REVERSIBLE REASONING IN MULTIPLICATIVE SITUATIONS: CONCEPTUAL ANALYSIS, AFFORDANCES AND CONSTRA1NTS

by

AJAY RAMFUL

(Under the Direction of John Olive)

ABSTRACT

While previous research studies have focused primarily on the additive structure $a \pm x = b$, relatively fewer attempts have been made to explore the multiplicative structure $ax = b$ from a reversibility perspective. The aim of this exploratory study was to investigate, at a fine-grained level of detail, the strategies and constraints that middle-school students encounter in reasoning reversibly in the multiplicative domains of fraction and ratio. In the first phase of the study a mathematical analysis of reversibility situations was conducted. In the second empirical phase, three pairs of above-average students at the grades 6, 7 and 8 were interviewed in a rural middle-school in the United States. Three selected data sets were analyzed using Vergnaud's theory of the multiplicative conceptual field, the concept of units, and notion of quantitative reasoning. The findings put into perspective the importance of the theorem-in-action $ax = b$ as a key element for reasoning reversibly in multiplicative situations. Further, the results show that reversibility is context-sensitive, with the numeric feature of problem parameters being a major factor. Relatively prime numbers and fractional quantities acted as inhibitors preventing the cueing of the invariant 'division as the inverse of multiplication', thereby constraining students from reasoning reversibly. Moreover, the form of reversible reasoning was found to be dependent on the type of multiplicative structures.

Among others, two key resources were identified as being essential for reasoning reversibly in fractional contexts: interpreting fractions in terms of units, which enabled the students to access their whole number knowledge and secondly, the coordination of 3 levels of units. Similarly, interpreting a ratio as a quantitative structure together with the coordination of 3 levels of units was found to be essential for reasoning reversibly in ratio situations. This study also shows that students can articulate mathematical relationships operationally but may not necessarily be able to represent them algebraically. The constraints that the students encountered in the multiplicative comparison of two quantities substantiate previous research that multiplicative reasoning is not naturally occurring. Failure to conceptualize multiplicative relations in reverse constrained the students to use more primitive fallback mechanisms, like the building-up strategy and guess-and-check strategy, leading them to solve problems non-deterministically and at higher computational cost.

INDEX WORDS: Division, Fraction, Multiplication, Multiplicative Comparison, Quantitative Reasoning, Ratio, Reversibility, Theorem-in-action, Units, Vergnaud's Theory

REVERSIBLE REASONING IN MULTIPLICATIVE SITUATIONS: CONCEPTUAL ANALYSIS, AFFORDANCES AND CONSTRAINTS

by

AJAY RAMFUL

B.S (Hons) Mathematics, University of Mauritius, Mauritius, 1997

M.Phil (Mathematics), University of Mauritius, Mauritius, 2000

MA, University of Brighton, England, 2004

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2009

© 2009

AJAY RAMFUL

All Rights Reserved

REVERSIBLE REASONING IN MULTIPLICATIVE SITUATIONS: CONCEPTUAL ANALYSIS, AFFORDANCES AND CONSTRAINTS

by

AJAY RAMFUL

Major Professor: John Olive

Committee: Denise S. Mewborn
Andrew G. Izsák

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
December 2009

DEDICATION

I dedicate this work to my daughter, Jaylina Aislee who is now supposedly at the concrete operational stage and who, at every given opportunity, teaches me children's mathematics.

ACKNOWLEDGEMENTS

I express my sincere gratitude and thanks to the members of my committee. I am deeply indebted towards my major professor, Dr Olive who has diligently and dedicatedly supervised this project from start to finish. I cannot forget his constant support and guidance starting from the independent study that I did in Spring 2007 when the current research was just in its embryonic stage. I express my sincere appreciation for his willingness to help and for always giving value to my ideas. I thank him for devoting so much time and effort in painstakingly reading my analysis over and over again, making insightful comments that prompted me to rethink about various parts of the dissertation. The three years that I have spent working with him shaped my thinking about children's mathematics. Thank you Dr Olive.

I am equally indebted to Dr Mewborn whose decision helped to shape critical aspects of this study. Her comments and critiques led me to reorient my thinking at various points. I acknowledge her prompt and quick support as a pragmatic head of department. I thank her for the various assistantships (sometimes at difficult moments) and for providing me teaching opportunities.

My interest in student's thinking was very much stimulated by Dr Izsák's video-recording class (in Spring 2007) that served as the springboard for the current study. I am very thankful to Dr Izsák who has a major contribution in the realization of this dissertation. His sharp and pointed questions served as triggers in designing this piece of research. His writings influenced me in a number of ways, from methodology to data analysis.

I chose to come at The University of Georgia in 2006 to work with people that I had been reading about from a distant island. I have a special thank to Dr Les Steffe who allowed me to collect data for this dissertation under his project. He taught me a lot about children's mathematics in the two EMAT classes. Dr Jeremy Kilpatrick has also been an inspiring educator. I would also like to thank the significant others – Dr Jim Wilson, Dr Patricia Wilson, Dr Larry Hatfield, Dr Nicholas Oppong, Dr Dorothy White, and other faculties from other departments who allowed me to audit their classes. I am also thankful to Helen, Rita, Valerie, and Ms Boswell who have always been on the forefront to help with administrative matters.

This dissertation would not have been possible without the six participants of the study, Ted, Cole, Brian, Aileen, Jeff and Eric. I would like to thank these six thinkers for allowing me to 'read' their minds while they were zealously involved in thinking about the problems I posed to them. I would also like to thank the school principal and one teacher who went out of her way to facilitate the conduct of this study. Unfortunately, anonymity should be maintained. I thank Katy for negotiating access to the school.

My thanks also go to Fulbright and the Mauritius Institute of Education for financially supporting me for the first two years of my study. I express my sincere appreciation to Pastor Jeff Thompson, the Clarke's family and friends who continuously supported me and my family here in Athens. I thank my father and mother who have supported me in various ways throughout the study. Finally, I express my sincere gratitude to my wife, Lovelina who has been by my side during those difficult moments and who provided me with the best conditions to complete my studies. I am particularly thankful to her for her patience. Lovelina and Aislee gave me the strength and motivation to go to the end.

TABLE OF CONTENTS

Page

LIST OF TABLES

Page

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

One of the ways to improve mathematics teaching and learning is to understand how students make sense of mathematics or construct mathematical structures and relations. Research in mathematics education has come a long way to understand how students attempt to make such fine-grained construction of mathematical structures and relations, generating a number of theories of knowledge acquisition in the process. Currently, constructivism is a widely accepted philosophy of mathematics teaching and learning, initiated by the foundational work of Piaget. Likewise, this study is situated within this philosophy and focuses on one aspect of Piaget's theory, namely reversibility of thought which essentially involves reasoning from a given result to the source producing the result.

Reversibility can be considered a key requirement in a number of problems in mathematics (Inhelder & Piaget, 1958). It can be a requirement in either a single operation, as in finding the missing addend in an addition/subtraction problem or in a multi-step task, as in constructing the whole corresponding to a given fraction. Reversibility can be a cognitively demanding form of reasoning, especially in multiplicative situations. However, despite being a central component of mathematical reasoning, reversibility is not prominent in current mathematics education research, partly because it is an implicit process. In the present study, reversibility has been investigated in the domains of whole number multiplication/division, fraction, and ratio at the middle school level, a period where students are required to expand and differentiate their competence to reason multiplicatively.

The inspiration of this investigation stemmed from Piaget's concept of reversibility of thought and has been motivated by the recent plea made by Lamon (2007) for analyzing such a process: "Researchers know very little about reversibility or about multiplicative operations and inverses, and these could be subjects for a valuable microanalysis research agenda" (p. 661). Another theoretical thrust to conduct this research is that, in contrast to additive situations (Carpenter & Moser, 1983, Fuson, 1992; Nesher et al., 1982), reversibility has not been given much attention in the multiplicative domain.

Furthermore, as a teacher I observed several critical events that prompted me to delve deeper in understanding how students reason in reverse. For example, over 90% of pre-service teachers out of 120 could not solve the following problem: 'Shade three more squares in Figure 1.1 so that the completed square grid has rotational symmetry of order 4.' In contrast, students could more easily determine the order of rotational symmetry when a completed figure was given.

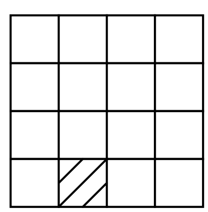


Figure 1.1. 4×4 grid

Another observation is that students could readily apply the power law of indices as $(\frac{a}{b})^x = \frac{a^x}{b^x}$ but it was more challenging for them to apply the reverse property $\frac{a^x}{b^x} = (\frac{a}{b})^x$. Such types of 'doing' and 'undoing' processes are widespread in mathematics, starting with early number experiences in counting forward and backward; in adding and subtracting or multiplying and dividing; in algebraic expansion and factorization at the middle school level; in deriving the inverse of functions and in finding derivative and anti-derivative in Calculus at the high school or college level.

Since Piaget's development of the notion of reversibility, researchers working in different areas (Behr & Post, 1992; Herscovics, 1996 ; Lamon, 1994; Olive, 1999; Olive & Steffe, 2002; Steffe, 1994; Tzur, 2004; Vergnaud, 1988) have been evoking this idea but it has not been a central point or focus, especially at the middle school level. Further, the way the concept of reversibility has been used in mathematics education is operatively different from Piaget's initial definition. In the section that follows, I present two conceptions of reversibility. In the first conception, I trace the concept based on Piaget's idea of negation and compensation. The second conception is based on research in mathematics education.

Conception 1: Piaget's Notion of Negation and Compensation

Piaget (1970) conceptualized reversibility as one of the four characteristics of an operation, the other three being regarded as follows: (i) as an internalized action (i.e., an action that can be carried out in thought as well as executed materially), (ii) as involving conservation/ invariance (e.g., in the case of addition, we can transform the way we group together 5+1, 4+2 or 3+3 but what is invariant is the sum) (iii) as related to a system of operations in a structure. Piaget defined reversibility operationally on the basis of the distinction that he made between classes and relations (Flavell, 1963). He characterized reversibility in two different forms: negation in the case of classes and reciprocity or compensation in the case of relations. In the case of classes, reversibility expresses the idea that every direct operation has an inverse which cancels or negates it. For example, addition is cancelled by subtraction and multiplication by division. Piaget (1970) argued that "subtraction is simply the reversal of addition – exactly the same operation carried out in the other direction" (p.22). Further, in terms of changes of position, he states that negation involves the understanding that a movement in one direction can be cancelled by a movement in the opposite direction.

On the other hand, reciprocity or compensation deals with relational structures. For example, understanding the relation embodied in the equation $2 + 2 = 4$ as a whole collection of 4 objects as well as two partial collections of two objects each (Chapman, 1992). Reversible reasoning is required to conceptualize the assimilation of the parts to the whole and viewing the whole as consisting of parts. Such an additive part-whole schema represents a major conceptual achievement in the early school years (Resnick, 1992). The part-whole schema can also be observed in the multiplicative domain. For example, in a situation such as 'if 2/5 of a parking lot holds 30 cars, how many cars can the parking lot contain?', one is working with the part-to-whole relation between a fraction and its referent whole (i.e., given a part make the whole) instead of the whole-to-part relation (i.e., given the whole make a part). Similarly, reciprocity is involved in the coordination of the relation 'less than' and 'more than'. One has to reason reversibly to deduce that if $b > a$ (the relation between *b* and *a*) then $a < b$ (the relation between *a* and *b*). Piaget also used the term reciprocity to refer to a reversal of order. For instance, if A = B then the reciprocal B = A is also true. In reciprocity, there is nothing negated. Piaget (1970) used the following seriation task to illustrate reciprocity. Suppose a child is asked to arrange a set of sticks of different length in order, starting from smallest to biggest. First, s/he starts with the very smallest stick, then s/he looks through the remaining sticks for the smallest ones left and continues this process until the whole series has been built. Piaget claims that this task involves reciprocity:

> When the child looks for the smallest stick of all those that remain, he understands at one and the same time that this stick is bigger than all the ones he has taken so far and smaller than all the ones he will take later. He is coordinating here at the same time the relationship “bigger than” and the relationship “smaller than” (p.29).

In summary, it appears that Piaget used the term reciprocity to denote the coordination between the two-sidedness of a relation.

To gain further insight on the processes of negation and reciprocity, I have consulted a range of Piaget’s books (Inhelder & Piaget, 1958; Piaget, Inhelder & Szeminska, 1960; Piaget & Inhelder, 1974; Piaget, 1985) as well as books written on Piaget’s theory (Beilin & Pufall, 1992; Flavel, 1963; Gallagher & Reid, 1981; Gruber & Vonèche, 1995; Siegel & Brainerd, 1978; Sigel & Cocking, 1977). One of the definitions which illuminates the notion of reversibility is given by Inhelder & Piaget (1958), where reversibility is defined as:

> … the permanent possibility of returning to the starting point of the operation in question. From a structural standpoint, it can appear in two distinct and complementary forms. First, one can return to the starting point by canceling an operation which has already been performed –i.e., by inversion or negation. In this case the product of the direct operation and its inverse is the null or identical operation. Secondly, one can return to the starting point by compensating a difference (in the logical sense of the term) –i.e., by reciprocity. In this case, the product of the two reciprocal operations is not a null operation but an equivalence (pp. 272-273).

The essence of this definition is that it emphasizes the fact that reversibility involves the operation of returning to the starting point. The interpretation provided by Taylor (1971, cited in Siegel & Brainerd, 1978) provides more insight into the above definition.

> Thought is “reversible” when it can operate [upon] transformations and still recover its point of departure. Now, properly to understand something is to be able to follow the changes it undergoes, or could undergo, and to grasp well enough what is involved in these changes so that one can say what would be required to return the object to its initial state; and this either by simple reversal, or by compensating operations of some kind whose relations to the original one understands … . (p.137)

Piaget also distinguished between perceptual, empirical and operational reversibility. His explanation of perceptual reversibility is not explicit and the only source I could access is the following quote: “At the perceptual level already (even though we have not yet complete reversibility) the inversions correspond to addition or elimination of elements, and the reciprocities correspond to symmetries and similarities” (Inhelder & Piaget, 1958, p.273). On the other hand, empirical reversibility is concerned with the physical action of undoing a prior action and has been termed revertibility (Pinard, 1981) and “operational reversibility is the inverse of the direct (or thetic) operation” (Piaget, Inhelder, & Szeminska, 1960, p.330). I infer that operational reversibility is concerned with reversing direct processes. In mathematics, this may mean reversing operations like doubling and halving, composing and decomposing, stretching and shrinking and so forth. Piaget’s emphasis on reversing *direct* operations for operational reversibility can also be deduced from his work on proportion. He does not consider the building-up strategy for solving proportional problems as operational reversibility because this strategy does not involve the inverse of a direct operation. As an example of the building-up strategy, consider the following problem: A candy store sells 2 pieces of candy for 8 cents. How much do 6 pieces of candy cost? The building-up strategy works as follows: 2 pieces for 8 cents, 4 pieces for 16 cents, and 6 pieces for 24 cents. In other words, the building-up strategy does not necessitate a multiplicative inverse and does not require one to reason reversibly. “Reversibility is demonstrated if, when a change is made in one of the four variables in a proportion, the student is able to compensate by changing one of the other variables by an appropriate amount” (Lesh & Doer, 2003, p. 20). Even, in the contexts of fractions, Piaget, Inhelder, & Szeminska (1960), analyzed reversibility from the perspective of the inverse of a direct operation. For instance, they assert that “ Re-grouping the pieces to form a unique whole is the inverse of cutting-out only if the latter operation follows a fixed and pre-ordered path, for only then can that path be unambiguously reversed. Such is the condition of true reversibility, …” (p. 331). This conception shows that for Piaget, reversibility is regarded as the capacity to operate to and fro on a given path by using precisely the same direct and inverse operations.

Still another form of reversibility (called semantic reversibility) has been identified in language comprehension studies (Sigel & Cocking, 1977). It has been observed that changing sentences from active to passive voice (i.e., altering the syntactic sequence of subject-verb-object) led children to attribute different meaning to the same sentence. For instance, changing the sentence “The boy is chasing the girl” and “The girl is being chased by the boy” (Sigel & Cocking, 1977, p.129) may lead children to confuse between subject and object.

It should be pointed out that reversibility occupied a central position in Piaget’s theory due to its association with the concept of conservation. Reversibility was regarded as the capacity to reverse a transformation from its final state to its initial state. For instance, children were asked to compare two strings that were shown to be of equal length, after which one was arranged in the form of a loop and the other was kept straight. Children who could deduce that the two strings were still of equal length showed evidence of conservation (i.e., the conserving child could mentally open the looped string). Reversibility was regarded as necessary for such conservation. Furthermore, through the use of physical apparatus like the balance beam, Piaget showed how negation and compensation operate in proportional tasks, using the principle of mechanical equilibrium. For example, if a balance beam is in disequilibrium, then there are two ways to establish the equilibrium in terms of the weight, either by removing weight from the heavier side (negation) or by adding weight to the lighter side (compensation). Further, Piaget used the notion of reversibility of thought in relation to stages of intellectual development. For instance, he argued that children at the formal level of operations are capable to coordinate negation and compensation simultaneously while the two processes operate independently at the concrete operational stage (Inhelder & Piaget, 1958; Furth, 1969). Moreover, Piaget considered reversibility to be "a necessary by-product of the equilibration-of-structures process; a psychological system which is strongly equilibrated must entail the balancing and compensation functions supplied by negation and reciprocal operations" (Flavell, 1963, p. 243).

Research on reversibility with young children in the 1960s and 70's was essentially focused on the concept of conservation within the Piagetian framework as can be inferred from the *Child Development* journal. For instance, Wallach, Wall & Anderson (1967) studied the roles of reversibility and misleading perceptual cues on conservation. In the context of research with young children, Sparks, Brown, & Bassler (1970) assert that reversibility refers to children's realization that "even before a set or an object is changed, there exists an inverse operation that will restore the original state" (p. 136). In most cases, these investigations involved experiments with concrete objects and pupils were required to carry out the reverse process. It is thus important to differentiate between the empirical analogues of reversibility or empirical return as compared to operational or logical reversibility (Schnall et al., 1972). In the case of reversibility experiments with concrete objects, the child may give a correct answer by virtue of sensory-motor action and not necessarily by logical deduction. However, in operational reversibility, the child is required to reverse a thought process by virtue of the properties of the system (e.g., using the mathematical relationship linking two quantities).

In mathematics education, Adi (1978) used the concept of negation and compensation to study the relationship between college students' developmental level and their performance on equation solving. She provides the equation $14 - \frac{15}{7 - x} = 9$ to illustrate her interpretation of negation and compensation. In solving this algebraic equation, negation is involved when one is asked to make the following inferences: 'Fourteen minus what equals nine?', 'Fifteen divided by what equals five?', and 'Seven minus what equals three?'. On the other hand, compensation is involved when one multiply both sides of the equation by $(7 - x)$ to obtain $98 - 14x - 15 = 63 - 9x$. Further, compensation occurs when one adds $14x$ to either side of the equation to obtain $83 = 63 + 5x$ after which 63 can be subtracted on either side to yield $20 = 5x$. In the final step, compensation is involved when one divides either side by 5 to get the answer as $x = 4$.

Conception 2: Reversibility in Mathematics Education Research

After the 1960's experiments on conservation, the concept of reversibility is apparent in mathematics education research in relation to children's acquisition of number, arithmetic operations, fractions, and to some extent in ratio and proportion. Rather than using Piaget's notion of negation and compensation, researchers working in different mathematical domains have used the idea of reversibility essentially as 'working from the output of a problem to the input' as illustrated by the following examples (though the appellation reversibility has not always been evoked). A first case in point in additive contexts is missing-addend problems of the form $a \pm x = b$ or $x \pm a = b$, where a and b are known and x is unknown. These problems have been termed "find-the-initial-state case" (Vergnaud, 1997, p. 16) or "Start-Unknown" (Riley, Greeno, & Heller, 1983, p. 160) and they have been found to be more difficult among students compared to other forms of addition and subtraction types. In her extensive analysis of addition and subtraction problems, Fuson (1992) points out that reversibility is needed to deal with addition and subtraction problems that cannot be solved by direct modeling. For instance, Jim has 5 marbles. He has 8 fewer marbles than Connie. How many marbles does Connie have? (Carpenter & Moser, 1983, p. 16). She further adds that "Reversibility knowledge enables one to think about a situation in reverse, such as solving the same problem by counting down the known Change quantity from the known End quantity" (p. 257). For instance, Bob got 2 cookies, and now he has 5 cookies. How many cookies did Bob have in the beginning?

The occurrence of reversibility can also be observed in the domain of algebra. Nathan & Koedinger (2000a) used the appellation "start-unknown problems" to refer to algebraic problems, where the end result is known but the starting quantity is unknown. One example from their paper reads as follows: "When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make?" (p. 170). Thinking with an unknown quantity either symbolically or non-symbolically is a key constituent of algebraic reasoning. As the foregoing example shows, reversibility often involves working with an unknown quantity. Previous studies (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Linchevski & Herscovics 1996; Nathan & Koedinger, 2000a, 2000b) have documented students' difficulties working with unknowns when investigating the development of algebraic reasoning. For instance, Linchevski & Herscovics (1996) identified the existence of a cognitive gap that middle-school students experienced when operating on unknowns. They highlighted the inverse reasoning that is involved in solving a linear equation such as $ax + b = c$ that involve both inverse operations as well as the inversion of the order of operations. Further, Nathan & Koedinger (2000a) analyzed the influence of the position of the unknown in an algebraic equation on solution strategies. They observed a significantly lower performance on start-unknown algebra problems compared to result-unknown problems. This gives an indication that reversibility problems are cognitively demanding. The importance of reversibility in algebraic thinking has also been pointed out by Driscoll (1999):

> Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them). In effect, it is the capacity not only to use a process to get to a goal, but able to understand the process well enough to work backward from the answer to the starting point." (p.1)

Taken together, these studies have focused primarily on the ways in which students operated on unknowns expressed in symbolic forms. In the current study, I identify ways in which students operated on non-symbolized unknowns as they occur implicitly in the form of theorems-in-action (as explained in Chapter 2).

Now, I turn to reversibility situations as observed in multiplicative contexts which can be characterized by $ax = b$ (compared to $a \pm x = b$ in additive situations). A theoretical analysis of reversibility situations in the multiplicative domain is made in Chapter 4. As an illustration consider the domain of fractions where a unit, non-unit, or improper fraction is given and the objective is to construct the whole. This type of problem has been investigated by Behr & Post (1992), Hackenberg (2005), Herscovics (1996), Olive (1999), Olive and Steffe (2002), Tzur

(2004), and Lamon (2007). Behr & Post (1992) use the term 'construct-the-unit' to refer to such fractional problems in which students are required to construct the unit whole from a given fractional part in either discrete or continuous contexts. A similar conception of reversibility is shared by Herscovics (1996). One additional example can be gleaned from Lamon's notion of "reasoning up and down" to determine the fractional part of a set when the unit is given implicitly (Lamon, 2007). For instance, "If 6 books are 2/3 of all the books on Robert's shelf, figure out how many books are 5/9 of the books on his shelf" (p. 653). It entails reasoning up from 2/3 to one whole/unit (consisting of 9 books) and then back down from the unit to 5/9 (i.e., 5/9 of 9 books). Another multiplicative context where reversibility has been shown to occur is in the domain of ratio. Using the perspective of units, Lamon (1993) observed the difficulties that students experience in decomposing a given quantity in terms of the components of a ratio. This is the only piece of research that I could access in the domain of ratio that explicitly refers to reversible reasoning.

Steffe's and Olive's Conception of Reversibility: A Fined-grained Definition Based on Schemes

Steffe's (1994) notion of reversibility can be inferred from his research on numbers and arithmetic operations. His conception of reversibility is deeply rooted in the very act of counting:

> I understand the construction of reversibility as a product of reinteriorization of the initial number sequence that occurs when a child takes an associated verbal number sequence as material in a review of counting. After constructing the number sequence at two levels of interiorization, the child can take the contents of the first level as the material of the operations at the second level, which is the constitutive operation of reversibility. (p.16)

Along the same lines, Olive (2001) adds that: "reversible operations are established through recursive applications of the activity of a scheme to the results of the scheme" (p. 5). Steffe's and Olive's conception of reversibility is essentially tied to their definition of scheme. They consider a scheme as a goal-directed anticipatory system of actions and operations that consist of three parts: (a) an experiential situation which serves as a trigger, (b) an activity or procedure (e.g., counting) and (c) a result. Steffe (1994) considers a scheme to be reversible if the first part (situation) and third part (result) of a scheme can be used interchangeably. To clarify this notion of reversible scheme, consider an example from one of his teaching experiments, where the student Johanna was initially asked to find the number of groups of three blocks in a container having 12 blocks. She correctly gave the answer as four, showing evidence of having a units-coordinating scheme. In the second step of the experiment, some more blocks were placed in a second container. The student was asked to find the number of groups of three in the second container if there were 27 blocks altogether. This is a reversibility situation as the output (27) is known and the aim is to find the missing addend (15). The student correctly gave five as the answer, explaining that there were a total of $4 + 5$ groups of three. In other words, she solved the problem using the composite unit of three elements. In the first part of the experiment, Johanna used her units-coordinating scheme to make four groups of 3 elements each. In the second part of the experiment, the composite unit of three was iterated 5 times, starting from 12 until 27 is reached. Another way of looking at this scenario is that the result of her first unit-coordinating scheme (i.e., 4 groups of 3) has been fed back in its 'situation' as Johanna has been *counting-on* from 4 groups of three to 9 groups of three. Steffe (1994) calls this scheme a reversible units-coordinating scheme. Two observations are in order here. Firstly, the result of a scheme is being fed back into its situation and secondly there is recursive applications (the iteration of the composite unit of 3 elements) of the activity of a scheme to the results of the scheme.

Steffe's and Olive's conception of reversibility is based on fine-grained processes like unit composition and unit segmentation. The different examples presented in their research on fractions (e.g., Olive, 1999; Olive & Steffe, 2002) indicate that they consider the construction of the unit whole from the fractional part as the reverse of finding the fractional part of a unit whole. A person having a *reversible fraction scheme* can construct a fraction from a given whole and a whole from a given part. Following their research orientation, Tzur (2004) defines a reversible fraction conception as "the learner's partitioning of a non-unit fraction (n/m) into n parts to produce the unit fraction ($1/m$) from which the non-unit fraction was composed in the first place" (p.93).

These researchers have been primarily experimenting with situations of the form $ax = b$, where a and b are given whole numbers or rational numbers and the aim is to find x. Most of their teaching experiments require students to operate with a computer microworld designed on the basis of fraction bars (a continuous quantity). Further, they do not use the idea of negation that Piaget refers to as the reverse operation involved in canceling a particular grouping operation (e.g., subtraction negates addition). Unlike Piaget, Steffe and Olive are primarily concerned with mental operations rather than reverting physical actions. They describe both reversibility of operations and reversible schemes.

Steffe (1992) also asserts that not all schemes are reversible, illustrating this claim with the following example. A student named Maya was asked to find how many piles of two she could make from 12 pennies and she immediately answered 6. Soon after, she was asked how many pennies would be in 6 piles of 2 and she gave the answer 3 rather than 12. Steffe claimed that Maya had a one-way unit-coordinating scheme because there was a lack of inversion between the 'situation' and the 'result' of such a scheme.

Other Definitions of Reversibility

In this section, I present other definitions of reversibility that I could gather from the mathematics education literature. Dreyfus & Eisenberg (1996) use the phrase 'reversal of thinking' to denote the process of thinking "from the result, to the source or sources causing the result" (p. 274). They provide examples from elementary, secondary and even collegiate mathematics, where such reversal of thinking is required. They assert that many textbooks have not given enough consideration to this type of thinking unlike test designers. Another interpretation is provided by Krutetskii (1976) who considers reversibility from two different but related perspectives. Firstly, reversibility is considered as the establishment of "two-way (or reversible) associations (bonds) of the type $A \leftrightarrow B$, as opposed to one-way bonds of the type $A \rightarrow B$, which function only in one direction" (p. 287). Secondly, reversibility is regarded as "thinking in reverse direction from the result or the product to the initial data" (p. 287). He further points out that in reversing one's thought process, it is not necessary to always travel through the same route but what matters is the order (i.e., in a reverse problem one starts from the output to the input). He considers reverse problems as "problems in which the subject matter of the original (direct) problem is retained, with the original unknown becoming part of the terms, and one or several elements of the original terms becoming unknown" (p. 143).

Though seemingly related, there is a difference between reversibility and 'working backwards'. Polya makes a distinction (1945/1988) between 'working forwards' and 'working backwards'. In 'working forwards', we start from the given initial situation to the desired final goal, from data to unknown. In 'working backwards', "we start from what is required and assume what is sought as already found" or "from what antecedent the desired result could be derived" (p. 227). This may involve the consideration of sequences of antecedents. In other words, in working backwards, we assume that "we have the thing we are looking for and work backwards from that assumption until we reach something we do know" (Grabiner, 1995, p. 85). The case of finding the inverse of a function is an example of the problem solving procedure of working backwards (which involves both inverting operations and their order). Consider the problem of finding the inverse of $f(x) = 4x + 1$. One customary procedure to find the inverse is as follows:

Let the inverse be denoted by y (i.e., $y = f^{-1}(x)$). Then $x = f(y) = 4y + 1$ which gives $y = (x-1)/4$. It can be observed that in finding the inverse of a function, the operations are reversed (or negated) and the sequence of the operations is also reversed as is graphically illustrated in Figure 1.2 below:

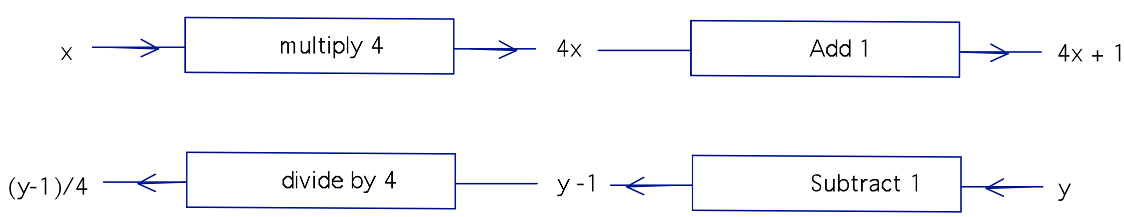


Figure 1.2. Diagrammatic representation of a function and its inverse

Reversibility and 'working backwards' in Polya's sense are different in that in 'working backward' we assume what has to be found is already known. Working backward may be regarded as a problem solving procedure that may involve successive reversal of operations.

To sum up, the different perspectives of reversibility presented above share one common feature: they attempt to capture the operation of reversing a thought process. In essence, the central task in a reversibility problem is to find an unknown piece of information (or condition) in a mathematical relation such that the given 'end' is satisfied.

Clarification of terms: reciprocity, reciprocal relationship of relative size, and reciprocal

Because this study is primarily concerned with multiplicative relationships, it is important to clarify the concept of 'reciprocity' as used by Piaget, the notion of 'reciprocal relationship of relative size' as used by Thompson & Saldanha (2003) and the mathematical term 'reciprocal.' Piaget used the term *reciprocity* to refer to the second type of reversibility (i.e., compensation, the first type being negation). Piaget's reciprocity is not restricted to multiplicative structures but encompasses a wider range of concepts as explained earlier. In essence, he used the term reciprocity to refer to the coordination between the 'two-sidedness' of a relation. For example, if $a > b$, then $b < a$ or if $a = b$, then $b = a$. On the other hand, Thompson & Saldanha (2003) consider a fraction as consisting of two quantities which are in *reciprocal relationship of relative size*. For instance, amount A is 1/5 the size of amount B means that amount B is 5 times as large as amount A. Amount A being 5 times as large as amount B means that amount B is 1/5 as large as amount A. Hackenberg (2005) uses the term *reciprocity* to refer to the conceptualization of a quantitative relation as bi-directional where one can appropriate any of the two quantities in the quantitative relationship as the basis by which another quantity is produced. For instance, it entails knowing that if A is two-thirds of B, then B must be three-halves of A. Such a simultaneous conceptualization entails the construction of a reciprocal multiplicative relationship. The third term ‘reciprocal’ refers to the multiplicative inverse of a number as is conventionally used in mathematics.

Differentiating Between ‘Reversibility Situations’ and ‘Operational Reversibility’

A reversibility problem defined on the basis of its mathematical structure does not necessarily imply that the student has to reverse his/her thought process to solve it. For instance, in the building-up strategy example given previously, students bypass the need for coordinating two ratios for solving specific proportion problems when they choose to use such an additive strategy. In fact, Greer (1992) brought a similar point to our attention: “… it is important to realize that the way in which a situation is interpreted is not inherent in the situation, but depends on the students’ construal of it” (p. 279). The distinction between the mathematical structure and children’s reasoning creates the necessity to delineate reversibility in two dimensions: ‘reversibility situations’ as a function of problem structure and ‘operational reversibility’ as a function of cognitive operation. The definition of ‘reversibility situations’ emanates from the inherent structures in mathematics (with regard to objects, operations, relations, and transformations). For instance, processes like composition and decomposition, halving and doubling, stretching and shrinking, expanding and factorizing, or squaring and taking square roots constitute inverses by virtue of their mathematical definition or mathematical conventions.

The way in which a mathematical problem is formulated may confer to it reversibility characteristics. We can either pose a problem straight forward where all the inputs are given and the task is to reflect conceptually and feed in the input through the necessary procedures, heuristics or algorithms to find the solution. Conversely, the problem can be stated in an indirect way where some conditions have to be satisfied and where the objective is to find the 'missing' information. As an illustration, consider the following tasks:

Example 1

Primal: Four objects have weights 3kg, 4kg, 6kg, and 7 kg. What is the average weight of the four objects?

Dual: The average weight of 4 objects is 5kg. Three of the objects have respective weights 3kg, 4kg, and 6 kg. What is the weight of the fourth object? (Zazkis and Hazzan, 1999, p.433)

Example 2:

Primal: Find the gradient of the ramp in Figure 1.3.

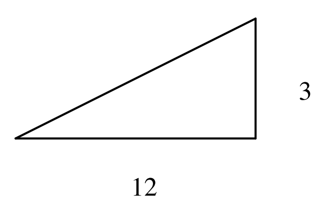


Figure 1.3. Ramp of base 12 units and height 3 units

Dual: Create a ramp with the same steepness as the one shown above but which has a base of 13 cm (Lobato & Thanheiser, 2002, p.172)

In answering the first research question in Chapter 4, I characterize the range of situations which afford opportunities for students to reason reversibly. In summary, a 'reversibility situation' is a characteristic of the problem, determined by its structure (syntactic and semantic) or formulation. On the other hand, operational reversibility is a characteristic of students and implies the cognitive operations they use to reverse a thought process (a metaphor for reversing a mental operation).

A Working Definition for Reversibility

Based on the different instances of reversibility that can be observed from the literature as well as the mathematical structure of reversibility situations, we can categorize such situations in essentially three classes: reversibility of operations, reversibility of transformations, and reversibility of relations, though these are not mutually exclusive. Reversibility of operations involves the reversal of arithmetic operations like addition and subtraction, multiplication and division, or at a fined-grained level operations such as making units of units of units and recursive partitioning (Steffe, 2003) or unit composition and decomposition. The second class of reversibility situations (i.e., reversibility of transformation) involves problems where a particular transformation is effected and the question is how to 'return back' to the original situation. For example: If the length of a given rectangle is increased two and half times how should the width be adjusted so that the area is the same? A second example is: If a diagram is reduced by 75%, what reverse transformation should be carried out to return the diagram to its original size? The third category of reversibility situations involves reversing relations. For example, in a part-to-whole reversibility situation like '2/5 of a parking lot holds 30 cars, how many cars can the parking lot contain?', one is working with the part-to-whole relation between a fraction and its referent whole (i.e., given a part make the whole) instead of the whole-to-part relation (i.e., given the whole make a part). This type of problem does not involve the reversal of a transformation; however it may involve the reversal of operation.

As mentioned earlier, Piaget defined reversibility as a property of an operation (Piaget, 1970) and conceptualized it in terms of two concepts, namely negation and reciprocity. Though these two concepts are relevant in understanding reversibility, they do not offer an adequate explanatory mechanism to explain when people show evidence of reversible reasoning, especially when researchers analyze empirical data. In this study, I have investigated different types of multiplicative situations, each of which involves the articulation of a *relation* between two or more quantities. I could interpret Piaget's concept of reciprocity (by which he meant the coordination between the 'two-sidedness' of a relation) in these situations. For example, in Chapter 6, the students were presented with the following multiplicative comparison problem: There are 21 marbles in a box. The number of marbles in the box is three times the number of marbles Paul has. How many marbles does Paul have? (problem 2.11). They readily deduced the inverse relationship (or reciprocity) between 'three times' and 'one third'. Another interpretation of reciprocity in Set 2 (fraction problems) can be given in terms of the coordination between the part-to-whole and whole-to-part relation.

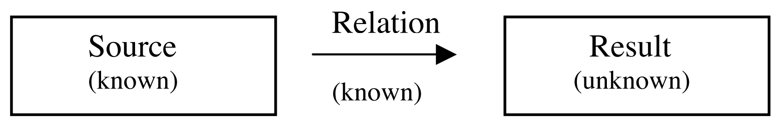
On the other hand, Olive's and Steffe's concept of reversibility based on scheme theory is fairly high-powered because it is based on the very basic element of cognition, namely a scheme. However, the application of this definition necessitates their conception of scheme as a three-part structure (situation, activity, and result). Further, to use their definition to identify whether students have reversible schemes also necessitates extensive collection of data over prolonged period of time in teaching experiments. Such a definition based on a scheme is more appropriate from a cognitive theoretical perspective but it may be pedagogically restrictive in terms of its interpretation by teachers. My aim was to formulate a definition that is pedagogically accessible and that captures the essence of Piaget's idea and Steffe's and Olive's definition. One of the definitions which highlights the pedagogical aspect is provided by Dreyfus & Eisenberg (1996) who define 'reversal of thinking' as the process of thinking from the result to the source causing the result.

My definition of reversibility

I shall consider reversibility as the ***deductive reconstruction of the "source, input, initial state, or initial condition" using the "result, output, final state, or given end" as raw material based on the structural relation between the source and the result; or the reconstruction of the structural relation based on the source and the result.*** It can be regarded as a process of recovery, where the reverse operation, transformation, or relation aims at restoring the 'source'. The term *deductive* emphasizes the fact that while reversing the operation (or sequence of operations), transformation or relation, the subject is not aware of the 'source' or 'relation' he or she is looking for. The term *reconstruction* emphasizes that the subject recognizes that s/he has a 'result' that has been constructed from a certain 'source' that is unknown in the problem. S/he has to reconstruct the 'source'. Alternately, the relation between the source and the result may need to be reconstructed. In this sense, a reversibility situation may involve reasoning with unknown quantities or an unknown relation among known quantities.

I shall be distinguishing between two types of problems: primal and dual. In the primal problem, the source and multiplicative relation are specified and the aim is to find the result. In the dual problem either the result and multiplicative relation are specified and the aim is to find the source or the result and source are specified and the aim is to find the multiplicative relation. Figure 1.4 provides a diagrammatic illustration of the primal/dual pair.

Primal problem



Dual problem

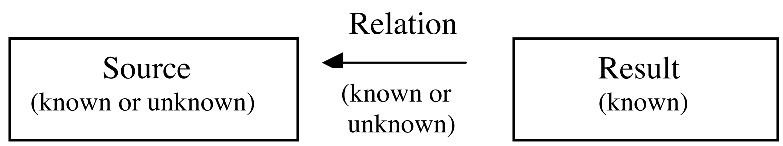


Figure 1.4. Structure of a primal and dual problem

In this research, I was primarily interested in studying the ways in which students reasoned reversibly or were constrained to do so in three types of multiplicative situations. The following examples show how a problem can be formulated as a primal/dual pair in these three types of situations.

Example 1: Set 1- Multiplicative comparison of two quantities in a measurement division situation

Primal problem: There are $2\frac{1}{2}$ times as many red counters as blue counters in a box. If there are 2 blue counters, how many red counters are there?

Dual problem: There are 5 red counters and 2 blue counters in a box. The number of red counters are how many times as large as the number of blue counters?

To solve the dual task, one may ask the question ‘which number when multiplied by 2 gives 5 or which number do I have to multiply 2 by to get 5’. This interpretation requires reversible reasoning because we have a result (5) and we want to know what multiplier or multiplicand produced the result starting from 2, a theorem-in-action algebraically equivalent to $2 \times x = 5$.

Example 2: Set 2-Multiplicative comparison of two quantities in a partitive division situation

I will refer to this category of problems as fraction situations.

Primal problem: A parking lot can contain a maximum of 45 cars. How many cars can 2/3 of the parking lot contain? (i.e., given the whole find a part)

Dual problem: There are 30 cars in a parking lot. This is 2/3 of the number of cars that the parking lot can contain. How many cars can the parking lot contain? (i.e., given a part, find the whole)

Figure 1.5 illustrates the reversible relationship between the primal and dual problem.

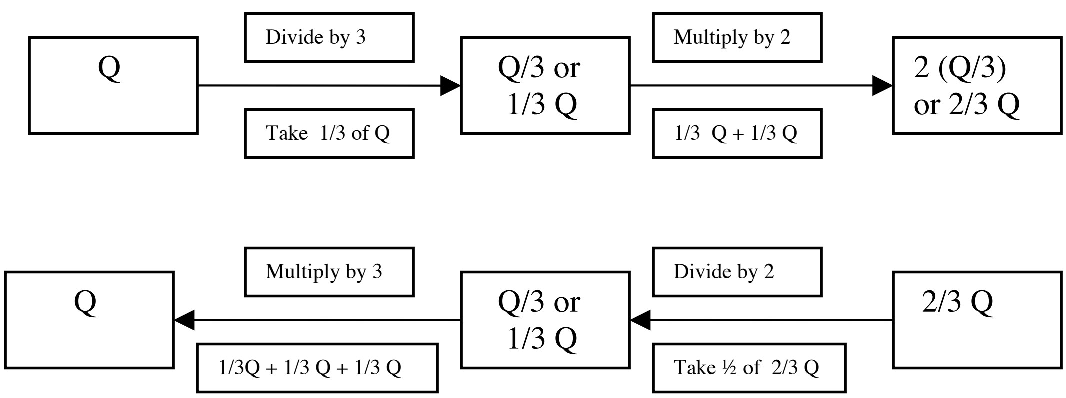


Figure 1.5. Constructing 2/3 from one unit and one unit from 2/3

Example 3: Set 3-Multiplicative comparison in a ratio situation

I will refer to this category of problems as ratio situations.

Primal problem: In a class, for every 2 boys, there are 3 girls. If there are 12 boys, how many children are in the class?

Dual problem: In a class there are 30 students. For every 2 boys, there are 3 girls. How many boys are there in the class?

Here, the 30 students represent the result of combining two quantities (in the ratio 2:3) and the objective is to find the two quantities (a problem algebraically equivalent to $(2+3)x = 30$).

As the students' responses show in the empirical part of the study, reversibility of thought involves the articulation of an unknown quantity (which may either be implicit or explicit). More precisely, in multiplicative situations it involves the articulation of the theorem-in-action $ax = b$ (as explained in Chapter 2). It should be pointed out that because reversibility situations involve working from the result to the source, often such problems involve a conditional statement (if-then). I would like to highlight that in this study I shall use the terms inverse reasoning, reversible reasoning and reversibility of thought interchangeably.

Problem Statement: The Research Gap

Research on number acquisition (Steffe et al., 1983), multiplication (Greer, 1992; Steffe , 1994), fractions (Behr & Post, 1992; Hackenberg, 2005; Olive 1999; Olive & Steffe, 2002; Post et al., 1986; Steffe, 2002; Tzur, 2004) and proportion (Lamon, 1993; Lesh et al., 1988; Lesh et al., 2000) have shown the manifestation of reversibility to variable extent. For instance, the process has been studied at the micro-analytic level in fractional problems by Olive & Steffe (2002) and in ratio situations by Lamon (1993). However, it appears that reversibility has been a by-product (or a part) of these research projects and not the central element or focus. Another observation is that reversibility has not been studied along different multiplicative domains in one study. Furthermore, since Piaget's time not much consideration has been given to reversibility in multiplicative situations unlike the attention given to the domain of addition and subtraction. Moreover, the development of analytical tools like the concept of units and the notion of quantities in mathematics education (Steffe, 1991, 1992, 1994) offer much potential to analyze reversibility at a micro-analytic level.

A number of questions crop up when we consider the concept of reversibility in multiplicative situations:

- What are the different forms of reversibility situations in the multiplicative conceptual field?
- How do students' ability to reverse their thought process vary across different multiplicative domains?
- What are some of the common cognitive operations that are involved in reversing thought processes across different multiplicative domains?
- What type of constraints do students' encounter when they are required to solve reversibility problems? What type of cognitive conflicts do they encounter?
- What effect do problem representations (e.g., discrete vs. continuous situations, diagrammatic representations) have on students' ability to reverse their thought process?
- To what extent do different curricula provide opportunities for reversibility?
- What are the various strategies that students deploy to reverse their thought process? What types of parsimonious strategies (curtailment) do they employ (Krutetskii, 1976)?
- Are students bound by enactive prototypes (Fishbein et al., 1985) for particular reversibility operations?
- What are the informal or intuitive strategies (Mack, 1993; Tirosh & Stavy, 1999) that students deploy as they reverse their thought process in multiplicative tasks?
- How does students' ability to reverse their thought processes vary as a function of mathematical maturity?

- In what ways do reversibility of thought provide students with flexibility to reason multiplicatively?

In the next section, I choose three of these questions that I shall address in this study.

Research questions

The purpose of this study is to investigate how reversibility operates within multiplicative situations. The endeavor is both theoretical and practical – developing a theoretical understanding of the problem and deriving curricular and instructional implications. To this end, the following research questions have been formulated.

Question 1

One of the first hurdles encountered in conceptualizing the present study was to trace the prominence of reversibility in the mathematics education literature since Piaget developed the idea. The only multiplicative domain where I could explicitly encounter research in this area was in fraction situations. As discussed earlier, reversibility has been described in different ways in the mathematics education literature though all of them refer to the process of reasoning from the result to the source causing the result. I argue that at this point, although a common conceptualization of reversibility seems to exist among mathematics education researchers, what seems to be missing is an adequate theoretical structure that may enhance our understanding of its very nature. The absence of this theoretical structure can partly be attributed to the fact that reversibility has not been analyzed in its own right, but has rather been a by-product of studies. The first aim of this research is to address this lacuna. Specifically, the aim is to analyze a range of multiplicative situations within the domain of whole number multiplication/division, fraction, ratio, proportion and percentage in order to trace the key features of ‘reversibility situations’. The first research question has thus been formulated as follows:

What are the different types of reversibility situations in multiplicative contexts and what are the structural relationships among these situations?

This question is meant to be theoretical in nature and attempts to characterize the different forms of reversibility across a range of multiplicative situations. This is an analytic activity based on the mathematical properties of the multiplicative conceptual field. This question also aims at setting up the structures on which the other research questions are to be investigated. In other words, question 1 will define the ‘reversibility situations’ from an adult mathematical perspective.

Question 2

After the theoretical analysis in question 1, I interviewed students on a range of reversibility situations in the domain of fraction, ratio, and proportion. I also collected some data on percentage. I chose to collect data over a range of multiplicative domains because I was aware that students may not necessarily reason reversibly to solve these problems. In this dissertation I present the data only for the domains of fraction and ratio. My aim was to identify the fine-grained mechanisms that are involved in reversing multiplicative relations. The second research question has been formulated as follows:

In what ways do students reason reversibly in multiplicative situations and what constructive resources do they deploy in such situations?

Question 3

The third aim of this study was to identify the constraints that students encounter in reversing their thought process and how such constraints impede them at the level of operation in terms of manipulating multiplicative relationships. In that sense, I was looking for those critical steps where students were constrained by such factors like the cognitive load involved (e.g., the number of pieces of information to manipulate), numeric features of the data, and resources available to the problem solver. As pointed out by Vergnaud (1988): "the complexity of problems depends on the structure of the problem, on the context domain, on the numerical characteristics of the data, and on the presentation; but the meaning and the weight of these factors depend heavily on the cognitive level of the students" (p. 143). Fischbein et al. (1985) also list a similar set of factors: familiarity of context, quantities involved, size and type of numbers involved, relation between the situation referred to and the appropriate operation, rigidity effects associated with specific operations and intuitive intervening models. Along the same lines, Kaput & West (1994) point out that there are four broad categories of task variables to take into account: (a) semantic structure of the situation depicted in the problem statement, (b) the numerical structure of the problem (c) the tools and representations available to the problem solver, and (d) the form of the text (e.g., written text versus a microworld) in which the problem statement is presented (p. 243).

As I mentioned earlier, I started my study focusing primarily on the strategies and constraints that students encountered in reversing their thought process in multiplicative comparison situations. As the data collection and analysis proceeded, I could observe the strong influence of problem conceptualization on problem solving strategies. Situations that were formulated to observe reversible reasoning were often conceptualized differently by the students. This prompted me to ask the following question:

What constraints do students encounter in conceptualizing multiplicative relations from a reversibility perspective?

Rationale: The Importance of Studying Reversibility

This study has been motivated by a tandem of theoretical concerns and practical needs. The literature indicates that reversibility as a process has not been studied extensively in mathematics education. It may not even be explicitly and widely known among mathematics teachers. It seems that, unlike a number of other Piagetian ideas, reversibility has been given secondary importance. From a theoretical perspective, a domain definition is lacking (especially with the increasing understanding gained on students' thinking in recent years). Besides, a coherent framework for studying reversibility is missing. This also accounts for the lack of empirical work in this direction. Ultimately, the inadequate attention given to reversibility may have resulted in its meager consideration in curriculum design as well as in instruction. The rationale is structured along three dimensions.

Students' learning and reasoning

This investigation is educationally significant with the greater emphasis being paid to the piece-by-piece understanding of children's mathematical concepts. The understanding of the way reversibility operates will contribute to extend the existing knowledge-base on students' thinking, especially in terms of multiplicative reasoning. Another reason that justifies why it is important to conduct this study is that it will provide indications of how children's intuitive and spontaneous methods can be possibly exploited in teaching and learning of mathematics. Such knowledge is didactically important to improve mathematics learning.

The importance of reversibility can also be thought of in terms of concept formation. Reversibility problems allow students to construct relations bi-directionally. For instance, in part-to-whole reversibility, students are prompted to articulate the multiplicative relationship between the part and the whole. From this perspective, reversibility problems generate situations that allow students to develop meaningful concepts as well as exploring them in depth. Analogous to the role of counterexamples in concept formation, reversibility problems can be regarded as providing situations for triggering accommodation and the equilibration of mathematical processes. These problems may also be regarded as generating conflicts to their instrumentally-packed experiences. In line with Skemp's idea of relational understanding (1987), reversibility may be a facilitator in bridging concepts.

As a coordinating operation, reversibility offers students the capability to develop flexibility in thinking (Dreyfus and Eisenberg, 1996). Such flexibility is required in the complex configuration of multiplicative structures like fraction, percentage, ratio, proportion and rate. Reversibility can be regarded as a key component of multiplicative reasoning. It is also an important component in reasoning algebraically from known to unknown, for instance, in situations that can be modeled by $ax = b$. Further, by their nature, reversibility problems are cognitively demanding and these readily provide situations for problem solving. As challenging problems, these may readily capture the interests of students and may also be regarded as intrinsically rewarding.

Enhancing teachers' repertoire

What are teachers to gain from such a study? The very first anticipated contribution of this research is that it will give visibility to reversibility and as such it will increase teachers' awareness of this form of reasoning. Since Shulman's seminal paper (1986) on different types of teacher knowledge, considerable research attention has been driven towards understanding the mathematical knowledge necessary for teaching (Ball et al., 2001). It can be envisaged that the cognitively demanding reversibility problems will prompt teachers to rethink the relationships among closely related concepts in the multiplicative domain and as such enhance their pedagogical content knowledge. Reversibility problems are higher-order problems and these can be used as a resource to exploit students' "funds-of-knowledge" (Franke, Kazemi, & Battey, 2007, p.245). This study describes children's thinking processes (as well as creative and parsimonious strategies) and thus contributes in generating more understanding about ways in which the curriculum in the multiplicative conceptual field can be made more accessible to them as well as providing more meaningful learning experiences. To sum up, detailed knowledge of reversibility will enhance teachers' repertoire of knowledge to support students' reasoning and problem solving skills and their intellectual and dispositional growth.

Implications for assessment and curricular design

This research is also important from an assessment perspective. A main objective in designing assessment tasks is to select problems that are deep and rich. Reversibility problems can be regarded as belonging to the 'Analysis' and 'Synthesis' levels in Bloom's taxonomy (Bloom, 1968). In these problems, students are required to disentangle the component parts in the problem and analyze the content and structure to find the required 'input' of the problem. The cognitively demanding reversibility tasks provide some additional advantages as compared to conventional tasks. They are potentially capable of revealing the depth and flexibility of students' thinking. Consider the following problem (primal) and its modified formulation (dual).

Primal: A tank has a capacity of 40 liters. Water was poured into the tank to 2/5 its volume. How many liters of water were poured into the tank?

Dual: Sixteen liters of water were poured into a tank, filling the tank to 2/5 of its volume. What is the volume of the tank? (Krutetskii, 1976, p.144).

The dual reversibility problem is not merely a fraction multiplication problem. It requires thinking of a given fraction (2/5) in relation to the desired whole. It also requires students to use an iterative conception of fraction involving the splitting operation (Steffe, 2003). The primal problem does not reveal students' conceptual understanding to the same extant that the dual problem does. This investigation puts into perspective the importance of formulating reversibility problems.

In general, reversibility problems do not constitute a common feature of a conventional mathematics curriculum. In fact, Dreyfus & Eisenberg (1996) assert that many textbooks have not given enough consideration to this type of problem unlike test designers. The role of tasks and their consequences about how students engage in sense-making have been pointed out by Stein & Lane (1996) and Franke, Kazemi & Battey (2007). The level and kind of thinking that students engage in, determine what they will learn. Reversibility problems can be regarded as a specific type of task with considerable thought-revealing power. By their very nature, they can contribute to algebraic thinking. Reversibility problems may also be used as a resource or strategy in rewording problems.

As pointed out by Resnick & Singer (1993), multiplicative reasoning develops slowly. Thus, the curriculum should provide children with problem situations that give them an experiential base for internalizing the multiplicative concepts like fraction, ratio, and proportion. This study is grounded in the multiplicative conceptual field with specific types of tasks designed to foster multiplicative reasoning. Further, Behr at al. (1993) point out that “curricular reform should be guided by extensive content analysis of the domain of multiplicative structures” (p. 33). This study can be regarded as making a specific content analysis of the core multiplicative domains of fraction and ratio.

CHAPTER 2

THEORETICAL FRAMEWORK

This chapter describes and justifies the combination of analytical tools that I used to study the ways in which students reason reversibly in multiplicative situations. Vergnaud's (1983;1988;1994;1996;1997;1998) notion of operational invariants (concepts-in-action and theorems-in-action) is the principal analytical tool that has been used across the three types of situations (multiplicative comparison, fraction, and ratio). I have also attempted to access the ways in which students articulate multiplicative relations through the idea of units (Steffe, 1992, 1994, 2003; Olive & Steffe, 2002) and through the concept of quantities (Thompson, 1994). I started the analysis of multiplicative situations using Vergnaud's approach as it is specifically designed for multiplicative structures. Though the notion of concepts-in-action and theorems-in-action proved to be useful in analyzing students' responses, more insight into students thinking could be obtained by using the concept of units and quantitative reasoning. This motivated me to add these constructs in my analytical framework.

Vergnaud's Approach as a Port of Entry

Vergnaud's theory provides a theoretical framework that permits the articulation between the mathematical problems to be solved, knowledge deployed, schemes, concepts and symbols involved in the solution procedure. Its usefulness resides essentially in conducting semantic and conceptual analyses and in the three central ideas that constitute the core of the theory, namely 'measure spaces', 'concepts-in-action', and 'theorems-in-action'. This theory is consonant with my study, where the aim is to conduct both semantic analyses of reversibility situations in the multiplicative conceptual field from a mathematical perspective as well as to carry out conceptual analyses of reversibility, starting from students' ways of operating.

Mathematical concepts exist in relation to each other and draw their meaning from a variety of situations. To analyze the complexity of the interrelatedness of concepts, Vergnaud (1988) introduced the theory of *conceptual fields*. He defines a conceptual field as "a set of situations, the mastering of which requires the mastery of several concepts of different natures" (p. 141). He considers the *conceptual field of multiplicative structures* as "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply and divide" (p. 141). This field can be regarded as consisting of a range of concepts and operations including multiplication, division, fraction, ratios, rational numbers, simple and multiple proportions, linear and n-linear functions, vector spaces and dimensional analysis. Apart from mathematical concepts, it also encompasses students' ideas (both competencies and misunderstanding), "procedures, problems, representations, objects, properties, and relationships that cannot be studied in isolation" (Lamon, 2007, p. 642).

The suitability of the framework to understand the operation of reversibility

The main aim of my study is to identify the affordances and constraints that students encounter in different forms of reversibility situations. The theory of multiplicative conceptual fields provides a useful framework for achieving this goal. Vergnaud (1996) points out two main advantages that the theory of conceptual fields offers to study the cognitive development of concepts. Firstly, it provides "a way to identify the similarities and differences between situations, their hierarchical structure, and also the continuities and discontinuities that organize the repertoire of schemes that is progressively developed to master these situations" (p. 225). Secondly, it provides us with the possibility to analyze students' implicit reasoning through concepts-in-action and theorems-in-action.

Another reason why Vergnaud's theory is suitable for my study is that it is specifically defined for the multiplicative conceptual field, a setting where my research is located. My first research question is concerned with the theoretical analysis of one form of reasoning (i.e., reversible reasoning) across several multiplicative domains (whole number multiplication/ division, fraction, ratio, proportion, and percentage). My study involves multiplicative situations that require participants to work with concepts-in actions like ratio, fraction, multiplication and division. Further, the tools developed by Vergnaud, for instance, Rules-of-action, Concepts-in-action, and Theorems-in-action are particularly useful to analyze students reasoning.

Key concepts in Vergnaud's framework

(i) Scheme

Vergnaud (1997) considers a scheme as "the invariant organization of behavior for a certain class of situations" (p.12). He outlines the following elements of a scheme:

1. goals and expectations: A scheme allows the problem solver to anticipate the objective to be reached and the effects to be expected;
2. operational invariants: to grasp and select the relevant information (concepts-in-action) and treat this information (theorems-in-action);

   Operational invariants refer to objects, properties, and relations that are consistently used by the problem solver in a set of similar situations. They decide how and which type of knowledge will be used in performing a task. What makes access to operational invariants possible, is their use in problem solving. For instance, when the problem solver shows the same mathematical behavior consistently over a range of situations, this gives evidence of a theorem-in-action.

3. rules to generate actions according to the evolution of the different variables of the situation and therefore rules to pick up information and check;
   The rules of action dictate what step-by-step action has to be effected to achieve the goal. It may involve conditional 'if-then' decision steps from which a specific sequence of actions are generated (Andres et al., 2006).
4. inference possibilities "that make it possible for the subject to infer what to do and to control as action and time goes on" (Vergnaud, 1996, p.22). A scheme is not a stereotype but a universal organization; it is relevant for a class of situations and not for one situation only.

Vergnaud (1998) considers problem conceptualization as the keystone of cognition. He asserts that: "a major part of students' behavior, in problem solving, is generated by hypotheses, analogies, metaphors, extensions of former knowledge or reductions" (p. 173). To characterize how students conceptualize problem situations he developed the notion operational invariants (concepts-in-action and theorems-in-action). Figure 2.1 (modified from Andres et al., 2006) shows the essential components of Vergnaud's theory and how they are related to each other.

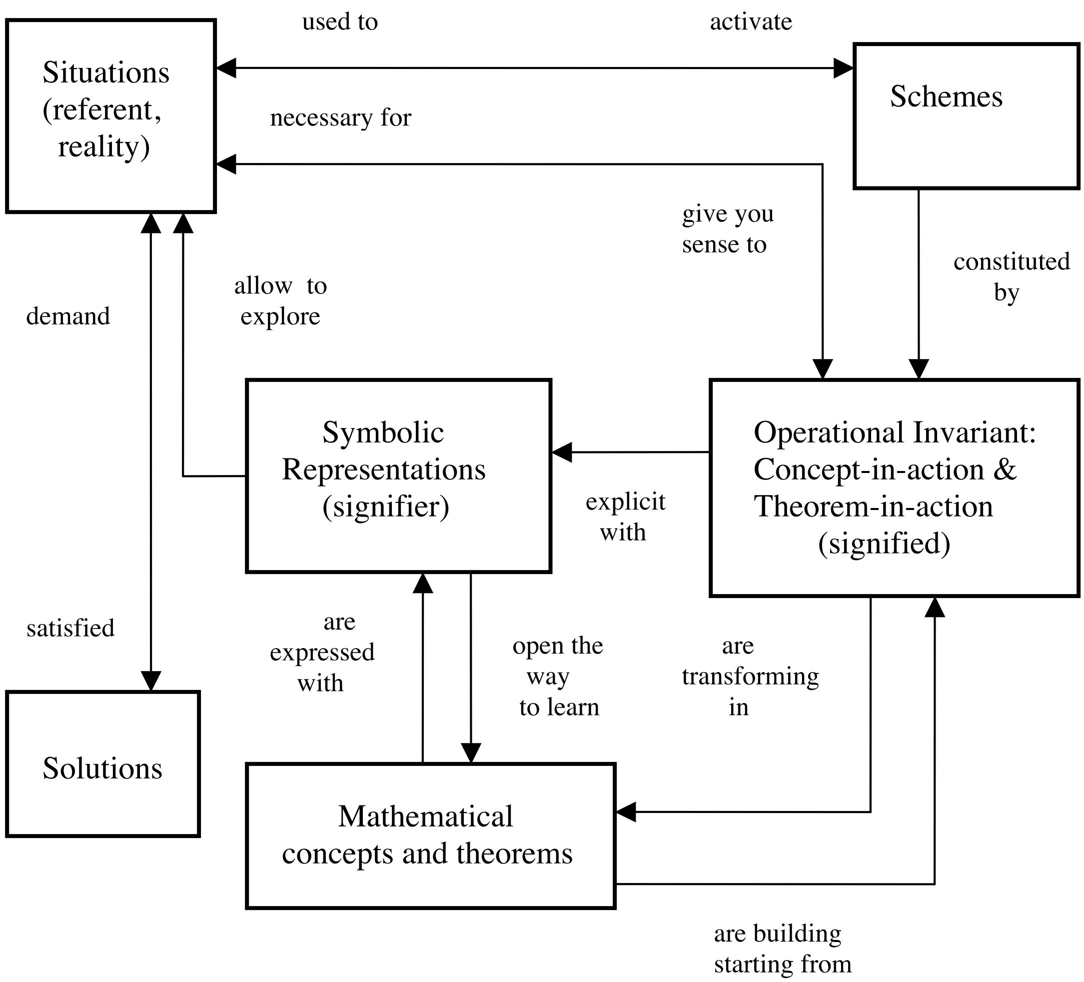


Figure 2.1. Key components of Vergnaud's Theory

Invariants

Invariants are logical requirements that must be respected in thinking mathematically. It also includes "relationships which are introduced in mathematics by conventions, but once introduced, must be kept constant" (Nunes & Bryant, 1996, p.10). For instance, once the relationship 'one centimeter is equivalent 10 millimeters' is specified, it must be kept constant, (i.e., invariant). Once the user adopts these conventions, they become logically compelling to use them and he/she finds it difficult to think in ways different from these conventions. In this study, I am primarily concerned with tracking one specific invariant: the inverse relationship between multiplication and division. Due to the centrality of this invariant in my study (which involves multiplicative situations), I shall label it as the multiplication/division invariant. Over the course of experience (through logic and conventions) one builds the understanding that division is the inverse of multiplication. This logical relationship or transformation relationship (invariant) is not readily cued in problem solving situations. Consider the following problems: 'Bar A weighs 6 pounds. Bar A weighs 5 times as much as bar B. What is the weight of bar B?' (problem 1) versus 'Bar A weighs 6 pounds. Bar A weighs 3 times as much as bar B. What is the weight of bar B?' (problem 2). The multiplication/division invariant is more likely to be cued in the second problem because of the divisibility relationship between the numbers. Even in the second problem before computing the result of the arithmetic operation (i.e., division), one needs to carry out an operation of thought based on the inverse property of multiplication and division. In Piaget's term, such articulation of the inverse property of multiplication and division is referred to as negation.

(ii) Concepts-in-action

"Concepts-in-action are categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved" (Vergnaud, 1996, p. 225). Examples of concepts-in-action include quantity and magnitude, unit value, ratio and fraction, function and variable, constant rate, dependence and independence, quotient and product of dimensions and so forth (Vergnaud, 1994). A student will choose particular concepts-in-action in a problem-solving situation, depending on whether he or she finds them relevant to the context.

(iii) Theorems-in-action

Verganud (1988) defines *Theorems-in-action* as the "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (p. 144). They are "held to be true propositions" (Vergnaud, 1997, p. 14) for a certain range of situations and may even be flawed. As clearly defined by Hiebert & Behr (1988), Theorems-in-action may also be viewed as an action (physical or mental) on the part of the cognizing subject "that provides behavioral evidence of implicit knowledge of more formal property or method or 'theorem' of mathematics" (p. 11). In addition, these relationships may not be expressed verbally by the students. In other words, Theorems-in-action are the implicit mathematical operations that students use in solving mathematical problems. In one sense, they can be regarded as the mental counterpart of the symbolic operations, relations or transformations. The Theorems-in-action allow the researcher to follow the mental operations (in symbolic form) that the student uses in solving problems. They provide a way to diagnose what students know or do not know and to access students' intuitive strategies.

As an example, consider the possible 'Theorems-in-action' involved in solving the following problem (denoted as the 'Shoes problem' for future reference): Shoes are being sold at a discounted price of 5% and George saved $10 on the purchase of his shoes. What was the original price of his shoes? One possibility is to reason as follows: 5% is equivalent to $10. Therefore, 1% is equivalent to $2 and 100% is equivalent to $\$2 \times 100 = \$200$ (unit-rate method). Alternatively, realizing that 5% multiply by 20 is 100%, it can be concluded that $10 should be multiplied by 20 to get the original price of the shoes as $200 (factor-of-change method). The two approaches to solve the same problem involve two different 'Theorems-in-action'.

Furthermore, the multiplicative relationship in the Theorems-in-action may be articulated mentally without recourse to symbols.

Vergnaud makes the distinction between operational invariants (concepts-in-action and theorems-in-action) contained in schemes and explicit mathematical concepts and theorems. The operational invariants described above are referred to as 'signified' knowledge and emphasizes conceptual meanings and operations. This form of knowledge has an implicit nature. In contrast, he uses the term 'signifier' to refer to explicit knowledge expressed in the form of symbolic representations and operations (e.g., representations like numerals, algebraic expressions, axes, dots, graphs, symbolic manipulation)

(iv) Definition of concept

Vergnaud (1996) considers a concept as a combination of three elements, denoted as $C = (S, I, R)$. S refers to the set of situations that make the concept meaningful; I is the set of operational invariants contained in the schemes to deal with the set of situations; and R denotes the symbolic (linguistic, graphic, gestural) representations that can be used to represent the relationships involved.

