Components of Springer Fibers for the Exceptional Groups G_2 and F_4

by

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(Under the direction of William Graham)

Abstract

Let G be the complex connected simply connected simple Lie group of type G_2 or F_4. Let K denote the fixed point subgroup relative to an involution of G that is lifted from a Cartan involution. We give a description of certain components of Springer fibers associated to closed K-orbits contained in the flag variety of G. Then we will describe certain multiplicity polynomials associated to discrete series representations of the real form G_2^2 of G_2 and the two real forms F_4^4 and F_4^{-20} of F_4. The goals for this paper are motivated by the descriptions of Springer fiber components for type $SU(p,q)$ described in a paper of Barchini and Zierau.

Index words: Springer fibers, Cartan involutions, Real forms, Flag varieties, Exceptional groups

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Dedication

I dedicate this dissertation to my best friend and incredible wife Katie whose support has made achieving my dreams possible.

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First, I must thank my family for their support, patience, and love throughout this important time in my life. With a love for a demanding subject comes great sacrifice, and my family has been very understanding every step of the way. I must also thank my advisor Bill Graham for his willingness to work with me and his patience as I completed this dissertation. His advice and support have been very much appreciated ever since he agreed to take me on as a student. I feel very blessed and know that I could not have asked for a better Ph.D. advisor. I also extend a thanks to Roger Zierau for his willingness to help me understand many important concepts related to my research interests.

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Contents

1

Introduction

Our goal is to give a description of certain components for Springer fibers in the exceptional types G_2 and F_4, so we need to introduce Springer fibers. Before describing the fibers, we begin with a discussion of important terminology. Let G denote a connected complex simple Lie group, and let $\mathfrak{g}$ denote the Lie algebra of G. Then let $\mathcal{B}$ denote the collection of all Borel subalgebras of $\mathfrak{g}$, and let $\mathcal{N}$ denote the cone of nilpotent elements of $\mathfrak{g}$. Recall that an element $x \in \mathfrak{g}$ is nilpotent if for any finite dimensional representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, $\pi(x)$ is a nilpotent operator on V. For a given Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ contained in $\mathfrak{b}$, we can write $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ for some maximal nilpotent ideal $\mathfrak{n}^-$. Let $Ad : G \to GL(\mathfrak{g})$ denote the adjoint representation of G, and let $g.x$ represent $Ad(g).x$ for any $g \in G$ and $x \in \mathfrak{g}$. If we define $B = Stab_G(\mathfrak{b})$ to be the stabilizer of $\mathfrak{b}$ in G, then we have an isomorphism of G/B with the flag variety $\mathcal{B}$ given by $gB \to g.\mathfrak{b}$.

With these objects in place, we can introduce a variety from which Springer fibers are built. Since the subspace $\mathfrak{n}^-$ of $\mathfrak{g}$ is B-stable, we can define the algebraic variety $G \times^B \mathfrak{n}^-$ to be the quotient of $G \times \mathfrak{n}^-$ by the equivalence relation defined in the following way. For any pair of elements (g_1, n_1) and (g_2, n_2) in $G \times \mathfrak{n}^-$, $(g_1, n_1) \sim (g_2, n_2)$ if there exists some element b in B such that $(g_1, n_1).b = (g_1 b, b^{-1}.n_1) = (g_2, n_2)$. To define Springer fibers, we introduce the Springer resolution

$$\mu : G \times^B \mathfrak{n}^- \to \mathcal{N},$$

which is a proper morphism given by $\mu\big((g,n)\big) = g.x$ for any $g \in G$ and $x \in \mathfrak{n}^-$. Details about the variety structure of $G \times^B \mathfrak{n}^-$ including various isomorphic descriptions of the variety can be found in [Spr81] and [Jan04]. One such interpretation is that $G \times^B \mathfrak{n}^-$ can be identified with the cotangent bundle $T^*\mathcal{B}$ of $\mathcal{B}$, so μ gives a map from the cotangent bundle onto the set of nilpotent elements of $\mathfrak{g}$.

Definition 1.1. Given an arbitrary nilpotent element $f \in \mathcal{N}$, the preimage of f with respect to μ is called a Springer fiber.

To outline the specific components of $\mu^{-1}(f)$ that we wish to describe, we need to introduce Cartan decompositions and real forms. These Cartan decompositions will play an important role because they help us describe certain components of Springer fibers, and then these components will tell us information about certain real forms of G. Let G_0 denote a noncompact real form of G with associated Lie algebra $\mathfrak{g}_0$ obtained from an involution θ of $\mathfrak{g}$. Since θ is an involution, i.e., a Lie algebra automorphism such that $\theta^2 = 1$, we have a decomposition of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is the $+1$ eigenspace and $\mathfrak{p}$ is the -1 eigenspace with respect to θ. Define Θ to be the involution of G that is the lift of θ to G, and let $K = \{g \in G : \Theta(g) = g\}$ denote the fixed point subgroup of G. In general, the connected subgroup of G with Lie algebra $\mathfrak{k}$ and the fixed point subgroup K may differ by a non-trivial discrete group, although [KR71, Proposition 1] tells us that they have the same Lie algebra $\mathfrak{k}$. There are many complex simple Lie groups including those considered in this paper for which K is necessarily connected.

Since we will be focusing on nilpotent elements in $\mathfrak{p}$, let $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}$ denote the set of those elements. Refer to [KR71], [Dok88], and [Kin92] for more information on nilpotent G-orbits and nilpotent K-orbits for elements $f \in \mathcal{N}_\theta$. Our goal is to describe certain irreducible components of $\mu^{-1}(f)$ associated to *closed* K-orbits in $\mathcal{B}$ when f is a specialized element called a "generic" element. The remainder of this chapter will involve an outline of the ingredients necessary to understand these components, which can be found in [BZ08, Section 2].

The fixed point subgroup K acts on $\mathcal{B}$ with finitely many orbits, so let $\mathcal{O} = K \cdot \mathfrak{b}$ denote one of these orbits. If we let K_B denote the intersection of K with B, then the conormal bundle to $\mathcal{O}$ in the cotangent bundle of $\mathcal{B}$ is identified with

$$T^*_{\mathcal{O}}\mathcal{B} = K \times^{K_B} (\mathfrak{n}^- \cap \mathfrak{p}).$$

Define a map $\psi_{\mathcal{O}} : \overline{T^*_{\mathcal{O}}\mathcal{B}} \to \mathcal{N}$ to be the restriction of μ to the closure of the conormal bundle $T^*_{\mathcal{O}}\mathcal{B}$ contained in $T^*\mathcal{B}$. Understanding the image is important and leads us to one of two lemmas, which will be used throughout this paper. Lemmas 1.2 and 1.6 are clearly known and used frequently in [BZ08], but we include original proofs here for completeness.

Lemma 1.2. *The image of $\psi_{\mathcal{O}}$ is the subset of $\mathcal{N}_\theta$ given by the closure of $K.(\mathfrak{n}^- \cap \mathfrak{p})$, and is the closure of a single K-orbit having the greatest possible dimension.*

Proof. Since μ restricted to $T^*_{\mathcal{O}}\mathcal{B}$ has image $K.(\mathfrak{n}^- \cap \mathfrak{p})$ and μ is a closed map, we know that

$$\overline{K.(\mathfrak{n}^- \cap \mathfrak{p})} = \overline{\mu(T^*_{\mathcal{O}}\mathcal{B})} \subset \overline{\mu(\overline{T^*_{\mathcal{O}}\mathcal{B}})} = \mu(\overline{T^*_{\mathcal{O}}\mathcal{B}}) = \psi_{\mathcal{O}}(\overline{T^*_{\mathcal{O}}\mathcal{B}}).$$

On the other hand, μ is a continuous map, so

$$\psi_{\mathcal{O}}(\overline{T^*_{\mathcal{O}}\mathcal{B}}) = \mu(\overline{T^*_{\mathcal{O}}\mathcal{B}}) \subset \overline{\mu(T^*_{\mathcal{O}}\mathcal{B})} = \overline{K.(\mathfrak{n}^- \cap \mathfrak{p})},$$

hence the image of $\psi_{\mathcal{O}}$ is the closure of $K.(\mathfrak{n}^- \cap \mathfrak{p})$. To prove the second part of the lemma, define the map

$$\lambda : K_\circ \times (\mathfrak{n}^- \cap \mathfrak{p}) \to K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})$$

by $(k, v) \to k.v$ where $K_\circ$ represents the identity component of K. Since $K_\circ \times (\mathfrak{n}^- \cap \mathfrak{p})$ is an irreducible variety, the image of λ is as well, i.e., $K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})$ is an irreducible variety.

Now, Kostant and Rallis show in [KR71] that there are finitely many K-orbits in $\mathcal{N} \cap \mathfrak{p}$, and also finitely many $K_\circ$-orbits in $\mathcal{N} \cap \mathfrak{p}$, hence $K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})$ is a finite union of nilpotent $K_\circ$-orbits. As a result, write $K_\circ.(\mathfrak{n}^- \cap \mathfrak{p}) = \mathcal{O}_1 \cup \mathcal{O}_2 \cdots \cup \mathcal{O}_n$, and note that the closure is $\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})} = \overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2} \cdots \cup \overline{\mathcal{O}_n}$. Since $\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})}$ is irreducible there must be some $\overline{\mathcal{O}_j}$ that is equal to $\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})}$. In particular, $\mathcal{O}_j$ is dense in $\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})}$, and the dimension of $\mathcal{O}_j$ is the same as the dimension of $\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})}$. Let $\{k_1, \ldots, k_m\}$ denote a set of coset representatives for $K/K_\circ$. Since $\mathcal{O}_j = K_\circ.f$ is dense in $\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})}$ and $k_iK_\circ.f$ is dense in $k_i\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})}$, it follows that

$$\bigcup_{i=1}^{m} k_iK_\circ.f = K.f$$

is dense in

$$\bigcup_{i=1}^{m} k_i\overline{K_\circ.(\mathfrak{n}^- \cap \mathfrak{p})} = \bigcup_{i=1}^{m} \overline{k_iK_\circ.(\mathfrak{n}^- \cap \mathfrak{p})} = \overline{K.(\mathfrak{n}^- \cap \mathfrak{p})}.$$

Finally, any two distinct K-orbits in $\overline{K.(\mathfrak{n}^- \cap \mathfrak{p})}$ must have empty intersection, so any other K-orbit lives in the boundary $\overline{K.f} - K.f$. By Proposition 8.3 in [Hum75], the orbits in the boundary have strictly smaller dimension, so the maximal dimensional dense K-orbit $K.f$ is unique. □

The above lemma actually proves more than we need since K happens to be connected as a subgroup of G_2 and F_4, but it is nevertheless true in a more general setting. Understanding the components of the Springer fibers this dissertation will describe hinges on exploring this particular orbit, so it is appropriate to name the elements of this K-orbit as they are defined in [BZ08].

Definition 1.3. The elements $f \in \mathfrak{n}^- \cap \mathfrak{p}$ such that the image of $\psi_{\mathcal{O}}$ is $\overline{K \cdot f}$ are referred to as generic elements.

We are going to be looking at closed K-orbits, so we need to consider how this affects the map $\psi_{\mathcal{O}}$. If $\mathcal{O}$ is a closed K-orbit of $\mathcal{B}$, then $T^*_{\mathcal{O}}\mathcal{B}$ is closed in $T^*\mathcal{B}$ and the image of

$\psi_{\mathcal{O}}$ is $K.(\mathfrak{n}^- \cap \mathfrak{p})$. Moreover, the closed K-orbit $\mathcal{O}$ is necessarily a flag variety for the fixed point subgroup K. To prove this fact, observe that the stabilizer subgroup of $\mathfrak{b}$ in K is the subgroup K_B. Then consider the isomorphism of varieties from K/K_B to $K.\mathfrak{b}$ given by $kK_B \to k.\mathfrak{b}$. Since $\mathcal{O}$ is closed in $\mathcal{B}$, it follows that K/K_B is closed in $\mathcal{B}$ and projective, so K_B is a parabolic subgroup of K. Then $K_B = K \cap B$ is solvable, so it follows that K_B is a Borel subgroup. For a generic element $f \in \mathfrak{n}^- \cap \mathfrak{p}$, we have the following description of the preimage $\psi_{\mathcal{O}}^{-1}(f)$:

$$\begin{aligned}\psi_{\mathcal{O}}^{-1}(f) &= \{(k,n) \in T^*_{\mathcal{O}}\mathcal{B} \mid k.n = f\} \\ &= \{(k, k^{-1}.f) \in T^*_{\mathcal{O}}\mathcal{B} \mid k^{-1}.f \in \mathfrak{n}^- \cap \mathfrak{p}\}.\end{aligned}$$

Now, the natural projection from $T^*\mathcal{B} \to \mathcal{B}$ takes $\psi_{\mathcal{O}}^{-1}(f)$ isomorphically onto its image, so we have

$$\psi_{\mathcal{O}}^{-1}(f) \simeq \{(k.\mathfrak{b} \mid k^{-1}.f \in \mathfrak{n}^- \cap \mathfrak{p}\} \text{ [BZ08, Section 2].}$$

If we let $N(f, \mathfrak{n}^- \cap \mathfrak{p})$ be the subset of K defined by

$$N(f, \mathfrak{n}^- \cap \mathfrak{p}) = \big\{k \in K : k.f \in \mathfrak{n}^- \cap \mathfrak{p}\big\},$$

then

$$\psi_{\mathcal{O}}^{-1}(f) \simeq N(f, \mathfrak{n}^- \cap \mathfrak{p})^{-1}.\mathfrak{b} \subset \mathcal{O}. \tag{1.1}$$

Therefore, $\psi_{\mathcal{O}}^{-1}(f)$ for any $f \in \mathfrak{n}^- \cap \mathfrak{p}$ is identified with a subvariety of the flag variety $\mathcal{O}$ for the fixed point subgroup K.

At this point, we should discuss the relationship between $\mu^{-1}(f)$ and $\psi_{\mathcal{O}}^{-1}(f)$. By [BZ, Section 1], it turns out that each irreducible component of $\mu^{-1}(f)$ for $f \in \mathcal{N}_\theta$ is contained in the closure of a single conormal bundle $T^*_{\mathcal{O}}\mathcal{B}$ associated to a K-orbit $\mathcal{O}$ in the flag variety of G. Moreover, if the K-orbit $\mathcal{O}$ is closed, then all of the components of $\mu^{-1}(f)$ contained in the closed conormal bundle $T^*_{\mathcal{O}}\mathcal{B}$ are permuted by the elements of a component group associated to f. Therefore, we will use the above isomorphism (1.1) frequently when we describe the irreducible components of $\mu^{-1}(f)$ contained in the closed conormal bundle $T^*_{\mathcal{O}}\mathcal{B}$. The following proposition illustrates the transitivity of the irreducible components under the action of the component group, which will help us describe $\mu^{-1}(f) \cap T^*_{\mathcal{O}}\mathcal{B}$, but first we need to define these component groups.

Definition 1.4. Given an element $f \in \mathcal{N}_\theta$, let K^f denote the stabilizer subgroup of f in K. Then the component group of f in K, denoted $A_K(f)$, is defined to be the quotient group $K^f/K^f_\circ$ where $K^f_\circ$ denotes the identity component.

Proposition 1.5. *If $\mathcal{O} = K.\mathfrak{b}$ is a closed K-orbit in $\mathcal{B}$ and C_f is an irreducible component of $\mu^{-1}(f)$ contained in $T^*_{\mathcal{O}}\mathcal{B}$ for $f \in \mathcal{N}_\theta$, then the component group $A_K(f)$ acts transitively on the irreducible components of $\mu^{-1}(f)$ contained in $T^*_{\mathcal{O}}\mathcal{B}$. In other words,*

$$\psi^{-1}_{\mathcal{O}}(f) = \bigcup_{z \in A_K(f)} z \cdot C_f.$$

Moreover, the irreducible components all have the same dimension: dim $\mathcal{B} - \dfrac{1}{2}$ dim $G.f$.

Proof. See [BZ] for proofs and a complete list of references. □

The fibers $\psi^{-1}_{\mathcal{O}}(f)$ for a generic element $f \in \mathfrak{n}^- \cap \mathfrak{p}$ associated to closed K-orbits for the noncompact real forms of G_2 and F_4 have a nice description. Most of the components are homogeneous, but there are few closed K-orbits for which the associated Springer fiber components are non-homogeneous. We will spend several chapters building all of the necessary tools to prove that $\psi^{-1}_{\mathcal{O}}(f)$ takes the form of exactly one of

$$L.\mathfrak{b}, \mathbb{Z}_2 L.\mathfrak{b}, ZL.\mathfrak{b} \simeq Z \times^{Z\cap Q} Q/K_B, \text{ or } \overline{ZL.\mathfrak{b}} \qquad (1.2)$$

where L is a reductive subgroup of K, Z is a centralizer subgroup of K^f, and Q is a parabolic subgroup of K containing L. It is important to note that the specific groups above depend on the closed K-orbit and will be described in the subsequent chapters. Moreover, we will know for each closed K-orbit, the specific isomorphism type of $\psi_{\mathcal{O}}^{-1}(f)$ among the possibilities given in (1.2). This structure will be important for the applications to real forms discussed in Chapter 8.

The last preliminary piece of information that we need involves describing the closed K-orbits. From now on, we will be considering Lie algebras $\mathfrak{g}$ such that the rank of K is the same as the rank of G. As a result, we will choose our Cartan subalgebra such that $\mathfrak{h} \subset \mathfrak{k}$. Let $\Phi(\mathfrak{h}, \mathfrak{g})$ (resp., $\Phi(\mathfrak{h}, \mathfrak{k})$) represent a system of roots (resp., compact roots) for $\mathfrak{h}$ in $\mathfrak{g}$ (resp., $\mathfrak{h}$ in $\mathfrak{k}$), let $\Delta(\mathfrak{h}, \mathfrak{g})$ represent a system of simple roots, and fix a positive system $\Phi^+(\mathfrak{h}, \mathfrak{g})$ (resp., $\Phi^+(\mathfrak{h}, \mathfrak{k})$) for $\mathfrak{h}$ in $\mathfrak{g}$ (resp., $\mathfrak{h}$ in $\mathfrak{k}$). The following lemma will tell us how to find and count the possible K-orbits.

Lemma 1.6. *Suppose that the rank of K equals the rank of G. There is a one to one correspondence between positive systems Φ^+ for $\Phi(\mathfrak{h}, \mathfrak{g})$ such that $\Phi^+(\mathfrak{h}, \mathfrak{k}) \subset \Phi^+ \subset \Phi(\mathfrak{h}, \mathfrak{g})$ and closed K-orbits in $\mathcal{B}$. Moreover, the number of such positive systems is given by $|\mathcal{W}/\mathcal{W}_K| = |\mathcal{W}_K \backslash \mathcal{W}|$ where $\mathcal{W}$ (resp., $\mathcal{W}_K$) denotes the Weyl group of G (resp., K).*

Proof. We need to define a map from

$$\{\text{Closed } K\text{-orbits } \mathcal{O}\} \text{ to } \{\Phi^+ \mid \Phi^+(\mathfrak{h}, \mathfrak{k}) \subset \Phi^+ \subset \Phi(\mathfrak{h}, \mathfrak{g})\}$$

where Φ^+ is a positive system for $\Phi(\mathfrak{h}, \mathfrak{g})$. Given a closed K-orbit $\mathcal{O}$, choose $\mathfrak{b}$ in $\mathcal{O}$ such that $\mathfrak{h}$ is contained in $\mathfrak{b}$, and write $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ for some maximal nilpotent ideal $\mathfrak{n}^-$. This is possible because $\mathfrak{h} \subset \mathfrak{k}$, so we can conjugate the Cartan subalgebra for a Borel in $\mathcal{O}$ to our fixed Cartan $\mathfrak{h}$ by an element of K. Then let $\Phi^+(\mathfrak{b})$ be the set consisting of those roots α such that $\mathfrak{g}_{-\alpha}$ is a root space contained in $\mathfrak{n}^-$. Choose the unique element $[w] \in \mathcal{W}_K (w \in K)$ such that $[w].(\Phi^+(\mathfrak{b}) \cap \Phi(\mathfrak{h}, \mathfrak{k})) = \Phi^+(\mathfrak{h}, \mathfrak{k})$. Then the Borel subalgebra $w.\mathfrak{b}$ is the unique element of $\mathcal{O} = K.\mathfrak{b}$ for which $w.\mathfrak{b} \supset \mathfrak{h}$ and $\Phi^+(w.\mathfrak{b})$ contains $\Phi^+(\mathfrak{h}, \mathfrak{k})$. To see the uniqueness, note that the only elements of K that fix $\mathfrak{h}$ are elements w of the normalizer of $H = Exp(\mathfrak{h})$ in K. Then acting by any nontrivial element $[w] \in \mathcal{W}_K$ will permute the roots so that $\Phi^+(\mathfrak{b})$ no longer contains $\Phi^+(\mathfrak{h}, \mathfrak{k})$. Now, we need to define the inverse mapping. Observe that from such a positive system Φ^+, we obtain a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ by letting $\mathfrak{n}^-$ consists of the root spaces $\mathfrak{g}_\xi$ for all roots ξ in $-\Phi^+$. Since the positive system used to construct $\mathfrak{b}$ contains the positive system for $\mathfrak{h}$ in $\mathfrak{k}$, $\mathfrak{k} \cap \mathfrak{b}$ is a Borel subalgebra in $\mathfrak{k}$. Therefore, $K \cap B$ is a Borel subgroup of K, so $K/(K \cap B)$ is isomorphic to the flag variety for K. Since $K.\mathfrak{b} \simeq K/(K \cap B)$, we see that the K-orbit $K.\mathfrak{b}$ is closed. The cardinality follows immediately because all positive systems for $\mathfrak{g}$ are obtained by applying Weyl group elements $[w] \in \mathcal{W}$ to a fixed positive system Φ^+. As in the proof above, there is a unique $[w'] \in \mathcal{W}_K$ such that the positive system $[w']([w]\Phi^+)$ contains $\Phi^+(\mathfrak{h}, \mathfrak{k})$. A simple computation shows that the map

$$\mathcal{W}_K \backslash \mathcal{W} \to \{\Phi^+ \mid \Phi^+(\mathfrak{h}, \mathfrak{k}) \subset \Phi^+ \subset \Phi(\mathfrak{h}, \mathfrak{g})\}$$

given by sending the right coset $\mathcal{W}_K[w]$ to the positive system $[w']([w]\Phi^+)$ gives a bijection.

□

The Borel subalgebras that appear in this work will always be put together in this manner. In the future, we may refer to the corresponding nilradical $\mathfrak{n}$ defined by the positive roots instead of the negative roots.

2

The Exceptional Lie Algebra $\mathfrak{g}_2$

In order to start computing some irreducible components of the Springer fibers for $G = G_2$, we begin by constructing the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{h}$ denote a fixed Cartan subalgebra of $\mathfrak{g}$, and let $\Phi(\mathfrak{h}, \mathfrak{g})$ denote a root system relative to $\mathfrak{h}$. Recall that relative to a choice of Cartan subalgebra $\mathfrak{h}$, we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\xi \in \Phi(\mathfrak{h}, \mathfrak{g})} \mathfrak{g}_\xi$$

for the root space decomposition of $\mathfrak{g}$ into its root spaces $\mathfrak{g}_\xi$ and two dimensional Cartan subalgebra $\mathfrak{h}$.

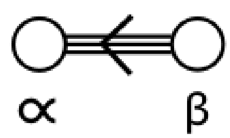


Figure 2.1: Dynkin Diagram for $\mathfrak{g}_2$

Let $\Delta(\mathfrak{h}, \mathfrak{g}) = \{\alpha, \beta\}$ denote a base for $\Phi(\mathfrak{h}, \mathfrak{g})$ where α is the short simple root and β is the long simple root. Then the positive roots relative to $\Delta(\mathfrak{h}, \mathfrak{g})$ for type $\mathfrak{g}_2$ are

$$\Phi^+(\mathfrak{h}, \mathfrak{g}) = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

Note that Figure 2.1 gives the Dynkin diagram for $\mathfrak{g}_2$. For our efforts, we will choose to work with a Chevalley basis, which is defined by the content of the following proposition in [Hum72, Prop. 25.2].

Proposition 2.1 (Existence of a Chevalley Basis)**.** *It is possible to choose root vectors* $x_\xi \in \mathfrak{g}_\alpha (\alpha \in \Phi(\mathfrak{h}, \mathfrak{g})$ *satisfying:*

(a) $[x_\xi, x_{-\xi}] = h_\xi$

(b) If ξ, ϵ *and* $\xi + \epsilon$ *are roots such that* $[x_\xi, x_\epsilon] = c_{\xi,\epsilon} x_{\xi+\epsilon}$*, then* $c_{\xi,\epsilon} = -c_{-\xi,-\epsilon}$*.*

We will give details about this basis for $\mathfrak{g}$, but let us first introduce a notation for such a basis. Let

$$\bigcup_{\xi \in \Phi(\mathfrak{h},\mathfrak{g})} \{x_\xi\} \cup \bigcup_{\xi \in \Delta(\mathfrak{h},\mathfrak{g})} \left\{ h_\xi = [x_\xi, x_{-\xi}] \right\}$$

represent a Chevalley basis for $\mathfrak{g}$ where x_ξ denotes a basis vector in $\mathfrak{g}_\xi$ for each

$$\xi = i\alpha + j\beta \in \Phi(\mathfrak{h}, \mathfrak{g}).$$

If $\xi = i\alpha + j\beta$, we will write $x_{(i,j)} = x_\xi$, and the notations will be freely interchanged whenever it is clear which root is being referenced. Lastly, we may refer to the elements of $\mathfrak{h}$ generated by the nonsimple root vectors, so let h_ξ denote those elements $[x_\xi, x_{-\xi}]$ for all $\xi \in \Phi^+(\mathfrak{h}, \mathfrak{g})$.

Now, it will be useful at this point to construct generators for a Chevalley basis for $\mathfrak{g}$. First, we choose the simple root vectors as prescribed by the Cartan matrix, and then use the Lie bracket to generate all of $\mathfrak{g}$. In other words, let

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

represent the Cartan matrix associated to $\mathfrak{g}$. Choose simple root vectors satisfying the relations in Table 2.1 below, and then use the Lie bracket to extend to the basis for $\mathfrak{g}$ given

$[h_{(1,0)}, x_{(1,0)}] = 2x_{(1,0)}$	$[h_{(1,0)}, x_{(0,1)}] = -3x_{(0,1)}$
$[h_{(0,1)}, x_{(1,0)}] = -x_{(1,0)}$	$[h_{(0,1)}, x_{(0,1)}] = 2x_{(0,1)}$
$[h_{(1,0)}, x_{(-1,0)}] = -2x_{(-1,0)}$	$[h_{(1,0)}, x_{(0,-1)}] = 3x_{(0,-1)}$
$[h_{(0,1)}, x_{(-1,0)}] = x_{(-1,0)}$	$[h_{(0,1)}, x_{(0,-1)}] = -2x_{(0,-1)}$

Table 2.1: Lie Brackets I

$x_{(1,1)} = -[x_{(1,0)}, x_{(0,1)}]$	$x_{(-1,-1)} = [x_{(-1,0)}, x_{(0,-1)}]$
$x_{(2,1)} = -1/2[x_{(1,0)}, x_{(1,1)}]$	$x_{(-2,-1)} = 1/2[x_{(-1,0)}, x_{(-1,-1)}]$
$x_{(3,1)} = -1/3[x_{(1,0)}, x_{(2,1)}]$	$x_{(-3,-1)} = 1/3[x_{(-1,0)}, x_{(-2,-1)}]$
$x_{(3,2)} = -[x_{(0,1)}, x_{(3,1)}]$	$x_{(-3,-2)} = [x_{(0,-1)}, x_{(-3,-1)}]$

Table 2.2: Lie Brackets II

in Table 2.2. Note that the basis given in Tables 2.1 and 2.2 is a Chevalley type basis that agrees with the mathematical software package GAP. The complete multiplication tables have been worked out, and the results are given in Appendix D for reference.

As we build some Springer fiber components, we will place an assortment of calculations that are necessary into the appendices as opposed to the main sections. From now on, if a calculation is included in Appendix X, then it will be followed by a "(see Appendix X)" marker. Most of the calculations can be done using the multiplication tables and linear algebra. Alternatively, some calculations involving the fixed point subgroup K can be made more explicit if we appeal to a representation of $\mathfrak{g}$ and G by matrices. By the Weyl dimension formula, we know that there exists a faithful, irreducible representation of $\mathfrak{g}_2$ of dimension 7. Before giving such a representation, we consider the following lemma.

Lemma 2.2. *There exists an embedding of $G = G_2$ as a closed subgroup of $GL(7, \mathbb{C})$.*

Proof. Let $\pi : \mathfrak{g} \to \mathfrak{gl}(7, \mathbb{C})$ be a faithful representation of $\mathfrak{g} = \mathfrak{g}_2$. Then viewing $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(7, \mathbb{C})$, we know from [Kna02, Chapter 1, Section 10] that there is a complex analytic subgroup G' of $GL(7,\mathbb{C})$ with Lie algebra isomorphic to $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, we know from [Hel78, Chapter 2, D.4.IV] that G' is necessarily a closed Lie subgroup of $GL(7,\mathbb{C})$. As there is only one connected complex simple Lie group of type G_2, G' is simply connected of type G_2 (see [Ale05, Proposition 4.1]). Therefore, G' is isomorphic to G, hence G can be viewed as a closed subgroup of $GL(7,\mathbb{C})$. □

To describe such a representation of $\mathfrak{g}$, we will use the notation $E^n_{i,j}$ to denote the $n \times n$ matrix with a 1 in the $\{i,j\}$ entry and 0's elsewhere. Following Howlett et al., we'll choose the representation $\pi : \mathfrak{g} \to \mathfrak{gl}(7,\mathbb{C})$ found in [HRT01, Section 3.6] with generators for a Chevalley basis given in Table 2.3. As one might hope, these generators completely agree with our first description of $\mathfrak{g}$ given abstractly in terms of the bracket relations on the generators. Moreover, the representation is faithful, so one can drop the π from the notation without encountering any problems. Since G is simply connected, π determines a unique map from $G \to GL(7,\mathbb{C})$ that commutes with the exponential mappings. Therefore, we will sometimes view the exponential mapping $Exp : \mathfrak{g} \to G$ as the matrix exponential, and the adjoint action in terms of matrix conjugation. The complete list of matrices is provided in Appendix D.

Root Vector x	$\pi(x)$
x_α	$E^7_{1,2} + 2E^7_{3,4} + E^7_{4,5} + E^7_{6,7}$
$x_{-\alpha}$	$E^7_{2,1} + E^7_{4,3} + 2E^7_{5,4} + E^7_{7,6}$
x_β	$E^7_{2,3} + E^7_{5,6}$
$x_{-\beta}$	$E^7_{3,2} + E^7_{6,5}$

Table 2.3: Generators for $\mathfrak{g}_2 \subset \mathfrak{gl}(7,\mathbb{C})$

3

Generic Elements: Split Real Form G_2^2

A real simple Lie algebra $\mathfrak{g}_0$ is called a *real form* of a complex simple Lie algebra $\mathfrak{g}$ if the complexification $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathfrak{g}$. In the theory of real Lie algebras, a real form $\mathfrak{g}_0$ can be constructed from an enhanced Dynkin diagram called a *Vogan diagram*. These Vogan diagrams are formed by taking the usual Dynkin diagram of a complex simple Lie algebra and attaching extra data to the simple roots, which is then used to construct an involution θ of $\mathfrak{g}$. From the associated complex Cartan decomposition of $\mathfrak{g}$, the real Lie algebra $\mathfrak{g}_0$ can be extracted.

Example 3.1. The Dynkin diagram for $A_1 = \mathfrak{sl}(2, \mathbb{C})$ consists of a single simple root. There are two non-isomorphic Vogan diagrams to consider for A_1. For the first Vogan diagram, the only simple root α is left *unpainted*, which means that the involution θ takes the corresponding simple root vector x_α to itself. As a result, θ is the trivial isomorphism and $\mathfrak{g}_0$ is the compact real form $\mathfrak{su}_2$. For the second, the only simple root α is *painted*, which means that the involution θ takes the corresponding simple root vector x_α to $-x_\alpha$. The resulting real Lie algebra $\mathfrak{g}_0$ is the split real form $\mathfrak{sl}(2, \mathbb{R})$. In general, a complex Lie algebra always admits at least two real forms: a *compact* real form and a *split* real form, although these are the only real forms for $\mathfrak{sl}(2, \mathbb{C})$ up to isomorphism.

Vogan diagrams are important because they are essential in the development of the classification theory of all real simple Lie algebras. Every real simple Lie algebra arises uniquely from a fixed Vogan diagram, although different Vogan diagrams can lead to the same real Lie algebra. To address this issue, a theorem of Borel and de Siebanthal can be used to eliminate redundant Vogan diagrams, so that no two Vogan diagrams give the same $\mathfrak{g}_0$ ([Kna02, cf. Chapter VI]). Cartan involutions will play an important role in this paper not only because of their relationship with real forms, but also with how they are essential in understanding Springer fibers and discrete series representations.

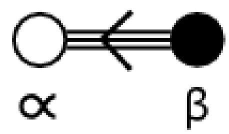


Figure 3.1: Vogan diagram for the split real form of $\mathfrak{g}_2$

According to Figure 6.2 in [Kna02, Chapter VI, Section 10], the split real form is the only noncompact real form for the exceptional simple Lie algebra of type $\mathfrak{g}_2$ up to isomorphism. This split real form, denoted G_2^2, will be the only real form of type G_2 for which we will describe Springer fiber components. To build the Cartan decomposition discussed in Chapter 1, define an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by first defining $\theta(x_\alpha) = x_\alpha$ and $\theta(x_\beta) = -x_\beta$ on the simple root vectors, and then extend to all of $\mathfrak{g}$ using $\theta([x_\xi, x_\epsilon]) = [\theta(x_\xi), \theta(x_\epsilon)]$. This is illustrated by a Vogan diagram (Figure 3.1) where a simple root vector is in $\mathfrak{k}$ whenever the associated simple root is not painted, and a simple root vector is in $\mathfrak{p}$ whenever the simple root is painted. Since $\mathfrak{g}$ is generated by the simple root vectors, it follows that $\theta(x_\xi) = x_\xi$ or $\theta(x_\xi) = -x_\xi$ for every root ξ in $\Phi(\mathfrak{h}, \mathfrak{g})$, so every root vector is in exactly one of $\mathfrak{k}$ or $\mathfrak{p}$. See [Kna02, Chapter VI] for more details.

Let $\Phi(\mathfrak{h}, \mathfrak{k})$ denote the set of roots for which the associated root vector is in $\mathfrak{k}$, and let $\Gamma_\mathfrak{p}$ denote the set of roots for which the associated root vector is in $\mathfrak{p}$. The roots in $\Phi(\mathfrak{h}, \mathfrak{k})$ are called *compact* roots, while the roots in $\Gamma_\mathfrak{p}$ are called *noncompact* roots. Since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we see that the positive roots in $\Phi(\mathfrak{h}, \mathfrak{k})$ are $\Phi^+(\mathfrak{h}, \mathfrak{k}) = \{\alpha, 3\alpha+2\beta\}$ and the positive roots in $\Gamma_\mathfrak{p}$ are $\Gamma_\mathfrak{p}^+ = \{\beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta\}$. Note that the roots associated to $\mathfrak{p}$ do not define a root system. For reference, the associated $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are given by

$$\mathfrak{k}_0 = \mathrm{Span}_{\mathbb{R}}\left(\bigcup_{\gamma\in\Delta(\mathfrak{h},\mathfrak{g})} \{ih_\gamma\} \cup \bigcup_{\gamma\in\Phi^+(\mathfrak{h},\mathfrak{k})} \{x_\gamma - x_{-\gamma}\} \cup \bigcup_{\gamma\in\Phi^+(\mathfrak{h},\mathfrak{k})} \{i(x_\gamma + x_{-\gamma})\}\right) \text{ and}$$

$$\mathfrak{p}_0 = \mathrm{Span}_{\mathbb{R}}\left(\bigcup_{\gamma\in\Gamma_{\mathfrak{p}}^+} \{i(x_\gamma - x_{-\gamma})\} \cup \bigcup_{\gamma\in\Gamma_{\mathfrak{p}}^+} \{x_\gamma + x_{-\gamma}\}\right)$$

although we will not need these subspaces in our description of the fibers.

With the partition of the positive roots, we have the decomposition of $\mathfrak{g}$ into a six dimensional subalgebra

$$\mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\xi\in\Phi(\mathfrak{h},\mathfrak{k})} \mathfrak{g}_\xi$$

that is isomorphic to

$$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{g}_{\pm\alpha} \oplus \mathfrak{g}_{\pm(3\alpha+2\beta)},$$

plus an eight dimensional subspace

$$\mathfrak{p} = \sum_{\xi\in\Gamma_{\mathfrak{p}}} \mathfrak{g}_\xi.$$

The Weyl group, $\mathcal{W}_K$, relative to $\Phi(\mathfrak{h},\mathfrak{k})$ is a dihedral group of order 4. Also, the Weyl group, $\mathcal{W}$, relative to $\Phi(\mathfrak{h},\mathfrak{g})$ is a dihedral group of order 12. Therefore, $\mathcal{W}/\mathcal{W}_K$ has order 3, so there are 3 positive systems Φ_j^+ such that $\Phi^+(\mathfrak{h},\mathfrak{k}) \subset \Phi_j^+ \subset \Phi(\mathfrak{h},\mathfrak{g})$. The three positive system along with their associated simple systems Δ_j are given in Table 3.1. It is clear that the sets Φ_j^+ in Table 3.1 contain $\Phi^+(\mathfrak{h},\mathfrak{k})$, but that they are actually positive systems with the corresponding simple system Δ_j requires a calculation (see Appendix A). When searching for the generic elements, one finds that there are an abundance of generic elements from which to choose. However, we will have many reasons for working with certain choices over others because some make determining the structure of the fibers more difficult. Although

Positive System	Roots	Simple Roots
Φ_1^+	$\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$	$\{\alpha, \beta\}$
Φ_2^+	$\{\alpha, -\beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$	$\{-\beta, \alpha+\beta\}$
Φ_3^+	$\{\alpha, -\beta, -\alpha-\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$	$\{-\alpha-\beta, 3\alpha+2\beta\}$

Table 3.1: Positive and Simple Systems Containing $\Phi^+(\mathfrak{h}, \mathfrak{k})$

Generic Element	dim $K.f_j$
$f_1 = x_{(-1,-1)} + x_{(-2,-1)}$	5
$f_2 = x_{(0,1)} + x_{(-1,-1)}$	6
$f_3 = x_{(-2,-1)} + x_{(0,1)}$	5

Table 3.2: Generic Elements $f_j \in \mathfrak{n}_j^- \cap \mathfrak{p}$

the basic root vectors given above are not generic elements, we can find generic elements that are linear combinations of basic root vectors with nice coefficients. Now, let us introduce some generic elements.

