# COGNITIVE DEMAND AND TECHNOLOGY USE IN HIGH SCHOOL MATHEMATICS TEACHERS' SELECTION AND IMPLEMENTATION OF TASKS

by

KYLE THOMAS SCHULTZ

(Under the Direction of James W. Wilson)

#### ABSTRACT

Teachers play a critical role in determining not only the mathematical content in their lessons but also the ways their students think about that content. Although many mathematics teachers want their students to develop the ability to problem solve and think critically about mathematics, the tasks they select and use for their lessons lack the cognitive demand—the kind and level of thinking needed by students to attain a solution—appropriate for developing these skills. In this qualitative case study, I investigated the practices of three high school mathematics teachers with respect to the tasks they selected and used in instruction. I paid particular attention to the relationship between cognitive demand and the use of instructional technology in these tasks. I observed one class period a day of each participant over a 2-week period. Each participant also participated in two in-depth interviews, one preceding and one following the observation period. Each participant had his or her own reasons for selecting the tasks used in class, and the cognitive demand of these tasks varied across participants. The strategies used by participants to maintain the cognitive demand of the tasks they used also varied. Two aspects of students' technology use affected cognitive demand: (a)

examining the mathematical features of the task to determine the appropriateness of technology use and (b) connecting the mathematical context of the task to a technological representation of it. From these findings, I suggest a technology-oriented addendum to Stein, Smith, Henningsen, and Silver's (2000) framework for analyzing the cognitive demand of mathematical tasks.

INDEX WORDS: Mathematics Teaching, Cognitive Demand, Mathematical Tasks, Instructional Technology, Mathematics Education

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KYLE THOMAS SCHULTZ B.A., Miami University, 1995

M.A.T., Miami University, 1997

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### KYLE THOMAS SCHULTZ

Major Professor: James W. Wilson

Committee:

Sybilla Beckmann Jeremy Kilpatrick Denise S. Mewborn

Electronic Version Approved:

Maureen Grasso Dean of the Graduate School The University of Georgia December 2009

## DEDICATION

For Ricki. Your love and patience have supported me on this journey.

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### TABLE OF CONTENTS

vi
¥ 1

Page

ACKNOW	WLEDGEMENTS	V
LIST OF	TABLES	viii
LIST OF	FIGURES	ix
CHAPTE	ER	
1	THE RESEARCH PROBLEM	1
	Rationale	5
	Benefits	7
2	REVIEW OF RELATED LITERATURE	10
	The Curriculum as Problems and Tasks	10
	Teachers' Influence on Tasks	14
	Level of Intellectual Engagement	17
	Technology Use and Cognitive Demand	24
3	METHODOLOGY AND CASES	28
	Participants	28
	Data Collection	30
	Data Analysis	35
	Validity and Reliability	40
	The Cases	41
4	RESULTS	56

		Question 1: Teachers' Selection of Mathematical Tasks	56
		Question 2: Implementation of Tasks	80
		Question 3: Technology and Cognitive Demand	
	5	DISCUSSION AND CONCLUSION	111
		Task Selection	112
		Task Implementation	114
		Technology	114
		Limitations	121
		Implications for Practice and Research	
RE	FERE	NCES	131
AP	PEND	ICES	
	А	PARTICIPANT SCREENING INSTRUMENT	139
	В	PRELIMINARY INTERVIEW GUIDE	146
	С	OBSERVATION INTERVIEW GUIDE	148
	D	FINAL INTERVIEW GUIDES	149
		Michael	149
		Sarah	150
		David	152
	E	SELECTED MATHEMATICAL TASKS	155
		Michael	155
		Sarah	157
		David	160

## LIST OF TABLES

Table 1: Connections Between the Data Sources and Research Questions	31
Table 2: Class Sessions Observed For Michael's Honors Calculus Course	45
Table 3: Class Sessions Observed For Sarah's Mathematics 1 Course	49
Table 4: Class Sessions Observed For David's Mathematics 1 Course	54
Table 5: Michael's Tasks For Honors Calculus	58
Table 6: Sarah's Tasks For Mathematics 1	68
Table 7: David's Tasks For Mathematics 1	75

## LIST OF FIGURES

Figure 1: The Mathematical Tasks Framework aligned with corresponding levels of	
curriculum	16
Figure 2: The modified version of the Task Analysis Guide	39
Figure 3: A technology addendum to the Task Analysis Guide	.127

Page

#### CHAPTER 1

#### THE RESEARCH PROBLEM

The development of high-speed communication, rapid innovations of digital technologies, and shifts in political climates have together unleashed an unprecedented level of global collaboration (Friedman, 2005). As a result of this trend toward globalization, the supply of workers who can perform routine technical tasks has increased and prompted a greater need for those who can adapt quickly to rapidly changing conditions in the workplace. These changes can be seen in many professions that make use of mathematics, where the requisite mathematical understanding and abilities have shifted to include higher degrees of creativity and analytical skills (Lesh & Zawojewski, 2007).

Policy makers, business leaders, and educators have responded to these changing conditions by calling for schools to provide students with cognitive skills that enable them to think critically and creatively, work collaboratively, balance multiple tasks, and learn new technologies (e.g., see Partnership for 21<sup>st</sup> Century Skills, 2002). This focus is not new in mathematics education and has been described in many different ways. For example, the National Council of Teachers of Mathematics [NCTM] (1989, 2000) emphasized the importance of students learning mathematics through—and simultaneously developing—problem-solving skills as well as the ability to reason, represent, connect, and communicate one's mathematical ideas. The National Research Council (2001) described mathematical proficiency as consisting of five interwoven

strands, including *strategic competence* (problem formulating and solving) and *adaptive reasoning*, "the ability to think logically about the relationships among concepts and situations" (p. 129).

For mathematics students to develop the kinds of mathematical thinking described above, they must have opportunities to practice them. Not all mathematical tasks, however, evoke the level of intellectual engagement needed for this development. Tasks requiring students to merely recall memorized facts or follow a prescribed procedure, for example, fail to provide the intellectual stimulus needed for students to gain experience thinking critically or flexibly. Sadly, many students' experiences in mathematics classrooms focus almost exclusively on these kinds of tasks.

Mathematics teachers play a critical role in providing their students with opportunities to develop the mathematical thinking that has become increasingly important in today's world. This role can be viewed as having three components. First, teachers determine the mathematical tasks that will be used in class. Schmidt, McKnight, and Raizen (1997) found that teachers had a large degree of latitude in the way that they use mathematics textbooks and, because of the unfocused nature of many books, were able to select from a wide array of material.

Second, mathematics teachers plan how selected tasks will be implemented in class. This process includes making real-time decisions during class that may affect students' work and thinking on those tasks. In this process, the mathematics content implemented in class and learned by students can change from what was originally intended (Robitaille, 1980; Travers & Westbury, 1989).

2

Third, teachers govern their students' use of instructional technology, including graphing calculators, computer spreadsheets, and dynamic geometry and statistical software. Although these resources enable students to interact with mathematics in ways that were previously impossible, this interaction can change the nature of the mathematical tasks with which the technology is used. For teachers, effectively using technology in the classroom entails understanding not only the mathematics being taught but also how it can be represented in a technological environment and how those representations can be used to develop students' mathematical understanding.

Howson, Keitel, and Kilpatrick (1981) suggested that curriculum "depends upon individual teachers, their methods and understanding, and their interpretation of aims, guidelines, texts, etc. The part played by the individual teacher must, therefore, be recognized" (p. 2). In this way, a curriculum designed to cultivate sophisticated mathematical thinking in students can do so only if teachers select and implement tasks from it that serve this purpose. Therefore, an investigation of the choices teachers make when selecting and implementing mathematical tasks would help us to better understand the mathematics teacher's role in cultivating students' mathematical thinking.

The purpose of this study was to investigate and describe how high school mathematics teachers select and implement tasks in their lessons and to describe the mathematical thinking required by those tasks. To better understand whether these tasks elicited the kinds of mathematical thinking consistent with the cognitive skills that are increasingly called for in our global society, I used the construct of cognitive demand to characterize the possible work (or task) that a student might perform to solve each task. In this process, I paid special attention to how the use of instructional technology influenced the cognitive demand of particular tasks and teachers' decisions about them. In particular, this study addressed the following questions:

- 1. How do high school mathematics teachers select the tasks that they implement during instruction? What is the cognitive demand of these tasks?
- 2. How do high school mathematics teachers maintain or fail to maintain the cognitive demand of the tasks they implement in their classrooms?
- 3. In what ways, if any, does the use of instructional technology influence how mathematical tasks are selected, the cognitive demand of those tasks, and how they are implemented?

Some of the concepts used in this study have a variety of meanings depending upon the context in which they are used or the scholar who is using them. To clarify the perspective taken by this study, I present my interpretations of these concepts here:

*Task.* A task is a classroom activity that focuses students' attention on and contributes to the development a particular mathematical idea. For this study, several tasks might be used in succession for a given mathematical idea. In this sense, a mathematical task includes items commonly referred to as *problems* or *exercises*.

*Cognitive demand.* Stein, Smith, Henningsen, and Silver (2000) defined cognitive demand of a mathematical task as "the kind and level of thinking required of students in order to successfully engage with and solve the task" (p. 11). Cognitive demand serves as a way of classifying the intellectual engagement that a task elicits from students.

*Curriculum*. My definition of curriculum is adapted from one proposed by NCTM (1989) in its *Curriculum and Evaluation Standards*. For me, a curriculum is an instructional plan that outlines goals for mathematics students' learning, how students are

to achieve those goals, what teachers are to do to help students achieve those goals, and the classroom context in which this process is to occur. In this vein, curriculum is more than a list of mathematical topics or even a mathematics textbook. It encompasses the pedagogy that teachers are directed to use and the kinds of work that students will do as they learn mathematics.

#### Rationale

During my time as a high school mathematics teacher, I often noticed a large discrepancy between the activities I had planned for my students and the way those activities unfolded in the classroom. At times, tasks I had perceived to be mathematically challenging when planning became something altogether different in the context of the classroom. In other cases, a challenging task I had devised degenerated into something that could scarcely be called mathematical. Although I could identify these discrepancies between what was planned and what was presented, I did not have any means of describing them in a way that was helpful to improving my practice.

Upon returning to graduate school, my interests focused on mathematical problem solving and factors that enable teachers to effectively engage students with challenging tasks. While reading *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development* (Stein et al., 2000), I found I could better describe many of my frustrations with teaching mathematics by focusing on how the tasks I had used and the cognitive demands of those tasks had evolved over the course of the lesson. More importantly, I became cognizant of the classroom-based factors that possibly influenced the tasks I used with my students and how my students thought about them.

The analysis of cognitive demand of mathematical tasks has been used as a professional development tool for practicing teachers (Arbaugh & Brown, 2004; Boston, 2006; Neumann, 2004; Stein et al., 2000). Teachers participating in professional development of this nature tended to focus more on the connection between tasks and student thinking, select more high-level tasks for instruction, and maintain high-level cognitive demand during implementation of those tasks. Further exploration of how teachers select and implement tasks for their practice, especially when technology is involved, may help to inform professional developers of ways to improve the programs they offer to both prospective and practicing teachers.

The use of technology in mathematics classrooms has the potential to enhance students' mathematics learning, support effective mathematics teaching, and enable the study of more sophisticated mathematical ideas (NCTM, 2000). In addition to these potential benefits, the use of technology in the mathematics classroom gives rise to additional considerations (Zbiek, Heid, Blume, & Dick, 2007). For example, teachers may govern when students use technology and privilege the use of technology for particular situations, such as only after a topic has been thoroughly studied. Teachers may not give their students much experience with a particular technological tool, preventing the student from developing effective strategies for using it. Finally, teachers may have varying familiarity with the capabilities of a technological tool as well as the implications of using that tool to represent different mathematical ideas. This lack of knowledge may potentially alter or limit what may be learned. Each of these considerations may affect the ways students use technology while working on mathematical tasks. As a consequence students' technology use may affect the cognitive demand of a task and limit their development of cognitive skills.

#### Benefits

The potential benefits of studying teachers' selection and implementation of tasks with respect to cognitive demand and technology use can be framed with respect to NCTM's (2007) standards for teaching and learning mathematics. In particular, the first five standards, which address what teachers know and how they implement that knowledge while teaching, play a crucial role in the phenomena I studied. I briefly describe each of these five standards below and state how my study may have contributed to our ways of thinking about them.

#### Standard 1: Knowledge of Mathematics and General Pedagogy

Included in this standard is that teachers should have a deep knowledge of mathematics content. By uncovering teachers' rationales for their decision-making with respect to selecting and implementing mathematical tasks, this study might help to further elaborate on the nature of specialized content knowledge (Ball, Thames, & Phelps, 2008), mathematics content knowledge used exclusively by teachers. In addition, this study might help us to understand teachers' knowledge of mathematics and of technology and how these domains of knowledge might interact.

#### Standard 2: Knowledge of Student Mathematical Learning

A key component of this standard with respect to this study is teachers' knowledge of "methods of supporting students as they struggle to make sense of mathematical concepts and procedures" (NCTM, 2007, p. 25). As teachers implement tasks in their classrooms, they need to make important decisions about how to respond to student questions about a task, evaluate student work, and suggest strategies to help those students who are struggling. These decisions can influence a task's cognitive demand (Henningsen & Stein, 1997). This standard also includes teacher knowledge of "a variety tools for use in mathematical investigation and the benefits and limitations of those tools" (NCTM, 2007, p. 25). By focusing on how the use of technological tools may influence a task's level of cognitive demand, this study addressed these tools' possible benefits and limitations.

#### Standard 3: Worthwhile Mathematical Tasks

This standard embodies the main theme of my study as it calls for teachers to implement tasks that, among other things, "engage students' intellect; ... stimulate students to make connections and develop a coherent framework of mathematical ideas; call for problem formulation, problem solving, and mathematics reasoning; ... and promote communication about mathematics (NCTM, 2007, pp. 32–33). These characteristics represent the essential cognitive skills described earlier in this chapter, and the main focus of this study was teachers' use of tasks with these characteristics.

#### Standard 4: Learning Environment

Mathematics teachers need to create an environment that optimizes student learning. NCTM (2007) described this environment as including "access and encouragement to appropriate technology" (p. 40). As part of this study, I examined how teachers incorporated technology into their students' exploration of mathematical tasks. Part of this incorporation of technology involved governing students' access to technology in order to preserve cognitive demand and evaluating whether students' use of appropriate technology altered the cognitive demand of tasks. In addition, this standard called for students to work collaboratively, another quality desired of students in today's world.

#### Standard 5: Discourse

This standard calls for mathematics teachers to facilitate discourse with and between their students. A way for teachers to facilitate this discourse is by "deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let students wrestle with a difficulty" (NCTM, 2007, p. 45). By focusing on teachers' implementation of mathematical tasks and its effect on cognitive demand, my study investigated the kinds of decisions described here and how teachers made these decisions.

This review of teaching standards pertaining to teacher knowledge and how it is implemented demonstrates that my investigation of how teachers select and implement mathematical tasks with respect to cognitive demand and technology use was situated within several areas of a broad framework used to describe good mathematics teaching. As a result, the study had the potential to contribute to a wide array of literature on mathematics teaching. The following chapter describes some of this literature and how the ideas therein helped to shape this study.

#### CHAPTER 2

#### **REVIEW OF RELATED LITERATURE**

The following review of literature addresses the key ideas that served as the foundation of my study. First, I describe how different scholars have used smaller units such as tasks or problems to describe and think about mathematics curricula. Second, I present literature describing the different stages of mathematics teachers' work and how a teacher's practice may influence the curriculum at these stages. Third, I describe how scholars have attempted to define and measure the level of intellectual engagement students must have in order to solve a given mathematics problem. Finally, I will present an analysis of research that investigated how instructional technology may affect mathematics teaching and learning and what this effect might have on my study.

#### The Curriculum as Problems and Tasks

Doyle (1983) suggested that one way to understand the inherent demands of a curriculum is to view it as a collection of *academic tasks*, which "influence learners by directing their attention to particular aspects of content and by specifying ways of processing information" (p. 161). If one is to focus on curriculum at this scale and study it by examining individual tasks, then it is crucial to understand and clearly define what a task, problem, or other arbitrary name for a unit of academic work, is. This goal is not new. Many mathematics educators have thought about the nature of students' mathematical work by focusing on and characterizing a basic unit of it. I present several of these characterizations that influenced this study below.

The terms *problem* and *problem solving* have been prominent in the lexicon of mathematics educators for nearly a century. Stanic and Kilpatrick (1989) provided a historical account of how problem solving has been studied, used, and championed as a vehicle for teaching and learning mathematics. As perhaps the most noted proponent of problem solving in the 20<sup>th</sup> century, George Pólya (1945/1973, 1962, 1965) wrote extensively about solving mathematical problems, which he described as occupying "the greater part of our conscious thinking" (1962, p. 117). To him, problems presented some sort of difficulty, obstacle, or elusive goal. Problem solving is the act of facing and meeting the challenges provided by a problem, which Pólya described as "the specific achievement of intelligence, ... the most characteristically human activity" (1962, p. v).

Pólya (1962) wrote about the utility of using problems in high school mathematics, suggesting that problem solving provided students with *know-how* in addition to simply possessing mathematical information. He also suggested that problems could be used not only to develop students' mathematical thinking but also to channel their motivations for studying mathematics at successively higher levels.

Although Pólya wrote extensively about problems and problem solving, he did not provide a clear definition of *problem*. In both *How To Solve It* (1945/1973) and *Mathematical Discovery* (1962, 1965) he began each preface discussing the art of problem solving but assumed that the meaning of problem was understood by the reader or implied by his description of what it meant to solve one. Although he defined specific types of problems (e.g. auxiliary problem, routine problem, problem to prove) in his glossary of terms related to heuristic, Pólya (1945/1973) defined each of these as a problem with particular features. Again, the assumption was that the reader had an understanding of the term *problem* to which these features could be applied. Further discussion of Pólya's notion of problem is provided below in the context of the level of students' intellectual engagement.

As mathematics educators increased their focus on problem solving as an effective medium for teaching and learning mathematics and researchers studied it in practice, many different definitions of *problem* arose. Wilson, Hernandez, and Hadaway (1993) provided a brief summary of these definitions. Synthesizing these views, a problem must have a goal to be attained, some sort of obstruction to the goal that makes the method for attaining it not readily apparent, and a willingness on the part of the individual to pursue a solution. There may be individual differences in what constitutes a problem (Schoenfeld, 1985). First, students have varying resources at their disposal in terms of prior knowledge and experiences, as well as varying degrees of pre-existing mathematical proficiency. These resources may alter the nature of the obstruction to a solution. Second, students' motivation and attitudes may vary, affecting their willingness to attempt looking for a solution. In other words, if there is no desire to work on a problem, then it is not a problem.

As mentioned above, Doyle (1983, 1988) used the term *academic task* to describe the work in which students engage. Four components constituted his description of academic task:

- 1. A product, such as numbers in blanks on a worksheet, answers to a set of test questions, oral responses in class, or a solution to a word problem
- 2. Operations to produce the product, for example, copying numbers off a list, remembering answers from previous lessons, applying a rule ..., or formulating an original algorithm to solve a problem.
- 3. Resources, such as notes from lectures, textbook information, conversations with other students, or models of solutions supplied by the teacher.

4. The significance or "weight" of a task in the accountability systems of a class (Doyle, 1988, p. 189)

Although these components describe a great deal about what students are to produce, how they are to produce it, what tools will they have available to them, and how prominent their work is in terms of the greater scheme of the course, it does not specifically address the content itself. The conceptual ideas of the mathematics underlying the academic task are only implicitly defined by this category scheme.

Stein, Grover, and Henningsen (1996) adapted Doyle's (1983) concept of academic task for the purpose of studying mathematics instruction. They defined *mathematical task* as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). For the purposes of their research, a new mathematical task began only when a new mathematical idea was addressed, making successive items dealing with the same idea a single task. They noted this approach differed sharply from research that examined performance assessment, where a single item constituted a single task.

Herbst (2006) built upon the work of Doyle (1988) and presented a way of thinking about tasks that differed from that of the work of Stein and her colleagues (Henningsen & Stein, 1997; Stein et al., 1996; Stein et al., 2000). In his study of a teacher's use of geometry problems, Herbst made a distinction between the mathematical content with which the students are engaged and the nature of this engagement by using two distinct concepts, *problem* and *task*. He defined problem as "a question whose answer hinges on bringing a mathematical theory within which a concept, formula, or method involved in answering the question is warranted" (p. 315). This definition does not make reference to students, their actions, or the context of the classroom. It confines its focus to the problem as a representation of mathematical knowledge.

Herbst's (2006) definition of *task* made use of the components of task described by Doyle (1983, 1988). He defined task as:

specific *units of meaning* (i.e., the actions and interactions with the symbolic environment) that constitute the intellectual context in which individuals think about the mathematical ideas at stake in a problem. A *task* is the universe of possible operations that an individual might or might not take while working on a problem, toward a certain product, with certain resources (p. 315)

Herbst's conceptualization presents a different view of the relationship between the problem and task. Whereas Stein et al. (1996) could incorporate several problems within a single task, Herbst's model describes a task as being particular to a single problem. This distinction became important when considering how to analyze the data of this study.

#### Teachers' Influence on Tasks

Teachers determine not only what part of the curriculum will be used in class, but also how it will be presented to students and how students will interact with it. Therefore, the teacher's role must be accounted for when considering the different conceptions of problem and task presented above.

To describe the influence of the teacher on the use of mathematical tasks in the classroom, Stein et al. (1996) presented the Mathematical Tasks Framework (MTF). The MTF uses three steps to describe how mathematical tasks develop in the classroom: (a) as the task appears in the curriculum materials available to the teacher, (b) as the task is introduced by the teacher to students, and (c) as the students work on it. From step to step in this model, the task may undergo changes depending upon what occurs in the

classroom. The changes in a task may affect its final manifestation and the knowledge gained by students from working on it.

As another way to think about teachers' influence on curriculum, Burkhardt, Fraser, and Ridgeway (1990) presented a framework that attended to how a mathematics curriculum varies according to the context in which it is defined. This framework has six different levels of curricula: (a) the *ideal* curriculum as described by experts, (b) the *available* curriculum for which teaching materials exist, (c) the *adopted* curriculum mandated by state or local policy, (d) the *implemented* curriculum actually taught by teachers, (e) the *achieved* curriculum learned by students, and (f) the *tested* curriculum measured through assessments. Ideally, these six curricula are aligned, but forces on a variety of scales, including the actions of teachers, prevent such an alignment. As this study proposed to examine teachers' selection and implementation of mathematical tasks, I focused on two levels of this curriculum framework, adopted and implemented.

If one accepts Doyle's (1983) conception of curriculum as a collection of tasks, then the potential metamorphosis of the mathematical tasks depicted by the MTF can be placed in correspondence with the levels of curricula (Burkhardt et al., 1990) described above (see Figure 1). Because this correspondence encompasses both the adopted and implemented curriculum, it provides a rationale for the MTF as an appropriate model to incorporate into this study.

Mathematical Task Framework Stages	Tasks appearing in instructional materials	Tasks selected and set up by teachers	Tasks worked on by students	Student learning from the tasks
Levels of Curriculum	The Available or Adopted Curriculum	The Implemented Curriculum		The Achieved Curriculum

*Figure 1*. The Mathematical Tasks Framework aligned with corresponding levels of curriculum. Adapted from Stein et al. (1996) and from Burkhardt et al. (1990).

Like the other models, Herbst's (2006) definition of task enables one to consider the relationship between a teacher's actions and the curriculum. By stipulating the product students must create and the resources students may access while working on a problem, teachers exert a great deal of influence on Herbst's notion of task, the possible operations students might consider or use. For example, allowing the students to use graphing calculators on a problem might enable them to consider different operations (such as using a brute force approach) that would not have been feasible had calculators not been permissible. Similarly, specifying the solution of the problem must include an explanation of the student's method for solving it rather than just simply an answer might affect the students' choice of operations as well. Furthermore, requiring students to provide an explanation for their solutions may increase or decrease the students' intellectual engagement (to be defined in detail below) with the problem.

Like the researchers above, I viewed the mathematics curriculum as a collection of tasks or problems that undergo potential transformations during the course of planning and instruction. The MTF described by Stein et al. (1996) provided a suitable way to describe this change. A correspondence can easily be drawn between it and the levels of curricula presented by Burkhardt et al. (1990). The conception of task offered by Stein et al., 1996, where a single task corresponds to an underlying mathematical ideal and could be linked to several items (problems, exercises, etc.), created difficulty in terms of considering the level of intellectual engagement (addressed below) inherent in the task. Because this level could vary from item to item, it became difficult to characterize the overall task. As a result, I viewed the MTF as a model for portraying the transformation of tasks during planning and instruction, but chose to adjust the definition of task suggested by Stein et al. so that multiple tasks could address a single mathematical idea.

#### Level of Intellectual Engagement

An important characteristic of any task is the amount of mental effort that a student must exert in order to successfully work on and solve it. I call this construct the level of intellectual engagement. To further define what is meant by this construct, I present several representations of it found in the mathematics education literature.

Pólya (1945/1973) made a distinction among the levels of difficulty found in problems. He defined *routine problems* as those that "can be solved either by substituting special data into a formerly solved general problem, or by following step by step, without any trace of originality, some well-worn conspicuous example" (p. 171). In characterizing a student's effort in solving such a problem, Pólya described it as "nothing but a little care and patience ... [and] no opportunity to use his judgment or his inventive faculties" (p. 171).

