ADDITIVE RELATIONSHIPS AND SIGNED QUANTITIES

by

CATHERINE LOUISE ULRICH

(Under the Direction of Leslie P. Steffe)

ABSTRACT

This study investigates the cognitive and social factors that influence how a student constructs quantities that can take on positive and negative values and how the student constructs sums and differences of these signed values. Four sixth-grade students worked in pairs during this five-month teaching experiment. We first determined the nature of the students' number sequences, additive reasoning, and their ability to reason about multiple levels of multiplicatively related units. The students then worked in three signed contexts: a game in which they kept track of how many points they were winning or losing by, changes in savings after combinations of withdrawals and deposits, and displacement on a vertical pole after combinations of trips up and down the pole. My findings were consistent with past research (e.g., Peled, 1991) that found that students start off constructing positive values as comprising one quantity and negative values comprising a different quantity. The students gradually move towards a unitary signed quantity that can take on both positive and negative values. My other findings included the fact that utilizing the idea of additive inverses in adding or subtracting signed quantities is facilitated by the ability to assimilate mathematical situations using three levels of units and that forming a unitary signed quantity requires the construction of an explicitly nested number sequence (Steffe, 2010b). I also discuss how gender, personal interaction style, and disparities in zones of potential construction may have affected the quality of my data and the amount of learning opportunities for the students.

INDEX WORDS: Mathematical Learning, Radical Constructivism, Teaching Experiment, Negative Numbers, Addition, Subtraction, Quantitative Reasoning, Scheme, Additive Inverse, Levels of Units

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DEDICATION

I dedicate this work to my father, Carl William Ulrich, who instilled in me a respect for scholarship, excellence, and service.

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I would first like to acknowledge all of my professors both at Rice University and The University of Georgia. I feel very lucky to have had a plethora of educators in my life who have not only exhibited a love of learning and teaching, but have always encouraged me to challenge myself intellectually both inside and outside of the classroom. This is exemplified by three of my professors who served on my doctoral committee: Les Steffe, Bettie St. Pierre, and Ted Shifrin. All three, along with Amy Hackenberg, who graciously agreed to be on my committee, have been great sources of support and advice during the dissertation process. In particular, I am greatly indebted to Les Steffe, who managed to get me through this process even though I was kicking and screaming the whole way.

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Finally, thank you to my students and participants. My experiences learning mathematics with you are what drive me as a mathematics educator.

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CHAPTER 1

PROBLEM STATEMENT AND RATIONALE

> Additive structures are a difficult conceptual field, more difficult than most mathematics teachers expect. Understanding additive structures is a long-term process that starts with some simple find-the-final-state problems and goes on into adolescence with subtraction of opposite-sign transformations. (Vergnaud, 1982, p. 58)

This study looks at how students construct and operate on additive structures of unsigned and signed quantities. As a classroom mathematics teacher, I found that my middle and high school students had trouble remembering the rules for adding and subtracting signed numbers. I could find ways to help them make sense of and remember these rules, but I often wondered what mathematical concepts the students were developing out of the common models for addition and subtraction of signed numbers—positive and negative chips, arrows on number lines, metaphors from banking or hot air balloon rides. Unfortunately, with the current emphasis on standardized tests, our evaluation of students' mathematics often ends with the correct answer. In the proposed study, I want to develop a more complete picture of what mathematics is involved in students' construction of signed quantities and their sums and differences. We, the mathematics education community, need to develop a new vision of what is important mathematically in instruction with signed numbers. We need to move from a focus on how to help students remember rules to a focus on helping students develop sophisticated number schemes that will serve them in algebra and beyond.

In the current study, I worked with four sixth-grade students over the course of several months. I wanted to distinguish between how students make sense of addition and subtraction with unsigned quantities and how they do so with signed quantities. Unsigned quantities are those that students construct before encountering the idea of positive and negative numbers. For example, an unsigned quantity might have values that are counting numbers. Signed quantities are quantities that can take on both positive and negative values. For example, a common unsigned quantity is distance, and a common signed quantity is displacement. After ascertaining some of the current mathematical ways of operating of the participants, I introduced them to three signed contexts—their relative scores in a game of luck, the results of deposits and withdrawals of money, and displacements from combinations of trips on a number line. We worked in these contexts over the course of multiple sessions in order to try to engender an increased sophistication in how successfully the students could operate and explain their thinking in each context. My goal was to investigate what factors, including the students' mathematical ways of operating from their unsigned work, affected their mathematical development in these new contexts.

My Previous Research

Beyond my experiences as a classroom teacher, the main impetuses for this study were my findings in an exploratory teaching experiment carried out under the umbrella of the Ontogenetic Analysis of Algebraic Knowing [OAK] project (Steffe, Olive, Hackenberg, & Lawler, 2002). In this exploratory teaching experiment, I was the main teacher/researcher for one pair of eighth grade students. Towards the end of the teaching experiment, I started to work with them on adding and subtracting signed numbers. I quickly found that both students could correctly evaluate numerical expressions, such as -3 – +8, in which two integers were being added or subtracted. This fit with the fact that both had passing scores on the state standardized tests and generally seemed competent with mathematical procedures. We also did several problems involving trips on a ladder, where a trip up the ladder was represented by a positive number and a trip down the ladder was represented by a negative number. Both students were able to solve, fairly consistently, problems such as, “Amanda climbed +101 feet, rested, and then climbed -13 feet. How far is she from where she started?” On the surface, the participants seemed to have satisfactorily mastered integer addition and subtraction. However, lacunas in their reasoning became evident once I asked them to visually represent or notate the situations. This experience prompted me to review the research and policy literature regarding student learning of signed numbers and develop my ideas for the current study. In the remainder of this chapter I will discuss how my earlier research and my reading of the literature informed my conceptual framework and research questions. The next chapter gives a more in-depth look at both my theoretical and conceptual framework. The third chapter gives a more in-depth view of my guiding methodology and the particulars of how the current study was conducted.

The research project OAK has had many participants working on various aspects of middle school mathematics. I report here only on my work with Amanda and Brad, two students who I worked with over the course of one school year. We started working on signed quantities in the context of taking trips up and down a ladder that went both above and below ground. Note that I use the word signed quantities to include fractional signed quantities. The students chose 0 to describe the position of the surface of the earth, and their drawn representation were often pseudo-number lines in that position in relation to 0 and length were the quantities at play. Within the first two sessions, the students had begun using positive numbers to refer to trips going up the ladder and negative numbers to refer to trips going down the ladder. I will discuss three findings that informed my planning and analysis of the current study.

Multiple Reference Points

The first finding relates to Brad's difficulties representing the quantities in the problem situations pictorially. His difficulties are not particular to situations in which signed quantities are being added and subtracted. In order to demonstrate this, I will give an example in which all the values are positive. Based on the fact that he does not draw the negative part of the ladder (see Figure 1.1) or using the term *positive* in his explanations, I hypothesize that he was thinking of this situation in terms of unsigned quantities.

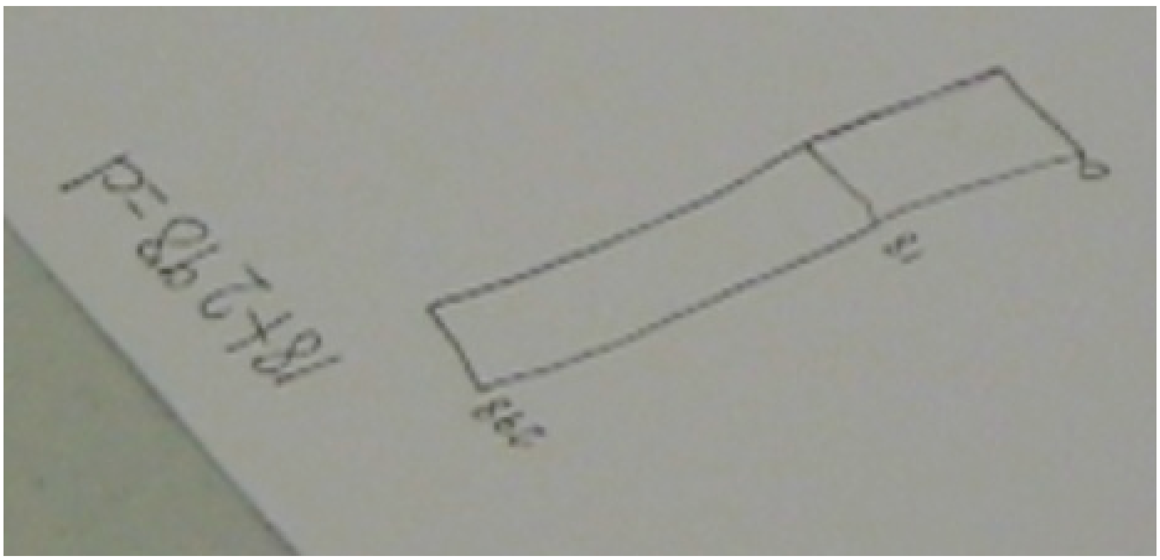


Figure 1.1. Brad draws a diagram for 18 + 298.

Given the endpoint of a trip that did not start at the origin, Brad characteristically conflated the position of the endpoint in relation to 0 with the length of the trip. For example, I gave the students the following problem: "Amanda climbed +18 and then +298. How could she have done that in one trip?" Brad drew the diagram in Figure 1.1, and he indicated that the top line was Amanda's final ending point. I asked him a series of questions about which distances the different parts of the diagram referred to in order to provoke him to re-label the top line as *18 + 298*, but in the course of his explanations he referred to the line segment from 0 to 298 as both "298" and "18 + 298." Similarly, he sometimes referred to the segment from 18 to 298 as "298," but at other times he said he would have to subtract to figure out how long it is. In the end, he seemed to settle on the top line as representing 298 (from 0) even though he had originally drawn it to represent Amanda's second trip. He then added another line to show her final ending point, as shown in Figure 1.2.

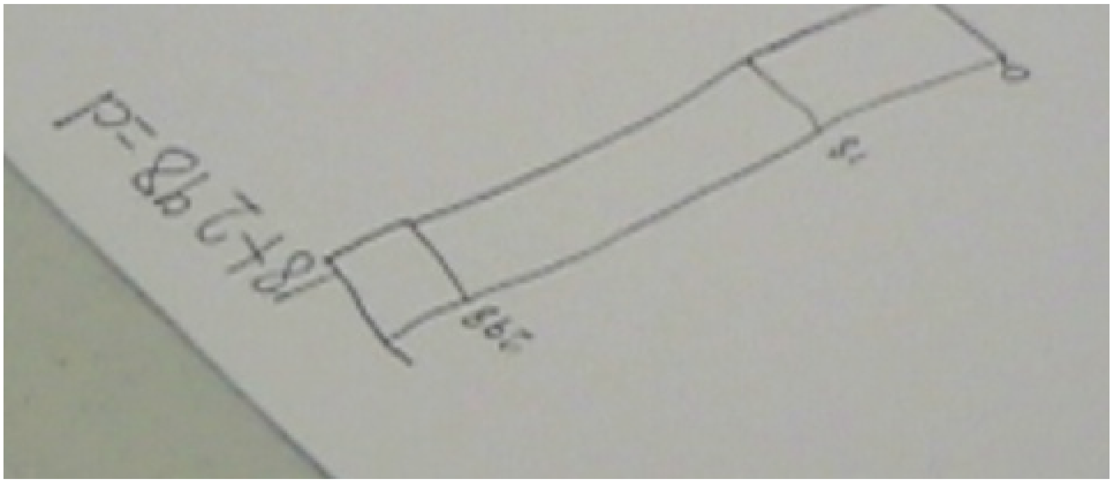


Figure 1.2. Brad changes his diagram in the course of his explanation.

His problem was not simply a matter of labeling; he was legitimately having trouble distinguishing between the different quantities in the problem. He behaved similarly on several occasions, including the determination of the result of a +132 trip and a -320 trip. In Figure 1.3, you can see his diagram of the situation: He has again labeled the ending point with the value of the second trip and then changed his mind, crossing out -*320*. In the explanation of this problem he insisted that he needed to subtract to find the distance between the upper and lower tick marks that he has drawn.

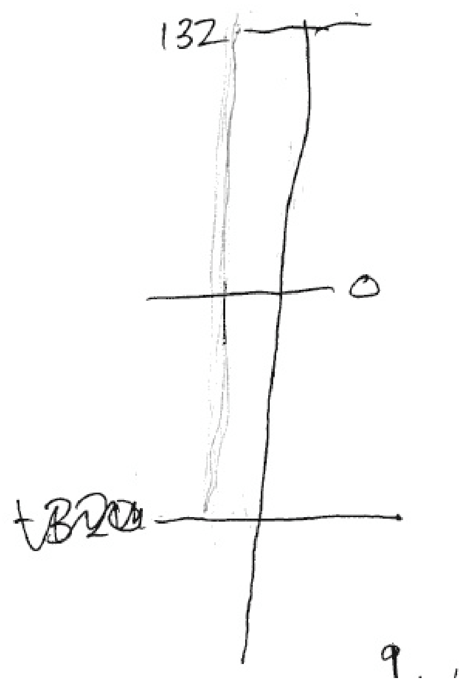


Figure 1.3. Brad’s diagram for (+132) + (-320).

My interpretation of Brad’s behavior is that he could intuitively re-create, either in his head or on paper, the situational quantities as he solved the problem, but that he did not retain an overarching structure of his construction to refer back to or reflect on. For example, as he is solving the first problem, he imagines going up 18 and he then knows that he goes up further 198, so he has some kind of intuitive sense that he should add to get the answer. However, once that decision is made, he does not hold on to the intuitive additive structure that he has built, and so when he adds, he can interpret the answer in the context of the problem, it gives how far he’s gone up, but he is not aware of the additive relationship between the 18, 298 and 18 + 298 trip. The result is that he cannot simultaneously consider the distances from the endpoint to 0 and to the ending point of the first trip. He wants all of the quantities in his diagram to have the same reference point, 0.

Brad’s conflation of how much his position had changed with its distance from a reference quantity has dire consequences for both using a number line and for attending to changes in quantity. My hypothesis was that in order for students to operate fluently in signed number situations, they would need to be able to deal with both of these quantities simultaneously, which involves being simultaneously aware of two reference points. One question this experience generated was how Brad's ways of operating with whole numbers could affect the way he operates in signed contexts.

Eventually Brad stopped conflating these quantities, at least when he was drawing situations of integer addition or subtraction. I designed an instructional intervention specifically to help him attend to both quantities simultaneously, but I was not sure if the change in his ability to represent and explain the situations was due to the instructional intervention or a re-conceptualization of the situation that would have come about from increased awareness of his own operations. In addition, I was not sure if his new ability to label his drawings differently reflected a change in his mental imagery and awareness of his operations. Therefore, coming into the current study, I also wanted to see how other students' explanations, both with and without pictorial representations, changed over time and try to isolate partial explanations for those changes.

Finally, Brad's difficulty brought home the inherent dual nature of signed numbers as representing both a change in quantity and the measure of quantity with relation to 0. In the current study, I tried to be more explicit with the participants about the different types of quantities that were present in the problem situations.

Attending to Changes in Position

Brad had two other unexpected behaviors that might be related to his conflation of quantities in the integer addition situations. The first was a tendency to attend to tick marks on a number line as opposed to the space between them. This was evidenced by the fact that he labeled only tick marks when I was consistently modeling labeling lengths, as in the previous examples. This was also evidenced by his solutions to number line pattern problems. His work for one pattern is shown in Figure 1.4. In this example, he was counting tick marks as opposed to spaces to figure out the pattern. In all of these cases, he is focused on positions on the number line and not distances. That is, he is not attending to the change in position between plotted points as a quantity. While he seemed intuitively aware of the distances between points in Figure 1.4, in that he appears to be attempting to quantify them, he only used the plotted points as experiential borders for sets of countable tick marks. As in "Multiple Reference Points," he is not coordinating the quantities representing a position on the number line (a directed distance from 0) and the quantities representing the changes in position.

Figure 1.4. Brad counts tick marks to find a number line pattern.

The second behavior is highly related to the first in that it has to do with reasoning about the difference between numbers and relating that to the original numbers. Brad was unable to solve problems in which he had to compare differences between numbers. One example of this is when he was given a pattern similar to the one in Figure 1.4, but with only numbers, no number line. This forced him to construct the first set of differences by comparing numbers as opposed to just counting. He was able to see a pattern in the differences, but once he was working with the differences, he was unable to relate them back to the original number pattern. In other words, he could form the differences in action, but once he took them as quantities to operate upon, he lost their relationship to the original numbers. This, as in his conflation of quantities, suggests that he is not able to attend to these two types of quantities simultaneously. I also gave the students a word problem from one of Pat Thompson's studies (1993) that involved reasoning with differences, and, as would be expected, Brad was unable to solve the problem. Amanda did not have trouble in any of the problems I have discussed so far.

As I analyzed the data from this exploratory teaching experiment, I began to suspect that the ability to work fluently with signed quantities as both a measure in relation to 0 and a change in quantity would relate to a student's ability to work with differences, represent differences, and reflect their operations with differences. Similarly to questions raised by Brad's original behavior in "Multiple Reference Points," my analysis led to a broader question of how a student's ways of conceptualizing unsigned numbers, sums and differences will affect their ability to conceptualize the addition and subtraction of signed numbers.

Notating Signed Addition and Subtraction

After Brad was no longer conflating quantities on his number line diagrams, I moved into notation. My intention, which I attempted to explain to the students in a very direct manner, was for us to use an addition expression, i.e., a sum, to describe a problem situation where we were combining trips, as in the example I gave in the Introduction. My intention was for us to use a subtraction expression, i.e., a difference, to describe missing addend problems; I instructed them to use subtraction expressions to model problems in which the students were trying to figure out one of the trips that is combined, as in the situation: "Amanda climbed +101 feet in the morning.

By the end of the day she was +88 feet from where she had started. How far did Amanda climb in the afternoon?"

Realizing that some of the addition situations could be modeled by related subtraction expressions, and vice versa, I delineated the two types of problems and how we would notate each on multiple occasions. I explained why the notation was consistent with what they had done with unsigned numbers: I called upon the idea of addition as combining two sets, and I called upon their experience notating missing addend situations using subtraction in their unsigned number work. The distinction seemed straightforward to me, and I knew from working with both students on unsigned quantities that they were comfortable describing missing addend situations using a subtraction expression.

However, given either signed word problems or signed numerical expressions, it became clear that the participants not only were not able to construct these two categories of problems, but also that the participants had meanings for addition and subtraction that would not generalize well to algebraic situations. For example, when given the word problem above, Amanda wrote +101 – -31 "because I am going down negative 31 feet." In algebraic situations, a student would not necessarily know whether they are going down or up, so a more general meaning for addition and subtraction is needed. After several days in which the students were not able to consistently model addition and subtraction appropriately, I concluded that I was encountering a *necessary error* (see the discussion in the next section or in Steffe, 2010f, p.90).

The situation with Amanda was perplexing to me because she seemed comfortable with similar definitions of addition and subtraction in whole number contexts: She had demonstrated that she considered a number sentence like 23 + 11 = 34 to represent the same situation as 34 – 23 = 11, which implies that she could think of subtraction as the inverse of addition and not just as a “take-away” situation. Even more to the point, she evaluated expressions like 34 – 23 by counting up from 23 to 34 to determine the missing part, which is using a missing addend approach to determine the numerical value of the expression. She insisted that she had never been taught that method of solution in the classroom, but that she almost always solved subtraction problems that way. So here, again, she does not limit subtraction situations to ones in which she was counting down or taking away, she also knew a subtraction situation might involve counting up her number sequence, which easily translates to a missing addend situation.

As a teacher, I was stumped. I did not know how to engender constructions of additive relationships among the integers that would allow Amanda and her partner, Brad, to see the situation as I did. This pedagogical problem spurred my interest in examining how student construct sums and differences of integers, which is the goal of the current study. Specifically, while both students could calculate sums and differences with unsigned numbers, I was not sure to what extent they were aware of the sums and differences as quantities that related back to the original numbers. Moreover, this pedagogical problem strengthened my commitment to a radical constructivist view of learning (see von Glasersfeld, 1995), which differentiates between my mathematics and the mathematics of each student. In this view of learning, I cannot teach students to think the way I do. I can only put them in situations where they are given the opportunity and motivation to organize their mathematical experiences in a way that is compatible with the how I organize my own mathematical experiences. During the spring of 2009, I was experiencing this inability to transmit my way of thinking to my students.

Foundational Theory and Concepts

The exploratory teaching experiment that formed part of OAK, the current study, and a closely related study that I would like to discuss by Dreyfus and Thompson (1985), all have very similar theoretical and conceptual frameworks. In order to discuss Dreyfus and Thompson's findings, my own findings from the exploratory teaching experiment, and my specific research questions, I will now introduce several concepts and terms that are important for understanding all three. For a more detailed discussion of my theoretical framework and conceptual framework, see Chapter 2.

In a radical constructivist view of learning, we have no way of accessing a transcendental Reality. We only have access to our experienced environment, which we have, in turn, constructed. Our construction of reality is self-regulating in that we start off with our current constructions of reality, and then when we perceive something unexpected, we attempt to modify our constructions in order to take the unexpected event into account. Put in terms of students' mathematics, students construct their mathematical reality in such a way that it will not contradict their experience of teachers' mathematics, textbooks' mathematics, etc., and they construct a mathematical reality that strives to be internally consistent. That is, if their mathematical reality leads to a paradox, they will attempt to modify that reality so that the paradox is resolved. Therefore, the students' mathematical constructions are made through a negative feedback system in which ways of operating that lead to results that the teacher reacts negatively to or that lead to paradoxes will be discarded or modified. The students are constructing a mathematical reality in terms of what the teacher's mathematical reality is *not*. Therefore, the students' mathematical realities do not necessarily reflect the teacher's mathematical reality or even approach it, although it will tend towards not contradicting it. As von Glasersfeld (1979) says, "a description in negative terms cannot be turned into a positive description, because the exclusion of some possibilities, in a field of infinite possibilities, does not make that infinity finite" (p. 115).

For a radical constructivist researcher studying student learning, the implication is that we do not have access to students' mathematical constructions. All we can do is build viable models of their mathematical constructions based on our experiences of them. I follow Steffe (2010e) in referring to a student's mathematical reality as *the student's mathematics* and my model of their mathematics as *mathematics of the student* in order to differentiate between the two. Because the *mathematics of the student* is only a model of student thinking, when I or other radical constructivist researchers make claims about the mathematics of a student, it is always understood to be a claim that we are presenting a viable model of the student's thinking, not a truth about the student's thinking.

In radical constructivist theory, knowledge is defined in terms of coordinations or associations between elements of our constructed reality. In order to talk about the mathematical knowledge of a student, many mathematics education researchers working from a radical constructivist perspective utilize the concept of a *scheme* because it captures the idea of coordination and association. The idea of schemes in radical constructivism comes out of Piaget's (e.g., 1977/2001) genetic epistemology and was elaborated in detail by von Glasersfeld (e.g., 1979). Von Glasersfeld (1981) went so far as to give a detailed account of his scheme theories regarding the development of early conceptions of number. In his work, a scheme consists of an assimilated stimulus that has been associated with the activity of the scheme, the activity or procedure of the student, and the result of the activity. Their relationship is circular in that the assimilated stimuli associated with the action of a scheme change in reaction to expected or unexpected outcomes. Hereafter, instead of using the term *stimulus,* I will use the concept of an *assimilating structure*. For example, a plurality of objects might be the necessary feature in a child's experience to allow the child to initiate his or her counting scheme. Clearly the assimilating structure is constructed. In this case, the child has to abstract out the concepts of *object* and *plurality* and then construe their intersection as a situation of their counting scheme. In that the countable items for a student change over time, the assimilated stimulus for their counting scheme is changing. An assimilating structure often needs to be paired with a goal, such as finding out how many objects are in a set, in order to activate a specific scheme.

Assimilating structures are, more generally, ways of organizing experience that are open to a student. For example, when I say a child has constructed object concepts, I mean that the child can look at the world in terms of objects. The child does not have to actively differentiate objects from sensory experience anymore, the experience now *includes* objects. As another example, the claim that I can assimilate with a two-level place value structure implies that I have constructed coordinated structures of units of ten and units of one with which I assimilate a situation with a two-digit number. Unlike a young child, I do not have to divide up the quantity representing 23 into 2 tens and 3 ones before I am aware of the place-value structure. New assimilating structures make possible new schemes because there are new aspects to the child's mathematical experience that can be operated on. When I talk about the mathematics of a student as being more *sophisticated* than the mathematics of another student, I mean that their assimilating structures are more elaborated and perhaps are at a higher level of abstraction, allowing the construction of more powerful schemes.

Finally, when I used the term *necessary error* in reference to my participants' method of notating addition and subtraction of integers, I understand the *error* to be defined from my perspective, the perspective of the observer. In this case, Amanda's actions were not compatible with my mathematical ways of thinking (or the generally accepted mathematics). The term *necessary* represented the fact that the error is an outgrowth of the students' current mathematical schemes and is a rational way of operating within their mathematical realities. *Necessary errors* are generally resilient in the face of instructional interventions. They usually serve as indications that the student has not yet developed a key mathematical operation, or way of organizing mathematical experience, that the observer has developed, and, hence, cannot reach a way of operating that is compatible with the observers' ways of operating.

