widehat{\otimes}} (D)$, we have $\langle A\rangle = \langle (D)\rangle$ and $\langle (D)\rangle = i[D]$ since (D) is concentrated in degree 0. Now, D and A are Brauer equivalent ($[D] = [A]$) since $A \cong \mathbb{M}_r(D)$. So, $i[D] = i[A]$ and hence $\langle A\rangle = i[A]$.

□

Theorem 5.2.5. *There is a group isomorphism $B(F) \cong GB_{E,1}(F)$*

Proof. By Proposition 5.2.2 we have a well defined, injective homomorphism $i' : B(F) \to GB_{E,1}(F) = \ker(j)$, where j is the well defined homomorphism from $GB(F)$ to $Q(F)$ in Proposition 5.2.3 and Proposition 5.2.4. To see this map is surjective, let $\langle A\rangle \in GB_{E,1}(F) = \ker(j)$. Then, by Proposition 5.2.4, since $\langle A\rangle \in \ker(j)$, $i([A]) = \langle A\rangle$, where $[A]$ is the Brauer class in $B(F)$. Thus we have found a preimage of $\langle A\rangle$ in $B(F)$, and our map is surjective. Thus, we have an isomorphism. □

The above theorem allows us to recognize the classic Brauer group as a subgroup of the graded Brauer group. Moreover, applying this result to Theorem 5.2.1 we see $Q(F) \cong GB(F)/B(F)$.

Bibliography

[1] Childs, L.N, Garfinkel, G, and Orzech, M. *The Brauer Group of Graded Azumaya Algebras*. Trans. Amer. Math. Soc. **175** (1973), no. 1, 299–326. MR0349652.

[2] Kac, V.G. *Lie Superalgebras*. Advances in Math **26** (1977), no. 1, 8–96. MR0486011.

[3] Kac, V.G., Martinez, C., and Zelmanov, E. *Graded Simple Jordan Superalgebras of Growth One*. Mem. American Mathematical Society **150** (2001), no. 711. MR1810856.

[4] Knus, Max A. *Algebras Graded by a Group*. Category Theory, Homology Theory and their Applications, II (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Two) pp. 117–133 Springer, Berlin. MR0242895.

[5] Koç, Cemal and Kurtulmaz, Yosum. *Structure Theory of Central Simple $\mathbb{Z}_d$-Graded Algebras*. Turkish J. Math. **36** (2012), no.4, 560–577. MR2993587.

[6] Lam, T.Y. *Introduction to Quadratic Forms over Fields*. American Mathematical Society (2005).

[7] Long, F.W. *A Generalization of the Brauer Group of Graded Algebras*. Proc. London Math. Soc. (3) **29** (1974), no. 1, 237–256. MR0354753.

[8] Manin, Yuri I. *Gauge Field Theory and Complex Geometry, 2^{nd} ed.* Translated by N. Koblitz and J.R. King. Springer-Verlag, Berlin.(1997). MR1632008.

[9] Musson, Ian M. *Lie Superalgebras and Enveloping Algebras*. Graduate Studies in Mathematics, vol. 131 (2012). MR1615916.

[10] Oystaeyen, F. Van and Zhang, Y.H, *Brauer Groups of Actions*, Rings, Hopf Algebras and Brauer Groups (1998), 299-324.

[11] Oystaeyen, Fred Van and Zhang, Yinhou. *The Brauer Group of a Braided Monoidal Category*. J. Algebra **202** (1998), no. 1, 96–128. MR1614178.

[12] Pierce, Richard S. *Associative Algebras*. Graduate Texts in Mathematics, vol. 88 . Springer-Verlag (1982).

[13] Wall, C. T. C. *Graded Brauer Groups*. J. Reine Angew. Math. **213** (1963/64), 187–199. MR294771.

[14] Varadarajan, V.S. *Supersymmetry for Mathematicians: An Introduction*. Courant Lecture Notes, 11. American Mathematical Society (2004). MR2069561.

[15] Vela, Montserrat. *Central Simple $\mathbb{Z}/n\mathbb{Z}$-Graded Algebras*. Comm. Algebra **30** (2002), no. 4, 1995-2001. MR1894056.