Proposition 3.2. *Let $\mathfrak{n}_j^-$ represent the sum of the root spaces for all roots in $-\Phi_j^+$. Table 3.2 lists a generic element f_j in $\mathfrak{n}_j^- \cap \mathfrak{p}$ along with the dimension of $K.f_j$.*

Naturally, with three possible positive systems, we will separate the discussion and proofs about these generic elements into three cases. As a remark, we will keep a consistent notation using subscripts to relate the closed K-orbits, the generic elements chosen, the $\mathfrak{sl}_2$ triple containing f_j, the Springer fiber components relative to these generic elements, and any other objects that relate to a particular choice of positive system.

Positive System Φ_1^+: As described above, define $\mathfrak{n}_1^-$ to be the sum of the root spaces for roots in $-\Phi_1^+$. Therefore, $\mathfrak{n}_1^- \cap \mathfrak{p} = \mathrm{Span}_{\mathbb{C}}\big(\{x_{(0,-1)}, x_{(-1,-1)}, x_{(-2,-1)}, x_{(-3,-1)}\}\big)$. We will show that f_1 is a generic element by giving an upper bound for the dimension of any orbit in $K.(\mathfrak{n}_1^- \cap \mathfrak{p})$, and then show that f_1 necessarily attains this upper bound.

Proof of Proposition 3.2 (f_1 is generic). The dimension of an orbit $K.f$ is equal to dim $K-$ dim K^f where K^f is the stabilizer subgroup of f in K. Moreover, the dimension of K^f is equal to the dimension of $\mathfrak{k}^f$ where $\mathfrak{k}^f = \{v \in \mathfrak{k} | [v, f] = 0\}$ is the centralizer of f in $\mathfrak{k}$. For the remainder of this chapter, let

$$k = c_1 h_\alpha + c_2 h_\beta + \sum_{\xi \in \Phi(\mathfrak{h},\mathfrak{k})} c_\xi x_\xi$$

denote an arbitrary element of $\mathfrak{k}$. Let us compute the dimension of $\mathfrak{k}^{f_1}$ with f_1 as above. Using the Lie brackets above, $[k, f_1] = 3c_{(-1,0)}x_{(-3,-1)} + (2c_{(-1,0)} - c_1)x_{(-2,-1)} + (c_1 + 2c_{(1,0)} - c_2)x_{(-1,-1)} + 3c_{(1,0)}x_{(0,-1)} + c_{(3,2)}x_{(1,1)} - c_{(3,2)}x_{(2,1)}$, so $[k, f_1] = 0$ if and only if $c_1 = c_2 = c_{(1,0)} = c_{(-1,0)} = c_{(3,2)} = 0$. As the dimension of $\mathfrak{k}^{f_1} = 1$, it follows that $K.f_1$ is a 5-dimensional K-orbit. On the other hand, $\mathfrak{k}^f$ is at least one dimensional for all elements $f \in \mathfrak{n}_1^- \cap \mathfrak{p}$ since any element of the root space $\mathfrak{g}_{-3\alpha-2\beta}$ is in $\mathfrak{k}^f$. Specifically, any negative root added to $-3\alpha - 2\beta$ is not a root and $f \in \mathfrak{n}_1^- \cap \mathfrak{p}$ is a sum of root vectors associated to negative roots only. The dimension of $K.f_1$ is maximal, so by Lemma 1.2, $K.f_1$ is dense in $K.(\mathfrak{n}_1^- \cap \mathfrak{p})$. □

Positive System Φ_2^+: For Φ_2^+, we have $\mathfrak{n}_2^- \cap \mathfrak{p} = \text{Span}_{\mathbb{C}}\big(\{x_{(0,1)}, x_{(-1,-1)}, x_{(-2,-1)}, x_{(-3,-1)}\}\big)$. Since f_1 lives in $\mathfrak{n}_2^- \cap \mathfrak{p}$, one might hope that it defines a generic element in this case as well. However, the dimension of $K.(\mathfrak{n}_2^- \cap \mathfrak{p})$ increases, which forces us to consider a different generic element. In this case, $K.(\mathfrak{n}_2^- \cap \mathfrak{p})$ meets the principal nilpotent orbit in $\mathfrak{g}$, so we chose a principal nilpotent element.

Proof of Proposition 3.2 (f_2 is generic). First, we begin with a general observation about orbits $K.f$ with $f \in \mathfrak{n}^- \cap \mathfrak{p}$ whose dimension happens to be equal to the dimension of K. Since $K.(\mathfrak{n}^- \cap \mathfrak{p})$ has a dense K-orbit, it is always bounded above by the dimension of K.

Hence, it follows that if $K.f$ has the same dimension as K, then it must be the unique orbit in $K.(\mathfrak{n}^- \cap \mathfrak{p})$ of maximal dimension. Therefore, f is necessarily a generic element. In any such case, our discussion in the previous proposition implies that it suffices to show $\mathfrak{k}^f$ has dimension zero. Since $[k, f_2] = -c_{(-3,-2)}x_{(-3,-1)} + 2c_{(-1,0)}x_{(-2,-1)} + (c_1 - c_2)x_{(-1,-1)} + 3c_{(1,0)}x_{(0,-1)} + (2c_2 - 3c_1)x_{(0,1)} - c_{(1,0)}x_{(1,1)} - c_{(3,2)}x_{(2,1)} = 0$ if and only if $c_1 = c_2 = c_{(1,0)} = c_{(-1,0)} = c_{(3,2)} = c_{(-3,-2)} = 0$, it follows that $\mathfrak{k}^{f_2}$ has dimension zero. □

Positive System Φ_3^+: For Φ_3^+, we have $\mathfrak{n}_3^- \cap \mathfrak{p} = \mathrm{Span}_{\mathbb{C}}\big(\{x_{(0,1)}, x_{(1,1)}, x_{(-2,-1)}, x_{(-3,-1)}\}\big)$. It turns out that $K.(\mathfrak{n}_3^- \cap \mathfrak{p})$ has dimension 5, but proving this will require a bit more work than the previous cases. Showing that f_3 is generic will serve as a good illustration for some of the methods that will be used when we pass to F_4.

Proof of Proposition 3.2 (f_3 is generic). Given an arbitrary element

$$f = \sum_{\xi \in -\Phi_3^+ \cap \Gamma_{\mathfrak{p}}} a_\xi x_\xi$$

of $\mathfrak{n}_3^- \cap \mathfrak{p}$, we have $[k, f] = (-3c_1 a_{(-3,-1)} + c_2 a_{(-3,-1)} + 3c_{(-1,0)} a_{(-2,-1)} - c_{(-3,-2)} a_{(0,1)})x_{(-3,-1)} + (-c_1 a_{(-2,-1)} + c_{(1,0)} a_{(-3,-1)} + c_{(-3,-2)} a_{(1,1)})x_{(-2,-1)} + (2c_{(1,0)} a_{(-2,-1)})x_{(-1,-1)} + (-3c_1 a_{(0,1)} + 2c_2 a_{(0,1)} - c_{(3,2)} a_{(-3,-1)} - 3c_{(-1,0)} a_{(1,1)})x_{(0,1)} + (-c_1 a_{(1,1)} + c_2 a_{(1,1)} - c_{(1,0)} a_{(0,1)} + c_{(3,2)} a_{(-2,-1)})x_{(1,1)} + (-2c_{(1,0)} a_{(1,1)})x_{(2,1)}$. Now, the equation $[k, f] = 0$ translates into the matrix equation

$$\begin{bmatrix} -3a_{(-3,-1)} & a_{(-3,-1)} & 0 & 0 & 3a_{(-2,-1)} & -a_{(0,1)} \\ -a_{(-2,-1)} & 0 & a_{(-3,-1)} & 0 & 0 & a_{(1,1)} \\ 0 & 0 & 2a_{(-2,-1)} & 0 & 0 & 0 \\ -3a_{(0,1)} & 2a_{(0,1)} & 0 & -a_{(-3,-1)} & -3a_{(1,1)} & 0 \\ -a_{(1,1)} & a_{(1,1)} & -a_{(0,1)} & a_{(-2,-1)} & 0 & 0 \\ 0 & 0 & -2a_{(1,1)} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_\alpha \\ c_{3\alpha+2\beta} \\ c_{-\alpha} \\ c_{-3\alpha-2\beta} \end{bmatrix} = 0.$$

Since being an element of $\mathfrak{k}^f$ is equivalent to being an element of the nullspace of the relations matrix, the dimension of $K.f$ is completely determined by the rank of the matrix. Observe that rows 3 and 6 are linearly dependent, so the rank is bounded above by 5. Therefore, $K.(\mathfrak{n}_3^- \cap \mathfrak{p})$ is at most 5 dimensional. For our choice of f_3, the matrix becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 3 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and it is clear that the matrix has rank exactly five. Therefore, we have found an element whose K-orbit dimension is maximal, so f_3 is generic. □

4

Springer Fiber Components: Split Real Form G_2^2

In order to describe certain components of the Springer fibers for each generic element defined above, we need to introduce some parabolic subgroups associated to the three positive systems given in the previous chapter. Recall our discussion from Chapter 1 which illustrates that understanding $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ is key to describing the Springer fiber components. It follows that we want to explore which elements of K live in the subset $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$. The defining property for $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ does not imply that it is necessarily a subgroup of K, but we can hope to find certain groups contained in $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ from which we can construct the fibers.

In many classical cases, there are instances where $N(f, \mathfrak{n}^- \cap \mathfrak{p})^{-1}.\mathfrak{b}$ is built up from several groups, and the fibers take the form

$$\psi_{\mathcal{O}}^{-1}(f) = L_n L_{n-1} \cdots L_1.\mathfrak{b}$$

for particular choices of reductive subgroups L_i of K. Such descriptions can be found in [BZ08] for the groups $SU(p,q)$. However, the structure of the fibers can be quite complicated for other types. For type G_2, the fibers are necessarily homogeneous and built from a well-chosen parabolic subgroup of K. Let us now introduce the procedure for constructing parabolic subgroups contained in $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$.

To build these parabolic subgroups, let $I_j = \Phi(\mathfrak{h}, \mathfrak{k}) \cap \Delta_j$ denote the simple compact roots in Φ_j^+, and let $\Phi_{I_j} \subset \Phi(\mathfrak{h}, \mathfrak{k})$ denote the root system associated to I_j with positive roots $\Phi_{I_j}^+ = \Phi_{I_j} \cap \Phi^+(\mathfrak{h}, \mathfrak{k})$. Then define a parabolic subalgebra

$$\mathfrak{q}_j = \left(\mathfrak{h} + \sum_{\gamma \in \Phi_{I_j}} \mathfrak{g}_\gamma\right) + \sum_{\gamma \in \Phi^+(\mathfrak{h},\mathfrak{k}) \setminus \Phi_{I_j}^+} \mathfrak{g}_{-\gamma} = \mathfrak{l}_j + \mathfrak{u}_j^-$$

of $\mathfrak{k}$, and let Q_j (resp., L_j) denote the connected subgroups of K with Lie algebras $\mathfrak{q}_j$ (resp., $\mathfrak{l}_j$). The Springer fiber components for each closed K-orbit $\mathcal{O}$ in the flag variety of G_2 have a nice description, and the remainder of Section 4 will be devoted to proving the following theorem.

Theorem 4.1. *Let $\mathfrak{b}_j$ denote the Borel subalgebra constructed from the positive system Φ_j^+, and let $\mathcal{O}_j = K.\mathfrak{b}_j$ denote the associated closed K-orbit in the flag variety of G_2. For each generic element f_j, $\psi_{\mathcal{O}_j}^{-1}(f_j)$ is isomorphic to $L_j.\mathfrak{b}_j$.*

It was noted above that $\psi_{\mathcal{O}_j}^{-1}(f_j) \simeq N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})^{-1}.\mathfrak{b}_j$, so it suffices to show that

$$L_j.\mathfrak{b}_j = N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p}).\mathfrak{b}_j.$$

Recall that our strategy consists of locating an irreducible component C_f, and then determining how the component group acts on C_f. Using the group L_j defined above, we want to show that $C_{f_j} = L_j.\mathfrak{b}_j$ constitutes an irreducible component in $\psi_{\mathcal{O}_j}^{-1}(f_j)$ for each generic element f_j. The necessary steps to prove this fact represent the content of the following lemma.

Lemma 4.2. *The following properties hold.*

(a) L_j stabilizes $\mathfrak{n}_j^- \cap \mathfrak{p}$, so $L_j \subset N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$

(b) $L_j.\mathfrak{b}_j$ is a closed, irreducible subset of $\psi_{\mathcal{O}_j}^{-1}(f_j)$

(c) The dimension of $L_j.\mathfrak{b}_j$ is equal to the dimension of $\psi^{-1}_{\mathcal{O}_j}(f_j)$.

Proof. To begin, consider the parabolic subgroup of $\mathfrak{g}$ given by

$$\tilde{\mathfrak{q}}_j = \left(\mathfrak{h} + \sum_{\gamma\in\Phi_{I_j}} \mathfrak{g}_\gamma\right) + \sum_{\gamma\in\Phi_j^+\setminus\Phi_{I_j}^+} \mathfrak{g}_{-\gamma} = \tilde{\mathfrak{l}}_j + \tilde{\mathfrak{u}}_j^-.$$

The corresponding parabolic subgroup $\tilde{Q}_j$ of G stabilizes $\tilde{\mathfrak{u}}_j^-$ and K stabilizes $\mathfrak{p}$, so the group $\tilde{Q}_j \cap K$ stabilizes $\tilde{\mathfrak{u}}_j^- \cap \mathfrak{p}$. Since the nilradical $\mathfrak{n}_j^-$ consists of the root spaces for all roots in $-\Phi_j^+$ and $\tilde{\mathfrak{u}}_j^-$ consists of the root spaces for all roots in $-(\Phi_j^+ \setminus \Phi_{I_j}^+)$, it follows that intersecting with $\mathfrak{p}$ corresponds to removing all compact roots from both sets. As a result, $\mathfrak{n}_j^- \cap \mathfrak{p} = \tilde{\mathfrak{u}}_j^- \cap \mathfrak{p}$ ([BZ08, cf. Remark 3.5]), so the group $\tilde{Q}_j \cap K$ stabilizes $\mathfrak{n}_j^- \cap \mathfrak{p}$. Finally, $\tilde{\mathfrak{l}}_j = \mathfrak{l}_j$ and $L_j \subset \tilde{Q}_j \cap K$, so L_j stabilizes $\mathfrak{n}_j^- \cap \mathfrak{p}$. To prove part (b), note that L_j is connected, hence irreducible, so part (a) implies that $L_j.\mathfrak{b}_j$ is an irreducible subvariety of $\psi^{-1}_{\mathcal{O}_j}(f_j)$. Since $\mathfrak{l}_j \cap \mathfrak{b}_j$ is a Borel subalgebra of $\mathfrak{l}_j$ relative to the reductive subgroup L_j, it follows that $L_j.\mathfrak{b}_j$ is a flag variety for L_j, hence $L_j.\mathfrak{b}_j$ is closed in $\mathcal{B}$. For part (c), we begin with the observation that $1/2 \dim G.f = \dim K.f$ ([CM93, cf. Remark 9.5.2]), so the dimension formula of Proposition 1.5 only requires us to know the dimension of $\mathfrak{k}^f$. Specifically, $\dim K = \dim \mathcal{B} = 6$, so $\dim \psi^{-1}_{\mathcal{O}_j}(f_j) = 6 - \dim \mathfrak{k}^f$. From Proposition 3.2, $\psi^{-1}_{\mathcal{O}_2}(f_2)$ has dimension zero, so part (c) is trivial for that case. As for the cases where $j = 1$ or 3, we need to show that $L_j.\mathfrak{b}_j$ is one dimensional. Let γ_j denote the only positive root in Φ_{I_j}, and let $A_j = Exp(\mathbb{C} \cdot x_{\gamma_j})$ denote the subgroup of L_j having dimension one. Since A_j completely misses $B_j = Stab_G(\mathfrak{b}_j)$, the dimension of $A_j.\mathfrak{b}_j$ is at least one, hence the dimension of $L_j.\mathfrak{b}_j$ is at least one as well. By Propositions 3.2 and 1.5, $L_j.\mathfrak{b}_j$ has the same dimension as $\psi^{-1}_{\mathcal{O}_j}(f_j)$. □

Now that we have found these irreducible pieces, we know from Proposition 1.5 that $\psi^{-1}_{\mathcal{O}_j}(f_j)$ consists of the orbit under the component group of f_j. Fortunately, the components groups are classified and will be extensively referenced in what follows. However, even though the component groups are known finite groups, finding coset representatives is quite challenging. Specifically, the component groups are known abstractly, but we do not know the elements explicitly for a given generic element. Without knowing more about them, we will not know how many irreducible components occur in a given fiber. One idea is to hope that finding elements of the stabilizer subgroup amounts to finding solutions to or making deductions from some algebraic equations. Indeed, we do have embeddings in terms of matrices, but the large dimensions can complicate things.

Alternatively, one may hope to find at least where the coset representatives are located inside of K in order to understand the fibers. Unfortunately, the exceptional groups carry an added level of difficulty given that they lack some of the algebraic clarity that the classical types possess. In many cases, we will need to utilize as much of the Lie algebra structure as possible. For G_2, the fixed point subgroup K is very nice because we will be able to use some SL_2 theory, whereas for F_4, we will need to dig a bit deeper. Before embarking on a journey to understand K, we state a useful theorem found in [Loo69, Chapter IV].

Theorem 4.3. *Let G be a connected Lie group and let θ be an involutive automorphism of G. Then the fixed point set K of θ has finitely many connected components, and the quotient of K by its identity component $K_\circ$ is isomorphic to the direct product of cyclic groups of order two. If G is simply connected, then K is connected.*

To compute the component groups, we will utilize our matrix representation π. Since $G = G_2$ is necessarily simply connected, we know from the above theorem that K is connected as a subgroup of G. Therefore, K must be isomorphic to the connected subgroup K' of $GL(7, \mathbb{C})$ generated by the exponential map whose Lie algebra $\pi(\mathfrak{k})$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. From our knowledge of covering groups, this implies that K is isomorphic to the quotient of $SL_2 \times SL_2$ by some discrete subgroup C of the center of $SL_2 \times SL_2$. The center of SL_2 is $\pm I_2$ where $I_2 \in SL_2$ denotes the 2×2 identity matrix, so C must be one of $\{I_2, I_2\}, \{(\pm I_2, \pm I_2)\}$, $\{(\pm I_2, I_2)\}$, $\{(I_2, \pm I_2)\}$, or $\{\pm(I_2, I_2)\}$.

Since we will utilize an assortment of $\mathfrak{sl}_2$ triples throughout this paper, we need to introduce a consistent notation for such triples. From now on, such triples will be written as a set $\{h, e, f\}$ or $\{h_j, e_j, f_j\}$ with an $\mathfrak{sl}_2$ correspondence given by:

$$h \leftrightarrow h_j \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad e \leftrightarrow e_j \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } \quad f \leftrightarrow f_j \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $\mathfrak{k}_\alpha$ (resp., $\mathfrak{k}_{3\alpha+2\beta}$) denote the copy of $\mathfrak{sl}_2$ in $\mathfrak{k}$ associated to the root α (resp., $3\alpha + 2\beta$). Using some SL_2 theory, let us define maps from SL_2 into $GL(7, \mathbb{C})$ whose image lies in K'. For $\mathfrak{k}_\alpha$, let $e = x_{(1,0)}$, $f = x_{(-1,0)}$, and $h = h_{(1,0)}$ denote such a triple, and recall the following identifications under π:

$$\pi(e) = \pi(x_{(1,0)}) = E^7_{1,2} + 2E^7_{3,4} + E^7_{4,5} + E^7_{6,7} \text{ and}$$
$$\pi(f) = \pi(x_{(-1,0)}) = E^7_{2,1} + E^7_{4,3} + 2E^7_{5,4} + E^7_{7,6}.$$

By using linearity of the bracket and the equation $[E^n_{i,j}, E^n_{k,l}] = \delta_{j,k}E^n_{i,l} - \delta_{l,i}E^n_{k,j}$ ([Hum72, Section 1]), an easy calculation gives

$$\pi(h) = \pi(h_{(1,0)}) = E^7_{1,1} - E^7_{2,2} + 2E^7_{3,3} - 2E^7_{5,5} + E^7_{6,6} - E^7_{7,7}.$$

Now, $\mathbb{C}^7$ decomposes into 2 irreducible $\mathfrak{sl}_2$ representations each of dimension two and 1 irreducible $\mathfrak{sl}_2$ representation of dimension three. If $\{b_1, b_2, \ldots, b_n\}$ represents a standard basis for $\mathbb{C}^n$, then this decomposition is given by

$$\mathrm{Span}_{\mathbb{C}}\big(\{b_1, b_2\}\big) \oplus \mathrm{Span}_{\mathbb{C}}\big(\{b_3, b_4, b_5\}\big) \oplus \mathrm{Span}_{\mathbb{C}}\big(\{b_6, b_7\}\big).$$

Since SL_2 is simply connected, we know that there is a corresponding SL_2 representation $\Pi_1 : SL_2 \to GL(7, \mathbb{C})$ with image in K' defined by $\Pi_1(Exp(X)) = Exp(\pi(X))$ for $X \in \mathfrak{k}_\alpha$.

By [FH96, Section 23], this map is uniquely determined by its derivative map $\pi|_{\mathfrak{k}_\alpha}$ because SL_2 is simply connected. Motivated by the $\mathfrak{sl}_2$ representations above, define a map $\phi_1 : SL_2 \to GL(7,\mathbb{C})$ where

$$\phi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 2ab & b^2 & 0 & 0 \\ 0 & 0 & ac & ad+bc & bd & 0 & 0 \\ 0 & 0 & c^2 & 2cd & d^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & c & d \end{bmatrix}.$$

Thinking of SL_2 representations in terms of SL_2 acting on homogenous two variable polynomials of dimension n motivates defining ϕ_1 in this manner. To show that we have found the correct part of K' lying above $\mathfrak{k}_\alpha$, we will need to show that $\phi_1 = \Pi_1$. It suffices to show that $d\phi_1$ is equal to $d\Pi_1 = \pi|_{\mathfrak{k}_\alpha}$ since a homomorphism between closed linear groups is determined by its derivative. Since $d\phi_1$ and π are both linear, it is only necessary to show this property on a basis. As a result, let

$$c_e(t) = Exp\left(t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

$$c_f(t) = Exp\left(t\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \text{ and}$$

$$c_h(t) = Exp\left(t\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Applying ϕ_1 gives us the following matrices in $GL(7,\mathbb{C})$:

$$\phi_1(c_e(t)) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2t & t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \phi_1(c_f(t)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 1 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 2t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 1 \end{bmatrix}, \text{ and}$$

$$\phi_1(c_h(t)) = \begin{bmatrix} e^t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \end{bmatrix}.$$

Differentiating with respect to t and evaluating at $t = 0$, we see that $d\phi_1(e) = \pi(e)$, $d\phi_1(f) = \pi(f)$, and $d\phi_1(h) = \pi(h)$, hence $d\phi_1 = \pi|_{\mathfrak{k}_\alpha}$. Since $\phi_1 = \Pi_1$, the image of ϕ_1 lies inside of K' and is isomorphic to SL_2.

Now, consider the subalgebra $\mathfrak{k}_{3\alpha+2\beta}$. The $\mathfrak{sl}_2$ correspondence here is $e = x_{(3,2)}$, $f = x_{(-3,-2)}$, and $h = h_{(3,2)}$, so we have

$$\pi(e) = \pi(x_{(3,2)}) = -E^7_{1,6} - E^7_{2,7},$$

$$\pi(f) = \pi(x_{(-3,-2)}) = -E^7_{6,1} - E^7_{7,2}, \text{ and}$$

$$\pi(h) = \pi(h_{(3,2)}) = E^7_{1,1} + E^7_{2,2} - E^7_{6,6} - E^7_{7,7}.$$

We have $\mathbb{C}^7$ expressed as two copies of the natural representation and three copies of the trivial representation:

$$\operatorname{Span}_{\mathbb{C}}\big(\{b_1, b_6\}\big) \oplus \operatorname{Span}_{\mathbb{C}}\big(\{b_3\}\big) \oplus \operatorname{Span}_{\mathbb{C}}\big(\{b_4\}\big) \oplus \operatorname{Span}_{\mathbb{C}}\big(\{b_5\}\big) \oplus \operatorname{Span}_{\mathbb{C}}\big(\{b_2, b_7\}\big).$$

Again, there is a corresponding SL_2 representation $\Pi_2 : SL_2 \to GL(7,\mathbb{C})$ defined by $\Pi_2(Exp(X)) = Exp(\pi(X))$ for $X \in \mathfrak{k}_{3\alpha+2\beta}$. Define $\phi_2 : SL_2 \to GL(7,\mathbb{C})$ where

$$\phi_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & a & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & d & 0 \\ 0 & -c & 0 & 0 & 0 & 0 & d \end{bmatrix}.$$

A similar argument shows that the image of ϕ_2, which is also isomorphic to SL_2, lies inside of K' since $\phi_2 = \Pi_2$.

To determine C and therefore $K \simeq (SL_2 \times SL_2)/C$, we need to understand the solutions to the equation $Exp(\pi|_{\mathfrak{k}}(X)) = I$ for $X \in \mathfrak{h} \subset \mathfrak{k}$. Exploring these solutions is sufficient since the discrete subgroup must lie inside of the Cartan subgroup $H = Exp(\pi(\mathfrak{h}))$ ([FH96, cf. Section 23]). An arbitrary element of this Cartan subgroup takes the form

$$\begin{bmatrix} ab & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^{-1}b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ab^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^{-1}b^{-1} \end{bmatrix}$$

for a and b in $\mathbb{C}^*$. Then the equation $Exp(\pi|_{\mathfrak{k}}(X)) = I$ gives us equations $ab = 1, a^{-1}b = 1, a^2 = 1, a^{-2} = 1, ab^{-1} = 1$, and $a^{-1}b^{-1} = 1$ whose only solutions are $a = b = 1$ or $a = b = -1$. It follows that $C = \pm(I_2, I_2) \subset SL_2 \times SL_2$. The following proposition summarizes this discussion.

Proposition 4.4. *The fixed point subgroup K is isomorphic to $(SL_2 \times SL_2)/\pm(I_2, I_2)$ and the map $\phi : SL_2 \times SL_2 \to GL(7, \mathbb{C})$ given by $\phi(x, y) = \phi_1(x)\phi_2(y)$ induces an isomorphism of K with the connected subgroup K' of $GL(7, \mathbb{C})$ with Lie algebra $\pi(\mathfrak{k}) \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.*

Proof. The first statement about K has already been proved in the discussion above. For the second statement, note that representations of K are completely determined by representations of $SL_2 \times SL_2$, which are trivial on C. It is clear that ϕ is trivial on C, so it induces a representation $\phi_K : K \to GL(7, \mathbb{C})$. The injectivity of ϕ_K follows from the computation for C given above, and the surjectivity of ϕ_K onto K' comes from the fact that the images of ϕ_1 and ϕ_2 are the two copies of SL_2 that cover K'. Therefore, ϕ_K yields the desired isomorphism, and provides the embedding of K into $GL(7, \mathbb{C})$. □

For each generic element f_j, let us identify the nilpotent orbit associated to it. This will allow us to look up which type of component group is associated to f_j. By the Jacobson-Morosov theorem described in [KR71], f_j may be embedded in an $\mathfrak{sl}_2$ triple $\{h_j, e_j, f_j\}$ with $e_j, f_j \in \mathfrak{p}$ and $h_j = [e_j, f_j] \in \mathfrak{k}$. Note that such triples with the nilpotent elements in $\mathfrak{p}$ and semisimple element in $\mathfrak{k}$ are refered to as *normal* triples. Dokovic gives a complete classification of all possible nilpotent K-orbits in $\mathfrak{p}$ in a series of tables found in [Dok88]. Since there are instances of multiple K-orbits in $\mathcal{N}_\theta$ with the same dimensions, we will need to introduce another piece of data from these tables that distinguishes the orbits. Altering the notation from Dokovic's slightly to keep track of our generic elements, define the subset

$$\mathfrak{g}(j, 2) = \{X \in \mathfrak{p} : [h_j, X] = 2X\}.$$

Then the dimension of the orbit together with the dimension of $\mathfrak{g}(j,2)$ will tell us precisely which orbit contains f_j. When one references the tables, observe that Dokovic denotes the subset $\mathfrak{g}(j,2)$ simply by $\mathfrak{g}(1,2)$ since there is no need to give a notation to any specific nilpotent in that case. Once we identify the orbit for f_j, we can cross-reference with [Kin92] to determine the related component group.

From the multiplication tables, we see that Table 4.1 contains choices for the nilpositive and semisimple elements of $\mathfrak{sl}_2$ triples containing f_j. For each generic element f_j, we have $dim\ \mathfrak{g}(1,2) = 4, dim\ \mathfrak{g}(2,2) = 2$, and $dim\ \mathfrak{g}(3,2) = 2$ (see Appendix A), so it follows from [Kin92] that the component groups $A_K(f_j)$ are as listed in Table 4.2. In general, there are several nilpotent K-orbits of a particular dimension, so such data is usually required in order to identity the component group. Since there is only one maximal 6-dimensional nilpotent K-orbit according to [Dok88], $A_K(f_2)$ is trivial (see [Kin92]). Consequently, we could have omitted the computation of the corresponding $\mathfrak{sl}_2$ triple and the dimension of $\mathfrak{g}(2,2)$. Now, let us begin the search for representatives of the component groups. To describe the component group for f_1, define generators r and s for $S_3 \subset (SL_2 \times SL_2)/\pm(I_2, I_2)$ where

$$r = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } s = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \times \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

and note that the relations $r^3 = s^2 = 1$ and $srs = r^{-1}$ for S_3 are satisfied. There is no need

j	Nilpositive Element e_j	Semisimple Element h_j
1	$-4/3x_{(0,1)} + 2/3x_{(1,1)} + 2/3x_{(2,1)} - 4/3x_{(3,1)}$	$2h_\alpha + 4h_\beta$
2	$10x_{(0,-1)} + 6x_{(1,1)}$	$6h_\alpha + 8h_\beta$
3	$x_{(0,-1)} + x_{(2,1)}$	$2h_\alpha + 2h_\beta$

Table 4.1: $\mathfrak{sl}_2$ triples $\{e_j, f_j, h_j\}$

Component Group	Isomorphism Type
$A_K(f_1)$	S_3 (Symmetric Group)
$A_K(f_2)$	1
$A_K(f_3)$	$\mathbb{Z}_2$

Table 4.2: Component Groups $A_K(f_j)$

to work hard for f_2 since the component group is trivial. As for f_3, choose the generator

$$z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \times \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

for a subgroup

$$\mathbb{Z}_2 \subset (SL_2 \times SL_2)/\pm(I_2, I_2).$$

To motivate these elements, recall that our effort to describe K and f_j in terms of matrices has its benefits because we are able to use linear algebra to unearth these generators. Different choices of generic elements yield isomorphic components groups, but the generic elements provided above simplify the algebra and produce nice representatives. There are actually other methods to discover coset representatives and work with the component groups that will be used for type F_4, but the fixed point subgroup K for G_2 allows us to exploit SL_2 representations.

Now, it is easy to show that embedding these generators according to Proposition 4.4 give elements which stabilize f_j (see Appendix A). However, we must ask ourselves whether these elements actually give a complete set of coset representatives for the component group. We must be careful that we have not found elements that belong to the same connected component. Fortunately, the following lemma will allow us to navigate this issue, but first we need a definition. Note that the definition and lemma are true for any fixed point subgroup K of an involution of G.

Definition 4.5. For any $\mathfrak{sl}_2$ triple $\{h, e, f\}$, the subgroup $K^{\{h,e,f\}}$ will denote the stabilizer subgroup in K of the triple $\{h, e, f\}$.

Lemma 4.6. *The component group* $A_K(f) \simeq K^{\{h,e,f\}}/K_\circ^{\{h,e,f\}}$ *where* $K_\circ^{\{h,e,f\}}$ *denotes the identity component of* $K^{\{h,e,f\}}$.

Proof. See Lemma 1.5 in [Kin92] for the details. Note that in his paper the fixed point subgroup is denoted G^θ. The idea is that the stabilizer subgroup can be separated into a semidirect product of $K^{\{h,e,f\}}$ with a connected unipotent part, which disappears in the quotient group. □

The lemma illustrates that finding elements stabilizing f_j amounts to finding elements that stabilize a normal triple containing f_j. One advantage to this approach is that finding $K^{\{h_j,e_j,f_j\}}$ is generally easier than finding K^{f_j}. Our motivation involves the fact that the tables in [Kin92] actually give us the isomorphism type of $K^{\{h,e,f\}}$ for most nilpotent K-orbits when K is connected. In our case, $K^{\{h_j,e_j,f_j\}}$ is the same as the component group S_3 (resp., 1, $\mathbb{Z}_2$) for the generic element f_1 (resp., f_2, f_3). The elements r, s, and z stabilize their respective triples (see Appendix A), so these elements are precisely coset representatives for $A_K(f_j)$. Now, we have all of the tools necessary to finish our main theorem in this section.

Proof of Theorem 4.1 . The only part we have left to prove according to Proposition 1.5 and Lemma 4.2 is that

$$\bigcup_{k \in A_K(f_j)} k.L_j.\mathfrak{b}_j = L_j.\mathfrak{b}_j.$$

First, notice that $Q_j.\mathfrak{b}_j = L_j.\mathfrak{b}_j$ since the part of Q_j that is disjoint from L_j is contained in the stabilizer of $\mathfrak{b}_j$ in K. Therefore, it suffices to show that the component group is contained in Q_j. For $j = 2$, there is nothing to prove since $A_K(f_2)$ is trivial. For $j = 1$, the direct factors of r and s coming from the portion of K lying above $\mathfrak{k}_\alpha$ live in $L_1 \subset Q_1$ since α is in I_1. On the other hand, the direct factors of r and s coming from the portion of K lying above $\mathfrak{k}_{3\alpha+2\beta}$ live in the Cartan subgroup $H \subset Q_1$ corresponding to $\mathfrak{h}$. Since both factors in the direct products for r and s are contained in Q_1, it follows that r, s, and any powers are as well. For $j = 3$, observe that z is actually an element of H, hence an element of Q_3. $\square$

5

The Exceptional Lie Algebra $\mathfrak{f}_4$

To begin work on the Springer fiber components for the two noncompact real forms for type F_4, we need to build the Lie algebra. Let $\mathfrak{h}$ denote a fixed Cartan subalgebra of $\mathfrak{g} = \mathfrak{f}_4$, and let $\Phi(\mathfrak{h}, \mathfrak{g})$ denote a root system relative to $\mathfrak{h}$. Then write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\xi \in \Phi(\mathfrak{h},\mathfrak{g})} \mathfrak{g}_\xi$$

for the root space decomposition of $\mathfrak{g}$ into its root spaces $\mathfrak{g}_\xi$ and four dimensional Cartan subalgebra $\mathfrak{h}$. Let $\Delta(\mathfrak{h}, \mathfrak{g}) = \{\alpha, \beta, \gamma, \delta\}$ denote a base for $\Phi(\mathfrak{h}, \mathfrak{g})$ where α and β constitute the long roots, while γ and δ constitute the short roots. Figure 5.1 gives the corresponding Dynkin diagram for $\mathfrak{f}_4$.

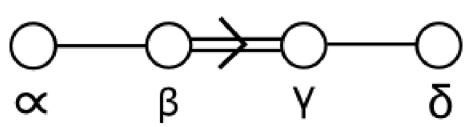


Figure 5.1: Dynkin Diagram for $\mathfrak{f}_4$

Using the algorithm for determining root strings found in [Hum72, Section 10.1], we see that the positive roots relative to $\Delta(\mathfrak{h}, \mathfrak{g})$ for type $\mathfrak{f}_4$ are precisely those roots in Table 5.1.

Let

$$\bigcup_{\xi \in \Phi(\mathfrak{h},\mathfrak{g})} \{x_\xi\} \cup \bigcup_{\xi \in \Delta(\mathfrak{h},\mathfrak{g})} \left\{ h_\xi = [x_\xi, x_{-\xi}] \right\}$$

α	β	γ
δ	$\alpha+\beta$	$\beta+\gamma$
$\gamma+\delta$	$\alpha+\beta+\gamma$	$\beta+2\gamma$
$\beta+\gamma+\delta$	$\alpha+\beta+2\gamma$	$\alpha+\beta+\gamma+\delta$
$\beta+2\gamma+\delta$	$\alpha+2\beta+2\gamma$	$\alpha+\beta+2\gamma+\delta$
$\beta+2\gamma+2\delta$	$\alpha+2\beta+2\gamma+\delta$	$\alpha+\beta+2\gamma+2\delta$
$\alpha+2\beta+3\gamma+\delta$	$\alpha+2\beta+2\gamma+2\delta$	$\alpha+2\beta+3\gamma+2\delta$
$\alpha+2\beta+4\gamma+2\delta$	$\alpha+3\beta+4\gamma+2\delta$	$2\alpha+3\beta+4\gamma+2\delta$

Table 5.1: Roots $\Phi(\mathfrak{h}, \mathfrak{g})$ for $\mathfrak{f}_4$

represent a Chevalley basis for $\mathfrak{g}$ where x_ξ denotes a basis vector in $\mathfrak{g}_\xi$ for each

$$\xi = i\alpha + j\beta + k\gamma + \ell\delta \in \Phi(\mathfrak{h}, \mathfrak{g}).$$

Since we may find it useful to write $x_{(i,j,k,\ell)}$ instead of x_ξ, the two notations should be freely interchanged. Lastly, we may refer to the elements of $\mathfrak{h}$ generated by the nonsimple root vectors, so let h_ξ denote those elements $[x_\xi, x_{-\xi}]$ for all $\xi \in \Phi^+(\mathfrak{h}, \mathfrak{g})$.