Why Signed Quantities Are Important

Based on my exploratory study, I became interested in student's addition and subtraction schemes, especially with regards to signed quantities. However, interesting student behavior emerged in several mathematical domains during the exploratory study. My review of the research literature on student learning with signed quantities was another impetus for choosing the topic of signed quantities for my dissertation study. My review of the literature showed that most research focused on integers, as did my research, instead of more general signed quantities, including rational numbers. The teaching focus in research is often on providing a way for students to remember computational tricks. The results of instructional interventions are almost always tested by the students' ability to give the correct number in response to an addition or subtraction problem, rarely contextualized (see Chapter 2 for a more thorough discussion). This focus on procedure over quantitative relationships may reflect many mathematics educators' lack of clarity about what important mathematical concepts students can gain out of their work with signed quantities. Thompson and Dreyfus (1988; Dreyfus & Thompson, 1985) discuss the importance of broadening the concept of number to include transformations when in signed contexts and, correspondingly, broadening the concept of addition to include the composition of transformations. In addition, several authors (Thompson & Dreyfus, 1988; Flores, 2008) discuss the importance of constructing the additive inverse as a mathematical object when working with signed numbers. However, in many cases, researchers focus on getting students to a numerical answer instead of focusing on what important cognitive operations could develop out of signed number contexts.

In the four standards documents I am most familiar with, the 2000 *Standards* from the National Council of Teachers of Mathematics (NCTM, 2000), NCTM *Focal Points* (2006), *Georgia Performance Standards* (Georgia Department of Education, 2008), and the *Common Core Standards* (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), seventh grade is the last place where negative numbers are explicitly mentioned. However, we know that negative numbers do show up in mathematics and science courses after seventh grade. In addition, the way that addition and subtraction signs are used in the secondary years changes, most noticeably when working with variables that represent signed quantities. In this section, I will outline some of the reasons that work with signed quantities is important for future mathematical success.

Thompson (1993) has highlighted the importance of reasoning with differences and he and Dreyfus highlight the importance of constructing changes in quantity as signed quantities (Dreyfus & Thompson, 1985; Thompson & Dreyfus, 1988). In particular, they note that Vergnaud (1982) has shown the difficulty students have dealing with addition and subtraction even with unsigned quantities when the quantities involved are expressed as changes in quantity. The ability to quantify and reason with differences will be essential in developing ideas of rate and function. This is not just important in upper-level mathematics classes. Having an intuitive understanding of how the rates of change in linear versus exponential growth is important for anyone with a credit card and fixed minimum payment. This kind of intuitive understanding develops on a foundational ability to attend to and reason with differences.

Numerous researchers have discussed how signed quantities can be introduced in the context of changes or comparisons in unsigned quantities (e.g., Linchevski & Williams, 1999; Moses, Kamii, Swap, & Howard, 1989; Streefland, 1996; Thompson & Dreyfus, 1988). In these approaches, students are constructing changes and differences as mathematical objects that can be mentally operated on and symbolized: Differences become key elements in reasoning. For that reason, I have used such an approach both in this study (see Chapter 3 for more details on my tasks) and my exploratory study.

Apart from being a good context for reasoning with differences, working with signed quantities could reorganize students' conceptions of addition to include the composition of transformations of quantities. Composing transformations in a signed context is a preliminary, special case of composing functions, which is a key operation in upper-level mathematics courses. Similarly, making sense of subtraction in relation to addition as the composition of transformations of quantities allows a firmer foundation for dealing with subtraction in algebraic expressions in which the sign of the variable or unknown quantity is not known.

Signed quantities, both integral and fractional, are generally the last new type of number concept introduced before the formal unknowns and variables of algebra. In Georgia, for example, square roots of perfect squares and irrational numbers are introduced in the same year as formal unknowns and variables, and fractional quantities have already been introduced in elementary school (GA DoE, 2008). In some ways, work with signed quantities is the students' last chance to enrich their quantitative reasoning schemes before they will be required to reflect on their quantitative reasoning in order to embody it in algebraic notation. In their work with unsigned quantities in elementary school, some students are able to avoid certain types of issues, such as developing subtraction as the inverse operation to addition. Other students may not have been ready for such mathematical developments. Mathematical situations involving the addition and subtraction of signed quantities provide an excellent opportunity for students to advance their conceptions of number, addition and subtraction, before the complications of notating variable quantities is introduced in secondary algebra work.

Considerations for Teaching and Researching Signed Quantities

The extant research on signed quantities includes only one study that uses scheme theory (Dreyfus & Thompson, 1985; Thompson & Dreyfus, 1988) to understand students' cognitive operations. There is certainly nothing as detailed as the extensive work that has been done on the development of students' fraction schemes (e.g., Hackenberg, 2007; Lesh, Behr & Post, 1987; Norton, 2008; Steffe & Olive, 2010; Tunc Pekkan, 2008), another important area in middle school students' mathematics. Fraction research has resulted in a rich conceptual framework that theorizes underlying mental operations involved in the construction of fractions. The current study lays the groundwork for developing a similarly rich conceptual framework of the mental operations involved in the development of signed quantity schemes. Although the current study looks mainly at how students work with integral signed quantities, I hope to expand this work to look at signed quantity schemes that include fractional quantities in the future.

Questioning Assumptions

The dearth of research on student understanding of signed quantity operations might also exist, in part, because many mathematics educators assume that integers are so similar to natural numbers (and signed quantities are similar to unsigned quantities) that student understanding of unsigned numbers should require only slight modifications in signed contexts. This assumption seems justified in that, for us, unsigned quantities (positive numbers) are simply a special case, or subset of signed quantities, and all that we need to add notationally to describe signed quantities given unsigned quantities is add a negative sign from time to time. However, this is coming at the matter from the perspective of someone who has already made sense of signed quantities, as I did both as a classroom teacher and as a researcher. I began to question my assumption during the exploratory teaching experiment with Amanda and Brad. For those students, the transition was clearly trickier than I had anticipated. I will now share instances in mathematics education literature on both research and policy in which I found evidence of such assumptions about student conceptions of number, addition and subtraction.

The nature of quantities. The first assumption I found in the literature relates to the nature of signed quantities. The *Common Core State Standards* (Natl. Governors Assn., 2010)[1] call for sixth-grade students to "extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates" (p. 43) and "understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge)" (p. 43). In all of these quotes, a signed number represents a point on a number line or a value of a familiar quantity. Certainly these understandings of signed number are important. However, students do not automatically differentiate between positions and changes in position, or a bank account balance and a change in the balance, etc. This is implied by Brad's conflation of position with changes in position. Dreyfus and Thompson (1985) also found this to be an issue with their sixth-grade participants. In their study, they interpreted participants' solution strategies involving a number line representation in terms of van Hiele levels (Burger & Shaughnessy, 1986) and

[1] Although I use excerpts from the *Common Core Standards* to illustrate some questionable assumptions many mathematics educators have about learning to add and subtract signed quantities, I do not mean to imply that the writers were more culpable in holding these assumptions than the rest of us have been. In fact, I commend the writers for the thought and detail with which they outlined important aspects of signed number conceptions.

differentiating between position and transformation of position constituted the base level, Level 0, of reasoning. Like Brad and unlike Amanda, their participants struggled with this at first.

In the second quote from the Common Core Standards, positive and negative numbers describe different quantities, specifically, “quantities having opposite directions or values.” However, in each example we can conceptualize a single quantity that can take on positive or negative values. For example, temperature above/below zero could be thought of as two quantities: the number of degrees above zero and the number of degrees below zero. Alternatively, we can think of one quantity, temperature. The “above/below zero” descriptors would refer to the fact that it can take on positive or negative values *in the current reference system*. With signed quantities, students now need to attend to the reference system to interpret values, just like they need to attend to the units of measurement with unsigned quantities to interpret value.

In line with the differentiation I have made between considering the temperature situation as one or two related quantities, some students will initially construct separate positive and negative quantities that are related by context, but are not seen as part of one overarching quantity, such as temperature. Peled (1991) discusses students’ use of separate rules of actions depending on which of these two quantities, positives or negatives, that they begin their situation in: “There are two worlds: a positive world on the right and a negative world on the left” (p. 147). In addition, Janvier (1985) briefly notes that at one time in the history of mathematics, the real number line was split into two, with the positive portion separate from the negative portion even by experts. To get a sense of how a signed situation could be thought of by a student who has not constructed a unitary signed quantity, consider the chip model (e.g., Flores, 2008) for teaching addition and subtraction of signed numbers: The positive numbers and negative numbers are presented as representing two separate quantities, the number of black chips and the number of red chips, respectively. These quantities are linked by the ability to add or take-away *zero pairs* of equal numbers of both quantities, but it would be hard to verbalize a meaning for the overall quantity that the positive and negative numbers are describing. In contrast, in the context of elevation above/below sea level, there is a clearer unitary signed quantity, elevation, that is evaluated with reference to sea level. Yet even in the context of elevation, some students will initially understand the elevations below sea level and above sea level as separate quantities. My hypothesis is that the construction of a unitary signed quantity is not trivial for most students, but involves a progressive awareness of the relationships between the related, but separate, negative and positive quantities, including their shared reference value. Recognition that the reference value is simply defining 0 in our reference system and could be arbitrarily changed would seem to imply the unitary nature of the signed quantity, as would the ability to assimilate signed quantities as changes starting from an arbitrary reference value.

Dreyfus and Thompson's (1985) findings give another indication that students do not automatically understand what are, to us, signed contexts, as involving a signed quantity.

> Minus signs were not part of numerals; rather, they qualified numerals. Similarly, the direction of a transformation was not part of the transformation; rather, it qualified the transformation. To add two of *their* transformations, one would add the number of turtle steps [the magnitude of a number line trip] in each. A direct analogy is college students' common mistake of adding two vectors' magnitudes to determine the magnitude of the sum. (p. 7)

In other words, their participants were thinking of everything in terms of an unsigned quantity—the size of a trip—and only using the negative sign to interpret what the, in their case, computer-generated representation would look like.

Addition and subtraction with natural numbers. The second assumption I find inherent in various Standards documents and some research literature is about the nature of students' addition and subtraction schemes on entering seventh grade. In the *Common Core Standards* (Natl. Governors Assn., 2010), students are expected to "Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers" (p. 48). Other standards documents, including the NCTM (2006) *Focal Points* use similar language: "Students extend understandings of addition, subtraction, multiplication, and division, together with their properties, to all rational numbers, including negative integers" (p. 19). I found the same language in various researchers' work (Linchevski & Williams, 1999; Peled, 1991). The use of the word *extend* implies that the same ideas of addition and subtraction that have worked for students in elementary school will work with signed numbers. In fact, there are ways that students think about addition and subtraction that are not easily generalizable to a signed number context. In the elementary grades, addition is often characterized as increasing the amount in a set, and subtraction is often characterized as taking away items from a set or decreasing the amount in a set. As I discussed in "Notating Signed Addition and Subtraction," this does not translate well to signed or algebraic situations. For some students, addition and subtraction are actions as opposed to indications of subset relationships. Certainly, the *notation* of addition and subtraction does not necessarily indicate the presence of sums and differences as quantities that can be operated on.

Thompson (2011) discusses the need for students to have more experience operating on and reasoning about notated sums and differences ($a + b$ or $a – b$) without carrying out the operations needed to determine a numerical value. Given the opportunities to reason about notated sums and differences, students might indeed come to middle school with more structural conceptions of addition and subtraction, particularly with respect to the notation. Dreyfus and Thompson (1985) hypothesized that the highest van Hiele level students could indicate in their teaching experiment would involve the ability to operate on the corresponding sum and difference structures for signed quantities. I hypothesize that this would be equivalent to students being able to use my notation system for addition and subtraction of signed quantities, since use of the system requires an awareness of problem structure. Neither Dreyfus and Thompson's participants nor my participants in the exploratory study reached this level of sophistication, which indicates that dealing with sums and differences as structures is significantly more difficult than evaluating sums and differences. Interestingly, Dreyfus and Thompson argue that operating on these types of integer sum and difference structures would correspond to early stages of algebraic thinking, implying that sums and differences of integers are relatively advanced structures.

In order to build a helpful body of knowledge about student signed quantity constructions, we need to rethink our assumption about the student mathematics involved in signed number addition and subtraction. In the next section I will discuss in more detail distinctions I have made between the notions of number, addition and subtraction some students will be bringing to middle school and the sophisticated concepts that I would like them to build by the end of middle school.

New Kinds of Quantities

In the first of the four examples of signed quantities in the *Common Core State Standards* (Nat'l Governors Assn., 2010, p. 43), "temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge," virtually any addition situation would involve working with changes in values of the quantity. For example, if a student were to attempt to develop a problem situation in a context of temperature involving the sum of two signed numbers, such as (-38) + (+12), then the second addend is almost certainly going to be a change in temperature, not a reading on the thermometer. An example of a question would be the following: *The temperature outside is 38 degrees Fahrenheit below 0 in the morning, but increases 12 degrees Fahrenheit by 4pm. What is the temperature at 4pm?* The second addend does not represent a reading of a thermometer in the same way the first addend and sum could (although the first addend and sum could also be thought of as changes starting from 0). We can come up with addition situations that do not involve an explicit change, such as the combination of charged particles: 4.6 moles of positively charged sodium ions and 0.5 moles of negatively charged chlorine ions are combined in a neutral solution. What will be the overall electrical charge of the solution (in faradays)? However, many situations involving addition of signed quantities do require operating with changes in quantity.

In fact, in all of the first three examples, every signed quantity can be thought of as a change in quantity. This is perhaps most easily seen when working with another common signed quantity, directed distance from 0 when carrying out trips on a number line. If you are modeling (+3) + (-7) on a number line, then your starting position could be +3, or you could think of your starting position as the reference point and then see the +3 as representing an increase of 3 units (see Figure 1.5).

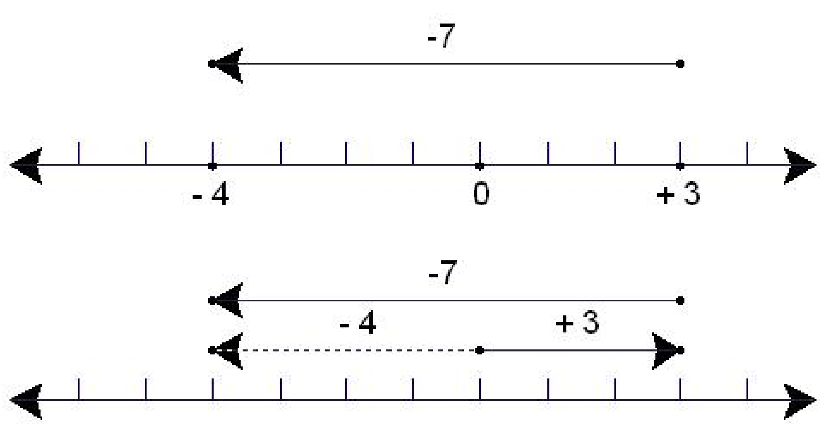


Figure 1.5. Two representations of (+3) + (-7) = (-4).

Having students work in contexts where all values represent changes in quantity has many benefits for their future mathematical development with regards to functions, rates of change, etc. Another benefit is that the commutative property of signed addition becomes more intuitive when the student is able to interpret signed numbers as changes in quantity. Students tend to intuit, purportedly based on their experiences, that the order of increases and decreases is irrelevant when a series of increases and decreases is made. However, the situation is less intuitive when some of the signed values represent positions: Recognizing that starting at +3 and decreasing 7 gets you to the same value as starting at -7 and increasing 3 requires greater reflection on the situation. In addition, when working with changes, you can have an unspecified reference point[2]. When students eventually work with vectors and matrices, the ability to deal with an unknown reference point is crucial. In addition, when interpreting slopes and rates of change in functions, the actual values of the underlying quantity are not as important as the way in which those values change.

This leads into an important conception of signed numbers that has helped orient me as a teacher and researcher: the idea of signed numbers as one-dimensional vectors. One of the key models used for representing signed quantities is movement on a number line. In this model, arrows represent signed quantities. This makes more explicit the vector nature of these signed quantities. Moving towards a vector conception of number is important for several reasons. First, students' ideas of addition and subtraction should be easily transformable into corresponding vector notions for students continuing on in higher science or mathematics. Second, a number line model involving one-dimensional vectors will provide an opportunity for students to work

[2] By *unspecified reference point*, I mean that you know what relationships the reference point has to the changes in quantity, but its value in terms of the underlying quantity is unknown. For example, if you put 30 cents in a piggy bank and take out 40 cents, you can figure out the overall change in the piggy bank's value without knowing how much was in it originally.

with the number line, which is clearly important in mathematics and science, yet tends to trip students up. This is in part because the number line forces the student to explicitly differentiate between the value of a number, which represents the directed difference from 0, and the transformation that brought you to that number, which represents the directed difference from the previous value; this is exactly what Brad was struggling with. Third, just as vectors are equated with reified transformations at the post-secondary level, we want students to gain experience in reifying signed numbers as transformations (i.e., constructing transformations as mathematical objects that can be mentally operated on). The use of arrows on the number line as a model for integer operations provides opportunities for this kind of abstraction. Fourth, it seems reasonable to assume that the cognitive operations necessary for assimilating changes in quantities as mathematical objects would be necessary for such common situations as operating with changes in function values to form conceptions of rate, slope and derivative. Hence the focus on developing integers as one-dimensional vectors has both a theoretical and pedagogical foundation.

Note that in the same situations in which changes in quantity are themselves a quantity that is operated upon, such as defining average or instantaneous rate of change, these changes in quantity are usually determined through subtraction, through taking the differences of two values of the underlying quantity. This brings me again to the idea that students' ability to reflect on differences and operate with differences is important. In Linchevski and William's work (1999), they try to engender construction of integer quantities out of the directed differences of dice rolls. Similarly, in my research, I used a card game in which students construct integer quantities out of pairs of natural number values. Initially, each student drew a number card (the base quantities) and the holder of the winning card got the difference of the cards' values added onto their score.

So far we are still dealing with natural numbers. However, once the students got the hang of the game, we just kept track of how much each player was winning or losing by instead of their actual score. Now the students were constructing integer quantities by keeping track of the directed difference of the numbers on the cards, as well as the sums of these directed differences.

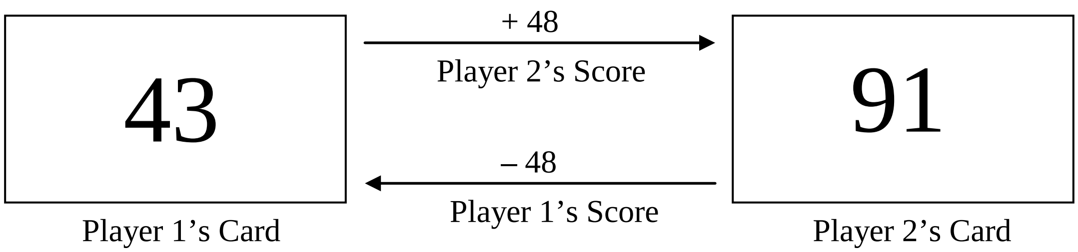


Figure 1.6. The card game.

So far I have discussed the new types of quantities that can be constructed in a signed quantity context, namely, additive transformations of (oftentimes continuous) quantities and additively derived quantities (often differences). Now I will switch to discussing how we might reconceptualize addition and subtraction when working with these new quantities.

New Addition and Subtraction Concepts

In dealing with changes in quantity, addition becomes a composition of transformations (Thompson & Dreyfus, 1988; Vergnaud, 1982). In particular, if I am focusing on situations in which signed quantities are referring to changes in quantity, students will need to construct an addition scheme that assimilates compositions of transformations as addition. With signed quantities in general, and with quantities that represent change in particular, subtraction is defined with respect to addition. In fact, there are two ways in which subtraction is generally defined. The first is subtraction as the determination of a missing addend, which I will call MAS. MAS is equivalent to thinking of subtraction as finding the directed distance between two values or how to get from one value to another. The second definition equates subtraction with addition of the additive inverse of the subtrahend, which I will call AAIS.

Both of these characterizations contrast with the other definition of subtraction, “take-away” subtraction, which we often use when dealing with unsigned quantities. Subtraction-as-taking-away is useful in many situations, but is not as useful when working with signed quantities or when working with more general vectors. Consider, for example, the frequently encountered use of subtraction to denote the distance between positions (here the absolute value of a difference is the usual notation). There is no intuitive sense of anything being removed in this situation. It is true that “take-away” situations with natural numbers can be fairly easily reformulated as situations of a decrease, so that students may be able to assimilate situations with a downward movement, a decrease, etc., as subtraction situations, but this reformulation of “take-away” subtraction still would not be adequate for making sense of all subtraction situations involving a negative number.

Briefly, I will give an example of MAS and AAIS in two dimensions. I use two dimensions here because it is easier to illustrate, I did not use two-dimensional situations with my participants. Note that given the graphical representation of two two-dimensional vectors, **a** and **b**, **a – b** generally represents one of two different vectors. The first possibility is that **a – b** represents the vector that is composed with **b** to result in **a**, represented by putting **a** and **b** “tail-to-tail” and then drawing an arrow from the endpoint of **b** to the endpoint of **a**, as in the left-hand drawing in Figure 1.7. The other possibility is that **a – b** represents the composition of **a** and the additive inverse of **b**, represented by the arrow formed by drawing **a** and **–b** (**b** pointing in the opposite direction) “head-to-tail” and then connecting the tail of **a** to the head of **–b**, as in the right-hand drawing in Figure 1.7.

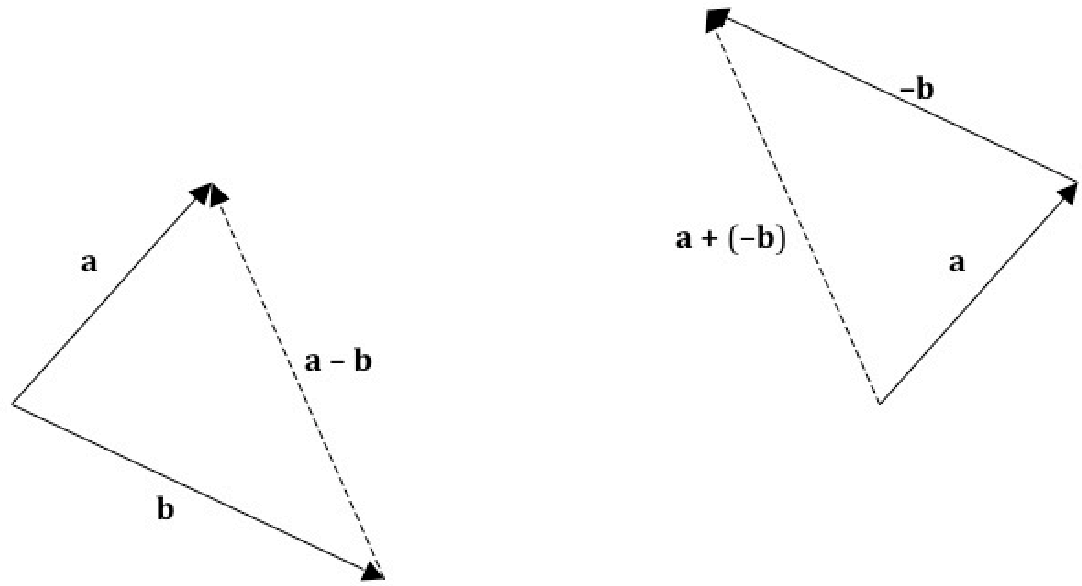


Figure 1.7. MAS and AAIS example.

These two vectors are isomorphic, but they come out of different characterizations of subtraction: MAS and AAIS, respectively. Using Figure 1.7, the recognition of a logical necessity that the two vectors always be isomorphic could follow from the application of various geometric theorems. For example, I could use the fact that any quadrilateral with two pairs of congruent sides is a parallelogram in order to put the two figures in Figure 1.7 together to form Figure 1.8. Then the congruence of **a + (-b)** and **a – b** follows from the fact that they both describe the same directed length of the diagonal of the parallelogram.

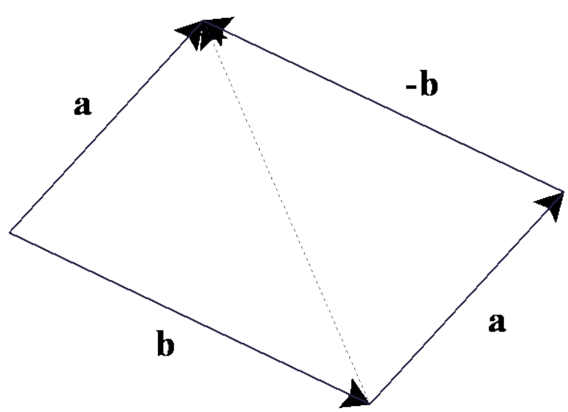


Figure 1.8. Congruence of **a + (-b)** and **a – b**.