Vergnaud's theory is based both on the epistemological content of knowledge and the conceptual analysis of this knowledge domain. Thus, the application of Vergnaud's theory requires an analysis of the mathematical structure of problem situations. In Chapter 4, I analyze the reversibility situations from a mathematical perspective while in Chapters 5, 6, and 7, I analyze students' responses.

A Second Path: Unit Structures

Previous studies have demonstrated how the concept of units can be an insightful explanatory tool to interpret students' reasoning. For instance, Steffe, Olive and colleagues have used the notion of units in the context of scheme theory to explain how students articulate whole numbers and fractions (Hackenberg, 2005; Olive & Steffe, 2002; Steffe, 2003, 2004; Tzur, 2004). Further, as part of the Rational Number Project, Behr et al. (1992) developed a symbolic system to represent units and to illustrate different ways in which they can be manipulated from a mathematical perspective. On the other hand, Lamon (1993a, b; 1996) used a 'bottom-up' approach, starting from children's responses in clinical interviews, to show the different ways in which students implicitly use units in their solution attempts. Furthermore, in the domain of ratio and proportion, Lamon observed that some of the participants in her study used a ratio as a unit, an observation also supported by Singh (2000). In his summary of research on multiplicative structures, Kieren (1994) underlines the role of units of quantity and their transformation "as a key and unifying activity in a child building a multiplicative structure" (p. 396).

Lamon (1999) defines unitizing "as the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity" (p. 42). For instance, there are different ways one can look at the number 12: (a) 12 = 1 (12-unit), (b) 12 = 2 (6-units), (c) 12 = 4 (3-units), (d) 12= 6 (2-units), (e) 12 = 3 (4-units), and (f) 12 =12 (1-units) as illustrated diagrammatically in Figure 2.2. The unitizing operation enables us to form composite units in the service of more complex conceptual maneuvers as will be illustrated in the empirical part of the study.

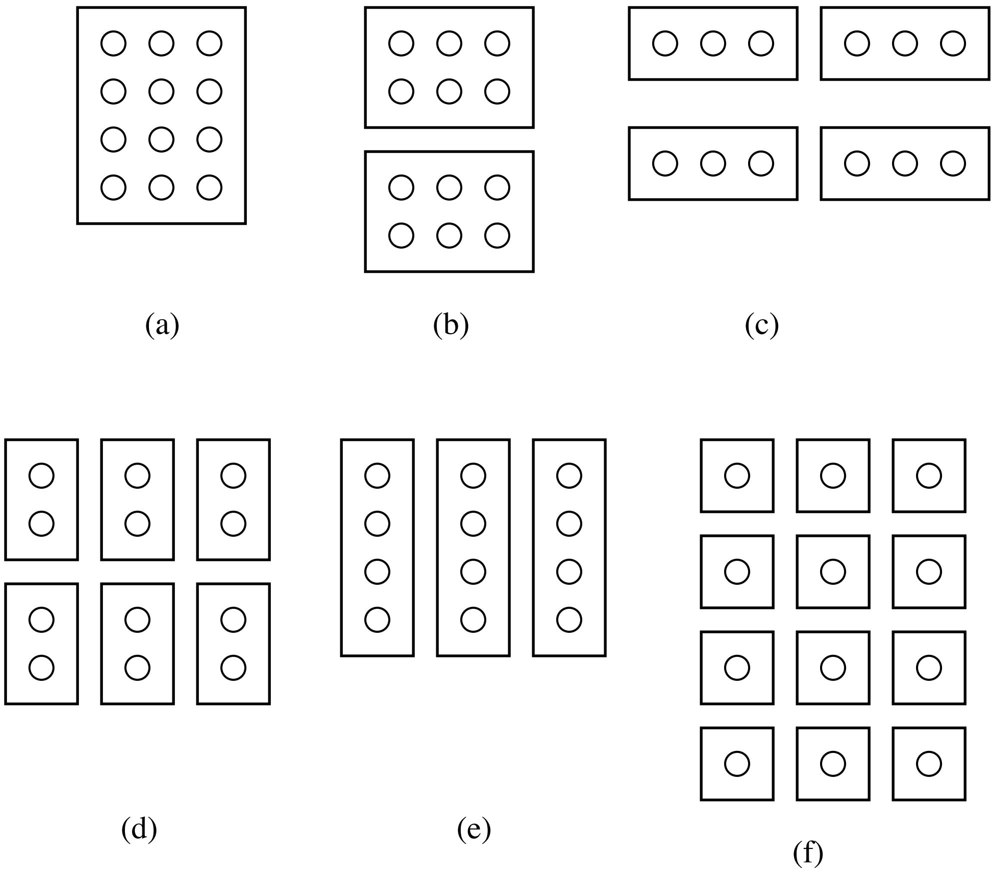


Figure 2.2. Interpreting an array of 12 elements in terms of different composite units

The concept of unitizing as defined by Lamon above is different from that used by Olive & Steffe who consider unitizing as the mental act of forming unit items out of sensory experiences. Instead they consider *uniting* to be the mental operation of compounding or joining together unit items (formed as a result of unitizing). When the joined objects can be acted on as a single object, while still maintaining its constituent parts, this results in the formation of *composite units*, e.g., five ones can be taken as one five and one five can be decomposed as five ones (Olive & Lobato, 2008).

Units can assume different forms, at varying levels of complexity. Homogeneous units can be united to form two levels of units (or units of units) or three levels of units (or units of units of units) (Steffe, 1994). In turn, composite units can be decomposed to form lower level units. Level of units has to do with the different number of units that one can coordinate simultaneously at one moment. I present two examples to illustrate the idea of two levels of units (also called unit-of-units): one in the context of whole numbers and the second in the context of fractions. I chose the number twelve to illustrate the difference between units-of-units and units-of-units-of-units. The number twelve can be conceptualized in different ways from the perspectives of units. Firstly, twelve can be regarded as a single entity, as a number (in its own right) independent of the twelve units that constitute it. Such a conception implies only one level of unit. Secondly, one can conceptualize twelve at two levels of units. At the first level, twelve is regarded as one entity in its own right and at the second level, twelve is regarded as consisting of twelve ones. This simultaneous conceptualization of twelve requires the coordination of two levels of units (Figure 2.3).

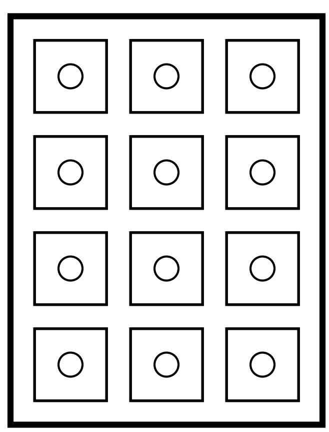


Figure 2.3. Conceptualizing 12 at two levels of units

The number 12 can also be interpreted at 3 levels of units as consisting of 3 fours, each of which is made up of 4 ones (Figure 2.4): Twelve elements viewed as a single entity (the whole group is one level) composed of 3 units (a second level), each of which is composed of four units (a third level of units).

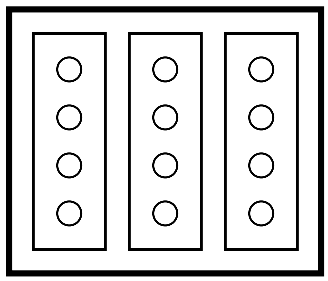


Figure 2.4. Conceptualizing 12 at three levels of units

In other words, being able to look at twelve simultaneously as one twelve consisting of three fours, each of which is made up of four ones involves three levels of units. The important point to emphasize is that it is the simultaneous coordination of these nested units that characterize them as two levels or three levels of units. The conception of a quantity at two or three levels of units depends upon the person conceptualizing the quantity and not the mathematical structure per se.

The concept of two and three levels of units in the interpretation of fractions

I use the example of fraction multiplication ($\frac{1}{2}$ of $\frac{1}{4}$) to illustrate the concept of two and three levels of units. Conceptually, the solution to this problem may either involve the coordination of two or three level of units, depending on the strategy one uses. To find $\frac{1}{2}$ of $\frac{1}{4}$, one may first consider one whole and partition it into 4 equal parts to produce 4 one-fourth units (Figure 2.5(a)). Then one may partition the first quarter into 2 (Figure 2.5 (b)). It is at this step

(Figure 2.7 (b)) that one may either reason with two or three levels of units to determine the size of the new unit (i.e., one eighth).

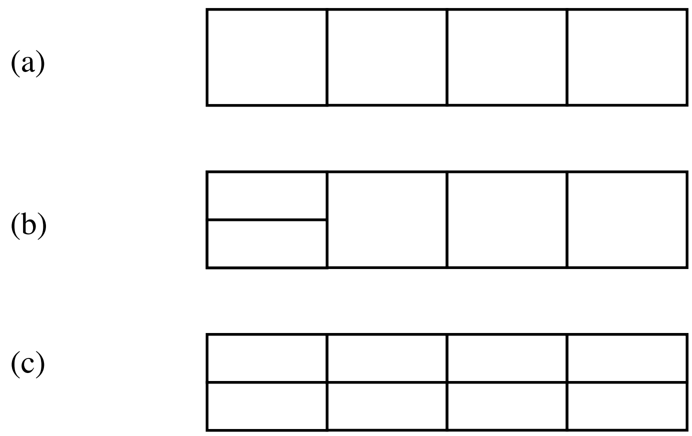


Figure 2.5. Finding $\frac{1}{2}$ of $\frac{1}{4}$ using 2 levels of units

We can determine the size of the new unit by *iterating* it within the given whole and counting how many such units fit the whole. This gives the size of the new unit as one eighth as 8 such units fit the whole as shown in Figure 2.5(c); we have 8 units within one whole. One does not explicitly use the fourth or quarters to determine the size of the unit. This is why it is referred to as two-level of units. In other words, two levels of units is based on the part-to-whole structure of fractions.

Alternatively, one may determine the size of one unit by considering each quarters as one unit (Figure 2.6 (a)). Then each of the four '1/4 units' are partitioned into 2 (Figure 2.6 (b)).

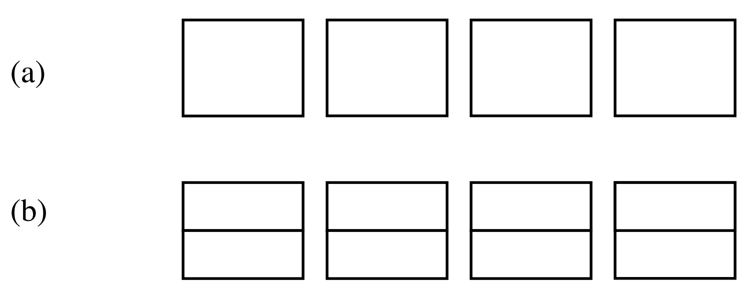


Figure 2.6. Finding $\frac{1}{2}$ of $\frac{1}{4}$ using 3 levels of units

So, here we are using the quarters explicitly. We are coordinating eighth units within fourth units within a one-whole unit. The whole is one level of unit, the quarters are a second level of unit and the eighths are a third level of unit.

The interpretation of a fraction at two or three levels of units depends on how one operates with the fraction in the given problem situation. For instance, one can interpret five-sevenths as five one-seventh, giving evidence of coordinating two levels of units. If one has the goal of producing one whole from five sevenths and interprets five sevenths as consisting of five one-seventh units, each of which can be iterated seven times to make a whole, then this gives evidence of coordinating 3 levels of units. In other words, when one looks at the whole that has been constructed (in activity or imagination), one sees two pieces of information: five-sevenths and seven-sevenths. Similarly if one has the goal of producing one whole from six-fifths, one can interpret six-fifths as a unit of 6 one-fifth units, any of which can be iterated 5 times to make the whole, another unit nested/embedded within the six-fifths. Thus, six-fifths can be regarded as a three levels of units structure.

Coordinating two three-levels units in the same bar

There are also problem situations where one has to coordinate two 3 levels of units in the same bar as illustrated by the following example: ‘Bar A is 7 units long. Bar A is 5 times as long as bar B. What is the length of bar B?’ One strategy to solve this problem is to partition each of the 7 units (in bar A) into 5 parts to get a total of 35 parts. Then one can re-interpret 7 small partitions as being one fifth of bar A (or one unit of bar B). This re-interpretation requires the coordination of 3 levels of units in two ways: (i) in terms of the multiplicative relationship and (ii) in terms of the measure. Firstly, the whole bar (i.e., bar A) of 35 partitions represents two levels of units, 35 one-thirty-fifth within one whole. Embedded in bar A is another unit consisting of 7 small partitions (i.e., bar B) which constitute the third level of units. Secondly, 3 levels of units can be interpreted in terms of the measure: Bar A is 7 units long and this constitute two levels of units, one partition has a measure of one unit and is part of the measure of 7 units. Bar B (consisting of 7 small partitions) is embedded in bar A and has a measure of $\frac{7}{5}$ units. Note that one small partition carries three pieces of information: It is 1/5 of one unit (as a measure), it is 1/35 (in terms of multiplicative relationship) of bar A and 1/7 of bar B.

Mental Operations

Reversibility of operations can be interpreted at two different levels. First, it can be regarded in terms of arithmetic operations and their inverses like adding and subtracting or multiplying and dividing (i.e., in terms of operational invariants) – 'Subtraction as the inverse of addition' or 'Division as the inverse of multiplication'. At a fine-grained level, it can be interpreted in terms of mental operations and their corresponding inverses like forming composite units (i.e., uniting and unitizing) and decomposing composite units (i.e., partitioning or segmenting). In this section, I identify mental operations and their corresponding inverses as observed in research in the domain of whole number and fractions. The empirical part of the study shall attempt to identify such instances of reversibility.

Fragmenting

Fragmenting is the mental act of breaking a discrete set or a continuous quantity in parts. The parts need not be necessarily equal (Olive & Lobato, 2008).

Segmenting

Segmenting is the act of successively marking off equivalent portions of a quantity. In other words, segmenting (or measuring) refers to the action of putting an amount into parts of a given size, and there is a pre-established group size (Thompson & Saldanha, 2003). For example, finding how many bundles of 4 pencils can be made from 12 pencils.

Partitioning/Sharing

Partitioning/sharing refers to the mental operation of dividing a quantity into equal parts, either dividing a region into equal parts or of separating a set of discrete objects into equivalent subsets (Behr & Post, 1992). It involves separating the quantity into a specified number of equal parts while the quantity remains as a whole (Olive & Lobato, 2008). Sharing (or partitioning) is the action of distributing an amount of something among a number of recipients so that each recipient receives the same amount, and there is a pre-established number of groups (Thompson & Saldanha, 2003). In other words, sharing (or partitioning) refers to partitive division and segmenting (or measuring) refer to measurement division. Another observation is that the composite unit segmentation operation is the reverse of iterating operation. The iterating operation produces multiple copies of a unit whereas the composite unit segmentation operation results in segmenting a composite unit into sub-units.

Reversible versus non-reversible partitioning operation

Consider the following example from Olive (1999) where two students were asked to find $\frac{1}{5}$ of $\frac{1}{6}$. They could produce this composition of fractions using the TIMA computer microworld in activity by first partitioning a stick into 6 parts, then partitioning one of the 6 parts into 5 parts and disembedding one of these small parts but they could not readily determine what fraction of the original whole was this resulting small piece. To determine the size of $\frac{1}{5}$ of $\frac{1}{6}$, they iterated the resulting small piece 30 times to deduce that it was $\frac{1}{30}$ of the whole, reasoning with two levels of units. Olive adds "they were not able to reverse their partitioning operations and project the partition of five parts in $\frac{1}{6}$ into each of the six parts of the unit whole from which they made $\frac{1}{6}$" (p. 292). In a second problem, the students were asked to find $\frac{3}{4}$ of $\frac{1}{7}$ at a later point in his teaching experiment. To solve this problem, one of the students decomposed $\frac{3}{4}$ of $\frac{1}{7}$ as 3 of $\frac{1}{4}$ of $\frac{1}{7}$ (reasoning with 3 levels of units). He then used the recursive partitioning operation to find $\frac{1}{4}$ of $\frac{1}{7}$ and used the uniting and unitizing operations to take 3 of these $\frac{1}{28}$ as one thing. The fact that the student could feed back or project the result $\frac{1}{28}$ from $\frac{1}{3}$ of 3 partitions to 3 partitions shows that his partitioning operation was reversible.

Measuring

Measuring involves comparing a given quantity to a specified unit using iterations or partitions of the unit (Olive & Lobato, 2008).

Disembedding operation

This mental operation allows a part from a partitioned whole to be lifted from its referent whole while keeping in mind its relation to the whole. Thus, both the part and the whole can be discerned as separate entities. For example, after partitioning a bar into 8 parts, a child can mentally focus on just one of those parts and compare that one part to the eight, realizing that the bar is 8 times as long as this part (Olive & Lobato, 2008).

Iterable units

Another accompanying feature of unit structure that is important to highlight is the notion of iterable units. Both single and composite units can be iterated. Iteration can be regarded as the mental or physical (either in the form of drawing or in a microworld) operation of "repeatedly instantiating an amount in order to produce another amount" (Hackenberg, 2007, p. 28). An iterative conception of fraction highlights the inherent multiplicative relationship between a non-unit fraction and the units that are iterated to generate it. For instance, 7/8 can be conceived as iterating 1/8 seven times. In addition, this is what characterizes an iterative conception of multiplication rather than an additive conception.

The splitting operation

Olive & Steffe (2002) assert that the splitting operation (as a mental operation) is fundamental in solving a reversibility problem where one is required to construct a whole from a given part. The splitting operation is regarded as the simultaneous application of the partitioning and iterating operations. For instance, given a diagrammatic representation of candy bar B which is 5 times as long as candy bar A, construct candy bar A. Solving such a problem may require a person to posit a segment in thought (by making a conceptual partition) that can be iterated five times to produce a segment equal to B. The simultaneous nature of partitioning and iterating operation is highlighted by Steffe (2003):

> Realizing that the desired segment can be produced by simply splitting the given segment into five parts implies the composition of the two operations of iterating and partitioning, where the partitioning produce the parts any of which can be iterated 5 times to produce the partitioned segment (i.e., bar B), and where any segment (i.e., the chosen bar A) for which

> the given segment is five times longer can be used to partition the given segment (i.e., bar B) (p. 240).

Students need to posit a bar (i.e., the chosen bar A) which stands in relationship to the given bar (i.e., bar B) but is also separate from it. To solve this problem one needs to be aware of the multiplicative relationship between the required bar (bar A) and the given bar (bar B); bar A taken five times produces bar B. The splitting operations allows us to reason 'in reverse.'

Recursive partitioning and *forming units of units of units*

Steffe (2003, 2004) makes the subtle difference between taking the partition of a partition and taking the partition of a partition in the service of a non-partitioning goal (what he defines as recursive partitioning). Such type of partitioning is involved in measuring the result of finding the composition of two fractions, for instance $\frac{1}{2}$ of $\frac{1}{4}$. To solve such a problem, one may first partition one unit into 4 equal parts and recursively partition each fourth into 2. Thus, recursive partitioning involves reasoning with three levels of units: One unit containing 4 fourth, each of which containing 2 eighths. It also involves aspects of distribution. Steffe (2003) asserts that "Recursive partitioning is the inverse operation of first producing a composite unit, multiple copies of this composite unit, and them uniting the copies into a unit of units of units" (p.240). He also points out that the splitting operation is the basic operation that is involved in the construction of recursive partitioning.

Distributive splitting

Hackenberg (2005) extended the splitting operation to define a new operation termed 'distributive splitting' as illustrated by the following example: "This 2-foot candy bar is three times longer than another bar; make the other bar and determine its length" (p. 317). One strategy to solve this problem is as follows: Split each foot into three equal parts and take two of these parts in order to split the entire 2 feet into three equal parts as shown by the 4 successive steps in Figure 2.7.

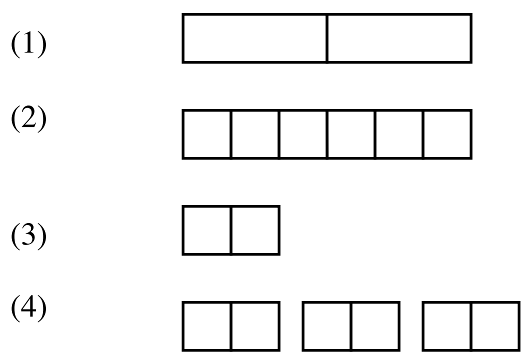


Figure 2.7. Distributive Splitting

She further asserts that students may split distributively without being aware of a distributive pattern in their activity.

Distributive reasoning

Finding the solution to a problem like $\frac{2}{5}$ of $\frac{3}{4}$ might involve taking two fifths of each of the three one-fourths (i.e., taking $\frac{2}{5}$ of $\frac{1}{4}$ three times, algebraically equivalent to $\frac{2}{5}\times\frac{3}{4}=\left(\frac{2}{5}\times\frac{1}{4}\right)+\left(\frac{2}{5}\times\frac{1}{4}\right)+\left(\frac{2}{5}\times\frac{1}{4}\right)=\frac{2}{5}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)$, using the fourths as the reference unit).

One may also interpret this fraction multiplication problem by taking $\frac{1}{5}$ of $\frac{3}{4}$ two times

$\frac{2}{5} \times \frac{3}{4} = \left(\frac{1}{5} \times \frac{3}{4}\right) + \left(\frac{1}{5} \times \frac{3}{4}\right) = 2\left(\frac{1}{5} \times \frac{3}{4}\right)$, using $\frac{1}{5}$ as the reference unit.

Reasoning with distribution may also be required while working with unknowns (Hackenberg, 2005). For instance, a 2-inch candy bar is $\frac{3}{4}$ of my candy bar, find the length of my candy bar ($2 = \frac{3}{4}x$). One may first reason that $\frac{1}{4}$ of my candy bar is $\frac{2}{3}$ inch long by considering $\frac{3}{4}$ of the unknown as $\frac{1}{4}$ taken three times, algebraically equivalent to $\frac{3}{4}x = \frac{1}{4}x + \frac{1}{4}x + \frac{1}{4}x$, where x denotes the unknown.

Norming

Another concept that was useful in explaining students' reasoning in Set 1 (multiplicative comparison of two quantities) was norming which is the process of re-conceptualizing a system in relation to some fixed unit or standard unit (Freudenthal (1983), Lamon, 1993b). Lamon describes the occurrence of the norming process in the domains of multiplicative comparison, fraction division and ratio. For instance, to compare 5 red counters in terms of 2 blue counters multiplicatively, one has to posit the two blue counters as one unit (as one thing, i.e., unitizing) and re-conceptualize the 5 counters in terms of the referent composite unit of 2 elements. The norming process is illustrated diagrammatically in Figure 2.8.

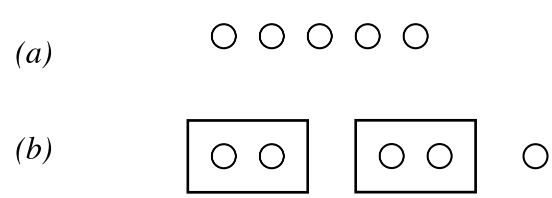


Figure 2.8. Norming 5 units in terms of units of 2

This process results in the scalar decomposition of 5 in units of 2 (i.e., $5 = 2(2 - \text{unit}) + \frac{1}{2}(2 - \text{unit}) = 2\frac{1}{2}(2 - \text{unit})$), analogous to the division of 5 by 2. The norming process is compatible with Steffe's conception (1992) of division in terms of units. He states that: "For a situation to be established as divisional, it is always necessary to establish at least two composite units, one composite unit to be segmented (or fragmented) and the other composite unit to be used in segmenting or fragmenting" (p. 267). A multiplicative comparison situation is different from a division situation though one may use division to compare two quantities. A multiplicative comparison situation involves a compared and a referent quantity. The norming process emphasizes the idea of re-conceptualization where one composite unit is chosen as a norm or referent to interpret the other quantity in the multiplicative comparison of two quantities.

The norming process may also be involved when the difference between the two quantities being compared is larger than the smaller quantity. For instance, when comparing 5 red and 2 blue counters using the comparative 'more', the difference of 3 counters can be normed in terms of 2 red counters to deduce that there are $1\frac{1}{2}$ times more red counters compared to blue counters. In terms of scalar decomposition this process can be written as: $3 = 1(2\text{ - unit}) + \frac{1}{2}(2\text{ - unit}) = 1\frac{1}{2}(2\text{ - unit})$.

In unitizing/uniting we construct composite units to be used as referents to interpret a situation. For example one may compare 6 red and 10 blue counters by unitizing in units of 2 or units of 6 as the data in Set 1 (situation 3) shows. Unitizing does not involve an element of multiplicative comparison of two quantities. For example, looking at 6 as 2 composite units of 3 elements or 3 composite units of 2 elements is a subjective process based on one's flexibility to coordinate units. However, norming involves the comparison of two quantities where one considers one quantity as a unit (through unitizing) to measure the other quantity. The unitizing operation precedes the norming operation. The norming action is particularly apparent when comparing a larger quantity in units or in terms of a smaller quantity. On the other hand, comparing a smaller quantity (e.g., 6 red counters) in terms of the larger quantity (e.g., 10 blue counters) may, however, be easier as the referent quantity is larger than the compared quantity. One may use the part-to-whole knowledge of fractions to observe that the red counters are $\frac{6}{10}$ or $\frac{3}{5}$ of the blue counters. The norming process may also be involved in the multiplicative comparison of two fractions. For instance, the comparison of $\frac{3}{4}$ in terms of $\frac{1}{2}$ (the norming unit) can be interpreted by the scalar decomposition: $\frac{3}{4} = 1(\frac{1}{2}) + \frac{1}{2}(\frac{1}{2})$.

A Third Path to Study Reversibility: The Notion of Quantities

Another analytical tool that was useful in understanding how the participants articulated the relationship between the known and unknown quantities in the reversibility situations (especially in Set 3) was the notion of quantitative reasoning as used by Thompson (1990, 1994,1995). Quantitative reasoning involves analyzing the quantities and relationships among quantities in a situation, creating new quantities, and making inferences with quantities. Thompson (1995) further points out that: “Quantitative reasoning is not about numbers, it is reasoning about objects and their measurements (i.e., quantities) and relationships among quantities” (p. 8). A quantity is a conceptual entity that has four dimensions:(a) it is an object (b) it has a quality (e.g., length) (c) it has an appropriate unit and (d) it involves a process to assign a numerical value to the quality. As highlighted by Olive & Çağlayan (2008), a quantity is named by its quality and measured by some identified unit and can be thought of in terms of the pair (name, unit) e.g., (dimes, number of dimes) or (dimes, value of dimes). The unit of measurement of the quantity may either be explicit or implicit. It may be explicit in terms of a specific unit that may be arbitrary (e.g., 10 paper clips) or standard (10 cm). It may be implicit in that no specific unit is given but the relation between two quantities are given (e.g., the length of candy bar A is 10 times the length of bar B).

Steffe (1991) characterizes a quantity as the outcome of unitizing or segmenting operations. Quantities are a product of people's conceptions of situations rather than the situations themselves. A quality of an object becomes quantified when the person can view the quality as being subdivided into some number of equal units. In other words, a quantity has a referent unit that gives it a measurable dimension. For example the tallness of a person becomes his/her height when a given unit is chosen (arbitrary or standard) as a referent. Similarly, the sum of two quantities or the difference between two quantities in a ratio situation get the status of a quantity (rather than a numerical sum or difference) when they are interpreted in relation to the two starting quantities and not merely as the result of arithmetic. For example: the ratio of boys to girls in a class is 3:4. If there are 28 students in the class, how many more girls are there than boys?

Thompson (1990) defines a quantitative operation (a mental operation) as "the conception of two quantities being taken to produce a new quantity" (p.11). For example, a multiplicative comparison such as "Paul is twice as old as his brother" is an example of a quantitative operation. It involves reasoning with two quantities (Paul's age and his brother's age) without using their numerical values but rather using the relationship between the quantities. In other words, one can reason about two (or more) related quantities without knowing their actual numerical values. The quantitative relation may assume different forms like additive and multiplicative comparison, additive and multiplicative combination, as well as in terms of instantiation, composition and generalization (Thompson, 1990). Further, Thompson (1994) asserts that "the quantitative operation of comparing two quantities additively originates in the action of matching two quantities with the goal of determining excess or deficit and the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing" (p. 186). In this study, I am particularly interested in investigating quantities that are in multiplicative relationship and where one of the quantities in the relationship is unknown.

Understanding reversibility from a quantitative orientation

Because I am studying reversibility in terms of multiplicative relationships among quantities, I am motivated to consider the link between reversibility and quantitative reasoning. Further, the tasks that I have chosen involve the multiplicative comparison of two quantities that, according to Thompson (1990), is one form of a quantitative operation. Thompson defines a quantitative relationship "as the conception of three quantities, two of which determine the third by a quantitative operation" (p.10). For communication purposes, I use algebraic notations to describe a quantitative relationship as $Q_1 * Q_2 \to Q_3$, where Q_1 and Q_2 are the two quantities operated upon to produce the quantity Q_3 and * denotes the quantitative operation. The quantities Q_1, Q_2 and Q_3 have a measure and the quantitative operation '*' may involve mental operations like combining or comparing quantities additively, combining or comparing quantities multiplicatively, generalizing a ratio and composing ratios (Thompson, 1990). One may also be required to operate on Q_1 and Q_3 in the quantitative structure $Q_1 * Q_3 \to Q_2$ to produce the quantity Q_2 (having an unknown unit).

In a multiplicative comparison situation, one can encounter the following three types of quantitative structures:

(i) When comparing one quantity in terms of the other, say Q_1 in terms of Q_2 (or comparing Q_2 in terms of Q_1) to produce Q_3 (with the corresponding arithmetic operation being division or measurement division). One may think of the quantities Q_1, Q_2 and Q_3 in terms of the pair (quality, unit): Q_1(quality, explicit unit), Q_2(quality, explicit unit) and Q_3(quality, relationship between Q_1 and Q_2 or implicit unit). For example, when comparing 5 red counters in terms of two blue counters, one may think of the quantities in terms of the pair (quality, unit) as follows: Q_1(marbles, number of red marbles), Q_2(marbles, number of blue marbles) and Q_3(marbles, relationship between Q_1 and Q_2). In the current dual problem:

$$Q_1\text{(marbles, 5)} * \ Q_2\text{(marbles, 2)} \rightarrow Q_3\ \text{(marbles, } Q_1 = \tfrac{5}{2}Q_2\text{)}$$

This category of situation has been investigated in Chapter 5 (Set 1).

(ii) When using a known quantity (say Q_1) and the quantitative relation between the known and an unknown quantity to produce the unknown quantity (with the corresponding arithmetic operation being division or partitive division). Consider the following dual problem: ‘Candy bar A is $\frac{4}{3}$ units long. Its length is $\frac{2}{5}$ of candy bar B. What is the length of candy bar B?’ In this problem, Q_1 and Q_2 refer to length of candy bars A and B respectively and Q_3 denotes the quantity produced by relating Q_1 and Q_2. Using the notations developed above, this situation can be represented as follows:

$$Q_1\text{(length, } \tfrac{4}{3} \text{ units)} * Q_3\text{(length, } Q_1 = \tfrac{2}{5}Q_2\text{)} \rightarrow Q_2\ \text{(length, unknown magnitude)}$$

This category of situation has been investigated in Chapter 6 (Set 2). Reversibility involves working with known and unknown quantities and consequently requires reasoning about relationships among quantities. Referring to the candy bar problem described above (algebraically equivalent to $\frac{2}{5}x = \frac{4}{3}$), one may reason quantitatively as follows: Divide $\frac{2}{5}$ of bar B (i.e., Q_3) into two equal parts to produce $\frac{1}{5}$ part of the unknown quantity (Q_2). Then use five of the $\frac{1}{5}$ parts to make one unit of the unknown quantity, thereby reversing the making of two fifths of bar B. Equivalently, one may reason quantitatively as follows: $\frac{2}{5}$ of quantity Q_2 has length $\frac{4}{3}$ units, therefore $\frac{1}{5}$ of quantity Q_2 has length $\frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$ units, hence $\frac{5}{5}$ of quantity Q_2 has length $5 \times \frac{2}{3} = \frac{10}{3}$ units, coordinating the quantitative relationship and the measure simultaneously.

(iii) When combining two quantities Q_1 and Q_3 to form Q_2 (with the corresponding arithmetic operation being multiplication). This category of situations has been referred to as 'multiplicative compare' by Nesher (1988). Consider the following primal problem: Dan has 12 marbles. Ruth has 6 times as many marbles as Dan has. How many marbles does Ruth have? This situation can be represented as follows:

Q_1(marbles, 12) * Q_3(marbles, $Q_2 = 6Q_1$) $\rightarrow Q_2$ (marbles, unknown magnitude)

Having outlined the theoretical framework, I now turn to describe the methodology used in designing the study. The next chapter outlines the methods, justification of design decisions, instruments, participants, overview of the chosen tasks, data analysis, and possible limitations of the study.

CHAPTER 3

METHODOLOGY

Building on Current Methods

Regarding a research method as a lens through which a research phenomenon is viewed, Schoenfeld (2002) points out that “one’s framing assumptions shape what one will attend to in research” (p. 447). He emphasizes the necessity to be mindful about the theoretical perspective to be adopted, the questions to be asked, the claims that one expects to make, the appropriateness of methods as well as the warrants that the chosen method or methods provide in substantiation of the claims. I used these principles as guidelines to frame my research by providing a justification for every design decision and reflecting on possible limitations or restrictions as well as assumptions being made.

A range of research methods can be identified in the mathematics education literature. Kelly & Lesh (2000) underline this methodological diversity by pointing out the array of methods: “teaching experiments, clinical interviews, analyses of videotapes, action research studies, ethnographic observations, software development studies, computer modeling studies” (p. 18). Apart from these qualitative approaches, one also encounters experimental designs and statistical studies, though the qualitative paradigm seems to be more and more prominent. Specifically, in the multiplicative domain of rational number, ratio, and proportion, Lamon (2007) points out the following prevalent methods – “empirical analysis, rational task analysis, clinical interviews, interventions, design experiments, and longitudinal studies” (p. 640).

The primary aim of this study was to understand the ways in which students reason reversibly in multiplicative contexts at a fine-grained level of detail and the constraints that they encountered in these situations. Such an endeavor necessarily called for a qualitative approach. The suitability of the qualitative paradigm in understanding students' reasoning in mathematics can be inferred by the pervasiveness of such a research orientation in publications on mathematics cognition (Ginsburg, 1997; Izsák, 2004, 2005; Olive & Steffe, 2002; Schoenfeld, Smith, & Arcavi, 1993).

The two research methods that were suitable for my research were Clinical interviews (Ginsburg, 1997; Goldin, 2000; Hunting, 1997) and Teaching experiments (Steffe & Thompson, 2000). Both of these approaches are commonly used in the study of students' cognition in mathematics education. In fact, several studies dealing with reversibility in fractional contexts (e.g., Hackenberg, 2005; Olive & Steffe, 2002; Tzur, 2004) have been done through Teaching experiments, with the aim of tracing the progressive reorganization of students' schemes over time. As pointed out by Steffe (1994), "The essential virtue of the teaching experiment is that it allows the study of constructive processes – those critical moments when restructuring takes place and is indicated by alterations in the child's behavior" (p. 32). Considering the virtues that the Teaching experiment method affords the researcher, initially I planned to use this approach. However, Teaching experiments are based on hypotheses as mentioned by Steffe & Thompson (2000): "one does not embark on the intensive work of a Teaching experiment without having major research hypotheses to test" (p. 277), though these hypotheses may be modified as the teaching experiment proceeds. I did not have any major explicit hypotheses as my study was essentially exploratory. Teaching experiments involve experimentation with the ways and means of influencing students' mathematical knowledge by engaging interactively with the participants.

My aim was not to promote extended growth of knowledge over time but rather to observe reversible reasoning across four related domains (multiplicative comparison, ratio, fraction and proportion). I chose Clinical interviews, more specifically ‘task-based interviews’ (Goldin, 2000) that refers to structured/semi-structured interviews developed on the basis of tasks. I often revisited tasks that proved to be problematic to students. As argued by Clement (2000), “carefully done clinical interview studies are an essential and irreplaceable part of the scientific enterprise of investigating a student’s mental processes” (p. 547). But the clinical interview is not without limitations. While teaching experiments allow extensive scenarios of evolving construction of knowledge over an extended period of time, the clinical interview is more restricted in terms of amount of interactions with the participants. One should also be cautious about false positive (attributing to children knowledge which they do not possess) and false negative (not attributing to children knowledge which they do possess) (Smith, 1993).

Once the decision to use Clinical interviews was chosen, my next task was to select instruments, participants, and tasks. The three sections that follow describe and justify these three design elements.

Instruments

> Among the most intriguing human activities that can never be directly observed is thinking and reflecting. … But the actual process of thinking remains invisible and so do the concepts it uses and the raw material of which they are composed. (von Glasersfeld, 1995/2002, p. 77)

As the above quote by von Glasersfeld suggests, understanding cognition is a complex task due to its unobservable nature and often ingenious methods are needed to reveal students' thinking. We cannot directly observe student's thinking, reasoning, cognitive processes, internal representations, meanings, knowledge structures, or affective states (Goldin, 2000). Indirect methods are used to access students' mathematical thinking by analyzing both verbal (spoken words, interjections) and non-verbal behavior (movements, writings, drawings, actions with external materials, gestures, facial expression). In this study data were gathered primarily through semi-structured, task-based interviews. I collected data in two phases. In phase I (from the 6th -20th of May 2008), I started with a broad range of multiplicative situations involving multiplicative comparison, ratio, fraction and proportion. The aim of the first phase of data collection was to explore different reversibility situations selected from the theoretical analysis made in Chapter 4. After analyzing selected portions of the data, I designed new tasks and conducted the second phase of data collection in December 2008 (from the 2nd to the 17th) and January 2009 (from 15th to the 30th). Table 3.1 shows the number of interviews conducted at the three grade levels. I provide more details about these interviews in the section on task design.

Following Hall (2000), two cameras were used to record the interviews so as to focus on students' responses as well as their inscriptions. During the interviews, the pair of students were provided with worksheets containing the problem statements. They could use the worksheet to make any calculation that they wanted and this also served as a record of their inscriptions.

Table 3.1 Number of interviews conducted during the two phases of data collection

Phase	Date	No. of interviews by grade		
		Grade 6	Grade 7	Grade 8
I	6th - 20th May 2008	9	7	9 *
II	2nd - 17th Dec. 2008	0	11	10
	15th – 30th Jan 2009	0	6	6

Key: (diagonal hatching) Interviews with students in grade 6 in Phase I

(vertical hatching) Interviews with students in grade 7 in Phase I

* Students in grade 8 in Phase I did not participate in Phase II

I was motivated to use a microworld due to its potential to elicit students' reasoning as highlighted by Greer (1992) and Lamon (2007). I chose to use the JavaBars microworld (Biddlecomb & Olive, 2000) as it was productively used to investigate students' reasoning (including reversible reasoning) in the domain of fraction (Hackenberg, 2007; Olive & Steffe, 2002, Tzur, 2004). Furthermore, JavaBars makes students' articulation of unit structures explicit, consonant with one of the analytical tools used to analyze the data. The JavaBars microworld is a program designed to support students' manipulation of quantities through elementary operations like partitioning, recursive partitioning, and iterating. Quantities are represented in the form of a rectangle. The user can copy parts of a bar, repeat it or join it to another bar. The program has also a 'clear', a 'break' and a 'pull-out' command that give considerable flexibility to the user compared to paper-and-pencil representations of fractional quantities. I will comment on the observed shortcomings of JavaBars in the limitations section. I also gave the students some of the primal problems to work at home.

Participants and selection criteria

This study was conducted in a rural middle school and involved a total of six students at the grade levels 6, 7 and 8. Students' mathematical experiences, competencies and ways of acting and operating vary in terms of a number of dimensions. It is desirable to have a varied sample of students, so as to observe a diversity of interactions with the chosen tasks in the clinical interviews. I started with 3 pairs of students (1 pair at each of the Grades 6, 7 and 8) in the first phase of the study. In choosing to focus the study at three levels instead of one, I hoped to obtain more variability in the findings. I interviewed pairs of students who were familiar with each other and who had worked together before. I asked the class teacher to select students who were above-average performers because reversibility situations are regarded as particularly demanding.

The first phase of the study gave me ample opportunity to know the 6 students in terms of their ability, ways of acting and operating and computational flexibility. It should be mentioned that the Grade 8 students were no longer available after the first phase of data collection. I chose to focus exclusively on the Grade 6 and 7 students who shifted to Grades 7 and 8 after the first phase.

Conducting the interview

Students were interviewed in the school during their Extended Learning Time. I wanted to benefit from the interactions between the two students in each interview. Further, as indicated by Schoenfeld (1983): "dialog between students often serves to make managerial decisions overt, whereas such decisions are rarely overt in single student protocols" (p. 350). Another justification is that when an interview is conducted in pairs, the participants may experience less tension. I paid attention to both the verbal and non-verbal behavior as well as what the six participants wrote during the interviews. I occasionally adjusted the wording, content, sequence or structure of questions after observing the reactions of the students to the task posed. I also paid careful attention to spontaneous behaviors.

I encouraged the students to work in pairs and allowed them to solve the problems to the point where they were satisfied with their answers. Whenever required, I prompted them to check their answers to help them detect any anomalies in their responses. When they experienced conflict, I formulated simpler problems to prompt them to rethink about the problematic situations. Besides reading the questions in the interview, I also provided each student a written statement of the different questions in the form of a worksheet to be explicit about the question being asked.

As pointed out by Ginsburg (1997), it is important to inform the participants at the outset what the aim of the research is and what their roles are. Students tend to reveal their thinking including, doubts and ignorance when an environment of trust and mutual respect is created and where power imbalance (Hunting, 1997) is promptly negotiated – issues to which I carefully paid attention. The suggestions given by Ginsburg (1997) were used as guiding principles for conducting the interviews, like recognizing the child's autonomy, establishing and monitoring motivation of the participants, using different probing devices (e.g., silent probe, tell-me-more probe and echo probe), looking for spontaneous responses, phrasing of questions, and so forth. At the start of the study, I informed the students that I was interested in their thinking process and not in the correct or incorrect answers. I can assert that they understood my objective clearly as can be deduced by Cole's comment to Ted: "He wants to know how you get this and not getting them right or wrong."

Tasks Design and Description

Well-designed tasks create the necessary condition for eliciting students' reasoning (Simon & Tzur, 2004). One of the demanding tasks in this study was to design and select problems that could prompt students to reason reversibly. I specifically focused on tasks that involve a multiplicative comparison relation, either explicitly or at one stage of the solution process. I selected prerequisite problems in the domain of ratio to have a sense of what the participants already knew and could perform.

I devised tasks, paying particular attention to several criteria. Firstly, these problems emanate from the theoretical analysis made in Phase I of the research. Different categories of problems were identified (see Chapter 4). The problem formulation framework allowed me to generate tasks that were progressively more complex. Simultaneously, this framework allowed me to make systematic variations in the tasks so that the third research question about the constraints that students experience in reversing their thought process could be investigated. In line with Olive and Steffe's (2002) suggestions "providing a realistic situation can provide the student with a more meaningful context" (p. 431), I selected problems that I assumed to be meaningful to students, where problem context does not tax extra cognitive load. The tasks were progressively sequenced by varying the order of complexity of the problem as well as the numeric feature of the problem parameters. Each set of tasks started with an initial question meant to familiarize students with the type of problems to be solved. While formulating the questions, I wrote a rationale to ensure that they were precise and no overlap among them occurred. Following Zazkis & Hazzan (1999), the majority of questions have been formulated to reveal “student strategies, approaches and conceptions, rather than their performance” (p. 431). I also paid attention to select tasks that were engaging.

However, because reversibility is a characteristic that very much depends on the individual solving the problem, these tasks were meant to be possible pathways to identify the mechanisms students use to reverse their thought process and there was no guarantee that students would solve the problem in ways that were being hypothesized, as the data shall show. I used preceding video recordings to build and modify new tasks between interviews. For instance, situations 6, 7 and 8 in Set 1 were formulated after observing the conflict that the students encountered earlier.

Overview of tasks: Problem formulation framework

I started by designing a range of tasks as I was attempting to answer the first research question (i.e., identifying different forms of reversibility situations in the multiplicative domain). In the empirical part of the study, I focused on three sets of tasks as described below. I will comment on the specific rationale for each set of tasks in greater detail in Chapters 5, 6, and 7. Below, I provide an overview of the three types of multiplicative situations selected for this study.

Set 1: Multiplicative comparison of two quantities in a measurement division situation

In Set 1, two types of situations were presented to the students. Firstly, they were required to compare two sets of counters multiplicatively (a discrete situation). Secondly, they were asked to compare the length of two candy bars (a continuous situation) in a microworld environment.

Set 1 can be regarded as involving measurement division situations, algebraically equivalent to $ax = b$.

Set 2: Multiplicative comparison of two quantities in a partitive division situation

Set 2 involves the multiplicative comparison of a known and an unknown quantity. A multiplicative comparison relation between two quantities is given as well as the measure of one of the two quantities and the students are asked to construct the second quantity. This set of problems deals primarily with fractional relations and are often referred to as ‘construct-the-unit’ problems. Set 2 problems are algebraically equivalent to $ax = b$.

Set 3: Multiplicative comparison in ratio situations

Set 3 involves three types of situations. Type I problems require the decomposition of a given quantity in terms of the components of a ratio (algebraically equivalent to $(a+b)x = q_1 + q_2$). Type II problems involve the decomposition of a given quantity based on two quantitative relations: a multiplicative comparison and a difference (algebraically equivalent to $(a-b)x = q_1 - q_2$). Type III problems require the participants to establish the equality between two quantities when an amount (*e*) is exchanged (algebraically equivalent to $(a-b)x = 2e$).

I also collected data on four other situations: (a) Gear-wheel problem involving the coordination of two gear-wheels as they turn synchronously, (b) Rice-cooker problem involving the coordination of two co-varying quantities, (c) ratio problems involving the multiplicative comparison of 3 quantities, and (d) percentage problems of the form $\frac{x}{100} \times a = b$ and $\frac{a}{100} \times x = b$.

Although I investigated a range of multiplicative situations, I report only selected portions to highlight the cognitive resources that the participants appear to have deployed and the conflicts that they encountered in reasoning reversibly. In this dissertation, I focus on 3 of the 7 problem types that I investigated. Now that the design of the study has been made explicit, I explain how I analyzed the data and the criteria that I used to ensure the credibility of the study.

Data analysis

A previous study taught me a myriad of issues underlying data analysis, which I see as being theoretically echoed in the research methodology literature, like data reduction, selection (highlighting specific parts of the transcript and obscuring other portions), categorization (sometimes with fudging boundaries), identification of instances with particular characteristics as well as looking for repetition, stability or change (Hardy & Bryman, 2004).

I analyzed the videotaped data using the procedure described by Cobb and Whitenack (1996), where episode-by-episode chronological analysis served to generate initial conjectures that were revised in subsequent episodes. For instance, in Set 1 (involving the multiplicative comparison of two quantities), the first set of interviews showed that the students experienced constraints with the comparatives 'more' and 'less'. Thus, in designing the second set of multiplicative comparison situations, I explicitly formulated questions to explore further the causes of this constraint. Further, the analysis of the data was carried out concurrently with the data collection as I proceeded inductively towards the more demanding reversibility tasks. As highlighted by Denzin & Lincoln (1994), this dynamic approach also allowed me to rectify errors made in previous interviews and to adjust questions posed in subsequent interviews. The data were parsed into phases and episodes on the basis of the type of tasks, strategies and constraints so as to provide coherent accounts of students' response.

Apart from the videotaped excerpts, a second source of data that I used was the worksheets given to the students during the interviews. Students' writings, as they attempt to solve mathematical problems, have been given considerable attention in recent publications in mathematics education journals. Izsák (2003) suggests that to capture students' thinking processes, they need to leave a trace of what they are thinking moment-by-moment. I paid careful attention to what students wrote on the worksheets and the drawings they used to externalize their thinking process. These inscriptions were particularly helpful in interpreting students' responses.

The audio and video recordings were combined using an audiovisual mixer to create a picture-in-picture. I transcribed all of the interviews myself as the 'recursive' nature of this task allowed me to be more involved with the data. Moving to and fro during the process of transcription provided me with an enduring recollection of what the participants said and this proved to be useful in deriving interpretations and linking events that were not apparent initially. The qualitative analysis (Transana) software (Woods, 2008) allowed me to move back and forth from students' responses (both in audio and video format) to local and global conjectures. Such a flexibility allowed greater sensitivity in making hypotheses and claims besides minimizing the possibility that accounts were artifacts of my adult understanding of mathematics.

I analyzed the data on a problem-by-problem basis. Then I looked at recurring strategies (or constructive resources) and constraints along the different problem situations. This yielded the categories that I have used to present the data. Such categorization was important to be able to identify the theorems-in-action, both flawed and correct.

I answered the second and third research questions in each of the three data sets separately, because the type of reversible reasoning is different in the three situations (Set 1, Set 2, and Set 3). In the last chapter (conclusion), I bring together commonalities and differences that I could observe across these problem situations.

The six participants in the study are: Ted and Cole (the pair of sixth-graders), Aileen and Brian (the pair of seventh-graders), and Jeff and Eric (the pair of eighth-graders). All names are pseudonyms. The first letter of the pseudonyms is used as abbreviations while presenting the interview transcripts, e.g., T for Ted and C for Cole. 'I' denotes the interviewer. The following convention has been used in the transcription of the interview (Izsák, 2005, p. 376):

// … // denotes concurrent talk;

[…] denotes a comment I think a student made;

(inaudible) denotes a time when I could not understand what a student said;

(…) denotes a comment that I inserted while preparing transcripts;

"…" denotes something that a student wrote down.

Additionally, I shall use three vertically-arranged dots (i.e., ⋮) to denote omitted segments of transcripts for the sake of clarity and brevity. Consecutive lines of transcript are labeled as L1, L2, and so forth. Moreover, I used […] for two purposes. First, I used it to clarify what the students intended to say (e.g., in episode 7.14, L8, Brian said 28 minus 16 would be 10; he actually meant 28 minus 18 would be 10). Secondly, I used […] to include comments that the students may have made when the tape was not very audible. I used (…) to include comments to guide the reader.

Quality Criteria

Because I conducted my study on a selected number of cases in a qualitative setting, it was important to identify criteria or safeguards to present viable accounts of students' responses and to ensure that findings and interpretations were "not accidental, spurious, one-time consequences of unobserved, uncontrolled particulars of the situation" (Goldin, 2000, p. 527). In interpretive research, where standardized quantitative parameters like reliability and validity cannot be easily defined, artifacts based on the researchers' interpretation, imagination or even design may crop up. I drew upon the research methodology literature in qualitative research to find quality criteria to ensure the trustworthiness of my study.

Guba & Lincoln (1989) defined a set of criteria, parallel to the positivistic notion of reliability and validity for qualitative studies, one of which is of immediate relevance in my study, namely credibility (or trustworthiness) or the degree of believability of the claims made in a research. Credibility is regarded as the "isomorphism between constructed realities of respondents and the reconstructions attributed to them" (p. 237) by the researcher. Guba and Lincoln provide six criteria to ensure the credibility of qualitative research – prolonged engagement, persistent observation, peer debriefing, negative case analysis, progressive subjectivity and member checks. As explained earlier, two of the three pairs of students were interviewed for an extended period of time (from May 2008 to January 2009). A total of 58 interviews were conducted (as shown in Table 3.1), each of duration of about 40-60 minutes. This prolonged engagement together with the combination of two video cameras ensured that I made persistent observations. I also interviewed the students over a range of tasks that allowed me to make an adequate assessment of their mathematical competencies in the domain of multiplication. Another useful quality criteria that Guba and Lincoln advocated is called a 'confirmability audit' as can be inferred by the following explanation:

> This means that data (constructions, assertions, facts, and so on) can be tracked to their sources, and that the logic used to assemble the interpretations into structurally coherent and corroborating wholes is both explicit and implicit in the narrative of a case study (p. 243).

Extended segments of interview transcripts have been provided in the data analysis section to show how interpretations have been made. Further, I have used three analytical tools (Vergnaud's notion of Theorems-in-action, the idea of units, and quantitative reasoning) to look at the findings to ensure that the data are scrutinized in as much depth as possible.

Apart from the audit trail described above, Guba and Lincoln also mention that another approach to ensure the credibility of qualitative research is to analyze data dynamically as they are being collected such that hypotheses being put forward can be verified and interpretation can be refined. As explained in the data analysis section above, I analyzed the data as they were being collected. After each interview, I watched the videotapes to modify or change any subsequent preplanned tasks to adapt to the level of the students. I analyzed a major portion of the data collected in May 2008 before I started the second phase of data collection in December 2008 and January 2009. I would also like to mention that the students were engaged in concentrated problem solving in the different situations that were posed to them as can be observed from the videotapes and this adds more credibility to the findings.

Limitations

As researchers we make assumptions about the nature of mathematics (ontology), about the acquisition of mathematical knowledge (epistemology), and the methods of gaining access to this knowledge (methodology), all of which impact the claims that we make about the particular research phenomenon. The researchers' ontological, epistemological, methodological, axiological and rhetorical assumptions inherently introduce limitations in the research process.

The interpretations given in this research are limited to my own belief system and assumptions. This research is situated within a constructivist perspective that is the dominating epistemology in contemporary mathematics education. The design of my study, the literature that I read and eventually the interpretations of the findings are all influenced by this constructivist orientation. In approaching this study from a constructivist perspective, I subscribe to the belief that knowledge is not transmitted but the cognizing subject has to build it for himself or herself. I also value Piaget's idea that the mind organizes the world by organizing itself. In other words, I believe that the mind interprets experience and shapes it into a structured world.

In his/her research endeavor, the researcher chooses to portray some aspects of the study and obscures other aspects. This act of selection, together with other design decisions like the research tools selected and theoretical commitments (Schoenfeld, 2007) impose limitations in the study. As highlighted by Cobb (2007), there is no neutral algorithm of theory choice. By choosing individuals as the unit of analysis, I did not focus extensively on social interaction in knowledge construction. Further, by taking a cognitive orientation, I did not consider issues of culture, power, affect and other variables that might have also been important in explaining students' behavior. I used Clinical interviews to collect data and this choice imposes limitations on what I could observe. As pointed out by Lamon (2007): "a persistent problem in snapshot research is that a child's actions at any particular time really depend on many causes – constraints and affordances – in his immediate environment. What the child might have done under different conditions, over longer periods of time, goes undiscovered" (p. 660). While the task-based interviews enabled me to categorize the students' various approaches to solving these types of problems and identify the constraints they encountered, this method does not enable me to construct models of how students might progress in their constructions of more powerful strategies. In order to construct such models of students' learning with respect to reversibility, longitudinal teaching experiments (Steffe & Thompson, 2000) are necessary. Further, the interpretations that I have given to the data are limited to the analytical tools that I have used. The sample chosen involves above-average students and as such it is important to interpret the findings in this study as being for this category of students. Understanding cognition inherently involves an element of subjectivity as it depends on inferences based on students' behavior because one cannot directly observe such thinking. Observations are taken as proxies of internal cognitive structures and processes or as Kaput & West (1994) state: "observable performance is understood as cognitively mediated." (p. 235). Thus, the description presented in this study should be interpreted with these limitations in mind.

Using microworlds to access knowledge construction

Undoubtedly, microworlds as mediating tools provide cognitive access to children's conceptual structures as they allow meaning to be externalized as has been pointed out by a number of research studies (Greer, 1992; Jones et al., 2002; Lamon, 2007; Olive, 2000; Olive & Steffe, 2002; Thompson, 1994). They provide a window on how children construct powerful mathematical ideas and enable the researcher to analyze such fine-grained constructions which may not always be apparent in a paper-and-pencil non-dynamic setting. However, by providing concrete embodiments to mathematical ideas, they also implicitly or explicitly direct the problem-solver to work in particular ways. In other words, a problem may be conceptualized differently and the solution may take a particular path when solved in a microworld compared to a paper-and-pencil approach. A microworld environment may change the nature of a problem and may impose additional constraints. This is one of the limitations that I observed when using the Javabars microworld. For instance, Aileen (and Brian) was asked to solve the following problem with and without JavaBars: "Candy bar A is $\frac{2}{3}$ unit long. Its length is $\frac{4}{7}$ of candy bar B. What is the length of candy bar B?" ($\frac{4}{7}x = \frac{2}{3}$). Aileen gave the following response without using JavaBars: "four sixths is equal to four sevenths and so you would add three more sixths" (L24). When asked to solve the problem on JavaBars, she drew a bar, divided it into 3 parts and shaded one part, thus representing $\frac{2}{3}$ as the unshaded part. Then she divided each of the two thirds in candy bar A into 3 (rather than 2) to make 6 partitions and pulled out 7 pieces as shown in Figure 3.1.

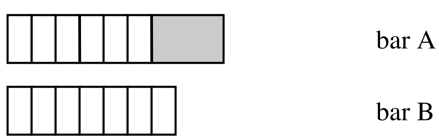


Figure 3.1. Aileen's solution to the problem $\frac{4}{7}x = \frac{2}{3}$

When asked to explain her answer, she gave the following response: "You can see that like if two thirds is equal to four sevenths that means that four sixths is equal to four sevenths. So one sixth equals one seventh. So to get the 7 total pieces, you have to add two more sixths or one third." She incorrectly divided each third in candy bar A into three parts each rather than 2 parts, probably guided by the 'sixth' in $\frac{4}{6}$. She assumed that one small subdivision in bar A is $\frac{1}{6}$ (which is actually $\frac{1}{9}$ according to Figure 3.1). This incorrect assumption led her to pull out 7 parts from bar A to make bar B. Aileen's explanation that "you have to add two more sixths or one third" shows that she was confounding the measure of the quantity ($\frac{2}{3}$ or $\frac{4}{6}$) and the quantitative relation ($\frac{4}{7}$). She should have added $\frac{3}{7}$ (with corresponding measure $\frac{3}{6}$) rather than $\frac{2}{6}$ as she correctly did in L24 without the JavaBars. The graphic representation imposed an additional load for Aileen in this situation. In previous problems, she mentioned that the visual representation on JavaBars was occasionally confusing for her.

In another situation (problem 2.63, $\frac{4}{5}x = 3$), Brian mentioned "The diagram just makes it more confusing for me." In fact, in various episodes he did not use the JavaBars. Further, when solving problem 2.51 ($1\frac{1}{2}x = 48$) on JavaBars, the students first represented the 48 partitions and as a result attempted to split it into one whole and a half by guess-and-check rather than focusing on $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$. I also hypothesize that the quantity-measure conflicts (Chapter 6, Set 2) reported in this study were partially generated by the JavaBars microworld. If I were to do this study again I would have interviewed the participants without JavaBars first and then in the second phase I would have used it only in the problematic situations.

CHAPTER 4

MATHEMATICAL ANALYSIS OF REVERSIBILITY SITUATIONS

In this chapter, I answer the first research question by analyzing different types of multiplicative situations within the domain of whole number multiplication/division, fraction, ratio, proportion and percentage in order to trace the key features of 'reversibility situations'. This chapter is also meant to set up the theoretical structure on which the empirical part of the study is based. The first research question reads as follows: *What are the different types of reversibility situations in multiplicative contexts and what are the structural relationships among these situations?*

The application of Vergnaud's theory demands that one performs a mathematical analysis of the problem situations as well as carrying out a conceptual analysis based on empirical data collected from interviewing students. It is necessary to understand what type of mathematical structures give rise to reversibility situations so as to identify the relationships among these situations. Such an analysis may also have instructional and curricular implications besides being useful for further research. As pointed out by Nesher (1988): "The significance of any theoretical analysis is that it enables us to hypothesize the major parameters that have explanatory power for the observed phenomena" (p. 38). The multiplicative domain has been analyzed from a number of theoretical perspectives by different researchers (Behr & Harel, 1990; Greer, 1992; Lamon, 2007; Steffe, 1992; Vergnaud, 1983). In this study, I use Vergnaud's (1983) categorization of multiplicative situations and its extension by Greer (1992) as a framework to show how reversibility situations are generated in the multiplicative conceptual field. I highlight the primal and dual formulations of the different categories of situations as they arise as a result of the inherent structure of multiplicative situations.

Several groups that occur in mathematics such as the integers, the rationals, matrices, or continuous functions allow us to combine two elements of the set to generate a third element. The binary operations used to combine the elements go by different names, such as addition, multiplication, composition, and so forth. The elements of these sets together with the defined operation generate different structures in mathematics. For instance, the set of integers forms a group under addition but not under multiplication due to the absence of a multiplicative inverse. Thus, it is necessary to define the set of rational numbers to ensure the existence of multiplicative inverses. The repercussion of such a mathematical structure is that finding the multiplicative inverse is not naturally occurring as in the case of the additive inverse. This may be one of the reasons why students fail to reason reversibly in multiplicative contexts as the empirical part of the study will show. Another point worth highlighting is that from a mathematical perspective, divisibility is defined via multiplication (i.e., d divides a if $a = kd$, where k is an integer) or in other words division gets its meaning only after multiplication has been defined. The problem of finding the multiplicative inverse has been a challenge since historical times. For instance, the Babylonians used a table of reciprocals constructed from multiplication to perform division (Eves,1990).

The Theorem-in-Action $ax = b$

I consider multiplicative reasoning to be the articulation of the relation $a \times c = b$ in a variety of contexts and situations (where a, b, and c can be integers, fractions, or decimals). Reversibility of thought is involved in coordinating the relationship between the multiplicand (or multiplier) and the product when the multiplier (or multiplicand) is unknown. Partitive division situations arise when the multiplier and product are known and the unknown is the multiplicand. Measurement/quotitive division situations arise when the multiplicand and product are known and the multiplier is unknown. These division situations can be characterized by the theorem-in-action $ax = b$, where x denotes the unknown quantity in the multiplicative relationship.

The different interpretations of $ax = b$

Vergnaud's notion of theorem-in-action prompts us to look at the different ways in which a particular mathematical concept may be conceptualized. The statement $ax = b$ can be interpreted from at least five perspectives as described below, each of which calls for different theorems-in-action.

(i) $ax = b$ *as a statement of multiplication*

This interpretation was observed in Set 1. For instance, in comparing 6 red counters in terms of 4 blue counters, Aileen asked the question: 'What do I have to multiply 4 by to get 6?', a theorem-in-action equivalent to $4x = 6$.

(ii) $ax = b$ as a statement of multiplicative comparison involving measurement division

$ax = b$ can also be interpreted in terms of measurement division. For instance, comparing 6 red counters in terms of 4 blue counters involves measuring the 6 red counters in terms of units of 4 blue counters. Here, the solution to the equation $4x = 6$ can be expressed by the scalar decomposition: $6 = 1(4) + \frac{1}{2}(4)$. This interpretation could be seen in Set 1 when the students applied the norming process.

(iii) $ax = b$ as a statement of multiplicative comparison involving partitive division

Consider the following problem: 'Candy bar A is $\frac{1}{4}$ unit long. Its length is $\frac{2}{5}$ of candy bar B. What is the length of candy bar B?' ($\frac{2}{5}x = \frac{1}{4}$). Here, a multiplicative comparison relationship between a known and an unknown quantity is given: Candy bar A is $\frac{2}{5}$ of candy bar B. One is required to find the size of one unit/whole or to construct one whole given part of one whole. This interpretation was particularly apparent in Set 2 (fraction situations).

(iv) $ax = b$ as a statement of proportionality

$ax = b$ can be interpreted as a statement of proportionality. The parameters *a* and *x* are directly proportional to *b* (i.e., $a \alpha b$ and $x \alpha b$) while *a* and *x* are inversely proportional to each other $a \alpha \frac{1}{x}$ or $x \alpha \frac{1}{a}$. I use Brian's response from Set 2 to show how the direct proportionality interpretation can be observed in students reasoning. In solving the following problem: 'Candy bar A is $\frac{1}{4}$ unit long. Its length is $\frac{2}{5}$ of candy bar B. What is the length of candy bar B?,' Brian used the following strategy:

$$\tfrac{1}{4} \to \tfrac{2}{5}$$
$$\tfrac{2}{8} \to \tfrac{2}{5}$$
$$\tfrac{1}{8} \to \tfrac{1}{5}$$
$$5(\tfrac{1}{5}) \to 5(\tfrac{1}{8})$$

(v) $ax = b$ as an algebraic statement

The interpretations of $ax = b$ presented above involve the articulation of a multiplicative relationship between a known and unknown quantity. However, the unknown quantity may not have an explicit status or may not always be symbolized algebraically as will be shown in Set 3.

More generally, when *a* and *b* are fractional quantities, $ax = b$ can be written as $\frac{p}{q}x = \frac{s}{t}$, where *p*, *q*, *s*, and *t* are integers. This equation can be manipulated in a number of ways (which in Vergnaud's terms represent different theorems-in-action) to find the unknown quantity *x*. I highlight some possible strategies. Assume $\frac{p}{q} < 1$.

Strategy 1: $x = (\frac{s}{t}) \div (\frac{p}{q})$,

Strategy 2: $\frac{1}{q}x = \frac{s}{pt}$, and $x = q(\frac{s}{pt})$ [unit-rate strategy]

Strategy 3: $px = \frac{qs}{t}$ and $x = (\frac{qs}{t}) \div p$

Strategy 4: $\frac{q}{p} \times \frac{p}{q}x = \frac{q}{p} \times \frac{s}{t}$

Strategy 5: $\frac{p}{q}x + (1 - \frac{p}{q})x = \frac{s}{t} + [(1 - \frac{p}{q}) \div \frac{p}{q}] \times \frac{s}{t}$. I illustrate this strategy by the following example: Candy bar A is $\frac{1}{4}$ unit long. Its length is $\frac{2}{5}$ of candy bar B. What is the length of candy bar B? This problem is algebraically equivalent to $\frac{2}{5}x = \frac{1}{4}$. Adding $\frac{3}{5}x$ on the left hand side of the equation corresponds to a compensation of $1\frac{1}{2}$ of $\frac{2}{5}$. Thus the right hand side of the equation has to be equivalently compensated by $1\frac{1}{2}(\frac{1}{4})$. In other words, $\frac{2}{5}x + 1\frac{1}{2}(\frac{2}{5}x) = \frac{1}{4} + 1\frac{1}{2}(\frac{1}{4})$ or

$\frac{2}{5}x + (1 - \frac{2}{5})x = \frac{1}{4} + [(1 - \frac{2}{5}) \div \frac{2}{5}] \times \frac{1}{4}$. One of the participants, Aileen, attempted to use this strategy in Chapter 6, episode 6.29, though she did not perform the compensation correctly.

When $1 < \frac{p}{q} < 2$, the following compensation may be effected

$\frac{p}{q}x - (\frac{p}{q} - 1)x = \frac{s}{t} - [(\frac{p}{q} - 1) \div \frac{p}{q}] \times \frac{s}{t}$. I illustrate this strategy by the following example: Candy bar A is $\frac{1}{4}$ unit long. Its length is $1\frac{3}{5}$ (or $\frac{8}{5}$) of candy bar B. What is the length of candy bar B? This problem is algebraically equivalent to $\frac{8}{5}x = \frac{1}{4}$. Subtracting $\frac{3}{5}x$ on the left hand side of the equation corresponds to a reduction of $\frac{3}{8}$ of $\frac{8}{5}$. Thus the right hand side of the equation has to be equivalently reduced by $\frac{3}{8}(\frac{1}{4})$. In other words, $\frac{8}{5}x - \frac{3}{8}(\frac{8}{5}x) = \frac{1}{4} - \frac{3}{8}(\frac{1}{4})$ or

$\frac{8}{5}x - (\frac{8}{5} - 1)x = \frac{1}{4} - [(\frac{8}{5} - 1) \div \frac{8}{5}] \times \frac{1}{4}$.

