Barcodes and Quasi-Isometric Embeddings into Hamiltonian Diffeomorphism Groups

by

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(Under the Direction of Michael Usher)

Abstract

We construct an embedding Φ of $[0,1]^\infty$ into $Ham(M,\omega)$, the group of Hamiltonian diffeomorphisms of a suitable closed symplectic manifold (M,ω). We then prove that Φ is in fact a quasi-isometry. After imposing further assumptions on (M,ω), we adapt our methods to construct a similar embedding of $\mathbb{R} \oplus [0,1]^\infty$ into either $Ham(M,\omega)$ or $\widetilde{Ham}(M,\omega)$, the universal cover of $Ham(M,\omega)$. Along the way, we prove results related to the filtered Floer chain complexes of radially symmetric Hamiltonians. We conclude by proving the boundedness of the boundary depth function β restricted to the set of autonomous Hamiltonian diffeomorphisms of (S^2,ω). The majority of our proofs rely heavily on a continuity result for barcodes (as presented in [36]) associated to filtered Floer homology viewed as a persistence module.

Index words: Hamiltonian diffeomorphisms, Hofer's metric, Hamiltonian Floer theory, persistence modules, barcodes, boundary depth

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Chapter 1

Preliminaries and Statement of Results

Suppose we are given a closed symplectic manifold (M, ω), (i.e. a $2n$-dimensional smooth manifold M equipped with a closed, non-degenerate 2-form $\omega(\cdot,\cdot)$), and let $H : M \to \mathbb{R}$ be a smooth function. The differential $dH(\cdot)$ of H is a 1-form on M, and the non-degeneracy of ω implies the existence of a unique vector field X_H on M with $\omega(X_H, \cdot) = -dH(\cdot)$. More generally, we may consider a smooth function $H : [0,1] \times M \to \mathbb{R}$; for each t, $H_t := H(t, \cdot)$ is a smooth function on M, and so we may use the above procedure to find, for each t, a vector field X_{H_t} on M. We have therefore constructed a *time-dependent* vector field on M. Since M is closed, we can use the existence and uniqueness of solutions to ordinary differential equations to find an isotopy ϕ_H^t of M satisfying

$$\frac{d}{dt}\Big|_{t=s} \phi_t(\cdot) = X_{H_s} \circ \phi_s(\cdot);$$

In the above, H is called a *Hamiltonian*, while X_{H_t} and ϕ_H^t are the corresponding *Hamiltonian vector field* and *Hamiltonian isotopy*, respectively. The time-1 map of a Hamiltonian isotopy is simply called a *Hamiltonian diffeomorphism*, and we use $Ham(M, \omega)$ to denote the space of all Hamiltonian diffeomorphisms of (M, ω). Important to later discussions is the fact that any Hamiltonian diffeomorphism ϕ can be generated by a 1-periodic Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, which in turn makes the Hamiltonian isotopy ϕ_t a 1-periodic function from $\mathbb{R}/\mathbb{Z} \times M$ into M. We also note here that if M is open or has boundary, we will instead let $Ham(M, \omega)$ consist only of diffeomorphisms generated by F whose support is compact and contained in $[0,1] \times \mathrm{int}(M)$.

Among the more noteworthy properties of Hamiltonian diffeomoprhisms is their preservation of symplectic forms. Indeed, where α is a differential k-form and v_t is a time-dependent vector field generating the flow ρ_t, we have the formula

$$\tfrac{d}{dt}\rho_t^*\alpha = \rho_t^*\mathcal{L}_{v_t}\alpha, \tag{†}$$

where $\mathcal{L}$ is the usual Lie derivative. Letting v_t be the Hamiltonian vector field X_{H_t} and ρ_t the Hamiltonian isotopy ϕ_H^t, we have

$$\begin{aligned}\tfrac{d}{dt}\rho_t^*\omega &= \rho_t^*\left(\mathcal{L}_{v_t}\omega\right)\\ &= \rho_t^*\left(d\iota_{v_t}\omega + \iota_{v_t}d\omega\right)\\ &= \rho_t^*\left(d(-dH_t)+0\right)\\ &= 0,\end{aligned}$$

meaning $(\phi_H^t)^*\omega$ is constant. Note our use of Cartan's formula in the second line.

For *autonomous*, meaning time-independent, smooth functions $F, H : M \to \mathbb{R}$, their *Poisson bracket* $\{F, H\}$ is defined as $\omega(X_F, X_H)$. By a simple application of (†) to $\frac{d}{dt}\left(F \circ \phi_H^t\right)$, we see that F is constant along the integral curves of X_H if and only if F and H Poisson commute (meaning their Poisson bracket vanishes):

$$\begin{aligned}\tfrac{d}{dt}\left(F\circ\phi_H^t\right) &= \tfrac{d}{dt}(\phi_H^t)^*F\\ &= \rho_t^*\left(\mathcal{L}_{X_H}F\right)\\ &= \rho_t^*\left(dF(X_H)\right)\\ &= \rho_t^*\left(-\omega(X_F, X_H)\right).\end{aligned}$$

The space $Ham(M,\omega)$ is in fact a group under composition: If F and H generate ϕ_F and ϕ_H, the Hamiltonian $-F_t\circ\phi_F^t$ generates ϕ_F^{-1} while $F_t + H_t\circ(\phi_F^t)^{-1}$ generates the composition $\phi_F\phi_H$. However, supposing F and H are both autonomous, their Poisson bracket vanishing implies the equality $H \circ (\phi_F^t)^{-1} = H$, so the composition $\phi_F \phi_H$ is in fact generated by the Hamiltonian $F + H$. Similarly, F being autonomous implies that ϕ_F^{-1} is generated by $-F$ and that the flow ϕ_F^t preserves F's level sets. These properties will prove useful to us in later chapters.

To every ϕ of $Ham(M, \omega)$, we may associate its *Hofer norm*

$$||\phi||_H = \inf \left\{ \int_0^1 \left(\max_M (H_t) - \min_M (H_t) \right) \, dt \, | \, \phi_H^1 = \phi \right\}.$$

It is easy to deduce that the Hofer norm is a pseudo-norm, meaning it satisfies all properties of being a norm except possibly non-degeneracy. Moreover, it is invariant under conjugation by other elements of $Ham(M, \omega)$ since the Hamiltonian $H \circ \phi_F^{-1}$ generates the element $\phi_F \phi_H \phi_F^{-1}$. Hofer was able to demonstrate the non-degeneracy of $|| \cdot ||_H$ for $M = \mathbb{R}^{2n}$ in [12]. Lalonde and McDuff demonstrated it for arbitrary symplectic manifolds in [17] by showing that any open $B \subset M$ has positive *displacement energy* $e(B)$, where $e(B)$ is defined by

$$e(B) = \inf \{ ||\phi||_H \mid \phi \in Ham(M, \omega) \;\; \phi(\bar{B}) \cap \bar{B} = \emptyset \},$$

The non-degeneracy of $|| \cdot ||_H$ easily follows, for any $\phi \in Ham(M, \omega)$ not equal to Id_M must displace some small open ball $B \subset M$, making $||\phi||_H$ positive. With $|| \cdot ||_H$ being a full-fledged norm, we may therefore define the metric d_H, called *Hofer's metric*, on $Ham(M, \omega)$ by

$$d_H(\phi, \psi) = ||\phi^{-1} \psi||_H$$

for any $\phi, \psi \in Ham(M, \omega)$. The Hofer norm being invariant under conjugation by other elements of $Ham(M, \omega)$ makes d_H not only left-invariant but biinvariant:

$$\forall \eta, \phi, \psi \in Ham(M, \omega), \quad d_H(\eta\phi, \eta\psi) = d_H(\phi, \psi) = d_H(\phi\eta, \psi\eta).$$

The additional assumption we must impose on our symplectic manifold is a relation between what we shall call the *Chern* and *area* homomorphisms restricted to $\pi_2(M)$. Recall that an *almost complex structure* J on a manifold M is a smooth automorphism $J : TM \to TM$ such that each $J_p : T_pM \to T_pM$ satisfies $J_p^2 = -\mathbb{1}$. A symplectic manifold (M, ω) always admits such a J, and by choosing one we may consider TM as a complex vector bundle over M. We restrict our attention, however, to those J which are ω-compatible (i.e. those J which make $\omega(\cdot, J\cdot)$ a Riemannian metric), for the set $\mathcal{J}(M,\omega)$ of all such J is path-connected, making any pair of associated complex vector bundles (TM, J_0) and (TM, J_1) with $J_0, J_1 \in \mathcal{J}(M,\omega)$ isomorphic. Thus, we may associate to (M,ω) Chern classes $c_i(TM) \in H^{2i}(M,\omega)$ by first choosing any $J \in \mathcal{J}(M,\omega)$ and defining $c_i(TM)$ to be those Chern classes associated to the complex vector bundle (TM, J). We refer to the map $c_1(TM) : H_2(M,\mathbb{Z}) \to \mathbb{Z}$ as simply the *Chern homomorphism*.

The *area homomorphism* is much simpler to define. With (M,ω) a closed symplectic manifold, ω determines a class $[\omega] \in H^2(M,\mathbb{Z})$ and so we get a map $[\omega]$ from $H_2(M,\mathbb{Z})$ to $\mathbb{R}$.

Definition 1.1. We call a closed symplectic manifold (M,ω) *monotone* if there exists a real number $\lambda \geq 0$ with

$$[\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)},$$

where $[\omega]|_{\pi_2(M)}$ and $c_1(TM)|_{\pi_2(M)}$ are the maps $[\omega]$ and $c_1(TM)$ precomposed with the Hurewicz homomorphism $h : \pi_2(M) \to H_2(M,\mathbb{Z})$. If the same relation above holds but with $\lambda < 0$, then we call (M,ω) *negative monotone*.

Since the image of $c_1(TM)|_{\pi_2(M)}$ is a subgroup of $\mathbb{Z}$, the image of $[\omega]|_{\pi_2(M)}$ forms a discrete subgroup of $\mathbb{R}$ when M is (negative) monotone. The *minimal Chern number* N is the non-negative generator of the image of $c_1(TM)|_{\pi_2(M)}$, and the *rationality constant* γ of M is the non-negative generator of the image of $[\omega]|_{\pi_2(M)}$. In particular, this work has $N = 0$, $\gamma = 0$ when $c_1(TM)|_{\pi_2(M)} = 0$, $[\omega]|_{\pi_2(M)} = 0$, respectively; while this is a break from the convention of setting N, γ equal to ∞ in such cases, we find it to be a worthwhile one for the present work as it simplifies the discussion of the various cases considered in the proof of Theorem 1.2 (particularly Lemma 5.2).

Finally, let $[0,1]^\infty$ denote the set of all $[0,1]$-valued sequences with only finitely many non-zero entries, and for $a = \{a_i\}_{i\geq 1}, b = \{b_i\}_{i\geq 1} \in [0,1]^\infty$, let

$$||a-b||_{\ell^\infty} = \mathrm{max}_i |a_i - b_i|.$$

Theorem 1.2. *Let M be a closed symplectic manifold which is either monotone or negative monotone. Suppose we may symplectically embed a ball $B(2\pi R)$ of radius $\sqrt{2R}$ into M, where if M's rationality constant γ is non-zero, we require $4\pi R \leq \gamma$. Then for any $\varepsilon > 0$, there exists an embedding $\Phi : [0,1]^\infty \to Ham(M,\omega)$ satisfying*

$$2\pi R||a-b||_{\ell^\infty} - \varepsilon \leq d_H(\Phi(a), \Phi(b)) \leq 4\pi R||a-b||_{\ell^\infty}$$

for any $a, b \in [0,1]^\infty$. That is, Φ is a quasi-isometric embedding of $[0,1]^\infty$ into $Ham(M,\omega)$.

Upon the introduction of Hofer's metric, it was natural to ask which symplectic manifolds (M,ω) yield $(Ham(M,\omega), d_H)$ with infinite diameter, and the appearances of results in this direction form a rich history; see [18], [25], [31], [23], [6], and [19], for example. Similarly, one may instead ask the broader question of which $Ham(M,\omega)$ admit quasi-isometric embeddings of multi-dimensional normed vector spaces. This question already has partial answers, among which are results appearing in [28] and [35]. Provided the existence of a closed Lagrangian $L \subset M$ which admits a Riemannian metric of non-positive curvature and has the inclusion-induced map $i_* : \pi_1(L) \to \pi_1(M)$ injective, Py shows that for any $m \in \mathbb{N}$ there exists a constant $C_m > 0$ and an embedding $\phi : \mathbb{Z}^m \to Ham(M,\omega)$ satisfying

$$C_m^{-1}||a-b||_{\ell^\infty} \leq d_H(\phi(a), \phi(b)) \leq C_m||a-b||_{\ell^\infty}$$

for any $a, b \in \mathbb{Z}^m$. This result was generalized in [35], in which Usher proves that if M admits an autonomous Hamiltonian $H : M \to \mathbb{R}$ whose flow has all of its contractible periodic orbits constant, then there exists an embedding of $\mathbb{R}^\infty$ into $Ham(M,\omega)$ similar to the one presented in [28]. It should be noted that Py's assumptions imply the existence of such an H, as explained in [35].

While the conclusion of Theorem 1.2 is much weaker than Usher's result and somewhat weaker than that of Py, its assumptions are quite mild and indeed do not lie entirely within the scope of these previous results. For instance, Usher points out in [35] that any closed toric manifold M will not admit an autonomous Hamiltonian H as described in the previous paragraph, and so any such manifold which is also (negative) monotone (for example, (S^2, ω)) is one for which Theorem 1.2 asserts something new about the geometry of $(Ham(M, \omega), d_H)$.

The Hamiltonian diffeomorphisms which define our embedding are generated by radially symmetric functions $\bar{F}_i$ which are zero outside of $B(2\pi R)$ and of the form $\bar{f}_i\left(\frac{|z|^2}{2}\right)$, with $\bar{f}_i : [0, R] \to \mathbb{R}$, for $z \in B(2\pi R)$. Each of our functions $\bar{f}_i$ are to have disjoint supports, each contained in $[R - \varepsilon, R]$, so that the induced functions $\bar{F}_i$ have supports contained in the thin $2n$-dimensional annulus $\{z \in B(2\pi R) \,|\, 2R - 2\varepsilon < |z|^2 < 2R\}$ near the boundary of $B(2\pi R)$. (We note that using ε in this manner to construct our functions yields the inequality from Theorem 1.2 with ε replaced by an appropriate scalar multiple.) See Figure 1.1 for a piecewise linear version of one of our $\bar{f}_i$. For an earlier application of such functions to questions of Hamiltonian dynamics, one may refer to [32], where Seyfaddini uses them to construct "spectral killers." In fact, our proofs employ several of the same strategies as [32], from the careful choices of perturbations of continuous, radially symmetric Hamiltonians, to the explicit enumerations of their actions.

In an effort to build an analogous embedding of $\mathbb{R} \oplus [0,1]^\infty$ (where $\mathbb{R} \oplus [0,1]^\infty$ is defined similarly to $[0,1]^\infty$), we wish to find symplectic manifolds (M, ω) whose $Ham(M, \omega)$ admit a one-parameter family of diffeomorphisms ϕ_s satisfying $||\phi_s||_H \geq K \cdot s$ for some positive constant K. Such families can be shown to exist whenever there is a *stable homogeneous Calabi quasi-morphism* $\mu : Ham(M, \omega) \to \mathbb{R}$; definitions and details are given in Chapter 6. We have the following.

Theorem 1.3. *Let (M, ω) and $B(2\pi R)$ be as in the statement of Theorem 1.2, and further assume that*

- *there exists a stable homogeneous Calabi quasi-morphism $\mu : Ham(M, \omega) \to \mathbb{R}$.*

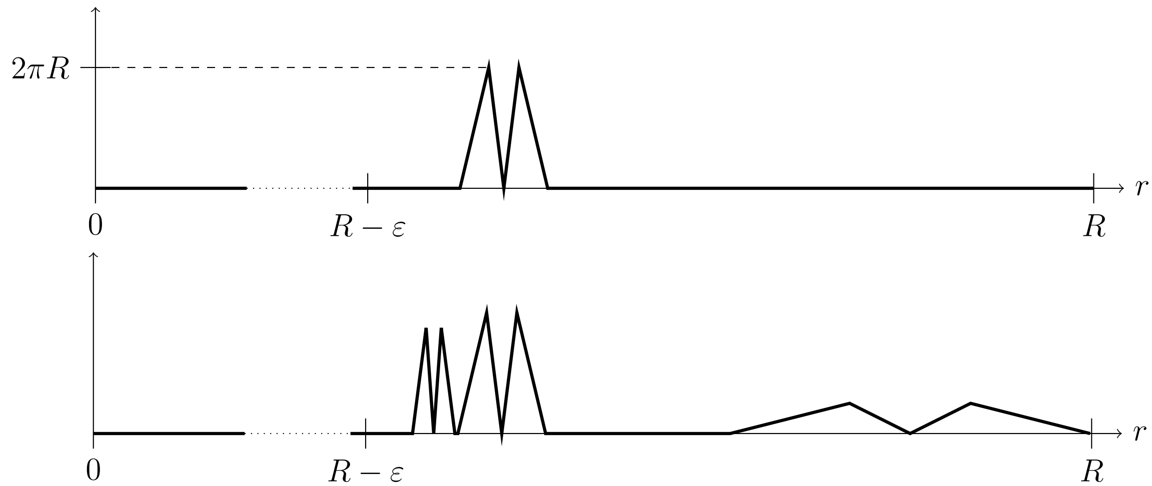

Figure 1.1: The top figure is a piecewise linear version of one of our functions $\bar{f}_i$. The bottom figure is a piecewise linear version of some $\sum_i^\infty a_i\bar{f}_i$, which will induce the Hamiltonian diffeomorphism $\Phi(a)$ with $a = \{a_i\}_{i\geq 1} \in [0,1]^\infty$.

- *$B(2\pi R)$ is displaceable in M, i.e. there exists a Hamiltonian diffeomorphism $\phi : M \to M$ such that $\phi(B(2\pi R)) \cap B(2\pi R) = \emptyset$.*

Then for any $\varepsilon > 0$, there exits an embedding $\overline{\Phi} : \mathbb{R} \oplus [0,1]^\infty \to Ham(M,\omega)$ so that for any $a, b \in \mathbb{R} \oplus [0,1]^\infty$,

$$C||a-b||_{\ell^\infty} - \varepsilon \leq d_H(\overline{\Phi}(a), \overline{\Phi}(b)) \leq 4\pi R||a-b||_{\ell^\infty},$$

where

$$C = \left(\frac{2\pi R \cdot Vol(B(2\pi R))}{Vol(M)} - \varepsilon\right).$$

Here, $Vol(B(2\pi R))$ and $Vol(M)$ are the symplectic volumes of $B(2\pi R)$ and M, respectively.

In [6], Entov and Polterovich explicitly construct a stable homogeneous Calabi quasi-morphism on $Ham(M,\omega)$ and outline sufficient conditions for which their construction holds. The authors therein also elaborate on the existence of such quasi-morphisms for a few specific (M,ω).

Example 1.4. Let $\varepsilon > 0$, and consider (S^2, ω), the 2-sphere with the area form ω such that $\int_{S^2} \omega = 4\pi$. We may symplectically embed a displaceable disk of radius $\sqrt{2(1-\varepsilon)}$ into the Northern hemisphere, and [6] shows that $Ham(S^2, \omega)$ admits a stable homogeneous Calabi quasi-morphism. Moreover, (S^2, ω) is monotone with rationality constant 4π. We may therefore apply Theorem 1.3 to say that there exists an embedding $\overline{\Phi} : \mathbb{R} \oplus [0,1]^\infty \to Ham(S^2, \omega)$ satisfying

$$(\tfrac{2\pi(1-\varepsilon)}{4\pi} - \varepsilon)||a-b||_{\ell^\infty} - \varepsilon \leq d_H(\overline{\Phi}(a), \overline{\Phi}(b)) \leq 4\pi(1-\varepsilon)||a-b||_{\ell^\infty}.$$

Remark 1.5. While it is again deduced in [6], $Ham(S^2, \omega)$ having infinite diameter with respect to Hofer's metric dates back earlier to [25]. However, it is still unknown whether a multi-dimensional normed vector space may be quasi-isometrically embedded into $Ham(S^2, \omega)$. In fact, there is nothing as of yet which rules out the possibility of $Ham(S^2, \omega)$ lying inside an infinitely long cylinder of a fixed radius. If this is the case, Theorem 1.3 and the example above therefore give a lower bound on what this radius can be.

We may also consider embeddings of $\mathbb{R} \oplus [0,1]^\infty$ into $\widetilde{Ham}(M, \omega)$, the universal cover of $Ham(M, \omega)$. Elements of this universal cover are homotopy classes $\{\phi_t\}$ of paths (rel. endpoints) of Hamiltonian diffeomorphisms. Similar to the case of $Ham(M, \omega)$, we may define the *Hofer pseudo-norm* $\widetilde{||\cdot||}_H$ by

$$\widetilde{||\{\phi_t\}||}_H = \inf\left\{\int_0^1 \left(\max_M(H_t) - \min_M(H_t)\right)\, dt \mid H_t \,\text{generates the path}\, \{\phi_t\}\right\},$$

after which we may define the *Hofer pseudo-metric* $\tilde{d}_H$ as in the case of $Ham(M, \omega)$. Again based on results from [6] concerning stable homogeneous Calabi quasi-morphisms, as well as a result from [26] about *stably non-displaceable* Lagrangians, we have the following result.

Theorem 1.6. *Let (M, ω) and $B(2\pi R)$ be as in the statement of Theorem 1.2. Further assume one of the following:*

- *M has a Lagrangian submanifold L which is stably non-displaceable, and $B(2\pi R) \cap L = \emptyset$.*

- *there exists a stable homogeneous Calabi quasi-morphism* $\tilde{\mu} : \widetilde{Ham}(M, \omega) \to \mathbb{R}$*, and* $B(2\pi R)$ *is displaceable in* M*.*

Then for any $\varepsilon > 0$*, there exits an embedding* $\widetilde{\Phi} : \mathbb{R} \oplus [0,1]^\infty \to \widetilde{Ham}(M, \omega)$ *so that for any* $a, b \in \mathbb{R} \oplus [0,1]^\infty$*,*

$$C||a - b||_{\ell^\infty} - \varepsilon \leq \tilde{d}_H(\widetilde{\Phi}(a), \widetilde{\Phi}(b)) \leq 4\pi R||a - b||_{\ell^\infty},$$

where C *is as in Theorem 1.3.*

See [6], [7], or [34] for more information concerning stable homogeneous Calabi quasi-morphisms on $\widetilde{Ham}(M, \omega)$, as well as some examples of closed (M, ω) whose $\widetilde{Ham}(M, \omega)$ admit such a quasi-morphism; for instance, it is shown in [34] that such (M, ω) include all closed toric manifolds, as well as any point blowup of an arbitrary closed symplectic manifold. For the definition of "stably non-displaceable," we refer the reader to [8]. For examples of stably non-displaceable Lagrangians, one may refer to [8] or [26], where in the latter, a Lagrangian L being stably non-displaceable is referred to as satisfying the *stable Lagrangian intersection property.*

Our last major theorem focuses on Hamiltonian diffeomorphisms of (S^2, ω) generated by autonomous Hamiltonians. The majority of the results already stated rely on the *boundary depth* function $\beta : Ham(M, \omega) \to \mathbb{R}$, first introduced by Usher in [33], which gives a lower bound on a Hamiltonian diffeomorphism's Hofer norm. As we explain in Chapter 7, finding a 1-parameter family of Hamiltonian diffeomorphisms ψ_t in $Ham(S^2, \omega)$ such that $\beta(\psi_t)$ grows arbitrarily large with t would show that $Ham(S^2, \omega)$ does not in fact lie in a cylinder of bounded radius (see Remark 1.5). The following theorem states that any hope of finding such a ψ_t lies in analyzing time-dependent Hamiltonians:

Theorem 1.7. *Where* (S^2, ω) *is the symplectic* 2*-sphere with total area* 4π*, let* $Aut(S^2, \omega)$ *denote the set of Hamiltonian diffeomorphisms generated by autonomous Hamiltonians. Then any* $\phi \in Aut(S^2, \omega)$ *satisfies* $\beta(\phi) \leq 14\pi$*.*

We note here, and explain in Chapter 7, how the existence of a ψ_t as above and Theorem 1.7 would also imply unbounded distance from $Aut(S^2, \omega)$ in $Ham(S^2, \omega)$; this property is proven in [27] to be possessed, for example, by $Ham(\Sigma, dA)$ when Σ is a closed oriented surface of genus $g \geq 4$ with area form dA.