Regardless of the method used to establish the logical necessity of isomorphism, it is not immediately obvious for the learner. In a one-dimensional situation, like the number line, the corresponding recognition that MAS and AAIS interpretations result in the same answer to a problem is equally, if not more, profound. In the one-dimensional situation (assuming the learner is unfamiliar with the two-dimensional situation), the recognition that the magnitudes are the same would seem to follow from a decomposition of the subtrahend, minuend or difference, depending on which of these has the largest magnitude, but then an additional analysis would be necessary to recognize that the directions of the differences would always match up. In fact, my own sense of logical necessity derived from breaking up all possible subtraction problems into several cases (the signs of the minuend and the subtrahend are the same or different, if the same, I have different cases depending on which magnitude is larger). Hence, the logical necessity of MAS and AAIS resulting in the same difference is less intuitively obvious for me in the one-dimensional than the two-dimensional case. I assume the logical necessity would be at least as unclear to my students.

In many models of integer subtraction, a subtraction sign is taken to be equivalent to an addition sign followed by the additive inverse of the subtrahend (AAIS). This view of subtraction more easily matches up with AAIS in that a take-away situation can often be reconceptualized as the composition of a positive change in quantity and a negative change in quantity, i.e., it can be reconceptualized as an addition of an additive inverse. For example, “If Verna has 9 apples and eats 3, how many are left?” is certainly a “take away” situation. It could also be thought of as a (prior) gain of 9 apples followed by a loss of three apples with a resulting net gain of 6 apples. Bob Moses’s Algebra Project was the only place I found subtraction foundationally characterized as MAS (Moses, Kamii, Swap & Howard, 1989). In the Algebra Project curriculum, the context was subway trips and integers represented the directed difference from one stop to another.

Regardless of which meaning of subtraction is taken as basic, we need students to be able to use them interchangeably. Therefore, students should have experiences with both characterizations of subtraction, and, ultimately, we would want students to develop a sense of why MAS and AAIS are equivalent. This equivalence is tied into a larger realization that subtraction situations and addition situations are both additive in nature. That is, the underlying additive relationship in either type of problem can be conceptualized as addition or subtraction, just as a natural number subtraction problem can be easily mapped to a missing addend situation and vice versa.

Research Questions

In the current study, I examine students' ways of operating in various additive situations in order to gain a better understanding of difficulties they might encounter in their path to constructing sophisticated schemes and operations when adding and subtracting signed quantities. In particular, I answer the following questions with respect to the current study:

1. What difficulties do students encounter in situations of signed addition?
2. What aspects of the *mathematics of* each participant, including schemes for unsigned quantities, impede or facilitate the participant's ability to work in situations of signed addition?
3. What aspects of the social interactions among participants and the teacher/researchers impede or facilitate participants' ability to work in situations of signed addition and the

researchers' ability to construct the *mathematics of* each participant in these mathematical contexts?

4. What, if any, changes are there in how the participants assimilate situations of signed addition?

CHAPTER 2

THEORETICAL AND CONCEPTUAL FRAMEWORKS

My general goal as a mathematics education researcher is to characterize ways in which students learn in order to improve teaching efficacy. My point of departure in this endeavor is the epistemological theory of radical constructivism (e.g., von Glasersfeld, 1995). I have chosen this framework because I think it best reflects the assumptions I am willing to work with. In the first part of this chapter I will discuss my theoretical framework, radical constructivism, before going on to discuss my conceptual framework[3]. I will start by elaborating on the summary of radical constructivism in "Foundational Theory and Concepts" in Chapter 1. I will then discuss the relationship of radical constructivism with Rorty's pragmatism and post-structuralist ideas of the subject and rationality.

Radical Constructivism

Radical constructivism is an epistemological theory developed by Ernst von Glasersfeld and influenced, in part, by Piaget's (1970) genetic epistemology and Ceccato's views of mental operations (Thompson & Saldanha, 2000). It has been applied to mathematics learning by many researchers including von Glasersfeld himself, who wrote about, among other things, the formation and modifications of a unitizing operation that is behind both object formation and constructions of numerical units (von Glasersfeld, 1981). In the radical constructivist theory of learning, the term *knowledge* would refer to a learner's attempts to organize experience: Knowledge represents "any coordination or association of individually recurring elements of

[3] I use the phrase *conceptual framework* to refer to theoretical constructs that inform the way that I construct my data and analysis.

experience that an organism uses in the ordering and systematization of its experience" (von Glasersfeld, 1979, p. 117). This knowledge is not thought to reflect Reality, but to provide a good fit within the learner's experiential reality. That is, the knowledge is a viable and useful way of organizing experience. Because knowledge does not reflect Reality, theories generated from a radical constructivist viewpoint do not make a claim of truth, but of viability. Hence a theory can be considered viable insofar as it does not contradict the researchers' experience. In addition to viability, radical constructivist researchers aim for intersubjectivity (Steffe & Thompson, 2000a): *Intersubjectivity* refers to this mutual acceptance of the theory as being viable with respect to each person's experience.

When I use the term *experience*, I refer to the physical and social environment the learner experiences as a result of his or her constructions, not a transcendent experience of an objective Reality. In fact, a learner's constructions of experiential reality and knowledge are constantly affecting each other: Interactions within the experiential reality engender modifications in knowledge that, in turn, allow new ways of organizing the experience. For example, at some point in my life, I developed a counting scheme, which is a knowledge structure that allows me to count groups of objects. My reality was thereby changed to include the experience of counting groups of objects. Repeated experience of counting sets of three objects, say, allows me to abstract the concept of *three-ness*, which is a new knowledge structure.

First- and Second-Order Models

A learner's knowledge structures constitute *first-order models*: "the models the observed subject constructs to order, comprehend, and control his or her experience" (Steffe, 1996, p. 91). These first-order models are what I refer to when I use terminology such as *my mathematics* or *a student's mathematics*, following Steffe's (2010e) precedent. When a student makes a statement, for example, “The capital of Texas is Austin,” that statement might activate some of my mental constructs. In this case, the student’s statement might provoke me to visualize a *re-presentation* (von Glasersfeld, 1995) of a map of Texas and a dot on that re-presented map representing Austin. My awareness of my ability to recreate Austin’s relationship to other cities and Texas’ relationship to other states and countries may be inherent in my re-presentation. However, I cannot impute any of these visualizations or geographical schemes to the speaker. All I can conclude is that the speaker has learned how to utter the phonemes making up the sentence. Similarly with any statement or action on the part of the learner, I can never truly assess what knowledge a given observable behavior reflects. The only assumption I start with is that these *first-order models* of the speaker are a construct that forms part of my reality, which I can form theories about, but never experience in a transcendental way.

Any theories I form about the speaker’s first-order models represent *second-order models* of the speaker’s experiential reality. Within mathematics education research, second-order models are formed by the teacher/researcher to explain and predict student behavior: “They are necessarily constructed through social interaction and are the models of primary concern in mathematics education” (Steffe, 1996, p. 91). These second-order models are clearly influenced by the first-order models of the student, but they are also contingent on the first-order models of the teacher/researcher. This happens in several ways: The teacher/researcher can only impute ways of operating to the student that exist within the realm of the researcher’s mathematics, hence a limited understanding of the mathematics at hand will greatly impede the construction of viable second-order models. In addition, we tend to impute ways of operating to others that mirror our favored ways of operating, so our second-order models will inevitably bear some resemblance to our first-order models. I am referring to a second-order model when I use phrases such as *mathematics of a student*, following Steffe and Cobb (1988; Steffe, 2010e). In this study, my goal is to construct second-order models of student's knowledge structures involving additive relationships between signed quantities.

Claims to Truth

Some people consider, explicitly or otherwise, mathematics to exist apart from human construction. Two well-known scholars, centuries apart, who put forth this view are Plato (*Republic*, Book VII, Jowett ed.) and Paul Erdős. Both Plato and Erdős were realists in the sense that they believed that we could gain partial access to reality through mathematical and/or philosophical pursuits. Plato held that mathematics points us to, or is the reflection of, a perfect form in Reality. With regards to number theory, he writes that "this knowledge may be truly called necessary, necessitating as it clearly does the use of the pure intelligence in the attainment of pure truth" (526), making explicit his belief in mathematics as leading to a transcendental truth. Similarly, he writes of plane geometry, "the knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient" (527). The great mathematician, Paul Erdős, would often refer to *The Book,* a book containing the most elegant proof of each theorem, held by God. He famously said that "You don't have to believe in God…but you should believe in *The Book*" (Erdős quoted in Babai & Spencer, 1998, p. 66). Regardless of whether Erdős actually believed *The Book* exists, he and others clearly see mathematics as transcendent. As Joel Spencer writes of Erdős's view of mathematics, "This platonic ideal spoke strongly to those of us in his circle. The mathematics was there; we had only to discover it" (Babai & Spencer, 1998, p. 65).

As a radical constructivist, I do not deny the existence of a transcendent reality. However, I do not believe that we have direct access to it, or even that our first-order models of our experiences reflects or resembles reality. Instead, I can claim the viability of my theoretical models in that they do not contradict my experience (see Chapter 1, “Foundational Theory and Concepts”). The viability of second-order models, the mathematics of students, is tested by thinking through what implications for student behavior there would be if a model were a match for the student’s mathematics and then making and testing hypotheses based on those implications. Hence, any claims I make to truth are claims of viability and not of correspondence to reality.

This idea of truth echoes William James’s (1907/1995) discussion of truth and pragmatism. He judged truth based on the practical consequences there would be from taking a statement as true. Truth is what is useful. Substituting *viability* for *utility*, the pragmatic idea of truth is essentially the same as the idea in radical constructivism. The substitution does seem necessary, though, in that viability in radical constructivism is weaker than the utility of pragmatism because it is negatively determined as that which is not not-useful. Nonetheless, pragmatists, like radical constructivists are interested in truth as a tool, not a correspondence to Reality.

Any such idea of truth does have its critics. A usual charge is that it is relativistic. For example, Burr (2007) discusses Andrew Collier’s critique coming from the perspective of critical realism. He differentiates between open practices, which open themselves up to testing against reality, and closed practices, which make up their own sense of reality and therefore cannot be questioned. He argues that pragmatism is a closed practice, thereby implying that critical realism is preferable. However, I think that Collier draws a false dichotomy. In fact, from a radical constructivist perspective, any practice assumes an initial vision of reality and creates new aspects of reality, meaning that all practice is closed in some sense. Yet practices within the tradition of any of the three theories in questions—critical realism, pragmatism and radical constructivism—are all open to change based on interaction with our experienced reality. As Steffe has said, although you construct your own reality, there are aspects of that constructed reality that “kick back” at you (Steffe, 2010d). The construction of reality does not proceed haphazardly: Mirroring Piaget’s (1964) four factors that contribute to development, there are biological, physical, social, and rational factors that influence our constructions. In other words, both the utility and viability of a practice are dependent on the learner’s experiences of reality.

The Nature of the Learner

As in Collier’s critique of pragmatism as a closed practice, one could wonder if the assumption that individual constructions of reality are all distinct and inaccessible to others would imply the impossibility of communication or a unified scientific project. That is, if our individual constructions are not becoming more like reality, how do we know there is sufficient convergence to communicate? However, as I wrote in the previous section, our constructions of reality are influenced by biological, physical, social, and rational factors. Given that we can communicate with each other, and, not only that, but similar patterns of behavior emerge (cf. Steffe & Cobb’s 1988 account of five patterns of counting behavior), I can only conclude that these influences are enough to allow intersubjective interaction. In fact, I agree with the position that these influences, particularly the social influence of discourses, do not allow for as much agency as we tend to attribute to the learner (e.g., Burr, 2007; Marcuse, 1964; Spivak, 1994; St. Pierre, 2000). Later, I will discuss how this limited agency is connected to the (ir)rationality of the learner. I will now discuss the four factors in shaping our constructions, including my characterization of *discourse*.

For me, *discourse* is constituted by the following three aspects: The first is Maturana's (1988) domain of explanation and rationality. Using this perspective, a discourse would include a set of assumptions, i.e., arguments that can be pointed to without explanation, as well as rules of rational argumentation, i.e., an accepted way of reasoning. The second aspect of discourse is a set of connotations of language, including gestural language. Referring to meaning-laden units of language as *signs* (Derrida, 1967/1978), the connotations of signs are based on reference to past usage and formation. Therefore, in any discourse, there is a self-referential quality to language that augments the definition of signs that would be intersubjective with other discourses. The third aspect of discourse is fluidity: A given discourse is not static and is constantly subject to the effects of subversive repetition (Butler, 1990) and the effects of related discourses.

As an example of discourse, consider the discourses around gender. DeBeauvoir (1952) discusses how we (in modern, Western society) view males as the normative sex and females as the "marked" sex, a special case of humanity. I will call this the *normative male* discourse. This discourse is evident in the debates in the last century about whether women should be able to vote and work. The male's right to do so is assumed. Similarly, she notes how older medical studies would only examine male subjects, assuming that men were representative of everyone. One example she gives that is still active in the American culture's discourse around gender is that women are "hormonal," i.e., that their behavior is controlled by hormones. In fact, as deBeauvoir points out, men have glands that secrete hormones as well. And modern neuroscience argues that these hormones do, in fact, affect their behavior (e.g., Sapolsky, 2005). However, women's hormones are seen as irregular, even though there is no reason to take the male hormone levels and cycles as normative; we could just as well take the female hormone levels and cycles as normative and considering males to be the abnormal beings.

Of course, there are other discourses around gender (cf. Butler, 1990), such as the discourse that gender is socially constructed as opposed to biologically determined, which both overlaps (assuming that there are distinct male and female behaviors) with the *normative male* discourse and clashes with it (assuming that physical make-up does not determine gender and, therefore, assumed rights). The important point is that these discourses, particularly the *normative male* discourse, are not something we consciously choose or reject. It is inherent in our language (the use of *he* as a generic personal pronoun being one of the most striking examples) and our customs (women's surnames are defined by men: her father and her husband). We run up against these language uses and customs from birth, and they both affect our constructions of reality, and themselves become reinforced in our own repetition of these languages and customs. In this way, I can speak of discourses as social factors *as if* they exist outside of the individual, although they are in fact co-constructed by the individual and others in the discursive community: They are dependent on social interactions (including interactions with social signs, such as writing) and draw their meaning from our interpretation of interactions with other people. In mathematics education, discourses include the norms for argumentation in a classroom or assumptions about the nature of mathematics.

Piaget, Inhelder and Szeminska's (1960) book, *The Child's Conception of Geometry*, details ways in which a toddler constructs space, and, indeed, I think of space and other aspects of the "physical world" as being constructed. However, these constructions are certainly limited by viability. In fact, our ability to survive as organisms depends on viable ways of maneuvering that which "kicks back" at us in our experience of the physical world. In mathematics, early conceptions of *number*, *more than*, *less than*, *addition* and *subtraction* are often tied to physical objects that have been constructed as countable objects by the child. Hence there can be unexpected physical outcomes of a child's solution strategy that will induce the child to modify some aspect of the scheme involved.

Maturana (1988) and Piaget (1964, 1970, 2001) both have backgrounds as biologists, so it is perhaps unsurprising that they both emphasize biological considerations when discussing human constructions. In particular, both start from the assumption that we are self-regulating organisms. This self-regulation is a large part of what drives the modification of schemes in Piaget's theory. Maturana, for his part, is explicit in his assumption that our biological make-up, both genotype and phenotype, will affect how we experience reality and are experienced by others: He often emphasizes that we humans are animals and living systems. Biological influences on individual constructions include our physical mechanisms for experiencing sensory data, what von Glasersfeld (1995) calls the *manifold*. In other words, we use sight, touch, taste, etc., to experience the world, even though, like social influences, our senses are affected by our mental and physical actions. Using an example from Chapter 1, once we have developed object concepts, we experience visual data as including objects, whereas visual data would have initially been experienced as amorphous by babies.

I believe that these biological influences also include developmental influences on constructions. For example, the nerves in our brains are undergoing a process of myelination from before birth until the process is completed, around the age of 20 (Aslin & Schlaggar, 2006; Sapolsky, 2005). This has behavioral implications: The pre-frontal cortex, thought to be used in impulse control, is the last part of the brain to be completely myelinated. That is thought to account, in part, for the lag in teenagers' development of judgment and impulse control when compared with a relatively strong intellectual development. However, there are also possible educational implications of the myelination process. For example, myelination of the regions of the brain thought to be involved in language use reaches its median level at around 18 months of age (as compared with the 6 months of age for the median of myelination in the brain in general), which has been hypothesized as the impetus for the "commonly observed 'spurt' in vocabulary growth that emerges around 18 to 24 months of age" (see Aslin & Schlaggar for a more detailed discussion and further references, quote on p. 304). Similarly, I suspect that there are similar biological structures that mature to provide a necessary, but not sufficient, basis for the development of various mathematical schemes and operations. I say this because there seem to be aspects of students' mathematics that are not viable in the eyes of the student or teacher, but that cannot be overcome by any clever sequencing of curricular activities (cf. Steffe & Olive, 2010).

The idea of learners as self-regulating organisms also informs my assertion that there are rational influences to our constructions. By rational influences, I mean our desire to reconcile internal contradictions in our constructions when such contradictions come to light. However, I believe that, for the most part, we seek this coherence within the discourses we are drawing upon at the moment. This leaves room for a lack coherency between discourses. Maturana refers to the learner as the observer and he makes the following claims about rationality:

> The operational coherences of the observer that constitute reason are the operational coherences of the observer in his or her praxis of living in language….therefore, rationality is not a property of the observer that allows him or her to know something that exists independently of what he or she does, but it is the operation of the observer according to the operational coherences of languaging in a particular domain of reality. And, accordingly, there are as many domains of rationality as there are domains of reality brought forth by the observer in his or her praxis of living as such….every rational system is founded on non-rational premises, and it is enough to specify some initial elements that through their properties specify a domain of operational coherences to specify a rational domain. (p. 42)

In other words, beyond a possible lack of coherence across "domains of reality," I am also aware that the assumptions that are acceptable within a given discourse are not necessarily rational.

Therefore, I do not think we, or students, are rational in the sense that we adhere to a transcendental system of logic. Of course, a sixth-grade student, such as one of my participants, would already have certain well-established discourses that they use to guide and make sense of their interactions in a formal mathematical setting. Therefore, students might not be experiencing different domains of reality in the context of a single teaching episode, implying that I can assume a certain level of internal coherence to their ways of reasoning. On the other hand, mathematics educators see time and again that students will behave very differently in different mathematical contexts, to the extent that answers will end up contradicting each other in the eyes of the observer. This often happens, in particular, when students use memorized procedures that they did not develop themselves in one situation and use informal reasoning in another.

Conceptual Constructs

I have finished my general discussion of my theoretical framework. Now I will introduce concepts from my conceptual framework that helped me to formulate my research questions, analyze my data, and describe my findings.

Schemes

One of the theoretical constructs I will use to describe my second-order models is an elaboration on Piaget’s (1977/2001) *scheme*. Schemes are traditionally described as being comprised of a stimulus, action and anticipated result. To understand how schemes might come to be developed in an individual it is important to understand some basic mental operations. From birth, and before, we receive input from nerves (tactile, olfactory, visual, auditory, gustatory, kinesthetic) that is analogous to raw data; it is filtered only in the sense that we have limited physical capabilities of receiving this data. This data leads to an originally undifferentiated mass of experience (von Glasersfeld, 1981). Immediately, the individual begins to structure this experience, attending to some stimuli more than others, eventually leading to groupings of pieces of experience through the unitizing operation. Von Glasersfeld (1981) elaborates the development of the unitizing operation from attentional pulses that he sees as an innate facet of sensory processing in the brain. The beginning and ending of these attentional pulses will cut off pieces of experience, and, eventually, the beginning and ending of a series of attentional pulses will serve to unitize a piece of experience. Once a piece of experience has been set apart in this manner, the opportunity opens for a recognition template (Steffe & Olive, 2010), to be formed. A recognition template is activated when a certain kind of stimulus is experienced (attended to) repeatedly. The mechanism for the formation of this recognition template is currently unelaborated. I think of a recognition template as a precursor of a concept or abstraction.

Assimilation and accommodation. The first stage in a scheme is assimilation of a stimulus for the scheme. Assimilation can be thought of as stimulation of a recognition template for a particular configuration of experience. For example, on seeing a set of like discrete objects, such as a collection of marbles, a child may set off a recognition template for "countable items" that would serve as a stimulus to start the action of counting. Note that something is not a stimulus unless it activates the individual's recognition template for a situation of that particular scheme being activated. Therefore, a given problem situation in a mathematics classroom may stimulate different mathematical schemes for different students or may fail to stimulate any mathematical schemes for some students. As a mathematics teacher, I often experienced this when introducing a contextualized problem situation to students.

Accommodation refers to a revision of a scheme. The most basic form of an accommodation is a generalizing assimilation, in which the learner enlarges the domain of stimuli for a scheme. For example, a student could have a counting scheme for discrete objects and recognize a connected but segmented item (like bricks on a wall) as a new place that the counting scheme can be deployed. This would be a generalizing assimilation of the child's counting scheme. Accommodations can also involve modifying actions of a scheme. For example, a student may have a scheme for adding n items that consists of counting up n (more) number words in the number sequence. The student may modify the action of the scheme if n is bigger than 10 by first using number word patterns to count by 10 b times, where b is a counting number and $n = b \cdot 10 + m$, and then counting up m more number words, thereby reducing the time needed to count up. These kinds of accommodations can happen based on imitation or can happen spontaneously based on the student's composition of skip counting operations, operations for decomposing numbers into decimal formation, and the student's previous counting up scheme. This spontaneous reorganization of a scheme usually requires some level of reflection, as described below.

The development of schemes is dependent on the learner's ability to abstract out from experience what will become the stimulus of a scheme—common aspects of situations that allow for the scheme action—and what will become the anticipated result of the scheme— common results of the scheme's action. Recognition templates contain the records of these abstractions. In addition, Steffe and Cobb (1988) describe two processes that are important parts of larger reorganizations of schemes: *internalization* and *interiorization*. In their discussion they also utilize von Glasersfeld's (1991) idea of re-presentation. All of these concepts are further elaborations of the reflecting abstraction of Piaget (1977/2001). First, I will give an example of internalization and interiorization.

Mechanisms of reflecting abstraction. The action of a child's early counting scheme involves the recitation of a number word sequence. Initially the sequence can be spoken, but the child is not explicitly aware of the makeup of the sequence. The utterance of one number word sets off the utterance of the next number word in the sequence, but the child would not be able to count backwards or tell you which words come before and after another without running through the sequence again from 1. We have all experienced this with songs, poems, etc., that we have memorized through repeated experience. Those experiences give an intuitive feeling for what it means to carry out a scheme's action without being aware of its structure, which, in the case of the early counting scheme, is something the child will eventually need to develop an awareness of. This awareness is brought about in two steps, internalization and interiorization.

The *internalization* of an action is the ability to re-present the action in visualized imagination. In this case, a child may have a counting scheme whose action is to point to each object in a set of objects, accompanying each pointing act with the utterance of a word in the number word sequence. The child may eventually be able to re-present these acts in visualized imagination, like a movie playing. With regards to a middle school student's behavior, I often find that a student seems more aware of how they are operating once they have internalized the action of a given scheme. They can usually give clearer explanations of what they are doing and why, for instance, then a student who has not internalized the action of the scheme. Re-presentation (von Glasersfeld, 1991), allowed by internalization, opens up the possibility of turning one's attention to this re-presentation and taking it as an object to operate on. In the case of the counting child, the child may able to imagine counting backwards or isolating and counting a certain segment of the counting acts (when keeping track of "how many more," for example). This is the *interiorization* of the activity of counting (Steffe & Cobb, 1988; Steffe & Olive, 2010).

The interiorization described above is the result of numerous reflective abstractions that a child engages in while constructing her or his number sequence. I am using the term *reflective abstraction* similarly to, but more broadly than, Campbell uses *reflecting abstraction* in his translation of Piaget (1977/2001), and my conception is a descendant of Piaget's. The term *reflective abstraction* is currently used by researchers (e.g., Steffe & Olive, 2010) and, I believe, better captures the connotation of awareness of structure. Nonetheless, the term can be misleading since every act of reflective abstraction does not require what would normally be called reflection, nor does it require an explicit awareness of structure. For Piaget, a reflecting abstraction consists of taking an action and "reflecting" it up onto a "higher plane" of operations in which action itself is now the raw material available for operating upon. I characterize a reflective abstraction as the taking of a mental or physical action and simultaneously setting it apart and turning attention to it in order to operate upon it. Interiorization, then, would be made possible by this kind of reflective abstraction. When I refer to *mathematical objects,* I mean the result of an interiorized scheme that points to the action of the scheme without having to carry it out. So, for example, *five* would be a mathematical object once it represented both the outcome of counting 5 objects and it pointed to the counting activity without needing to carry it out.

Levels of Units

An important construct for describing the mathematics of students is *levels of units* (cf. Hackenberg, 2010). In the following discussion, I use examples to clarify what constructing or assimilating with various numbers of levels of units means. Before giving examples, I will describe the order in which and rate at which people develop the ability to work with multiple levels of units.

General ranges of unit structures. A person can construct zero, one, two, or three levels of units and can assimilate with zero, one, two, or three levels of units. Some students, even in elementary school, can deal with problem situations involving more than three levels of units (cf. Olive & Steffe, 2010), which can be explained through the recursive use of a three-levels-of-units structure (and often notation). I will not go into my analysis of this recursive application because it is not central to my analysis in this study. Currently, no one hypothesizes that any research subjects work concurrently with four levels of units.