Multiplication and Division as Inverse Operations

Understanding the reverse relation between multiplication and division requires an analysis of multiplication and division situations as they occur in their most basic forms. In its most primitive form, a multiplication/ division situation involves a relation of the type $a \times c = b$, where a, b, and c may be integers, rational numbers or reals in general. Different multiplicative situations can be generated by casting this primitive equation in different forms. For instance, if a or c is unknown, we have a 'missing-factor' problem of the form $a \times x = b$ or $x \times c = b$, where x is to be determined. In other words, if either a or c is unknown, it becomes a reversibility problem, depending on the syntactic arrangement of the multiplier, multiplicand and product. In most cases, for every multiplicative situation, one can formulate two types of division problems: measurement (or quotitive) division and partitive division. The multiplier states the number of sets while the multiplicand states the size of each set and the product states the total number of elements. In a partitive division problem the whole is shared equally among a known number of sets and one is required to determine the size of each share. In a measurement/ quotitive division situation, the size of one share is known and what is required is the number of sets. Interpreting a multiplicative situation in the form 'multiplier $\times$ multiplicand = product' or 'number of groups $\times$ number per group = total number', allows one to observe that when the multiplier is unknown we have a measurement division situation and when the multiplicand is unknown we have a partitive division situation.

Vergnaud (1983) categorizes multiplicative situations in terms of three classes, namely isomorphism of measures, product of measures, and multiple proportion. Isomorphism of measures includes situations in which there is a direct proportion between two measure spaces M_1 and M_2. If we denote *a* as the multiplier, *c* the multiplicand and *b* the product ($a \times c = b$), then this category of situations can be schematically represented in Figure 4.1. The quantities *a*, *b*, and *c* are generally integers, fractions or decimals. The interpretation of multiplicative situations using such a schematic representation highlights the link between multiplication/division and the inherent proportionality.

M_1	M_2
1	c
a	x

(i)

M_1	M_2
1	c
x	b

(ii)

M_1	M_2
1	x
a	b

(iii)

Figure 4.1. (i) product unknown (ii) multiplier unknown (measurement division)
(iii) multiplicand unknown (partitive division)

Multiplicative situations like finding the area of a rectangle and a Cartesian product cannot be described in terms of isomorphism of measures and this motivated the definition of the second class of problems, namely product of measures. In this category of situations two measure spaces M_1 and M_2 are mapped onto a third measure space, M_3. For instance, length and width are mapped into area. Greer(1992) enumerates 10 classes of multiplication/division situations and distinguish between the multiplicative task and its complementary reverse problem in terms of partitive and quotitive divisions. I analyze these problems in terms of the primal/dual distinction that I made in Chapter 1 and in terms of Vergnaud's dimensional analysis. The ten classes include equal groups, equal measures, rate, measure conversion, multiplicative comparison, part/whole situations, multiplicative change, Cartesian product, rectangular area and product of measures.

It should be mentioned that Vergnaud considers the category 'product of measures' as the Cartesian composition of two measure-spaces to produce a third measure space. He includes in this category the concepts of area, volume, Cartesian product, work done as well as other physical concepts. On the other hand, it appears that Greer (1992) considers product of measures to involve two different measure spaces and as such classifies area and Cartesian products in different categories.

Equal groups

One of the earliest experiences of multiplication that children encounter is the 'equal group' situation in which there is a number of groups of objects, each having the same number of elements as illustrated by the following example.

(i) Primal problem: Paul has 4 boxes of pencils. There are 3 pencils in each box. How many pencils does Paul have altogether? (Figure 4.2 (i))

(ii) Dual problem (measurement division): Paul has 12 pencils. They are packed in boxes such that each box has 3 pencils. How many boxes of pencils does he have? (algebraically equivalent to $3x = 12$) (Figure 4.2(ii))

(iii) Dual problem (partitive division): Paul has 12 pencils. He distributes them equally in 4 boxes. How many pencils are there per box? (algebraically equivalent to $4x = 12$) (Figure 4.2(iii))

Boxes	Pencils
1	3
4	x

(i)

Boxes	Pencils
1	3
x	12

(ii)

Boxes	Pencils
1	x
4	12

(iii)

Figure 4.2. Known and unknown quantities in 'equal groups' situations

Equal measures

This class of situation is an extension of the 'equal class' situations where a measure is replicated some number of times.

(i) Primal problem: Eric has 3 pencils. Each pencil is $5\frac{1}{2}$ inches long. What is the total length of the three pencils? (Figure 4.3 (i))

(ii) Dual problem: (measurement division): The total length of his pencils is $16\frac{1}{2}$ inches. Given that each pencil is $5\frac{1}{2}$ inches long, how many pencils does he have? (Figure 4.3 (ii))

(iii) Dual problem (partitive division): The total length of his three pencils is $16\frac{1}{2}$ inches. Given that the three pencils are of the same length, what is the length of one pencil? (Figure 4.3 (iii))

Pencils	length
1	$5\frac{1}{2}$
3	x

(i)

Pencils	length
1	$5\frac{1}{2}$
x	$16\frac{1}{2}$

(ii)

Pencils	length
1	x
3	$16\frac{1}{2}$

(iii)

Figure 4.3. Known and unknown quantities in 'equal measures' situations

Rate problems

Rate situations involve speed, price problems, and mixture problems. Greer (1992) defines the general structure of this category of problems as follows:

$$x\ [\text{measure}_1] \times y[\text{measure}_2 \text{ per measure}_1] = xy\ [\text{measure}_2]. \qquad \text{(Equation 4.1)}$$

(i) Primal problem: A car travels at a speed of 60 miles per hour. How much distance does it cover in $1\frac{1}{2}$ hours? (Figure 4.4 (i))

(ii) Dual problem: (measurement division): Given that a car travels at a speed of 60 miles per hour, how much time does it take to travel a distance of 90 miles? (Figure 4.4 (ii))

(iii) Dual problem (partitive division): A car travels a distance of 90 miles in $1\frac{1}{2}$ hours. What is the average speed of the car? (Figure 4.4 (iii))

Time	Distance
1	60
$1\frac{1}{2}$	x

(i)

Time	Distance
1	60
X	90

(ii)

Time	Distance
1	x
$1\frac{1}{2}$	90

(iii)

Figure 4.4. Known and unknown quantities in rate problems

Measure conversion

Greer (1992) considers measure conversion problems as a special case of rate problems where $measure_1$ and $measure_2$ in equation (4.1) are alternative measures of the same quantity. Conversion problems may also involve situations where a particular unit is converted to its subunits or vice versa (e.g., in converting meters to centimeters).

(i) Primal problem: One kilometer is 5/8 of a mile. How long is five kilometers in miles? (Figure 4.5 (i))

(ii) Dual problem: (measurement division): Given that one kilometer is 5/8 of a mile, how long is 25/8 miles in kilometers? (Figure 4.5 (ii))

(iii) Dual problem (partitive division): Five kilometers correspond to 25/8 miles. How many miles in one kilometer? (Figure 4.5 (iii))

Kilometer	Mile
1	$\frac{5}{8}$
5	x

(i)

Kilometer	Mile
1	$\frac{5}{8}$
x	$\frac{25}{8}$

(ii)

Kilometer	Mile
1	x
5	$\frac{25}{8}$

(iii)

Figure 4.5. Known and unknown quantities in measure conversion problems

Multiplicative comparison

The multiplicative comparison of two quantities is another set of situations in the multiplicative domain. For instance, John has 3 times as many apples as Mary and Mary has 1/3 as many apples as John. In general, these situations involve the phrase '*n* times as many' where *n* can be regarded as the multiplier. In other words, multiplicative comparison problems involve a comparison of two quantities in which one quantity is described as a multiple of the other. This multiple can be an integer or a real number as the examples below illustrate. The relation between the two quantities is described in terms of how many times one is larger than the other. The number identifying this relationship is not an identifiable quantity (Carpenter et al., 1999). As pointed out by Greer, one can also view this category of problems as a many-to-one correspondence (for every x unit of quantity A there are y units of quantity B). I provide two examples of such type of multiplicative comparison problems below, the first involving fractions and the second involving decimals.

Example 1

(i) Primal problem: Tom is $1\frac{1}{2}$ times as tall as Kelly. Given that Kelly is 3 feet tall, what is Tom's height? (Figure 4.6 (i))

(ii) Dual problem: (measurement division): Kelly's height is 3 feet and Tom's height is $4\frac{1}{2}$ feet. How tall is Tom compared to Kelly? (Figure 4.6 (ii))

(iii) Dual problem (partitive division): Tom is $1\frac{1}{2}$ times as tall as Kelly. Given that Tom is $4\frac{1}{2}$ feet tall, what is the height of Kelly? (Figure 4.6 (iii))

Kelly	Tom
1	$1\frac{1}{2}$
3	x

(i)

Kelly	Tom
1	x
3	$4\frac{1}{2}$

(ii)

Kelly	Tom
1	$1\frac{1}{2}$
x	$4\frac{1}{2}$

(iii)

Figure 4.6. Known and unknown quantities in multiplicative comparison situations involving fractions

Example 2 (Greer, 1992)

(i) Primal problem: Iron is 0.88 times as heavy as copper. If a piece of copper weighs 4.2 kg, how much does a piece of iron the same size weigh? (Figure 4.7 (i))

(ii) Dual problem: (measurement division): If equally sized pieces of iron and copper weigh 3.7 kg and 4.2 kg respectively, how heavy is iron relative to copper? (Figure 4.7 (ii))

(iii) Dual problem (partitive division): Iron is 0.88 times as heavy as copper. If a piece of iron weighs 3.7 kg, how much does a piece of copper the same size weigh? (Figure 4.7 (iii))

Copper	Iron
1	0.88
4.2	x

(i)

Copper	Iron
1	x
4.2	3.7

(ii)

Copper	Iron
1	0.88
x	3.7

(iii)

Figure 4.7. Known and unknown quantities in multiplicative comparison situations involving decimals

In contrast to the other categories of multiplicative situations, in multiplicative comparison problems, the tables are reversed for partitive and measurement divisions.

Nesher (1988) analyzed multiplicative comparison problems on the basis of their textual structure. She provides the following example to illustrate that there is a one-directional scalar function which links the referent and compared quantities in this category of problem: “Dan has 5 marbles. Ruth has 4 times as many marbles as Dan. How many marbles does Ruth have?” (p. 23). The first sentence (string) points out the reference quantity and the second sentence (string) points out the one-directional scalar function which links the referent and compared quantity. Building on Nesher’s study, Harel et al. (1988) mentioned that a “division compare” problem like ‘Steve has 72 pizzas. Steve has 6 times as many pizzas as John. How many pizzas does John have?' can be interpreted as involving partitive or quotitive division depending on the sense that one attribute to the phrase 'as many as'. The statement 'Steve has 6 times as many pizzas as John has' can be interpreted as a 'unit-rate-per-statement' (i.e., for each pizza that John has, there are 6 pizzas for Steve) and this is equivalent to a measurement/quotitive division situation. This is because knowing that one set (of Steve) has 6 pizzas, the question is how many such sets can be obtained from 72 pizzas. The statement 'Steve has 6 times as many pizzas as John' can also be interpreted as a "lot-per-statement" (i.e., for one set of pizza that John has, Steve has 6 such sets). The question is if Steve has 72 pizzas, how many pizzas does John have? This interpretation is equivalent to a partitive division situation as it calls for finding the number of pizzas per set.

Harel et al. also pointed out that the nature of quantities (discrete or continuous) involved in a multiplicative comparison situation influence the interpretation of the phrase 'as many as' as either involving 'unit-rate-per-statement' or 'lot-per-statement'. Continuous quantities are more likely to invoke a 'lot-per-statement' interpretation. They also provide the following contrasting problem to highlight how changing the multiplier from 6 to 6.3 influence students' choice of division over multiplication: "Steve has 72 pizzas. Steve has 6.3 times as many pizzas as John. How many pizzas does John have?"

One of the recent studies involving pairs of multiplicative comparisons (Nesher et al., 2003) emphasized the role of the linguistic structure, mathematical model chosen by solvers, and underlying schemes involved in such situations. The researchers pointed out that the syntactic structure of the problem statement and the lexical items 'more' and 'less' influenced the participants' ability to solve multiplicative comparison problems.

Part-whole situations

The relation between the part-whole schema and reversibility in the domain of addition/subtraction has been observed by different researchers but it appears that relatively less attention has been paid to such relation in the multiplicative domain. For instance, Resnick (1992) made the following statement, while commenting on students' strategies to solve 9 – 7 = _ by changing it to 7 + _ = 9:

> Children's willingness to treat these two problems as equivalent means that, at least implicitly, they understand the additive inverse property. This property, in turn depends on an additive composition interpretation of the problem in which 9 is understood to be a whole that is decomposed into two parts, one of which is 7 (p. 387).

Analogously, in the domain of multiplication, Vergnaud (1988) made the following observation:

> Some children, because mental inversion of the relationship $\times b$ into / b is difficult, prefer to find x such that $x \times b = c$ (eventually by trial and error). This *missing factor procedure*, which is similar to *missing addend procedures* in subtraction problems, avoids the conceptual difficulty raised by inversion (p. 131).

In multiplicative contexts the part-whole relationship is more complex and diverse due to the different forms of multiplicative structures: whole number multiplication and division, fraction, ratio, proportion, percentage, and rate. The question is how do such part-whole or part-part-whole schemas operate in multiplicative contexts? Singer & Resnick (1992) explicitly investigated the part-whole schema in the domain of ratio. Considering ratios as consisting of three quantities, a whole and two parts, the authors point out that situations involving the term 'for every' calls for a part-part schema while those involving the term 'out of' calls for a part-whole schema and these terms generate particular forms of reasoning. The relationship between the part-whole schema and reversible thinking in the domain of ratio has not been explicitly explored. In contrast, the part-whole schema has received much more attention in the domain of fractions. In a part-to-whole fractional reversibility situation one is working with the part-to-whole relation between a fraction and its referent whole (i.e., given a part make the whole) instead of the whole-to-part relation (i.e., given the whole make a part). The following example illustrates this type of reversibility situation.

(i) Primal problem: Parking lot B can hold 33 cars. Parking lot A can hold $1\frac{2}{3}$ as many cars as parking lot B. How many cars can parking lot A hold? (Figure 4.8 (i))

(ii) Dual problem: (measurement division): Parking lot A can hold 55 cars while parking lot B can hold 33 cars. The number of cars that parking lot A can hold is what fraction of that of parking lot B? (Figure 4.8 (ii))

(iii) Dual problem (partitive division): Parking lot A can hold 55 cars. It can hold $1\frac{2}{3}$ as many cars as parking lot B. How many cars can parking lot B hold? (Figure 4.8 (iii))

Parking B	Parking A
1	$1\frac{2}{3}$
33	x

(i)

Parking B	Parking A
1	x
33	55

(ii)

Parking B	Parking A
1	$1\frac{2}{3}$
x	55

(iii)

Figure 4.8. Known and unknown quantities in part-whole situations

It should be highlighted that part-to-whole problems can also be looked from the perspective of a multiplicative comparison. For example, the statement ‘Parking lot A can hold $1\frac{2}{3}$ as many cars as parking lot B’ involves an element of multiplicative comparison between a known and an unknown quantity. Furthermore, part-whole reversibility situations can also be posed in terms of diagrammatic representations as shown by the following two examples.

Example 1 (discrete situation)*:* The beads shown in Figure 4.9 represents $1\frac{1}{2}$, how many of them represent one whole? This is essentially a partitive division problem as we are finding the size of the unit.

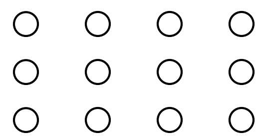


Figure 4.9. Representation of discrete quantity

Example 2 (continuous situation): The candy bar shown in Figure 4.10 represents $\frac{3}{5}$. Make a diagram to represent one whole.

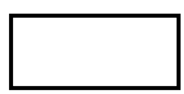


Figure 4.10. Representation of continuous quantity

Part-whole relation in the domain of percentage

Like fractions, one of the interpretations of a percent is in terms of the part-whole construct (Parker & Leinhardt, 1995). Percent problems have traditionally been categorized into three classes depending on which two of the three values in the following equation are known: Rate $\times$Base = Part (Lembke, 1991). These situations are algebraically equivalent (a) $\frac{a}{100} \times b = x\ \%$, (b) $\frac{x}{100} \times a = b$, (c) $\frac{a}{100} \times x = b$. As pointed out by Risacher (1992), the third type of percent problems "is comparable to the 'missing-addend' problem in that a known action has been performed on an unknown quantity with the result known" (p. 42). He also points out that an understanding of inverse operations or the use of an algebraic approach is necessary to solve this type of problem.

(i) Primal problem 1: Elizabeth scored 75% of the 800 marks that can be obtained in the examination. How much did she score in the examination? (75% of 800 = x) (Figure 4.11 (i))

(ii) Primal problem 2: Elizabeth scored 600 out of the 800 marks in the examination. What percentage of marks did Elizabeth score? ($\frac{600}{800} \times 100\% = x\%$) (Figure 4.11 (ii))

(iii) Dual problem 1: Elizabeth scored 600 marks in the examination and that represents 75% of the maximum marks. What is the maximum score that one can obtain in the examination? (75% of x = 600) (Figure 4.11 (iii))

(iv) Dual problem 2: How many marks should Elizabeth score out of 800 to get 75% of the marks? ($\frac{x}{800} \times 100\% = 75\%$) (Figure 4.11 (iv))

Percentage	Score
100%	800
75%	x

(i)

Percentage	Score
100%	800
x%	600

(ii)

Percentage	Score
100%	x
75%	600

(iii)

Percentage	Score
100%	800
75%	x

(iv)

Figure 4.11. Known and unknown quantities in percentage problems

Complementation and construct-the-unit problems (algebraically equivalent to $(1-a)x = b$ or $(1+a)x = b$)

A variation of part-whole problems can also be formulated by presenting the information in a complementary form. Consider the problem 'Joe had some marbles. Then he gave 1/5 of his marbles to Tom. Now Joe has 20 marbles. How many marbles did he have at the beginning?' This problem is algebraically equivalent to $(1-\frac{1}{5})x = 20$. One can also formulate such complementation problems involving percentage. Consider the following problem 'Tom thinks his collection of goldfish got chickenpox. He lost 70% of his collection of goldfish. If he has 60 survivors, how many did he have originally?' (Rayner, 1986). This situation is algebraically equivalent to $(100\% - 70\%)x = 60$. Another common example often encountered in textbook is in price reduction or price increase in consumer mathematics. For example, 'After a price reduction of 10%, a television set costs $360. What was the price before the reduction?'

Part-part-whole structure in ratios

Ratio is another multiplicative structure where reversible reasoning may be observed. I have identified three categories of ratio situations, which may involve reversible reasoning.

Type I: $(a+b)x = q_1 + q_2$

Primal problem: Joe has 7 marbles. Then his friend gives him 5 times as many marbles as he has. How many marbles does he have in total?

Dual problem: Joe had some marbles. Then his friend gave him 5 times as many marbles as he had initially. Now Joe has 42 marbles. How many marbles did Joe have initially? (algebraically equivalent $(1+5)x = 42$)

Type II: $(a-b)x = q_1 - q_2$

Primal problem: For every \$5 that Alan receives, Bill receives \$3. Given that Alan receives \$25, how much does Bill receive?

Dual problem: A sum of money was divided between Alan and Bill. For every \$5 that Alan received, Bill received \$3. Given that Alan received \$10 more than Bill, calculate how much Bill received? (algebraically equivalent to $(5-3)x = 10$, where Bill receives $3x$ dollars)

Type III: $(a-b)x = 2e = q_1 - q_2$ (where e is the amount exchanged)

Primal problem: Right now, for every 3 marbles that Richard has, John has 1 marble. Given that Richard has 54 marbles, how many of those should he give to John so that they have an equal number of marbles?

Dual problem: Right now, for every 3 marbles that Richard has, John has 1 marble. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?

Multiplicative change

This type of situation involves the multiplicative comparison of the same quantity before and after a transformation. These situations can be regarded as requiring reversibility of transformations.

(i)Primal problem: A picture is increased to 2.5 times of its original length. If the original length is 10 inches, what is its new length? (Figure 4.12 (i))

(ii) Dual problem (measurement division): The length of a picture is increased from 10 inches to 25 inches. By what factor is it lengthened? (Figure 4.12 (ii))

(iii) Dual problem: (partitive division): A picture is increased to 2.5 times of its original length. The new length of the picture is 25 inches long. What is the length of the original picture? (Figure 4.12 (iii))

Original Length	New length
1	2.5
10	x

(i)

Original length	New length
1	x
10	25

(ii)

Original length	New length
1	2.5
x	25

(iii)

Figure 4.12. Known and unknown quantities in 'multiplicative change' problems

Scaling problems

Another set of situations which fall into the category of multiplicative change is shrinking and stretching problems, algebraically equivalent to $ax = 1$, where a is the scaling factor. These situations can be regarded as a specific instance of $ax = b$, where b is 1 (i.e., $ax = 1$).

Example 1:

Primal problem: What is the area of the resulting rectangle if a unit square's width is reduced to $\frac{3}{4}$ unit and its length is increased to $1\frac{1}{3}$ unit?

Dual problem: If the width of a unit square is reduced to $\frac{3}{4}$ unit, how should the length be adjusted so that the area is the same? ($\frac{3}{4}x = 1$).

Example 2:

Primal problem: The length of a rectangle is $\frac{2}{5}$ unit long. It is increased by 250% of its present size. What is the length of the transformed rectangle?

Dual problem: The length of a rectangle is one unit long. It is reduced to 40% of its original size. By what factor should its present size be changed to restore it to its original size?

$(40\% \, x = 100\%)$

Product of measures

In this category of situations, two measure spaces M_1 and M_2 are mapped into a third measure space M_3. Vergnaud (1983) gives a range of situations that fall under this category like area, volume, Cartesian product, work, and physical concepts.

Cartesian product

This category of situations correspond to the number of ordered pairs that can be formed from two given sets. The distinction between partitive and measurement division does not arise here due to the symmetry in the situation.

(i) Primal problem: How many possible shirt-trousers combination are possible if there are 4 shirts and 3 trousers? (Figure 4.13 (i))

(ii) Dual problem 1: How many shirts are required to make 12 shirt-trousers combinations if 3 trousers are available? (Figure 4.13 (ii))

(iii) Dual problem 2: How many trousers are required to make 12 shirt-trousers combinations if 4 shirts are available? (Figure 4.13 (iii))

M_1 (shirt)	4
M_2 (Trousers)	3
M_3 (Shirt-Trousers combination)	x

(i)

M_1 (shirt)	x
M_2 (Trousers)	3
M_3 (Shirt-Trousers combination)	12

(ii)

M_1 (shirt)	3
M_2 (Trousers)	x
M_3 (Shirt-Trousers combination)	12

(iii)

Figure 4.13. Known and unknown quantities in 'Cartesian product' problems

Rectangular area

Here also the two numbers being multiplied play equivalent roles and they are not distinguishable as multiplier and multiplicand and hence there is only one type of division.

(i)Primal problem: What is the area of a rectangle 3.3 meters long by 4.2 meters wide? (Figure 4.14 (i))

(ii) Dual problem: If the area of a rectangle is 13.86 m^2 and the length is 3.3 meters, what is the size of its width? (Figure 4.14 (ii))

(iii) Dual problem: If the area of a rectangle is 13.86 m^2 and the width is 4.2 meters, what is the size of its length? (Figure 4.14 (iii))

M_1 (length)	3.3
M_2 (width)	4.2
M_3 (area)	x

(i)

M_1 (length)	3.3
M_2 (width)	x
M_3 (area)	13.86

(ii)

M_1 (length)	x
M_2 (width)	4.2
M_3 (area)	13.86

(iii)

Figure 4.14. Known and unknown quantities in area problems

Simple proportion

As pointed out by Vergnaud (1988), multiplicative situations can also be interpreted as a proportion. More generally, if the quantities $x_1, x_2, f(x_1),$ and $f(x_2)$ correspond to the measure spaces M_1 and M_2 shown in Figure 4.15, then proportional problems can be characterized by the equation $\frac{x_1}{x_2} = \frac{f(x_1)}{f(x_2)}$. In a missing-value problem (Harel & Behr, 1989) three of the quantities are stated and the aim is to find the fourth quantity.

M_1	M_2
x_1	$f(x_1)$
x_2	$f(x_2)$

Figure 4.15. Representation of simple proportion problems in terms of measure spaces

Multiple proportion

A multiple proportion problem involves three or more variables. Figure 4.16 (Vergnaud, 1983) shows that if the quantity x_1 changes to x_1' and x_2 changes to x_2', then the quantity $f(x_1, x_2)$ changes proportionally to $f(x_1', x_2')$.

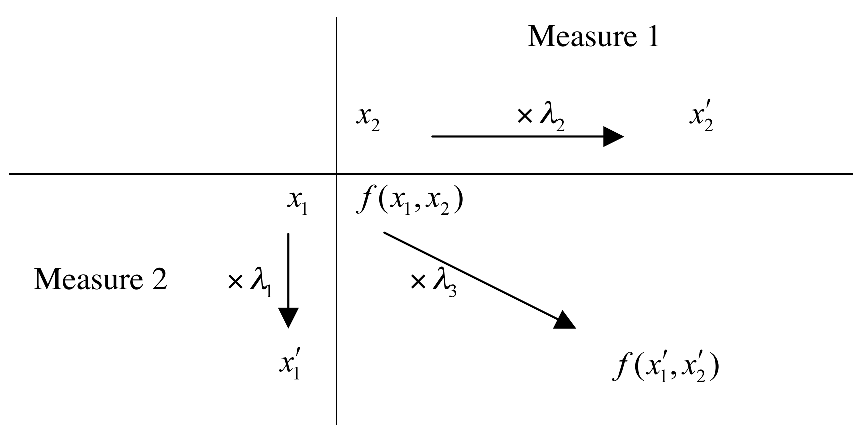


Figure 4.16. Representation of multiple proportion problems in terms of measure spaces

A range of problems can be formulated by specifying different parameters. I illustrate one situation in the following example.

Primal problem: A group of 50 persons goes to a holiday camp for 28 days. They need to buy enough sugar. They read in a book that the average consumption of sugar is 3.5 kg per week for 10 persons. How much sugar do they need? (Vergnaud, 1988) (Figure 4.17)

Dual problem: A group of 50 persons goes to a holiday camp. They read in a book that the average consumption of sugar is 3.5 kg per week for 10 persons. They bought 70 kg of sugar. For how many days can they survive with the 70 kg of sugar?

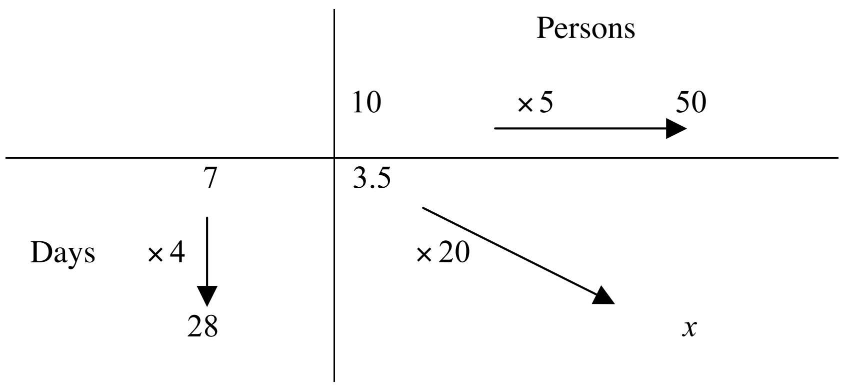


Figure 4.17. An example of a multiple proportion problem

Inverse proportion problems

In inverse proportion problems, there is an inverse relationship between the two measure spaces generating the proportion. Let the measure spaces be M_1, and M_2 with corresponding quantities $x_1, x_2, f(x_1)$, and $f(x_2)$ as shown in Figure 4.18. Here, $x_1 f(x_1) = x_2 f(x_2)$.

M_1	M_2
x_1	$f(x_1)$
x_2	$f(x_2)$

Figure 4.18. Representation of inverse proportion problems in terms of measure spaces

Example 1: Six men take 4 days to paint a house. (i) How many days will be required to paint the house if eight men are available? (Figure 4.19 (i)), (ii) How many men are required if the house is to be painted in 3 days? (Figure 4.19 (ii))

No. of men	No. of days
6	4
8	x

(i)

No. of men	No. of days
6	4
x	3

(ii)

Figure 4.19. An example of an inverse proportion problem

Example 2: The balance beam problem

Consider the inverse variation

$$wd = k, \qquad \text{(Equation 4.2)}$$

where w and d denote weight and distance respectively and k is a constant ($k \neq 0$).

Let (w_1, d_1) and (w_2, d_2) be solutions to equation (4.2). Then

$$w_1 d_1 = k \text{ and } w_2 d_2 = k.$$

Thus, $w_1 d_1 = w_2 d_2$ or $w_1/w_2 = d_2/d_1$ at equilibrium. The equations $w_1 = k/d_1$ and $w_2 = k/d_2$ show that weight and distance are inversely proportional on either side of the fulcrum. The equation $w_1 d_1 = w_2 d_2$ can be interpreted from at least four different ways or Theorems-in-action, all of which are mathematically equivalent but not necessarily conceptually (or cognitively) equivalent:

$$w_1 d_1 = w_2 d_2 \qquad \text{(Equation 4.3)}$$

$$w_1/w_2 = d_2/d_1 \text{ or } w_2/w_1 = d_1/d_2 \qquad \text{(Equation 4.4)}$$

$$w_1 = (d_2/d_1)w_2 \text{ or } w_2 = (d_1/d_2)w_1 \qquad \text{(Equation 4.5)}$$

$$d_1 = (w_2/w_1)d_2 \text{ or } d_2 = (w_1/w_2)d_1 \qquad \text{(Equation 4.6)}$$

In this problem, two or three out of the four parameters are stated and one is required to determine the fourth parameter to maintain an equilibrium. I studied how students reasoned reversibly to determine the third or fourth parameter in an earlier study (Ramful & Olive, 2008).

Example 3: The gear-wheel problem

The gear-wheel problem consists of coordinating the number of teeth and the number of turns as two gears with different number of teeth turn synchronously. The four variables (S, L, n_s, n_L) in the problem are in multiplicative relation; S and L denote the number of teeth that the small and large gear have respectively; n_s and n_L denote the number of turns that the small and large gear make respectively. Further, the gear-wheel problem involves an inverse proportion relation between the number of teeth and the number of turns as can be inferred from $n_s S = n_L L$ or $n_s / n_L = L / S$ or $n_s : n_L = L : S$. Compared to the balance beam problem that also involves an inverse relation between weight and distance ($w_1 d_1 = w_2 d_2$), this task has the additional feature of imposing the condition that the number of teeth in the gears should be integers. In other words, working with teeth involves whole numbers while working with number of turns gives the possibility to work with fractions. Problems with different levels of complexity can be generated by varying the four parameters of the situation.

Reversibility in qualitative situations

During this analysis, I could also observe multiplicative situations that have a reversibility structure but which do not necessarily involve computations. The following problems illustrate such qualitative situations.

Example 1: In an apartment, ½ of the number of men is married to ¼ of the number of women. Are there more men or women in the apartment?

Example 2: 1/6 of egg carton A is to form 2/3 of another egg carton B. Which egg carton can hold more eggs?

Example 3: The length of a table is being measured using two strips of paper (strip A and strip B) of different length. It requires 8 units of strip A to measure the length of the table. On the other hand, if we use strip B, we require 6 units. Which of the two strips is longer?

Example 4: Four US dollars can be exchanged for three British pounds. Which currency has a higher value?

Example 5: The ratio of students to teachers in a school is 30: 1. The school headmaster decided to reduce this ratio to 25:1. Does s/he have to increase or decrease the number of teachers?

Example 6: Ron takes 5 days to paint a house. If Ron and Kelly work together, they can paint the house in 3 days. Will Kelly take more time, less time, or the same amount of time as Ron if she works alone?

Example 7: One kilometer is 5/8 of a mile. The distance between France and Mauritius is approximately 5750 miles. Is the distance in kilometers larger or smaller than the value 5750?

Summary: Looking at Reversibility Situations across Multiplicative Domains

Building on Vergnaud's (1983;1988) and Greer's (1992) categorization of multiplicative situations, I have analyzed different forms of reversibility situations. This theoretical endeavor has been achieved by analyzing a rather extensive set of problems in the multiplicative domain from research reports (e.g., Krutetskii, 1976), middle school textbooks, and specific textbooks on rational numbers and proportion (e.g., Lamon, 1999), together with available NAEP reports. The two main questions that have been guiding this theoretical analysis are: What are the different types of reversibility situations in multiplicative contexts and what are their characteristics? What are the structural relationships between these situations?

Reversibility situations in multiplicative contexts have their roots in the equation $a \times c = b$ which characterizes multiplicative and division problems in their most primitive form. Different reversibility situations can be generated by casting this primitive equation in different forms. For instance, if *a* or *c* is unknown, we have a 'missing-factor' problem of the form $x \times c = b$ or $a \times x = b$, where *x* is to be determined. In other words, if either *a* or *c* is unknown, it becomes a reversibility situation, depending on the syntactic arrangement of the multiplier, multiplicand and product. However, this relation can take a variety of forms, depending on the complexity of the multiplicative situation. The first class of multiplicative situations that I discussed was the isomorphism of measures problem (Vergnaud, 1983) that involve equal groups, equal measures, rate problems, measure conversion, multiplicative comparison, part-whole situations and multiplicative change and all these classes can be represented by the theorem-in-action $ax = b$. Another category of problem that I could identify are complementation situations, algebraically equivalent to $(1-a)x = b$. Still, another variation of the equation $ax = b$ can be observed in the multiplicative change situations that are algebraically equivalent to $ax = 1$. Percentage problems give rise to situations of the form $\frac{x}{100}a = b$, $\frac{a}{100}x = b$, and $(100 \pm a)x = b$. Ratio situations are algebraically equivalent to $(a+b)x = q_1 + q_2$, $(a-b)x = q_1 - q_2$ and $(a-b)x = 2e$. Furthermore, a proportion situation such as $a:b = c:x$ involves the equation $\frac{a}{b} = \frac{c}{x}$. The examples that I have looked at are primarily from the middle school curriculum. By casting the different types of reversibility situations in algebraic form, the aim is to show how these multiplicative situations are special cases of the linear equation $ax = b$. Such a categorization is not meant to be exhaustive.

I recognize that the constitution of such situations as either requiring reversibility or not from a cognitive perspective is a function of how the problem solver conceptualizes such situations. The present analysis is meant to show the different ways in which multiplicative relationships arise and how they can possibly create a situation where one may be required to reason reversibly. Besides serving a theoretical purpose, the current analysis may be useful to systematically examine how students reason reversibly across different situations. In this study, I specifically investigate three types of situations from this analysis in the next three chapters. I now turn to present the data, analysis, and discussion in Chapters 5, 6, and 7 in the three categories of tasks.

CHAPTER 5

SET 1: MULTIPLICATIVE COMPARISON OF TWO QUANTITIES IN A MEASUREMENT DIVISION SITUATION

As the interview data will reveal, there are at least four ways one may proceed to compare two quantities multiplicatively, which in Vergnaud's (1988) framework represent different 'theorems-in-action'. For instance, one may perform such a comparison by dividing one quantity by the other. Another approach is to measure one quantity in terms of the other by finding how many units of one quantity fits the other using the norming process (as described in the theoretical framework). Still another approach is to compare the two quantities by referring to the theorem-in-action $ax = b$: Comparing a larger quantity in terms of a smaller quantity means determining by what one should multiply the smaller quantity to produce the larger quantity. Similarly, comparing the smaller quantity in terms of the larger quantity means determining by what one should multiply the larger quantity to produce the smaller quantity. I use the example of the comparison of 5 red and 2 blue counters to show when reversible reasoning is seen to occur in such multiplicative comparison tasks. When comparing the larger number of counters in terms of the smaller number of counters, one may ask the question 'which number do I multiply 2 by to give me 5?'. This interpretation requires reversible reasoning because we have a result (5) and we want to know what multiplier produced the result starting from 2, a theorem-in-action algebraically equivalent to $2 \times x = 5$ (where x denotes an unknown number, i.e., the multiplier). The different ways of comparing two quantities multiplicatively described above are mathematically equivalent, but each of these are operationally different.

Overview and justification of tasks

Table 5.1 shows the eight situations that were formulated to analyze at a fine-grained level of detail how students multiplicatively compare one quantity in units of the other and to identify the ways in which they reasoned reversibly. In discrete situations 1 to 5, the students were asked to compare a set of red (R) and blue (B) Unifix cubes. They could join or separate the unifix cubes if they wanted to. In continuous situations 6 to 8, they were asked to compare the length of two candy bars (Tom's and Kelly's candy bar) in the JavaBars microworld (Biddlecomb & Olive, 2000). This microworld gives the flexibility to represent and partition continuous quantities. Different problem situations were created by varying the ratio of the two quantities. Although I investigated a range of situations, I report only eight of them to highlight the cognitive resources that the participants appear to have deployed and the conflicts that they encountered.

Table 5.1. Structure of multiplicative comparison tasks

Situation	Structure	Algebraic equivalence
Situation 1	$R:B=6:3$	$R=2B$ and $B=\frac{1}{2}R$
Situation 2	$R:B=6:4$	$R=1\frac{1}{2}B$ and $B=\frac{2}{3}R$
Situation 3	$R:B=6:10$	$R=\frac{3}{5}B$ and $B=1\frac{2}{3}R$
Situation 4	$R:B=7:5$	$R=1\frac{2}{5}B$ and $B=\frac{5}{7}R$
Situation 5	$R:B=5:4$	$R=1\frac{1}{4}B$ and $B=\frac{4}{5}R$
Situation 6	$T=\frac{3}{4}$ and $K=\frac{2}{3}T$ or $T:K=3:2$	$T=1\frac{1}{2}K$ and $K=\frac{2}{3}T$
Situation 7	$T=1\frac{3}{4}$ and $K=\frac{6}{7}T$ or $T:K=7:6$	$T=1\frac{1}{6}K$ and $K=\frac{6}{7}T$
Situation 8	$T=1\frac{2}{3}$ and $K=\frac{3}{10}T$ or $T:K=10:3$	$T=3\frac{1}{3}K$ and $K=\frac{3}{10}T$

Situation 1 was meant to give the participants an entrée into the multiplicative comparison problems. It was meant to serve as a reference; I often referred to this problem whenever I observed any form of constraints that the students encountered, especially when using the term ‘as many as’ (e.g., in situation 3). In situations 2 to 8, the smaller quantity does not divide the larger quantity, and this was meant to allow me to observe how they measure the larger quantity in terms of the smaller quantity. In situations 2 and 3, I chose the ratio of red to blue counters as 6:4 and 6:10 rather than 3:2 and 3:5 so as to create opportunities to “observe” the application of the unitizing operation because these situations require the construction of a composite unit. In contrast, in situations 4 and 5, the components of the ratio are relatively prime, and this requires using the smaller quantity as the composite unit.

The continuous situations (6, 7 and 8) were originally designed to be the primal problems in Set 2 (Chapter 6) but because they involve measurement division and further illustrate the ‘faulty-remainder’ theorem-in-action (see next section), I included them in this problem set for comparison with the discrete situations. Situation 6 is similar to situation 2 but in a continuous context. The selection of situation 7 was motivated by the constraints observed in situation 4. Furthermore, situation 8 was specifically selected because the difference between Tom’s and Kelly’s share is larger than the latter (i.e., the smaller quantity). Thus, measuring the difference ($10 - 3 = 7$) in terms of Kelly’s share (referent unit) requires more than one iteration of the referent unit. The measurement of the difference between the two quantities occurs when the comparatives ‘less’ and ‘more’ are used. In all the previous situations (1 to 7), the difference between the compared and the referent quantity is less than the referent quantity. The data for situations 1 to 5 were collected in the first phase of the study, and the resulting analysis highlighted the necessity to explore students’ responses on absolute and relative comparisons. I explored how students used the comparative terms ‘more’ and ‘less’ from a relative perspective in situations 6 to 8. The dates and order in which the tasks were posed are shown in Table 5.2.

Table 5.2. Chronological order of problems in Set 1

Date	Grade 6	Grade 7	Grade 8
5/6/08	Situation 1	Situation 1	Situation 1
	Situation 2	Situation 2	Situation 2
	Situation 3	Situation 3	Situation 3
5/7/08	Situation 4	Situation 5	Situation 4
		Situation 4	
12/2/08	Situation 6	Situation 6	NA
	Situation 7		
12/4/08	Situation 6	Situation 7	NA
	Situation 7	Situation 8	
	Situation 8		

The analysis of the data is organized as follows. I first illustrate how Aileen multiplicatively compared two quantities using the theorem-in-action $ax = b$ in episodes 5.1, 5.2, and 5.3. Next, I present the data for the ‘faulty-remainder’ theorem-in-action in episodes 5.4 to 5.7. Then I show how the students interpreted a multiplicative comparison situation from an absolute and relative perspective. Finally, I illustrate the 4 strategies that the students used in the multiplicative comparison of two quantities to show the resources they deployed in such situations.

Aileen's Theorem-in-Action

In this study, I was particularly interested in the ways in which students articulate the theorem-in-action $ax = b$, involving an unknown quantity. Only one of the participants, Aileen, interpreted the multiplicative comparison of two quantities using such a theorem-in-action. In most instances, the other participants used the norming process as will be shown in the data analysis. In this section I present a first set of data to show how Aileen used the relationship between the multiplicand, multiplier, and product to reason reversibly. In various problems, she compared the larger number of counters (say *b*) in terms of the smaller number of counters (say *a*) by finding which number she had to multiply *a* by to produce *b*. She used a systematic guess-and-check procedure to determine the multiplier/multiplicand *x*. For example, to express the 6 red counters (i.e., $b = 6$) in terms of the 4 blue counters (i.e., $a = 4$) in situation 2, she deduced that *x* should be 1.5 as $x = 1$ and $x = 2$ did not satisfy her implicit equation $4 \times x = 6$. Similarly, in situation 3, she compared 10 blue counters in terms of 6 red counters using the implicit rule $6 \times x = 10$. Her guess-and-check strategy to determine the multiplier can also be observed in situations 4 and 5. The element of reversible reasoning in Aileen's strategy is that she posits an unknown multiplier between the given multiplicand and product. Though Aileen could set the structural relationship between the known quantities and unknown multiplier in these situations, she did not deploy the arithmetic operation of division to solve the problem. In other words, these situations did not cue the multiplication/division invariant (i.e., the inverse relationship between multiplication and division) for Aileen as illustrated in episodes 5.1-5.3.

Episode 5.1: ***Aileen*** **[Situation 2,** $R : B = 6 : 4$ **]**

Data. In this situation I associated the name Paul to the red counters and Mary to the blue counters.

L1 I: You said that Mary has two third the number of counters as Paul. But if I want to compare Paul to Mary, Paul has how many times?

L2 B: One, one third. (B stands for Brian, Aileen's partner in the interview)

L3 A: One point, one point five, [Is that right?]. He has 4, she has 4 and he has 6, so it will be 1.5. (Aileen multiplied 1.5 by 4 to check if she has 6.)

L4 B: Yeah.

L5 A: OK, so, he would have 1.5 times as many as she has.

L6 I: How do you get this 1.5, and I can see you have multiplied by 4, right, what does that mean?

L7 A: Uh, that's 1.5, because like if you see, if you did 4 times one, it is still going to give you 4, so you know it cannot be [it]. If you do 4 times 2, it gives you 8, which is too many. So it has to be somewhere between one and two, and if you try 1.5, it gives you 6, which is how many he has.

Analysis. Aileen used a systematic guess-and-check procedure to determine which number when multiplied by 4 gives 6. Such an interpretation of the multiplicative comparison of 4 blue and 6 red counters is characteristic of reversible reasoning. She situated her answer between 1 and 2 and multiplied 1.5 by 4 to check if she got 6 (L7). This procedure of finding x in $4 \times x = 6$, where instead of doing division (i.e., 6/4) one uses multiplication has been termed the "missing factor" procedure by Vergnaud (1983). This situation did not cue the multiplication/division invariant for Aileen and as such she had to find the multiplier by guess-and-check using multiplication.

Episode 5.2: ***Aileen*** **[Situation 3,** $R : B = 6 : 10$ **]**

Data. As in Episode 5.1, I associated the name Paul to the red counters and Mary to the blue counters.

L1 I: And if I want to compare the other way, like Mary has how many times as compared to Paul?

⋮

L2 A: I think that he, he, [one second], she has one and two third more than he does.

L3 I: How do you get one and two third?

L4 A: Uh, I, pretty much started at 1.75 times 6 to see if it got me anywhere close to that (inaudible) of 10 and that gave me 10 and a half.

L5 I: Where is the 10 and a half, can you show me on your paper? (see Figure 5.1)

L6 A: Right here, it is 10 and a half.

L7 I: Yes.

L8 A: And, so that is more than 10. So [I started] going down by 5 hundredths, each time. So, the second time I multiplied this 1.70 and that gave me, uh, times 6, wait, gave me 10.2. And then I went down 5 more hundredths which is 1.65 times 6 and it gave me 9 and 9 tenths. So, I knew that I was a little bit too, uh, far down. And I went up, and the closest I could get, was right, is right at one and two third, which is 1.67. And that gave me 10.02 (see Figure 5.1).

Analysis. Aileen used the same systematic guess-and-check procedure that she had been using in situation 2. Her strategy was to look for a number which when multiplied by 6 gives 10, a theorem-in-action equivalent to $6x = 10$. She proceeded in a systematic way by going down by "5 hundredths" (L8). She adjusted her answer from 1.75 to 1.67 as shown in Figure 5.1.

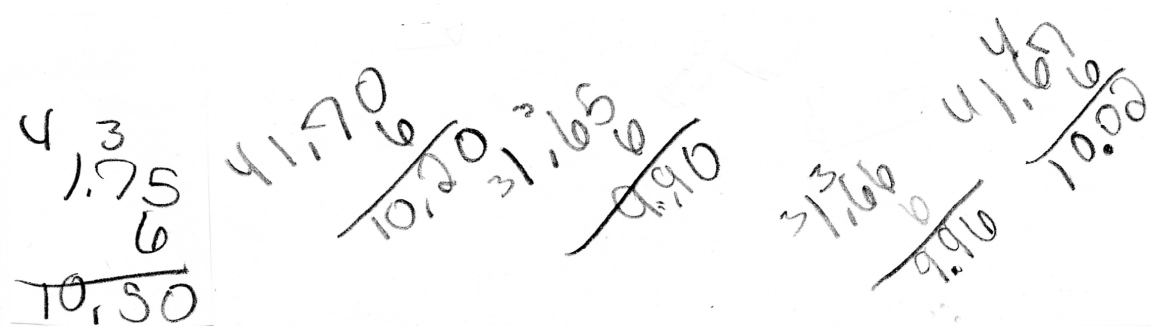


Figure 5.1. Aileen's successive approximations

Episode 5.3: ***Aileen*** **[Situation 4,** $R:B=7:5$ **]**

Data.

L1 I: Suppose you have 7 red and you have 5 blue. Right. Now compare the red and the blue.

⋮

L2 A: I was thinking one and a half times bigger, but it's not.

L3 I: How many times it's bigger?

L4 B: $3\frac{1}{2}$.

L5 A: It's not, it's not $3\frac{1}{2}$.

L6 I: How many times is the red counters larger than the blue counters, I mean in terms of number?

L7 A: That's $1\frac{1}{2}$ (pointing to one blue and '$1\frac{1}{2}$' red counters, then another set of one blue and '$1\frac{1}{2}$' red counters, until she exhausted 4 out of the 5 counters). It's not $1\frac{1}{2}$. It can't be $1\frac{1}{2}$, it can't be $1\frac{1}{4}$.

L8 B: (inaudible)

L9 A: It is between one and a fourth and one and a half.

L10 B: It would be a decimal.

L11 I: What did you say?

L12 B: It would be a decimal.

L13 I: How much of a decimal would that be?

L14 A: That's the question.

L15 A: 1.3, is it one and a third?

L16 B: Uh, (inaudible)

L17 A: 1.33. I think it is one and a third (pointing to 1 blue and '$1\frac{1}{3}$' red counters). I think it is one a third.

⋮

L18 I: And how did you get one and one third?

L19 A: Uh, I know that it has to be between one and one fourth and one and a half. One third just kind of popped in my head and (inaudible).

Analysis. Aileen used her fingers to coordinate the blue and red counters (L7) and this allowed her to deduce that the number of red counters cannot be $1\frac{1}{2}$ or $1\frac{1}{4}$ of that of the blue counters, possibly because she could observe that $1\frac{1}{2}\times 5 > 7$ and $1\frac{1}{4}\times 5 < 7$. Thus, she chose the fraction

$\frac{1}{3}$, which lies between $\frac{1}{4}$ and $\frac{1}{2}$. Aileen's response shows that she set the multiplicative relationship between the two quantities using the theorem-in-action $5 \times x = 7$ as she did in previous situations. She used a guess-and-check procedure to obtain $1\frac{1}{3}$, rather than $1\frac{2}{5}$ as the solution.

Faulty-Remainder Theorem-in-Action

One unintended outcome of this study is the conflict that the participants experienced in coordinating the comparative "less than" versus "more than" as well as the comparative "times as many" versus "a fraction of." A multiplicative comparison of two quantities in units of the other can be made in two ways, either in terms of comparatives "more or less" or "times as many or a fraction of." For instance, in comparing 6 red counters to 4 blue counters multiplicatively, the following four comparatives are involved: "a third less", "a half more", "$1\frac{1}{2}$ times", and "$\frac{2}{3}$ of." The number of blue counters is one *third less* than the number of red counters while the number of red counters is one *half more* than that of the blue counters. Similarly, the number of red counters is $1\frac{1}{2}$ *times* the number of blue counters while the number of blue counters is $\frac{2}{3}$ *of* the number of red counters as diagrammatically illustrated in Figure 5.2 (modified from Parker & Leinhardt, 1995).

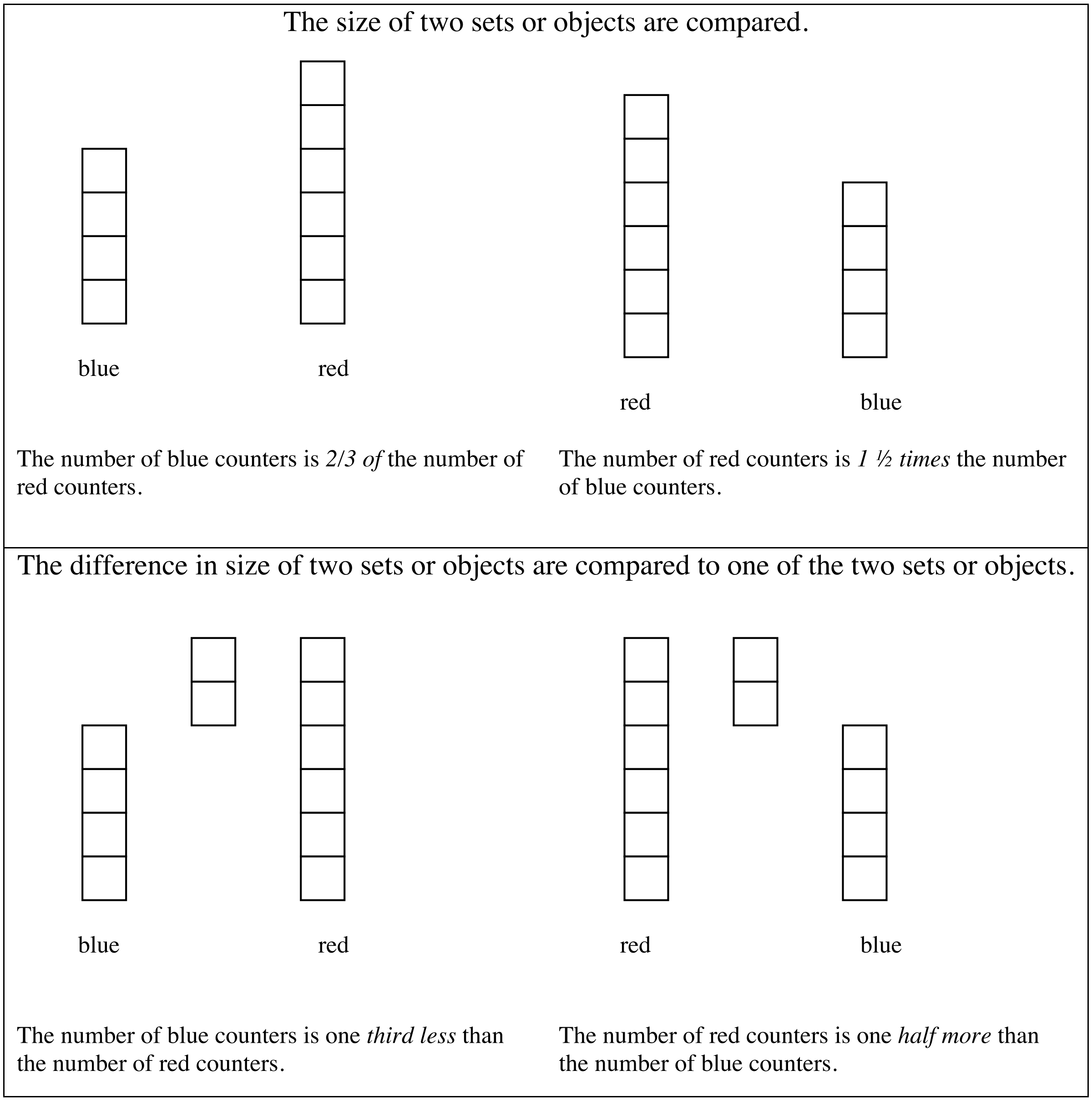


Figure 5.2. Multiplicative comparison of two quantities

As highlighted earlier, a multiplicative comparison may be regarded as the solution to the equation $a \times x = b$, where *a* and *b* are the two numbers (or quantities) to be compared; when $b < a$, *x* is a proper fraction and when $b > a$, *x* is an improper fraction or a mixed number. For the case $b > a$, one may determine *x* by trial and error (as Aileen did) or one may divide *b* by *a*.

One may also express b as $b = qa + r$, where q is the number of times a divides b and r is the remainder. For example, if we are comparing 7 red counters and 2 blue counters, (i.e., $a = 2$ and $b = 7$), then we can write $b = qa + r$ as $7 = 3(2) + 1 = 3(2) + \frac{1}{2}(2)$ and thus the number of red counters is $3\frac{1}{2}$ times the number of blue counters. However, for some students the remainder is viewed in relation to the quantity b (i.e., in terms of the whole of which it is part). In the present example, with this flawed conception, one will interpret the number of red counters as $3\frac{1}{7}$ times the number of blue counters. I have termed this flawed interpretation of the remainder as the 'faulty remainder' theorem-in-action. I provide 4 episodes (5.4-5.7) to illustrate its occurrence in the multiplicative comparison of a larger quantity in terms of a smaller quantity.

Episode 5.4: *Ted and Cole* [Situation 4, $R : B = 7 : 5$]

Data.

L1 I: Now let us take a different number of counters. Take 7 red counters and take 5 blue counters. Right. Now can you compare the red to the blue and then the blue to the red as we have been doing?

L2 T: The blue is 5/7 times the red and the red is ... Uh..

L3 C: One and two ...

L4 T: One and two seventh times of the blue.

L5 I: Can you say that again?

L6 T: The blue is 5/7 of the red and the red is 2/7 times the blue.

L7 I: The red is 2/7 times?

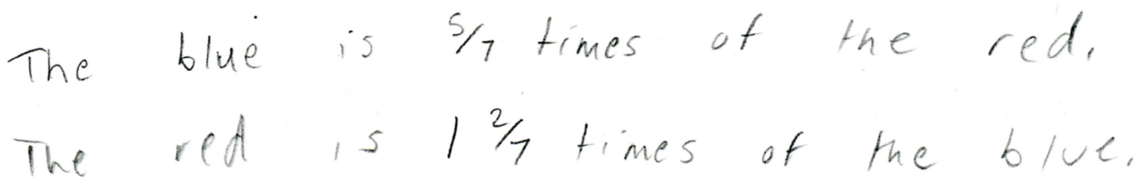
L8 C: One and 2/7 times.

L9 T: Yeah.

L10 I: Can you, on the page that you have below, can you write this sentence for me? Both of you just write what you have just said.

Ted's and Cole's answer are shown in Figure 5.3.

Ted's answer:



Cole's answer:

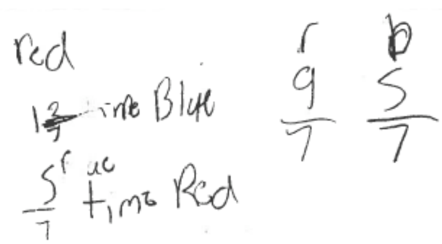


Figure 5.3. Ted's and Cole's comparison of 7 red and 5 blue counters

L11 I: How did you get $1\frac{2}{7}$, from the counters can you show me, on the counters how did you get $1\frac{2}{7}$?

L12 C: Because if you take off this (meaning 2 red counters), it will be one and 2/7 counters, so it will be 2/7 (holding the two red counters)

L13 I: So you are saying that these 2 counters that you removed, can you show this again? So show me the 2/7 that you are explaining.

L14 C: All right, so this is one because we took off two counters and then you would add these two counters to this and that will be 7 and that will be 2, so it will be $1\frac{2}{7}$.

Analysis: Both students readily deduced that the number of blue counters is 5/7 of the red counters. However, when comparing the larger number of counters (7 red) in terms of the smaller number of counters (5 blue), they gave the answer $1\frac{2}{7}$ rather than $1\frac{2}{5}$. After equating the 5 red and 5 blue counters, the students interpreted the two extra red counters in terms of the 7 red counters of which they are a part, rather than the 5 blue counters. It appears that this step (choosing the right referent to quantify the remainder) requires a significant conceptual leap. In this situation, it seems that the visual representation of the two sets of counters (L12) may have prompted them to intuitively give the answer $1\frac{2}{7}$ rather than $1\frac{2}{5}$. They did not justify their answer by multiplying $1\frac{2}{7}$ by 5, for instance. However, in situation 2 ($R:B=6:4$), they were able to measure the larger quantity in terms of the smaller quantity by creating a second unit of the smaller quantity as illustrated by the data below.

L15 T: There will be one times more because if you add one more (a composite unit of two) to the red block it will be a times bigger than it is.

L16 I: So the red counters are ...

L17 T: One times a half of it. Would it?

L18 C: It would be ...

L19 T: Because this (pointing to the extra 2 red counters) is one half of this (pointing to 4 blue counters). So if this (pointing to the 4 blue counters) is equal to two, then one half, one half times of the one (pointing to the 4 blue counters) would be this (pointing to 2 extra red counters).

⋮

L20 C: Because one, one would be these two (pointing to the 4 red and 4 blue counters in parallel) and half (pointing to the two red counters) will be this one.

L21 I: Is the number of red counters greater than one times the number of blue counters?

L22 T and C: //Yes//

L23 C: It could be one and a half times as much, more than the, uh, blue counters because if you took two of them off, that would be even and you add two more red counters it would be a whole, another whole, so that would be half.

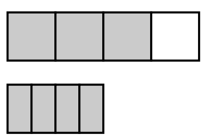
Ted's strategy in this situation was to unitize in composite units of 2. His statement in L15 "if you add one more to the red block it will be a times bigger than it is" shows that he added a composite unit of 2 elements to the 6 red counters (interpreted as 4 and 2 counters) to construct another whole. Similarly, Cole added a composite unit of 2 elements to norm the two extra red counters in terms of the referent unit (4 blue counters).

The question that arises is why Ted and Cole correctly measured the 6 red counters in terms of the 4 blue counters in situation 2 while here (7 red and 5 blue counters) they did not do so. A close look at their response to situation 2 gives a possible clue to the incorrect choice of referent (i.e., they chose 7 as the referent instead of 5) in the current problem. In situation 2, the difference between the counters (i.e., $6-4=2$) divides the smaller number of counters (i.e., $\frac{2}{4}=\frac{1}{2}$) and one can verify that $\frac{1}{2}$ of 4 is 2 and $4+2=6$ (i.e., the smaller number of counters added to the difference between the counters is equal to the larger number of counters). In other words, when divisibility relationships are available one can plug back the solution in the situation to verify that it satisfies the conditions of the problem. In contrast, here the difference between the counters ($7-5=2$) does not divide 5 or even 7, and this prevented them from plugging-back their solution to check the soundness of their answer.

Episode 5.5: ***Aileen and Brian*** **[Situation 6,** $T=\frac{3}{4}$ **and** $K=\frac{2}{3}T$ **or** $T:K=3:2$**]**

Data.

L1 I: Tom has three quarter of a candy bar and Kelly's candy bar is two third of that of Tom. (They made a bar, divided it into 4 and considered the first 3 partitions as Tom's share. Then they pulled out one part but inadvertently divided it into 2 and repeated it to construct Kelly's share, Figure 5.4 (b)).



(a) Tom's share (3 partitions)

(b) Kelly's share

Figure 5.4. Representation of Tom's and Kelly's share on JavaBars ($T:K=3:2$)

L2 I: So Tom's share is how many times as large as Kelly's share?

L3 A: One and one third.

L4 B: One and one third?

L5 A: Because she has two thirds.

L6 B: He has three thirds. Wouldn't it be one and a half?

L7 A: Hum. One and one third.

L8 B: Because two (pointing to Kelly's share) and that's two (pointing to two partitions of Tom's share), so it would be half of this (pointing to Kelly's share) would be added on to that, so it's one and a half.

L9 A: It would be times one and one third because this is two thirds (pointing to Kelly's share). That's (pointing to Tom's part) a three thirds. So two thirds times one, so two thirds just (inaudible).

L10 B: But this (pointing to Kelly's part) times one and one half would make the third (pointing to Tom's part). So it's two or two fourth. Look at it like this, don't see these dividing (referring to the subdivisions in Kelly's share). It's just two boxes and two boxes (pointing to Tom's share). So for this (Tom's share) it's three boxes. So you would divide, uh, times this (inaudible)

L11 A: Or you would time the half (inaudible). And there is one and one half.

L12 I: What will it be?

L13 A: So, Tom's is one and one half times larger than Kelly's.

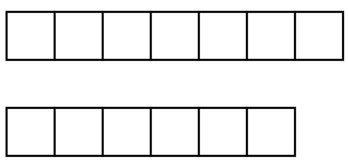
Analysis. The question that this episode raises is why Aileen chose $1\frac{1}{3}$ rather than $1\frac{1}{2}$ when comparing Tom's share to Kelly's share. She deduced that Tom's share is $1\frac{1}{3}$ times as large as Kelly's share by interpreting the one additional share that Tom has as being $\frac{1}{3}$ because each partition in Tom's share in Figure 5.4 is of size $\frac{1}{3}$ when $\frac{3}{4}$ is taken as a whole/unit. Further, the problem statement (L1) involves thirds and this may have been a pointer leading her to focus on thirds. This temporary conflict can also be explained in terms of coordinating 3 levels of units. It seems that Aileen did not interpret the two partitions that Kelly has as a unit in its own right within the unit of 3 partitions that Tom has. Instead she focused on the additive difference between Tom's and Kelly's share. In contrast, Brian measured the extra share that Tom has as being $\frac{1}{2}$ of Kelly's share in L8 by explicitly considering the latter as the referent quantity. He could view the 3 units that Tom has as consisting of 2 different units: a 2-unit within a 3-unit. He also multiplied Kelly's share by $1\frac{1}{2}$ to produce Tom's share in L10 to justify his solution. This justification prompted Aileen to observe the multiplicative relationship between the two quantities being compared. The data also show that the comparative terms 'larger' and 'as large as' tend to be used interchangeably as can be inferred from Aileen's statement in L13.

Episode 5.6: *Ted and Cole* [Situation 7, $T = 1\frac{3}{4}$ and $K = \frac{6}{7}T$ or T: K = 7:6]

Data.

L1 I: Tom's candy bar is one three quarter units long. And Kelly's candy bar is six seventh of that of Tom.

They made the representation shown in Figure 5.5.



(a) Tom's share

(b) Kelly's share

Figure 5.5. Representation of Tom's and Kelly's share on JavaBars ($T : K = 7 : 6$)

L2 I: Tom's share is how many times as large as Kelly's share?

L3 C: As large as Kelly's, it would be one and one ...

L4 T: Seventh

L5 C: Seventh.

L6 T: Inaudible.

L7 C: Yeah it will be one and one seventh bigger than Kelly's. No. It will be one seventh bigger than Kelly's.

L8 I: It will be one seventh bigger. How many times as large as? You said one and one seventh?

L9 C: I think.

L10 I: What about Ted?

L11 C: It might just be a seventh because there would be six (pointing to Kelly's share), that would be seven (pointing to Tom's share).

L12 I: There is one seventh more, this is what you are saying?

L13 C: Yes.

L14 I: But, if I want to compare it like how many times, in terms of how many times as large as?

L15 C: One and one seventh.

L16 T: No, uh, what I think it's one and one sixth because if we make Kelly's a whole, then Tom's got a whole and that's six which will be like one and one sixth times. Right. Right. Because we make Kelly's a whole and then Tom's get a whole and (inaudible).

L17 C: I still say one and one seventh.

L18 T: I still say one and one sixth. OK.

L19 C: Look, if it is one and one seventh, there is 7 boxes, that six boxes.

L20 T: But then Kelly's got six. Kelly's got six.

L21 C: Six, so it will be one (pointing to the diagram) and then you would have one seventh.

L22 T: Sixth. One sixth. We got six of this (pointing to the Kelly's share). We have six of this and Tom's got a full (inaudible) of this it would be one time as much as Kelly's with this. The last piece over here (the extra part) will be like a sixth of Kelly's, too. That would be like (inaudible) Kelly's got 6 pieces you know so it will be one and one sixth times as large.

L23 I: What about Cole?

L24 C: One and one sixth.

L25 I: You have changed your idea? Why?

L26 C: Ted persuaded me to.

L27 I: Why?

L28 C: Because he said that there is 7 up here and you can compare it to Kelly's then this (apparently referring to the extra part) would be a sixth instead of seventh. So it would be one and one sixth.

L29 I: Why one and one seventh is not good?

L30 C: I still think it is good but I don't…

Analysis. Situation 7 gives further evidence of the ongoing conflict in measuring the larger quantity in terms of the smaller quantity. Both of them initially interpreted the extra share that Tom had as being one seventh (L4 and L5). After explicitly positing Kelly's share as the referent unit as can be deduced from his statement "if we make Kelly's a whole" (L16), Ted correctly measured the extra one share as $\frac{1}{6}$ rather than his initial answer $\frac{1}{7}$. Though Cole could follow Ted's argument, his response (L30) shows that he was not convinced that Tom's share is $1\frac{1}{6}$ of Kelly's share. He interpreted the difference between the two shares additively rather than multiplicatively. It is also important to highlight the distinction that Cole made in L7 when he changed the terms "one and one seventh bigger" to "one seventh bigger" as it shows that he made the difference between the comparative terms 'larger' and 'as large as', though both of them are incorrect in this situation.