First, we choose the simple root vectors as prescribed by the Cartan matrix, and then use the Lie bracket to generate all of $\mathfrak{g}$. Let

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

be the Cartan matrix for $\mathfrak{f}_4$. Then choose simple root vectors satisfying the relations in Table 5.2, and then use the Lie bracket to extend to a basis for $\mathfrak{g}$ (see Table 5.3). Note that this basis is a Chevalley type basis that agrees with the mathematical software package

$[h_{(1,0,0,0)}, x_{(1,0,0,0)}] = 2x_{(1,0,0,0)}$	$[h_{(1,0,0,0)}, x_{(0,1,0,0)}] = -x_{(0,1,0,0)}$
$[h_{(1,0,0,0)}, x_{(0,0,1,0)}] = 0$	$[h_{(1,0,0,0)}, x_{(0,0,0,1)}] = 0$
$[h_{(0,1,0,0)}, x_{(1,0,0,0)}] = -x_{(1,0,0,0)}$	$[h_{(0,1,0,0)}, x_{(0,1,0,0)}] = 2x_{(0,1,0,0)}$
$[h_{(0,1,0,0)}, x_{(0,0,1,0)}] = -x_{(0,0,1,0)}$	$[h_{(0,1,0,0)}, x_{(0,0,0,1)}] = 0$
$[h_{(0,0,1,0)}, x_{(1,0,0,0)}] = 0$	$[h_{(0,0,1,0)}, x_{(0,1,0,0)}] = -2x_{(0,1,0,0)}$
$[h_{(0,0,1,0)}, x_{(0,0,1,0)}] = 2x_{(0,0,1,0)}$	$[h_{(0,0,1,0)}, x_{(0,0,0,1)}] = -x_{(0,0,1,0)}$
$[h_{(0,0,0,1)}, x_{(1,0,0,0)}] = 0$	$[h_{(0,0,0,1)}, x_{(0,1,0,0)}] = 0$
$[h_{(0,0,0,1)}, x_{(0,0,1,0)}] = -x_{(0,0,1,0)}$	$[h_{(0,0,0,1)}, x_{(0,0,0,1)}] = 2x_{(0,0,1,0)}$

Table 5.2: Relations for $\mathfrak{f}_4$

GAP. Recall that if ξ, ϵ and $\xi + \epsilon$ are roots such that $[x_\xi, x_\epsilon] = c_{\xi,\epsilon} x_{\xi+\epsilon}$, then $c_{\xi,\epsilon} = -c_{-\xi,-\epsilon}$. Therefore, we will omit half of the bracket relations. The complete multiplication tables are given in Appendix E for reference.

Recall that in Chapter 2, we introduced a faithful representation of $\mathfrak{g}_2 \subset \mathfrak{gl}(7, \mathbb{C})$ as a means to better understand the fixed point subgroup K. This representation proved useful because we were able to classify K as well as find the nontrivial component groups in a concrete manner. Naturally, we would like to implement that strategy for F_4 as well. However, the jump in dimension makes viewing $K \subset F_4$ in terms of matrices more complicated as the smallest faithful irreducible representation of $\mathfrak{f}_4$ is 26-dimensional. Of course, we know the isomorphism type of the Lie algebra of K, but finding an actual embedding of K in $GL(26, \mathbb{C})$ becomes an enormous task. Fortunately, the fibers can be described without relying heavily on a representation of F_4 provided we make good choices for the generic elements.

The methods that will be used for $\mathfrak{f}_4$ could have been implemented in $\mathfrak{g}_2$ as well, but we would have lost the information gained about $K \subset G_2$. As for G_2, many of the fibers $\psi_{\mathcal{O}}^{-1}(f)$ for F_4 have one component, but we will prove this with other strategies. If we can show that nontrivial component groups are necessarily contained in the groups used to build the fibers

$x_{(1,1,0,0)} = -[x_{(1,0,0,0)}, x_{(0,1,0,0)}]$	$x_{(0,1,1,0)} = [x_{(0,1,0,0)}, x_{(0,0,1,0)}]$
$x_{(0,0,1,1)} = [x_{(0,0,1,0)}, x_{(0,0,0,1)}]$	$x_{(1,1,1,0)} = [x_{(1,1,0,0)}, x_{(0,0,1,0)}]$
$x_{(0,1,2,0)} = 1/2[x_{(0,1,1,0)}, x_{(0,0,1,0)}]$	$x_{(0,1,1,1)} = [x_{(0,1,1,0)}, x_{(0,0,0,1)}]$
$x_{(1,1,2,0)} = 1/2[x_{(1,1,1,0)}, x_{(0,0,1,0)}]$	$x_{(1,1,1,1)} = [x_{(1,1,1,0)}, x_{(0,0,0,1)}]$
$x_{(0,1,2,1)} = [x_{(0,1,2,0)}, x_{(0,0,0,1)}]$	$x_{(1,2,2,0)} = [x_{(1,1,2,0)}, x_{(0,1,0,0)}]$
$x_{(1,1,2,1)} = [x_{(1,1,2,0)}, x_{(0,0,0,1)}]$	$x_{(0,1,2,2)} = 1/2[x_{(0,1,2,1)}, x_{(0,0,0,1)}]$
$x_{(1,2,2,1)} = [x_{(1,1,2,1)}, x_{(0,1,0,0)}]$	$x_{(1,1,2,2)} = 1/2[x_{(1,1,2,1)}, x_{(0,0,0,1)}]$
$x_{(1,2,3,1)} = [x_{(1,2,2,1)}, x_{(0,0,1,0)}]$	$x_{(1,2,2,2)} = 1/2[x_{(1,2,2,1)}, x_{(0,0,0,1)}]$
$x_{(1,2,3,2)} = [x_{(1,2,3,1)}, x_{(0,0,0,1)}]$	$x_{(1,2,4,2)} = 1/2[x_{(1,2,3,2)}, x_{(0,0,1,0)}]$
$x_{(1,3,4,2)} = -[x_{(1,2,4,2)}, x_{(0,1,0,0)}]$	$x_{(2,3,4,2)} = -[x_{(1,3,4,2)}, x_{(1,0,0,0)}]$

Table 5.3: Lie Brackets for $\mathfrak{f}_4$

$\psi_{\mathcal{O}}^{-1}(f)$, then we will eliminate the need to actually find the elements representing the component group. However, one of the fibers $\psi_{\mathcal{O}}^{-1}(f)$ for the split real form of F_4 consists of two irreducible components, so we will be looking for generators of the component group. The generator will be defined independent of any representation, but building a representation of $\mathfrak{f}_4$ does offer some insight into finding the elements of $\mathfrak{k}$ defining the generator.

There is only one complex connected simple Lie group of type F_4 ([Ale05, cf. Proposition 4.1]). It follows that this group is necessarily simply connected. Therefore, we can build a representation of $G = F_4$ in $GL(26, \mathbb{C})$ from a faithful representation of $\mathfrak{g} = \mathfrak{f}_4$ in $\mathfrak{gl}(26, \mathbb{C})$. Let M^{T} represent the matrix transpose for any matrix M. Following [HRT01], we'll choose the representation $\pi : \mathfrak{g} \to \mathfrak{gl}(26, \mathbb{C})$ with generators for a Chevalley basis given in Table E.4. Just like the $\mathfrak{g}_2$ case, these generators agree completely with the generators given above purely in terms of bracket relations.

6

Generic Elements for F_4

6.1 Generic Elements: Split Real Form F_4^4

In Chapter 5, we discussed that there are two different noncompact real forms of the complex Lie algebra of type $\mathfrak{f}_4$. Our next goal for this paper will be to give results similar to the results given in type G_2 for both of these forms. Let us begin by building the Cartan decomposition for the split real form F_4^4. To build the split real form, define an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by first defining $\theta(x_\alpha) = -x_\alpha$, $\theta(x_\beta) = x_\beta$, $\theta(x_\gamma) = x_\gamma$, and $\theta(x_\delta) = x_\delta$ on the simple root vectors, and then extend to all of $\mathfrak{g}$ so that θ commutes with the Lie bracket. Again, this is illustrated by a Vogan diagram (see Figure 6.2) where a root vector is in $\mathfrak{k}$ whenever a simple root is not painted, and a root vector is in $\mathfrak{p}$ whenever a simple root is painted.

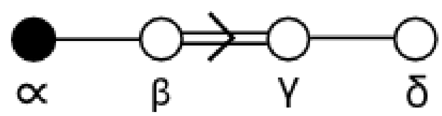


Figure 6.1: Vogan diagram for the split real form of $\mathfrak{f}_4$

As in the $\mathfrak{g}_2$ case, we have a partition of the roots into the set of compact roots $\Phi(\mathfrak{h}, \mathfrak{k})$ and the set of noncompact roots $\Gamma_{\mathfrak{p}}$. A simple calculation shows that the positive roots in $\Phi(\mathfrak{h}, \mathfrak{k})$ are given in Table 6.1 and the positive roots in $\Gamma_{\mathfrak{p}}$ are given in Table 6.2. Finally,

β $\beta+2\gamma$	γ $\beta+\gamma+\delta$	δ $\beta+2\gamma+\delta$	$\beta+\gamma$ $\beta+2\gamma+2\delta$	$\gamma+\delta$ $2\alpha+3\beta+4\gamma+2\delta$

Table 6.1: Compact Roots $\Phi^+(\mathfrak{h},\mathfrak{k})$

α $\alpha+\beta+\gamma+\delta$ $\alpha+\beta+2\gamma+2\delta$ $\alpha+2\beta+4\gamma+2\delta$	$\alpha+\beta$ $\alpha+2\beta+2\gamma$ $\alpha+2\beta+3\gamma+\delta$ $\alpha+3\beta+4\gamma+2\delta$	$\alpha+\beta+\gamma$ $\alpha+\beta+2\gamma+\delta$ $\alpha+2\beta+2\gamma+2\delta$	$\alpha+\beta+2\gamma$ $\alpha+2\beta+2\gamma+\delta$ $\alpha+2\beta+3\gamma+2\delta$

Table 6.2: Noncompact Roots $\Gamma_{\mathfrak{p}}^+$

note that we have a decomposition of $\mathfrak{g}$ into a 24-dimensional subalgebra

$$\mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\xi\in\Phi(\mathfrak{h},\mathfrak{k})} \mathfrak{g}_\xi \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(6,\mathbb{C}),$$

plus a 28-dimensional subspace

$$\mathfrak{p} = \sum_{\xi\in\Gamma_{\mathfrak{p}}} \mathfrak{g}_\xi.$$

See [Kna02] for the complete details about this real form. In particular, the subspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ will be omited as they are defined the same way as for type G_2^2.

The Weyl group for $\mathfrak{f}_4$ relative to $\Phi(\mathfrak{h},\mathfrak{g})$, denoted $\mathcal{W}$, is a group of order 1152. Also, the roots corresponding to the copy of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{k}$ are the two roots $2\alpha+3\beta+4\gamma+2\delta$ and $-2\alpha-3\beta-4\gamma-2\delta$ whose root spaces $\mathfrak{g}_{2\alpha+3\beta+4\gamma+2\delta}$ and $\mathfrak{g}_{-2\alpha-3\beta-4\gamma-2\delta}$ commute with the other root spaces $\mathfrak{g}_\xi$ for roots $\xi \in \Phi(\mathfrak{h},\mathfrak{k}) \setminus \pm\{2\alpha+3\beta+4\gamma+2\delta\}$. As a result, the Weyl group relative to $\Phi(\mathfrak{h},\mathfrak{k})$, denoted $\mathcal{W}_K$, is built from the Weyl group of A_1 together with the Weyl group of C_3. Since $\mathcal{W}_k$ is a group of order $2! \times (2^3 \times 3!) = 96$, $\mathcal{W}/\mathcal{W}_K$ has order 12, so there are twelve positive systems Φ_j^+ such that $\Phi^+(\mathfrak{h},\mathfrak{k}) \subset \Phi_j^+ \subset \Phi(\mathfrak{h},\mathfrak{g})$. Since the positive

Root System	$(n_{\xi_1}, n_{\xi_2}, \dots, n_{\xi_{14}})$
Φ_2^+	$(-1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_3^+	$(-1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_4^+	$(-1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_5^+	$(-1, -1, -1, 1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_6^+	$(-1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_7^+	$(-1, -1, -1, -1, 1, -1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_8^+	$(-1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_9^+	$(-1, -1, -1, -1, -1, 1, -1, 1, 1, 1, 1, 1, 1, 1)$
Φ_{10}^+	$(-1, -1, -1, -1, -1, 1, -1, 1, -1, 1, 1, 1, 1, 1)$
Φ_{11}^+	$(-1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1)$
Φ_{12}^+	$(-1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1)$

Table 6.3: Scalars n_{ξ_j} for Φ_j^+

systems must contain $\Phi^+(\mathfrak{h}, \mathfrak{k})$, we know that such a positive system has the form

$$\Phi^+(\mathfrak{h}, \mathfrak{k}) \cup \bigcup_{\xi \in \Gamma_{\mathfrak{p}}^+} \{n_\xi \xi\}$$

for particular scalars $n_\xi \in \{1, -1\}$. Use the ordering for $\Gamma_{\mathfrak{p}}^+$ introduced across the rows of Table 6.2 to enumerate the roots $\{\xi_1, \xi_2, \dots, \xi_{14}\}$. Omitting the positive system $\Phi_1^+ = \Phi^+(\mathfrak{h}, \mathfrak{g})$ with simple system Δ_1, the possible scalars $(n_{\xi_1}, n_{\xi_2}, \dots, n_{\xi_{14}})$ that define the remaining eleven positive systems are given in Table 6.3. The fact that these are actually positive systems with the corresponding simple system in Table 6.4 requires some work (see Appendix B). With so many choices of coefficients, these positive systems are computed by running a simple loop in Mathematica® based on the definition of a positive system.

Now, let us introduce a set of generic elements. Root vectors associated to simple roots will not usually serve as generic elements, but we have worked hard to give relatively simple

Simple System	Simple Roots
Δ_2	$\{-\alpha, \gamma, \delta, \alpha+\beta\}$
Δ_3	$\{\beta, \delta, -\alpha-\beta, \alpha+\beta+\gamma\}$
Δ_4	$\{\beta, -\alpha-\beta-\gamma, \alpha+\beta+2\gamma, \alpha+\beta+\gamma+\delta\}$
Δ_5	$\{\beta, \delta, -\alpha-\beta-\gamma-\delta, \alpha+\beta+2\gamma\}$
Δ_6	$\{\gamma, -\alpha-\beta-2\gamma, \alpha+\beta+\gamma+\delta, \alpha+2\beta+2\gamma\}$
Δ_7	$\{\beta, \gamma, \alpha+\beta+\gamma+\delta, -\alpha-2\beta-2\gamma\}$
Δ_8	$\{-\alpha-\beta-2\gamma, -\alpha-\beta-\gamma-\delta, \alpha+2\beta+2\gamma, \alpha+\beta+2\gamma+\delta\}$
Δ_9	$\{\gamma, -\alpha-\beta-2\gamma-\delta, \alpha+2\beta+2\gamma, \alpha+\beta+2\gamma+2\delta\}$
Δ_{10}	$\{\gamma, \delta, -\alpha-\beta-2\gamma-2\delta, 2\alpha+3\beta+4\gamma+2\delta\}$
Δ_{11}	$\{\beta, -\alpha-\beta-\gamma-\delta, -\alpha-2\beta-2\gamma, \alpha+\beta+2\gamma+\delta\}$
Δ_{12}	$\{\gamma, -\alpha-2\beta-2\gamma, -\alpha-\beta-2\gamma-\delta, 2\alpha+3\beta+4\gamma+2\delta\}$

Table 6.4: Simple Systems for Φ_j^+

linear combinations of basis elements. We will discuss why certain choices for generic elements are preferred later when we begin building the Springer fiber components. The proofs that these elements have the corresponding K-orbit dimension can be found in Appendix B. However, the proofs that these elements are actually generic will appear in the next chapter once we introduce the subgroups of F_4 used to build the fibers. Naturally, we would like to carry out the method used for the positive system Φ_3^+ in type $\mathfrak{g}_2$ where we compute an upper bound for the rank of the relations matrix resulting from the equation $[k, f] = 0$. However, these matrices are significantly larger for $\mathfrak{f}_4$, so finding these ranks can be a challenge. Fortunately, the dimension of $K.(\mathfrak{n}_j^- \cap \mathfrak{p})$ can be computed for most of the fibers independent of finding these relations matrices, so we will only need to utilize this method for a few cases.

Proposition 6.1. *Let $\mathfrak{n}_j^-$ represent the sum of the root spaces for all roots in $-\Phi_j^+$. Table 6.5 lists a generic element f_j in $\mathfrak{n}_j^- \cap \mathfrak{p}$ along with the dimension of the orbit $\mathcal{O} = K.f_j$.*

Generic Element	dim $K.f_j$
$f_1 = x_{(-1,0,0,0)} + x_{(-1,-2,-2,-1)} + x_{(-1,-2,-4,-2)}$	15
$f_2 = x_{(1,0,0,0)} + x_{(-1,-1,0,0)} + x_{(-1,-1,-2,-1)}$	21
$f_3 = x_{(1,1,0,0)} + x_{(-1,-1,-1,-1)} + x_{(-1,-2,-2,0)}$	21
$f_4 = f_6 = x_{(1,1,0,0)} + x_{(1,1,1,0)} + x_{(-1,-1,-1,-1)} + x_{(-1,-2,-2,0)}$	23
$f_5 = x_{(1,1,1,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,0)} + x_{(-1,-1,-2,-2)}$	22
$f_7 = x_{(1,1,1,0)} + x_{(1,2,2,0)} + x_{(-1,-2,-2,-1)} + x_{(-1,-2,-4,-2)}$	20
$f_8 = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,0)} + x_{(-1,-1,-2,-1)}$	24
$f_9 = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,0)} + x_{(-1,-1,-2,-2)}$	23
$f_{10} = x_{(1,1,1,0)} + x_{(1,1,2,0)} + x_{(-1,-2,-2,0)} + x_{(-1,-2,-4,-2)}$	20
$f_{11} = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-1,-2,-1)} + x_{(-1,-3,-4,-2)}$	22
$f_{12} = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,-1)} + x_{(-1,-1,-2,-2)}$	22

Table 6.5: Generic Elements $f_j \in \mathfrak{n}_j^- \cap \mathfrak{p}$

6.2 Generic Elements: Real Form F_4^{-20}

Every complex simple Lie algebra has both a compact and split real form. In the case of $\mathfrak{g}_2$, these are the only two real forms of $\mathfrak{g}_2$ up to isomorphism. For $\mathfrak{f}_4$, there is precisely one additional real form that is noncompact and nonsplit, which we will now describe. Consider the involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by first defining $\theta(x_\alpha) = x_\alpha$, $\theta(x_\beta) = x_\beta$, $\theta(x_\gamma) = x_\gamma$, and $\theta(x_\delta) = -x_\delta$ on the simple root vectors, and then extend to all of $\mathfrak{g}$ so that θ commutes with the Lie bracket. For this real form, the Vogan diagram is similar to the split case except that the paintings are reversed on the α and δ roots (see Figure 6.2).

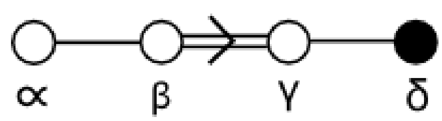


Figure 6.2: Vogan diagram for the noncompact and nonsplit real form of $\mathfrak{f}_4$

α	β	γ	$\alpha+\beta$
$\beta+\gamma$	$\alpha+\beta+\gamma$	$\beta+2\gamma$	$\alpha+\beta+2\gamma$
$\alpha+2\beta+2\gamma$	$\beta+2\gamma+2\delta$	$\alpha+\beta+2\gamma+2\delta$	$\alpha+2\beta+2\gamma+2\delta$
$\alpha+2\beta+3\gamma+2\delta$	$\alpha+2\beta+4\gamma+2\delta$	$\alpha+3\beta+4\gamma+2\delta$	$2\alpha+3\beta+4\gamma+2\delta$

Table 6.6: Compact Roots $\Phi^+(\mathfrak{h},\mathfrak{k})$

δ	$\gamma+\delta$	$\beta+\gamma+\delta$	$\alpha+\beta+\gamma+\delta$
$\beta+2\gamma+\delta$	$\alpha+\beta+2\gamma+\delta$	$\alpha+2\beta+2\gamma+\delta$	$\alpha+2\beta+3\gamma+\delta$

Table 6.7: Noncompact Roots $\Gamma_{\mathfrak{p}}^+$

We have a partition of the roots into the set of compact roots $\Phi(\mathfrak{h},\mathfrak{k})$ given in Table 6.6 and the set of noncompact roots $\Gamma_{\mathfrak{p}}$ given in Table 6.7. For this real form, we have a decomposition of $\mathfrak{g}$ into a 36-dimensional subalgebra $\mathfrak{k}$ isomorphic to $\mathfrak{so}(9,\mathbb{C})$ obtained from the 16 positive compact roots plus a 16-dimensional subspace $\mathfrak{p}$ associated to the 8 positive noncompact roots. The Weyl group for type $\mathfrak{f}_4$ relative to $\Phi(\mathfrak{h},\mathfrak{g})$, denoted $\mathcal{W}$, is a group of order 1152, and the Weyl group relative to $\Phi(\mathfrak{h},\mathfrak{k})$, denoted $\mathcal{W}_K$, is the Weyl group associated to $B_4=\mathfrak{so}(9,\mathbb{C})$. Therefore, $\mathcal{W}_k$ is a group of order $2^4\times 4! = 384$, hence $\mathcal{W}/\mathcal{W}_K$ has order 3. Consequently, there are three positive systems Φ_j^+ such that $\Phi^+(\mathfrak{h},\mathfrak{k})\subset\Phi_j^+\subset\Phi(\mathfrak{h},\mathfrak{g})$. Since the positive systems must contain $\Phi^+(\mathfrak{h},\mathfrak{k})$, we know that such a positive system has the form

$$\Phi^+(\mathfrak{h},\mathfrak{k})\cup\bigcup_{\xi\in\Gamma_{\mathfrak{p}}^+}\{n_\xi\xi\}$$

for particular scalars $n_\xi\in\{1,-1\}$. Use the ordering for $\Gamma_{\mathfrak{p}}^+$ introduced across the rows in Table 6.7 to enumerate the roots $\{\xi_1,\xi_2,\ldots,\xi_8\}$. Omitting the positive system Φ_1^+ with simple system Δ_1 such that $\Phi_1^+=\Phi^+(\mathfrak{h},\mathfrak{g})$, the possible scalars $(n_{\xi_1},n_{\xi_2},\ldots,n_{\xi_8})$ that define

Root System	$(n_{\xi_1}, n_{\xi_2}, \ldots, n_{\xi_8})$
Φ_2^+	$(-1, 1, 1, 1, 1, 1, 1, 1)$
Φ_3^+	$(-1, -1, 1, 1, 1, 1, 1, 1)$

Table 6.8: Scalars n_{ξ_j} for $\Phi^+(\mathfrak{h}, \mathfrak{k})$

Simple System	Simple Roots
Δ_2	$\{\alpha, \beta, -\delta, \gamma + \delta\}$
Δ_3	$\{\alpha, \gamma, -\gamma - \delta, \beta + 2\gamma + 2\delta\}$

Table 6.9: Simple Systems for Φ_j^+

the other two positive systems are given in Table 6.8. Refer to Appendix C to see that Δ_j given in Table 6.9 constitutes a simple system.

We conclude this chapter with a list of generic elements for the noncompact and nonsplit real form. What is interesting to note for this real form is that the number of nilpotent K-orbits in $\mathcal{N}_\theta$ has considerably dropped in comparison to the split case. In particular, the K-orbits in $\mathcal{N}_\theta$ miss the principal nilpotent orbit, so we will always have nontrivial Springer fiber components. As the dimension of $\mathfrak{k}$ has increased, while the dimension of $\mathfrak{p}$ has decreased, we would expect a change in the number of nilpotent K-orbits. However, there are actually only two such orbits besides the zero orbit, which is perhaps a little surprising. This real form presents some different challenges than the split case. The component groups for this nonsplit case are trivial, but $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ becomes more complicated to understand.

Proposition 6.2. *Let $\mathfrak{n}_j^-$ represent the sum of the root spaces for all roots in $-\Phi_j^+$. Table 6.10 lists a generic element f_j in $\mathfrak{n}_j^- \cap \mathfrak{p}$ along with the dimension of the orbit $\mathcal{O} = K.f_j$.*

Proof of Proposition 6.2. The nilpotent K-orbit in $\mathcal{N}_\theta$ with largest possible dimension is 15 according to the tables in [Dok88]. As a result, verifying these orbit dimensions of the elements listed in Table 6.10 will automatically imply that these elements are generic (see Appendix C). □

Generic Element	dim $K.f_j$
$f_1 = x_{(-1,-1,-1,-1)} + x_{(0,-1,-2,-1)}$	15
$f_2 = x_{(0,0,0,1)} + x_{(0,0,-1,-1)}$	15
$f_3 = x_{(-1,-1,-1,-1)} + x_{(0,-1,-2,-1)}$	15

Table 6.10: Generic Elements $f_j \in \mathfrak{n}_j^- \cap \mathfrak{p}$

7

Springer Fiber Components for F_4

To build the Springer fiber components for the real forms F_4^4 and F_4^{-20}, we need to introduce several parabolic subgroups of K contained in $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$. In the G_2 case, one parabolic subgroup constituted a subset of $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ whose dimension was large enough to describe the whole fiber. For the most part, this same behavior holds true in F_4^4 and F_4^{-20}, but there are a few fibers where additional groups in $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ must be included. Before describing the fibers specific to each real form, we begin this chapter with a review of the useful constructs in both cases.

As before, let $I_j = \Phi(\mathfrak{h}, \mathfrak{k}) \cap \Delta_j$ denote the simple compact roots in Φ_j^+, and let $\Phi_{I_j} \subset \Phi(\mathfrak{h}, \mathfrak{k})$ denote the root system associated to I_j with positive roots $\Phi_{I_j}^+ = \Phi_{I_j} \cap \Phi^+(\mathfrak{h}, \mathfrak{k})$. Then define a parabolic subalgebra

$$\mathfrak{q}_j = \left(\mathfrak{h} + \sum_{\gamma \in \Phi_{I_j}} \mathfrak{g}_\gamma\right) + \sum_{\gamma \in \Phi^+(\mathfrak{h},\mathfrak{k}) \setminus \Phi_{I_j}^+} \mathfrak{g}_{-\gamma} = \mathfrak{l}_j + \mathfrak{u}_j^-$$

of $\mathfrak{k}$, and let Q_j (resp., L_j) denote the connected subgroups of K with Lie algebras $\mathfrak{q}_j$ (resp., $\mathfrak{l}_j$). Regardless of the structure of the fibers for each of the closed K-orbits, these groups will always be used in constructing a substantial part of the fiber or the whole fiber. As we proceed, we will need to verify that we have found a subset of $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ of large enough dimension. The following well-known lemma adapted from [BZ, Lemma 4.12] will be used to count dimensions.

Lemma 7.1. *Let $\mathfrak{n}_j$ denote the sum of the root spaces $\mathfrak{g}_\xi$ for all $\xi \in \Phi_j^+$, and let N_j denote the connected subgroup of G with Lie algebra $\mathfrak{n}_j$. There exists a map*

$$T : \mathfrak{n}_j \to N_j.\mathfrak{b}_j$$

yielding an isomorphism of varieties.

Proof. First, the map $T_1 : N_j/\text{Stab}_{N_j}(\mathfrak{b}_j) \to N_j.\mathfrak{b}_j$ in [Jan04, Section 2.1] given by $n \to n.\mathfrak{b}_j$ yields an isomorphism of varieties. Since our Borel subalgebras are built from $-\Phi_j^+$ rather than Φ_j^+, it follows that $\text{Stab}_{N_j}(\mathfrak{b}_j)$ is trivial. By [Spr81, Section 8.2], there exists a B_j-equivariant isomorphism $T_2 : \mathfrak{n}_j \to N_j$ given by

$$x = \sum_{\xi \in \Phi_j^+} c_\xi x_\xi \to \prod_{\xi \in \Phi_j^+} Exp(c_\xi x_\xi) \in N_j.$$

Therefore, the composition $T = T_1 \circ T_2$ gives our desired isomorphism. □

Finally, when we introduce candidates for the components of the Springer fibers, we will want to know that they are closed subvarieties. In some cases, we will have to take closures in $\mathcal{B}$ to get the components. However, there are a few cases where we can utilize the next proposition.

Proposition 7.2. *Let G be an algebraic group acting rationally on a variety V. Suppose that a parabolic subgroup P of G stabilizes a closed subset A of the variety V. Then the union*

$$G.A = \bigcup_{g \in G} g.A$$

is closed in V.

Proof. See §0.15 in [Hum95] for the details. □

7.1 Springer Fiber Components: Split Real Form F_4^4

Our first order of business is to argue that the elements listed in Proposition 6.1 are generic. To accomplish this, we take advantage of part (a) from Lemma 4.2. The idea is that because L_j stabilizes $\mathfrak{n}_j^- \cap \mathfrak{p}$ regardless of a choice of generic element $f \in \mathfrak{n}_j^- \cap \mathfrak{p}$, we will automatically obtain an upper bound on the dimension of $K.(\mathfrak{n}_j^- \cap \mathfrak{p})$. Observe that Lemma 4.2 was cast in terms of $\mathfrak{g}_2$, but the result holds for $\mathfrak{f}_4$ as well with the same proofs.

Proof of Proposition 6.1. Begin by using Lemma 7.1 to see that

$$\left|\Phi_{I_j}^+\right| \leq dim\ \psi_{\mathcal{O}}^{-1}(f) = dim\ \mathcal{B} - dim\ K.f \tag{7.1}$$

for any generic element f in $\mathfrak{n}_j^- \cap \mathfrak{p}$. Indeed, the span of the root vectors for roots in $\Phi_{I_j}^+$ is contained in $\mathfrak{l}_j$, so the span has dimension $\left|\Phi_{I_j}^+\right|$ and maps via T into $\psi_{\mathcal{O}}^{-1}(f)$. Hence for any generic element f, $dim\ K.f \leq dim\ \mathcal{B} - \left|\Phi_{I_j}^+\right|$. This implies that for any $f' \in \mathfrak{n}_j^- \cap \mathfrak{p}$, if dim $K.f' =$ dim $\mathcal{B}-$ dim $\left|\Phi_{I_j}^+\right|$, then $K.f'$ is a K-orbit of maximal dimension in $\mathfrak{n}_j^- \cap \mathfrak{p}$, so f' is generic. Since the cardinalities of $\Phi_{I_3}^+$ and $\Phi_{I_{11}}^+$ are not large enough to make (7.1) above an equality, we prove that f_3 and f_{11} are generic in Appendix B. For the positive system $\Phi_{I_8}^+$, $K.f_8$ has the maximum possible dimension (dim $K.f_8 = 24 =$ dim K), so f_8 is automatically generic. For all of the remaining cases, $dim\ \mathcal{B} - \left|\Phi_{I_j}^+\right|$ listed in Table 7.1 equals the dimension of $K.f_j$ listed in Table 6.5, so f_j is necessarily generic. □

Most of the fibers can be described right away based on what has already been proved in Chapter 4. For the generic elements f_j with $j \neq 3, 11$, the subset $C_{f_j} = L_j\mathfrak{b}_j$ of $\psi_{\mathcal{O}_j}^{-1}(f_j)$ will again represent closed irreducible components of the Springer fibers. For the remaining fibers though, we need to introduce a few more subgroups because L_j is not large enough by itself to build irreducible components of the fibers. The most natural place to look is the stabilizer subgroup K^f since K^f is contained in $N(f, \mathfrak{n}^- \cap \mathfrak{p})$. For $\Phi_{I_3}^+$, consider the $\mathfrak{sl}_2$

Positive System	Roots	$dim\ \mathcal{B} - \left\|\Phi^+_{I_j}\right\|$
$\Phi^+_{I_1}$	$(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,1,1,0),(0,0,1,1),$ $(0,1,1,1),(0,1,2,0),(0,1,2,1),(0,1,2,2)$	15
$\Phi^+_{I_2}$	$(0,0,1,0),(0,0,0,1),(0,0,1,1)$	21
$\Phi^+_{I_3}=\Phi^+_{I_5}$	$(0,1,0,0),(0,0,0,1)$	22
$\Phi^+_{I_4}=\Phi^+_{I_{11}}$	$(0,1,0,0)$	23
$\Phi^+_{I_6}=\Phi^+_{I_9}$	$(0,0,1,0)$	23
$\Phi^+_{I_7}$	$(0,1,0,0),(0,0,1,0),(0,1,1,0),(0,1,2,0)$	20
$\Phi^+_{I_8}$	None	24
$\Phi^+_{I_{10}}$	$(0,0,1,0),(0,0,0,1),(0,0,1,1),(2,3,4,2)$	20
$\Phi^+_{I_{12}}$	$(0,0,1,0),(2,3,4,2)$	22

Table 7.1: Positive Systems $\Phi^+_{I_j}$

subalgebra

$$\mathfrak{z}_3 = \mathrm{Span}_{\mathbb{C}}(\{X_{f_3} = 2x_{(0,1,1,0)} + 2x_{(0,0,0,1)}, Y_{f_3} = x_{(0,-1,-1,0)} + x_{(0,0,0,-1)},$$
$$H_{f_3} = [X_{f_3}, Y_{f_3}] = 4h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + 2h_{(0,0,0,1)}\})$$

contained in $\mathfrak{k}^{f_3}$. Then use the connected SL_2 subgroup $Z_{f_3} \subset K^{f_3}$ whose Lie algebra is $\mathfrak{z}_{f_3}$ to define $C_{f_3} = Z_{f_3}L_3.\mathfrak{b}_3 \subseteq \psi^{-1}_{\mathcal{O}_j}(f_3)$. It turns out that C_{f_3} is precisely the fiber, but this will require some work. For $\Phi^+_{I_{11}}$, consider the one-parameter subgroup $Z_{f_{11}} = \{Exp(tX_{f_{11}}) : t \in \mathbb{C}\}$ contained in $K^{f_{11}}$ where

$$X_{f_{11}} = x_{(0,1,1,0)} + x_{(2,3,4,2)} - 3x_{(0,-1,-2,0)} + x_{(0,-1,-1,-1)}.$$

Let $C_{f_{11}} = \overline{Z_{f_{11}}L_{11}.\mathfrak{b}_{11}}$ denote the subset of $\psi^{-1}_{\mathcal{O}_{11}}(f_{11})$ where $\overline{U}$ denotes the closure of U in $\mathcal{B}$. In contrast to the previous case, we include the closure here because $\mathfrak{k}^{f_{11}}$ is too small to afford a subgroup from which $C_{f_{11}}$ will necessarily be closed. We omit closures in defining C_{f_3} since we will prove momentarily that it is closed. Our first step toward describing the fibers is the content of the following proposition.

Proposition 7.3. *For each generic element f_j, $C_{f_j} \subset \mathcal{B}$ constitutes a closed, irreducible subvariety of $\psi_{\mathcal{O}_j}^{-1}(f_j)$ whose dimension is the same as that of the corresponding $\psi_{\mathcal{O}_j}^{-1}(f_j)$.*

Proof. For $j \neq 3, 11$, the same proofs used for Lemma 4.2 prove that $C_{f_j} = L_j\mathfrak{b}_j$ is a closed, irreducible subvariety of $\psi_{\mathcal{O}_j}^{-1}(f_j)$. The dimension follows from Lemma 7.1 because $\Phi_{I_j}^+$ defines a subspace of $\mathfrak{n}_j$ of the same dimension as $\psi_{\mathcal{O}_j}^{-1}(f_j)$ that maps via T into C_{f_j}. To finish the proof, we need to work on $\Phi_{I_3}^+$ and $\Phi_{I_{11}}^+$.

It is clear that $Z_{f_{11}}L_{11} \subset N(f_{11}, \mathfrak{n}_{11}^- \cap \mathfrak{p})^{-1}$ since $Z_{f_{11}}$ fixes f_{11} and L_{11} stabilizes $\mathfrak{n}_{11}^- \cap \mathfrak{p}$. As a result, $C_{f_{11}}$ is a closed subvariety of $\psi_{\mathcal{O}_{11}}^{-1}(f_{11})$. The vectors $X_{f_{11}}$ and x_β are linearly independent, so $\mathrm{Span}_{\mathbb{C}}(\{X_{f_{11}}, x_\beta\})$ represents a 2-dimensional subspace that maps via T to a 2-dimensional subvariety of $C_{f_{11}}$. Since $K.f_{11}$ has dimension 22, it follows that $\psi_{\mathcal{O}_{11}}^{-1}(f_{11})$ is 2-dimensional, hence $C_{f_{11}}$ has the same dimension as $\psi_{\mathcal{O}_{11}}^{-1}(f_{11})$. To show that $C_{f_{11}}$ is irreducible, consider the morphism $Z_{f_{11}} \times L_{11} \to \mathcal{B}$ given by $(z, \ell) \to z\ell.\mathfrak{b}_{11}$. Then the image is an irreducible topological space since $Z_{f_{11}} \times L_{11}$ is irreducible, although the image need not be a variety itself. Taking closures, we see that $C_{f_{11}}$ is irreducible, hence represents an irreducible component of $\psi_{\mathcal{O}_{11}}^{-1}(f_{11})$.

For C_{f_3}, $Z_{f_3}L_3 \subset N(f_3, \mathfrak{n}_3^- \cap \mathfrak{p})^{-1}$ since Z_{f_3} fixes f_3 and L_3 stabilizes $\mathfrak{n}_3^- \cap \mathfrak{p}$. To show that C_{f_3} is closed, define the two one-parameter subgroups $T_{f_3} = \{Exp(tH_{f_3}) : t \in \mathbb{C}\}$ and $U_{f_3}^- = \{Exp(tY_{f_3}) : t \in \mathbb{C}\}$ that together yield a Borel subgroup $B_{f_3}^- = T_{f_3}U_{f_3}^-$ of Z_{f_3}. Since $B_{f_3}^- \subset Q_3$ and

$$B_{f_3}^- L_3.\mathfrak{b}_3 = B_{f_3}^- Q_3.\mathfrak{b}_3 = Q_3.\mathfrak{b}_3 = L_3.\mathfrak{b}_3,$$

Proposition 7.2 tells use that $C_{f_3} = Z_{f_3}L_3.\mathfrak{b}_3$ is closed because the Borel subgroup $B_{f_3}^-$ stabilizes the closed subset $L_3.\mathfrak{b}_3$ of $\mathcal{B}$. The same argument as above shows that C_{f_3} is irreducible and 3-dimensional since Z_{f_3} and L_3 are irreducible, and $\{X_{f_3}, x_\beta, x_\gamma\}$ is a linearly independent set. Since $K.f_3$ has dimension 21, the dimension of C_{f_3} is the same as the dimension of $\psi_{\mathcal{O}_3}^{-1}(f_3)$. □

Next, we need to understand the component groups in order to complete the picture of the fibers. As mentioned above, working with K is more challenging with the jump in dimension. Most of the component groups are non-trivial, but it turns out that many of them act trivially on the irreducible components C_{f_j}. Rather than compute all of the component groups directly, we will switch our attention to the semisimple element h_j of a normal $\mathfrak{sl}_2$ triple containing f_j, and argue using different techniques that the component group is either trivial or essentially irrelevant. To accomplish this, we will use the following proposition.