The ability to work with levels of units develops in the expected way: A child starts without any (mathematical) unit constructions, develops one level of units, starts assimilating with that one level of units, begins operating on those units to construct two levels of units, starts assimilating with the two levels of units, operates on them to form three levels of units, and, ultimately, begins assimilating with three levels of units. By middle school, the range of unit structures in a regular (as opposed to a special education resource or self-contained) mathematics classroom probably starts with students who can assimilate with one levels of units and cannot yet construct two levels of units. Hackenberg (2010) documents encountering four such students in selection interviews. In all, her research team interviewed 20 sixth-graders, about half of the sixth-graders at the research site, and she did not encounter any students who could not assimilate with one level of units. In fact, as is clear from my descriptions below of how students who have yet to assimilate with one level of students can operate, such students would have noticeable difficulty in the middle grades mathematics classroom, and so I would guess that such students would be placed in a pull-out resource or remedial class. Several researchers have reported to me that they have worked with college students who they are confident cannot assimilate with three levels of units (A. J. Hackenberg, A. Norton, L. P. Steffe, E. Tillema, personal communication), so there is probably great variation in the development of levels of units even in the adult population.

Constructing one level of units. When I say that a child has constructed *n* levels of units, I mean the child can be simultaneously aware of *n* quantities and the quantitative relationships that link them. If the child needs perceptual or experiential aids in order to do this, then I would not attribute the same levels-of-units structure. At birth, a child has not constructed any units. As von Glasersfeld (1981) pointed out, a child will have started using the *unitizing operation* very early on in life, for example, in order to construct object concepts. However, using the unitizing operation is not the same as creating one level of units. In particular, the child will at first apply the unitizing operation only to perceptual material. Also, the results of early unitizing are not mathematical in that they are not necessarily involved in producing the concept of numerosity or counting. Instead, the early application of the unitizing operation results in what can call object concepts. The stage at which I am willing to say that a child can construct *one level of units* in action is when the child can construct *arithmetical unit items*.

Both Steffe (2010b) and von Glasersfeld (1981) describe a type of unit that the child would have constructed prior to the arithmetical unit, namely, the *perceptual unit item*. In the case of children who are limited to the construction of *perceptual unit items*, the children need to actually see items in order to count them, i.e., they require perceptual aids in order to create units. The next type of unit items both discuss are *figurative unit items*. Children who are limited to the construction of figurative unit items can create mentally re-presented countable units, but the re-presentation of figurative unit items must be activated by a perceptual situation. Therefore, I do not yet consider this to reflect *one level of units*. They will often use substitutes, such as fingers or patterns, in order to count their re-presentations, but they are mentally producing a re-presentation. It is important to point out that children are still pre-numerical at this stage. They are not aware of the quantitative property of numerosity before counting. Instead, "the child is aware of figurative plurality, a child at this level can give meaning to a number word by producing visualizing [sic] images of hidden items. This extensive meaning of the number word remains indefinite, and to make it definite, the child has to actually count" (Steffe, 2010b, p. 34).

When student begins re-processing figurative unit items to form arithmetical units, then the child has constructed one level of units. In order for the student to construct arithmetical units, the student must abstract out the counting act from the specific re-presented perceptual material that is being counted, i.e., the student goes from counting re-presentations of 5 cookies to being aware of counting 5 times. Steffe (2010b) gives an example of how this can happen through an attempt to monitor the counting of figurative unit items. In the example, a student named Jason[4] is attempting to count up the union of a hidden collection of 8 cookies and a hidden collection of 10 cookies. He counts as he makes 8 pointing actions, presumably pointing at re-presented figural unit items, and then monitors himself counting 10 more figural unit items by forming a visual pattern, called a *numerical pattern*, for 10 as he continues counting and pointing. In monitoring counting action, the child is taking their counting of figurative unit items as an object of reflection, which leads to *interiorization* of counting acts, but this is happening in action: the child is aware of counting *while* counting. Before the child can assimilate with one level of units, the arithmetical unit items "decay" in that the arithmetical units that were formed during counting may no longer be available for reflection after counting.

[4] Student names are pseudonyms throughout all chapters.

Assimilating with one level of units. Once the arithmetical units no longer decay, and the child is aware of the a numerosity of counting acts *before engaging in counting activity*, then the child is assimilating with arithmetical units. Pinpointing the precise time when students go from assimilating situations with figurative unit items and constructing arithmetical units in action to assimilating situations with arithmetical units is quite hard using behavioral indicators because, in either case, the students are able to form countable items so that they do not have to count re-presented *perceptual* unit items in order to make sense of a number word, For example, Steffe gives examples of two students who count the union of two covered collections, but attributes the use of figurative unit items to one and arithmetic unit items to the other based on subtle differences in behavior (Steffe & Cobb, 1988). As in the example in the previous subsection, monitoring is an important instigator of this re-processing. Monitoring behavior involves a disembedding behavior in which a sequence of counting acts can be experientially bounded and monitored. As with the formation of arithmetic units, monitoring can lead to interiorization of segments of counting acts. The interiorization of the counting acts allows the child "run through a counting activity and produce its results in thought without motor action and without given sensory material to act on" (Steffe, 2010b, p. 35), resulting in an INS (see "Number Sequences") in which students can *count on* instead of *counting all*. However, composite units are generally experientially bounded and not attentionally bounded by the student. That is, the student has not yet applied the unitizing operation to these experiential composite units to form a second level of unit.

Two levels of units. The unitization of these experiential composite units corresponds to the beginning stages of Steffe's tacitly nested number sequence (TNS), which I describe in a later section. In a fractional (to the observer) situation, this could involve fragmenting a unit into equal-sized sub-units. In the fractional situation, the containing unit does not need to be unitized, since it was the starting point for the construction. However, in either case, the student must be able to attend to both levels of units simultaneously. That is, once the fractional whole has been fragmented, the student must be able to mentally retain the whole as made up of the fragmented segments. When a student cannot yet construct two levels of units, the student might have trouble coordinating the number and size of fragments if attempting to fragment the whole into a certain number of equal-sized segments (Steffe, 2010a).

While still in the TNS stage, students can begin to operate on their two-levels-of-units structures, i.e., assimilate with two levels of units. One consequence of this is an ability to use composite units as input for creating counting acts. Previously, students could *monitor* their counting acts using numerical composites, but now the student can count using the elements of their numerical composite as a template. If a student were trying to solve an addition problem, such as 8 + 6, the student who cannot assimilate with two levels of units could count, “8, (pause) 9, 10, 11, 12, 13, 14,” and use a numerical pattern to intuitively keep track of their continued counting. That student would become aware of the six-ness of the continued counting segment when they get to 14. However, a student who could assimilate with two levels of units would be aware of the six-ness of the continued counting segment before beginning their counting on activity.

Three levels of units. Also within the TNS stage, a student can begin constructing three levels of units. I now give an example to illustrate the different kind of thought processes these students would have as opposed to a student who could not yet construct three levels of units: If a student who can assimilate with two levels of units (but not three) were trying to figure out how many piles of 4 pennies can be made from 28 pennies (with no actual pennies present), the student would be aware of two kinds of units before operating. The student would understand the problem situation in terms of a number of units of 1 (28), and a number of units of 4 (an as yet indefinite number of piles). The student might then mentally add piles of 4 together, keeping track of the number of piles added so far and the accumulated number of pennies, until the seventh pile gives the goal of 28 pennies.

The difference between the students who can only work with two levels of units and those that can work with three is found in the nature of their thought processes when “keeping track” of the repeated additions. The students working with two levels of units would use their number sequence to *monitor* the repeated additions. However, the student who could construct three levels of units could actually unitize the accumulated number of pennies and piles to form a three-level-of-unit structure that included a correspondence between the two number sequences being used, the monitoring sequence of 1’s and the sequence of 4’s that represents the total number of pennies. As these students counted up by 4, they could be explicitly aware that 1 pile results in 4 pennies, 2 piles results in 8 pennies, 3 piles results in 12 pennies, etc. In other words, the two-levels-of-units students would be limited to running up and down a sequence of 4’s, whereas the students constructing three levels of units would be aware of a coordinated sequence of 1’s. So if the student were then asked how many pennies there would be if six more piles were added, the student with two levels of units, would monitor counting up by 4, six more times. The students with three levels of units would have the possibility of recalling that six piles corresponded to 24 pennies and then adding 24 to 28. Note that this is only a possibility and that even if a student is constructing three levels of units in this context, the behavioral indicators may only be a more explicit awareness of the coordination between the number of piles and the number of pennies as opposed to any kind of strategic reasoning involving the number of piles to answer further questions.

In summary, a student who could construct three levels of units would be able to unitize the resulting seven groups of 4 to construct 28 as being made up of 7 groups of 4. This is not to say that they would visualize 28 pennies or counting acts, but that the records of counting by 4 would be contained in each of the 7 groups. The student who did not unitize the 7 would be aware of 7 piles after counting, and could focus on one pile and be aware of 4 pennies in the pile, but the student would not be aware of both quantities simultaneously to form a unit of units of units structure.

Once a student can construct three levels of units in action, the next, and final, step I will discuss is the possibility that the student will take the two-levels-of-units structure as an object of reflection and interiorize it to form a three-levels-of-units assimilating structure. Three levels of quantitatively related quantities and their quantitative relationships is a lot of quantitative information to analyze concurrently. That is probably why we do not see students develop the ability to assimilate with three levels of units before they have abstracted out an *iterable unit* from the units of their TNS. The units in a TNS still contain records of counting and are seen as equivalent, in that each represents one counting act, but are not seen as identical: There is still a sense of sequentiality in a TNS student's concept of composite unit, for example. If a student abstracts out the "sameness" in a series of arithmetical units, the units becomes interchangeable. Instead of having a sense of counting to *n*, the student would now have an awareness of the co-occurrence of *n* identical units. This allows the student to further curtail the operations needed to assimilate a composite unit. The student can be merely aware of one unit that is iterated a bunch of times instead of being aware of a sequence of distinct.

The ability to use three levels of units in assimilation means that the student can be aware of three levels of related quantities in a mathematical situation and be aware of quantitative relationships between them *before operating*. This means that the student can form a solution strategy that takes into account all three levels. In addition, I restrict attributing three levels of units as assimilatory until the student can assimilate re-presented material using three levels of units, without any visual aids, for example. Although the student who can construct, but not assimilate with, three levels of units is aware of these quantitative relationships *after acting*, such a student would not experience the statement of the problem situation in the same way as a student who can assimilate with three levels of units. For example, in the penny problem, a student assimilating with three levels of units would be aware of the multiplicative relationship between the total number of pennies, the number of pennies in a stack, and the number of stacks before doing any counting or computations. In general, an increased facility in working with multiplicative relationships is characteristic of such a student. I will discuss this further in the section, "Number sequences." In that section, I also point out that the assimilation with three levels of units implies the operations for constructing a GNS.

Ambiguity in interpretation. The levels of units a student is using cannot be detected from a mathematical analysis of the problem solved. This is evident from the fact that students can use multiple thought processes utilizing levels of units at differing levels of sophistication and get the same numerical answer. The quantitative relationships between levels of units must be embedded in the student's mental representation of the situation.

Even when a researcher is attempting to analyze the mathematics of the student, and not the problem, differences in behavior can be subtle. An example of this comes out of Steffe's work on fractions (Steffe & Olive, 2010): Two students had determined that a stick that was 6 units long had been iterated to form a stick that was 24 units long. They knew that there were 4 copies of the 6-stick in the 24-stick. Based on these facts, a teacher/researcher might suppose the student was aware of a multiplicative relationship between the 6-stick and 4-stick, which would imply the construction of three levels of units. However, the students were not able to recognize this as a situation of “one-fourth,” even though they did have a fractional meaning for *one-fourth*. Steffe’s interpretation of the situation is that the students were not able to disembed a copy of the 6-stick and use that as a reference quantity with which to form a multiplicative relationship to the 24-stick. In other words, the 24 had not been unitized by the students and so the students could not see the 24 as a whole made up of the units of 6.

Number Sequences

I use Steffe’s natural number schemes (Steffe & Cobb, 1988) to help me characterize student behavior. In particular, different participants utilize a tacitly nested number sequence (TNS), an explicitly nested number (ENS), and/or a generalized number sequence (GNS) at various points during data collection. I will describe all three of these number sequences. In addition, I will describe a less sophisticated number sequence, the initial number sequence (INS), in order to emphasize the relative power of my participants’ number sequences. The order of the number sequence constructions is stable and proceeds from INS to TNS to ENS to GNS. Within those larger schemes there are important mathematical objects and operations that play an important role in defining stages, many of which I have already discussed. These include the development of an iterable unit and the ability to use two or three levels of units in action and in assimilation.

Initial number sequence. A student has constructed an initial number sequence when their arithmetical units have been interiorized, giving a one-level-of-units assimilating structure.

In other words, the student can re-present counting activity and operate on this re-presentation. This implies that number words are assimilated by INS students as pointing to records of counting activity. Therefore, the student can use a single number word to stand in for the counting sequence from *one* up to that number, being aware of the potential to count from *one* without having to carry out the counting action in order to give meaning to the number word. For example, given 11 jellybeans and asked how many would result from adding 3 more, the student could simply count, “11 (pause), 12, 13, 14!” This is referred to as *counting on*. Another positive result of the interiorization of the number sequence is that the student is poised to act upon re-presentations of counting in order to form two levels of units in action. Once the student start doing this, I may refer to them as having constructed the tacitly nested number sequence.

Tacitly nested number sequence. Similarly to the relatively quick transition from a student being able to construct one level of unit in action (like Jason when he was monitoring his counting of figurative unit items) to being able to assimilate the that level of units, the transition from being able to construct two levels of units, i.e., construct a composite unit, to being able to assimilate situations with composite units happens quickly. Students at the level of the tacitly nested number sequence, are characterized by their ability to construct, and usually assimilate with, composite units. In addition, the name of the number sequence reflects that students start intuitively, or tacitly, monitoring composite subsets of their composite units, thereby constructing three levels of units in action. For example, given a subtraction problem, such as 17 – 8, the student is aware of the 8 inside the 17 and the goal of enumerating the remainder of the 17 counting sequence. However, because of the nature of units at the TNS stage, which are not yet at the level of abstraction of ENS units, the students at this level will not be able to assimilate with three levels of units. Furthermore, although a TNS student could construct a more explicit subset relationship, as in the case of 8 and 9 being constituent subsets of 17, *in action*, the student cannot disembed the subsets. So, for example, a student can answer a question such as "How many threes are in twelve?" by counting to twelve but keeping track, implicitly, of how many times a group of three number words are uttered: *one two THREE four five SIX seven eight NINE ten eleven TWELVE*. However, the student would not be able to disembed one of these composite units of three number words from the number sequence for twelve and become aware of the multiplicative relationship between the number of units in each sequence.

As I discussed in the subsection, "Two Levels of Units," earlier in this chapter, arithmetical unit items still contain records of counting, so that the number word *seven* consists of seven equivalent, yet distinct, units. Hence, even though the student would not have to actually re-present all seven units when assimilating a situation of *seven*, an awareness of *seven* would require an awareness of multiple distinct counting acts. Once students have constructed the explicitly nested number sequence, this is no longer the case.

Explicitly nested number sequence. An explicitly nested number sequence (ENS) consists of iterable units. As I discussed in an earlier section, "Assimilating With Three Levels of Units," the consequence of forming iterable units is that the information contained by a composite unit has been stripped down to its bare essentials: Whereas a TNS student might have a sense of eight sequential units evoked by the concept of eight, an ENS student would simply have a sense of one unit that is iterated eight times. Therefore, when an ENS student constructs three levels of units, each of the two levels of composite units is holding information about one iterated abstract unit item, whereas TNS students' composite units would imply the presence of all the items in the indicated section of their number sequence; numbers and numerical relationships that underlie the composite units have been further curtailed. This economy in reasoning with composite units is probably what allows students to reflect on the three-levels-of-units structures they have been constructing in order to interiorize them and form a three-levels-of-units assimilatory structure.

The construction of an iterable unit is particularly powerful in fractional situations, where unit fractions can now derive their meaning from the ability to recreate the whole through iteration. The iterability of fractional units is apparent in how students work with fractions. For example, after dividing a continuous whole into six equal-sized pieces, a TNS student may not recognize that the first sixth is the same as the fourth sixth in that it can be copied to replace the fourth sixth. Once the fractional units are iterable, the student is can disembed any of the sixths from the original bar and iterate it five more times to form a bar that is equal in length to the original bar. I discuss the relationship between ENS and fractional schemes in the upcoming section, "Research on Student Learning of Fractions," but one important caveat to the relationship is that students who have constructed an ENS *can* assimilate fractional situations with their ENS scheme, but they will not necessarily do so. Therefore, a lack of iterable unit fraction, for example, is not a contraindication to the construction of an ENS.

Iterable units also have significance for the ability to decompose whole numbers. TNS students would be able to explicitly double count in order to ascertain the numerosity of a section of the number word sequence: Asked to find the total number of marbles after adding 5 marbles to 18 marbles, a student who has constructed three levels of units could say, "18…19, that's one, 20, that's two, 21, that's three, 22, that's four, 23, that's five…23 marbles." For a TNS student, the number sequence from 1 to 23 is *segmented* into two parts in this case, the first 18 and the last 5, but those parts cannot be mentally pulled out of the contained unit without breaking awareness of the containing unit. The ENS student is able to mentally *disembed* the number sequence from 19 on and count its units, while still being explicitly aware of the disembedded sequence as a subset of the larger number sequence from 1 to 23. The ability to disembed subsets, while still maintaining an awareness of the containing composite unit, leads to the ability to multiplicatively compare composite units. As discussed the composite units themselves are now defined in relation to an iterative, or multiplicative, relationship with the iterable units of 1.

ENS is also indicated by the ability to decompose whole numbers into addends and freely interchange those addends. These parts are interchangeable precisely because all of their units are identical. This would seem to have implications for constructing a difference as a missing addend. In fact, Steffe (2004) pointed out that a student would not be able to develop this idea of subtraction until after the student had constructed an ENS. The ability to decompose a number into addends and operate on those addends also makes *strategic reasoning* (Steffe & Cobb, 1988) possible when a student is adding or subtracting. *Strategic reasoning* refers to the use of related addition or subtraction results to solve the problem at hand. For example, if a student were adding 23 and 68, the student might reason, “If 68 were 70, then the answer would be 93. Because 68 is 2 less than 70, I’m adding 2 less and the answer will be only 91.”

Generalized number sequence. The generalized number sequence (GNS) is constructed through the coordination of two explicitly nested number sequences. The archetype of this type of reasoning is Nathan’s solution to the following problem in Olive and Steffe’s chapter in their 2010 book: Use strings of 3 toys and strings of 4 toys to make a string of 24 toys. Nathan reasoned, “Three and four is seven; three sevens is twenty-one, so three more to make twenty-four. That’s four threes and three fours!” (p. 278). In this example, Nathan is not only iterating composite units, but he is also coordinating iterations of three and iterations of four to get iterations of seven, while keeping track of the threes and the fours in the iterated sevens, as evidence by his ability to operate further on the total number of strings of three to account for the three to get from 21 to 24 toys.

The ability to assimilate with three levels of units actually implies the construction of a GNS. This is because an ENS involves taking two levels of units as given and operating on them to form three-levels-of-units structures in action. Therefore coordinating two explicitly nested number sequences would imply operating on these three-levels-of-units structures.

Research on Student Learning of Fractions

Steffe (2010a) hypothesizes, based on his research on fraction learning in children from third to fifth grades, that children use their already constructed numerical counting schemes when constructing their fraction schemes. In particular, the students construct various types of mathematical objects important to fraction reasoning, such as connected numbers and partitioned wholes, and, in the process, learn to modify their existing numerical schemes in order to assimilate these new mathematical situations involving novel mathematical objects. This leads to novel ways of operating and eventually to rich conceptualizations of fractions. Steffe calls this his *reorganization hypothesis,* and he summarizes it as claiming that "children's fractional knowing can emerge as accommodations in their natural number knowing" (Steffe & Olive, p. vii).

There are important corollaries to this hypothesis. The first is that the learning of fractions should not be thought of (presented as) something separate from the learning of whole number operations. Steffe feels that fraction instruction should explicitly draw upon the pre-existing whole number knowledge of students since the student learning can use that knowledge to facilitate fraction understanding. Another important corollary to the reorganization hypothesis implicit in Steffe and Olive's work is that poorly developed whole number schemes impede development of advanced fractional schemes. Or, more precisely, the lack of concepts and operations underlying certain number schemes may imply the impossibility of developing certain fraction schemes that build on similar mental operations. Examples of mental operations and concepts that underlie schemes for both natural numbers and fractions are distributive reasoning, reciprocal reasoning in multiplicative situations, developing an iterable unit of one, and the ability to assimilate a situation using three levels of units.

Going into the current study on student signed number addition and subtraction, I considered the possibility that second-order models of students' signed mathematics may lead to a parallel to the reorganization hypothesis that links the development of signed number schemes with a reorganizations of unsigned number schemes. In particular, as I explain in Chapter 3, I end up using mainly integer values with the students, so I was interested in whether natural number schemes were being reorganized to construct signed number schemes. Certainly, whole number schemes are inextricably linked with integer schemes in that we sometimes consider whole number schemes as a subset of integer schemes. There would be parallel corollaries to such a hypothesis. One would be that integer instruction should build on existing whole number operations in the students. The second parallel corollary is that there will be a necessary relationship between student constraints when working with whole numbers and integers: the inability to construct certain schemes with whole numbers could signify a necessary inability to construct corresponding schemes with integers. This led me to consider students' fractional and natural number (unsigned) operations and schemes before working with them on signed quantities. As I hypothesized constraints for the students' signed number operations, I attempted to explain these constraints first in terms of their natural number schemes.

Because I use students' fractional schemes and operations to make hypotheses about more general schemes and operations that underlie natural number reasoning and possibly signed reasoning, I outline here a few of the key fractional terms I borrow from Steffe and Olive (2010).

Fragmenting schemes. In order to describe the fragmenting operations available to my participants, I need to describe simultaneous, equi- and distributive partitioning. Those three operations form a hierarchy of sophistication from simultaneous partitioning as the least sophisticated fragmenting operation out of the three and with distributive partitioning as the most sophisticated fragmenting operation described by Steffe (2010a). For the purpose of contrast, I will describe equi-segmenting, which is a common fragmenting scheme middle school students and is a level lower in sophistication than the partitioning operations.

The behavior of a child who is applying an equi-segmenting operation and a child who is applying a simultaneous or equi-partitioning operation can look similar. If given an unmarked bar and asked to divide up the bar into *n* fair shares, students with any of these fragmenting operations would usually start by marking off an estimate for 1/*n* and then using that estimate to segment the rest of the bar. However, the mental operations behind the making of the estimate and the segmenting are distinct for the three fragmenting operations. This can be observed in characteristic mathematical behavior of the students. One commonality they share is that the goal of making equal-sized pieces would be present throughout the activity of children utilizing any of these three fragmenting operations. This is in contrast to the child who has not yet constructed two levels of units, as I describe in an earlier section, "Two Levels of Units." Therefore, an equi-segmenting or partitioning operation implies the ability to assimilate with two levels of units, which implies the operations necessary to construct a TNS.

As for the differences between the mental operations, a child who is equi-segmenting is thinking of the fair shares sequentially. That is, the child would construct an estimate and then *construct* the other shares. A child who is partitioning is able to project an awareness of *n* shares simultaneously onto the bar. Naturally, if *n* is big enough, the child might not be able to actually visualize all of the partitions simultaneously, but the child would be attempting to coordinate *n* shares on the bar *before* making the estimate. The sequential nature of equi-segmenting means that the estimates of these children are often much less accurate than the estimate of a partitioner.

An equi-segmenter and a simultaneous partitioner are not assimilating lengths as iterable units (Steffe, 2010c). Therefore, the shares they are forming, through sequential segmenting in the case of an equi-segmenter and through simultaneous projection in the case of the simultaneous partitioner, are not identical. These children are attempting to make them the same size, so they are in some sense equivalent, like the units of a TNS. Furthermore, like the units of one making up a composite unit in a TNS, the partitions made by equi-segmenters or simultaneous partitioners cannot be disembedded from the composite unit in order for the purpose of comparison, in this case with the other partitions, and the partitions cannot be used to regenerate the original bar. Both the disembedding and potential iteration are inherent in the equi-partitioner's assimilation of a fair share situation. All of this leads to several differences in behavior between equi-partitioners and the other two types of fragmenting. The first is that if asked to mark off just one fair share, the equi-partitioner will often independently measure the length of the estimate with their fingers and then mark off $n - 1$ iterations above the remainder of the bar. At the end, the equi-partitioner would compare the length of the hypothetical bar that would be formed by the iterations with the length of the original bar. This shows an awareness that the first partition should regenerate the length of the original bar upon disembedding and iteration.