Episode 5.7: ***Aileen and Brian*** **[Situation 7,** $T = 1\frac{3}{4}$ **and** $K = \frac{6}{7}T$ **or** $\boldsymbol{T}$**:** $\boldsymbol{K}$ **= 7:6]**

Data.

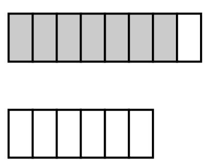
L1: I: Tom's candy bar is one three quarter units long. Can you represent this?

They made a bar and divided it into 8 partitions (Figure 5.6(a)).

L2 A: And then he has the first seven.

L3 I: And Kelly's candy is six seventh of Tom. Can you draw Kelly's candy bar?

They pulled out one part and repeated it 5 times (Figure 5.6(b)).



(a) Tom's share (7 partitions)

(b) Kelly's share

Figure 5.6. Representation of Tom's and Kelly's share on JavaBars ($T : K = 7 : 6$)

L4 I: So once again the same question, Tom's share is how many times as large as Kelly's share?

L5 A: One and ... one seventh.

L6 B: Yeah.

L7 A: One and one seventh times larger.

L8 I: What about Brian? Why do you say one and one seventh?

L9 B: Because it has to be one to make it equal one and one seventh of that to have the extra (inaudible).

L10 I: Where is the one seventh, can you show me the one and seventh?

L11 B: That (pointing to the seventh partition in Figure 5.6(a)) is one seventh.

L12 I: So how much more candy bar does Tom have compared to Kelly?

L13 B: One seventh.

L14 A: One seventh more.

L15 I: And how much less?

L16 A & B: //One seventh//

Analysis. In contrast to situation 6 ($T:K=3:2$) where Brian could justify that the extra share that Tom had was half of Kelly's share (episode 5.5), here he did not deduce that one sixth of Kelly's share is equal to the extra share. He justified his incorrect answer (one seventh) by focusing on the additive difference (L9) rather than norming the larger quantity in terms of the smaller quantity. I hypothesize that because $\frac{1}{7}$ is not a common fraction, this may have prevented him from verifying that $1\frac{1}{7}\times 6 \neq 7$. It should also be noted that both of them incorrectly mentioned that Tom has one seventh more than Kelly. Looking at their response, one may also argue that they deduced that Kelly has $\frac{1}{7}$ less candy bar than Tom by merely changing "one seventh more" in L14 to one seventh less in L16.

In the next section, I give empirical evidence to show how students use the comparatives 'more' and 'less' from an absolute and a relative perspective.

Absolute versus Relative Comparisons

A common mathematical principle is that if A > B then B < A and the difference |A-B| is the same for both inequalities. This is a common invariant that one encounters in many situations involving the quantities A and B. However, this is not the case when making a multiplicative comparison. One can make a multiplicative comparison directly by taking the ratio A/B or B/A or by considering the relative difference $|A-B|/B$ or $|A-B|/A$ (using the comparatives smaller or larger). The absolute difference $|A-B|$ is the same, but the relative comparison is with respect to the chosen referent and $|A-B|/A \neq |A-B|/B$. The comparative terms 'less' and 'more' tend to be intuitively used from an absolute perspective rather than in a relative sense. For example, in situation 2 ($R:B=6:4$), Eric compared the two extra red counters from an absolute perspective as follows. The names Mary and Paul were associated with the 6 red and 4 blue counters respectively.

E: Kind of how I see is if you break these (breaking the 6 red counters into 3 groups of 2) evenly you will have like 1/3 will equal 2 and over here like (breaking the 4 blue counters into 2 groups of 2) if you do the same thing, you would have, Mary would have a third more than Paul because Paul has 2 thirds and Mary has 3 thirds. That's how I see it.

By grouping the counters into sets of two (i.e., unitizing in sets of 2), Eric reduced the multiplicative comparison problem from 6 red and 4 blue counters to 3 red and 2 blue counters. He interpreted each composite unit of 2 counters as being equivalent to 1/3 because he considered the 6 red counters as one unit. This led him to interpret the difference in counters (i.e., 2) as being 1/3 more.

Similarly, in situation 3 ($R : B = 6 : 10$), he interpreted the difference in counters from an absolute perspective. The names Mary and Paul were associated with the 6 red and 10 blue counters respectively.

E: I am seeing these by twos again (as in situation 2), but like if you break these (pointing to the 10 blue counters) into twos like the last problem you would get 5 fifths because if you break them in half like two by twos, you would have 5 sets of two and you would have 3 sets of twos, so, Uh, let's see. Paul has like 2/5 more and that, yeah 2/5 more than Mary because Mary only has 3/5 compared to Paul; he has five fifths.

Eric grouped the counters in sets of 2 as in situation 2 and associated each such composite unit with a fifth. This led him to observe that the difference of 4 counters (equivalent to two composite units of 2) is equal to 2 fifths. He interpreted the difference with respect to the larger number of counters (of which it is part). Such a comparison is not multiplicative in nature. Further, he did not view the composite unit of 6 elements as a unit in its own right within the composite unit of 10 elements (i.e., a 3-levels of units structure). Eric's response prompted me to look at the subtle difference in the additive and multiplicative interpretation of the comparative *more*. His statement 'Paul has 2/5 more' can be interpreted as a correct response if the referent is considered as an extant quantity – a referent that is common to both Paul and Mary. Paul has 5/5 of that quantity and Mary has 3/5 of the quantity, thus Paul has 2/5 more. However, viewed from a multiplicative perspective his response is incorrect because the difference between the two sets of counters (i.e., $10 - 6 = 4$) does not represent 2/5 more than the number of counters that Mary has (the referent quantity); it actually represents 2/3 more than that of Mary.

Furthermore, reversing the syntactic order of words in a multiplicative comparison relation does not reverse the relation (i.e., if B is $\frac{2}{7}$ less than A then A is not $\frac{2}{7}$ more than B; it is $\frac{2}{5}$ more than B). In situation 7 (Tom: Kelly = 7:6), Brian and Aileen mentioned that Tom had $\frac{1}{7}$ more candy bar than Kelly and Kelly had $\frac{1}{7}$ less than Tom by changing the comparatives 'more' and 'less'.

The following salient episode illustrates how Brian's absolute conception of difference and Aileen's relative conception of difference led to a conflict in perspectives.

Episode 5.8: ***Aileen and Brian*** **[Situation 5,** $R:B=5:4$ **]**

Data.

L1 I: Now take 5 red counters and 4 blue counters. Can you compare the red and the blue counters?

L2 B: There is one fifth more red than there is blue.

L3 A: One fourth.

L4 B: One fifth, because there is 5, it should be 4/5, five fifths (pointing to the 5 red counters).

L5 A: I know but if you went, if you are going, are you going from red to blue or blue to red?

L6 B: The red is 5 and that will be whole (inaudible). So this (pointing to the blue counters) is 4/5 of this (pointing to the red counters).

L7 A: Yeah, you could do that or you could do, uh, one fourth larger. The (number of) red (counters) is one fourth larger than the (number of) blue (counters), because each one of those (pointing to the blue) is ...

L8 B: Is that one fourth of the red or one fourth of the blue?

L9 A: It would be one…, if this one (pointing to one of the blue counters) equal to one and one fourth and one (and) one fourth, and one (and) one fourth (and) one and one fourth and this is going (pointing to the 5 red counters) to be 5 (the four one and one-fourth quantities).

L10 I: So, the red is, when you compare the red to the blue, what is your..., what are you saying?

L11 B: Uh, from red to blue that is ...

L12 A: One and one fourth larger than the blue.

L13 B: Yeah.

L14 I: And from the blue to the red?

L15 B: It's one fifth.

L16 A: It's one and one fourth smaller.

L17 B: No

L18 A: One fourth.

L19 I: You are saying one fifth? (talking to Brian)

L20 B: It's one fifth larger, the red is.

L21 A: It's one fourth.

L22 I: You are saying it's one fourth larger? (talking to Aileen)

L23 B: That depends what you are using as the whole. Because if you are using this (pointing to the 4 blue counters) as they are, which you are using as the denominator is 4, then that's what it is, but if you are using this (pointing to the 5 red counters) as the denominator which is 5, then it would be 1/5 larger.

L24 A: If each one of these (pointing to the 4 blue counters) equal one and one fourth (of the red counters), then if you add it all together, it would equal 5 whole cubes and his is, this (pointing to the 5 red counters) is a whole, then that (pointing to the 4 blue counters) is 4/5 (of the whole).

L25 B: That would be one fifth larger. [Referring to the red counters]

L26 A: But if this (pointing to the 4 blue counters) is the whole, then that is ...

L27 I: OK, I can see what you are saying. Suppose that the red one, you want to take the red one as the whole, this is what you want to say.

L28 B: Whichever.

L29 I: OK, if the red one is the whole, then what is the blue, what does the blue represent?

L30 B: Uh, 4/5 of the whole.

L31 A: 4/5 of the red.

L32 I: And if the blue is the whole?

L33 A: It is, the (number of) red (counters) is one and one fourth times larger

L34 B: Inaudible.

L35 A: than the (number of) blue. (The number of) blue (counters) would be one and one fourth smaller (than the number of red counters).

L36 I: Just like the example that you used previously, like when we had 3 red counters and 6 blue, you said that the red counters are 1/2 of the blue counters. Right. And then you said that the blue counters is twice than the red counters. So, if you, and there you did not make the difference between the ..., which one is the whole. Right. Can you use a similar statement here? The red one, the red counters are how many times the blue counters?

L37 A: One and one fourth. They are one and fourth (inaudible) times larger than the blue.

L38 B: It is the same way ...

L39 I: And the blue ...

L40 B: If you are counting which one is the whole. So,...

L41 A: I still think that the red is one and one fourth times larger than the blue.

L42 I: And the blue in comparison to the red?

L43 A: Is one and one fourth times smaller than the red.

Analysis. Brian's answer "one fifth more" (L2) and Aileen's answer "one fourth" (L3) brought to the fore the issue of referent unit as pointed out by Brian: "That depends what you are using as the whole" (L23). Further, Aileen's question "are you going from red to blue or blue to red" (L5) implicitly emphasizes that she kept track of a referent quantity. In contrast to the previous multiplicative comparison situations, Aileen had no conflict about the referent unit here. By quantifying each of the blue counters as corresponding to "one and one fourth" (L9) red counters and summing them to 5 (which is equivalent to the number of red counters), Aileen could check that her answer of 1 ¼ was correct. This shows that Aileen was looking for a number (an unknown) which when multiplied by 4 gives 5, a theorem-in-action equivalent to $4x = 5$. Brian for his part used the total number of either red or blue counters to decide whether the answer should be $\frac{1}{4}$ *more* or $\frac{1}{5}$ *more* as can be deduced from his justification in L23.

Aileen tended to use the terms *as large as* and *larger than* interchangeably as can be inferred from L12 and L41, where she used the comparative term *larger* to mean *as large as*. Further, she used the pair of comparatives "less than' and 'greater than' from an additive perspective (i.e., in absolute terms). She interchanged the terms "one and one fourth times larger" (L33) and "one and one fourth smaller" (L35). The number of blue counters is actually

$\frac{1}{5}$ smaller than the number of red counters. It is intuitively appealing to replace smaller by greater and to interchange the red and blue counters. This is another aspect of reversibility that is worth noting, and it has been found to be problematic in other situations. For example, if a picture is reduced by $\frac{1}{4}$ of its size (i.e., $\frac{3}{4}$ reduction), what magnification should be used to return it to its original size (Lamon, 1999)? Students may only say $\frac{1}{4}$ more (or $\frac{5}{4}$ magnification). Another typical problem is concerned with percentage increase and decrease (Parker & Leinhardt, 1995). For example, if $80 is the cost of an item that has been reduced by 10%, what was the original cost? Many students will add 10 % of $80 to find the original cost.

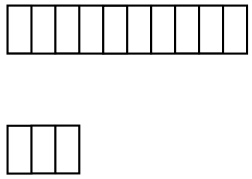
In the previous episode, I showed how an absolute and a relative conception of a multiplicative comparison influence students' responses. Now, I present one episode to show how the participants (Ted and Cole) correctly normed the larger quantity in terms of the smaller quantity. It also highlights the correct use of the comparatives 'more' and 'less'.

Episode 5.9: *Ted and Cole* [Situation 8, $T = 1\frac{2}{3}$ and $K = \frac{3}{10}T$ or *T*: *K* = 10:3]

Data.

L1 I: Tom's candy bar is one two thirds units long. Kelly's candy bar is three tenth of that of Tom.

They made a bar, partitioned it into 5 and recursively partitioned each of the fifths into 2 as shown in Figure 5.7 (a). Then they pulled out one partition and repeated it two times to construct Kelly's share (Figure 5.7 (b)).



(a) Tom's share

(b) Kelly's share

Figure 5.7. Representation of Tom's and Kelly's share on JavaBars ($T:K=10:3$)

L2 I: Tom's share is how many times as large as Kelly's share?

L3 C: It would be

L4 T: Three times and one third.

L5 C: Yeah.

L6 I: Right. And how much more does Tom have compared to Kelly?

L7 T: Seven ...

L8 C: 1,2,3,4,5,6,7 (counting the difference between the two quantities). Seven thirds.

L9 T: Thirds.

L10 C: Seven thirds.

L11 T: Or two and one third.

L12 I: Which is?

L13 T: Same thing still.

L14 I: And how much less does Kelly have compared to Tom?

L15 C: Seven ...

L16 T: Tenth.

L17 C: Seven tenth.

Analysis. Ted's response "three times and one third" suggests that he measured the larger quantity in terms of the smaller quantity by iterating the smaller quantity over the larger quantity.

This is evidence of the application of the norming process. However, when asked how much Tom has more compared to Kelly, he started his response with “seven” (L7) and paused for few seconds. Had he given the answer ‘2 times and one third,’ this would have shown that he measured the difference (7) by segmenting it in units of 3. Ted’s and Cole’s answer “seven thirds” (L7 to L10) suggests that they may have viewed the 7 extra partitions as 7 iterations of $\frac{1}{3}$ of Kelly’s bar or they may have divided 7 by 3. In this question the referent unit is relatively smaller than the compared quantity and such a structure may have facilitated the application of the unitizing and norming process guided by the visual representation on JavaBars. This episode also shows that they used the comparative ‘less’ with respect to the correct referent as can be inferred from L14-L17.

Four Strategies for Comparing Two Quantities Multiplicatively

(i) Division

All six participants divided the larger quantity by the smaller quantity in situation 1 ($R : B = 3 : 6$) due to the divisibility relations between the two quantities being compared. For instance, Ted made the following statement: “There are 6 (blue) of them and the number of red blocks are 3. So there is kind like finding a common number to them like 3 times 2 equals 6. And 6 divided by 2 equals 3.” However, such direct division could not be observed in the other problem situations. It should be pointed out that the multiplicative comparison tasks were presented to the students in terms of Unifix cubes and JavaBars, and this may have influenced their strategies in comparing the two quantities. The visual display of these manipulatives made the unit structure of the two quantities being compared more apparent, and may have prompted the students to use a form of visually-induced/figurative comparison.

(ii) Aileen's strategy

Aileen multiplicatively compared the two quantities using the theorem-in-action $ax = b$ as illustrated in episodes 5.1 to 5.3. She interpreted the multiplicative comparison situation as a missing-factor problem (Vergnaud, 1988).

(iii) Norming

Episode 5.10: *Jeff* [Situation 3, $R : B = 6 : 10$]

Jeff's response in situation 3, involving the comparison of 10 blue counters in terms of 6 red counters, is shown in the following segment. I associated the name Paul to the red counters and Mary to the blue counters.

J: Paul has 3 (pointing to the 3 sets of 2 red counters), so, (inaudible) six which you can break them into sets of 2 and then you put all those together, three will make one, and take that (3 sets of 2 blue counters), that would be one, so that (pointing to the 5 sets of 2 counters) would be $1\frac{2}{3}$.

Jeff's response shows that first he made composite units consisting of 2 individual units. Then he reasoned with 3 levels of units as illustrated in Figure 5.8. He united 3 composite units (of 2 elements) to form a new unit of 6 elements: the unit of 6 red counters made up of 3 sets of 2 units, each 2-unit being one third of the unit whole (of 6 reds). This led him to deduce that the 4 extra blue counters are $\frac{2}{3}$ of the red counters. Jeff's response illustrates the norming process because he re-conceptualized the 10 units in terms of a composite unit of 6 elements.

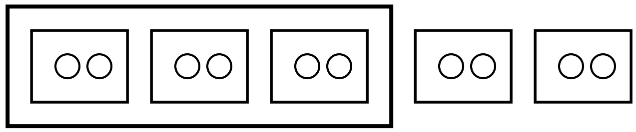


Figure 5.8. Interpreting 10 counters at 3 levels of units

Another illustration of the norming process can be inferred from Jeff's response in situation 4 ($R:B=7:5$):

J: Because that being how many, like if you are comparing those two (pointing to the 5 red and 5 blue counters) that's my whole unit and that would be, this would be one and that (pointing to the remaining two red counters) would be 4/10 of another one.

He explained that he obtained $\frac{4}{10}$ by multiplying the numerator and denominator of $\frac{2}{5}$ by 2 and gave the following clarification: "It's mainly because I didn't want to use the decimal. Instead of using 1.4, I used $1\frac{4}{10}$."

(iv) Creating a copy of the smaller quantity to compare the larger quantity

In situation 2 ($R:B=6:4$), Ted and Cole added 2 red counters to the 6 red counters to measure the red counters in terms of the blue counters (episode 5.4). Another instance of this strategy could be observed from Brian's response to situation 3 as illustrated below.

Episode 5.11: ***Aileen and Brian*** **[Situation 3,** $R : B = 6 : 10$**]**

Data.

L1 I: Suppose you have 6 red counters and you have 10 blue counters. Right. And now if I ask you to compare the red and the blue counters or let us say, Paul has 6 red counters and Mary has 10 blue counters.

L2 B: So, it would be three fifths. Whichever one has the smaller amount has three fifths of the other amount.

L3 A: Yeah.

L4 I: What is, I mean Paul has ...

L5 A: Paul has three fifth of what Mary has.

L6 I: And if I want to compare the other way, like Mary has how many times as compared to Paul?

L7 B: One one third which will be 1.3333 and so on.

L8 I: How do you get one one third?

L9 B: Because if you have three fifth and you double 3 which gives you 6, that's too much, if you take away a third of it [which was to be 5].

L10 I: Why do you take a third?

L11 B: Because if you take it back to half of it, then a third would be 1/5 and 6 over 5 take away 1/5 [would be 5 over 5]

L12 A: I think, it's one and two third.

L13 I: So, Brian what is your answer, once again?

L14 B: Uh, one one third.

L15 I: One one third? You are saying that Mary has one one third more counters as compared to Paul.

L16 B: One one third as much as.

⋮

L17 B: Uh, if you look at this one (pointing to the 10 blue counters), if you take away half of both of them, then you have to take away 5 from this (the blue counters), and.

L18 A: three (of the red counters)

L19 B: Then it ends up being three fifth but if you double this (pointing to the red), then you still have one more. So out of this (pointing to 6 red counters) you have to take away one of these (pointing to red counters) for it to be equal. So it has to be one one third.

L20 I: I think I did not follow what you said, could you please say it again.

L21 B: [If you take one] half of it, it would be

L22 I: You are dividing it into half?

L23 B: Yeah.

L24 I: OK.

L25 B: Then it would be three fifth or a ratio of 3:5 whatever. If you double the 3 that should give (inaudible) you as close you can with a whole number to the whole 5. But you are still a little bit too far. You have to take away one of the six. And in comparison down if you [go back down] to this (pointing to the three red counters) and that would equal to one third.

L26 I: And, Aileen?

(She arranged the 10 blocks in two groups of 5.)

L27 A: Since you take away 5 from this group (one group of 5 blue counters), five (one group of 5 red counters) it is going to give you 1 to 5.

L28 B: Remember, with (inaudible) ratio you can't subtract (inaudible)

L29 A: So, I can't.

L30 B: (inaudible)

L31 A: Something is one and two thirds though. How, I don't really know.

Analysis. Brian's strategy is remarkable in his attempt to measure the 10 blue counters in terms of the 6 red counters, giving the same explanation at three points in the interview (L9, L19, and L25). First, he reduced the number of counters by half. He expressed this by saying "if you take it back to half of it" (L11). His next move was to double the three red counters and, he could observe that this produced one red counter more than the blue counters as illustrated in Figure 5.9. In other words, starting from 3 red counters, he attempted to determine how 'far' he should go to make 5 (blue) counters. The fact that he deduced that the one extra red counter is 1/3 (L9) shows that he conceptualized the three red counters as one unit. He associated this 1/3 as being 1/5 of the blue counters as can be inferred from his response "Because, then a third would be 1/5" (L11). Figure 5.9 helps to explain what he means by "6 over 5 take away 1/5 [would be 5 over 5]" (L11). Though he could equate the red and blue counters by subtracting 1/3 from the two composite units of 3 elements, he stated that the number of blue counters is $1\frac{1}{3}$ of the red counters rather than $1\frac{2}{3}$. Brian's response shows that although he could conceptualize the 3 red counters as one unit, he was constrained in using it to norm the 5 blue counters.

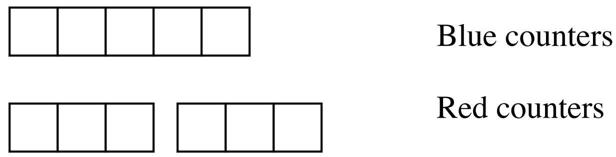


Figure 5.9. Brian's correspondence between 1/5 and 1/3

Discussion

The aim of this first problem set was to track down the ways in which students reason reversibly in comparing two quantities multiplicatively and to identify the resources that they deploy in such situations. The discrete situations involving the comparison of red and blue counters, together with the continuous situations involving the comparison of the length of candy bars were explored in depth to track at a fine-grained level of detail the resources deployed as well as the constraints that the middle school students encountered. The students' responses show that when comparing two quantities multiplicatively norming the larger quantity in units of the smaller quantity can be a demanding task for middle school students. A multiplicative comparison is constitutively a basic type of quantitative operation where one quantity is chosen as a unit to measure the other quantity. In this data set I showed how this measurement takes place through the unitizing and norming process and the conflict that the students encountered.

What does this first data set suggest about the second research question: *In what ways do students reason reversibly in multiplicative situations and what constructive resources do they deploy in such situations?*

I identified two ways in which the students reasoned reversibly in the multiplicative comparison situations.

1. Interpreting a multiplicative comparison as $ax = b$

In situation 2, 3, 4, and 5, Aileen set the structural relationship between the two quantities being compared using the theorem-in-action $ax = b$. (i) In situation 2 (episode 5.1), Aileen compared the larger number of counters in terms of the smaller number of counters by finding which number she had to multiply 4 by to produce 6. She knew that she had a result (6) which had been produced by multiplying 4 by a number. She determined this missing factor by using a guess-and-check strategy. (ii) In situation 3 (episode 5.2), her strategy was to look for a number which when multiplied by 6 gives 10, a theorem-in-action equivalent to $6x = 10$. She again used a systematic guess-and-check procedure to find the unknown multiplier/multiplicand. (iii) In situation 5 (episode 5.8), involving the multiplicative comparison of 5 red and 4 blue counters, Aileen showed evidence of working with the theorem-in-action $4x = 5$, where she determined x by 'partitioning' the 5 red counters into four $1\frac{1}{4}$ units. She coordinated each such $1\frac{1}{4}$ unit with one blue counter. These situations did not trigger the multiplication/division invariant for her (i.e., she did not deduce that the division operation would yield the answer).

2. Feeding the result back into the situation

A second way in which reversible reasoning could be observed was in terms of feeding the result produced from the multiplicative comparison back in the situation to verify that it satisfies the relation $ax = b$. For example, in situation 6 ($T = \frac{3}{4}$ and $K = \frac{2}{3}T$ or $T : K = 3 : 2$), after deducing that Tom's share is $1\frac{1}{2}$ times of Kelly's share, Brian multiplied Kelly's share (represented by 2 partitions on JavaBars) by $1\frac{1}{2}$ to prove to Aileen that Tom's share corresponds to 3 partitions. In situation 8 (multiplicative comparison of 10 units and 3 units), he multiplied $3\frac{1}{3}$ by 3 to justify that Tom's share is $3\frac{1}{3}$ times as large as Kelly's share. Similarly, in problem 2 ($R:B = 6:4$) and problem 3 ($R:B = 6:10$), Aileen plugged back her solution in the situation to verify that $1\frac{1}{2} \times 4 = 6$ and $1\frac{2}{3} \times 6 = 10$, respectively.

Constructive resource for reasoning reversibly when comparing two quantities multiplicatively

Isolating the known-unknown structure, $ax = b$, is a key element for coordinating the known quantities (multiplicand and product) and unknown quantity (multiplier) or known quantities (multiplier and product) and unknown quantity (multiplicand) in a multiplicative comparison situation. Such a structure is also necessary to justify the result of the multiplicative comparison. As mentioned earlier, Aileen is the only participant who explicitly interpreted the multiplicative comparison of two quantities, *a* and *b* in terms of the theorem-in-action $ax = b$, $b > a$, though the numeric feature of these parameters prevented the cueing of the multiplication/division invariant. Brian similarly articulated this structure to verify the answer he produced after making the multiplicative comparison. The 'faulty-remainder' theorem-in-action observed in this data set can be attributed partly to the inability of the participants to isolate the structure $ax = b$ to observe the relationship between the computed multiplier and the given multiplicand and product or the computed multiplicand and the given multiplier and product.

Constructive resources for comparing two quantities multiplicatively

I comment on two key resources necessary for comparing two quantities multiplicatively.

Resource 1: Unitizing and norming

At various points of the interview the students explicitly formed composite units in their attempt to compare the two quantities. The data show two ways in which the participants used their unitizing operation: (i) in terms of the smaller number of counters (e.g., unitizing in units of 3 when comparing 10 and 3 partitions in situation 8) (ii) in terms of the highest common factor of the two numbers being compared (e.g., unitizing in units of 2 when comparing 4 blue and 6 red counters).

Though the students could unitize, they did not always measure the larger quantity in terms of the smaller quantity. This led me to infer that forming composite units is necessary but not sufficient for comparing two quantities. The data in episodes 5.4-5.7 show that the key element missing in the students' thinking was the norming process (i.e., re-conceptualizing the larger quantity in terms of the smaller quantity). In other words, the interpretation of the larger quantity in terms of the smaller quantity requires one to posit the latter as the referent unit with which to measure the larger quantity. The norming process requires the coordination of 3 levels of units as Jeff did in episode 5.10. It could also be observed that the norming operation was correctly applied when divisibility relationships were available between the two quantities as in situations 1, 2 and 3. Regarding the conflict that the above-average students encountered to perform such a norming operation, it can be argued that this re-conceptualization requires a significant conceptual leap.

Resource 2: Coordination between the compared and referent quantity

The coordination of the referent and compared quantity is another essential resource for comparing two quantities multiplicatively. A comparative statement inherently involves a referent quantity. For instance, if Q_1 is k times as large as Q_2 then it is implied that Q_2 is the referent unit. Because Q_2 comes after Q_1 when we express this comparison in terms of written or spoken language, the referent quantity tends to be held in the background and may even be suppressed or become implicit. To conceptualize Q_1 one has to simultaneously posit Q_2 in thought. Students may not readily posit Q_2 as the referent quantity. For instance, in situation 6 ($T = \frac{3}{4}$ and $K = \frac{2}{3}T$ or T: K = 3:2), observing the constraint that Ted was experiencing in comparing the two quantities, Cole gave him the following suggestion to focus his attention on the referent: "You compare it to the person you are comparing it to. Like Tom will compare to Kelly because she has two, so it will be half and Kelly comparing to Tom so it will be thirds. You get it now."

When asked how much larger is Q_1 compared to Q_2, the students looked at the additive difference in terms of an extant referent. They did not interpret such a difference with respect to the implied referent Q_2. One reason for the 'faulty remainder' theorem-in-action is that the participants did not coordinate the difference between the two quantities to the referent quantity but interpreted such a difference in terms of the compared quantity of which it is part. Brian's question "Is that (i.e., the difference of one counter) one fourth of the red or one fourth of the blue?" (episode 5.8, L8) when comparing 5 red and 4 blue counters in situation 5 highlights the internal conflict (in terms of differentiating between the compared and the referent quantity) that students may encounter in such multiplicative comparison situations.

What does this first data set suggest about the third research question: *What constraints do students encounter in conceptualizing multiplicative relations from a reversibility perspective?*

One way to perform a multiplicative comparison is to ask the following questions: "By what should I multiply the larger quantity to produce the smaller quantity?" (a multiplicative relationship less than one) and "By what should I multiply the smaller quantity to produce the larger quantity?" (a multiplicative relationship greater than one). This interpretation of a multiplicative comparison situation involves the theorem-in-action $ax = b$. As mentioned earlier, though Aileen could set such a relationship between the two given quantities and inferred that she was looking for an unknown, this situation did not trigger the multiplication/division invariant for her. In other words, this situation was a multiplicative problem for her and she did not deduce that she could obtain the unknown multiplier by division.

Numeric feature of the quantities being compared

The prompt responses of the students to situations 1 and 2 (in contrast to the other situations) show that the presence of divisibility relationships facilitated the multiplicative comparison of two quantities. In situation 2 ($R:B = 6:4$, episode 5.4), by adding 2 red counters to the existing 6 red counters, Ted and Cole constructed a composite unit of 4 from which they could norm the 4 blue counters. Such reversible reasoning was possible due to the divisibility relationship between the extra number of counters (2 red counters) and the norming unit (4 blue counters). In contrast, in situation 4 ($R:B = 7:5$, episode 5.4), they did not deploy such a verification criteria because the difference between the two sets of counters (i.e., 2) does not divide either 5 or 7. Similarly, in situation 7 (Tom : Kelly = $7:6$), Brian did not reason reversibly to deduce that Tom has $\frac{1}{6}$ more candy bar than Kelly by verifying that $\frac{1}{6}$ of her candy bar is equal to the difference between the shares in contrast to situation 6 (Tom : Kelly = $3:2$), where he could feed back the result in the situation. The articulation of the relation 'multiplier $\times$ multiplicand = product' and its counterpart 'multiplier = product $\div$ multiplicand' is sensitive to the numeric feature of the quantities involved.

Multiplicative comparison in discrete and continuous situations

Though I did not plan to explore the difference in multiplicative comparison between discrete and continuous situations, Aileen's response in situation 6 ($T = \frac{3}{4}, K = \frac{2}{3}T$ or $T : K = 3 : 2$) suggests that continuous situations can impose more demand than discrete situations. She represented Tom's and Kelly's candy bars with 3 and 2 partitions, respectively. When asked to compare the two quantities (episode 5.5, L1) she gave the following response as the problem statement involved thirds:

L1 I: Tom has three quarter of a candy bar and Kelly's candy bar is two thirds of that of Tom.

⋮

L9 A: It would be times one and one third because this is two thirds (pointing to Kelly's share). That's (pointing to Tom's part) a three thirds. So two thirds times one, so two thirds just (inaudible).

The denominator of $\frac{2}{3}$ acted as a pointer leading her to interpret the extra quarter that Tom had as being $\frac{1}{3}$. In contrast, in situation 2 ($R : B = 6 : 4 = 3 : 2$) she did not encounter such conflict in the discrete context (involving integer quantities), where she could readily deploy the theorem-in-action $4x = 6$. Similarly, Aileen (and Brian) gave the answer $1\frac{1}{7}$ in situation 7 ($T = 1\frac{3}{4}$ and $K = \frac{6}{7}T$ or $T : K = 7 : 6$) because the problem was presented as follows: 'Tom's candy bar is one and three quarter units long. And Kelly's candy bar is six sevenths of that of Tom'. She did not encounter such a conflict in an analogous discrete situation (e.g., situation 4, $R : B = 7 : 5$). The influence of the denominator as a pointer can also be observed in situation 5

($R : B = 5 : 4$), episode 5.8, where Brian mentioned that there are one fifth more red counters than blue counters, justifying his response by the following quote: "one fifth, because there is 5."

One of the limitations of the findings in continuous situations 6, 7, and 8 is that they may have been biased by the formulation of the problem. I first asked the students to represent Tom's candy bar and then using his candy bar to construct that of Kelly. For instance, in situation 7, I asked them to represent Tom's candy bar having a length of $1\frac{3}{4}$ units. Then I asked them to construct Kelly's bar as $\frac{6}{7}$ of that of Tom. The problems should have been posed by asking the students to represent two candy bars having length $1\frac{3}{4}$ and $1\frac{1}{2}$ units, respectively, so that the intended ratio $T : K = 7 : 6$ is maintained. The multiplicative comparison of two quantities in discrete and continuous situations is a problem that needs further exploration. My analysis also shows that, besides being context-sensitive (discrete versus continuous), multiplicative comparison situations are also syntactically sensitive.

Multiplicative comparison: A syntactically-sensitive situation

The comparatives *as large as* and *larger* tend to be used interchangeably in multiplicative comparison situations. For instance, in situation 5 ($T : K = 5 : 4$, episode 5.8), Aileen mentioned that the number of red counters are "one and one fourth larger" than the number of blue counters when she actually meant *one and one fourth times as large*. Moreover, the phrase *how much more* tends to be intuitively interpreted as the additive difference between two quantities. For example, while comparing 6 red and 4 blue counters in situation 2, Eric mentioned that the number of red counters is 1/3 more than the number of blue counters. In this example, the number of red counters is a half time more than the blue counters, but the number of blue counters is one third less than the red. Such subtle syntactic/semantic distinction that the comparative terms *more than*, *less than*, *as large as* or *how much more* require, possibly accounts for some of the incorrect answers that the students gave in the interviews. Parker & Leinhardt (1995) highlighted such syntactic sensitivity in multiplicative comparison situations in the domain of percentage. The interpretation of the multiplicative comparison terms used in the interviews imposes another limitation on the findings of this study as the students may have given different interpretations to the question asked.

Absolute versus relative comparisons

Two types of multiplicative comparisons were of interest in this data set. Firstly, the students were asked to compare the larger quantity in terms of the smaller quantity involving the comparative *as large as*. Secondly, the difference between the two quantities was compared with either the smaller or larger quantity in situations 5 to 8.

Aileen's and Brian's responses in situation 5 ($R:B=5:4$), episode 5.8, brought to the fore the necessity to differentiate between an absolute and a relative comparison. This led me to select situations 6, 7, and 8 in a continuous context to specifically focus on relative comparisons where I more carefully pointed students' attention to the comparatives terms (e.g., as large as versus how much larger) in the interviews.

The comparative terms *less* and *more* are customarily used in relation to an absolute difference. For instance, in comparing 6 red counters to 4 blue counters in situation 2, Eric interpreted the 2 extra red counters as being 1/3 more than the blue counters rather than a 1/2 time more. Similarly, in situation 3 ($R:B:6:10$), he interpreted the difference of 4 counters as 2/5 more than the red counters rather than 2/3 more. In situation 4 ($R:B=5:4$), Brian interpreted the one extra red counter as being 1/5 more than the number of blue counters in contrast to ¼ more. These interpretations involve an absolute conception of a difference, and the students assumed that the referent was the larger quantity. Eric did not realize that 1/3 of the number of blue counters does not amount to the difference of 2 counters in situation 2, or 2/5 of the number of red counters does not amount to the difference of 4 counters. Similarly, Brian did not realize that 1/5 of the number of blue counters does not amount to the difference of one counter.

Conventionally, the order relation or invariant "if A < B, then B > A" is used from an additive perspective. The interpretation of such order relation from a relative perspective requires more demanding coordination. For instance, if A is 2/7 larger than B, then B is 2/9 smaller than A. Altering the lexical terms larger to smaller does not reverse the multiplicative relation. In situation 4 ($R:B = 7:5$), after listening to Brian's response: "The red is 2/7 larger than the blue", Aileen stated that: "The blue would be 2/7 smaller." In situation 5 ($R:B = 5:4$), she mentioned that "the red is one and one fourth times larger than the blue" and the blue "is one and one fourth times smaller than the red." Similarly, in situation 6 ($T:K = 3:2$), Cole mentioned that "Kelly has one third less than Tom, and Tom has one third more than Kelly". Taken together, these responses show that relative comparisons are not naturally occurring, like other forms of multiplicative reasoning, as substantiated by previous research (Clark & Kamii, 1996; Sowder et al., 1998).

CHAPTER 6

SET 2: MULTIPLICATIVE COMPARISON OF TWO QUANTITIES IN A PARTITIVE DIVISION SITUATION

My focus in this study was primarily to identify the strategies that students use and the constraints that they encounter in reasoning reversibly in multiplicative situations. In the previous chapter, I illustrated the constraints that some of the participants encountered in comparing two quantities multiplicatively, where given a source and a result, they had to find the multiplicative relation between them. In this chapter, I illustrate further the strategies and constraints that the students encountered in constructing one whole, starting from a given fractional part (referred to as construct-the-unit problems). From an algebraic perspective this type of situation can be modeled by $ax = b$ or as a multiplicative relationship between a known and unknown quantity. It can also be regarded as a statement of division, as a statement of multiplicative comparison, or a statement of proportionality.

As stated in the literature review, reversibility of thought in the domain of fractions has been studied at the micro-analytic level by a number of researchers (Hackenberg, 2005; Olive & Steffe, 2002; and Tzur, 2004) where their focus was oriented toward fractional or algebraic reasoning rather than explicitly exploring reversible reasoning as a central element. These research studies have used scheme theory to identify the schemes and mental operations that may be involved in reasoning reversibly. In this study, I explicitly focused on reversible reasoning as the participants were required to construct one unit of a quantity, starting from part of the quantity. I paid particular attention to how the students articulated the multiplicative comparison relation between the known and unknown quantities. By systematically varying the numeric structure of the problems, my aim was to identify the conditions under which students could or could not reason reversibly. Table 6.1 shows how such numeric variations were made. The problems have been graduated along seven levels based on the assumed level of complexity. The order in which students engaged these problems was not necessarily in the same sequence. Table 6.2 indicates the chronological order in which I presented the problems to the students.

Overview and Justification of Tasks

The justification for the choice of the parameters *a* and *b* in $ax = b$ is given in the last column of Table 6.1. I use the algebraic notation to show the justification behind the formulation of the problem, but in no way does this suggest that the students were expected to look at these situations from an algebraic perspective. The values of *a* and *b* were progressively varied, starting from integers and unit fractions to the more demanding situations in levels 6 and 7 with relatively prime numerators. I provide additional justifications for the tasks in levels 5, 6 and 7 below.

Level 5: This category of problems involves the multiplicative comparison of a mixed number and an integer quantity, where *a* is the mixed number and *b* is the integer quantity in $ax = b$. Two questions (2.51 and 2.52) were formulated in this category with the mixed numbers $1\frac{1}{2}$ (involving halves) and $1\frac{2}{3}$ (involving thirds). In both cases, the fractional quantities were presented in terms of mixed-number representation ($1\frac{1}{2}$ and $1\frac{2}{3}$) rather than improper-fraction representation ($\frac{3}{2}$ and $\frac{5}{3}$). These situations have been motivated by an observation made in the first phase of the study where the response of one of the students highlighted the necessity to differentiate between mixed number and improper fraction representation. I observed that the representation $1\frac{1}{3}$ and $\frac{4}{3}$ cued different resources in reversibility contexts.

Level 6: The problems at Level 6 involve the multiplicative comparison of a fraction and an integer quantity, where the numerator of the fraction and the integer quantity are relatively prime. Problems 2.61, 2.62, 2.63, and 2.64 involve a non-unit fraction while problems 2.65 and 2.66 involve a mixed number.

Level 7: This last category of problems represents the more demanding situations in Set 2, where both of the quantities being compared are fractional quantities with relatively prime numerators. Four questions were formulated in this category, varied by changing the type of fractions (proper and improper fractions). These problems require the coordination of two units-of-units. For example, question 2.71 ($\frac{4}{5}x = \frac{3}{4}$) involves the multiplicative comparison of 3 one-fourth units of one quantity and 4 one-fifth units of another quantity. Further, in comparison to problem 2.63 ($\frac{4}{5}x = 3$), problem 2.71 required the coordination of an additional level of unit (a fourth). I initially planned to use problems 2.72, 2.73 and 2.74 to investigate the effect of mixed number representation (versus improper fraction representation), but observing the constraints that the students encountered with such numeric representations, I presented the students with both representations (e.g.,$1\frac{2}{5}$ as well as $\frac{7}{5}$ in problem 2.72). It should also be highlighted that the same problem context was used in levels 6 and 7.

Table 6.1. Structure of fraction tasks

Problem no.	Question structure	Justification of level based on $ax = b$
	LEVEL 1	
2.11 ($3x = 21$)	There are 21 marbles in a box. The number of marbles in the box is three times the number of marbles Paul has. How many marbles does Paul have?	*a* divides *b*
2.12 ($5x = 7$)	Bar A weighs 7 pounds. Bar A weighs 5 times as much as bar B. What is the weight of bar B?	*a* and *b* are relatively prime whole numbers
	LEVEL 2	
2.21 ($\frac{1}{4}x = 5$)	Emily has $5. She has one fourth as much money as Rachel. How much money does Rachel have?	*a* is a unit fraction and *b* is a whole number
2.22 ($\frac{1}{7}x = 3$)	Emily has $3. She has one seventh as much money as Rachel. How much money does Rachel have?	*a* is a unit fraction and *b* is whole number
	LEVEL 3	
2.31 ($\frac{2}{5}x = 30$)	There are 30 marbles in a box. This is $\frac{2}{5}$ of the number of marbles you have. How many marbles do you have?	*a* is a non-unit fraction and *b* is a whole number
2.32 ($\frac{3}{7}x = 18$)	Eighteen liters of water were poured into a tank, filling the tank to $\frac{3}{7}$ of its volume. What is the volume of the tank?	*a* is a non-unit fraction and *b* is a whole number
	LEVEL 4	
2.41 ($\frac{1}{5}x = \frac{1}{4}$)	Candy bar A is $\frac{1}{4}$ unit long. Its length is $\frac{1}{5}$ of candy bar B. What is the length of candy bar B?	*a* and *b* are unit fractions
2.42 ($\frac{2}{5}x = \frac{1}{4}$)	Candy bar A is $\frac{1}{4}$ unit long. Its length is $\frac{2}{5}$ of candy bar B. What is the length of candy bar B?	Divisibility relationship between numerators
2.43 ($\frac{8}{5}x = \frac{4}{3}$)	Candy bar A is $1\frac{1}{3}$ (or $\frac{4}{3}$) unit long. Its length is $\frac{8}{5}$ of candy bar B. What is the length of candy bar B?	Divisibility relationship between numerators

(*table continues*)

Table 6.1. (continued)

	LEVEL 5	
2.51 ($1\frac{1}{2}x = 48$)	In a seventh-grade survey of lunch preferences, 48 students said they prefer pizza. This is one and one half times the number of students who prefer fries. How many students prefer fries?	*a* is presented as a mixed number
2.52 ($1\frac{2}{3}x = 55$)	Parking lot A can hold 55 cars. It can hold $1\frac{2}{3}$ as many cars as parking lot B. How many cars can parking lot B hold?	*a* is presented as a mixed number
	LEVEL 6	
2.61 ($\frac{7}{8}x = 5$)	Candy bar A is 5 units long. Its length is $\frac{7}{8}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; b is a whole number
2.62 ($\frac{3}{4}x = 2$)	Candy bar A is 2 units long. Its length is $\frac{3}{4}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; b is a whole number
2.63 ($\frac{4}{5}x = 3$)	Candy bar A is 3 units long. Its length is $\frac{4}{5}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; b is a whole number
2.64 ($\frac{3}{4}x = 5$)	Candy bar A is 5 units long. Its length is $\frac{3}{4}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; b is a whole number
2.65 ($1\frac{1}{2}x = 2$)	Candy bar A is 2 units long. Its length is $1\frac{1}{2}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; b is a whole number
2.66 ($1\frac{2}{5}x = 3$)	Candy bar A is 3 units long. Its length is $1\frac{2}{5}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; b is a whole number
	LEVEL 7	
2.71 ($\frac{4}{5}x = \frac{3}{4}$)	Candy bar A is $\frac{3}{4}$ unit long. Its length is $\frac{4}{5}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; $a < 1, b < 1$
2.72 ($\frac{7}{5}x = \frac{4}{9}$)	Candy bar A is $\frac{4}{9}$ unit long. Its length is $1\frac{2}{5}$ (or $\frac{7}{5}$) of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; $a > 1, b < 1$
2.73 ($\frac{3}{5}x = \frac{7}{4}$)	Candy bar A is $1\frac{3}{4}$ (or $\frac{7}{4}$) unit long. Its length is $\frac{3}{5}$ of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; $a < 1, b > 1$
2.74 ($\frac{12}{7}x = \frac{5}{3}$)	Candy bar A is $1\frac{2}{3}$ (or $\frac{5}{3}$) unit long. Its length is $1\frac{5}{7}$ (or $\frac{12}{7}$) of candy bar B. What is the length of candy bar B?	Numerators of *a* and *b* are relatively prime; $a > 1, b > 1$

The solution to these problems involves two essential components: (i) setting the quantitative relationship among the quantities and (ii) finding the measure of the unknown quantity using the measure of the given (known) quantity. Looked at from the perspective of quantities, Set 2 can be regarded as consisting of a known quantity and a quantitative/multiplicative relation where the aim is to find the second quantity.

I analyzed the data by categorizing the strategies and constraints that the students encountered. First, I present the three types of strategies that I could identify from the students' responses: (1) Measure strategy (episodes 6.1-6.3) (2) 'Unit-rate' strategy (episodes 6.4-6.7), and (3) Guess-and-check strategy (episodes 6.8-6.10). Then, I illustrate how the interpretation of mixed numbers in terms of fractional units allows students to use their whole number knowledge to reason reversibly in episodes 6.11-6.14. Further, I highlight one fallback strategy that occurred on a number of occasions where the denominator of the given fraction (expressing the multiplicative relationship between the known and unknown quantities) acted as a pointer for division (episodes 6.15-6.19).

Moreover, I observed three categories of constraints: (1) Quantity-measure conflict (episodes 6.20-6.24), (2) constraints emanating from the numeric feature of the problems (episodes 6.25-6.27), and (3) constraints emanating from the syntactic structure of the problems (episode 6.28). The last episode (6.29) shows the inference of additive reasoning in multiplicative situations. Table 6.3 (presented at the end of this chapter) summarizes the strategies used and conflicts encountered by the participants for problems 2.31 to 2.74. If the students used different attempts in finding the solution, all of them are reported in the order in which they were used. The number in the square bracket in the table indicates the episode number. The dates and order in which the tasks were posed are shown in Table 6.2.

Table 6.2. Chronological order of problems in Set 2

Date	Grade 7 (Ted & Cole)	Grade 8 (Aileen & Brian)
12/11/08	2.11, 2.12	2.11, 2.12, 2.21, 2.22, 2.31, 2.32, 2.61
12/12/08	2.12, 2.21, 2.22, 2.31, 2.32, 2.61, 2.51, 2.52	Grade 8 students were not interviewed.
12/15/08	2.41, 2.42, 2.71, 2.72, 2.73	2.61, 2.51, 2.52, 2.41, 2.42, 2.71
12/16/08	2.43, 2.74	2.72, 2.73
12/17/08	Grade 7 students were not interviewed.	2.43, 2.74
1/23/09	2.72, 2.74, 2.62, 2.63, 2.64, 2.65, 2.66	2.62, 2.63, 2.64, 2.65, 2.66

I have interpreted reversible reasoning in Set 2 as the construction of one whole from part of a whole or a part larger than a whole. Reversible reasoning is also involved in deducing that if quantity A is *m* times as large as quantity B, then quantity B is 1/*m* times as large as quantity A. Such form of reasoning may occur at different points in the construction of one whole. For instance, in problem 2.32 ($\frac{3}{7}x = 18$), Aileen reasoned reversibly at the end of the situation while Ted did so at the start. Aileen constructed one whole from $\frac{3}{7}$ as follows: $\frac{3}{7} + \frac{3}{7} + \frac{1}{7}$ (using the 'measure strategy' as described in episode 6.3). After deducing that two $\frac{3}{7}$-units fit in one whole, she inferred that she needed $\frac{1}{7}$ more to make one whole. She reasoned reversibly to construct $\frac{1}{7}$ from $\frac{3}{7}$. On the other hand, in the same problem, Ted constructed one whole by first finding $\frac{1}{7}$ right at the start of the problem and constructed one whole as $\frac{7}{7}$ (using the 'unit-rate' strategy as described in episode 6.4).

Measure Strategy

The measure strategy consists of building-up one whole from a given fractional part of the unknown quantity by using the fractional part as a unit with which to measure the unknown quantity. Simultaneously, it involves coordinating the fractional part with the measure of the known quantity. This strategy was used only in problems 2.31 and 2.32 by Cole, Brian and Aileen. The three interview protocols that follow illustrate this measure strategy.

Episode 6.1: *Cole* [Problem 2.31]

Data.

L1 I: There are 30 marbles in the box. The number of marbles in the box is two fifths of the number of marbles that you have.

L2 C: Oh, that's easy. That would be ...

L3 T: It would be 75 marbles. (T stands for Ted, Cole's partner)

L4 C: 75.

L5 I: How do you get this?

L6 C: Thirty, thirty would be sixty. Fifteen would be.

He gave the following explanation at a later point:

L7 C: I just took it because it said two fifths and that said it was two fifths. Thirty equals two fifths. So multiply thirty by two equals sixty. Half thirty would be 15. Add 15 to 60 will be 75

Analysis. Cole considered $\frac{2}{5}$ as a quantity (or as a unit of quantity as Behr et al. (1992) refer to such type of units) having a measure of 30 units. He interpreted a whole as $\frac{2}{5}+\frac{2}{5}+\frac{1}{5}$ as can be deduced from L6 and L7. This type of building-up reasoning can be regarded as involving the coordination of two different units: a composite fraction unit ($\frac{2}{5}$) and an integer composite unit having a measure of 30 units.

Episode 6.2: *Aileen and Brian* [Problem 2.31]

Data. Brian gave the following justification for this problem:

L1 B: 75.

L2 A: Are you sure, 75?

L3 B: Because it's two fifths, so times two makes sixty and that's four fifths and half of 30 is 15. Sixty plus fifteen is 75.

L4 A: Yeah, you are right.

Analysis. Brian's justification in L3 shows that the strategy he used to construct one whole is similar to that of Cole. By considering the measure of two one-fifth units as 30, he constructed $2\frac{1}{2}$ such units additively ($\frac{2}{5}+\frac{2}{5}+\frac{1}{5}$) to obtain the measure of one whole unit as $30+30+15=75$.

Episode 6.3: *Aileen* [Problem 2.32]

Problem 2.32: Eighteen liters of water were poured into a tank, filling the tank to $\frac{3}{7}$ of its volume. What is the volume of the tank?

Data.

L1 A: What I did is, if 18 liters is three sevenths, you can multiply that by 2 and get 6 seventh and that brings you to 36 and then I did 18 divided by 3 gives you 6 and 36 plus 6 is 42.

Analysis. Aileen worked with the composite unit $\frac{3}{7}$ (as a quantity) and constructed one whole additively as $\frac{3}{7}+\frac{3}{7}+\frac{1}{7}$ (similar to Cole). Simultaneously, she coordinated $\frac{3}{7}$ with 18 liters of water. She interpreted $\frac{3}{7}$ as 3 one-seventh units and this allowed her to find the measure of $\frac{1}{7}$ of the tank as 6 liters.

‘Unit-rate’ Strategy

In solving the fractional situations in Set 2 (algebraically equivalent to $ax = b$ or $\frac{p}{q}x = \frac{s}{t}$), the ‘unit-rate’ strategy consists of first finding the measure of one fractional unit ($\frac{1}{q}$) of the unknown quantity x and then determining the measure of one whole by multiplying by q. For example, if a rod measuring 18 inches is $\frac{3}{4}$ of some other rod ($\frac{3}{4}x = 18$), then $\frac{1}{4}$ of the other rod is 6 inches, and the total length is $4 \times 6 = 24$ inches. Beckmann (2008) refers to this strategy as “Going through 1” (p. 357). Such a strategy tends to be prompted when a divisibility relation between a and b is available or could be set up. Besides the divisibility relation, the interpretation of the given fractional quantities in terms of unit fractions is also a necessary resource to set the multiplicative relation between a and b. I present four interview protocols to show how the interpretation of the problem situations using a ‘unit-rate’ (a concept-in-action) afforded the students the necessary resources to solve the problem. Episodes 6.4 to 6.7 also show that $ax = b$ can be interpreted as a statement of proportionality.

Episode 6.4: *Ted* [Problem 2.32]

Data:

L1 I: Eighteen liters of water were poured into a tank, filling the tank to three seventh of its volume. What is the volume of the tank?

⋮

L2 T: Well, I multiply, I mean I divide 18 by 3, is that right or was it 3 by 18? I got 6 and I multiplied 6 by 7 to get 42.

⋮

L3 T: 18 was three seventh of it.

L4 I: Right.

L5 T: So I thought may be I could divide 18 by 3 to get 6 and well 6...

L6 I: Six represents what?

L7 T: Six represents one seventh of the whole tank. So I multiplied by 7 to get the whole tank.

Analysis. Ted's strategy was to find the size of $\frac{1}{7}$ from which he could find seven sevenths (L2). He interpreted $\frac{3}{7}$ as 3 one-seventh units, and this prompted him to observe the divisibility relation between 18 and 3. Such an interpretation allowed him to construct one whole starting from part of the unknown quantity.

Episode 6.5: ***Aileen and Brian*** **[Problem 2.42]**

Problem 2.42: Candy bar A is $\frac{1}{4}$ unit long. Its length is $\frac{2}{5}$ of candy bar B. What is the length of candy bar B?

Data.

L1 A: I was thinking that if one fourth equal two fifths then one eighth would equal one fifth and so one eighth is one half of one fourth. And so you could double them for if (inaudible), and the two fifths which will give you two fourths equal four fifths and then you would add the one eighth that would give you five fifths.

L2 I: What about Brian?

L3 B: I just saw that if it was one fourth and two fifths, then it's the same thing as two eighths, uh, putting that down just makes one fifth, so one eighth would equal one fifth. Then since there is five fifths in total, there are going to be five eighths.

Analysis. Both Aileen and Brian reasoned quantitatively to establish the relationship between the measure of the given quantity (Bar A as 1/4 unit long) and the given quantitative relationship (Bar A is 2/5 of candy bar B). They could observe that one fifth of candy bar B has a length of one eighth. Aileen constructed the whole by repeating $\frac{2}{5}$ and its associated measure $\frac{2}{8}$ two and a half times using a measure strategy (L1) while Brian multiplied $\frac{1}{8}$ five times to get $\frac{5}{8}$. This episode shows that the divisibility relationship between $\frac{1}{4}$ (interpreted as $\frac{2}{8}$) and $\frac{2}{5}$ prompted the students to set the multiplicative relationship between the known and unknown quantities.

Aileen's strategy can be characterized as follows:

$$\tfrac{1}{4} \rightarrow \tfrac{2}{5}$$
$$\tfrac{1}{2}(\tfrac{1}{4}) \rightarrow \tfrac{1}{2}(\tfrac{2}{5}) \text{ and } \tfrac{1}{8} \rightarrow \tfrac{1}{5}$$
$$\tfrac{2}{5} + \tfrac{2}{5} + \tfrac{1}{5} \rightarrow \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8}$$

Brian's strategy can be characterized as follows:

$$\tfrac{1}{4} \rightarrow \tfrac{2}{5}$$
$$\tfrac{2}{8} \rightarrow \tfrac{2}{5}$$
$$\tfrac{1}{8} \rightarrow \tfrac{1}{5}$$
$$5(\tfrac{1}{5}) \rightarrow 5(\tfrac{1}{8})$$

Thinking about a known quantity (i.e., bar A has a measure of $\frac{1}{4}$ unit) that is two times (two one-fifth units) as much as an unknown quantity caused them to reverse their thinking to deduce that half of the unknown quantity (i.e., one one-fifth unit) corresponds to half of the measure (i.e., $\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$).

Episode 6.6: *Brian* [Problem 2.62]

Problem 2.62: Candy bar A is 2 units long. Its length is $\frac{3}{4}$ of candy bar B. What is the length of candy bar B?

Data.

L1 B: Two and two thirds.

⋮

L2 I: How do you get two and two thirds?

L3 B: Because if you divide two by three, then each individual one third of the two should be two third of the whole unit. And so you just, if that's already three fourths then you just add an extra two third to it (inaudible) to be two and two third.

At a later point, I asked Brian to illustrate his solution on JavaBars. He made a bar 2 units long and divided each of them into three. He shaded the partitions as shown in Figure 6.1 so that the three composite units of two thirds are apparent. This form of splitting where the 2-unit bar is partitioned in 2 successive stages has been termed distributive splitting by Hackenberg (2005).

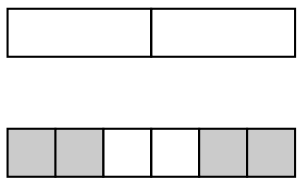


Figure 6.1. Brian's strategy to split a 2-unit bar in 3 equal parts

He gave the following explanation:

L4 B: So of these three, this one is going to be two thirds of that piece (the first one unit); this is going to be two thirds of this piece (the second one unit). The individual extra one is two thirds because that's split into three. But it has to be four of those, so you would add on an extra two of those thirds.

Analysis. This is another key episode where there is clear evidence of reversible reasoning. Brian conceptualized $\frac{3}{4}$ as 3 one-fourth units (i.e., the known amount is 3 times $\frac{1}{4}$ of the unknown amount). This led him to reverse his thinking to deduce that $\frac{1}{3}$ of the unknown amount (i.e., one one-fourth unit) has a measure of $\frac{2}{3}$ units. Another way to look at Brian's reasoning is that he interpreted 2 units as 3 two-third units to be able to set a one-to-one correspondence with $\frac{3}{4}$ or 3 one-fourth units. This allowed him to deduce that the quantity one-fourth unit has a measure of

$\frac{2}{3}$. He then added $\frac{1}{4}$ to $\frac{3}{4}$ and correspondingly $\frac{2}{3}$ to 2 to obtain $2\frac{2}{3}$. In terms of a theorem-in-action, Brian's reasoning can be explained as follows:

$$\tfrac{3}{4} \rightarrow 2$$

$$\tfrac{3}{4} \rightarrow 3(\tfrac{2}{3})$$

$$\tfrac{1}{4} \rightarrow \tfrac{2}{3}$$

$$\tfrac{3}{4} + \tfrac{1}{4} \rightarrow 3(\tfrac{2}{3}) + \tfrac{2}{3} = 2 + \tfrac{2}{3}$$

It is important to highlight that though there is no divisibility relationship between 2 and 3, Brian could re-conceptualize 2 units as 3 two-third units. He reasoned with two levels of unit when he interpreted $\frac{3}{4}$ as 3 one-fourth units. However, the interpretation of 2 in $\frac{2}{3}$-units to construct one whole of the unknown shows that he reasoned with three levels of units: Each $\frac{2}{3}$ unit is $\frac{1}{3}$ of the 2-unit (so its measure is two thirds) and at the same time each $\frac{2}{3}$ unit is $\frac{1}{4}$ of the unknown quantity (its relationship to the unknown). In other words, he coordinated two different fractions ($\frac{1}{3}$ and $\frac{1}{4}$) having different referents (or wholes) at the same time.

Another explanation for Brian's response can be given in terms of the 'dual view' (Hackenberg, 2005) or commutativity with which he interpreted bar A. The 2-unit long candy bar is a unit of two units into which three units have been inserted (the first 3 levels of units). The resulting 6 partitions are then reinterpreted as a bar consisting of 3 units, each of which contains 2 units (the second 3 levels of units). This procedure involves recursive partitioning and distributive splitting.

Episode 6.7: ***Aileen and Brian*** **[Problem 2.73]**

Problem 2.73: Candy bar A is $1\frac{3}{4}$ (or $\frac{7}{4}$) unit long. Its length is $\frac{3}{5}$ of candy bar B. What is the length of candy bar B?

Data:

L1 I: So which one would be longer?

L2 A: B.

L3 I: Brian?

L4 B: If that is seven fourths then it would be three fifths, so 7 divided by 3 is

L5 A: Is …

L6 B: One and one sixth. So, if one and one sixth is one fifth, then one and one sixth times 5 would be 5 and 5 sixths would the whole B.

L7 I: 7 divide by 3 is how much?

L8 B: One and one seventh.

L9 I: It's two and …

L10 A: It's two and …

L11 I: Two and one third.

L12 A: So (you) would do two and one third times two, which is going to be four and two thirds added on to that so it's going to be 11 and two thirds.

L13 I: Can you represent this in the diagram? Can you explain what you did or you want to make the diagram?

L14 A: I'll explain. What I said was seven fourth is equal to three fifths. So I did seven divided by 3 to try to get one fifth by itself. And if you (do) seven divided by three you get two and one third. And then you do two and one thirds times 2 to get the remaining two pieces and that gave me 4 and 2 thirds and then I added the four and two thirds to the original seven and that gave me 11 and two thirds pieces.

She made the following construction:

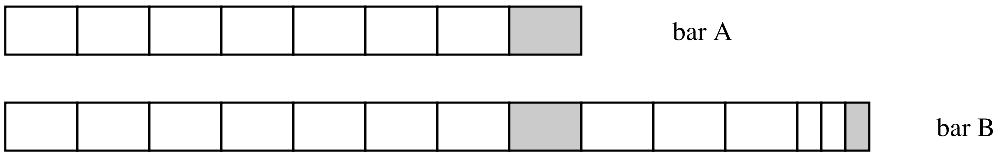


Figure 6.2. Aileen's construction of candy bar B from candy bar A

Analysis. Brian interpreted $\frac{7}{4}$ as 7 one-fourth units and $\frac{3}{5}$ as 3 one-fifth units, and this allowed him to use his whole number knowledge to promptly compare 7 units in terms of 3 units to deduce the measure of $\frac{1}{5}$ unit of the unknown quantity. In other words, he set a one-to-one correspondence between $\frac{7}{4}$ and $\frac{3}{5}$ to deduce the measure of $\frac{1}{5}$ unit of the unknown quantity. However, he mistakenly computed 7/3 as $1\frac{1}{6}$ and multiplied this result by 5 to get $5\frac{5}{6}$ (L6). In his calculation, he considered 7 as 7 whole units rather than 7 one-fourth units. Building on Brian's 'unit-rate' strategy, Aileen inferred that the unit of quantity $\frac{1}{5}$ would have a magnitude of $2\frac{1}{3}$ units (rather than $2\frac{1}{3}$ one-fourth units). Thus, she constructed one whole as $\frac{3}{5}+2(\frac{1}{5})$ with corresponding measure $7+2(2\frac{1}{3})=11\frac{2}{3}$ rather than $7(\frac{1}{4}\text{-unit})+2[2\frac{1}{3}(\frac{1}{4}\text{-unit})]=11\frac{2}{3}(\frac{1}{4}\text{-unit})$. Both participants lost track of the one-fourth unit as they were involved in multiplicatively comparing the 7 one-whole units to the 3 one-fifth units. This problem requires the multiplicative comparison of two composite fractional units or two units of units: 7 one-fourth units and 3 one-fifth units. The result is a non-integer quantity ($2\frac{1}{3}$) to be interpreted in terms of fourths, and this explains why the current problem can be demanding when such a multiplicative approach is used. By dividing 7 by 3, Brian gave evidence of reversing his thought process in this situation though he lost track of the one-fourth unit.

Guess-and-Check Strategy

In this section, I illustrate how the students used guess-and-check strategies to solve particular problems. This strategy arose when they did not compare the two quantities multiplicatively. I present three such episodes.

Episode 6.8

Problem 2.52: Parking lot A can hold 55 cars. It can hold $1\frac{2}{3}$ as many cars as parking lot B. How many cars can parking lot B hold?

Data. Aileen's response to problem 2.52

L1 A: So she would start, I mean parking lot B would have 33 cars and then parking lot A would have 22 extra because it's two thirds.

L2 I: But how did you get the 33?

L3 A: I have no clue. I was kind of thinking numbers divisible by 3 and 33 just kind popped into my head and then 22. But I don't really know why.

Analysis. Aileen's response shows that she used the third in $1\frac{2}{3}$ as a pointer to consider numbers that are divisible by 3. She exploited her knowledge of multiples to determine the size of the unit that would give the measure of 55 as $1\frac{2}{3}$ of that unit, thus performing the partitive division by guess-and-check. She did not appear to interpret $1\frac{2}{3}$ as $\frac{5}{3}$ or 5 one-third units as she did in other problems where the multiplicative approach could be cued because of the ready availability of divisibility relation.

Data. Ted's and Cole's response to problem 2.52

L4 I: Let's listen to what Ted is saying. How do you get 33 cars? You said 33? OK. Go ahead.

L5 T: Well, it just came up to me when I divided 55 by 5 to get 11 and I thought that if eleven multiply by 3.

L6 C: Oh.

L7 T: is 33, to make 33 to be a whole, there will be 22 left ...

L8 C: I should have done that.

L9 T: which will make it two thirds of that. Like, OK, let's see. If I divide 5 by, I mean 55 by 5 to get 11 and if it is one and two thirds, still (inaudible) he multiply 11 by 5 will be 55. Well, I thought, it came up to me if I could like multiply 11 by 3 to get 33 and. So, I thought that 33 could be the one whole and 22 could be, and then the rest of the pieces would be 22 over 33 because I am making 33 as the whole. So it will like 22 out of 33, so I just divide those two by 11 to get two thirds.

L10 I: But where does the five come from? It's just like this?

L11 T: The five came from this, the 55. The 5 and the 5 cars and well I know that 11 times 5 was 55, so I just thought that was a solution to like getting it.

L12 I: A trial?

L13 T: Yeah, like a trial to get it. So, I got it. I have done estimation (inaudible).

Analysis. In contrast to Aileen who looked for numbers divisible by 3, Ted focused on numbers divisible by 5. He determined the number of cars in parking lot B as 33 by dividing 55 by 5 and multiplying the result by 3. His knowledge of multiples afforded him the path to this solution as can be inferred from L5 and L11. Ted explicitly explained in L9 that he chose 33 partitions as the unit and interpreted the remaining 22 partitions as 2 thirds of that unit. He also explicitly mentioned that he used an estimate (L13). This episode also shows that Ted could solve such a reversibility situation not by explicitly reasoning reversibly but by using his knowledge of multiples and estimation.

Episode 6.9: ***Aileen and Brian*** **[Problem 2.72]**

Problem 2.72: Candy bar A is $\frac{4}{9}$ unit long. Its length is $1\frac{2}{5}$ (or $\frac{7}{5}$) of candy bar B. What is the length of candy bar B?

Data. Aileen partitioned the four pieces representing $\frac{4}{9}$ into five pieces each as shown in Figure 6.3 below.