Proposition 7.4. *Let $\{e_j, f_j, h_j\}$ denote a normal $\mathfrak{sl}_2$ triple containing f_j. Then the stabilizer subgroup K^{h_j} of h_j in K is a connected group containing $K^{\{h,e,f\}}$.*

Proof. By Theorem 2.3.3 in [CM93], the stabilizer subgroup $\tilde{K}^{h_j}$ of the simply connected group $\tilde{K}$ is connected because $\mathfrak{k}$ is reductive and h_j is semisimple. By the theory of covering groups, there is a surjective map $p : \tilde{K} \to K$ that commutes with the adjoint action of G on $\mathfrak{g}$. Since p is continuous, K^{h_j} is the continuous image of a connected set, so K^{h_j} is also connected. □

The data from Table 7.2 lists a normal $\mathfrak{sl}_2$ triple containing each generic element f_j. Part of the challenge is to find normal $\mathfrak{sl}_2$ triples containing the generic elements so that the semisimple element of the triple lies in $\mathfrak{h}$. Searching for such triples is worth the effort because $\mathfrak{k}^h$ becomes easier to describe, which expedites the process of finding the component groups in K. As we proceed, we will learn that $A_K(f_4)$ is isomorphic to $\mathbb{Z}_2$ and yields two components for the corresponding fiber. However, the other component groups will either be trivial or stabilize the irreducible components C_{f_j} introduced above. To complete the discussion of the fibers, we finish this section with our desired theorem.

j	Nilpositive Element e_j	Semisimple Element h_j
1	$x_{(1,0,0,0)} + x_{(1,2,2,1)} + x_{(1,2,4,2)}$	$4h_\alpha + 6h_\beta + 4h_\gamma + 2h_\delta$
2	$6x_{(1,1,0,0)} + 6x_{(1,1,2,1)} + 10x_{(-1,0,0,0)}$	$8h_\alpha + 18h_\beta + 12h_\gamma + 6h_\delta$
3	$6x_{(1,1,1,1)} + 6x_{(1,2,2,0)} + 10x_{(-1,-1,0,0)}$	$8h_\alpha + 14h_\beta + 12h_\gamma + 6h_\delta$
$4, 6$	$10x_{(1,1,1,1)} + 14x_{(1,2,2,0)} + 10x_{(1,1,2,1)} + 10x_{(-1,-1,0,0)} +$ $8x_{(-1,-1,-1,0)} - 8x_{(-1,-1,-2,0)}$	$8h_\alpha + 22h_\beta + 16h_\gamma + 10h_\delta$
5	$10x_{(1,2,2,0)} + 4x_{(1,1,2,1)} + 4x_{(1,2,2,1)} + 10x_{(1,1,2,2)} +$ $4x_{(-1,-1,-1,0)} + 4x_{(-1,-1,-1,-1)}$	$4h_\alpha + 14h_\beta + 12h_\gamma + 6h_\delta$
7	$3x_{(1,1,1,1)} + 3x_{(1,2,2,1)} + x_{(1,2,4,2)} - x_{(-1,0,0,0)} +$ $x_{(-1,-1,-1,0)} + 3x_{(-1,-2,-2,0)}$	$2h_\alpha + 6h_\beta + 4h_\gamma + 4h_\delta$
8	$22x_{(1,2,2,0)} + 30x_{(1,1,2,1)} + 42x_{(-1,-1,-2,0)} + 16x_{(-1,-1,-1,-1)}$	$8h_\alpha + 30h_\beta + 24h_\gamma + 14h_\delta$
9	$14x_{(1,2,2,0)} + 18x_{(1,1,2,2)} + 8x_{(-1,-1,-2,0)} + 10x_{(-1,-1,-1,-1)}$	$4h_\alpha + 18h_\beta + 14h_\gamma + 8h_\delta$
10	$4x_{(1,2,2,0)} + 4x_{(1,2,2,2)} + 2x_{(1,2,3,2)} + 2x_{(1,2,4,2)} - 2x_{(-1,-1,0,0)} +$ $2x_{(-1,-1,-1,0)} + 2x_{(-1,-1,-2,0)} + 2x_{(-1,-1,-2,-2)}$	$6h_\beta + 4h_\gamma + 2h_\delta$
11	$8x_{(1,1,2,1)} + x_{(1,3,4,2)} + 9x_{(-1,-1,-2,0)} + 5x_{(-1,-1,-1,-1)}$	$-2h_\alpha + 4h_\gamma + 4h_\delta$
12	$5x_{(1,2,2,1)} + 9x_{(1,1,2,2)} + x_{(-1,-1,-2,0)} + 8x_{(-1,-1,-1,-1)}$	$2h_\alpha + 12h_\beta + 10h_\gamma + 6h_\delta$

Table 7.2: $\mathfrak{sl}_2$ triples $\{e_j, f_j, h_j\}$

Theorem 7.5. *Let $\mathfrak{b}_j$ denote the Borel subalgebra constructed from the positive system Φ_j^+, let $\mathcal{O}_j = K.\mathfrak{b}_j$ denote the associated closed K-orbit in the flag variety of F_4, and let C_{f_j} denote the irreducible components listed in Proposition 7.3. For each f_j with $j \neq 4$, the fiber $\psi_{\mathcal{O}_j}^{-1}(f_j)$ is isomorphic to C_{f_j}, while $\psi_{\mathcal{O}_4}^{-1}(f_4)$ is isomorphic to $\mathbb{Z}_2.C_{f_4}$. Moreover, $\psi_{\mathcal{O}_3}^{-1}(f_3)$ is isomorphic to the fiber bundle*

$$Z_{f_3} \times^{(Z_{f_3} \cap Q_{f_3})} Q_{f_3}/K_{B_3}$$

over $Z_{f_3}/(Z_{f_3} \cap Q_{f_3})$.

Proof. We begin by considering the fiber associated to the element f_8. There is only one nilpotent K-orbit of dimension 24 (see [Dok88]), so we know immediately from [Kin92] that the component group is trivial. For f_3, the subset $\mathfrak{g}(3,2)$ has dimension 7 (see Appendix B), so we know that the component group is again trivial (see [Kin92]). For f_4, the orbit $K.f_4$ has dimension 23, so the component group $A_K(f_4)$ must be isomorphic to $\mathbb{Z}_2$. Moreover, we know from [Kin92] that $K^{\{h_4,e_4,f_4\}}$ is $\mathbb{Z}_2$, so by Lemma 4.6, it suffices to find the generator for $K^{\{h_4,e_4,f_4\}}$ in order to obtain $A_K(f_4)$. By working carefully to choose generic elements and normal triples, we have the luxury of finding a generator which acts by $e^{\pm\pi i} = -1$ on certain root vectors. Focusing our attention on $\mathfrak{k}^{h_4}$, the only root vectors in $\mathfrak{k}$ commuting with h_4 are x_γ and $x_{-\gamma}$. As for the generator, consider the element of K given by

$$k_4 := Exp(\pi i(h_\gamma + h_\delta))Exp(2x_\gamma).$$

It is clear that k_4 is not the identity element, since $2x_\gamma$ is nilpotent. However, it is not obvious that k_4 has order two. In Appendix B, we determine the matrix representing k_4 within the Adjoint representation and show that it has order two. Now, observe that

$$Exp(2x_\gamma).f_4 = x_{(1,1,0,0)} - x_{(1,1,1,0)} + x_{(-1,-1,-1,-1)} + x_{(-1,-2,-2,0)}$$

since

$$\begin{aligned} Exp(2x_\gamma).f_4 &= f_4 + [2x_\gamma, f_4] + 1/2[2x_\gamma,[2x_\gamma,f_4]] + 1/6[2x_\gamma,[2x_\gamma,[2x_\gamma,f_4]]] + \dots \\ &= f_4 + [2x_\gamma, f_4] + 1/2[2x_\gamma,[2x_\gamma,f_4]]. \end{aligned}$$

By the equations

$$\begin{aligned} &[\pi i(h_\gamma + h_\delta), x_\alpha] = 0, [\pi i(h_\gamma + h_\delta), x_\beta] = -2\pi i x_\beta, \\ &[\pi i(h_\gamma + h_\delta), x_\gamma] = \pi i x_\gamma, \text{ and } [\pi i(h_\gamma + h_\delta), x_\gamma] = \pi i x_\delta, \end{aligned}$$

$Exp(\pi i(h_\gamma + h_\delta)).x_{(i,j,k,\ell)} = (-1)^k(-1)^\ell x_{(i,j,k,\ell)}$, so it follows that k_4 stabilizes f_4. We already know by construction that $k_4 \in K^{h_4}$, so we just need to verify that it stabilizes e_4 as well. Similar to the case of f_4,

$$\begin{aligned} Exp(2x_\gamma).e_4 = {} & 10x_{(1,1,1,1)} + 14x_{(1,2,2,0)} - 10x_{(1,1,2,1)} + \\ & 10x_{(-1,-1,0,0)} - 8x_{(-1,-1,-1,0)} - 8x_{(-1,-1,-2,0)} \end{aligned}$$

since

$$\begin{aligned} Exp(2x_\gamma).e_4 &= e_4 + [2x_\gamma, e_4] + 1/2[2x_\gamma,[2x_\gamma,e_4]] + 1/6[2x_\gamma,[2x_\gamma,[2x_\gamma,e_4]]] + \dots \\ &= e_4 + [2x_\gamma, e_4] + 1/2[2x_\gamma,[2x_\gamma,e_4]]. \end{aligned}$$

Again, we have

$$Exp(\pi i(h_\gamma + h_\delta)).x_{(i,j,k,\ell)} = (-1)^k(-1)^\ell x_{(i,j,k,\ell)},$$

so it follows that k_4 stabilizes e_4. Since k_4 stabilizes the whole triple, we know that it represents the generator for the component group $\mathbb{Z}_2$. As k_4 is not in $K_{B_4} \cup L_4$, it acts non-trivially on C_{f_4}, so

$$\psi_{\mathcal{O}_4}^{-1}(f_4) \simeq C_{f_4} \sqcup k_4.C_{f_4}.$$

For the fibers associated to f_j with $j \neq 3,4,8$, we see that $\mathfrak{k}^{h_j}$ is contained in the levi factor $\mathfrak{l}_j$ (see Appendix B), so K^{h_j} is contained in L_j. By Proposition 7.4, it follows that the group $K^{\{h,e,f\}}$ is contained in L_j. Using Lemma 4.6, $A_K(f_j)$ is contained in $K^{\{h,e,f\}}/K_\circ^{\{h,e,f\}}$, so $A_K(f_j)$ stabilizes C_{f_j}. For the components C_{f_j} built using only L_j, the fiber $\psi_{\mathcal{O}_j}^{-1}(f_j)$ is precisely C_{f_j}. For $C_{f_{11}}$, we must prove that the generator of $A_K(f_{11})$ normalizes $Z_{f_{11}}$ in $Z_{f_{11}}L_{11}.\mathfrak{b}_{11}$ in order to know that $\psi_{\mathcal{O}_{11}}^{-1}(f_{11})$ is $C_{f_{11}}$. Since the orbit $K.f_{11}$ has dimension 22, the component group $A_K(f_{11})$ must be isomorphic to $\mathbb{Z}_2$. Moreover, we know from [Kin92] that $K^{\{h_{11},e_{11},f_{11}\}}$ is also $\mathbb{Z}_2$, so it suffices to find the generator for $K^{\{h_{11},e_{11},f_{11}\}}$. As for the generator, consider the element of K given by

$$k_{11} := Exp(\pi i(h_\alpha + h_\beta + h_\delta)).$$

Observe that

$$[\pi i(h_\alpha + h_\beta + h_\delta), x_\alpha] = \pi i x_\alpha, [\pi i(h_\alpha + h_\beta + h_\delta), x_\beta] = \pi i x_\beta,$$
$$[\pi i(h_\alpha + h_\beta + h_\delta), x_\gamma] = -2\pi i x_\gamma, \text{ and } [\pi i(h_\alpha + h_\beta + h_\delta), x_\gamma] = 2\pi i x_\delta,$$

so $k_{11}.x_{(i,j,k,\ell)} = (-1)^i(-1)^j x_{(i,j,k,\ell)}$. Therefore, k_{11} stabilizes both f_{11} and e_{11}. Since $k_{11} \in H = Exp(\mathfrak{h}) \subset K^{h_{11}}$, we see that k_{11} stabilizes the whole triple, so it represents the generator for the component group $\mathbb{Z}_2$. Now, to show that k_{11} normalizes $Z_{f_{11}}$, we begin by observing that $k_{11}.X_{f_{11}} = -X_{f_{11}}$. By the formula, $Exp(k.X) = k(Exp(X))k^{-1}$ in [Kna02, Section 1.10], it follows that $k_{11}Z_{f_{11}} = Z_{f_{11}}k_{11}$. As a result,

$$k_{11}C_{f_{11}} = k_{11}\overline{Z_{f_{11}}L_{11}.\mathfrak{b}_{11}} \subseteq \overline{k_{11}Z_{f_{11}}L_{11}.\mathfrak{b}_{11}} = \overline{Z_{f_{11}}k_{11}L_{11}.\mathfrak{b}_{11}} = \overline{Z_{f_{11}}L_{11}.\mathfrak{b}_{11}} = C_{f_{11}}$$

since $k_{11} \in L_j$, so we know that $C_{f_{11}}$ represents the entire fiber $\psi_{\mathcal{O}_{11}}^{-1}(f_{11})$.

Finally, we need to prove that C_{f_3} is isomorphic to $Z_{f_3} \times^{(Z_{f_3}\cap Q_{f_3})} Q_{f_3}/K_{B_3}$. It suffices to prove that the map

$$Z_{f_3} \times^{(Z_{f_3}\cap Q_{f_3})} Q_{f_3}/K_{B_3} \to Z_{f_3}Q_{f_3}.\mathfrak{b}_3 \simeq Z_{f_3}Q_{f_3}.\mathfrak{b}_3$$

given by $[z, qK_{B_3}] \to zq.\mathfrak{b}_3$ is bijective ([GZ11, Theorem 2.10]) since $\psi_{\mathcal{O}_3}^{-1}(f_3)$ is a smooth variety ([GZ11, Lemma 2.9]). The map is clearly surjective, so we just need to check injectivity. Suppose we have elements $[z_1, q_1K_{B_3}]$ and $[z_2, q_2K_{B_3}]$ of $Z_{f_3} \times^{(Z_{f_3}\cap Q_{f_3})} Q_{f_3}/K_{B_3}$ such that $z_1q_1.\mathfrak{b}_3 = z_2q_2.\mathfrak{b}_3$. Then $(q_2)^{-1}(z_2)^{-1}z_1q_1 \in K_{B_3}$, so there exists some $b \in K_{B_3}$ such that $(z_2)^{-1}z_1 = g_2b(q_1)^{-1}$. As a result, $(z_2)^{-1}z_1 = g_2b(q_1)^{-1}$ is an element of $Z_{f_3} \cap Q_{f_3}$. Thus, the equality

$$[z_2, q_2K_{B_3}] = [z_2(z_2)^{-1}z_1, g_1b^{-1}(q_2)^{-1}q_2K_{B_3}] = [z_1, g_1K_{B_3}]$$

gives the desired isomorphism. □

7.2 Springer Fiber Components: Real Form F_4^{-20}

To build the component $\psi_{\mathcal{O}_4}^{-1}(f_4)$ for the Springer fibers of this real form, we need to rely heavily on the stabilizer subgroup K^{f_j} in order to supplement L_j. It turns out that L_j contributes very little to the full dimension of the fiber in two of the three closed K-orbits, so we have to hope that $\mathfrak{k}^{f_j}$ is relatively large. We need to find out the dimension of L_j in order to know how much we may be missing in building $\psi_{\mathcal{O}_j}^{-1}(f_j)$. We can follow the same procedure that we implemented for the fibers associated to the generic elements f_3 and f_{11} in the previous real form. Table 7.3 lists the roots in $\Phi_{I_j}^+$ along with the cardinality.

Positive System	Roots	$\lvert\Phi_{I_j}^+\rvert$
$\Phi_{I_1}^+$	$(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,0,0),(0,1,1,0),$ $(1,1,1,0),(0,1,2,0),(1,1,2,0),(1,2,2,0)$	9
$\Phi_{I_2}^+$	$(1,0,0,0),(0,1,0,0),(1,1,0,0)$	3
$\Phi_{I_3}^+$	$(1,0,0,0),(0,0,1,0),(0,1,2,2),(1,1,2,2)$	4

Table 7.3: Positive Systems $\Phi_{I_j}^+$

Let us now introduce some components that will represent $\psi_{\mathcal{O}_j}^{-1}(f_j)$. For the generic element f_1, define the component $C_{f_1} = L_1.\mathfrak{b}_1$. For the generic elements f_2 and f_3, define the components $C_{f_2} = \overline{K^{f_2}L_2.\mathfrak{b}_2}$ and $C_{f_3} = K^{f_3}L_3.\mathfrak{b}_3$. Showing that these represent irreducible components utilizes the same arguments as in the split real form cases. Observe the lack on closure on the third component similar to the case $\psi_{\mathcal{O}_3}^{-1}(f_3)$ in the split real form F_4^4. The following theorem completes our goal of describing Springer fiber components for G_2 and F_4.

Theorem 7.6. *Let $\mathfrak{b}_j$ denote the Borel subalgebra constructed from the positive system Φ_j^+, and let $\mathcal{O}_j = K.\mathfrak{b}_j$ denote the associated closed K-orbit in the flag variety of F_4. Then the fiber $\psi_{\mathcal{O}_j}^{-1}(f_j)$ is isomorphic to the irreducible component C_{f_j}. Moreover, $\psi_{\mathcal{O}_3}^{-1}(f_3)$ is isomorphic to the fiber bundle*

$$K^{f_3} \times^{(K^{f_3} \cap Q_{f_3})} Q_{f_3}/K_{B_3}$$

over $K^{f_3}/(K^{f_3} \cap Q_{f_3})$.

Proof. The component groups $A_K(f_j)$ relative to this real form are all trivial (see [Kin92]), so each of the fibers is automatically irreducible. Moreover, every $\psi_{\mathcal{O}_j}^{-1}(f_j)$ has dimension nine since $dim\ K.f_j = 15$ and $dim\ \mathcal{B} = 24$. As a result, the same proofs used in Lemma 4.2 prove that the theorem is true for $j = 1$. The only modification involves determining the dimension of $L_1.\mathfrak{b}_1$. As the nine root vectors associated to the positive roots for $\mathfrak{l}_1$ map via T into a nine dimensional subspace of $\psi_{\mathcal{O}_j}^{-1}(f_j)$, it follows that C_{f_1} must have dimension nine. The proofs that C_{f_j} is an irreducible component of the corresponding Springer fiber of the correct dimension for cases f_2 and f_3 are identical to those in Proposition 7.3. Indeed, focusing our attention on the basis elements built from root vectors associated to positive roots, we see that C_{f_2} is irreducible of dimension nine. Three dimensions coming from the connected subgroup L_2, while the remaining dimensions come from the part of K^{f_2} associated to the linearly independent set of elements

$$x_{(0,1,1,0)} - x_{(0,1,2,2)}, x_{(1,1,1,0)} - x_{(1,1,2,2)}, x_{(1,2,2,0)} - x_{(1,2,3,2)}, x_{(1,2,2,2)}, x_{(1,3,4,2)}, \text{ and } x_{(2,3,4,2)}$$

in $\mathfrak{k}^{f_2}$ disjoint from $\mathfrak{l}_2$. The same holds true for C_{f_3} as L_3 contributes four dimensions, while the remaining dimensions come from the part of K^{f_3} associated to the linearly independent set of elements

$$x_{(0,1,0,0)}, x_{(1,1,0,0)} - x_{(0,1,1,0)}, x_{(1,1,1,0)} - x_{(0,1,2,0)}, x_{(1,1,2,0)}, \text{ and } x_{(1,2,2,0)}$$

in $\mathfrak{k}^{f_3}$ disjoint from $\mathfrak{l}_3$. We close this chapter by discussing the closure of the last fiber C_{f_3}. The stabilizer subgroup K^{f_3} has Lie algebra given by

$$\begin{aligned}\mathfrak{k}^{f_3} = \operatorname{Span}_{\mathbb{C}}(\{&x_{(1,0,0,0)} - x_{(0,0,1,0)}, x_{(0,1,0,0)}, x_{(1,1,0,0)} - x_{(0,1,1,0)}, x_{(1,1,1,0)} - x_{(0,1,2,0)},\\ &x_{(1,1,2,0)}, x_{(1,2,2,0)}, x_{(-1,0,0,0)} - x_{(0,0,-1,0)}, x_{(0,-1,0,0)}, x_{(-1,-1,0,0)} - x_{(0,-1,-1,0)},\\ &x_{(-1,-1,-1,0)} - x_{(0,-1,-2,0)}, x_{(-1,-1,-2,0)}, x_{(-1,-2,-2,0)}, x_{(0,-1,-2,-2)},\\ &x_{(-1,-1,-2,-2)}, x_{(-1,-2,-2,-2)}, x_{(-1,-2,-3,-2)}, x_{(-1,-2,-4,-2)}, x_{(-1,-3,-4,-2)},\\ &x_{(-2,-3,-4,-2)}, h_{(1,0,0,0)} + h_{(0,0,1,0)}, h_{(0,1,0,0)}\}).\end{aligned}$$

To apply Proposition 7.2, we need to find an appropriate Borel subgroup of K^{f_3} that stabilizes $L_3.\mathfrak{b}_3$. Consider the subalgebra

$$\begin{aligned}\mathfrak{b}_{f_3} = \operatorname{Span}_{\mathbb{C}}(\{&x_{(-1,0,0,0)} - x_{(0,0,-1,0)}, x_{(0,-1,0,0)}, x_{(-1,-1,0,0)} - x_{(0,-1,-1,0)},\\ &x_{(-1,-1,-1,0)} - x_{(0,-1,-2,0)}, x_{(-1,-1,-2,0)}, x_{(-1,-2,-2,0)}, x_{(0,-1,-2,-2)},\\ &x_{(-1,-1,-2,-2)}, x_{(-1,-2,-2,-2)}, x_{(-1,-2,-3,-2)}, x_{(-1,-2,-4,-2)}, x_{(-1,-3,-4,-2)},\\ &x_{(-2,-3,-4,-2)}, h_{(1,0,0,0)} + h_{(0,0,1,0)}, h_{(0,1,0,0)}\})\end{aligned}$$

of $\mathfrak{k}^{f_3}$. Because $\mathfrak{b}_3$ is solvable in $\mathfrak{g}$, it follows that $\mathfrak{b}_{f_3}$ is solvable in $\mathfrak{k}^{f_3}$. The maximality follows from the fact that a larger such subalgebra $\mathfrak{b}_{f_3}'$ would necessarily contain elements from which an $\mathfrak{sl}_2$ triple of $\mathfrak{b}_{f_3}'$ can be constructed. The Jacobson-Morosov theorem tells us how to find such triples, but we have to verify that it can be used here. First, note that a larger subalgebra would contain elements of the subspace

$$\begin{aligned} \operatorname{Span}_{\mathbb{C}}(\{x_{(1,0,0,0)} - x_{(0,0,1,0)}, x_{(0,1,0,0)}, x_{(1,1,0,0)} - x_{(0,1,1,0)}, \\ x_{(1,1,1,0)} - x_{(0,1,2,0)}, x_{(1,1,2,0)}, x_{(1,2,2,0)}\}. \end{aligned} \tag{7.2}$$

By [Dok88], the Levi factor $\mathfrak{l}$ of $\mathfrak{k}^{f_3}$ is isomorphic to $\mathfrak{g}_2$. In this case, the Levi factor is

$$\begin{aligned} \mathfrak{l} = \operatorname{Span}_{\mathbb{C}}(\{& x_{(1,0,0,0)} - x_{(0,0,1,0)}, x_{(0,1,0,0)}, x_{(1,1,0,0)} - x_{(0,1,1,0)}, x_{(1,1,1,0)} - x_{(0,1,2,0)}, \\ & x_{(1,1,2,0)}, x_{(1,2,2,0)}, x_{(-1,0,0,0)} - x_{(0,0,-1,0)}, x_{(0,-1,0,0)}, x_{(-1,-1,0,0)} - x_{(0,-1,-1,0)}, \\ & x_{(-1,-1,-1,0)} - x_{(0,-1,-2,0)}, x_{(-1,-1,-2,0)}, x_{(-1,-2,-2,0)}, h_{(1,0,0,0)} + h_{(0,0,1,0)}, h_{(0,1,0,0)}\}, \end{aligned}$$

and the elements $x_{(1,0)} := x_{(1,0,0,0)} - x_{(0,0,1,0)}$, $x_{(-1,0)} := x_{(-1,0,0,0)} - x_{(0,0,-1,0)}$, $x_{(0,1)} := x_{(0,1,0,0)}$, and $x_{(0,-1)} := x_{(0,-1,0,0)}$ represent a generating set. Since $\mathfrak{l}$ is semisimple, we can apply the Jacobson-Morosov theorem to obtain an $\mathfrak{sl}_2$ triple containing any nilpotent element in (7.2). As a result, the maximality claim is proved. Let B_{f_3} denote the connected subgroup of K^{f_3} with Lie algebra $\mathfrak{b}_{f_3}$. Since $B_{f_3} \subset Q_3$ and

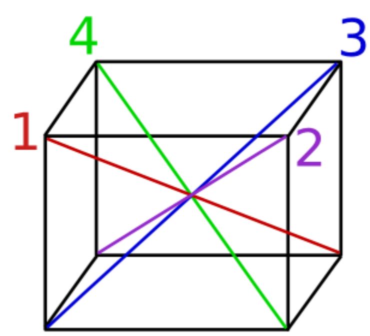
$$B_{f_3} L_3.\mathfrak{b}_3 = B_{f_3}^{-} Q_3.\mathfrak{b}_3 = Q_3.\mathfrak{b}_3 = L_3.\mathfrak{b}_3,$$

Proposition 7.2 tells use that $C_{f_3} = K^{f_3} L_3.\mathfrak{b}_3$ is closed because the Borel subgroup B_{f_3} stabilizes the closed subset $L_3.\mathfrak{b}_3$ of $\mathcal{B}$. The additional structure as a fiber bundle follows the exact same proof as in Theorem 7.5. □

8

Multiplicity Polynomials

To motivate the final chapter, let us recall an example from elementary group theory. Consider the group of symmetries of a cube arising from rotations of $\mathbb{R}^3$. Looking at vertices, edges, and faces of the cube one can learn something about the group by considering the geometry of the vertices, edges, and faces. The point of view is that we can better understand the group under consideration by looking at the way it acts on various sets. For example, we discover by looking at the faces of a cube that the group has 24 elements. Indeed, there are six faces and the stabilizer subgroup has order four, so the group has order 24. Also, by looking at the diagonals of the cube, we learn that the group is isomorphic to the permutation group S_4.



The Springer fiber components considered in the paper give an example of this general philosophy. From the real form $G_{\mathbb{R}}$, we understand the portions of the Springer fiber via an action of a subgroup of G on the flag variety $\mathcal{B}$. In return, the structure of the fiber is going to indicate additional information about $G_{\mathbb{R}}$. To understand this behavior, we need a few definitions and theorems whose details are contained in [BZ08].

Let $\mathcal{O}$ denote a closed K-orbit in $\mathcal{B}$ associated to some positive system Φ^+ where $\Phi^+(\mathfrak{h},\mathfrak{k}) \subset \Phi^+ \subset \Phi^+(\mathfrak{h},\mathfrak{g})$, and let ρ and ρ_c denote the weights given by

$$2\rho = \sum_{\xi \in \Phi^+} \xi \qquad \text{and} \qquad 2\rho_c = \sum_{\xi \in \Phi^+(\mathfrak{h},\mathfrak{k})} \xi.$$

Then there exists a family of discrete series representations X_λ for $\lambda \in \mathfrak{h}^*$ where λ is a regular, dominant weight such that

$$\tau = \lambda + \rho - 2\rho_c$$

is analytically integral, i.e., the derivative of a character on H. Our descriptions for certain Springer fiber components given in the previous chapters will allow us to give the multiplicity of $\overline{K.f}$ in the associated cycles of X_λ for certain generic elements f. To understand this multiplicity, note that the weight τ yields a line bundle $\mathcal{L}_\lambda \to \mathcal{O}$ where $\mathcal{L}_\lambda = K \times^{K_B} \mathbb{C}_\tau$. Then the multiplicity is a polynomial in λ that is given by

$$M_{\mathcal{O}}(\lambda) = dim\Big(H^0\big(\psi_{\mathcal{O}}^{-1}(f), \mathcal{F}\big(\mathcal{L}_\lambda|_{\psi_{\mathcal{O}}^{-1}(f)}\big)\big)\Big)$$

where $\mathcal{F}\big(\mathcal{L}_\lambda|_{\psi_{\mathcal{O}}^{-1}(f)}\big)$ denotes the structure sheaf on $\mathcal{L}_\lambda|_{\psi_{\mathcal{O}}^{-1}(f)}$ (see [BZ08]). The following theorem will allow us to better understand these polynomials.

Theorem 8.1. *Let $W_{-\tau}$ denote the irreducible representation of K having lowest weight $-\tau$. If $w_{-\tau}$ denotes a lowest weight vector of $W_{-\tau}$, then*

$$M_{\mathcal{O}}(\lambda) = dim(Span_{\mathbb{C}}\big(\{k^{-1}w_{-\tau} : k \in N(f, \mathfrak{n}^- \cap \mathfrak{p}\}\big)$$

for λ sufficiently dominant.

Proof. See [BZ08] for a discussion and complete list of references. Note that for λ sufficiently dominant, τ is $\Phi^+(\mathfrak{h},\mathfrak{k})$-dominant, hence $\mathrm{Span}_{\mathbb{C}}\big(\{k^{-1}w_{-\tau} : k \in N(f, \mathfrak{n}^- \cap \mathfrak{p}\}\big)$ is necessarily finite dimensional. $\square$

Using the results of the previous chapters, we can now determine some of these multiplicity polynomials. Consider the homogeneous fibers associated to a generic element f_j for the real forms G_2^2, F_4^4, and F_4^{-20} introduced in the previous chapters. For each generic f_j with $\psi_{\mathcal{O}_j}^{-1}(f_j) = L_j.\mathfrak{b}_j$, Theorem 8.1 tells us that if λ is sufficiently dominant, then

$$dim(\mathrm{Span}_{\mathbb{C}}\big(\{k^{-1}w_{-\tau} : k \in N(f, \mathfrak{n}^- \cap \mathfrak{p}\}\big) = dim(\mathrm{Span}_{\mathbb{C}}\big(\{k^{-1}w_{-\tau} : k \in L_j\}\big)$$

is the dimension of the irreducible representation of L_j of lowest weight $-\tau$. To simplify the computation, we can restrict to the semisimple part of L_j, denoted $(L_j)_{ss}$, when computing the dimension because the difference between L_j and $(L_j)_{ss}$ lies within the center of G. Using the Weyl dimension formula and the results above, we have the final theorem of this dissertation. Following the theorem, we will discuss the specific polynomials for which the theorem applies.

Theorem 8.2. *Let $M_{\mathcal{O}_j}(\lambda)$ denote the multiplicity polynomial associated to a closed K-orbit $\mathcal{O}_j$ where λ is a sufficiently dominant weight such that the analytical integral weight $\tau = \lambda + \rho - 2\rho_c$ is $\Phi^+(\mathfrak{h},\mathfrak{k})$-dominant. Then the fibers $\psi_{\mathcal{O}_j}^{-1}(f_j)$ isomorphic to the variety $L_j.\mathfrak{b}_j$ have multiplicity polynomials given by*

$$M_{\mathcal{O}_j}(\lambda) = \prod_{\xi \in \Phi^+_{I_j}} \frac{\langle \tau + \rho_{\mathfrak{l}_j}, \xi \rangle}{\langle \rho_{\mathfrak{l}_j}, \xi \rangle}$$

where $\rho_{\mathfrak{l}_j}$ denotes the half sum of the roots in $\Phi^+_{I_j}$. Moreover, the multiplicity polynomial associated to fibers $\psi_{\mathcal{O}_j}^{-1}(f_j)$ composed of n disjoint connected components $k.L_j.\mathfrak{b}_j$ with $k \in$

$A_K(f_j)$ is given by

$$M_{\mathcal{O}_j}(\lambda) = n \prod_{\xi \in \Phi^+_{I_j}} \frac{\langle \tau + \rho_{\mathfrak{l}_j}, \xi \rangle}{\langle \rho_{\mathfrak{l}_j}, \xi \rangle}.$$

Proof. The first part is immediate from the discussion above because if τ is $\Phi^+(\mathfrak{h}, \mathfrak{k})$-dominant, then τ is I_j dominant, so the Weyl dimension formula readily applies to the irreducible representation of $(L_j)_{ss}$. We need to prove the last claim about $M_{\mathcal{O}_j}(\lambda)$ when there are n irreducible components. By Theorem 7.5, the fiber is composed of n irreducible disjoint components since it is known that the fiber is smooth. As a result, $H^0\big(\psi_{\mathcal{O}}^{-1}(f), \mathcal{F}\big(\mathcal{L}_\lambda|_{\psi_{\mathcal{O}}^{-1}(f)}\big)\big)$ decomposes as a direct sum

$$\bigoplus_{k \in A_K(f_j)} H^0\big(kL_j.\mathfrak{b}_j, \mathcal{F}\big(\mathcal{L}_\lambda|_{kL_j.\mathfrak{b}_j}\big)\big).$$

Since $H^0\big(L_j.\mathfrak{b}_j, \mathcal{F}\big(\mathcal{L}_\lambda|_{L_j.\mathfrak{b}_j}\big)\big)$ is isomorphic to $H^0\big(kL_j.\mathfrak{b}_j, \mathcal{F}\big(\mathcal{L}_\lambda|_{kL_j.\mathfrak{b}_j}\big)\big)$ for any $k \in A_K(f_j)$, it follows that the multiplicity polynomial is n times the dimension of the irreducible L_j representation with lowest weight $-\tau$. □

Armed with the previous theorem, we can finish our goal of computing the multiplicity polynomials for most of the closed K-orbits. The multiplicity polynomials associated to non-homogeneous fibers are still unknown to the author, but represent a future research goal. In fact, the Springer fiber components above which are described in terms of a fiber bundle are very likely to yield the multiplicity polynomials. Table 8.1 (resp., Table 8.2, Table 8.3) lists the known multiplicity polynomials for the real forms G_2^2 (resp., F_4^4, F_4^{-20}) along with the non-negative integer values x_j for which $\tau = \sum x_j \lambda_j + \rho - 2\rho_c$ is $\Phi^+(\mathfrak{h}, \mathfrak{k})$-dominant.

$\Phi^+_{I_j}$	Multiplicity Polynomial	Coefficients $\{x_1, x_2\}$
$\Phi^+_{I_1}$	x_1	$x_1 \geq 1, x_2 \geq 0$
$\Phi^+_{I_2}$	1	$x_1 \geq 0, x_2 \geq 0$
$\Phi^+_{I_3}$	$x_1 + 2x_2$	$x_1 + 2x_2 \geq 1$

Table 8.1: Multiplicity Polynomials for G_2^2

$\Phi^+_{I_j}$	Multiplicity Polynomial	Coefficients $\{x_1, x_2, x_3, x_4\}$
$\Phi^+_{I_1}$	$1/720x_2x_3x_4(x_2+x_3)(x_2+x_3+x_4)(x_3+x_4)\cdot$ $(2x_2+x_3+x_4)(2x_2+x_3)(2x_2+2x_3+x_4)$	$x_1 \geq 0, x_2 \geq 1, x_3 \geq 1, x_4 \geq 1$
$\Phi^+_{I_2}$	$1/2x_3x_4(x_3+x_4)$	$x_1 \geq 0, x_2 \geq 0, x_3 \geq 1, x_4 \geq 1$
$\Phi^+_{I_4}$	$2x_2$	$x_1 \geq 0, x_2 \geq 1, x_3 \geq 0, x_4 \geq 0$
$\Phi^+_{I_5}$	x_2x_4	$x_1 \geq 0, x_2 \geq 1, x_3 \geq 0, x_4 \geq 1$
$\Phi^+_{I_6}$	x_3	$x_1 \geq 0, x_2 \geq 0, x_3 \geq 1, x_4 \geq 0$
$\Phi^+_{I_7}$	$1/6x_2x_3(x_2+x_3)(2x_2+x_3)$	$x_1 \geq 0, x_2 \geq 1, x_3 \geq 1, x_4 \geq 0$
$\Phi^+_{I_8}$	1	$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$
$\Phi^+_{I_9}$	x_3	$x_1 \geq 0, x_2 \geq 0, x_3 \geq 1, x_4 \geq 0$
$\Phi^+_{I_{10}}$	$1/2x_3x_4(x_3+x_4)(2x_1+3x_2+2x_3+x_4)$	$x_1 \geq 0, x_2 \geq 0, x_3 \geq 1, x_4 \geq 1$
$\Phi^+_{I_{12}}$	$x_3(2x_1+3x_2+2x_3+x_4)$	$x_1 \geq 0, x_2 \geq 0, x_3 \geq 1, x_4 \geq 0$

Table 8.2: Multiplicity Polynomials for F_4^4

$\Phi^+_{I_j}$	Multiplicity Polynomial	Coefficients $\{x_1, x_2, x_3, x_4\}$
$\Phi^+_{I_1}$	$1/720x_1x_2x_3(x_1+x_2)(x_2+x_3)(x_1+x_2+x_3)\cdot$ $(x_1+2x_2+x_3)(2x_2+x_3)(2x_1+2x_2+x_3)$	$x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, x_4 \geq 0$

Table 8.3: Multiplicity Polynomials for F_4^{-20}

A

Computations for G_2

A.1 The Set Δ_j Forms a Simple System for Φ_j^+

For the given positive system, the tables indicate how each root $\xi \in \Phi_j^+$ can be written as a nonnegative linear combination $\xi = i\xi_1 + j\xi_2$ of the listed simple roots $\Delta_j = \{\xi_1, \xi_2\}$. Naturally, the case $\Phi_1^+ = \Phi^+(\mathfrak{h}, \mathfrak{g})$ will be omitted.