Pólya (1945/1973) conceded that routine problems are necessary and appropriate for some educational goals such as to develop students' ability to perform routine mathematical operations quickly and accurately. However, he stressed routine problems should by no means constitute the entire curriculum. His three principles of learning (Pólya, 1965)—(a) active learning, where optimal learning is based upon discovery; (b) best motivation, where problems should stimulate students' interests; and (c) consecutive phases, where exploration, formalization, and assimilation occur successively—reiterated the importance of including nonroutine problems in the high school mathematics curriculum. More recent scholars have echoed this reasoning in describing students' development of mathematical proficiency (National Research Council, 2001).

Madaus, Clarke, and O'Leary (2003) identified Ralph Tyler as an early proponent for considering the "types of mental action" (p. 1326) that pupils should exhibit when taking mathematics tests. Tyler proposed educational objectives should be defined according to both the mathematical content students should master along with a *behavioral component*. This idea of considering what students did with respect to the mathematics content became the foundation for the taxonomy of educational objectives developed by Bloom et al. (1956). This taxonomy, commonly referred to as Bloom's Taxonomy, contains six hierarchical levels of student behavior (listed from lowest to highest): knowledge, comprehension, application, analysis, synthesis, and evaluation. Thus, by classifying the behaviors associated with curricular objectives, one could define and compare the intellectual engagement needed by students on various mathematics assessments. Researchers tailored the ideas of Tyler and Bloom to develop models for examining aspects of mathematics curricula. Wilson (1971) described several of these models. For example, the School Mathematics Study Group (SMSG) developed and used a two-dimensional classification model in the National Longitudinal Study of Mathematical Abilities during the 1960s. This model embodied Tyler's idea of looking beyond the nature of a task's mathematical content and incorporated levels of behavior adapted from Bloom's Taxonomy. The SMSG model classified mathematics test items according to one of three content areas (number systems, algebra, and geometry) and one of four behavior levels (computation, comprehension, application, and analysis). Wilson noted the behavior levels were intended to "reflect the cognitive complexity of the task (*not* simply the difficulty of the task)" (p. 648). This distinction could be found in later models and was important within the context of this study, as I was primarily interested in cognitive processes needed by students to solve a task.

The National Assessment of Educational Progress (NAEP) was established in the late 1960s with the goals to "report what the nation's citizens know and can do and then to monitor changes over time" (Jones, 1996, p. 15). The early development of mathematics test items used a category scheme that included both content and behavior. Wilson (1971) described some of the early development of the NAEP framework and compared it to the SMSG model:

The cognitive levels [of NAEP] were later reorganized as follows:

- 1. Recall and/or recognize definitions, facts, and symbols
- 2. Perform mathematical manipulations
- 3. Understand mathematical concepts and processes
- 4. Solve mathematical problems—social, technical, and academic
- 5. Use mathematics and mathematical reasoning to analyze problem situations, define problems, formulate hypotheses

6. Appreciate and use mathematics (Committee on Assessing the Progress of Education, 1969, p. 6)

Level 6 is notable in this scheme in that it is, in part, a noncognitive category. Level 1 is a separate knowledge category. Except for this separation of knowledge-level behaviors, the cognitive levels are very much like the ones given [by the SMSG model]. (p. 652)

In the time since these early iterations of a model for classifying mathematics test items,

NAEP has refined its model while still focusing on more than merely the content area

addressed by a given item. The most current version of the framework (National

Assessment Governing Board, 2008) used a three-level model of mathematical

complexity measuring the demands a test item placed on student thinking, described as

"what [the item] asks the students to recall, understand, reason about, and do" (p. 37).

This framework articulated the three levels of mathematical complexity as follows:

Low-complexity items expect students to recall or recognize concepts or procedures specified in the framework. Items typically specify what the student is to do  $\dots$  (p. 38)

Moderate-complexity [items] involve more flexibility of thinking and choice among alternatives  $\dots$  The student is expected to decide what to do and how to do it  $\dots$  (p. 42)

High complexity items make heavy demands on students, because they are expected to use reasoning, planning, analysis, judgment, and creative thought. (p. 46)

These categories focus on the types of thinking a test item asks students to do, but not the particular approaches students might take to formulate a solution. This distinction was aptly described in a discussion of in the 2005 framework: "[mathematical complexity] focuses on the characteristics of items rather than on inferred cognitive abilities of students, which may vary widely from student to student" (Neidorf, Binkley, Gattis, & Nohara, 2006, D-1). This separation of the mathematical content from student strategies

is similar to that which Herbst (2006) described when distinguishing between his concepts of *problem* and *task*.

As another way of describing students' intellectual engagement in mathematics, Doyle (1983) and Stein et al. (1996) used the term *cognitive demand* to characterize the thinking by students needed to solve a particular task and, as a result, that task's appropriateness for a given learning objective. Stein et al. (2000) defined cognitive demand as "the kind and level of thinking required of students in order to successfully engage with and solve the task" (p. 11). Different tasks place different cognitive demands on students, depending upon their content and the processes needed to complete them (Doyle, 1983). According to this model, tasks with a high level of cognitive demand (subsequently referred to as high-level tasks) were characterized as requiring complex mathematical thinking or having connections to other concepts. Tasks with a low level of cognitive demand (low-level) encompassed recalling facts or completing procedures isolated from any context.

Stein et al. (2000) developed a *Task Analysis Guide* (TAG) as a tool for describing and classifying the cognitive demand of a mathematical task. Consisting of a list of general criteria describing students' work with a given task, the TAG enables one to classify tasks according to one of four levels of cognitive demand (listed from lowest to highest level):

(a) *Memorization* [which I call *Recall and Recognition*], characterized by the recall of basic facts and operations and explicit direction given to the student;

- (b) Procedures Without Connections, such as using algorithms in an explicitly prescribed manner with no connection to the underlying mathematical ideas;
- (c) Procedures With Connections, where, for example, where broad or general procedures encourage students to create connections between different mathematical representations or to underlying mathematical ideas; and
- (d) *Doing Mathematics*, which requires complex and non-algorithmic thinking, self-regulation of problem-solving behavior, and a great deal of effort.

The first pair of categories represents low-level tasks, and the second pair represents high-level tasks.

Stein et al. (2000) described two important considerations when assessing the cognitive demand of a task. First, they stressed the importance of going beyond the superficial features of a task because these features often do not indicate the level of mathematical complexity found in the task. For example, tasks presented in a complicated manner requiring use of manipulatives, diagrams, or real-world contexts can often require the students to apply simple, well-rehearsed procedures whereas tasks that are stated simply can require complex mathematical thought. Second, the level of cognitive demand is dependent upon the students who work on the task. The options students have for strategies to solve a given task are defined by their age, grade level, and prior mathematical experiences. For example, a student in an Algebra 1 class might

expend considerable energy thinking about a task that an AP Calculus student could solve with a simple, easily recalled procedure.

Engaging students with high-level mathematical tasks is important, but teachers face several obstacles in achieving this goal. First, teachers may not select high-level tasks. The selection of appropriate tasks is crucial for fostering students' conceptions of mathematics and the ability to think mathematically, as "tasks convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991, p. 24). Failure to include high-level tasks could occur for a variety of reasons including limitations of available classroom materials; pressure to prepare students for high-stakes assessments that do not include high-level tasks; and the knowledge, goals, and beliefs of teachers.

Even if teachers select high-level mathematical tasks, they may inadvertently alter the nature of these tasks during instruction (Doyle, 1983). As a result, the level of students' thinking demanded by these tasks may change, particularly at junctures of the MTF. Pólya (1965) also addressed this phenomenon as he described the dilemma teachers face when helping students to solve problems:

This situation requires careful handling. If the teacher helps too little, there will be no progress. If the teacher helps too much, there will be no opportunity for the students to learn from their own work. (pp. 135–136)

Pólya suggested the key question to ask in this dilemma is not *how much* help should be given to students, but rather *how* that help should be given.

Scholars (e.g. Bennett & Desforges, 1988; Blumenfield & Meece, 1988; Doyle, 1983, 1986, 1988) have investigated this key question by describing ways that high-level tasks have either remained at a high level or evolved into less-demanding forms of student activity. This research also includes findings by the QUASAR project (see Silver & Stein, 1996) with respect to mathematical tasks. One result of this research was an augmented version of the MTF that included factors of mathematics lessons that may affect the cognitive demand. For example, during the set up of a task, teachers' goals, content knowledge, and knowledge of students may compel them to modify a task from its original form. During the lesson, classroom norms, features of a task, and dispositions of the teacher and students may affect how the task is implemented. This framework accommodates the assertion that the nature of a mathematical task, as represented in curricular materials, may change during classroom instruction.

Because in the present study I strove to analyze possible changes in the level of cognitive demand before and during classroom instruction, this framework aided in the identification of classroom-based factors that had such an effect. I also considered one additional factor, the use of instructional technology, which is examined in the next section.

#### Technology Use and Cognitive Demand

As an available resource, technology may change the substance of a task and, as a result, its cognitive demand. For example, technology may shift the cognitive demand from a symbolic analysis to one that is more visual or from analyzing the features of a series of static representations to determining the influence of parameters driving a representation that is dynamic and fluid.

Zbiek et al. (2007) provided a thorough summary of existing research on technology in mathematics education. Instead of trying to characterize the literature on technology in its entirety, I focused on one framework that aided my thinking with respect to technology and cognitive demand, Salomon and Perkins's (2005) distinction between different kinds of effects technology has on intelligence. A broad definition of technology includes any practical application of knowledge, including a wide array of tools that are implemented in mathematics classrooms such as manipulatives, measurement tools, and calculators. Although Salomon and Perkins proposed their framework in this general sense, in this study I used a narrower definition of technology, focusing solely on computing technology such as calculators, computers, and the software used by teachers and students to facilitate mathematics learning.

Salomon and Perkins (2005) explored the relationship between technology and intelligence. In doing so, they attempted to better understand whether the use of technology expands our cognitive capabilities. They classified the technological effects on intelligence in three ways: (a) effects *with* technology, (b) effects *of* technology, and (c) effects *through* technology. These effects were briefly defined in the following way:

Considered are *effects with* technology, how use of technology often enhances intellectual performance; *effects of* technology, how using a technology may leave cognitive residues that enhance performance even without the technology; and *effects through* technology how technology sometimes does not just enhance performance but fundamentally reorganizes it. (p. 72)

To place these definitions in the context of my study, I offer the following examples of how each type of effect could be observed in the mathematics classroom.

Effects with technology occur when certain intellectual operations are exclusively carried about by the technology, dividing the labor between user and technology. Salomon and Perkins (2005) noted that this division of labor "frees the user from distractions of lower-level cognitive functions or ones that simply exceed mental capacity" (p. 74). In the mathematics classroom, students' use of calculators to perform
computations involving large or tedious numbers or graphing complicated functions in conjunction with solving a task exemplifies effects with technology.

Effects of technology are those that remain after a period of using the technology but when the technology is no longer in use. Salomon and Perkins (2005) described the manifestation of this type of effect as the acquisition of a new skill or the improved mastery (or atrophy) of an existing skill. In the mathematics classroom, dynamic geometry software packages give students the opportunity to test a vast number of geometric cases very quickly. Whereas students who have not used software packages like this may have trouble envisioning more than a few cases, those who have used the software may be able to envision many, even without the use of the software. Another example of effects of technology in the mathematics classroom might be the deterioration of basic computation skills owing to overdependence on use of calculators.

Effects through technology occur when use of the technology gradually reshapes the way we perform an activity. Salomon and Perkins (2005) stated such effects do not emerge quickly, because new technologies are often assimilated into already established systems where the technologies cannot be used to their fullest potential. With respect to the mathematics classroom, Herrera, Preiss, and Riera (2008) provided example of effects through technology by describing how recent mathematical technologies have aided in reshaping the structure of mathematics teaching, from one that began with the formulization of the result and ended with visualization and analysis to another where that process is reversed.

Although my study asked a different question with respect to technology than that of Salomon and Perkins (2005), there is a relationship between the depictions of these

different technological effects on intelligence and how technology, as a tool in solving a mathematical task, affects both the mathematical constraints of the task and the ways that students might formulate a solution. By considering the use of technology observed during my study with respect to the three classifications of technological effects, I was better able to identify instances where the cognitive demand may have been affected by technology use.

The review of literature in this chapter provided a foundation for developing the research methodology of my study as well as a framework for interpreting the results. A description of the methodology is presented in the following chapter, along with descriptions of the three teachers whose practices I studied.

## CHAPTER 3

## METHODOLOGY AND CASES

Marshall and Rossman (2006) listed several types of research that warrant the use of qualitative methods. Three types of research included in their list could describe this study: (a) research eliciting tacit knowledge and subjective understanding and interpretations; (b) research delving in depth into complexities and process; and (c) research on real, as opposed to stated, organizational goals. By attempting to understand how mathematics teachers select and implement mathematical tasks, I explored the teachers' understanding of the mathematics they teach, monitored how they manage the implementation of tasks in a real-time environment, and examined differences between expected and actual outcomes. Hence, this study warranted a qualitative approach to collecting and analyzing the data.

More specifically, a case study methodology was used to gather data, as I sought "to answer focused questions by producing in-depth descriptions and interpretations over a relatively short period of time" (Hays, 2004, p. 218). Particular details about the methods used are provided below.

## Participants

This study examined the practices of three high school mathematics teachers who incorporated instructional technology in their teaching. In particular, I sought veteran teachers who were regarded in the local mathematics education community as effective educators. In this way, I hoped to observe teaching representing best practices with respect to choosing and implementing high-level tasks as well as incorporating technology into that process. To locate these teachers, I made a preliminary list of those I knew who might be suitable candidates. In addition, I consulted my major professor and other colleagues who suggested other teachers I might consider.

At this point, I contacted and, when possible, met five of these teachers to ascertain both their availability and willingness to work with me. Several of these meetings were fruitless, as either the teacher was not in a situation conducive to participate or felt the use of instructional technology at their school would not meet my needs. For example, one teacher who had recently earned her doctorate in mathematics education was struggling to reacclimate to the demands of high school teaching and felt she could not provide the time needed outside of class to facilitate my research. Another had transitioned from being a classroom teacher to a mathematics instruction specialist in his district. A third potential participant said his school mandated uniformity in how and what mathematics teachers taught with very little consideration made for the use of technology in the classroom.

From each of these setbacks I gained connections to other potential participants. Some of the teachers recommended other teachers in their district. One sent an email to calculus teachers from other schools through an electronic discussion group. I followed each of these leads in a way similar to that with my original contacts, making contact with five additional teachers.

From my search, four teachers expressed a willingness to participate in my study. I used a criterion-based sampling method (deMarrais, 2004) to select participants from this group. I focused on the teachers' experience (how long and in what settings they had taught), educational background (the amount and diversity of training they had received), and the kinds of technology used and with what frequency, and their teaching philosophy in general and with respect to technology. After being informed of the general nature of the study, I asked the four potential participants to complete a brief questionnaire (Appendix A) adapted with permission from an instrument designed by Becker and Anderson (1998). This questionnaire collected data with respect to the above criteria and informed my selection of the participants.

I selected three participants based upon the questionnaire responses. The responses of the excluded teacher indicated, unlike the three included teachers, he used instructional technology minimally, far below the degree that I had hoped to observe. In addition, his school was nearly 90 miles from my residence, creating potential logistical difficulties with respect to my teaching and other responsibilities.

Prior to the participant-selection process, I had obtained approval from the University of Georgia's Institutional Review Board to conduct my study and followed their guidelines throughout the study. In addition, once I had selected the three participants, I contacted an administrator at each school for permission to collect data in the participants' classrooms. Prior to data collection I visited each participant at his or her school to introduce myself, schedule interviews, and determine the logistics of my daily visits.

## Data Collection

I used a variety of data sources to address my research questions. Table 1 lists these sources as well as the how data collected from each source corresponded to my three research questions:

Table 1

Connections Between the Data Sources and Research Questions

Research Question	Preliminary	Observation	Mathematical	Classroom	Final Interview	Student Data
	Interview	Interviews	Tasks	Observation, Notes		
1. How are tasks	Establish teacher's	Establish teacher's	Confirmation of			
selected?	general rationale	rationale for	teacher's rationale			
	with respect to	selection of	for selection			
	selecting tasks	particular tasks				
1a. CD of selected	~	Establish teacher's	Establish CD	Confirmation of		Confirmation of
tasks?		perceived CD of	through analysis of	CD		8
		tasks	tasks			
2. Maintenance of			Establish CD prior	CD monitored	Teachers' rationale	Confirmation of
cognitive demands?			to instruction	during instruction	for actions that	CD maintained
					maintained CD (or	(or not)
		1.101			not)	
3. Influence of	Establish teacher's	Establish teacher's	Confirmation of	CD of TOTs	Teachers' rationale	Confirmation of
technology on task	general rationale	rationale for	teacher's rationale	confirmed,	for actions that	CD of TOTs,
selection, CD, and	with respect to	choosing particular	for selection of	monitored during	maintained CD (or	whether CD was
maintenance of CD?	using technology,	TOTs	TOTs; Establish	instruction	not) for TOTs	maintained
	selecting TOTs		CD			

Abbreviations. CD: Cognitive Demand TOT: Technology-Oriented Task

Note. The final row of the table, dealing with the use of technology, parallels the collective content of the rows above. Whereas the data collected for this purpose was from the same sources, distinguishing between technology-oriented tasks and nontechnology-oriented tasks enabled a comparison that provided evidence to address the third question.

- 1. How do high school mathematics teachers select the tasks that they implement during instruction? What is the cognitive demand of these tasks?
- 2. How do high school mathematics teachers maintain or fail to maintain the cognitive demand of the tasks they implement in their classrooms?
- 3. In what ways, if any, does the use of instructional technology influence how mathematical tasks are selected, the cognitive demand of those tasks, and how they are implemented?

Further detail on how data were collected from each source is described below.

## Preliminary Interview

I conducted a semi-structured preliminary interview with each participant. Interviews are used "when researchers want to gain in-depth knowledge from participants about particular phenomena, experiences, or sets of experiences" (deMarrais, 2004, p. 53). The preliminary interview (detailed in Appendix B) had several purposes. First, it allowed me to gain insight with respect to the teacher's general methods and rationale for selecting tasks to be used in class. Second, it established for me the teacher's general practices in using technology during instruction and any special considerations the teacher may make when choosing technology-oriented tasks (TOTs). Finally, the preliminary interview provided me with a look at the teacher's background, personality, and general habits that would allow me to better understand what was happening in the classes I observed. Each preliminary interview was audio recorded and transcribed. The duration of the preliminary interview varied by participant, with none exceeding 45 minutes. Excerpts from preliminary interviews appearing in this report are labeled with the acronym *PI* followed by the line numbers.

## **Observation Interviews**

Immediately before each observed lesson, I interviewed the participant informally to determine the goals for that particular lesson, the understanding of the mathematics needed to reach those goals, the mathematical tasks selected to meet those goals, and the rationale for selecting those tasks. An interview guide for these interviews can be found in Appendix C. Field notes were taken to capture the participant's responses.

## Classroom Observations

For each participant, one class period was selected for observation based upon its content and potential for producing viable data. I observed this class each day over the course of a typical (not including standardized testing, end-of-course examinations, etc.) 2-week period. Each observed class session was videotaped, with the camera located in the back of the classroom and focused on the teacher. During times when the class was engaged in individual or small group work, the camera followed the teacher's movement around the room. A portable microphone carried by the teacher was used to capture conversations between the teacher and individual students or groups of students. Excerpts of classroom discourse appearing in this report are labeled with the date of the lesson followed by the time interval of the lesson during which the discourse occurred.

After each observed lesson, I wrote field notes to reflect upon what had occurred during the lesson, focusing on the following questions guided by the MTF, when appropriate:

• How did the teacher initially present the task? Was this presentation consistent with how it is found in the textbook or curriculum materials?

33

- How did the students work on the task? Was this implementation consistent with the teacher's presentation?
- How did the teacher interact with students while they were working?
- How did the teacher and students collectively resolve their work on the task? Was this resolution faithful to the teacher's original intent for the task?

In my field notes, I also recorded my general impressions of what had occurred during the lesson, as well as any comments the teacher made about it.

## Final Interview

Following the 2-week period of observation of a participant, I reviewed the transcript of the preliminary interview, watched each recorded lesson again, and reviewed all field notes I had written. During this process, I noted particular events as well as general trends in the data pertaining to my research questions. From this analysis, I wrote questions focused on capturing the teacher's perspective of what I had observed during the lessons. I compiled these questions to create a final interview protocol customized for each participant. (Each participant's final interview protocol is provided in Appendix C.)

The final interview was conducted with each participant to provide further detail, clarify, and get a general response to the events that transpired during the observations. Because some questions referred to specific incidents occurring during the observed lessons, the participant was invited to watch a video excerpt of that incident to refresh his or her memory. Although I began asking the questions in the order they appeared on the protocol, I often changed the order of questions based on the participant's responses. When time did not permit me to ask all questions on the interview protocol, I prioritized questions according to their perceived strength with respect to generating worthwhile data and maintaining a balance in addressing my three research questions. Each final interview was audio recorded and transcribed. The duration of the final interview varied by participant, with none exceeding 70 minutes. Excerpts from final interviews appearing in this report are labeled with the acronym *FI* followed by the line numbers.

## Student Data

Finally, student data, including recordings of discussions with the teacher during class and written work on class handouts, were collected. The handouts given to students during the observed lessons provided a record of the tasks used and established a basis for determining the cognitive demand associated with those tasks. Student work on these handouts was collected from a sample of students. Although the study did not focus on student learning (the achieved curriculum), examining student discussion with the teacher and written work aided me in confirming the analysis of the other sources of data. In particular, it helped to clarify my thinking with respect to cognitive demand and, in some cases, identified additional methods or resources students might use when working on a task. Examining how students chose to approach tasks also confirmed whether teacher actions in the classroom influenced the cognitive demand.

## Data Analysis

To ensure that the collected data are not unfocused, repetitious, or overly voluminous, case studies require an ongoing data analysis while they are collected (Merriam, 1988). This ongoing analysis informed subsequent data collection in an attempt to ensure the relevance of data. In addition, the framework for analysis changed after each pass through the data as I refined my concepts of task, problem, and cognitive demand. Data were analyzed in the following manner.

## Mathematical Tasks

Each observed lesson was partitioned according to the mathematical tasks the teacher used. To determine where this partitioning should occur, I consulted the videotaped lessons I observed as well as data obtained from the observation interviews. I initially used the definition of task offered by Stein et al. (2000), "a segment of classroom activity devoted to the development of a mathematical idea" (p. 7), but this way of thinking presented difficulties in my initial pass over the data. It entailed identifying the underlying mathematical ideas, some of which encompassed multiple problems or exercises. For example, one task in a classroom might have been to review the factoring of quadratic equations. The activity occurring during this task might take the form of a review of homework problems or students working on a warm-up problem to begin the lesson. My dilemma then became how to characterize the cognitive demand of the task associated with that mathematical idea, especially when the problems and exercises varied in cognitive demand. Furthermore, the boundaries between mathematical ideas examined during the same lesson sometimes proved to be nebulous, especially when one idea built on another.

In addition to these difficulties, I initially struggled with the idea of a mathematical task, defined in this sense, as having a varying cognitive demand according to the students involved. This variation stemmed from the lack of separation between the mathematical elements of the task and the students' ways of developing a solution. As increasingly more advanced students work on a task, the task itself does not change, even though the approach students take might.

My conceptual and practical difficulties in using the Stein et al. (2000) definition of task with my data led me to adopt Herbst's (2006) organization, in which each mathematical problem has its own associated task. Having a one-to-one correspondence between problems and tasks, I could establish clear boundaries between my units of analysis and easily assign a single level of cognitive demand to a problem and its associated task. Furthermore, I found that when considering different levels of student ability, the problem remained static while its associated task, the possible operations students might use to solve it, could change.

Despite these apparent advantages, I struggled with this organization as well. I had difficulty finding the boundary between the domains of problem and task. Where did mathematical components of the problem end and the mathematical strategies that students might use begin? In addition, the use of the terms *problem* and *task* in this sense created difficulties in communicating my findings and ideas, as these terms could be perceived as interchangeable.

As a result of my struggle to adopt the terminology for task used in other studies, I modified the term *task* this study. For my purposes, a task was a segment of classroom activity devoted to the development of a mathematical idea, as Stein et al. (2000) defined it, but a task did not necessarily have to represent the entirety of that development. In other words, multiple tasks could address a single mathematical idea. In this way, the scope of my conception of task is similar to Herbst's (2006) conception of problem in terms of relative size as a unit of curriculum.

## Task Selection and Cognitive Demand

I analyzed the data from the preliminary interview and the observation interviews to determine how the teachers selected the mathematical tasks they used in instruction. This analysis entailed looking at the teacher's purpose for each lesson and rationale for how the tasks chosen addressed that goal. In addition, I analyzed the collection of lessons and tasks as a whole (for the entire period of observation) to identify overarching trends in task selection (e.g., the source material for tasks with respect to particular goals).

Despite modifying the Stein et al. (2000) definition of task, I chose their construct of cognitive demand for my analysis. Once individual tasks were identified, the cognitive demands was characterized at the three stages of the MTF: (a) initially, as written in the curricular materials (if applicable, as some tasks were created by the teachers themselves); (b) as set up by the teacher in class (determined by the observation video and field notes); and (c) as implemented by the students (observation video, field notes, and student data).