This dissertation is organized as follows. Chapter 2 recalls the basic construction of filtered Floer homology. We then use Chapter 3 to discuss persistence modules, barcodes, and their application to filtered Floer homology, including how the boundary depth of a Hamiltonian diffeomorphism can be recovered from its barcode. Chapter 4 reviews radially symmetric Hamiltonians, discusses how to associate barcodes to radially symmetric C^0 functions, and proves certain lemmas concerning these barcodes. Chapter 5 proves Theorem 1.2, while Chapter 6 proves Theorems 1.3 and 1.6. Chapter 7 concludes this dissertation with a proof of Theorem 1.7.

Chapter 2

Hamiltonian Floer homology

Below, we recall the basic construction of the filtered Hamiltonian Floer homology $HF_*^\tau(H)$ associated to a non-degenerate Hamiltonian H on a closed (negative) monotone manifold M. For more details, we refer the reader to [10] for the monotone case, [13] for the semipositive case, and [24] for the case of a general closed symplectic manifold. For the remainder of this work, "monotone" will include the case of negative monotone.

For a smooth $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, let ϕ_H^t be the induced Hamiltonian isotopy as defined in Chapter 1. Let $x \in M$ be a fixed point of ϕ_H^1 such that the 1-periodic orbit of H given by $x(t) = \phi_H^t(x)$ is contractible in M. We call $x(t)$ *non-degenerate* if the time 1 map of the linearization of its flow has all eigenvalues not equal to 1 (i.e. $det(\mathbb{1} - d_{x(1)}(\phi_H^1)) \neq 0$), and we call H non-degenerate if all contractible 1-periodic orbits of H are non-degenerate. A Hamiltonian H being non-degenerate makes all of its fixed points isolated, so if M is compact, the set $\mathcal{P}(H)$ of H's contractible 1-periodic orbits must be finite.

Each $x(t) \in \mathcal{P}(H)$ can be capped by gluing a disk to $x(t)$ via a map $v : \mathbb{D}^2 \to M$ satisfying $v(e^{(2\pi\sqrt{-1})t}) = x(t)$. We let either $[x(t), v]$ or $\bar{x}$ denote an equivalence class of capped $x(t)$, where two capped periodic orbits $[x(t), v]$ and $[y(t), w]$ are considered equivalent if $x(t) = y(t)$ and $c_1(TM)|_{\pi_2(M)}([v\#\overline{w}])$ and $\int_{S^2}(v\#\overline{w})^*\omega$ are both zero; here, $v\#\overline{w}$ is the sphere created by gluing w to v by an orientation-reversing map on their boundary.

Given a capped periodic orbit $[x(t), v]$, we may symplectically trivialize $v^*(TM)$ and use this trivialization to express the linearization $d_{x(0)}(\phi_H^t)$ of ϕ_H^t along $x(t)$ as a path in $Sp(2n, \mathbb{R})$, the space of $2n \times 2n$ symplectic matrices. Recall that the space of $n \times n$ unitary matrices $U(n)$ can be included into $Sp(2n, \mathbb{R})$ via the map

$$X + iY \;\mapsto\; \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix},$$

The determinant gives a map from $U(n)$ into $S^1 \subset \mathbb{C}$, and a continuous extension ρ of this map to $Sp(2n, \mathbb{R})$ is constructed in [30]. The authors of [30] then use ρ to associate to every path $\gamma : [0,1] \to Sp(2n, \mathbb{R})$ with $det(\mathbb{1} - \gamma(1)) \neq 0$ an integer measuring the rotation of specific eigenvalues as we move along $\gamma(t)$. The linearization $d_{x(t)}(\phi_H^t)$ of our flow along $x(t)$ is such a path precisely when $x(t)$ is non-degenerate, and the associated integer is called the *Conley-Zehnder* index $\mu_{CZ}([x(t), v])$ of $[x(t), v]$. If v and w are two different cappings for $x(t)$, then $\mu_{CZ}([x(t), v]) - \mu_{CZ}([x(t), w]) = -2c_1(TM)|_{\pi_2(M)}([v\#\overline{w}])$ (see [22]). Different conventions are used in different works when defining the Conley-Zehnder index of a capped periodic orbit. Our conventions are the same as those used in [30] so that if f is a C^2-small Morse function on the $2n$-dimensional M, a critical point of Morse index j will have Conley-Zehnder index $j - n$ when treated as a trivially capped periodic orbit.

We note here that under our monotonicity condition, two capped periodic orbits $[x(t), v]$ and $[y(t), w]$ are equivalent if and only if $x(t) = y(t)$ and $\mu_{CZ}([x(t), v]) = \mu_{CZ}([y(t), w])$. Indeed, we would have $c_1(TM)|_{\pi_2(M)}([v\#\overline{w}]) = 0$ by the previous paragraph, so

$$\int_{S^2} (v\#\overline{w})^*\omega = \lambda c_1(TM)|_{\pi_2(M)}([v\#\overline{w}]) = 0$$

as well. Hence, for every periodic orbit $x(t) \in \mathcal{P}(H)$ and $d \in \mathbb{Z}$, there exists at most one equivalence class $[x(t), v]$ so that $\mu_{CZ}([x(t), v]) = d$. This and $\mathcal{P}(H)$ being finite implies that $\widetilde{\mathcal{P}}_d(H)$, the set of equivalence classes of capped periodic orbits of H with Conley-Zehnder index d, is a finite set. We may therefore construct a finite dimensional vector space over $\mathbb{Q}$ with generators the elements of $\widetilde{\mathcal{P}}_d(H)$, and we let $CF_d(H)$ denote this vector space. This represents the d-th graded portion of H's *total Floer chain complex*, denoted by $CF_*(H)$.

Remark 2.1. We see that, generally, the total Floer chain complex is infinite dimensional over $\mathbb{Q}$. One way of getting around this is by considering $CF_*(H)$ as a finite dimensional vector space over a *Novikov ring* (see [13], for instance). The previous paragraph shows why we have no need for a Novikov ring in our construction of the Floer chain complex, for we assume monotonicity and restrict our attention to each degree d-th portion.

Remark 2.2. Under our monotonicity assumption, every capping for a fixed periodic orbit $x(t)$ can be obtained by first fixing a capping v and then attaching a multiple of an appropriate element of $\pi_2(M)$ to v. To be precise, let $[A] \in \pi_2(M)$ and a capped periodic orbit $[x(t), v]$ be given. Where $[x(t), v\#A]$ is the capped periodic orbit created by attaching the sphere A to v, we have

$$\mu_{CZ}([x(t), v\#A]) = \mu_{CZ}([x(t), v]) - 2c_1(TM)|_{\pi_2(M)}([A]).$$

So choosing $[A]$ with $c_1(TM)|_{\pi_2(M)}([A]) = -N$, every possible capping of $x(t)$ is given by

$$\{[x(t), v\#kA]\}_{k\in\mathbb{Z}},$$

while the set of possible Conley-Zehnder indices is given by

$$\{\mu_{CZ}([x(t), v]) + 2Nk\}_{k\in\mathbb{Z}}.$$

Here, $v\#kA$ means k copies of A attached to v. (Note that if $N = 0$, every capped periodic orbit in the first set is equivalent.)

To describe the boundary operator ∂_H of $CF_*(H)$, we first let $\widetilde{\mathcal{L}_0(M)}$ denote the space of all capped, contractible loops in M endowed with the same equivalence relation used on capped periodic orbits. For a given Hamiltonian H on M, we can define the *action functional* $\mathcal{A}_H$ on $\widetilde{\mathcal{L}_0(M)}$ by

$$\mathcal{A}_H([\gamma(t), v]) = -\int_{\mathbb{D}^2} v^*(\omega) + \int_0^1 H(t, \gamma(t))dt,$$

which is well-defined by our equivalence relation on capped periodic orbits. The critical points of this action functional are precisely the capped periodic orbits of H, and when H is non-degenerate, the boundary operator ∂_H for $CF_*(H)$ is defined by a count of isolated (formal) negative gradient flowlines of $\mathcal{A}_H$ on $\widetilde{\mathcal{L}_0(M)}$. (In the case that M is semipositive, these may be more concretely defined, for generic choices of non-degenerate H and time-dependent

ω-compatible almost-complex structure J_t, as isolated solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ to the Hamiltonian Floer equation

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right) = 0$$

with finite energy $E(u) = \int_{\mathbb{R}} \int_0^1 ||\partial_s u||^2 dt\, ds$. If the capped periodic orbit $[y(t), w]$ has a non-zero coefficient in $\partial_H([x(t), v])$, then there exists such a u which limits on $x(t)$ (resp. $y(t)$) as s goes to negative (resp. positive) infinity and such that $[x(t), v] = [x(t), u\#w]$. The resulting filtered homology, defined below, is independent of our choice of J_t. The case of M being semipositive includes all monotone manifolds with $\lambda \geq 0$, though with λ as such, this construction of the Floer chain complex works for any non-degenerate Hamiltonian.) It is true, though highly nontrivial to prove, that ∂_H defined in this way gives well-defined maps $\partial_{H,d} : CF_d(H) \to CF_{d-1}(H)$ satisfying $\partial_{H,d-1} \circ \partial_{H,d} = 0$ for all degrees d.

After restricting $\mathcal{A}_H$ to $\cup_{d\in\mathbb{Z}} \widetilde{P}_d(H)$, we may extend it to a function ℓ on all of $CF_*(H)$ by setting

$$\ell(0) = -\infty$$

and

$$\ell(c) = \max_{\{i \,|\, q_i \neq 0\}} (\mathcal{A}_H([x_i(t), v_i])$$

for $c = \sum q_i[x_i(t), v_i]$ a non-zero chain in $CF_*(H)$. It is known that $\ell(\partial(c)) < \ell(c)$ for such non-zero chains, so we may create the subcomplex $CF_*^\tau(H)$ of $CF_*(H)$ (where $\tau \in \mathbb{R}$) generated by capped periodic orbits with action less than or equal to τ. Letting ∂_H^τ denote the boundary operator of this subcomplex, we set $HF_*^\tau(H) = [\ker(\partial_H^\tau)]/[\mathrm{Im}(\partial_H^\tau)]$ to get the *filtered Floer homology* of H; we write $HF_*(H)$ for $HF_*^\infty(H)$ and call it the *total* Floer homology.

We take a final moment to recall that the *action spectrum* *Spec*(H) of H is simply the set $\cup_{d\in\mathbb{Z}} \mathcal{A}_H(\widetilde{\mathcal{P}}_d(H))$. Later on, we may refer to a *degree d action of H*, by which we mean an element of $\mathcal{A}_H(\widetilde{P}_d(H))$.

Remark 2.3. It is important to note here the effect of recappings on actions. Where $[x(t), v]$ and $[A]$ are as from our previous remark, we have

$$\mathcal{A}([x(t), v\#A]) = \mathcal{A}([x(t), v]) - [\omega]|_{\pi_2(M)}([A])$$

which is equal to $\mathcal{A}([x(t), v]) + \sigma(\lambda)\gamma$ under our monotonicity condition; furthermore,

$$\mathcal{A}([x(t), v\#kA]) = \mathcal{A}([x(t), v]) - k[\omega]|_{\pi_2(M)}([A]) = \mathcal{A}([x(t), v]) + k\sigma(\lambda)\gamma.$$

Here, $\sigma(\lambda)$ is the sign of the monotonicity constant λ (with $\sigma(0) = 0$). It is this fact that will allow us to enumerate all possible actions and degrees for the capped periodic orbits of certain non-degenerate Hamiltonians on monotone manifolds.

Chapter 3

Persistence Modules, Barcodes, and Boundary Depth

For our discussion of persistence modules and barcodes, we mainly follow the expositions provided in [36] and [27].

3.1 Persistence modules and barcodes

Let K be a field. A *persistence module* $\mathbb{V} = (V, \sigma)$ consists of a K-module V_t for each $t \in \mathbb{R}$ and morphisms $\sigma_{st} : V_s \to V_t$, for each pair s, t with $s \leq t$, such that $\sigma_{ss} = \mathrm{Id}|_{V_s}$ and $\sigma_{tu} \circ \sigma_{st} = \sigma_{su}$.

For an easy example of a persistence module, we may construct an *interval module* $\mathbb{M}(I) = (M(I), \sigma)$ by choosing an interval $I \subset \mathbb{R}$ and defining each $M(I)_t$ by

$$M(I)_t = \begin{cases} K, & t \in I \\ 0, & \text{otherwise;} \end{cases}$$

our maps $\sigma_{st} : M(I)_s \to M(I)_t$ in this case will be the identity when $s, t \in I$ and the zero map otherwise.

As well as being an easy example of a persistence module, interval modules turn out to be the building blocks of other persistence modules satisfying certain conditions. One such condition (as the following theorem asserts) is $\mathbb{V}$ being *pointwise finite-dimensional*, where each V_t is a finite-dimensional vector space. (Another sufficient condition is $\mathbb{V}$ being of *finite type* as in [37].)

Theorem 3.1. *([5]) Any pointwise finite-dimensional persistence module $\mathbb{V}$ can be uniquely expressed as a direct sum of interval modules $\mathbb{M}(I_\alpha)$.*

Thus, for a pointwise finite dimensional persistence module $\mathbb{V}$, we can define its *barcode* as the collection $\mathcal{B} = \{(I_\alpha, m_\alpha)\}$, where each I_α is an interval appearing in $\mathbb{V}$'s interval module decomposition with multiplicity $m_\alpha > 0$. We may sometimes refer to an I_α with $(I_\alpha, m_\alpha) \in \mathcal{B}$ as a *bar* or *interval of* $\mathcal{B}$, while by a *left* or *right-hand endpoint of* $\mathcal{B}$ we mean the left or right-hand endpoint of a bar of $\mathcal{B}$.

Remark 3.2. Let H be non-degenerate on closed monotone M and fix a degree d. Referring to Section 2, one sees that $CF_*^s(H)$ is a subcomplex of $CF_*^t(H)$ whenever $s \leq t$, and it is easily verified from here that we get a pointwise finite-dimensional persistence module by setting $V_t = HF_d^t(H)$ and $\sigma_{st} : HF_d^s(H) \to HF_d^t(H)$ equal to the map induced by inclusion on the chain level. It therefore has an associated barcode $\mathcal{B}^d(H)$, which we call the *degree d barcode of H*. Theorem 6.2 of [36] asserts that any I_α for $(I_\alpha, m_\alpha) \in \mathcal{B}^d(H)$ will have a degree d action as its left-hand endpoint and a degree $d+1$ action (or infinity) as its right-hand endpoint. We say that two actions of degrees d and $d+1$ *pair* with each other if they are endpoints of the same interval in $\mathcal{B}^d(H)$. Combining Proposition 5.5, Theorem 6.2, and the beginning of the proof of Theorem 12.3 of [36] gives that every degree d action c of H will appear as an endpoint of $\mathcal{B}^d(H) \cup \mathcal{B}^{d-1}(H)$ with multiplicity equal to the number of elements $[x(t), v] \in \widetilde{P}_d(H)$ such that $\mathcal{A}_H([x(t), v]) = c$.

Remark 3.3. Since a subset of the finite-valued degree d actions of H comprise the left-hand endpoints of $\mathcal{B}^d(H)$, and since our interest in persistence modules and barcodes lies only in their application to this context of Hamiltonian Floer thoery, all barcodes $\mathcal{B}$ will be assumed from now on to have finite-valued left-hand endpoints.

Given a barcode $\mathcal{B} = \{(I_\alpha, m_\alpha)\}$, create a set of indexed intervals $\langle \mathcal{B} \rangle = \{I_\alpha^{i_\alpha}\}_{1 \leq i_\alpha \leq m_\alpha}$ which treats an interval I_α with multiplicity m_α as m_α separate copies of I_α. For $\varepsilon > 0$ and a barcode $\mathcal{B}$, let $\langle \mathcal{B} \rangle_\varepsilon$ denote the subset of $\langle \mathcal{B} \rangle$ consisting of all intervals of length less than or equal to 2ε. A function μ from a subset of $\langle \mathcal{B} \rangle$ to a subset of $\langle \mathcal{C} \rangle$ is called an *ε-matching* between barcodes $\mathcal{B}$ and $\mathcal{C}$ if:

- $\langle \mathcal{B} \rangle \backslash \langle \mathcal{B} \rangle_\varepsilon$ is contained in the domain of μ.

- $\langle \mathcal{C} \rangle \backslash \langle \mathcal{C} \rangle_\varepsilon$ is contained in the image of μ.
- If $\mu([a,b)) = [a',b')$, where $[a,b) \in \langle \mathcal{B} \rangle \backslash \langle \mathcal{B} \rangle_\varepsilon$ or $[a',b') \in \langle \mathcal{C} \rangle \backslash \langle \mathcal{C} \rangle_\varepsilon$, then $|a-a'| < \varepsilon$ and b and b' are either both infinity or both finite with $|b-b'| < \varepsilon$.

Finally, the *bottleneck distance* d_b between barcodes $\mathcal{B}$ and $\mathcal{C}$ is defined as

$$d_b(\mathcal{B}, \mathcal{C}) = \inf\{\varepsilon > 0 \,|\, \text{there exists an } \varepsilon\text{-matching between } \mathcal{B} \text{ and } \mathcal{C}\}.$$

In our context of Hamiltonian Floer theory, we have the following result, which is a much weaker version of Theorem 12.2 from [36].

Theorem 3.4. *Let H_0 and H_1 be two non-degenerate Hamiltonians on closed and symplectic M. Then for any degree d,*

$$d_b(\mathcal{B}^d(H_0), \mathcal{B}^d(H_1)) \leq \int_0^1 ||H_0(t,\cdot) - H_1(t,\cdot)||_{L^\infty} dt.$$

Though barcodes so far have only been defined for non-degenerate Hamiltonians, the above theorem may occasionally be applied to define barcodes for degenerate or even (as we will do later) merely continuous functions on M.

3.2 Boundary depth

The boundary depth of a Hamiltonian diffeomorphism is our main motivation for studying barcodes, so we pause very briefly to remind the reader of its definition, its relation to barcodes, and a few of its key properties. See [33], [35], and [36] for more details.

Let ϕ be a Hamiltonian diffeomorphism generated by non-degenerate H which, as we recall, we may assume to be 1-periodic in time. After constructing its Floer chain complex $CF_*(H)$, we may define the quantities $\beta_d(\phi) \in \mathbb{R}$ as

$$\beta_d(\phi) = \sup_{0 \neq x \in \partial(CF_d(H))} \inf\{\ell(y) - \ell(x) \,|\, \partial_H(y) = x\},$$

which is independent of the choice of such an H. The *boundary depth* $\beta(\phi)$ of ϕ may be defined as

$$\beta(\phi) = \sup_{d \in \mathbb{Z}} \beta_d(\phi)$$

and is a finite quantity. The relationship between $\beta(\phi)$ and the barcodes $\mathcal{B}^d(H)$ becomes clear when one deduces from Theorems 4.11 and 6.2 of [36] that $\beta_d(\phi)$ is simply the length of the longest finite-length bar in $\mathcal{B}^d(H)$. The quantity $\beta(\phi)$ is of particular interest to us because it gives a lower bound on ϕ's Hofer norm; we refer the reader to [1] and [35] for previous instances in which the boundary depth is used to answer questions of Hofer's geometry.

Similar to our continuity result for barcodes, we have the following continuity result for boundary depth which will prove useful in a later argument.

Theorem 3.5. *([33], [35]) For a Hamiltonian H, set*

$$||H|| = \int_0^1 \left(\underset{M}{max}(H_t) - \underset{M}{min}(H_t) \right) dt.$$

If $\phi, \psi \in Ham(M, \omega)$ are generated by non-degenerate Hamiltonians H and K, respectively, then

$$|\beta(\phi) - \beta(\psi)| \leq ||H - K||.$$

It is this continuity result that allows us to define the boundary depth of a degenerate $\phi \in Ham(M, \omega)$ (namely, if ϕ is generated by H, choose a sequence of non-degenerate H_k's which C^0-converges to H and let $\beta(\phi)$ be the limit of the $\beta(\phi_{H_k})$).

We conclude this chapter by recalling the following additional properties of the boundary depth function.

Theorem 3.6. *([35]) Let β be the boundary depth function on $Ham(M, \omega)$.*

(i) For $\phi \in Ham(M, \omega)$, we have

$$\beta(\phi) = \beta(\phi^{-1}).$$

(ii) For $\phi, \psi \in Ham(M, \omega)$*, we have*

$$|\beta(\phi) - \beta(\psi)| \leq ||\phi^{-1}\psi||_H.$$

(iii) For every $\phi \in Ham(M, \omega)$ *and symplectomorphism* σ*,*

$$\beta(\phi) = \beta(\sigma^{-1}\phi\sigma).$$

Chapter 4

Radially symmetric Hamiltonians

Suppose that we may symplectically embed a ball $B(2\pi R) = \left\{(\vec{x}, \vec{y}) \mid \sum_i (x_i^2 + y_i^2) \leq 2R\right\}$ of radius $\sqrt{2R}$ into M. Let $f(r)$ be a smooth function on $[0, R]$ which has vanishing derivatives of all orders (except for possibly the 0-th) at $r = R$ and which has $f'(0)$ not an integer multiple of 2π. Letting $z = (\vec{x}, \vec{y})$ be coordinates on our symplectic ball $B(2\pi R)$ with symplectic form $\sum_i dx_i \wedge dy_i$, we may define a smooth function $F : M \to \mathbb{R}$ by

$$F(p) = \begin{cases} f\left(\frac{|z|^2}{2}\right), & p = z \in B(2\pi R) \\ \\ f(R), & \text{otherwise.} \end{cases}$$

Calculating the flow of this function (away from the origin) becomes much simpler if we first express our symplectic form in polar coordinates as $\sum_i r_i dr_i \wedge d\theta_i$. In which case, we get

$$dF = \sum_i \left(r_i f'(\tfrac{r_1^2 + \ldots + r_n^2}{2}) dr_i \right).$$

Our corresponding Hamiltonian vector field is therefore $\sum_i f'(\frac{r_1^2+\ldots+r_n^2}{2})\partial_{\theta_i}$, so our flow ϕ_F^t is defined by

$$\phi_F^t(p) = \begin{cases} e^{\sqrt{-1} f'(\frac{|z|^2}{2})t} z, & p = z \in B(2\pi R) \\ \\ p, & \text{otherwise.} \end{cases}$$

Assuming for now that there are only a finite number of r_i for which $f'(r_i)$ is an integer multiple of 2π, the above formula for our flow tells us that we will have an S^{2n-1}'s worth of periodic orbits at every radius equal to $\sqrt{2r_i}$, and any orbit (with capping contained in $B(2\pi R)$) in such an S^{2n-1} family will have action

$$f(r_i) - f'(r_i)r_i.$$

Indeed, the value of the function f along this orbit is the constant $f(r_i)$, while the symplectic area of any capping disk will be precisely $f'(r_i)r_i$. All points outside of $B(2\pi R)$ are constant periodic orbits of action precisely $f(R)$. We will also have a constant periodic orbit occurring at the center of our symplectic ball whose action (with trivial capping) will be precisely $f(0)$. Our condition on $f'(0)$ implies that this capped orbit will be non-degenerate and, assuming $2\pi l < f'(0) < 2\pi(l+1)$, will have Conley-Zehnder index $-2ln-n$ (one arrives at this formula by following the reasoning provided in [21], while keeping in mind that they compute the negative version of our μ_{CZ}).

As can be seen by the presence of non-isolated periodic points, our F is degenerate, so we perturb it to a non-degenerate $\tilde{F}$ for which we may construct a barcode. A very specific perturbation is chosen as follows.