Root	i	j	Root	i	j	Root	i	j
$(1,0)$	1	1	$(0,-1)$	1	0	$(1,1)$	0	1
$(2,1)$	1	2	$(3,1)$	2	3	$(3,2)$	1	3

Table A.1: Positive System $\Phi_2^+ : \Delta_2 = \{-\beta, \alpha + \beta\}$

Root	i	j	Root	i	j	Root	i	j
$(1,0)$	2	1	$(0,-1)$	3	1	$(-1,-1)$	1	0
$(2,1)$	1	1	$(3,1)$	3	2	$(3,2)$	0	1

Table A.2: Positive System $\Phi_3^+ : \Delta_3 = \{-\alpha - \beta, 3\alpha + 2\beta\}$

A.2 Computing the Dimensions of $\mathfrak{g}(j, 2)$

We need to show that $dim\ \mathfrak{g}(1,2) = 4$, $dim\ \mathfrak{g}(2,2) = 2$, and $dim\ \mathfrak{g}(3,2) = 2$. First, $dim\ \mathfrak{g}(2,2) = 2$ is clear from the tables in [Dok88] because there is only one nilpotent K-orbit of dimension 6. For the other two cases, let $p = c_{(0,1)}x_{(0,1)} + c_{(0,-1)}x_{(0,-1)} + c_{(1,1)}x_{(1,1)} + c_{(-1,-1)}x_{(-1,-1)} + c_{(2,1)}x_{(2,1)} + c_{(-2,-1)}x_{(-2,-1)} + c_{(3,1)}x_{(3,1)} + c_{(-3,-1)}x_{(-3,-1)}$ be an arbitrary element of $\mathfrak{p}$. Since $[h_1, p] = 2p$ if and only if $2c_{(0,1)}x_{(0,1)} - 2c_{(0,-1)}x_{(0,-1)} + 2c_{(1,1)}x_{(1,1)} - 2c_{(-1,-1)}x_{(-1,-1)} + 2c_{(2,1)}x_{(2,1)} - 2c_{(-2,-1)}x_{(-2,-1)} + 2c_{(3,1)}x_{(3,1)} - 2c_{(-3,-1)}x_{(-3,-1)} = 2p$ if and only if $c_{(0,-1)} = c_{(-1,-1)} = c_{(-2,-1)} = c_{(-3,-1)} = 0$, we have $\dim \mathfrak{g}(1,2) = 4$. Finally, $[h_3, p] = 2p$ if and only if $-2c_{(0,1)}x_{(0,1)} + 2c_{(0,-1)}x_{(0,-1)} + 0c_{(1,1)}x_{(1,1)} + 0c_{(-1,-1)}x_{(-1,-1)} + 2c_{(2,1)}x_{(2,1)} - 2c_{(-2,-1)}x_{(-2,-1)} + 4c_{(3,1)}x_{(3,1)} - 4c_{(-3,-1)}x_{(-3,-1)} = 2p$ if and only if $c_{(0,1)} = c_{(1,1)} = c_{(-1,-1)} = c_{(-2,-1)} = c_{(3,1)} = c_{(-3,-1)} = 0$, so $\dim \mathfrak{g}(3,2) = 2$.

A.3 Generators for the Component Groups

The elements r and s generate $K^{\{h_1,e_1,f_1\}}$, and the element z generates $K^{\{h_3,e_3,f_3\}}$. Using the matrix representations, we will omit the maps ϕ and π. The fact that these elements stabilize their respective triples can easily be checked using conjugation.

$$f_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \end{bmatrix} \qquad 3e_1 = \begin{bmatrix} 0 & 0 & -2 & -4 & 4 & 0 & 0 \\ 0 & 0 & -4 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad r = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$r^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad s = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$f_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_3 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

B

Computations for the Split Real Form F_4^4

B.1 The Set Δ_j Forms a Simple System for Φ_j^+

The following tables indicate how each root $\xi \in \Phi_j^+$ can be written as a nonnegative linear combination $\xi = i\xi_1 + j\xi_2 + k\xi_3 + \ell\xi_4$ of the listed simple roots $\Delta_j = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. Again, the case $\Phi_1^+ = \Phi^+(\mathfrak{h}, \mathfrak{g})$ will be omitted.

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	0	1	$(0,0,1,0)$	0	1	0	0	$(0,0,0,1)$	0	0	1	0
$(0,1,1,0)$	1	1	0	1	$(0,0,1,1)$	0	1	1	0	$(0,1,2,0)$	1	2	0	1
$(0,1,1,1)$	1	1	1	1	$(0,1,2,1)$	1	2	1	1	$(0,1,2,2)$	1	2	2	1
$(2,3,4,2)$	1	4	2	3	$(-1,0,0,0)$	1	0	0	0	$(1,1,0,0)$	0	0	0	1
$(1,1,1,0)$	0	1	0	1	$(1,1,2,0)$	0	2	0	1	$(1,1,1,1)$	0	1	1	1
$(1,2,2,0)$	1	2	0	2	$(1,1,2,1)$	0	2	1	1	$(1,2,2,1)$	1	2	1	2
$(1,1,2,2)$	0	2	2	1	$(1,2,3,1)$	1	3	1	2	$(1,2,2,2)$	1	2	2	2
$(1,2,3,2)$	1	3	2	2	$(1,2,4,2)$	1	4	2	2	$(1,3,4,2)$	2	4	2	3

Table B.1: $\Delta_2 = \{-\alpha, \gamma, \delta, \alpha + \beta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	0	0	$(0,0,1,0)$	0	0	1	1	$(0,0,0,1)$	0	1	0	0
$(0,1,1,0)$	1	0	1	1	$(0,0,1,1)$	0	1	1	1	$(0,1,2,0)$	1	0	2	2
$(0,1,1,1)$	1	1	1	1	$(0,1,2,1)$	1	1	2	2	$(0,1,2,2)$	1	2	2	2
$(2,3,4,2)$	1	2	2	4	$(-1,0,0,0)$	1	0	1	0	$(-1,-1,0,0)$	0	0	1	0
$(1,1,1,0)$	0	0	0	1	$(1,1,2,0)$	0	0	1	2	$(1,1,1,1)$	0	1	0	1
$(1,2,2,0)$	1	0	1	2	$(1,1,2,1)$	0	1	1	2	$(1,2,2,1)$	1	1	1	2
$(1,1,2,2)$	0	2	1	2	$(1,2,3,1)$	1	1	2	3	$(1,2,2,2)$	1	2	1	2
$(1,2,3,2)$	1	2	2	3	$(1,2,4,2)$	1	2	3	4	$(1,3,4,2)$	2	2	3	4

Table B.2: $\Delta_3 = \{\beta, \delta, -\alpha-\beta, \alpha+\beta+\gamma\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	0	0	$(0,0,1,0)$	0	1	1	0	$(0,0,0,1)$	0	1	0	1
$(0,1,1,0)$	1	1	1	0	$(0,0,1,1)$	0	2	1	1	$(0,1,2,0)$	1	2	2	0
$(0,1,1,1)$	1	2	1	1	$(0,1,2,1)$	1	3	2	1	$(0,1,2,2)$	1	4	2	2
$(2,3,4,2)$	1	2	2	2	$(-1,0,0,0)$	1	2	1	0	$(-1,-1,0,0)$	0	2	1	0
$(-1,-1,-1,0)$	0	1	0	0	$(1,1,2,0)$	0	0	1	0	$(1,1,1,1)$	0	0	0	1
$(1,2,2,0)$	1	0	1	0	$(1,1,2,1)$	0	1	1	1	$(1,2,2,1)$	1	1	1	1
$(1,1,2,2)$	0	2	1	2	$(1,2,3,1)$	1	2	2	1	$(1,2,2,2)$	1	2	1	2
$(1,2,3,2)$	1	3	2	2	$(1,2,4,2)$	1	4	3	2	$(1,3,4,2)$	2	4	3	2

Table B.3: $\Delta_4 = \{\beta, -\alpha-\beta-\gamma, \alpha+\beta+2\gamma, \alpha+\beta+\gamma+\delta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	0	0	$(0,0,1,0)$	0	1	1	1	$(0,0,0,1)$	0	1	0	0
$(0,1,1,0)$	1	1	1	1	$(0,0,1,1)$	0	2	1	1	$(0,1,2,0)$	1	2	2	2
$(0,1,1,1)$	1	2	1	1	$(0,1,2,1)$	1	3	2	2	$(0,1,2,2)$	1	4	2	2
$(2,3,4,2)$	1	2	0	2	$(-1,0,0,0)$	1	2	2	1	$(-1,-1,0,0)$	0	2	2	1
$(-1,-1,-1,0)$	0	1	1	0	$(1,1,2,0)$	0	0	0	1	$(-1,-1,-1,-1)$	0	0	1	0
$(1,2,2,0)$	1	0	0	1	$(1,1,2,1)$	0	1	0	1	$(1,2,2,1)$	1	1	0	1
$(1,1,2,2)$	0	2	0	1	$(1,2,3,1)$	1	2	1	2	$(1,2,2,2)$	1	2	0	1
$(1,2,3,2)$	1	3	1	2	$(1,2,4,2)$	1	4	2	3	$(1,3,4,2)$	2	4	2	3

Table B.4: $\Delta_5 = \{\beta, \delta, -\alpha-\beta-\gamma-\delta, \alpha+\beta+2\gamma\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	0	1	0	1	$(0,0,1,0)$	1	0	0	0	$(0,0,0,1)$	1	1	1	0
$(0,1,1,0)$	1	1	0	1	$(0,0,1,1)$	2	1	1	0	$(0,1,2,0)$	2	1	0	1
$(0,1,1,1)$	2	2	1	1	$(0,1,2,1)$	3	2	1	1	$(0,1,2,2)$	4	3	2	1
$(2,3,4,2)$	2	1	2	1	$(-1,0,0,0)$	2	2	0	1	$(-1,-1,0,0)$	2	1	0	0
$(-1,-1,-1,0)$	1	1	0	0	$(-1,-1,-2,0)$	0	1	0	0	$(1,1,1,1)$	0	0	1	0
$(1,2,2,0)$	0	0	0	1	$(1,1,2,1)$	1	0	1	0	$(1,2,2,1)$	1	1	1	1
$(1,1,2,2)$	2	1	2	0	$(1,2,3,1)$	2	1	1	1	$(1,2,2,2)$	2	2	2	1
$(1,2,3,2)$	3	2	2	1	$(1,2,4,2)$	4	2	2	1	$(1,3,4,2)$	4	3	2	2

Table B.5: $\Delta_6 = \{\gamma, -\alpha-\beta-2\gamma, \alpha+\beta+\gamma+\delta, \alpha+2\beta+2\gamma\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	0	0	$(0,0,1,0)$	0	1	0	0	$(0,0,0,1)$	1	1	1	1
$(0,1,1,0)$	1	1	0	0	$(0,0,1,1)$	1	2	1	1	$(0,1,2,0)$	1	2	0	0
$(0,1,1,1)$	2	2	1	1	$(0,1,2,1)$	2	3	1	1	$(0,1,2,2)$	3	4	2	2
$(2,3,4,2)$	1	2	2	0	$(-1,0,0,0)$	2	2	0	1	$(-1,-1,0,0)$	1	2	0	1
$(-1,-1,-1,0)$	1	1	0	1	$(-1,-1,-2,0)$	1	0	0	1	$(1,1,1,1)$	0	0	1	0
$(-1,-2,-2,0)$	0	0	0	1	$(1,1,2,1)$	0	1	1	0	$(1,2,2,1)$	1	1	1	0
$(1,1,2,2)$	1	2	2	1	$(1,2,3,1)$	1	2	1	0	$(1,2,2,2)$	2	2	2	1
$(1,2,3,2)$	2	3	2	1	$(1,2,4,2)$	2	4	2	1	$(1,3,4,2)$	3	4	2	1

Table B.6: $\Delta_7 = \{\beta, \gamma, \alpha+\beta+\gamma+\delta, -\alpha-2\beta-2\gamma\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	1	0	$(0,0,1,0)$	0	1	0	1	$(0,0,0,1)$	1	0	0	1
$(0,1,1,0)$	1	1	1	1	$(0,0,1,1)$	1	1	0	2	$(0,1,2,0)$	1	2	1	2
$(0,1,1,1)$	2	1	1	2	$(0,1,2,1)$	2	2	1	3	$(0,1,2,2)$	3	2	1	4
$(2,3,4,2)$	1	0	1	2	$(-1,0,0,0)$	2	2	1	2	$(-1,-1,0,0)$	1	2	0	2
$(-1,-1,-1,0)$	1	1	0	1	$(-1,-1,-2,0)$	1	0	0	0	$(-1,-1,-1,-1)$	0	1	0	0
$(1,2,2,0)$	0	0	1	0	$(1,1,2,1)$	0	0	0	1	$(1,2,2,1)$	1	0	1	1
$(1,1,2,2)$	1	0	0	2	$(1,2,3,1)$	1	1	1	2	$(1,2,2,2)$	2	0	1	2
$(1,2,3,2)$	2	1	1	3	$(1,2,4,2)$	2	2	1	4	$(1,3,4,2)$	3	2	2	4

Table B.7: $\Delta_8 = \{-\alpha-\beta-2\gamma, -\alpha-\beta-\gamma-\delta, \alpha+2\beta+2\gamma, \alpha+\beta+2\gamma+\delta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	0	2	1	1	$(0,0,1,0)$	1	0	0	0	$(0,0,0,1)$	0	1	0	1
$(0,1,1,0)$	1	2	1	1	$(0,0,1,1)$	1	1	0	1	$(0,1,2,0)$	2	2	1	1
$(0,1,1,1)$	1	3	1	2	$(0,1,2,1)$	2	3	1	2	$(0,1,2,2)$	2	4	1	3
$(2,3,4,2)$	0	0	1	1	$(-1,0,0,0)$	2	4	1	2	$(-1,-1,0,0)$	2	2	0	1
$(-1,-1,-1,0)$	1	2	0	1	$(-1,-1,-2,0)$	0	2	0	1	$(-1,-1,-1,-1)$	1	1	0	0
$(1,2,2,0)$	0	0	1	0	$(-1,-1,-2,-1)$	0	1	0	0	$(1,2,2,1)$	0	1	1	1
$(1,1,2,2)$	0	0	0	1	$(1,2,3,1)$	1	1	1	1	$(1,2,2,2)$	0	2	1	2
$(1,2,3,2)$	1	2	1	2	$(1,2,4,2)$	2	2	1	2	$(1,3,4,2)$	2	4	2	3

Table B.8: $\Delta_9 = \{\gamma, -\alpha-\beta-2\gamma-\delta, \alpha+2\beta+2\gamma, \alpha+\beta+2\gamma+2\delta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	0	2	2	1	$(0,0,1,0)$	1	0	0	0	$(0,0,0,1)$	0	1	0	0
$(0,1,1,0)$	1	2	2	1	$(0,0,1,1)$	1	1	0	0	$(0,1,2,0)$	2	2	2	1
$(0,1,1,1)$	1	3	2	1	$(0,1,2,1)$	2	3	2	1	$(0,1,2,2)$	2	4	2	1
$(2,3,4,2)$	0	0	0	1	$(-1,0,0,0)$	2	4	3	1	$(-1,-1,0,0)$	2	2	1	0
$(-1,-1,-1,0)$	1	2	1	0	$(-1,-1,-2,0)$	0	2	1	0	$(-1,-1,-1,-1)$	1	1	1	0
$(1,2,2,0)$	0	0	1	1	$(-1,-1,-2,-1)$	0	1	1	0	$(1,2,2,1)$	0	1	1	1
$(-1,-1,-2,-2)$	0	0	1	0	$(1,2,3,1)$	1	1	1	1	$(1,2,2,2)$	0	2	1	1
$(1,2,3,2)$	1	2	1	1	$(1,2,4,2)$	2	2	1	1	$(1,3,4,2)$	2	4	3	2

Table B.9: $\Delta_{10} = \{\gamma, \delta, -\alpha-\beta-2\gamma-2\delta, 2\alpha+3\beta+4\gamma+2\delta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	1	0	0	0	$(0,0,1,0)$	0	1	0	1	$(0,0,0,1)$	1	0	1	1
$(0,1,1,0)$	1	1	0	1	$(0,0,1,1)$	1	1	1	2	$(0,1,2,0)$	1	2	0	2
$(0,1,1,1)$	2	1	1	2	$(0,1,2,1)$	2	2	1	3	$(0,1,2,2)$	3	2	2	4
$(2,3,4,2)$	1	0	0	2	$(-1,0,0,0)$	2	2	1	2	$(-1,-1,0,0)$	1	2	1	2
$(-1,-1,-1,0)$	1	1	1	1	$(-1,-1,-2,0)$	1	0	1	0	$(-1,-1,-1,-1)$	0	1	0	0
$(-1,-2,-2,0)$	0	0	1	0	$(1,1,2,1)$	0	0	0	1	$(1,2,2,1)$	1	0	0	1
$(1,1,2,2)$	1	0	1	2	$(1,2,3,1)$	1	1	0	2	$(1,2,2,2)$	2	0	1	2
$(1,2,3,2)$	2	1	1	3	$(1,2,4,2)$	2	2	1	4	$(1,3,4,2)$	3	2	1	4

Table B.10: $\Delta_{11} = \{\beta, -\alpha-\beta-\gamma-\delta, -\alpha-2\beta-2\gamma, \alpha+\beta+2\gamma+\delta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(0,1,0,0)$	0	0	2	1	$(0,0,1,0)$	1	0	0	0	$(0,0,0,1)$	0	1	1	1
$(0,1,1,0)$	1	0	2	1	$(0,0,1,1)$	1	1	1	1	$(0,1,2,0)$	2	0	2	1
$(0,1,1,1)$	1	1	3	2	$(0,1,2,1)$	2	1	3	2	$(0,1,2,2)$	2	2	4	3
$(2,3,4,2)$	0	0	0	1	$(-1,0,0,0)$	2	1	4	2	$(-1,-1,0,0)$	2	1	2	1
$(-1,-1,-1,0)$	1	1	2	1	$(-1,-1,-2,0)$	0	1	2	1	$(-1,-1,-1,-1)$	1	0	1	0
$(-1,-2,-2,0)$	0	1	0	0	$(-1,-1,-2,-1)$	0	0	1	0	$(1,2,2,1)$	0	0	1	1
$(1,1,2,2)$	0	1	0	1	$(1,2,3,1)$	1	0	1	1	$(1,2,2,2)$	0	1	2	2
$(1,2,3,2)$	1	1	2	2	$(1,2,4,2)$	2	1	2	2	$(1,3,4,2)$	2	1	4	3

Table B.11: $\Delta_{12} = \{\gamma, -\alpha-2\beta-2\gamma, -\alpha-\beta-2\gamma-\delta, 2\alpha+3\beta+4\gamma+2\delta\}$

B.2 Understanding $\mathfrak{k}^{h_j}$

For each of the semisimple elements h_j with $j \neq 3, 4$, or 8, we determine which basis vectors commute with h_j in order to show that $\mathfrak{k}^{h_j} \subset \mathfrak{l}_j$. Beside each h_j below, the roots associated to root vectors commuting with h_j are listed. Since $h_j \in \mathfrak{h}$, it follows that $\mathfrak{h}$ is contained in $\mathfrak{k}^{h_j}$. The span of the root vectors associated to the roots listed below are a subset of the roots in Table 7.1, hence the span of those root vectors is also contained in $\mathfrak{k}^{h_j}$.

Semisimple Element	Roots
h_1	$\pm(0,1,0,0)$, $\pm(0,0,1,0)$, $\pm(0,0,0,1)$, $\pm(0,1,1,0)$, $\pm(0,0,1,1)$, $\pm(0,1,1,1)$, $\pm(0,1,2,0)$, $\pm(0,1,2,1)$, $\pm(0,1,2,2)$
h_2	$\pm(0,0,1,0)$, $\pm(0,0,0,1)$, $\pm(0,0,1,1)$
h_5	$\pm(0,1,0,0)$, $\pm(0,0,0,1)$
h_6	$\pm(0,0,1,0)$
h_7	$\pm(0,1,1,0)$
h_9	None
h_{10}	$\pm(0,0,1,0)$, $\pm(0,0,0,1)$, $\pm(0,0,1,1)$, $\pm(2,3,4,2)$
h_{11}	None
h_{12}	None

B.3 Supplement to Theorem 7.5

We need to prove that $k_4^2 = 1$ for $k_4 := Exp(\pi i(h_\gamma + h_\delta))Exp(2x_\gamma)$. We will do this by using working within the Adjoint representation. If the matrix representing k_4 has order two, then it follows that k_4 generates the desired $\mathbb{Z}_2$.

$k_4.x_{(1,0,0,0)} = x_{(1,0,0,0)}$, $k_4.x_{(0,1,0,0)} = x_{(0,1,0,0)} + 2x_{(0,1,1,0)} + 4x_{(0,1,2,0)}$, $k_4.x_{(0,0,1,0)} = -x_{(0,0,1,0)}$

$k_4.x_{(0,0,0,1)} = -x_{(0,0,0,1)} + 2x_{(0,0,1,1)}$, $k_4.x_{(1,1,0,0)} = x_{(1,1,0,0)} + 2x_{(1,1,1,0)} + 4x_{(1,1,2,0)}$

$k_4.x_{(0,1,1,0)} = -x_{(0,1,1,0)} - 4x_{(0,1,2,0)}$, $k_4.x_{(0,0,1,1)} = x_{(0,0,1,1)}$, $k_4.x_{(1,1,1,0)} = -x_{(1,1,1,0)} - 4x_{(1,1,2,0)}$,

$k_4.x_{(0,1,2,0)} = x_{(0,1,2,0)}$, $k_4.x_{(0,1,1,1)} = x_{(0,1,1,1)} + 2x_{(0,1,2,1)}$, $k_4.x_{(1,1,2,0)} = x_{(1,1,2,0)}$

$k_4.x_{(1,1,1,1)} = x_{(1,1,1,1)} + 2x_{(1,1,2,1)}$, $k_4.x_{(0,1,2,1)} = -x_{(0,1,2,1)}$, $k_4.x_{(1,2,2,0)} = x_{(1,2,2,0)}$

$k_4.x_{(1,1,2,1)} = -x_{(1,1,2,1)}$, $k_4.x_{(0,1,2,2)} = x_{(0,1,2,2)}$, $k_4.x_{(1,2,2,1)} = -x_{(1,2,2,1)} - 2x_{(1,2,3,1)}$

$k_4.x_{(1,1,2,2)} = x_{(1,1,2,2)}$, $k_4.x_{(1,2,3,1)} = x_{(1,2,3,1)}$, $k_4.x_{(1,2,2,2)} = x_{(1,2,2,2)} + 2x_{(1,2,3,2)} + 4x_{(1,2,4,2)}$

$k_4.x_{(1,2,3,2)} = -x_{(1,2,3,2)} - 4x_{(1,2,4,2)}$, $k_4.x_{(1,2,4,2)} = x_{(1,2,4,2)}$, $k_4.x_{(1,3,4,2)} = x_{(1,3,4,2)}$

$k_4.x_{(2,3,4,2)} = x_{(2,3,4,2)}$, $k_4.x_{(-1,0,0,0)} = x_{(-1,0,0,0)}$, $k_4.x_{(0,-1,0,0)} = x_{(0,-1,0,0)}$

$k_4.x_{(0,0,-1,0)} = -x_{(0,0,-1,0)}$, $k_4.x_{(0,0,0,-1)} = -x_{(0,0,0,-1)}$, $k_4.x_{(-1,-1,0,0)} = x_{(-1,-1,0,0)}$

$k_4.x_{(0,-1,-1,0)} = 4x_{(0,-1,0,0)} - x_{(0,-1,-1,0)}$, $k_4.x_{(0,0,-1,-1)} = 2x_{(0,0,0,-1)} + x_{(0,0,-1,-1)}$

$k_4.x_{(-1,-1,-1,0)} = 4x_{(-1,-1,0,0)} - x_{(-1,-1,-1,0)}$

$k_4.x_{(0,-1,-2,0)} = 4x_{(0,-1,0,0)} - 2x_{(0-1,-1,0)} + x_{(0,-1,-2,0)}$, $k_4.x_{(0,-1,-1,-1)} = -x_{(0,-1,-1,-1)}$

$k_4.x_{(-1,-1,-2,0)} = 4x_{(-1,-1,0,0)} - 2x_{(-1,-1,-1,0)} + x_{(-1,-1,-2,0)}$, $k_4.x_{(-1,-1,-1,-1)} = x_{(-1,-1,-1,-1)}$

$k_4.x_{(0,-1,-2,-1)} = 2x_{(0,-1,-1,-1)} - x_{(0,-1,-2,-1)}$, $k_4.x_{(-1,-2,-2,0)} = x_{(-1,-2,-2,0)}$

$k_4.x_{(-1,-1,-2,-1)} = 2x_{(-1,-1,-1,-1)} - x_{(-1,-1,-2,-1)}$, $k_4.x_{(0,-1,-2,-2)} = x_{(0,-1,-2,-2)}$

$k_4.x_{(-1,-2,-2,-1)} = -x_{(-1,-2,-2,-1)}$, $k_4.x_{(-1,-1,-2,-2)} = x_{(-1,-1,-2,-2)}$

$k_4.x_{(-1,-2,-3,-1)} = -2x_{(-1,-2,-2,-1)} + x_{(-1,-2,-3,-1)}$, $k_4.x_{(-1,-2,-2,-2)} = x_{(-1,-2,-2,-2)}$

$k_4.x_{(-1,-2,-3,-2)} = 4x_{(-1,-2,-2,-2)} - x_{(-1,-2,-3,-2)}$

$k_4.x_{(-1,-2,-4,-2)} = 4x_{(-1,-2,-2,-2)} - 2x_{(-1,-2,-3,-2)} + x_{(-1,-2,-4,-2)}$

$k_4.x_{(-1,-3,-4,-2)} = x_{(-1,-3,-4,-2)}$, $k_4.x_{(-2,-3,-4,-2)} = x_{(-2,-3,-4,-2)}$

$k_4.h_{(1,0,0,0)} = h_{(1,0,0,0)}$

Since the matrix is so large, we will give it in terms of our matrices $E_{i,j}^{52}$.

$E_{5,5}^{52} + E_{6,6}^{52} + 2E_{10,6}^{52} + 4E_{13,6}^{52} - E_{7,7}^{52} - E_{8,8}^{52} + 2E_{11,8}^{52} + E_{9,9}^{52} + 2E_{12,9}^{52} + 4E_{15,9}^{52} - E_{10,10}^{52} - 4E_{13,10}^{52} + E_{11,11}^{52} - E_{12,12}^{52} - 4E_{15,12}^{52} + E_{13,13}^{52} + E_{14,14}^{52} + 2E_{17,14}^{52} + E_{15,15}^{52} + E_{16,16}^{52} + 2E_{19,16}^{52} - E_{17,17}^{52} + E_{18,18}^{52} - E_{19,19}^{52} + E_{20,20}^{52} - E_{21,21}^{52} - 2E_{23,21}^{52} + E_{22,22}^{52} + E_{23,23}^{52} + E_{24,24}^{52} + 2E_{25,24}^{52} + 4E_{26,24}^{52} - E_{25,25}^{52} - 4E_{26,25}^{52} + E_{26,26}^{52} + E_{27,27}^{52} + E_{28,28}^{52} + E_{29,29}^{52} + E_{30,30}^{52} - E_{31,31}^{52} - E_{32,32}^{52} + E_{33,33}^{52} + 4E_{30,34}^{52} - E_{34,34}^{52} + 2E_{32,35}^{52} + E_{35,35}^{52} + 4E_{33,36}^{52} - E_{36,36}^{52} + 4E_{30,37}^{52} - 2E_{34,37}^{52} + E_{37,37}^{52} + E_{38,38}^{52} + 4E_{33,39}^{52} - 2E_{36,39}^{52} + E_{39,39}^{52} + E_{40,40}^{52} + 2E_{38,41}^{52} - E_{41,41}^{52} + E_{42,42}^{52} - E_{43,43}^{52} + E_{44,44}^{52} - E_{45,45}^{52} + E_{46,46}^{52} - 2E_{45,47}^{52} + E_{47,47}^{52} + E_{48,48}^{52} + 4E_{48,49}^{52} - E_{49,49}^{52} + 4E_{48,50}^{52} - 2E_{49,50}^{52} + E_{50,50}^{52} + E_{51,51}^{52} + E_{52,52}^{52} + E_{1,1}^{52} + E_{2,2}^{52} - 2E_{7,2}^{52} + E_{3,3}^{52} - 4E_{7,3}^{52} + E_{4,4}^{52} - 2E_{7,4}^{52}$

Embedding the data into a matrix algebra system, one sees by direct computation that the matrix has order two, so k_4 is indeed our desired component group generator.

B.4 Computing the Dimension of $\mathfrak{g}(3,2)$

We need to compute $\mathfrak{g}(3,2)$ in order to determine the component group $A_K(f_3)$. Given an arbitrary element

$$p = \sum_{\xi \in \Gamma_{\mathfrak{p}}} m_\xi x_\xi$$

of $\mathfrak{p}$, we see that

$$[h_3, p] = 2p$$

if and only if $m_{(1,1,0,0)} = 0$, $m_{(1,1,2,0)} = 0$, $m_{(1,1,2,1)} = 0$, $m_{(1,1,2,2)} = 0$, $m_{(1,2,3,1)} = 0$, $m_{(1,2,3,2)} = 0$, $m_{(1,2,4,2)} = 0$, $m_{(1,3,4,2)} = 0$, $m_{(-1,0,0,0)} = 0$, $m_{(-1,-1,-1,0)} = 0$, $m_{(-1,-1,-2,0)} = 0$, $m_{(-1,-1,-1,-1)} = 0$, $m_{(-1,-2,-2,0)} = 0$, $m_{(-1,-1,-2,-1)} = 0$, $m_{(-1,-2,-2,-1)} = 0$, $m_{(-1,-1,-2,-2)} = 0$, $m_{(-1,-2,-3,-1)} = 0$,

$m_{(-1,-2,-2,-2)} = 0$, $m_{(-1,-2,-3,-2)} = 0$, $m_{(-1,-2,-4,-2)} = 0$, $m_{(-1,-3,-4,-2)} = 0$, so $\mathfrak{g}(3,2)$ has dimension 7.

B.5 Supplement to Proposition 6.1

For the remainder of this appendix, let

$$k = c_1 h_\alpha + c_2 h_\beta + c_3 h_\gamma + c_4 h_\delta + \sum_{\xi \in \Phi(\mathfrak{h},\mathfrak{k})} c_\xi x_\xi$$

be an arbitrary element of $\mathfrak{k}$. Since $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, there exists coefficients m_ξ such that

$$[k,f] = \sum_{\xi \in \Gamma_{\mathfrak{p}}} m_\xi x_\xi \text{ for any } f \in \mathfrak{p}.$$

Proof of Proposition 6.1 (Cont.) We now prove that the K-orbit dimensions are correct for each element f_j listed in Proposition 6.1.

Positive System Φ_1^+ (dim $K.f_1 = 15$) For this first case, Φ_1^+ is equal to $\Phi^+(\mathfrak{h},\mathfrak{g})$, so the nonzero coefficients m_ξ with $f_1 = x_{(-1,0,0,0)} + x_{(-1,-2,-2,-1)} + x_{(-1,-2,-4,-2)}$ are given by $m_{(1,1,0,0)} = -c_{(2,3,4,2)}$, $m_{(1,1,2,1)} = c_{(2,3,4,2)}$, $m_{(1,3,4,2)} = -c_{(2,3,4,2)}$, $m_{(-1,0,0,0)} = -2c_1 + c_2$, $m_{(-1,-1,0,0)} = -c_{(0,-1,0,0)} + 2c_{(0,1,2,1)}$, $m_{(-1,-1,-1,0)} = -c_{(0,-1,-1,0)} - c_{(0,1,1,1)}$, $m_{(-1,-1,-2,0)} = -c_{(0,-1,-2,0)} + c_{(0,1,2,2)}$, $m_{(-1,-1,-1,-1)} = -c_{(0,-1,-1,-1)} - c_{(0,1,1,0)}$, $m_{(-1,-2,-2,0)} = 2c_{(0,0,0,1)}$, $m_{(-1,-1,-2,-1)} = -c_{(0,-1,-2,-1)} + c_{(0,1,0,0)} - c_{(0,1,2,1)}$, $m_{(-1,-2,-2,-1)} = -c_2 + c_3$, $m_{(-1,-1,-2,-2)} = -c_{(0,-1,-2,-2)} + c_{(0,1,2,0)}$, $m_{(-1,-2,-3,-1)} = c_{(0,0,-1,0)} - c_{(0,0,1,1)}$, $m_{(-1,-2,-2,-2)} = 2c_{(0,0,0,-1)}$, $m_{(-1,-2,-3,-2)} = -c_{(0,0,-1,-1)} + c_{(0,0,1,0)}$, $m_{(-1,-2,-4,-2)} = c_2 - 2c_3$, and $m_{(-1,-3,-4,-2)} = 2c_{(0,-1,-2,-1)} - c_{(0,-1,0,0)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,2,1)} = 1/2c_{(0,1,0,0)}$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = c_{(0,1,0,0)}$, $c_{(0,0,-1,0)} = c_{(0,0,1,1)}$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = -c_{(0,1,1,1)}$, $c_{(0,0,-1,-1)} = c_{(0,0,1,0)}$, $c_{(0,-1,-2,0)} = c_{(0,1,2,2)}$, $c_{(0,-1,-1,-1)} = -c_{(0,1,1,0)}$, $c_{(0,-1,-2,-1)} = 1/2c_{(0,1,0,0)}$, and

$c_{(0,-1,-2,-2)} = c_{(0,1,2,0)}$. Therefore, $\mathfrak{k}^{f_1}$ has dimension 9, hence $K.f_1$ has dimension 15.

Positive System Φ_2^+ (dim $K.f_2 = 21$) For this case, the nonzero coefficients m_ξ with $f_2 = x_{(1,0,0,0)} + x_{(-1,-1,0,0)} + x_{(-1,-1,-2,-1)}$ are $m_{(1,0,0,0)} = 2c_1 - c_2$, $m_{(1,1,0,0)} = c_{(0,1,0,0)}$, $m_{(1,1,1,0)} = c_{(0,1,1,0)}$, $m_{(1,1,2,0)} = c_{(0,1,2,0)}$, $m_{(1,1,1,1)} = c_{(0,1,1,1)}$, $m_{(1,1,2,1)} = c_{(0,1,2,1)}$, $m_{(1,2,2,1)} = -c_{(2,3,4,2)}$, $m_{(1,1,2,2)} = c_{(0,1,2,2)}$, $m_{(1,2,4,2)} = c_{(2,3,4,2)}$, $m_{(-1,0,0,0)} = -c_{(0,1,0,0)} - 2c_{(0,1,2,1)}$, $m_{(-1,-1,0,0)} = -c_1 - c_2 + 2c_3$, $m_{(-1,-1,-1,0)} = c_{(0,0,-1,0)} + c_{(0,0,1,1)}$, $m_{(-1,-1,-2,0)} = 2c_{(0,0,0,1)}$, $m_{(-1,-1,-1,-1)} = c_{(0,0,-1,-1)} + c_{(0,0,1,0)}$, $m_{(-1,-2,-2,0)} = c_{(0,-1,-2,0)}$, $m_{(-1,-1,-2,-1)} = -c_1 + c_2 - c_3$, $m_{(-1,-2,-2,-1)} = c_{(0,-1,-2,-1)} + c_{(0,-1,0,0)}$, $m_{(-1,-1,-2,-2)} = 2c_{(0,0,0,-1)}$, $m_{(-1,-2,-3,-1)} = c_{(0,-1,-1,0)}$, $m_{(-1,-2,-2,-2)} = c_{(0,-1,-2,-2)}$, $m_{(-1,-2,-3,-2)} = -c_{(0,-1,-1,-1)}$, $m_{(-1,-2,-4,-2)} = -2c_{(0,-1,-2,-1)}$, and $m_{(-1,-3,-4,-2)} = c_{(-2,-3,-4,-2)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = 0$, $c_{(0,0,-1,0)} = -c_{(0,0,1,1)}$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = 0$, $c_{(0,0,-1,-1)} = -c_{(0,0,1,0)}$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = 0$, $c_{(0,-1,-2,-1)} = 0$, $c_{(0,-1,-2,-2)} = 0$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_2}$ has dimension 3, hence $K.f_2$ has dimension 21.

Positive System Φ_3^+ (dim $K.f_3 = 21$) For this case, the nonzero coefficients m_ξ with $f_3 = x_{(1,1,0,0)} + x_{(-1,-1,-1,-1)} + x_{(-1,-2,-2,0)}$ are $m_{(1,0,0,0)} = c_{(0,-1,0,0)}$, $m_{(1,1,0,0)} = c_1 + c_2 - 2c_3$, $m_{(1,1,1,0)} = -c_{(0,0,1,0)}$, $m_{(1,1,1,1)} = -c_{(0,0,1,1)}$, $m_{(1,2,2,0)} = -c_{(0,1,2,0)}$, $m_{(1,2,2,1)} = -c_{(0,1,2,1)}$, $m_{(1,1,2,2)} = -c_{(2,3,4,2)}$, $m_{(1,2,3,1)} = c_{(2,3,4,2)}$, $m_{(1,2,2,2)} = -c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -2c_{(0,1,1,1)}$, $m_{(-1,-1,0,0)} = 2c_{(0,0,1,1)} + c_{(0,1,2,0)}$, $m_{(-1,-1,-1,0)} = c_{(0,0,0,1)} - c_{(0,1,1,0)}$, $m_{(-1,-1,-2,0)} = c_{(0,1,0,0)}$, $m_{(-1,-1,-1,-1)} = -c_1 + c_3 - c_4$, $m_{(-1,-2,-2,0)} = -c_2 + 2c_4$, $m_{(-1,-1,-2,-1)} = c_{(0,0,-1,0)}$, $m_{(-1,-2,-2,-1)} = -c_{(0,-1,-1,0)} + c_{(0,0,0,-1)}$, $m_{(-1,-1,-2,-2)} = 2c_{(0,0,-1,-1)}$, $m_{(-1,-2,-3,-1)} = -c_{(0,-1,-2,0)} - c_{(0,0,-1,-1)}$, $m_{(-1,-2,-2,-2)} = -2c_{(0,-1,-1,-1)}$, $m_{(-1,-2,-3,-2)} = -c_{(0,-1,-2,-1)}$, $m_{(-1,-2,-4,-2)} = -c_{(-2,-3,-4,-2)}$, and $m_{(-1,-3,-4,-2)} = -c_{(0,-1,-2,-2)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_3 = c_2/2$, $c_4 = c_2/2$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,1,1,0)} = c_{(0,0,0,1)}$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$,

$c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = 0$, $c_{(0,0,-1,0)} = 0$, $c_{(0,-1,-1,0)} = c_{(0,0,0,-1)}$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = 0$, $c_{(0,-1,-2,-1)} = 0$, $c_{(0,-1,-2,-2)} = 0$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_3}$ has dimension 3, hence $K.f_3$ has dimension 21.