To identify the cognitive demand for each task at these three stages, I developed and used a modified version of the Task Analysis Guide (Stein et al., 2000). Because the language of the original Task Analysis Guide varied in focus between elements of the task and the thought processes of students, I adjusted the language to focus consistently on the thought processes of students. By making this adjustment, the characteristics of the task were consistently linked to potential student thinking as called for by the definition of cognitive demand. My modified Task Analysis Guide is presented in Figure 2.

Once I classified each task's cognitive demand according to one of the four levels of the Modified Task Analysis Guide for each of three stages described above, I

# Task Analysis Guide (Modified)

Lower-Level Demands The activity of students to obtain an answer is primarily characterized by:

## Recall or Recognition

- formulae, or definitions OR committing these elements to The reproduction of previously learned facts, rules, memory i,
- suitable procedure or a time frame too short for procedures Failure to use a procedure due to the nonexistence of a to be used N
  - Reproduction of previously seen operations (as directed by the exercise) to achieve an answer ŝ
    - A lack of connection to the exercise's underlying mathematical ideas 4

## Procedures Without Connections

- The use of algorithms, which are explicitly called for by the exercise or apparent due to prior instruction, experiences, or placement of the exercise
- Little confusion as to what is to be done or how it is to be done due to the clarity of the exercise di
  - No connection between the procedure being used and the underlying mathematical meaning e.
- A focus on producing a correct answer rather than on mathematical understanding 4
- No explanation of why the procedure was appropriate for the context of that exercise ŝ

Higher-Level Demands The activity of students to obtain an answer is primarily characterized by:

## Procedures With Connections

- Use of procedures to develop deeper understanding of mathematical concepts and ideas
- Tailoring broad general procedures (suggested explicitly or implicitly by the task) for use in the task context and with underlying conceptual ideas di
  - Making connections between multiple representations of the task situation (such as visual diagrams, manipulative, svmbols) e.
- Following general procedures that requires monitoring of elements of the task and engagement with the conceptual ideas that underlie the procedures 4

## Doing Mathematics

- Complex and nonalgorithmic thinking
- Exploration and understanding of the nature of -i di
- mathematical concepts, processes, or relationships
- Accessing relevant knowledge and experiences and using Self-monitoring or self-regulating of cognitive processes ė 4
- them appropriately
- Analysis of the task and active examination of its constraints ŝ
- A great deal of cognitive effort and possibly some anxiety due to the unpredictable nature of the solution process 9

Figure 2. The modified version of the Task Analysis guide, adapted from Stein et al. (2000)

compared these demands across the stages. This comparison served two purposes. First it served as a way to verify the accuracy of my description of each task's cognitive demand. For example, examination of the cognitive demand for a given task as implemented by students may reveal ways of thinking not considered when initially analyzing the task. In that case, the cognitive demand of the task at earlier stages of the MTF was reconsidered.

The second purpose of comparing the cognitive demand of a task at each of the three stages of the MTF was to determine whether the original cognitive demand had changed during instruction. This information helped me to identify actions taken by the teacher that maintained or failed to maintain the cognitive demand. These actions, when identified, were used in the final interview to elicit further information about the teachers' analysis of those instances and the decisions leading to those actions.

## Technology

I gave special consideration to every task that included instructional technology as an available resource for students. For each of these tasks, I considered how the availability of technology might influence the decisions students made with respect to solving the task, the role the technology played in the mathematical work done to solve that task, and the level of thinking needed for students to use the technology appropriately when working on the task. From these considerations, I made an assessment of how the availability and use of technology might affect the cognitive demand of that particular task.

## Validity and Reliability

As displayed in Table 1, multiple sources of data were used to answer each research question. In this way, the data from one source corroborated the data from

another. This triangulation supported the validity of the study's findings. The ongoing analysis of data as it was collected helped to ensure that it remained relevant to the research questions.

In addition, a member-checking procedure, "one of the most needed forms of validation in qualitative research" (Stake, 2005, p. 462), was used. I asked participants to read pertinent sections of a preliminary draft of the research report to corroborate the descriptions and eliminate any misunderstanding. One of the three participants, Michael, commented on the draft by correcting a minor detail about his educational background and offering an alternative interpretation of two occurrences I reported.

## The Cases

This section provides an introduction to the three teachers who participated in this study and a description of each teacher's class I observed.

The Case of Michael

Michael was a veteran mathematics teacher at the Creekbend Academy, a private K-12 Christian school located in northern Georgia. In his nineteenth year of teaching, Michael attained a doctorate in education and National Board Certification, and his teaching was recognized at the state and national level. His interest in technology and, in particular, computer algebra systems (CAS) afforded him opportunities to participate in professional experiences beyond the classroom such as contributing to textbooks and journals and organizing professional development opportunities for mathematics teachers to learn about technology.

*Background*. The first from his family to attend college after graduating from high school, Michael began his studies focused on chemistry, but soon gravitated toward

mathematics and philosophy. He enrolled in mathematics courses as electives and realized that, by tutoring others in mathematics, he could make the money needed to support his studies. In addition, his experiences as a tutor fostered his desire to be a teacher. While discussing this experience, Michael described, "The light that went off as somebody understood something because of whatever it was that I was doing. ... I just got such a high off of other people finally understanding something that had been eluding them" (PI, 35–38). Upon completing his teaching certification, Michael began teaching mathematics at Creekbend.

Michael's interest in technology began well before teaching, but he cited his first year of teaching as being paramount to shaping his teaching with it. "My department chair, one or two days prior to school starting, ... handed me a TI-81 ... and said you're going to be teaching that, teaching with this in your Algebra 2 class this year" (PI, 236– 239). At the time, graphing calculators were a new technology, and Michael encouraged his students to experiment with the TI-81 and share what they discovered. This classroom environment, according to Michael, shaped his practice:

[The students] noticed patterns that I hadn't seen before. They offered theorems that I had never considered. And now I had to go back as a learner. I was the teacher but I had to go back and figure: Is this something legit? ... If it is, what in the world is going on here? ... This is how math and science really does grow. Somebody notices a pattern, and you've got to figure out: Is it real? ... And that's what technology gave me. (PI, 264–270)

The contributions of students provided Michael with a standard for success in his teaching. When asked to describe a lesson he had taught that was, in his opinion, particularly successful, he stated that his best lessons involved working on "one or two problems and [solving] each one of them three or four different ways" (PI, 179), with some of these methods involving novel techniques or approaches. "I've been doing this

for 20 years, and ... it's a tremendous day when a student offers a perspective that I hadn't quite considered before" (PI, 181–183).

Michael also attributed his development as a mathematics teacher to opportunities for professional development in which he had participated. Early in his career, Michael's department chair sent him to conferences focusing on instructional technology, often specifying which sessions he must attend. Michael described the advantage of his situation: "That I was free to move about, that I was free to explore, that I was hungry. I was very fortunate to have right mind and the right mentor in the right place" (FI, 386– 388).

In addition to exposure to new ideas relatively early in his career as influencing his professional growth, Michael also stated the importance of working and interacting with his colleagues. At the time of my observation, Michael and a colleague were revising an honors calculus textbook that they had co-written. Speaking of this and other interactions with his colleagues, Michael stated, "just being around people can spur your own mind to be creative and keep exploring" (FI, 429–430).

*Observed class.* For my study, I observed Michael's Honors Calculus class during a 2-week period late in the school year. This class comprised seven 10<sup>th</sup>- and 11<sup>th</sup>-grade students: three girls and four boys. During my observation, I found these students to be enthusiastic towards the mathematics Michael taught (mirroring his own enthusiasm), as well as very creative in their explorations of it.

The Honors Calculus course designed by Michael was part of a 2-year sequence encompassing the content of both precalculus and an Advanced Placement (AP) Calculus BC course. Explaining the course's design, Michael stated, "Essentially the precalculus year is condensed into slightly more than the first semester, and then differential [calculus] fills out all but a month of the spring" (PI, 90–92). In the second year, the students explore integral calculus in preparation for the AP exam.

The students in Michael's Honors Calculus course were expected to use the Texas Instruments TI-*N*spire calculator model, which features a CAS platform capable of performing many algebraic and calculus-based operations including solving equations and inequalities, factoring and simplifying expressions, differentiation, and integration. Michael viewed the calculator use in his class not for explicitly defined times but rather as "an organic separate tool" (PI, 97) to be used when appropriate. In order to use the calculator in this way, he stressed the importance of his students learning to distinguish when calculator use is appropriate. Although common practice might suggest this goal entails encouraging students to recognize opportunities where technology use is unnecessary, Michael stated that, for his students, the situation is the opposite:

It's really difficult for students to know when to use the machine and when not to. ... With the honors students, ... I'm not worried about calculator dependency with them because they've been trained so hard away from the machine that I'm trying to pull them back. (FI, 119–122)

Michael also explained that his students were expected to understand all topics in both technology- and nontechnology-based contexts. This expectation was evident in his tests, which comprised two parts, one in which calculator use was allowed and one in which it was not.

Honors Calculus met four times a week, each for 55 minutes, on a rotating schedule. The subject matter of the class sessions I observed is provided in Table 2. These class sessions contained the last novel material that the students would study prior to reviewing for the final exam. Table 2

Class	Sessions	Observed	for	' Michael's	Honors	Calculus	Course
CIGOD	565510115	0000011000	101		11011010	Culculus	000000

Date	Topic
Thursday, April 23	Particle motion
Friday, April 24	Chapter review
Wednesday, April 29	Parametric differentiation
Thursday, April 30	Parametric differentiation
Monday, May 4	Review of homework
Thursday, May 7	Roots of tangent lines, Newton's method
Friday, May 8	Newton's method, basins of attraction

Note. Honors Calculus did not meet on Tuesdays. Class sessions on April 28 and May 6 were not observed because of testing. The class on April 30 was videotaped and analyzed, though I was not present during class. Michael was absent from school on May 1.

When asked to describe his role in the classroom, Michael said he would like to be more of a facilitator but that he spends a lot of time during the lesson at the front of the classroom. He quickly added there is also a great deal of discussion taking place in his lessons: "It's not just pure lecture from my end. It's always interacting. But my classes will often be pretty noisy because they're not just having conversations with me" (PI, 215–217). He said this unstructured form of interaction, although "random and chaotic" (PI, 224) at times, facilitated his students' learning.

The Case of Sarah

Like Michael, Sarah was a veteran mathematics teacher, having taught at the secondary level for 15 years. Sarah and David (the third participant, to be introduced below) taught at Mountain Lake High School located in a small city in northern Georgia. Sarah's experiences in mathematics education extended beyond the secondary classroom; she had earned a doctoral degree in mathematics education and had taught at the college level before returning to the high school classroom.

*Background*. Sarah had initially enrolled in college as a chemistry major with aspirations of being a veterinarian. After deciding that she did not like chemistry and reflecting on her own teachers' positive influence on her life, she gravitated toward being a mathematics teacher. Describing her choice of mathematics, she said, "I picked math because I thought it would be the easiest thing to teach" (PI, 13–14), referencing her experiences as a good student in a traditional classroom. Her courses in mathematics education as an undergraduate, however, challenged her initial conception of what it meant to teach mathematics. Throughout her coursework, she embraced a student-centered approach to teaching mathematics. "I just thought, 'Why didn't somebody teach it to me like this? And how did I never get to ask these questions?' And I just completely fell in love with it" (PI, 31–32).

As a mathematics teacher, Sarah focused on her how her students worked together and discussed the mathematics at hand. She described her idea of a successful mathematics lesson as "when kids are talking to each other about mathematics and … they're getting somewhere with their conversation" (PI, 137–138). When elaborating on facilitating student discourse, she also noted an inherent difficulty between balancing the sharing of ideas and maintaining the opportunity for individual students to make their own discoveries or mathematical connections:

The hardest part for me is to balance that kind of lesson where it is exciting when you figure it out. ... I want the kids to be enthusiastic about sharing, [but] when they all have the opportunity to get it themselves, that's the best lesson to me. (PI, 144–147)

Sarah viewed her job as not only cultivating her students' interest and enthusiasm in mathematics but also channeling that enthusiasm in a way that each student had an opportunity for personal discovery.

*Observed class.* I observed Mathematics 1, Sarah's first class of the day. Mathematics 1, designed for students in their first year of high school, was a part of the newly implemented state curriculum framework called the Georgia Performance Standards (GPS). Designed to address process as well as content, GPS provided teachers with content standards, suggested mathematical tasks, and examples of student work with accompanying commentary.

In their first year of implementing Mathematics 1, Sarah and her colleagues struggled with implementing the GPS as it differed considerably from what teachers and students alike were used to. "This is by far the most difficult year I've ever had teaching just in terms of planning and classroom management. I think it's just a different kind of mathematics [and] kids who just didn't get prepared" (PI, 261–263). In addition to striving for more of a balance between conceptual and procedural understanding, the GPS was also organized in an integrated manner with algebra, geometry, and other strands of mathematics spread across the entire high school curriculum.

The Mathematics 1 class I observed had 24 students (13 boys and 11 girls) and was representative of the diverse population of students attending Mountain Lake (12 Latino, 8 Black, and 4 White, with several English language learners). Sarah described her Mathematics 1 students as below average for their grade level, compounding the challenges provided by the new curriculum. "They are, by whatever standards we've set, these are the weak math students. And I guess our standard is they haven't been successful in math in the past" (FI, 387–389). Sarah viewed her job as important because learning in her class was a way for her students to overcome their fear of mathematics and boost self-esteem. However, she admitted this task was incredibly difficult:

I feel bad for a lot of them because they're just struggling. So they act out. A lot of my planning is really taking into account what my kids can handle, what their attention span is, and if I give them something, are they going to throw it? They're really sweet, [but] they haven't learned that this is how you behave in school. (PI, 263–268)

This struggle to acclimate to a productive learning environment was evident as few students had completed their homework assignments during the course of my observation. As a result, Sarah had to rely on class time alone to attain her goals for the class.

To aid her students' learning, Sarah had a classroom set of Texas Instruments TI-83 calculators and computer software that projects the calculator for the class to see, as well as listing the keystrokes Sarah would use during a demonstration. Sarah viewed her student's use of the calculator as important because it performed many of the calculations with which the students struggled:

I just put the calculator in their hand and beg them to use it. I've taught them how to put fractions in the calculator. They're missing the questions on probability because they can't add a half and a third. Well, at this point I care way more about if they know when to add a half and a third than can they do it by hand. (FI, 182–185)

Sarah was enthusiastic about the ways that calculators could help her students, but she also noted that they were a source of tension at her school. Despite the benefits of using graphing calculators to learn mathematics, students were not allowed to use them when taking Georgia's high school graduation test. (The test prohibited graphing calculators and those with data storage capabilities.) Sarah said this policy was especially frustrating because many of the tasks written for the GPS framework encouraged students to use graphing calculators for exploring the mathematics at hand. As a result, Sarah said she had to balance her teaching so her students would not become overly dependent on using their TI-83s.

Sarah's Mathematics 1 class met each day for 90 minutes over the course of the semester. Table 3 displays the subject matter addressed during my observation. These lessons represented part of a unit on quadratic functions, focusing on the relationship between a function's factorization and the *x*-intercepts of its graph.

Table 3Class Sessions Observed for Sarah's Mathematics 1 Course

Date	Торіс
Thursday, February 19	Describing families of parabolas
Friday, February 20	Characteristics of quadratic graphs
Monday, February 23	Factors of quadratic functions
Tuesday, February 24	Solving a quadratic equation by factoring
Wednesday, February 25	The relationship between solutions and factors
Thursday, February 26	Fitting a quadratic curve to real-world data
Friday, February 27	Maxima and minima of quadratic functions
Tuesday, March 3	Finding the maximum in a real-world context
Wednesday, March 4	Review of factored and expanded form
Thursday, March 5	Comparing the minimum points and <i>x</i> -intercepts

Note. Though class met on March 2, I did not observe it as my return to Georgia from weekend travel was delayed due to inclement weather.

During our conversations, Sarah did not explicitly discuss her role as a teacher in the classroom. Her lessons focused on activities students did in groups of three or four. During group work, Sarah moved around the room, listening to student discussion, asking questions, and providing feedback. She intermittently interrupted group work to discuss what the students had found, where they struggled, and how it was connected to what they had previously learned.

## *The Case of David*

As mentioned above, David was a colleague of Sarah's, teaching Mathematics 1 students at Mountain Lake High School. Unlike Sarah, David was relatively new to teaching mathematics, having taught for 3 years. A second-career teacher, he had earned a masters degree in education and had an interest in using technology to teach mathematics.

*Background*. David entered college out of high school with the intention of becoming a mathematics teacher. After enrolling in college, however, he decided to look at more financially lucrative career paths, including electrical engineering. He did not finish his degree initially and spent several years in the freight industry driving a semitrailer truck before returning to school. David described this experience as important in shaping his views of the importance of precise communication with respect to learning mathematics:

I would have a much harder time teaching if I had gone into teaching right out of high school, having worked with people who couldn't communicate. If you drive a truck, and you have to talk to people, and they're trying to give you directions, you get really good at understanding how important precision is. So that's one of the things I stress more to [my students] than a lot of other teachers do. [FI, 331–335]

David's experiences outside of the field of education also shaped his view of the value of learning mathematics:

[My students] tell me, "I'll never have to solve any of this stuff in real life." That's probably true. ... So the big thing that we're trying to teach is more of the thinking skills than the math. The math is a vehicle to teach [students] what they need to know. [FI, 336–341]

For David, the process of learning mathematics helped students shape their ability to think critically and solve problems in nonmathematical as well as mathematical contexts.

David chose mathematics because it had been the easiest subject for him to learn. The ease with which he learned mathematics, however, posed a challenge for him as a teacher: "What I'm finding out teaching [mathematics] is, because it came easiest to me, it makes it probably the hardest to teach. If you learn something right away, it's hard to explain it to someone else" (PI, 21–23). This challenge was compounded by the implementation of the GPS Mathematics 1 curriculum. He noted that because the topics found in an Algebra 1 course were now distributed across several grade levels, it was more difficult for him to make assumptions about the knowledge his students brought from their previous mathematics classes. In addition, he admitted struggling to devise hands-on activities to support students' learning of mathematics, the kind of activities emphasized by the GPS.

Despite the challenges presented by the GPS curriculum, David thought that adopting it was a step in the right direction. When asked to describe a successful lesson he had recently taught, he cited a GPS-inspired lesson he and his colleagues had implemented in which the students constructed different sized cubes with sugar cubes in order to study linear, quadratic, and cubic relationships. Describing his students' work, David said, "I was impressed with what they were able to learn and actually retained it for later assessment" (PI, 86–87). Although he could see the potential of the GPS curriculum, David was also candid in his assessment of it: "I don't think it's going to get easier to teach as you go on" (FI, 391–392).

David's teacher education program had exposed him to a variety of instructional technologies that mathematics teachers could use in their classrooms including Microsoft Excel, Key Curriculum Press's Geometer's Sketchpad and Fathom software, as well as the Texas Instruments Calculator-Based Ranger and Calculator-Based Laboratory. At the time of my observation, he and one of his colleagues at Mountain Lake were working to develop an elective mathematics course featuring these technologies. Noting the constraints of the mainstream curriculum, David envisioned his course as being offered to non-college-bound students:

It's more hands-on, and it's more practical. The things you can do in Excel that would help the average student, we don't offer, ... because we have a curriculum, and we have to cover things. Unfortunately, Excel doesn't mesh really well with what we have to get in. (PI, 175–179)

Despite the pressure to cover certain topics in the GPS curriculum, David did manage to incorporate some forms of technology in the Mathematics 1 course I observed.

*Observed class.* For my study, I observed David's Mathematics 1 class that met during the second block of the school day. As in Sarah's class, the 17 students (7 male, 10 female) David taught represented a diverse set of backgrounds (8 Latino, 1 Asian, 4 Black, 4 White), and many lacked the mathematical competencies expected at their grade level. David commented on how his students' mathematical abilities interfered with learning more advanced concepts: "They all struggle with ... the math facts, the multiplication facts. It's hard to factor when you don't know how to get 12 other than by 3 and 4" (FI, 48–50). In reflecting upon the challenges of trying to use the GPS

framework with his students, he said:

We do have a different population than the rest of the state. [The GPS framework] is written for the state. So, because of the language barriers that we have, more so with our students, there are things that we have to change. (FI, 178–180)

As an example of the changes he had made to the GPS framework, David said that he

sometimes modified or omitted material that might be problematic for his students.

David had a classroom set of Texas Instruments TI-84 graphing calculators for

student use. In many cases, however, he refrained from allowing the students to use them,

because of Georgia's mathematics assessment:

I'm trying to move away from using graphing calculators. It's a great tool for them to have, and it does help give them access to some of the higher-level thinking that they are going to have to do. The problem is they can't rely on it too much because, when they get to the assessment at the end, the state doesn't let them use a graphing calculator. (PI, 108–112)

Although he had reduced the use of graphing calculators in his classes, David did say that he relied on the graphing calculator to show his students how functions could be represented in different ways, as a graph, a table, or an equation. He also mentioned that his school was working to obtain scientific calculators without graphing capabilities for students to use in class.

Table 4 displays the subject matter addressed during each 90-minute block of

David's class that I observed. Like Sarah's Mathematics 1 class, David's class was engaged in a unit on quadratic functions.

When talking about his class's discussions, David described his own role as one who synthesized the contributions that students made in class. During our final interview,

Table 4

Date	Topic
Thursday, February 19	Modeling quadratic functions: Graphs
Friday, February 20	Modeling quadratic functions: Average rate of change
Monday, February 23	Factoring trinomials
Tuesday, February 24	Factoring trinomials
Wednesday, February 25	Quiz: Solving quadratic equations by factoring
Thursday, February 26	Solving quadratic equations by factoring, with graphs
Friday, February 27	Modeling quadratic functions: Solving with graphs
Tuesday, March 3	Solving quadratic equations with graphs; odd and even functions
Wednesday, March 4	Odd and even functions; modeling rational functions
Thursday, March 5	Modeling rational functions

Class Sessions Observed for David's Mathematics 1 Course

Note. Though class met on March 2, I did not observe it as my return to Georgia from weekend travel was delayed due to inclement weather.

he described how students would present different partially correct solutions to the task at

hand. To reconcile the differences, he would

show the commonalities [between student contributions] or actually just merge them into one. ... A lot of times, one group will ... explain the input part really great, and they'll just sort of [say], "It gives you something." Another group will ... give this nice long eloquent answer on this is how you found [the output], this is what you did, and this is what it means. So if I take their input and their output and put them together, I've got a better answer than either of the groups had. (FI, 311–316)

In this way, David thought that he helped his students see how their work fit into the

larger context of the task at hand and how collaboration with others was helpful in

solving tasks.

The profiles of each participant and his or her observed class above provide a foundation for presenting the results of the study. In the next chapter, I attempt to answer the three research questions with respect to each participant.

## CHAPTER 4

## RESULTS

The results of the study are organized according to the research questions. Each of the three participants is discussed in turn in the context of each question. The participants and their background were introduced in chapter 3.

Question 1: Teachers' Selection of Mathematical Tasks

To examine the mathematical tasks selected by teachers to be used in instruction, I posed the following research questions: (a) How do high school mathematics teachers select the mathematical tasks that they implement during instruction? (b) What is the cognitive demand of these tasks?

To address the first question, I asked the participants to describe their planning practices and to clarify how they produced the tasks they used. In addition, I used the observation interviews and final interview as a means of confirming their rationale for selecting the specific tasks from the lessons I observed. To address the cognitive demand of each selected task, I used my modification of the Task Analysis Guide (Stein et al., 2000) and assigned one of four levels of cognitive demand: (a) recall or recognition, (b) procedures without connections, (c) procedures with connections, and (d) doing mathematics. A selection of the tasks used by the teachers during my observations appears in Appendix E.

## Michael

*Selection of tasks*. When asked how he planned his mathematics lessons, Michael replied that he did not plan nearly as much as he used to early in his career, relying on his experience. To put it in his words, "I just know the courses cold" (PI, 87). He cited the Honors Calculus course he created and I observed as evidence of his familiarity with the content.

Michael described his lesson plans as centered on three to four tasks, each preceded by a brief introduction when a new topic was introduced. A list of the tasks he used during my observation along with where he found them is provided in Table 5. These tasks came from a variety of sources: "I pick my [tasks] by always exploring, reading journals, reading textbooks. I very rarely just follow a textbook" (PI, 119–120). An example of this practice occurred on May 7, when Michael used a task (Roots of a Cubic, 5/7) he had found in a book on CAS (Fey, Cuoco, Kieren, McMullen, & Zbiek, 2003). This task required students to explore a specific relationship between the roots of a general cubic function. In another situation, Michael used information he found in a book on chaos theory (Gleick, 1988 to modify a two-dimensional equation illustrating basins of attraction into a one-dimensional model for use in class (Newton's Method 2, 5/8).