We start with the S^{2n-1} families of periodic orbits. Along with our assumption that all points r_i where $f'(r_i)$ is an integer multiple of 2π are isolated, we further assume that $f''(r_i) \neq 0$ for each such r_i so that we may perform the standard perturbation of F around the S^{2n-1} families of periodic orbits (see [4], [21], [32]). In particular, define a perfect Morse function h_i on the S_i^{2n-1} corresponding to r_i, and smoothly extend it to a small tubular neighborhood in $B(2\pi R)$ by having it decrease in either radial direction. Calling these extended functions h_{r_i}, the time-dependent function $F + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1}$ with δ small enough will have each S_i^{2n-1} splitting into two periodic orbits z_1, z_2. The existence of these two periodic orbits can be seen by noting that the Hamiltonian diffeomorphism generated by $F + \delta h_{r_i} \circ (\phi_F^t)^{-1}$ is the composition $\phi_F \phi_{r_i}$, where ϕ_{r_i} is the Hamiltonian diffeomorphism generated by δh_{r_i}. The function δh_{r_i} has two Morse critical points z_1 and z_2 occurring on the original S^{2n-1} family of periodic orbits, which makes them fixed points for ϕ_{r_i}. We therefore have two fixed points for $\phi_F \circ \phi_{r_i}$ on said S^{2n-1}; the authors in [4] demonstrate why $\phi_F \phi_{r_i}$ has no other fixed points in our tubular neighborhood when δ is chosen small enough.

If $f'(r_i) = 2\pi l$ and $[z_j, v_j]$ denotes these orbits with cappings contained in $B(2\pi R)$, their indices will be given by

$$\mu_{cz}([z_1, v_1]) = \begin{cases} -2ln + n, & f''(r_i) < 0 \\ -2ln + n - 1, & f''(r_i) > 0 \end{cases}$$

$$\mu_{cz}([z_2, v_2]) = \begin{cases} -2ln - n + 1, & f''(r_i) < 0 \\ -2ln - n, & f''(r_i) > 0 \end{cases}$$

Their actions will be approximately $f(r_i) - f'(r_i)r_i$, with the error term going to zero with δ.

We now deal with the periodic orbits outside of $B(2\pi R)$. Choose once and for all a Morse function $g : M \to [-1, 0]$ that has a unique critical point (a maximum, where g attains the value 0) in $B(2\pi R)$, and choose a sufficiently small collar neighborhood C of $\partial(B(2\pi R))$ so that C contains no periodic points of F which occur in $\text{int}(B(2\pi R))$; the existence of such a C is guaranteed by our finiteness assumption on the number of r_i. Then define $\tilde{g}$ to be equal to g on $M\backslash(B(2\pi R) \cup C)$, 0 on $B(2\pi R)\backslash C$, and to be smoothly extended to all of M so that it has no critical points in C. If f is decreasing right before R, then the final step in our perturbation of F will be to $F + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1} + \varepsilon\tilde{g}$, with ε small enough so that the only periodic points in $M\backslash\text{int}(B(2\pi R))$ of this new function are critical points of g. (If f is increasing right before R, our final perturbation is instead to $F + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1} - \varepsilon\tilde{g}$.) We refer to these periodic orbits as *exterior orbits*. By our choice of g, the exterior orbits (with trivial cappings) of $F + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1} + \varepsilon\tilde{g}$ will have Morse indices lying in $[0, 2n-1]$ so that their Conley-Zehnder incices will lie in $[-n, n-1]$ by our convention, again assuming that ε is small enough. Their actions will lie in $[f(R) - \varepsilon, f(R))$. (Such trivially capped orbits will have Conley-Zehnder indices in $[-n+1, n]$ and actions in $(f(R), f(R) + \varepsilon]$ if our final perturbation is instead to $F + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1} - \varepsilon\tilde{g}$.)

We let $\tilde{F} = F + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1} + \varepsilon \tilde{g}$ denote this non-degenerate Hamiltonian. To get all possible actions and indices of $\tilde{F}$'s capped periodic orbits, we must only consider the actions and indices already described and how they change under recappings. Our monotonicity condition implies that any such change can only occur when $N \neq 0$, in which case increasing the index by $k2N$ (with $k \in \mathbb{Z}$) via recapping will increase its action by $k\sigma(\lambda)\gamma$, where $\sigma(\lambda)$ is as in Chapter 2. As noted earlier, these indexed actions give us all finite-valued endpoints of all bars in $\tilde{F}$'s barcode.

Moving our focus away from smooth functions, suppose that $f : [0, R] \to \mathbb{R}$ is piecewise linear. We say that f satisfies the *slope condition* if all of its slopes are not integer multiples of 2π and if the slope s going into the line $r = R$ satisfies $|s| < 2\pi$. Assuming f satisfies the slope condition, and letting $F : M \to \mathbb{R}$ be the C^0 function induced by f, our goal now is to show how we may associate to this non-differentiable F a barcode in any degree.

We first describe a specific kind of perturbation of F, which we will refer to as *standard* (or more commonly as a *standard perturbation of f*); this perturbation is the same as that described in [32]. Pick small enough ε'-neighborhoods around the r-values where f is not differentiable so that no two neighborhoods intersect, and pick a smoothing $f_{\varepsilon'}$ which has strictly monotonic first derivative on these ε' neighborhoods and which is equal to f elsewhere. (We choose our smoothing at $r = R$ so that our function $f_{\varepsilon'}$ has vanishing derivatives of all orders, except possibly the 0-th, at $r = R$.) Where $F_{\varepsilon'}$ is the function on M induced by $f_{\varepsilon'}$, choose ε and δ small enough to construct the non-degenerate, time-dependent perturbation $\tilde{F}_{\varepsilon'}$ of $F_{\varepsilon'}$ as described above:

$$\tilde{F}_{\varepsilon'} = F_{\varepsilon'} + \delta \sum_i h_{r_i} \circ (\phi_F^t)^{-1} + \varepsilon \tilde{g}.$$

If ε'_k is a sequence converging to zero, we may choose similar smoothings $f_{\varepsilon'_k}$ and appropriate sequences δ_k, ε_k (both converging to zero) to create a sequence of standard perturbations $\tilde{F}_{\varepsilon'_k}$ (abbreviated as $\tilde{F}_k$) of F which C^0-converges to F when F is regarded as a function with domain $\mathbb{R}/\mathbb{Z} \times M$. Our assumption that $|s| < 2\pi$ ensures the existence of a collar neighborhood C such that, for any ε'_k small enough, $C \cap \text{int}(B(2\pi R))$ contains no periodic orbits of $F_{\varepsilon'_k}$. We may therefore use the same function $\tilde{g}$ for every entry in our sequence $\tilde{F}_k$, a fact which will aid us momentarily.

Letting d be any degree, each $\tilde{F}_k$ has the same number of actions in degree d by the monotonic behavior of each $f_{\varepsilon'_k}$'s derivative on the ε'_k neighborhoods. Furthermore, the set of degree d actions for $\tilde{F}_k$ forms a sequence converging to a specific set of real numbers. (To see why these statements are true, let $\bar{r}$ be a point of non-differentiability for f with s_1 and s_2 being the slopes of f immediately before and after $\bar{r}$, and suppose $2\pi l$ for some $l \in \mathbb{Z}$ is between s_1 and s_2. By our choice of smoothings, every $f_{\varepsilon'_k}$ has a unique r value $r_{i,k}$ in $(\bar{r} - \varepsilon'_k, \bar{r} + \varepsilon'_k)$ for which $f_{\varepsilon'_k}$'s derivative is $2\pi l$. Letting $[x, v]_k$ be the corresponding capped orbit of $\tilde{F}_k$ of lower (or higher) index d (with capping contained in $B(2\pi R)$), we form the sequence of degree d actions $\mathcal{A}([x, v]_k)$, which converges to $-2\pi l\bar{r} + f(\bar{r})$. Such convergence statements clearly apply to recappings of the $[x, v]_k$, as well as actions coming from the y-intercept and from exterior orbits, since we are using the same function $\tilde{g}$ for every entry $\tilde{F}_k$ in our limiting sequence.) This fact is essential in proving the following:

Claim 4.1. *Abbreviate $\mathcal{B}^d(\tilde{F}_k)$ as $\mathcal{B}^d_k$. The sequence $\mathcal{B}^d_k$ converges in the bottleneck distance to a unique barcode $\mathcal{B}^d$.*

Proof. Let $A = \{a_i\}_{i=1}$ be the limiting set of degree d actions, and let $B = \{b_j\}_{j=0}$ be the limiting set of degree $d+1$ actions unioned with $\{\infty\}$ (set $b_0 = \infty$). Choose ε so that 4ε is less than the minimal positive distance between all elements of $A \cup (B\backslash\{\infty\})$. For fixed elements $a_i \in A$ and $b_j \in B$, define the integer $m^\varepsilon_k(a_i, b_j)$ to be the number of bars $[a', b')$ in $\langle \mathcal{B}^d_k \rangle$ with $|a_i - a'| < \varepsilon$ and either $|b_j - b'| < \varepsilon$ or $b' = \infty$ in the case that $j = 0$.

For all k big enough, every finite-valued endpoint of $\mathcal{B}^d_k$ is contained within the union of intervals $\cup_{i,j\neq 0}\{(a_i - \varepsilon, a_i + \varepsilon), (b_j - \varepsilon, b_j + \varepsilon)\}$, and since our sequence of functions $\tilde{F}_k$ is Cauchy with respect to the C^0 norm, we may assert the existence of ε-matchings $\mu^{k_1,k_2}_\varepsilon$ between $\mathcal{B}^d_{k_1}$ and $\mathcal{B}^d_{k_2}$ for all k_1, k_2 big enough. Moreover, for such k_1 and k_2, the bars in $\langle \mathcal{B}^d_{k_1} \rangle$, $\langle \mathcal{B}^d_{k_2} \rangle$ which define $m^\varepsilon_{k_1}(a_i, b_j)$, $m^\varepsilon_{k_2}(a_i, b_j)$, are of length at least 2ε when $a_i \neq b_j$; these bars are therefore in the domains and ranges of our $\mu^{k_1,k_2}_\varepsilon$. From this, we deduce that the sequence

$m_k^\varepsilon(a_i, b_j)$ is eventually constant and so converges to some integer $m^\varepsilon(a_i, b_j)$ when $a_i \neq b_j$. Define $\mathcal{B}^d$ to be the collection

$$\{([a_i, b_j), m^\varepsilon(a_i, b_j)) \mid a_i \in A, \, b_j \in B, \, a_i \neq b_j, \, m^\varepsilon(a_i, b_j) \neq 0\}.$$

From here, it is easy to conclude that $\mathcal{B}^d$ is in fact the limit of the $\mathcal{B}_k^d$. Indeed, let $\varepsilon' > 0$ be less than ε. Then for any k large enough, there is clearly an injection $\mu_{\varepsilon'}^k : \langle \mathcal{B}^d \rangle \to \langle \mathcal{B}_k^d \rangle$ so that

- $\mu_{\varepsilon'}^k$ satisfies the third condition of being an ε'-matching (see Section 3).
- any bar not in the range of $\mu_{\varepsilon'}^k$ has endpoints contained in an interval of the form $(c - \varepsilon', c + \varepsilon')$, where $c \in A \cup (B \backslash \infty)$.

(The second condition holds since $\varepsilon' < \varepsilon$.) In particular, $\mu_{\varepsilon'}^k$ is an ε'-matching.

□

Letting H_k be any other sequence of non-degenerate Hamiltonians which C^0-converge to F gives another sequence of barcodes $\mathcal{B}^d(H_k)$ which must also necessarily converge to B^d. Assuming otherwise, we could fix k' big enough and compare $\mathcal{B}^d(H_{k'})$ with $\mathcal{B}_{k'}^d$ from the proof of Claim 4.1 to arrive at a contradiction of the continuity of barcodes. Our function F may therefore be attributed a well-defined barcode in any degree d, though we abuse notation and refer to it as the degree d barcode of f, or $\mathcal{B}^d(f)$. It is clear from our construction of $\mathcal{B}^d(f)$ and Theorem 3.4 that for two piecewise linear functions f_1 and f_2 satisfying our slope condition, we have

$$d_b(\mathcal{B}^d(f_1), \mathcal{B}^d(f_2)) \leq ||f_1 - f_2||_{L^\infty}.$$

We pause to define some terms. In the following definitions, f refers to a piecewise linear function satisfying our slope condition, $\{r_i\}_{i \geq 0}$ are the r-values of f's points of non-differentiability in decreasing order with $r_0 = R$, and $\{m_i\}_{i \geq 1}$ are the slopes of f as we move from right to left (so m_{i+1} and m_i are the slopes on the left and right, respectively, of the point $(r_i, f(r_i))$).

Definition 4.2. A number $c \in \mathbb{R}$ is a *degree d action of f* if it is the limit of a sequence of degree d actions arising from a sequence of standard perturbations of f.

Definition 4.3. The *degree d action spectrum of f with multiplicity*, denoted by $Spec_m^d(f)$, is the collection of all degree d actions of f considered with multiplicity. Similarly, the *action spectrum of f with multiplicity $Spec_m(f)$* refers to the union over all degrees d of the $Spec_m^d(f)$.

In light of Section 2, it is clear that right-hand endpoints of $\mathcal{B}^d(f)$ are either infinity or elements of $Spec_m^{d+1}(f)$, while left-hand endpoints are elements of $Spec_m^d(f)$.

Definition 4.4. If $m_{i+1} < m_i$ (resp. $m_i < m_{i+1}$), then we call $(r_i, f(r_i))$ a *concave up* (resp. *down*) *kink* of f.

By the comments immediately preceding Claim 4.1, $Spec_m^d(f)$ and $Spec_m(f)$ are well-defined, and we enumerate the elements of $Spec_m(f)$ with their degrees below.

(1) If $(r_i, f(r_i))$ is a concave up kink of f with $m_{i+1} < 2\pi l < m_i$ for some $l \in \mathbb{Z}$, then $-2\pi l r_i + f(r_i)$ will be a degree $-2ln + n - 1$ and a degree $-2ln - n$ action of f. Furthermore, for any integer k, $-2\pi l r_i + f(r_i) + k\sigma(\lambda)\gamma$ will be a degree $-2ln + n - 1 + k2N$ and a degree $-2ln - n + k2N$ action of f if $N \neq 0$.

(2) If $(r_i, f(r_i))$ is a concave down kink of f with $m_{i+1} > 2\pi l > m_i$ for some $l \in \mathbb{Z}$, then $-2\pi l r_i + f(r_i)$ will be a degree $-2ln + n$ and a degree $-2ln - n + 1$ action of f. Furthermore, for any integer k, $-2\pi l r_i + f(r_i) + k\sigma(\lambda)\gamma$ will be a degree $-2ln + n + k2N$ and a degree $-2ln - n + 1 + k2N$ action of f if $N \neq 0$.

(3) If the slope s of the line coming out of the y-axis satisfies $2\pi l < s < 2\pi(l+1)$, then $f(0)$ will be a degree $-2ln - n$ action of f, and for any integer k, $f(0) + k\sigma(\lambda)\gamma$ will be a degree $-2ln - n + k2N$ action of f if $N \neq 0$.

(4) If g has a critical point of Morse index j outside of $B(2\pi R)$, then $f(R)$ will be a degree $j - n$ action of f, and for any integer k, $f(R) + k\sigma(\lambda)\gamma$ will be a degree $j - n + k2N$ action of f if $N \neq 0$.

Note that a sequence of standard perturbations of f might have some sequence of bars whose lengths go to zero as the sequence progresses, so there is no guarantee that any single action from the above enumeration has to appear in any $\mathcal{B}^d(f)$. However, it should be clear from our construction of $\mathcal{B}^d(f)$ that if any degree d action from our enumeration has multiplicity one in $Spec_m(f)$, then it must appear in either $\mathcal{B}^d(f)$ or $\mathcal{B}^{d-1}(f)$.

Definition 4.5. A *kink action* of f is an action coming from either (1) or (2) above, while an *exterior action* of f is one coming from (4).

Our final piece of terminology is only to be applied in the case that $N \neq 0$, i.e. that M is monotone but not symplectically aspherical. Where f satisfies our slope condition with $\{r_i\}_{i\geq 0}$ and $\{m_i\}_{i\geq 1}$ as before, let S^i be the collection of integers l with $2\pi l$ between m_i and m_{i+1}.

Definition 4.6. If N and γ are both non-zero, we say that f has *distinct kink actions* if

(1a) for any two triples (r_i, l, k) and $(r_{i'}, l', k')$, with $l \in S^i$, $l' \in S^{i'}$, and $k, k' \in \mathbb{Z}$, we have the equalities

$$r_i = r_{i'},\ l = l',\ k = k'$$

holding whenever

$$-2\pi l r_i + f(r_i) + k\sigma(\lambda)\gamma = -2\pi l' r_{i'} + f(r_{i'}) + k'\sigma(\lambda)\gamma\,;$$

(1b) for any triple (r_i, l, k) with $l \in S^i$ and $k \in \mathbb{Z}$, $-2\pi l r_i + f(r_i) + k\sigma(\lambda)\gamma$ does not equal $f(0) + k'\sigma(\lambda)\gamma$ or $f(R) + k'\sigma(\lambda)\gamma$ for any integer k'.

In the case that $N \neq 0$ and $\gamma = 0$, we say that f has distinct kink actions if

(2a) for any two pairs (r_i, l) and $(r_{i'}, l')$ with $l \in S^i$, $l' \in S^{i'}$, we have the equalities

$$r_i = r_{i'},\ l = l'$$

holding whenever

$$-2\pi l r_i + f(r_i) = -2\pi l' r_{i'} + f(r_{i'});$$

(2b) for any pair (r_i, l) with $l \in S^i$, $-2\pi l r_i + f(r_i)$ does not equal $f(0)$ or $f(R)$.

Conditions (1b) and (2b) ensure that no kink action equals any exterior action or any action coming from the y-axis.

With our terminology established, we may conclude this section with a few key lemmas and theorems concerning barcodes of piecewise linear functions.

Lemma 4.7. *Let $\varepsilon > 0$ be given, and let f_1 and f_2 be two piecewise linear functions satisfying our slope condition and the following:*

- $||f_1 - f_2||_{L^\infty} < \varepsilon$.
- *the minimal distance between finite a and any action of f_1 or f_2 outside of $I_\varepsilon(a) := (a - \varepsilon, a + \varepsilon)$ is at least 3ε.*

Then for a fixed degree d, the number of degree d actions in $I_\varepsilon(a)$ which pair with degree $d+1$ actions outside of $I_\varepsilon(a)$ is the same for f_1 and f_2; this conclusion with $d+1$ replaced by $d-1$ also holds.

Proof. The proof of either implication is the same, so we restrict our attention to the first. By the assumption that $||f_1 - f_2||_{L^\infty} < \varepsilon$, we know that an ε-matching μ_ε exists between $\mathcal{B}^d(f_1)$ and $\mathcal{B}^d(f_2)$. Any pairing between a degree d action in $I_\varepsilon(a)$ with a degree $d+1$ action outside of $I_\varepsilon(a)$ gives rise to a bar of length at least 2ε and so is in the domain (or range) of μ_ε. Moreover, our second condition implies that μ_ε must match such a bar to a bar whose degree d (resp. $d+1$) endpoint also lies inside (resp. outside) of $I_\varepsilon(a)$. Hence, μ_ε gives a bijection between the set of intervals of the form $[c^d, c^{d+1})$, with $c^d \in I_\varepsilon(a)$ and $c^{d+1} \notin I_\varepsilon(a)$, for $\mathcal{B}^d(f_1)$ and the set of such intervals for $\mathcal{B}^d(f_2)$.

$\square$

Lemma 4.7 is helpful in proving Theorem 4.8, which is key to proving Theorem 1.2. Before proving Theorem 4.8 in full generality, however, we prove it in the case of Lemma 4.9, where f is assumed to have distinct kink actions.

Theorem 4.8. *Let f be any piecewise linear function satisfying our slope condition, and let c^{n+1} be an action which in degree $n+1$ only comes from concave down kinks of f. Then c^{n+1} does not enter into $\mathcal{B}^{n+1}(f)$, and if no degree n action equals c^{n+1}, then c^{n+1} must appear in $\mathcal{B}^n(f)$.*

Lemma 4.9. *Let f be a piecewise linear graph satisfying our slope condition and having distinct kink actions. Let c^{n+1} denote a degree $n+1$ action coming from a concave down kink in f's graph. Then c^{n+1} must appear in $\mathcal{B}^n(f)$.*

Proof of Lemma 4.9. We restrict our attention to the case of $N \neq 0$, since nearly identical (and even simpler) reasoning applies to the case of $N = 0$. We also must separate our proof into the cases that $\lambda \neq 0$ (so $\gamma \neq 0$) and $\lambda = 0$ (so $\gamma = 0$ but M is not symplectically aspherical).

The case that $\lambda \neq 0$.

Let f be such a function with $\{r_i\}_{i\geq 0}$ and $\{m_i\}_{i\geq 1}$ as previously defined. Our goal is to choose an appropriate homotopy ending in f for which it will be easy to keep track of the corresponding continuum of barcodes. Our homotopy of choice is performed by connecting the zero function to f through the series of intermediate functions g_i defined by

$$g_i(r) = \begin{cases} f(r), & r \geq r_i \\ f(r_i) + m_i(r - r_i), & 0 \leq r \leq r_i \end{cases}$$

for $i \geq 1$. We connect these intermediate functions via straight-line homotopies

$$h_i(t, r) = tg_i + (1-t)g_{i-1},$$

where we take g_0 to be the zero function, and we call the concatenation of these homotopies h_t. Geometrically, this homotopy is taking the graph of the zero function and folding it along the kinks of f's graph from the outside in until f's graph is created (see Figures 4.1a - 4.1d).

A few comments about the homotopy h_t are in order. Note that for all but finitely many values of time $T_0 = \{t_\alpha\}$, each function h_t satisfies our slope condition; the times it does

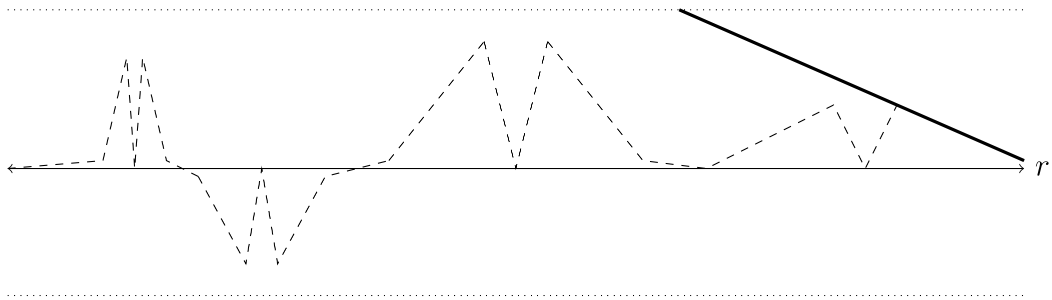

Figure 4.1a: One of our functions g_i, with the graph of f represented by the dashed lines.

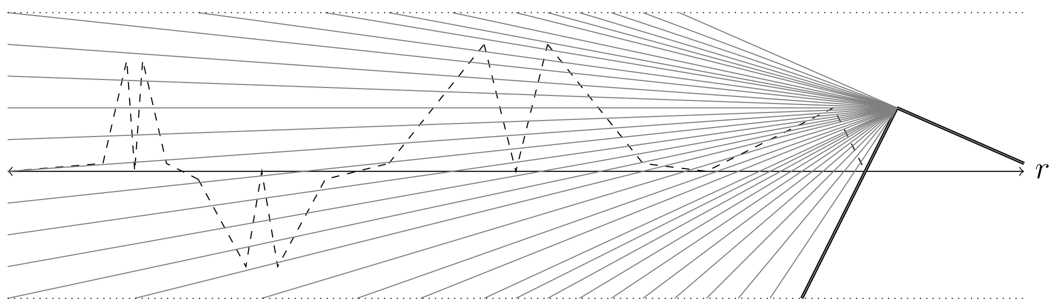

Figure 4.1b: The function g_i bending down at $(r_{i+1}, f(r_{i+1}))$ to make g_{i+1}.