In contrast, equi-segmenters would probably independently check their estimate of one fair share also, but they would do so somewhat intuitively. Often, instead of using a substitute for the first partition's length to measure off the rest of the partitions, as equi-partitioners often do using their fingers as the substitute, equi-segmenters will use visual motion, or perhaps an even more intuitive visual estimate to make the other partitions. Simultaneous partitioners may not feel the need to check their estimate at all because they are already aware of the other partitions, and they do not have an easy way to think about checking the partition if they cannot physically mark off other partitions on the bar.

If a simultaneous partitioner or an equi-segmenter actually draws in the rest of the partitions, another distinguishing behavior is that they are clearly drawing the partitions *in* the given bar. So they will often adjust the size of the estimates with regards to the remaining length and the number of partitions they have left to draw. They also will look at *all* of the resulting partitions and focus on whether some are too big or too small in their explanation for whether their estimate was a fair share. An equi-partitioner would attempt to draw all of the partitions in the same length as the estimate also, but usually measuring off as opposed to visually estimating. An even clearer behavioral distinction is that the equi-partitioner would usually judge whether or not the estimate was fair by referring to just the estimate, "No, it would need to be bigger/smaller," or the fact that the last partition does not line up with the end of the bar, "No, we would need this [endpoint of bar] to bigger," or, "That's [the last partition] too small." The equi-partitioner would not usually refer to the intermediate partitions.

Given that equi-partitioners are constructing *iterable* fractional units, equi-partitioning behavior indicates that a student has constructed an ENS. Simultaneous partitioning behavior does not necessarily imply construction of an ENS, but the simultaneous nature of the partitioning behavior, as opposed to the sequential behavior in the case of an equi-segmenter, implies that the student is at least in the process of abstracting out the counting records in their arithmetical unit items in order to form iterable unit items. Therefore, simultaneous partitioning would indicate at least that the student is in the late stages of reorganizing their TNS into an ENS. Also, as always, the student might have the operations necessary to construct a more sophisticated way of operating, equi-partitioning in this case, but might not have done so yet. For example, Steffe (2010c) worked with a student named Laura who had constructed an ENS, and yet did not assimilate lengths units as iterable units.

Steffe (2010a) described distributive partitioning as the most sophisticated form of fragmenting. An example of a distributive partitioning task would be to share *n* bars among *m* people fairly. If the student uses a template for *m* to project partitions onto each of the *n* bars, with an awareness that each person would get one partition from each bar and that their overall share would be *n/m* of one bar (assuming the bars are equal sizes), then the student is engaging in distributive partitioning. Distributive partitioning requires distributive reasoning in that the student needs to understand that dividing the set of bars into *m* equal parts is equivalent to dividing each of the constituent bars in the set into *m* equal parts. More generally, a multiplicative operation on the whole can be carried out equivalently on each of its constituent parts.

In order to come up with this strategy, the student must be able assimilate the situation with the potential to partition each bar, requiring a two-levels-of-units assimilating structure, and the student must assimilate the partitions as iterable so that it will not matter which *m* of the resulting *mn* partitions the student uses to make fair shares. The partitions must be interchangeable. Therefore, construction of the distributive partitioning operation implies the construction of an ENS. If a student forms an improper fraction in solving a distributive partitioning task, i.e., if *n* is greater than *m*, then this is a strong indication that the student has constructed an iterative fraction scheme (see next sub-section), and, hence has constructed a GNS.

Fraction schemes. The fragmenting behavior can be used, as I have indicated in my examples, without the use of fractional language. I now present some of the fraction schemes students use to make non-unit fractions of a whole in explicitly fractional situations.

A part-whole fraction scheme is the first and least sophisticated fraction scheme I discuss here. In this scheme, fractional units are not taken as iterable, and, hence there is not the sense of an iterative, or multiplicative, relationship between the fractional unit and the fractional whole, as there would be if the student were equi-partitioning to construct the fractional unit. Therefore, a part-whole fraction scheme is a contraindication that the student is equi-partitioning, although, as always, the student could have an equi-partitioning operation, but not deploy it in a fractional situation.

Characteristic behavior for a student with a *part-whole* scheme is for them to find the numerator in a fraction name by counting up the number of partitions in the region being named, and find the denominator by counting up all the regions in the "whole." For example, if a student with a part-whole scheme were to be asked to find a third of a bar and then find fourths of that third, as is shown in Figure 2.1, then the student might interpret two of the smallest partitions as 2/6 of the bar. Although the student would probably know, if asked, that the partitions should all be the same size, because the thirds are not iterable, and therefore identical, the student is not likely to see the possibility of finding fourths of the other two thirds in order to get equal-sized pieces.

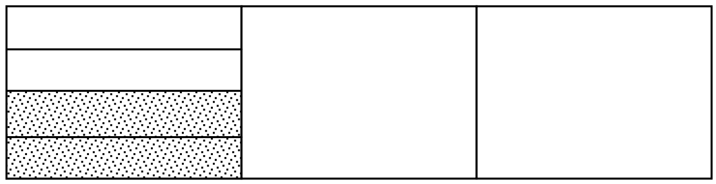


Figure 2.1. Two fourths of one third.

Another key distinction between a part-whole fraction scheme and the other fraction schemes I discuss is that the student does not unitize the whole, meaning that the student does not see the necessity of the whole to be the same size if they are comparing two fractional numbers. For example, if a student had drawn a bar and segmented it into thirds, if asked to find one fourth of the bar, the student might add another third, so that there are four pieces in all.

The next scheme I discuss is the *partitive fraction scheme*. As the name implies, the partitive fraction scheme implies that the student is equi-partitioning to form fractional units. Therefore, the fractional units are in an iterative relationship with the fractional whole. The fraction one-fourth would imply one partition of a whole that can be iterated four times to form the whole. However, the partitive fraction scheme does not yet allow the student to form improper fractions. This is because, although the student can mentally disembed a fractional unit from a fractional whole without destroying the fractional whole, the student still understands the numerator of a fraction as describing how many parts they have *out of* the iterable fractional units that make up the whole.

Once the student assimilates the numerator as describing the number of iterations of an iterable fractional unit, the student is no longer constrained by the fractional whole and can make sense of improper fractions. This double use of iteration results in the *iterative fraction scheme*.

The iterative fraction scheme requires three levels of units. Hence to use it in an anticipatory manner implies the ability to assimilate with three levels of units and the construction of a GNS. The three levels of units and their quantitative relationships are especially clear in this case: The unit fraction (1/*b*) is in an iterative relationship with the fractional whole, giving two levels of units, but the unit fraction is also in an iterative relationship with the composite fractional amount (*a*/*b*), giving a third level of unit that is created out of the other two.

Research on Student Learning of Addition and Subtraction

Task analysis. Much of the literature on natural number addition and subtraction that I have read focuses on classifying addition and subtraction problems through mathematical or linguistic means and testing the difficulty of the different types of problems with students. An example of this is a study by Carpenter and Moser (1982). They note that their work is consistent in theory and findings with numerous other researchers (e.g., Glendon Blume, Jim Hiebert, E. Glenadine Gibb, Pearla Nesher, and Tamar Katriel). I agree and use it here, along with the work of Gérard Vergnaud that I cited in the first chapter, to illustrate some concepts and findings from this group of addition and subtraction researchers that informed my analysis.

Carpenter and Moser (1982) carried out a three-year longitudinal study with 150 students, conducting three interviews with each student each year. Their focus was on the relationship between solution strategies and problem type, but they also examined statistics on how often students correctly solved the different problem types. One of the distinctions in tasks that I found intriguing was between *active* and *static* relationships between sets or objects.

> Some problems contain an explicit reference to a completed or contemplated action causing a change in the size of a quantity given in the problem. In other problems no action is implied; that is, there is a static relationship between quantities given in the problem. (Carpenter & Moser, 1982, p. 10)

This same idea is followed up in Vergnaud's (1982) work, which differentiates between addition situations where one addend is a change in quantity and situations where both addends are static collections being combined. In the first case, adding would be increasing a collection and, in the second, adding would be combing two different collections to produce a third. In his research, the first kind of addition situation is easiest for students to work with. The observer can still view this as combining collections in that you are combining the original items with the new arrivals, but I think the primacy of adding as increasing a quantity is important.

Vergnaud also discussed the situation where both addends are a change in quantity. For example, "Jack earned three dollars in the morning and five dollars in the afternoon. How much money did Jack earn in the whole day?" would be such a situation. He noted that these kinds of situations involve a *composition of transformations*.

Both studies also differentiated between situations where the change involved is an increase or a decrease, and which type of quantity is the missing quantity that must be determined by the student. Problems that would be solved through addition were easier for students to solve than those that are solved through subtraction. For problems that would be solved through a subtraction of natural numbers, both studies found that situations of a given starting state and decrease were easiest to solve. The "missing addend" types of problems were hardest to solve. Vergnaud (1982) gives some interesting statistics:

Problem A. There are 4 boys and 7 girls around the table. How many children are there altogether?

Problem B. John just spent 4 francs. He now has 7 francs in his pocket. How much did he have before?

Problem C. Robert played two games of marbles. On the first game, he lost 4 marbles. He played the second game. Altogether, he now has won 7 marbles. What happened in the second game?

> Although a simple addition, 4 + 7, is needed in all three cases, Problem B is solved 1 or 2 years later than A, and C is failed by 75% of 11-year-old students. (p. 39)

Problem C would correspond to a missing addend subtraction problem in a situation of composition of transformations. This is exactly the type of problem that I will do with my participants. Note that fifth-graders find these problems very challenging. This adds credence to my claim that some students will not have been ready to deal with transformations of changes in elementary school and that this type of problem should be reintroduced in the context of signed quantities to give students another opportunity at constructing viable schemes to deal with these kinds of mathematical situations.

When Carpenter and Moser (1982) looked at students' solution strategies for problems, they found that solution strategies for addition did not vary by student across different types of problems. However, solution strategies for subtraction problems did vary. They hypothesize that "at first, children do not recognize the interchangeability of these strategies….Our data suggest that younger children have several independent conceptions of subtraction" (p. 21). They did not work with students above third grade, but they note that other researchers have found that older students are able to choose the most efficient subtraction strategy, implying that they are aware of the interchangeability. My interpretation of these findings is that students do not assimilate all of these different subtraction situations (and addition situations) as related until long after they can solve them all. Similarly, my pilot work implies that the same is true for integer addition and subtraction. In the current study, I continue to look for indications about students' underlying assimilating structures.

Links to number sequences. Steffe (2004) discusses various ways in which students at the INS stage might try to solve subtraction problems, all of which lead to possible advancements in the students number sequence, since solving a subtraction problem involves quantifying a (non-initial) part of the number sequence. His underlying meaning for subtraction at this stage would be *taking away*. In fact, he does not see the possibility of a student constructing subtraction as a difference (comparison) between two numbers until an ENS is constructed. This is because this view of subtraction equates a difference with a missing addend. This kind of relationship involves the ability to decompose the sum quantity into numerical composites (whose order does not matter, thanks to iterable units) representing addends and construct an additive relationship between the three quantities. This allows the student to recognize that the additive relationship remains the same regardless of which of the three quantities is missing. Vergnaud's (1982) finding that *take-away* subtraction situations are the easiest for students to deal with supports Steffe's hypothesis. This led me to seek ENS or GNS students for my study, because I was interested in how students could construct coherent assimilatory structures for signed addition and signed subtraction situations.

Complex additive relationships. In Thompson's (1993) study on complex additive relationships, he demonstrates the inherent difficulties in reasoning about differences. He finds that students have trouble solving and discussing the quantities involved in problems such as,

> Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B. The two fellows argued about which one stood taller over his sister. It turned out that Brother A won by 17 centimeters. Brother A was 186 cm tall. Sister A was 87 cm tall. Brother B was 193 cm tall. How tall was Sister B? (p. 175)

In particular, they have trouble talking about, reasoning about, and representing the difference between differences represented by 17cm in this problem. Similarly, I put students in situations where they have to reason about differences and compare differences. I would eventually like students to be able to think of signed quantities as generalized differences, therefore I infer from Thompson's work that this will be both difficult to do and an interesting area to study do to the student difficulties.

Early algebra research. Researchers who are interested in the development of early algebraic reasoning in the elementary school have done some interesting work with unknown quantities, additive structures, and number line representations. In particular, Carraher, Schliemann, Brizuela, and Earnest (2006) give results from two classes during an early algebra intervention. The participants were 69 students in third grade and the classes focused on using a variable number line to represent additive relationships. They introduce the student to a number line with negative numbers already plotted in the second lesson, and they introduce the use of an arrow to represent changes in position on the number line. They do so with the explicit purpose of "distinguishing between numbers as points and numbers as intervals" (p. 96). Although they do not explain in this article why they feel the need to distinguish between position and interval, I assume it is from their prior teaching and research experience. This fits with my pilot study findings and Thompson and Dreyfus's work that I discuss in Chapter 1. Another parallel between their ideas and Thompson's (1993) is that "even in so 'simple' an area as additive structures, children need to be able to reify differences so that they can be treated as bona fide quantities with their own properties and subject to arithmetical operations….They may confuse a height with a difference between two heights" (p. 111). Finally, they agree with Vergnaud (1982), Thompson, and others in their assertion that students need to learn to focus on quantitative relationships as opposed to specific calculations: "We view the introduction of algebra in elementary school as a move from particular numbers and measures toward relations among sets of numbers and measures, especially functional relations" (p. 88).

Carraher and colleagues (2006) have several promising results: They found that *some* students could explain why the addition of additive inverses gives 0, implying both that this should be trivial for some middle school students and that there were students in their classroom who could not articulate why the sum is 0. In my data analysis, I look for the extent to which participants can articulate explanations involving additive inverses. They also found that students readily took up the differentiation in notation between positions as points on the number line and changes as arrows on the number line. This implies that the difficulty I encountered with Brad in my pilot study (see Chapter 1) could be an anomaly. Even more impressive than the consistent use of arrows for changes, is that the students were able to not only operate on a number line containing negative numbers and representing negative changes, but they were also able to do so on a variable number line. The variable number line had N instead of 0 as the label for the origin. Other values were labeled in relation to N: …, $N - 3$, $N - 2$, $N - 1$, N, $N + 1$, $N + 2$, $N + 3$,….

However, as I read about the instructional interventions, I began to doubt the sophistication of the students' understandings. Although the authors write, "Our work has been guided by the ideas that … meaning and children's spontaneous notations should provide a footing for syntactical structures during initial learning, even though syntactical reasoning should become relatively autonomous over time" (p. 110), they introduce all of the notational and procedural conventions I discuss above. There are several places where the authors claim students are dealing with variables because the students were, for example, able to give various (pairs of) possibilities for N and $N + 3$. However, based on their description of the student behavior, most students are dealing with unknowns as opposed to what I would call variables (cf. Wagner, 1983). In my experience, students can be aware of the fact that there are other *possibilities* for values, while still needing to instantiate a specific value in order to reason about the unknown. In this case, the students would be dealing with the variable as if it represented a particular unknown quantity, and, furthermore, it would not necessarily represent a generalization, but a specific calculation. The authors note that the students reason on the number line, not directly with the algebraic script, which I think is important because it implies that these algebraic notations do not represent a tool to them, but a way of giving an answer. That is, they are not being used in quantitative reasoning.

In their statement, “generalizing lies at the heart of algebraic reasoning” (p. 88), I found confirmation that the authors may be thinking about the students understandings differently than I do. Their statement implies the well-known assumption that patterns are at the heart of algebra. I would agree that patterns do play an important role in the development of algebra, but *noticing* number patterns is a very small (necessary) step that can involve pseudo-empirical abstraction (Piaget, 1977/2001) instead of a larger realization about quantitative relationships. In addition to recognizing patterns, the students need to figure out *why* the pattern works and what the necessary numerical properties are that yield the pattern. An excellent example of the teacher/researchers’ limiting of attention to the *finding* of patterns is the following excerpt: “With an overhead projector, we sometimes employed two number lines: the variable number line and a standard number line with an origin at 0. By placing one line over the other (they shared the same metric), students could determine the value of N; it was the integer aligned with N. They also gradually realized they could infer the values of, say, $N + 3$ from seeing that $N + 7$ sat above 4 on the regular number line. The connections to algebraic equations should be obvious to the reader” (p. 97). Although I agree that the *reader* should find connections to algebraic equations obvious, the *student* would surely not. Furthermore, if I look at this teaching intervention from a student’s perspective, I conclude that students are being asked to abstract a procedural pattern. The authors do not claim that the students are able to move away from using the number lines eventually or that the students came up with this idea themselves. Therefore, I cannot conclude from this excerpt that the students are algebraically reasoning or that they “can work with" a variable number line in a conceptual sense. Some students could have been forming valuable mathematical schemes, but there simply is not enough evidence here to conclude that.

The authors do write, "Although students expressed their personal understandings in drawings and explanations, we do not suggest that their behavior was completely spontaneous. Clearly, their thinking was expressed through culturally grounded systems, including mathematical representations of various sorts. Number line representations and the use of letters to represent unknown amounts or variables are examples of cultural representations we explicitly introduced to the students. The issue is not whether they invent such representations fully on their own but rather whether they embrace them as their own—that is, whether they incorporate them into their repertoire of expressive tools" (p. 108). I agree, and they do show evidence of students doing using these tools in exciting ways. For example, "the students increasingly came to use algebraic notations and number line representations as a natural means of describing the events of problems they were presented with" (p. 108). This is promising. However, the student understandings demonstrated by developing a tool or technique and the student understandings demonstrated by assimilating an externally observed notation or technique are not isomorphic. In my own study, I take a more conservative approach to attributing sophisticated ways of operating to students based on learned procedures.

Research on Student Learning of Integers

As I discuss in Chapter 1, "Why Do Students Need to Learn Signed Quantities?," the research studies on student learning of addition and subtraction of signed quantities focuses, as I do in this study[5], on integral quantities. Within this research, there are several research categories: research on the historical development of integers, research on the effectiveness of particular interventions or models, and research that aims to form hypotheses about student

[5] I discuss the choice to exclude fractional quantities, for the most part, in Chapter 5.

conceptions. My study fits into the last category. I introduce some of the researchers and findings from the first two categories, and then I discuss the importance of further research into the development of student conceptions in integer operations.

Historical development of integers. At least one researcher has applied the historical-critical method (Gallardo, 2002) to the mathematical concept of integers. The idea behind this method is that the historical development of the concept could provide insight into an individual's stages of construction. In her study, Gallardo first outlines some aspects of our contemporary concept of integers that earlier mathematicians had difficulty accepting. Interesting to me was the confusion between absolute zero and relative zero. This was not expounded on in the article, but I conjecture that this refers to the conflation between understanding the end state of a series of changes in quantity with respect to 0 and with respect to the starting point for the final change in quantity. In other words, this may be related to Brad's issues with number line representation that I present in Chapter 1.

Gallardo's main findings are that there are four discernible stages in acceptance of negative quantities. In the first stage, no negative quantities are ever allowed. In the second stage, negative quantities can be construed as directed or relative numbers, as in the chip or number line models I have described. In the third stage, a negative number can refer to a unitary quantity. And in the fourth stage, negative numbers are part of our contemporary system of integers (or real numbers). I look particularly for parallels in the two middle stages when working with my students.

Instructional interventions. The majority of studies on student learning of integers focus on the efficacy of various instructional interventions when teaching integer addition and subtraction. In order to choose which interventions to investigate in their study, the authors often categorized extant models for teaching integers. Some of the categorizations are built on visual differences in the models, such as categorizing models for integer operations as number line models versus chip models (e.g., Bolyard, 2005), without attending to the aspects of integers, the aspects of addition and the aspects of subtraction that these models were most likely to engender. Indeed, many comparison studies measured learning through standardized tests as opposed to focusing on patterns in the differences in the mathematics constructed by students in the different treatment groups (e.g., Bolyard, 2005; Smith, 1995). Therefore, such studies could be said to measure the efficacy of a model for helping students memorize procedures. As I discuss in Chapter 1, I found during my pilot study that students who could numerically solve addition and subtraction problems involving positive and negative numbers did not necessarily have a viable quantitative understanding of related signed contexts. Therefore, I do almost no work with evaluating decontextualized number expressions, which is what these kinds of studies tend to rely on for assessment.

Some authors did discuss aspects of the models that might encourage different types of integer conceptions. For example, in picking his intervention, Smith (1995) focused on the distinction between “integers-as-sets” and “integers-as-transformations,” which seems to translate to my idea of integers as referring to quantitative states versus integers as referring to changes in quantitative states. However, he did not examine the nature of addition and subtraction in the different models that he felt highlighted the transformation or set nature of integers. In fact, I was only able to find two studies (Thompson & Dreyfus, 1988; Vergnaud, 1982) involving addition and subtraction of integer quantities that commented on the changing nature of addition and subtraction as students move from whole number situations to integer situations. In both cases, their focus was on the changing nature of number from representing static quantities to representing transformations and the subsequent change from addition as combining to addition as composing transformations. I propose that another important distinction is between models that are meant to lead to operations consistent with whole number addition and “take away” subtraction and models that are meant to make operations consistent with vector addition and missing addend subtraction. As for instructional interventions, an interesting aspect to look at would be whether the interventions attempted to explicitly address the issue of position (or value) versus transformation (or change in value). I only found evidence of this distinction in Dreyfus and Thompson’s (1985) study on student conceptions.

Most authors, however, focused on neither superficial aspects nor the deeper mathematical structures of the models. Instead, most authors focused on finding models in which the models allowed for some kind of sense-making on the part of the student with respect to the rules for addition and subtraction with integers. Along the same lines, other researchers focused on the need to create models in which constraints on integer operations followed logically from the situation of the model as opposed to falling back on the necessity of imposing artificial constraints, such as being allowed to add or take away zero pairs in a chip model (Janvier, 1985; Streefland, 1995). The general findings seemed to be that models in which rules could be deduced from the context were more effective at helping students reason through problems.

Janvier (1985) and Streefland (1996) both did a good job of thinking about what aspects of their models may have been helpful, but they were both focusing in on sense making, without focusing on the aspects of integers and operations that could be engendered by their interventions. In fact, both were looking for models that allowed students to reason out rules for computation themselves, but, in doing so, both focused on models that used integers to represent transformations. Streefland seemed somewhat aware of this, but Janvier was more focused on whether models represented static or movement models that lines up fairly well, but not perfectly, with the distinction between integers as transformation and integers as quantitative measures. For example, in many number line models, only one of the integers (the second addend) represents a transformation and in static models one could probably develop integers as referring to comparisons between quantitative states, which fits in with the idea of integers as transformations. Smith (1995) had the idea of studying whether making the artificial constraints part of a microworld so that the constraints became necessary helped with student performance. However, his methodology was problematic in that he had only two instructors per treatment group and, unfortunately, found that the instructor had a larger effect on student performance than the intervention method.

Student conceptions. In Chapter 1, I discuss several results related to student conceptions that have influenced my study. What follows is an overview of aspects of that research that I did not cover in Chapter 1.

Several studies have been done that attempt to look more deeply at student thinking when working with integers. However, there are large gaps in this literature. Gallardo (2002) looked at the work of 35 students and conducted twenty-minute interviews with 15 of those students to look at their use of negative quantities in different situations. However, she seemed most interested in showing that some students would not accept negative quantities in situations in which I do not feel negative quantities would make sense. For example, one question was: “A person has a certain amount of money and receives $100.00. If he now has $50.00, how much money did he have initially?” (p. 182). She saw a student’s inability to come up with and/or interpret a negative number in this context as evidence of a lower level of negative number concepts. However, the way the question is framed does not lend itself to a negative monetary quantity, like debt. Apart from this issue, though, she is not looking so much at why students have the difficulties as whether they had certain difficulties. While this data has potential utility, a more detailed analysis of students' construction of integers would be necessary to fill in the *why* of the difficulties.

Similarly, Peled (1991) conducted an analysis of stages in student construction of integers using a change-in-quantity model and using a static model. In both cases, her highest levels involved a consistent use of procedure regardless of problem type. However, the conceptual understanding reflected in her description of these final stages is negligible. For example, with regards to the highest stage in the change-in-quantity model, she wrote: The student "can simply refer to the second number to determine whether the movement on the number line will be to the right or to the left. When the second number is positive, one faces the positive direction, thus going right for addition and left for subtraction. When it is negative, one faces the negative world, going towards it to the left for addition and away from it to the right for subtraction" (p. 147). Hence, I would argue that she is pointing out characteristic difficulties students face in the earlier stages, but that a deeper analysis needs to be done to figure out both why the students are having those difficulties and what additional constructs are necessary for students to develop true integer concepts. There are similar studies that look to find characteristic ways of operating without problematizing the underlying conceptual constructs (e.g., Peled, Mukhopadhyay, & Resnick, 1989).

Chiu (2001) did a large-scale assessment of the mental models that children and adults used when evaluating arithmetic expressions that involved negative numbers. Her main finding was that the adults were more likely to employ more metaphors and be more flexible in their use of metaphors and rules of arithmetic than children. However, this does not explain why certain metaphors were useful, or what aspects of the models the problem solver was abstracting out when working with them. In short, her objective was not to develop a model of the mental operations students are using.

Thompson and Dreyfus (1988) did a teaching experiment where they engaged a pair of students in a computer microworld that used their number line model described above. They did theorize about student thinking, and I include a discussion of some of their results in Chapter 1. However, I felt that the model in their computer microworld, like the number line model in Smith's (1995) study, treated of subtraction and negative signs quite procedurally. My study will add to their research on how students move to a view of integers as changes (or comparisons) of quantitative measures and addition as a composition of transformations.