In addition to using tasks from a variety of sources, Michael also sought to bring unique challenges to his students:

A lot of times I'll give them a [task] that has something subtle or a particular characteristic that will force them to pause and figure out how do I get myself past this one point. ... So it really is just trying to be creative in trying to get the students to think. (PI, 33–37)

Many of the tasks Michael chose for his students were from past AP Calculus exams. During my observation of his classes, five of the eight homework tasks that

## Table 5

			Initial level of
Date	Task name and source code	Class context	cognitive demand
			<del>_</del>
04/23	Lamppost Shadow (U)	Homework review	PWC
	Circle and Square (A)	Homework review	PWC
	Bug on a Line (I)	Topic introduction	PWOC
04/24	Critical Points (T)	Student exploration	PWC
	Particle on Line 1 (A)	Homework review	PWOC (a, b), PWC (c)
	Total Distance (U)	Homework review	PWC
04/29	Unit Circle (I)	Topic introduction	PWOC
	Inverse Sine (T)	Topic introduction	PWC
	Parametric Function (I)	Topic introduction	PWOC
04/20	Town out Line (LD)	TT	DUVC
04/30	Tangent Line (U)	Homework review	
	Parametric Ellipse (U)	Homework review	PWOC (a, b), PWC (c, d)
05/04	Particle on L ine $2(\Delta)$	Homework review	PWC
05/04	$\frac{1}{1} \frac{1}{1} \frac{1}$	Homework review	PWOC(a b) PWC(a d)
	Dertiale on Circle (A)	Homework review	$\frac{1}{2} WOC(a, b), 1 WC(c, d)$
	Faiticle off Clicle (A)	nonnework review	FWOC(a), FWC(0, c)
05/07	Roots of a Cubic $(O)$	Student exploration	PWC
00/07	Newton's Method 1 (I)	Topic introduction	PWC
			1
05/08	Newton's Method 2 (O)	Topic introduction	PWC

Michael's Tasks for Honors Calculus

Note. The parenthetical notations after each task indicate its source: (A) AP task, (I) improvised task, (O) outside source, (T) past test item recast in a new context, (U) unknown source. The codes for level of cognitive demand are: (PWOC) procedures without connections, (PWC) procedures with connections. Multiple levels of cognitive demand listed for a given task indicates a multiple-part task where the parts differed.

students asked to review were from past AP exams. (This ratio was consistent with the overall ratio between the AP tasks and all tasks used in Michael's assignments.) When asked how he selected the AP tasks to be included on assignments, Michael stated he solved the tasks released each year, looking for "anything that strikes me as interesting" (FI, 10) which he described as a task that has

multiple solutions, a clever twist or an idea that will let you shortcut something, pulls together ideas in an uncommon way, sometimes maybe just a little bit more challenging, sometimes may involve a subtle point in a definition that students will often take for granted. (FI, 10–13)

Michael added that if he were to rate tasks on a scale of 0 to 10 in terms of difficulty, he selected only tasks that had a difficulty of 5 or greater for his students.

On four occasions during my observation, Michael improvised a task used in his lesson, choosing the function to be used at the moment of instruction. For example, when introducing the concept of particle motion (Bug on a Line, 4/24), he said, "I've got some particle, some little bug, ... moving along the *x*-axis. ... The bug's position along the *x*-axis will be defined as  $x(t) = t^3 - 6t + 5$ " (CT, 4/23). There was a distinct hesitation before Michael stated the function describing the bug's position. During the final interview, Michael confirmed that he had improvised this function. When asked if there was a rationale for deciding what function to use, he replied, "I chose a cubic so that the range would be all [real numbers], so that no matter where I was looking there would be a solution" (FI, 29–31). He also purposefully selected the terms included in the function. Michael said he eliminated the quadratic term and used a negative linear term to ensure that the derivative would be a quadratic equation with real solutions. In addition to this technique, Michael said he sometimes eliminated the linear term in a cubic function to ensure that the derivative would always be factorable.

Furthermore, Michael said his improvisation was intentional. He was not particularly concerned if the functions he used in class were easy to work with:

A lot of times I'll just call for random coefficients from the students so that they even know they're making up the function as we go along. Part of the point is that it's not special functions. It's not like these are the little unique cases where this works. This is a general relationship. ... I really don't care if the numbers are pretty or the numbers are ugly. In fact, ... having ugly numbers come out in the very first shot also keeps students from believing that answers have to be pretty. (FI, 33–40)

Although Michael found improvisation to be a good way to emphasize the generality of mathematical techniques, he also emphasized the importance of adhering to the central goal of the task at hand. For example, given his choice of equation for Bug on a Line, he stated, "A quadratic is always solvable, but the effort to do so can be distracting when the goal is understanding, not manipulation" (email communication, 6/30/09).

In another example of improvisation, Michael used the function of  $y = x^2 - 5$  to demonstrate Newton's method for finding the roots of a function (Newton's Method 1, 5/7–5/8). Here again, although Michael generated this equation spontaneously, he had specific mathematical goals:

I didn't want it to be rational. ... Partly also because the whole nature of being able to use this as a technique for approximating the square root of 5, the fact that this thing converged so quickly ... Isn't this a really cool technique for getting— how many decimal places of accuracy do you want? Well, it won't take you that many iterations to get there. (FI, 276–282)

*Cognitive demand.* When examining the cognitive demand of the tasks Michael selected for his Honors Calculus class, I was aware of his students' relatively advanced mathematical knowledge and training. This knowledge provided them a vast and easily accessible arsenal of mathematical procedures and strategies for implementation. Often, the students quickly determined an appropriate approach for solving the task and skillfully implemented it. The timing of my observation, near the end of the school year, took advantage of their proficiency, as they transferred much of what they had already learned to new tasks.

Because of this proficiency, my assessment of cognitive demand was more conservative (toward the lower end of the cognitive demand hierarchy) to account for the students' ability. Nevertheless, the tasks Michael selected for his students tended to exhibit the characteristics of procedures-with-connections tasks (see Table 5). For example, consider Lamppost Shadow, a task from an AP exam that Michael discussed with the students on April 23:

A student walks past a lamppost that is 15 feet high. The student is 5 feet tall and walks at a constant rate of 6 feet per second.

- (a) How fast is the length of the shadow changing?
- (b) How fast is the tip of the shadow moving?

Although the handout on which this task was found was entitled *Related Rates Supplement II*, and procedures exist for approaching tasks of this type (for example, see Kelley, 2002, pp. 148–150), these procedures are general and must be tailored to the specific characteristics of the task at hand (procedures with connections, Characteristic 2). In this case, the student must visually represent the situation of a student walking past a lamppost; determine an appropriate labeling scheme for the relevant variable quantities such as the distance from the student to the base of the lamppost, the length of the student's shadow, and the distance from the tip of the student's shadow to the base of the lamppost; and devise a way to relate the given information to the quantity sought (such as using similar triangles). When the students discussed this task in class, their comments suggested conceptual difficulties inherent in the task such as determining the direction of the walking student and the preservation of similarity despite a change of the right triangles' shape.
Another task I classified as procedures with connections was Critical Points, a task used by Michael as a warm-up activity on April 24. This task required the students to find all critical points of the following function:

$$y = x^{\frac{1}{\ln x}}$$

Although this task is stated very simply, the means to determine the critical points is not readily apparent. Consistent with the second characteristic of procedures with connections in the Task Analysis Guide, the students had a general procedure for finding the critical points of a function (differentiate with respect to x, set y equal to zero, and solve for x), but that procedure must be tailored to a particular task context and in connection with the use of logarithms to eliminate a variable from an exponent.

Despite the students' ability to quickly recognize the context for particular procedures and apply them, they struggled initially with this task and failed to recognize this context. Even once they recognized logarithmic differentiation as an appropriate way to tackle the step of differentiating the function, they struggled to make sense of the solution (every point is a critical point). This struggle prompted the students to reconsider their conceptualization of critical point, consistent with the first characteristic of procedures-with-connections tasks: the use of procedures to develop deeper understanding of mathematical concepts and ideas.

Several of the tasks Michael used during my observation could be described as procedures without connections. In these cases, the students readily recognized and performed the procedure that the task context demanded. For example, consider the three parts of Bug on a Line that Michael used to introduce linear motion:

I've got some particle, some little bug, ... moving along the x-axis. ... The bug's position along the x-axis will be defined as  $x(t) = t^3 - 6t + 5$ .

- (a) Can you tell me when the bug is at the origin, if ever?
- (b) When is the bug at rest?
- (c) Let's say I'm only interested in the time interval from 0 to 4. When is the bug moving the fastest?

In the case of each part, the students immediately recognized the procedure necessary to attain an answer, most likely because of their frequent exposure to similar task contexts during previous instruction (procedures without connections, Characteristic 1). For example, determining when the bug is at rest (a thin disguise for setting the related velocity function equal to zero) is a familiar procedure for students who are completing their first year of differential calculus. Observation of the students during the class when this task was presented confirmed the classification of this task as procedures without connections. The students instantly supplied a correct strategy for each part of the task and correct answers (procedures without connections, Characteristic 4).

A second example of a procedures-without-connections task is Parametric Function, which required students to differentiate the following function:

$$x(t) = \cos(t) + t$$
$$y(t) = t^{3} + \sqrt{t}$$

Even though Michael had just introduced parametric differentiation to his students, they could also readily differentiate this function because of their familiarity with differentiation rules for powers, roots, and trigonometric functions (procedures without connections, Characteristics 1 and 2). As a result, differentiating each part of the function with respect to *t* was trivial. As with Bug on a Line, when the students quickly and correctly calculated the derivatives, that indicated that this task, as stated, could be more accurately described as low-level.

As Table 5 shows, the level of cognitive demand within multi-part tasks often varied. In these cases, the preliminary parts asked the students to produce data for use with the more complex subsequent parts. For example, the task Kite (5/04) demanded different levels of operations from the students:

*Kite*. A kite flies according to the parametric equations  $x = \frac{t}{8}$  and  $y = -\frac{3}{64}t(t-128)$  where *t* is measured in seconds.  $0 \le t \le 90$ (a) How high is the kite above the ground at t = 32 seconds? (b) At what rate is the kite rising at t = 32 seconds? (c) At what rate is the string being reeled out at t = 32 seconds? (d) At what time does the kite start to lose altitude?

As with the procedures-without-connections tasks described above, the students could easily recognize and use well-rehearsed procedures to solve the first (solve for *y* at t = 32) and second (solve for the derivative of *y* with respect to *t* at t = 32) parts of Kite. The third and fourth parts, however, presented a more substantial challenge. For (c), the students had to represent the length of the string (a function of *x* and *y*) as a function of *t* (procedures with connections, Characteristic 2). For (d), the students had to recognize that the point where the kite begins to lose altitude is when it reaches a maximum height and determine when that situation might occur.

In general, the level of cognitive demand of the tasks selected by Michael for use in his class corresponded to the context in which each task was used. For example, the tasks he used to introduce a new topic or concept tended to be procedures without connections, a lower level. Tasks used for exploration or homework tended to have a higher level of cognitive demand, procedures with connections. As many of the homework tasks were from AP exams, this higher level of cognitive demand confirms Michael's effort to implement more difficult AP tasks (5 or above on his 10-point scale).

#### Sarah

*Task selection*. Sarah described her lesson-planning process as beginning with the GPS framework and examining the state-provided tasks for the content she was to teach. She said the quality of these tasks varied and that she had to be selective with the GPS material:

I want [a task] to be open-ended enough that kids have a chance to at least talk about it or explore it a little bit before all of these very specific guided questions come. ... While they say that they're [open-ended] tasks, [the tasks] tend to just be very long word problems with ... a context, and ... fifteen different sequential questions that sort of lead [students] to the answer. I just don't think there's a lot—a lot of tasks just don't allow kids to explore at the beginning. (PI, 51–56)

To illustrate this point, Sarah described a GPS task in which students were to explore the relationship between the area and perimeter of a rectangle. This task immediately asked students to find side measurements for specific cases without allowing them an opportunity to explore the possible dimensions a rectangle might have for some fixed perimeter.

Despite these shortcomings, Sarah said it was still possible to use the GPS tasks in a productive way: "You can use the context and then not use all the little subquestions. … We've rewritten some of [the tasks] where we've taken out all of those little tiny questions" (FI, 48–51). At other times, however, Sarah abandoned the GPS task provided for a given mathematical objective and relied on other resources. That was the case for the unit on quadratic functions during which my observation took place.

The GPS-provided task for quadratic functions was called *Paula's Peaches* (for a complete description, a link to the task on the Georgia Department of Education website is provided in David's section of Appendix E). It required students to explore the context of a peach farm where the number of trees planted per acre correlated negatively with the

number of peaches that could grow on each tree. As a result, the yield for the entire farm when expressed as a function of the number of trees per acre behaved in a quadratic manner. Students explored this context by examining the yield for different amounts of trees per acre and identifying the number of trees that provided the maximum yield. Through this exploration, they also explored the symmetry of quadratic graphs and solved quadratic equations through factoring.

Sarah elected not to use Paula's Peaches. Her reasoning for this choice provided insight as to what she valued in the tasks she uses in her classes. First, she felt that it did not adequately address the standard in question. Although Paula's Peaches did address graphs of quadratic functions and included practice in solving quadratic equations and factoring trinomials, Sarah felt that the context focused more on optimization. Second, she felt the numbers used in Paula's Peaches were unfriendly for students. As a result of these factors, Sarah doubted whether the task would have been effective with her students:

[The context] would have gotten in the way of them being able to visualize *x*-intercepts on the graph with factors. And I don't think that connection was made strongly enough in that task. ... You can see from my lesson that I spent a lot more time trying to make that connection. (FI, 15–20)

Sarah also added that Paula's Peaches included only algebraic representations of factoring and said an area model or other graphical representations should also be used.

When not using the tasks provided by GPS, Sarah consulted other resources. First, her school had adopted a textbook published by Carnegie Learning (Hadley, Snyder, & Sinopoli, 2008) that had been written to exclusively address the GPS Mathematics 1 curriculum. Although this textbook was convenient because each student had a copy, Sarah found, like those provided by GPS, the tasks were composed of "a context with a series of really leading questions following it" (PI, 112–113). As a result, Sarah said she relied upon her past experiences as a teacher, drawing on effective tasks and activities used in prior years and modifying them to suit her needs. In particular, she mentioned her prior work with the Connected Mathematics Project materials (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006), from which she used the investigations I have labeled Parabola Maxima and Minima (2/27).

The tasks Sarah used during my observation are displayed in Table 6. Students' work on some of the tasks extended over multiple days. In this case, the task is relisted on each day it appeared. As noted from the table, Sarah spent little time during class reviewing homework. This practice can be attributed to only a few of her students having their homework completed on a regular basis. Instead, she reviewed the concepts that the students were supposed to study by providing new tasks for them to think about. *Cognitive demand*. Assigning the cognitive demand for the tasks Sarah used with her students was challenging because of the students' difficulty in recalling the procedures she had taught during previous lessons (or, in extreme cases, the same day). I classified many of the tasks in Table 6 as procedures without connections because they were intended to serve as a review of previously learned concepts and procedures. These tasks explicitly called for specific algorithms (procedures without connections, Characteristic 1) and were not connected to the underlying mathematical meaning (Characteristic 3).

Despite the apparently low level of cognitive demand, these tasks created difficulty for Sarah's students. They either could not recall the definitions and procedures specified by the task or could not recognize what to do from the directions. I describe these challenges in more detail below in the results for the second research question.

## Table 6

			Initial level of
			cognitive
Date	Task name and source code	Class context	demand
	_		
02/19	Domain and Range of $y = x^2$ (I)	Concept review	PWC
	Families of Parabolas (U)	Student exploration	PWOC/PWC
	Quadratic Data Table (U)	Student exploration	PWOC/DM
02/20	Quadratic Data Table (U)	Student exploration	PWOC/DM
02/23	Quadratic Data Table (U)	Student exploration	PWOC/DM
	Factoring Trinomials (I)	Concept review	PWOC
02/24	Domain and Range With Graphs (I)	Concept review	PWOC
	Quadratic Characteristics Review (I)	Procedure review	PWOC
02/25	$p^2 + 10p + 25$ (T)	Homework review	PWOC
02,20	$v^{2} + 17v + 72 = 0$ (T)	Homework review	PWOC
	$r^2 - 9r - 36 = 0$ (T)	Homework review	PWOC
	$f(r) = r^2 - 9r - 36 = 0$ (T)	Homework review	PWOC
	Sports Arena (O)	Homework review	PWC
02/26	Sports Arena (O)	Student exploration	PWC
02/27	Parabola Maxima and Minima (O)	Student exploration	PWOC/DM
03/03	Polynomial Skill Review (U)	Procedure review	PWOC
	Dog Run (T)	Student exploration	PWOC/PWC
03/04	Dog Run (T)	Student exploration	PWOC/PWC
03/05	Quadratic Modeling (O)	Concept application	PWC

Note. The parenthetical notations after each task indicate its source: (T) Textbook, (I) improvised task, (O) outside source, (U) unknown source. The codes for level of cognitive demand are: (PWOC) procedures without connections, (PWC) procedures with connections, (DM) doing mathematics. Multiple levels of cognitive demand listed for a given task indicates a multiple-stage task where the stages differed.

Sarah used several tasks I classified as procedures with connections. Sports Arena

(2/25-26) challenged students to approximate a quadratic curve that would fit provided

data representing the relationship between wind speed and the pressure wind would exert on the surface of a building. Once this curve had been defined, the students were expected to use its equation to extrapolate additional data points. To accomplish these goals, the students were expected to use a process of trial and error, choosing values of a, b, and c for the coefficients of the quadratic equation and testing their guess by graphing it in the same window as a scatter plot of the data. An established procedure for finding the quadratic equation had been learned, but in using that procedure, students had to consider how changes in the coefficients of a quadratic equation would influence its graph. Thus, the use of the procedure enabled the development of a deeper understanding of quadratic functions (procedures with connections, Characteristic 1). In addition, students were asked to make connections between two different representations of a quadratic function, its equation, and its graph (procedures with connections, Characteristic 3).

I also classified Families of Parabolas (2/19) as high level because it required students to graph groups of related quadratic functions (each group featured a particular transformation, such as a vertical translation) and provide an "observation" based upon the graphs. Although the process of graphing the functions could be characterized as low level, producing an observation with respect to the group of graphs as a whole required the students to connect the change occurring in the symbolic representation (the change of a numerical parameter) with the change occurring in the graphical representation. This connection is representative of procedures with connections (Characteristic 3).

Like Families of Parabolas, several of the tasks used during my observations had both low- and high-level cognitive demand components. One of these tasks, Quadratic Data Table, spanned three class periods (2/19–20, 2/23). During this time, students collected data in the form of different characteristics (coefficients, intercepts, and graphs) for 11 quadratic functions and recorded them in a table Sarah provided. Once these data had been collected, Sarah asked the students to write down any patterns they found or observations they could make about the data. This open-ended exploration required students to build an understanding of the nature of quadratic functions (doing mathematics, Characteristic 4) when they did not have a prescribed method for determining these patterns (doing mathematics, Characteristic 1). Thus, Quadratic Data Table required students to use higher-level thinking to analyze data collected using lower-level procedures.

This combination of low- and high-level cognitive demand could also be found in Parabola Maxima and Minima (2/27), a task Sarah adapted from the Connected Mathematics Project materials (Lappan et al., 2006, pp. 26–28). In this task, students again were asked to generate and examine data with respect to quadratic functions, this time focusing on the intercepts, line of symmetry, and coordinates of the maximum or minimum point. Once these data were collected, the students were then expected to look for and describe patterns within them. The prompts provided specifically asked for students to make predictions and explain their thinking:

Describe how you can predict the *x*- and *y*-intercepts from the equation. ... Can you predict the location of the line of symmetry from the equation? Explain. ... Can you predict the maximum and minimum point from the equation? Explain. (Lappan et al., 2006, p. 29)

Like Quadratic Data Table, Parabola Maxima and Minima expected students to use lowlevel procedures to collect data that could be used for more complex and nonalgorithmic thinking (doing mathematics, Characteristic 1) as they explored and attempted to establish a basic conceptual understanding of quadratic functions (doing mathematics, Characteristic 4). In this way, both procedural and conceptual knowledge could be developed.

Although Sarah indicated that she did not like to use tasks that could be characterized by a series of "very specific guided questions" (PI, 52–53), she did use Dog Run (3/3–4), a task that could be described in that manner. In Dog Run, the students examined the area of a rectangular space built on three sides with a fixed amount of fencing, and the fourth side was a preexisting structure. The task directed the students to initially use specific values of one linear dimension to determine the other linear dimension as well as the area. The students then filled in a provided table with these results and used the table to answer specific questions about the relationships between the variables. For example:

Describe what happens to the length as the width of the dog run increases. Why do you think this happens? ... Describe what happens to the width and area as the length of the dog run *increases*. Describe what happens to the width and area as the length of the dog run *decreases*. (Hadley et al., 2008, pp. 134–135)

After answering these questions, the task directed the students to plot the data with respect to length and width (a linear relationship) and width and area (a quadratic relationship). A series of questions regarding the graphs followed, including one that asked students to identify when the area was maximized: "What is the greatest possible area? What are the length and width of the dog run with the greatest possible area? Use complete sentences to explain how you determined your answer" (p. 137).

Dog Run had elements that could be characterized as either procedures without connections or procedures with connections. Initially, when students were asked to determine the length of the dog run when its width was 2 yards, the students had to tailor a known procedure, calculating the perimeter of a rectangle and adapting it to think about a three-sided rectangular figure where one side and the perimeter were known (procedures with connections, Characteristic 4). When the students were prompted to repeat this activity using different values for the width, the needed procedure was apparent because of their experience with the initial example (procedures without connections, Characteristic 1). In addition, although this task asked students to describe in detail the relationships they saw in the data, the task itself brought attention to these relationships. For example, one question asked students to "describe what happens to the length and area as the width of the dog run decreases" (Hadley et al., 2008, p. 135). This prompt explicitly asked the students to analyze the relationship between width and the other two measures. However, by focusing on the width as a decreasing quantity, it suggested a means for analyzing the other two measures: increasing, decreasing, or static. In addition, the students were not prompted to consider how this relationship was connected to the task's context.

Furthermore, by answering the questions asked by the task, the students were shown and asked to consider specific mathematical relationships. Dog Run, however, does not raise awareness of the underlying mathematical argument that was developing because it relieved the students of the burden of making that argument. In this way, I viewed Dog Run's cognitive demand as mostly low level (procedures without connections) because its focus was not on eliciting high-level student thinking. *David* 

*Task selection*. David's method for selecting tasks for his lessons began with the GPS framework explorations. His approach to teaching these explorations, however,

changed as a result of his initial experiences with them. During the fall semester, he found his students could not make the connections between mathematics and context that the GPS tasks intended. "When I tried it without the background and [by] doing discovery-based learning, it's not scaffoldable for them. ... They can't just go from what they know to where the state thinks they should be" (PI, 40–43). To address his students' difficulties, he adjusted his teaching during the spring semester to include "preteaching" each topic: teaching the basic skills to be developed in the GPS task before doing the task itself. Through preteaching, David reported his students could navigate through the tasks in a more independent manner.

When selecting tasks for his students, David relied mainly on the three most immediately available resources: (a) the activities provided by the GPS framework, (b) the Carnegie Learning textbook (Hadley et al., 2008), and (c) an additional textbook published by McDougal Littell (Long, 2008). Once he determined the topic to be studied, David said he usually consulted the textbooks first:

I'll look through the two textbooks and find the one I think is best for doing the background on [the topic]. Generally, I'll start with the MacDougal book because it's the most basic stuff. It's the most straightforward; has the most drill and practice. ... Then, if I have to, I'll do a few sections out of Carnegie book, which is more hands-on, more like the frameworks. (PI, 35–40)

Once David felt he had laid a sufficient foundation of basic skills in a particular area, he would proceed to having his students work on the GPS-provided tasks.

In addition to the state- and school-provided resources, David said he also

searched the Internet and other outside sources for tasks to use:

Often I'll steal parts of Power Points by going on the Web and finding a topic. ... Beyond that, it's just things that have accumulated from 3 years of teaching Algebra 1. So it will be different worksheets I've used in that for the same topic. (FI, 195–198) He described one source of worksheets he had, in which students could match their answers to those in a provided list, as particularly useful: "That often works well as a way to introduce a topic because it lends itself to [students] checking their work. I'll use that for the first couple days of a topic and then build into more rigorous tasks" (PI, 202–204).

During my observations, David told me about a new resource he had found on the Web, the North Georgia Regional Educational Service Agency (RESA) Web site (http://www.ngresa.org/MathResources.htm). From this site, David could download instructional units for Mathematics 1 written by other high school mathematics teachers. "They've already done what we're doing. This should help make our work easier" (Observation interview, 2/20). During my observation, David used supporting materials found on this Web site to support his teaching of selected GPS tasks.

Although David consulted a variety of sources for finding the tasks he used in preteaching the basic skills, his students needed to succeed with the GPS-provided tasks, I did not have the opportunity to observe this process. David's class began work on Paula's Peaches (see Appendix E for a link to the Georgia Department of Education Web site containing this task), a GPS exploration focused on modeling, factoring, and solving quadratic equations, on the first day of my observation. Work on this exploration continued through the majority of my visit (see Table 7). Once Paula's Peaches had been completed, David's class spent the majority of the remaining 3 days exploring two other tasks, Logo Symmetry and Resistance (GPS framework tasks).