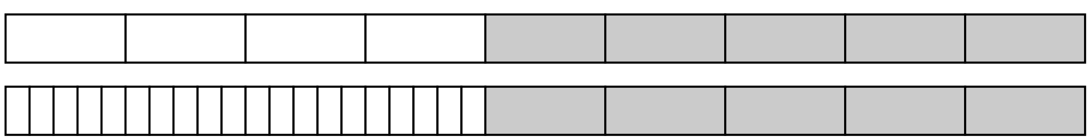


Figure 6.3. Aileen's subdivision of 4 ninths into fifths

Then she started to guess-and-check how many pieces out of the 20 should she choose as the unit (partitive division - looking for the unit) so that A is $1\frac{2}{5}$ of B. The following segment illustrates her guesses and Brian's evaluation of Aileen's suggested unit:

L1 A: Would be like two big pieces (referring to $\frac{2}{9}$ or half of candy bar A) and like half. OK. I think I just figured it out (inaudible). No, I didn't.

L2 I: You can always try.

L3 A: May be that (the 4 cells representing $\frac{4}{9}$) could be the one. One and two fifths. So it would be like from here over (moving the cursor over the 4 cells representing $\frac{4}{9}$) would be the one and then you add those two pieces for the two fifths. Right.

L4 B: That would be two fifths of one not two fifths of the B one.

L5 A: Say that (pointing to the 12 squares shaded in bold in Figure 6.4) was your one. You, if, bring that up (pointing to the first bar in Figure 6.4), you are going to have these two pieces (the first two cells in the first bar) and two pieces of that one (meaning the two small pieces). And then you can multiply by two fifths and two fifths is equal to 8 tenths. I mean 8 twentieths. So you would have to add the 8 onto it and that would give you the extra one (referring to the fourth cell in the first bar).

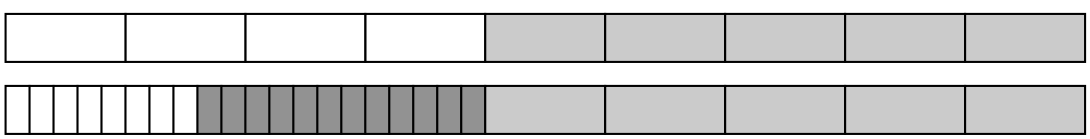


Figure 6.4. Aileen's selection of 12 partitions as a whole

Analysis. Because Aileen did not interpret $\frac{4}{9}$ as 4 one-ninth units and $\frac{7}{5}$ as 7 one-fifth units, and because of lack of a divisibility relation between the numerators 4 and 7, she did not compare the two quantities multiplicatively. Using the denominator of seven fifth as a pointer, she partitioned the 4 one-ninth units into 5 parts each and attempted to guess the size of one whole. She chose a different number of partitions as one whole and verified if $1\frac{2}{5}$ corresponded to the length of bar A ($\frac{4}{9}$). Initially, she chose $\frac{2}{9}$ or half of bar A as a whole (L1), then she considered $\frac{4}{9}$ unit (bar A) as a whole (L3). Finally, she used 12 partitions in bar B as one whole (L5). Because $\frac{8}{20}$ is equivalent to $\frac{2}{5}$, she was convinced that her answer was correct and did not realize that she was using two different units: 12 units as one whole and 8 units out of 20.

Episode 6.10: *Ted* [Problem 2.74]

Problem 2.74: Candy bar A is $1\frac{2}{3}$ (or $\frac{5}{3}$) unit long. Its length is $1\frac{5}{7}$ (or $\frac{12}{7}$) of candy bar B. What is the length of candy bar B?

Data. Ted and Cole created a bar, partitioned it into 6 and shaded the last partition. Then they divided each of the thirds into seven as shown in Figure 6.5.

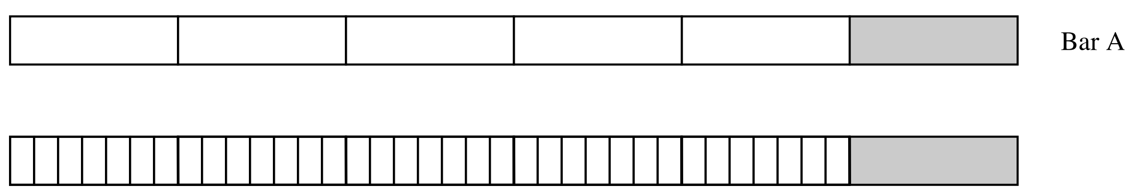


Figure 6.5. Ted's and Cole's subdivision of thirds into sevenths

L1 T: Uh, well, since it's one and five seventh of bar B, then I thought about if it's got to be like at least bigger than a half of that part.

L2 I: At least half of the red part (the partitioned part in the second bar) you mean?

L3 T: Yeah. Because if not then the red part is going be like at least like two times or three times, something (inaudible) like that. So we got to make at least one times bigger than that. So I thought that if we could make the bar, bar B, candy bar B be at least bigger than half of 35 we can at least have more chances to find (it).

They tried other guesses like 14, 12 and 21 partitions to represent a whole. Cole explicitly mentioned “I usually guess a lot by estimates. I estimate a lot.”

Analysis. The two mixed numbers or improper fractions with relatively prime numerators in the problem prevented them from constructing candy bar B from candy bar A or from setting the multiplicative relation between the two quantities. After dividing each of the thirds in bar A into sevenths as dictated by the denominator in $1\frac{5}{7}$, they chose different number of partitions as a whole and checked whether the relation between bar A and bar B held. Ted could also observe that the length of bar B should be greater than half of bar A because the length of bar A is less than 2 times that of bar B. This response is a form of reversible reasoning.

Constructive Resource: Interpreting Mixed Numbers in Terms of Fractional Units

A mixed number and an improper fraction can represent two different concepts-in-action and can cue different resources in a problem solving situation. Interpreting $1\frac{2}{5}$ as $\frac{7}{5}$ (seven one-fifth units) or as a composite of fractional units enables one to use his/her whole number knowledge to conceptualize such a quantity. Such a conceptualization in terms of fractional units is even more productive in multiplicative comparison tasks as is the case in the present study. The four episodes presented below illustrate how the interpretation of mixed numbers in terms of fractional units served as a resource in constructing a whole from a quantity greater than one.

Episode 6.11: *Ted and Cole* [Problem 2.51]

Problem 2.51: In a seventh-grade survey of lunch preferences, 48 students said they prefer pizza. This is one and one half times the number of students who prefer fries. How many students prefer fries?

Data.

L1 I: One and one half times, one and a half times. This 48 pizza is one and a half times.

L2 C: Divide by 3.

L3 T: It's 16.

L4 I: Why do you say divide by 3?

L5 C: 16, 16 is 32, right.

L6 T: Yeah.

L7 C: So, 32.

L8 I: Do it and then we will verify. Why did you divide by 3? Just dividing?

L9 C: (inaudible)

L10 T: Because if we, if we. Well, he did that because he divided the one whole in two which we got (inaudible) into halves it will three pieces of it, three halves.

L11 I: One whole divide into…?

L12 T: Yeah. It will be three halves with the other. See, one whole and one half.

L13 I: Right.

L14 T: You divide the one whole into halves.

L15 I: You have three halves.

L16 T: Yes. I think that's how he divided it.

Analysis. A multiplicative comparison relation involving a mixed number (e.g., quantity A is $1\frac{1}{2}$ times the amount of quantity B) may be less intuitive compared to one involving whole numbers (e.g., quantity A is 2 times as large as quantity B). Interpreting $1\frac{1}{2}$ as three units of $\frac{1}{2}$ (i.e., unitizing in fractional units) is the key resource that makes the multiplicative relationship between the known and unknown quantities apparent. Such an interpretation allowed Ted and Cole to use their knowledge of whole number multiplication to compare quantities involving fractional relationships.

Episode 6.12: *Aileen and Brian* [Problem 2.51]

Problem 2.51: In a seventh-grade survey of lunch preferences, 48 students said they prefer pizza. This is one and one half times the number of students who prefer fries. How many students prefer fries?

Data.

L1 I: These 48 students who prefer pizza, right. This is one and a half.

L2 A: What is 48 times 2? Or.

L3 B: No, it’s 48 divided by three.

L4 A: How do you get 48 divided by 3?

L5 B: Because, if it is one and a half times then that means that two thirds of that is how many for fries. So, it has to be divided by 3.

L6 I: 16, right.

L7 A: So 16 plus 24.

L8 B: 16 times 2.

L9 A: 32.

L10 I: 32. How do you get the two thirds, Brian?

L11 B: Because it's one and a half times this, so it's one and (inaudible). The halves have to be separated into three halves which would be three thirds of the total 48.

Analysis. Brian's interpretation of one and a half as three halves (L5) offered him the necessary resource to reverse his thought process to undo the making of $\frac{3}{2}$ (i.e., to construct one whole from $1\frac{1}{2}$). He interpreted one half as 'one third' of 'one and a half' as evidenced by his statement: "three halves which would be three thirds of the total 48" (L11). Brian's re-conceptualization of $1\frac{1}{2}$ as 3 halves, two thirds of which produce one whole, shows he reasoned with three levels of units, coordinating two fractions within the same bar – halves and thirds.

Episode 6.13: *Brian* [Problem 2.52]

Problem 2.52: Parking lot A can hold 55 cars. It can hold $1\frac{2}{3}$ as many cars as parking lot B. How many cars can parking lot B hold?

Data.

L1 B: I was thinking that, that would be 5 thirds and so it's 5 thirds, 5 goes into 55 eleven times.

⋮

L2 I: Brian, you were saying that 5 thirds and 55.

L3 B: One and two thirds is the same thing as five thirds and so put that into 11, I mean 55, then 55 divided by 5 is 11.

L4 A: And then 11 times 3.

L5 B: Each one third is 11 and it's 5 thirds.

Analysis. Brian's interpretation of the mixed number $1\frac{2}{3}$ as $\frac{5}{3}$ or 5 thirds (L1) allowed him to coordinate the given measure (55 cars) and the quantitative relationship (5 thirds) to deduce that one third would correspond to 11 cars. Such an interpretation of 5 thirds was not readily available to the other participants, and they did not use such a multiplicative approach.

Episode 6.14: *Ted* [Problem 2.65]

Problem 2.65: Candy bar A is 2 units long. Its length is $1\frac{1}{2}$ of candy bar B. What is the length of candy bar B?

Data.

L1 T: Yeah, we would divide them by threes.

After representing 2 units on JavaBars (Figure 6.6(a)), Ted divided each of the two units into three parts (Figure 6.6(b)) and pulled out one part and repeated it 3 times as shown in Figure 6.6(c).

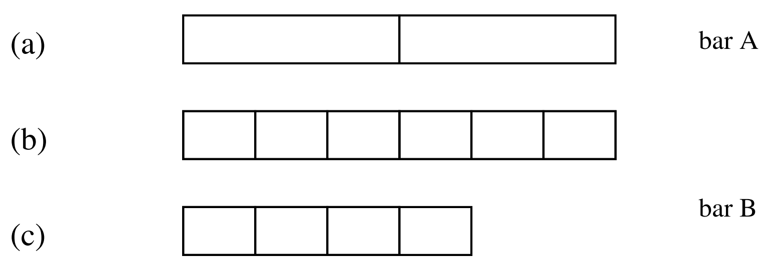


Figure 6.6. Constructing one whole from $1\frac{1}{2}$ units

L2 I: So what is the length of candy bar B? The length of the first one is two units long. What is the length of the other?

L3 C: One and one third.

Analysis. Ted's statement "we would divide them by threes" in L1 shows that he interpreted $1\frac{1}{2}$ as three half-units which led him to divide each of the two units of candy bar A into 3 partitions each. After obtaining 6 total partitions (Figure 6.6 (b)), Ted re-conceptualized the bar in terms of both thirds and halves: $\frac{2}{3}$ of 3 halves produces 2 halves or one whole, thus giving evidence of coordinating 3 levels of units. This allowed him to undo the making of $1\frac{1}{2}$, though the visual representation may have facilitated the unitizing procedure.

In summary, episodes 6.11-6.14 show that the representation of a rational number (as a mixed number or as an improper fraction) influences the type of resources that may be cued in a problem-solving situation. For example, Brian's interpretation of $1\frac{2}{3}$ as 5 thirds in episode 6.13 allowed him to reason reversibly compared to Aileen's interpretation of $1\frac{2}{3}$ as a mixed number in episode 6.8, where the denominator served as a pointer leading her to use a guess-and-check strategy.

Denominator as a Pointer for Division: A Faulty Theorem-in-action

In various situations, the participants focused on the denominator of the given fractional relationship as a pointer to solve the problem, as a theorem-in-action. They used the denominator of the fractional relationship in two ways: (i) either they split the known quantity in terms of the denominator (ii) or they attempted to make the denominator of the measure and quantitative relationship equal so as to have something in common, which from their perspective would facilitate the comparison. Tzur (2004) made similar observations in his analysis of two fourth graders' reversible fraction conceptions where the denominator acted as a mental pointer to partition the known quantity to produce the unknown quantity. He asked the two students to construct the whole starting from $\frac{5}{8}$ represented in a non-partitioned JavaBar. The students divided the bar by 8 rather than 5. Such a focus on the denominator (which seems to be due to the intuitive part-whole conception of fraction) skews students' attention and they get entangled in constraining situations. I present one episode (6.15) to illustrate how the denominator acts as a pointer for division and another four episodes (6.16-6.19) to show how the students attempted to make the denominators of the measure and the quantitative relationship equal as a fallback strategy to compare the known and unknown quantities.

Episode 6.15: ***Aileen and Brian*** **[Problem 2.61]**

Problem 2.61: Candy bar A is 5 units long. Its length is $\frac{7}{8}$ of candy bar B. What is the length of candy bar B?

Data. They started the problem by making a bar of 5 partitions (Figure 6.7).

L1 A: You divide them each into eighths

L2 B: Eighths?

L3 A: And that would give you 40 total pieces, it will be 40 eighths.

L4 B: Then how much is each piece?

L5 A: That would mean that each eighth is 5 pieces. So you would add, another one (pointing to one partition) of these, 5 little pieces, so like.

L6 B: Each one is one and two sevenths, I think, right.

L7 A: I was thinking you divide it like that and you would add 5 of these pieces to it. What do you think?

She pulled out one of the partitions and divided it into 8.

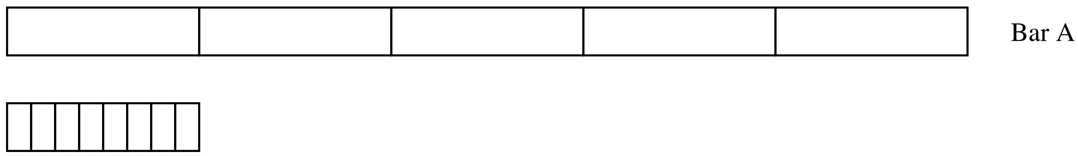


Figure 6.7. Aileen's subdivision of one unit into eighth

L8 I: What are you doing Aileen?

L9 A: I was thinking you could divide each like fifth into eighth and then we could do 40 total eighths and then if 40 eighths is equal 7 eighths.

Analysis. Aileen's response shows that she divided each of the 5 units in candy bar A into 8 pieces to get 40 pieces. She showed this subdivision for only one of the fifths in JavaBars in Figure 6.7. When Brian asked her what is the size of one small piece (i.e., $\frac{1}{40}$ of the 5-unit bar) in L4, she mentioned that "each eighth is 5 pieces" (L5). She intended to add 5 small partitions to bar A to construct bar B (L5 and L7). The 5 small pieces actually represents $\frac{5}{40}$ or $\frac{1}{8}$ of the given bar A of length 5 units rather than $\frac{1}{8}$ of the unknown bar B whose multiplicative relation to bar A is given as $\frac{7}{8}$. In other words, the two eighths have different referents: $\frac{1}{8}$ of the known quantity A and $\frac{1}{8}$ of the unknown quantity B. She tried to relate $\frac{40}{8}$ to $\frac{7}{8}$ in L9 but did not proceed further probably because of the absence of a direct proportional relationship between the numerators 40 and 7. I have termed this type of conflict where one confounds the measure of the known quantity (here 5 units = 40 one-eighth units) and the given quantitative relationship (candy bar A is $\frac{7}{8}$ of candy bar B) as the quantity-measure conflict and this will be discussed more thoroughly in the next section. The point to note here is that the denominator 8 in $\frac{7}{8}$ acted as a pointer for division. Brian's interjection in L2 shows that he could observe that bar A should not have been divided by 8 as Aileen did. He compared 7 units in terms of 5 units in L6 to get $1\frac{2}{7}$ rather than $1\frac{2}{5}$, probably an instance of the faulty-remainder theorem-in-action (as discussed in Chapter 5).

Episode 6.16: *Aileen and Brian* [Problem 2.72]

Problem 2.72 Candy bar A is $\frac{4}{9}$ unit long. Its length is $1\frac{2}{5}$ (or $\frac{7}{5}$) of candy bar B. What is the length of candy bar B?

Data. I asked them to represent $\frac{4}{9}$ on JavaBars. After a relatively long pause, I asked Brian and Aileen if they had any idea for how to start the problem. They gave the following response:

L1 B: I was thinking how many times would 7 go into 4 and yeah.

L2 A: I was thinking that you could divide each section into 5 so you would have 20 as a common denominator because there are four pieces and you want it into fifth. So each piece would equal five fifths and then the whole thing has to equal 7 fifths.

Aileen partitioned the four pieces representing $\frac{4}{9}$ into five pieces each as shown by Figure 6.3 below.

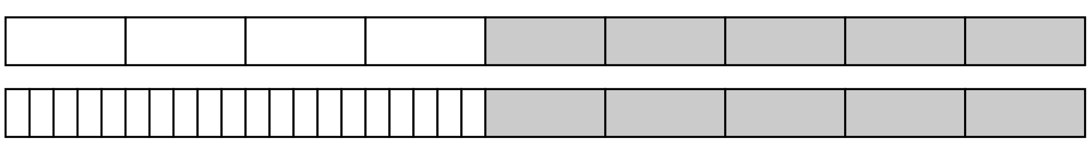


Figure 6.3. Aileen's subdivision of 4 ninths into fifths

Analysis. Like episode 6.15, the denominator acted as a pointer for division and this together with the indivisibility relation between 4 and 7 blocked the solution path. Aileen's theorem-in-action was to bring in a divisibility relation by transforming the given fractional quantities into another equivalent fraction, as can be inferred from L2 by her use of the term "common denominator". She changed the numerator of $\frac{4}{9}$ into 20 so that she could split it into fifths as her focus was on fifths rather than sevenths. Her goal was to split 4 one-ninth units into fifths. On the other hand, Brian's statement "how many times would 7 go into 4" suggests that he interpreted $\frac{4}{9}$ as 4 one-ninth units and $\frac{7}{5}$ as 7 one-fifth units that led him to compare 7 and 4 multiplicatively.

Episode 6.17: *Aileen and Brian* [Problem 2.71]

This episode further highlights how the theorem-in-action 'making the denominator equal' arises as a fallback strategy. Again the denominator of the fractional quantity expressing the multiplicative relationship acted as a pointer for division.

Problem 2.71: Candy bar A is $\frac{3}{4}$ unit long. Its length is $\frac{4}{5}$ of candy bar B. What is the length of candy bar B?

Data. They started the problem by constructing bar A. Aileen drew a bar and divided it into four partitions and shaded one of the parts (i.e., she represented $\frac{3}{4}$ by the three unshaded partitions). She made a copy of the representation for $\frac{3}{4}$ and divided each of the three partitions into 5. Then she pulled out one of the small subdivisions as shown in Figure 6.8(b).

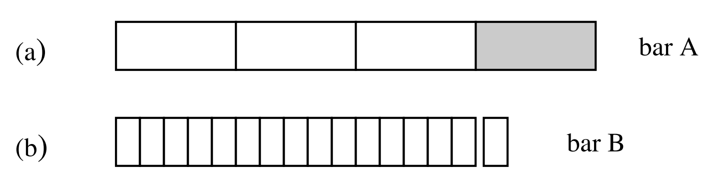


Figure 6.8. Aileen's construction of bar B from bar A

The following interview segment shows their response:

L1 I: This is candy bar B? How many pieces do you have?

L2 A: 16 twentieths, I think.

L3 I: 16 twentieths? How does it represent four fifths? OK. Why did you divide by, you divide each of them by 4.

L4 A: By 5.

L5 I: By 5, yes.

L6 A: So that they could have a common denominator of 20.

L7 I: Right.

L8 A: And so.

L9 I: Common, you mean…

L10 A: Like the same, you could compare them easier.

L11 I: Which one easier you are saying?

L12 A: Both of them. So like, for three fourths you would multiply numerator and denominator by 5 which would give you 15 twentieths and then four fifths would be 16 twentieths. And you could see that you would just add, wait, you would add four pieces. Not just one.

L13 I: Brian?

L14 B: I was doing something kind like that except for I divided, (inaudible). No I did this. I got 6 and then 8 and then times them by 5, so 6 over that (meaning 8), that would be 30 and 80. I had 40, and 4 and 5 times 8 were 32 and 40. So they still have a common denominator. I don't know where I was going with that.

(His worksheet showed the following representations: $\frac{6}{8} = \frac{30}{40} = \frac{32}{40}$.)

Analysis. Both Aileen and Brian attempted to make the denominator of the two given fractions (with relatively prime numerators) equal, thereby turning an uncongenial multiplicative comparison problem into a congenial one. Brian's strategy was to change $\frac{3}{4}$ to $\frac{30}{40}$ and $\frac{4}{5}$ to $\frac{32}{40}$. Similarly, Aileen's strategy was to make both fractions stand on the same denominator by rewriting $\frac{3}{4}$ as $\frac{15}{20}$ and $\frac{4}{5}$ as $\frac{16}{20}$ (L12). Initially, she added one partition to bar A ($\frac{3}{4}$ unit) to make bar B by comparing $\frac{15}{20}$ and $\frac{16}{20}$ and later added 4 small partitions so that bar B and the first bar (Figure 6.8(a)) were equal. By attempting to equate the two bars additively she lost track of the multiplicative relationship between the two quantities.

Episode 6.18: *Ted and Cole* [Problem 2.74]

This is one of the most demanding tasks in the current problem set. I presented the problem by reading both the mixed numbers as well as the improper fractions.

Problem 2.74: Candy bar A is $1\frac{2}{3}$ (or $\frac{5}{3}$) unit long. Its length is $1\frac{5}{7}$ (or $\frac{12}{7}$) of candy bar B. What is the length of candy bar B?

Data. They represented candy bar A on JavaBars by first partitioning a bar into 6 parts and shading the sixth partition. In other words, the 5 unshaded partitions represented $\frac{5}{3}$.

L1 T: Can we just find a denominator thing for (inaudible)?

L2 C: You can't (Talking to Ted). Because you divide them by two, it would be ten, you divide them by three, it would be like 16 or something.

L3 I: You are trying to divide by? Three?

L4 C: No we are trying to find one that we can divide by to…

L5 I: You can divide by what?

L6 C: We don't know. We are trying to find one that we can divide by, so we can take the 12 out of it.

L7 I: What did you write?

L8 T: I am just like to try a, (inaudible). Like equivalent denominator for both of them. So I just multiply 3 by 7 and 7 by 3 so it will be like 5 thirds times 7 seventh (doing the calculation in his worksheet) and 12 seventh times 3 thirds which I got for the 12 seventh will be 36, 21; and 5 thirds will be 35, 35, twenty-ones.

Analysis. This episode shows that the indivisibility relationship between the two fractional quantities prompted Ted to make the denominators equal, like Aileen. He multiplied $\frac{5}{3}$ by $\frac{7}{7}$ to get $\frac{35}{21}$ and $\frac{12}{7}$ by $\frac{3}{3}$ to get $\frac{36}{21}$. His strategy was to make the two given fractions ($\frac{5}{3}$ and $\frac{12}{7}$) comparable because they were incommensurate for him in the present form. On the other hand, Cole attempted to divide the 5 partitions representing candy bar A into 2 and 3 to verify if he could get a multiple of 12 out of them, but none of these partitioning strategies worked. His goal, however, indicates that he was relating the numerators multiplicatively rather than the denominators of the two fractions.

Episode 6.19: *Aileen and Brian* [Problem 2.74]

Problem 2.74: Candy bar A is $1\frac{2}{3}$ (or $\frac{5}{3}$) unit long. Its length is $1\frac{5}{7}$ (or $\frac{12}{7}$) of candy bar B. What is the length of candy bar B?

Data.

L1 I: OK, so the length of candy bar A is one and five seventh of candy bar B or the length of candy bar A is 12 sevenths of candy bar B. Can you construct candy bar B?

They represented candy bar A as shown in Figure 6.9(a).

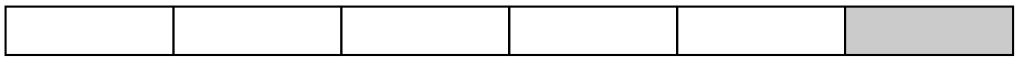


Figure 6.9(a) Representation of bar A

L2 A: What are you trying to do? (talking to Brian)

L3 B: Trying to get that (each third in bar A) to seventh. So, it would be 45 in total.

(He divided each partition by 7 as shown below).

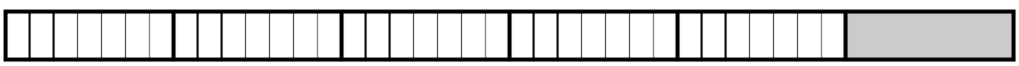


Figure 6.9 (b) Partitioning each thirds into 7

L4 A: Hum.

L5 B: Yeah, because it is divisible by both seven and five.

L6 A: 45 is not divisible by 7.

L7 B: Oh, 42. No wait.

L8 A: 42 is, no.

L9 B: 35 perhaps.

L10 A: 35 is not divisible by 3.

L11 B: Three!

L12 A: You have thirds, not fifth, or sixth.

L13 B: But, it's five thirds.

L14 A: If you use 21, it's divisible.

L15 B: It's five thirds, though. So if that equals to (inaudible)

L16 A: So you have thirds.

L17 B: It's not divisible by.

L18 A: That still is (inaudible) divisible by 3. Because if you are going to make it smaller, it has to be a factor or a multiple of 3.

L19 B: But, then it is not divisible by 5.

L20 A: (inaudible)

L21 I: You are trying to find a number which is divisible by?

L22 A: 3, 5, and 7 and I don't think there is?

They cleared the partitions in bar B and constructed bar A again as initially represented in Figure 6.9 (a).

L23 B: You split, you do 5 times 12.

L24 A: That equals 60.

L25 B: or something divisible by both.

L26 A: 60 is divided by 3 and 5 but not 7 but it is by 12.

⋮

Long pause.

L27 I: You can try to divide it by one of the numbers that you are suggesting and try to see if that makes sense.

L28 B: I don't think it has to be divisible by 3. It should just be divisible by 12 and 5. Because once you split like that isn't there any more the seventh.

L29 A: (inaudible) 21. So five thirds is equal to 35 twenty-first and 12 sevenths is equal to 36 twenty-first.

L30 I: You are making both of them in terms of twenty-one, twenty-first or whatever? So what do you observe? You have what?

L31 A: You would divide the top one, each piece into seven and then B would have one tiny piece less.

L32 I: You can try to divide and see if that makes sense?

She made the following construction by partitioning each of the five thirds in bar A by seven to get 35 partitions and deleting one of the partition to get a total of 34 partitions.

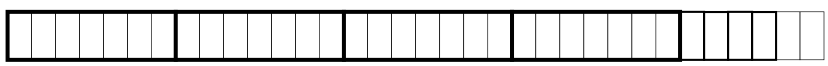


Figure 6.9(c) Representation of 34 partitions

L33 A: I think that's what B would be if it was on twenty-first. It would have one less because when you compare, woah, I just realized something here.

L34 I: Aileen, what did you say, one less?

L35 A: Yeah, but I think that's wrong because you are going to go.

L36 B: That's 20 twenty-first, not 12 sevenths.

L37 A: It's how many, what?

L38 B: Twenty twenty-first not twelve sevenths.

L39 A: Yeah, it is. But it is equal somehow. It should be 36, I mean 35.

L40 I: Why did you remove one of the pieces Aileen? Is there any reason, I am trying to figure out. Because you have 35 pieces in the red part. Right.

L41 A: What I did was 5 thirds times 7 which give you 35 twenty-first. And then I did 12, 12 sevenths times 3, so they would have a common denominator and that give you 36 over 21 which would mean that candy bar A is going to have 36. I don't know. They have.

L42 I: As if, you are using the 35 and the 36.

L43 A: Hum.

L44 I: What about Brian, any idea?

L45 B: I am pretty much lost, right now.

L46 A: Yeah, I cannot think about this one.

Analysis. This episode gives further evidence that the denominator (7) of the fractional quantity ($\frac{12}{7}$) expressing the multiplicative relation between the known and the unknown quantities acted as a pointer for division in the absence of divisibility relationships. It also shows that the two students focused on different concepts-in-action in their interpretations of the current problem. For Brian $\frac{5}{3}$ represented 5 one-third units (as can be deduced from L13 and L15). Similarly, for Brian $\frac{12}{7}$ represented the concept-in-action 12 one-seventh units whereas for Aileen $\frac{12}{7}$ was a representation of an improper fraction. Brian's statement in L28 "isn't there any more the seventh" shows that he conceptualized the seventh as a unit. Thus, Brian focused on the numerator while Aileen focused on the denominator. With this differential interpretation, each of the students looked for different routes in search of divisibility. Brian searched for a number that was divisible by 5 (from $\frac{5}{3}$) and 12 (from $\frac{12}{7}$). He was trying to multiplicatively compare 12 one-seventh units and 5 units as can be inferred from L28.

After failing to find a number that is divisible by 3, 5, and 7, Aileen changed her strategy by making the denominators of the two fractions equal. Thus, she converted $\frac{5}{3}$ to $\frac{35}{21}$ and $\frac{12}{7}$ to $\frac{36}{21}$ to get a better hold on the problem (L29). Again, for Aileen the denominator acted as a pointer for division because she divided each of the five thirds (bar A) into seven as can be observed from Figure 6.9 (c). She deleted one of the partitions because of the difference between $\frac{35}{21}$ and $\frac{36}{21}$. This task proved to be problematic for both of them.

In the remaining part of this data set, I highlight three other forms of constraint: (1) quantity-measure conflict, (2) constraint due to the numeric feature of the data, and (3) constraint due to the syntactic structure of the reversibility situations.

Constraint 1: Quantity-Measure Conflict

One pertinent observation across various problem situations is that the students could use the quantitative relationship between the known (bar A) and unknown (bar B) to construct the unknown quantity but, they could not find the measure of the unknown quantity. After constructing one unit of the unknown quantity, they measured its length as one unit, confounding the quantitative relationship and the measure of the known quantity. I have termed this conflict as the *quantity-measure conflict*. This conflict arises because two pieces of information have to be coordinated with one bar: a measure and a multiplicative relationship. In some cases the students also confounded the given multiplicative relationship between the two quantities as the measure (as illustrated in episode 6.21) right at the beginning of the problem, especially when a and b were both improper fractions in $ax = b$. The lack of flexibility to switch between the measure and quantitative relationship has also been observed in earlier research (Hackenberg, 2005). The incorrect measurement of the unknown quantity (bar B) can be explained in terms of unit structures as a lack of coordination of three levels of units. The reversibility situations may involve the coordination of two three-levels of units, one associated with the measure and the second with the multiplicative relationship within the same bar.

Episode 6.20 illustrates the quantity-measure conflict when two relatively prime integers are compared.

Episode 6.20: ***Ted and Cole*** **[Problem 2.12]**

Problem 2.12: Bar A weighs 7 pounds. Bar A weighs 5 times as much as bar B. What is the weight of bar B? ($5x = 7$)

Data. They made a bar of seven partitions and divided each into 5 to get 35 pieces as shown in Figure 6.10.

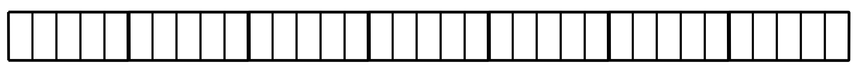


Figure 6.10. Subdivision of 7 units into fifths

L1 T: OK. Now. That is bar A, which we convert the seventh into 35.

L2 C: And that would make the one seventh and then you simplify and it will be 5.

L3 T: Yes.

L4 C: And that one (bar B) would be one.

L5 T: Yes.

L6 C: So bar B is one pound.

L7 I: Bar B is one pound?

L8 T: No.

L9 I: If bar B is one pound, bar A is 5 times as heavy, it is only 5 pounds. Ted, let’s look at what you have done.

L10 T: OK.

L11 I: You have divided the 7 pounds ...

L12 T: 7 pounds into 35 and so now 5 times which we could divide 35 by 5 ...

L13 C: It will be one and a half pound.

L14 T: which will be 7 and so we will just do like. We just pull out one of these. OK. Then we repeat this. OK. 1,2,3,4, it's 5, yeah, 6, 7. Right.

(He pulled out one of the partitions and repeated it 6 times.)

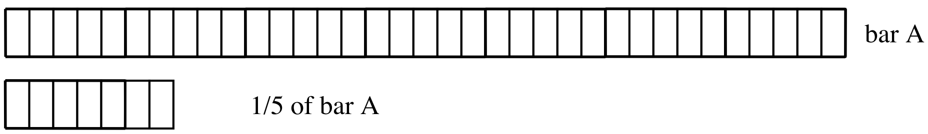


Figure 6.11. Construction of 1/5 of bar A

L15 I: OK, so what does the second one represent?

L16 T: The second one represents one fifth of the 35, which makes the 35 five times as much as the second one (reversible reasoning).

L17 I: Is bar B, is bar A five times heavier than bar (B)?

L18 T: Hum (agreeing by nodding his head)

L19 I: Can you see this from the diagram (Figure 6.11)? How?

L20 T: Because 35 and you divide that by 5, because it is 5 times as much as the bar B, so it will be 7.

L21 I: What about Cole?

L22 C: I get what he is saying.

L23 I: So what is the weight of bar B?

L24 T: OK.

L25 C: You have to simplify it. It will be one and something. It would be like one and a half.

Looking at the conflict in finding the measure of bar B at this point, I changed the problem from weight to length.

L26 I: OK. Suppose bar A, this is seven units long, can you see the first one is 7 units long and bar A is 5 times as long as bar B. Right. And you have constructed bar B here (referring to Figure 6.11). This is bar B. So what is the length of bar B?

L27 T: Seven thirty-fifths.

L28 I: This is seven units long. Is this one fifth?

L29 T: Yes.

L30 C: It's a seventh.

L31 T: It is, of seventh and so it will be seven over 35 or 35 over seven in case. And so if it is 35 over 7, it will be 5 times as much, it will be 5. But then if was 7 over 35, it will be one fifth because you will divide 7 and 35 by 7 to get one fifth. One into 5, yeah.

L32 I: What I want to know, what is the length of this bar B. How many units long?

L33 T: One and two fifth.

L34 I: OK. So, you are saying bar B, this is one and two fifths? The length of this one?

L35 C: I don't know.

L36 C: That's a fifth (referring to bar B). That's a fifth because if you had seven out of 35, 35, so you divided it and that would be 5.

L37 I: Cole, I did not follow. Can you ...

L38 C: You take 7 and multiply it by, I mean, multiplied by 5 will be 35. So you divide it, it's, it would be five and it would basically be one fifth because that thing (bar B) would go into this (bar A) five times.

L39 I: So, is it one fifth unit long?

L40 C: Yeah.

L41 I: What about Ted? Is this one fifth unit long?

L42 T: Yeah.

Analysis. By recursively partitioning the seven units into 5 each in Figure 6.10, Ted opened the way for the division of 7 by 5. Following the subdivision of the original bar of 7 pieces into 35 pieces, Cole mentioned “And that would make the one seventh and then you simplify and it will be 5” (L2), confounding the quantitative relationship (5 times) and the measure (7 units). This led them to deduce that the seven partitions in bar B is one pound (as can be inferred from L4 to L6) rather than one and two fifths pounds. Ted’s construction in Figure 6.11 confirm that he correctly performed the quantitative operation in redistributing the 7 groups of 5 partitions each as 5 groups of 7 partitions each. This is also supported by his statement in L16 and L20. However, they did not assign the right measure to the new unit (bar B) in Figure 6.11. They considered the quantitative relationship (5 times) as a measure. Cole made an approximation of the new unit of 7 small partitions as one and a half pounds in L13 and L25. Ted stated the answer one and two fifths in L33 by comparing five partitions of bar A to the seven partitions of bar B in Figure 6.11. As I prompted them to explain their answer, Ted did not differentiate between the quantitative relationship and measure (L37-L42).

The problem statement in the current situation presents two pieces of information: a multiplicative relationship between a known and an unknown quantity (5 times) and the measure of one quantity (Candy bar A is 7 units long). The ‘measure’ aspect is directly apprehensible from the pictorial representation of 7 partitions, each representing one pound. The second aspect of the problem (bar A weighs five times as much as candy bar B) has to be constructed operationally in relation to the unknown quantity B. The solution to this problem on JavaBars requires the interpretation of 1/5 with two different referents - Firstly, 1/5 as a measure with referent as one pound in bar A (and one pound being 1/7 of bar A) and secondly 1/5 as a multiplicative relationship with referent as bar A. Similarly, the fraction 1/7 can be interpreted with two different referents: Each pound (i.e., 5 partitions of bar A) is 1/7 of bar A. Each small partition in bar B is 1/7 of bar B.

Being able to look at one small partition as carrying two pieces of information is the crucial step to solve this problem. One small partition is 1/5 of one pound and at the same time it is 1/35 of bar A. The students constructed bar B by using one partition from bar A. They transferred the multiplicative relationship '1/35' rather than the measure '1/5 pound.' I hypothesize that the visual appearance of the bars, together with the intuitively appealing part-whole conception of fraction, made the 1/35 more readily available and this might have influenced the students to transfer 1/35 rather than 1/5 pound as the measure. Thus, they measured bar B as 7/35 or 1/5 rather than $\frac{7}{5}$ pounds. Being able to conceptualize the measure of one partition of bar A as 1/5 pound requires the coordination of 3 levels of units: one unit containing 7 sevenths, each of which contains 5 one-fifth units. Similarly, being able to conceptualize the multiplicative relationship '1/35' requires the coordination of three levels of units: one unit containing 7 sevenths, each of which contains 5 one-thirty-fifth units.

In summary, when solved in JavaBars, this problem requires the coordination of two fractions within the same bar, a fifth and a thirty-fifth after distributive/recursively partitioning the bar. Each of these fractions has different referents. Switching from one unit to the other may require a significant conceptual leap and is the cause of the quantity-measure conflict (i.e., conflict in coordinating the quantitative relationship and the measure simultaneously).

The analysis of the current problem situation and constraints that the students encountered bring to the fore three key resources for constructing and measuring the unknown quantity (B) from the known quantity (A): unitizing, distributive/recursive partitioning, and coordinating two pieces of information within one bar by keeping track of the different referents of the given fractions.

Episode 6.21: ***Aileen and Brian*** **[Problem 2.43]**

This episode provides another illustration of the quantity-measure conflict where Brian confounded the multiplicative relationship and measure right at the start of the problem.

Data.

L1 I: Candy bar A is one and one third or four third unit long. Its length is eight fifth of candy bar B. Can you construct candy bar B?

I asked them to represent candy bar A on JavaBars. They made the first bar shown in Figure 6.12, which represents 2 wholes with 4/3 unshaded.

L2 B: It's six. Because if it's three fourths or one and one third, it's six and then one third of 6, there is 8, so 8 over 5.

L3 A: So how many pieces will candy bar B have?

L4 B: Five, I mean, it would be, five fifths would be 6.

L5 I: Can you explain again?

L6 B: Because it's one and one third, so, uh, like that's just 8 split into 4 pieces because its 4 thirds at the same time. And so three pieces of that four is six and then one third of that, we need an extra two which is 8.

I asked Brian to represent what he said in L6 on JavaBars. He made a copy of bar A and divided the four cells representing $\frac{4}{3}$ into two each as shown by the second bar in Figure 6.12.

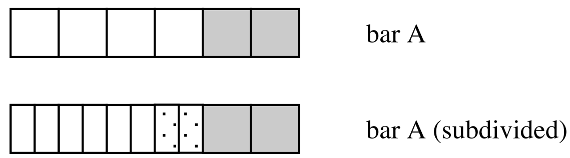


Figure 6.12. Subdivision of each of the 4 thirds into 2

L7 B: Yeah. And then six of them would be the one and the other would be two, would be the not one. (He shaded two of the subdivided partitions.)

L8 A: But you are adding.

L9 B: I need to break it.

L10 A: You are adding three fifths.

L11 B: I am not.

L12 A: Like you are adding one and three fifths to candy bar B to get candy bar A. And 8 is not divisible by 5.

L13 B: But, it is one and one third as much. So if that's one.

L14 A: No, it's not one and one third as much. He has one and one third. It's one, one and three fifths more.

L15 I: Eight fifths or one and three fifths.

L16 B: But, wait.

L17 A: He has one third.

L18 B: Yeah.

L19 A: One and one third, that's the total length of this right here (referring to bar A), the red (the unshaded part).

L20 B: Oh. Yeah. See that's the one (pointing to the first 6 partitions in the second bar in Figure 6.12) and then one third of the one is these two (the remaining two partitions).

L21 A: I know but you are not adding one third. You are adding...

L22 B: I know this (pointing to the six pieces) is what B has and that's (pointing to the 8 partitions) the total what A has. Just the (inaudible)

L23 A: There is no way that you are adding three fifths.

L24 B: Three thirds. That's three thirds

L25 A: I know (inaudible)

L26 B: We are adding two sixths.

L27 A: No, look. The total length is one and one third. It's length is 8 fifths of candy bar B. That means it has one and three fifths times more or like one and three fifths pieces more. It's not one and one third more, it's one and three fifths more. No, not one and one third. One and one third is the length, I mean.

L28 B: But it is one and one third as large.

L29 A: That's not. One and one third units long. It's length is 8 fifths of candy bar B.

L30 B: Oh. Never mind. I get it.

L31 I: So, what were you thinking about, Brian? You thought that it was one and one third?

L32 B: I thought that it was one and one third times larger and that candy bar B was just one unit and there was one and one third units, so…

Pause (45 seconds)

L33 B: It's 6 fifths. Wait. I was going somewhere with that. I (inaudible). Because if that is four thirds, then you divided by 4 to get one third and you divide that by 4 to get two fifths. So one third equals two fifths. And then you times it to get three thirds and that times three, right, yeah.

L34 A: So, it will be 4, it will be 5 sixths.

L35 B: Yeah.

L36 A: OK. I follow that.

She constructed bar B as shown in Figure 6.13.

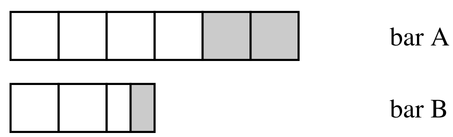


Figure 6.13. Construction of bar B from bar A

L37 A: So that would be what candy bar B would have.

L38 B: Yeah.

L39 I: How did you figure out for B?

L40 A: Well, if you have 4 thirds and you want to get to one third you would divide by 4 and so you have to do the same thing to the 8 fifths which will give you two fifths.

L41 I: Right.

L42 A: And so you have one third equals

L43 B: Two fifths.

L44 A: Two fifths. So if you add another third that equals four fifths.

L45 I: Right.

L46 B: And then it's six fifths, not five sixths.

L47 A: Is it 6 fifth or 5 sixth?

L48 B: No, no. Two thirds of the unit thing. Because three thirds is 6 fifth. So we need to do.

L49 A: If one third, one equals two fifth, so two thirds equals four fifth, three thirds equal six fifth (hesitating).

L50 B: Six sixths equal six over five.

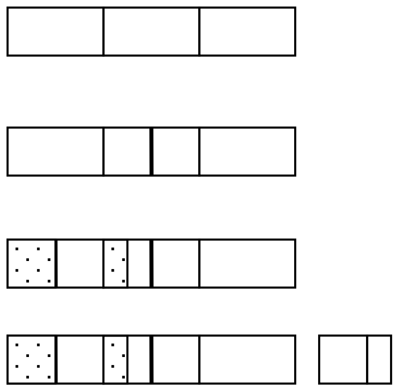
Analysis. Brian's response (L2-L6) shows that he took the measure (length) of quantity A ($\frac{4}{3}$) as the quantitative relationship. He interpreted $\frac{8}{5}$ in terms of 4 two-fifth units (i.e., $\frac{8}{5} = 4(\frac{2}{5}) = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5}$) showing that he reasoned with two levels of units. He then considered 3 of those two-fifth units as a whole and the fourth unit as one third (L6 and L7), giving evidence that he coordinated 3 levels of units because $\frac{2}{5}$ is still part of the original $\frac{8}{5}$ (i.e., $\frac{1}{4}$ of 4 two-fifth units or $\frac{1}{4}$ of a unit of units) as well as part of another whole ($\frac{1}{3}$ of 3 two-fifth units, i.e., $\frac{1}{3}$ of a unit of units). The unmaking of $\frac{4}{3}$ into a whole shows that he reasoned reversibly in L2-L6, though using the incorrect quantity. Actually, the second bar in Figure 6.12 represents two pieces of information: Each small partition represents one fifth of bar B, and two partitions have a measure of one third. Aileen could observe that Brian was confounding the multiplicative relationship $\frac{8}{5}$ and the length of candy bar A (L14 and L27) and thus prompted him to realize his incorrect interpretation (L29), which he acknowledged (L32). Brian reinterpreted the proportional relationship between $\frac{8}{5}$ and $\frac{4}{3}$ in L33 to conclude that "one third equals two fifths". Similarly, Aileen set the relationship between one third and two fifths (L42 and L44). Brian's answer of "six fifths" in L46 can be interpreted in two ways. Firstly, he may have referred to the 6 small partitions in the second bar in Figure 6.13 as it appears to be the case from L50 "six sixths equal six over five". Secondly, he may have mixed the multiplicative relation and the measure a second time, constructing one whole from $\frac{1}{3}$ rather than $\frac{2}{5}$. Brian's response in L46 and L48 may have focused Aileen's attention on thirds rather than fifths in L47 and L49.

In summary, Brian's response shows that the subtle coordination of the two pieces of information (i.e., the measure of quantity A and the quantitative relationship between A and B) in such reversibility situations may be cognitively demanding. Although Brian could articulate the proportional relations between the two given fractions, he did not keep track of the measure of the known quantity A and the quantitative relationship between quantity A and unknown quantity B. He constructed quantity B using thirds rather than fifths. This episode shows that flexibility to switch between the two pieces of information (i.e., the quantitative relationship and the measure) is a necessary resource for reversible reasoning. Conversely, reversible reasoning could be a necessary resource for being able to flexibly switch between the two pieces of information.

Episode 6.22: *Ted and Cole* [Problem 2.63]

Problem 2.63: Candy bar A is 3 units long. Its length is four fifths of candy bar B. What is the length of candy bar B?

Data. Cole made a bar and divided it into three to represent candy bar A, 3 units long. Then he divided the bar into 2 to make two $\frac{2}{5}$ pieces, and in turn divided one $\frac{2}{5}$ by 2 to make $\frac{1}{5}$ as shown by the successive diagrams below.



(a) bar A (3 units long)

(b) dividing bar A into 2 to produce
2/5 + 2/5 (of bar B)

(c) dividing 2/5 into 2 (distributively)

(d) adding 1/5 to 4/5 (to make
5/5 of bar B)

Figure 6.14. Cole's construction of one whole from $\frac{4}{5}$

L1 I: Right. So what is the length of this Ted and Cole (referring to the above Figure)?

L2 C: Five fifths.

L3 T: Five fifths.

L4 I: No, the whole length. I do understand it is making one whole, five fifths. But this is three units. If this is three units long, what will be the length if this?

L5 T: Three and

L6 C: Three and one fifth. No.

L7 I: Why?

L8 C: No, that's wrong.

L9 T: (inaudible)

L10 C: How many times four fifth go into that?

L11 T: (Referring to the diagram) Four fifths, five fifths and that's three unit. So three and five fifths, three and one fifth, yeah.

L12 C: Hold on. One, two, three, four. (Cole using the bar to measure)

L13 T: Three and one fourth.

L14 C: Three and one fourth.

Analysis. This problem requires the partitioning of a prime number (3) into an even number of parts (i.e., 4 parts). Cole first divided the 3 units into two to make two $\frac{2}{5}$-pieces (Figure 6.14 (b)) and in turn divided the resulting $1\frac{1}{2}$ units into two by partitioning one unit into $\frac{1}{2}$ and $\frac{1}{2}$ unit into $\frac{1}{2}$ (Figure 6.14 (c)), which gives evidence of distributive reasoning. This allowed him to produce $\frac{1}{5}$ of the unknown quantity (i.e., bar B), which he added to the given $\frac{4}{5}$ (bar A) to produce one unit of bar B. This shows that he articulated the quantitative relationship between the two quantities to undo the making of $\frac{4}{5}$ to construct one whole. The visual representation on JavaBars and the divisibility of the numerator of $\frac{4}{5}$ by 2 facilitated the coordination between the known (bar A) and unknown (bar B) quantities.

When asked to find the length of candy bar B, Ted and Cole used the visual representation in Figure 6.14 (d) to measure quantity B as 'five-fifths" in L2 and L3, as "three and one fifth" in L6 and L11, and as "three and one fourth" in L13 and L14 rather than $3\frac{3}{4}$. They did not coordinate the referents of the two fractional quantities (the 'fourth' in the measure of bar A and the 'fifth' in the quantitative relationship). The $\frac{1}{5}$ added to bar A (to make bar B) is $\frac{1}{4}$ of candy bar A, but at the same time it is $\frac{1}{5}$ of bar B and also $\frac{3}{4}$ unit long. In other words, the same quantity (the added part) has three referents: It has a measure expressed in terms of a fourth of bar A, and it is quantitatively related to bar B in terms of a fifth ($\frac{1}{5}$); further, $\frac{1}{4}$ of candy bar A is $\frac{1}{4}$ of 3 units and not $\frac{1}{4}$ by itself, and this coordination requires three levels of units: 3 one-whole units, each containing $\frac{1}{4}$ units. Though they could observe that the $\frac{1}{5}$ they added to bar A to make bar B is $\frac{1}{4}$ of bar A (using the visual representation), they left out the third level of units, giving the answer 3 and $\frac{1}{4}$ rather than 3 and $\frac{1}{4}$ of 3 units or $3\frac{3}{4}$ units.

Episode 6.23: *Ted and Cole* [Problem 2.64]

Problem 2.64: Candy bar A is 5 units long. Its length is $\frac{3}{4}$ of candy bar B. What is the length of candy bar B?

After representing bar A in terms of 5 units, Cole successively partitioned it into three parts. First, he divided 2 of the 5 partitions into 2 each (Figure 6.15 (b)) and distributed 3 of the 4 resulting partitions under each of the first 3 partitions (Figure 6.15 (c)). Then he divided the fourth partition into three parts to complete the division and equal distribution process. This allowed him to produce $\frac{1}{4}$ of the unknown quantity (bar B), which he repeated four times to produce one unit (Figure 6.15 (d))

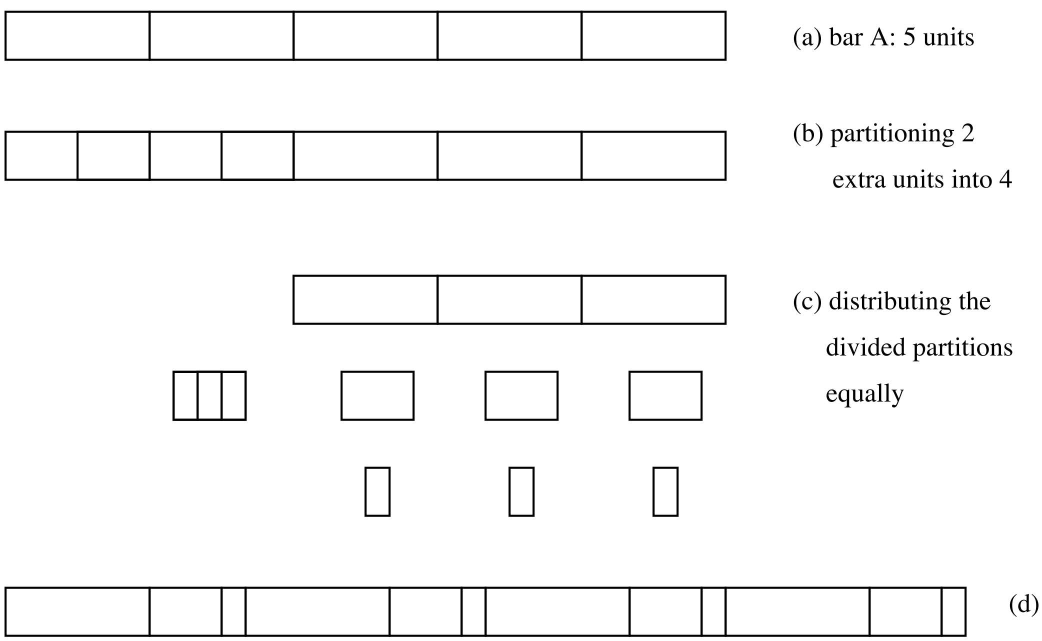


Figure 6.15. Division of 5 units into 3 equal parts and construction of one whole from $\frac{3}{4}$

When asked what is the length of candy bar B, Cole gave the answer $6\frac{3}{4}$ by comparing bar A and bar B visually after adding one partition to bar A as shown below.

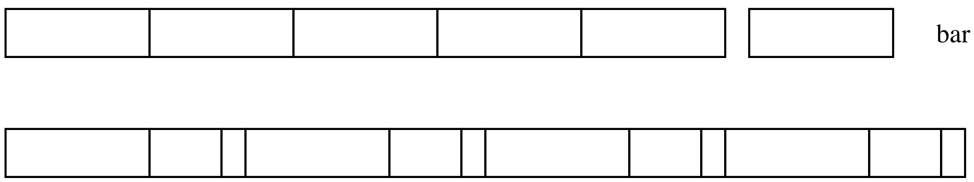


Figure 6.16. Cole's comparison of bar A and bar B

Analysis. Cole's strategy to construct the unknown quantity (bar B) from the known quantity (5 units) and the quantitative relationship ($\frac{3}{4}$) was to divide the 5 units into 3 units of one quarter (of bar B). To be able to divide 5 units into 3 equal parts, he divided the two (5-3 = 2) extra units into four parts and then successively distributed these parts to the remaining three units in bar A. This form of sharing of a continuous quantity based on exhausting the quantity by successive fair distribution/partitioning is characteristically a partitive division interpretation (finding the number of parts per group).

The visual representation on the Javabars allowed him to undo the making of $\frac{3}{4}$ of bar B. However, he did not measure the resulting one fourth unit of bar B. Actually, the measure of one fourth of candy bar B is $1+\frac{1}{2}+\frac{1}{6}=\frac{5}{3}$. Cole used the visual comparison of bar A and bar B in Figure 6.16 to estimate the length of bar B as $6\frac{3}{4}$ rather than $6\frac{2}{3}$. The measurement of bar B may be demanding because it requires keeping track of the referent of the successively partitioned bars.

Quantitatively, this problem requires the unfolding of the given multiplicative relationship '$\frac{3}{4}$' into $\frac{1}{4}$ and simultaneously coordinating it with $\frac{1}{3}$ of the measure of the known quantity A (5 units). This coordination may be described as follows:

$\frac{3}{4}$ of the unknown quantity $\rightarrow$ 5 units

$\frac{1}{3}$ of [$\frac{3}{4}$ of the unknown quantity] $\rightarrow$ $\frac{1}{3}$ (5 units)

$\frac{1}{4}$ of the unknown quantity $\rightarrow$ $\frac{1}{3}$ (5 units)

$4 \times$[$\frac{1}{4}$ of the unknown quantity] $\rightarrow$ $4 \times$[$\frac{1}{3}$ (5 units)]

Episode 6.24: *Aileen and Brian* [Problem 2.73]

Problem 2.73: Candy bar A is $1\frac{3}{4}$ (or $\frac{7}{4}$) unit long. Its length is $\frac{3}{5}$ of candy bar B. What is the length of candy bar B?

Data.

L1 A: So (you) would do two and one third times two, which is going to be four and two thirds added on to that, so it's going to be 11 and 2 thirds.

L2 I: Can you represent this in the diagram? Can you explain what you did or you want to make the diagram?

L3 A: I'll explain. What I said was seven fourths is equal to three fifths. So I did seven divided by 3 to try to get one fifth by itself. And if you do seven divided by three you get two and one third. And then you do two and one third times 2 to get the remaining two pieces and that gave me 4 and 2 thirds and then I added the four and two thirds to the original seven and that gave me 11 and two thirds pieces.

She constructed Bar B as $11\frac{2}{3}$ of a $\frac{1}{4}$ unit as represented below.

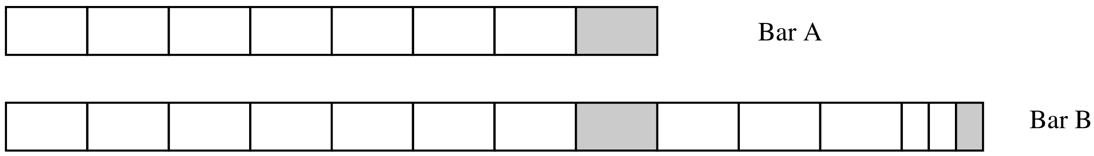


Figure 6.17. Aileen's construction of one whole from $\frac{3}{5}$

L4 I: So, right. What will be? This is candy bar B. What will be the length of candy bar B?

L5 A: Eleven and two thirds.

L6 I: OK, let's try to see. You are saying 11 and 2/3 by counting this, right. But what is one of this partition here (referring to Figure 6.17), what is the length of one?

L7 A: The length of one would be …

L8 B: Two and one third.

L9 A: Yeah, two and one third.

L10 I: What is the whole length?

L11 A: The whole length is seven fourths.

L12 I: Is seven fourths. What is one?

L13 A: One fourth.

L14 I: So, what is the length of this one (Bar B)?

L15 B: 11 and 2 thirds over 4.

L16 A: You can't have that.

L17 B: Eleven fourths and one twelfth.

L18 I: So what is one, the one piece is, what is the length of one piece?

L19 A: One fourth.

L20 I: Right. So what will be the length of the whole?

L21 A: It would be 11 and two thirds.

L22 B: You have to split them into threes.

L23 A: So I guess you have to split them all into three.

L24 B: which would be 12 times 3, 36.

L25 A: So that's 36 pieces and then

L26 B: 35 thirty-sixths.

L27 A: 35 thirty-sixths and that would, 7 times [3] 21 thirty-sixths. Right. ([3] means what the student may have said but was not audible from the tape.)

L28 B: Yeah.

L29 I: Can you explain what you are saying?

L30 A: If you split all of these pieces into thirds. Then you would have 36 total pieces in the entire thing.

⋮

L31 I: So what will be the length?

L32 A: The length will be 21 thirty-sixths and 35 thirty-sixths.

L33 I: 35 thirty-sixths.

⋮

L34 I: One second, what is the length of one?

L35 A: One fourth.

L36 I: So when you divide this into three, how much, what is it?

L37 A: One twelfth.

L38 I: So what will be the whole?

L39 A: It would be, or you could do into twelfths which would be 35 twelfths. Because if you divide fourths by 3, each one is going to be a twelfth. So you could do 35 twelfths.

L40 B: No. Because once you divide them into three they are still (inaudible) part of the same thing. So it will be one thirty-sixth, each one. Not one twelfth any more. Three of them would be one twelfth.

Analysis. Aileen first observed that she had to add $\frac{2}{5}$ to $\frac{3}{5}$ to construct one whole. She interpreted $\frac{3}{5}$ as three one-fifth units that led her to reason reversibly to divide 7 by 3. Thus, she inferred that $\frac{1}{5}$ of the unknown quantity (bar B) corresponds to $\frac{7}{3}$ or $2\frac{1}{3}$ of the known quantity (bar A). Then she produced $\frac{2}{5}$ of the unknown quantity as $4\frac{2}{3}$ and added it to 7 units to obtain $11\frac{2}{3}$ (L1 and L3). She lost one level of unit (the fourth in $\frac{7}{4}$) in coordinating the 3 one-fifth units to the measure of candy bar A that she considered simply as 7 units.

When I prompted them to deduce that the length of one partition in bar A is $\frac{1}{4}$ (L12), Brian mentioned that bar B is “11 and 2 thirds over 4” (L15) and simplified it to “eleven fourth and one twelve” (L17). It appears that he divided 11 by 4 and $\frac{2}{3}$ by 4 separately to get $\frac{11}{4}$ and $\frac{1}{12}$ (he probably made a computational mistake and obtained $\frac{1}{12}$ rather than $\frac{1}{6}$). Their next step to find the measure of bar B was to recursively partition it into thirds (L22). This led Aileen to deduce that the length of bar B was $\frac{35}{36}$. This shows that they determined the length of bar B using the visual representation in Figure 6.17 using bar B as a whole, independent of bar A. I referred them again to consider the length of one partition of bar A as one fourth (L34) and Aileen deduced that the length of bar B was $\frac{35}{12}$. A comparison of their previous solution (i.e., $\frac{35}{36}$) and the prompted solution (i.e., $\frac{35}{12}$) shows that what was missing in the students’ thinking was the coordination of a third level of unit: $11\frac{2}{3}$ one-fourth units versus $11\frac{2}{3}$ units. They considered bar B only at two levels of units to deduce that it was $\frac{35}{36}$ units long (a part-whole relationship) rather than 11 one-fourth units, each containing 3 one-third units (of $\frac{1}{4}$) plus 2 more one-third units (of $\frac{1}{4}$).

Constraint 2: Numeric Feature of Data

In the previous episodes (e.g., episode 6.20), I observed that when *a* and *b* (in $ax = b$) were relatively prime, some of the participants were constrained from articulating the multiplicative relation between the known and unknown quantities. Furthermore, constraints in the manipulation of the multiplicative relation were observed when *a* and/or *b* were fractional quantities (proper and improper) and involved relatively prime numerators (e.g., episodes 6.22, 6.23, and 6.24). In this section, I present three further episodes to show how the numeric structure of the two problem parameters (*a* and *b*) constrained the students from conceptualizing the multiplicative relationship. I also compare Ted's and Cole's response to problem 2.63 ($\frac{4}{5}x = 3$) and problem 2.71 ($\frac{4}{5}x = \frac{3}{4}$) to show how changing the measure of the known quantity from the integer 3 to the composite fraction $\frac{3}{4}$ influenced their response.

Episode 6.25: *Cole* [Problem2.65]

Problem 2.65: Candy bar A is 2 units long. Its length is $1\frac{1}{2}$ of candy bar B. What is the length of candy bar B?

Data.

L1 I: Candy bar A is 2 units long. This one is probably easy. Can you draw that?

Cole made a bar of 2 units to represent candy bar A (Figure 6.18).

L2 I: This is two units long and its length is one and a half, one and a half of candy bar B. Can you construct candy bar B?

L3 C: Is it bigger or smaller, Ted?

L4 T: OK. So if we turn the one and one half into an improper fraction, it will be.

L5 I: Three over two.

L6 T: Three over two.

L7 I: This is one and a half times of candy bar B. Can you construct candy bar B?

Cole divided the second partition in bar A into two and pulled out half of the partition to construct bar B as 3 units long.

L8 C: So I get it (referring to the diagram that he made).

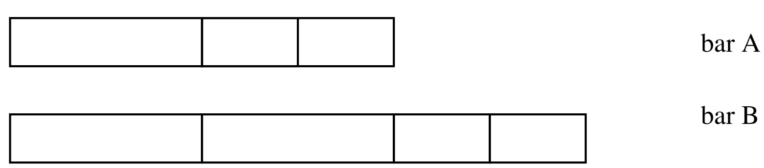


Figure 6.18. Cole's construction of candy bar B

L9 I: No, which one is. This one (referring to bar A) is one and a half of B. Can you follow that?

L10 C: So it would be smaller.

Analysis. Cole's question "is it (bar B) bigger or smaller, Ted?" at the start of the problem in L3 shows that when the multiplicative comparison statement was expressed in terms of mixed numbers, he did not readily conceptualize which of the two bars was longer. The reversibility situation here involves a known and an unknown quantity and a non-integer multiplicative relationship between the two quantities. Conceptualizing a non-integer relationship may not be as intuitive as an integer relationship and is a possible explanation for the constraint that Cole experienced here. Furthermore, positing an unknown quantity in thought may be a demanding task. Cole constructed $1\frac{1}{2}$ of the known quantity (bar A) rather than conceptualizing the known quantity as being $1\frac{1}{2}$ of the unknown quantity (bar B). A similar observation was made in problem 2.52 ($1\frac{2}{3}x = 55$) and problem 2.74 ($1\frac{5}{7}x = 1\frac{2}{3}$).

Episode 6.26: *Ted and Cole* [Problem 2.66]

Problem 2.66: Candy bar A is 3 units long. Its length is $1\frac{2}{5}$ of candy bar B. What is the length of candy bar B?

Data. Cole made a bar 3 units long to represent candy bar A (Figure 6.19 (a)). Then he divided the third unit into 5. He did not proceed any further with the problem. Ted mentioned "my head is going to blow up with it now." The following interview segment shows their response to this problem.

L1 I: Candy bar A is three units long. It's length is one and two fifths or seven fifths of candy bar B.

L2 T: So candy bar B will be a lot smaller.

L3 C: One and two fifths.

L4 I: One and two fifths.

L5 C: Got it.

Cole made a copy of bar A and appeared to use some form of visual estimation. He made Figure 6.19 (b) by dividing the last partition into 5.

L6 I: Ted, any idea.

L7 T: No, not yet. I am so stuck on this.

L8 I: You are dividing by what, by five?

L9 C: Hum.

L10 T: I am not getting any of this any more. So if it is three units long and it is one and two fifths. Fifth is the key word, seven fifths.

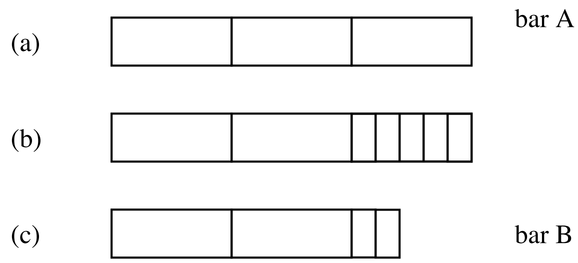


Figure 6.19. Cole's attempt to construct bar B from bar A

Analysis. This episode shows that the students were constrained in conceptualizing the multiplicative comparison relationship between the known (bar A) and unknown (bar B) quantities due to the numeric feature of the problem parameters (3 and $1\frac{2}{5}$). The numbers 3 and $1\frac{2}{5}$ were incommensurate for them (i.e., 3 could not be compared with $1\frac{2}{5}$ due to the unavailability of a divisibility relationship). Failure to conceptualize this relationship led the participants to use the fallback strategy of dividing the known quantity by the denominator of $1\frac{2}{5}$. This problem requires the interpretation of $1\frac{2}{5}$ as 7 one-fifth units so that the multiplicative relation between the known and unknown quantities is apparent. Such an interpretation allows 3 units to be compared to 7 units and opens the way for the division of 3 units in 7 parts. One and two-fifths ($1\frac{2}{5}$) and seven-fifths ($\frac{7}{5}$) are two different concepts-in-action and cue different resources, especially in the context of multiplicative comparison.