CHAPTER 3

METHODOLOGY AND METHODS

Based on my research questions and my theoretical framework, I determined that a small-scale constructivist teaching experiment (Steffe & Thompson, 2000b) would be the best methodology to utilize. The teaching experiment allows for longitudinal hypothesis building about student thinking that, in turn, allows the researcher to study changes in the students' thinking during the course of the teaching intervention. I chose to work with a small number of students in order to have the time to develop a deep understanding of each student's mathematics. My goal is to develop *epistemic student* (Steffe, 2010d) out of my experience with my participants. An epistemic student is the interiorization of abstracted patterns in how students operate. An epistemic student can be used not only to predict the behavior of the students from whose operations the epistemic student was abstracted, but also can be used to assimilate the mathematical operations of new students who have similar ways of operating. Perhaps because of commonalities in both mathematical discourses and our ways of self-organizing as genetically-related organisms, only a limited number of epistemic students are needed for a teacher to assimilate the general types of counting schemes of a classroom of children (Steffe & Cobb, 1988). In fact, this idea that the ways of operating of one student will be mirrored in the actions of other students' is one of the reasons that teaching experiments with a very small number of participants can still inform us about children's mathematics more generally. I hope that such a convergence emerges from the mathematics of students involving signed quantities. Building on that hope, I looked for commonalities and differences in my participants in order to begin the construction of epistemic students with regard to schemes of signed addition and subtraction.

Based on my pilot study, I knew that I wanted at least 18 teaching episodes of between 30 and 45 minutes with each participant. Furthermore, I knew that I might need more time with the participants if I had not yet found any changes in their ways of operating during the teaching episodes. Given the amount of time required of the participants, finding a site that had a non-academic place in the schedule that would allow sufficient time for teaching episodes proved difficult, and, once I did find a site, the constraints of the school I would be working at were such that they needed me to work with the students three days a week, every week, for the first six week of the teaching experiment. This arrangement was not ideal because I would not have very much time in between teaching sessions to do ongoing analysis and think about tasks for the students. More importantly, though, I had to work with all of the students at once, which meant that if I wanted to work with more than one pair of students, I would need the help of several other researchers, both for teaching and for witnessing, which I will describe later in this chapter. Luckily, I found three graduate students that wanted experience with this methodology and were excited to help with data collection and some analysis. My advisor committed to helping with data collection as well. All of these people were interested in constructing second-order models of middle school students' mathematics, and all of them carried out additional analyses on the data that were separate from the current analysis. I will refer to us as the *research team* throughout this chapter.

During each data collection session, I, along with one of the other researchers, had a dual role as a teacher/researcher. The duality of the role reflects two intertwined goals I had during the data collection phase of the study. The first goal was a teaching goal: to provoke student construction of complex additive and signed additive relationships; in particular, I wanted to provoke the construction of additive relationships between differences of and changes in quantities in order to begin the construction of signed numerical operations. The second goal was a research goal: to build second-order models of student thinking.

In this chapter, I first describe the methodology that guided my study: the constructivist teaching experiment. In particular, I will discuss how this methodology is meant to address both goals—eliciting change and building second-order models. I will then give a brief overview of the participants as a lead-in to my research methods during data collection, including a discussion of unexpected problems that occurred. I conclude this chapter with a description of the retrospective data analysis carried out in order to answer my research questions.

Constructivist Teaching Experiment

A constructivist teaching experiment is similar to a series of clinical interviews (cf. methodology in Piaget, 1977/2001) in that each session of the teaching experiment consists of engaging the participants in a series of semi-structured tasks. The term *semi-structured* refers to the fact that the researcher goes in with a written list of tasks or questions as in a structured interview but plans on changing tasks and/or questions in response to the student's responses. The set of tasks planned before each session seeks to test the researchers' working hypotheses—temporary conjectures about the participants' mathematical schemes and their zones of potential construction. However, the tasks are freely modified, extended and/or replaced during the course of the session as the researcher develops new working hypotheses in response to unexpected student behavior.

One key difference between clinical interviews and a teaching experiment is that the researcher in a teaching experiment has a joint teacher/researcher role: In addition to describing a student's current ways of operating, the teacher/researcher attempts to provoke changes in the student's ways of operating that could help the researcher better understand the nature of the students' schemes, represented by characteristic mathematical behavior in this case, and relationships between those schemes. Hence the researchers' working hypotheses will involve not just current schemes, but also *zones of potential construction*. In a clinical interview, working hypotheses might focus on a student's current schemes or on a developmental hierarchy of schemes, but not the zone of pontential construction for an individual student.

Zone of Potential Construction

The second goal of the teacher/researcher, to build second-order models of student thinking, directly affects the first goal, to provoke new mathematical thinking. I needed the second-order models to inform each student's *zone of potential construction*, which is crucial when attempting to provoke new mathematical constructions. The zone of potential construction (ZPC; Steffe, 2010e, pp. 17-18; cf. *zone of proximal development* in Vygotsky, 1978) consists of all types of novel (to the student) modifications of mathematical schemes that a researcher hypothesizes a student might be able to make. All such modifications are important stepping stones in the student's mathematical journey, but some, such as a generalizing assimilation, in which a new type of mathematical situation activates an existing scheme, can be relatively minor. Whereas others, such as the coordination of the action of two existing schemes or the reprocessing of a scheme's action as a new object to be operated upon, can have far-reaching implications for the student's future mathematical success (cf. Steffe & Olive, 2010).

If all solution paths for a given task require operations that are not within a student's current ZPC, then, with regards to the first goal of engendering new mathematical thinking, presenting the student with such a task would be unproductive. In fact, presenting the student with such a task may be counterproductive, as it has the potential to lower the students' self-confidence as a mathematical thinker and/or lower the child's trust in the teacher to give solvable tasks. On the other extreme, if I were to give a task whose solution requires no modifications of a student's current ways of operating, then I am unlikely to witness new mathematical behavior. Therefore, both building a second-order model of students' current thinking and narrowing down students' ZPC's were essential to eliciting the constructions of novel types of additive relationships.

My construction of ZPC's is not only based on a student's current mathematical ways of operating. I also take into account the student's trust in him or herself as a mathematical thinker and his or her trust in the teacher to provide appropriate problems. There were instances in which I felt that a student had the mental tools available to operate in a given mathematical situation, but I did not feel that the student's self-confidence or confidence in the appropriateness of the task allowed him or her to persist in trying current mathematical operations in an unfamiliar way. I give an example of this with one student, Michelle, during her initial interview (see Protocol 4.13).

Flexibility in Task Selection

The flexibility of a semi-structured sequence of tasks is particularly important in the teaching experiment. If the student responds to a given task in an unexpected way, then the researcher needs the flexibility of giving unplanned tasks in order to further explore the reasons for the unusual behavior. For example, in my pilot work, I posed the following planned task: "Amanda climbed +18 feet and then +298 feet. How could she have done that in one trip? Draw a diagram to explain your answer." One of the students, Brad, unexpectedly labeled only the ending of trips (tick marks) with the trip length, instead of labeling line segments, when representing motions on the number line, as shown in Figure 3.1. He then became confused about whether the 298 on his diagram represented 298 from 0 or 298 from the end of the first trip. I formed a working hypothesis that his confusion was due solely to poor notation. The flexibility to adjust my tasks allowed me to follow up by asking him to act out the situation again using his sketch in order to reorient him to the space between the ending positions instead of the positions. His confusion remained, even after he attempted to fix the situation by adding in a line for 298 + 18, as shown in Figure 3.2. This added information allowed me to infer in my retrospective analysis that his behavior in this instance was indicative of his potential for conflating the size of a change with the result of a change. The conflation led me to hypothesize that he was not able to keep track of all the quantities involved, not because of notation, but because he could not assimilate the situation with three levels of units. Had I noted his incorrect answer and moved on, I would not have been able to disconfirm my incorrect working hypothesis and I may not have become interested in the relationship between levels of units and signed numerical operations.

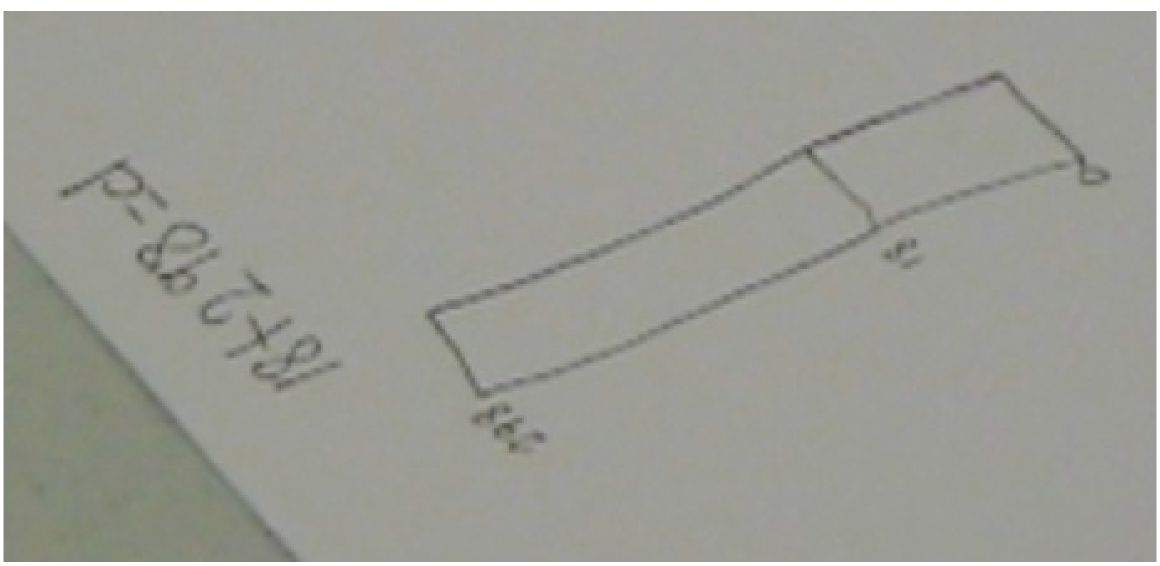


Figure 3.1. Brad draws a diagram for 18 + 298.

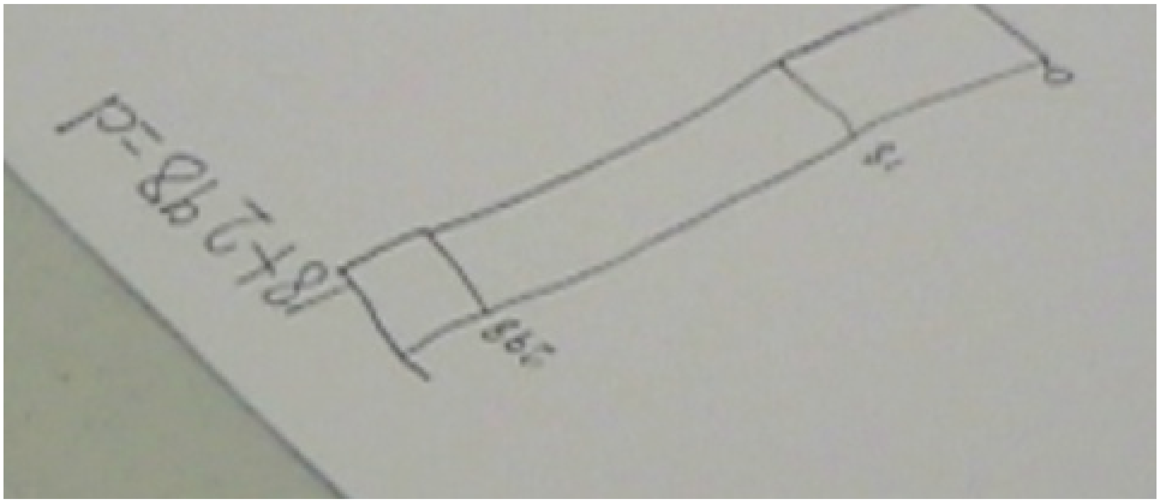


Figure 3.2. Brad changes his diagram in the course of his explanation.

Overview of Data Collection

The research team worked with four sixth-grade students during the participants' spring semester. Whenever possible, the students worked in pairs. Data collection took place over the course of five months. During the first month we selected participants through individual, initial interviews. During the next two months we met with the participants Tuesday through Thursday for 30 to 40 minutes during a portion of their schedule set aside for enrichment learning activities. During the last two months, we were only able to work with the students sporadically due to conflicts with standardized testing and other end of the year activities. In the end, we worked with students six days in the last two months, for a total of 23 meetings with two of the participants and 20 meetings with the other two participants. In the rest of this overview, I describe how we selected the participants, give a brief portrait of each participant, describe the ongoing analysis and task generation process, describe the set-up of the teaching episodes, and, finally, describe the retrospective analysis process.

Participant Selection

Participant selection was carried out in two phases. First, the four sixth-grade mathematics teachers each chose two students from their classes to invite to participate in the study, for a total of eight invitations. The criteria they were told to use by the assistant principal was the student's likely interest in participating, an ability to articulate their thinking, and an "average" ability in mathematics. None of these students were in the gifted mathematics class at the school. In fact, our intention was not necessarily to get average sixth-grade mathematics students. We were focused on getting students at least at a comparable level of operating as the students in my pilot work so that I could investigate some of the same issues as in my pilot work. In fact, I was hoping for at least one pair of students more advanced than the students I had worked with because neither of the students in my pilot study was able to construct integer addition or subtraction schemes. Based on the assumption that average to above average sixth graders would operate on the same level as below-average to average eighth graders, I asked for a selection of advanced and average students. However, through a miscommunication we received only students identified as average mathematics students. As it turned out, after analyzing the initial interviews we identified four of these students that I hypothesized to be at a sufficiently sophisticated level of mathematical operating—they had constructed an ENS—and so I did not end up using students from the gifted class.

Of the eight students invited to participate by their teachers, seven expressed interest in participating and met with the researchers for an initial interview. In this initial interview, we worked with the students individually using the protocol in Appendix A. Too great of a differential in a pair of students' abilities to successfully operate in mathematical situations can lead to impatience on the part of the more advanced student and frustration or embarrassment on the part of the less advanced student. Therefore, I wanted pairs of students who had similar mathematical behavior on the tasks in the interviews. I also looked for how easy or hard it was to get the student to engage in problem solving, and I looked for students who I felt would feel comfortable talking about their thinking.

During the initial interviews, there was a list of questions that the interviewer chose from based on student responses. Although the interviewer could improvise ways to re-word the questions or elicit explanations of solution strategies, the tasks themselves were set beforehand, unlike later teaching episodes in the teaching experiment. This is because we were not trying to provoke changes in the students' ways of operating during this initial interview; we were trying to determine whether they had constructed well-researched mathematical schemes and operations such as a TNS, an ENS, a GNS, equi-partitioning operations, recursive partitioning operations, a splitting operation, and the ability to distributively reason (see Chapter 2 for more details on these terms).

Each interview lasted about 30 minutes and was videotaped. Either I or my advisor was present as the teacher or as an observer, called a *witness*, in each of the initial interviews. The role of a witness is to provide a second perspective on the teaching episode and to make suggestions to the teacher/researcher for further tasks or questions during the course of the teaching episode. I watched the tape of interviews I did not conduct directly afterwards with either the teacher/researcher or witness, and I take notes of their impressions of the student's mathematics during these sessions. The initial interviews spanned two weeks. Once all of the initial interviews had been conducted, the entire research team met up and reviewed our conjectures about the current mathematics of each of the possible participants. We ended up selecting the four students who we felt demonstrated the highest level of mathematical sophistication. In particular, we had tentatively attributed the construction of an ENS to all four of the chosen participants. The most advanced pair showed some indications of having constructed a GNS. Indeed, during the course of the teaching experiment, both of these participants gave sufficient indications for me to impute to them the construction of a GNS. However, there were notable differences in the participants' mathematics that were apparent during the initial interviews, which I will expand on in Chapter 4.

Description of Participants

I provide a non-mathematical description of each participant so that the reader can get a better sense of the participants' personalities and appearance during the teaching episodes. I will also give a brief mathematical overview of our conjectures regarding each participant after the initial interview and at the end of the study. This section is meant to both set the scene for the data collection and provide a summary of participants that the reader can refer back to.

Adam[6]. Adam almost always came to us dressed in his Georgia Bulldog red sweatshirt. He had blond, bowl haircut and thick glasses. He often had allergies, was sick, or seemed tired, but once he got going he was very jovial and quite a jokester, even mathematically. For example, when I asked him to make up a problem he would often laugh as he handed me a problem with very small or very big quantities. On the videos of his teaching episodes, I've noticed that he often has a library book. The book he brought to his first session was titled, "Evil Genius."

During Adam's initial interview he displayed an amazing persistence in problem solving, sitting still for periods of over a minute with a look of concentration on his face. He also came to the study confident in his basic operations on natural numbers and seemed to enjoy working in problem situations involving natural numbers. I hypothesized after the initial interview that he had constructed an ENS, but, in retrospect, I only feel confident in attributing the construction of

[6] The names of all participants are pseudonyms.

a TNS to him. In either case, he was definitely assimilating mathematical situations with two levels of units, and due to his confidence and persistence in problem solving, he probably maximized his chances for success in natural number situations, explaining why I originally attributed more advanced schemes and operations to him than was appropriate.

However, he seemed much less confident when working in any fraction situation. He did seem to have many of the operations that have been found to be important to fraction scheme development in other studies (Steffe & Olive, 2010; Norton & Wilkins, 2009, 2010). For example, he could partition, showed an understanding of multiplicative relationships in whole number contexts, and he could solve splitting tasks. However, like an ENS student, Laura, in one of Steffe's (2010f) teaching experiments, the lack of iteration in his continuous units limited his progress in constructing fraction schemes. Throughout the study, he appears to have a part-whole fraction scheme, making temporary modifications at times.

Once we moved into work with signed quantities, he encountered more difficulties than the other participants did in developing viable ways of operating. In a number line context, it became apparent that he was having trouble assimilating situations that involved the addition of two directed quantities such as (-17) + (+32) where the second quantity had the opposite orientation and a larger magnitude than the first. This was not something I had experienced with the other participants in this study or the pilot study.

As I wrote earlier in this section, during the initial interview, Adam demonstrated an amazing ability to concentrate. However, the amount of mental energy he was willing to invest in problems waned as the teaching experiment progressed. This may have been partly due to a feeling that he was not performing well, given that his original partner, Michelle, was more adept at solving the tasks we were posing. Unfortunately, both Michelle and Adam noticed this disparity and this seemed to contribute to an increasing antagonism in their behavior towards one another. We attempted to alleviate this by changing the pairings. Although Justin and Lily both had constructed more sophisticated mathematical operations than Adam or Michelle, Adam seemed more comfortable with Lily than he did with the other two participants, and we hoped that Lily's patient and unassuming demeanor would make the disparity in the sophistication of their ways of operating less noticeable. Although the relationship between Adam and Lily never grew antagonistic, the disparity in their ways of operating was evident to Adam. Despite these dynamics, he continued to show a propensity for intensive problem solving under the right circumstances. He also was willing to argue his answer if he and his partner disagreed, although this dropped off as well in the last four sessions of his time in the teaching experiment.

Adam was the only participant to leave the study after the first three months, which was the original length of time I had expected to collect data. The week before his last session, he expressed a desire to participate in extra band practices during our usual meeting times. Although he may have been willing to continue, I felt that he was unlikely to make very much more progress with us given both his available schemes and operations and his probable decrease of confidence due to the unfortunate match-up of his ways of operating and the more mathematically powerful ways of operating available to our other participants. Therefore, we did not attempt to convince him to continue. After Adam left the study, we decided to have Lily, his partner, participate in another pilot study.

Justin. Justin is a slight, pale boy with jet black hair and eyes. He had angular features and long bangs crossing his forehead at a sharp angle. His clothing was not attention-grabbing from afar, but he both his clothes and haircuts were notably stylish. Whenever we remarked on either he would give the same shy, but triumphant, smirk that he gave when he successfully solved a difficult problem. The few interactions we observed with his classmates made it clear that he was well-liked and confident. Every time I saw him in his regular classroom or in the hallway, he was smiling and talking to a group of boys.

Justin was reserved and serious during his work with us. When working on problems he would usually look off into the distance and crack his knuckles distractedly. Even if he made eye contact during these moments of concentration, it was clear he was still working away in his head and not distracted by the movements and actions of others. He was easy to keep engaged during teaching episodes because he would always actively follow along with what his partner was doing if he himself had finished a problem. This was in contrast to Michelle and Adam, both of whom would usually stop paying attention to the discussion if we did not directly ask them a question.

Like Adam, Justin was very comfortable thinking at length about a problem. Unlike Adam, he did not generally get flustered if his partner finished before him or got a different answer. Of course, as one of the stronger problem solvers in the group, he had occasions to feel successful quite often, which probably mitigated any potential frustration or embarrassment. However, I think another reason that Justin did not get as flustered as other participants when he was challenged is because he seemed more interested in understanding the mathematics than he was in comparing himself to his partner. He would not, for example, adopt someone else’s successful solution method unless directly told to do so or unless he understood why it works. Both Adam and Michelle were generally both content with getting the correct answer, even if they seemed unsure about why it is correct. Therefore, Justin was a very good participant in that his ZPC was determined by his cognitive operations, not by frustration or a lack of confidence.

For the first seven teaching episodes, Justin was paired with Lily because of similar mathematical behavior in the initial interview. However, as I discussed in the description of Adam, I decided to switch the pairings after one month, and so Justin was paired with Michelle for the duration of the study, which consisted of 16 more teaching episodes for him. With Michelle, who he knew from his classes, he was more animated. He would interact with her more, both during problem solving and in general bantering.

Of all the participants, Justin and Lily seemed most explicitly aware of quantitative relationships in that they were able to talk about the quantitative reasoning behind their solution strategies more fluently than Adam and Michelle. This was probably made possible by their advanced mathematical operations. During the initial interview, Justin demonstrated that he could assimilate a situation with two levels of units and had constructed an ENS. Furthermore, during the first month of the teaching episodes, he showed indications of assimilating with three levels of units.

By the end of the teaching experiment, Justin was able to solve signed addition problems as well as signed missing addend problems and describe how he determined the sign as well as the absolute value of his solution. That is in contrast to Michelle, his partner at that point in the teaching experiment, who could also solve the signed problems. She could give a convincing reason for the sign of her answer, but not for why her method to find the absolute value worked. Neither Justin nor Michelle was able to recognize the difference between the structure of a signed addition problem or signed missing addend problem. In addition, neither was able to operate with a variable addend. I believe that this difficulty is based on an inability for Justin to reify the actions of his signed addition schemes.

Lily. Lily had blond hair that was almost always pulled back in a low ponytail. She had very pale skin and light blue eyes. She had a round face, and she usually wore a loose-fitting hooded sweatshirt and jeans. She seemed shy, and she generally spoke quietly when she did speak. However, she seemed to take pride in explaining how she solved mathematics problems, and she seemed even more comfortable talking to us about her solutions as the teaching experiment went along.

Lily did not show frustration when faced with a problem situation that she could not solve, like Michelle did. However, she would not persist in individual mental activity for as long as Adam and Justin if she did not yet have a solution strategy. However, like Justin, she was very attentive to what was going on. In fact, she was the best of the participants at understanding her partners' solutions. This may be, in part, because she had strong mathematical understandings with which to make sense of their solutions. In addition, I think she was more willing than the other participants to de-center from her own strategy and try to make sense of her partners' explanations. This may have been made possible by the fact that she was often experiencing less mathematical perturbation than her partners.

Lily was the only participant who gave a strong indication of being able to assimilate with three levels of units, and therefore of using a GNS, during her initial interview. This assimilatory structure was also confirmed during the first month of teaching episodes. She did not struggle with any of the signed addition problems I gave her, but she was moved to another research study before I had gotten to the more challenging problems that Adam and Michelle struggled with, so I do not have enough data to determine the full extent of her signed addition schemes. Nonetheless, before she began the other research study, she had already started explaining why her method for determining both the sign and absolute value of her answer worked. This is similar to Justin's ability to explain signed addition situations.

Michelle. Michelle was in the same homeroom and many, if not all, of the same classes as Justin. She had dark brown hair and eyes, and a tan complexion. For her age, she was tall and had an athletic build. She either wore her long hair down with a headband or pulled back in a ponytail. Much of her wardrobe appeared to be from Abercrombie and Fitch, and her shirts were fashionably fitted. Her features were rounded and she had a large and personable smile that she used often to show she was listening or appreciated our jokes.

Michelle became flustered if she did not understand what she was supposed to do in a problem situation. A typical reaction would be for her to raise her eyebrows and say with exasperation, "How is that even possible?" After the first month, she seemed to develop more trust that the teacher/researchers would give her problem situations she had the means to understand. Nonetheless, she was less likely to engage in problem solving than either Justin or Lily for the duration of the study. This could partly be due to the fact that the problem solving process was more frustrating for her than it was for Justin or Lily, given that she had less powerful mathematical tools to use when problem-solving, or it may have been a characteristic she brought with her to the study. (She was fairly persistent in the initial interview, making the second hypothesis less plausible.) In either case, her aversion to problem solving may have also impeded her progress in constructing new mathematical ways of operating.