*Cognitive demand.* Paula's Peaches occupied the majority of my observation of David's class. In this task, the students were introduced to a scenario where Paula, a peach farmer in Georgia, had to make decisions about her farm based upon data from an

# Table 7

### David's Tasks for Mathematics 1

			Initial level of
			cognitive
Date	Task name and source code	Class context	demand
00/10			
02/19	Paula's Peaches, 1 (G)	Student exploration	PWOC/PWC
	Paula's Peaches, 2, parts a–c (G)	Student exploration	PWOC/PWC
02/20	Paula's Peaches, 2 (G)	Student exploration	PWOC/PWC
02/20	Paula's Peaches 3 (G)	Student exploration	PWOC/PWC
		Student exploration	1
02/23	Paula's Peaches, 4 (G)	Student exploration	PWOC/PWC
	Paula's Peaches, 5–8, all parts (G)	Procedure review	PWOC
00/04			DULOC
02/24	Paula's Peaches, 9 (G)	Procedure review	PWOC
	Paula's Peaches, 10–12 (G)	Student exploration	PWOC/PWC
02/25	Quadratic equations quiz (U)	Assessment	PWOC/PWC
02/26	Review of Factoring	Procedure review	PWOC
	Paula's Peaches, 13, parts a–b (G)	Student exploration	PWOC/PWC
02/27	Paula's Peaches $13-14(G)$	Student exploration	PWOC/PWC
02/27		Student exploration	1
03/03	Paula's Peaches, 15 (G)	Student exploration	PWOC
	Odd or Even (G)	Topic introduction	PWOC
	Logo Symmetry 4, 11 (G)	Student practice	PWOC
03/04	Odd or Even (G)	Topic introduction	PWOC
03/04	Logo Symmetry 11 (C)	Concept review	
	Dogo Symmetry, 11 (C)	Student exploration	
	Resistance (G)	Student exploration	PWUC
03/05	Resistance (G)	Student exploration	PWOC

Note. The parenthetical notations after each task indicate its source: (G) Georgia Performance Standards, (I) improvised task, (U) unknown source. The codes for level of cognitive demand are: (PWOC) procedures without connections, (PWC) procedures with connections. Multiple levels of cognitive demand listed for a given task indicates a multiple-stage task where the stages differed.

agricultural experiment. The key relationship presented was, for every tree per acre planted over 30, the average number of peaches grown on each tree decreased by 12 from the average 30-tree yield of 600 peaches. When considering the average yield of peaches per acre, where the average yield per tree is multiplied with the number of trees per acre, a quadratic relationship was apparent.

Paula's Peaches was broken into 15 separate items, each having as many as ten subdivisions. When viewed in their entirety, some of the items have characteristics of high-level tasks. For example, consider the opening sentence for Item 3:

Paula wants to know whether there is a different number of trees per acre that will give the same yield per acre as the yield when she plants 30 trees per acre. (Paula's Peaches, 3)

Prior to this item, the students had established a formula for both the average yield of peaches per tree planted (T) and for the average number of peaches per acre (Y), with both functions having the number of trees (x) as the independent variable:

$$T(x) = -12x + 960$$
$$Y(x) = -12x^{2} + 960x$$

At this point, the instructions for Item 3 could have been presented in a way that would present quite a challenge. Previously, the students had found the value of *Y* for prescribed values of *x* and had seen that Y(38) = Y(42) = 19,152 peaches, but no clear method had been presented to find different values for the number of trees that would produce the same yield. The students would most likely have had to experiment with further output values of *Y* and, as a result of this experimentation, may have encountered the inherent symmetry by either representing the data in a table or as a graph. These possible student actions can be characterized as procedures with connections because they developed deeper understanding of the quadratic functions (Characteristic 1), made connections between multiple representations (Characteristic 3), and followed the general procedure of testing different numbers of trees while also engaging with the behavior of quadratic functions (Characteristic 4).

Although the opening sentence of Item 3 of Paula's Peaches provided the

potential for students to work on a high-level task, the actual task was subdivided into ten

parts that removed much of the experimentation and uncertainty. Here is a partial list of

these subdivisions:

- (a) Write an equation that expresses the requirement that *x* trees per acre yields the same total number of peaches per acre as planting 30 trees per acre.
- (b) Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in x on one side of the equation and 0 on the other.
- (c) Multiply this equation by an appropriate rational number so that the new equation is of the form  $x^2 + bx + c = 0$ . Explain why this new equation has the same solution set as the equations from parts (a) and (b).
- (d) [Identify the values of *b* and *c*.]
- (e) Find integers *m* and *n* such that mn = c and m + n = b. (Paula's Peaches, 3)

The instructions of these parts, taken as a whole, delineated a clear path toward a solution. When included with the opening sentence, these parts reduced the potential cognitive demand significantly. The students used explicitly called for steps (procedures without connections, Characteristic 1), had little confusion as to what was to be done (Characteristic 2), and made little connection between the procedure they were using and the underlying symmetry of the quadratic function (Characteristic 4).

Although the cognitive demand of Paula's Peaches 3 was reduced by the explicit instructions found its subdivided parts, it still contained elements that would require students to use higher-level thinking. Several parts asked the students to identify and explain connections between the steps they had taken. For example, Part (h) asked the students to connect the zero-product property to their factored equation, and Part (j) asked them to relate their solution back to the context of the task. By focusing students' attention on connecting their work to both the context of the task and the underlying mathematics, these questions may have given students an opportunity to engage in higher-level thinking.

The process of solving a quadratic equation by factoring delineated in the steps of Paula's Peaches 3 was repeated in a similar way in 4. These items set the stage for Items 5–9, where students practiced factoring trinomials and solving quadratic equations by factoring independent of the peach farm context. The directions for these tasks were explicit as to what procedure the students were to use (procedures without connections, Characteristics 1 and 2).

The practice of the solving-by-factoring procedure was followed by Items 10–12, which returned to the peach farm context and asked students to investigate particular yields of average peaches per acre. For example, 12 stated:

At the same auction where Paula heard about the peach grower who was getting a low yield, she talked to the owner of a major farm supply store in the area. Paula began telling the store owner about her plans to expand her orchard, and the store owner responded by telling her about a local grower that gets 19,200 peaches per acre. Is this number of peaches per acre possible? If so, how many trees were planted? (Paula's Peaches, 12)

This task required students to set Y(x) = 19,200 and solve the equation, revealing a singular solution of 40 trees per acre. After several previous tasks requiring students to find the number of trees for a given average yield of peaches per acre, students would have a clear idea of what the task was asking them to do (procedures without connections, Characteristic 2). Although Item 12 did not probe further for the meaning of the singular solution, the following task, Item 13, asked, "What points on the graph [of Y(x)] correspond to the answers to items 10, 11, and 12?" (Paula's Peaches, Item 13, Part (b)). In this way, students may have connected the singular solution they found in Item 12

with the maximum point of the graph, but no part of Item13 called for further exploration of this relationship.

One element of Paula's Peaches that may have inadvertently increased the cognitive demand of its tasks was the language used in the instructions to students. As seen in Item 3, Part (b) above, the focus on asking mathematically precise and pedagogically supportive questions caused the language to become wordy and, given the background of the students I observed, inaccessible. As a result, the language found in these tasks may have increased the cognitive demand in an unintentional and nonmathematical way.

Logo Symmetry asked students to investigate the logos on military uniforms. By doing so, the students were to become familiar with even and odd functions. From this task, David selected Items 4 (exploring even functions) and 11 (odd functions). Consistent with his purpose to have students initially explore mathematical concepts out of context in order to establish basic skills, these tasks were detached from the logo context. For Item 4, students were given points (3, 5), (-2, 4), and (a, b) on the graph of an even function and asked to identify other points that must be part of the graph, describe the symmetry of an even function, and complete the sketch of a graph in which only the positive values of *x* had been drawn. Given that the directions explicitly stated what the students were to do and that David had presented these concepts to his students prior to work on this task, the cognitive demand was procedures without connections (Characteristics 1 and 2).

#### Question 2: Implementation of Tasks

My second research question asked how teachers maintained or failed to maintain the level of cognitive demand found in the tasks they selected. To address this question, I examined whether each teacher's interactions with her or his students changed the potential ways that the students might use to solve a particular task. Once again, I referred to my modified version of the Task Analysis Guide (see chapter 3) to classify how students worked on or thought about a given task. These classifications were compared to those made for each task prior to instruction. To get a sense of why the level of cognitive demand changed or remained unchanged, I examined teacher's actions when implementing the task in class. In general, I found that a teacher's actions could raise or lower the level of cognitive demand for the class, sometimes simultaneously. As a result, it was often difficult and at times impossible to characterize an overall effect a teacher had on the way students thought about a task. Instead, I identify particular instances and characterize the teacher behavior that precipitated them.

*Michael.* Michael's actions during class directly affected the way students considered and worked on the tasks he selected. His interactions with his students had a varied effect on the level of cognitive demand, sometimes lowering it and at other times elevating it. I present below several tasks that illustrate these effects.

Recall Lamppost Shadow, a task Michael discussed with his students and I classified as procedures with connections:

A student walks past a lampost that is 15 feet high. The student is 5 feet tall and walks at a constant rate of 6 feet per second.

(a) How fast is the length of the shadow changing?

(b) How fast is the tip of the shadow moving?

After the task was read aloud by a member of the class, Michael initiated a discussion of

how to solve it:

Michael: What do you know? [Draws a horizontal segment on the whiteboard.]

Jenny: I kept coming up with ideas and then getting zero equals zero.

Paul: You know that the height of the lamppost is 15 feet high.

- Michael: Okay, start drawing a picture of the lamppost. [Draws a vertical segment extending up from the right endpoint (labeled L) of the horizontal segment and denotes its length as 15.]
- Paul: I did that. It doesn't make sense. [Continues naming the constraints of the task.] (04/23, 0:00:45 0:01:03)

In this excerpt of class discussion, Michael placed the burden on his students to

determine an opening course of action for solving the task. In this way, the level of

cognitive demand was maintained. Although the student's response to his suggestion,

"Start drawing a picture," indicated drawing a picture was routine in the context of work

on a task, Michael's suggestion could be interpreted as an overt suggestion that a

productive picture was important and attainable for Lamppost Shadow. Thus, this action

may have lowered the level of cognitive demand. Later on in the discussion, Michael

suggested an element of this productive picture that had not occurred to his students:

Michael: Now what?

Alex: He is walking at a constant rate of 6 feet per second.

Michael: [Pointing to the left and then to the right] It doesn't say which direction? [The students debate what it means to walk *past* the lamppost and what that means in terms of direction.] Think about this. ... As soon as you walk past the lamppost, where is the shadow? [He models this using the clock above the whiteboard as the lamp and walking away from it.] So if I'm heading off to the left, the shadow is here? [He points to the left side of the person in his diagram. From this demonstration, he draws a dotted line from the top of the lamppost in his diagram through the top of the student and continuing to the ground. The resulting figure has two similar triangles that share a common angle.] Jenny: Similar! Paul: They are congruent— Jenny: No, they're similar! Michael: Are they similar triangles? Alex: They look to be. Yes.

Michael: Convince me. (04/23, 0:01:18–0:02:55)

In this excerpt, Michael used a physical representation (himself) to model the movement of the person in the task. However, by completing the diagram without the input of his students, he provided them with the important connection they had struggled to attain on their own. His students became excited when they recognized similar triangles embedded in the diagram, indicating that this was a new discovery for them and that they had not modeled the task situation in this way before. By showing the students this relationship, Michael decreased the amount of thought the students needed to give in order to attain a satisfactory model for solving the task.

Despite this instance where the cognitive demand was lowered while solving the task, the subsequent discussion orchestrated by Michael required the students to engage the ideas underlying the task situation as well as consider multiple ways to represent it. For example, he prompted the students to contextualize their solution:

Does the length of the shadow—does how quickly it grows have anything to do with where that person is standing? Think about that. ... If the length of my shadow has absolutely nothing to do with where I'm standing, why is it that my shadow seems to run away from me faster and faster and faster? (4/23, 0:10:05-0:11:30)

By pressing his students to apply their solution to the task's context, he forced them to focus on more than merely providing a correct answer. Even though a general procedure was applied to solve the task, using it enabled Michael and his students to discuss

mathematical ideas at a level deeper than what it would take to simply provide an answer (procedures with connections, Characteristic 1).

In addition to attaining and interpreting a solution to Lamppost Shadow, Michael encouraged his students to think about different ways to express it mathematically. When discussing how to calculate the rate at which the tip of the shadow is moving, Michael said:

One way to play this is [points to one student] your suggestion here. It's just relativity, right? ... If you want to take a more analytical approach, which I think is what you were saying [points to a second student and proceeds to use implicit differentiation to analyze the movement of the shadow's tip.] (0:14:30–0:15:30)

By attending to different students' ideas for ways to represent the task and making connections between them (procedures with connections, Characteristic 3), Michael ensured that his students' level of thinking with respect to Lamppost Shadow remained high.

Although Michael generally used procedures-without-connections tasks in conjunction with introducing a new topic in class, the questions he posed to his students during these introductions challenged them to think about the mathematics at a higher level. For example, in Bug on a Line, Michael connected the students' act of solving the cubic position function set equal to zero to previous class discussions on using Cardano's formula for solving cubic equations and generalized solutions for other orders of polynomials (procedures with connections, Characteristic 4).

In Bug on a Line, Michael also asked his students to interpret the solutions they attained with relative ease in the context of the task. When his students concluded the bug would pass over the origin three times, he asked how that was possible. When they produced answers with negative values, he challenged them to consider how those values might be interpreted. In this way, Michael connected the answers obtained using routine procedures to elements of the task and underlying conceptual ideas (procedures with connections, Characteristic 4).

During student explorations (Critical Points and Roots of a Cubic), Michael struggled to maintain the cognitive demand inherent in these tasks. Although each was classified as procedures with connections, Michael's eagerness to have students use a viable method of solution led him to suggest possible methods to his students. During Critical Points, the students initially struggled to determine an appropriate way to differentiate the function. When one student said, "It's the same thing we did for x to the x," Michael smiled and said "Oh" in a drawn out and knowing way, implicitly affirming the viability of that method. This affirmation may have preempted any attempts to find alternative methods by other students.

As mentioned above, Michael found Roots of a Cubic in an NCTM book focused on the use of CAS in schools (Fey et al., 2003). Michael set up this task in the following manner:

Michael: You have a cubic function. By the fundamental theorem of algebra, how many zeros do you have?

Alex: Three.

Michael: Three. Are they all real?

Nathan: Not necessarily.

Michael: Not necessarily. Are any of them real?

Allison: Yes, at least one.

Michael: Assuming?

Allison: There are no imaginary coefficients.

Michael: Assuming you have all real coefficients. Okay. But all of that doesn't matter. Let's just say that you have a cubic that has three zeros. What do you want to call them?

Allison: *a*, *b*, and *c*.

Michael: All right, that was creative. So your cubic zeros are *a*, *b*, and *c*. [Speaks while writing on the board] One, I want you to ... write the equation of a tangent at *x* equals the mean of two of the zeros. Make sense?

Allison: Are you saying—is that specifically *a* and *b*, or is it perhaps *a* and *c*?

Michael: Or *b* and *c*.

Paul: It does not matter.

Michael: It doesn't matter. I want you to draw a tangent or get an equation for a tangent for that cubic at the average of any two of the zeros. [Writes on board while speaking. Two.] Find the *x*-intercept of that tangent. And [Three.] Why is that answer cool? [He then encourages the students to work together. The students begin working.] (5/7, 0:00:25–0:03:30]

Michael initially walked around the room monitoring his students' attempts to solve the

task. After 3 minutes, a student commented that he was going to use a "brute force

approach." Shortly after this comment, Michael stopped the class to discuss their work so

far and steer them toward using CAS:

Michael: All right now, folks, what do you notice about this problem already, just in your early thoughts?

Nathan: Well, there are a lot of variables that we don't know.

Michael: Yes. In fact, you don't know any of them, anything.

Alex: You know it's cubic.

Michael: Okay. Good. Is this going to be a pleasant problem to manipulate?

Paul: [Sarcastic laughter indicating "no."]

Michael: Why? [Long pause from students with no reply] So? [After another significant pause, sarcastically.] Oh, please tell me you learned something this year. What should you be doing? [Another long pause with no response.] Is this a problem to work by hand?

Alex: [Exasperated.] Oh, I hate you so much right now. I've already started [doing the task by hand], and I'm not going back.

Michael: Okay.

In this limiting of potential student operations to solve the task and suggesting an optimal way to approach the solution, Michael may have lowered the level at which students thought about the task. When asked to describe the factors he considered before intervening, Michael attributed his actions partially to his "exuberance" (FI, 103) and enthusiasm toward the mathematics in the task but also to what he saw when he walked around the room:

I start getting one question repeated a lot in different places or ... see half to two thirds of the students just stop writing or stop really making any kind of progress. At that point, ... clearly, no one was picking up their machine and doing anything. (FI, 108–111)

When responding to a draft of this chapter, Michael suggested that his students' tendency to resist CAS as a viable method had also prompted him, in this case, to encourage calculator use. In this case, Michael justified lowering the level of cognitive demand in order to enable his students to continue with their investigation.

As in the preceding example, Michael placed great importance on assessing and responding to his student's mathematical understanding. During the preliminary interview, Michael emphasized the importance of being flexible with respect to the lesson plan and the tasks to be used in class, emphasizing the need to follow the interests of his students:

You need to be as flexible as you possibly can to allow ... the course to follow the natural inquisitiveness of the students. When they ask questions, you need to follow the moment because that is when they are the most ready. And if you hold yourself to the textbook, ... then you lose a lot of the motivation for the students to even learn in the first place because you're ignoring their needs and going with some external need. (PI, 122–128)

As indicated by the representative sample of results provided above, I noted Michael's attention to the students' ideas and questions during my visit to his classroom. By carefully considering student questions and providing responses that further provoked their curiosity, Michael was able to provide students with the motivation to work on and remain persistent with high-level tasks.

### Sarah

In her interviews, Sarah placed a great deal of importance on student exploration and discussion as a goal for her class. During my observation, her actions in class supported these goals. These actions also helped to maintain the cognitive demand of the high-level tasks that Sarah used with her students.

In Sarah's class, the students were responsible for making mathematical explanations, answering questions, and determining the validity of the contributions made by themselves and others. By trying "not to ever just give a flat answer" (FI, 132) to her students' questions, Sarah allowed them to think further about their own ideas. She did not accomplish that goal, however, by simply not responding to students' questions and ideas. Instead, she redirected the question or idea back to its originator or to other students in the class. For example, during Quadratic Data Table, Sarah discussed with a group of students an observation made by one of its members:

Sarah [in response to Vince's observation]: You multiply the two *x*-intercepts together, and that gives you the *y*-intercept? Is that what you're saying?

Rosa: That's how you do it?

Sarah [to Rosa]: Are you asking if [Vince's] observation is true? Is it true for you?

Rosa: I don't know.

Sarah: You don't know? Well, look at your two *x*-intercepts and look at your *y*-intercept. (2/23, 0:31:30–0:32:18)

Despite Rosa's request that Sarah validate Vince's observation, Sarah deflected the request back to Rosa, asking her to test the observation with her own data. In another discussion of the same task, Sarah listened to a student's explanation and encouraged her to engage the other members of her group: "I think that's an excellent observation. Is that true on everybody else's paper, or just yours?" (2/23, 0:39:47–0:40:00).

When each group shared one of their observations with the entire class, Sarah still insisted that the students validate those observations. She would call on students and ask if they agreed with the observation. At one point, a student shared his conjecture that the sum of the *x*-intercepts is the same as the coefficient *b*. After some other students had agreed that it was true, Sarah responded, "I disagree with his statement" (2/23, 1:09:20–1:09:24). She then asked the students to check specific examples. During their work, she asked further questions aimed at helping them resolve the dilemma: "Is any part of it true? How could I make it true?" (2/23, 1:12:07–1:12:15) By requiring the students to validate their own or each other's work, Sarah ensured that the level of cognitive demand would remain at the high level intended by the task. The students were forced to test their observations using the data they had gathered and make their own conclusions about the relationships inherent in quadratic functions.

As another way of placing the responsibility of mathematical thinking on her students, Sarah often encouraged students who were struggling with a task to seek out help from those whom she knew had an understanding of what to do. During the second day of work on Quadratic Data Table (2/20), Sarah noticed one student in a group, Jacob, had made the connection that the constant value of a quadratic function was equal to the

*y*-intercept. When Margo, another student in Jacob's group, was struggling to identify the *y*-intercept of a particular parabola, Sarah said, "Jacob can tell you because he knows how" (1:07:15–1:07:25). Shortly after, Sarah advised a student in another group was also struggling with determining the *y*-intercept: "Jacob is about to explain that to Margo, so if you want to listen, that would be good" (1:07:40–1:07:50).

During Quadratic Characteristics Review (2/24), Sarah noticed one of her students, DeShawn, had succeeded in determining the characteristics for the parabola  $y = x^2 + 5x - 6$ . At this point, she announced to the class, "If you are on DeShawn's side of the room, he's figured all of this out. So, if you need help, you can ask him" (2/24, 0:50:55–0:51:05). Later, when she noticed Marta, a student in DeShawn's group, was struggling, she said: "DeShawn, come over here and tell her why that's not right. [To Marta]: He's going to help you. I promise." (2/24, 0:52:50–0:53:01)

Because of Sarah's encouragement, students in her class often took the initiative to help each other when working on tasks. During work on Factoring Trinomials (2/23) a group of students summoned her for assistance. Sarah, however, ended up not needing to say anything of substance:

Jason: We don't know how to do this. Sarah: Do what? Steve: That. Sarah: How to factor? [Gasps in surprise.] Steve: Yeah. Jason: Just a little bit. William: I'll help you. Jason: Don't you put *x*—? William: Yeah, you're right.

Jason: Don't you put *x* minus *x* in there and *x* minus 20? Ain't that how you're supposed to do it?

William: No, it's got to equal -1, those two numbers.

Jason: Oh, duh. Okay, I know how to do it. [Sarah walks away from the group without having said anything more.] (2/23, 1:00:50–1:01:30)

William's intervention into Jason and Steve's conversation with Sarah allowed her to remain silent and not provide assistance that might have lowered the cognitive demand of the task. In a similar situation during work on Dog Run, Sarah was about to help a student when another said, "I get it. Can I help him?" This request placed the responsibility of solving the task on the students, rather than Sarah. As a result, she did not have the opportunity to say something that might have lowered the cognitive demand of the task for her students.

When asked about her efforts to get students to work together, Sarah described several strategies:

I try really hard to do that. If the student is stuck, I try to get that student to ask someone in their group a question. Or, I see that Laura had something to say about that. Laura, could you tell Joseph what you just did, what you just said to me. Or, Joseph could you ask your group what you're asking me? But then you have to make sure to listen that they do. (FI, 104–108)

As the above examples from my observation indicate, the strategies Sarah described here were consistent with those in her practice.

Although Sarah was reluctant to provide assistance to her students when they asked her questions, she did find other ways to support their work on the tasks used in class. At several points during my observation, she provided some sort of aid that gave structure to the students' thinking about a task. During the students' work on Families of Parabolas (2/19), she wrote on the board: "As the coef. of  $y = x^2$  increases, the parabolas

\_\_\_\_\_." In discussing this sentence stem with the students, she said, "This is what your assignment might look like" (0:29:00–0:29:40). In her final interview, Sarah described her rationale for this strategy:

I had heard that's a good ELL [English Language Learner] strategy. You always give them a start. ... And I thought that's a really good idea that you give kids beginnings because then it sparks their thinking. (FI, 179–183)

By showing this template to her students, Sarah provided a way for them to structure their written observations. This assistance may have lowered the cognitive demand with respect to the students' communication of mathematical ideas and also by pointing attention to the relationships between the two mathematical representations needing examination. By doing so, however, Sarah enabled her students to begin productive work on the task and raised the level of cognitive demand in the ensuing discussion of it.

About 15 minutes later in the same task, Sarah typed a list of words on the projected computer screen, "reflect, narrow, wide, approaches *y*-axis, approaches *x*-axis, vertical shift, horizontal shift," identifying them as words the class had used in the fall semester to describe transformations. These words provided further structure for the students in aiding the communication of their ideas. This word list may have aided the students in identifying particular relationships between the equations of the functions and the corresponding graphs. In this way, the list of words also may have lowered the cognitive demand of the task.

During the investigation of Sports Arena (2/26), Sarah modeled the guess-andcheck strategy that the students were to use when trying to determine the coefficients of a quadratic equation that would fit the provided wind speed and pressure data. After the students worked on their own for 2 minutes, she wrote on the board "a = 1: too narrow. a

91

= -5: reflected—not good." She drew a graph showing the parabolas resulting from these choices for the value of *a* and explained why neither of these choices was suitable for modeling the data. By modeling the guess-and-check procedure, Sarah provided structure to the method that the students were to use and the thought process for evaluating each guess. Although Sarah's decision to model this procedure lowered the cognitive demand by indicating students should (a) use a guess-and-check method, (b) make a guess, and (c) evaluate the accuracy of that guess, her students were able to focus on a fourth component of the task: determining how to refine their guess to obtain a more accurate graph with respect to the data. In this way, lowering the cognitive demand of some components of the task enabled students to work at a high level on another component.

Each of Sarah's interventions during her students' work on a task came after she had walked around the classroom monitoring their individual or group work. When asked her rationale for determining when it was appropriate to intervene, Sarah described what she typically saw in the students' work:

I'm looking for either consistent misconceptions or an interesting way to approach the problem. If I see something several times in a short amount of time, then that's an indication to me that I need to pull the class back together. There's something that everybody's not seeing or not getting. Or they don't understand what I've asked them to do. (FI, 97–100)

Sarah intervened with her students only when she perceived that their current understanding of the mathematics at hand would bar further progress with the task. By reserving her interventions only for these occasions, she ensured that the level of cognitive demand of the tasks she gave her students would remain as high as possible.