Positive System Φ_4^+ (dim $K.f_4 = 23$) For this case, the nonzero coefficients m_ξ with $f_4 = x_{(1,1,0,0)} + x_{(1,1,1,0)} + x_{(-1,-1,-1,-1)} + x_{(-1,-2,-2,0)}$ are $m_{(1,0,0,0)} = 2c_{(0,-1,-1,0)} + c_{(0,-1,0,0)}$, $m_{(1,1,0,0)} = c_1 + c_2 - 2c_3 - 2c_{(0,0,-1,0)}$, $m_{(1,1,1,0)} = c_1 - c_4 - c_{(0,0,1,0)}$, $m_{(1,1,2,0)} = -2c_{(0,0,1,0)}$, $m_{(1,1,1,1)} = -c_{(0,0,0,1)} - c_{(0,0,1,1)}$, $m_{(1,2,2,0)} = 2c_{(0,1,1,0)} - c_{(0,1,2,0)}$, $m_{(1,1,2,1)} = -c_{(0,0,1,1)}$, $m_{(1,2,2,1)} = c_{(0,1,1,1)} - c_{(0,1,2,1)}$, $m_{(1,1,2,2)} = -c_{(2,3,4,2)}$, $m_{(1,2,3,1)} = -c_{(0,1,2,1)} + c_{(2,3,4,2)}$, $m_{(1,2,2,2)} = -c_{(0,1,2,2)}$, $m_{(1,2,3,2)} = -c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -2c_{(0,1,1,1)}$, $m_{(-1,-1,0,0)} = 2c_{(0,0,1,1)} + c_{(0,1,2,0)}$, $m_{(-1,-1,-1,0)} = c_{(0,0,0,1)} - c_{(0,1,1,0)}$, $m_{(-1,-1,-2,0)} = c_{(0,1,0,0)}$, $m_{(-1,-1,-1,-1)} = -c_1 + c_3 - c_4$, $m_{(-1,-2,-2,0)} = -c_2 + 2c_4$, $m_{(-1,-1,-2,-1)} = c_{(0,0,-1,0)}$, $m_{(-1,-2,-2,-1)} = -c_{(0,-1,-1,0)} + c_{(0,0,0,-1)}$, $m_{(-1,-1,-2,-2)} = 2c_{(0,0,-1,-1)}$, $m_{(-1,-2,-3,-1)} = -c_{(0,-1,-2,0)} - c_{(0,0,-1,-1)}$, $m_{(-1,-2,-2,-2)} = -2c_{(0,-1,-1,-1)}$, $m_{(-1,-2,-3,-2)} = c_{(-2,-3,-4,-2)} - c_{(0,-1,-2,-1)}$, $m_{(-1,-2,-4,-2)} = -c_{(-2,-3,-4,-2)}$, and $m_{(-1,-3,-4,-2)} = -c_{(0,-1,-2,-2)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,0,-1,0)} = 0$, $c_{(0,0,0,-1)} = -1/2c_{(0,-1,0,0)}$, $c_{(0,-1,-1,0)} = -1/2c_{(0,-1,0,0)}$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = 0$, $c_{(0,-1,-2,-1)} = 0$, $c_{(0,-1,-2,-2)} = 0$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_4}$ has dimension 1, hence $K.f_4$ has dimension 23.

Positive System Φ_5^+ (dim $K.f_5 = 22$) For this case, the nonzero coefficients m_ξ with $f_5 = x_{(1,1,1,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,0)} + x_{(-1,-1,-2,-2)}$ are $m_{(1,0,0,0)} = 2(c_{(0,-1,-1,-1)} + c_{(0,-1,-1,0)})$, $m_{(1,1,0,0)} = -2(c_{(0,0,-1,-1)} + c_{(0,0,-1,0)})$, $m_{(1,1,1,0)} = c_1 - c_4 - c_{(0,0,0,-1)}$, $m_{(1,1,2,0)} = -2c_{(0,0,1,0)}$, $m_{(1,1,1,1)} = c_1 - c_3 + c_4 - c_{(0,0,0,1)}$, $m_{(1,2,2,0)} = 2c_{(0,1,1,0)} + c_{(2,3,4,2)}$, $m_{(1,1,2,1)} = -c_{(0,0,1,0)} - c_{(0,0,1,1)}$, $m_{(1,2,2,1)} = c_{(0,1,1,0)} + c_{(0,1,1,1)}$, $m_{(1,1,2,2)} = -2c_{(0,0,1,1)} - c_{(2,3,4,2)}$, $m_{(1,2,3,1)} = c_{(0,1,2,0)} - c_{(0,1,2,1)}$, $m_{(1,2,2,2)} = 2c_{(0,1,1,1)}$, $m_{(1,2,3,2)} = c_{(0,1,2,1)} - c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = c_{(0,1,2,0)}$, $m_{(-1,-1,-1,0)} = -c_{(0,1,1,0)}$,

$m_{(-1,-1,-2,0)} = c_{(0,1,0,0)}$, $m_{(-1,-1,-1,-1)} = c_{(0,0,1,1)}$, $m_{(-1,-2,-2,0)} = -c_2 + 2c_4, m_{(-1,-1,-2,-1)} = c_{(0,0,0,1)}$, $m_{(-1,-2,-2,-1)} = c_{(0,0,0,-1)}$, $m_{(-1,-1,-2,-2)} = -c_1 + c_2 - 2c_4$, $m_{(-1,-2,-3,-1)} = -c_{(-2,-3,-4,-2)} - c_{(0,0,-1,-1)}$, $m_{(-1,-2,-2,-2)} = c_{(0,-1,0,0)}$, $m_{(-1,-2,-3,-2)} = c_{(-2,-3,-4,-2)} + c_{(0,-1,-1,0)}$, $m_{(-1,-2,-4,-2)} = c_{(0,-1,-2,0)}$, and $m_{(-1,-3,-4,-2)} = -c_{(0,-1,-2,-2)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = 0$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = -c_{(0,0,-1,0)}$, $c_{(0,0,-1,-1)} = -c_{(0,0,-1,0)}$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = c_{(0,0,-1,0)}$, $c_{(0,-1,-2,-2)} = 0$, and $c_{(-2,-3,-4,-2)} = c_{(0,0,-1,0)}$. Therefore, $\mathfrak{k}^{f_5}$ has dimension 2, hence $K.f_5$ has dimension 22.

Positive System Φ_6^+ (dim $K.f_6 = 23$) We have already determined the dimension because $f_4 = f_6$. Since Φ_4^+ and Φ_6^+ intersect in such a way that f_6 lives in $\mathfrak{n}_4^- \cap \mathfrak{n}_6^- \cap \mathfrak{p}$, it follows that $\mathfrak{k}^{f_6}$ has dimension 1, hence $K.f_6$ has dimension 23. It is interesting to note that although the generic elements are the same, the corresponding Springer fibers are quite different.

Positive System Φ_7^+ (dim $K.f_7 = 20$) For this case, the nonzero coefficients m_ξ with $f_7 = x_{(1,1,1,0)} + x_{(1,2,2,0)} + x_{(-1,-2,-2,-1)} + x_{(-1,-2,-4,-2)}$ are $m_{(1,0,0,0)} = 2c_{(0,-1,-1,0)}$, $m_{(1,1,0,0)} = -c_{(0,-1,-2,0)} - 2c_{(0,0,-1,0)} - c_{(2,3,4,2)}$, $m_{(1,1,1,0)} = c_1 - c_4 + c_{(0,-1,-1,0)}$, $m_{(1,1,2,0)} = -c_{(0,-1,0,0)} - 2c_{(0,0,1,0)}$, $m_{(1,1,1,1)} = -c_{(0,0,0,1)}$, $m_{(1,2,2,0)} = c_2 - 2c_4 + 2c_{(0,1,1,0)}$, $m_{(1,1,2,1)} = -c_{(0,0,1,1)} + c_{(2,3,4,2)}$, $m_{(1,2,2,1)} = -c_{(0,0,0,1)} + c_{(0,1,1,1)}$, $m_{(1,2,3,1)} = c_{(0,0,1,1)} - c_{(0,1,2,1)}$, $m_{(1,2,3,2)} = -c_{(0,1,2,2)}$, $m_{(1,3,4,2)} = c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = 2c_{(0,1,2,1)}$, $m_{(-1,-1,-1,0)} = -c_{(0,1,1,1)}$, $m_{(-1,-1,-2,0)} = c_{(0,1,2,2)}$, $m_{(-1,-1,-1,-1)} = -c_{(0,1,1,0)}$, $m_{(-1,-2,-2,0)} = 2c_{(0,0,0,1)}$, $m_{(-1,-1,-2,-1)} = c_{(0,1,0,0)} - c_{(0,1,2,1)}$, $m_{(-1,-2,-2,-1)} = -c_2 + c_3$, $m_{(-1,-1,-2,-2)} = c_{(-2,-3,-4,-2)} + c_{(0,1,2,0)}$, $m_{(-1,-2,-3,-1)} = c_{(0,0,-1,0)} - c_{(0,0,1,1)}$, $m_{(-1,-2,-2,-2)} = 2c_{(0,0,0,-1)}$, $m_{(-1,-2,-3,-2)} = c_{(-2,-3,-4,-2)} - c_{(0,0,-1,-1)} + c_{(0,0,1,0)}$, $m_{(-1,-2,-4,-2)} = c_2 - 2c_3$, and $m_{(-1,-3,-4,-2)} = 2c_{(0,-1,-2,-1)} - c_{(0,-1,0,0)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$,

$c_{(0,1,0,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = -2c_{(0,0,1,0)}$, $c_{(0,0,-1,0)} = 0$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = 0$, $c_{(0,0,-1,-1)} = c_{(0,0,1,0)} - c_{(0,1,2,0)}$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-2,-1)} = -c_{(0,0,1,0)}$, and $c_{(-2,-3,-4,-2)} = -c_{(0,1,2,0)}$. Therefore, $\mathfrak{k}^{f_7}$ has dimension 4, hence $K.f_7$ has dimension 20.

Positive System Φ_8^+ (dim $K.f_8 = 24$) For this case, the nonzero coefficients m_ξ with $f_8 = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,0)} + x_{(-1,-1,-2,-1)}$ are $m_{(1,0,0,0)} = c_{(0,-1,-2,0)} + 2c_{(0,-1,-1,-1)}$, $m_{(1,1,0,0)} = -2c_{(0,0,-1,-1)}$, $m_{(1,1,1,0)} = -c_{(0,0,-1,0)} - c_{(0,0,0,-1)}$, $m_{(1,1,2,0)} = c_1 - c_2 + 2c_3 - 2c_4$, $m_{(1,1,1,1)} = c_1 - c_3 + c_4$, $m_{(1,2,2,0)} = -c_{(0,1,0,0)}$, $m_{(1,1,2,1)} = -c_{(0,0,0,1)} - c_{(0,0,1,0)}$, $m_{(1,2,2,1)} = c_{(0,1,1,0)} - c_{(2,3,4,2)}$, $m_{(1,1,2,2)} = -2c_{(0,0,1,1)} - c_{(2,3,4,2)}$, $m_{(1,2,3,1)} = c_{(0,1,1,1)} + c_{(0,1,2,0)}$, $m_{(1,2,2,2)} = 2c_{(0,1,1,1)}$, $m_{(1,2,3,2)} = c_{(0,1,2,1)}$, $m_{(1,2,4,2)} = -c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -2c_{(0,1,2,1)}$, $m_{(-1,-1,0,0)} = c_{(0,1,2,0)}$, $m_{(-1,-1,-1,0)} = c_{(0,0,1,1)} - c_{(0,1,1,0)}$, $m_{(-1,-1,-2,0)} = 2c_{(0,0,0,1)} + c_{(0,1,0,0)}$, $m_{(-1,-1,-1,-1)} = c_{(0,0,1,0)}$, $m_{(-1,-2,-2,0)} = -c_2 + 2c_4$, $m_{(-1,-1,-2,-1)} = -c_1 + c_2 - c_3$, $m_{(-1,-2,-2,-1)} = c_{(0,-1,0,0)} + c_{(0,0,0,-1)}$, $m_{(-1,-1,-2,-2)} = 2c_{(0,0,0,-1)}$, $m_{(-1,-2,-3,-1)} = -c_{(-2,-3,-4,-2)} + c_{(0,-1,-1,0)} - c_{(0,0,-1,-1)}$, $m_{(-1,-2,-2,-2)} = -c_{(-2,-3,-4,-2)}$, $m_{(-1,-2,-3,-2)} = -c_{(0,-1,-1,-1)}$, $m_{(-1,-2,-4,-2)} = -2c_{(0,-1,-2,-1)}$, and $m_{(-1,-3,-4,-2)} = -c_{(0,-1,-2,-2)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = 0$, $c_{(0,0,-1,0)} = 0$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = 0$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = 0$, $c_{(0,-1,-2,-1)} = 0$, $c_{(0,-1,-2,-2)} = 0$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_8}$ has dimension 0, hence $K.f_8$ has dimension 24.

Positive System Φ_9^+ (dim $K.f_9 = 23$) For this case, the nonzero coefficients m_ξ with $f_9 = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,0)} + x_{(-1,-1,-2,-2)}$ are $m_{(1,0,0,0)} = c_{(0,-1,-2,0)} + 2c_{(0,-1,-1,-1)}$, $m_{(1,1,0,0)} = -2c_{(0,0,-1,-1)}$, $m_{(1,1,1,0)} = -c_{(0,0,-1,0)} - c_{(0,0,0,-1)}$, $m_{(1,1,2,0)} = c_1 - c_2 + 2c_3 - 2c_4$, $m_{(1,1,1,1)} = c_1 - c_3 + c_4$, $m_{(1,2,2,0)} = -c_{(0,1,0,0)} + c_{(2,3,4,2)}$, $m_{(1,1,2,1)} = -c_{(0,0,0,1)} - c_{(0,0,1,0)}$, $m_{(1,2,2,1)} = c_{(0,1,1,0)}$, $m_{(1,1,2,2)} = -2c_{(0,0,1,1)} - c_{(2,3,4,2)}$, $m_{(1,2,3,1)} = c_{(0,1,1,1)} + c_{(0,1,2,0)}$, $m_{(1,2,2,2)} = 2c_{(0,1,1,1)}$, $m_{(1,2,3,2)} =$

$c_{(0,1,2,1)}$, $m_{(1,2,4,2)} = -c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = c_{(0,1,2,0)}$, $m_{(-1,-1,-1,0)} = -c_{(0,1,1,0)}$, $m_{(-1,-1,-2,0)} = c_{(0,1,0,0)}$, $m_{(-1,-1,-1,-1)} = c_{(0,0,1,1)}$, $m_{(-1,-2,-2,0)} = -c_2 + 2c_4$, $m_{(-1,-1,-2,-1)} = c_{(0,0,0,1)}$, $m_{(-1,-2,-2,-1)} = c_{(0,0,0,-1)}$, $m_{(-1,-1,-2,-2)} = -c_1 + c_2 - 2c_4$, $m_{(-1,-2,-3,-1)} = -c_{(-2,-3,-4,-2)} - c_{(0,0,-1,-1)}$, $m_{(-1,-2,-2,-2)} = -c_{(-2,-3,-4,-2)} + c_{(0,-1,0,0)}$, $m_{(-1,-2,-3,-2)} = c_{(0,-1,-1,0)}$, $m_{(-1,-2,-4,-2)} = c_{(0,-1,-2,0)}$, and $m_{(-1,-3,-4,-2)} = -c_{(0,-1,-2,-2)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,-1,0,0)} = 0$, $c_{(0,0,-1,0)} = 0$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = 0$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = 0$, $c_{(0,-1,-2,-2)} = 0$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_9}$ has dimension 1, hence $K.f_9$ has dimension 23.

Positive System Φ^+_{10} (dim $K.f_{10} = 20$) For this case, the nonzero coefficients m_ξ with $f_{10} = x_{(1,1,1,0)} + x_{(1,1,2,0)} + x_{(-1,-2,-2,0)} + x_{(-1,-2,-4,-2)}$ are $m_{(1,0,0,0)} = c_{(0,-1,-2,0)} + 2c_{(0,-1,-1,0)}$, $m_{(1,1,0,0)} = -2c_{(0,0,-1,0)} - c_{(2,3,4,2)}$, $m_{(1,1,1,0)} = c_1 - c_4 - c_{(0,0,-1,0)}$, $m_{(1,1,2,0)} = c_1 - c_2 + 2c_3 - 2c_4 - 2c_{(0,0,1,0)}$, $m_{(1,1,1,1)} = -c_{(0,0,0,1)}$, $m_{(1,2,2,0)} = -c_{(0,1,0,0)} + 2c_{(0,1,1,0)}$, $m_{(1,1,2,1)} = -c_{(0,0,0,1)} - c_{(0,0,1,1)}$, $m_{(1,2,2,1)} = c_{(0,1,1,1)}$, $m_{(1,1,2,2)} = -c_{(2,3,4,2)}$, $m_{(1,2,3,1)} = c_{(0,1,1,1)} - c_{(0,1,2,1)}$, $m_{(1,2,3,2)} = -c_{(0,1,2,2)}$, $m_{(1,2,4,2)} = -c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = c_{(0,1,2,0)}$, $m_{(-1,-1,-1,0)} = -c_{(0,1,1,0)}$, $m_{(-1,-1,-2,0)} = c_{(0,1,0,0)} + c_{(0,1,2,2)}$, $m_{(-1,-2,-2,0)} = -c_2 + 2c_4$, $m_{(-1,-1,-2,-1)} = -c_{(0,1,2,1)}$, $m_{(-1,-2,-2,-1)} = c_{(0,0,0,-1)}$, $m_{(-1,-1,-2,-2)} = c_{(0,1,2,0)}$, $m_{(-1,-2,-3,-1)} = -c_{(0,0,-1,-1)} - c_{(0,0,1,1)}$, $m_{(-1,-2,-2,-2)} = -c_{(-2,-3,-4,-2)}$, $m_{(-1,-2,-3,-2)} = c_{(-2,-3,-4,-2)} + c_{(0,0,1,0)}$, $m_{(-1,-2,-4,-2)} = c_2 - 2c_3$, and $m_{(-1,-3,-4,-2)} = -c_{(0,-1,-2,-2)} - c_{(0,-1,0,0)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,0,-1,0)} = 0$, $c_{(0,0,0,-1)} = 0$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = -2c_{(0,-1,-1,0)}$, $c_{(0,-1,-2,-2)} = -c_{(0,-1,0,0)}$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_{10}}$ has dimension 4, hence $K.f_{10}$ has dimension 20.

Positive System Φ^+_{11} (dim $K.f_{11} = 22$) For this case, the nonzero coefficients m_ξ with

$f_{11} = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-1,-2,-1)} + x_{(-1,-3,-4,-2)}$ are $m_{(1,0,0,0)} = c_{(0,-1,-2,0)} + 2c_{(0,-1,-1,-1)} + c_{(2,3,4,2)}$, $m_{(1,1,0,0)} = -2c_{(0,0,-1,-1)}$, $m_{(1,1,1,0)} = -c_{(0,0,-1,0)} - c_{(0,0,0,-1)}$, $m_{(1,1,2,0)} = c_1 - c_2 + 2c_3 - 2c_4$, $m_{(1,1,1,1)} = c_1 - c_3 + c_4$, $m_{(1,2,2,0)} = -c_{(0,1,0,0)}$, $m_{(1,1,2,1)} = -c_{(0,0,0,1)} - c_{(0,0,1,0)}$, $m_{(1,2,2,1)} = c_{(0,1,1,0)} - c_{(2,3,4,2)}$, $m_{(1,1,2,2)} = -2c_{(0,0,1,1)}$, $m_{(1,2,3,1)} = c_{(0,1,1,1)} + c_{(0,1,2,0)}$, $m_{(1,2,2,2)} = 2c_{(0,1,1,1)}$, $m_{(1,2,3,2)} = c_{(0,1,2,1)}$, $m_{(1,2,4,2)} = -c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -2c_{(0,1,2,1)}$, $m_{(-1,-1,-1,0)} = c_{(0,0,1,1)}$, $m_{(-1,-1,-2,0)} = 2c_{(0,0,0,1)}$, $m_{(-1,-1,-1,-1)} = c_{(0,0,1,0)}$, $m_{(-1,-2,-2,0)} = -c_{(0,1,2,2)}$, $m_{(-1,-1,-2,-1)} = -c_1 + c_2 - c_3$, $m_{(-1,-2,-2,-1)} = c_{(0,-1,0,0)} + c_{(0,1,2,1)}$, $m_{(-1,-1,-2,-2)} = 2c_{(0,0,0,-1)}$, $m_{(-1,-2,-3,-1)} = -c_{(-2,-3,-4,-2)} + c_{(0,-1,-1,0)} - c_{(0,1,1,1)}$, $m_{(-1,-2,-2,-2)} = -c_{(-2,-3,-4,-2)} - c_{(0,1,2,0)}$, $m_{(-1,-2,-3,-2)} = -c_{(0,-1,-1,-1)} + c_{(0,1,1,0)}$, $m_{(-1,-2,-4,-2)} = -2c_{(0,-1,-2,-1)} - c_{(0,1,0,0)}$, and $m_{(-1,-3,-4,-2)} = c_1 - c_2$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = c_{(0,1,1,0)}$, $c_{(0,-1,0,0)} = 0$, $c_{(0,0,-1,0)} = 0$, $c_{(0,0,0,-1)} = 0$, $c_{(0,-1,-1,0)} = 0$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = -3c_{(0,1,1,0)}$, $c_{(0,-1,-1,-1)} = c_{(0,1,1,0)}$, $c_{(0,-1,-2,-1)} = 0$, and $c_{(-2,-3,-4,-2)} = 0$. Therefore, $\mathfrak{k}^{f_{11}}$ has dimension 2, hence $K.f_{11}$ has dimension 22.

Positive System Φ_{12}^+ (dim $K.f_{12} = 22$) For this last case, the nonzero coefficients m_ξ with $f_{12} = x_{(1,1,2,0)} + x_{(1,1,1,1)} + x_{(-1,-2,-2,-1)} + x_{(-1,-1,-2,-2)}$ are $m_{(1,0,0,0)} = c_{(0,-1,-2,0)} + 2c_{(0,-1,-1,-1)}$, $m_{(1,1,0,0)} = -2c_{(0,0,-1,-1)}$, $m_{(1,1,1,0)} = -c_{(0,0,-1,0)} - c_{(0,0,0,-1)}$, $m_{(1,1,2,0)} = c_1 - c_2 + 2c_3 - 2c_4$, $m_{(1,1,1,1)} = c_1 - c_3 + c_4$, $m_{(1,2,2,0)} = -c_{(0,1,0,0)} + c_{(2,3,4,2)}$, $m_{(1,1,2,1)} = -c_{(0,0,0,1)} - c_{(0,0,1,0)} + c_{(2,3,4,2)}$, $m_{(1,2,2,1)} = c_{(0,1,1,0)}$, $m_{(1,1,2,2)} = -2c_{(0,0,1,1)}$, $m_{(1,2,3,1)} = c_{(0,1,1,1)} + c_{(0,1,2,0)}$, $m_{(1,2,2,2)} = 2c_{(0,1,1,1)}$, $m_{(1,2,3,2)} = c_{(0,1,2,1)}$, $m_{(1,2,4,2)} = -c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = 2c_{(0,1,2,1)}$, $m_{(-1,-1,-1,0)} = -c_{(0,1,1,1)}$, $m_{(-1,-1,-1,-1)} = c_{(0,0,1,1)} - c_{(0,1,1,0)}$, $m_{(-1,-2,-2,0)} = 2c_{(0,0,0,1)}$, $m_{(-1,-1,-2,-1)} = c_{(0,0,0,1)} + c_{(0,1,0,0)}$, $m_{(-1,-2,-2,-1)} = -c_2 + c_3$, $m_{(-1,-1,-2,-2)} = -c_1 + c_2 - 2c_4$, $m_{(-1,-2,-3,-1)} = -c_{(-2,-3,-4,-2)} + c_{(0,0,-1,0)}$, $m_{(-1,-2,-2,-2)} = -c_{(-2,-3,-4,-2)} + c_{(0,-1,0,0)} + 2c_{(0,0,0,-1)}$, $m_{(-1,-2,-3,-2)} = c_{(0,-1,-1,0)} - c_{(0,0,-1,-1)}$,

$m_{(-1,-2,-4,-2)} = c_{(0,-1,-2,0)}$, and $m_{(-1,-3,-4,-2)} = 2c_{(0,-1,-2,-1)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$, $c_{(0,1,0,0)} = 0$, $c_{(0,0,1,0)} = 0$, $c_{(0,0,0,1)} = 0$, $c_{(0,1,1,0)} = 0$, $c_{(0,0,1,1)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(0,1,1,1)} = 0$, $c_{(0,1,2,1)} = 0$, $c_{(0,1,2,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,0,-1,0)} = 1/3c_{(0,-1,0,0)}$, $c_{(0,0,0,-1)} = -1/3c_{(0,-1,0,0)}$, $c_{(0,-1,-1,0)} = 0$, $c_{(0,0,-1,-1)} = 0$, $c_{(0,-1,-2,0)} = 0$, $c_{(0,-1,-1,-1)} = 0$, $c_{(0,-1,-2,-1)} = 0$, and $c_{(-2,-3,-4,-2)} = 1/3c_{(0,-1,0,0)}$. Therefore, $\mathfrak{k}^{f_{12}}$ has dimension 2, hence $K.f_{12}$ has dimension 22.

To finish Proposition 6.1, we will now prove that the dimension of $K.(\mathfrak{n}_3^- \cap \mathfrak{p})$ is at most 21, and the dimension of $K.(\mathfrak{n}_{11}^- \cap \mathfrak{p})$ is at most 22. Let ξ_i denote *ith* root of

$$\begin{aligned}-\Phi_3^+ \cap \Gamma_{\mathfrak{p}} = \{&(1,0,0,0),(1,1,0,0),(-1,-1,-1,0),(-1,-1,-2,0),(-1,-1,-1,-1),(-1,-2,-2,0),\\ &(-1,-1,-2,-1),(-1,-2,-2,-1),(-1,-1,-2,-2),(-1,-2,-3,-1),(-1,-2,-2,-2),\\ &(-1,-2,-3,-2),(-1,-2,-4,-2),(-1,-3,-4,-2)\},\end{aligned}$$

and let

$$f = \sum_{i=1}^{14} a_i x_{\xi_i}$$

be an arbitrary element of $\mathfrak{n}_3^- \cap \mathfrak{p}$. The 14 coefficients $m_{(1,1,1,0)} = -a_2c_{(0,0,1,0)} + a_1c_{(0,1,1,0)} + a_{12}c_{(2,3,4,2)}, m_{(1,1,2,0)} = a_1c_{(0,1,2,0)} - a_{11}c_{(2,3,4,2)}, m_{(1,1,1,1)} = -a_2c_{(0,0,1,1)} + a_1c_{(0,1,1,1)} - a_{10}c_{(2,3,4,2)}, m_{(1,2,2,0)} = -a_2c_{(0,1,2,0)} + a_9c_{(2,3,4,2)}, m_{(1,1,2,1)} = a_1c_{(0,1,2,1)} + a_8c_{(2,3,4,2)}, m_{(1,2,2,1)} = -a_2c_{(0,1,2,1)} - a_7c_{(2,3,4,2)}, m_{(1,1,2,2)} = a_1c_{(0,1,2,2)} - a_6c_{(2,3,4,2)}, m_{(1,2,3,1)} = a_5c_{(2,3,4,2)}, m_{(1,2,2,2)} = -a_2c_{(0,1,2,2)} + a_4c_{(2,3,4,2)}, m_{(1,2,3,2)} = -a_3c_{(2,3,4,2)}$, $m_{(1,2,4,2)} = 0, m_{(1,3,4,2)} = 0, m_{(-1,0,0,0)} = -2a_3c_{(0,1,1,0)} - 2a_5c_{(0,1,1,1)} - a_4c_{(0,1,2,0)} - 2a_7c_{(0,1,2,1)} - a_9c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = 2a_3c_{(0,0,1,0)} + 2a_5c_{(0,0,1,1)} + a_6c_{(0,1,2,0)} + 2a_8c_{(0,1,2,1)} + a_{11}c_{(0,1,2,2)}$ determine a 14×24 matrix (Figure B.1) whose rank is at most 7. This matrix illustrates 14 rows of the full relations matrix R associated to the equation $[k, f] = 0$. Since the rank of R is at most

$7 + 14 = 21$, it follows that $dim\ K.(\mathfrak{n}_3^- \cap \mathfrak{p}) \leq 21$.

For the other case, let ξ_i denote the root in the ith position of

$$-\Phi_{11}^+ \cap \Gamma_{\mathfrak{p}} = \{(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,2,0),(1,1,1,1),(1,2,2,0),(-1,-1,-2,-1),$$
$$(-1,-2,-2,-1),(-1,-1,-2,-2),(-1,-2,-3,-1),(-1,-2,-2,-2),$$
$$(-1,-2,-3,-2),(-1,-2,-4,-2),(-1,-3,-4,-2)\},$$

and let

$$f = \sum_{i=1}^{14} a_i x_{\xi_i}$$

be an arbitrary element of $\mathfrak{n}_{11}^- \cap \mathfrak{p}$. The 14 coefficients $m_{(1,1,2,1)} = -a_4c_{(0,0,0,1)} - a_5c_{(0,0,1,0)} - a_3c_{(0,0,1,1)} + a_1c_{(0,1,2,1)} + a_8c_{(2,3,4,2)}$, $m_{(1,2,2,1)} = -a_6c_{(0,0,0,1)} + a_5c_{(0,1,1,0)} + a_3c_{(0,1,1,1)} - a_2c_{(0,1,2,1)} - a_7c_{(2,3,4,2)}$, $m_{(1,1,2,2)} = -2a_5c_{(0,0,1,1)} + a_1c_{(0,1,2,2)}$, $m_{(1,2,3,1)} = a_6c_{(0,0,1,1)} + a_4c_{(0,1,1,1)} + a_5c_{(0,1,2,0)} - a_3c_{(0,1,2,1)}$, $m_{(1,2,2,2)} = 2a_5c_{(0,1,1,1)} - a_2c_{(0,1,2,2)}$, $m_{(1,2,3,2)} = a_5c_{(0,1,2,1)} - a_3c_{(0,1,2,2)}$, $m_{(1,2,4,2)} = -a_4c_{(0,1,2,2)}$, $m_{(1,3,4,2)} = a_6c_{(0,1,2,2)}$, $m_{(-1,0,0,0)} = -2a_7c_{(0,1,2,1)} - a_9c_{(0,1,2,2)}$, $m_{(-1,-1,0,0)} = 2a_8c_{(0,1,2,1)} + a_{11}c_{(0,1,2,2)}$, $m_{(-1,-1,-1,0)} = a_7c_{(0,0,1,1)} - a_8c_{(0,1,1,1)} + a_{10}c_{(0,1,2,1)} + a_{12}c_{(0,1,2,2)}$, $m_{(-1,-1,-2,0)} = 2a_7c_{(0,0,0,1)} - 2a_{10}c_{(0,1,1,1)} + a_{13}c_{(0,1,2,2)}$, $m_{(-1,-1,-1,-1)} = a_7c_{(0,0,1,0)} + a_9c_{(0,0,1,1)} - a_8c_{(0,1,1,0)} - a_{11}c_{(0,1,1,1)} - a_{10}c_{(0,1,2,0)} - a_{12}c_{(0,1,2,1)}$, $m_{(-1,-2,-2,0)} = 2a_8c_{(0,0,0,1)} - 2a_{10}c_{(0,0,1,1)} - a_{14}c_{(0,1,2,2)}$ determine a 14×24 matrix (Figure B.2) whose rank is at most 8. This matrix illustrates 14 rows of the full relations matrix R associated to the equation $[k, f] = 0$. Since the rank of R is at most $8 + 14 = 22$, $dim\ K.(\mathfrak{n}_{11}^- \cap \mathfrak{p}) \leq 22$.

□

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -a_2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & -a_{11} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0 & a_1 & 0 & 0 & -a_{10} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0 & 0 & 0 & a_9 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & a_8 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0 & -a_7 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & -a_6 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & a_4 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_3 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_3 & 0 & -a_4 & -2a_5 & -2a_7 & -a_9 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 2a_3 & 0 & 0 & 2a_5 & a_6 & 0 & 2a_8 & a_{11} & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

Figure B.1: Submatrix of Relations Matrix Associated to Φ_3^+

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -a_5 & -a_4 & 0 & -a_3 & 0 & 0 & a_1 & 0 & a_8 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_6 & a_5 & 0 & 0 & a_3 & -a_2 & 0 & -a_7 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_5 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & -a_3 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_5 & 0 & -a_2 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & -a_3 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_7 & -a_9 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_8 & a_{11} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & -a_8 & a_{10} & a_{12} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2a_7 & 0 & 0 & 0 & -2a_{10} & 0 & a_{13} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & a_7 & 0 & -a_8 & a_9 & -a_{10} & -a_{11} & -a_{12} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2a_8 & 0 & -2a_{10} & 0 & 0 & 0 & -a_{14} & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

Figure B.2: Submatrix of Relations Matrix Associated to Φ_{11}^+

C

Computations for the Real Form F_4^{-20}

C.1 The Set Δ_j Forms a Simple System for Φ_j^+

The tables indicate how each root $\xi \in \Phi_j^+$ can be written as a nonnegative linear combination $\xi = i\xi_1 + j\xi_2 + k\xi_3 + \ell\xi_4$ of the listed simple roots $\Delta_j = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. The case $\Phi_1^+ = \Phi^+(\mathfrak{h}, \mathfrak{g})$ will be omitted.

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(1,0,0,0)$	1	0	0	0	$(0,1,0,0)$	0	1	0	0	$(0,0,1,0)$	0	0	1	1
$(1,1,0,0)$	1	1	0	0	$(0,1,1,0)$	0	1	1	1	$(1,1,1,0)$	1	1	1	1
$(0,1,2,0)$	0	1	2	2	$(1,1,2,0)$	1	1	2	2	$(1,2,2,0)$	1	2	2	2
$(0,1,2,2)$	0	1	0	2	$(1,1,2,2)$	1	1	0	2	$(1,2,2,2)$	1	2	0	2
$(1,2,3,2)$	1	2	1	3	$(1,2,4,2)$	1	2	2	4	$(1,3,4,2)$	1	3	2	4
$(2,3,4,2)$	2	3	2	4	$(0,0,0,-1)$	0	0	1	0	$(0,0,1,1)$	0	0	0	1
$(0,1,1,1)$	0	1	0	1	$(1,1,1,1)$	1	1	0	1	$(0,1,2,1)$	0	1	1	2
$(1,1,2,1)$	1	1	1	2	$(1,2,2,1)$	1	2	1	2	$(1,2,3,1)$	1	2	2	3

Table C.1: $\Delta_2 = \{\alpha, \beta, -\delta, \gamma + \delta\}$

Root	i	j	k	ℓ	Root	i	j	k	ℓ	Root	i	j	k	ℓ
$(1,0,0,0)$	1	0	0	0	$(0,1,0,0)$	0	0	2	1	$(0,0,1,0)$	0	1	0	0
$(1,1,0,0)$	1	0	2	1	$(0,1,1,0)$	0	1	2	1	$(1,1,1,0)$	1	1	2	1
$(0,1,2,0)$	0	2	2	1	$(1,1,2,0)$	1	2	2	1	$(1,2,2,0)$	1	2	4	2
$(0,1,2,2)$	0	0	0	1	$(1,1,2,2)$	1	0	0	1	$(1,2,2,2)$	1	0	2	2
$(1,2,3,2)$	1	1	2	2	$(1,2,4,2)$	1	2	2	2	$(1,3,4,2)$	1	2	4	3
$(2,3,4,2)$	2	2	4	3	$(0,0,0,-1)$	0	1	1	0	$(0,0,-1,-1)$	0	0	1	0
$(0,1,1,1)$	0	0	1	1	$(1,1,1,1)$	1	0	1	1	$(0,1,2,1)$	0	1	1	1
$(1,1,2,1)$	1	1	1	1	$(1,2,2,1)$	1	1	3	2	$(1,2,3,1)$	1	2	3	2

Table C.2: $\Delta_3 = \{\alpha, \gamma, -\gamma-\delta, \beta+2\gamma+2\delta\}$

C.2 Supplement to Proposition 6.2

Proof of Proposition 6.2: (Cont.) We now prove that the K-orbit dimensions are correct for each element f_j listed in Proposition 6.2. As mentioned above, these calculations will automatically imply that these elements are generic.

Positive System Φ_1^+ (dim $K.f_1 = 15$) Using the multiplication tables, the nonzero coefficients m_ξ with $f_1 = x_{(-1,-1,-1,-1)} + x_{(0,-1,-2,-1)}$ are given by $m_{(0,0,0,1)} = -c_{(0,1,2,2)}$, $m_{(0,0,1,1)} = -c_{(1,1,2,2)}$, $m_{(0,1,1,1)} = c_{(1,2,2,2)}$, $m_{(1,1,1,1)} = -c_{(1,2,3,2)}$, $m_{(0,1,2,1)} = c_{(1,2,3,2)}$, $m_{(1,1,2,1)} = -c_{(1,2,4,2)}$, $m_{(1,2,2,1)} = c_{(1,3,4,2)}$, $m_{(1,2,3,1)} = c_{(2,3,4,2)}$, $m_{(0,0,0,-1)} = -c_{(0,1,2,0)} - c_{(1,1,1,0)}$, $m_{(0,0,-1,-1)} = -c_{(0,1,1,0)} - c_{(1,1,0,0)}$, $m_{(0,-1,-1,-1)} = c_{(0,0,1,0)} + c_{(1,0,0,0)}$, $m_{(-1,-1,-1,-1)} = -c_1 + c_3 - c_4$, $m_{(0,-1,-2,-1)} = c_1 - c_3$, $m_{(-1,-1,-2,-1)} = c_{(-1,0,0,0)} + c_{(0,0,-1,0)}$, $m_{(-1,-2,-2,-1)} = -c_{(-1,-1,0,0)} - c_{(0,-1,-1,0)}$, and $m_{(-1,-2,-3,-1)} = -c_{(-1,-1,-1,0)} - c_{(0,-1,-2,0)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_3 = c_1$, $c_4 = 0$, $c_{(0,0,1,0)} = -c_{(1,0,0,0)}$, $c_{(0,1,1,0)} = -c_{(1,1,0,0)}$, $c_{(0,1,2,0)} = -c_{(1,1,1,0)}$, $c_{(0,1,2,2)} = 0$, $c_{(1,1,2,2)} = 0$, $c_{(1,2,2,2)} = 0$, $c_{(1,2,3,2)} = 0$, $c_{(1,2,4,2)} = 0$, $c_{(1,3,4,2)} = 0$, $c_{(2,3,4,2)} = 0$, $c_{(0,0,-1,0)} = -c_{(-1,0,0,0)}$, $c_{(0,-1,-1,0)} = -c_{(-1,-1,0,0)}$, and $c_{(0,-1,-2,0)} = -c_{(-1,-1,-1,0)}$. Therefore, $\mathfrak{k}^{f_1}$ has dimension 21, hence $K.f_1$ has dimension 15.