During her initial interview, she indicated the construction of an ENS, but she did not give strong indications of the construction of a GNS. I looked for indications of a GNS throughout the rest of the teaching experiment, but all of the indications were weak. By the end of the study, in signed addition settings, Michelle operated close to the level of Justin and Lily, but several of her mathematical conceptions seemed more fragile. For example, she did not seem as explicitly aware of the quantitative relationships she constructed as Lily and Justin did. Although she could determine the sign and a comparative size for a missing signed addend or sum, she did not independently reference the subset relationship inherent in the signed addition problems like Justin and Lily did.

At the beginning of the study, Michelle was paired with Adam. Adam had significantly less sophisticated fractional operations than Michelle. This caused a severe imbalance in their ability to operate in the problem situations at the beginning of the teaching experiment, when we were still trying to get a handle on their natural number and fraction schemes. As I explained in Adam’s description, both students seemed aware of the discrepancy, which was frustrating for Adam and seemed to be causing him to lose confidence in himself as a problem solver. Unfortunately, she also developed the tendency to ignore what Adam was working on, perhaps because it was sometimes different than what she was working on and seemed easy to her. For those reasons, we switched pairs so that Michelle was working with Justin, but even as she worked with Justin, she had a propensity to only engage in problem solving when we directly asked her too; she would not usually try to follow along with Justin’s explanations or try out problems directed towards him. Although Justin was a better problem solver, she did not seem to notice the discrepancy, and the discrepancy was much less than it had been with Adam.

Ongoing Analysis and Task Generation

After each teaching episode, starting with the initial interviews, I formed working hypotheses about the students’ current mathematical schemes and operations. Some schemes and operations are fairly well-defined, with generally agreed upon indicators. For example, if I give a student a rectangle and ask the student to use that rectangle to make another rectangle so that the rectangle I drew is 5 times as long as the rectangle they draw, I can generally attribute the splitting operation to a student who solves this task. This type of task has been used as an indication of construction of the splitting operation by many researchers (e.g., Hackenberg, 2005; Norton & Wilkins, 2010; Olive & Steffe, 2010) and the validity of using this task as an indicator is backed up by cogent conceptual analysis (Olive & Steffe, 2010).

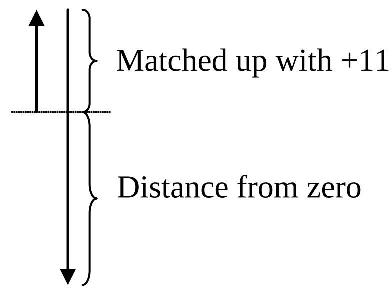


Figure 3.3. Decomposing one trip in terms of the other quantities in a problem.

However, all of these well-defined, well-analyzed and well-researched schemes and operations were within the domain of whole numbers and fractions. As I moved into situations that involved operating with or on differences and directed changes in unsigned quantities, my hypotheses were more context-specific. Often in these situations, I would form a plausible explanation of how the student might be thinking about a problem in order to explain observable behaviors. I would then construct tasks that would test my hypothesis by putting the student in problem situations that would require similar solution strategies. For example, in one teaching episode Michelle and Justin modeled climbing up and down a ladder on a pseudo-number line on which every 10 meters marked off with tick marks. She would give her final directed distance from 0 by reading her position off the number line, however, she seemed tacitly aware of subset relationships within the trips. For example, if the first trip in Figure 3.3 represented a +11m trip and then it was followed by a -34m trip, then she would seem aware that the total distance from zero was also the length of the part of the -34 trip that was not matched up with the +11 trip.

Based on her explanations, I hypothesized that she would be able to solve these problems even when the tick marks were not present on the pseudo-number line.

Input from the research team. This ongoing analysis was not carried out in isolation. After a teaching episode I taught or witnessed, I would debrief with all other researchers present. At times those discussions were recorded, but often I simply took notes on any conjectures, task suggestions or critical incidents that we discussed. These debriefings would often start as we packed up equipment. However, we usually would follow these short debriefings (about five to ten minutes) with a longer debriefing, which involved researchers that had been working with both pairs of students. These meetings could range anywhere from ten minutes to half an hour, depending on the number of unexpected student behaviors that were observed. The insights of the other researchers were essential in my analysis. For example, in the example above, I had concluded that Michelle was not able to reason using the subset relationships based on her use of tick marks to determine the final distance from zero. However, other members of the research team pointed out details of her explanation that had escaped me. Based on their insights I revised my hypothesis. As it turned out, the new hypothesis was correct; Michelle was able to solve the problems without the tick marks present.

After these meetings, I watched the teaching episodes from the day (with another researcher present, whenever possible). I would note any behaviors that provided corroboration or contraindications to current hypotheses. If there were additional incidents that provided new information about a student's operations that had not yet been discussed in the group, I would form additional hypotheses to test. Every week or two, as many members of the research team as possible would meet for about an hour and a half to discuss hypothetical learning trajectories

(Simon, 1995) for each of the participants moving forward. These discussions required that we revisit hypotheses from the previous week.

There were two ways in which I sought to improve the viability of my interpretations of student behavior. One was continual testing of my hypotheses when gathering new data. The second is the continual input from other members of the research team. In addition to the input I have described, I also had indirect feedback on my hypotheses through feedback from my advisor on proposed tasks that I would share with him before the teaching episodes.

Developing tasks. An integral part of the data collection process was the task development before each encounter with the students. This task development relies on ZPC's as well as my own conceptual analysis (von Glasersfeld, 1978) of the operations required to successfully solve a particular task. I did all the initial designs of tasks for both groups. In order to develop the tasks, I constructed hypothetical learning trajectories for each pair of students, where my final goal was variable, but always involved the ability to use notation to represent additive relationships among directed quantities. These learning trajectories were a tentative plan for the rest of the teaching experiment and were meant to help ensure that each day's tasks were taking us towards the goal of constructing additive relationships among directed quantities. In order to develop the actual tasks for the following teaching episode, I would develop tasks that fit into the hypothetical learning trajectory for both students and had the potential to be in the ZPC of each student. Often, ideas for tasks had come out of brainstorming sessions with the other researchers during either our debriefing sessions or our longer research meetings.

After developing the tasks, I would type up a sample protocol, complete with the wordings of questions I could ask the students. Particularly with new tasks, I would sometimes include a description of what I hoped to find out from the students' solutions to the tasks and what I hypothesized the students would do. After I shared the tasks with my co-researchers orally and/or in written form, I would incorporate their feedback to modify the tasks. I sent the tasks out to all researchers, but generally got feedback only from my advisor. If the researcher who would serve as the other teacher/researcher on a given day was not present as I developed the tasks for that day, then, at some point before the teaching session, I would discuss the goal of the session with the other teacher/researcher and discuss modifications that could be made depending on students' reactions to the tasks.

I will briefly give an example of how the daily tasks and the learning trajectories fit together: One of the end goals I had during beginning sessions was for students to be able to explore why A – B = A + (-B) by the end of the study, where A and B are directed quantities (reified differences). In my conceptual analysis of this problem, I started by analyzing what mathematical knowledge students would need to construct in order to explore this identity. I felt that the problem situation would be hard to understand unless the students were able to notate addition and subtraction situations involving directed quantities. This, in turn, required that the students construct concepts of subtraction and addition in the directed quantity context. In order to construct these concepts, they needed to have experienced combining directed quantities (addition) and comparing directed quantities (subtraction). And all of this certainly required that students were able to construct directed quantities, where the quantities represented a change (or transformation) in a base quantity. Based on this hypothetical learning trajectory, I focused on tasks that would involve determining differences between two quantities and then comparing or combining these quantities. In fact, we started with a game in which students each drew a number card (the base quantities), earned a score that was a directed difference of the numbers on the cards, and then combined those differences to keep track of their total score.

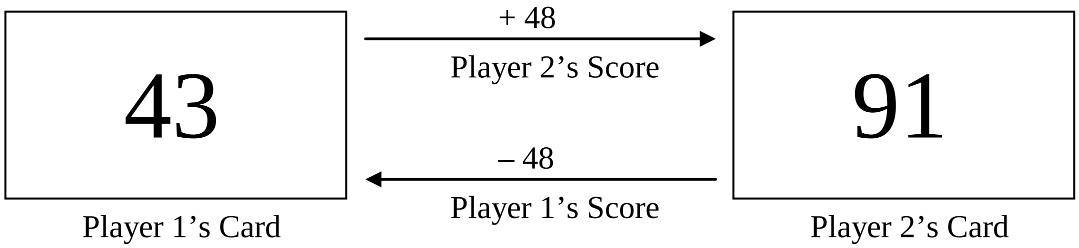


Figure 3.4. The card game.

As I will explain in the next section, these tasks mainly provided a starting place for the teacher/researcher. In reality, the teacher/researcher would react “on the fly” to students’ responses, sometimes resulting in the use of unplanned tasks in the teaching episodes. Nonetheless, the analysis involved in developing the tasks informed the unplanned tasks as well and thus was not carried out in vain.

Teaching Episodes

We worked with the pairs of students in different rooms. One or two researchers would go meet both groups of students in their home rooms and have a chance to talk informally with the students, building a sense of rapport. In addition, the teachers would often start off the session with several minutes of informal discussion. The groups met for a period of 30 to 40 minutes. In each room there was at least one witness and one teacher. In a few episodes there were two witnesses. In all but two days of data collection I was one of the two teachers. Both witnesses and teachers interacted with all four students during the teaching experiment. In fact, the three members of the research team that went out all three days a week during the first two months made it a point to observe or work with both pairs of students in any given two-week period.

Teaching episodes took place in the same two rooms for the duration of the teaching experiment, with the exception of two teaching episodes that had to be moved to a nearby computer lab due to special school activities. In one room, the students were seated at a small, round table next to each other so that the teacher/researcher was seated next to one student and across from the other. In the other room, a conference room, the students were seated at a large rectangular table. In the last two teaching episodes, involving Justin and Michelle, the students were seated on adjacent sides of the table, so that the corner of the table was between them. The teacher/researcher sat on the longer side of the table next to the students, not between. This set-up ended up allowing a better view for the video camera that we were using to capture the student work. However, for the majority of the teaching episodes held in the conference room, the students were seated next to each other with the teacher across from them if they were not using a computer and the teacher between and behind them if they were using computers.

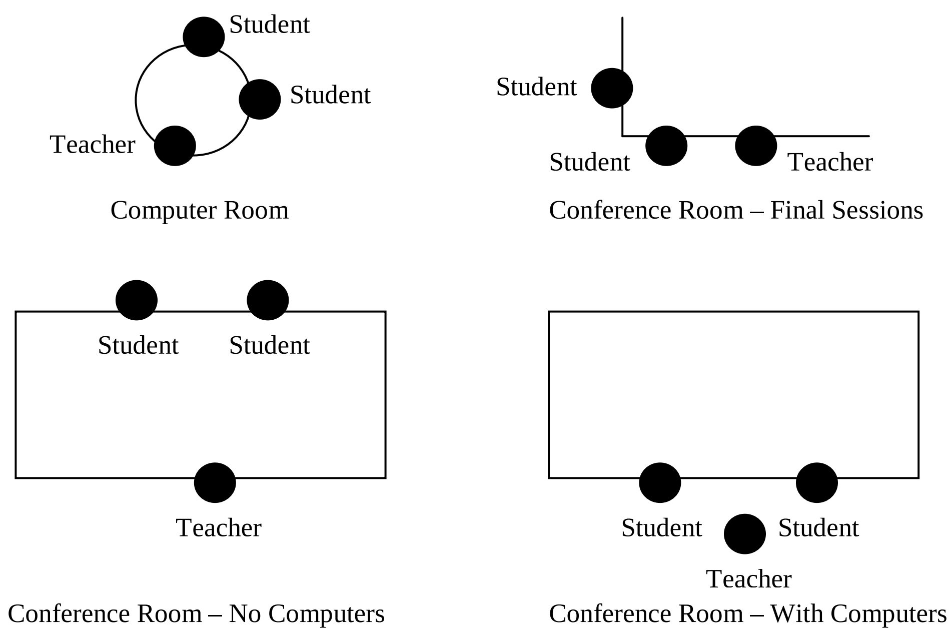


Figure 3.5. Physical set-up of teaching sessions.

In each of these situations we videotaped from two perspectives. One provided a view of both students and the teacher, with the goal of capturing facial expressions and hand gestures. The other video camera was focused on the students. The goal was for this camera view to capture the students' writing or interactions with other materials, like paper strips. For the last two weeks, a screen capture program was used in place of the student view camera when the students were working on computers.

The teaching episodes usually involved a few tasks that each took a significant amount of time to complete as opposed to many, quick tasks. There was generally a lot of time during which the students were problem solving individually, with occasional questions or prompts from the teacher/researcher. Many of the teaching episodes, involving Adam in particular, had different tasks for each participant, although the tasks were always in the same general domain. For example, Adam might be asked to find 1/5 of an unmarked rectangle whereas Michelle would be asked to find 2/5 of a bar marked into 3 equal parts. Our hope was that the tasks would not appear too different to the students, yet would allow us to challenge both students, while remaining in their respective ZPC's.

The teacher/researcher in the teaching episode was very responsive to the students' actions and comments. If a student started yawning, for example, we quickly found this to be (somewhat counterintuitively) an indication that the task at hand was too hard, outside their current ZPC. Hence, a yawn might act as a signal to modify the task or move into a different type of task from the list of suggested tasks. In addition, students would often solve a task in an unexpected way. Either the student might use what appeared to be sophisticated reasoning that we had not been in our second-order model of the student's mathematical thinking, or the student might be stumped on a task whose solution was thought to be well within the student's ZPC. In either of these events, follow-up questions would be asked and the teacher/researcher would develop new tasks on the spot that would either push the student even further or hone in on the source of their difficulty. The witness would also interject suggestions at times, either by asking questions of the students or by giving written or whispered suggestions to the teacher. Sometimes, the planned tasks would include variations depending on student behavior, but teacher improvisation was a huge reality of the teaching experiment. Details on specific tasks used will be given in Chapters 5, 6 and 7, where I present my findings.

Retrospective Analysis

At the end of the data collection process, I had notes on the content of each teaching episode video, including the tasks posed and general solution strategies of each participant. The next step was for the research team to make sure the videos and collected student work were easily accessible to everyone involved in analysis. This involved converting all the videos so that we had one Quicktime video containing both video views for each teaching episode. Some of these Quicktime videos use a picture-in-picture design, with the close-up of student work as the background, while others had the two videos views side-by-side. In episodes when screen capture programs were used for two different computers, there was a separate Quicktime video made for each participant. In addition, any artifacts such as written student work, paper strips or strings were filed by date.

After this organization of data, I re-watched all the videos, in chronological order, for each participant (not including Lily) and noted critical events in each episode for that participant. A critical event was one in which I did not yet have a plausible explanation for the students' behavior, when the students' scheme or operations appeared to have undergone a shift, or when a student made a novel *necessary error*. There were not usually more than one or two critical events in each episode. During this process, which lasted about five weeks, I had five meetings with Dr. Steffe and other members of the research team to get their feedback and insights into what operations or schemes we could attribute to the participants based on the critical events I had identified.

At the end of this overview of the data, I identified a set of related critical events and narrowed the goal of my analysis down to forming second-order models of the students' mathematics in these critical events. As I developed plausible hypotheses, I would reexamine relevant teaching episodes outside the critical events to test these hypotheses. This process lasted for several months. I came up with some useful categories of mathematical behavior to help me make sense of the data, but I was not satisified with my hypotheses, and I kept finding contraindications when I started to write up my findings. Therefore, I decided to re-analyze all of the additive reasoning episodes along with sections of the students' fraction work using my new categories. At the end of this process, I wrote up a summary of critical events for each student in the signed addition episodes. I then abstracted out more general statements I could make about the students' behavior as a group. These formed the anchor for my conclusions section, and my findings section is built up around the indications, and occasionally possible contraindications, of my theories.

Naturally, as I began writing up my findings again, analysis continued. Although I would still find new interpretations for the data as I was writing, I gathered confidence in my new conclusions because they generally required very little modification in order to accommodate the new data. At all stages, I strove to form alternative hypotheses. When appropriate I have included alternative hypotheses in my analysis that I was not able to disconfirm. However, I have hopefully given convincing arguments of why my current hypotheses are more viable.

Organization of Data Analysis Chapters

The remainder of the chapters center on the results of my data analysis. Chapter 4 describes my finding about each student based on their initial interview. Chapter 5 describes findings from the first month of the teaching episodes in which Adam and Michelle were paired together and Justin and Lily were paired together. The bulk of the analysis occurs in Chapters 6 and 7 in which I analyze the new pairings working on signed addition problems. I end with a final chapter in which I synthesize my analysis of the four participants' constructions and the cognitive and social factors that affected these constructions.

CHAPTER 4

INITIAL INTERVIEWS

Based on my pilot study, I hypothesized that the construction of ENS is necessary, but not sufficient, to reorganize the concept of addition and subtraction to encompass additive relationships among signed quantities. Hence I wanted to make sure to select participants who had at least constructed an explicitly nested number sequence (ENS). One of my main goals during the initial interviews was to determine the students' current number sequence (TNS, ENS, or GNS). In addition, I was interested in how many levels of units the students could assimilate a mathematical situation with. Finally, I was interested in putting the students in missing addend situations to determine their facility with formal and informal subtraction schemes.

The first two goals were approached both from discrete and continuous fractional situations. Steffe and others (Hackenberg, 2005; Norton & Hackenberg, 2010; Norton & Wilkins, 2010; Steffe & Olive, 2010; Tunc-Pekkan, 2008) have investigated the reorganization of number sequences for the construction of fraction schemes as well as the relationships between assimilatory level-of-units structures and various fraction operations as discussed in Chapter 2. Hence I am using fraction tasks to get information about the students' underlying number sequences and other assimilatory structures.

Adam's Initial Interview

I conducted Adam's initial interview on January 26. Adam gave an indication on one task that he had constructed the *splitting operation* and, therefore, had constructed an ENS. At the time, however, I thought that he had given both an indication and confirmation of having constructed an ENS because I had inappropriately attributed strategic reasoning to him in two of the natural number tasks involving money and because I had inappropriately attributed equi-partitioning operations to him in a fraction task. Here I will discuss his work on those three tasks, on the task in which he indicates the construction of a splitting operation, and on a third fraction task that was meant to encourage distributive partitioning. I provide his behavior on the distributive partitioning task in order to facilitate comparison of his mathematical behavior with the other three participants.

Adam Assimilates With Two Levels of Units

Protocol 4.1 reports the first natural number task, and after it I explain why I thought his behavior indicated construction of an ENS at the time and why I have since changed my mind. While this task technically deals with decimal amounts of dollars, the students are all clearly thinking of it in terms of natural number values of cents and dollars. Therefore, I think it is fair to refer to it as a whole number task. (In the protocols throughout this chapter, A, J, L, M, T, and W will indicate that Adam, Justin, Lily, Michelle, the teacher/researcher, or the witness is the speaker or actor, respectively. The symbol, […], indicates that dialogue has been omitted. Ellipses between lines of dialogue indicate omitted dialogue. Actions will be set apart in brackets.)

Protocol 4.1: Adam makes change for one dollar.

T: I'm going to buy some gum [holding a dollar bill]. You're like the store owner, and you're selling me some gum. […] I give you a dollar [handing him the dollar], but the gum only costs 63 cents. How are you going to give me my change?
A: [After 8s] How much I'm going to give you?
T: How much change, yeah, what change.
A: 37 cents.
T: How did you get that?
A: Because you gave me a dollar, if it [the cost of the gum] was 60 cents, I'd be giving you 40 cents back, but since it's 63 cents, you get 37 cents back.
T: And how did you know it was gonna be 37? You said if it was 60 cents, it'd be 40.

A: I subtracted 100 minus 63.
T: Really?! You did it pretty quickly.
A: [Nods.]

When the participants in my pilot study said that they added or subtracted without further elaboration, as when Adam says, “I subtracted 100 minus 63,” it often meant that they had mentally performed a *traditional addition or subtraction algorithm*. I use *traditional subtraction algorithm* to refer to the method in which any digit of the larger number which is larger than the corresponding digit of the smaller number is increased by 10, which is borrowed from the next largest place value’s digit, followed by subtracting each place value to find the difference. The *traditional addition algorithm* is, similarly, the process of adding up each place value, right to left, and recording 10 of any value by adding 1 to the next place value (carrying). Although some of the participants may have understood the mathematics behind the traditional addition algorithm, I am almost certain that Adam, Justin, and Michelle did not understand the mathematics behind the traditional subtraction algorithm. In my analysis of Justin’s initial interview, I will give two examples (one is in Protocol 4.6) of Justin struggling with this algorithm, and we can see there that he is aware of the place values, but does not know how he coordinates the place values in the algorithm. Adam and Michelle never refer to place value when discussing the traditional subtraction algorithm. Therefore, when I say that a student is using the traditional subtraction algorithm, I simply mean the procedure involving digits and their positions, not an awareness of how place value is involved in “borrowing.”

My incredulous reply when Adam said, “I subtracted 100 minus 64,” indicates my disbelief that he had performed the traditional subtraction algorithm. If Adam had been using a traditional subtraction algorithm, he would have determined the units’ digit of the difference before determining the tens’ digit. However, in his explanation in Protocol 4.1, he determined the tens' digit first. Therefore, I do not think he used the traditional algorithm. At the time, I thought that Adam's explanation was referring to the use of *strategic reasoning*. In particular, I thought that his statement, "if it was 60 cents, I'd be giving you 40 cents back, but since it's 63 cents, you get 37 cents back," meant that he had found a number close to 63, namely 60, solved the problem using that number and then adjusted the result appropriately for the actual given value of 63. This would imply that he could operate on the embedded composite unit of 40 to decompose it into 37 and 3 and then add the 3 to the embedded unit of 60, which would require an explicit awareness of the nested structure of his number sequence as well as the ability to disembed embedded composite units. In other words, such strategic reasoning would imply the construction of an ENS (cf. Steffe, 2010b, p. 46).

I now realize that he may have been counting down from 100 by six 10's and three 1's in order to determine 100 – 63. He took long enough to answer to allow such counting down, and it would make sense for him to reference the difference of 100 and 60 as an intermediate step. The way he said it seems a little unusual in that he seemed to be comparing 100 – 60 to 100 – 63, but he could have been explicitly monitoring his counting down as opposed to strategically reasoning. Furthermore, when I asked for further clarification, it would make sense that he would simply say, "I subtracted 100 minus 63," because, for Adam, both that phrase and the activity of counting down from 100 to 63 probably implied taking away 63 from 100. As I will discuss further in "Justin's Initial Interview," such counting down by 10's and 1's would be possible even if Adam had only constructed a tacitly nested number sequence (TNS), although it does imply that he assimilates his number sequence with two levels of units, 10's and 1's. Therefore, Adam has at least constructed a TNS and can assimilate with two levels of units.

The second whole number task was directly after the first. In it, Adam gives further indications about his method of determining differences and his limitations.

Protocol 4.2: Adam makes change for 10 dollars.

T: I give you 10 dollars [handing him a 10-dollar bill], and the gum only costs 7 dollars and 41 cents. What change will you give me?
A: 3 dollars and 19 cents [smiling].
T: 3 dollars and 19 cents? All right. Tell me how you got the 19 cents.
A: What was the total?
T: 7 dollars and 41 cents.
A: Wait. It wouldn't be 19.
T: [After 90s] What do you think about your $3.19?
A: [Shaking head and smiling] It's not right.
T: How do you know it's not right?
A: Because change goes up to 99 cents before it's a dollar, and I only went up to 60.
T: Ohhh. OK. All right. So you have a new answer?
A: [Shakes his head.] Not yet. I'm still trying to figure it out.
T: [After 40s] This is a harder one. How are you trying to do it in your head?
A: 100 minus 41.
T: A hundred minus 41? And you're trying to do the paper and pencil in your head?
A: [Nods.]
T: That one's hard to do in your head like that. Can you think of a different way to figure out 100 minus 41?
A: [After 8s] It'd be 59.
T: How'd you get that?
A: 'Cause 100 minus 40 is 60, and then you just take 1 off of that. It's…59 [nodding].
T: Why did you take 1 off of it?
A: Because it's 41 cents and not 40.

First, note that when Adam attempted to use the traditional algorithm to find 100 – 41, it was taking him at least 40s to finish and he possibly spent over 2 minutes on the algorithm. However, when he did not use it, he finished his calculation in only 8s. This confirms that his quick answer in Protocol 4.1 would not have been the result of the traditional algorithm. In his statement, "Because change goes up to 99 cents before it's a dollar, and *I only went up to 60*," Adam uses counting up language. Hence I hypothesize that he got 19 cents by counting up from 41 to 60. His switch to the traditional algorithm after his original answer may have been prompted by a distrust of his reasoning skills, given his recent mistake of using 60 in place of

100. However, once he started reasoning through the problem again, he seems to subtract 40 and then "just take 1 off of that…because it's 41 cents." Again, he could be engaging in strategic reasoning that adjusts the difference, 100 – 40, to get the difference, 100 – 41. However, his language is just as compatible with the idea of counting down by four 10's and then a 1. He may have enough practice with multiples of 10 to know 100 – 40 as a subtraction fact (similarly for 100 – 60 in Protocol 4.1). In either case, if he is not strategically reasoning, his behavior indicates the construction of a TNS but not necessarily an ENS.