The examples of Sarah's interaction with her students described above are only a few of many that exhibited her restraint in validating student thinking. When she

encountered a situation in which she felt she had to provide some sort of assistance, she generally did so in ways that preserved her students' level of mathematical thinking for some element of the task.

#### David

In his class's work on Paula's Peaches, David tried to ensure his students could successfully complete each task or part of a task by choosing whether students would work on each in groups or individually or if he would lead the whole class through the steps necessary for a solution. For example, after handing out Paula's Peaches (2/19), he instructed the students to work on (a), (b), and (c) of Item 1 in their groups. After walking around the room to assess the students' progress and to provide assistance where needed, David reviewed these three items with the class. When giving instructions for (d), however, David did not give the class an opportunity to work on the task independently, saying, "Part (d) is a little bit more difficult. ... Now I need to write an equation for this." (2/19, 0:32:30–0:33:10).

When asked how he decided whether certain tasks or parts of tasks were to be worked on independently by students or as an entire class, David described his rationale for this example:

Well, the big part of [(d)] is it's a fairly long question. There's a lot of verbiage in it. There's a lot of notation in it. ... If it's something they could struggle and succeed with, I'll give them 15 minutes before we do it as a class. That looked like something they would struggle with and really not be able to get any further on. So, I thought they would be better served to just lead them through it. (FI, 269–274)

In this selection process, David assessed the difficulty of the tasks or parts of the tasks and, according to his assessment, implemented these tasks in different ways (I observed this varied implementation at several points during my 10-day visit.) In this way, David could ensure the level of cognitive demand was low enough so that his students would understand the inherent mathematics. This practice, however, also prevented some opportunities for his students to engage in high-level thinking.

For Paula's Peaches Item 1, Part (d), the students were to write an expression to define T(x), the average yield of peaches per tree. After stating that this part of the task was difficult, David proceeded to guide the class through it by asking a series of short questions:

David: Function notation. We're going to write it as what? This is the easy part, what are we calling it? ...

Students: T.

David: T, because it stands for what?

Students: Trees.

David: What's our variable going to be?

Students: *x*.

David: ... Now what is this function going to give us? [Calls on student; no response.] This is a function to determine what? [Pause.]

Manuel: The number of peaches grown.

David: The number of peaches per tree. (2/19, 0:33:25-0:34:20)

David then wrote on the overhead projector the results from (b) and (c), which

determined the average number of peaches per tree when 36 and 42 trees were planted,

respectively, to provide examples how the average yield per tree was calculated:

1 (b). 528 600 - 6(12)

1 (c). 600 - (42 - 30)12

David used these expressions to scaffold his students' thinking about what a general equation might look like. From these expressions, they recognized the function for average yield per tree would need to begin with the value 600, from which a multiple of 12 was subtracted. With this information, David then led his students to write T(x) in two forms:

$$T(x) = 600 - (x - 30)12$$
$$T(x) = 960 - 12x$$

Using these forms of T(x), David revisited (a)–(c), using the function to verify the answers they had initially given.

Although David had lowered the cognitive demand of Paula's Peaches Item 1, Part (d) by using short closed questions to guide his students through the formulation of T(x), he also pointed their attention to how (d) was connected to the other parts of Item 1. For example, he encouraged his students to use of the structure of their calculations in (b) and (c) to aid their formulation of T(x). Furthermore, he asked them to use their formulation of T(x) to confirm their results from (a)–(c). By encouraging students to examine the relationships between the different parts of Item 1, David's students may have been able to make connections between the mathematics and the context of the task that they may not have uncovered by simply working through the packet on their own. If David's students made connections, then the cognitive demand for this part of the task increased.

During my observation of David's class, his students had several opportunities to work on tasks individually or in small groups. During these times, David walked about the class, observing his students' progress and providing assistance. When asked his intervention during a task that dealt with the average rate of change (Paula's Peaches Item 2, Part (d), 2/20), David described the group composition in that class and how he used it to assess the class's progress:

In that particular class, I didn't have them ability grouped. I had groups with very differing abilities. ... I knew if I needed an answer, I had a group I could go to. And then I had people in each of the other groups who were basically supposed to be leading those groups. So when I went around, I looked at the people who were the stronger people in the weaker groups. And I saw that they were struggling with it. That told me that ... they're leading the rest of the group down the wrong path. So, it was from that that I decided to go back and redo it a different way. (FI, 150–159)

David used the understanding of his group leaders to determine whether a particular task was too difficult for the class as a whole. If he ascertained that a task was too difficult for the more advanced students in his class, he would lead the class as a whole through the task, lowering the cognitive demand to a level that enabled them to solve the task more readily.

If, during his assessment of student work, however, only certain students were struggling, David would engage these students in a one-on-one conversation similar to those used to address the entire class. For example, when Louisa was struggling to solve the equation  $3x^2 = 21x - 30$  (Paula's Peaches Item 9, Part (c)), David intervened:

David: Make the right side zero first.

Louisa: I don't know how to do that.

David: Well, what's on the right side?

Louisa: 30. Add 30, right?

David: Okay.

Louisa: To what?

David: To both sides because you want the right side to be zero.

Louisa: What about the 21x?

David: It stays there, unless you want to subtract it in the same step.

Louisa: No.

David: It stays there, 21x.

Louisa: Minus zero?

David: You don't really need it. Whether it's a minus or a plus, you don't really need it, because it's a zero. Right? So what do we have on this side now? (2/24, 0:23:40–0:24:30)

In this excerpt of their conversation, David assisted Louisa by providing the initial step needed to solve the equation by factoring. In addition, he helped her make sense of what she was doing by offering a procedural rationale for that step and interpreting the result (21x - 0). During his final interview, David described the process through which he helps students during individual or group work:

I'll go around and talk to people, and they're working on a problem. ... When I'm going around the second time, and they're on the same problem, and they haven't written anything down yet, I'll try to give them a lead toward the first step. And then I'll leave them to work on that. [After going around the class again], if they're still on the first step, I'll give them a lead toward the second step. So I try not to give them more than they need. (FI, 240–245)

As observed in the dialogue between David and Louisa above and confirmed by the excerpt of his interview, David thought it was important to show struggling students the immediate step or steps needed to progress in their work on a task and at times an explanation for why that immediate step was appropriate. Although the cognitive demand of solving an equation like the one that challenged Louisa is most readily classified as procedures without connections, David felt it necessary to lower the level of cognitive demand further in order for her to be successful with it.
Question 3: Technology and Cognitive Demand

The third research question that I addressed in my study dealt with the influence instructional technology might have on how teachers select the tasks they use in class or on the way that students think about and work on those tasks. To answer this question, I focused on the tasks for which students used (or were expected to use) instructional technology in their solution. For these technologically oriented tasks (TOTs), I analyzed the teachers' goals for the use of technology, how the students were directed to use the technology, and what role the technology played in helping the students think about the task.

*Michael.* As mentioned in the previous chapter, Michael viewed technology as one tool students could use in the course of working on a mathematical task. In this manner, he did not call on the students to use the calculator at specifically defined times but rather left it to them to determine when the use of technology was appropriate. In this way, the inclusion of TOTs presented an additional hurdle to students—they had to recognize from the mathematical context of the task whether or not the use of technology was appropriate. This extra hurdle can be interpreted as increasing the task's cognitive demand relative to a task that explicitly states whether or not technology should be used. Michael aptly described this process during his preliminary interview:

You've got to be intelligent enough to know when it makes sense to pick it up and when it makes a lot more sense to understand, as a human brain, the possibilities, and you don't have to slow yourself down in the technology. (PI, 288–290)

Examples of Michael's students determining whether or not to use their TI-*N*spires while working on a TOT occurred frequently within the data. Often, in situations where the students opted not to use a calculator, Michael would ask them if using their

"machines" would be appropriate in that situation:

Michael: Now, could we do [the differentiation step of Lamppost Shadow] with our machines?

Students: Sure.

Michael: Could I actually take that derivative with my machine?

Jenny: Does our machine do implicit differentiation?

Michael: Yes, but let's see if it's worth it.

Jenny: Probably not, because it's kind of easy to take the derivative [by hand].

During the final interview, Michael provided several types of mathematical activities for

which he expected his students to use their calculators. Along with generalizing a

mathematical relationship, he listed:

When it's hard. When it's repetitious. When the algebra just gets annoying. When you need to create a lot of data quickly. ... Anything where the solving or the manipulation or the mathematics involved is not the point of the moment. (FI, 172–186)

At several points during my observations, these reasons were explicitly stated in the context of the TOT being discussed. For example, recall my description of Michael's intervention during his students' work on Roots of a Cubic. After several minutes of the students futilely trying to work on the task with pencil and paper, Michael steered them toward the use of CAS.

Even though it was possible (but very tedious) to find the equation of the tangent line and its *x*-intercept using algebraic techniques by hand, it was more important for Michael that his students be able to recognize in Roots of a Cubic an appropriate opportunity to use technology as a tool in their work. This decision may have also stemmed from a pedagogical consideration. During his final interview, Michael also cited "where the solving or manipulation or the mathematics involved is not the point of the moment" (FI, 185–186) as a reason to use technology. In this case, it was more important for Michael that his students be able to recognize and apply the general process needed to find the desired tangent line and its *x*-intercept than that they manually perform the associated algebraic drudgery. When responding to a draft of these results, Michael elaborated on how using CAS can maintain focus on high-level thinking: "What math technologies (especially CAS) enable is the big step over all that drudgery so that students can remain focused entirely on the thinking and interpretation of the problem" (personal communication, 10/9/09).

A second example of Michael considering pedagogical implications of students' technology use and cognitive demand occurred during Critical Points. In this TOT, he forbade the students from using their calculators, saying, "It gives it away." (4/24, 0:03:05–0:03:15). After his students had used logarithmic differentiation and were struggling with their result, that every point was a critical point, Michael allowed them to use a calculator to test this result. However, he restricted their calculator use to numbers only (no variables) as he did not want them to immediately see that the function's graph was a horizontal line. Throughout this episode, Michael carefully controlled his students' calculator use to preserve the cognitive demand of the task.

When asked about his decisions during Critical Points, Michael said that his initial experimentation with the task led him to restrict calculator use because it would have made certain elements of the mathematics transparent:

It was far better for them to catch that the derivative was always zero and to try to understand what that meant ... Knowing y = e would have told them that this was

a constant so the derivative equals zero really wouldn't have been all that impressive to them. To work the other direction from the derivative is always zero going back to the function must have been horizontal is much deeper thinking. (FI, 322–328)

Michael's awareness of technology's potential impact on the way his students thought about mathematics enabled him to maintain the cognitive demand for this TOT.

In addition to recognizing when to use technology, Michael's students also struggled with effectively communicating with the technology they used. This struggle to communicate occurred both in instructing the calculator what to do and in interpreting the result of what the calculator had done. When using CAS to compute a solution to Lamppost Shadow, the class encountered the result 0 = 0. Michael used this result as an opportunity for his students to consider the mathematics with respect to the technology:

Paul: 0 = 0

Allison: 0 = 0

Alex: I got "True."

Michael: You got true. Well, 0 = 0.

Allison: Well, it has a little warning thing at the bottom. "Differentiating an equation may produce a false—"

Michael: All right. We have issues. Is this at all what you expected?

Allison: No.

- Michael: Okay, so explain it. You asked a question. It answered the question you asked. What did you ask?"
- Paul: When you plugged in dD over dt, that's the change in something with respect to t. And there's no t in your equation.

Michael: Sure there is. *D* is a function of *t*. [Points at the board.]

Paul: It doesn't know that.

Michael: How do you know it doesn't know that?

Allison: The only thing you told it is that *D* is related to *A* and *B*.

Michael: How could I let the calculator know that *D* is a function of time? [The class works together to rewrite the equation to include *t*. (4/23, 0:16:45-0:17:50)

Because the calculator did not return the expected the result, the students were forced to reconsider the parameters they had provided for it to make its calculations. Through this process, interfacing with the technology made Michael's students consider some of the underlying mathematical principles of differentiation.

During Roots of a Cubic, the students asked the calculator to find the *x*-intercept of a particular tangent line. When the calculator returned the result x = c, the students were confused by it (possibly because of the complexity of the algebraic approach they had tried before using their CAS) and suggested other possible steps to take. Only after further examination (and some reassurance from Michael), were his students satisfied that CAS had indeed returned the result they had sought. In both of these examples, the students' interaction with technology forced them to reexamine the mathematical processes they had used in their solution of the task and the mathematical ideas underlying these processes. Michael elaborated on the importance of this process:

Learning how to phrase questions so that a CAS can give you a useful answer is FAR more cognitively demanding at a much higher level than almost all algebraic manipulation. Some say tools like a CAS make students dependent and erode ability. I counter that appropriate and clever use of a CAS actually requires more, not less mathematical understanding. (Personal communication, 10/9/09)

As described by these examples and argued by Michael, the use of technology can be viewed as a way to increase the cognitive demand of tasks.

### Sarah

Sarah's students used the classroom set of Texas Instruments TI-83 graphing calculators often during my observations. As mentioned in the previous chapter, her students' low proficiency with basic computation prompted Sarah to encourage them to use the calculator as a means to alleviate procedural difficulties so they could tackle the conceptual ideas underlying the mathematics. I observed this strategy for calculator use during my visits to Sarah's classroom.

For example, Families of Parabolas (2/19) required the students to graph three quadratic functions related by a particular transformation and then make an observation connecting the algebraic and graphical representations of the functions. Sarah directed her students to use their calculator to graph the functions and to copy the graphs onto their papers. Once this task had been completed, the students could examine the graphs for notable trends. In this task, the students relied on the calculator to alleviate the burden of graphing each parabola by hand. Though the students knew how to create a table of values, plot these values on the Cartesian plane, and estimate a smooth curve from these points (I observed this on another occasion), they could not perform these steps quickly or accurately.

By directing the students to use the calculators in this way, Sarah reduced the already low level of cognitive demand associated with graphing the functions further. At the same time, however, she enabled her students to extend their focus on the high-level component of the task: looking for patterns in the data and connecting the two representations of the functions. Sarah's students approached Quadratic Data Table (2/19–20, 2/23) and Sports Arena (2/25–26) in the same manner, using the calculator to

replace low-level activity to quickly generate data that would support high-level activity. In this way, Sarah could optimize the amount of time her students spent on developing conceptual understandings of quadratic functions.

The TOTs Sarah used included both low- and high-level thinking but sometimes varied in structure. For example, although Sports Arena included both low- and high-level thinking like Families of Parabolas and Quadratic Data Table, it was subtly different. During work on this task, the students' guess-and-check methodology cycled between low- and high-level activity as they alternated between graphing quadratic equations with their calculator and evaluating their choice of coefficients and refining subsequent guesses.

Sarah provided opportunities for her students to interpret both what they inputted into their calculators and what the calculator returned to them. For example, when walking around the room during Families of Parabolas, she noticed several students (she spoke to four students explicitly about this; she may have observed others) having difficulties with the order of operations when squaring a term with a negative coefficient. To address this difficulty, she interrupted the class's work and wrote two pairs of expressions on the board:  $(-2)^2$  and  $-2^2$ ,  $(-3)^2$  and  $-3^2$ . To begin the discussion, she said, "I want you to understand what your calculator is doing when you type in [these expressions] ... Your calculator does not read those as the same thing" (2/24, 0:29:06 – 0:29:26). Similarly, Sarah spent time discussing how the window (the range of values that are displayed on the calculator screen) would affect the students' interpretation of a graph. By making the students consider their interaction with the graphing calculator, Sarah forced them to think about related underlying mathematical ideas of a TOT (procedures with connections, Characteristic 1). In this way, her students' use of technology raised the cognitive demand of the procedures they performed.

To support her students' work with the calculator, Sarah provided explicit instructions for different procedures. These procedures included finding an appropriate window for viewing a graph (Families of Parabolas, Quadratic Data Table) calculating the zeros of a graphed function (Quadratic Data Table), calculating the value of a function for a given input (Sports Arena), and creating a scatter plot for a set of bivariate data and finding a quadratic regression to approximate it (Sports Arena).

From my observations, it was evident that many students were unfamiliar with these procedures and, interestingly, Sarah was surprised by this unfamiliarity. For example, during her explanation of entering bivariate data into the calculator for Sports Arena, Sarah asked, "Are y'all remembering this?" (2/26, 0:27:10–0:27:20). From the lack of response and subsequent struggles of the students, it was apparent they did not remember. When discussing this lapse during her final interview, Sarah provided some insight into the situation:

Not all of them worked with a graphing calculator last semester. I asked [the other teachers], did you do this with the calculator? We had all talked about it and said we were going to do it. They both chose not to. (FI, 402–404)

Because some of the students did not have Sarah as a teacher during the fall semester, much of what she was showing them was completely new. As a result, these students did not have the experience with the calculator that Sarah normally provides:

If I'm just introducing the graphing calculator, we spend a lot of time talking about what happens if you push that button. ... I usually talk about the three different main screens and three different representations. I just made some wrong assumptions about what they had seen. (FI, 414–418)

This lack of introduction, coupled with the students' limited access to calculator use, may have created difficulty for these students in being able to use their calculators effectively or, if they were able to perform the desired procedures, understand what they were asking the calculator to do. For these students, calculator use may not have been connected to the related mathematical algorithms (procedures without connections, Characteristic 3). As a result, the tasks would have a lower level of cognitive demand for these students.

Sarah chose the moments in class when her students could use graphing calculators. As a result, her students may not have developed the ability to decide when calculator use was appropriate. At one point during my observation, she asked the class for a strategy they could use when they needed to know what a graph looks like. After 3 minutes of discussion focusing on using the textbook as a resource, a student suggested using the calculator, to which Sarah replied, "I think that's a great idea" (2/24, 0:11:45– 0:11:49). In this example, Sarah wanted her students to recognize from the mathematical context of the task an opportunity where their calculator could help them. Their inability to recognize the mathematical features of the task context that called for calculator use, however, may have prevented them from doing so. This additional cognitive demand of interpreting the context of the task with respect to the advantages of technology use became an important feature of the task.

### David

David incorporated several forms of technology into his mathematics instruction. First and foremost, David's students had access to the classroom set of TI-84 graphing calculators and were free to use them when they chose. At several points during my observation, I observed students getting out of their seats during group work and returning with a calculator to use in their work. Although David allowed his students to choose the moments during work on Paula's Peaches that were appropriate for using calculators, the task itself sometimes specified instances where technology use should occur. For example, Paula's Peaches Item 3, Part (i) (2/20) and Item 4, Part (i) (2/23) share the same instructions: "Verify that the answers to part (h) are solutions to the equation written in part (a). It is appropriate to use a calculator for the arithmetic." In each case, the students were to substitute large values of *x* into the yield-per-acre function to verify that the designated yield was produced. By specifying to students that technology use was appropriate in this situation, the wording of the task might have deprived the students of an opportunity to refine their own judgment about appropriate calculator use with respect to the mathematical context.

During the preliminary interview, David said he wanted his students to learn mathematical procedures by hand before they were allowed to do them with a calculator:

You've done all this work by hand. Here's a much easier way to do it. And I'll show them how to put in the equation, how to find the table, how to find the graph for it. So they're doing all the work by hand first, grinding it out. And then I tell them, "Okay, you know how to do it." (PI, 139–142)

This practice of learning a procedure by hand followed by technology use occurred during my observation with respect to graphing quadratic functions. The result, however, was different from what David had intended.

When investigating Items 13–15 of Paula's Peaches, David and his students used the graphing capabilities of their calculators. On February 27, David demonstrated how to model the average-yield-per-acre function for a particular yield as a system of equations to find the average number of trees per acre that produced that yield. The following Monday (3/2, the day I did not observe), David's students worked with the calculator to graph the quadratic equations found in Item15. The following day (3/3), David said his students struggled considerably with graphing functions on the calculator, noting in particular they could not determine an appropriate window for viewing the parabola.

David began class that day (3/3) by returning to Item 15, saying, "You will definitely appreciate the easy way now" (0:05:10–0:05:15). He then demonstrated the procedure for graphing a parabola by hand through identifying its intercepts, whether it opened up or down, and identifying the coordinates of its maximum or minimum point, all of which were skills the students had previously learned. Because of length of the process, a student prompted David to justify why this process was indeed the easy way:

Jenny: I thought you said this was the easy way.

David: It is because, when you do it on the calculator, you had to put [the function] in, and then you had to get the window right, and then you had to try to figure out where each of the points was. The only hard one to find [when graphing by hand] is the vertex. (3/3, 0:11:45–0:12:05)

Later in the lesson, David spoke further to the students about their difficulties with the calculator: "The calculator way will get a lot easier for you. The reason the calculator way was hard for you was because this is the first class where they've had you use those calculators" (3/3, 0:25:40–0:25:50). He went on to explain that the students' experiences in future mathematics classes would help them to develop those skills. In this situation, the cognitive demand of using the calculator, deciding how to encode the mathematical input and interpret the output, was too great for David's students. Because they had not learned to represent the mathematics using technology, they were forced to rely on the more familiar noncalculator procedures for graphing quadratic functions.

David also used technology to demonstrate during my observations. In addition to projecting the TI-84 on the board to graph several of the quadratic functions explored in

class, he used an applet showing an area model for factoring quadratic equations (2/23, Paula's Peaches, Item 6) and connected it to work his students had done the previous semester on expanding the product of two binomials. In this way, David's connection between the factoring procedure his students were expected to do and its inverse procedure, multiplying binomials, may have helped the students to achieve a deeper understanding of factoring (procedures with connections, Characteristic 1). The introduction of an additional representation of factoring, an area model, may have also aided his students' understanding (procedures with connections, Characteristic 3). Although the inclusion of this model may have provided his students deeper insight into the particular example that he demonstrated, that insight may have been limited because the students were not required to adopt this model or practice it in their work on factoring.

Finally, David used technology in his classroom as a measurement tool for gathering data. During the students' work on Resistance (3/4–5), David's students used an ohmmeter to measure the electrical resistance of different circuit configurations. These measurements were used to generate a table of data on the board from which the students could identify relationships between the different columns. During this discussion, David assisted his students by providing his own interpretation of some of the data for which the relationship would not hold:

David: My contention is that, with the exception of that one [line of data] and that one, there is a particular relationship between the first two columns. Can anyone see what that is? [During a long pause, David restates his question and erases the two errant rows.]

Reggie: Hey, you erased my data! Why did you do that?

David: Because they don't fit the data, which means there was an equipment problem. (3/5, 0:17:35 - 0:18:50)

Maria: It doubles?

David: It doubles.

By eliminating the two errant lines of data, David enabled his students to more readily identify the relationship between the resistances afforded by a circuit with a single resistor and two resistors wired in a series. In doing so, however, he eliminated the opportunity for his students to interpret and consider what the errant data might mean, resulting in a lowered cognitive demand.

The results described in this chapter summarize my analysis of the data with respect to my three research questions. In the next chapter, I summarize the important ideas in the results and describe what those ideas might suggest for teachers, teacher educators, and researchers.

## CHAPTER 5

# DISCUSSION AND CONCLUSION

My interest in the cognitive demand found in mathematics classrooms arose from reading about the QUASAR project's work on mathematical tasks and cognitive demand (Stein et al., 2000) during my doctoral studies. As I read about cognitive demand and how it could fluctuate during instruction, I recalled my struggles as a high school mathematics teacher and lessons that to me had a lot of promise at the time but had somehow gone very wrong. I found I could explain the failures of many of these lessons using the framework proposed by Stein and her colleagues. Furthermore, I could identify facets of my teaching that contributed to those failures.

Through this research, I attempted to better understand the process of how effective mathematics teachers chose the tasks they used in class and how they implemented those tasks. In particular, I wanted to see how they integrated technology into the process and what effect that technology had on the ways students considered the mathematics in those tasks. In particular, the research study described in the preceding chapters aimed at answering the following research questions:

- 1. How do high school mathematics teachers select the tasks that they implement during instruction? What is the cognitive demand of these tasks?
- 2. How do high school mathematics teachers maintain or fail to maintain the cognitive demand of the tasks they implement in their classrooms?

3. In what ways, if any, does the use of instructional technology influence how mathematics tasks are selected, the cognitive demand of those tasks, and how they are implemented?

To answer these questions, I sought out experienced mathematics teachers who regularly incorporated technology into their lessons. Using case study methodology, I conducted a series of interviews and observations for each of the three participating teachers, examining the tasks they selected and implemented during the 2 weeks I spent with each of them. From these data, I determined the cognitive demand of each task and analyzed how the teacher's actions during a lesson may have affected it. For tasks in which students used instructional technology, I analyzed how technology use may have affected the ways in which students thought about those tasks. My observations and conclusions below draw upon the results of these analyses.

# Task Selection

The interviews I conducted provided insight into each participant's view of what constituted important mathematics for students to learn. Handal (2003) argued the conception of mathematics held by teachers, ranging on a continuum between traditional and progressive (socioconstructivist), influences their instructional practice and that inconsistencies in this relationship can be attributed to factors outside of their control such as administrative policies, standardized assessments, textbooks, and expectations of students. For each participant, I observed the interplay of these forces in his or her selection of mathematical tasks and how it might affect the possible level of cognitive demand students would encounter in class. Michael held a progressive view of mathematics, as seen by his enthusiasm towards his students' ideas about mathematics and his encouragement for them to look at mathematical situations from multiple perspectives. This view, coupled with the freedom afforded by his environment (such no pressures from standardized tests, the ability to use a textbook he himself had written) allowed him the opportunity to select high-level tasks for his Honors Calculus class.