Positive System Φ_2^+ (dim $K.f_2 = 15$) For this case, the nonzero coefficients m_ξ with $f_2 = x_{(0,0,0,1)} + x_{(0,0,-1,-1)}$ are $m_{(0,0,0,1)} = -c_3 + 2c_4$, $m_{(0,0,1,1)} = c_{(0,0,1,0)}$, $m_{(0,1,1,1)} = c_{(0,1,1,0)} + c_{(0,1,2,2)}$, $m_{(1,1,1,1)} = c_{(1,1,1,0)} + c_{(1,1,2,2)}$, $m_{(0,1,2,1)} = c_{(0,1,2,0)}$, $m_{(1,1,2,1)} = c_{(1,1,2,0)}$, $m_{(1,2,2,1)} = c_{(1,2,2,0)} - c_{(1,2,3,2)}$, $m_{(1,2,3,1)} = -c_{(1,2,4,2)}$, $m_{(0,0,0,-1)} = -c_{(0,0,1,0)}$, $m_{(0,0,-1,-1)} = c_2 - c_3 - c_4$, $m_{(0,-1,-1,-1)} = -c_{(0,-1,0,0)}$, $m_{(-1,-1,-1,-1)} = -c_{(-1,-1,0,0)}$, $m_{(0,-1,-2,-1)} = -c_{(0,-1,-2,-2)} - c_{(0,-1,-1,0)}$, $m_{(-1,-1,-2,-1)} = -c_{(-1,-1,-2,-2)} - c_{(-1,-1,-1,0)}$, $m_{(-1,-2,-2,-1)} = -c_{(-1,-2,-2,-2)}$, and $m_{(-1,-2,-3,-1)} = -c_{(-1,-2,-3,-2)} + c_{(-1,-2,-2,0)}$. Then the equations $m_\xi = 0$ yield the relations on the coefficients c_ξ given by $c_3 = 2c_2/3$, $c_4 = c_2/3$, $c_{(0,0,1,0)} = 0$, $c_{(0,1,2,0)} = 0$, $c_{(1,1,2,0)} = 0$, $c_{(0,1,2,2)} = -c_{(0,1,1,0)}$, $c_{(1,1,2,2)} = -c_{(1,1,1,0)}$, $c_{(1,2,3,2)} = c_{(1,2,2,0)}$, $c_{(1,2,4,2)} = 0$, $c_{(0,-1,0,0)} = 0$, $c_{(-1,-1,0,0)} = 0$, $c_{(0,-1,-2,-2)} = -c_{(0,-1,-1,0)}$, $c_{(-1,-1,-2,-2)} = -c_{(-1,-1,-1,0)}$, $c_{(-1,-2,-2,-2)} = 0$, and $c_{(-1,-2,-3,-2)} = c_{(-1,-2,-2,0)}$. Therefore, $\mathfrak{k}^{f_2}$ has dimension 21, hence $K.f_2$ has dimension 15.

Positive System Φ_3^+ (dim $K.f_3 = 15$) For this last case, we have already shown that this case is true in light of the fact that f_3 is the same as f_1. We could have chosen all three of the generic elements to be the same for this real form, but the choices given make $N(f_j, \mathfrak{n}_j^- \cap \mathfrak{p})$ easier to describe. $\square$

D

Elements of $\mathfrak{g}_2$

D.1 Multiplication Tables for $\mathfrak{g}_2$

These multiplication tables are constructed using GAP and were verified using Mathematica® software. For simplicity, we will replace the Lie bracket with juxtaposition of elements. Naturally, the antisymmetry allows us to present half of the multiplication.

$x_{(1,0)}x_{(1,0)} = 0$	$x_{(1,0)}x_{(0,1)} = -x_{(1,1)}$
$x_{(1,0)}x_{(1,1)} = -2x_{(2,1)}$	$x_{(1,0)}x_{(2,1)} = -3x_{(3,1)}$
$x_{(1,0)}x_{(3,1)} = 0$	$x_{(1,0)}x_{(3,2)} = 0$
$x_{(1,0)}x_{(-1,0)} = h_{(1,0)}$	$x_{(1,0)}x_{(0,-1)} = 0$
$x_{(1,0)}x_{(-1,-1)} = 3x_{(0,-1)}$	$x_{(1,0)}x_{(-2,-1)} = 2x_{(-1,-1)}$
$x_{(1,0)}x_{(-3,-1)} = x_{(-2,-1)}$	$x_{(1,0)}x_{(-3,-2)} = 0$
$x_{(1,0)}h_{(1,0)} = -2x_{(1,0)}$	$x_{(1,0)}h_{(0,1)} = x_{(1,0)}$
$x_{(0,1)}x_{(0,1)} = 0$	$x_{(0,1)}x_{(1,1)} = 0$
$x_{(0,1)}x_{(2,1)} = 0$	$x_{(0,1)}x_{(3,1)} = -x_{(3,2)}$
$x_{(0,1)}x_{(3,2)} = 0$	$x_{(0,1)}x_{(-1,0)} = 0$
$x_{(0,1)}x_{(0,-1)} = h_{(0,1)}$	$x_{(0,1)}x_{(-1,-1)} = -x_{(-1,0)}$
$x_{(0,1)}x_{(-2,-1)} = 0$	$x_{(0,1)}x_{(-3,-1)} = 0$
$x_{(0,1)}x_{(-3,-2)} = x_{(-3,-1)}$	$x_{(0,1)}h_{(1,0)} = 3x_{(0,1)}$
$x_{(0,1)}h_{(0,1)} = -2x_{(0,1)}$	$x_{(1,1)}x_{(1,1)} = 0$
$x_{(1,1)}x_{(2,1)} = 3x_{(3,2)}$	$x_{(1,1)}x_{(3,1)} = 0$
$x_{(1,1)}x_{(3,2)} = 0$	$x_{(1,1)}x_{(-1,0)} = 3x_{(0,1)}$

$x_{(1,1)}x_{(0,-1)} = -x_{(1,0)}$ $\quad$ $x_{(1,1)}x_{(-1,-1)} = h_{(1,0)} + 3h_{(0,1)}$

$x_{(1,1)}x_{(-2,-1)} = -2x_{(-1,0)}$ $\quad$ $x_{(1,1)}x_{(-3,-1)} = 0$

$x_{(1,1)}x_{(-3,-2)} = -x_{(-2,-1)}$ $\quad$ $x_{(1,1)}h_{(1,0)} = x_{(1,1)}$

$x_{(1,1)}h_{(0,1)} = -x_{(1,1)}$ $\quad$ $x_{(2,1)}x_{(2,1)} = 0$

$x_{(2,1)}x_{(3,1)} = 0$ $\quad$ $x_{(2,1)}x_{(3,2)} = 0$

$x_{(2,1)}x_{(-1,0)} = 2x_{(1,1)}$ $\quad$ $x_{(2,1)}x_{(0,-1)} = 0$

$x_{(2,1)}x_{(-1,-1)} = -2x_{(1,0)}$ $\quad$ $x_{(2,1)}x_{(-2,-1)} = 2h_{(1,0)} + 3h_{(0,1)}$

$x_{(2,1)}x_{(-3,-1)} = -x_{(-1,0)}$ $\quad$ $x_{(2,1)}x_{(-3,-2)} = x_{(-1,-1)}$

$x_{(2,1)}h_{(1,0)} = -x_{(2,1)}$ $\quad$ $x_{(2,1)}h_{(0,1)} = 0$

$x_{(3,1)}x_{(3,1)} = 0$ $\quad$ $x_{(3,1)}x_{(3,2)} = 0$

$x_{(3,1)}x_{(-1,0)} = x_{(2,1)}$ $\quad$ $x_{(3,1)}x_{(0,-1)} = 0$

$x_{(3,1)}x_{(-1,-1)} = 0$ $\quad$ $x_{(3,1)}x_{(-2,-1)} = -x_{(1,0)}$

$x_{(3,1)}x_{(-3,-1)} = h_{(1,0)} + h_{(0,1)}$ $\quad$ $x_{(3,1)}x_{(-3,-2)} = -x_{(0,-1)}$

$x_{(3,1)}h_{(1,0)} = -3x_{(3,1)}$ $\quad$ $x_{(3,1)}h_{(0,1)} = x_{(3,1)}$

$x_{(3,2)}x_{(3,2)} = 0$ $\quad$ $x_{(3,2)}x_{(-1,0)} = 0$

$x_{(3,2)}x_{(0,-1)} = x_{(3,1)}$ $\quad$ $x_{(3,2)}x_{(-1,-1)} = -x_{(2,1)}$

$x_{(3,2)}x_{(-2,-1)} = x_{(1,1)}$ $\quad$ $x_{(3,2)}x_{(-3,-1)} = -x_{(0,1)}$

$x_{(3,2)}x_{(-3,-2)} = h_{(1,0)} + 2h_{(0,1)}$ $\quad$ $x_{(3,2)}h_{(1,0)} = 0$

$x_{(3,2)}h_{(0,1)} = -x_{(3,2)}$ $\quad$ $x_{(-1,0)}x_{(-1,0)} = 0$

$x_{(-1,0)}x_{(0,-1)} = x_{(-1,-1)}$ $\quad$ $x_{(-1,0)}x_{(-1,-1)} = 2x_{(-2,-1)}$

$x_{(-1,0)}x_{(-2,-1)} = 3x_{(-3,-1)}$ $\quad$ $x_{(-1,0)}x_{(-3,-1)} = 0$

$x_{(-1,0)}x_{(-3,-2)} = 0$ $\quad$ $x_{(-1,0)}h_{(1,0)} = 2x_{(-1,0)}$

$x_{(-1,0)}h_{(0,1)} = -x_{(-1,0)}$ $\quad$ $x_{(0,-1)}x_{(0,-1)} = 0$

$x_{(0,-1)}x_{(-1,-1)} = 0$ $\quad$ $x_{(0,-1)}x_{(-2,-1)} = 0$

$x_{(0,-1)}x_{(-3,-1)} = x_{(-3,-2)}$ $\quad$ $x_{(0,-1)}x_{(-3,-2)} = 0$

$x_{(0,-1)}h_{(1,0)} = -3x_{(0,-1)}$ $\quad$ $x_{(0,-1)}h_{(0,1)} = 2x_{(0,-1)}$

$x_{(-1,-1)}x_{(-1,-1)} = 0$ $\quad$ $x_{(-1,-1)}x_{(-2,-1)} = -3x_{(-3,-2)}$

$x_{(-1,-1)}x_{(-3,-1)} = 0$ $\quad$ $x_{(-1,-1)}x_{(-3,-2)} = 0$

$x_{(-1,-1)}h_{(1,0)} = -x_{(-1,-1)}$ $\quad$ $x_{(-1,-1)}h_{(0,1)} = x_{(-1,-1)}$

$x_{(-2,-1)}x_{(-2,-1)} = 0$ $\quad$ $x_{(-2,-1)}x_{(-3,-1)} = 0$

$x_{(-2,-1)}x_{(-3,-2)} = 0$ $\quad$ $x_{(-2,-1)}h_{(1,0)} = x_{(-2,-1)}$

$x_{(-2,-1)}h_{(0,1)} = 0$ $\quad$ $x_{(-3,-1)}x_{(-3,-1)} = 0$

$x_{(-3,-1)}x_{(-3,-2)} = 0$ $\quad$ $x_{(-3,-1)}h_{(1,0)} = 3x_{(-3,-1)}$

$x_{(-3,-1)}h_{(0,1)} = -x_{(-3,-1)}$ $\quad$ $x_{(-3,-2)}x_{(-3,-2)} = 0$

$x_{(-3,-2)}h_{(1,0)} = 0$ $\quad$ $x_{(-3,-2)}h_{(0,1)} = x_{(-3,-2)}$

$h_{(1,0)}h_{(1,0)} = 0$ $\quad$ $h_{(1,0)}h_{(0,1)} = 0$

$h_{(0,1)}h_{(0,1)} = 0$

D.2 Basis for $\mathfrak{g}_2 \subset \mathfrak{gl}(7, \mathbb{C})$

Root Vector x	$\pi(x)$
$x_{(1,0)}$	$E^7_{1,2} + 2E^7_{3,4} + E^7_{4,5} + E^7_{6,7}$
$x_{(-1,0)}$	$E^7_{2,1} + E^7_{4,3} + 2E^7_{5,4} + E^7_{7,6}$
$x_{(0,1)}$	$E^7_{2,3} + E^7_{5,6}$
$x_{(0,-1)}$	$E^7_{3,2} + E^7_{6,5}$
$x_{(1,1)}$	$-E^7_{1,3} + 2E^7_{2,4} - E^7_{4,6} + E^7_{5,7}$
$x_{(-1,-1)}$	$-E^7_{3,1} + E^7_{4,2} - 2E^7_{6,4} + E^7_{7,5}$
$x_{(2,1)}$	$-2E^7_{1,4} + E^7_{2,5} + E^7_{3,6} - E^7_{4,7}$
$x_{(-2,-1)}$	$-E^7_{4,1} + E^7_{5,2} + E^7_{6,3} - 2E^7_{7,4}$
$x_{(3,1)}$	$-E^7_{1,5} + E^7_{3,7}$
$x_{(-3,-1)}$	$-E^7_{5,1} + E^7_{7,3}$
$x_{(3,2)}$	$-E^7_{1,6} - E^7_{2,7}$
$x_{(-3,-2)}$	$-E^7_{6,1} - E^7_{7,2}$
$h_{(1,0)}$	$E^7_{1,1} - E^7_{2,2} + 2E^7_{3,3} - 2E^7_{5,5} + E^7_{6,6} - E^7_{7,7}$
$h_{(0,1)}$	$E^7_{2,2} - E^7_{3,3} + E^7_{5,5} - E^7_{6,6}$

Table D.2: Basis for $\mathfrak{g}_2 \subset \mathfrak{gl}(7, \mathbb{C})$

E

Elements of $\mathfrak{f}_4$

E.1 Multiplication Tables for $\mathfrak{f}_4$

These multiplication tables are constructed using GAP, and were verified using Mathematica® software. Recall that if ξ, ϵ and $\xi+\epsilon$ are roots such that $[x_\xi, x_\epsilon] = c_{\xi,\epsilon}x_{\xi+\epsilon}$, then $c_{\xi,\epsilon} = -c_{-\xi,-\epsilon}$. These relations together with the antisymmetry allows us to omit some of the bracket relations. Since the multiplication tables are significantly large, we also omit most zero products.

Products contained in $\mathfrak{g} \setminus \mathfrak{h}$

$x_{(1,0,0,0)}x_{(0,1,0,0)} = -x_{(1,1,0,0)}$ $\quad$ $x_{(1,0,0,0)}x_{(0,1,1,0)} = -x_{(1,1,1,0)}$

$x_{(1,0,0,0)}x_{(0,1,2,0)} = -x_{(1,1,2,0)}$ $\quad$ $x_{(1,0,0,0)}x_{(0,1,1,1)} = -x_{(1,1,1,1)}$

$x_{(1,0,0,0)}x_{(0,1,2,1)} = -x_{(1,1,2,1)}$ $\quad$ $x_{(1,0,0,0)}x_{(0,1,2,2)} = -x_{(1,1,2,2)}$

$x_{(1,0,0,0)}x_{(1,3,4,2)} = x_{(2,3,4,2)}$ $\quad$ $x_{(1,0,0,0)}x_{(-1,-1,0,0)} = x_{(0,-1,0,0)}$

$x_{(1,0,0,0)}x_{(-1,-1,-1,0)} = x_{(0,-1,-1,0)}$ $\quad$ $x_{(1,0,0,0)}x_{(-1,-1,-2,0)} = x_{(0,-1,-2,0)}$

$x_{(1,0,0,0)}x_{(-1,-1,-1,-1)} = x_{(0,-1,-1,-1)}$ $\quad$ $x_{(1,0,0,0)}x_{(-1,-1,-2,-1)} = x_{(0,-1,-2,-1)}$

$x_{(1,0,0,0)}x_{(-1,-1,-2,-2)} = x_{(0,-1,-2,-2)}$ $\quad$ $x_{(1,0,0,0)}x_{(-2,-3,-4,-2)} = -x_{(-1,-3,-4,-2)}$

$x_{(0,1,0,0)}x_{(0,0,1,0)} = x_{(0,1,1,0)}$ $\quad$ $x_{(0,1,0,0)}x_{(0,0,1,1)} = x_{(0,1,1,1)}$

$x_{(0,1,0,0)}x_{(1,1,2,0)} = -x_{(1,2,2,0)}$ $\quad$ $x_{(0,1,0,0)}x_{(1,1,2,1)} = -x_{(1,2,2,1)}$

$x_{(0,1,0,0)}x_{(1,1,2,2)} = -x_{(1,2,2,2)}$ $\quad$ $x_{(0,1,0,0)}x_{(1,2,4,2)} = x_{(1,3,4,2)}$

$x_{(0,1,0,0)}x_{(-1,-1,0,0)} = -x_{(-1,0,0,0)}$ $\quad$ $x_{(0,1,0,0)}x_{(0,-1,-1,0)} = -x_{(0,0,-1,0)}$

$x_{(0,1,0,0)}x_{(0,-1,-1,-1)} = -x_{(0,0,-1,-1)}$ $\quad$ $x_{(0,1,0,0)}x_{(-1,-2,-2,0)} = x_{(-1,-1,-2,0)}$

$x_{(0,1,0,0)}x_{(-1,-2,-2,-1)} = x_{(-1,-1,-2,-1)}$ $\quad$ $x_{(0,1,0,0)}x_{(-1,-2,-2,-2)} = x_{(-1,-1,-2,-2)}$

$x_{(0,1,0,0)}x_{(-1,-3,-4,-2)} = -x_{(-1,-2,-4,-2)}$ $x_{(0,0,1,0)}x_{(0,0,0,1)} = x_{(0,0,1,1)}$

$x_{(0,0,1,0)}x_{(1,1,0,0)} = -x_{(1,1,1,0)}$ $x_{(0,0,1,0)}x_{(0,1,1,0)} = -2x_{(0,1,2,0)}$

$x_{(0,0,1,0)}x_{(1,1,1,0)} = -2x_{(1,1,2,0)}$ $x_{(0,0,1,0)}x_{(0,1,1,1)} = -x_{(0,1,2,1)}$

$x_{(0,0,1,0)}x_{(1,1,1,1)} = -x_{(1,1,2,1)}$ $x_{(0,0,1,0)}x_{(1,2,2,1)} = -x_{(1,2,3,1)}$

$x_{(0,0,1,0)}x_{(1,2,2,2)} = -x_{(1,2,3,2)}$ $x_{(0,0,1,0)}x_{(1,2,3,2)} = -2x_{(1,2,4,2)}$

$x_{(0,0,1,0)}x_{(0,-1,-1,0)} = 2x_{(0,-1,0,0)}$ $x_{(0,0,1,0)}x_{(0,0,-1,-1)} = -x_{(0,0,0,-1)}$

$x_{(0,0,1,0)}x_{(-1,-1,-1,0)} = 2x_{(-1,-1,0,0)}$ $x_{(0,0,1,0)}x_{(0,-1,-2,0)} = x_{(0,-1,-1,0)}$

$x_{(0,0,1,0)}x_{(-1,-1,-2,0)} = x_{(-1,-1,-1,0)}$ $x_{(0,0,1,0)}x_{(0,-1,-2,-1)} = x_{(0,-1,-1,-1)}$

$x_{(0,0,1,0)}x_{(-1,-1,-2,-1)} = x_{(-1,-1,-1,-1)}$ $x_{(0,0,1,0)}x_{(-1,-2,-3,-1)} = x_{(-1,-2,-2,-1)}$

$x_{(0,0,1,0)}x_{(-1,-2,-3,-2)} = 2x_{(-1,-2,-2,-2)}$ $x_{(0,0,1,0)}x_{(-1,-2,-4,-2)} = x_{(-1,-2,-3,-2)}$

$x_{(0,0,0,1)}x_{(0,1,1,0)} = -x_{(0,1,1,1)}$ $x_{(0,0,0,1)}x_{(1,1,1,0)} = -x_{(1,1,1,1)}$

$x_{(0,0,0,1)}x_{(0,1,2,0)} = -x_{(0,1,2,1)}$ $x_{(0,0,0,1)}x_{(1,1,2,0)} = -x_{(1,1,2,1)}$

$x_{(0,0,0,1)}x_{(0,1,2,1)} = -2x_{(0,1,2,2)}$ $x_{(0,0,0,1)}x_{(1,2,2,0)} = -x_{(1,2,2,1)}$

$x_{(0,0,0,1)}x_{(1,1,2,1)} = -2x_{(1,1,2,2)}$ $x_{(0,0,0,1)}x_{(1,2,2,1)} = -2x_{(1,2,2,2)}$

$x_{(0,0,0,1)}x_{(1,2,3,1)} = -x_{(1,2,3,2)}$ $x_{(0,0,0,1)}x_{(0,0,-1,-1)} = x_{(0,0,-1,0)}$

$x_{(0,0,0,1)}x_{(0,-1,-1,-1)} = x_{(0,-1,-1,0)}$ $x_{(0,0,0,1)}x_{(-1,-1,-1,-1)} = x_{(-1,-1,-1,0)}$

$x_{(0,0,0,1)}x_{(0,-1,-2,-1)} = 2x_{(0,-1,-2,0)}$ $x_{(0,0,0,1)}x_{(-1,-1,-2,-1)} = 2x_{(-1,-1,-2,0)}$

$x_{(0,0,0,1)}x_{(0,-1,-2,-2)} = x_{(0,-1,-2,-1)}$ $x_{(0,0,0,1)}x_{(-1,-2,-2,-1)} = 2x_{(-1,-2,-2,0)}$

$x_{(0,0,0,1)}x_{(-1,-1,-2,-2)} = x_{(-1,-1,-2,-1)}$ $x_{(0,0,0,1)}x_{(-1,-2,-2,-2)} = x_{(-1,-2,-2,-1)}$

$x_{(0,0,0,1)}x_{(-1,-2,-3,-2)} = x_{(-1,-2,-3,-1)}$ $x_{(1,1,0,0)}x_{(0,0,1,1)} = x_{(1,1,1,1)}$

$x_{(1,1,0,0)}x_{(0,1,2,0)} = x_{(1,2,2,0)}$ $x_{(1,1,0,0)}x_{(0,1,2,1)} = x_{(1,2,2,1)}$

$x_{(1,1,0,0)}x_{(0,1,2,2)} = x_{(1,2,2,2)}$ $x_{(1,1,0,0)}x_{(1,2,4,2)} = -x_{(2,3,4,2)}$

$x_{(1,1,0,0)}x_{(-1,0,0,0)} = x_{(0,1,0,0)}$ $x_{(1,1,0,0)}x_{(0,-1,0,0)} = -x_{(1,0,0,0)}$

$x_{(1,1,0,0)}x_{(-1,-1,-1,0)} = -x_{(0,0,-1,0)}$ $x_{(1,1,0,0)}x_{(-1,-1,-1,-1)} = -x_{(0,0,-1,-1)}$

$x_{(1,1,0,0)}x_{(-1,-2,-2,0)} = -x_{(0,-1,-2,0)}$ $x_{(1,1,0,0)}x_{(-1,-2,-2,-1)} = -x_{(0,-1,-2,-1)}$

$x_{(1,1,0,0)}x_{(-1,-2,-2,-2)} = -x_{(0,-1,-2,-2)}$ $x_{(1,1,0,0)}x_{(-2,-3,-4,-2)} = x_{(-1,-2,-4,-2)}$

$x_{(0,1,1,0)}x_{(0,0,1,1)} = x_{(0,1,2,1)}$ $x_{(0,1,1,0)}x_{(1,1,1,0)} = 2x_{(1,2,2,0)}$

$x_{(0,1,1,0)}x_{(1,1,1,1)} = x_{(1,2,2,1)}$ $x_{(0,1,1,0)}x_{(1,1,2,1)} = -x_{(1,2,3,1)}$

$x_{(0,1,1,0)}x_{(1,1,2,2)} = -x_{(1,2,3,2)}$ $x_{(0,1,1,0)}x_{(1,2,3,2)} = -2x_{(1,3,4,2)}$

$x_{(0,1,1,0)}x_{(0,-1,0,0)} = -x_{(0,0,1,0)}$ $x_{(0,1,1,0)}x_{(0,0,-1,0)} = 2x_{(0,1,0,0)}$

$x_{(0,1,1,0)}x_{(-1,-1,-1,0)} = -2x_{(-1,0,0,0)}$ $x_{(0,1,1,0)}x_{(0,-1,-2,0)} = -x_{(0,0,-1,0)}$

$x_{(0,1,1,0)}x_{(0,-1,-1,-1)} = -x_{(0,0,0,-1)}$ $x_{(0,1,1,0)}x_{(0,-1,-2,-1)} = -x_{(0,0,-1,-1)}$

$x_{(0,1,1,0)}x_{(-1,-2,-2,0)} = -x_{(-1,-1,-1,0)}$ $x_{(0,1,1,0)}x_{(-1,-2,-2,-1)} = -x_{(-1,-1,-1,-1)}$

$x_{(0,1,1,0)}x_{(-1,-2,-3,-1)} = x_{(-1,-1,-2,-1)}$ $x_{(0,1,1,0)}x_{(-1,-2,-3,-2)} = 2x_{(-1,-1,-2,-2)}$

$x_{(0,1,1,0)}x_{(-1,-3,-4,-2)} = x_{(-1,-2,-3,-2)}$ $x_{(0,0,1,1)}x_{(1,1,1,0)} = -x_{(1,1,2,1)}$

$x_{(0,0,1,1)}x_{(0,1,1,1)} = -2x_{(0,1,2,2)}$ $x_{(0,0,1,1)}x_{(1,1,1,1)} = -2x_{(1,1,2,2)}$

$x_{(0,0,1,1)}x_{(1,2,2,0)} = x_{(1,2,3,1)}$ $x_{(0,0,1,1)}x_{(1,2,2,1)} = x_{(1,2,3,2)}$

$x_{(0,0,1,1)}x_{(1,2,3,1)} = 2x_{(1,2,4,2)}$ $x_{(0,0,1,1)}x_{(0,0,-1,0)} = -x_{(0,0,0,1)}$

$x_{(0,0,1,1)}x_{(0,0,0,-1)} = x_{(0,0,1,0)}$ $x_{(0,0,1,1)}x_{(0,-1,-1,-1)} = 2x_{(0,-1,0,0)}$

$x_{(0,0,1,1)}x_{(-1,-1,-1,-1)} = 2x_{(-1,-1,0,0)}$ $x_{(0,0,1,1)}x_{(0,-1,-2,-1)} = x_{(0,-1,-1,0)}$

$x_{(0,0,1,1)}x_{(-1,-1,-2,-1)} = x_{(-1,-1,-1,0)}$ $x_{(0,0,1,1)}x_{(0,-1,-2,-2)} = x_{(0,-1,-1,-1)}$

$x_{(0,0,1,1)}x_{(-1,-1,-2,-2)} = x_{(-1,-1,-1,-1)}$ $x_{(0,0,1,1)}x_{(-1,-2,-3,-1)} = -2x_{(-1,-2,-2,0)}$

$x_{(0,0,1,1)}x_{(-1,-2,-3,-2)} = -x_{(-1,-2,-2,-1)}$ $x_{(0,0,1,1)}x_{(-1,-2,-4,-2)} = -x_{(-1,-2,-3,-1)}$

$x_{(1,1,1,0)}x_{(0,1,1,1)} = -x_{(1,2,2,1)}$ $x_{(1,1,1,0)}x_{(0,1,2,1)} = x_{(1,2,3,1)}$

$x_{(1,1,1,0)}x_{(0,1,2,2)} = x_{(1,2,3,2)}$ $x_{(1,1,1,0)}x_{(1,2,3,2)} = 2x_{(2,3,4,2)}$

$x_{(1,1,1,0)}x_{(-1,0,0,0)} = x_{(0,1,1,0)}$ $x_{(1,1,1,0)}x_{(0,0,-1,0)} = 2x_{(1,1,0,0)}$

$x_{(1,1,1,0)}x_{(-1,-1,0,0)} = -x_{(0,0,1,0)}$ $x_{(1,1,1,0)}x_{(0,-1,-1,0)} = -2x_{(1,0,0,0)}$

$x_{(1,1,1,0)}x_{(-1,-1,-2,0)} = -x_{(0,0,-1,0)}$ $x_{(1,1,1,0)}x_{(-1,-1,-1,-1)} = -x_{(0,0,0,-1)}$

$x_{(1,1,1,0)}x_{(-1,-2,-2,0)} = x_{(0,-1,-1,0)}$ $x_{(1,1,1,0)}x_{(-1,-1,-2,-1)} = -x_{(0,0,-1,-1)}$

$x_{(1,1,1,0)}x_{(-1,-2,-2,-1)} = x_{(0,-1,-1,-1)}$ $x_{(1,1,1,0)}x_{(-1,-2,-3,-1)} = -x_{(0,-1,-2,-1)}$

$x_{(1,1,1,0)}x_{(-1,-2,-3,-2)} = -2x_{(0,-1,-2,-2)}$ $x_{(1,1,1,0)}x_{(-2,-3,-4,-2)} = -x_{(-1,-2,-3,-2)}$

$x_{(0,1,2,0)}x_{(1,1,1,1)} = x_{(1,2,3,1)}$ $x_{(0,1,2,0)}x_{(1,1,2,2)} = -x_{(1,2,4,2)}$

$x_{(0,1,2,0)}x_{(1,2,2,2)} = x_{(1,3,4,2)}$ $x_{(0,1,2,0)}x_{(0,0,-1,0)} = x_{(0,1,1,0)}$

$x_{(0,1,2,0)}x_{(0,-1,-1,0)} = -x_{(0,0,1,0)}$ $x_{(0,1,2,0)}x_{(-1,-1,-2,0)} = -x_{(-1,0,0,0)}$

$x_{(0,1,2,0)}x_{(0,-1,-2,-1)} = -x_{(0,0,0,-1)}$ $x_{(0,1,2,0)}x_{(-1,-2,-2,0)} = x_{(-1,-1,0,0)}$

$x_{(0,1,2,0)}x_{(-1,-2,-3,-1)} = -x_{(-1,-1,-1,-1)}$ $x_{(0,1,2,0)}x_{(-1,-2,-4,-2)} = x_{(-1,-1,-2,-2)}$

$x_{(0,1,2,0)}x_{(-1,-3,-4,-2)} = -x_{(-1,-2,-2,-2)}$ $x_{(0,1,1,1)}x_{(1,1,2,0)} = x_{(1,2,3,1)}$

$x_{(0,1,1,1)}x_{(1,1,1,1)} = 2x_{(1,2,2,2)}$ $x_{(0,1,1,1)}x_{(1,1,2,1)} = x_{(1,2,3,2)}$

$x_{(0,1,1,1)}x_{(1,2,3,1)} = 2x_{(1,3,4,2)}$ $x_{(0,1,1,1)}x_{(0,-1,0,0)} = -x_{(0,0,1,1)}$

$x_{(0,1,1,1)}x_{(0,0,0,-1)} = x_{(0,1,1,0)}$ $x_{(0,1,1,1)}x_{(0,-1,-1,0)} = -x_{(0,0,0,1)}$

$x_{(0,1,1,1)}x_{(0,0,-1,-1)} = 2x_{(0,1,0,0)}$ $x_{(0,1,1,1)}x_{(-1,-1,-1,-1)} = -2x_{(-1,0,0,0)}$

$x_{(0,1,1,1)}x_{(0,-1,-2,-1)} = -x_{(0,0,-1,0)}$ $x_{(0,1,1,1)}x_{(0,-1,-2,-2)} = -x_{(0,0,-1,-1)}$

$x_{(0,1,1,1)}x_{(-1,-2,-2,-1)} = -x_{(-1,-1,-1,0)}$ $x_{(0,1,1,1)}x_{(-1,-2,-3,-1)} = -2x_{(-1,-1,-2,0)}$

$x_{(0,1,1,1)}x_{(-1,-2,-2,-2)} = -x_{(-1,-1,-1,-1)}$ $x_{(0,1,1,1)}x_{(-1,-2,-3,-2)} = -x_{(-1,-1,-2,-1)}$

$x_{(0,1,1,1)}x_{(-1,-3,-4,-2)} = -x_{(-1,-2,-3,-1)}$ $x_{(1,1,2,0)}x_{(0,1,2,2)} = x_{(1,2,4,2)}$

$x_{(1,1,2,0)}x_{(1,2,2,2)} = -x_{(2,3,4,2)}$ $x_{(1,1,2,0)}x_{(-1,0,0,0)} = x_{(0,1,2,0)}$

$x_{(1,1,2,0)}x_{(0,0,-1,0)} = x_{(1,1,1,0)}$ $x_{(1,1,2,0)}x_{(-1,-1,-1,0)} = -x_{(0,0,1,0)}$

$x_{(1,1,2,0)}x_{(0,-1,-2,0)} = -x_{(1,0,0,0)}$ $x_{(1,1,2,0)}x_{(-1,-2,-2,0)} = -x_{(0,-1,0,0)}$

$x_{(1,1,2,0)}x_{(-1,-1,-2,-1)} = -x_{(0,0,0,-1)}$ $x_{(1,1,2,0)}x_{(-1,-2,-3,-1)} = x_{(0,-1,-1,-1)}$

$x_{(1,1,2,0)}x_{(-1,-2,-4,-2)} = -x_{(0,-1,-2,-2)}$ $x_{(1,1,2,0)}x_{(-2,-3,-4,-2)} = x_{(-1,-2,-2,-2)}$

$x_{(1,1,1,1)}x_{(0,1,2,1)} = -x_{(1,2,3,2)}$ $x_{(1,1,1,1)}x_{(1,2,3,1)} = -2x_{(2,3,4,2)}$

$x_{(1,1,1,1)}x_{(-1,0,0,0)} = x_{(0,1,1,1)}$ $x_{(1,1,1,1)}x_{(0,0,0,-1)} = x_{(1,1,1,0)}$

$x_{(1,1,1,1)}x_{(-1,-1,0,0)} = -x_{(0,0,1,1)}$ $x_{(1,1,1,1)}x_{(0,0,-1,-1)} = 2x_{(1,1,0,0)}$

$x_{(1,1,1,1)}x_{(-1,-1,-1,0)} = -x_{(0,0,0,1)}$ $x_{(1,1,1,1)}x_{(0,-1,-1,-1)} = -2x_{(1,0,0,0)}$

$x_{(1,1,1,1)}x_{(-1,-1,-2,-1)} = -x_{(0,0,-1,0)}$ $x_{(1,1,1,1)}x_{(-1,-2,-2,-1)} = x_{(0,-1,-1,0)}$

$x_{(1,1,1,1)}x_{(-1,-1,-2,-2)} = -x_{(0,0,-1,-1)}$ $x_{(1,1,1,1)}x_{(-1,-2,-3,-1)} = 2x_{(0,-1,-2,0)}$

$x_{(1,1,1,1)}x_{(-1,-2,-2,-2)} = x_{(0,-1,-1,-1)}$ $x_{(1,1,1,1)}x_{(-1,-2,-3,-2)} = x_{(0,-1,-2,-1)}$

$x_{(1,1,1,1)}x_{(-2,-3,-4,-2)} = x_{(-1,-2,-3,-1)}$ $x_{(0,1,2,1)}x_{(1,1,2,1)} = 2x_{(1,2,4,2)}$

$x_{(0,1,2,1)}x_{(1,2,2,1)} = -2x_{(1,3,4,2)}$ $x_{(0,1,2,1)}x_{(0,0,-1,0)} = x_{(0,1,1,1)}$

$x_{(0,1,2,1)}x_{(0,0,0,-1)} = 2x_{(0,1,2,0)}$ $x_{(0,1,2,1)}x_{(0,-1,-1,0)} = -x_{(0,0,1,1)}$

$x_{(0,1,2,1)}x_{(0,0,-1,-1)} = x_{(0,1,1,0)}$ $x_{(0,1,2,1)}x_{(0,-1,-2,0)} = -x_{(0,0,0,1)}$

$x_{(0,1,2,1)}x_{(0,-1,-1,-1)} = -x_{(0,0,1,0)}$ $x_{(0,1,2,1)}x_{(-1,-1,-2,-1)} = -2x_{(-1,0,0,0)}$

$x_{(0,1,2,1)}x_{(0,-1,-2,-2)} = -x_{(0,0,0,-1)}$ $x_{(0,1,2,1)}x_{(-1,-2,-2,-1)} = 2x_{(-1,-1,0,0)}$

$x_{(0,1,2,1)}x_{(-1,-2,-3,-1)} = x_{(-1,-1,-1,0)}$ $x_{(0,1,2,1)}x_{(-1,-2,-3,-2)} = -x_{(-1,-1,-1,-1)}$

$x_{(0,1,2,1)}x_{(-1,-2,-4,-2)} = -x_{(-1,-1,-2,-1)}$ $x_{(0,1,2,1)}x_{(-1,-3,-4,-2)} = x_{(-1,-2,-2,-1)}$

$x_{(1,2,2,0)}x_{(0,1,2,2)} = -x_{(1,3,4,2)}$ $x_{(1,2,2,0)}x_{(1,1,2,2)} = x_{(2,3,4,2)}$

$x_{(1,2,2,0)}x_{(0,-1,0,0)} = x_{(1,1,2,0)}$ $x_{(1,2,2,0)}x_{(-1,-1,0,0)} = -x_{(0,1,2,0)}$

$x_{(1,2,2,0)}x_{(0,-1,-1,0)} = -x_{(1,1,1,0)}$ $x_{(1,2,2,0)}x_{(-1,-1,-1,0)} = x_{(0,1,1,0)}$

$x_{(1,2,2,0)}x_{(0,-1,-2,0)} = x_{(1,1,0,0)}$ $x_{(1,2,2,0)}x_{(-1,-1,-2,0)} = -x_{(0,1,0,0)}$

$x_{(1,2,2,0)}x_{(-1,-2,-2,-1)} = -x_{(0,0,0,-1)}$ $x_{(1,2,2,0)}x_{(-1,-2,-3,-1)} = x_{(0,0,-1,-1)}$

$x_{(1,2,2,0)}x_{(-1,-3,-4,-2)} = x_{(0,-1,-2,-2)}$ $x_{(1,2,2,0)}x_{(-2,-3,-4,-2)} = -x_{(-1,-1,-2,-2)}$

$x_{(1,1,2,1)}x_{(1,2,2,1)} = 2x_{(2,3,4,2)}$ $x_{(1,1,2,1)}x_{(-1,0,0,0)} = x_{(0,1,2,1)}$

$x_{(1,1,2,1)}x_{(0,0,-1,0)} = x_{(1,1,1,1)}$ $x_{(1,1,2,1)}x_{(0,0,0,-1)} = 2x_{(1,1,2,0)}$

$x_{(1,1,2,1)}x_{(0,0,-1,-1)} = x_{(1,1,1,0)}$ $x_{(1,1,2,1)}x_{(-1,-1,-1,0)} = -x_{(0,0,1,1)}$

$x_{(1,1,2,1)}x_{(-1,-1,-2,0)} = -x_{(0,0,0,1)}$ $x_{(1,1,2,1)}x_{(-1,-1,-1,-1)} = -x_{(0,0,1,0)}$

$x_{(1,1,2,1)}x_{(0,-1,-2,-1)} = -2x_{(1,0,0,0)}$ $x_{(1,1,2,1)}x_{(-1,-2,-2,-1)} = -2x_{(0,-1,0,0)}$

$x_{(1,1,2,1)}x_{(-1,-1,-2,-2)} = -x_{(0,0,0,-1)}$ $x_{(1,1,2,1)}x_{(-1,-2,-3,-1)} = -x_{(0,-1,-1,0)}$

$x_{(1,1,2,1)}x_{(-1,-2,-3,-2)} = x_{(0,-1,-1,-1)}$ $x_{(1,1,2,1)}x_{(-1,-2,-4,-2)} = x_{(0,-1,-2,-1)}$