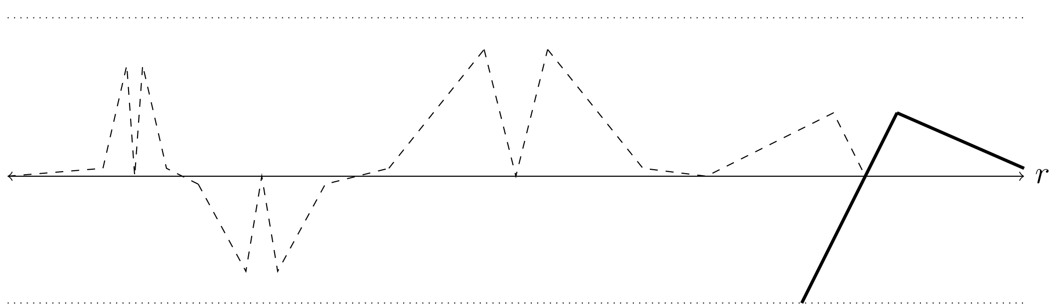

Figure 4.1c: The function g_{i+1}.

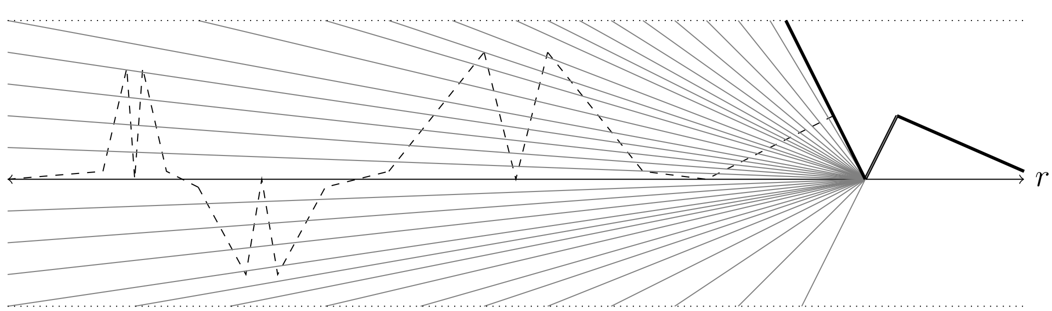

Figure 4.1d: The function g_{i+1} bending up at $(r_{i+2}, f(r_{i+2}))$ to make g_{i+2}.

not correspond to when the slope out of the y-axis is a multiple of 2π. Hence, for any t in an interval of the form $(t_\alpha, t_{\alpha+1})$, the function h_t has a well-defined barcode. Next, let $(r^\alpha, f(r^\alpha))$ be the point of non-differentiability for f at which h_t is bending for $t \in (t_\alpha, t_{\alpha+1})$. For all times in this interval, we see that the slope on the right of $(r^\alpha, f(r^\alpha))$ stays constant while the slope $s(t)$ (the slope of the line coming out of the y-axis at time t) on its left is between $2\pi l$ and $2\pi(l+1)$ for some integer l. This, in conjunction with the kink $(r^\alpha, f(r^\alpha))$ being stationary, implies that the set of all actions coming from this kink is the same for all such h_t, and the same is clearly true for all such actions coming from kinks $(r_i, f(r_i))$ with $r_i \geq r^\alpha$. From this we can conclude that any change in $Spec_m(h_t)$ with t lying in $(t_\alpha, t_{\alpha+1})$ can only come from recappings of the y-intercept. The degree of any such action does not change with time since $s(t)$ does not cross a multiple of 2π. So for a fixed degree d, we may further conclude that $\#|Spec_m^d(h_t)|$ stays the same as t varies in $(t_\alpha, t_{\alpha+1})$, and moreover, that the actions of h_t may be parametrized as functions of time with domain $(t_\alpha, t_{\alpha+1})$. Finally, our formulae for the possible degrees of actions coming from the y-intercept tell us that they are all of the same parity as n, so whenever d has parity differing from n, $Spec_m^d(h_t)$ is the same for all t in $(t_\alpha, t_{\alpha+1})$, i.e. these actions are constant as functions of time.

Now examine what happens at a time $t_\alpha \in T_0$. For $t \in (t_{\alpha-1}, t_\alpha)$, we can parametrize the action (with trivial capping) coming from the y-intercept of h_t as $h_t(0)$, while recappings of this action will be of the form $h_t(0) + k\sigma(\lambda)\gamma$ with $k \in \mathbb{Z}$. This parametrization will also hold for times in $(t_\alpha, t_{\alpha+1})$, though the degrees of these actions may differ.

Suppose for now that the function $s(t)$ is increasing, implying that $h_t(0)$ is decreasing and $(r^\alpha, f(r^\alpha))$ is a concave down kink for f. If $s(t_\alpha) = 2\pi l$ for $l \in \mathbb{Z}$, then $s(t)$ lies between $2\pi(l-1)$ and $2\pi l$ for $t \in (t_{\alpha-1}, t_\alpha)$, so our enumeration of actions and their degrees tells us that $h_t(0) + k\sigma(\lambda)\gamma$ has index $-2(l-1)n - n + k2N = -2ln + n + k2N$ and limits on $-2\pi l r^\alpha + f(r^\alpha) + k\sigma(\lambda)\gamma$ as t goes to t_α. Examining h_t for times $t \in (t_\alpha, t_{\alpha+1})$, we note that our kink at $(r^\alpha, f(r^\alpha))$ has an extra multiple of 2π lying between the slopes on its left ($s(t)$) and right, so we have infinitely many new pairs of actions $\{c_{1,k},\, c_{2,k}\}_{k\in\mathbb{Z}}$ with

$$c_{1,k} = -2\pi l r^{\alpha} + f(r^{\alpha}) + k\sigma(\lambda)\gamma = h_{t_\alpha}(0) + k\sigma(\lambda)\gamma, \quad \text{of degree} - 2ln + n + k2N$$
$$c_{2,k} = -2\pi l r^{\alpha} + f(r^{\alpha}) + k\sigma(\lambda)\gamma = h_{t_\alpha}(0) + k\sigma(\lambda)\gamma, \quad \text{of degree} - 2ln - n + 1 + k2N$$

coming from the set of kinks in our graph. In particular, note that $h_t(0) + k\sigma(\lambda)\gamma$ for $t \in (t_{\alpha-1}, t_\alpha)$ has index and limiting action equal to the index and action of $c_{1,k}$, while for times $t \in (t_\alpha, t_{\alpha+1})$ it has index $-2ln - n + k2N$ and action limiting on $-2\pi l r^\alpha + f(r^\alpha) + k\sigma(\lambda)\gamma$ as t decreases to t_α.

Finally, observe that for any $t \notin T_0$, the kink actions of h_t are a subset of the kink actions of f.

With these observations about h_t out of the way, we continue with our proof. Set $T = [0,1] \backslash T_0$, and let $4\varepsilon > 0$ be the smaller of the minimal positive distance between all of f's kink actions and γ. By our analysis of our homotopy, we know that the minimal distance between h_t's kink actions will be greater than 4ε for all $t \in T$ (where we consider the minimum of the empty set to be infinity, in this case).

Now let t_0 be the time of c^{n+1}'s inception in $Spec_m^{n+1}(f)$. Since $t_0 \neq 0$, we may find small enough intervals of time $(t_{-1}, t_0), (t_0, t_1) \subset T$ so that for any $t_- \in (t_{-1}, t_0)$ and $t_+ \in (t_0, t_1)$ we have $||h_{t_-} - h_{t_+}||_{L^\infty} < \varepsilon$, which in particular implies that $|h_{t_+}(0) - h_{t_-}(0)| < \varepsilon$. So let t_- and t_+ be any two such times. With t_0 being the time of c^{n+1}'s inception, we must have $t_0 \in T_0$, so we can write $s(t_0) = 2\pi l$. The function $s(t)$ must be increasing on the interval (t_{-1}, t_1) because c^{n+1} comes from a concave down kink. Hence, $s(t_+) > s(t_0)$, meaning the possible actions coming from the y-intercept at time t_+ will be of degree

$$-2ln - n + k2N$$

and have the form

$$h_{t_+}(0) + k\sigma(\lambda)\gamma.$$

Moreover, since t_0 is the time of c^{n+1}'s inception and $s(t_0) = 2\pi l$, we must have a solution k to the equations

$$-2ln - n + 1 + k2N = n + 1,$$

and

$$h_{t_0}(0) + k\sigma(\lambda)\gamma = c^{n+1}.$$

This same value of k will give us an action $h_{t_+}(0) + k\sigma(\lambda)\gamma$ of degree n coming from the y-intercept. By our choice of ε, this degree n action does not equal any other actions from h_{t_+} and therefore exists in either $\mathcal{B}^{n-1}(h_{t_+})$ or $\mathcal{B}^n(h_{t_+})$. The number of degree n actions in $I_\varepsilon(c^{n+1})$ at time t_- is zero, so we may apply Lemma 4.7 to say that $h_{t_+}(0) + k\sigma(\lambda)\gamma$ must pair with an action in $I_\varepsilon(c^{n+1})$ at time t_+. The only other actions in $I_\varepsilon(c^{n+1})$ at time t_+ are two actions equal to

$$c^{n+1} = h_{t_0}(0) + k\sigma(\lambda)\gamma$$

with one of degree $n+1$ and the other of degree $3n$. Hence, this degree n action pairs with our degree $n+1$ action, implying c^{n+1} is an endpoint in $\mathcal{B}^n(h_{t_+})$ and therefore that Lemma 4.9 holds for any h_t with $t \in (t_0, t_1)$. See Figure 4.2 for a depiction of this evolution of $\mathcal{B}^n(h_t)$, where in the picture for $\mathcal{B}^n(h_{t_+})$ appearing on the right, the red endpoint represents the degree n action $h_{t_+}(0) + k\sigma(\lambda)\gamma$, while the blue (resp. lightly shaded) endpoint represents the degree $n+1$ (resp. $3n$) action c^{n+1}. In the picture for $\mathcal{B}^n(h_{t_-})$, the lightly shaded endpoint represents the degree $3n$ action $h_{t_-}(0) + k\sigma(\lambda)\gamma$.

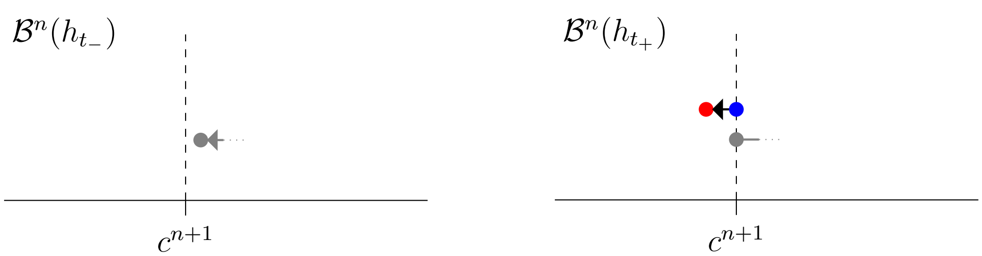

Figure 4.2: The evolution of $\mathcal{B}^n(h_t)$ with respect to time. The lower bar in each picture does not appear in $\mathcal{B}^n(h_t)$ as its left-hand endpoint is of degree $3n$; we indicate this absence from $\mathcal{B}^n(h_t)$ by shading it.

Remark 4.10. The conventions in this work for the visualization of a degree n barcode are as follows: red endpoints correspond to degree n actions, blue to degree $n+1$ actions, black to an action of any other degree, and any bars belonging to a barcode of another degree will be lightly shaded. Furthermore, an endpoint which moves left or right as we move through a family of barcodes will have an arrow next to it. Action values are measured along the horizontal axis; a bar's height bears no significance.

We complete our proof via contradiction, and towards this end we let $t_0' \in [0,1]$ be the infimum of all times $t > t_0$ in T such that c^{n+1} does not appear in $\mathcal{B}^n(h_t)$. First, note that $t_0' \notin T_0$. Assuming otherwise, t_0' would be an infimum of times belonging to a set not including it, so there would exist a sequence of time values $t_+^i \in T$ decreasing towards t_0' for which c^{n+1} does not appear in $B^n(h_{t_+^i})$. We know by the reasoning above that there exists a non-empty interval of time (t_{-1}', t_0') with c^{n+1} appearing in $\mathcal{B}^n(h_t)$ for times $t \in (t_{-1}', t_0')$; for such t, define c_t^n to be the degree n action at time t with $[c_t^n, c^{n+1}) \in \mathcal{B}^n(h_t)$. We may use the existence of the t_+^i to assert that c_t^n must go to c^{n+1} as t goes to t_0. (Indeed, given any $\varepsilon' > 0$, choose times $t_- \in (t_{-1}', t_0')$ and t_+^i which give $||h_{t_+^i} - h_{t_-}||_{L^\infty} < \varepsilon'$ so that an ε'-matching $\mu_{\varepsilon'}$ exists between the two degree n barcodes; since no bar with c^{n+1} as a right endpoint exists in $\mathcal{B}^n(h_{t_+^i})$, the bar $[c_{t_-}^n, c^{n+1})$ must have length less than $2\varepsilon'$.) With c_t^n getting arbitrarily close to c^{n+1} as t increases to t_0', our assumption on f's actions therefore says c_t^n is an action coming from the y-intercept of h_t. Yet if $t_0' \in T_0$, our analysis of our homotopy says this implies h_t will have a degree n action equal to c^{n+1} coming from a kink for all times past t_0' and hence for $t = 1$. This contradiction of our assumptions on f's actions allows us to conclude that $t_0' \notin T_0$.

So t_0' has to be in T. Moreover, arguments similar to the ones given above show that c^{n+1} does not appear in $\mathcal{B}^n(h_{t_0'})$ and that c_t^n as previously defined must still be an action coming from the y-intercept (and so of the form $h_t(0) + k\sigma(\lambda)\gamma$) which converges to c^{n+1} as t goes to t_0'. We must therefore have $t_0' \neq 1$ to avoid contradicting our assumptions on f's actions. We can thus find a non-empty interval of time $(t_{-1}', t_1') \subset T$ containing t_0' with

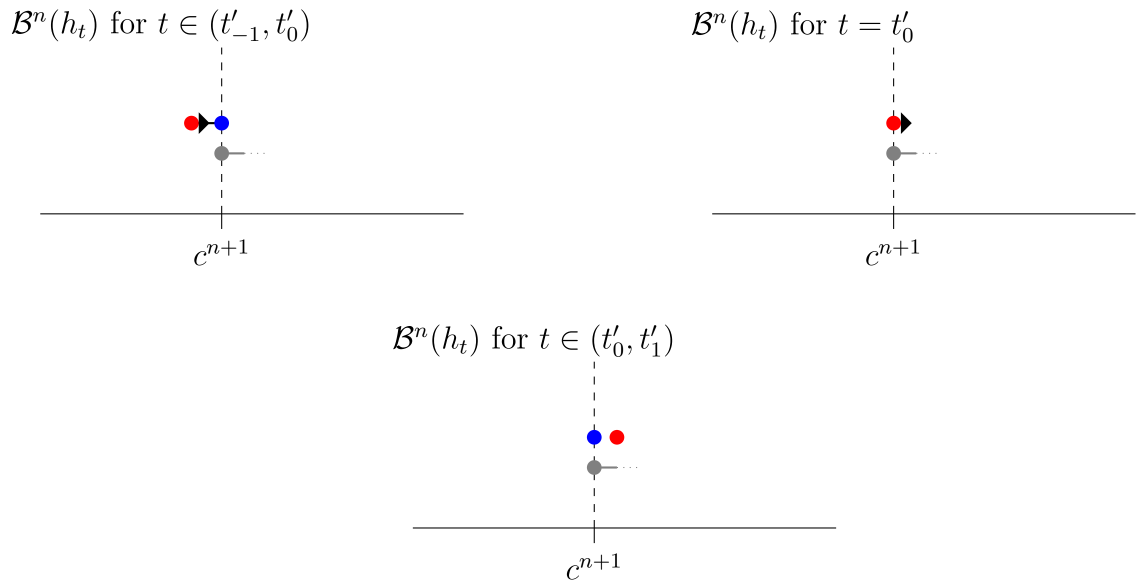

Figure 4.3: The evolution of $\mathcal{B}^n(h_t)$ for times close to t'_0. The red, degree n action having nothing to pair with for times immediately following t'_0 contradicts the existence of t'_0.

$||h_{t_-} - h_{t_+}||_{L^\infty} < \varepsilon$ for every $t_-, t_+ \in (t'_{-1}, t'_1)$, implying the existence of ε-matchings for the various pairs $\mathcal{B}^n(h_{t_-})$, $\mathcal{B}^n(h_{t_+})$. Moreover, $h_t(0) + k\sigma(\lambda)\gamma$ will continue to increase past c^{n+1} as t increases past t'_0 while remaining an action of index n.

Our proof of Lemma 4.9 for the case of $\lambda \neq 0$ is nearly complete. Let $t_+ \in (t'_0, t'_1)$ be given. At this time, we have our degree n action $h_{t_+}(0) + k\sigma(\lambda)\gamma$ not equal to any degree $n+1$ or $n-1$ actions, so it must appear in either $\mathcal{B}^{n-1}(h_{t_+})$ or $\mathcal{B}^n(h_{t_+})$. Moreover, this action is higher in action but lower in degree than any other actions in $I_\varepsilon(c^{n+1})$ (c^{n+1} of degrees $n+1$ and $3n$), so it must pair with something outside of $I_{3\varepsilon}(c^{n+1})$. But the number of degree n actions in $I_\varepsilon(c^{n+1})$ pairing with anything outside of $I_{3\varepsilon}(c^{n+1})$ was zero for any time in (t'_{-1}, t'_0), so Lemma 4.7 implies the same should be true for t_+. We therefore have a contradiction of the definition of t'_0; see Figure 4.3.

The case that $\lambda = 0$.

Before explaining the changes we make for the proof of Lemma 4.9 in the case that $\lambda = 0$, we take the time to explain their necessity. There were several instances in the proof of our previous case where we applied Lemma 4.7 to compare the barcodes of h_{t_-} with those of h_{t_+} for some times t_-, t_+, and then noted that the number of degree n actions in $I_\varepsilon(c^{n+1})$ which paired with actions outside of $I_{3\varepsilon}(c^{n+1})$ was zero for h_{t_-}. Such claims do not always carry over in the case of $\lambda = 0$.

Indeed, for any $\varepsilon > 0$, let t_0, (t_{-1}, t_0), t'_0, and (t'_{-1}, t'_0) be as previously defined. For $t_- \in (t_{-1}, t_0)$, all actions coming from the y-intercept will be of the form $h_{t_-}(0)$ (which is within ε of $c^{n+1} = h_{t_0}(0)$), and if $N = n$, then one such action coming from the y-intercept will be of degree n. We therefore cannot say as before that the number of degree n actions in $I_\varepsilon(c^{n+1})$ is zero at time t_-. Furthermore, we cannot assert as before that, for times in (t'_{-1}, t'_0), the number of degree n actions in $I_\varepsilon(c^{n+1})$ which pair with something outside of $I_{3\varepsilon}(c^{n+1})$ is zero. In particular, the case that $N = n$ gives a degree n action precisely equal to c^{n+1} for all times past t_0, and this action may very well pair with something outside of $I_{3\varepsilon}(c^{n+1})$ for times immediately preceding t'_0.

We proceed with the case $\lambda = 0$. As discussed in Chapter 2, the Floer differential in this case may be described by a count of solutions to the Hamiltonian Floer equation (though we must choose a generic almost complex structure first). If such a solution u connects two capped periodic orbits $\bar{x}$ and $\bar{y}$, then the *energy* of the strip u, defined as

$$E(u) = \int_{\mathbb{R}} \int_0^1 ||\partial_s u||^2 dt\, ds,$$

is precisely equal to the difference in the actions of $\bar{x}$ and $\bar{y}$. We make use of the following lemma, presented here as in [14], nearly verbatim.

Lemma 4.11. *Let V denote an open subset of M with (at least) two distinct smooth boundary components W_1, W_2. Consider a Hamiltonian H which is autonomous in V whose time-1 map ϕ^1_H has no fixed points in V. Further assume that W_1 and W_2 are contained in two distinct level sets of H. Then there exists a constant $\varepsilon(V, H|_V, J|_V) > 0$, depending on the domain V*

and the restrictions of the Hamiltonian H and the almost complex structure J to the domain V, such that if $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ is a solution to the Hamiltonian Floer equation and intersects W_1 and W_2, then

$$E(u) \geq \varepsilon(V, H|_V, J|_V).$$

Recalling the definition of $r_0 = R$ and r_1, we choose V to be the subset of $B(2\pi R)$ defined by

$$V = \left\{ z \in B(2\pi R) \;\middle|\; r_1 + \frac{R - r_1}{4} \leq \frac{|z|^2}{2} \leq R - \frac{R - r_1}{4} \right\},$$

and we fix on M a time-dependent ω-compatible almost complex structure J_0.

Redefine T in this case to be those values of time for which h_t satisfies our slope condition and equals $m_1(r - R) + f(R)$ on (r_1, R) (this corresponds to having completed the first leg of our homotopy). We know h_t is the same on the set $(r_1 + \frac{R-r_1}{4}, R - \frac{R-r_1}{4})$ for any $t \in T$, so the same may be said for any standard perturbation $\tilde{H}_t$ of such h_t on V.

As noted in the introduction, we must pick a regular almost complex structure J for each non-degenerate $\tilde{H}_t$ to have the differential for the Floer chain complex well-defined. This may lead one to believe that our choices for J may differ on V from perturbation to perturbation. However (as remarked in [14]), for every $t \in T$ and any standard perturbation $\tilde{H}_t$ of h_t, the periodic orbits of $\tilde{H}_t$ do not enter V, implying that our choice of regular almost complex structure may be chosen to equal J_0 on V. Hence our $\varepsilon(V, H|_V, J|_V)$ from Lemma 4.11 will work for any standard perturbation $\tilde{H}_t$ of h_t and any $t \in T$.

With Lemma 4.11 introduced, we continue with our proof. The first part is essentially the same as when $N, \lambda \neq 0$. We let t_0 be the time of c^{n+1}'s inception and say that $s(t_0) = 2\pi l$. However, we now choose 4ε to be the minimum of the $\varepsilon(V, H|_V, J|_V)$ from Lemma 4.11 and the minimal positive distance between all kink actions of f. We then choose appropriate time intervals $(t_{-1}, t_0), (t_0, t_1)$ as before. Again, $s(t_+) > s(t_0)$ for any $t_+ \in (t_0, t_1)$, so the possible actions coming from the y-intercept will be of degree

$$-2ln - n + k2N \tag{**}$$

and have the form

$$h_{t_+}(0).$$

These, along with the actions $h_{t_0}(0)$ of degree either

$$-2ln + n + k2N \text{ or } -2ln - n + 1 + k2N \tag{***}$$

comprise all actions in $I_\varepsilon(c^{n+1})$ at time t_+. By our definition of t_0, $h_{t_0}(0) = c^{n+1}$.

We claim that:

Claim 4.12. *Our c^{n+1} pairs with $h_{t_+}(0)$ of degree n. Hence, c^{n+1} appears in $\mathcal{B}^n(h_t)$ for $t \in (t_0, t_1)$.*

Proof of Claim 4.12. Let $t_- \in (t_{-1}, t_0)$ and $t_+ \in (t_0, t_1)$ as always, and let ε' be small enough so that $I_{\varepsilon'}(c^{n+1}) \cap I_{\varepsilon'}(h_{t_+}(0)) = \emptyset$. Note that $\varepsilon' < \varepsilon$ by our assumption on f's actions. Since $||h_{t_+} - h_{t_-}||_{L^\infty} < \varepsilon$, we can find standard perturbations $\tilde{H}_-$ and $\tilde{H}_+$ with $\int_0^1 ||\tilde{H}_+ - \tilde{H}_-||_{L^\infty} dt < \varepsilon$ (where t in this expression is the $\mathbb{R}/\mathbb{Z}$ parameter and not the homotopy parameter), and we can choose our perturbations $\tilde{H}_\pm$ so that every $c^\pm \in \mathit{Spec}(\tilde{H}_\pm)$ is within ε' of its corresponding action in $\mathit{Spec}_m(h_{t_\pm})$, with the correspondence being clear when one reviews the proof of Claim 4.1 and the discussion preceding it. Our choice of ε and perturbations further guarantees that actions of $\tilde{H}_\pm$ not in $I_\varepsilon(c^{n+1})$ are outside of $I_{3\varepsilon}(c^{n+1})$. With $\tilde{H}_-$ not having any degree $n+1$ actions in $I_\varepsilon(c^{n+1})$, we can therefore use a variation of Lemma 4.7 to say that any degree $n+1$ actions of $\tilde{H}_+$ in $I_\varepsilon(c^{n+1})$ must pair with something in $I_\varepsilon(c^{n+1})$.