Like Michael, Sarah had a progressive view of mathematics because of her focus on student discussion as the vehicle for developing mathematical ideas. Her teaching environment, however, may have interfered with that view, to some degree, as her students had to be prepared for state-mandated standardized tests. This need may have led Sarah to implement a more traditional view of teaching and learning. This conflict between a view of mathematics and the reality of her school environment could be seen in her balance between low- and high-level tasks. Although she valued tasks that enabled her students to have productive conversations about mathematics, she also spent time in class reviewing and practicing procedures her students would need to know for future testing. Although Sarah included some low-level tasks for her students, she planned her lessons so that many of these tasks supported the high-level tasks she used by generating data supporting high-level operations.

David held a traditional view of mathematics than the other two participants. For David, knowledge is transmitted from teacher to students and more value is placed on final products than on the processes that produced them. This view was confirmed by his practice of preteaching the skills to be developed during a task before work on that task and his focus on correct answers in class over how the students produced them. Although he wanted to develop his students' "thinking skills," the tasks he selected were predominantly low level. Given this view of mathematics and his stated purpose of developing students mathematical thinking, David may have believed that the tasks he selected were high level. This may be attributed to his expectations of the GPS curriculum tasks he used during my observations. These tasks (such as Paula's Peaches) were claimed to be designed and promoted as facilitating student development of mathematical habits consistent with those described by the NCTM Process Standards (Georgia Department of Education, 2006). The manner in which each task was written, however, made it very difficult for David to realize his goal for students' mathematical thinking in his classroom. In Paula's Peaches, for example, the partitioning of each task into many small steps confined the students to a single well-marked path toward its goal, a mathematical model for the yield of a peach farm. As a result, these tasks were predominantly low-level (procedures without connections), and David's use of them may have undermined his overall goal for his students to develop high-level mathematical thinking and reasoning.

#### Task Implementation

When introducing tasks to students and monitoring their work on those tasks, the participants made decisions and intervened in ways that affected cognitive demand. Although there were similarities in the situations triggering these interventions, such as students struggling with a task to the point where they were no longer progressing toward a solution, the typical response varied from participant to participant. These responses can be connected to Henningsen and Stein's (1997) description of classroom factors that affect the use of high-level mathematical tasks.

At several points during the observations, Michael suggested (explicitly or implicitly) particular solution paths students might try. He attributed this practice in part to his own enthusiasm toward the mathematics. Examining these actions in the context of Henningsen and Stein's (1997) research, Michael's suggestion of particular solution paths might be considered a shift in focus to a correct answer because he was favoring one solution path over others. Although the cognitive demand was lowered in these instances, Michael also improved the cognitive demand in other instances by asking students to consider multiple representations of a particular solution, demanding detailed interpretations of how the mathematics in a particular task related to its real-world context, and extending the mathematics in the task to novel situations. These behaviors could be classified as drawing conceptual connections, a factor Henningsen and Stein described as supporting high-level thinking.

Sarah's actions in situations where her students' progress had ebbed were more effective at maintaining the cognitive demand of some part of the task than those of the other two participants. In response to the questions of struggling students, she rarely provided explicit answers but rather scaffolded their thinking by providing templates for appropriate responses, modeling the provision of an appropriate reason in limited examples related to the task, or encouraging them to help each other. Scaffolding was one of the most frequent occurring factors noted by Henningsen and Stein (1997) in their study.

David attempted to avoid situations that would cause his students to struggle by selecting the tasks for which his students would work independently as opposed to those for which he would lead them through the steps to a solution. In this case, the tasks or

115

parts of tasks may have been inappropriate for the students' current level of understanding (Henningsen and Stein, 1997). As a result, when David's students would potentially encounter a task with cognitive demand higher than what he perceived as his students' capabilities, he lowered the cognitive demand. Although this practice ensured that students' work would proceed with minimal frustration, it also denied the students the opportunity to work on tasks with high cognitive demand.

In addition, David would preteach the skills that were supposed to be developed during work on a task because of his students' past difficulty in developing a conceptual foundation for the mathematics they were to learn during those tasks. In that way, his students were prepared for the mathematical challenges demanded by tasks such as Paula's Peaches. At the same time, however, his students lost the opportunity to generalize and formulate mathematical relationships developed in these tasks because David had already done this work for them by focusing on the procedure to be followed in order to attain correct answers.

As described by Stein, Grover, and Henningsen (1996), student dispositions may also be a factor in how teachers implement tasks in their classrooms. Because of the differences between Michael's students and those of Sarah and David, this factor became important when making comparisons between the three classrooms. As Michael's students had a high aptitude for and past successes with mathematics, it may be reasonable to conclude that they were more apt to push beyond initial frustration when solving a task and persevere until all ideas in their mathematical repertoires had been exhausted. As a result, their struggles may have signaled to Michael that a more direct form of intervention was necessary. On the other hand, Sarah and David's students had less mathematical knowledge and fewer successful experiences in mathematics to draw upon when solving mathematical tasks. This limited proficiency may have limited their confidence when solving tasks, and small obstacles may have caused them to completely shut down their efforts in class. As a result, it is possible that the timing and nature of interventions necessary in Sarah and David's classrooms was different, as their students needed more scaffolding than Michael's students. This potential difference in student needs, however, does not account for the large difference between Sarah and David's styles of intervention.

## Technology

When considering the use of technology in this study, I identified two ways technology can affect the cognitive demand of a mathematics task: (a) assessing the mathematical context of a task to determine the appropriateness of technology use as a solution strategy and (b) translating between the mathematical and technological contexts of the task. In this section, I elaborate on these ideas as well as relate them to the technological effects framework (Salomon & Perkins, 2006).

#### Appropriateness of Technology Use.

At many times during the study, students had to assess the mathematical context of the task to determine whether technology use was appropriate. Much as for the nontechnological strategies they might have used, students had to identify mathematical cues within the task calling for technology use as a strategy. These cues, such as needing many repetitions of the same operation or series of operations or needing to make tedious computations, however, were unique to the particular technological tool at hand and had to be recognized by students.

As a result of the unique factors that determine the appropriateness of using a particular technological tool, the teachers in the study needed to provide opportunities for their students to recognize these factors and, when necessary, model appropriate thinking and decision-making for their students. The participants provided these opportunities to varying degrees. Michael's students, who were the most proficient in using their calculators, also had the most opportunity to practice making appropriate decisions with respect to technology, and Michael often prompted them to provide a rationale for their students when to use their calculators, but she often waited until they had exhausted all other options (as opposed to considering technology as an initial option). David's students were able to access their calculators at any time during class for basic computations, but he explicitly dictated when calculators should or should not be used for more advanced operations.

#### Translating Between Mathematics and Technology.

Technology also contributed to the cognitive demand of a task by requiring students to encode the mathematical context of the task into a format that their technological tool could process and then decode the returned result back into that context. This encoding, such as when David's students needed to specify an appropriate window to view their graphed function, and decoding, such as Michael's students' need to interpret the meaning of 0 = 0 while working on Lamppost Shadow, was critical to successful technology use and, ultimately, a successful and meaningful solution of the

task. To aid in his students' mastery of using their calculators, Michael often asked them to describe in their own words what they had asked the calculator to do and what the calculator had returned to them. Sarah, in demonstrating procedures that students were to carry out with their calculators, frequently explained the meaning of each step she had taken. She also asked her students to make conjectures about what values might be appropriate to input (such as the parameters for the window of a graph). David used the calculator only for demonstrations during my observations and limited the discussion to what should be inputted.

I interpreted the translation of a mathematical situation into something that can be programmed into processed in a technological environment as a unique mathematical representation. Though one could argue that graphing an equation by hand versus with a calculator could constitute the same graphical representation of a mathematical relationship, using the calculator requires the student to make different considerations. For example, students using the calculator must determine an appropriate window through which to view the graph and know how to instruct the calculator to show that window. In addition, the student must take into account specific protocols of calculator function, such as requiring functions graphed in the Cartesian plane to be inputted with respect to *y*, when programming mathematical processes. As an additional way to represent a task, technology requires students to use potentially unique ways of thinking about the underlying mathematics and to consider how it can be connected to nontechnological representations. In this way, use of technology to solve a mathematics task can change the cognitive demand involved.

### Technological Effects.

Salomon and Perkins's (2006) framework for technological effects on intelligence can also be used to examine the relationship between technology use in this study and cognitive demand. Effects *with* technology, where the student delegates certain jobs to be carried out by the calculator could be seen in all three classrooms, where the calculator was given computations and procedures to perform that would have been difficult or time consuming to do by hand. For example, Sarah's students used their calculators to determine the coordinates of the zeros of quadratic functions they had graphed, David's students used their calculators to compute the average yield per acre for given numbers of peach trees, and Michael's students used CAS to simplify and solve equations with multiple parameters to represent fixed values.

Effects *of* technology, where a student's understanding or ability with respect to mathematics (independent of the calculator) had changed through extended calculator use, was more difficult to observe because of the relatively limited amount of time I spent in each classroom. Given this limitation, I hypothesize that the students in Michael's class may have an improved understanding of the structure of functions due to their use of CAS throughout the year. By having to carefully consider how their calculators implicitly differentiated the equation they had written for Lamppost Shadow, Michael's students needed to rewrite that equation so it had functions expressed with respect to the correct parameter. This activity was instigated by trying to decode the calculator's output from their initial equation, and the students' work on it could help them in formulating future equations that might or might not be solved with CAS.

Effects *through* technology, where the use of technology fundamentally transforms the way an activity is conducted, could be identified by examining the nature of the tasks the participants used and considering whether the solution strategies for these tasks were fundamentally different from those that would have been used prior to the introduction of that calculator. For Michael's class, most of the tasks could have been solved using techniques that did not involve CAS. Using CAS, however, allowed his students to represent and think about the mathematical context in ways that previously were not possible. Sarah's students were able to use a guess-and-check strategy to approximate a quadratic curve to fit a set of data in a matter of minutes. This activity would not have been possible without the calculator because Sarah's students would have struggled to understand the mathematical procedures and would not have been able to complete them in a reasonable amount of time.

# Limitations

While formulating and gathering data for this study, I identified moments when, if given the ability to go back and revise my decisions or actions, I would have taken a different path. Below is a description of some of these moments. My hope is to provide some perspective on how my findings might be limited and points of caution for those who might choose similar paths of inquiry in the future.

### Participant Selection

For this study I sought experienced high school teachers who were adept at using technology with their classes. I had difficulty locating a large enough population from which to choose suitable candidates. As a result, I had to select participants based more on availability than on the needs of the study. I was fortunate to recruit the teachers I did. They were extremely professional in sharing their practice and accommodating my needs as a researcher.

Because I did not have many potential participants to choose from, I did not have to rely heavily on the questionnaire instrument for participant selection. This proved to be inconsequential, as I found that the responses given on the questionnaire did not provide an accurate representation of the classrooms I observed. The instrument was not effective in obtaining the information I wanted to know. Classroom observations prior to the design of the instrument might have improved its development. In addition, observing each participant's class for several days prior to selection would have improved my effectiveness in determining their suitability for my study.

The stark differences between the courses, students, and teaching I observed at the two schools made comparisons between the case of Michael and those of Sarah and David difficult. As a result, variables affecting teaching practice other than those that I wished to study may have influenced what I observed. Limiting the boundaries of this study to include a group of participants homogeneous in experience, courses taught, school, or students' mathematical proficiency (or any combination of these variables) might have provided a wider basis for comparison.

### Cognitive Demand

I selected cognitive demand as the theoretical construct to describe the intellectual engagement afforded by the tasks selected for and implemented in the classrooms I observed. Although analyzing cognitive demand using the Task Analysis Guide served as a way to consider the intellectual engagement of an idealized "typical" student for a given grade level (or perhaps for assessing individual students in an individual setting), I found it difficult to assess the cognitive demand for a class of students of varying ability. As Schoenfeld (1985) found that what constituted a problem varied from student to student, I discovered that some the level of cognitive demand for some tasks could vary according to particular students in a given class.

Henningsen and Stein (1997) described a set of classroom-based factors that contributed to a decline in cognitive demand of tasks implemented by mathematics teachers. Although I recognized some of these factors in the classrooms I observed, the decline in cognitive demand I observed often served as a means to continue the activity to a stage where high-level thinking about the task occurred. This lowering and raising of the level of student's thinking and raising of the level of cognitive demand within the same task made a definitive characterization difficult and suggests a need for a more dynamic characterization of cognitive demand in classroom settings.

### Implications For Practice and Research

The results of this study and the findings I have described above have prompted me to reflect upon how they might be extended to different facets of our work as mathematics educators as well how they contribute to our body of knowledge of teaching and learning mathematics. To organize the implications of this study, I have divided this section into two parts: (a) implications for the practice of mathematics teachers and teacher educators, and (b) implications for current and future research.

# Practice of Mathematics Teachers and Teacher Educators

My interviews of Michael revealed that he had developed an informal way of thinking about the level at which a task would challenge his students. In particular, he had developed a 10-point scale on which he rated potential tasks for his class and, as a rule, chose only those that rated 5 or higher. In rating a task, he considered both the mathematical characteristics of the task and his students' potential to solve it. Michael's system for selecting tasks demonstrated a keen knowledge of both the mathematics he taught and his students.

This facet of Michael's practice suggests mathematics teachers can benefit from having some sort of structured way to consider and evaluate the potential of mathematical tasks for classroom use. This way of thinking must honor both the mathematical characteristics of the task and the cognitive characteristics of the students. Research (e.g., see Boston, 2006) has shown that a framework for task selection based on cognitive demand as a way to classify mathematical tasks for students at a given grade level can scaffold mathematics teachers' thinking about the tasks they select for their students, prompting them to select more tasks demanding a high level of thinking.

Michael had developed his own framework for judging the worthiness of tasks for use in his classrooms, but this development may have been attributed to his experience or his exceptional knowledge of mathematics. For other mathematics teachers, especially those with little experience or limited mathematical knowledge, this development of a way to effectively evaluate the challenges inherent in a mathematical task may not occur. As a result and because of the research described above, teacher educators should provide experiences for prospective and practicing teachers to learn about frameworks that examine the intellectual engagement provided by different mathematical tasks. Arbaugh and Brown (2004) describe activities of this nature, such as task sorting and textbook analysis, which they have used with teachers for this purpose.

The mathematics teachers in this study used a variety of strategies to ensure a

high level of cognitive demand was maintained for the tasks they used. Many of these strategies corresponded to factors identified by Henningsen and Stein (1997) as maintaining a high level of cognitive demand. In addition, Michael extended some of the tasks he used by challenging students to make connections between multiple representations of the task situation. By doing so, he provided opportunities for his students to think at a high level. Henningsen and Stein identified the teacher's act of drawing conceptual connections as a way that cognitive demand can be maintained. Given the effectiveness and general nature of these actions, teachers should consider how they might widely incorporate them into their own practice. Similarly, mathematics teacher educators must consider how to help teachers toward this goal.

David's struggle to facilitate high-level thinking among his students while using GPS framework tasks could be attributed to the way these tasks were written as well as they way David implemented them. As a consequence, mathematics teachers must carefully consider tasks written for large populations of students, such as those for state frameworks like the GPS or even those found in mathematics textbooks, and adapt them when necessary to meet the particular needs of their students. In addition, teachers must be mindful of the intent of a task with respect to implementation and carefully consider the consequences for modifying the way it is used in class., Mathematics teacher educators could support teachers in this way by providing teachers with opportunities to investigate tasks provided by state frameworks, assess their appropriateness for particular groups of students, and discuss the consequences of various means of implementation.

Although technology use in mathematics classrooms is often looked upon as making students' work easier by removing the need to perform mathematical procedures by hand, this study identified two contexts in which technology use can increase the level of cognitive demand for students: (a) examining the mathematical features of a task to determine when technology should be used and (b) translating mathematical ideas into and out of a technological context. Teachers need to understand that their students might struggle in these contexts when using technology to solve mathematical tasks. Therefore, teachers should create opportunities for their students to provide rationales for choosing when and when not to use technology as well as interpret in their own words the commands they give to a technological tool and the result of those commands. Teacher educators should give explicit attention to these contexts so that teachers are aware of their importance.

### Research in Mathematics Education

The findings of this study presented me with further questions about the relationships between teaching mathematics, the cognitive demand of mathematical tasks, and the use of instructional technology in mathematics classes. These questions suggest directions that further research might take.

*Cognitive demand.* This study focused in depth on three high school mathematics teachers in order to describe how their practices affected the cognitive demand of the tasks they used. To improve the analysis of their practice and to clarify my thinking about the distinction between the mathematical properties of a task and the cognitive abilities that students use to solve it, I modified the Task Analysis Guide presented by Stein et al (2000). This modification allowed me to consistently analyze each task with respect to students' ways of thinking when solving the task. Although the modified version of the Task Analysis Guide was a valuable guide in this study, it still needs further scrutiny to

determine its applicability to other contexts.

My findings with respect to technology suggest the need for a modification of the Task Analysis Guide to include student operations involving technology or perhaps a simple addendum that accommodates mathematical tasks where technology is used. A proposed addendum appears in Figure 3. These guidelines may prove useful in future investigations of cognitive demand when technology is involved. For example, this study suggests that there may be a relationship between the implementation of high-level technology problems (as defined by Figure 3) and students' proficiency in using technology.

# Cognitive Demand For Technology-Oriented Problems An Addendum to the Task Analysis Guide

The operations that students use to obtain an answer to the problem while using technology are primarily characterized by:

## Low-Level Demands

- 1. An explicit call for technology use by the problem or apparent need for technology use due to prior instruction
- 2. No requirement for technology use to make a connection between the procedure being used and the underlying mathematical ideas
- 3. No explanation of *why* the selected technological representation of the problem and the result of that representation are appropriate for the mathematical context of the problem

High-Level Demands

- 1. An implicit (not readily apparent) call for technology use in the problem
- 2. An explanation of why technology use is necessary and/or preferred
- 3. Attention to connections between the procedure being used and the underlying mathematical ideas
- 4. Attention to connections between the technological representation of the problem and the mathematical context of the problem

*Figure 3*. A technology addendum to the Task Analysis Guide. These criteria, modeled after those in the Task Analysis Guide (Stein et al, 2000) incorporate findings from this study with respect to cognitive demand and technology.

Although cognitive demand provides a useful way to think about the challenge to students afforded by mathematical tasks, it needs to be developed further. Because cognitive demand depends on a student's knowledge and prior mathematical experiences, examining and characterizing a definitive level of cognitive demand for class of students with varying abilities and experiences is problematic. Further refinement of this concept will enable researchers to better understand classroom dynamics during students' work on a task and provide teacher educators with a framework that provides a differentiated view of student understanding.

*Teacher knowledge.* Another factor that may have influenced the participants' selection of mathematical tasks is their mathematical knowledge for teaching (MKT) (Bass, 2005). This knowledge includes specialized mathematical content knowledge that is used exclusively in the work of teaching mathematics and content-specific pedagogical knowledge. For example, Michael used a personal scale to consistently classify and select high-level AP tasks indicating a detailed knowledge of both the mathematics represented by each task and his students' potential ability to solve it. An investigation of task selection with respect to MKT might identify elements of teachers' knowledge that are necessary for identifying appropriate high-level tasks for particular students.

In addition, an investigation of MKT with respect to task selection might also explain the disparity between Sarah's and David's use of high-level problems. As a veteran teacher with a doctorate in mathematics education, Sarah perhaps had a more fully developed repertoire of ways to represent mathematical ideas, diagnose and address student misconceptions, and, most important in the context of this study, identify good tasks to use with students for the purpose of developing high-level student thinking.

Adding further complexity, the knowledge teachers need to effectively teach mathematics with technology is fundamentally different from that of simply teaching mathematics. Mishra and Koehler (2006) presented a model defining technological pedagogical content knowledge (TPCK, but also referred to as TPACK. Niess, 2008, describes the impetus to modify the acronym). Much like Shulman's (1986) identification of pedagogical content knowledge as the intersection of content and pedagogical knowledge, the TPCK model identifies specialized forms of knowledge that occur in the overlaps between content, pedagogical, and technological knowledge. TPCK represents the intersection of all three forms of knowledge and includes the specialized knowledge mathematics teachers must have in order to select appropriate technologies to represent the mathematics they are teaching, integrate the use of those technologies into their lessons, and understand how use of those technologies may affect the way students understand a particular mathematical concept. When a teacher is forced to consider the effect that technology might have on the cognitive demand of a mathematical task, this knowledge incorporates knowledge of mathematics, teaching students, and technology. As a result, further examination of how teachers consider the ways technology might affect the cognitive demand of a mathematical task might further define the TPCK needed for teaching mathematics.

*A final consideration.* When considering other ways to understand the relationship between cognitive demand and technology use, I considered Artigue's (2002) description of instrumental genesis in the context of French students using CAS to learn mathematics. In this view of technology use, a technological tool such as a calculator or software package does not become an *instrument* for students until students construct cognitive schemes (either personal or pre-existing) for using the tool. The process of developing these schemes, *instrumental genesis*, is bidirectional, where the users both shape and are shaped by the technological tool they are using.

This study identified two ways in which the use of technology influenced the cognitive demand of mathematical tasks, analyzing the mathematical features of a task to determine the appropriateness of technology use and translating between the mathematical context of a task and its technological representation. These skills could be interpreted as two schemes necessary for instrumental genesis to occur. Thus, a study framed using the construct of instrumental genesis and investigating the ways that technology use contributes to how students' work on mathematical tasks might provide further illumination of the results of this study pertaining to cognitive demand. In doing so, we would enhance our knowledge of how the use of instructional technology could best enhance mathematics teaching and learning.

#### REFERENCES

- Arbaugh, F., & Brown, C. A. (2004). What makes a mathematical task worthwhile?
  Designing a learning tool for high school mathematics teachers. In R. N.
  Rubenstein & G. W. Bright (Eds.), *Perspectives on the teaching of mathematics* (Sixty-sixth yearbook of the National Council of Teachers of Mathematics, pp. 27–41). Reston, VA: NCTM.
- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245–274.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society*, *42*(4), 417–430.
- Becker, H. J., & Anderson, R. E. (1998). Teaching, learning, and computing, 1998: A national survey of schools and teachers describing their best practices, teaching philosophies, and uses of technology. Retrieved October 2, 2008, from http://www.crito.uci.edu/tlc/ questionnaires/teachers\_qs.pdf
- Bennett, N., & Desforges, C. (1988). Matching classroom tasks to students' attainments. *Elementary School Journal, 88*(3), 220–234.

- Bloom, B. S., Engelhart, M. D., Furst, E. J., Hill, W. H., & Krathwohl, D. R. (1956). *Taxonomy of educational objectives: Handbook 1, The cognitive domain*. New York: David McKay.
- Blumenfeld, P. C., & Meece, J. L. (1988). Task factors, teacher behavior, and students' involvement and use of learning strategies in science. *Elementary School Journal*, 88(3), 235–250.
- Boston, M. D. (2006). Developing secondary mathematics teachers' knowledge of and capacity to implement instructional tasks with high level cognitive demands.
   *Dissertation Abstracts International*, 67(06). (UMI No. 3223943)
- Burkhardt, H, Fraser, R., & Ridgeway, J. (1990). The dynamics of curriculum change. In
  I. Wirszup & R. Streit (Eds.), *Development in school mathematics education around the world* (Vol. 2, pp. 3–29). Reston, VA: National Council of Teachers of
  Mathematics.
- DeMarrais, K. (2004). Qualitative interview studies: Learning through experience. In K. deMarrais & S. D. Lapan (Eds.), *Foundations for research: Methods of inquiry in education and the social sciences* (pp. 51–68). Mahwah, NJ: Erlbaum.
- Doyle, W. (1983). Academic work. Review of Educational Research, 53(2), 159–199.
- Doyle, W. (1986). Classroom organization and management. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.; pp. 392–431). New York: Macmillan.
- Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167–180.
- Fey, J. T., Cuoco, A., Kieran, C., McMullin, L., & Zbiek, R. M. (Eds.). (2003). Computer algebra systems in secondary school mathematics education. Reston, VA:

National Council of Teachers of Mathematics.

- Friedman, T. L. (2005). *The world is flat: A brief history of the twenty-first century*. New York: Farrar, Straus, and Giroux.
- Georgia Department of Education. (2006). *Georgia performance standards, K–12*. Atlanta: Author.

Gleick, J. (1988). Chaos: Making a new science. New York: Penguin.