$x_{(1,1,2,1)}x_{(-2,-3,-4,-2)} = -x_{(-1,-2,-2,-1)}$ $x_{(0,1,2,2)}x_{(0,0,0,-1)} = x_{(0,1,2,1)}$

$x_{(0,1,2,2)}x_{(0,0,-1,-1)} = x_{(0,1,1,1)}$ $x_{(0,1,2,2)}x_{(0,-1,-1,-1)} = -x_{(0,0,1,1)}$

$x_{(0,1,2,2)}x_{(0,-1,-2,-1)} = -x_{(0,0,0,1)}$ $x_{(0,1,2,2)}x_{(-1,-1,-2,-2)} = -x_{(-1,0,0,0)}$

$x_{(0,1,2,2)}x_{(-1,-2,-2,-2)} = x_{(-1,-1,0,0)}$ $x_{(0,1,2,2)}x_{(-1,-2,-3,-2)} = x_{(-1,-1,-1,0)}$

$x_{(0,1,2,2)}x_{(-1,-2,-4,-2)} = x_{(-1,-1,-2,0)}$ $x_{(0,1,2,2)}x_{(-1,-3,-4,-2)} = -x_{(-1,-2,-2,0)}$

$x_{(1,2,2,1)}x_{(0,-1,0,0)} = x_{(1,1,2,1)}$ $x_{(1,2,2,1)}x_{(0,0,0,-1)} = 2x_{(1,2,2,0)}$

$x_{(1,2,2,1)}x_{(-1,-1,0,0)} = -x_{(0,1,2,1)}$ $x_{(1,2,2,1)}x_{(0,-1,-1,0)} = -x_{(1,1,1,1)}$

$x_{(1,2,2,1)}x_{(-1,-1,-1,0)} = x_{(0,1,1,1)}$ $x_{(1,2,2,1)}x_{(0,-1,-1,-1)} = -x_{(1,1,1,0)}$

$x_{(1,2,2,1)}x_{(-1,-1,-1,-1)} = x_{(0,1,1,0)}$ $x_{(1,2,2,1)}x_{(0,-1,-2,-1)} = 2x_{(1,1,0,0)}$

$x_{(1,2,2,1)}x_{(-1,-2,-2,0)} = -x_{(0,0,0,1)}$ $x_{(1,2,2,1)}x_{(-1,-1,-2,-1)} = -2x_{(0,1,0,0)}$

$x_{(1,2,2,1)}x_{(-1,-2,-3,-1)} = -x_{(0,0,-1,0)}$ $x_{(1,2,2,1)}x_{(-1,-2,-2,-2)} = -x_{(0,0,0,-1)}$

$x_{(1,2,2,1)}x_{(-1,-2,-3,-2)} = x_{(0,0,-1,-1)}$ $x_{(1,2,2,1)}x_{(-1,-3,-4,-2)} = -x_{(0,-1,-2,-1)}$

$x_{(1,2,2,1)}x_{(-2,-3,-4,-2)} = x_{(-1,-1,-2,-1)}$ $x_{(1,1,2,2)}x_{(-1,0,0,0)} = x_{(0,1,2,2)}$

$x_{(1,1,2,2)}x_{(0,0,0,-1)} = x_{(1,1,2,1)}$ $x_{(1,1,2,2)}x_{(0,0,-1,-1)} = x_{(1,1,1,1)}$

$x_{(1,1,2,2)}x_{(-1,-1,-1,-1)} = -x_{(0,0,1,1)}$ $x_{(1,1,2,2)}x_{(-1,-1,-2,-1)} = -x_{(0,0,0,1)}$

$x_{(1,1,2,2)}x_{(0,-1,-2,-2)} = -x_{(1,0,0,0)}$ $x_{(1,1,2,2)}x_{(-1,-2,-2,-2)} = -x_{(0,-1,0,0)}$

$x_{(1,1,2,2)}x_{(-1,-2,-3,-2)} = -x_{(0,-1,-1,0)}$ $x_{(1,1,2,2)}x_{(-1,-2,-4,-2)} = -x_{(0,-1,-2,0)}$

$x_{(1,1,2,2)}x_{(-2,-3,-4,-2)} = x_{(-1,-2,-2,0)}$ $x_{(1,2,3,1)}x_{(0,0,-1,0)} = x_{(1,2,2,1)}$

$x_{(1,2,3,1)}x_{(0,-1,-1,0)} = x_{(1,1,2,1)}$ $x_{(1,2,3,1)}x_{(0,0,-1,-1)} = -2x_{(1,2,2,0)}$

$x_{(1,2,3,1)}x_{(-1,-1,-1,0)} = -x_{(0,1,2,1)}$ $x_{(1,2,3,1)}x_{(0,-1,-2,0)} = -x_{(1,1,1,1)}$

$x_{(1,2,3,1)}x_{(0,-1,-1,-1)} = -2x_{(1,1,2,0)}$ $x_{(1,2,3,1)}x_{(-1,-1,-2,0)} = x_{(0,1,1,1)}$

$x_{(1,2,3,1)}x_{(-1,-1,-1,-1)} = 2x_{(0,1,2,0)}$ $x_{(1,2,3,1)}x_{(0,-1,-2,-1)} = x_{(1,1,1,0)}$

$x_{(1,2,3,1)}x_{(-1,-2,-2,0)} = x_{(0,0,1,1)}$ $x_{(1,2,3,1)}x_{(-1,-1,-2,-1)} = -x_{(0,1,1,0)}$

$x_{(1,2,3,1)}x_{(-1,-2,-2,-1)} = -x_{(0,0,1,0)}$ $x_{(1,2,3,1)}x_{(-1,-2,-3,-2)} = -x_{(0,0,0,-1)}$

$x_{(1,2,3,1)}x_{(-1,-2,-4,-2)} = x_{(0,0,-1,-1)}$ $x_{(1,2,3,1)}x_{(-1,-3,-4,-2)} = x_{(0,-1,-1,-1)}$

$x_{(1,2,3,1)}x_{(-2,-3,-4,-2)} = -x_{(-1,-1,-1,-1)}$ $x_{(1,2,2,2)}x_{(0,-1,0,0)} = x_{(1,1,2,2)}$

$x_{(1,2,2,2)}x_{(0,0,0,-1)} = x_{(1,2,2,1)}$ $x_{(1,2,2,2)}x_{(-1,-1,0,0)} = -x_{(0,1,2,2)}$

$x_{(1,2,2,2)}x_{(0,-1,-1,-1)} = -x_{(1,1,1,1)}$ $x_{(1,2,2,2)}x_{(-1,-1,-1,-1)} = x_{(0,1,1,1)}$

$x_{(1,2,2,2)}x_{(0,-1,-2,-2)} = x_{(1,1,0,0)}$ $x_{(1,2,2,2)}x_{(-1,-2,-2,-1)} = -x_{(0,0,0,1)}$

$x_{(1,2,2,2)}x_{(-1,-1,-2,-2)} = -x_{(0,1,0,0)}$ $x_{(1,2,2,2)}x_{(-1,-2,-3,-2)} = -x_{(0,0,-1,0)}$

$x_{(1,2,2,2)}x_{(-1,-3,-4,-2)} = x_{(0,-1,-2,0)}$ $x_{(1,2,2,2)}x_{(-2,-3,-4,-2)} = -x_{(-1,-1,-2,0)}$

$x_{(1,2,3,2)}x_{(0,0,-1,0)} = 2x_{(1,2,2,2)}$ $x_{(1,2,3,2)}x_{(0,0,0,-1)} = x_{(1,2,3,1)}$

$x_{(1,2,3,2)}x_{(0,-1,-1,0)} = 2x_{(1,1,2,2)}$ $x_{(1,2,3,2)}x_{(0,0,-1,-1)} = -x_{(1,2,2,1)}$

$x_{(1,2,3,2)}x_{(-1,-1,-1,0)} = -2x_{(0,1,2,2)}$ $x_{(1,2,3,2)}x_{(0,-1,-1,-1)} = -x_{(1,1,2,1)}$

$x_{(1,2,3,2)}x_{(-1,-1,-1,-1)} = x_{(0,1,2,1)}$ $x_{(1,2,3,2)}x_{(0,-1,-2,-1)} = -x_{(1,1,1,1)}$

$x_{(1,2,3,2)}x_{(-1,-1,-2,-1)} = x_{(0,1,1,1)}$ $x_{(1,2,3,2)}x_{(0,-1,-2,-2)} = x_{(1,1,1,0)}$

$x_{(1,2,3,2)}x_{(-1,-2,-2,-1)} = x_{(0,0,1,1)}$ $x_{(1,2,3,2)}x_{(-1,-1,-2,-2)} = -x_{(0,1,1,0)}$

$x_{(1,2,3,2)}x_{(-1,-2,-3,-1)} = -x_{(0,0,0,1)}$ $x_{(1,2,3,2)}x_{(-1,-2,-2,-2)} = -x_{(0,0,1,0)}$

$x_{(1,2,3,2)}x_{(-1,-2,-4,-2)} = -x_{(0,0,-1,0)}$ $x_{(1,2,3,2)}x_{(-1,-3,-4,-2)} = -x_{(0,-1,-1,0)}$

$x_{(1,2,3,2)}x_{(-2,-3,-4,-2)} = x_{(-1,-1,-1,0)}$ $x_{(1,2,4,2)}x_{(0,0,-1,0)} = x_{(1,2,3,2)}$

$x_{(1,2,4,2)}x_{(0,0,-1,-1)} = -x_{(1,2,3,1)}$ $x_{(1,2,4,2)}x_{(0,-1,-2,0)} = x_{(1,1,2,2)}$

$x_{(1,2,4,2)}x_{(-1,-1,-2,0)} = -x_{(0,1,2,2)}$ $x_{(1,2,4,2)}x_{(0,-1,-2,-1)} = -x_{(1,1,2,1)}$

$x_{(1,2,4,2)}x_{(-1,-1,-2,-1)} = x_{(0,1,2,1)}$

$x_{(1,2,4,2)}x_{(0,-1,-2,-2)} = x_{(1,1,2,0)}$

$x_{(1,2,4,2)}x_{(-1,-1,-2,-2)} = -x_{(0,1,2,0)}$

$x_{(1,2,4,2)}x_{(-1,-2,-3,-1)} = x_{(0,0,1,1)}$

$x_{(1,2,4,2)}x_{(-1,-2,-3,-2)} = -x_{(0,0,1,0)}$

$x_{(1,2,4,2)}x_{(-1,-3,-4,-2)} = x_{(0,-1,0,0)}$

$x_{(1,2,4,2)}x_{(-2,-3,-4,-2)} = -x_{(-1,-1,0,0)}$

$x_{(1,3,4,2)}x_{(0,-1,0,0)} = -x_{(1,2,4,2)}$

$x_{(1,3,4,2)}x_{(0,-1,-1,0)} = x_{(1,2,3,2)}$

$x_{(1,3,4,2)}x_{(0,-1,-2,0)} = -x_{(1,2,2,2)}$

$x_{(1,3,4,2)}x_{(0,-1,-1,-1)} = -x_{(1,2,3,1)}$

$x_{(1,3,4,2)}x_{(0,-1,-2,-1)} = x_{(1,2,2,1)}$

$x_{(1,3,4,2)}x_{(-1,-2,-2,0)} = x_{(0,1,2,2)}$

$x_{(1,3,4,2)}x_{(0,-1,-2,-2)} = -x_{(1,2,2,0)}$

$x_{(1,3,4,2)}x_{(-1,-2,-2,-1)} = -x_{(0,1,2,1)}$

$x_{(1,3,4,2)}x_{(-1,-2,-3,-1)} = x_{(0,1,1,1)}$

$x_{(1,3,4,2)}x_{(-1,-2,-2,-2)} = x_{(0,1,2,0)}$

$x_{(1,3,4,2)}x_{(-1,-2,-3,-2)} = -x_{(0,1,1,0)}$

$x_{(1,3,4,2)}x_{(-1,-2,-4,-2)} = x_{(0,1,0,0)}$

$x_{(1,3,4,2)}x_{(-2,-3,-4,-2)} = x_{(-1,0,0,0)}$

$x_{(2,3,4,2)}x_{(-1,0,0,0)} = -x_{(1,3,4,2)}$

$x_{(2,3,4,2)}x_{(-1,-1,0,0)} = x_{(1,2,4,2)}$

$x_{(2,3,4,2)}x_{(-1,-1,-1,0)} = -x_{(1,2,3,2)}$

$x_{(2,3,4,2)}x_{(-1,-1,-2,0)} = x_{(1,2,2,2)}$

$x_{(2,3,4,2)}x_{(-1,-1,-1,-1)} = x_{(1,2,3,1)}$

$x_{(2,3,4,2)}x_{(-1,-2,-2,0)} = -x_{(1,1,2,2)}$

$x_{(2,3,4,2)}x_{(-1,-1,-2,-1)} = -x_{(1,2,2,1)}$

$x_{(2,3,4,2)}x_{(-1,-2,-2,-1)} = x_{(1,1,2,1)}$

$x_{(2,3,4,2)}x_{(-1,-1,-2,-2)} = x_{(1,2,2,0)}$

$x_{(2,3,4,2)}x_{(-1,-2,-3,-1)} = -x_{(1,1,1,1)}$

$x_{(2,3,4,2)}x_{(-1,-2,-2,-2)} = -x_{(1,1,2,0)}$

$x_{(2,3,4,2)}x_{(-1,-2,-3,-2)} = x_{(1,1,1,0)}$

$x_{(2,3,4,2)}x_{(-1,-2,-4,-2)} = -x_{(1,1,0,0)}$

$x_{(2,3,4,2)}x_{(-1,-3,-4,-2)} = x_{(1,0,0,0)}$

Products contained in $\mathfrak{h}$

$$x_{(1,0,0,0)}x_{(-1,0,0,0)} = h_{(1,0,0,0)}$$

$$x_{(0,1,0,0)}x_{(0,-1,0,0)} = h_{(0,1,0,0)}$$

$$x_{(0,0,1,0)}x_{(0,0,-1,0)} = h_{(0,0,1,0)}$$

$$x_{(0,0,0,1)}x_{(0,0,0,-1)} = h_{(0,0,0,1)}$$

$$x_{(1,1,0,0)}x_{(-1,-1,0,0)} = h_{(1,0,0,0)} + h_{(0,1,0,0)}$$

$$x_{(0,1,1,0)}x_{(0,-1,-1,0)} = 2h_{(0,1,0,0)} + h_{(0,0,1,0)}$$

$$x_{(0,0,1,1)}x_{(0,0,-1,-1)} = h_{(0,0,1,0)} + h_{(0,0,0,1)}$$

$$x_{(1,1,1,0)}x_{(-1,-1,-1,0)} = 2h_{(1,0,0,0)} + 2h_{(0,1,0,0)} + h_{(0,0,1,0)}$$

$$x_{(0,1,2,0)}x_{(0,-1,-2,0)} = h_{(0,1,0,0)} + h_{(0,0,1,0)}$$

$$x_{(0,1,1,1)}x_{(0,-1,-1,-1)} = 2h_{(0,1,0,0)} + h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,1,2,0)}x_{(-1,-1,-2,0)} = h_{(1,0,0,0)} + h_{(0,1,0,0)} + h_{(0,0,1,0)}$$
$$x_{(1,1,1,1)}x_{(-1,-1,-1,-1)} = 2h_{(1,0,0,0)} + 2h_{(0,1,0,0)} + h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(0,1,2,1)}x_{(0,-1,-2,-1)} = 2h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,2,2,0)}x_{(-1,-2,-2,0)} = h_{(1,0,0,0)} + 2h_{(0,1,0,0)} + h_{(0,0,1,0)}$$
$$x_{(1,1,2,1)}x_{(-1,-1,-2,-1)} = 2h_{(1,0,0,0)} + 2h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(0,1,2,2)}x_{(0,-1,-2,-2)} = h_{(0,1,0,0)} + h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,2,2,1)}x_{(-1,-2,-2,-1)} = 2h_{(1,0,0,0)} + 4h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,1,2,2)}x_{(-1,-1,-2,-2)} = h_{(1,0,0,0)} + h_{(0,1,0,0)} + h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,2,3,1)}x_{(-1,-2,-3,-1)} = 2h_{(1,0,0,0)} + 4h_{(0,1,0,0)} + 3h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,2,2,2)}x_{(-1,-2,-2,-2)} = h_{(1,0,0,0)} + 2h_{(0,1,0,0)} + h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,2,3,2)}x_{(-1,-2,-3,-2)} = 2h_{(1,0,0,0)} + 4h_{(0,1,0,0)} + 3h_{(0,0,1,0)} + 2h_{(0,0,0,1)}$$
$$x_{(1,2,4,2)}x_{(-1,-2,-4,-2)} = h_{(1,0,0,0)} + 2h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(1,3,4,2)}x_{(-1,-3,-4,-2)} = h_{(1,0,0,0)} + 3h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + h_{(0,0,0,1)}$$
$$x_{(2,3,4,2)}x_{(-2,-3,-4,-2)} = 2h_{(1,0,0,0)} + 3h_{(0,1,0,0)} + 2h_{(0,0,1,0)} + h_{(0,0,0,1)}$$

Action of $\mathfrak{h}$ on $\mathfrak{g}$

$$h_{(1,0,0,0)}x_{(1,0,0,0)} = 2x_{(1,0,0,0)}$$
$$h_{(1,0,0,0)}x_{(0,1,0,0)} = -x_{(0,1,0,0)}$$
$$h_{(1,0,0,0)}x_{(0,0,1,0)} = 0$$
$$h_{(1,0,0,0)}x_{(0,0,0,1)} = 0$$
$$h_{(1,0,0,0)}x_{(1,1,0,0)} = x_{(1,1,0,0)}$$
$$h_{(1,0,0,0)}x_{(0,1,1,0)} = -x_{(0,1,1,0)}$$
$$h_{(1,0,0,0)}x_{(0,0,1,1)} = 0$$
$$h_{(1,0,0,0)}x_{(1,1,1,0)} = x_{(1,1,1,0)}$$
$$h_{(1,0,0,0)}x_{(0,1,2,0)} = -x_{(0,1,2,0)}$$
$$h_{(1,0,0,0)}x_{(0,1,1,1)} = -x_{(0,1,1,1)}$$
$$h_{(1,0,0,0)}x_{(1,1,2,0)} = x_{(1,1,2,0)}$$
$$h_{(1,0,0,0)}x_{(1,1,1,1)} = x_{(1,1,1,1)}$$
$$h_{(1,0,0,0)}x_{(0,1,2,1)} = -x_{(0,1,2,1)}$$
$$h_{(1,0,0,0)}x_{(1,2,2,0)} = 0$$
$$h_{(1,0,0,0)}x_{(1,1,2,1)} = x_{(1,1,2,1)}$$
$$h_{(1,0,0,0)}x_{(0,1,2,2)} = -x_{(0,1,2,2)}$$
$$h_{(1,0,0,0)}x_{(1,2,2,1)} = 0$$
$$h_{(1,0,0,0)}x_{(1,1,2,2)} = x_{(1,1,2,2)}$$
$$h_{(1,0,0,0)}x_{(1,2,3,1)} = 0$$
$$h_{(1,0,0,0)}x_{(1,2,2,2)} = 0$$
$$h_{(1,0,0,0)}x_{(1,2,3,2)} = 0$$
$$h_{(1,0,0,0)}x_{(1,2,4,2)} = 0$$

$h_{(1,0,0,0)}x_{(1,3,4,2)} = -x_{(1,3,4,2)}$

$h_{(1,0,0,0)}x_{(2,3,4,2)} = x_{(2,3,4,2)}$

$h_{(1,0,0,0)}x_{(-1,0,0,0)} = -2x_{(-1,0,0,0)}$

$h_{(1,0,0,0)}x_{(0,-1,0,0)} = x_{(0,-1,0,0)}$

$h_{(1,0,0,0)}x_{(0,0,-1,0)} = 0$

$h_{(1,0,0,0)}x_{(0,0,0,-1)} = 0$

$h_{(1,0,0,0)}x_{(-1,-1,0,0)} = -x_{(-1,-1,0,0)}$

$h_{(1,0,0,0)}x_{(0,-1,-1,0)} = x_{(0,-1,-1,0)}$

$h_{(1,0,0,0)}x_{(0,0,-1,-1)} = 0$

$h_{(1,0,0,0)}x_{(-1,-1,-1,0)} = -x_{(-1,-1,-1,0)}$

$h_{(1,0,0,0)}x_{(0,-1,-2,0)} = x_{(0,-1,-2,0)}$

$h_{(1,0,0,0)}x_{(0,-1,-1,-1)} = x_{(0,-1,-1,-1)}$

$h_{(1,0,0,0)}x_{(-1,-1,-2,0)} = -x_{(-1,-1,-2,0)}$

$h_{(1,0,0,0)}x_{(-1,-1,-1,-1)} = -x_{(-1,-1,-1,-1)}$

$h_{(1,0,0,0)}x_{(0,-1,-2,-1)} = x_{(0,-1,-2,-1)}$

$h_{(1,0,0,0)}x_{(-1,-2,-2,0)} = 0$

$h_{(1,0,0,0)}x_{(-1,-1,-2,-1)} = -x_{(-1,-1,-2,-1)}$

$h_{(1,0,0,0)}x_{(0,-1,-2,-2)} = x_{(0,-1,-2,-2)}$

$h_{(1,0,0,0)}x_{(-1,-2,-2,-1)} = 0$

$h_{(1,0,0,0)}x_{(-1,-1,-2,-2)} = -x_{(-1,-1,-2,-2)}$

$h_{(1,0,0,0)}x_{(-1,-2,-3,-1)} = 0$

$h_{(1,0,0,0)}x_{(-1,-2,-2,-2)} = 0$

$h_{(1,0,0,0)}x_{(-1,-2,-3,-2)} = 0$

$h_{(1,0,0,0)}x_{(-1,-2,-4,-2)} = 0$

$h_{(1,0,0,0)}x_{(-1,-3,-4,-2)} = x_{(-1,-3,-4,-2)}$

$h_{(1,0,0,0)}x_{(-2,-3,-4,-2)} = -x_{(-2,-3,-4,-2)}$

$h_{(0,1,0,0)}x_{(1,0,0,0)} = -x_{(1,0,0,0)}$

$h_{(0,1,0,0)}x_{(0,1,0,0)} = 2x_{(0,1,0,0)}$

$h_{(0,1,0,0)}x_{(0,0,1,0)} = -x_{(0,0,1,0)}$

$h_{(0,1,0,0)}x_{(0,0,0,1)} = 0$

$h_{(0,1,0,0)}x_{(1,1,0,0)} = x_{(1,1,0,0)}$

$h_{(0,1,0,0)}x_{(0,1,1,0)} = x_{(0,1,1,0)}$

$h_{(0,1,0,0)}x_{(0,0,1,1)} = -x_{(0,0,1,1)}$

$h_{(0,1,0,0)}x_{(1,1,1,0)} = 0$

$h_{(0,1,0,0)}x_{(0,1,2,0)} = 0$

$h_{(0,1,0,0)}x_{(0,1,1,1)} = x_{(0,1,1,1)}$

$h_{(0,1,0,0)}x_{(1,1,2,0)} = -x_{(1,1,2,0)}$

$h_{(0,1,0,0)}x_{(1,1,1,1)} = 0$

$h_{(0,1,0,0)}x_{(0,1,2,1)} = 0$

$h_{(0,1,0,0)}x_{(1,2,2,0)} = x_{(1,2,2,0)}$

$h_{(0,1,0,0)}x_{(1,1,2,1)} = -x_{(1,1,2,1)}$

$h_{(0,1,0,0)}x_{(0,1,2,2)} = 0$

$h_{(0,1,0,0)}x_{(1,2,2,1)} = x_{(1,2,2,1)}$

$h_{(0,1,0,0)}x_{(1,1,2,2)} = -x_{(1,1,2,2)}$

$h_{(0,1,0,0)}x_{(1,2,3,1)} = 0$

$h_{(0,1,0,0)}x_{(1,2,2,2)} = x_{(1,2,2,2)}$

$h_{(0,1,0,0)}x_{(1,2,3,2)} = 0$

$h_{(0,1,0,0)}x_{(1,2,4,2)} = -x_{(1,2,4,2)}$

$h_{(0,1,0,0)}x_{(1,3,4,2)} = x_{(1,3,4,2)}$

$h_{(0,1,0,0)}x_{(2,3,4,2)} = 0$

$h_{(0,1,0,0)}x_{(-1,0,0,0)} = x_{(-1,0,0,0)}$

$h_{(0,1,0,0)}x_{(0,-1,0,0)} = -2x_{(0,-1,0,0)}$

$h_{(0,1,0,0)}x_{(0,0,-1,0)} = x_{(0,0,-1,0)}$

$h_{(0,1,0,0)}x_{(0,0,0,-1)} = 0$

$h_{(0,1,0,0)}x_{(-1,-1,0,0)} = -x_{(-1,-1,0,0)}$

$h_{(0,1,0,0)}x_{(0,-1,-1,0)} = -x_{(0,-1,-1,0)}$

$h_{(0,1,0,0)}x_{(0,0,-1,-1)} = x_{(0,0,-1,-1)}$

$h_{(0,1,0,0)}x_{(-1,-1,-1,0)} = 0$

$h_{(0,1,0,0)}x_{(0,-1,-2,0)} = 0$

$h_{(0,1,0,0)}x_{(0,-1,-1,-1)} = -x_{(0,-1,-1,-1)}$

$h_{(0,1,0,0)}x_{(-1,-1,-2,0)} = x_{(-1,-1,-2,0)}$

$h_{(0,1,0,0)}x_{(-1,-1,-1,-1)} = 0$

$h_{(0,1,0,0)}x_{(0,-1,-2,-1)} = 0$

$h_{(0,1,0,0)}x_{(-1,-2,-2,0)} = -x_{(-1,-2,-2,0)}$

$h_{(0,1,0,0)}x_{(-1,-1,-2,-1)} = x_{(-1,-1,-2,-1)}$

$h_{(0,1,0,0)}x_{(0,-1,-2,-2)} = 0$

$h_{(0,1,0,0)}x_{(-1,-2,-2,-1)} = -x_{(-1,-2,-2,-1)}$

$h_{(0,1,0,0)}x_{(-1,-1,-2,-2)} = x_{(-1,-1,-2,-2)}$

$h_{(0,1,0,0)}x_{(-1,-2,-3,-1)} = 0$

$h_{(0,1,0,0)}x_{(-1,-2,-2,-2)} = -x_{(-1,-2,-2,-2)}$

$h_{(0,1,0,0)}x_{(-1,-2,-3,-2)} = 0$

$h_{(0,1,0,0)}x_{(-1,-2,-4,-2)} = x_{(-1,-2,-4,-2)}$

$h_{(0,1,0,0)}x_{(-1,-3,-4,-2)} = -x_{(-1,-3,-4,-2)}$

$h_{(0,1,0,0)}x_{(-2,-3,-4,-2)} = 0$

$h_{(0,0,1,0)}x_{(1,0,0,0)} = 0$

$h_{(0,0,1,0)}x_{(0,1,0,0)} = -2x_{(0,1,0,0)}$

$h_{(0,0,1,0)}x_{(0,0,1,0)} = 2x_{(0,0,1,0)}$

$h_{(0,0,1,0)}x_{(0,0,0,1)} = -x_{(0,0,0,1)}$

$h_{(0,0,1,0)}x_{(1,1,0,0)} = -2x_{(1,1,0,0)}$

$h_{(0,0,1,0)}x_{(0,1,1,0)} = 0$

$h_{(0,0,1,0)}x_{(0,0,1,1)} = x_{(0,0,1,1)}$

$h_{(0,0,1,0)}x_{(1,1,1,0)} = 0$

$h_{(0,0,1,0)}x_{(0,1,2,0)} = 2x_{(0,1,2,0)}$

$h_{(0,0,1,0)}x_{(0,1,1,1)} = -x_{(0,1,1,1)}$

$h_{(0,0,1,0)}x_{(1,1,2,0)} = 2x_{(1,1,2,0)}$

$h_{(0,0,1,0)}x_{(1,1,1,1)} = -x_{(1,1,1,1)}$

$h_{(0,0,1,0)}x_{(0,1,2,1)} = x_{(0,1,2,1)}$

$h_{(0,0,1,0)}x_{(1,2,2,0)} = 0$

$h_{(0,0,1,0)}x_{(1,1,2,1)} = x_{(1,1,2,1)}$

$h_{(0,0,1,0)}x_{(0,1,2,2)} = 0$

$h_{(0,0,1,0)}x_{(1,2,2,1)} = -x_{(1,2,2,1)}$

$h_{(0,0,1,0)}x_{(1,1,2,2)} = 0$

$h_{(0,0,1,0)}x_{(1,2,3,1)} = x_{(1,2,3,1)}$

$h_{(0,0,1,0)}x_{(1,2,2,2)} = -2x_{(1,2,2,2)}$

$h_{(0,0,1,0)}x_{(1,2,3,2)} = 0$

$h_{(0,0,1,0)}x_{(1,2,4,2)} = 2x_{(1,2,4,2)}$

$h_{(0,0,1,0)}x_{(1,3,4,2)} = 0$

$h_{(0,0,1,0)}x_{(2,3,4,2)} = 0$

$h_{(0,0,1,0)}x_{(-1,0,0,0)} = 0$

$h_{(0,0,1,0)}x_{(0,-1,0,0)} = 2x_{(0,-1,0,0)}$

$h_{(0,0,1,0)}x_{(0,0,-1,0)} = -2x_{(0,0,-1,0)}$

$h_{(0,0,1,0)}x_{(0,0,0,-1)} = x_{(0,0,0,-1)}$

$h_{(0,0,1,0)}x_{(-1,-1,0,0)} = 2x_{(-1,-1,0,0)}$

$h_{(0,0,1,0)}x_{(0,-1,-1,0)} = 0$

$h_{(0,0,1,0)}x_{(0,0,-1,-1)} = -x_{(0,0,-1,-1)}$

$h_{(0,0,1,0)}x_{(-1,-1,-1,0)} = 0$

$h_{(0,0,1,0)}x_{(0,-1,-2,0)} = -2x_{(0,-1,-2,0)}$

$h_{(0,0,1,0)}x_{(0,-1,-1,-1)} = x_{(0,-1,-1,-1)}$

$h_{(0,0,1,0)}x_{(-1,-1,-2,0)} = -2x_{(-1,-1,-2,0)}$

$h_{(0,0,1,0)}x_{(-1,-1,-1,-1)} = x_{(-1,-1,-1,-1)}$

$h_{(0,0,1,0)}x_{(0,-1,-2,-1)} = -x_{(0,-1,-2,-1)}$

$h_{(0,0,1,0)}x_{(-1,-2,-2,0)} = 0$

$h_{(0,0,1,0)}x_{(-1,-1,-2,-1)} = -x_{(-1,-1,-2,-1)}$ $h_{(0,0,1,0)}x_{(0,-1,-2,-2)} = 0$

$h_{(0,0,1,0)}x_{(-1,-2,-2,-1)} = x_{(-1,-2,-2,-1)}$ $h_{(0,0,1,0)}x_{(-1,-1,-2,-2)} = 0$

$h_{(0,0,1,0)}x_{(-1,-2,-3,-1)} = -x_{(-1,-2,-3,-1)}$ $h_{(0,0,1,0)}x_{(-1,-2,-2,-2)} = 2x_{(-1,-2,-2,-2)}$

$h_{(0,0,1,0)}x_{(-1,-2,-3,-2)} = 0$ $h_{(0,0,1,0)}x_{(-1,-2,-4,-2)} = -2x_{(-1,-2,-4,-2)}$

$h_{(0,0,1,0)}x_{(-1,-3,-4,-2)} = 0$ $h_{(0,0,1,0)}x_{(-2,-3,-4,-2)} = 0$

$h_{(0,0,0,1)}x_{(1,0,0,0)} = 0$ $h_{(0,0,0,1)}x_{(0,1,0,0)} = 0$

$h_{(0,0,0,1)}x_{(0,0,1,0)} = -x_{(0,0,1,0)}$ $h_{(0,0,0,1)}x_{(0,0,0,1)} = 2x_{(0,0,0,1)}$

$h_{(0,0,0,1)}x_{(1,1,0,0)} = 0$ $h_{(0,0,0,1)}x_{(0,1,1,0)} = -x_{(0,1,1,0)}$

$h_{(0,0,0,1)}x_{(0,0,1,1)} = x_{(0,0,1,1)}$ $h_{(0,0,0,1)}x_{(1,1,1,0)} = -x_{(1,1,1,0)}$

$h_{(0,0,0,1)}x_{(0,1,2,0)} = -2x_{(0,1,2,0)}$ $h_{(0,0,0,1)}x_{(0,1,1,1)} = x_{(0,1,1,1)}$

$h_{(0,0,0,1)}x_{(1,1,2,0)} = -2x_{(1,1,2,0)}$ $h_{(0,0,0,1)}x_{(1,1,1,1)} = x_{(1,1,1,1)}$

$h_{(0,0,0,1)}x_{(0,1,2,1)} = 0$ $h_{(0,0,0,1)}x_{(1,2,2,0)} = -2x_{(1,2,2,0)}$

$h_{(0,0,0,1)}x_{(1,1,2,1)} = 0$ $h_{(0,0,0,1)}x_{(0,1,2,2)} = 2x_{(0,1,2,2)}$

$h_{(0,0,0,1)}x_{(1,2,2,1)} = 0$ $h_{(0,0,0,1)}x_{(1,1,2,2)} = 2x_{(1,1,2,2)}$

$h_{(0,0,0,1)}x_{(1,2,3,1)} = -x_{(1,2,3,1)}$ $h_{(0,0,0,1)}x_{(1,2,2,2)} = 2x_{(1,2,2,2)}$

$h_{(0,0,0,1)}x_{(1,2,3,2)} = x_{(1,2,3,2)}$ $h_{(0,0,0,1)}x_{(1,2,4,2)} = 0$

$h_{(0,0,0,1)}x_{(1,3,4,2)} = 0$ $h_{(0,0,0,1)}x_{(2,3,4,2)} = 0$

$h_{(0,0,0,1)}x_{(-1,0,0,0)} = 0$ $h_{(0,0,0,1)}x_{(0,-1,0,0)} = 0$

$h_{(0,0,0,1)}x_{(0,0,-1,0)} = x_{(0,0,-1,0)}$ $h_{(0,0,0,1)}x_{(0,0,0,-1)} = -2x_{(0,0,0,-1)}$

$h_{(0,0,0,1)}x_{(-1,-1,0,0)} = 0$ $h_{(0,0,0,1)}x_{(0,-1,-1,0)} = x_{(0,-1,-1,0)}$

$h_{(0,0,0,1)}x_{(0,0,-1,-1)} = -x_{(0,0,-1,-1)}$ $h_{(0,0,0,1)}x_{(-1,-1,-1,0)} = x_{(-1,-1,-1,0)}$

$h_{(0,0,0,1)}x_{(0,-1,-2,0)} = 2x_{(0,-1,-2,0)}$ $h_{(0,0,0,1)}x_{(0,-1,-1,-1)} = -x_{(0,-1,-1,-1)}$

$h_{(0,0,0,1)}x_{(-1,-1,-2,0)} = 2x_{(-1,-1,-2,0)}$ $h_{(0,0,0,1)}x_{(-1,-1,-1,-1)} = -x_{(-1,-1,-1,-1)}$

$h_{(0,0,0,1)}x_{(0,-1,-2,-1)} = 0$ $h_{(0,0,0,1)}x_{(-1,-2,-2,0)} = 2x_{(-1,-2,-2,0)}$

$h_{(0,0,0,1)}x_{(-1,-1,-2,-1)} = 0$ $h_{(0,0,0,1)}x_{(0,-1,-2,-2)} = -2x_{(0,-1,-2,-2)}$

$h_{(0,0,0,1)}x_{(-1,-2,-2,-1)} = 0$ $h_{(0,0,0,1)}x_{(-1,-1,-2,-2)} = -2x_{(-1,-1,-2,-2)}$

$h_{(0,0,0,1)}x_{(-1,-2,-3,-1)} = x_{(-1,-2,-3,-1)}$ $h_{(0,0,0,1)}x_{(-1,-2,-2,-2)} = -2x_{(-1,-2,-2,-2)}$

$h_{(0,0,0,1)}x_{(-1,-2,-3,-2)} = -x_{(-1,-2,-3,-2)}$ $h_{(0,0,0,1)}x_{(-1,-2,-4,-2)} = 0$

$h_{(0,0,0,1)}x_{(-1,-3,-4,-2)} = 0$ $\qquad$ $h_{(0,0,0,1)}x_{(-2,-3,-4,-2)} = 0$

E.2 Generators for $\mathfrak{f}_4 \subset \mathfrak{gl}(26, \mathbb{C})$

Root Vector x	$\pi(x)$
x_α	$E^{26}_{4,5} + E^{26}_{6,7} + E^{26}_{8,10} + E^{26}_{18,20} + E^{26}_{19,21} + E^{26}_{22,23}$
$x_{-\alpha}$	$\pi(x_\alpha)^{\mathsf{T}}$
x_β	$E^{26}_{3,4} + E^{26}_{7,9} + E^{26}_{10,12} + E^{26}_{16,18} + E^{26}_{17,19} + E^{26}_{23,24}$
$x_{-\beta}$	$\pi(x_\beta)^{\mathsf{T}}$
x_γ	$E^{26}_{2,3} + E^{26}_{4,6} + E^{26}_{5,7} + E^{26}_{9,11} + E^{26}_{12,13} + 2E^{26}_{12,14} + E^{26}_{14,16} +$ $E^{26}_{15,17} + E^{26}_{19,22} + E^{26}_{21,23} + E^{26}_{24,25}$
$x_{-\gamma}$	$E^{26}_{3,2} + E^{26}_{6,4} + E^{26}_{7,5} + E^{26}_{11,9} + E^{26}_{14,12} + E^{26}_{16,13} + 2E^{26}_{16,14} +$ $E^{26}_{17,15} + E^{26}_{22,19} + E^{26}_{23,21} + E^{26}_{25,24}$
x_δ	$E^{26}_{1,2} + E^{26}_{6,8} + E^{26}_{7,10} + E^{26}_{9,12} + 2E^{26}_{11,13} + E^{26}_{11,14} + E^{26}_{13,15} +$ $E^{26}_{16,17} + E^{26}_{18,19} + E^{26}_{20,21} + E^{26}_{25,26}$
$x_{-\delta}$	$E^{26}_{2,1} + E^{26}_{8,6} + E^{26}_{10,7} + E^{26}_{12,9} + E^{26}_{13,11} + 2E^{26}_{15,13} + E^{26}_{15,14} +$ $E^{26}_{17,16} + E^{26}_{19,18} + E^{26}_{21,20} + E^{26}_{26,25}$

Table E.4: Generators for $\mathfrak{f}_4 \subset \mathfrak{gl}(26, \mathbb{C})$

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