We know $\tilde{H}_+$ has a degree $n+1$ action in $I_{\varepsilon'}(c^{n+1})$ coming from a capped periodic orbit of the form $[x, v\#k_1A]$, with v the capping contained in $B(2\pi R)$ and $[A]$ as chosen in Remark 2.2. Since this action must pair with an action in $I_\varepsilon(c^{n+1})$, and since the Floer differential in the present case is given by a count of solutions to the Hamiltonian Floer equation, there must exist a Floer trajectory u between $[x, v\#k_1A]$ and some other capped periodic orbit $[y, w\#k_2A]$ (again, with w contained in $B(2\pi R)$) whose action lies in $I_\varepsilon(c^{n+1})$; see Theorem 6.2 and the beginning of the proof of Theorem 12.3 from [36] for more on why such a trajectory should exist. The energy of any such u is less than 2ε, which by Lemma 4.11 means it must be contained within our symplectic ball. Such a u must also satisfy $[u\#w\#k_2A] = [v\#k_1A]$ (or possibly $[w\#k_2A] = [u\#v\#k_1A]$) as elements of $\pi_2(M) \cong \pi_2(M, B(2\pi R))$. With u, v, and w all lying in $B(2\pi R)$, we conclude $k_1 = k_2$.

The integer k_1 plugged into the expression on the right in (***) gives $n+1$, the degree of c^{n+1}. Hence, the capped periodic orbits of $\tilde{H}_+$ having k_1 copies of A attached and action in $I_\varepsilon(c^{n+1})$ must have degree n or $3n$ according to the remaining expressions from (**) and (***). Noting that $n \neq 1$ for the case of $N \neq 0, \lambda = 0$ so that $3n \neq n+2$, $[x, v\#k_1A]$'s action must pair with the degree n action lying in $I_{\varepsilon'}(h_{t_+}(0))$. Since such a pairing holds for arbitrarily small ε' and perturbations of h_{t_+}, our claim holds.

$\square$

Next, define t_0' (as in the $\lambda \neq 0$ case) as the infimum of all times $t \in T$ for which c^{n+1} is not in $B^n(h_t)$. Recalling that the degrees of actions coming from the y-intercept are of the form

$$-2l'n - n + k2N,$$

we may use our reasoning from the $\lambda \neq 0$ case to conclude the following:

- $t_0' \notin T_0$, so the various degrees for the $h_t(0)$ use the same integer l' for time values close to t_0'.
- $h_t(0)$ is increasing for times close to t_0'.
- One value of k makes $-2l'n - n + k2N = n$.
- $h_{t_0'}(0) = c^{n+1}$.

Moreover, we may use another energy argument as in the proof of our previous claim to say that our k value from the third item above must be k_1, while yet another such energy argument gives us our contradiction: For any time t_+ sufficiently close to but greater than t_0', any small enough perturbation $\tilde{H}_+$ of h_{t_+} must have the action c which corresponds to

c^{n+1} pairing with the action of an orbit that has k_1 recappings by A. This other action must lie in $I_\varepsilon(c^{n+1})$, and the only orbits for $\tilde{H}_{t_+}$ which satisfy all of these properties either have an incompatible degree ($3n$ with $n \neq 1$) or have degree n with action higher than c. This concludes the proof of Lemma 4.9 in the case $\lambda = 0$ and thus in general.

$\square$

With Lemma 4.9 in hand, we may now prove Theorem 4.8 with ease.

Proof of Theorem 4.8. Consider f's kinks $\{(r_i, f(r_i))\}_{i\geq 1}$. By moving these points slightly in the r and y directions and connecting them with straight lines, we may create for any $\varepsilon > 0$ a piecewise linear function g such that

- g satisfies our slope condition and has distinct kink actions.
- $||f - g||_{L^\infty} < \varepsilon$.
- For every degree d, a natural bijection ν^d exists between $Spec_m^d(f)$ and $Spec_m^d(g)$ with $|\nu^d(c) - c| < \varepsilon$ for all $c \in Spec_m^d(f)$.

Let $\{c_i^{n+1}\}_{i=1}^{\eta} \subset Spec_m^{n+1}(f)$ be the set of index $n+1$ actions of f which are equal to c^{n+1} (so c^{n+1} has multiplicity η in $Spec_m^{n+1}(f)$). By Lemma 4.9, every $\nu^{n+1}(c_i^{n+1})$ must appear in $\mathcal{B}^n(g)$ and hence not in $\mathcal{B}^{n+1}(g)$. Choose a sequence $\varepsilon_k \to 0$ and a corresponding sequence of g_k, then apply the continuity of barcodes to conclude that c^{n+1} cannot appear in $\mathcal{B}^{n+1}(f)$.

For the second part of the theorem, let 4ε be the minimal distance between c^{n+1} and any degree n action of f, and choose a function g as above corresponding to ε. Then again, any $\nu^{n+1}(c_i^{n+1})$ appears in $\mathcal{B}^n(g)$, and by our choice of ε, it will be the endpoint of a bar of length at least 2ε. Hence, the ε-matching μ_ε between $\mathcal{B}^n(f)$ and $\mathcal{B}^n(g)$ has this bar in its range, and its preimage must be a bar in $\mathcal{B}^n(f)$ with right endpoint c^{n+1} and length at least 4ε. $\square$

Finally, making slight alterations to the proofs of Lemma 4.9 and Theorem 4.8 yields the following, which are just as essential as Theorem 4.8 to proving Theorem 1.2.

Theorem 4.13. *Let f be any piecewise linear function satisfying our slope condition, and let c^{-3n+1} be an action which in degree $-3n+1$ only comes from concave down kinks of f. Then c^{-3n+1} does not enter into $\mathcal{B}^{-3n+1}(f)$, and if no degree $-3n$ action equals c^{-3n+1}, then c^{-3n+1} must appear in $\mathcal{B}^{-3n}(f)$.*

Theorem 4.14. *Let f be any piecewise linear function satisfying our slope condition, and let c represent an action which, in degree $n+1$ (respectively, $-3n+1$), only comes from concave up kinks in f's graph. Then c does not enter into $\mathcal{B}^{n}(f)$ (resp. $\mathcal{B}^{-3n}(f)$). Furthermore, if no degree $n+2$ (resp. $-3n+2$) actions equal c, then c appears as the left-hand endpoint of a bar in $\mathcal{B}^{n+1}(f)$ (resp. $\mathcal{B}^{-3n+1}(f)$).*

Chapter 5

Proof of the main theorem

Now suppose we have symplectically embedded our ball $B(2\pi R)$ and let $\varepsilon > 0$. Separate the interval $(R-\varepsilon, R]$ into the union of intervals

$$\bigcup_{i=1}^{\infty}[R-\varepsilon+\varepsilon(1/2)^i, R-\varepsilon+\varepsilon(1/2)^{i-1}].$$

To define the functions which will later define our embedding, we start by defining for each i a piecewise linear function $f_i : [0, R] \to \mathbb{R}$ which is supported in the interval $I_i = [R-\varepsilon+\varepsilon(1/2)^i, R-\varepsilon+\varepsilon(1/2)^{i-1}]$. These f_i are defined as follows:

- f_i is 0 at the midpoint $r_{i,2}$ of I_i and on $U_{i,1}$, $U_{i,2}$, where $U_{i,1}$, $U_{i,2}$ are small neighborhoods of I_i's left and right endpoints, respectively.
- f_i is $2\pi R$ at points $r_{i,1}$, $r_{i,3}$, where the interval $(r_{i,1}, r_{i,3})$ is centered at $r_{i,2}$.
- f_i is linear and increasing, with slope an irrational multiple of 2π, from the right-hand endpoint of $U_{i,1}$ to $r_{i,1}$ and from $r_{i,2}$ to $r_{i,3}$.
- f_i is linear and decreasing, with slope an irrational multiple of 2π, from $r_{i,1}$ to $r_{i,2}$ and from $r_{i,3}$ to the left-hand endpoint of $U_{i,2}$.

See Figure 5.1.

Choose for each i a smooth function $\bar{f}_i$, also supported in I_i, which is less than $\varepsilon(1/2)^i$ away from f_i in the C^0 norm and has maximum less than $2\pi R$. Each such $\bar{f}_i$ induces a Hamiltonian $\bar{F}_i : M \to \mathbb{R}$, and we define our embedding $\Phi : [0,1]^\infty \to Ham(M,\omega)$ by

$$\Phi(a) = \phi_{\sum_{i=1}^{\infty} a_i \bar{F}_i},$$

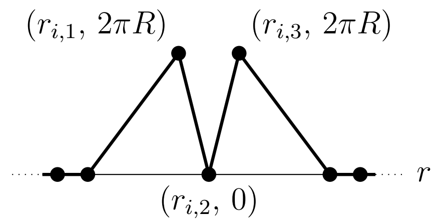

Figure 5.1: One of our f_i's. The neighborhoods $U_{i,1}$ and $U_{i,2}$ have endpoints marked by the first two and last two nodes, respectively.

i.e. the sequence $a = \{a_i\}_{i\geq 1}$ is sent to the Hamiltonian diffeomorphism generated by $\sum_{i=1}^{\infty} a_i \bar{F}_i$. We will sometimes abuse notation and refer to such diffeomorphisms as being generated by $\sum_{i=1}^{\infty} a_i \bar{f}_i$ instead.

By the definition of the Hofer distance between two Hamiltonian diffeomorphisms, the chain of inequalities from our theorem is equivalent to

$$2\pi R||a-b||_{\ell^\infty} - \varepsilon \leq ||\Phi(a)^{-1} \circ \Phi(b)||_H \leq 4\pi R||a-b||_{\ell^\infty}.$$

With $\Phi(a)$ being generated by the *autonomous* $\sum_{i=1}^{\infty} a_i \bar{F}_i$, $\Phi(a)^{-1}$ is generated by $\sum_{i=1}^{\infty} -a_i \bar{F}_i$, and since the functions $\sum_{i=1}^{\infty} -a_i \bar{F}_i$ and $\sum_{i=1}^{\infty} b_i \bar{F}_i$ Poisson commute, $\Phi(a)^{-1} \circ \Phi(b)$ is generated by the function $\sum_{i=1}^{\infty}((b_i - a_i)\bar{F}_i)$ (see Chapter 1). This expression makes the right-most inequality above trivial. Indeed, by definition, the Hofer norm of any Hamiltonian diffeomorphism generated by an autonomous function H will be less than or equal to the difference between H's maximum and minimum values, which in turn is less than twice the maximum of its absolute value. For our function $\sum_{i=1}^{\infty}((b_i - a_i)\bar{F}_i)$, this quantity is bounded above by $4\pi R||a-b||_{\ell^\infty}$.

Furthermore, note that $\{b_i - a_i\}_{i\geq 1}$ is a sequence with entries in $[-1, 1]$, so that the left inequality from our theorem is implied by the following:

Theorem 5.1. *Any function $\sum_{i=1}^{\infty} a_i \bar{f}_i$ with $\{a_i\} \in [-1,1]^\infty$ and $\bar{f}_i$ as above induces a diffeomorphism ϕ whose boundary depth $\beta(\phi)$ satisfies $\beta(\phi) \geq 2\pi R(max_i|a_i|) - (4\pi + 7)\varepsilon$.*

As discussed earlier, this proves the desired inequality since the boundary depth of a Hamiltonian diffeomorphism provides a lower bound for its Hofer norm. Theorem 5.1 is a direct consequence of the following (to be proven momentarily) and Theorem 3.5.

Lemma 5.2. *Let $a = \{a_i\}_{i\geq 1}$ be a sequence in $[-1,1]^\infty$ with $a_k = \pm 1$ for some k, and let ϕ be the Hamiltonian diffeomorphism generated by $\sum_{i=1}^\infty a_i \bar{f}_i$. Then $\beta(\phi) \geq 2\pi R - (4\pi + 7)\varepsilon$.*

Proof of Theorem 5.1, assuming Lemma 5.2. Let ϕ be generated by $\sum_{i=1}^\infty a_i \bar{f}_i$ and let a_k be such that $|a_k| = \max_{i\geq 1}(|a_i|)$. We may assume that a_k is positive by the following: According to Theorem 3.6 (i), $\beta(\phi) = \beta(\phi^{-1})$ for every Hamiltonian diffeomorphism ϕ, and if ϕ is generated by autonomous g, its inverse is generated by $-g$. Hence we may replace each a_i with $-a_i$ while leaving the boundary depth unchanged.

Define $b \in [-1,1]^\infty$ by setting $b_i = a_i$ for $i \neq k$ and $b_k = 1$, and let ϕ' be the Hamiltonian diffeomorphism induced by $\sum_{i=1}^\infty b_i \bar{f}_i$. By Lemma 5.2, $\beta(\phi') \geq 2\pi R - (4\pi + 7)\varepsilon$. Let h_t be the straight-line homotopy between the functions $\sum_{i=1}^\infty b_i \bar{f}_i$ and $\sum_{i=1}^\infty a_i \bar{f}_i$, and let ϕ_t be the induced path of diffeomorphisms with $\phi_0 = \phi'$ and $\phi_1 = \phi$. Theorem 3.5 then tells us that

$$|\beta(\phi') - \beta(\phi_t)| \leq \max_{[0,R]}\left(\left(\sum_{i=1}^\infty b_i f_i\right) - h_t\right) - \min_{[0,R]}\left(\left(\sum_{i=1}^\infty b_i f_i\right) - h_t\right)$$

(compare the above upper bound to the upper bound from Theorem 3.5).

The function $\left(\sum_{i=1}^\infty b_i \bar{f}_i\right) - h_t = [t(1-a_k)]\bar{f}_k$ has maximum less than $[t(1-a_k)]2\pi R$ and minimum zero, so the above inequality becomes

$$|\beta(\phi') - \beta(\phi_t)| \leq [t(1-a_k)]2\pi R,$$

leading us to

$$\beta(\phi') - [t(1-a_k)]2\pi R \leq \beta(\phi_t).$$

Taking $t = 1$ and using that $\beta(\phi') \geq 2\pi R - (4\pi + 7)\varepsilon$ finishes the proof.

∎

The rest of this section is devoted to proving Lemma 5.2.

Proof. Our goal is to eventually find a bar of the appropriate length in a degree d barcode of some function which is C^0-close to $\sum_{i=1}^{\infty} a_i \bar{f}_i$. From this, we get a lower bound on our boundary depth and our lemma is proved. We work out the case where M is monotone with $\lambda > 0$, $n\gamma - 2\pi NR \geq 0$, and $N \neq 0$ in detail, while a brief discussion of the (slight) modifications necessary for the remaining cases is reserved for the end of the proof.

Case 1 $N \neq 0, \lambda > 0$, and $n\gamma - 2\pi NR \geq 0$.

Instead of working directly with the function $\sum_{i=1}^{\infty} a_i \bar{f}_i$, we first pass to its piecewise linear counterpart $\sum_{i=1}^{\infty} a_i f_i$, and then to a piecewise linear function g which satisfies our slope condition.

Let $\sum_i^{\infty} a_i f_i$ be given with a satisfying our hypothesis. Assume without loss of generality as in the proof of Theorem 5.1 that $a_k = 1$, and let $r_1 = r_{k,1}$, $r_2 = r_{k,2}$, and $r_3 = r_{k,3}$ be the r-values near the center of f_k's support where f_k has kinks (as labeled in Figure 5.1). We may C^0 perturb our graph $\sum_i^{\infty} a_i f_i$ by less than ε to a new piecewise linear function g which has kinks at precisely the same values of r as $\sum_i^{\infty} a_i f_i$, satisfies our slope condition, and leaves the points $(r_\alpha, f_k(r_\alpha)), \alpha = 1, 2, 3$ unchanged. For convenience, we further assume that the the slopes m_0, m_1 of the line coming out of the y-axis and of the line going into the line $r = R$ are, respectively, negative and positive. We also assume $g(R) > 0$.

Define functions g_0 and g_1 by

$$g_0(r) = \begin{cases} m_0(r - r_2), & 0 \leq r \leq r_2 \\ f_k(r), & r_2 \leq r \leq r_3 \\ m_1(r - r_3) + 2\pi R, & r_3 \leq r \leq R \end{cases}$$

and

$$g_1(r) = \begin{cases} m_0(r - r_1) + 2\pi R, & 0 \leq r \leq r_1 \\ f_k(r), & r_1 \leq r \leq r_3 \\ m_1(r - r_3) + 2\pi R, & r_3 \leq r \leq R. \end{cases}$$

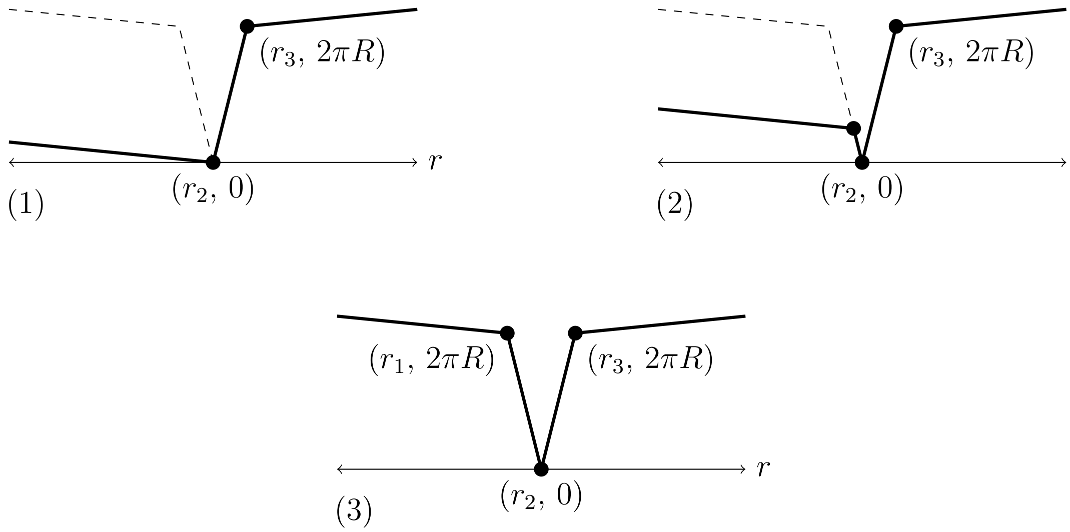

Figure 5.2: The homotopy h_t^1. The solid graphs in the first and third pictures are g_0 and g_1, respectively. The r-coordinate of the leftmost kink in the second picture is $r(t)$.

The graphs of these functions are displayed in the first and third graphs of Figure 5.2. The function g_0 is a C^0 approximation of a function which starts off as a constant 0, then exhibits the rapidly increasing behavior of f_k right after the midpoint of its support, then becomes a constant $2\pi R$ for the rest of our interval. The function g_1 is a C^0 approximation of a similar function which exhibits the interesting behavior of f_k on $[r_1, r_3]$ instead.

Let $r(t) = (1-t)r_2 + tr_1$. We connect g_0 to g_1 via the following homotopy:

$$h_t^1(r) = \begin{cases} m_0(r - r(t)) + 2\pi Rt, & 0 \leq r \leq r(t) \\ g_1(r), & r(t) \leq r \leq R. \end{cases}$$

Notice that the number of kinks in the graph of h_t^1 stays the same once the homotopy starts. Moreover, the slopes around each kink are the same throughout the homotopy, implying that we may parametrize the actions of the h_t^1 as functions of time.

First, we give an explicit parametrization of the degree $n+1$ actions which can occur at r_3. Since h_t^1 is concave down at r_3, we know that the possible degrees occurring here are of the form $-2ln+n+k2N$ or $-2ln-n+1+k2N$. Only the latter of these expressions has values l and k which give it a degree of $n+1$, leading us to focus only on solutions to the equation

$$-2ln-n+1+k2N = n+1$$

or

$$2n(-l)+k2N = 2n.$$

Letting D represent the greatest common divisor of $2n$ and $2N$, we may therefore parameterize our solutions $-l$ and k to the above equation as

$$-l = 1-\tfrac{2N}{D}z, \qquad k=\tfrac{2n}{D}z \tag{*}$$

where z is an integer. Using our enumeration of actions from Chapter 4 when $\lambda > 0$, we conclude that any such action has the form

$$2\pi\Big(1-\tfrac{2N}{D}z\Big)r_3+2\pi R+\tfrac{2n}{D}z\gamma;$$

setting $r_3 = R-\delta_3$ for some $\delta_3 > 0$ and simplifying the above expression yields

(A1) $4\pi R - 2\pi\delta_3 + \frac{2}{D}z(n\gamma - 2\pi N(R-\delta_3))$

We know that $n\gamma - 2\pi N(R-\delta_3) > 0$ since $n\gamma - 2\pi NR \geq 0$, and since $-l < 0$ at $(r_3, f_k(r_3))$, we must have $z > 0$ in (*) and hence in (A1). Meanwhile, the inequality $\frac{2}{D}(n\gamma - 2\pi NR) \geq 2\pi(1-\frac{2N}{D})\delta_3$ implies $\frac{2}{D}(n\gamma - 2\pi N(R-\delta_3)) \geq 2\pi\delta_3$. This discussion allows us to conclude that any degree $n+1$ action coming from $(r_3, f_k(r_3))$ is at least as big as $4\pi R$.

A similar analysis gives

(A2) any degree n action coming from r_3 will be of the form

$$2\pi R + \tfrac{2}{D}z(n\gamma - 2\pi N r_3),$$

with $z > 0$. Hence, all such actions are strictly greater than $2\pi R$.

Next, we parameterize the relevant actions at r_2 and $r(t)$. Again using our enumeration of actions from Chapter 4 and calculations similar to those above, we conclude the following.

(A3) Any degree $n+1$ action at time t occurring at $r(t)$ has the form

$$2\pi r(t) + 2\pi Rt + \tfrac{2}{D} z(n\gamma - 2\pi N r(t)).$$

(A4) Any degree n action at time t occurring at $r(t)$ has the form

$$2\pi Rt + \tfrac{2}{D} z(n\gamma - 2\pi N r(t)).$$

We must have $z < 0$. Using reasoning similar to the case of (A1), we may say that all actions here are no more than $2\pi r_2$ for all $t \in (0, 1]$.

(A5) Any degree n action at time t occurring at r_2 has the form

$$2\pi r_2 + \tfrac{2}{D} z(n\gamma - 2\pi N r_2).$$

(A6) We have a degree n action coming from the y-intercept of the form $2\pi Rt - m_0 r(t)$. By adjusting m_0 if necessary, we may assume that $0 < -m_0 r(1) < \varepsilon$.

(A7) Any exterior degree n actions will be at least as big as $2\pi R$.

(A8) Any exterior degree $n+1$ actions will be at least as big as $2\pi R + \gamma \geq 6\pi R$.

With this new enumeration of actions out of the way, we may continue with our proof. Consider the degree $n+1$ action $2\pi r(t) + 2\pi Rt$ from (A3) with $z = 0$, which is easily verified to be a possible value of z if $|m_0|$ was chosen small enough. This action does not equal any of the previously calculated degree $n+1$ actions for all $t > 0$, so if there exists a time t for which another degree $n+1$ action equals our chosen one, it must come from the concave up kink occurring at r_2. There are only finitely many of these. Similarly, we see that there are only finitely many times where our degree $n+1$ action can equal any degree n action. Hence, we can find an interval of time right after $t=0$ in which this action is unique among all degree $n+1$ and n actions for h_t^1. Apply Theorem 4.8 to conclude that our degree $n+1$ action must appear in $\mathcal{B}^n(h_t^1)$ for this small interval of time.