Episode 6.27: A comparison of Ted's and Cole's response to $\frac{4}{5}x = 3$ (problem 2.63) and $\frac{4}{5}x = \frac{3}{4}$ (problem 2.71)

Ted's and Cole's response to problem 2.63 ($\frac{4}{5}x = 3$) was presented in episode 6.22 where they constructed one whole ($\frac{5}{5}$), by partitioning the given 3 units into 4 parts and added $\frac{1}{5}$ to the existing $\frac{4}{5}$. They measured the resulting $\frac{5}{5}$-bar (Figure 6.14 (d)) as $3\frac{1}{4}$ (rather than $3\frac{3}{4}$). In other words, they left out the third level of units in measuring the $\frac{5}{5}$-bar as 3 and $\frac{1}{4}$ rather than 3 and $\frac{1}{4}$ of 3 units (i.e., $3\frac{3}{4}$). At one point they also gave the answers "five fifths" and "three and one fifth," confounding the multiplicative relationship and the measure given in the problem. Their response to problem 2.71 ($\frac{4}{5}x = \frac{3}{4}$) is presented below.

Problem 2.71: Candy bar A is $\frac{3}{4}$ unit long. Its length is $\frac{4}{5}$ of candy bar B. What is the length of candy bar B?

Ted made a bar and divided it into four and shaded one partition to represent three quarters as shown in Figure 6.20(a). Then he divided each of the three partitions into 4.

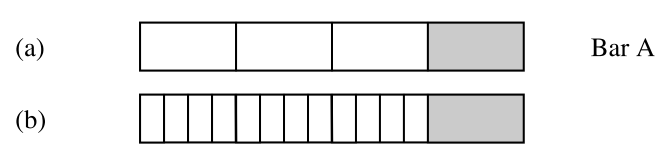


Figure 6.20. Ted's subdivision of bar A into fourths

L1 C: What would that be, 3 divide by 4 be?

L2 T: Three divided by four. I could find that out but.

L3 C: Because you can't just divide that into.

L4 T: Four pieces.

L5 C: It would not be like that Ted.

L6 I: One second, one second Cole. If you want to try something different you can do it on the right hand side so that we can come to Ted's solution.

Cole made a copy of Figure 6.20(a) and cleared the 3 partitions representing three quarters and divided it into four (see Figure 6.21(a)).

L7 I: You have divided this into 4.

L8 C: Yeah. And then, this is. So it said that is what, four

L9 I: Four fifths of candy bar B.

L10 C: That's four fifths. So just take that and copy.

L11 I: Ted, can you follow what he is doing.

L12 T: Hum.

(Cole pulled out one of the four parts that he made and repeated it four times).

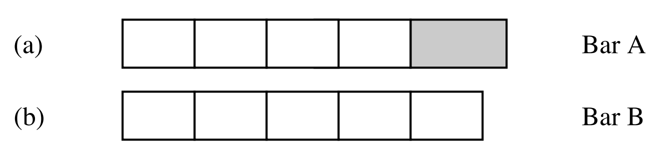


Figure 6.21. Cole's construction of bar B

L13 I: What will be its length?

L14 C: Four fourths.

L15 I: How do you get this?

L16 C: Because it is just as big as that one (Bar A).

Ted's solution:

L17 I: Let's take Ted's solution again and try to see if we can figure out something. He has divided this into 12. Ted, you have got 12 parts, right?

L18 T: Yeah.

L19 I: And these 12 parts represent how much?

L20 T: Uh, one twelfth.

L21 I: No, I mean, it represents four fifths of candy bar B. Right. Cole. These twelve parts represent four fifths of candy bar B. So can you construct candy bar B?

L22 T: I think. Let's see.

Ted started to count the pieces in Figure 6.20(b).

L23 C: It should be 12 parts (helping Ted). Four times three would be 12.

L24 T: That's 12, it should be 15.

He made the following diagram from Figure 6.20(b).

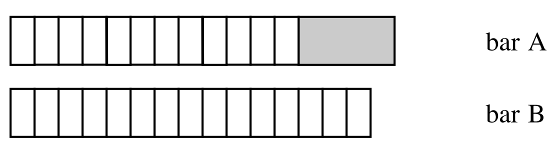


Figure 6.22. Ted's construction of bar B

L25 I: Ted, the three represents (the 3 parts that he added) what?

L26 T: One fifth.

Problem 2.63 ($\frac{4}{5}x = 3$) and problem 2.71 ($\frac{4}{5}x = \frac{3}{4}$) differs in the measure of candy bar A, 3 units versus $\frac{3}{4}$ or 3 one-fourth units. The first difference in the approach used in the two problems is in terms of the partitioning strategy. In problem 2.63, Cole partitioned the given 3 units into 4 equal parts by successive partitioning (Figure 6.14) whereas in problem 2.71 he deleted the three partitions and divided the resulting bar into four pieces (Figure 6.21) and pulled out one part, representing one fifth of bar B. The change from 3 to $\frac{3}{4}$ altered his partitioning strategy because he did not interpret $\frac{3}{4}$ as 3 one-fourth units.

On the other hand, Ted partitioned each of the 3 one-fourths into four parts each (Figure 6.20). It is more likely that Ted partitioned each fifth into 4 parts in Figure 6.20 using the denominator of $\frac{3}{4}$ as a pointer; otherwise he would have deduced that $\frac{1}{5}$ of candy bar B is equivalent to 3 partitions without prompting. Cole measured bar B in Figure 6.21 as “four fourths” (L14) while Ted measured each partition in Figure 6.22 as “one twelfth” (L20), which shows that their focus was on the part-whole relation represented on JavaBars. My statement in L21: “These twelve parts represent four fifths of candy bar B” prompted Ted to observe the divisibility relationship between 4 (i.e., 4 one-fifth units) and 12 (i.e., 12 partitions). This allowed him to reason reversibly (L23 and L24) to find the size of $\frac{1}{5}$ unit of bar B as being equivalent to 3 small partitions by dividing 12 by 4.

Constraint 3: Syntactic Structure of Problem

On various occasions, the participants interpreted the problem statements of the reversibility situations as multiplication situations and as such worked in a forward direction by applying the given multiplier on the known quantity. Similar observations were made by Olive & Steffe (2002) who found that some of their fourth-grade participants multiplied by nine instead of dividing by 9 when asked to produce a stick such that a given stick was 9 times longer than the required stick. In this section, I present 5 examples to show how the syntactic structure of the problems cued the multiplication operation. By syntactic structure of problem, I mean the phrasing of the problem as involving a multiplicative statement such as “times as much” or “as many as.” Furthermore, in Set 2 problems the first sentence gives the measure of the known quantity, the second sentence states the multiplicative relationship, and the third sentence asks for the measure of the unknown quantity.

Episode 6.28

Example 1: [Problem 2.12] Bar A weighs 7 pounds. Bar A weighs 5 times as much as bar B. What is the weight of bar B?

Ted, Cole, and Aileen multiplied 7 by 5 to obtain the weight of bar B as 35 pounds. Aileen’s response is presented below.

L1 A: 35 pounds.

L2 B: 35!

L3 A: 7 times 5.

Example 2: [Problem 2.31] There are 30 marbles in a box. This is $\frac{2}{5}$ of the number of marbles you have. How many marbles do you have?

Cole multiplied $\frac{2}{5}$ by 30 to get 12 as can be deduced from the following interview segment:

L4 C: It would be 12.

L5 I: How do you get 12?

L6 C: Because take 30 divided by 5 which would be 6, multiplied by 2 which would be 12.

Example 3: [Problem 2.51] In a seventh-grade survey of lunch preferences, 48 students said they prefer pizza. This is one and one half times the number of students who prefer fries. How many students prefer fries?

Ted, Aileen and Brian computed $1\frac{1}{2}$ of 48 to get 72. Their response (72) once again suggests that it is intuitively appealing to interpret this type of situation as a multiplicative problem, taking a fraction of the given known quantity rather than the unknown.

Example 4: [Problem 2.52] Parking lot A can hold 55 cars. It can hold $1\frac{2}{3}$ as many cars as parking lot B. How many cars can parking lot B hold?

Aileen started the problem by dividing 55 by 3 and multiplied the result by 2 to obtain $\frac{110}{3}$ as shown in Figure 6.23. Aileen's computation suggests that she was trying to find $1\frac{2}{3}$ of 55, rather than finding a number which when multiplied by $1\frac{2}{3}$ yields 55.

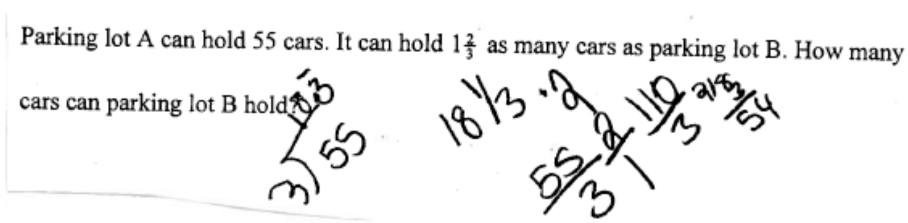


Figure 6.23. Aileen's computation

Similarly, Cole also took $1\frac{2}{3}$ of the known quantity rather than associating $1\frac{2}{3}$ to the unknown quantity. He computed $\frac{2}{3}$ of 55 as $36\frac{2}{3}$ and added it to 55 to get $91\frac{2}{3}$.

Example 5: [Problem 2.72] Candy bar A is $\frac{4}{9}$ unit long. Its length is $1\frac{2}{5}$ (or $\frac{7}{5}$) of candy bar B. What is the length of candy bar B?

I asked Ted and Cole to represent candy bar A ($\frac{4}{9}$) on JavaBars (Figure 6.24 (a)). Cole pulled out the four shaded squares representing $\frac{4}{9}$, cleared the partitions and divided it into five parts (Figure 6.24(b)). In the next step, he made another copy of the $\frac{4}{9}$ and pulled out two fifths from the previous bar he made.

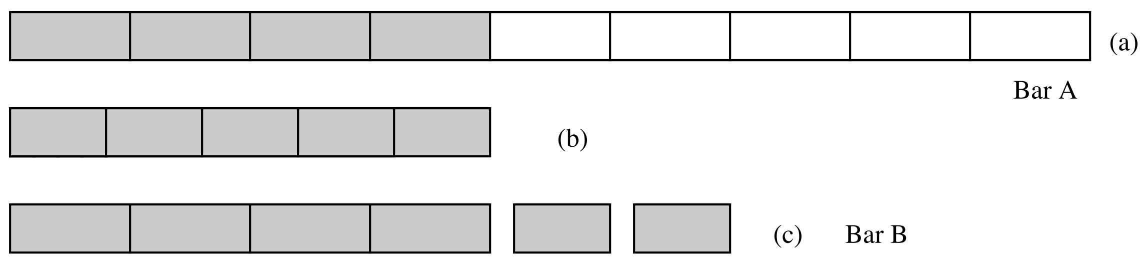


Figure 6.24. Cole's construction of bar B from bar A

The example further substantiates that the syntactic structure of reversibility situations tends to prompt students to take a fraction (especially when mixed numbers are involved) of the known quantity rather than the unknown quantity. Using the denominator of $\frac{7}{5}$ as a pointer, Cole divided $\frac{4}{9}$ into 5 parts (Figure 6.24) rather than 7 parts. He then pulled out two parts to construct $\frac{2}{5}$. His construction shows that his goal was to take $1\frac{2}{5}$ of the known quantity rather than the unknown quantity.

Inference of additive reasoning

In this last episode before the discussion, I illustrate the way additive reasoning interferes in multiplicative situations. Episode 6.29 shows how the interpretation of the problem situation from an additive perspective skewed Aileen's strategy.

Episode 6.29: *Aileen* [Problem 2.62]

Problem 2.62: Candy bar A is 2 units long. Its length is $\frac{3}{4}$ of candy bar B. What is the length of candy bar B?

Data.

L1 I: Candy bar A is 2 units long and its length is three quarter of candy bar B. What is the length of candy bar B?

L2 B: Two and two thirds.

L3 A: I was thinking that it was like three and a half or like two and a half.

⋮

L4 A: You could add. But if you are adding a fourth to get the candy bar B, then you add a fourth to candy bar A. And a fourth of 2 is the half. It would be two and a half, not two and two thirds or two and whatever you said.

Analysis. Observing that $\frac{1}{4}$ has to be added to bar A to make one whole, Aileen correspondingly added $\frac{1}{4}$ of 2 units (i.e., $\frac{1}{2}$) to obtain the measure of bar B as $2\frac{1}{2}$ units. The $\frac{1}{4}$ that Aileen added to bar B in fact represents $\frac{1}{3}$ of $\frac{3}{4}$, and this implies that $\frac{1}{3}$ of 2 units (i.e., $\frac{2}{3}$) should have been added to the measure. Because Aileen interpreted the difference between one whole and $\frac{3}{4}$ from an additive perspective, she compensated the corresponding quantity by $\frac{1}{4}$ of 2 units.

Discussion

As mentioned at the start of this chapter, I had three motives behind this set of reversibility situations. The first objective was to capture those instances where the participants appeared to be reversing their thought processes so as to trace the resources (and theorems-in-action) that they may have used to reverse the given multiplicative relationships at a fine-grained level. I also paid attention to the flawed theorems-in-action they deployed in these situations. Secondly, my objective was to identify the constraints that such multiplicative situations may impose on students in terms of coordinating the involved quantities. Thirdly, my objective was to investigate the influence of the numeric feature of problem parameters on the cueing of cognitive resources.

Strategies

The data show three types of strategies: measure strategy, 'unit-rate' strategy, and guess-and-check strategy. The measure strategy was used in problem 2.31 ($\frac{2}{5}x = 30$) and problem 2.32 ($\frac{3}{7}x = 18$). The structure of these two problems is distinct from the remaining problems in that the multiplicative relationship between the known and unknown quantity (i.e., $\frac{2}{5}$ and $\frac{3}{7}$) involves proper fractions with numerators dividing the measure of the corresponding known integer quantity (30 and 18). Further, the two known quantities have a relatively large measure (30 and 18) compared to the fractional quantities. Cole constructed one whole from $\frac{2}{5}$ as $\frac{2}{5} + \frac{2}{5} + \frac{1}{2}(\frac{2}{5})$ while Aileen constructed one whole from $\frac{3}{7}$ as $\frac{3}{7} + \frac{3}{7} + \frac{1}{3}(\frac{3}{7})$.

The 'unit-rate' strategy was primarily cued when a divisibility relationship between the two given fractions was present as in episode 6.5 (problem 2.42, $\frac{2}{5}x = \frac{1}{4}$). However, Brian's response in episode 6.6 (problem 2.62, $2 = \frac{3}{4}x$) shows that even when relatively prime numbers were involved, he could reinterpret one quantity in terms of the other. He interpreted the given 2-unit bar as 3 two-thirds units, which allowed him to construct $\frac{1}{4}$ of the unknown quantity. Thirdly, the guess-and-check strategy, as presented in episode 6.8 ($1\frac{2}{3}x = 55$) and episode 6.9 ($\frac{7}{5}x = \frac{4}{9}$) arose as a fallback strategy when the students did not relate the two quantities multiplicatively as a result of the numeric feature of the data.

What does this data set suggest about the second research question: *In what ways do students reason reversibly in multiplicative situations and what constructive resources do they deploy in such situations?*

The problems presented in the current data set have a structure where a fractional part (an end result) is given, and one has to deductively construct the whole that produced the result. They can also be regarded as involving the coordination between a known and an unknown quantity. For example, in problem 2.61: 'Candy bar A is 5 units long. Its length is 7/8 of candy bar B. What is the length of candy bar B?', one is required to coordinate the relationship between candy bar A as a known quantity and candy bar B as an unknown quantity. This problem situation can also be interpreted in terms of a measure and a multiplicative relationship. The fraction 7/8 expresses the multiplicative relationship between the two bars, and 5 units represent the measure of the known quantity.

The commonality among the current set of problems is that they involve a multiplicative comparison relation. Thinking about a known amount that is n times as large as an unknown amount allowed the participants to reverse their thinking to deduce that the unknown amount is $1/n$ times as large as the known amount. To solve these problems the participants had to use the inverse relationship between multiplication and division (an invariant or a theorem-in-action) and carry out an operation of thought (that is, apply this inverse transformation) before calculating the result of the arithmetic operation. In other words, before doing the arithmetic operation, the mental operation of inverting the transformation multiplication to division must be carried out in order to connect the multiplicative situation with a divisive situation.

Instances of reversible reasoning were particularly apparent when a multiplicative approach was used to solve the problem situations in episodes 6.1 to 6.7. For example, in episode 6.4 ($\frac{2}{5}x = \frac{1}{4}$), Aileen and Brian undid the making of two fifths by dividing by two to produce one fifth, from which they constructed one unit of the unknown quantity as five fifths. In episode 6.6 (problem 2.62, $\frac{3}{4}x = 2$), Brian produced one whole from $\frac{3}{4}$ by dividing by 3 to produce $\frac{1}{4}$ and constructed one whole by adding $\frac{3}{4}$ to $\frac{1}{4}$. Similarly, Aileen and Brian reasoned reversibly in episode 6.7 ($\frac{3}{5}x = \frac{7}{4}$) to make one whole starting from part of a whole.

Constructive resources deployed

The strategies and constraints that the students encountered in this problem set allow me to identify several resources that may be necessary for reasoning reversibly in fractional contexts. Firstly, being able to conceptualize the given problem situation as a multiplication situation is a foremost requirement to reason reversibly. The theorem-in-action/invariant 'division as the inverse of multiplication' is cued when students are able to use their whole number knowledge of multiplication and division in fractional situations. This is possible when fractional quantities are interpreted in terms of their corresponding unit fractions. For example, a problem like $\frac{4}{5}x = 3$ is interpreted as a multiplicative comparison problem only when $\frac{4}{5}$ is regarded as 4 one-fifth units. Such an interpretation of fractions is just as important in the case of mixed fractions. For example, the interpretation of $1\frac{1}{2}$ as three one-half units in problem 2.51 is the concept-in-action that allowed Brian to solve the problem deterministically. Therefore, an important resource for reversible reasoning in fractional contexts is the interpretation of such quantities in terms of unit fractions and also having an iterative conception of fraction.

Another essential resource for reasoning reversibly in multiplicative comparison situations involving relatively prime numbers (or fractions with relatively prime numerators), is the coordination of 3-levels of units as Brian did in episode 6.22 ($2 = \frac{3}{4}x$). In this problem, Brian re-conceptualized a 2-unit structure with each unit having three sub-units as 3 units of 2 sub-units each. However, problems involving two composite fractions as in problem 2.73 ($\frac{3}{5}x = \frac{7}{4}$) required more demanding coordination. For instance, Brian compared 7 units to 3 one-fifth units rather than 7 one-fourth units to 3 one-fifth units. Thus, he obtained the measure of the unknown quantity as $11\frac{2}{3}$ units rather than $11\frac{2}{3}$ one-fourth units. The students' solution on JavaBars also brought to the fore the importance of recursive partitioning and reasoning with distribution.

The reversibility situations explored in this chapter require the coordination of a quantitative relationship (expressed as a fraction) and a measure (i.e., two distinct pieces of information). Being able to switch between the measure and the multiplicative relationship is a flexibility that is necessary for measuring one whole unit from the given fractional part. The quantity-measure conflict arose when students had difficulties in coordinating these two pieces of information.

The 'unit-rate' strategy in episodes 6.4-6.7 shows that the simultaneous coordination of quantities is another important resource for reasoning reversibly. For instance, in episode 6.5 (problem, 2.43, $\frac{2}{5}x = \frac{1}{4}$), Aileen mentioned that "if one fourth equal two fifths then one eighth would equal one fifth and so one eighth is one half of one fourth" (L1). Being able to set a one-to-one correspondence between the known and unknown quantities and work simultaneously with the two quantities is what triggers reversible reasoning. In this study, such simultaneous coordination of two quantities were deployed in a limited number of problems.

What does this data set suggest about the third research question: *What constraints do students encounter in conceptualizing multiplicative relations from a reversibility perspective?*

Failure to interpret the fractional quantities in terms of units prevented the students from using their whole number knowledge to perform the multiplicative comparison of the two quantities. The immediate consequence of such an unavailability of unit fraction interpretation (a concept-in-action) is that the participants were led to use guess-and-check strategies. Often the denominator of the given fractional quantities acted as a pointer for division. For instance, in episode 6.16 (problem 2.72, $\frac{7}{5}x = \frac{4}{9}$), Aileen divided each of the 4 one-ninth units into 5 rather than 7. After dividing the measure of the known quantity by the denominator of the fraction expressing the multiplicative relation, they had no option but to use guess-and-check strategies by considering different numbers of partitions as one whole. In other words, the absence of a divisibility relation between the quantities a and b in $ax = b$ led the participants to focus on the denominator of the fraction (expressing the multiplicative relation between the known and unknown quantities) as a pointer for division. As illustrated in episodes 6.16 to 6.19, the students attempted to introduce divisibility relations between the two given fractional quantities by making the denominators equal. For example, in episode 6.17 (problem 2.71, $\frac{4}{5}x = \frac{3}{4}$), Aileen changed $\frac{4}{5}$ to $\frac{16}{20}$ and $\frac{3}{4}$ to $\frac{15}{20}$ and compared $\frac{15}{20}$ to $\frac{16}{20}$ additively. In summary, some of the students did not readily compare the fractional quantities because they did not always interpret the fractions in terms of units and as such did not use their whole number knowledge to perform the multiplicative comparison. The common denominator strategy was cued as a fallback strategy to turn an uncongenial problem to a congenial one.

One of the recurring constraints that some of the students encountered was the quantity-measure conflict (i.e., measuring the unknown quantity either in terms of the fraction representing the quantitative relation or in terms of the part-whole relation after constructing the unknown quantity). The participants did not always transfer the measure from the known quantity to the unknown quantity. Though they could use the given multiplicative relationship to construct one unit of the unknown quantity (bar B) from the known quantity (bar A), they did not readily measure the unknown quantity. The visual representation on the JavaBars may have also been influential in leading the students to incorrectly measure the unknown quantity. In one instance (episode 6.21), Brian even interchanged the measure and the quantitative relationship. I attribute the quantity-measure conflict to two factors: failure to switch flexibly between the measure and the quantitative relationship and coordination of three levels of units consistent with Hackenberg's (2005) study.

The numeric feature of problem parameters

The data collected in the current problem set (Set 2) show that the numeric characteristic of the parameters a and b in $ax = b$ has a strong influence on cueing the necessary resources to solve the reversibility situations. A first observation is that when a is a factor of b in $ax = b$, the division operation is readily cued. The four participants could readily deduce that they had to divide in question 2.11 ($3x = 21$). In other words, the divisibility relationship among the numbers facilitated/cued the theorem-in-action 'division as the inverse of multiplication'. In question 2.12 $(5x = 7)$, a and b are relatively prime, and this proved to be a challenging task for three of four participants when asked to carry out this partitive division task on JavaBars. Level 2 problems (2.21, $\frac{1}{4}x = 5$ and 2.22, $\frac{1}{7}x = 3$) were readily solved by all the four participants. These problems involve unit fractions and as such they were interpreted as multiplicative problems. Level 3 problems (2.31, $\frac{2}{5}x = 30$ and 2.32, $\frac{3}{7}x = 18$) were solved by both measure and ‘unit-rate’ strategies and here the divisibility relationship between the numerator of the given fraction and the integer quantity facilitated the construction of one unit of the unknown quantity. Similarly, the divisibility relationship between the numerators of the fractional quantities facilitated the manipulation of the quantitative relationship between the known and unknown quantities in Level 4 problems (2.41, $\frac{1}{5}x = \frac{1}{4}$; 2.42, $\frac{2}{5}x = \frac{1}{4}$; 2.43, $\frac{8}{5}x = \frac{4}{3}$).

The influence of mixed number representation of fractional quantities on reversible reasoning was made explicit by Level 5 problems (2.51, $1\frac{1}{2}x = 48$; 2.52, $1\frac{2}{3}x = 55$). In episode 6.11, Ted, Cole and Brian interpreted $1\frac{1}{2}$ as three $\frac{1}{2}$-units and this allowed them to perform the multiplicative comparison between the known and unknown quantities using their knowledge of whole numbers. Failure to interpret the mixed numbers in terms of fractional units led to guess-and-check strategies. For example, in episode 6.8 (problem 2.52, $1\frac{2}{3}x = 55$), Aileen did not interpret $1\frac{2}{3}$ as $\frac{5}{3}$ and attempted to split 55 by guess-and-check.

To further explore the influence of numeric variations on the cueing of cognitive resources, Level 6 problems were formulated where the numerators of the measure of the known quantity and the fraction expressing the multiplicative relation (between the two bars) were relatively prime. The first four problems involve the multiplicative comparison of a proper fraction and a whole number: 2.61, $\frac{7}{8}x = 5$; 2.62, $\frac{3}{4}x = 2$; 2.63, $\frac{4}{5}x = 3$ and 2.64, $\frac{3}{4}x = 5$. These tasks proved to be problematic for the participants. For instance, Aileen did not see problem 2.61 ($\frac{7}{8}x = 5$) as involving the multiplicative comparison of 7 one-eighth units and 5 units. Thus she divided the 5 units into 8 (i.e., the denominator acted as a pointer for division). In problem 2.63 ($\frac{4}{5}x = 3$), though Ted and Cole constructed one unit of the unknown quantity, they measured its length incorrectly because of lack of coordination of 3 levels of units. Similarly, in problem 2.64 ($\frac{3}{4}x = 5$), though they could use JavaBars to find one unit of the unknown quantity, they were constrained in finding its measure. In problem 2.66 ($1\frac{2}{5}x = 3$), the numbers $1\frac{2}{5}$ and 3 were incommensurate for Ted and Cole.

Similar observations were made in Level 7 problems. For instance, in problem 2.71 ($\frac{4}{5}x = \frac{3}{4}$), Aileen attempted to make the denominator of the two given fractions equal so as to make the fractions comparable from her perspective. Another fallback strategy besides making the denominator equal, was to partition the known quantity by the denominator of the fraction representing the multiplicative relationship (e.g., problem 2.72 ($\frac{7}{5}x = \frac{4}{9}$) in episode 6.9). Still, another impact of the numeric feature of the problem parameters was observed in problem 2.74 ($1\frac{5}{7}x = 1\frac{2}{3}$),where the mixed numbers constrained the students from conceptualizing which of the two quantities being compared was larger.

Positing an unknown as a quantity and syntactic structure of problem

The question that crops up after analyzing episode 6.28 ($5x = 7$, $\frac{2}{5}x = 30$, $1\frac{1}{2}x = 48$, $1\frac{2}{3}x = 55$, $\frac{7}{5}x = \frac{4}{9}$) is why the students associated the fraction (expressing the multiplicative relationship) with the known quantity rather than the unknown quantity. Two possible explanations seem to be plausible. Firstly, the reversibility situations involve an unknown quantity, and the students did not readily conceptualize such an unknown having an implicit nature. These problems require positing an unknown (referent) quantity in thought (i.e., imagining an unknown quantity in multiplicative relationship to a known quantity). Secondly, the syntactic structure of these situations tends to intuitively favor multiplication. Consider problem 2.31: 'There are 30 marbles in a box. This is $\frac{2}{5}$ of the number of marbles you have. How many marbles do you have?' Cole interpreted this problem as finding $\frac{2}{5}$ of the 30 marbles in the box. The word 'this' in the second sentence of the problem statement does not readily make the unknown quantity (the number of marbles you have) apparent as might have been the case if the problem had been stated as follows: 'There are 30 marbles in a box. These 30 marbles are $\frac{2}{5}$ of the number of marbles you have. How many marbles do you have?' Another observation is that the phrase '*m* times as many as' is a statement that tends to cue multiplication by *m* as shown in episode 6.28 where 3 of the 4 participants performed multiplication rather than division.

How do the findings of this study corroborate previous research on reversible reasoning?

This study substantiates Hackenberg's (2005) and Tzur's (2004) research in terms of the resources necessary for reversible reasoning in fractional contexts as well as the constraints that students may encounter in such situations. Both Hackenberg and Tzur showed that an iterative conception of fraction is a necessary element for reversible reasoning. Moreover, Hackenberg highlighted the following resources: positing an unknown as a quantity, unit coordination (coordinating 3 levels of units), reasoning with distribution, and the mental operations of splitting, distributive splitting and recursive partitioning. Her focus was on identifying the schemes necessary to solve situations of the form $ax = b$ with the aim of characterizing algebraic reasoning. Consequently, she analyzed how students coordinated the known and unknown quantities in $ax = b$. She did not explicitly consider $ax = b$ as a statement of multiplicative comparison. In my study, I analyzed the reversibility situations from a multiplicative comparison perspective. By systematically varying the quantities *a* and *b* in $ax = b$, I identified two conditions under which the theorem-in-action 'division as the inverse of multiplication' was cued: (i) when divisibility relations between *a* and *b* were available or could be readily set up, and (ii) when fractional quantities were interpreted in terms of their corresponding unit fractions such that whole number knowledge of multiplication and division was available. The two constraints emanating from the problem feature, namely numeric characteristics of the parameters *a* and *b* and syntactic structure of the reversibility situation, were not of major concern in Hackenberg's study. One of the main findings of this study is that reversibility is strongly sensitive to the numeric features of the task.

In terms of constraints emanating from the problem solver, my findings are in line with Hackenberg's study. The quantity-measure conflict (i.e., the conflict arising as a result of the demand for coordinating two pieces of information - a measure of a known quantity and a multiplicative relation between the known and unknown quantities) was also observed by Hackenberg. Consistent with Tzur's (2004) findings, I also observed that the denominator expressing the multiplicative relation between the two quantities in $ax = b$ acted as a pointer for division, but this occurred specifically in the absence of divisibility relations. Further, the participants in this study attempted to bring in divisibility relationships between *a* and *b* by making the denominators equal. My study also shows how additive reasoning tends to interfere in multiplicative situations (episode 6.29), an issue not explicitly highlighted by the two previous studies involving reversible reasoning in fractional contexts.

Table 6.3. Summary of Strategies Used by the Four Participants

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.31 ($\frac{2}{5}x = 30$)	He used $\frac{2}{5} = \frac{30}{x}$ and obtained *x* by comparing 2 and 30.	He constructed a whole as $\frac{2}{5} + \frac{2}{5} + \frac{1}{5}$ and added 30 + 30+15. [6.1];	She acknowledged Brian's solution. [6.2]	He constructed a whole as $\frac{2}{5} + \frac{2}{5} + \frac{1}{5}$ and added 30 + 30+15. [6.2]
2.32 ($\frac{3}{7}x = 18$)	He divided 18 by 3 and multiplied the result by 7. [6.4]	He added 18 + 18 + $\frac{1}{3}$ of 18.	She added 18 + 18 + $\frac{1}{3}$ of 18. [6.3]	He divided 18 by 3 and multiplied the result by 7. [6.4]
2.41 ($\frac{1}{5}x = \frac{1}{4}$)	He temporarily experienced the quantity-measure conflict in measuring bar B, confusing between $\frac{1}{4}$ as a measure and $\frac{1}{5}$ as a multiplicative relationship.	After representing $\frac{1}{4}$ for candy bar A on JavaBars, he repeated it 5 times to construct candy bar B and measured it as $\frac{5}{4}$.	A: "Uh, I just thought that if this one fourth (candy bar A) is one fifth of something else, you have to have four more of those pieces to make 5 total."	B: "It will be one fourth times 5, it will be one and one fourth or five fourth. There will be five fourth in candy bar B."
2.42 ($\frac{2}{5}x = \frac{1}{4}$)	After representing $\frac{1}{4}$ as one of four partitions in a bar, he split all the partitions into 2 to get 8 partitions and pulled out 5 to make bar B. He did not measure bar B without prompting.	Cole constructed bar B by repeating $\frac{2}{5}$, two and a half times and measured its length as $2\frac{1}{2}$ fourth.	$\frac{1}{4} \to \frac{2}{5}$ $\frac{1}{2}(\frac{1}{4}) \to \frac{1}{2}(\frac{2}{5})$ $\frac{1}{8} \to \frac{1}{5}$ and $\frac{2}{5} + \frac{2}{5} + \frac{1}{5}$ corresponds to $\frac{1}{4} + \frac{1}{4} + \frac{1}{8}$ [6.5]	$\frac{1}{4} \longrightarrow \frac{2}{5}$ $\frac{2}{8} \longrightarrow \frac{2}{5}$ $\frac{1}{8} \longrightarrow \frac{1}{5}$ $5(\frac{1}{5}) \longrightarrow 5(\frac{1}{8})$ [6.5]

(*table continues*)

Table 6.3. (*continued*)

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.43 ($\frac{8}{5}x = \frac{4}{3}$)	Referring to the 8 subdivisions that Cole made (see Cole's response), he mentioned: "I was just thinking that the 8 could be like the numerator, so I just like add 5 under, it will be five eighths equivalent thing."	After representing $\frac{4}{3}$ as 4 out of 6 partitions (2 wholes), he divided each of them into 2 to get 8 small partitions. Then he pulled out 5 of them to represent one whole of the unknown quantity and stated: "So I divided each one by 2 to keep it 8 because (inaudible) eight fifths so as to take 5 out of 8."	She set the quantitative relationship between $\frac{4}{3}$ and $\frac{8}{5}$, stating that "if you have 4 thirds and you want to get to one third you would divide by 4 and so you have to do the same thing to the 8 fifth which will give you two fifth" , confounding between the measure and the quantitative relationship. [6.21]	Brian took the measure (length) of quantity A ($\frac{4}{3}$) to be the quantitative relationship instead of $\frac{8}{5}$. He interpreted $\frac{8}{5}$ in terms of 4 two-fifth units, and constructed one whole with 3 of those two-fifth units. At a later point he set the quantitative relationship between $\frac{4}{3}$ and $\frac{8}{5}$ and deduced that "one third equals two fifths" and incorrectly constructed one whole by multiplying the measure $\frac{1}{3}$ by 3. [6.21]
2.51 ($1\frac{1}{2}x = 48$)	He divided 48 by 3 and added 16 + 16 to get 32 [6.11];	1st attempt: After representing 48 partitions on JavaBars, he divided 48 by 4 and multiplied the result by 3 to get 36. 2nd attempt: He divided 48 by 3 and added 16 + 16 to get 32 [6.11]	She first attempted to divide 48 by 2 but then followed Brian's solution. [6.12];	He interpreted $1\frac{1}{2}$ as 3 one-half units and divided 48 by 3 and multiplied the resulting 16 by 2 to get 32. [6.12];

(*table continues*)

Table 6.3. (*continued*)

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.52 ($1\frac{2}{3}x = 55$)	1st attempt: Looking at the 55 partitions on Javabars, like Cole he made the suggestion to divide by 4. 2nd attempt: He used a guess-and-check strategy to suggest 33 as the answer. [6.8]	He divided 55 by 3. He also made the suggestion to divide by 6. [6.8]	1st attempt: She divided 55 by 8. [6.28] 2nd attempt: She used a guess-and-check strategy. [6.8]	1st attempt: He divided 55 by 8. 2nd attempt: he interpreted $1\frac{2}{3}$ as 5 one-third units and divided 55 hy 5 and multiplied the result by 3 [6.13]
2.61 ($\frac{7}{8}x = 5$)	He added 5 partitions to the 35 partitions that Cole made (see Cole's response) and mentioned "If I multiply seven eighth by 5, I just do seven eighths times 5 fifth, equal to the whole of it."	After representing bar A by 5 partitions, he divided each of them into 7 to get 35 small partitions and Ted took over from that point.	After making a bar of 5 partitions she divided each of them into 8 and did not proceed further after deducing that "I was thinking you could divide each like fifth into eighth and then we could do 40 total eighths and then if 40 eighths is equal 7 eighths…" [6.15]	He set the quantitative relationship between $\frac{7}{8}$ and 5 but because of indivisibility did not conceptualize $\frac{1}{8}$ as corresponding to a fractional quantity (i.e., $\frac{5}{7}$). They divided $\frac{5}{7}$ to obtain 0.71.
2.62 ($\frac{3}{4}x = 2$)	Cole took the lead in the problem (see his response) and Ted find the measure of bar B as $2\frac{2}{3}$ after initially giving the answer of $2\frac{1}{3}$	He made a bar of 2 partitions, cleared the partitions and divided it into 3. He then pulled out one of 2 quarters to construct bar B.	Observing that $\frac{1}{4}$ has to be added to bar B to make one whole, She correspondingly added $\frac{1}{4}$ of 2 units (i.e., $\frac{1}{2}$) to obtain the measure of bar B as $2\frac{1}{2}$ units. [6.29]	He interpreted 2 units as 3 two-third units and added one two-third unit to 2 to obtain $2\frac{2}{3}$. [6.6]

(*table continues*)

Table 6.3. (*continued*)

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.63 ($\frac{4}{5}x = 3$)	Cole took the lead (see his response) and Ted measured the resulting bar as $3\frac{1}{4}$ initially. [6.22]	After making a bar 3 units long, he partitioned it into 4 to produce $\frac{1}{4}$ that he added to bar A. He measured the resulting bar as $3\frac{1}{4}$ initially. [6.22]	After representing a bar of 3 partitions, she divided each of them into 4 and stated: "so each one of them represents a fifth now. And so you would add one more … you would have 15 pieces".	He re-conceptualized 3 in units of $\frac{3}{4}$ by dividing 3 by 4. Then he multiplied the resulting $\frac{3}{4}$ by 5 to obtain $3\frac{3}{4}$.
2.64 ($\frac{3}{4}x = 5$)	Cole took the lead in the problem(see his response) and Ted did not contribute much in the discussion. [6.23]	After representing bar A as 5 partitions, Cole divided them into 3 equal parts by successive partitioning on JavaBars. He measured constructed bar (B) as $6\frac{3}{4}$. [6.23]	After representing bar A as 3 partitions, she divided each of them into 3 each and stated: "each 5 equals a third. So, uh, you wanted to get to a total of 4 fourth. So you had to add 1 more piece and that gives you 20 and so then you do 20 divided by 6 and that gives you $6\frac{2}{3}$."	He re-conceptualized 5 into 3 units of $\frac{5}{3}$ as can be inferred from his statement: "Like 5 has to be three fourth of this. So five could be split by three and then another one of those thirds of the five will make it four fourth."
2.65 ($1\frac{1}{2}x = 2$)	He interpreted $1\frac{1}{2}$ as 3 one half units and divided each of the two units of candy bar A into 3 to construct one whole as 4 partitions. [6.14; 6.25]	He made a bar that was $1\frac{1}{2}$ of the known quantity (A) rather than $1\frac{1}{2}$ of the unknown quantity (B). [6.25]	Aileen's response: "I did , uh, 2 divided by one and a half or two times two thirds and that gave me two thirds, or, not two thirds, four thirds."	He plugged-in numbers to find which number times $1\frac{1}{2}$ gives 2. "I knew that it was $1\frac{1}{2}$ of one, so it had to be somewhere over 1." It appears that first, he tried $1\frac{3}{4}$. He obtained $1\frac{1}{3}$ in the 2nd trial.

(*table continues*)

Table 6.3. (*continued*)

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.66 ($1\frac{2}{5}x = 3$)	He mentioned "my head is going to blow up with it now". [6.26]	He made a bar 3 units long to represent candy bar A. Then he divided the third unit into 5. He did not proceed any further with the problem. [6.26]	She made a bar of 3 partitions and divided each of the partitions into 7 parts after interpreting $1\frac{2}{5}$ as $\frac{7}{5}$. After obtaining 21 pieces she stated: "and then you had to take away two of the seventh to get back to the original one. So if you take away two of them, you will get, left with 15 pieces, yeah, 15 pieces out of 21 pieces and then that's equal to 5 seventh, I think. Yeah, five seventh" (Quantity-measure conflict)	He started by making approximations and mentioned: "Higher than two but close to that."
2.71 ($\frac{4}{5}x = \frac{3}{4}$)	After representing $\frac{3}{4}$, he divided each of the three partitions into 4 each. On probing, he constructed bar B with 15 partitions. He measured the length of one small partition as $\frac{1}{12}$ rather than $\frac{1}{16}$. [6.27]	Starting from Ted's representation of $\frac{3}{4}$, he cleared the 3 partitions representing $\frac{3}{4}$ and divided it into four. Then he pulled out one of the four parts that he made and repeated it four times to construct candy bar B. He measured the length of bar B as "four fourth". [6.27]	She rewrote the two given fractions on the same denominator $\frac{3}{4}$ as $\frac{15}{20}$ and $\frac{4}{5}$ as $\frac{16}{20}$. [6.17]	He rewrote the two given fractions on the same denominator as $\frac{3}{4}$ to $\frac{30}{40}$ and $\frac{4}{5}$ to $\frac{32}{40}$. [6.17]

(*table continues*)

Table 6.3. (*continued*)

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.72 ($\frac{7}{5}x = \frac{4}{9}$)	Cole took the lead to solve this problem (see Cole's response) and Ted acknowledged his construction. [6.28]	He pulled out the 4 shaded squares representing $\frac{4}{9}$, cleared the partitions and divided it into 5 parts. In the next step, he made another copy of the $\frac{4}{9}$ and pulled out two fifths from the previous bar he made. [6.28]	She partitioned the four pieces representing $\frac{4}{9}$ into five pieces. Then she started to guess-and-check how many pieces out of the 20 should she choose as the unit so that A is $1\frac{2}{5}$ of B. [6.9; 6.16]	He mentioned that "I was thinking how many times would 7 go into 4" (N26) but did not proceed further with the calculation. [6.16]
2.73 ($\frac{3}{5}x = \frac{7}{4}$)	Cole took the lead to solve this problem (see Cole's response) and Ted acknowledged his construction. In addition, Ted stated: "Can we just divide it? Seven by 3. Yeah, can we just divide 7 by 3 which would be 2 point 5. No, forget that."	Cole represented $\frac{7}{4}$ by 7 partitions, cleared the partitions and divided it into 3. He then pulled out one part and iterated it 4 times to produce bar B as five fifths. He stated the length of bar B as "five thirds".	Building on Brian's proportion strategy (see his response) , she inferred that the unit of quantity $\frac{1}{5}$ would have a magnitude of $2\frac{1}{3}$ units (rather than $2\frac{1}{3}$ one-fourth units). Thus, she constructed one whole as $\frac{3}{5} + 2(\frac{1}{5})$ with corresponding measure $7 + 2(2\frac{1}{3}) = 11\frac{2}{3}$ rather than $11\frac{2}{3}$ fourth. [6.7; 6.24]	He set a one-to-one correspondence between $\frac{7}{4}$ and $\frac{3}{5}$ to deduce the measure of $\frac{1}{5}$ unit of unknown quantity. [6.7; 6.24]

(*table continues*)

Table 6.3. (*continued*)

Problem	*Ted*	*Cole*	*Aileen*	*Brian*
2.74 ($\frac{12}{7}x = \frac{5}{3}$)	He rewrote the two fractions on the same denominator, (i.e., $\frac{12}{7}$ as $\frac{36}{21}$ and $\frac{5}{3}$ as $\frac{35}{21}$). [6.18] 2nd attempt: Using guess-and-check on JavaBars [6.10]	He represented $\frac{5}{3}$ on JavaBars and divided each third into 7 parts and used guess-and-check to find bar B. [6.10]	1st attempt: She focused on the denominator and attempted to find common denominator for 3, 5, and 7. 2nd attempt: she converted $\frac{5}{3}$ to $\frac{35}{21}$ and $\frac{12}{7}$ to $\frac{36}{21}$. 3rd attempt: After representing $\frac{5}{3}$ by 5 partitions, she divided each of them by 7. [6.19]	He focused on the numerators of the two given fractions and attempted to multiplicatively compare 12 one-seventh units and 5 units. [6.19]

CHAPTER 7

SET 3: MULTIPLICATIVE COMPARISON IN RATIO SITUATIONS

In the previous chapter, I looked at reversibility situations in fractional contexts where the participants had to construct one whole starting from a fractional part or a part larger than one. In other words, in fractional contexts, it is the part-whole structure that generates reversibility situations. As Tzur (2004) pointed out "a reversible fraction conception refers to the learner's partitioning of a non-unit fraction (*n/m*) into *n* parts to produce the unit fraction (*1/m*) from which the non-unit fraction was composed in the first place" (p. 93). This allows us to produce one whole (*m/m*) starting from the unit fraction (*1/m*). In ratio contexts, reversibility is interpreted in terms of the part-part-whole structure involving the comparison of the cardinalities of two (or more) subsets constituting a set. *A reversible ratio conception involves the decomposition of a quantity in terms of the constituent parts that generated the quantity in the first place*. Let the quantity $q\ (= q_1 + q_2)$ be decomposed in the ratio $a : b$ such that $q_1 = ka$ and $q_2 = kb$. In a reversibility situation, the ratio $a : b$ (a multiplicative relation) and *q* are given, and the aim is to find q_1 and q_2 (algebraically equivalent to $(a + b)x = q = q_1 + q_2$). Alternately, the ratio and the difference between q_1 and q_2 are given and the aim is to find q_1 and q_2 (algebraically equivalent to $(a - b)x = q_1 - q_2$).

Ratios defined with the same measure space can be categorized as part-part and part-whole comparisons (Lamon, 1999). For instance, suppose a box of chocolates has two types of chocolates: 8 milk and 4 mint. In a part-part comparison, the components constituting the whole are compared; for instance, the ratio of milk to mint chocolate is 8:4 or 2:1. In a part-whole comparison, one component of the ratio is compared to the whole; for example, the ratio of milk chocolate to the whole box is 8:12 or 2:3. In ratio situations, composition and decomposition are two fundamental operations - combining parts to make a whole and splitting wholes to make parts. I use the following example (Apartment problem) from Lamon (1994) to illustrate how ratios constitute reversibility situations:

> In a certain town, the demand for rental units was analyzed and it was determined that, to meet the community's needs, builders would be required to build units in the following way: every time they build 3 one-bedroom units, they should build 4 two-bedroom units and 1 three-bedroom unit. Suppose the builder decides to build 40 units. How many one-bedroom units, two bedroom units, and three bedroom units would the apartment building contain? (p. 106)

This problem requires the composition and decomposition of composite units. The one-bedroom apartments, two bedroom apartments, and three bedroom apartments constitute one level of units (of different numerosity). The combination of 3 one-bedroom units, 4 two-bedroom units and 1 three-bedroom unit form a second level of unit, or a unit of units (a composite unit of 8 units). In other words, one has to conceptualize the three different units as one entity. The foregoing problem requires the decomposition of the given composite unit of 40 units in terms of the composite unit of 8 elements, *considered* as one entity (i.e., $40 = 5(8)$). Then another level of conceptualization is required, namely $40 = 5(3 + 4 + 1) = 15 + 20 + 5$. Algebraically, the Apartment problem is equivalent to solving the equation $(3 + 4 + 1)x = 40$.

Overview and Justification of Tasks

As pointed out by Thompson (1993), a problem situation may be regarded as complex by the problem solver because it requires him/her to keep multiple relationships in mind in order to constitute it. One of the characteristics of the problems chosen in this set of situations is that they involve the articulation of two quantitative relationships. For instance, in problem 3.2.1 (see Table 7.1), one needs to coordinate a multiplicative comparison relationship (5:3) and an additive comparison (the difference between the two quantities is 10) or a quantitative difference rather than a numerical difference. Thompson (1993) distinguished a quantitative difference from a numerical difference in that the first form of difference is the amount by which one quantity exceeds the other while the second form of difference is the result of subtraction. One characteristic of the problems in Table 7.1 is that the quantities have no stated or actual values, but rather the relations between the quantities are specified.

In this problem set, I investigated three types of reversibility situations: $(a+b)x = q_1 + q_2$ (Type I), $(a-b)x = q_1 - q_2$ (Type II), and $(a-b)x = 2e = q_1 - q_2$ (Type III), where e is the amount exchanged as shown in Table 7.1. Below I justify the choice of problems that were developed prior to the interviews and those that cropped up as a result of my interaction with the students.

Table 7.1. Structure of ratio tasks

Problem no.	*Problem statement*	*Algebraic Representation*
3.1.1 (Type I)	Dana has 3 times as many marbles as Clay. Together, Dana and Clay have 32 marbles. How many marbles does Clay have?	(a) $(1+3)x = 32$ (b) $(1+3)x = 88$ (7th graders only)
3.1.2 (Type I)	Joe had some marbles. Then his friend gave him 5 times as many marbles as he had initially. Now Joe has 42 marbles. How many marbles did Joe have initially?	$(1+5)x = 42$
3.1.3 (Type I)	For every $3 that Mac saves, his dad contributes $5 to his saving account. How much will Mac have to save and how much money his dad has to contribute so as to get enough money to buy (a) a coat for $32 (b) a bicycle for $120?	(a) $(3+5)x = 32$ (7th graders only) (b) $(3+5)x = 120$
3.2.1 (Type II)	A sum of money was divided between Alan and Bill. For every $5 that Alan received, Bill received $3. Given that Alan received $10 more than Bill, calculate how much Bill received?	(a) $(5-3)x = 10$ (b) $(5-3)x = 7$ (6th graders only)
3.2.2 (Type III)	Right now, for every R marbles that Richard has, John has J marbles. If Richard were to give N marbles to John, they would have the same number. How many marbles does each student have right now? (a) $R:J = 3:1$, $N = 18$ (b) $R:J = 4:1$, $N = 18$ (c) $R:J = 3:2$, $N = 8$	(a) $(3-1)x = 2\times 18$ (b) $(4-1)x = 2\times 18$ (c) $(3-2)x = 2\times 8$ (6th graders only)

In Type I problems, the sum of two quantities ($q_1 + q_2$) and the multiplicative relationship ($q_1 : q_2 = a : b$) between them are given, and the objective is to construct the two quantities. These problems are algebraically equivalent to $(a+b)x = q_1 + q_2$. For instance, problem 3.1.1 is algebraically equivalent to $(1+3)x = 32$. In this discrete situation, the multiplicative relationship ($a : b = 3 : 1$) is given and the end quantity ($q_1 + q_2 = 32$) is specified; the aim is to find the quantity corresponding to the components of the ratio. Because one of the seventh-graders used a guess-and-check procedure, I also asked them to solve the problem for a larger total ($q_1 + q_2 = 88$). Three problems (3.1.1, 3.1.2, and 3.1.3) were explored in this category.

In Type II problems, the difference between two quantities ($q_1 - q_2$) and the multiplicative relationship ($q_1 : q_2 = a : b$) between them are given, and the objective is to construct the two quantities. These problems are algebraically equivalent to $(a-b)x = q_1 - q_2$. Like Type I problems, this class of situations involves the decomposition of a whole in terms of the components of a ratio. However, here the problems involve the difference between the components of the ratio, rather than the sum. For instance, problem 3.2.1(a) is algebraically equivalent to $(5-3)x = 10$. The sixth graders were also required to solve $(5-3)x = 7$, a question that cropped up in the interview. Further, in this problem, the actual values of the quantities are not stated; only the relationships that link the quantities are given. In problem 3.2.1(a) the difference between Alan's and Bill's share $(5-3=2)$ divides the difference in sum ($10) while this is not the case in problem 3.2.1(b), where the difference in sum is $7. In other words, the worth of one share is $3.5 in the second case.

Type III problems involve a multiplicative and a difference relationship between two quantities but also require the consideration of a double difference. Three problems were formulated in this category, algebraically equivalent to $(3-1)x = 2\times 18$, $(4-1)x = 2\times 18$, and $(3-2)x = 2\times 8$. Problem 3.2.2(b) is parallel to problem 3.2.2(a) except that the share ($(4-1)/2 = 3/2$) that has to be exchanged is not an integer. Algebraically, this involves the solution to $1\frac{1}{2}x = 18$ or $1.5x = 18$ (a multiplicative comparison involving a fraction or decimal and an integer). Problem 3.2.2 (c) cropped up during the interview with the sixth-graders and is algebraically equivalent to $\frac{1}{2}x = 8$. Analogous to problem 3.2.2(b), here the share that has to be exchanged ($(3-2)/2 = 1/2$) is not an integer. The dates and order in which the tasks were posed are shown in Table 5.2.

Table 7.2. Chronological order of problems in Set 3

Date	Grade 6	Grade 7	Grade 8
5/7/08	NA	NA	3.1.1(a), 3.1.2, 3.13(b), 3.2.1(a)
5/8/08	3.1.1(a), 3.1.2	3.1.1(a), 3.1.1(b), 3.1.2, 3.1.3(a), 3.1.3(b), 3.2.1(a), 3.2.2(a), 3.2.2(b)	3.2.2(a), 3.2.2(b)
5/9/08	3.1.3(b), 3.2.1(a), 3.2.2(a), 3.2.2(b)	NA	NA
5/12/08	3.2.1(b), 3.2.2(c)	NA	NA

I have categorized the participants' responses on the basis of the type of strategies that they used in the problem situations. As Vergnaud (1988) pointed out, it is the strategy that the student use that shows evidence of particular theorems-in-action. Four main strategies are evident from the data: (i) additive building-up strategy (episodes 7.1-7.5), (ii) multiplicative strategy (episodes 7.6-7.10), (iii) algebraic approach (episodes 7.11-7.13), and (iv) guess-and-check strategy (episode 7.14) as summarized in Table 7.3. Within these four broad categories, I analyzed the fine-grained mechanisms that the participants appeared to use to identify subtle theorems-in-action. The analysis was guided by both the identification of strategies (concepts-in-action, theorems-in-action and rules-of-action in play) and constraints that the participants encountered. The focus was also on the identification of ways in which the students reasoned reversibly or failed to do so and the resultant consequences. I focus on both implicit and explicit theorems-in-action. I also consider the ways in which the students conceptualized the problem situations. Moreover, I show how the quantitative interpretation of a situation is a key resource for reasoning reversibly in the ratio problems.

Table 7.3 lists the strategies used on a problem-by-problem basis. If the students made different attempts in finding the solution, all of them are reported in the order in which they were used. The algebraic representation in the first column of the table is meant to show the structure of the problem. The number in the square bracket indicates the episode number.

Table 7.3. Summary of Strategies Used by the Six Participants

Problem no.	*Ted*	*Cole*	*Aileen*	*Brian*
3.1.1(a) $(1+3)x = 32$	Dividing the given quantity (32) by the sum of shares (4) [7.6]	Dividing the given quantity (32) by the sum of shares (4) [7.6]	Using the building-up strategy based on systematic guess and check	1st attempt: He divided 32 by 3. 2nd attempt (after Aileen's intervention), he used the ratio 3:1 and divided 32 by 4.
3.1.1(b) $(1+3)x = 88$	NA	NA	Using the building-up strategy based on systematic guess and check [7.1]	Using the building-up strategy [7.1]
3.1.2 $(1+5)x = 42$	Dividing the given quantity (42) by 5 rather than 6	Dividing the given quantity (42) by 5	Using the building-up strategy based on systematic guess and check	Using the ratio 5:1
3.1.3(a) $(3+5)x = 32$	NA	NA	Using the building-up strategy	He used the sum of shares and divided 32 by 8.
3.1.3(b) $(3+5)x = 120$	He added 3 + 5 to get 8 and multiplied 8 by 15 to get 120. [7.7]	He added 3 + 5 to get 8 and multiplied 8 by 15 to get 120. [7.7]	Using the building-up strategy [7.2]	Using the building-up strategy [7.2]

(*table continues*)

Table 7.3. (*continued*)

Problem no.	*Ted*	*Cole*	*Aileen*	*Brian*
3.2.1(a) $(5-3)x=10$	Using the building-up strategy [7.3]	Using the building-up strategy [7.3]	Using the building-up strategy [7.3]	By equating difference in shares to difference in amount [7.8]
3.2.1(b) $(5-3)x=7$	Using the building-up strategy [7.4]	Using the building-up strategy [7.4]	NA	NA
3.2.2(a) $(3-1)x=2\times 18$	Guess-and-check; After prompting they equated the shares to be exchanged to the given difference	Guess-and-check [7.14]; He did not obtain the final answer on his own.	1st attempt: Guess-and-check [7.14]; 2nd attempt: After prompting she used the ratio 3:1 [7.12]	1st attempt:Guess-and-check [7.14]; 2nd attempt: algebraic approach $R-J=36$ $J=1/3\ R$ $2/3\ R=36$ [7.12]
3.2.2(b) $(4-1)x=2\times 18$	By equating the shares to be exchanged to the given difference, though incorrectly. He did not obtain the final answer.	By equating the shares to be exchanged to the given difference (after prompting)	1st attempt: Algebraic approach [7.13] 2nd attempt: By equating the shares to be exchanged to the given difference (after prompting)	1st attempt: algebraic approach [7.13] 2nd attempt: By equating the shares to be exchanged to the given difference (after prompting) [7.16]
3.2.2(c) $(3-2)x=2\times 8$	Using the building-up strategy [7.5]	Using the building-up strategy [7.5]	NA	NA

(*table continues*)

Table 7.3. (continued)

Problem no.	*Eric*	*Jeff*
3.1.1(a) $(1 + 3)x = 32$	Guess-and-check procedure	Using the ratio 3:1 [7.6]
3.1.1(b) $(1 + 3)x = 88$	NA	NA
3.1.2 $(1 + 5)x = 42$	Guess-and-check procedure, dividing 42 by 5, followed by an algebraic procedure	Initially he divided 42 by 5. Then he used an algebraic approach $x+5x = 42$ [7.11]
3.1.3(a) $(3 + 5)x = 32$	NA	NA
3.1.3(b) $(3 + 5)x = 120$	He misinterpreted the problem.	Using the ratio 3:5 [7.7]
3.2.1(a) $(5 - 3)x = 10$	Using the building-up strategy [7.3]	By equating the difference in shares to the difference in amount [7.8]
3.2.1(b) $(5 - 3)x = 7$	NA	NA
3.2.2(a) $(3 - 1)x = 2 \times 18$	Plugging-in numbers in the ratio 3:1 [7.14]	1st attempt: Algebraic equation [7.12] 2nd attempt: He equated the difference between the shares to the difference in amount. [7.9]
3.2.2(b) $(4 - 1)x = 2 \times 18$	He did not solve the problem.	By equating the shares to be exchanged to the given difference and using division [7.10]
3.2.2(c) $(3 - 2)x = 2 \times 8$	NA	NA

Additive Building-up Strategy

The additive building-up strategy consists of iterating a given ratio a number of times until a required sum or difference is attained. It is very prominent in the current data set as has been observed in previous research involving ratio situations (e.g., Kaput & West, 1994). I provide five episodes from the ten problem situations (Table 7.1) to show the different ways in which the six participants reasoned additively, at different levels of sophistication.

Episode 7.1: ***Brian:*** **[Problem 3.1.1(b)]**

Dana has 3 times as many marbles as Clay. Together, Dana and Clay have 88 marbles. How many marbles does Clay have?

Data. Brian made a table (Figure 7.1) to build-up the given quantity (88 marbles) in terms of Dana's and Clay's share.

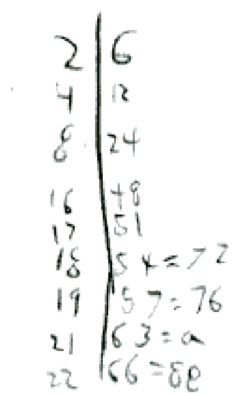


Figure 7.1. Brian's building-up strategy

The tabular representation enabled him to coordinate the values between the two quantities. He gave the following response:

B: It starts out as 1 to 3 and you just keep on multiplying upwards 2, 6; 4, 12; 8, 24 (pointing to his table) and so on. (Inaudible) you could not just double it again (referring to the number 17 in his table) because that would go way over it, so you just have to count up by ones (inaudible).

Analysis. From Figure 7.1, we observe that Brian incremented the values of the first quantity (for Clay) by doubling its value to 17. This form of computationally-efficient coordination between the two components of the ratio was observed by Kaput & West (1994). When the same problem was presented to Brian with a smaller number (32 rather than 88), he used a multiplicative strategy by dividing 32 by 4 and calculated Dana's and Clay's share as 3×8 and 1×8. The fact that Brian altered his strategy when a larger quantity (88) was presented suggests that different concepts-in-action are triggered when the numeric structure of even one quantity of a problem is changed. With the number 88, he did not consider the two shares as one quantity. This is turn cued an additive theorem-in-action, namely incrementing (by doubling) the values proportionally in a table. Rather than working from the given result 88 to the find the corresponding shares, he worked in a forward direction starting from the given share to reach the end quantity 88.

Similarly, Aileen used a systematic incrementing procedure, starting with 10 marbles, doubling it to 20, and moving in steps of one unit until she reached 22 marbles for Clay:

A: I started going up by like 10s, so I did, if Clay had 10, then he would have 30, and 30 plus 10 that's 40. So I knew it had to be higher than 10, so I went up to 20 and 20 times 3 is 60 and so 60 plus 20 is 80. Now it's really close to us, so I started going up by one which is 21 times 3 that's 63 plus 21 that would be 84. So I [went up] to 22 and 22 times 3 is 66 plus 22 equals 88. So Clay would have 22 and Dana would have 66.

Episode 7.2: ***Aileen and Brian*** **[Problem 3.1.3(b)]**

In episode 7.1, we observed that the additive strategy used by Brian was based on doubling the previous shares (as shown in Figure 7.1). In episode 7.2, we observe another abbreviated form of the building-up strategy used to solve problem 3.1.3(b). Furthermore, both of the participants generated the given quantity (120) by working in a forward direction by incrementing the given ratio rather than working backward to decompose the given quantity (i.e., starting from the end result of 120)

Data.

L1 I: So here, for every $3 that Mac saves, his dad will contribute $5 to his saving account. So, how much will Mac have to put into his account to get enough money to buy a bicycle for $120? So every time Mac saves 3 dollars, his dad will contribute 5 dollars and the bicycle costs 120 dollars. How much will Mac have to put into his account or will have to contribute and how much will his father contribute?

L2 A: He would have to put in 45 dollars.

Aileen first set up the ratio 3:5. Then, she constructed the ratio 36:60 followed by 45:75. She then added 45 to 75 to verify that she obtained 120 as shown in Figure 7.2.

L3 I: Yes, Brian?

L4 B: It would be 45 because 5 plus 3 is 8. So, (inaudible) 50 and 30. You put that together that's 80 and that's both times 10. You still know that there is 40 left. So if you times by 10, and that gives you twice as much as 40. So, if you divide in half, what you are multiplying by, in the first place (inaudible) five. Then you have 15, 3 times 15 is 45.

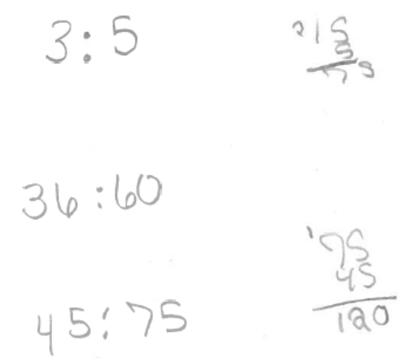


Figure 7.2. Aileen's building-up strategy

Analysis. Aileen explicitly wrote the multiplicative relation as a ratio, a concept-in-action. Brian first combined the two shares into a composite unit (3 + 5 = 8), which he multiplied by 10 to obtain 80. He observed that 80 did not meet the required end quantity ($120). The interpretation of the additional 40 units as $\frac{1}{2}$ of 80 led him to deduce that he had to use half of the initial scaling factor (i.e., $\frac{1}{2}\times 10 = 5$, L4). Thus, he obtained the scaling factor 15 (= 10 + 5) which allowed him to deduce that Mac would have $\$3\times 15$ or $45. In other words, instead of iterating the ratio 3:5 fifteen times in sequence, he iterated it 10 times and $\frac{1}{2}(10)$ times in two steps. What is remarkable here is that despite forming the total share $5+3=8$, Brian did not divide 120 by 8 explicitly but rather used a systematic and parsimonious building-up strategy. Aileen's strategy was to set up the ratio and to scale up until she reached the desired result as she did in problems 3.1.1 and 3.1.2. Relating the sum of shares to the total quantity to be decomposed from a multiplicative perspective necessitates the theorem-in-action $(a+b)x = n$, which neither of the two participants deployed (where *a* and *b* denote the ratio of shares and *n* denote the quantity to be decomposed) in the current problem.

Episode 7.3: [Problem 3.2.1(a)]

Compared to situations 3.1.1, 3.1.2, and 3.1.3, in problem 3.2.1 one is required to conceptualize a difference as a quantity rather than a sum as a quantity. In this problem, a sum of money was divided between Alan and Bill in the ratio 5:3, and they were asked to find how much money Bill received if Alan received $10 more than Bill. Looked at from a quantitative perspective, the difference between Alan's and Bill's share (i.e., 5 – 3 = 2) corresponds to $10 and this yields the size of one share as $5.

Data. Ted's and Cole's strategy for problem 3.2.1(a)

I asked Ted and Cole to interpret the ratio 5:3 and Ted mentioned that "Alan would have one and two thirds times as Bill" and "Bill has three fifth times as Alan". Ted's response shows that the concept-in-action 'ratio' represented a multiplicative comparison. The following transcript illustrates their strategies for finding Bill's share.

L1 I: Given that Alan received $10 more than Bill. OK. When you look at the ratio, who has more money? Alan or Bill?

L2 T and C: //Alan//

L3 I: Right. So given that Alan received $10 more than Bill, can you find how much Bill received?

Pause

Cole made a column of 10 squares while Ted made Figure 7.3 by generating the second, third and first column successively.

L4 C: How would you get it? (talking to Ted)

L5 T: (pointing to his diagram, Figure 7.3) 5, 10, 15, 20, 25. That's the ratio right here. I multiplied by 5 because, yeah. And so I multiplied $5 by 3 for Bill to get $15 and I multiplied 5 (meaning $5) by Alan's number which is 5 to get 25 and so I multiplied 25, I subtract 25 with 15 to get $10. So, Bill received $15.

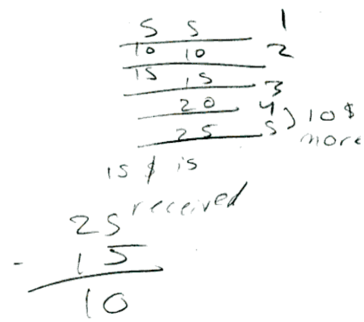


Figure 7.3. Ted's strategy

Analysis. Ted repeated $5 three times in the first column and $5 five times in the second column in line with what he said in L5, "I multiplied $5 by 3 for Bill to get $15" (first column) and "I multiplied 5 (meaning $5) by Alan's number which is 5 to get 25" (second column). He generated the first and second column one after the other and not simultaneously, which gives evidence that he unknowingly started from the assumption that the value of one share was $5. Ted validated his answer by subtracting $10 from $25 (a rule-of-action). He made the third column in Figure 7.3 to indicate the number of sets of 5 that he was using. Rather than building-up the corresponding sequences 5:3, 10:6, 15:9, 20:12, 25:15, Ted worked with the size of one share directly, which he assumed to be $5. The solution procedure may have been facilitated by the fact that the value of one share corresponds to one of the components of the ratio (i.e., Alan's share), as will not be the case in the next problem. On the other hand Cole made a column consisting of 10 squares and when asked to explain what his diagram represents, he mentioned: "Because it said given Alan received $10 more, so that means they have to use 10, but it did not work." This episode also shows that both students reasoned numerically rather than quantitatively in this situation. They did not generate a theorem-in-action to conceptualize the difference between the shares (5 – 3 =2) as one entity in relation to the given difference of $10. Thus, they had to work in a forward direction to hit a difference of $10.

Aileen's strategy for Problem 3.2.1(a)

Aileen's strategy in this problem was to scale the ratio up in increments of one unit until she reached a difference of 10:

A: I knew that it had to get to $10 more. So I did 5 times 2 and 3 times 2 and that gave me 10 over 6. And I did 5 times 3 and 3 times 3 and that gives me 15 over 9. And then I did 5 times 4 and 3 times 4 and that was 20 and 12. And then I got to 5 and 5 times 5 and 3 times 5 is 25 to 15.