- Hadley, W. S., Snyder, J., & Sinopoli, J. (2008). Georgia Mathematics 1. Pittsburgh: Carnegie Learning.
- Handal, B. (2003). Teachers' mathematical beliefs: A review. *The Mathematics Educator*, *13*(2), 47–57.
- Hays, P. A. (2004). Case study research. In K. deMarrais & S. D. Lapan (Eds.), Foundations for research: Methods of inquiry in education and the social sciences (pp. 217–234). Mahwah, NJ: Erlbaum.
- Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition:
  Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524-549.
- Herbst, P. G. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education*, 37(4), 313–347.
- Herrera, M., Preiss, R., & Riera, G. (2008, July). Intellectual amplification and other effects "with," "of," and "through" technology in teaching and learning mathematics. Paper presented at the meeting of the International Congress of
Mathematics Instruction, Monterrey, Mexico. Retrieved July 13, 2009, from dg.icme11.org/document/get/76

- Howson, G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum development in mathematics*. Cambridge: Cambridge University Press.
- Jones, L. V. (1996). A history of the National Assessment of Education Progress and some questions about its future. *Educational Researcher*, *25*(7), 15–22.
- Kelley, W. M. (2002). The complete idiot's guide to calculus. New York: Alpha Books.
- Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006). *Frogs, fleas, and painted cubes: Quadratic relationships*. New York: Pearson.
- Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 763–804). Reston, VA: National Council of Teachers of Mathematics.
- Long, A. (2008). *McDougal Littell Georgia high school mathematics 1*. Geneva, IL: McDougal Littell.
- Madaus, G. Clarke, M., & O'Leary, M. (2003). A century of standardized mathematics testing. In J. Kilpatrick & G. M. A. Stanic (Eds.), *A history of school mathematics* (Vol. 2, pp. 1311–1433). Reston, VA: National Council of Teachers of Mathematics.
- Marshall, C., & Rossman, G. B. (2006). *Designing qualitative research* (2<sup>nd</sup> ed.). Thousand Oaks, CA: Sage.
- Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. San Fransisco: Jossey-Bass.
- Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A

framework for teacher knowledge. *Teachers College Record*, *108*(6), 1017–1054. National Assessment Governing Board. (2008). *Mathematics framework for the 2009 National Assessment of Educational Progress*. Washington, DC: U.S. Department of Education.

- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (1991). Professional standards for school mathematics. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2007). *Mathematics teaching today: Improving practice, improving student learning*. Reston, VA: Author.
- National Research Council. (2001). Adding it up: Helping children learn mathematics.
  J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Mathematics Learning Study
  Committee, Center for Education, Division of Behavioral and Social Sciences and
  Education. Washington, DC: National Academy Press.
- Neidorf, T. S., Binkley, M., Gattis, K., & Nohara, D. (2006). Comparing mathematics content in the National Assessment of Educational Progress (NAEP), Trends in International Mathematics and Science Study (TIMSS), and Program for International Student Assessment (PISA) 2003 assessments. Washington, DC: National Center for Education Statistics.

- Neumann, A. M. (2004). The Mathematical Tasks Framework: A consequence of and condition for professional development. *Dissertation Abstracts International*, 65(07), 2533. (UMI No. 3139616)
- Niess, M. L. (2008). Knowledge needed for teaching with technologies—Call it TPACK. *AMTE Connections*, 17(2), 9–10.
- Partnership for 21<sup>st</sup> Century Skills. (2002). Learning for the 21<sup>st</sup> century: A report and mile guide for 21<sup>st</sup> century skills. Washington, DC: Author.
- Polya, G. (1962). Mathematical discovery: On understanding, learning, and teaching problem solving (Vol. 1). New York: Wiley.
- Polya, G. (1965). Mathematical discovery: On understanding, learning, and teaching problem solving (Vol. 2). New York: Wiley.
- Polya, G. (1973). *How to solve it: A new aspect of mathematical method*. Princeton, NJ:Princeton University Press. (Original work published 1945)
- Robitaille, D. F. (1980). Intention, implementation, realization: The impact of curriculum reform in mathematics. *Journal of Curriculum Studies*, *12*(4), 299–306.
- Salomon, G., & Perkins, D. (2005). Do technologies make us smarter? Intellectual amplification with, of, and through technology. In R. J. Sternberg & D. D. Preiss (Eds.), *Intelligence and technology: The impact of tools on the nature and development of human abilities* (pp. 71–86). Mahwah, NJ: Erlbaum.
- Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *A splintered vision: An investigation of U.S. science and mathematics education*. Hingham, MA: Kluwer.
- Schoenfeld, A. H. (1985). Mathematical problem solving. Orlando, FL: Academic Press.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching.

Educational Researcher, 15(2), 4–14.

- Silver, E. A., & Stein, M. K. (1996). The QUASAR Project: "The revolution of the possible" in mathematics instructional reform in urban middle schools. *Urban Education*, 30(4), 476–521.
- Stake, R. (2005). Qualitative case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (3<sup>rd</sup> ed., pp. 443-466). Thousand Oaks, CA: Sage.
- Stanic, G. M. A., & Kilpatrick, J. (1989). Historical perspectives on problem solving in the mathematics curriculum. In R. I. Charles & E. A. Silver (Eds.), *The teaching* and assessing of mathematical problem solving: Research agenda for mathematics education (Vol. 3, pp. 1–22). Hillsdale, NJ: Erlbaum.
- Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455-488.
- Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). Implementing standards-based mathematics instruction: A casebook for professional development. New York: Teachers College Press.
- Travers, K. J., & Westbury, I. (1989). The IEA study of mathematics I: Analysis of mathematics curricula. Oxford: Pergamon Press.
- Wilson, J. W. (1971). Evaluation of learning in secondary school mathematics. In B. S.
  Bloom, J. T. Hastings, & G. F. Madaus (Eds.), *Handbook on formative and summative evaluation of student learning* (pp. 643–696). New York: McGraw-Hill.

- Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving.
  In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57–78). New York: Macmillan.
- Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 257–315). Charlotte, NC: Information Age.

## APPENDIX A

## PARTICIPANT SCREENING INSTRUMENT

Secondary Mathematics Teachers' Interpretation and Implementation

of Mathematics Curricula

Dear Participant,

You are invited to participate in a study to be conducted in conjunction with my dissertation research. For this study, I will conduct interviews, observe classroom lessons, and collect classroom artifacts to learn more about how mathematics teachers interpret and implement their school's curriculum. In particular, I am interested in how instructional technology is incorporated into this process.

This questionnaire consists of items asking for information regarding your current teaching practice. I hope you will take this opportunity to share your views on teaching and your practice with me. Thank you for your help.

Sincerely,

Kyle T. Schultz

1. Name:

- 2. School:
- 3. For how many years have you taught mathematics at the secondary level (Grades 6–12)?

4. Teaching Philosophy: Different teachers have described very different teaching philosophies to researchers. For each of the following pairs of statements, check (✓) the box that best shows how closely your own beliefs are to each of the statements in a given pair. The closer your beliefs to a particular statement, the closer the box you check. Please check only one box for each set.

"I mainly see my role as a facilitator. I try to provide opportunities and resources for my students to discover or construct mathematics themselves."			"That's all nice, but students won't really learn mathematics unless you go over the material in a structured way. It's my job to explain, to show students how to do the work and to assign specific practice."
"The most important part of mathematics instruction is the content of the curriculum. That content is the community's judgment about what children need to know and do."			"The most important part of mathematics instruction is that it encourage "sense-making" or thinking among students. Content is secondary."
"It is useful for students to become familiar with many different mathematical ideas and skills even if their understanding, for now, is limited. Later, in college, they will learn these things in more detail."			"It is better for students to master a few complex mathematical ideas and skills well, and to learn what deep understanding is all about, even if the breadth of their knowledge is limited until they are older."
"It is critical for students to become interested in doing mathematics—interest and effort are more important than the particular mathematics they are working on."			"While student motivation is certainly useful, it should not drive what students study. It is more important that students learn the mathematics in their textbooks."
"It is good to have all sorts of activities going on in the mathematics classroom. It's hard to get the logistics right, but the successes are more important than the failures."			"It is more practical to give the whole mathematics class the same assignment, one that has clear directions, and one that can be done in short intervals that match students' attention spans and the daily class schedule."

5. Please use the table below to describe the courses you are currently teaching. Please write the specific course title, the number of students enrolled in that course, and the grade level(s) of those students. For the student achievement or ability levels, please check (✓) all achievement levels that apply to at least five students in that class. By very low or very high, I mean more than a year below or above average students enrolled in that class.

	# of	# of	# of	# of	t of Grada	Student Achievement or Ability Levels					
Name of Course	Students	Level(s)	Very	Below	Average	Above	Very				
			Low	Average	Ū	Average	High				

6. From those you listed in #5, identify the course that you are the most satisfied with teaching.

	Never	Some- times	1-3 times per month	1-3 times per week	Almost everyday
Work individually answering questions in the textbook or worksheets					
Do hands-on/laboratory activities					
Work on projects that take a week or more					
Work in small groups to come up with a joint solution or approach to a problem or task					
Write up a solution in which they are expected to explain their mathematical thinking or reasoning at some length					

## 7. About how often do students in the class identified in Item 6 take part in the following activities?

8. When you ask student questions in the class indicated in Item 6, how often are you trying to accomplish the following goals?

	Never	Sometimes	Often	Very Often	Always
See if students know the correct answer					
See if students have done the homework					
Elicit students' ideas and opinions					
Get students to justify and explain their reasoning					
Have students relate what they are working on to their own experiences					

Type of Technology	Avail	ability			Frequency of	Use	
	Teacher	Students	Never	Rarely: Less than once per week	Often: About once per week	Frequently : 2–3 times per week	<b>Daily</b> (or nearly every day)
Graphing Calculator							
Scientific Calculator							
CAS (Computer Algebra System)							
Geometer's Sketchpad							
CABRI Geometry							
Computer Spreadsheet							
Tutorial Software							
Other: (please specify)							

9. Please identify the different kinds of instructional technology that are accessible to you and your students and how often you utilize them during classroom instruction.

10. Where do students use computers during class and how many computers are available in each location? Please check (✓) the most common arrangement.

Location	# of Computers
Classroom	
Computer Lab	
Media Center	
Other (please specify):	

11.	Wh use	hich of the following are among the objectives you have for student technolog? Please check all that apply.	logy
	a.	Mastering skills just taught	
	b.	Remediation of skills not learned well	
	c.	Expressing their ideas in writing	
	d.	Communicating electronically with other people	
	e.	Finding out about ideas and information	
	f.	Analyzing information	
	g.	Presenting information to an audience	
	h.	Improving computer skills	
	i.	Learning to work collaboratively	
	j.	Learning to work independently	
	k.	Other (describe):	

12. Of the objectives listed in Item 11, have been the most important for you? (Please indicate the top three.)

1.	
2.	
3.	

	Not Useful	Slightly Useful	Moderately Useful	Very Useful	Essential
Short-answer and multiple- choice items					
Essay or extended-response items					
Open-ended problems					
Individual and group projects					
Student presentations					

# 13. How useful are each of the following kinds of assessment for you in judging how well your students are learning?

## APPENDIX B

## PRELIMINARY INTERVIEW GUIDE

Thank you for agreeing to participate in my doctoral dissertation study. The purpose of this preliminary interview is to get to know you and your practice prior to the two-week period of observation.

- 1. Warm-Up: *Please begin by telling me briefly about how you came to be a mathematics teacher*.
- 2. Planning: *This study will look at how you interpret and implement the curriculum adopted by your school and the curriculum materials that are provided. Describe how you typically plan a lesson for your* [name of course] *class?* [If not provided, prompt participant to elaborate on the following:
  - a. Sources for lessons, tasks?
  - b. Selection of tasks? (How do you select the problems/exercises that you will give to students?)
  - c. Lesson structure?
- 3. Implementation: *I would like you to recall a lesson that you've recently taught that, in your opinion, was particularly effective. Please describe that lesson and explain why you felt it was effective.* [If not provided, prompt participant to elaborate on the following:
  - a. Goals of the lesson

- b. Structure of the lesson
- c. The role of the teacher during the lesson
- d. The tasks given to students (What were the students expected to do?)
- e. The outcomes of the lesson]
- 4. Technology: One aspect of teaching that I would like to examine is how teachers incorporate classroom-based technology, such as computers and calculators, into their instruction. I have a few questions about your views of technology as a tool in mathematics teaching and learning.
  - a. When do you feel it is appropriate for students to use technology, such as calculators or computers, in mathematics classes? Why?
  - b. What are the advantages and disadvantages of students using technology to learn mathematics?
  - *c. How did you come to learn how to incorporate technology into your lessons?*
  - *d.* What difficulties have you encountered in trying to implement activities that require the use of technology in the classroom?
  - e. How, if at all, are you supported in using technology in the classroom?

## APPENDIX C

## **OBSERVATION INTERVIEW GUIDE**

This brief interview will help me to understand what you have planned for the upcoming lesson.

- 1. What are your goals for this particular lesson?
- 2. How do you propose to meet those goals?

[With particular attention to making sure each of the following are described:

- a. Overview of lesson activities
- b. Tasks that will be posed to enable the attainment of those goals
  - i. Anticipated products of student work
  - ii. Anticipated operations that students will use
  - iii. Available resources that students may use
- c. How progress towards those goals will be assessed]

#### APPENDIX D

### FINAL INTERVIEW GUIDES

#### Michael

- 1. Your assignments featured a lot of problems from past AP exams. What are the criteria you use to select AP problems for assignments, tests, etc.?
- 2. [View video segment: 04/23/09, 0:21:38 to 0:23:00] The function you wrote on the board to describe the bug's position appeared to be improvised. If possible, can you describe your rationale for using this particular function? Were there particular mathematical properties that you considered?
- 3. [View video segment: 05/04/09, 0:38:00 to 0:40:45] In this segment, it appears that a relatively simple process of dividing each term in the rational function by the largest power of x would quickly provided insight regarding the limit. Can you describe your thinking with regard to interpreting the students' responses and why you chose to respond the way you did? Why do you think the students struggled here as much as they did?
- 4. When investigating Newton's method with your students, you used the function  $y = x^2 5$ . Was there a particular rationale for choosing this function?
- 5. [View video segment: 05/07/09, 0:05:50 to 0:07:00] At the point where this segment begins, the students had been working independently on the cubic function task from the CAS book for about three minutes. One student mentioned he was using a "brute force" approach about thirty seconds prior to the start of the

segment. What was your rationale for intervening at this point?

- 6. [View video segment: 05/08/09, 0:26:40 to 0:27:00] It seemed that there were frequent contributions by students that demonstrated thinking well beyond the current moment in class. Given this level of talent and the conversational tone found in your class, how do you ensure that the students as a whole get the opportunity to consider or think about the ideas presented in class?
- 7. I noticed that you placed a great deal of importance on your students being able to determine when it is and when it is not appropriate to use their machines. Specific criteria I noted included the relative effort needed to complete the computations by hand, whether computational techniques were available, and if the calculations were repetitious. Are there other examples of situations when you will prompt students to consider when technology use is appropriate?
- 8. In my observation of your class, I noted an instance during the exploration of the critical points of  $y = x^{(1/\ln(x))}$  where you forbade the use of calculators. In another instance, the students opted to work a problem by hand and you steered them into using the calculator in order for them to consider what they might do if they had a more difficult equation to deal with. These instances point to your role as a regulator in terms of how the students use technology to interact with the mathematics. What factors influence how you fulfill this role?

#### Sarah

1. Earlier you had stated that you did not use the *Paula's Peaches* task provided by the GPS framework because you did not like it. Can you explain your rationale for why you avoid this task?

- 2. During my observation, you devoted a lot of class time making sure the students knew the domain and range of  $y = x^2$ . Why was there such a sustained focus on this problem? Why this particular function?
- 3. When your students are working during class and you're walking around the room monitoring their work, are there any general strategies you follow in terms of how you help them, what you'll do when they ask you a question, or what you'll do when they're having trouble getting started?
- 4. During class discussions, do you have any typical way of deciding when to answer student questions or not to answer them? Is there philosophy behind it?
- 5. I'm going to show you a video where you interact with a student who is trying to come up with a way to describe what's happening with a set of parabolas. Then you work with an ELL student who is struggling with the same thing, but there are two distinct approaches. [Plays video 2/19, 0:38:40–0:43:00] In what ways does the way that you work with English Language Learners differ from the way that you work with other students?
- 6. In this next video, you were trying to get the students to connect the solution of a quadratic equation by factoring and the x-intercepts of a graph. You went through several examples and spelled it out to the point that they were even saying that they were the same thing. But then you left it and transitioned into something else. [Plays video: 2/25, 1:02:12 1:05:00] You said that it's important for them to understand what the relationship is, but you never specifically said what it is. Why did you choose to end the discussion before the students had explicitly articulated that relationship?

- 7. In this next video, [2/26, 0:23:20–0:25:15, 0:38:50–0:40:30] you ask the students to distinguish whether the data involved are continuous or discrete. You take some time to graph it for them and become frustrated with their struggles and say, "Let me tell you a secret. This data is not discrete." What prompted you to resolve this issue for them?
- 8. Later on in that lesson, they were using the approximated curve to find data points that they didn't necessarily have at the beginning. [Plays video, 2/27, 1:04:10–1:09:04] I thought it was interesting that you used a different equation there as an example. You said that you just made it up. Was there any particular rationale for the equation that you used?
- 9. You were trying to have your students do high-level things like, after they filled out the grid with the intercepts, coefficients, and factored form, they tried to discover patterns or relationships in the data. They came up with some very good observations. I was surprised that they got stuck on the low-level part, filling out the table. It wasn't with the open-ended patterns, but more with remembering how to factor or remembering how to do the left-bound, right-bound, etc. To what do you attribute these struggles? Why do you think they had an easier time with the high-level component of the activity?

#### David

 The majority of the time in class during my visit was devoted to working on the *Paula's Peaches* task from the GPS framework. Prior to the first lesson I observed, you indicated that you worked with the students on factoring trinomials. Looking back at their work on the task, in what ways did this strategy affect what occurred? How is that different from the results of similar activities for which you took a different approach?

- 2. Thinking back to the students' work on *Paula's Peaches*, with what aspects of the task were the students most successful? What do you attribute this to? With what aspects did they struggle? What do you attribute this to? If you were to use *Paula's Peaches* again next year, what might you do differently to improve it or how students perform on it?
- 3. I found some of the mathematical elements of *Paula's Peaches* interesting, especially when considered in the context of your students. One was the treatment of average rate of change in items 2 (d) and (e). [View video segment: 02/20/09, 0:29:30] What aspect of this concept prompted students to struggle? Had your students worked with average rate of change before? At this point, you broke the problem down to demonstrate the unit-to-unit changes. Can you recall what factored into this decision? [Suggest displaying data as a table]
- 4. The day the students took the quiz was interesting to me. Prior to the quiz, you provided a few examples of quadratic equations to solve as a review. When the students began the quiz, however, they immediately ran into trouble solving an equation that had already been factored. [Play video segment: 02/25/09, 0:15:00] This problem persisted [0:19:00 and 0:34:30] even with students from before. To what do you attribute this struggle?
- 5. As the students increasingly asked questions about items on the quiz, it seemed that you shifted your focus from only providing hints or redirection to working for a greater level of understanding of what to do. What caused this shift? How does

this instance fit into your philosophy about quizzes and tests?

- What characteristics of a problem lead to group versus whole-class work? For example, you had the students work in groups for 1 (a), (b), and (c), but (d) was done as a class.
- What strategies do you employ when working individually with students, especially when they are struggling? How much help to you try to provide? [For examples, see video clips: 02/24/09 at 0:27:00, 02/24/09 at 0:54:00, and 02/19/09 at 0:28:00]

#### APPENDIX E

## SELECTED MATHEMATICAL TASKS

This section contains a selection of tasks used by the three participating teachers during my observation. I included these tasks so that the reader could refer to them while reading this report and better understand the mathematics involved. These tasks are listed chronologically, in the order in which they were presented to the students.

#### Michael

Flagpole Shadow. A student walks past a lamppost that is 15 feet high. The student is 5

feet tall and walks at a constant rate of 6 feet per second.

- (c) How fast is the length of the shadow changing?
- (d) How fast is the tip of the shadow moving?

Bug on a Line. I've got some particle, some little bug, ... moving along the x-axis ... The

bug's position along the x-axis will be defined as  $x(t) = t^3 - 6t + 5$ .

- (d) Can you tell me when the bug is at the origin, if ever?
- (e) When is the bug at rest?
- (f) Let's say I'm only interested in the time interval from 0 to 4. When is the bug moving the fastest?

*Critical Points*. Find all critical points of  $y = x^{\frac{1}{\ln(x)}}$ .

*Particle on Line 1.* A particle moves along the *x*-axis so that its velocity at any time  $t \ge 0$ is given by  $v(t) = 3t^2 - 2t - 1$ . The position x(t) is 5 when t = 2.

- (a) Write a polynomial expression for the position of the particle at any time  $t \ge 0$ .
- (b) For what values of t, 0 ≤ t ≤ 3, is the particle's instantaneous velocity the same as its average velocity on [0, 3]?
- (c) Find the total distance traveled by the particle from t = 0 to t = 3.

Tangent Line. Find an equation for the line tangent to the curve at the given point defined

by the given value of t.

$$x(t) = \sec^{2}(t) - 1$$
$$y(t) = \tan(t)$$
$$t = -\frac{\pi}{4}$$

Parametric Ellipse. Given the function defined by the parametric equations:

$$x(t) = 3\cos(t)$$
$$y(t) = 5\sin(t)$$

- (a) Use the calculator to graph this ellipse.
- (b) Find an equation for  $\frac{dy}{dx}$ .
- (c) Find the equation of the line tangent to the graph at  $t = \frac{\pi}{4}$ .
- (d) Using  $\frac{dy}{dx}$ , find all the points where the tangent line is vertical and horizontal.
- (e) Rewrite the function in Cartesian form by eliminating the parameter t.

*Kite*. A kite flies according to the parametric equations  $x = \frac{t}{8}$  and  $y = -\frac{3}{64}t(t-128)$ 

where *t* is measured in seconds.  $0 \le t \le 90$ 

- (e) How high is the kite above the ground at t = 32 seconds?
- (f) At what rate is the kite rising at t = 32 seconds?
- (g) At what rate is the string being reeled out at t = 32 seconds?

#### (h) At what time does the kite start to lose altitude?

*Particle on Circle*. A particle moves on a circle  $x^2 + y^2 = 1$  so that at time  $t \ge 0$  the

position is given by the vector  $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ 

- (a) Find the velocity vector.
- (b) Is the particle ever at rest?
- (c) Give the coordinates of the point that the particle approaches as *t* increases without bound.

*Roots of a Cubic* (found in Fey et al., 2003). Let f(x) be a cubic function with three real roots, *a*, *b*, and *c*, and let  $m = \frac{1}{2}(a + b)$  the average of any two roots. Prove that the tangent line to f(x) at x = m intersects the cubic at (c, 0), the third root of the cubic. Sarah

*Domain and Range of*  $y = x^2$ . What is the domain and range of  $y = x^2$ ?

- *Families of Parabolas.* [Students were asked to graph a group of quadratic functions that were all related by a particular transformation and then make an observation about the graphs.]
- *x-Intercept Data Table.* Students completed a table by identifying the following characteristics for provided quadratic functions: coefficients, intercepts, and graph.Once this table was complete, the students were challenged to find three patterns within the data in the table.
- *Domain and Range with Graphs*. Students were asked to graph the following functions and find the domain and range of each: y = |x|,  $y = x^3$ ,  $y = x^2$

Sports Arena. Accompanied a video on structural engineering presented by The Futures

Channel, this task asks students to fit a quadratic curve to provided data. Reproduced with permission.



#### SkyHighScrapers

To: Structural Engineering Team #5 From: R. Shreve, Chief Architect

We just got back the data from wind tunnel testing for our proposed "doubledecker sports arena" (see diagram 1).



Diagram 1: the "Double-Decker"

Table 1 shows the pressure when the wind blows from the east.

Table 2 shows the pressure when the wind blows from the north.

The pressure, of course, increases with higher wind speed. Theoretically, we expect that the relationship is quadratic:

$$p = av^2 + bv + c$$

Can you help us out?

1) For table 1, graph the data, then find the values of a, b, and c for the quadratic equation that gives a good fit to the data in the table. Draw the graph of your equation on top of the graph of the original data, so I can see how well it matches. Use that equation to give us an estimate for the pressure for wind speeds for 5 mph and 100 mph.

2) Do the same thing for table 2.

Thanks.

Rich

SkyHighScrapers

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Table 1: East Wind

Wind speed	Pressure
(miles per hour)	(pounds of force per
	square foot)
10	1
20	1.6
30	2.8
40	4.3
50	5.9
60	8.1
70	11
80	14

#### Table 2: North Wind

Wind speed	Pressure
(miles per hour)	(pounds of force
	per square foot)
10	2
20	3
30	4.5
40	6.5
50	10
60	13
70	16.5
80	22

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SkyHighScrapers

Parabola Maxima and Minima. Adapted from the Connected Mathematics Project materials (Lappan et al., 2006), this investigation required students to match eight quadratic functions in factored form with their graphs and analyze the relationships between them. The instructions for the students were:

- A. Do parts (1)–(5) for each equation.
  - 1. Match the equation to its graph.
  - 2. Label the coordinates of the *x* and *y*-intercepts of the graph. Describe how you can predict the *x* and *y*-intercepts from the equation.
  - 3. Draw the line of symmetry of the graph. Can you predict the location of the line of symmetry from the equation? Explain.
  - 4. Label the coordinates of the maximum or minimum point. Can you predict the maximum or minimum point from the equation? Explain.
  - 5. Describe the shape of the graph.

#### David

Paula's Peaches and Logo Symmetry. From the Georgia Performance Standards
Framework developed by the Georgia Department of Education, accessed March 8,
2006 from: https://www.georgiastandards.org/Frameworks/GSO%20Frameworks/912%20Math%20I%20Student%20Edition%20Unit%205%20Algebra%20in%20Cont
ext.pdf