Furthermore, note that if our degree $n+1$ action limits on a degree $n+1$ action from r_2 as t goes to 0 (implying that said action from r_2 has to be $2\pi r_2$), then we must have $2\pi(-l)r_2+k\gamma = 2\pi r_2$, or equivalently, $k\gamma = 2\pi(l+1)r_2$. If $k \neq 0$, we may break this equality by slightly shrinking ε and thus changing our value of r_2. On the other hand, if $k=0$, then l would have to be -1; again assuming $|m_0|$ was chosen small enough, this gives an impossible value of l at r_2 for $t=0$. We are therefore justified in assuming $2\pi r_2$ is not a degree $n+1$ action for h_0^1. This and the continuity in t of $\mathcal{B}^n(h_t^1)$ imply that our degree $n+1$ action must pair with a degree n action c_t^n which is close to it for our previously chosen small interval of time. In particular, c_t^n must satisfy $2\pi r(t)+2\pi Rt - c_t^n \to 0$ as $t \to 0$, and of the degree n actions enumerated above, the only one to do this is the one with $z=0$ from (A5).

In fact, we claim the following:

Claim 5.3. *Such a pairing persists until such time $\bar{t}$ that the degree n action coming from the y-axis, $2\pi R\bar{t} - m_0 r(\bar{t})$, equals our chosen degree n action coming from f_k's kink at r_2.*

Proof of Claim 5.3. The idea behind this proof is the following: Our degree n action will remain stationary for all time, so the only way in which the left-hand endpoint of our bar can change is if some degree n action which changes with time eventually equals our chosen one. However, the only one which can do this occurs at time $\bar{t}$ and is given by the degree n action coming from the y-intercept. On the other hand, our degree $n+1$ action increases with time, so our bar grows until possibly when our degree $n+1$ action equals another; see Figure 5.3 for a seemingly possible, and troublesome, depiction of how $\mathcal{B}^n(h_t^1)$ changes with time. But as we shall see, any other degree $n+1$ actions which can equal our chosen one must come from a concave up kink, which by Theorem 4.14 cannot enter into $\mathcal{B}^n(h_t^1)$. Hence, the scenario depicted in Figure 5.3 cannot occur.

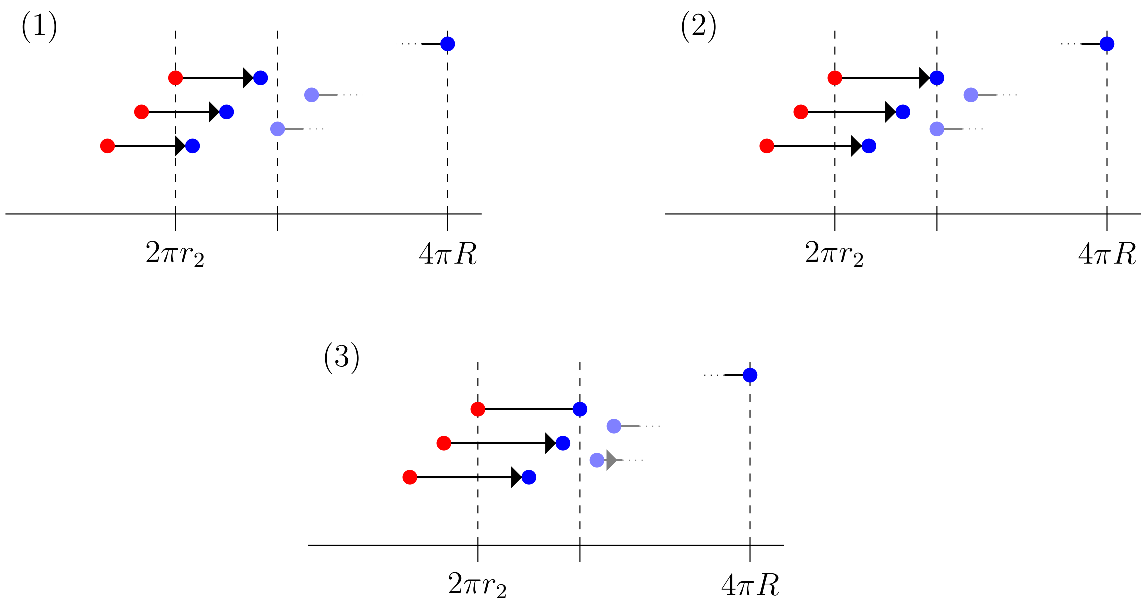

Figure 5.3: A troublesome evolution of $\mathcal{B}^n(h_t^1)$, with our bar of choice being the one with left-hand endpoint at $2\pi r_2$. In the second picture, our preferred bar has its increasing, degree $n+1$ action switch with a stationary degree $n+1$ action. This keeps our bar from growing, as depicted in the third picture. Theorem 4.14, however, assures us that this cannot happen.

We will first show that the set

$$T = \{t \in (0, \bar{t}) \,|\, [2\pi r_2, 2\pi r(t) + 2\pi Rt) \in \mathcal{B}^n(h_t^1)\}$$

is open and closed in $(0, \bar{t})$ and so must be equal to $(0, \bar{t})$ since, as we have already seen, T is non-empty. If $t_0 \in T$ with $t_0 \neq \bar{t}$, we assert the existence of a time interval (t_{-1}, t_1) and an $\varepsilon' > 0$ so that for all times $t \in (t_{-1}, t_1)$

- $(2\pi r(t_0) + 2\pi Rt_0) - 2\pi r_2 > 2\varepsilon'$.
- our degree $n+1$ action is at least ε' away from all other $n+1$ actions for h_t^1 *except* possibly for constant degree $n+1$ actions of the form $2\pi r(t_0) + 2\pi Rt_0$. In particular, the only other possible degree $n+1$ actions lying in the interval $(2\pi r(t_0) + 2\pi Rt_0 - \varepsilon', 2\pi r(t_0) + 2\pi Rt_0 + \varepsilon')$ come from the concave up kink of h_t^1.

- our degree n action is at least ε' away from all other degree n actions of h_t^1.
- $||h_t^1 - h_{t_0}^1||_{L^\infty} < \varepsilon'$.

Our third condition may be met since $t_0 \neq \bar{t}$. So choose $t' \in (t_{-1}, t_1)$. We know there should exist an ε'-matching $\mu_{\varepsilon'}$ between $\mathcal{B}^n(h_{t_0}^1)$ and $\mathcal{B}^n(h_{t'}^1)$. Our first condition above tells us $[2\pi r_2, 2\pi r(t_0) + 2\pi R t_0) \in \mathcal{B}^n(h_{t_0}^1)$ is in the domain of $\mu_{\varepsilon'}$. Furthermore, $\mu_{\varepsilon'}$ should match our degree $n+1$ action $2\pi r(t_0) + 2\pi R t_0$ at time t_0 with a degree $n+1$ action which is in $(2\pi r(t_0) + 2\pi R t_0 - \varepsilon', 2\pi r(t_0) + 2\pi R t_0 + \varepsilon')$, and the only such degree $n+1$ actions at time t' are $2\pi r(t') + 2\pi R t'$ and $2\pi r(t_0) + 2\pi R t_0$. The latter action, however, can only come from our concave up kink in $h_{t'}^1$'s graph and so cannot enter into $\mathcal{B}^n(h_{t'}^1)$ by Theorem 4.14. Hence, our degree $n+1$ action $2\pi r(t_0) + 2\pi R t_0$ at time t_0 must be matched with the degree $n+1$ action $2\pi r(t') + 2\pi R t'$ at time t'. Similar reasoning shows that our chosen degree n action must be matched with itself between times t_0 and t'. Hence, a bar of the form $[2\pi r_2, 2\pi r(t') + 2\pi R t')$ exists in $\mathcal{B}^n(h_{t'}^1)$ for all times $t' \in (t_{-1}, t_1)$, showing that T is an open subset $(0, \bar{t})$.

The set T is closed for a simpler reason: If $t_0 \in (0, \bar{t})$ is a limit point of T, then there are times t immediately prior to (or after) t_0 for which a bar of the appropriate form exists in $\mathcal{B}^n(h_t^1)$. With the lengths of these bars not limiting on something of zero-length as t approaches t_0, we may use the continuity of the barcode to say that a bar of the form $[2\pi r_2, 2\pi r(t_0) + 2\pi R t_0)$ must exist in $\mathcal{B}^n(h_{t_0}^1)$. So t_0 is in T and consequently $T = (0, \bar{t})$.

To finish the proof of our claim, we note that the same argument used in the previous paragraph shows that $\mathcal{B}^n(h_{\bar{t}}^1)$ must have a bar of the appropriate form.

$\square$

Similar reasoning shows that for times t bigger than $\bar{t}$, we either have a bar of the form $[2\pi r_2, 2\pi r(t) + 2\pi R t)$ or of the form $[2\pi R t - m_0 r(t), 2\pi r(t) + 2\pi R t)$. Taking $t = 1$, we have a bar of the form $[2\pi r_2, 2\pi r_1 + 2\pi R)$ or $[2\pi R - m_0 r_1, 2\pi r_1 + 2\pi R)$ in $\mathcal{B}^n(h_1^1) = \mathcal{B}^n(g_1)$. Figure 5.4 shows how $\mathcal{B}^n(h_t^1)$ can change with time so that $\mathcal{B}^n(h_1^1)$ will have $[2\pi R - m_0 r_1, 2\pi r_1 + 2\pi R)$ as a bar.

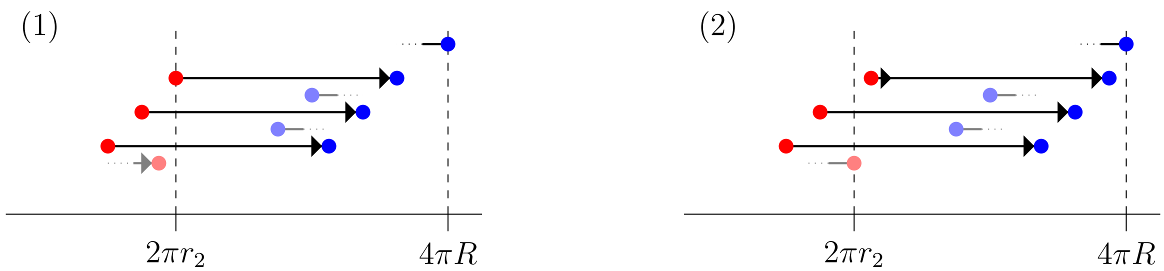

Figure 5.4: How $\mathcal{B}^n(h_t)$ can change between times before and after $\bar{t}$. In the first picture, the lowest red endpoint corresponds to the degree n action coming from the y-axis. The second picture shows this action eating into our bar after having switched places with the degree n action at $2\pi r_2$.

Remark 5.4. In fact, one may use the homotopy from the proof of Lemma 4.9 to show that the left-hand endpoint of our bar is *at least* $2\pi R - m_0 r_1$. In conjunction with the previous paragraph, we conclude that our bar is precisely of the form $[2\pi R - m_0 r_1, 2\pi r_1 + 2\pi R)$.

Next, consider the homotopy h_t^2 between g_1 and our function g given by

$$h_t^2(r) = \begin{cases} \max\{m_0(r - r_1) + 2\pi R(1-2t), g\}, & 0 \le r \le r_1 \\ g_1(r), & r_1 \le r \le r_3 \\ \max\{m_1(r - r_3) + 2\pi R(1-2t), g\}, & r_3 \le r \le R. \end{cases}$$

See Figure 5.5.

Recall that $2\pi R - m_0 r_1 < 2\pi R + \varepsilon$. Our next claim is:

Claim 5.5. *A bar of the form* $[c_t, 2\pi r_1 + 2\pi R)$*, where* $c_t < 2\pi R + \varepsilon$*, exists in* $\mathcal{B}^n(h_t^2)$ *for all time.*

Proof of Claim 5.5. The idea behind this proof is that, as can be seen from Figure 5.5, the only new degree $n+1$ actions which appear and change with time must be decreasing and either come from concave up kinks in our graph or are exterior actions. Actions of the former kind are unable to appear in the degree n barcode, and those of the latter are bigger than or equal to $4\pi R$ for all time, so our specified right-hand endpoint must be maintained. On

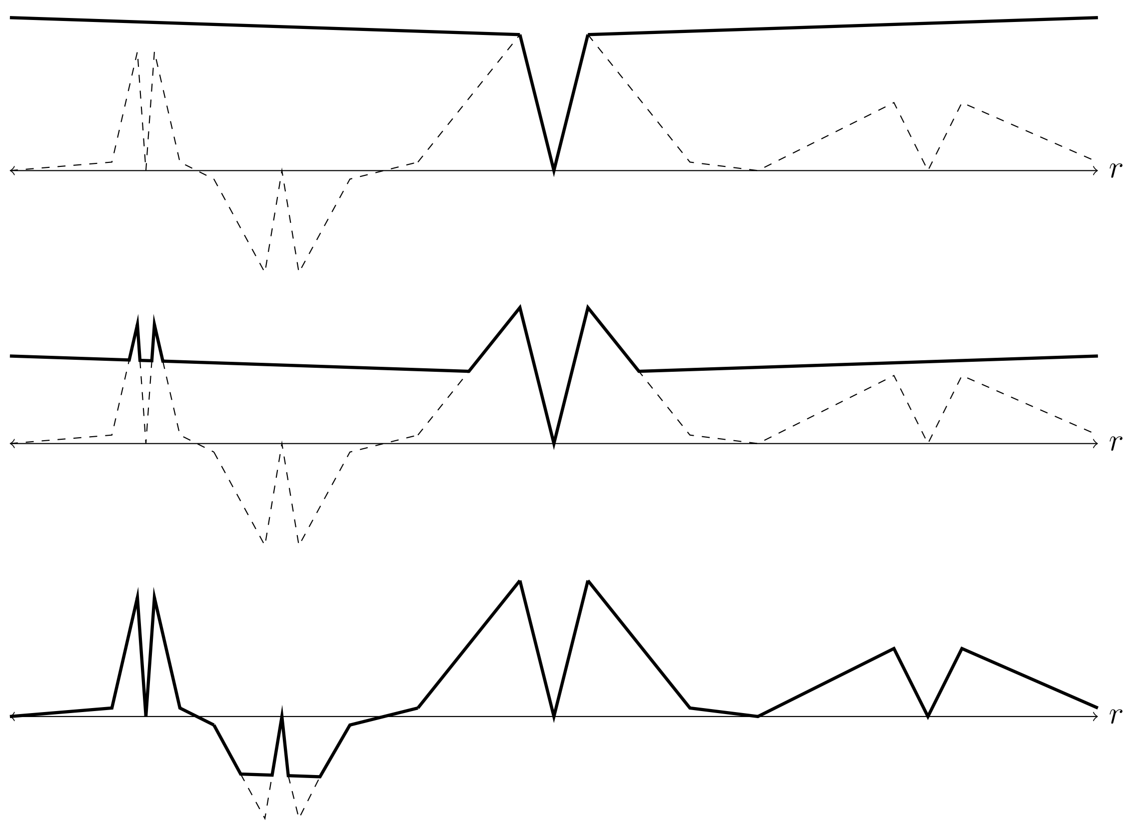

Figure 5.5: The solid graph is g_1 in the first picture, while the solid graphs in the second and third pictures are intermediate functions from our homotopy h_t^2.

the other hand, our left-hand endpoint can only decrease since all degree n actions which change with time are decreasing during this homotopy. It is an easy exercise to verify that the homotopy h_t^2 satisfies these properties. Figure 5.6 shows the evolution of $\mathcal{B}^n(h_t^2)$.

Remark 5.6. All degree $n+1$ exterior actions being greater than $4\pi R$ is due to our assumption that $4\pi R \leq \gamma$. Hence, we see here one instance of this assumption's necessity.

Define T by

$$T = \{t \in [0,1] \,|\, [c_t, 2\pi r_1 + 2\pi R) \in \mathcal{B}^n(h_t^2)\,,\, c_t \leq 2\pi r_2 + \varepsilon\}.$$

Similar to the previous claim's proof, we choose $\varepsilon' > 0$ and a small enough interval of time $(0, t_1)$ so that for any $t \in (0, t_1)$

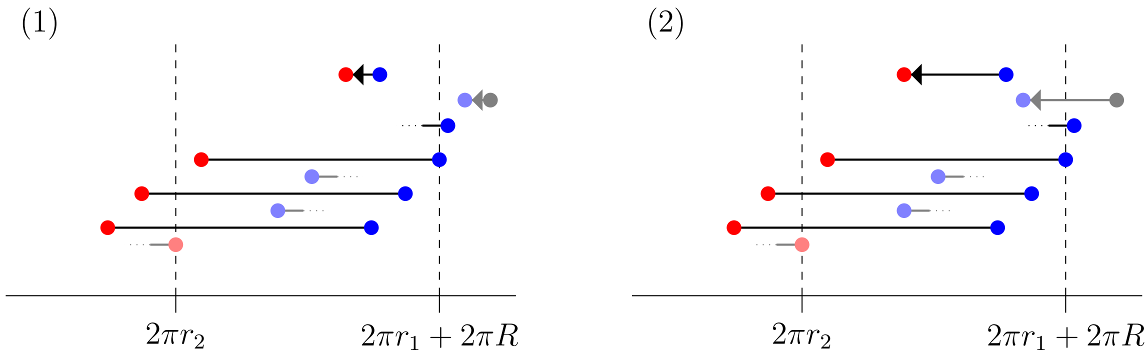

Figure 5.6: The evolution of $\mathcal{B}^n(h_t^2)$. Note that the bar on the far right does not have its left-hand endpoint switch with our chosen bar's right-hand endpoint because said left-hand endpoint is an action coming from a concave up kink.

- no new action values are created. Hence, we may parameterize all action values as functions of time with domain $(0, t_1)$.
- $||h_0^2 - h_t^2||_{L^\infty} < \varepsilon'$.
- If $c^{n+1}(t)$ is a parameterization of a degree $n+1$ action with domain $(0, t_1)$ which limits to a value in $(2\pi r_1 + 2\pi R - \varepsilon', 2\pi r_1 + 2\pi R + \varepsilon')$ as t goes to 0, then said limit is $2\pi r_1 + 2\pi R$. Furthermore, if such a $c^{n+1}(t)$ has $c^{n+1}(t') \neq 2\pi r_1 + 2\pi R$ for some $t' \in (0, t_1)$, then $c^{n+1}(t)$ is always less than $2\pi r_1 + 2\pi R$.
- Similarly, if $c^n(t)$ is a parameterization of a degree n action with domain $(0, t_1)$ which limits to a value in $(c_0 - \varepsilon', c_0 + \varepsilon')$ as t goes to 0, then said limit is c_0. Furthermore, if such a $c^n(t)$ has $c^n(t') \neq c_0$ for some $t' \in (0, t_1)$, then $c^n(t)$ is always less than c_0.

The third and fourth conditions may be met in this case because our actions are non-increasing with respect to time during $(0, t_1)$. Note our choice of notation c_t (instead of $c(t)$) for the left endpoint of our bar to avoid confusion with the parametrizations of our actions as mentioned in the first condition above.

From here, we prove that $T \neq \emptyset$. For any t' in $(0, t_1)$, we have an ε'-matching $\mu_{\varepsilon'}$ between $\mathcal{B}^n(h_0^2)$ and $\mathcal{B}^n(h_{t'}^2)$ which must match $2\pi r_1 + 2\pi R$ at time 0 with an action in the interval $(2\pi r_1 + 2\pi R - \varepsilon', 2\pi r_1 + 2\pi R + \varepsilon')$ at time t'. By the conditions above, the only degree $n+1$ actions in $(2\pi r_1 + 2\pi R - \varepsilon', 2\pi r_1 + 2\pi R + \varepsilon')$ which are not equal to $2\pi r_1 + 2\pi R$ at time t' are ones which change with time; such actions in our action window must correspond to concave up kinks in $h_{t'}$'s graph. Theorem 4.14 therefore states that these do not enter into $\mathcal{B}^n(h_{t'}^2)$, and so $2\pi r_1 + 2\pi R$ must be matched by $\mu_{\varepsilon'}$ with itself.

Similarly, $\mu_{\varepsilon'}$ must take c_0 to an action in $(c_0 - \varepsilon', c_0 + \varepsilon')$, and by construction the only degree n actions in this interval at time t' which are not equal to c_0 are those strictly less than it. Hence, any such $\mathcal{B}^n(h_{t'}^2)$ has a bar of the desired form, proving $T \neq \emptyset$.

With T non-empty, it has a supremum $t_s \in [0,1]$, and since T is closed (by an argument similar to the one presented at the end of Claim 5.3's proof), $t_s \in T$. Supposing $t_s \neq 1$, the above argument shows that there exists a $t_1 > t_s$ so that $[t_s, t_1) \in T$, contradicting t_s being a supremum. Hence $t_s = 1$, and our proof is complete.

□

Therefore, $\mathcal{B}^n(g)$ has a bar at least as big as $2\pi r_1 + 2\pi R - (2\pi R + \varepsilon) = 2\pi r_1 - \varepsilon$. Any standard perturbation $\tilde{G}$ of g which is less than ε away in the C^0 norm will therefore induce a diffeomorphism having boundary depth at least $2\pi r_1 - 2\varepsilon$. Using the notation of Theorem 3.5, such a standard perturbation will satisfy $|| \sum_{i=1}^{\infty} a_i \bar{f}_i - \tilde{G} || < 6\varepsilon$, and so Theorem 3.5 tells us that the Hamiltonian diffeomorphism generated by $\sum_{i=1}^{\infty} a_i \bar{f}_i$ will have boundary depth at least $2\pi r_1 - 8\varepsilon$, which is easily verified to be greater than $2\pi R - (4\pi + 7)\varepsilon$ since r_1 is greater than $R - \varepsilon$. This completes the proof of the case that $N \neq 0$, $\lambda > 0$, and $n\gamma - 2\pi NR \geq 0$.

Remark 5.7. It is now possible to see why our proofs cannot assert Theorem 1.2 with $[0,1]^\infty$ replaced by $[0, C]^\infty$ for some $C > 1$. Indeed, suppose we tried, so that our function g_1 has maximum $2C\pi R > 2\pi R$. Then as explained in Remark 5.4, our bar of choice in $\mathcal{B}^n(g_1)$ with right-hand endpoint $2\pi r_1 + 2C\pi R$ would have left-hand endpoint at least $2C\pi R - m_0 r_1$. In this best case scenario of $[2C\pi R - m_0 r_1, 2\pi r_1 + 2C\pi R)$ being a bar in $\mathcal{B}^n(g_1)$, the reasoning behind Claim 5.5 would still give $2\pi R$ as an approximate lower bound on the boundary depth of $\sum_{i=1}^{\infty} a_i \bar{f}_i$.

In fact, in the case that $B(2\pi R)$ is displaceable, we *must* have some impediment to $\sum_{i=1}^{\infty} a_i \bar{f}_i$ having arbitrarily large boundary depth, for the boundary depth of a Hamiltonian diffeomorphism is bounded above by twice the displacement energy of its support (see [35]).

The rest of this section is devoted to describing the changes necessary to the above proof when dealing with the various other cases. The only case necessitating any significant changes is the last one, when $N \neq 0$ and $\lambda = 0$.

Case 2 $N = 0$.

The proof given for case 1 can be applied almost directly to the case of $N = 0$, which by our monotonicity assumption implies that M is symplectically aspherical; indeed, the only difference is that our enumeration of h_t^1's actions would exclude those described by (A1), (A2), (A4) and (A8) and only consider the case $z = 0$ for those described by (A3) and (A5).

Case 3 $N \neq 0, \lambda > 0$, and $n\gamma - 2\pi NR < 0$.