Eric's strategy for Problem 3.2.1(a)

Eric's strategy was to plug in numbers like Aileen, constructing the ratios 15:9, 20:12, and 25:15 by incrementing the scale factor by one until the required difference between the two quantities was attained.

E: I kept in mind that every $5 Alan would get, Bill would get $3. So, I just started plugging in numbers like I would like maybe on a table or something like you would do. I plugged in 4 and Alan would have 15 and Bill would have 9, so that would not work. And I plugged in, no that was 3. I plugged in 4 and I got 20 to 12. And I plugged in 5 and I got 25 and 15. So 5 was how much it would get and Alan would have $ 25 and Bill would have $15.

Episode 7.4: ***Ted and Cole*** **[Problem 3.2.1(b)]**

Compared to episode 7.3, here the difference between Alan's and Bill's share is $7 (rather than $10). This difference is produced when Alan has $17.5 and Bill has $10.5. In other words, the difference between the two quantities does not correspond to a ratio with integer components. The non-integer ratio increased the level of complexity of the problem and led the participants to revise their building-up theorem-in-action to generate rules-of-action for the new class of situation. This was observed by three explicit adjustments that Cole made in the process of solving the problem.

Data.

L1 I: Suppose a sum of money is divided between Alan and Bill and for every $5 that Alan receives, Bill receives $3. Given that Alan receives $7 more than Bill, can you find how much Bill received?

Ted started by making the table shown in Figure 7.4.

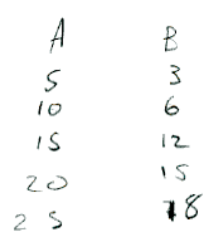


Figure 7.4. Ted's building-up strategy

L2 T: I added 5 plus 5 plus 5 plus 5 (referring to Figure 7.4) to get 25 and I added 3 plus 3 plus 3 plus 3 plus 3 plus 3 to get 18 and when I subtracted 25 by 18, it was 7 dollars.

L3 C: I have Alan on this side (first column of Figure 7.5) and Bill on this side (second column). So they started with 5 and 3, so I got 5 times 2, 10; 3 times 2, 6. Five times 3 is 20, no 5 times 4 is 20 and (3 times 4 is) 12 and then I got 5 times 5 will be 25 and 3 times 5 will be 18, and, you subtract ...

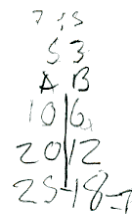


Figure 7.5. Cole’s building-up strategy

L4 T: Hum, hum, 3 times 5 is 15.

L5 C: And then you would subtract 25 by 18 and you would come up with 7. So that would mean that Alan had $7 more than Bill.

⋮

L6 I: Can you show me that in your table that you have made Ted, 25, 18.

L7 T: Yeah. I just multiplied it like 5 times 2, 5 times 3, 5 times 4, 5 times 5. Three times 2, 3 times 3, let see. 3 times 3 is what, 9. May be I skip some.

L8 C: Three times 4 is 12.

L9 T: I skip some.

L10 C: No, that’s how I got mine, too.

L11 T: But then is 9 a multiple of 3, that’s where I messed up.

L12 T: Yeah.

L13 C: (inaudible).

L14 I: So, what is your answer?

L15 T: Uh, let me, I got to do this thing again. (He made the table shown in Figure 7.6)

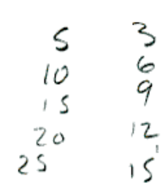


Figure 7.6. Ted’s reconstruction of the two sequences of numbers

L16 I: Let me see for Cole. Can I see your table (Figure 7.7)?

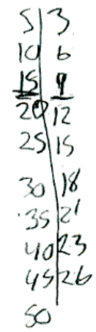


Figure 7.7. Cole's location of the difference of 7

L17 C: I just multiplied 3 and then multiplied 5.

L18 I: And what are you looking for? Alan received $7 more than Bill, right.

L19 C: That's 8 dollars pointing to 20 and 12.

L20 T: That's 10 (pointing to 25 and 15 in Figure 7.7), that's 8 (pointing to 20 and 12), that's 6 (pointing to 15 and 9), so. It would be, look, look, look, it is right here. Seven dollars more. Fifteen subtracted by 9 is 6. Twenty subtracted by 12 is 8. So, it will be ...

L21 C: $15\frac{1}{2}$ and $9\frac{1}{2}$. No.

L22 T: 15.50 and 9.50 (Ted subtracts 9.50 from 15.50 on his worksheet).

L23 C: No, we are multiplying by 5's and 3's.

L24 T: I don't care.

L25 C: So, it will have to be 9, 10, 11, 12. It would be 9, 10, $10\frac{1}{2}$. It will be $10\frac{1}{2}$ and 15 and $16\frac{1}{2}$. $10\frac{1}{2}$ and $16\frac{1}{2}$. Just write it down and see.

L26 T: Why is it $10\frac{1}{2}$? Why can't it be $10 and 50 cents.

L27 C: OK, same deal. No.

L28 I: Yes, how did you get the $10\frac{1}{2}$ dollars and $16\frac{1}{2}$ dollars? How did you get that? From your table...

L29 C: Yeah, right here (Figure 7.7). I took, I knew 5 and 12, I mean 5 and 9 that will be 9, 10, 11, 12, 13, 14, 15, that will be \$6 and then 12 and 20 that will be \$8, so it had to be in the middle. So I took 9 and added one because we are multiplying by threes. So I added one and it will be 10, then I had to add a half to make it in the middle of 9 and 12. And then added 1 to 15 and I added a half to make it in the middle of 20 and I got $10\frac{1}{2}$ and $16\frac{1}{2}$ and that would not be 7.

L30 I: That would not be, what?

L31 C: It would not be 7.

L32 I: Why?

L33 C: 10, 11, 12, 13, 14, 15, 16. It's only 6.

L34 I: How do you, know, sorry?

L35 C: It's only 6.

L36 T: So would it be 16...

L37 C: We just have to work it out.

L38 T: See this and this would be just like \$6. So we need do like add a dollar to this.

L39 C: Oh, I get it now. 15, 16, 17, 18 (inaudible). \$ $2\frac{1}{2}$ to 15, so 15, 16, 17 and a half. And then 9 and 12 has 3, so $1\frac{1}{2}$ with it. So that will be $10\frac{1}{2}$.

L40 T: One and a half with it?

L41 C: So $10\frac{1}{2}$. That is 7!

L42 T: What are you talking about?

L43 C: Oh, Ted.

L44 T: What are you talking about? Please tell me what you did.

L45 C: Look you got 15 and 20 and 9 and 12. All right, there is 3 in between 9 and 12. So to get, in the middle you have to have $1\frac{1}{2}$.

L46 T: 9, 12, 10, 11, ...

L47 C: What are you doing?

L48 T: $10\frac{1}{2}$, right.

L49 C: Yeah. And then there is 5 in between 15 and 20. So you would have to have $2\frac{1}{2}$ which would be half of 5. So that will be 7.

Analysis. Initially, although Ted and Cole made a table (Figures 7.4 and 7.5) to observe how Alan's and Bill's money varied, neither coordinated the two sequences of numbers consistently, probably in a rush to get a difference of 7 (based on the building-up theorem-in-action). Rather than working with the shares from a quantitative perspective, they generated members of the equivalence class 3:5 (although inconsistently) until they could observe a difference of 7, thus working in a forward direction, starting from known quantities.

When asked to explain their answer a second time, each of them generated another table (Figures 7.6 and 7.7). Using Cole's table (Figure 7.7), Ted deduced that the answer should be between the pair 15:9 and 20:12 (L20) because 7 lies in between the difference 15 and 9, and 20 and 12 as illustrated by the following schema:

Alan	Bill	Difference
15	9	6
?	?	7
20	12	8

This is one of the instances where they were specifically confronted with a reversibility situation in that knowing the difference (i.e., 7), they had to find a ratio that satisfies it. Cole made three adjustments until he could find a pair of numbers in Figure 7.7 with a difference of 7. In his first attempt, he added half to 15 and 9 respectively to obtain $15\frac{1}{2}$ and $9\frac{1}{2}$ or 15.50 and 9.50 (L21 and L22). Then Cole observed that there is a difference of 3 between 9 and 12 and half of this interval would be $1\frac{1}{2}$ which led him to obtain $\$10\frac{1}{2}$ (L25) for Bill. However, he added the same amount ($1\frac{1}{2}$) to 15 to obtain $16\frac{1}{2}$ (second adjustment). After observing that his solution ($16\frac{1}{2}$ and $10\frac{1}{2}$) did not produce the required difference of 7, he took half of the difference between 15 and 20 to deduce that the middle value of the interval 15 to 20 is $17\frac{1}{2}$ in L39 (third adjustment). The three adjustments show that Cole had to modify his building-up theorem-in-action or construct another theorem-in-action, namely finding the proportional difference between the intervals [15,20] and [9,12] in the two sequences of numbers he generated in Figure 7.7. Viewed from a mathematical perspective, Cole's approach to the problem is analogous to the mathematical concept of linear interpolation. This primitive form of linear interpolation was observed by Resnick & Singer (1993).

The participants solved the problem in a forward direction, generating pairs of numbers in the given ratio and then looking for the required difference. Failure to conceptualize the situation from a quantitative perspective (in terms of the equivalence between shares and corresponding quantities) may have prevented them from working from the given difference to find the components that create the difference. This observation leads me to infer that quantitative reasoning is an important resource for reversible reasoning in such ratio situations.

Episode 7.5: ***Ted and Cole*** **[Problem 3.2.2(c)]**

As illustrated in previous episodes, the building-up strategy was cued in problem situations where the relationships between the quantities were given but no particular values were stated. Problem 3.2.2(c) is another situation where a multiplicative and a difference relationship between two quantities were provided and the participants had to find the values of the quantities.

Data.

L1 I: Right now, for every $3 that Richard has, John has $2. If Richard were to give $8 to John, they would have the same amount.

Both of them made a table of values scaling the base ratio 3:2 successively to 6:4, 9:6, 12:8, and so forth as shown in Figure 7.8. The left hand table represents Ted's diagram while that on the right represents Cole's diagram.

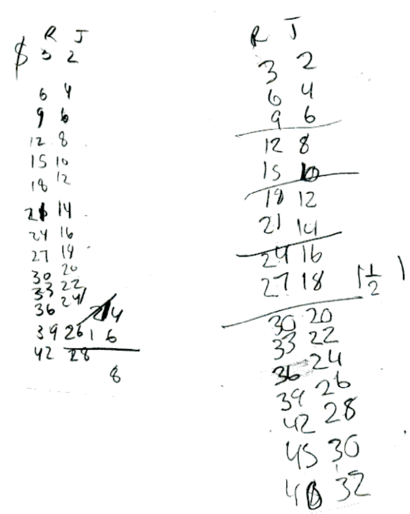


Figure 7.8. Building-up strategy used by Ted and Cole

L2 I: Cole, you said that answer is going to be between what?

L3 C: 24 and 27 and 16 and 18 (referring to Figure 7.8).

L4 I: How do you know?

L5 C: Because I [am using] the same method as I used just right here (referring to the previous problem) and I did. I multiplied this (referring to Figure 7.8) all the way to 24. I multiplied 3 all the way to 24 and 27 and multiplied 2 all the way to 16 and 18. And 16 and 24 that would be 7.

L6 I: 16 and 24 will be what?

L7 C: Seven.

Ted intervened and mentioned that 16 subtracted from 24 is 8.

L8 C: And 18 and 27. That would be 9. It has to be in the middle.

L9 I: In the middle of?

L10 C: 16 and 18 and 24 and 27 (referring to his diagram).

⋮

L11 T: I mean 24 and 16.

L12 I: Twenty four and sixteen.

L13 T: Yeah. Richard has $24 and John has $16. And...

L14 I: So, you are saying that right now Richard has how many?

L15 T and C: //$24//

L16 T: And John has $16.

L17 C: And that would be $8 more. So.

L18 T: But if you subtract that from Richard then they will have the same as $16 and $16 (writing his answer on the given worksheet).

L19 I: OK, right now Richard has 24 and ...

L20 C: Oh, he is talking about taking it from Richard and giving it to John.

L21 T: Yeah.

L22 I: Yes.

L23 T: They will still get the same.

L24 C: No.

L25 I: OK, let's take your solution, Ted. You said that Richard has 24 and John has 16. You are taking 8 from Richard and. So when you take 8 from Richard, how much does Richard have?

L26 T: Oh, OK. I did not get the first one. Because you were doing this (referring to Cole's table) and you are subtracting.

⋮

L27 C: Oh, I get it. We need to find it where $16 more (referring to his table). OK (talking to Ted). Right. So you can take off $8 (inaudible).

Cole extended his table in an attempt to find a difference of 16 while Ted multiplied 16 by 3 and 2 respectively to obtain 48 and 32.

L28 T: My solution is 48 which is Richard's and 32 to John.

L29 I: And how did you get that?

L30 T: So, when I found out that 8 would not go in correctly. Well it's like, uh Richard will need 8, well Richard needs like a number that will equal up to John. So, it's like, Richard will need $8 so that he will have more to John. Right, yeah. And so, if I gave, I gave it to John then John will have the same number as Richard has when he gave to him. So I went to the next number which 8 multiply by 2 equals 16 and I just multiply 16 by 2 to get 32 and 3 by 16 which equals 48. So it was like 48 subtracted by 8 is 40 and 32 plus 8 equals 40.

L31 I: Where did you get the 16?

L32 C: I know it is 16 because I thought if it was 16 over, like if, Richard was 16 over John, it could be, you could give away 8 and it (would be) still the same. So it will add 8 to John, but it would keep the same 8 with Richard.

At a later point he gave the following explanation why a difference of 16 should be considered:

L33 C: All right. Because if it was $8, if he had $8 more, if Richard had $8 more than John and he gave $8 away, he would be under $8. He would have $8 less than John. So you would have to add 8 and 8 which would be 16 and because if you had just like right here (pointing to his table) you have 48 and 32, it would be 16 over, and you could take 8 away but it will still have the other 8 they were adding to John.

Analysis. Their immediate rule-of-action (based on the building-up theorem-in-action) to approach such a problem was to make a table of values. This strategy also reduces the problem to a lower level of abstraction because the numerical values helped them to conceptualize the multiplicative and subtractive relationship between the quantities. Once the multiplicative relationship was set in the table of values, their next step was to look for a difference of 8 units. Thus, Ted deduced that Richard would have $24 and John would have $16 (L13). At that point neither of them realized that by transferring $8 from Richard to John, Richard would be deficient by 8. In other words, they focused on the difference only. On the other hand, based on the interpolation strategy that helped him to solve problem 3.2.1, Cole once again attempted to situate his answer between two adjacent ratios (L5). He incorrectly assumed that 24 minus 16 is 7 (L7), which led him to deduce that the difference of 8 units that he was looking for, at that moment of the interview was situated between 24:16 (difference of 8 units) and 27:18 (difference of 9 units) in Figure 7.8. Their current theorem-in-action was no longer sufficient for the given situation.

When I prompted them to verify their solution 24:16 (L25), they observed that by transferring $8 from Richard to John, the two component quantities would not be equal. The critical event in this episode is Cole's deduction that the difference between Richard's and John's share should have been 16 (L27), which also prompted Ted to multiply 16 by 3 and 2 to obtain his solution. His rule-of-action "Because if it was $8, if he had $8 more, if Richard had $8 more than John and he gave $8 away, he would be under $8" (L33) suggests the following theorem-in-action, where *x* denotes the amount that John has.

Richard	John
$x+8$	x
x	$x+8$

On the other hand, his statement "So you would have to add 8 and 8 which would be 16 … it would be 16 over, and you could take 8 away but it will still have the other 8 they were adding to John" (L33) suggests the following theorem-in-action:

Richard	John
$x+8+8$	x
$x+16-8$	$x+8$

Mathematically, this problem requires the solution to the equations $R:J = 3:2$ or $J = \frac{2}{3}R$ and $R - 8 = J + 8$ or $R - J = 16$. The second equation ($R - 8 = J + 8$) is apparent in Cole's theorem-in-action while the first equation $R:J = 3:2$ is represented in the table of values in Figure 7.8. Anticipating that the difference between the two quantities should be doubled is a necessary theorem-in-action to solve this problem that Cole could construct by reasoning to and fro between Richard's and John's share.

Multiplicative Strategy: Interpreting the Sum of Shares as a Quantity

Besides the additive strategies outlined in the previous section, another explicit category of approaches are the multiplicative strategies. Such an approach allowed the students to work from the result to the source producing the result deterministically and at a lower computational cost. I provide 2 episodes to show how the sum of the components of a ratio was taken as one quantity to decompose the given quantity using a multiplicative strategy. Here, the theorem-in-action $(a + b)x = n$ is apparent in the students' responses. It is in this category of multiplicative strategy that reversible reasoning could be observed.

Episode 7.6

Data. Ted's and Cole's strategy for problem 3.1.1(a)

L1 I: Dana has 3 times as many marbles as Clay. Together, Dana and Clay have 32 marbles. How many marbles does Clay have?

Ted and Cole spontaneously divided 32 by 4 to obtain 8. They gave the following justification for their solution:

L2 C: I divided 4 by 32 because it said Dana and Clay have 32 marbles and Dana has three times as many as Clay. So I would take 4 times because there is 4 times (inaudible) and 4 divided by 32 would be 8 remainder zero and 8 marbles would be what Clay has.

L3 I: How much will Dana have?

L4 T: Twenty-four.

L5 C: Yeah, 24. Because 32 divided by, I mean minus 8.

L6 T: Or pretty much it's 8 times 3 equals 24.

Analysis. Both Ted and Cole decomposed the given quantity (32 marbles) by dividing by 4 (i.e., the total number of shares, 3 + 1 = 4). This shows that they conceptualized Dana's share and Clay's share as one quantity. The word 'together' in the problem statement may have prompted them to add the two shares. Cole determined Dana's share by subtracting 8 from the given total (32) while Ted multiplied 8 times 3. Being able to conceptualize the two quantities as one entity is the key resource to interpret the problem multiplicatively compared to the building-up strategy where such a single entity may not be required.

Jeff's strategy for problem 3.1.1(a)

Like Ted, Cole and Brian, Jeff also used a multiplicative strategy as shown by the following quote:

J: I put, because Dana has 3 times as many, that makes their ratio 1 to 3. And to figure out how many they each one had, I added one and three to give you 4 which would be the total amount of marbles, were like each set I guess. Then I divided 32 by 4 and got 8 and then I multiplied 1 by 8 to give me how many Clay have and then multiplied 8 times 3 to give me how many Dana had.

The statement '3 times as many' cued the concept-in-action 'ratio' for Jeff. He interpreted the sum of the components of the ratio 1:3 as one set or one composite unit and determined the number of such sets in the given quantity (32).

Episode 7.7

The application of Vergnaud's theory of conceptual fields requires that one provide students with a range of situations of different degrees of complexity so as to create opportunities to observe the diversity of concepts-in-action and theorems-in-action that they may deploy in such situations. Problem 3.1.3 was formulated as a variation to problems 3.1.1 and 3.1.2, both of which involved a unit ratio.

Data. Ted's and Cole's strategy for problem 3.1.3(b)

L1 I: For every $3 Mac saves, his dad will contribute $5 to his saving account. How much will Mac have to put into his account to get enough money to buy a bicycle for $120?

L2 T: I multiplied 15 by 8 to get $120.

L3 I: So how much Mac will have to contribute and how much will his father contribute?

L4 T: Mac has to contribute. Let's see. Fifteen times.

L5 C: Mac will have to contribute $45 and his dad will have to contribute ...

L6 T: 75.

L7 C: It's $75.

⋮

L8 I: How did you get the 15?

L9 C: Because I know 15 times 8 will be 120. And then 15 times 3 and that will be 45 that Mac would have to contribute and you said his dad contributes 5 for every 3 so you multiply 15 by 5 to get how much his dad will contribute.

Analysis. As mentioned earlier, compared to problems 3.1.1 and 3.1.2, the current situation involves a non-unit ratio (3:5). Furthermore, the two quantities being compared are clearly distinguishable here. Neither Cole nor Ted used the term ratio in his response at any point of this episode. The phrasing of the problem: "for every $3 Mac saves, his dad will contribute $5 to his saving account" may have prompted them to add the two quantities. What is striking in this situation is that both Ted and Cole multiplied 15 by 8 to get 120 rather than dividing 120 by 8 to get 15. It appears that 15×8 may have been a known number fact to them as mentioned by Cole "I know 15 times 8 will be 120" (L9). Further, the context of the problem involves money and this may have also facilitated the solution procedure.

Viewed from the perspective of units this problem requires the formation of composite units and the decomposition of those composite units into their constituent parts. The sum $3 + 5 = 8$ can be regarded as a unit of units, and the multiplicative comparison of this composite unit with the given quantity (120) yields the number of times that such a unit 'fits' into 120. This can be characterized by the following theorem-in-action:

$$120 = 15(8) = 15(3 + 5) = 45 + 75$$

Jeff's strategy for problem 3.1.3(b)

Data.

L10 J: I got that he had to save $45. He would save $45 and then his dad would put in $75.

L11 I: How do you get this?

L12 J: I just made it in the ratio again. So it will be 3 to 5 and added it together to get 8. I then divided 120 by 8 which gave me, uh, 15. So then I multiplied 3 times 15 which gave me 45 which would be how much Mac would put in and then I multiplied 5 by 15 which gave me how much his dad put in.

L13 I: So, you divided, why did you divide by 8?

L14 J: Because, like. Because that would be like the whole, like set. Because for every, Mac put in $3, his dad would put in $5. So the total amount in the account then would be $8. So I divided by that to see how many like sets you would take to get there.

⋮

L15 I: And why did you multiply by 15?

L16 J: Because like I was saying, like one set would be 8. You divide 120 by 8 that would be 15. So that means there had to be 15 sets to get to 120 and then since it takes 15 then for each, every set Mac put in 3. So for 15 sets you multiply three by 15 to see how many, see how many, how much money he put in. And then to see how much money his dad put in, he puts in 5 dollars in every set. So you multiply 5 times 15 which will give you 75 for how much that his dad has.

Analysis. Jeff conceptualized the problem as a ratio situation. First, he added the two shares 3 and 5 to form a unit of units (8). Then he compared the 8 units to the end quantity (120) and deduced that he needed 15 such units of units. He explained that the 15 that he obtained on dividing 120 by 8 represents 15 sets of $8 (L14), where one set represents the combined amount that Mac and his father contribute. Jeff's strategy to find the number of sets of 8 in 120 is characteristic of measurement division. His solution also gave evidence of reversible reasoning when he decomposed the 8 unit back into $3 and $5 and used distributive reasoning to multiply each amount by 15. In other words, reversibility is illustrated in terms of unit composition and decomposition. Jeff's use of distributive reasoning with the elements of his decomposed set of 8 and his multiplier of 15 indicate his ability to coordinate three levels of units: 120 as fifteen 8-units, each of which is composed of a 3-unit and a 5-unit, producing 45 as fifteen 3-units and 75 as fifteen 5-units.

Multiplicative Strategy: Interpreting the Difference of Shares as a Quantity

In the next 3 episodes, I show the multiplicative strategies that the participants used in Type II and Type III situations, which involves two simultaneous quantitative relations: a multiplicative comparison and an additive comparison (a difference).

Episode 7.8

Compared to Type I situations, in Type II problems one is required to conceptualize the difference between two quantities as one entity. Consider problem 3.2.1(a): *A sum of money was divided between Alan and Bill. For every $5 that Alan received, Bill received $3. Given that Alan received $10 more than Bill, calculate how much Bill received*. Quantities q_1 (Alan's share) and q_2 (Bill's share) are related multiplicatively in the ratio 5:3, and the quantitative difference between them is \$10. Further, in this problem, the actual values of the quantities are not stated but only the relationships that link the quantities are specified. From a mathematical perspective, this problem is algebraically equivalent to $(5-3)x = 10$. Only two of the six participants (Brian and Jeff) conceptualized this problem from a multiplicative perspective. The interview segments that follow present their response.

Data. Brian's solution to problem 3.2.1(a)

L1 I: So, a sum of money was divided between Alan and Bill in the ratio 5:3. Given that Alan received \$10 more than Bill, calculate how much Bill received.

L2 B: Fifteen. Because, if there is \$10 more, then 5 is 2 more than 3, and 2 times 5 is 10. Five times 3 is 15.

L3 A: Why would you do 5 times 2 and 3 times 5? You are doing the same thing, so you do 5 times 5 and 3 times 5. That will give you 25 and 15. You can't do one five (inaudible).

L4 B: I just saw that 5 (shares) is two more than that (3 shares), if he had 10 (dollars) more, then you have to go up by 2 each time. Two times 5 will be 10. So this will have to go up times 5 too. So that would be 15.

Analysis. Brian conceptualized the difference between Alan's share and Bill's share (5 – 3 = 2) as one quantity and related it to the difference in amount (\$10). This led him to deduce that a difference of one unit between the two quantities corresponds to \$5 as can be inferred from his statement "You have to go up by 2 each time. Two times 5 will be 10." (L4). In this problem, Brian's theorem-in-action was to set the correspondence between two differences - difference in shares and difference in amount. Working with the relation between the quantities rather than specific values of the quantities is evidence of quantitative reasoning. Brian was able to find the values of the two quantities after setting such a quantitative relation. This shows that quantitative reasoning is an important resource for reversible reasoning. Aileen did not construct such a quantitative structure (L3) and had to use a building-up strategy starting from specific values of the two quantities.

Data. Jeff's solution to problem 3.2.1(a)

As in the previous situations Jeff interpreted the ratio 5:3 as one *set*, which shows that he considered the sum of the components as one quantity or as a unit of units. The following segment of transcript shows his multiplicative strategy.

L5 I: A sum of money was divided between Alan and Bill in the ratio 5:3. Given that Alan receives $10 more than Bill, calculate how much Bill received?

⋮

L6 J: I will go ahead. So what I got. I got that Alan would get $25 and Bill would have $15.

L7 I: And how did you figure out this?

L8 J: I was seeing (inaudible) through, like. I was kind of looking at it the way, that 'set' again and then how many like. For every little like set that is given for every. It's kind of hard to explain. But as you notice, every $8 that they gave that would be one set.

L9 I: Every $8 will be what?

L10 J: It will be like one set. Because Alan will get 5 and Bill will get 3. That will be one set. So each time Alan will get two more dollars. So it had to go through 5 sets to get 10 more dollars than Bill. And then I just multiplied that by each one of them to get, uh. Alan would get 25 and Bill will get 15.

L11 I: Can you explain it again?

L12 J: Anyway, [what] I got is like I was saying, the set would be 8. So in each set Alan will get 2 more dollars than Bill and then since Alan receives $10 more than Bill, then you have to divide 10 by 2. That gives you 5. So it is taking 5 sets to get 10 more dollars than Bill. And I just multiplied 5 times 5 which will give you the first part in the set, it will give you 25, Alan got $25. And I multiplied 5 times 3 to see how many Bill get which will be $15.

Analysis. By considering the sum of the components of the ratio of shares as one composite unit ($5 + 3 = 8$), Jeff coordinated the difference between the shares ($5 - 3 = 2$): "So in each set Alan will get 2 more dollars than Bill" (L12). This led him to deduce that the total difference ($10) will require the consideration of 5 such sets: "So it is taking 5 sets to get 10 more dollars than Bill". This interpretation allowed him to deduce that 5 such sets of 8 (= 5 + 3) composite units amount to $\$5 \times 5$ for Alan and $\$3 \times 5$ for Bill. In terms of rules-in-action, Jeff's reasoning (based on L12) may be described as follows:

$$\text{One set of } \$(5+3) \rightarrow \text{a difference of } \$(5-3) = \$2$$

$$\text{A difference of } \$10 \rightarrow 5 \text{ sets of } \$2$$

$$5 \text{ sets of } (\$2) \rightarrow 5 \text{ sets of } \$(5+3) = \$25 + \$15$$

Jeff's response shows that he used quantitative operations to conceive the situation in such a way that afforded reversible reasoning. His answer gives evidence of reversible reasoning in two ways. First, he constructed a composite unit (a set of 8) and in turn decomposed that composite unit into its constituent parts. Note that the decomposition of the composite unit involves distributive reasoning: $5(8) = 5(3+5) = 15 + 25$. His rule-of-action again gives evidence of his ability to reason with three levels of units simultaneously. Secondly, he reasoned reversibly when he deduced that he had to divide (measurement division) the difference of $10 by $2 to determine the number of sets of $8. Jeff's strategy is subtly different from that of Brian in this situation because the latter did not form the composite unit from the ratio of shares.

Episode 7.9: *Jeff's second strategy* [problem 3.2.2(a)]

Episode 7.9 presents the multiplicative strategies used by the participants in problem 3.2.2(a), the first problem in Type III situations. Note that in this category of problems a multiplicative relationship is to be coordinated with twice the amount exchanged (i.e., $(a-b)x = 2e = q_1 - q_2$) or half of the difference between the ratio of shares is to be coordinated with the amount exchanged (i.e., $\frac{1}{2}(a-b)x = e$). Algebraically, problem 3.2.2(a) can be represented by $(3-1)x = 2 \times 18$ or $\frac{1}{2}(3-1)x = 18$.

As in the previous situations, I kept the focus on the multiplicative comparison of two quantities, and my objective was to observe reversible reasoning. Jeff is the only participant who used a multiplicative strategy on his own initiative. All the other participants initially used an additive strategy or guess-and-check strategy. It was only after I prompted them to look at the difference between the components of the ratio that they considered the problem from a multiplicative perspective.

In problem 3.2.2(a) the multiplicative relation and quantitative difference specify how the two quantities (Richard's share and John's share) are related (in a network of relations as described by Thompson (1993)), but no particular values of the quantities are stated. One needs to work with these two quantitative relations to figure out their specific values, and both quantities are unknown. The theorem-in-action that triggered the solution was the realization that either one share (equivalent to 18 marbles) was to be exchanged for the components of the ratio to be equal or twice the exchanged amount (36 marbles) constituted the difference between the quantities.

Data.

L1 I: Right now, Richard has three times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?

Jeff's initial attempt was to model the problem algebraically, but he could not solve the equations. I will comment on the algebraic solution (Figure 7.10) at a later point in episode 7.12.

L2 J: Anyway, this ratio would be 3 to 1, how many like marbles they have. And so, then it says, if Richard were to give 18 to John then that would mean that they will have the same number. For that to work then. Right now Richard would have to have 36 more marbles than John. So, subtracting 1 from 3 ...

(Disturbance: Students entering the interview room)

L3 J: Anyway, it's like. The ratio will be 3 to 1. So to figure out how many for each like set that they are given, Richard would get 2 more than John. And so I divided. So it had to be 36. So I divided 36 by 2 to get 18 and I multiplied 18 times 3 to get how many Richard would have and then I multiplied 18 times 1 to get how many John would have.

L4 I: But what does the ..., how did you get the 36?

L5 J: Well, it says that if he were to give 18 marbles to John then he would, then that would make them even. So that mean that he would have to have, he would also have to have 18 along with giving John the 18. So, it had to be double that number which would be 36.

L6 I: So who would have to, who would need to have 36?

L7 J: Richard would need to have 36 more than John.

L8 I: OK. And then how do you proceed once you have got the 36.

L9 J: Well, it says for each that. Because he has 3 times as many then like the ratio would be 3 to 1 (pointing to the ratio that he had written on paper). So to figure out how many more marbles he got, he has (inaudible) Richard for each like, set that it moves up, like moves up in steps (referring to his ratio of 3:1) it goes like. One set would be 3, 3 and 1 and then the next set would be 6 and 2 and if it goes like that and then so. I just divided 36 by 2 because that's how many he moves up in each one and it gave me 18. So that means that there have to be 18 sets for more than it has (inaudible) the right amount of marbles. That just multiply 18 and 3 and 18 and 1 to get how many marbles each person has.

Analysis. As in the previous situations, Jeff interpreted the multiplicative relationship between Richard and John using a ratio as his concept-in-action. He observed that there was a difference of two shares between them (L3). He also deduced that the difference between Richard's and John's marbles was 2 times 18 as evidenced from his statement: "So, it had to be double that number, which would be 36" (L3). Realizing that the difference between the two quantities is twice the amount exchanged is a critical deduction in this problem.

His next step was to relate the difference in amount to the difference in ratio. Using his 'set' interpretation of ratio (i.e., considering the ratio 3:1 as one set or one unit), Jeff observed that for each replication of the ratio 3:1, a difference of two units is generated. He inferred that he had to make 18 such iterations that for him meant 18 units or sets of the ratio 3:1. This led him to find the number of marbles that Richard and John had as 3×18 and 1×18 (involving distributive reasoning). The interpretation of a ratio as a set (or unit) is the concept-in-action that afforded Jeff the flexibility to use the multiplicative strategy. It should be pointed out that Lamon (1994) and Olive & Lobato (2008) also highlighted the conceptualization of a ratio as a unit.

Jeff reasoned reversibly in this episode by coordinating the difference in shares with twice the quantity exchanged. His theorem-in-action can be characterized as $(3-1)x = 2 \times 18$. Though he did not posit an unknown quantity explicitly between the difference in shares (3-1) and the difference between the quantities (2×18), he interpreted (2×18) as the result of multiple iterations of the difference in shares (3-1). This led him to deduce that $36/2$ (a measurement division) would produce the required number of iterations. Again, Jeff's solution illustrates his facility in coordinating three levels of units.

Episode 7.10: *Jeff* [Problem 3.2.2(b)]

I formulated another question parallel to problem 3.2.2(a) by changing the ratio between the two quantities from 3:1 to 4:1.

Data.

L1 I: Right now, Richard has four times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?

Long Pause.

L2 I: You want to comment on your solution (referring to Jeff).

L3 J: Well, I got that, right now Richard would have 48 marbles and John would have 12. And to get this, it is saying that the ratio would be 4 to 1. So to make it equal, you have to give, you would have to move one and a half from this side (pointing to the 4) to this side

> (pointing to the 1). So I divided 18 by 1.5 and I got 12 and then I just multiplied 12 times 4 to get 48 which would be how many Richard has and I multiplied 12 times 1 to get how many John would have which is 12.

Analysis. Jeff conceptualized the multiplicative relation in the problem as a ratio (4:1). He could observe that 1.5 shares had to be exchanged (L3) to make the components of the ratio (4:1) equal. His ability to equalize the unknown quantities by operating on the ratio of these quantities indicates a concept-in-action of a ratio as a quantitative structure. Being able to conceptualize the ratio as a quantitative relation enabled him to observe the equivalence of 1.5 shares and 18 marbles as a division situation (to find the size of one share), and he did not experience the conflict that the other participants encountered. Being able to deploy the division operation as Jeff did involves conceptualizing the multiplicative relation between the non-integer quantity (1.5 shares) and the integer quantity (18 marbles), something that other participants did not do independently. Such reasoning is also evidence of a generalized number sequence (Olive, 1999; Steffe & Olive, 2009).

Algebraic Approach

The algebraic approach constitutes a third category of strategy besides the additive and multiplicative strategies. The data show that some of the students explicitly defined the unknown quantities to be able to articulate the simultaneous multiplicative and additive relationships between them. Moreover, in various instances, although the students' responses gave indication that they were using particular theorems-in-action, they could not symbolize them in terms of algebraic equations. I made similar observations in one of my previous studies (Ramful & Olive, 2008). I present three episodes to illustrate the algebraic approach used by the participants and the constraints that they experienced in representing their theorems-in-action algebraically.

Episode 7.11: *Jeff* [Problem 3.1.2]

Problem 3.1.2 reads as follows: *Joe had some marbles. Then his friend gave him 5 times as many marbles as he had initially. Now Joe has 42 marbles. How many marbles did Joe have initially?*

Data.

J: Uh, I got that initially he would have 7 marbles because it says that he had some marbles then his friend gave him 5 times as many. So that would be the same as, uh, well what I came out was $x + 5x = 42$. So I just added the x's together and divided 42 by 6 and got 7.

Analysis. By defining the unknown quantity as x (the number of marbles that Joe had initially), Jeff could articulate the quantitative multiplicative relation between the unknown and the end result. Such a quantification of the unknown lowers the cognitive load required in thinking about an unknown quantity in a quantitative relationship and allows one to solve the problem in a forward direction. Formulating such an equation may, however, involve reversible reasoning.

Episode 7.12

Problem 3.2.2(a) reads as follows: *Richard has three times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?*

*Data. Brian's approach to p*roblem 3.2.2(a)

In his first attempt to solve this problem, Brian (and Aileen) used a guess-and-check procedure (see episode 7.14 in the next section). Observing the constraint that they were experiencing to solve the problem, I prompted them to write the multiplicative relationship in terms of a ratio, which Aileen wrote as $3:1$ while Brian wrote the following algebraic equations:

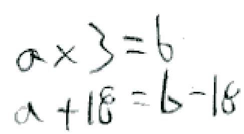


Figure 7.9. Brian's equations

L1 B: I tried to write an equation (but) it did not help me.

L2 I: You wrote an equation. What does the first equation represent?

L3 B: First equation is how Richard has three times as much, so *a* times 3 would equal to *b*. The second equation is *a* plus ...

L4 I: What does *a* and *b* represents?

L5 B: *a* is what John has, *b* is what Richard has.

L6 I: So, you are saying that *a* times 3 is equal to *b*. And then what you have written below?

L7 B: *a* plus 18 would equal *b* minus 18.

L8 I: And what does that mean?

L9 B: It would not mean, it's just 36 in between them. So it would have to be whatever is one third of something it still have 36 in between (them). So 36 would be two thirds. Oh, I could have gone from that. Couldn't I?

L10 A: Yeah.

L11 B: Because half of 36 would be ... 18.

L12 A: And we would have had the answer.

L13 I: Half of 36. You said two thirds of ...

L14 B: Yeah, if 18 [he actually meant 36] was two thirds, then one third would have to be half of 36.

L15 A: which is 18, we would have our answer.

Analysis. Brian wrote two equations in Figure 7.9, the first representing the multiplicative relationship and the second representing the difference relationship between Richard's and John's share. His statement "It's just 36 in between them. So it would have to be whatever is one third of something it still have 36 in between (them). So 36 would be two thirds." (L9) shows that he is referring to the theorems-in-action: $R - J = 36$ and $J = \frac{1}{3}R$, and the resulting powerful deduction $\frac{2}{3}R = 36$. He used the words "whatever" and "something" to implicitly refer to the two unknown quantities (i.e., John's share and Richard's share). His theorems-in-action do not correspond exactly to the two algebraic equations. Further, although he could write the two equations, he did not explicitly use them to solve the problem, probably because of two equations in two unknowns. Instead, he solved the problem mentally using his theorems-in-action: $R - J = 36$ and $J = \frac{1}{3}R$. This episode shows two instances where Brian reasoned reversibly or where the multiplication/division invariant was cued. First, he deduced that John has one third as many marbles as Richard from the problem statement: ‘Richard has three times as many marbles as John’. Secondly, he deduced that he should divide $\frac{2}{3}$ by 2 to produce $\frac{1}{3}$ in L14.

*Data. Jeff's initial strategy for p*roblem 3.2.2(a)

L16 I: Right now, Richard has three times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?

Pause.

L17 I: OK, let me look at your solution, Jeff in the mean time.

L18 J: I got that Richard would have 54 marbles right now and John would have 18.

L19 I: Yes, and how did you come out…?

L20 J: To get that, it said that Richard has 3 times as many as John, so that the ratio will be 3:1.

L21 I: Yeah, and I can see you have written 3*x*, *x*, what is it?

L22 J: Yeah, I tried to make it to equation (see Figure 7.10) but it did not work out right.

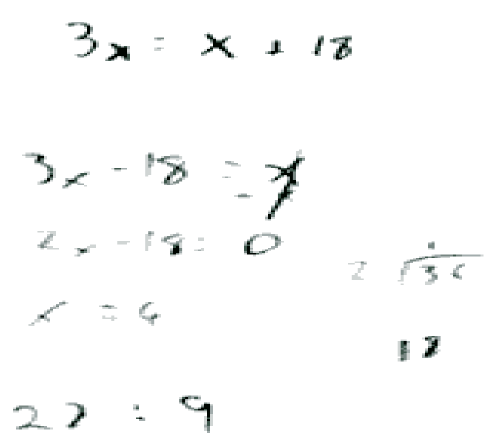


Figure 7.10. Jeff's equations

The following successive equations are shown in Figure 7.10: $3x = x + 18$;

$3x - 18 = x$; $2x - 18 = 0$; $x = 9$. The last line shows the ratio $27:9$.

Analysis. This problem involves a multiplicative and an additive relationship. By denoting John's share as *x*, Jeff could readily set the multiplicative relationship. To establish the additive relationship between the two quantities, he literally translated the problem statement 'if Richard were to give 18 marbles to John' into symbolic form as $x + 18$. He did not consider the fact that Richard would be deficient by 18 marbles after giving John 18 marbles. By solving the equation $3x = x + 18$, he obtained $x = 9$ and could observe that 27:9 did not satisfy the conditions of the problem and aborted this solution path (L22).

Episode 7.13: *Aileen's and Brian's first strategy* [Problem 3.2.2(b)]

Problem 3.2.2(b) reads as follows: *Right now, Richard has four times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?*

Data. Both of the seventh graders started the problem by writing the relations in terms of algebraic equations. Aileen wrote only one equation with two variables *a* and *b* as shown in Figure 7.11 and did not proceed further to solve it. In her equation, Aileen referred to '4*a*' as Richard's share and '*b*' as John's share, though she did not state it. I made this inference on the basis of her subtraction of 18 marbles from Richard's share.

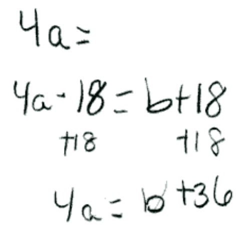


Figure 7.11. Aileen’s equation

Brian wrote the two equations correctly as shown in Figure 7.12 but could not solve them simultaneously (where the variables *J* and *R* denote John’s and Richard’s shares, respectively).

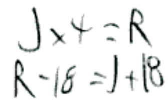


Figure 7.12. Brian’s equations

By including two variables in their equations, the students were constrained in solving them.

Guess-and-Check Strategy

The last category of strategy involves guess-and-check procedures. This form of strategy often arose as a fallback when the students’ previous attempts failed. It involves plugging-in values for the two quantities in the problem to verify if they fit the quantitative relationships. Often these guess-and-check strategies were systematic and were based on the numerical flexibility of the participants. I present one episode to illustrate the guess-and-check strategy that the students used.

Episode 7.14

Data. Aileen's and Brian's initial strategy for problem 3.2.2(a)

Both Aileen and Brian started this problem using a guess-and-check strategy as highlighted below.

L1 I: In this problem, right now, Richard has three times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?

Brian and Aileen used a guess-and-check procedure, and tried the numbers 6, 9, 12 and 14 for John's share and verified that they did not fit the constraints of the problem. For instance, Aileen gave the following justification when she tried the number 12:

L2 A: I knew that it has to be like a larger number to get where if you minus, if you subtracted 18 from each side, it would be, I mean subtracted 18 from Richard's it will still leave him with something and it will make them even. So I started at 10 and 10 times 3 is 30 and 30 minus 18 is 12, 12, yes 12 and that was not, not even to 18. So, it had to be 12 because 12 times 3 gives 36 and 36 minus 18 is 18. So, they would ..., never mind that can't be right because that did not come in. He does not have his original marbles.

L3 I: So, Richard has how many marbles, right now?

L4 A: I was going [to say], I was thinking 12. No. Richard would have 36. But that can't be right.

L5 I: You are saying that when you are subtracting 18, you should get an even number. This is what you said, no?

L6 A: Yeah. I thought that until I started thinking about it again because I did not think about John having his original marbles.

At one point Brian suggested that John has 14 marbles and Richard has 42 marbles. Aileen verified Brian's guess by adding 14 to 18 to obtain 32 and she subtracted 18 from 42 to obtain 24. Observing that 24 and 32 are not equal, she incremented 24 by 1 unit and reduced 32 by one unit successively as follows: 25, 31; 26, 30; 27, 29; 28, 28 after which she stated:

L7 A: It's going to be 28. They are both going to have 28. We need to figure out how to do that.

L8 B: So 28 minus 16 [he meant 18] would be 10 and 28 plus 18 would be 36. (inaudible).

L9 B: That would make 36 and 10. That would be (inaudible). I know it is going to be 24, uh, 32 difference, 36 difference. So. May be 17 and 40 ...

Analysis. The solution to this problem needs to satisfy two relations: a multiplicative relation ($R = 3J$) and an additive relation ($R - J = 2 \times 18$) simultaneously. Further, two unknown quantities are involved (Richard's share, *R* and John's share, *J*). Aileen's and Brian's responses show that they used a guess-and-check procedure, substituting different numbers to verify if they satisfy the two relationships. Initially (L2), Aileen used the multiplicative relationship '3 times as many' to produce her first trial solution for Richard's share as 30 and John's share as 10. Observing that 18 subtracted from 30 did not produce the same number of marbles for Richard and John, she incremented her initial guess from 10 to 12. She could observe that 12 also did not satisfy the two relations in the problem. Then she used Brian's guess of 14 and 42. Her next move was to adjust the difference (42 – 18 = 24) and sum (14 + 18 = 32) until she could get an equality by considering 25, 31; 26, 30; 27, 29; 28, 28. Responding to Aileen's equality at 28, Brian suggested the pair of values 17 and 40 for John's and Richard's share in L9, which does not satisfy the multiplicative relationship. This episode shows that the simultaneous coordination of two relations imposed considerable constraints for the two students.

Data. Eric's strategy for problem 3.2.2(a)

L10 I: Eric, any solution?

L11 E: I am just doing the original way I usually do it, just plug-in numbers. I just plugged it in to the ratio and, see I got up like 15, like the ratio is 3 to 1. So, right now, I just plugged-in like, uh, 15 into 3 and 15 into 3 is, let's see. That's 45, 45 to 15 but I just figured out now that it wouldn't be it (i.e., the difference would not be 18). So, I have not really completed the problem but in other words I will just keep going till I get the answer, that's pretty much what I do. (He also tried 16 on his worksheet). And then what I will do when I get my problem is, I would take away 18 from like. When I was at 15, I take away 18 from 45 and see (doing the subtraction on paper). And add 18 to the, uh, one like, to the 15 on the previous one and that wouldn't give me an even, they wouldn't both have even amount of marbles. So, it's now on 17 (he tried 17 on paper). Well, just I went on and on (inaudible) I just keep plug-in numbers in it until I find when they get even amounts (inaudible).

Analysis. Like Aileen, Eric used a guess-and-check procedure starting from the multiplicative relationship, which he wrote in terms of the ratio 3:1. Then he attempted to plug in different values starting from 15 until he could make the two quantities equal. He did not coordinate the two quantitative relations ('three times as many' and 'additive difference') simultaneously, and the guess-and-check strategy resulted as a fallback.

Positing an Unknown as a Quantity

Being able to posit an unknown as a quantity is an essential resource for reasoning reversibly as highlighted by the responses of the participants to problem 3.1.2 (*Joe had some marbles. Then his friend gave him 5 times as many marbles as he had initially. Now Joe has 42 marbles. How many marbles did Joe have initially?*). Four of the six participants first attempted to solve this problem by dividing 42 by 5. Looked at from the perspective of quantities, this problem requires one to posit an unknown quantity (Joe's initial amount of marbles, call it q_1) that is in multiplicative relationship to itself ($5q_1$). It also involves an additive relationship ($q_1 + 5q_1$) and the result of these two operations (42 marbles). In other words, in this problem the resulting end quantity (Joe's 42 marbles) is specified, and the aim is to find the initial quantity, q_1. In this sense this problem is regarded as a reversibility situation: 42 marbles is the result of adding Joe's initial marbles to the five times the quantity that his friend gave him. I highlight Ted and Cole's response in episode 7.15 below.

Episode 7.15: *Ted and Cole* [Problem 3.1.2]

Ted's and Cole's first attempt was to divide 42 by 5 to obtain $8\frac{2}{5}$. Cole immediately mentioned that they cannot have $\frac{2}{5}$ of a marble. In trying to correct Ted's response, he said that Joe would have 8 marbles and two left over because 42 cannot be divided evenly by 5. In this problem, Joe is both the referent and the compared quantity. Here, the referent quantity (Joe's initial amount) has an implicit nature, being an unknown quantity. It is the fractional remainder ($\frac{2}{5}$) that cued the students to realize that their solution was not correct.

Observing the conflict that they encountered, I suggested that they use a diagrammatic representation to make sense of the problem situation because in previous interviews involving multiplicative comparison situations, they had been comparing two sets of Unifix cubes. Ted drew 7 rows of 6 marbles while Cole represented the 42 marbles as 10 columns of 4 and 2 additional marbles. Ted made another representation of the 42 marbles by drawing a column of 42 unit squares and encircled groups of 5. Realizing that he would still get remainders, he said "So, I don't know how you solve it". My intention in asking them to draw a diagrammatic representation was to prompt them to consider the ratio 1:5, but instead of representing the relationship between the two quantities they represented the final quantity 42. The statement of the problem did not trigger 'ratio' as a concept-in-action for them.

Compared to problem 3.1.1, where they could articulate the relationship between the two quantities, here they did not set up a quantitative structure involving the unknown quantities, given that Joe's marbles are both the referent and compared quantity. This is further substantiated by Cole's statement in L6 (shown in transcript below) and Ted's statement in L9 (shown in transcript below), which show that they did not conceptualize 42 as being the result of combining the amount that Joe had initially and the five times the amount that he received.

At a later point Ted stated that the answer will either be 6 or 7. I prompted them to verify which of his guess 6 or 7 correspond to the number of marbles that Joe had initially and they could observe that the answer should have been 7. After obtaining the answer as 7, Ted mentioned "I did not do what he had initially, too," meaning that he did not consider the amount that Joe had initially as a quantity. This shows that he did not interpret the problem by positing an unknown quantity. I prompted them to write the situation in terms of ratio. They could write the ratio 1:5 but encountered further constraints as highlighted by the following interview segment:

L1 I: What does 1:5 represent in this problem? What does the 1 represent and what does the 5 represent?

L2 T: The 5 represents how many times he gave it to, uh, his friend gave it to Joe and the one represents.

L3 C: what he initially had.

L4 T: what he had.

L5 I: OK. Right. So, using this (ratio) can you solve the problem now?

L6 C: I don't think you could. Because you couldn't, you didn't know how many marbles there was. He did not know there were 42 marbles.

L7 T: But then you can't by multiplying, right. I don't know what to multiply. So, then his friend gives him five times as much more as he did. Now Joe has 42 marbles (reading the problem on the worksheet).

L8 C: No idea.

L9 T: So if he has 1 to 5 and the five equals to how many times and the one equals to what he has and so there .... Yeah, we would not find out, because we don't know what the total number of marbles he has.

Because they did not posit an unknown as a quantity, this prevented them from coordinating the known and unknown quantities in the multiplicative relation. They did not interpret 42 as being the result of combining Joe's initial amount of marbles (which was an unknown) and the five times the amount he received. Positing an unknown as a quantity is a key resource for reasoning reversibly in this situation.

Instances of Reversible Reasoning in a Fractional Context

Episode 7.16

Data. Cole's strategy for problem 3.2.2(b)

In question 3.2.2(b), Cole and Brian (separately) used a remarkable strategy to construct 4 units, starting from $1\frac{1}{2}$ units. The statement for question 3.2.2(b) reads as follows: *Right now, Richard has four times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now?* After deducing that $1\frac{1}{2}$ shares had to be exchanged from Richard to John to make their shares equal (because Richard had 4 shares and John had one share), Cole constructed 4 shares as follows:

C: I took 4 boxes (see Figure 7.13) and I knew one and a half would be 18. So I do 18, right here. And then 18 and then I knew that you would have to divide, you have to divide that by 3 and that will be 6. So you basically have thirds in that, in that part of the box, in that one part of the box and it would be divided by 6 and you would multiply 6 times 2 which would be 12. Add those up and that would be, that would be 48.

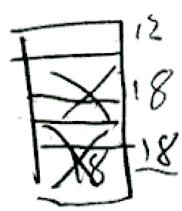


Figure 7.13. Cole's interpretation of 4 units in $1\frac{1}{2}$-units

With the knowledge that $1\frac{1}{2}$ shares were equivalent to 18 marbles, Cole used a diagrammatic representation to reverse his thought process. He partitioned the initial 4 shares (in Figure 7.13) as $1\frac{1}{2} + 1\frac{1}{2} + 1$. This partitioning allowed him to coordinate $1\frac{1}{2}$ to 18 marbles. To find the value of one share, he divided 18 by 3 and multiplied the result (6) by 2 to get 12. Figure 7.13 facilitated the unitizing procedure in terms of units of half in that he could observe that $1\frac{1}{2}$ corresponds to three half-squares (i.e., it is made up of three halves). This led him to him to deduce that 18 correspond to 3 half squares, from which he could infer that one half-square is equivalent to 6 marbles. To sum up, the diagrammatic representation (Figure 7.13) afforded him the necessary resource to reverse his thought process.

Data. Brian's strategy for problem 3.2.2(b)

Brian started this problem by writing two equations: $J \times 4 = R$ and $R - 18 = J + 18$, but he could not solve them simultaneously. Observing Aileen's ratio $4:1$ on her worksheet, I prompted them to represent it diagrammatically in terms of squares and to consider the number of shares that had to be exchanged to make them equal. They drew 4 squares to represent Richard's share and one square to represent John's share and mentioned that $1\frac{1}{2}$ shares had to be exchanged. The following transcript shows how Brian reasoned reversibly in this problem.

Brian first mentioned that Richard has 42 marbles using the following faulty theorem-in-action: "if 18 would be $1\frac{1}{2}$ of 4, then 18 times $2\frac{1}{2}$ would make it to where, that as much as Richard has. So 18 times 2 is 36 and half of 18 is 6 and 36 plus 6 is 42." Similarly, Aileen mentioned: "It equals 18 times 2.5 is 45." I prompted Brian to reconsider the equality between $1\frac{1}{2}$ and 18 marbles.

L1 I: Brian, let us come back to what you have said. You said that 18 represents what?

L2 A: One and a half of Richard's.

L3 I: Can you write this on the paper? So 18 represents one and a half, right. (He wrote $18 = 1.5R$ in Figure 7.14).

Figure 7.14. Brian's equation

L4 B: Yeah.

L5 I: And how do you proceed to find Richard's share from this point?

L6 B: Well we want that to equal 4 (pointing to 18 = 1.5R) and you (inaudible) times this, whatever we get it to equal 4. Times it by 2 and that will give you 3. And then times it by 3 that is just going to give you to 4.5. But you can't do that because 4.5 wouldn't, would go over that. So you need to go two thirds of that (pointing to the number 18 in his equation) which would be 12.

L7 I: Aileen? Do you follow the argument that Brian is saying, that one and a half is, represents 18 marbles?

L8 A: Yes. I see that and then ...

L9 I: Can you write this? (She wrote 1.5 = 18.)

L10 B: One and a half have to equal 18 because if he is giving one and a half from this diagram (probably referring to the squares representing the ratio 4:1) and then if he is giving 18 in words, [one and a half] would have to be 18.

L11 I: So you are saying that one and a half represents 18. Right.

L12 A: (inaudible) you are going to do that times 2.5 to get what Richard has. That gives you 45 and 45 minus 18...

L13 B: But now only 12 equals one because one and a half and that's in all sense 3 thirds, if you have 1.5. So if you take at that two thirds. Two thirds of 18 is just 12. So each of these boxes equal 12. Four times 12 is ...

L14 A: 46. No 4 times 12 is 48.

L15 B: So, it is 48 that he has, not 42. (He obtained 42 at the start by incorrectly multiplying 18 by $2\frac{1}{2}$ as $36 + 6$).

L16 A: 48, yeah that works. It's 48.

L17 I: How do you get 48?

L18 A: Richard has 48.

L19 B: Because if you have 18 as one and a half, then…

L20 A: that means that

L21 B: One and a half would be that; that's basically 3 thirds. So you take that under two thirds, take one third out of 18. One third of 18 is 6. Subtract 6 from 18 is 12. 12 times 4 is 48.

L22 A: What I did was if 18 is 1.5 then you can find the one by taking off, uh, a third of 18 which is 6 which gives you 12. So if you do 12 times 4 that gives you 48. 48 minus 18 is 30 and 12 plus 18 is 30 and then they are both equivalent.

Analysis. I brought to their attention (L3) the fact that 1.5 shares represent 18 marbles by asking Brian to write it on his worksheet (see Figure 7.14). He could observe that if he takes 1.5 two times, he would have only three shares and if he takes 1.5 three times, he would have 4.5 shares which "would go over that" (i.e., it would be greater than 4 shares) in L6. He reasoned in units of $1\frac{1}{2}$ to construct 4 units. In other words, he interpreted 4 shares in units of $1\frac{1}{2}$ shares as $1\frac{1}{2} + 1\frac{1}{2} + \frac{2}{3}(1\frac{1}{2})$) and coordinated these units of quantity with their corresponding measure $18 + 18 + 6$ (which should have been $18 + 18 + 12$). Further, his statement "one and a half would be that; that's basically 3 thirds. So you take that under two thirds, take one third out of 18. One third of 18 is 6. Subtract 6 from 18 is 12" (L21) strengthens the evidence that he reversed his thought process in this problem. On the other hand, Aileen did not readily make such a realization on her own, and she attempted to multiply 18 by 2.5 in L12. Following Brian's explanation, she interpreted $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$ and determined the corresponding measure as 12 marbles (L22).

This problem shows that the reason Brian could reverse his thought process to solve $1.5R = 18$ (in Figure 7.14) is because he could conceptualize $1\frac{1}{2}$ as three units of half. Working with such fractional units avoids the necessity to explicitly find the reciprocal or to perform division by decimals (i.e., $18 \div 1.5$). Another resource besides the flexibility to work with fractional units is being able to *conceptualize* the problem from such reversibility perspective, which Brian could do without much prompting, while Aileen took more time to make such a realization. Another observation is that Brian did not solve the algebraic equation in Figure 7.12 but used its corresponding counterpart as theorem-in-action to coordinate the quantities in the problem.

Discussion

The problem of interest in the current set of situations was to understand the ways in which students reason reversibly in ratio situations and to identify the resources that they deploy in reasoning from the result to the source causing the result. I was particularly interested in the ways in which the participants articulated the quantitative relations between two quantities to determine their specific values. The problems chosen here for understanding reversible reasoning share one common feature: They involve a multiplicative and an additive/difference relationship between two quantities. Given these two quantitative relationships, one is required to find the specific values of the two quantities. Variations to the problems were effected by altering the form of the additive relationship between the quantities as well as changing the numerical values of the quantities. Table 7.1 shows the 10 problem situations that were formulated. Three categories of multiplicative comparison problems were developed: (i) Type I : $(a+b)x = q_1 + q_2$ (ii) Type II: $(a-b)x = q_1 - q_2$ and (iii) Type III: $(a-b)x = 2e = q_1 - q_2$ or $\frac{1}{2}(a-b)x = e = \frac{1}{2}(q_1 - q_2)$, where e is the amount exchanged.

What does this data set suggest about the second research question: *In what ways do students reason reversibly in multiplicative situations and what constructive resources do they deploy in such situations?*

The first category of problem $(a+b)x = q_1 + q_2$ can be solved using either the primitive building-up additive strategy or using a multiplicative strategy. For instance, Brian solved the second part of problem 3.1.1, $(1+3)x = 88$ by iterating the ratio 1:3 in Figure 7.1 using an additive strategy. However, he worked out the first part of the same problem $(1+3)x = 32$ using the ratio 1:3 from a multiplicative perspective, determining how many sets of 4 fit in 32, similar to what Cole and Ted did. Iterating a given ratio a number of times (using repeated addition) until it fits/satisfies the end quantity versus determining the number of iterations by dividing the end quantity by the sum of the components of the ratio involves two different conceptualizations or theorems-in-action of the same situation. The additive and multiplicative conceptualizations are illustrated by the following schema, where a quantity q is decomposed in the ratio $a:b$ as $q_1 : q_2$ with $q_1 = x \times a$ and $q_2 = x \times b$, and x denoting the number of iterations.

Additive conception: building-up strategy

$$a : b$$
$$a + a : b + b$$
$$a + a + a : b + b + b \quad ,$$
$$\vdots$$
$$q_1 : q_2$$

Multiplicative conception

$$x = \tfrac{q}{(a+b)}$$

$$q_1 = x \times a \text{ and } q_2 = x \times b \, .$$

The decomposition (or undoing/unmaking) of a given quantity q in terms of the components of a ratio $a : b$ from the multiplicative perspective involves reversible reasoning as it requires one to think of the quantity q as the end result of multiple iterations of $a + b$. One has to determine the number of times that $a + b$ (interpreted as one entity or unit-of-units) is contained in the initial quantity q, and this may necessitate positing an unknown between the sum of the components of the ratio and the quantity q, a situation algebraically equivalent to $(a + b)x = q$. This decomposition problem may also be regarded as a measurement division situation, where the quantity $x = \tfrac{q}{(a+b)}$ is viewed as the number of sets of $(a + b)$ in q as interpreted by one of the participants (Jeff). Such an unknown quantity (x) is often implicit but offers the path to articulate the relation between the quantities in such multiplicative comparison situations. The multiplicative/division invariant is cued if one can think of the end result q as being formed from multiple iterations of the sum $(a + b)$. Another level of reversible reasoning occurs when one decomposes the composite unit $(a+b)$ back as a and b and uses distributive reasoning to multiply each amount by the unknown quantity x (or computed multiplier).

Simplistic as it appears from a numerical/algorithmic perspective, the multiplicative conceptualization requires a quantitative interpretation of the situation as one has to coordinate the relation among the quantities rather than the numerical values of the quantities. A numerical approach to solve problem 3.1.1 may involve plugging different values for Clay's share, correspondingly computing Dana's share, and verifying when the total share is 32. A quantitative interpretation, on the other hand, implies that one can identify the two relations in the problem (a multiplicative relation and an additive relation) and form a unit of units in relation to the end quantity. In other words, the sum or composite unit $a+b$ has to be regarded as one quantity in relation to the end quantity q. Failure to conceptualize such a quantitative relationship led the participants to work in a forward direction, by incrementing the ratio $a:b$ using the additive building-up strategy or other guess-and-check procedures until the required sum was obtained. For instance, Aileen in most cases used such a forward strategy rather than decomposing the end quantity.

The second category of reversibility situations that were under investigation in this study is algebraically equivalent to $(a-b)x = q_1 - q_2$, where the difference between two quantities as well as the multiplicative relation between the quantities are given and the objective is to find the two quantities. An additive strategy involves generating elements of the equivalence class of $a:b$ until the required difference is achieved. A multiplicative strategy calls for a different theorem-in-action where one is required to find the number of sets of $a-b$ in the given end result q. The scalar x denotes the number of times that the ratio $a:b$ is iterated.

Additive conception

$$\begin{array}{l} a:b \\ a+a:b+b \\ \vdots \\ q_1:q_2 \end{array},$$

where $q = q_1 - q_2$

Multiplicative conception

$$x = \frac{q}{(a-b)}$$

$q_1 = x \times a$ and $q_2 = x \times b$

This second type of problem proved to be more problematic for the participants. The prevalence of the building-up strategy in Table 7.3 shows that the students worked in a forward direction reasoning numerically using a building-up or guess-and-check strategy (e.g., problem 3.2.1(a)). Such numerical reasoning as a fallback strategy can be explained by their failure to conceptualize the problems in reverse by analyzing these situations from the perspective of two quantitative relations. The conceptualization of a difference (i.e., $a-b$) as a quantity was the missing element for the majority of the participants. By considering $a-b$ as a quantity, I mean conceptualizing it as an entity or object that is measurable or that can be used as a measure.

A third category of tasks was created as a variation to Type II problems that required the participants to establish the equality between the two quantities, given a multiplicative and a difference relationship. Type III problems are algebraically equivalent to $(a-b)x = 2e = q_1 - q_2$ or $\frac{1}{2}(a-b)x = e = \frac{1}{2}(q_1 - q_2)$. The data show that this category of situations (problems 3.2.2(a), 3.2.2(b) and 3.2.2(c)) was the most demanding for the students. Four of the six participants (except Jeff and Brian) used a building-up strategy for this set of situations. The solution to these tasks either requires one to consider the equality between the shares as Jeff did in problem 3.2.2(b) (episode 7.10) involving the theorem-in-action $\frac{1}{2}(a-b)x = e = \frac{1}{2}(q_1 - q_2)$ or by realizing that the amount exchanged should be doubled as Brian did in problem 3.2.2(a) (episode 7.12) involving the theorem-in-action $(a-b)x = 2e = q_1 - q_2$.

Interpreting a ratio as a quantitative structure

Previous research has given a range of interpretations to the concept of ratio. From a mathematical perspective, Freudenthal (1983) considered a ratio as "an equivalence relation in the set of ordered pairs of numbers (or magnitude values)" (p. 180). A ratio ($a:b$) can also be understood from the perspective of a fraction (*a*/*b*) and as a quotient (*a* divided by *b*). In terms of units-coordination, a ratio can be interpreted as requiring the coordination of two number sequences. From the point of view of quantities, a ratio can be regarded as a multiplicative comparison between two extensive quantities (Schwartz, 1988) in the same measure space. From a cognitive perspective, Thompson (1994) considered a ratio as "the result of comparing two quantities multiplicatively" (p. 190). Clark, Berenson, & Cavey (2003) distinguished between two conceptions of ratio: descriptive ratios and functional ratios. Examples like comparing the number of boys and girls in a class constitute a descriptive ratio as they are concerned with a static counts of objects in two sets. On the other hand, functional ratios deal with consistent linear relationships between two variables.

Jeff's response additionally teaches us that the concept-in-action ratio can be interpreted in terms of a quantitative structure where the sum and difference of the components are also meaningful. A ratio represented more than just a multiplicative comparison relation for him. He could interpret the problem situations quantitatively because he could make sense of the meaning of the sum of shares that he described as *sets*, referring to a unit of units in problems 3.1.1,3.1.2, and 3.1.3. Similarly, he could make sense of the difference of shares as one entity in problems 3.2.1 and 3.2.2 (by coordinating three levels of units), and this offered him a path to all the problems without observable constraints. In contrast, Aileen in most cases (although she symbolized the multiplicative relationship as a ratio) did not combine the two components of the ratio as one quantity or considered their difference as one quantity, and this unavoidably led her to use the building-up strategy.

Working with unknown quantities

At various points of the interview, the students' responses suggested that they worked with an implicit unknown to articulate the multiplicative and additive/difference relations among the quantities. I have characterized these ways of operating on unknowns in terms of theorems-in-action. The students referred to unknown quantities using the words 'whatever' and 'something,' which acted as a place-holder to permit the articulation of the multiplicative relation between them. As a first example, in problem 3.2.2(a), Brian used the terms 'whatever' and 'something' to refer to the two unknown quantities: "So it would have to be *whatever* is one third of *something* it still have 36 in between (them). So 36 would be two thirds" (episode 7.12, L9). Here, he was referring to the following theorems-in-action involving two unknown quantities, namely Richard's share and John's share: $R - J = 36$, $J = \frac{1}{3}R$, and $\frac{2}{3}R = 36$.