In the case that $N \neq 0$, $\lambda > 0$, and $n\gamma - 2\pi N < 0$, we first pick our ε to ensure that $n\gamma - 2\pi N(R - \varepsilon) < 0$ and construct our functions f_i. Again assuming that $a_k = 1$, we choose g_1 and g to be as before, but define, $r(t)$, g_0, and our first homotopy h_t^1 by

$$r(t) = (1-t)r_2 + tr_3,$$

$$g_0(r) = \begin{cases} m_0(r - r_1) + 2\pi R, & 0 \leq r \leq r_1 \\ f_k(r), & r_1 \leq r \leq r_2 \\ m_1(r - r_2), & r_2 \leq r \leq R. \end{cases}$$

and

$$h_t^1(r) = \begin{cases} g_1(r), & 0 \le r \le r(t) \\ m_1(r - r(t)) + 2\pi R t, & r(t) \le r \le R. \end{cases}$$

Our strategy is to now use the homotopy h_t^1 and the continuity of the barcode to establish the existence of a bar either of the form $[-2\pi R + 4\pi\delta_3, 2\pi\delta_3)$ or $[-2\pi r_2, 2\pi\delta_3)$ in $\mathcal{B}^{-3n}(g_1)$. We enumerate the relevant degree $-3n$ and $-3n+1$ actions for h_t^1 below, where our notation is as before except that $r(t)$ now represents the right-most kink of h_t^1's graph.

(B1) Degree $-3n+1$ actions from r_1 have the form

$$2\pi R - 2\pi r_1 + \tfrac{2}{D} z(n\gamma - 2\pi N r_1),$$

which equals

$$2\pi\delta_1 + \tfrac{2}{D} z(n\gamma - 2\pi N r_1)$$

if we let $R - \delta_1 = r_1$. Here, we must have $z < 0$. With $n\gamma - 2\pi N r_1 < 0$, any such action must be strictly greater than $2\pi\delta_1$.

(B2) Degree $-3n$ actions coming from r_1 are of the form

$$2\pi R - 4\pi r_1 + \tfrac{2}{D} z(n\gamma - 2\pi N r(t)),$$

which equals

$$-2\pi R + 4\pi\delta_1 + \tfrac{2}{D} z(n\gamma - 2\pi N r(t)),$$

with $z < 0$. Hence, all such actions are no less than $-2\pi R + 4\pi\delta_1$.

(B3) Degree $-3n+1$ actions from $r(t)$ are of the form

$$2\pi Rt - 2\pi r(t) + \tfrac{2}{D}z(n\gamma - 2\pi Nr(t)).$$

We must have $z \geq 0$. Choosing $z = 0$ here gives the action which will become the right endpoint of our bar. Note that this action is $2\pi\delta_3$ at $t = 1$, which is strictly less than any degree $-3n+1$ action from (B1).

(B4) A degree $-3n$ action coming from $r(t)$ will have the form

$$2\pi Rt - 4\pi r(t) + \tfrac{2}{D}z(n\gamma - 2\pi Nr(t)).$$

We must have $z \geq 0$ here (more work than usual must be done to conclude this; see the reasoning following this enumeration). In particular, any such action is less than the action we get when we take $z = 0$: $2\pi Rt - 4\pi r(t)$. At $t = 1$, this is $-2\pi R + 4\pi\delta_3$. This is the action which may overtake our initial choice of degree $-3n$ action as t gets close to 1.

(B5) A degree $-3n$ action coming from r_2 will be of the form

$$-2\pi r_2 + \tfrac{2}{D}z(n\gamma - 2\pi Nr_2).$$

Choose $z = 0$ (giving $-2\pi r_2 = -2\pi R + 2\pi\delta_2$) to get the left endpoint of our bar for times t far enough away from 1. Note that this is strictly less than any degree $-3n$ action from (B2).

(B6) Any degree $-3n$ action coming from the y-axis will be at most

$$2\pi R - m_0(r_1) - \gamma \leq -2\pi R - m_0(r_1)$$

since we assume that $4\pi R \leq \gamma$. Making $|m_0|$ smaller if necessary, we can ensure that this is strictly less than our chosen degree $-3n$ action from (B5).

(B7) An exterior degree $-3n$ action will be at most

$$2\pi Rt + m_1(R - r(t)) - \gamma \leq 2\pi Rt + m_1(R - r(t)) - 4\pi R$$

for all times t. Taking $t = 1$ gives $-2\pi R + m_1(R - r_3) = -2\pi R + m_1\delta_3$ as our upper bound, and with $m_1 < 1$, this too is strictly less than our degree $-3n$ action from (B5).

(B8) Similarly, any exterior degree $-3n + 1$ actions at time t will be at most

$$2\pi Rt + m_1(R - r(t)) - 4\pi R.$$

With m_1 small, this is strictly less than our chosen degree $-3n + 1$ action from (B3) for all time.

Our claim concerning the possible values of z for (B3) is due to the following. In the case of (B3), our parametrization for l (compare with the analysis preceding (A1)) is

$$-l = -2 - \tfrac{2N}{D}z;$$

since we are at $r(t)$, we must have $-l < 0$. This gives that

$$-2 - \tfrac{2N}{D}z < 0,$$

and since $\frac{2N}{D}$ is a positive integer, we conclude that $z \geq -1$. Note that we may only include the case $z = -1$ if $2N = D$, and the equality $2N = D$ contradicts the assumptions $n\gamma - 2\pi NR < 0$ and $4\pi R \leq \gamma$. Indeed, the latter assumption yields the first step in the following chain of inequalities:

$$
\begin{aligned}
0 &\leq \gamma - 2\pi R \\
&\leq \tfrac{2n}{D}\gamma - 2\pi R \\
&= \tfrac{2n}{D}\gamma - \tfrac{2N}{D}2\pi R \\
&= \tfrac{2}{D}(n\gamma - 2\pi N R),
\end{aligned}
$$

where the second inequality from above uses that $\frac{2n}{D}$ is a positive integer.

Choose the degree $-3n+1$ action occurring at $r(t)$ with $z = 0$ (so $2\pi Rt - 2\pi r(t)$). Arguing as before, we may assume that $-2\pi r_2$ is not a degree $-3n+1$ action for h_0^1. Use Theorem 4.13 and the continuity in t of $B^{-3n}(h_t^1)$ to pair our chosen action with the degree $-3n$ action occurring at r_2 $(-2\pi r_2)$ for values of t close to 0, then follow the same reasoning as before to conclude that a bar of the form $[-2\pi R + 4\pi\delta_3, 2\pi\delta_3)$ or $[-2\pi r_2, 2\pi\delta_3)$ exists in $\mathcal{B}^{-3n}(g_1)$. Finally, choose h_t^2 as before and follow the reasoning previously given, but employing Theorem 4.14 and the fact that all exterior degree $-3n+1$ actions are strictly less than our chosen one as we perform h_t^2, to deduce that $\mathcal{B}^{-3n}(g)$ has a bar of length at least $2\pi r_2 - 2\pi\delta_3$. We may conclude from here that the boundary depth of the Hamiltonian diffeomorphism generated by $\sum_{i=1}^{\infty} a_i \bar{f}_i$ is at least $2\pi R - (4\pi + 7)\varepsilon$.

Case 4 $N \neq 0$ and $\lambda < 0$.

For the case of $N \neq 0$ with $\lambda < 0$, we choose our piecewise linear functions and homotopies exactly as in the case of $N \neq 0, \lambda > 0$, and $n\gamma - 2\pi NR \geq 0$. We list out the relevant degree $n+1$ and n actions for h_t^1 below, from which it should be easy to deduce the appropriate lower bound for the boundary depth.

(C1) Any degree $n+1$ action coming from r_3 will be of the form

$$4\pi R - 2\pi\delta_3 + \tfrac{2}{D}z(-n\gamma - 2\pi N(R - \delta_3)).$$

and we must have $z > 0$. Since $\frac{2}{D}(-n\gamma - 2\pi N(R - \delta_3))$ is strictly less than $-\gamma \leq -4\pi R$, any action described here is negative.

(C2) Any degree n action coming from r_3 will be of the form

$$2\pi R + \tfrac{2}{D}z(-n\gamma - 2\pi N r_3),$$

with $z > 0$. Similar to the case of (B1), any action here is negative.

(C3) Any degree $n+1$ coming from $r(t)$ will be of the form

$$2\pi r(t) + 2\pi Rt + \tfrac{2}{D}z(-n\gamma - 2\pi N r(t))$$

where we must have $z \leq 0$. Choose $z = 0$ to get our right-hand endpoint.

(C4) Any degree n action coming from $r(t)$ will be of the form

$$2\pi Rt + \tfrac{2}{D}z(-n\gamma - 2\pi N r(t)),$$

where we must have $z < 0$. Hence, all actions here are at least as big as $2\pi Rt + \gamma \geq 2\pi Rt + 4\pi R$.

(C5) Any degree n action coming from r_2 will be of the form

$$2\pi r_2 + \tfrac{2}{D}z(-n\gamma - 2\pi N r_2).$$

Choose $z = 0$ here to get our initial left-hand endpoint.

(C6) We have a degree n action coming from the y-intercept of the form $2\pi Rt - m_0 r(t)$. By adjusting m_0 if necessary, we may assume that $0 < -m_0 r(1) < \varepsilon$. This might eventually overtake our initial choice of degree n action.

(C7) We have a degree n exterior action of $2\pi R + m_1(R - r_3)$, while all others will be no more than $2\pi R + m_1(R - r_3) - \gamma \leq -2\pi R + m_1(R - r_3)$. Note that these actions never equal our degree n action of choice.

(C8) Any degree $n+1$ exterior actions will be no more than $-2\pi R + m_1(R - r_3)$, which is strictly less than our degree $n+1$ action of choice for all time.

Case 5 $N \neq 0$ and $\lambda = 0$.

Finally, we deal with the case that $\lambda = 0$. Where g_1 is as always, our strategy is to again establish the existence of a bar of the appropriate length in $\mathcal{B}^{-3n}(g_1)$ via the homotopy h_t^1 given for the case of $N \neq 0$ and $n\gamma - 2\pi NR < 0$. The actions are given below.

(D1) Degree $-3n+1$ actions from r_1 have the form

$$2\pi R - 2\pi r_1 + \tfrac{2}{D} z(-2\pi N r_1),$$

which equals

$$2\pi\delta_1 + \tfrac{2}{D} z(-2\pi N r_1),$$

if we let $R - \delta_1 = r_1$. Here, we must have $z < 0$. With $-2\pi N r_1 < 0$, any such action must be strictly greater than $2\pi\delta_1$.

(D2) Degree $-3n$ actions coming from r_1 are of the form

$$2\pi R - 4\pi r_1 + \tfrac{2}{D} z(-2\pi N r(t)),$$

which equals

$$-2\pi R + 4\pi\delta_1 + \tfrac{2}{D} z(-2\pi N r(t)),$$

with $z < 0$. Hence, all such actions are no less than $-2\pi R + 4\pi\delta_1$.

(D3) Degree $-3n+1$ actions from $r(t)$ are of the form

$$2\pi Rt - 2\pi r(t) + \tfrac{2}{D}z(-2\pi N r(t)).$$

We must have $z \geq 0$. Choosing $z = 0$ here gives the action which will become the right endpoint of our bar.

(D4) A degree $-3n$ action coming from $r(t)$ will have the form

$$2\pi Rt - 4\pi r(t) + \tfrac{2}{D}z(-2\pi N r(t)).$$

We must have $z \geq -1$ here.

(D5) A degree $-3n$ action coming from r_2 will be of the form

$$-2\pi r_2 + \tfrac{2}{D}z(-2\pi N r_2).$$

Choose $z = 0$ to get the left endpoint of our bar for times t far enough away from 1.

(D6) Any degree $-3n$ action coming from the y-axis will be precisely

$$2\pi R - m_0(r_1).$$

(D7) An exterior degree $-3n$ action will be equal to

$$2\pi Rt + m_1(R - r(t)).$$

for all times t.

(D8) Similarly, any exterior degree $-3n+1$ actions will be equal to

$$2\pi Rt + m_1(R - r(t)).$$

What makes this case slightly more difficult than the others occurs when $N = 1$. Supposing so, the degree $-3n$ action given by (D4) with $z = -1$ will be equal to our chosen degree $-3n+1$ action for all time, so we may not apply Theorem 4.13 directly. However, another energy argument as presented in the proof of Lemma 4.9 when $\lambda = 0$ shows that we must still have our degree $-3n+1$ action pairing with the usual degree $-3n$ action for times t close to zero. The rest of the proof for this case matches those of the other cases, though now we must worry about an exterior degree $-3n+1$ action overtaking our chosen degree $-3n+1$ action as we perform h_t^2. But this would give a bar in $\mathcal{B}^{-3n}(g)$ of the form $[-2\pi R + 4\pi\delta_3, g(0))$, the length of which is at least $2\pi R - 4\pi\delta_3 > 2\pi R - (4\pi + 7)\varepsilon$.

$\square$

Chapter 6

Adaptation: Embeddings of $\mathbb{R} \oplus [0,1]^\infty$

We begin by defining the terms in "stable homogeneous Calabi quasi-morphism." Where G is a group, a *quasi-morphism* μ from G into $\mathbb{R}$ is a function for which there exists a finite constant C such that, for any $g, h \in G$, we have

$$|\mu(gh) - \mu(g) - \mu(h)| \leq C.$$

A quasi-morphism is *homogeneous* if we additionally have $\mu(g^m) = m\mu(g)$ for any $g \in G$ and $m \in \mathbb{Z}$. For us, the group G will be $Ham(M, \omega)$ for a suitable closed symplectic manifold (M, ω). A quasi-morphism μ on $Ham(M, \omega)$ is called *stable* (as in [9]) if for ϕ_F, ϕ_H generated by normalized F, H respectively, we have

$$\mathrm{Vol}(M) \cdot \int_0^1 \min_M (F_t - H_t)\, dt \ \leq\ \mu(\phi_H) - \mu(\phi_F) \ \leq\ \mathrm{Vol}(M) \cdot \int_0^1 \max_M (F_t - H_t)\, dt,$$

where $\mathrm{Vol}(M)$ is the symplectic volume of M. A Hamiltonian F being *normalized* means that $\int_M F_t \omega^n = 0$ for all $t \in [0,1]$. The stability assumption is important to us because it forces μ to be continuous with respect to the Hofer metric:

Claim 6.1. *If a quasi-morphism μ on $Ham(M, \omega)$ is stable, then*

$$|\mu(\phi) - \mu(\psi)| \leq \mathit{Vol}(M) \cdot ||\phi^{-1}\psi||_H.$$

Proof of Claim 6.1. Let general Hamiltonian diffeomorphisms ϕ_F and ϕ_H be generated by Hamiltonians F and H respectively, and note that the functions F', H' with $F'(t, \cdot) = F_t -$

$\frac{1}{\text{Vol}(M)}\int_M F_t\omega^n$, $H'(t,\cdot) = H_t - \frac{1}{\text{Vol}(M)}\int_M H_t\omega^n$ are normalized Hamiltonians generating ϕ_F, ϕ_H. Then the Hamiltonian $F' \circ \phi^t_{H'} - H'_t \circ \phi^t_{H'}$ generates the composition $\phi_H^{-1}\phi_F$. We can rearrange the terms in the definition of μ's stability and use the obvious equalities $\min_M(F'_t - H'_t) = \min_M\{(F'_t - H'_t)\circ\phi^t_{H'}\}$ and $\max_M(F'_t - H'_t) = \max_M\{(F'_t - H'_t)\circ\phi^t_{H'}\}$ to obtain

$$\mu(\phi_H) - \mu(\phi_F) \ \le\ \text{Vol}(M)\cdot\left(\int_0^1 \max_M\{(F'_t - H'_t)\circ\phi^t_{H'}\} - \min_M\{(F'_t - H'_t)\circ\phi^t_{H'}\}\,dt\right).$$

However, our integrand is precisely equal to $\max_M\{(F_t - H_t)\circ\phi^t_H\} - \min_M\{(F_t - H_t)\circ\phi^t_H\}$, so we make this replacement and take the infimum over all F, H generating ϕ_F, ϕ_H to get $\mu(\phi_H) - \mu(\phi_F) \le \text{Vol}(M)\cdot||\phi_H^{-1}\phi_F||_H$. Our proof is complete by similarly obtaining the inequality

$$\begin{aligned}\mu(\phi_F) - \mu(\phi_H) &\le \text{Vol}(M)\cdot||\phi_F^{-1}\phi_H||_H\\ &= \text{Vol}(M)\cdot||(\phi_H^{-1}\phi_F)^{-1}||_H\\ &= \text{Vol}(M)\cdot||\phi_H^{-1}\phi_F||_H.\end{aligned}$$

□

Where U is a proper, nonempty open subset of M, the subgroup $Ham(U,\omega)$ of $Ham(M,\omega)$ consists of all Hamiltonian diffeomorphisms ϕ generated by smooth $F : [0,1]\times M \to \mathbb{R}$ with support contained in $[0,1]\times U$. When ω restricts to U as an exact form, we get a well-defined homomorphism $Cal_U : Ham(U,\omega) \to \mathbb{R}$, the *Calabi homomorphism* [2], by setting

$$Cal_U(\phi) = \int_0^1 \left(\int_M F_t\omega^n\right) dt,$$

where F is any function as above which generates ϕ. As in [6], we call a quasi-morphism μ on $Ham(M,\omega)$ *Calabi* if, for any non-empty displaceable open subset U of M on which ω is exact, μ coincides with Cal_U when restricted to $Ham(U,\omega)$.

Therefore, if (M,ω) is such that $Ham(M,\omega)$ and $B(2\pi R)$ satisfy the hypotheses of Theorem 1.3, any autonomous function F supported in $B(2\pi R)$ will induce a Hamiltonian diffeomorphism ϕ_F with Hofer norm at least $\frac{1}{\mathrm{Vol}(M)}|\int_M F\,\omega^n|$. Indeed, with F as such, $B(2\pi R)$'s displaceability gives

$$\begin{aligned}\mu(\phi_F) =& Cal_U(\phi_F)\\ =& \int_0^1 \left(\int_M F_t\omega^n\right) dt\\ =& \int_M F\omega^n\end{aligned}$$

where μ is our assumed quasi-morphism, and μ's continuity with respect to d_H gives $|\int_M F\omega^n| \leq \mathrm{Vol}(M)\cdot ||\phi_F||_H$. Choosing such an F with $\int_M F\omega^n \neq 0$, we may construct the one-parameter family of Hamiltonian diffeomorphisms ϕ_{sF} whose Hofer norm grows at least linearly in s, i.e. $||\phi_{sF}||_H \geq \frac{s}{\mathrm{Vol}(M)}|\int_M F\,\omega^n|$.

Setting $\psi_s = \phi_{sF}$ for all $s \in \mathbb{R}$, results from [6] show that the path $\{\psi_s\}$ in $\widetilde{Ham}(M,\omega)$ also has $||\widetilde{\{\psi_s\}}||_H \geq \frac{s}{\mathrm{Vol}(M)}|\int_M F\,\omega^n|$ when $\widetilde{Ham}(M,\omega)$ admits a stable homogeneous Calabi quasi-morphism $\tilde{\mu}$. On the other hand, if $\widetilde{Ham}(M,\omega)$ does not admit such a $\tilde{\mu}$, we may use Proposition 7.1.A from [26] to assert that this linear growth still occurs if M has a *stably non-displaceable* Lagrangian L with *supp*$(F) \cap L = \emptyset$. We now begin to prove Theorems 1.3 and 1.6.

First, suppose $Ham(M,\omega)$ admits a Calabi quasi-morphism μ and that our symplectic ball $B(2\pi R)$ is displaceable in M. Pick an $\varepsilon > 0$ and construct the functions $\bar{f}_i$, $i \geq 1$, which define our embedding from Theorem 1.2. Afterwards, define a function $f_0 : [0,R] \to \mathbb{R}$ satisfying

- f_0 is 0 at $R-3\varepsilon$ and on the interval $[R-\varepsilon-\delta, R]$ with δ very small.
- f_0 is $2\pi R$ at $1-2\varepsilon$.
- f_0 is $2\pi R$ on the interval $[0, R-4\varepsilon]$.

- f_0 is linear and increasing on $[R-3\varepsilon, R-2\varepsilon]$ with slope an irrational multiple of 2π.

- f_0 is linear and decreasing, with slopes irrational multiples of 2π, on $[R-4\varepsilon, R-3\varepsilon]$ and $[R-2\varepsilon, R-\varepsilon-\delta]$.

The integral over M of the induced C^0 function $F_0 : M \to \mathbb{R}$ is strictly greater than $2\pi R \cdot \mathrm{Vol}(B_{R-4\varepsilon})$, where we use $B_{R-4\varepsilon}$ to abbreviate $B(2\pi(R-4\varepsilon))$. Therefore, we may choose a smoothing $\bar{f}_0$ of f_0, also supported in $[0, R-\varepsilon]$, so that the induced Hamiltonian $\bar{F}_0$ has its integral satisfying the same inequality.

Now consider $\mathbb{R} \oplus [0,1]^\infty$, the set of all sequences $a = \{a_i\}_{i\geq 0}$ with $a_0 \in \mathbb{R}$, $a_i \in [0,1]$ when $i \geq 1$, and with only finitely many non-zero entries. Similar to our definition of Φ, we may define a new map $\overline{\Phi} : \mathbb{R} \oplus [0,1]^\infty$ into $Ham(M,\omega)$ by making $\overline{\Phi}(a)$ equal to the Hamiltonian diffeomorphism generated by $\sum_{i=0}^\infty a_i \bar{f}_i$. We now prove Theorem 1.3.

Proof of Theorem 1.3. Let $\bar{F}_i$, $i \geq 0$ be the Hamiltonians induced by the $\bar{f}_i$, and use $V_{4\varepsilon}$ to denote the difference in the symplectic volumes of $B(2\pi R)$ and $B_{R-4\varepsilon}$. Similar to the proof of our main theorem, Theorem 1.3 will be established if we show

$$\frac{2\pi R \cdot \mathrm{Vol}(B_{R-4\varepsilon})}{\mathrm{Vol}(M)} \left(\max_{i\geq 0}\{|a_i|\} \right) - \max\left\{ (4\pi+7)\varepsilon,\ \frac{2\pi R \cdot V_{4\varepsilon}}{\mathrm{Vol}(M)} \right\} \leq ||\phi_{\bar{F}}||_H$$

for $\bar{F} = \sum_{i=0}^\infty a_i \bar{F}_i$, where $a = \{a_i\}_{i\geq 0} \in \mathbb{R} \oplus [-1,1]^\infty$. This will give us the left-hand inequality of our theorem, while the right-hand inequality is again obvious. In the case that $|a_0| \neq \max_{i\geq 0}\{|a_i|\}$, we may use the proofs of Theorem 5.1 and 5.2 to again arrive at the inequality

$$2\pi R(\max_{i\geq 0}\{|a_i|\}) - (4\pi+7)\varepsilon \leq ||\phi||_H,$$

from which our desired inequality follows. On the other hand, suppose $|a_0| = \max_{i\geq 0}|a_i|$. Then the Hamiltonian $\sum_{i=1}^\infty a_i \bar{F}_i$ on M is supported in a region of M with volume $V_{4\varepsilon}$ and has absolute value bounded above by $2\pi R$. From the above discussion, we therefore have

$$
\begin{aligned}
||\phi_{\bar F}||_H &\geq \frac{1}{\mathrm{Vol}(M)}\Bigg(|\textstyle\int \bar F\,\omega^n|\Bigg)\\
&\geq \frac{1}{\mathrm{Vol}(M)}\Bigg(|\textstyle\int a_0\bar F_0\,\omega^n| \;-\; |\int \displaystyle\sum_{i=1}^{\infty} a_i\bar F_i\,\omega^n|\Bigg)\\
&\geq \frac{1}{\mathrm{Vol}(M)}\Bigg(|a_0|\cdot \textstyle\int \bar F_0\,\omega^n \;-\; \int |\displaystyle\sum_{i=1}^{\infty} a_i\bar F_i|\,\omega^n\Bigg)\\
&\geq \frac{1}{\mathrm{Vol}(M)}\Bigg(|a_0|\cdot 2\pi R\cdot \mathrm{Vol}(B_{R-4\varepsilon}) \;-\; 2\pi R\cdot V_{4\varepsilon}\Bigg)\\
&= \frac{2\pi R\cdot \mathrm{Vol}(B_{R-4\varepsilon})}{\mathrm{Vol}(M)}\Bigg(|a_0|\Bigg) - \frac{2\pi R\cdot V_{4\varepsilon}}{\mathrm{Vol}(M)}.
\end{aligned}
$$

□

As for Theorem 1.6, take our functions $\bar F_i, i \geq 0$ as before, and assume that (M,ω) and $B(2\pi R)$ satisfy the appropriate hypotheses. Then for $a \in \mathbb{R} \oplus [0,1]^\infty$, set $\widetilde{\Phi}(a)$ equal to the *path* of Hamiltonian diffeomorphisms generated by the Hamiltonian $\sum_{i=0}^{\infty} a_i\bar F_i$. With $\widetilde{\Phi}$ so defined, the proof of Theorem 1.6 follows that of Theorem 1.3, thanks in part to the obvious inequality

$$||\phi_1||_H \leq ||\widetilde{\{\phi_t\}}||_H$$

for an element $\{\phi_t\}$ of $\widetilde{Ham}(M,\omega)$.