In trying to find how many sets of 6 there are in 42 in problem 3.1.2, Brian made the following statement: "you would have 6 of how many sets *whatever*. So 42 divided by 6 that's 7 because 6 times 7 equals 42." Here, he is referring to the theorem-in-action $6x = 42$, where the unknown *whatever* refers to the number of sets of 6 units. Another instance where Brian referred to an unknown quantity using the word *whatever* is in problem 3.2.2(b), episode 7.16, where he had to determine how many $1\frac{1}{2}$-units fit in 4 units as can be inferred from his quote: "Well we want that (referring to 1.5 in Figure 7.14) to equal 4 and you (inaudible) times this, *whatever* we get it to equal 4" (L6). His response is evidence of the theorem-in-action $1\frac{1}{2}x = 4$. The different examples highlighted in this section show that one may conceptualize an unknown in a multiplicative relation without the necessity for symbolization by using terms like 'whatever' and 'something' from common language.

The algebraic approach was used only by the seventh and eighth graders. It is most likely that the sixth graders (Ted and Cole) did not have enough exposure to algebra in their school curriculum. Jeff used an algebraic approach in only 2 of the 10 problems, namely 3.1.2 and 3.2.2a, though he did not write the algebraic equation correctly in the latter case. In all the other situations, he interpreted the problems from a ratio perspective and did not symbolize an unknown quantity to articulate the two given quantitative relations. Brian also used an algebraic approach in problem 3.2.2(a) and 3.2.2(b), and in both cases he used two variables to formulate two equations (Figures 7.9 and Figure 7.12), one for the multiplicative relationship and the other for the additive relationship. The two variables constrained him from solving the equations. Aileen and Eric also attempted to use the algebraic method without much success.

What does this data set suggest about the third research question: *What constraints do students encounter in conceptualizing multiplicative relations from a reversibility perspective?*

The question that arises is why the multiplicative strategy (which inherently involves reversible reasoning as explained earlier) was not readily cued in comparison to the more intuitive building-up strategy as clearly apparent from Table 7.3. The participants' responses to the three types of problem situations brought to the fore two forms of constraints that prevented them from reasoning reversibly: (i) quantitative interpretation of the problem situations in terms of a ratio (ii) positing an unknown as a quantity. In this section, I show how these two resources were critical to reason reversibly. It should also be highlighted that in all the problem situations, the relation among the quantities rather than their actual values are given. Furthermore, the building-up approach is a low-cost strategy consisting of generating successive pairs of numbers satisfying the given multiplicative relationship. It readily takes care of one of the relationships (i.e., the multiplicative relationship) and as such one has to keep the focus primarily on the additive/difference relationship (i.e., either the target sum or difference). In this sense, it provides students with an affordable means to coordinate the two relations. The building-up strategy was primarily used in Type II and Type III situations where a difference was the target. However, the limitation of such a strategy is when ratios with fractional components are involved as in episode 7.4.

Constraint 1: Interpreting a ratio as a quantitative structure

In most of the problems, Jeff gave evidence of reasoning reversibly as he determined the components constituting the given quantity using a multiplicative approach. For instance, in problem 3.2.1(a), he related the difference between the components of the ratio 5:3 (i.e., $5 - 3 = 2$) to the total difference of $10 (in terms of a multiplicative relation). The other participants (except Brian) opted for the more intuitive additive strategy by incrementing the ratio 5:3 (which ensures that the multiplicative relation is maintained) until they hit the required difference of $10. Jeff could conceptualize the problem in terms of shares. The other students did not think about the difference between the shares as representing an entity or a quantitative difference (i.e., how much Alan had in excess compared to Bill) as pointed out by Thompson

(1994). Setting and interpreting the multiplicative relation between the difference in shares and the difference in amount is the key step for reversible reasoning in such a problem, a flexibility shown by only two of the participants, namely Brian and Jeff. Ted and Cole experienced further constraints in problem 3.2.1(b) ($(5-3)x = 7$), where the difference between the quantities was not a multiple of the difference between the components of the ratio. The building-up strategy based on integer increments proved to be insufficient, and they were constrained to seek alternative approaches.

Problem 3.2.2 (Type III: $(a-b)x = 2e = q_1 - q_2$) is even more demanding as compared to problem 3.2.1 because it involves a multiplicative comparison between two quantities and twice the amount exchanged. Five of the six participants used a guess-and-check strategy. It is only after prompting them to represent the situation quantitatively in terms of a diagram (where they could observe the relation between the shares and the amount exchanged) that they could reason quantitatively. Besides the inadequate quantitative analysis of the situation, another possible explanation for their constraint is that the numerical information presented in the problem did not allow propagation of calculation as only relations rather than specific values of quantities are specified.

Constraint 2: Positing one of the unknown quantities as the referent quantity in the multiplicative comparison relation

The ten problem situations require the determination of two unknown quantities from two quantitative relations. However, being able to posit one of the two quantities given in the multiplicative comparison relation as a referent allows one to think of the two unknown quantities in terms of a single unknown. It appears that it is this step that allows one to form a composite unit from the two given quantities. For instance, in problem 3.1.1, the multiplicative comparison relation reads as follows: 'Dana has 3 times as many marbles as Clay'. If Clay has x amount then Dana has $3x$ amount. Thinking of Clay's marbles as the referent unknown quantity in this way allows one to combine the two quantities in a composite unit. Positing such an unknown quantity gives one the flexibility to articulate the relationship among the quantities. Jeff is one of the participants who in most cases could posit the referent quantity as an unknown and as such could articulate the two quantitative relationships. As mentioned earlier, his ability to coordinate three levels of units simultaneously appears to be a necessity for solving problems in this way.

The involvement of an unknown quantity (as part of the semantic structure of the problem situation) is probably more explicit in problem 3.1.2 by virtue of its formulation and as can be observed in episode 7.15 from the students' responses. Interestingly, 4 of the 6 participants divided 42 by 5 rather than 6 as their initial solution to this problem. The participants did not readily conceptualize an unknown (Joe's share) as a quantity. Brian also made a similar misinterpretation in problem 3.1.1, where he divided 32 by 3 rather than 4.

The necessity for thinking about an unknown quantity becomes even more important in Type III situations. Interpreting the statement "If Richard were to give 18 marbles to John, they would have the same number" as meaning that the difference between them is 2 times 18 is a demanding realization. It involves thinking about two unknown quantities and may be characterized by the theorem-in-action: $R - 18 = J + 18$ or $R - J = 36$. Failure to posit such unknown quantities led the participants to use the building-up strategy.

Problem conceptualization and cueing of cognitive resources

The different ways in which the participants solved the same problem led me to focus my attention on the ways in which they conceptualized the same problem situation differently. Vergnaud's notion of concepts-in-action prompted me to focus on the meaning that the students attributed to a multiplicative comparison relation. For example, in problem 3.1.1, the concept-in-action '3 times as many as' cued different resources for the 6 participants, resulting in different theorems-in-action for splitting a given quantity q in terms of q_1 and q_2. Three subtly different solution paths could be observed. For Ted and Cole, the comparative statement '3 times as many as' represented a total of 4 shares and as such they divided 32 by 4. For Aileen and Eric, it represented a multiplicative comparison between two quantities (i.e., if the first quantity is q_1 then the second quantity is $q_2 = 3q_1$), but they did not conceptualize the two quantities as one entity (a unit of units) and as such used the building-up strategy. For Jeff, '3 times as many as' explicitly represented the concept-in-action 'ratio' (i.e., $q_2 : q_1 = 3:1$). Brian used the comparative statement '3 times as many as' both as a ratio and as a multiplicative comparison between two quantities.

As another example, three different interpretations of problem 3.1.2 ($(1+5)x = 42$) were observed. Ted and Cole interpreted the problem as a multiplicative comparison situation and attempted to divide 42 by 5 rather than 6. Aileen interpreted the problem additively and used a systematic building-up strategy while Brian used the ratio 5:1. Jeff interpreted the problem algebraically and wrote the equation $x + 5x = 42$.

What the data of this study additionally highlight is that the problem solver may possess specific resources, but these may not be cued during problem solving situations. For instance, in problem 3.2.2(b), despite writing the equation 18=1.5*R* in Figure 7.14 (episode 7.16), Brian did not conceptualize the problem as requiring division to find *R*, though division is a familiar operation for him. In other words, this equation did not represent a division situation for him. As another example, looking at Jeff's solution, Eric realized that interpreting problem 3.2.1(a) as a ratio would have made it simpler to solve. Though Eric knew how to decompose a quantity in terms of a ratio, he did not interpret the problem as a ratio situation and as such resorted to guess-and-check strategies. Similarly, Ted and Cole did not interpret problem 3.1.2 as a ratio situation. In problem 3.2.2(a), I asked them to reinterpret the problem by considering the ratio between Richard's and John's share, and this led Ted to observe the equivalence between the quantity 'one share' and 18 marbles. Similarly, Aileen pointed out that "working out with the ratio just like clicked. But the 3 to 1 helped after I wrote that down, thinking about it." In other words, being able to conceptualize the problem as a ratio situation is an important step in solving the problem. Taken together, these observations suggest that the way a problem is conceptualized dictates what resources get deployed and in turn which solution strategies are used.

Intermediate success or failure as influencing problem conceptualization

Another feature that influenced the participant's conceptualization of the problem situations were the intermediate successes or failures that they encountered as they were involved in solving the problem. Often the students deployed verification criteria to check the soundness of their answers. For instance, in problem 3.1.2, Ted, Cole, Eric and even Jeff divided 42 by 5 and the non-integer remainder (as a piece of information that does not satisfy the problem) prompted them to change their strategy. Further, in problem 3.2.2(a), Jeff dropped his algebraic approach as the solution he obtained from this method (Figure 7.10) did not satisfy the conditions of the problem. Moreover, in problem 3.2.1(b), Cole's successive attempts to find the pair of numbers that were in a ratio of 5:3 and differed by 7 illustrate how the answer to a problem can itself be a cue to reorient the problem solver. After situating the answer between 15:9 and 20:12, he started at $15\frac{1}{2}$ and $9\frac{1}{2}$, and observed that the difference is not seven. This incorrect answer *cued* him to reconsider his thinking process and led him to produce his next solution, namely $16\frac{1}{2}$ and $10\frac{1}{2}$, which still did not produce a difference of 7. This chain of reasoning prompted him to look at his conceptualization of the difference again to obtain the pair of numbers $17\frac{1}{2}$ and $10\frac{1}{2}$ by finding the mid-points of the range of each component of the two ratios (15 and 20, and 9 and 12).

In summary, the way a problem gets conceptualized depends on the concepts-in-action that the students can identify from the problem situation. In turn, this dictates the cueing of particular theorems-in-action to solve the problem and consequently reversible reasoning is a function of problem conceptualization. In terms of Vergnaud's theory, the operational invariants identified with the problem situation constrain or cue the rules to generate action and the inference possibilities that make reversible reasoning possible.

CHAPTER 8

CONCLUSIONS

This study was motivated by a tandem of theoretical concerns and practical needs to understand how students reason reversibly in multiplicative situations and what implications there might be for instruction. By presenting the six participants in the study with a range of fraction and ratio situations, I attempted to capture the ways in which they manipulated multiplicative relationships, the constraints they faced, and the consequences of failing to reason reversibly. I also identified the conditions under which they could or could not reason reversibly. I now summarize the major conclusions of the study.

I considered multiplicative reasoning as the articulation of the relation $a \times c = b$ in a variety of contexts and situations (where a, b, and c can be integers, rational numbers, or real numbers in general). This ternary *relation* (English & Halford, 1995) between a multiplier, a multiplicand, and a product involves three sources of variation. The determination of the multiplier when the multiplicand and product are given or the determination of the multiplicand when the multiplier and product are given requires reversibility of thought. These missing-factor situations can be characterized by the theorem-in-action $ax = b$. One has to carry out an operation of thought based on the invariant ‘division as the inverse of multiplication’ before actually carrying out the arithmetic operation of division to find the missing factor. Similarly, reversibility of thought is required in deducing that if A is m times as large as B then B is $1/m$

times as large as A or if A is two-thirds of B, then B must be three-halves of A. The cueing of such invariants in problem solving situations are sensitive to the parameters in the problems.

Conclusion #1: Reversibility is Strongly Sensitive to the Numeric Feature of the Problem Parameters

The parameters a, b, and x in $ax = b$ can vary numerically as integers, rational numbers or real numbers. One of the salient conclusions of this study is that the articulation of the relation $ax = b$ is highly sensitive to the numeric features of the data. I asked the participants to find the missing factor in the following numerical problem in the first phase of data collection: $\frac{7}{12} = \frac{2}{7} \times$ ___ . This problem proved to be a constraining situation for all of them. For instance, Brian transformed the equation to $\frac{49}{84} = \frac{24}{84} \times$ __ to make the denominator equal and attempted to guess which number times 24 gives 49. Aileen mentioned: “Trying to see 2 times what gives you 7 and 7 times what gives you 12”. She plugged in 1.75 times 7 to get 12.25 and then adjusted 1.75 to 1.72 times 7 – working with the numerator and denominator separately so that she can use her whole number knowledge. This type of fallback strategy was observed at different instances in Set 2 (Chapter 6) where the students attempted to introduce divisibility relations in the problem by making the denominator of the product and multiplier or multiplicand equal. These observations show that the numeric structure of the problem conditions the way it is conceptualized. For the participants in this study the problem $\frac{7}{12} = \frac{2}{7} \times$ __ did not cue division or the multiplication/division invariant. These findings are consistent with previous research (e.g., Fischbein et al., 1985; Harel et al., 1988) in illustrating how changing the parameter ‘a’ in $ax = b$ from an integer to a decimal can impede students’ performance considerably. For example, changing the parameter '*a*' from the integer 6 to the decimal 6.3 requires a change in the way one thinks about the problem or may involve more cognitive load. In the section that follows, I summarize the ways in which the participants were constrained in reasoning reversibly as a result of the numeric feature of the problems in the three sets of tasks.

Set 1: Multiplicative comparison of two quantities in a measurement division situation

The students could reason reversibly when divisibility relationships between the two quantities being compared were available. For instance, in situation 1 (involving the comparison of 6 red and 3 blue counters), Ted mentioned that "so there is kind like finding a common number to them like 3 times 2 equals 6. And 6 divided by 2 equals 3". They did not give evidence of using such forms of reversible reasoning involving the multiplier, multiplicand, and product in the other situations in Set 1 due to the absence of divisibility relationships.

Aileen interpreted a multiplicative comparison relationship in terms of the theorem-in-action $ax = b$ in situations 2, 3, and 4. She compared the larger quantity in terms of the smaller quantity by determining which number to multiply the smaller quantity by to produce the larger quantity. Though she could set such a relation between the unknown multiplier, multiplicand and product, these situations did not cue the multiplication/division invariant for her. She determined the unknown multiplier/multiplicand by guess-and-check.

Set 2:Fraction situations

After their analysis of the impact of number type on multiplication and division problems when the multiplier or multiplicand is a decimal, Harel et al. (1994) raised the following point: "The question of whether subjects encounter similar difficulties with multiplication and division problems that involve fractions has never been directly addressed" (p. 381). In Set 2 (which is a division problem from the arithmetic point of view), the type of numbers (integers, relative prime numbers, unit fractions, non-unit fractions, improper fractions) used in the formulation of the multiplicative comparison problems were systematically varied to observe how such variations influence the type of resources that students deploy in this category of reversibility situations and when the theorem-in-action 'division as the inverse of multiplication' is cued. The data convincingly show that the numeric feature of problem parameters can either cue or block a particular solution path. Not surprisingly, the presence of a divisibility relationship between *a* and *b* readily cued reversible reasoning, independent of whether *a* and *b* were integer or fractional quantities as is the case in problems 2.11 ($3x = 21$); 2.31 ($\frac{2}{5}x = 30$); 2.32 ($\frac{3}{7}x = 18$); 2.42 ($\frac{2}{5}x = \frac{1}{4}$) ; 2.43 ($\frac{8}{5}x = \frac{4}{3}$). However, when *a* and *b* were relatively prime or involved prime numerators as in Level 6 and Level 7 problems, this theorem-in-action was not readily triggered. Unfamiliarity with the fractions resulting from the division of two prime integers imposed considerable constraints on the students as in problem 2.12 ($5x = 7$) and problem 2.61 ($\frac{7}{8}x = 5$). Absence of a divisibility relationship between *a* and *b* in $ax = b$ prevented the students from conceptualizing a non-unit fraction in terms of the corresponding unit fraction. The presence of two improper fractions with relatively prime numerators as in problem 2.74 ($\frac{12}{7}x = \frac{5}{3}$) suppressed the cueing of the multiplication/division invariant. Further, the mixed number representation and improper fraction representation cued different concepts-in-action as is contrastingly illustrated in episodes 6.8 and 6.11.

Conceptualizing a multiplicative relationship involving a non-integer quantity is less intuitive than one involving an integer quantity. Consider the following two problems where the multiplicative relationship is changed from '2 times' to '$1\frac{1}{4}$ times': 'Kelly's pencil is 6 inches long. It is 2 times as long as Susan's pencil. How long is Susan's pencil?' (problem 1) versus

'Kelly's pencil is 6 inches long. It is $1\frac{1}{4}$ times as long as Susan's pencil. How long is Susan's pencil?' (problem 2). In problem 1, the theorem-in-action 'division as the inverse of multiplication' is readily cued while this is not always the case in problem 2, which for some students may even cue multiplication.

Set 3: Ratio situations

The numeric characteristic of problem parameters can cue particular resources and possibly favor particular problem solving strategies. I report two examples where a change in strategy was observed as a result of a change in numbers. Firstly, when the quantity to be decomposed was increased from 32 to 88 in problem 3.1.1 ($(1+3)x = 32$), Brian changed his multiplicative strategy to an additive strategy, building-up successive pairs of values in the form of a table (Figure 7.1) until he obtained a sum of 88. Secondly, as expected, determining two non-integer quantities that satisfy a given ratio proved to be demanding for the participants. In problem 3.2.1(b), Ted and Cole were asked to determine two quantities in the ratio $5:3$ and which differ by 7. This problem led them to modify their building-up strategy to interpolate between the closest integer ratios, namely $15:9$ and $20:12$ to determine the two unknown quantities in the ratio $17.5:10.5$.

Conclusion #2: Reversible Reasoning Requires the Simultaneous Coordination of Quantities and Relations

Being able to reason from a given result to the source causing the result requires the simultaneous coordination of quantities and relations. For instance, in fraction situations, one has to coordinate two pieces of information at one time: a measure and a multiplicative relation. At a more fine-grained level, being able to think in reverse in fractional contexts involves the coordination of 3 levels of units. Similarly, to determine the parts that constitute a given quantity in ratio situations requires the coordination of two quantitative relationships: a multiplicative relationship and an additive or difference relationship. In this section, I show how such simultaneous coordination of information is important for reasoning reversibly in fraction and ratio situations.

Set 2: Fraction situations

Coordinating 3 levels of units is a key resource for reasoning reversibly in fractional situations. For instance, in episode 6.12 (problem 2.51, $1\frac{1}{2}x = 48$), Brian re-conceptualized $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$ and interpreted each $\frac{1}{2}$ as $\frac{1}{3}$ of $1\frac{1}{2}$. It is this re-conceptualization of the same quantity with respect to two different units that enabled him to reason reversibly. Such a form of reasoning requires the coordination of 3 levels of units because one is coordinating both halves and thirds within the same quantity – taking two thirds of three halves to construct one whole. Additional supporting evidence that shows the necessity for coordinating 3 levels of units can be observed in episode 6.6 (problem 2.62, $\frac{3}{4}x = 2$), where Brian could reason reversibly by re-conceptualizing the given 2-unit bar as three $\frac{2}{3}$-unit. Aileen did not perform such a re-conceptualization and added $\frac{1}{4}$ to $\frac{3}{4}$ and correspondingly $\frac{1}{4}$ of 2 to 2 to obtain the length of the unknown quantity as $2\frac{1}{2}$ rather than $2\frac{2}{3}$, using a faulty theorem-in-action. Similarly, Ted and Cole did not perform such a restructuring, and their strategy in this problem was to draw a bar of 2 units, clear the partitions and divide the bar into 3. Then they pulled out one partition and measured the whole length as $2\frac{1}{3}$.

The demand for the coordination of multiple pieces of information simultaneously is explicitly apparent in reversibility situations involving two non-unit fractions with relatively prime numerators. For instance, in episode 6.7 ($\frac{3}{5}x = \frac{7}{4}$), Brian multiplicatively compared 7 units in terms of 3 units to obtain $2\frac{1}{3}$. This led him to deduce that $\frac{1}{5}$ of the unknown quantity is $2\frac{1}{3}$ units long rather than $2\frac{1}{3}$ one-fourth units or $\frac{7}{12}$ units long. In other words, he lost track of the one-fourth. This problem is demanding as it requires the coordination of 3 one-fifth units of an unknown quantity and 7 one-fourth units of a known quantity.

In addition, the reversibility situations in Set 2 (fraction situations) require the coordination of two pieces of information simultaneously. For instance, in problem 2.63 (Candy bar A is 3 units long. It's length is $\frac{4}{5}$ of candy bar B. What is the length of candy bar B?, $\frac{4}{5}x = 3$), one has to interpret 3 units as a measure and $\frac{4}{5}$ as a multiplicative relationship. Lack of flexibility to switch between these two relationships led to the quantity-measure conflict as highlighted in episodes 6.20 to 6.24 in Chapter 6. For example, in the above problem (episode 6.22, $3 = \frac{4}{5}x$), Cole and Ted constructed one whole unit of candy bar B by adding $\frac{1}{5}$ unit to candy bar A on JavaBars after partitioning bar A into 4 parts. When asked to measure the length of the resulting whole, they gave the answer $3\frac{1}{4}$ rather than $3\frac{3}{4}$ because they confounded the measure and the multiplicative relationship.

Set 3: Ratio situations

The ratio situations presented in Set 3 require the simultaneous coordination of two unknown quantities and two relations (a multiplicative relation and an additive or difference relation). A multiplicative solution to the ratio problems requires the interpretation of a ratio in terms of shares. Being able to conceptualize such situations in terms of sum of shares (or composite units) or difference between shares (as one entity) reduces the problem to a single unknown quantity. Failure to do so prevented the students from conceptualizing the end result in multiplicative relation to the starting quantities (i.e., prevented the students from reasoning reversibly). Jeff could make such coordination, and this allowed him to decompose the given quantity in terms of its constituent shares. The consequence of lack of reversible reasoning is that one is led to use building-up and guess-and-check procedures as fallback strategies instead of solving the problem deterministically, at a lower computational cost, and with more parsimony.

Conclusion #3: Reversible Reasoning Involves Positing and Articulating Unknown Quantities

The three problem situations (Set 1, Set 2, and Set 3) involve working with an unknown quantity in a multiplicative relationship. Positing an unknown as a quantity either implicitly or explicitly permits the articulation of the multiplicative relationship and facilitates reversible reasoning. The data show different ways in which the participants articulated such an unknown quantity, either implicitly or explicitly. In the algebraic approach in Set 3 (Chapter 7, episodes 7.11 to 7.13), Aileen, Brian, and Jeff explicitly represented the unknown quantities though the seventh graders (Aileen and Brian) could not always solve the resulting equations.

In Set 1, Aileen posited an unknown quantity between the two quantities being compared. Her response in situations 2 and 3 shows that she explicitly worked with the theorems-in-action $4 \times x = 6$ and $6 \times x = 10$ respectively, without symbolizing the unknown. Another way in which the participants articulated an implicit unknown was in terms of the unit-rate proportionality schema in Set 2. For example, in problem 2.42 ($\frac{2}{5}x = \frac{1}{4}$, episode 6.5), Brian reasoned that if $\frac{2}{5}$ of the unknown quantity has a measure $\frac{1}{4}$ unit, then $\frac{1}{5}$ of the unknown quantity has a measure $\frac{1}{8}$ unit.

The unit-rate proportionality schema involves the articulation of the quantitative relationship between the known and unknown quantity and avoids the need for explicitly quantifying the unknown quantity symbolically. Similarly, working with drawn representations to represent unknown quantities (as in the fractional problems on JavaBars) avoids the need for symbolizing the unknown quantity. In solving Set 2 problems, the students first represented the known quantity on JavaBars and attempted to construct one unit of the unknown quantity from that bar. None of the students wrote an algebraic equation relating the known and unknown quantities in Set 2 problems.

Reasoning reversibly involves positing an unknown quantity in multiplicative relation to known quantities, a conceptual maneuver that may be cognitively demanding. Failure to posit such an unknown quantity led the participants to work with the known quantity. For instance, in problem 2.51 ($1\frac{1}{2}x = 48$) and 2.52 ($1\frac{2}{3}x = 55$), some of the students attempted to split 48 and 55 in such a way that it represents 'one and a half' and 'one and two thirds' respectively. This automatically led to guess-and-check strategies. In other words, they attempted to plug-in numbers by decomposing b (in $ax = b$) in different ways, often systematically, such that they could extract the unknown quantity from the known quantity when a mixed number was involved.

The use of the multiplicative strategy in Set 3 requires the articulation of an unknown quantity between either the sum or difference of shares and the quantity to be decomposed. For instance, after forming a composite unit of 4 marbles in problem 3.1.1, Jeff deduced that 8 such sets of marbles could be obtained from the total of 32 marbles (giving evidence of using the theorem-in-action $4x = 32$) and distributed the resulting 8 as $8(3+1) = 8 \times 3 + 8 \times 1$. Aileen, who used the building-up strategy, did not posit such an unknown. Further, episode 7.15 clearly shows that Ted and Cole could not find the solution to problem 3.1.2 because they did not define an unknown quantity in the problem. Another example of the involvement of unknowns in ratio situations can be observed in problem 3.2.2(a), which involves thinking of two unknown quantities (*R* and *J*) such that when 18 marbles are transferred from one quantity to the other, they are equal (i.e., $R - 18 = J + 18$). Being able to posit such unknowns is what helped Brian to solve problem 3.2.2(a) without explicit use of equations.

Representing theorems-in-action algebraically

I refer to episode 7.12, problem 3.2.2(a) to show how the algebraic representation of a problem situation may be subtly different from the theorems-in-action one gives evidence of using in such a situation. Problem 3.2.2(a) reads as: Richard has 3 times as many marbles as John. If Richard were to give 18 marbles to John, they would have the same number. How many marbles does each student have right now? Brian wrote the algebraic equations $a \times 3 = b$ and $a + 18 = b - 18$ (Figure 7.9), where *a* and *b* denote the number of marbles that John and Richard have respectively, but he could not proceed further to solve these equations. However, he could mentally determine the value of the two unknown quantities using the theorems-in-action $b - a = 36$ and $a = \frac{1}{3}b$. In other words, the algebraic representation he wrote was different from the theorem-in-action he used to solve the problem. Jeff's response to the same problem gives further evidence of the difference between theorems-in-action and their algebraic representations. Though he could realize that the difference between Richard's and John's share is 36, he could not express this information algebraically (episode 7.12, Figure 7.10). Vergnaud (1996) explains such lack of connection between theorems-in-action and their algebraic representations in terms of *signifier* and *signified*, whereas Kaput (1991) refers to such disparity in terms of 'referential relationships.'

Conclusion #4: The Form of Reversible Reasoning Varies Across Multiplicative Structures

In this study, I looked at two types of mathematical structures, namely fraction and ratio. The nature of reversible reasoning is different in these two situations. As illustrated in Chapter 4, from a mathematical perspective, reversibility in fractional contexts involves a part-whole structure compared to the part-part-whole structure in ratio contexts. Algebraically, the fractional problems are equivalent to $ax = b$ while the ratio situations can be characterized as $(a+b)x = q_1 + q_2$ or $(a-b)x = q_1 - q_2$. Fractional situations involve the construction of one whole from a given part while in ratio situations one has to construct the parts starting from a given quantity. Each of these two structures requires different forms of reasoning. In fractional situations, the students had to coordinate a multiplicative comparison relation between a known and an unknown quantity, and they had to find the measure of the unknown quantity. In the ratio situations, the students had to coordinate a multiplicative comparison relation and an additive or difference relation to find the value of the two unknown quantities. However, in ratio contexts the two unknown quantities could be combined in a single quantity, and these situations could be considered as involving only one unknown.

Conclusion #5: Multiplicative Comparison Situations are Syntactically Sensitive

The tasks that I have chosen in this study involve the multiplicative comparison of two quantities. The students' responses in Set 1 and Set 2 allow me to conclude that such multiplicative comparison situations are syntactically sensitive. As illustrated in episode 6.28 in Set 2 (Chapter 6), the students occasionally performed a multiplication operation after reading the statement '*n* times as many.' This substantiates Nesher's (1988) observation that the multiplicative statement '*n* times as many' is "syntactically very complicated and is learned as a holistic expression denoting the multiplication operation" (p. 23). Such a lexical term may be a verbal cue for multiplication. The repercussion of such a multiplicative bias of the terms '*n* times as many' in Set 2 is that students took a fraction of the known quantity rather than the unknown quantity. Thus, such multiplicative comparison lexical terms may influence students' interpretation of reversibly situations. Parker & Leinhardt (1995) also emphasize the critical role played by the wording of multiplicative comparison statements. Comparative terms like 'as large as' and 'as much as' serve as triggers in cueing the multiplication or division operation. In conducting the interviews, I could observe how these specific words acted as prompts in cueing particular operations. For instance, after reading the statement of problem 2.12 (Bar A weighs 7 pounds. Bar A weighs 5 times as much as bar B. What is the weight of bar B?) on the first occasion, Cole multiplied 7 by 5 to obtain 35. After I read the statement of the problem a second time, emphasizing the terms '5 times as much', he said: "Just divide by 5".

The comparative terms 'more' and 'less' are frequently used from an additive perspective or in absolute terms rather than from a multiplicative perspective. For instance, in situation 5 (Chapter 5, Set 1), where the ratio of red to blue counters is $5:4$, Brian mentioned that there are one fifth more red counters than blue counters rather than one fourth more (episode 5.8), interpreting 1/5 from an additive perspective. Further, reversing the syntactic order of the comparative terms 'less' and 'more' in a multiplicative comparison relation does not reverse the order of the relation as is the case in absolute comparison. If $A < B$ then $B > A$ but if A is $\frac{2}{7}$ less than B, then B is not $\frac{2}{7}$ more than A. Such a faulty conception could be observed in the responses of the participants.

The Consequences of Failing to Reason Reversibly

One of the aims of this study was to identify different instances where the students reasoned reversibly in multiplicative situations. However, in various cases they failed to do so, and this led to a number of fallback strategies. In this section, I highlight some of the resulting fallback strategies in Set 1, Set 2 and Set 3.

Fallback strategy #1: Failure to reason reversibly led to guess-and-check procedures

The data show that children tend to resort to more primitive strategies when they are not able to deploy the necessary resources to conceptualize a given problem from a reversibility perspective. For instance, In Set 1 (in situation 2 and 3), after deploying the theorems-in-action $4x = 6$ and $6x = 10$, Aileen used a guess-and-check strategy to find the value of the unknown quantity as $1\frac{1}{2}$ and $1\frac{2}{3}$, respectively. This situation did not cue the multiplicative/ division invariant for her. In episode 6.9 ($\frac{7}{5}x = \frac{4}{9}$) the presence of two composite fractions with relatively prime numerators seems to have prevented Aileen from setting the quantitative relationship between the known and unknown quantities. After representing $\frac{4}{9}$ on JavaBars, she divided each one ninth by 5 and used a guess-and-check strategy in attempting to find the size of one whole. Another fallback strategy is that when the two quantities being compared could not be related because of the numeric feature of the data, the students attempted to make the denominators of the two given fractions equal (as illustrated in episodes 6.16-6.19 in Chapter 6) so as to turn an uncongenial problem into a congenial one.

Fallback strategy #2: Failure to reason reversibly led students to seek for more primitive strategies

Failure to coordinate the quantities in the given relation(s) in a problem situation led the students to resort to more primitive numerical reasoning. The participants used their basic knowledge of number relations and arithmetic to systematically substitute different values for the quantities in the problem situation. Another consequence of failing to reason reversibly in multiplicative situations is that one is led to use the more intuitively-appealing additive strategies. The students resorted to building-up strategies when they could not relate the quantities into a quantitative structure based on the relations between the quantities. This was particularly apparent in the ratio situations where the participants used actual numerical values to build up to the given sum or difference, working forward rather than backward. A table of values allowed them to coordinate the two relations simultaneously.

Fallback strategy #3: Failure to reason reversibly led students to seek the support of an external representation

Another fallback strategy that could be observed is that the students used an external representation (e.g., a drawing) to coordinate the known and unknown quantities in the multiplicative relation when they did not conceptualize such an unknown internally. It appears that there are at least three ways one can think about an unknown quantity: internally as a quantity in a relation, symbolically in terms of x (as a place holder) and diagrammatically (e.g., in JavaBars). By making a drawing, the conceptualization of the unknown is brought to a more concrete level. For instance, the JavaBars microworld served as a support in helping the participants construct one whole from part of the unknown though the measurement of the resulting whole was occasionally constraining for some of them.

Cole used a diagrammatic representation (Figure 7.13, episode 7.16) to find the quantity, one and a half of which is 18.

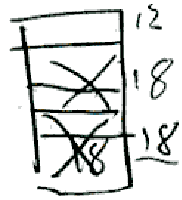


Figure 7.13. Cole's interpretation of 4 units in $1\frac{1}{2}$-units

In contrast, Brian re-conceptualized $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$ and chose two of those units to construct one whole (episode 7.16). Similarly, Jeff could reason reversibly in these situations and conceptualize the problem as requiring division and did not use such an external representation.

Other contributions of the study

By putting the concept of reversibility in the limelight and illustrating its operation in different multiplicative contexts, this study gives visibility to a form of reasoning that has not received much attention in mathematics education research. Further, this study specifically focuses on reversible reasoning in multiplicative situations, a domain where such form of reasoning has not been extensively studied unlike additive situations (e.g., Carpenter & Moser, 1983; Fuson, 1992). In the theoretical part of the study, I provided a trace of the history of reversibility in mathematics education since Piaget introduced the concept as a property of an operation. Further, the formulation of a definition of reversibility, based on both the review of the literature and empirical analyses contributes to giving another perspective to Piaget's initial definition of reversibility in terms of negation and reciprocity as merely a characteristic of an operation. In this research, I considered reversible reasoning to be the deductive reconstruction of the source or sources causing the result. In answering the first research question, I constructed a theoretical structure of reversibility situations that encompasses the range of multiplicative situations. This framework can be used to analyze how students reason reversibly in multiplicative domains like proportion and percentage, areas where reversibility has not been explicitly explored.

The theoretical analysis and empirical data allowed me to interpret $ax = b$ from five different perspectives: (i) as a statement of multiplication, (ii) as a statement of multiplicative comparison involving measurement division, (iii) as a statement of multiplicative comparison involving partitive division, (iv) as a statement of proportionality, and (v) as an algebraic statement. Another important point to mention is that the identification of such a structure ($ax = b$) in problem situations is a key resource that guides the strategies selected by the participants. My focus was to track down those instances where students articulated this central theorem-in-action in a variety of contexts. This study provides different instances where the multiplication/division invariant could or could not be cued. Previous research on multiplication and division (e.g., Anghileri, 2001; Clark & Kamii, 1996; Fishbein et al. 1985, Kouba, 1989; Mulligan & Mitchelmore, 1997) did not pay much attention to this aspect of multiplicative reasoning.

A close look at multiplicative comparison situations

The three sets of tasks chosen in this study involve a multiplicative comparison relation. In this section, I comment on the specificity of this type of multiplicative relation. As pointed out by Carpenter et al. (1999), the multiplicative comparison of two quantities results in a nonidentifiable quantity. For instance, if bar A is 5 units long and bar B is 3 units long, then bar A is $\frac{5}{3}$ as long as bar B. The length of bar A and bar B are measurable quantities, but the number quantifying the relationship ‘five thirds’ is not an identifiable quantity. In fact, Thompson (1994) considers a multiplicative comparison as one form of quantitative operation. In the previous example, one can only think of ‘five thirds,’ but we do not have a physical representation of this relationship, it is something that we have to construct. Moreover, when non-integer values are involved in such a multiplicative comparison relation, its conceptualization becomes even more taxing as illustrated in Chapter 6.

A multiplicative comparison statement such as ‘Ruth has 4 times as many marbles as Dan’ refers to one but no particular values of the two quantities (like the conceptualization of a variable). In other words, the ratio 4:1 represents only one element of the equivalence class. Moreover, the conceptualization of a multiplicative comparison relation where the two quantities involved have non-unit values are more demanding than one having unit values. For example, a statement such as ‘For every 4 marbles that Ruth has, Dan has 5 marbles’ is more demanding to conceptualize than a statement such as ‘Ruth has 4 times as many marbles as Dan’ because the first multiplicative relation involves the coordination of two composite units.

Set 1 involves the multiplicative comparison of two quantities and is constitutively a measurement division situation. Set 2 requires the construction of one unit given a fractional part or a part larger than one unit and is constitutively a partitive division situation. Both forms of multiplicative comparisons (i.e., partitive and measurement division problems) were problematic, though partitive division problems were more constraining. Measurement division problems gave rise to the ‘Faulty-remainder’ theorem-in-action (Chapter 5) while partitive division situations led to the quantity-measure conflict (Chapter 6). In Chapter 5 (Set 1), norming (i.e., the re-conceptualization of a quantity in terms of another quantity) was identified as a key process for the multiplicative comparison of two quantities. In Chapter 6 (Set 2), another form of re-conceptualization was observed that enabled some of the students to perform a partitive division (or construct one unit of the unknown quantity). For instance, in episode 6.6 ($\frac{3}{4}x = 2$), Brian re-conceptualized the measure 2 units as three $\frac{2}{3}$ units. This allowed him to produce $\frac{1}{4}$ of the unknown quantity, which he could add to the given two units to obtain $2\frac{2}{3}$. This form of re-conceptualization is more advanced than the norming process and involves partitioning (sharing) 2 units in 3 equal parts. In norming we know a priori the unit we are choosing to re-conceptualize the given quantity. For instance, a multiplicative comparison of 3 red counters in terms of 2 blue counters involves re-conceptualizing 3 units in terms of units of 2 to obtain $1\frac{1}{2}$ units of 2. In 'construct-the-unit' (partitive division) problems we are looking for the size of the unit.

Implications

The fine-grained analysis of children's thinking in terms of reversible reasoning in the domains of fraction and ratio allows me to make a number of suggestions with regard to instruction, curriculum design, and assessment. I formulate these suggestions based on the strategies and constraints that I could observe. I also use the review of past research and the mathematical analysis that I carried out in Chapter 4 to derive these implications.

Implications for instruction

What does this study suggest about ways of fostering reversible reasoning in fractional contexts?

Implication # 1: Experiences to work with fractional units

One of the reasons the participants encountered considerable constraints in solving the reversibility situations in Set 2 is that they did not interpret the fractional quantities in terms of fractional units. For instance, the interpretation of 4/7 from a part-whole perspective or as a number (one entity) and the interpretation of 4/7 as 4 one-sevenths involves two different concepts-in-action, each of which convey different information. The conceptualization of fractions in terms of units is an important resource for solving the different multiplicative comparison situations in Set 2. For instance, interpreting $1\frac{1}{2}$ as 3 halves (i.e., unitizing in halves) and $1\frac{2}{3}$ as 5 thirds (i.e., unitizing in thirds) makes the multiplicative relationship between the known and unknown quantities more explicit and facilitates the reversal of the multiplicative relationship. Such interpretation of composite units in terms of unit fractions enables the problem solver to use his/her whole number knowledge and consequently facilitates the cueing of the multiplication/division invariant in the multiplicative comparison situations. Failure to make such type of interpretation leads to approximation strategies based on guess-and-check procedures.

Implication # 2: Experiences in articulating the unit-rate proportionality schema

One of the ways to solve the reversibility problems in Set 2 is to use the unit-rate proportionality schema. For example, in episode 6.5 (problem 2.43, $\frac{2}{5}x = \frac{1}{4}$), Brian set a one-to-one correspondence between the two given quantities as follows:

$$\tfrac{1}{4} \rightarrow \tfrac{2}{5}$$
$$\tfrac{2}{8} \rightarrow \tfrac{2}{5}$$
$$\tfrac{1}{8} \rightarrow \tfrac{1}{5}$$
$$5(\tfrac{1}{5}) \rightarrow 5(\tfrac{1}{8})$$

The students articulated this form of quantitative reasoning only in few instances, especially when they used the 'unit-rate' strategy in episodes 6.4-6.7. It appears that they may not have had enough experience with this form of reasoning. The advantage that this schema offers is that it facilitates the coordination between the known and unknown quantities or the measure and the multiplicative relationship between the two quantities. Compared to the solution of the problem on JavaBars, where one has to articulate the multiplicative relation and the measure separately, the unit-rate proportionality schema allows these two pieces of information to be coordinated simultaneously. Further, this schema does not involve the multiplicative inverse of the fractional quantity in one step but rather in two steps: first the multiplicative inverse of the numerator is computed (by division) followed by multiplication by the denominator. Thus, the unit-rate proportionality schema is an operational mechanism that facilitates reversible reasoning and should be encouraged in instruction. Students should be given experiences to articulate this schema, especially in situations involving fractions with prime numerators.

Implication # 3: Counteracting the influence of additive reasoning in multiplicative situation

Another pedagogical suggestion that emanates from Aileen's response in episode 6.29 (problem 2.62, $\frac{3}{4}x = 2$) is the necessity to provide students with opportunities to observe how intuitively-appealing additive reasoning tends to interfere in multiplicative tasks. In problem 2.62 (Candy bar A is 2 units long. Its length is $\frac{3}{4}$ of candy bar B. What is the length of candy bar B?), Aileen added $\frac{1}{4}$ to $\frac{3}{4}$ to make a whole and consequently added $\frac{1}{4}$ of 2 units to bar A to obtain the length of bar B as $2\frac{1}{2}$. Such interference of additive reasoning in multiplicative situations has also been highlighted by previous research (e.g., Hart, 1984). Students may not readily realize the flaw (or faulty theorem-in-action) in this type of reasoning and instruction is necessary to limit this type of additive intrusion.

What does this study suggest about ways of fostering reversible reasoning in ratio situations?

Interpreting a ratio as a quantitative structure

Interpreting a multiplicative comparison relation as a ratio (a concept-in-action with a specific structure) is what afforded one of the participants (Jeff) the flexibility to solve the different problem situations in Set 3. He interpreted the problems in terms of ratio of shares. My analysis further shows that students do not readily interpret the sum of the components of a ratio as one entity or as a unit of units. Consequently, they cannot interpret the decomposition of a quantity in terms of its constituent parts from a multiplicative perspective. Instead, they use a building-up strategy to hit the target quantity. These observations suggest that students should be given opportunities to analyze multiplicative comparison situations in terms of ratios and interpret the components of the ratio in terms of shares. Students should be prompted to make sense of the sum of shares as one entity and relate it to the quantity to be decomposed. Similarly, the constraints that the students encountered in problems 3.2.1 and 3.2.2 show that they do not readily make sense of the difference of shares in a multiplicative comparison relation as one entity. Conceptualizing the difference between the components of a ratio as one entity is what allowed Jeff and Brian to solve problem 3.2.1 deterministically using a multiplicative strategy. Giving students experiences to interpret ratios in terms of shares can help them to coordinate a multiplicative comparison and a difference relation simultaneously. However, as Steffe & Olive

(2009) point out in their extensive study of children's fractional knowledge, the ability to coordinate such relations among quantities simultaneously requires the availability of three levels of units a priori. Such higher-level coordination cannot be forced but must, rather, emerge as the result of students' accommodations of their current schemes.

What does this study suggest about ways of fostering multiplicative reasoning?

In this study the chosen tasks led me to focus primarily on one of the ten categories of multiplicative situations described in Chapter 4, namely multiplicative comparison. As observed by Nesher (1988), multiplicative comparison exercises are not always explicit in instruction and are not even a point of concern. By making explicit the strategies and conflicts that the middle-school students encountered, the findings in Set 1 can enlighten teachers about students' reasoning in such types of multiplicative comparison situations. They bring to light some essential components that teachers need to foster in their instruction, for instance, making the distinction between the compared and the referent quantity and the application of the norming process. This study shows that comparing two quantities multiplicatively may not be an easy realization for students. The comparative terms 'less than' and 'more than' are not readily used in a multiplicative sense. The findings also alert teachers to be sensitive to the syntax/language used in making multiplicative comparisons. Further, the constraints encountered by the above-average students in Set 1 lead me to suggest the inclusion of such categories of task in the middle-school curriculum.

The necessity to provide students experiences in multiplicatively comparing relatively prime numbers from a conceptual perspective is motivated from the participants' response to problem 2.12 ($5x = 7$). This problem (Bar A weighs 7 pounds. Bar A weighs 5 times as much as bar B. What is the weight of bar B?) requires the multiplicative comparison of 7 units in terms of

5 units. Re-conceptualizing 7 units, each containing 5 units into 5 units, each containing 7 units and coordinating this transformation with the measure can be a cognitively demanding task. The division of the 7 units into 5 units each produces a partition having a multiplicative relationship 1/35 compared to the whole 7 units and each small partition in the original one unit has a measure of 1/5 unit. On re-conceptualizing the bar of 7 units, 7 small partitions represent 1/5 in terms of multiplicative relationship to the 7 units while the measure of the small partition remains 1/5. This problem involves two one-fifth units with different identities: 1/5 as a measure and 1/5 as a multiplicative relation. Students may not readily make such subtle coordination. Therefore, giving students experiences in multiplicatively comparing prime numbers and developing a conceptual understanding of such situations is an element that should be included in instruction. Such multiplicative comparisons should not be restricted to primes only but should also include quantities involving divisibility relations as these were just as constraining for some students as illustrated by the 'faulty-remainder' theorem-in-action in Set 1.

Students do not readily see the reciprocal relationship between two quantities in multiplicative contexts, and multiplicative inverses are not naturally occurring, especially when they involve rational numbers (a different world for the students), a system of numbers that are less intuitive compared to integers. When comparing two quantities, it did not naturally occur to the participants to deduce that they had to divide. The theorem-in-action 'division as the inverse of multiplication' is highly sensitive to the numeric feature of the data, and it may not be cued if the quantities being compared look incommensurable from their perspective. Previous research (e.g., Clark & Kamii, 1996; Resnick & Singer, 1993) has pointed out that multiplicative reasoning develops slowly. Further, Sowder et al. (1998) stressed that schooling is required to develop a thorough understanding of multiplication. This study provides further support to these claims that multiplicative reasoning is not naturally occurring, and instruction needs to pay more explicit attention to this form of reasoning.

Implications for curriculum design

The mathematical analysis that I carried out in Chapter 4 shows how multiplicative problems can be formulated from a reversibility perspective (which I have called ‘dual problem’). It shows how each multiplicative situation can be associated with a corresponding division situation. Students do not readily make such associations. Further, I have presented a range of diverse situations whose solution may require one to think in reverse starting from a given result to the source generating the result. The middle school curricula that I could access do not seem to have this range of multiplicative situations. Such deficiencies in curricular experiences may lead to reduced opportunities for making the connection between multiplicative and their corresponding divisive situations. Further, as the data in my study show, numeric variations of problem parameters strongly influence students’ conceptualization of multiplicative/divisive situations. For instance, multiplicative comparison problems involving relatively prime numbers and fractions may be challenging for middle-school students. Thus, these numeric variations need to be considered by curriculum designers. Furthermore, the analysis made in Chapter 4 may be practically useful to curriculum designers in terms of choice of tasks for developing curriculum materials.

The conflicts that the above-average students encountered in the first set of tasks involving the multiplicative comparison of two quantities came as a surprise. Indeed, this simple category of tasks is quite rare in the middle-school curriculum as observed by Nesher (1988). Further, all the fraction problems in Set 2 could have been solved as a one-step problem by performing division, but none of the students could conceptualize these situations as requiring division. Similarly, in Set 3, the students did not readily consider a sum and a difference of shares in a ratio as one entity. Informing curriculum design on the basis of children's mathematics as well as the mathematical structure of tasks rather than merely ordering the curriculum linearly and parceling it down into bits and pieces may be a better alternative in helping students construct mathematical knowledge.

Implications for Assessment

In this section, I highlight the importance of reversibility situations for assessing students' understanding of concepts. The domains of fraction and ratio were the focus of this study, and the interviews revealed that students experienced varying levels of conflict in solving such situations. Asking students to solve these types of problems where a result is given and one has to deductively construct the source can be informative to classroom teachers. During the data collection, I interacted with the students for more than 10 minutes on a single task in some situations as the students strived through different paths in their attempts to find the solution. Interviewing the students on the ratio situations, for instance, revealed that they did not always consider the sum or difference of shares as one entity, and this constrained them for reasoning reversibly. Consequently, they opted for the building-up strategy. Engaging students in such situations may provide teachers valuable information about students' extant conceptions. In choosing and designing assessment tasks, teachers may think about the reversibility principle where the end result is given and one has to construct the source. Students may readily take a fraction of a given quantity in a primal problem, but they may not readily construct one whole from a given part and determine its corresponding measure as explicitly shown in Set 2 in the dual problem. Further, the set of problems given in Chapter 4 can be a substantial resource to assess students' abilities to reason multiplicatively and reversibly.

Future Work

The theoretical analysis made in Chapter 4 involves a range of multiplicative situations, each of which offers possible avenues for studying reversible reasoning in terms of the strategies and constraints that students encounter. In this study I have analyzed only three categories of multiplicative structures: direct multiplicative comparison, fraction, and ratio situations. The fractional situations investigated in this study can complementarily be studied in the domain of percentage. As an example, consider the following problem: 'Steve pays 12% of the money he receives in his part-time job as tax. Given that he paid $36 as tax, how much money did he receive in his part-time job', a situation mathematically equivalent to $12\%x = 36$. This type of problem has been given much attention in the domain of percentage (Lembke, 1999; Parker & Leinhardt, 1995; Risacher, 1992) but these situations have not been studied in terms of coordinating an unknown quantity at a micro-analytic level.

In a previous study (Ramful & Olive, 2008), I investigated how students reason reversibly in the domain of proportion. Specifically, in that study I analyzed how a pair of middle-school students articulated the multiplicative relations in the balance beam problem that involve both a direct and an inverse proportion context. The gear-wheel problem (Lamon, 1999), which involves the coordination between the number of teeth and the number of turns as two gear wheels turn synchronously, is another situation that offers much opportunities to observe reversible reasoning and will be explored as an extension of the current work.

Another set of multiplicative situations that remain to be explored are in terms of reversibility of transformation, where a particular transformation is carried out and one has to determine what inverse transformation has to be effected to return back to the original situation. Greer (1992) defines this category of problems as 'multiplicative change'. For example, if a diagram is reduced by 75%, what reverse transformation should be carried out to return the diagram to its original size? One 'multiplicative change' situation that proved to be problematic for the participants in the preliminary stage of data collection is the following – The students were given a unit square and were asked to find the width of the resulting rectangle if the length is reduced by a fractional amount and the area is conserved. The exploration of these situations along with the other categories of multiplicative problems presented in Chapter 4 can reveal how students articulate relations and transformations at a fine-grained level and can contribute in enhancing our knowledge of reversible reasoning. They can potentially show how intuitive reasoning like additive interpretations of situations tends to interfere in problems that are characteristically multiplicative.

Reversible reasoning is also involved in other knowledge domains like calculus (e.g., in reasoning with functions), geometry, or statistics. One of the observations that I made in teaching algebra and calculus is that students were often constrained in deriving the graph of a function starting from the graph of the derivative of the function. Another example is in the transformation of graphs, where students could more readily draw the graph of a transformation (e.g., $y = 2\sin(x - \frac{\pi}{2}) + 3$) in contrast to identifying which one of a given set of visual representations of graphs showed $y = 2\sin(x - \frac{\pi}{2}) + 3$. Similarly, students could not readily construct the relationship between the derivative and the anti-derivative of functions. Reversible reasoning processes in calculus are not naturally occurring because the concepts are defined on the basis of mathematical conventions. For example, students have to be taught that the inverse of the logarithm function is the exponential function. They have to construct the invariant (or theorem-in-action) between a function and its inverse. Often functions have restricted domains and require more demanding considerations (e.g., $\sin^{-1}(\sin\frac{\pi}{6}) = \frac{\pi}{6}$ but $\sin^{-1}(\sin\frac{7}{6}\pi) = -\frac{\pi}{6}$ because the domain of $\sin^{-1}(x)$ is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$). In this study, I looked at reversible reasoning in multiplicative structures, but a whole range of mathematical structures remain to be explored at a fine-grained level.

End note

Of the different ways to address the problems in mathematics education, I believe that fine-grained analysis of children's mathematics offers productive ways to make the curriculum accessible to students, as such an approach minimizes the influences of adult conceptions and cultural influences of mathematical conventions. Such analysis allows us to identify those challenging and subtle coordination, the filiations and jumps in students' thinking, and the influence of intuitive thoughts, things that we do not always pay attention to in making curricular decisions. I have studied a minor fragment of students' thinking in the domain of multiplication, and this yielded a myriad of curricular and pedagogical issues. This study showed how students constructed theorems-in-action (both correct and flawed) as they were involved in deductively constructing the source from the given result. The path from the result to the source is often more challenging than from the source to the result. Effective mathematical thinking involves both doing and undoing processes within the range of structures that constitute the body of mathematics. Reversible reasoning provides flexibility in thinking, allows students to solve problems deterministically at lower computational cost and gives parsimony in thinking. Thus, it is highly desirable to include reversibility situations in instruction.

REFERENCES

Adi, H. (1978). Intellectual development and reversibility of thought in equation solving. *Journal for Research in Mathematics Education, 9*(3), 204-213.

Andrés Z, M., Pesa, M., & Meneses, J. (2006). A learning model (MATLaF) for conceptual development in laboratory: Mechanical waves. In E. van den Berg, A. L. Ellermeijer, O. Slooten (Eds.), GIREP Conference: *Modelling in Physics and Physics Education* (pp. 753-759). Amsterdam, University of Amsterdam. Retrieved August 27, 2009, from http://www.girep.org/proceedings/proceedings.html

Anghileri, J. (2001). Development of division strategies for year 5 pupils in ten English schools. *British Educational Research Journal, 27*(1), 85-103.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed.). New York: Macmillan.

Beckmann, S. (2008). *Mathematics for elementary teachers* (2nd ed.). Boston: Pearson/Addison-Wesley.

Behr, M., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis - emphasis on the operator construct. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, NJ: Lawrence Erlbaum Associates.

Behr, M. J., & Harel, G. (1990). Understanding the multiplicative structure. Retrieved April 25, 2007, from http://education.umn.edu/rationalnumber.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). NY: Mcmillan Publishing.

Behr, M., & Post, T. (1992). Teaching rational number and decimal concepts. In T. Post (Ed.), *Teaching mathematics in grades K-8: Research-based methods* (2nd ed., pp. 201-248). Boston: Allyn & Bacon.

Beilin, H., & Pufall, P. (1992). *Piaget's theory: Prospects and possibilities*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Biddlecomb, B., & Olive, J. (2000). JavaBars [Computer software]. Retrieved May 20, 2008 from http://math.coe.uga.edu/olive/welcome.html

Bloom, B. S. (Ed.). (1968). *Taxonomy of educational objectives: The classification of educational goals (Handbook 1: Cognitive domain)*. New York: David McKay Company.

Carpenter, T. P., & Moser, J. M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 7-44). New York: Academic Press.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth: Heinemann.

Chapman, M. (1992). Equilibrium and the dialectics of organization. In H. Beilin & P. Pufall (Eds.), *Piaget's Theory: Prospects and possibilities* (pp. 39-56). Hillsdale, NJ: Lawrence Erlbaum Associates.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51.

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of ratios and fractions and their roles as tools in proportional reasoning. *The Journal of Mathematical Behavior, 22*(3), 297-317.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & A. R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Whitenack, J. W. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics, 30*(3), 213-228.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 3-38). Charlotte, NC: Information Age Publishing.

Denzin, N. K., & Lincoln, Y. S. (Eds.). (1994). *Handbook of qualitative research*. Thousand Oaks: Sage Publications.

Dreyfus, T., & Eisenberg, T. (1996). On different facets of mathematical thinking. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 253-284). Mahwah, NJ: Lawrence Erlbaum Associates.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers, grades 6-10*. Portsmouth, NH: Heinemann.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum Associates.

Eves, H. (1990). *An introduction to the history of mathematics* (6th ed.). Philadelphia: Saunders College Publishing.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19-25.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Flavell, J. H. (1963). *The developmental psychology of Jean Piaget*. Princeton, NJ: Van Nostrand.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 225–256). Charlotte, NC: Information Age Publishing.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company.

Furth, H. G. (1969). *Piaget and knowledge: Theoretical foundations*. New Jersey: Prentice-Hall.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Gallagher, J. M., & Reid, D. K. (1981). *The learning theory of Piaget and Inhelder*. Monterey, California: Brooks/Cole Publishing Company.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & A. R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates.

Grabiner, J. (1995). Descartes and problem-solving. *Mathematics Magazine, 68*(2), 83-97.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

Gruber, H. E., & Voneche, J. J. (1995). *The essential Piaget: An interpretive reference and guide*. New Jersey: Jason Aronson Inc.

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth generation evaluation*. London: Sage Publications.

Hackenberg, A. J. (2005). *Construction of algebraic reasoning and mathematical caring relations*. Unpublished doctoral dissertation, The University of Georgia, Georgia.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior, 26*, 27-47.

Hall, R. (2000). Videorecording as Theory. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 647–664). Mahwah, NJ: Erlbaum.

Hardy, M., & Bryman, A. (2004). Introduction: Common threads among techniques of data analysis. In M. Hardy & A. Bryman (Eds.), *Handbook of data analysis* (pp. 1-13). London: SAGE Publications.

Harel, G., Post, T., & Behr, M. (1988). On the textual and semantic structures of mapping rule and multiplicative compare problems. In A. Borbas (Ed.) *Proceedings of the XII*

International Congress, Psychology of Mathematics Education (PME), vol. II (pp. 372-379). Budapest. Retrieved September 27, 2009 from http://www.cehd.umn.edu/rationalnumberproject/88_6.html

Harel, G., & Behr, M. (1989). Structure and hierarchy of missing-value proportion problems and their representation. *The Journal of Mathematical Behavior, 8*, 77-119.

Harel, G., Behr, M., Post, T., & Lesh, R. (1994). The impact of the number type on the solution of multiplication and division problems: Further considerations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 363-384). Albany: State University of New York.

Hart, K., M. (1984). *Ratio: Children's strategies and errors - A report of the strategies and errors in Secondary Mathematics Project*. Windsor, UK: Nfer-Nelson.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59-78.

Herscovics, N. (1996). The construction of conceptual schemes in mathematics. In L. P. Steffe & P. Nesher (Eds.), *Theories of mathematical learning* (pp. 351–379). Mahwah, NJ: Lawrence Erlbaum Associates.

Hiebert, J., & Behr, M. (1988). Introduction: Capturing the major themes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 1-18). Reston, VA: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior, 16*(2), 145-165.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence* (A. Parsons & S. Milgram, Trans.). New York: Basic Books, Inc.

Izsák, A. (2003). "We want a statement that is always true": Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education, 34*(3), 191-227.

Izsák, A. (2004). Students' coordination of knowledge when learning to model physical situations. *Cognition and Instruction, 22*(1), 81-128.

Izsák, A. (2005). "You have to count the squares": Applying knowledge in pieces to learning rectangular area. *The Journal of the Learning Sciences, 14*(3), 361–403.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Nisbet, S. (2002). Elementary students' access to powerful mathematical ideas. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 113-141). Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 53–74). The Netherlands: Kluwer Academic Publishers.

Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany: State University of New York Press.

Kelly, A. E., & Lesh, A. R. (2000). The need to address priority problems. In A. E. Kelly & A. R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 11-33). Mahwah, NJ: Lawrence Erlbaum Associates.

Kieren, T. E. (1994). Multiple views of multiplicative structure. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 387-397). Albany: State University of New York Press.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20*(2), 147-158.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children* (J. Teller, Trans.). Chicago: The University of Chicago Press.

Lamon, S. J. (1993a). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*(1), 41-61.

Lamon, S. J. (1993b). Ratio and proportion: Children's cognitive and metacognitive processes. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 131-156). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-120). Albany: State University of New York Press.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC: Information Age Publishing.

Lembke, L., O. (1991). *The development of concepts and strategies used in solving percent problems*. Unpublished doctoral dissertation, University of Missouri, Columbia.

Lesh, R., & Doerr, H. M. (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Research design in mathematics and science education* (pp. 591–646). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 93–118). Reston, VA: National Council of Teachers of Mathematics & Lawrence Erlbaum Associates.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics, 30*(1), 39-65.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation for slope. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions (2002 Yearbook)* (pp. 162–175). Reston, VA: The National Council of Teachers of Mathematics.

Mack, N. K. (1993). Learning rational numbers with understanding: the case of informal knowledge. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85–105). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309-330.

Nathan, M. J., & Koedinger, K. R. (2000a). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168-190.

Nathan, M. J., & Koedinger, K. R. (2000b). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction, 18*(2), 209-237.

Nesher, P., Greeno, J. G., & Riley, M. S. (1982). The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics, 13*(4), 373-394.

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 19-40). Reston, VA: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.

Nesher, P., Hershkovitz, S., & Novotna, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Educational Studies in Mathematics, 52*, 151-176.

Nunes, T., & Bryant, P. (1996). *Children doing mathematics*. Oxford, UK: Blackwell Publishers.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning, 1*(4), 279-314.

Olive, J. (2000). Computer tools for interactive mathematical activity in the elementary school. *International Journal of Computers for Mathematical Learning, 5*, 241-262.

Olive, J. (2001). Children's number sequences: An explanation of Steffe's constructs and an extrapolation to rational numbers of arithmetic. *The Mathematics Educator, 11*(1), 4-9.

Olive, J., & Steffe, L. P. (2002). The construction of an iterative fractional scheme: The case of Joe. *The Journal of Mathematical Behavior, 20*, 413-437.

Olive, J., & Çağlayan, G. (2008). Learners' difficulties with quantitative units in algebraic word problems and the teacher's interpretation of those difficulties. *International Journal of Science and Mathematics Education, 6*(2), 269-292.

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In K. Heid & G. Blume (Eds.), *Research on technology and teaching and learning of mathematics: Research synthesis* (Vol. 1, pp. 1-53). NC: Information Age Publishing.

Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. *Review of Educational Research, 65*, 421-482.

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York: Columbia University Press.

Piaget, J., & Inhelder, B. (1974). *The child's construction of quantities* (A. J. Pomerans, Trans.). London: Routledge & Kegan Paul.

Piaget, J. (1985). *The equilibration of cognitive structures* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago Press.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry* (E. A. Lunzer, Trans.). London: Routeledge & Kegan Paul.

Pinard, A. (1981). *The conservation of conservation*. Chicago: University of Chicago Press.

Polya, J. (1945/ 1988). *How to solve it: A new aspect of mathematical method* (2nd ed.). Princeton: Princeton University Press.

Post, T., Behr, M., Lesh, R., & Wachsmuth, I. (1986). Selected results from the Rational Number Project. Retrieved July 30, 2007, from http://www.education.umn.edu/rationalnumberproject/86_4.html

Ramful, A., & Olive, J. (2008). Reversibility of thought: An instance in multiplicative tasks. *The Journal of Mathematical Behavior, 27*, 138-151.

Rayner, D. (1986). *General mathematics - revision and practice* (Singapore ed.). Singapore: Oxford University Press.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373-429). Hillsdale, NJ: Lawrence Erlbaum Associates.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational Numbers: An integration of research* (pp. 107-130). Hillsdale, NJ: Lawrence Erlbaum Associates.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153-196). New York: Academic Press.

Risacher, B. F. (1992). *Knowledge growth of percent during the middle school years*. Unpublished doctoral dissertation, University of Delaware, Delaware.

Schnall, M., Alter, E., Swanlund, T., & Schweitzer, T. (1972). A sensory-motor context affecting performance in a conservation task: A closer analogue of reversibility than empirical return. *Child development, 43*(2), 1012-1023.

Schoenfeld, A. H. (1983). Episodes and executive decisions in mathematical problem-solving. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 345–395). New York: Academic Press, Inc.

Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 435-487). Mahwah, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (2007). Method. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 69-107). Charlotte, NC: Information Age Publishing.

Schoenfeld, A. H., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55-175). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 41-52). Reston, VA: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Siegel, L. S., & Brainerd, C. J. (1978). *Alternatives to Piaget: Critical essays on the theory*. New York: Academic Press.

Sigel, I., E., & Cocking, R. R. (1977). *Cognitive development from childhood to adolescence: A constructivist perspective*. New York: Holt, Rinehart and Winston.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Singer, J. A., & Resnick, L. B. (1992). Representations of proportional relationships: Are children part-part or part-whole reasoners? *Educational Studies in Mathematics, 23*(3), 231-246.

Singer, J. A., Kohn, A. M., & Resnick, L. B. (1997). Knowing about proportions in different contexts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 115-132). Sussex: Psychology Press.

Singh, P. (2000). Understanding the concepts of proportion and ratio constructed by two grade six students. *Educational Studies in Mathematics, 43*(3), 271-292.

Skemp, R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Smith, L. (1993). *Necessary knowledge: Piagetian perspectives*. Hove, UK: Lawrence Erlbaum Associates.

Sowder, J., Armstrong, B., Lamon, S. J., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education, 1*, 127-155.

Sparks, B. E., Brown, J. A., & Bassler, O. C. (1970). The feasibility of inducing number conservation through training on reversibility. *Journal for Research in Mathematics Education, 1*(3), 134-143.

Steffe, L. P., von Glasersfeld, E. V., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory and application*. New York: Praeger Publishers.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences, 3*(1), 61-82.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *The Journal of Mathematical Behavior, 22*, 237-295.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129-162.

Steffe, L. P. & Olive, J. (2009). *Children's fractional knowledge*. New York: Springer-Verlag.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50-80.

Thompson, P. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. Progress report to the National Science Foundation. San Diego State University, Center for Research in Mathematics and Science Education.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany: State University of New York Press.

Thompson, P. W. (1995). Notation, convention, and quantity in elementary mathematics. In J. Sowder & B. Schapelle (Eds.), *Providing a foundation for teaching middle school mathematics* (pp. 199-221). Albany, NY: SUNY Press.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Tirosh, D., & Stavy, R. (1999). Intuitive rules: A way to explain and predict students' reasoning. *Educational Studies in Mathematics, 38*(1/3), 55-66.

Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *The Journal of Mathematical Behavior, 23*(1), 93-114.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York: Academic Press, Inc.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 141-161). Reston, VA: National Council of Teachers of Mathematics & Lawrence Erlbaum Associates.

Vergnaud, G. (1994). Multiplicative conceptual field: Why and what? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41-59). Albany: State University of New York Press.

Vergnaud, G. (1996). The theory of conceptual fields. In L. P. Steffe & P. Nesher (Eds.), *Theories of mathematical learning* (pp. 219-239). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 5-28). Hove, UK: Psychology Press.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior, 17*(2), 167-181.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London: RoutledgeFalmer.

Wallach, L., Wall, J. A., & Anderson, L. (1967). Number conservation: The roles of reversibility, addition-subtraction, and misleading perceptual cues. *Child Development, 38*(2), 425-442.

Woods, D. (2008). Qualitative analysis software for video and audio data. (http://www.transana.org).

Zazkis, R., & Hazzan, O. (1999). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior, 17*(4), 429-439.