Chapter 7

Boundary depth for $Aut(S^2, \omega)$

The goal of this chapter is to prove Theorem 1.7. Recall that we use $Aut(S^2, \omega)$ to denote the collection of elements ϕ in $Ham(S^2, \omega)$ which are generated by autonomous smooth functions $F : S^2 \to \mathbb{R}$. We shall call such ϕ *autonomous Hamiltonian diffeomorphisms*.

As mentioned in Remark 1.5, it is still unknown whether $Ham(S^2, \omega)$ lies in some infinite cylinder of a fixed radius. In an attempt to answer this question in the negative, one may be motivated to seek a 1-parameter family ψ_t of Hamiltonian diffeomorphisms whose boundary depth (and therefore Hofer norm) grow to be arbitrarily large with t and combine this with the 1-parameter family from Chapter 6. Indeed, our function $\bar{F}_0$ from Chapter 6 is supported in a displaceable subset, so $\beta(\phi_{s\bar{F}_0})$ has a bound B independent of s as explained in Remark 5.7. Therefore, if we can find some family ψ_t satisfying $\beta(\psi_t) \to \infty$ with t, we can use Theorem 3.6 (ii) as follows

$$\begin{aligned} \beta(\psi_t) - B &\leq |\beta(\psi_t) - \beta(\phi_{s\bar{F}_0})| \\ &\leq d_H(\psi_t, \, \phi_{s\bar{F}_0}) \end{aligned}$$

to conclude that the distance from ψ_t to the 1-parameter family $\phi_{s\bar{F}_0}$ grows arbitrarily large as t increases, and so $Ham(S^2, \omega)$ could not be contained in such a cylinder. As previously remarked, Theorem 1.7 states that such a ψ_t must be generated by a time-dependent Hamiltonian.

We can also see at this point why our claim following the statement of Theorem 1.7 holds: simply apply Theorem 3.6 (ii) as above in conjunction with Theorem 1.7 to conclude that the distance from $Aut(S^2, \omega)$ in $Ham(S^2, \omega)$ is unbounded.

One of the tools for our proof of Theorem 1.7 is the *Lagrangian Hofer metric* $\delta_{\mathcal{L}}$. For a fixed closed Lagrangian submanifold L of a closed symplectic manifold (M, ω), we let $\mathcal{L}(L)$ denote the set of all closed Lagrangians $L' \subset M$ (identified by reparametrization) such that $L' = \phi(L)$ for some $\phi \in Ham(M, \omega)$. One may then define $\delta_{\mathcal{L}}(L', L'')$ for $L', L'' \in \mathcal{L}(L)$ as follows:

$$\delta_{\mathcal{L}}(L', L'') = \inf_{\phi \in Ham(M,\omega)} \{ ||\phi||_H \mid \phi(L') = L'' \}.$$

It is easy to see that $\delta_{\mathcal{L}}$ defines a pseudo-metric on any $\mathcal{L}(L)$, and its non-degeneracy when (M, ω) is *tame*, as is always the case when (M, ω) is closed, was proven by Chekanov in [3]. Furthermore, the conjugation invariance of the Hofer norm implies that $\delta_{\mathcal{L}}(\cdot, \cdot)$ is invariant under Hamiltonian diffeomorphisms, i.e. for any $\psi \in Ham(M, \omega)$ we have

$$\delta_{\mathcal{L}}(L', L'') = \delta_{\mathcal{L}}(\psi(L'), \psi(L'')).$$

We have the following:

Lemma 7.1. *Let L denote the equator in S^2. Then for any $\phi \in Ham(S^2, \omega)$, we have*

$$\beta(\phi) \leq \delta_{\mathcal{L}}(\phi(L), L) + 2\pi.$$

Proof. Let $\varepsilon > 0$. For a given $\phi \in Ham(S^2, \omega)$, let $\psi \in Ham(S^2, \omega)$ be such that $| \, ||\psi||_H - \delta_{\mathcal{L}}(\phi(L), L)| < \varepsilon$ and $\psi(L) = \phi(L)$. Assuming $\psi^{-1}\phi$ sends the Northern and Southern hemispheres N and S to themselves, we may Hofer-approximate $\psi^{-1}\phi$ to get a map α which is the identity on a small tubular neighborhood of L. On the other hand, if $\psi^{-1}\phi$ interchanges N and S, we may first apply a half-rotation R of S^2 with axis through L and then Hofer-approximate $R\psi^{-1}\phi$ to again get a map (which we also call α) equal to the identity on a neighborhood of L. Applying R to α from either case gives a Hamiltonian diffeomorphism of S^2 with precisely two fixed points, both of which lie on L and are non-degenerate. For a closed symplectic manifold (M, ω), Theorem 1.6 of [35] states that the boundary depth of a non-degenerate $\phi \in Ham(M, \omega)$ is zero precisely when the number of its fixed points equals $\sum_{k=0}^{2n} rank H_k(M, \mathbb{Q})$ (where the appearance of $\mathbb{Q}$ is due to our total Floer chain complex taking coefficients in this field). So $\beta(R\alpha) = 0$, which by continuity makes $\beta(R\psi^{-1}\phi)$ or $\beta(R^2\psi^{-1}\phi) = \beta(\psi^{-1}\phi)$ zero.

If $\beta(R\psi^{-1}\phi) = 0$, we use Theorem 3.6 (ii) to get

$$\begin{aligned}
|\beta(R\psi^{-1}\phi) - \beta(\phi)| &\leq ||R\psi^{-1}||_H \\
&\leq ||R||_H + ||\psi^{-1}||_H \\
&\leq ||R||_H + \delta_{\mathcal{L}}(\phi(L), L) + \varepsilon.
\end{aligned}$$

The first term in this chain of inequalities is $|\beta(\phi)|$ by assumption, and the half-rotation R has Hofer norm 2π. Let ε go to zero to achieve the desired inequality. The proof for when $\beta(\psi^{-1}\phi) = 0$ is similar.

□

Another key ingredient in our proof of β's boundedness on $Aut(S^2, \omega)$ is the notion of the *median* of a Morse function F on S^2, as presented by Entov and Polterovich in [6], and a brief review of this notion is necessary before we may prove Theorem 1.7. A measured tree (T, ρ) is a finite tree T with a non-atomic Borel probability measure ρ such that, for each open edge e of T, (e, ρ) is homeomorphic to the Lebesgue measure on an open interval. It is shown in [6] that every measured tree (T, ρ) has a unique point m such that every connected component of $T \setminus \{m\}$ has measure not exceeding $\frac{1}{2}$, and this point is called the measured tree's *median*.

Now recall that the *Reeb graph* associated to a Morse function F on a closed surface Σ is a finite graph T whose points correspond to connected components of F's level sets, which we refer to as level curves. Specifically, the vertices of this graph correspond to those level curves containing critical points, while the edges correspond to the connected open cylinders formed by those level curves without critical points. This graph is clearly a tree T when our surface is S^2, and as Entov and Polterovich do in [6], we may define a measure [1] ρ on T as follows: For x, y belonging to the same open edge in T, define $\rho([x, y])$ to be the area of the cylinder in S^2 whose boundary consists of the level curves corresponding to x and y, and set $\rho(T) = 4\pi$, the total area of the sphere. This makes $(T, \frac{1}{4\pi}\rho)$ into a measured tree with a median. In relation to our Morse function, the level curve of F corresponding to the median m of (T, ρ) is the one cutting S^2 into regions of area not exceeding 2π, and it is this curve γ_m which we shall refer to as the median of the Morse function F.

Proof of Theorem 1.7. Let ϕ be generated by the autonomous $F : S^2 \to \mathbb{R}$. By Hofer-approximating, we may take F to be a Morse function with critical points having distinct critical values; F therefore has a median γ_m, and we may assume that F is zero on γ_m since Hamiltonian isotopies are invariant under addition of a constant to the corresponding Hamiltonian. Assume for now that γ_m contains a critical point, making $S^2 \backslash \gamma_m$ consist of three regions each with area less than 2π. We may further Hofer-approximate ϕ by a $\tilde{\phi} \in Aut(S^2, \omega)$ generated by $\tilde{F} : S^2 \to \mathbb{R}$ for which there exist small neighborhoods $V \subset U$ of γ_m with $\tilde{F}$ zero on V and equal to F outside of U. Our new function $\tilde{F}$ is therefore expressable as the sum of three autonomous functions F_1, F_2, and F_3 with respective supports contained in disks D_1, D_2, and D_3, each of which is contained in one of the three regions of $S^2 \setminus \gamma_m$. Letting ϕ_1, ϕ_2, and ϕ_3 be the corresponding Hamiltonian diffeomorphisms, the pairwise disjointness of each of the D_i's implies $\tilde{\phi} = \phi_1\phi_2\phi_3$.

Choose three arcs α_i, $i = 1, 2, 3$ (half portions of great circles connecting a point p on L to its antipodal point) to divide S^2 into three open regions (slices) S_1, S_2, and S_3 with the area of each S_i being greater than the area of its respective D_i and smaller than 2π. We make use of the following claim but save its proof for the end of this chapter.

Claim 7.2. *There exists a symplectomorphism ψ of (S^2, ω) which sends each D_i into its respective S_i.*

[1] Technically, Entov and Polterovich take the sphere to have total area one, making ρ a probability measure and (T, ρ) a measured tree.

The function $\tilde{F} \circ \psi^{-1}$ generates the Hamiltonian diffeomorphism $\psi\tilde{\phi}\psi^{-1} = \psi\phi_1\phi_2\phi_3\psi^{-1}$. By β's invariance under conjugation and Lemma 7.1, we therefore have

$$\begin{aligned}\beta(\tilde{\phi}) = \beta(\psi\tilde{\phi}\psi^{-1}) &\leq \delta_{\mathcal{L}}(\psi\tilde{\phi}\psi^{-1}(L), L) + 2\pi \\ &= \delta_{\mathcal{L}}(\psi\phi_1\phi_2\phi_3\psi^{-1}(L), L) + 2\pi \\ &= \left(\sum_{i=1}^{3} \delta_{\mathcal{L}}(\psi\phi_i\psi^{-1}(L), L)\right) + 2\pi\end{aligned}$$

Theorem 1.7 for the case that γ_m contains a critical point then follows from each $\delta_{\mathcal{L}}(\psi\phi_i\psi^{-1}(L), L)$ being less than 4π. Indeed, where R_i is a rotation of the sphere displacing L from S_i, we have $\psi\phi_i^{-1}\psi^{-1}R_i(L) = R_i(L)$. The triangle inequality for $\delta_{\mathcal{L}}$ and its invariance under Hamiltonian diffeomorphisms then gives

$$\begin{aligned}\delta_{\mathcal{L}}(\psi\phi_i\psi^{-1}(L), L) &\leq \delta_{\mathcal{L}}(\psi\phi_i\psi^{-1}(L), R_i(L)) + \delta_{\mathcal{L}}(R_i(L), L) \\ &= \delta_{\mathcal{L}}(L, \psi\phi_i^{-1}\psi^{-1}R_i(L)) + \delta_{\mathcal{L}}(R_i(L), L) \\ &= \delta_{\mathcal{L}}(L, R_i(L)) + \delta_{\mathcal{L}}(R_i(L), L) \\ &\leq 2||R_i||_H,\end{aligned}$$

and each R_i has Hofer norm at most 2π since at most half of a rotation is needed to displace L from any of the S_i.

When γ_m does not contain a critical point of F (making it a simple closed curve), the proof of Theorem 1.7 is simpler. Since γ_m splits S^2 into two regions, its definition implies that each region has area equal to 2π. The proof of Claim 7.2 can be adapted to show that any two simple closed curves which divide S^2 into two equal areas are related by a symplectomorphism ψ, so there exists ψ with $\psi(\gamma_m) = L$. Recall from Chapter 1 that Hamiltonian diffeomorphisms leave invariant the level sets of the autonomous Hamiltonians which induce them. So $\phi(\gamma_m) = \gamma_m$, making $\psi\phi\psi^{-1}(L) = L$. Apply the conjugation invariance of β and Lemma 7.1 as in the previous case:

$$\begin{aligned}\beta(\phi) = \beta(\psi\phi\psi^{-1}) &\leq \delta_{\mathcal{L}}(\psi\phi\psi^{-1}(L), L) + 2\pi \\ &= \delta_{\mathcal{L}}(L, L) + 2\pi \\ &= 2\pi.\end{aligned}$$

$\square$

Proof of Claim 7.2. Denote by C_i the boundary of each of the D_i, and where σ is an orientation preserving diffeomorphism from S^2 to itself sending each D_i into its respective S_i, examine the symplectic form $\sigma^*\omega$. Our strategy is to use a relative version of a standard argument in symplectic topology (the Moser trick) to find another orientation preserving diffeomorphism ρ which fixes setwise each C_i and makes $\sigma \circ \rho$ a symplectomorphism.

Integral to the Moser trick is that the symplectic forms ω and $\sigma^*\omega$ are cohomologous in $H^2_{dR}(S^2, \cup_i C_i)$. For a closed manifold M with closed submanifold S, the relative de Rham cohomology $H^k_{dR}(M, S)$ may be defined as the cohomology of the complex $(\Omega^k(M, S), d)$, where $\Omega^k(M, S)$ is the set of smooth k-forms on M which vanish when restricted to S and d is the usual exterior derivative. Where $j^* : \Omega^k(M, S) \to \Omega^k(M)$ is the natural inclusion map and i^* is induced by the inclusion $i : S \to M$, we have a series of commutative diagrams of the following form, with each row being a short exact sequence (see [15]):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k(M,S) & \xrightarrow{j^*} & \Omega^k(M) & \xrightarrow{i^*} & \Omega^k(S) & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & \Omega^{k+1}(M,S) & \xrightarrow{j^*} & \Omega^{k+1}(M) & \xrightarrow{i^*} & \Omega^{k+1}(S) & \longrightarrow & 0 \end{array}$$

We may therefore build in the usual fashion (see [11], for instance) a long exact sequence

$$\ldots \xrightarrow{i^*} H^{k-1}_{dR}(S) \xrightarrow{\partial} H^k_{dR}(M, S) \xrightarrow{j^*} H^k_{dR}(M) \xrightarrow{i^*} H^k_{dR}(S) \xrightarrow{\partial} \ldots,$$

after which we restrict our attention to the following portion of this long exact sequence with $M = S^2$ and $S = \cup C_i$:

$$
\begin{array}{ccccccccc}
H^1_{dR}(S^2) & \xrightarrow{i^*} & H^1_{dR}(\cup C_i) & \xrightarrow{\partial} & H^2_{dR}(S^2, \cup C_i) & \xrightarrow{j^*} & H^2_{dR}(S^2) & \xrightarrow{i^*} & H^2_{dR}(\cup C_i) \\
\wr\| & & \wr\| & & & & \wr\| & & \wr\| \\
0 & & \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} & & & & \mathbb{R} & & 0
\end{array}
$$

The first (non-trivial) isomorphism from the left is given by $[\alpha] \mapsto (\int_{C_1} \alpha, \int_{C_2} \alpha, \int_{C_3} \alpha)$, while the isomorphism from $H^2_{dR}(S^2)$ to $\mathbb{R}$ is given by $[\beta] \mapsto \int_{S^2} \beta$.

The form $\sigma^*\omega - \omega$ being exact will follow if we can show that any cohomology class $[\beta] = H^2_{dR}(S^2, \cup C_i)$ with $\int_{U_i} \beta = 0$ for each i must be the trivial cohomology class. Such a $[\beta]$ must be in the image of ∂ since $[j^*\beta] = 0 \in H^2_{dR}(S^2)$. Recalling the construction of the map ∂, a class $[\alpha] \in H^1_{dR}(\cup C_i)$ maps to $[\beta]$ if there exists a 1-form $\alpha' \in \Omega^1(S^2)$ with $\alpha = i^*\alpha'$ and $d(\alpha') = j^*(\omega)$. We use this definition, our hypothesis on β, and Stokes' Theorem to conclude that any such $[\alpha]$ must be trivial:

$$\int_{C_i} \alpha = \int_{C_i} i^*\alpha' = \int_{U_i} j^*\omega = 0.$$

So $[\beta] = 0$.

Proceeding with the Moser trick, let $d\alpha = \sigma^*\omega - \omega$ with $\alpha \in \Omega^1(S^2, \cup C_i)$, and set $\omega_t = t\sigma^*\omega + (1-t)\omega$. For each t, ω_t is a symplectic form since $\sigma^*\omega$ and ω are area forms on S^2 corresponding to the same orientation. By the non-degeneracy of each ω_t, then, we can solve the following equation for the time-dependent vector field v_t:

$$\iota_{v_t}\omega_t + \alpha = 0.$$

Note here that such a v_t must be tangent to $\cup C_i$ since α vanishes on this submanifold, so the isotopy ρ_t generated by v_t fixes setwise each C_i. Apply formula (†) from Chapter 1 to $\frac{d}{dt}(\rho_t^*\omega_t)$:

$$\begin{aligned}
\frac{d}{dt}(\rho_t^*\omega_t) &= \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d}{dt}\omega_t\right) \\
&= \rho_t^*\left(d\iota_{v_t}\omega_t + \iota_{v_t}d\omega + \psi^*\omega - \omega\right) \\
&= \rho_t^*(d\iota_{v_t}\omega_t + d\alpha) \\
&= d\rho_t^*\left(\iota_{v_t}\omega_t + \alpha\right) \\
&= 0.
\end{aligned}$$

Thus $\rho_t^*\omega_t$ is equal to ω for all t, making $(\sigma \circ \rho_1)^*\omega = \rho_1^*\sigma^*\omega = \omega$ in particular; our desired symplectomorphism ψ is given by $\sigma \circ \rho_1$.

$\square$

Bibliography

[1] Alvarez-Gavela, D., Kaminker, V., Kislev, A., Kliakhandler, K., Pavlichenko, A., Rigolli, L., Rosen, D., Shabtai, O., Stevenson, B. and Zhang, J., *Embeddings of free groups into asymptotic cones of Hamiltonian diffeomorphisms*, arXiv:1602.05842.

[2] Calabi, E., *On the group of automorphisms of a symplectic manifold*, in *Problems in analysis*, Princeton Univ. Press, Princeton, N.J. 1970, 1-26.

[3] Chekanov, Y., *Invariant Finsler metrics on the space of Lagrangian embeddings*, Math. Z. **234** (2000), 605 - 619.

[4] Cieliebak, K., Floer, A., Hofer, H. and Wysocki, K., *Applications of symplectic homology II: Stability of the action spectrum*, Math. Z., **223** (1996), 27-45.

[5] Crawley-Boevey, W., *Decomposition of pointwise finite-dimensional persistence modules*, J. Algebra Appl. **14** (2015), 1550066.

[6] Entov, M. and Polterovich, L., *Calabi quasimorphism and quantum homology*, Int. Math. Res. Notices, no. **30** (2003), 1635-1676.

[7] Entov, M. and Polterovich, L., *Symplectic quasi-states and semi-simplicity of quantum homology*, in *Toric Topology*, pp. 47-70, Contemporary Mathematics **460**, AMS, 2008.

[8] Entov, M. and Polterovich, L., *Rigid subsets of symplectic manifolds*, Compositio Math. **145** (2009), 773-826.

[9] Entov, M., Polterovich, L. and Zapolsky, F., *Quasi-morphisms and the Poisson bracket*, Pure Appl. Math. Quarterly **3**:4, part 1 (2007), 1037-1055.

[10] Floer, A., *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575-611.

[11] Hatcher, A., *Algebraic Topology.* Cambridge University Press, New York, 2001.

[12] Hofer, H., *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 25 - 38.

[13] Hofer, H. and Salamon, D., *Floer homology and Novikov rings.* In *The Floer memorial volume.* Progr. Math. **133**, Birkhäuser, Basel, 1995, 483-524.

[14] Humilière, V., Le Roux, F. and Seyfaddini, S., *Towards a dynamical interpretation of Hamiltonian spectral invariants on surfaces*, arXiv:1502.03834v1, to appear in Geom. Topol.

[15] Karoubi, M. and Leruste, C., *Algebraic Topology via Differential Geometry*, Cambridge Univ. Press, Cambridge, 1987.

[16] Khanevsky, M., *Hofer's metric on the space of diameters*, J. Topol. Anal. **1** (2009), no. 4, 407 - 416.

[17] Lalonde, F. and McDuff, D. *The geometry of symplectic energy*, Ann. of Math. (2) **141** (1995), no. 2, 349-371.

[18] Lalonde, F. and McDuff, D. *Hofer's L^∞-geometry: energy and stability of flows II*, Invent. Math. **122** (1995), 1-69.

[19] McDuff, D., *Monodromy in Hamiltonian Floer theory*, Comment. Math. Helv. **85** (2010), no. 1, 95-133.

[20] McDuff, D. and Salamon, D., *J-holomorphic curves and symplectic topology.* AMS Colloquium Publications **52**, AMS, Providence, 2012.

[21] Oancea, A., *A survey of Floer homology for manifolds with contact type boundary or symplectic homology*, Ensaios Matematicos, volume 7 (2004), 51-91.

[22] Oh, Y.-G., *Lectures on Floer theory and spectral invariants of Hamiltonian flows*, in *Morse-theoretic methods in nonlinear analysis and in symplectic topology.* NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006, 321-416.

[23] Ostrover, Y., *A comparison of Hofer's metrics on Hamiltonian diffeomorphisms and Lagrangian submanifolds*, Commun. Contemp. Math. **5**:5 (2003), 803 - 811.

[24] Pardon, J., *An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves*, Geom. Topol. **20** (2016), 779-1034.

[25] Polterovich, L., *Hofer's diameter and Lagrangian intersections*, Internat. Math. Res. Notices 1998, no. 4, 217-223.

[26] Polterovich, L., *The Geometry of the group of symplectic diffeomorphisms*, Lectures in Mathematics ETH Zurich, Birkhäuser Verlag, Basel, 2001.

[27] Polterovich, L. and Shelukhin, E., *Autonomous Hamiltonian flows, Hofer's geometry and persistence modules*, Selecta Math. (2015), published online, doi: 10.1007/s00029-015-0201-2.

[28] Py, Pierre, *Quelques plat pour la métrique de Hofer*, J. Reine Angew. Math. **620** (2008), 185-193.

[29] Salamon, D., *Lectures on Floer Homology*, in *Symplectic geometry and topology (Park City, Utah, 1997).* AMS, Providence, 1999.

[30] Salamon, D. and Zehnder, E., *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math., **45** (1992), 1303-1360.

[31] Schwarz, M., *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), 419-461.

[32] Seyfaddini, S., *Spectral Killers and Poisson bracket invariants*, J. Mod. Dyn. **9** (2015), 51-66.

[33] Usher, M., *Boundary depth in Hamiltonian Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds*, Israel J. Math. **184** (2011), 1-57.

[34] Usher, M., *Deformed Hamiltonian Floer theory, capacity estimates, and Calabi quasimorphisms*, Geom. Topol. **15** (2011), 1313-1417.

[35] Usher, M., *Hofer's metrics and boundary depth*, Annales Scientifiques de l'École Normale Supérieure **46** (2013), no. 1, 57-128.

[36] Usher, M. and Zhang, J., *Persistent homology and Floer-Novikov theory*, arXiv:1502.07928v3, to appear in Geom. Topol.

[37] Zomorodian, A. and Carlsson, G., *Computing persistent homology*, Discrete Comput. Geom. **33** (2005), 249-274.