Adam did not coordinate his calculation regarding dollars and his calculation regarding cents. I assume that he gets the $3 by subtracting $7 from $10. When he then calculates 100 – 41 to find the number of cents, he clearly knows that the 100 cents represents a dollar because of his earlier statement, "change goes up to 99 cents before it's a dollar." However, he does not adjust his 10 – 7 calculation to 9 – 7 to account for the dollar he is using in the cents calculation. If Adam were able to assimilate mathematical situations with three levels of units, then he should be able to assimilate not only the 10's and 1's in the number of cents, but also the dollars as being made up of 100 cents with the embedded 10's and 1's structure. Instead, he seems to work with dollars and cents separately. This is an indication that he did not assimilate this particular situation with three levels of units, and a weak indication that he cannot assimilate with three levels of units and has not yet constructed a GNS. I will put him in many different situations during the teaching episodes to see if he shows indications of operating with a GNS.

Adam's Fraction Schemes

Protocol 4.3 gives Adam's mathematical behavior on a partitioning task. Because I did not see consistent segmenting behavior in the initial interview, and I knew that Adam had constructed composite units based on his work in the natural number tasks, I originally hypothesized that Adam was equi-partitioning in both tasks. However, I now hypothesize that he was *simultaneously partitioning*, as I will explain below. This is a significant difference in that simultaneously partitioning involves assimilation with composite units, but not with composite units made up of iterable units. Therefore, a student who has only constructed a TNS can develop a simultaneous partitioning operation, but not an equi-partitioning operation, which requires assimilation with iterable units. With regards to Adam, this means that I cannot conclude that he has constructed an ENS based on this partitioning task.

Protocol 4.3: Adam shares a candy bar among five people.

T: [Puts a strip of paper down.] Let's pretend this is a candy bar, and […] you have four people that you're going to share this with. So you have five people, you and four friends, that are sharing this candy bar. Can you try to draw me just where I should cut off *one* piece, one fair share.
A: [Stares at bar as he makes four vertical motions over the bar over the course of 6s.]
T: Try just drawing one line first.
A: [Partitions off a piece that is closer to 1/7.]
T: All right. That looks reasonable. Now how could you check and see if that's a fair share or not?
A: Keep drawing that and see if I get five.
T: You'd keep drawing that same one? OK. Go ahead and do that.
A: [Draws in two more lines using visual estimation. See Figure 4.1.]
T: All right. Before you draw another one, can you tell me if that's going to be a fair share?
A: [Shakes his head.] No.
T: It's not? Is it too big or too small?
A: One's going to get a bigger one than everyone else.
T: […] Should you make your piece smaller or bigger?
A: Bigger.
T: […] You want to try again on the back?
A: [Nods.]
T: [Flipping over strip.] OK. Try to just look at it real hard, think about it for a second, and then I want you to make another estimate.
A: [Stares at bar. Looks down at pencil. Looks at bar again. Marks off approximately 1/5.]
T: All right. You want to check that one?
A: [Moves pencil over the piece and then moves pencil over again as if segmenting off another piece. Erases estimate. Marks off approximately 1/6. Compares it to his estimate on the other side. Using visual estimates, marks off two more pieces, pauses and presses

lips together, and then divides the remaining bar in half. The five pieces are not of equal size. See Figure 4.2.]
T: That’s closer.
A: Yeah. [Dispirited.]
T: Do you think that one was too big or too small?
A: [Pointing to the first one piece.] Too small.
T: […] Your first one might have been a good guess [using fingers to measure off]. That’s a little too big. Well, you get the idea. I get the idea. You’re getting close.
A: [Smiles.]

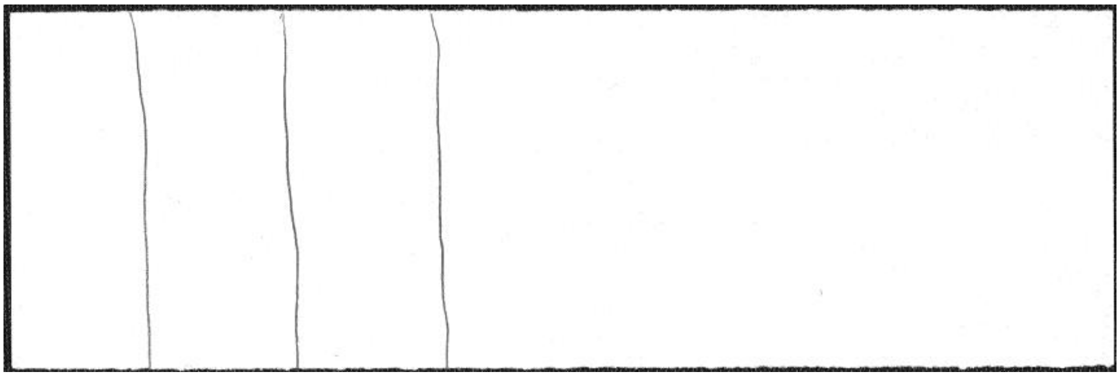


Figure 4.1. Adam’s original attempt at sharing one bar among five people.

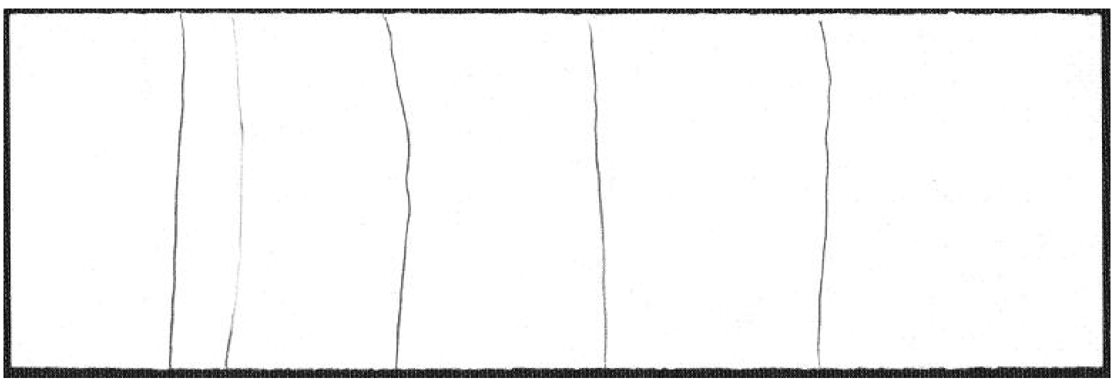


Figure 4.2. Adam’s second attempt at sharing one bar among five people.

In Steffe’s (2010f) account of a student, Laura, who had constructed a simultaneous partitioning scheme, one of the characteristics of her behavior that allowed him to attribute partitioning to her was her ability to make uncannily accurate estimates for partitions involving ten or less pieces. As you can see in Figures 4.1 and 4.2, this is not the case with Adam. However, neither in Protocol 4.3, nor elsewhere in the initial interview, does Adam display classic equi-segmenting behavior. By that I mean that he does not appear to be measuring things off, either with his eyes or fingers in order to make his original estimates and does not feel the necessity to segment off using his estimate or make the other divisions. In fact, when he makes his first estimate in Protocol 4.3, he is staring intently and unmovingly at the *bar* as he makes motions mimicking the drawing of vertical lines over the bar. He is not looking at the individual partitions he would be making with his imaginary lines. Nor does he appear to look at his hand as he is partitioning. My interpretation of this behavior is that he is focusing on the totality of the bar, using his composite unit of 5 to mentally project a partitioning onto the bar, and then indicating with his hands where those imagined partitions would be. In other words, he is partitioning, not segmenting. That implies that he has constructed at least a simultaneous partitioning operation and possibly an equi-partitioning operation.

One of the key distinctions between simultaneous partitioning and equi-partitioning is the ability, after equi-partitioning, to disembed an estimate and iterate it to form a new bar that can be compared to the original bar. In the current setting, when the students have only a strip of paper and a pen, an equi-partitioner would still be able to mentally disembed the estimate and mentally iterate it. This thinking might manifest itself in the student's use of the length of an estimate to measure off the indicated number of lengths and compare the resulting length to the original bar. For example, see Justin's behavior in Protocol 4.8 in which he uses his fingers to measure the bar using the length of his estimate. Furthermore, an equi-partitioner might use language to indicate a comparison of the total bar to iterations of the estimate. For example, consider Justin's explanation of how to check an estimate from Protocol 5.8: "You'd have to count it and see if it [the original bar] equals it [the estimate] eight times."

Adam’s ways of operating do not indicate that he is mentally disembedding his estimate from the bar. Justin’s language contrasts with Adam’s language in Protocol 4.3: “Keep drawing that [the estimate] and see if I get five.” On first look, both participants seem to be expressing the same general idea. However, Justin explicitly refers to the original bar and compares it multiplicatively with the estimate. Adam’s wording avoids an explicit reference to the whole bar and is additive in the sense that he gives a sense of sequential, not simultaneous, action.

Justin’s reference to the original bar implies that he is able to step back from his operations within the bar and take the bar as an object of comparison. A student who is simultaneously partitioning, on the other hand, is never operating outside of the bar: The estimate is not seen as a separate bar or length that can be compared to the original bar. Instead, if segmenting operations are activated at all, the student would only be aware of using the estimate to form partitions *within* the bar. Not only does Adam use language that contraindicates an ability to disembed and compare the estimate to the bar, his gestures imply that he is working in the bar in that he actually draws in further partitions as opposed just measuring off the length of the estimate, as Justin does.

Justin’s use of multiplicative language implies an iteration of the estimate. Adam does seem aware that he is drawing something equivalent to the estimate each time in that he says, “Keep drawing *that*,” but equivalence is different than the sense of equality that comes out of iterating. In fact, two times when Adam checks his estimates, his behavior indicates that he does not see the partitions as identical or as separate from the original bar. The first time is when he notes that, “One’s going to get a bigger one than everyone else,” implying that his goal is to make the partitions and compare their size. Similarly, after we flip the bar over and he is making an estimate on the second side, he starts drawing in more partitions, and it appears that he intends for the first three partitions (including the estimate) to be the same size. Then he pauses and looks upset when he sees that a lot of the bar still remains for the last two pieces. Instead of continuing to measure off with the estimate's length, he divides the remaining segment of the bar in two approximately pieces. This implies to me that his goal is, again, to draw the partitions *within* the bar and see if they are equal. He experiences perturbation when he has the opposing goals of making the partitions the same size as the estimate and using up the whole bar, so he divides up the last segment in half in order to use up the whole bar and avoid having one partition that is much bigger than the others. This shows, though, that he is attempting to make five partitions within the *given* bar that are the same size. He is not attempting to compare the combined length of five iterations of the estimate to the length of the original bar.

Adam's lack of indications of mentally disembedding and iterating his estimate and his contraindications of disembedding and iterating his estimate make it impossible for me to conclude that he is equi-partitioning. In fact, even though he sees me use my fingers to measure off one of his estimates, he never incorporates this strategy into his own mathematical ways of operating in later partitioning tasks in the initial interview. Therefore, I only attribute the ability to simultaneously partition a continuous bar at this point, which only implies the construction of a TNS.

Protocol 4.4 presents the only task in which Adam gives a strong indication of having constructed an ENS. In particular, he uses the splitting operation to solve a problem, and the splitting operation implies the use of iterable units, which are a hallmark of an ENS.

Protocol 4.4. Adam splits.

T: [Puts a piece of paper in front of Adam with an unmarked rectangle drawn underneath the question, "This is your bar. Your bar is 4 times as long as my bar. Draw my bar."] We're going to pretend this is your candy bar [outlining the candy bar on the sheet], and your bar is four times as long as my bar. Can you draw me *my* candy bar?

A: [After 14s, makes three vertical marks in the air above the printed bar. He then draws a bar underneath the printed bar that is a little smaller than 1/4 of the bar.]
T: Excellent! [Talking in part to the witness] And I saw how he made marks here to check? Is that what you were doing?
A: [Nods.]
T: Very nice. How many of my candy bars would I need to get a candy bar as big as yours?
A: 4.

This is a commonly used task to determine whether a student has constructed the splitting operation. When a student, such as Adam, has constructed a splitting operation, he simultaneously applies his partitioning and iteration schemes in order to form partitions that contain records of their potential iteration in order to re-form the whole; iteration and partitioning have now become inverse operations to the student and the potential for both operations is implied when carrying out either operation. For example, in Protocol 4.4, I present the task as an iteration task: I present the iterative statement, “Your bar is *4 times as long as* my bar.” In order for Adam to solve this task, he needs to be able to generate the image of iterating a bar four times. However, he has to do more. He has to be able to imagine iterating a hypothetical bar that produces the given bar, which means that he has to *partition* the given bar in order to solve what at first appears to be an iterating task. If Adam can do this, we can assume that he has constructed the splitting operation. Often, a student who has not constructed the splitting operation will draw a bar bigger than the given bar.

In Protocol 4.4, Adam thinks for quite a while, 14 seconds, before making any motions, so it might be possible that he tried visualizing iterations of the given bar first, saw that that answer did not work and realized the new bar must be smaller. However, in the video, I can see that Adam brings the paper close to him and seems to stare directly at the bar without moving his head or fingers during the 14s. Therefore, I think it is unlikely that he started out with an unreasonably large estimate in that he does not have to make any noticeable movements in his line of sight in order to imagine combining four bars the same length as his estimate together. Nonetheless, it is possible that he determines the new bar must be smaller than the drawn bar without splitting. In order to actually come up with an estimate, he could either use some trial and error visualized segmenting or he could partition simply because it is an operation that will make a smaller bar. Using his composite unit of 4 from his TNS as his partitioning template would be natural given that 4 is the only number given in the problem.

Right before he draws his answer, he quickly makes three vertical slashes in the air above the bar. He did not seem to be segmenting; he appeared to be projecting a composite unit onto the bar through simultaneous or equi-partitioning. But he then uses that partitioning to make a *separate bar* that can be repeated to form the first, which implies that he was engaging in equi-partitioning as opposed to simultaneous partitioning. If he had been simultaneously partitioning he would not be likely to disembed a partition and use it to form a new bar to compare to the original partitioned bar. As discussed in Chapter 2, equi-partitioning implies construction of an ENS, and hence iterable units. Therefore, Adam appears to be combining partitioning and iteration to solve the problem, which implies splitting. However, it is possible that he was simultaneously partitioning using a composite unit from his TNS to do so, and that he did not disembed his estimate to solve the bar, but simply realized that if all the partitions were the same length, then it is as if the given bar is 4 times as long. He would then have drawn a new bar only because the directions were to do so and the new bar did not represent a disembedding.

In conclusion, it would be unusual for a student to solve the task in Protocol 4.4 with only a TNS, but I can imagine such a solution strategy, so I will not conclude from this one task that Adam has constructed the splitting operation. If he had appeared to disembed a partition and iterate it, then there would be a stronger indication that he is splitting, but because we only see partitioning behavior, my interpretation of the data is necessarily ambiguous.

In Protocol 4.5, Adam gives contraindications of having constructed a distributive partitioning scheme. I am including this example because it is one of the tasks that seemed to differentiate Justin and Lily from Adam and Michelle, and therefore it was an important task with respect to my decision-making about the original student pairings in the teaching experiment.

Protocol 4.5: Adam tries to share two candy bars among three people.

T: [Putting two identical paper strips in front of Adam] You have these two candy bars, and you have three friends that want to share the two candy bars. How are you going to find a fair share for one person?
A: [Puts the strips end-to-end. After 20s, draws a vertical line down the middle of one bar.] One of them would get half of one of them [pointing to the bar he divided], and the other two would split the other [pointing to unpartitioned bar].
T: And would that be fair?
A: ‘Cause they’re getting the same amount.
T: There’s *three* people.
A: Right! If you split that one [pointing to partitioned bar]—
T: So one person gets…
A: and save that one [right-hand partition on first bar] and split the other [pointing to unpartitioned bar]…
T: Oh. And you just had this left over where no one gets that one [covering the right-hand partition]?
A: [Nods.]
T: They have to eat all of the candy bar.
A: [Erases line.]
T: You were right, but now they have to eat all the candy bar.
A: [For one minute, alternates between looking intently at the bars and looking up as if thinking.]
T: Are you stuck?
A: [Nods.]
T: [Goes on to help him split the remaining half into 3 equal pieces.]

Adam does divide both bars into halves and in that way he is partitioning distributively, but when I say that a student can *distributively partition*, I mean it in a technical sense of the phrase in which the distribution refers to the realization that dividing a whole up into *n* parts and taking one of those shares is equivalent to dividing each partition of the whole into *n* parts and combining one share from each partition. In that sense, Adam is not distributively partitioning.

Summary of Adam's Mathematics

Unfortunately, Protocol 4.4 is the strongest indication we have that Adam has constructed an ENS. Given that there is a possibility he managed to solve the splitting task without splitting, I looked for confirmation of an ENS throughout his teaching episodes, but I did not find any strong indications of ENS elsewhere. Based on the initial interview itself, I can conclude only that he has constructed a TNS and can assimilate with a two-levels-of-units structure. It would be unusual for a student to solve the splitting task, explicitly keep track of counting by tens and ones, and simultaneously partition with a TNS as the most sophisticated assimilatory number sequence. However, I think that there is a good chance that this is the case with Adam. One of the characteristics of Adam as a problem solver that was apparent even, and especially, in the initial interview, is the willingness to concentrate intently on a task for long periods of time without any intervention or prodding from the teacher. For example, in Protocol 4.2, he thinks silently for 90s before I interrupt him and after my intervention he thinks silently for another 40s. During this time he appears to be deeply concentrating, not day dreaming, sulking, etc. Again, in Protocol 4.5, he thinks for an entire minute silently before I interrupt. In both of these situations, he probably did not solve the problems he was attempting to solve during those silent periods. Nonetheless, these long periods of intense concentration imply a willingness to endure perturbation and engage in problem solving. This is in contrast to Michelle, for example, who gets easily nervous and defensive when she is unsure of a solution strategy. I hypothesize that Adam's willingness to engage in problem solving for long periods of time has allowed him to maximize his mathematical operations, thereby appearing at times to have more sophisticated ways of operating than he actually does.

Justin's Initial Interview

Another researcher conducted the initial interview with Justin on January 19. Justin gave several indications of having constructed an ENS, but did not give any clear indications that he could assimilate with three levels of units. The first indication of construction of an ENS was in the second task of the interview, described in Protocol 4.6.

Protocol 4.6: Justin counts up by 10's and 1's.

T: Suppose we have, say, 29 pennies. OK?
J: [Nods.]
T: And somebody else came along and gave us some money and now you have 41. So how many more pennies would you have?
J: [Over the course of 4 seconds he moves his pencil to the right in little dips as if counting tick marks along a number line. He starts with one big dip, does two fast but distinct dips followed by about four less distinct dips and then slows down again to end with two distinct dips at the end.] 12.
T: How'd you get that, for heaven's sake?
J: 'Cause I counted in my head.
T: Well, tell me how you did it in your head.
J: I added 1 to 9 and I know you add 10 'cause that's 40, and then another 1 and it was 12.

Justin's Number Sequence and Subtraction Algorithms

The first aspect of Justin's behavior in this task that I would like to highlight is his explicit use of double counting that includes a composite unit in order to determine the number of units in Justin's (whole) number sequence that starts after 29 and ends at 41. Double counting in and of itself is only a strong indication of the construction of a TNS (Olive, 2001). However, Justin's counting motions and explanation imply that he assimilates with a number sequence that is segmented into both units of 10 and units of 1, which is an indication of an ENS. His first counting motion represented an addition (or move) of 1 that got him up to 30 in the number sequence. This first counting motion was differentiated in its quality from the other counting motions. He then begins to count up from 30 to 40, but he curtails his counting motions after about four dips. The less distinct dips referred to in Protocol 4.6 seemed to signify the act of counting 10 times without the need of actually doing so. The fact that he does not need actually count 10 times is further indicated by his use of the composite unit 10 in his explanation in the last line of Protocol 4.6. At the end, he uses distinct dips to indicate the end of the composite unit of 10 counting acts and the extra 1 needed to get to his goal. Taken together, I interpret this behavior to indicate that Justin was aware of multiples of 10 as strategic elements in his number sequence before he began operating in this situation, supporting my hypothesis that composite units of 10's and 1's were an *assimilating structure* (see Chapter 2) for his number sequence.

A TNS student can count up by 10's when asked to determine how many 10's are in a number, for example, and could potentially have decided to try counting up 10 as a shortcut. Justin, however, does not seem to be testing out the use of 10, he seems aware that there is exactly one 10 between 29 and 41 even before he has counted to 40. I am concluding this in part because of the lack of hesitation in any of his actions. He does not have to think if he should add 10 or 1 when he gets to 40, he seems to know before he gets there that he will be 1 away from his goal because there is no hesitation in between the final two dips he makes. In addition, Justin takes the result of counting up by 1, counting up by 10, and counting up by 1, and unites them to form a total of 12. Again, a TNS student can take a composite unit that results from counting counting acts as material for mental operations, but Justin's fluency in switching between units of 1 and 10 implies to me that a structure of 10's and 1's inhabits his number sequence before action. The fact that he did all of this quickly and was able to explicitly talk about the quantities implies that he has is aware of the general structure of the situation before he begins counting.

Therefore, I would tentatively hypothesize that Justin has constructed an ENS based on his mathematical ways of operating up to this point.

Justin's double counting is one way to keep track of how much he is counting up to get from one number in his number sequence to another. *Counting up* using a combination of units of 10 and units of 1 was a common strategy that Justin uses in additive situations. For example, consider this example from the March 16 episode in which he explains how he found 172 – 39:

> I added 70 to it [the 39]. I pretended like there was no 9 there [he ignored the ones' digit for now], so I added 30 to it, the 70, which would be a hundred. It'd be up to a hundred. Then I added 60 to it [the 70], and I got 130, then I added the 9 back [took the 9 back into consideration, giving 39 + 130 = 169, so far]. The [one hundred sixty-] 9 plus 3 would be 172, which would be 133 plus 39.

In this example, he determines the difference by adding 70 + 60 + 3 and coordinating the sum at each juncture with the number sequence from 39 to 172. Further examples of Justin explaining how he used 10's and 1's to count up can be found in Protocols 6.15, 6.18, 6.22, and 6.23. However, there are many more examples in the data that I do not document here. In fact, the only times he describes how he determined a difference or sum and does not refer to counting by 10's and 1's are when he describes using a *traditional addition or subtraction algorithm*.

Justin does give further indications that he has constructed an ENS in the initial interview. For example, in the task following the discussion in Protocol 4.6, Justin is asked how much change he should give if the teacher/researcher pays him a dollar for a 63-cent pack of gum, he immediately says, "37." The speed of the answer again implies that his assimilating number sequence is segmented into both units of 1 and units of 10 simultaneously and that he uses that structure to quickly solve this problem. Based on his future predilection for the counting up strategy, I imagine that he did use a quick counting up strategy here in which he counted up first by tens and then by ones (or by 7 ones and then 3 tens) to get from 63 to 100. Using the traditional subtraction algorithm, on the other hand, would have required borrowing, which would take some time to think about and keep track of. Therefore, I infer that Justin was reasoning additively with 10’s and 1’s as opposed to relying on the traditional subtraction algorithm. Based on the numerous indications of the construction of an ENS, I hypothesize that Justin had constructed an ENS, but I look for further corroborations in the teaching episodes.

Later in the interview, the teacher/researcher asked Justin three similar questions that involved making change in a money context. The first one I have discussed. It asked how much change he would give if someone paid $1 for a 63¢ item. The second task asked how much change he would give if someone paid $10 for a $7.41 item. This time he immediately began writing with paper and pencil to attempt the traditional subtraction algorithm. Although he did attempt to “borrow,” he ended up with $3.58 by way of two errors: He changed 10.00 to 9.99, losing 1 cent from his answer, but did not add the 1 cent back in elsewhere or later, and forgot to subtract the 7 from 9 instead of 10, even though he had 9 written on his paper. (See Figure 4.3.) Justin’s classroom teacher consistently represented him as a good student who did not struggle with mathematics. Therefore, I suspect that Justin’s computational mistakes may have been due to nerves both here and elsewhere in the initial interview. I very much doubt he was consistently making the same errors in his math class based on his teacher’s appraisal. Nonetheless, the first error, in particular, seems to imply a lack of understanding about the place value concepts behind the algorithm.

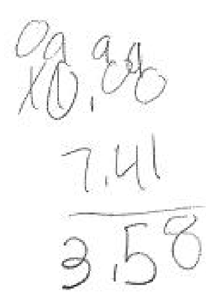


Figure 4.3. Justin attempts the traditional subtraction algorithm.

Given the teacher/researcher's personal experience that students often do not understand the mathematics behind the subtraction algorithm, the teacher/researcher did not investigate the source of these errors, but instead decided to see how Justin would try to solve the problem if he were asked not to use paper and pencil. Therefore, the teacher/researcher gave a new problem without any paper or pencil available to see if he could get richer data on Justin's additive reasoning. Protocol 4.7 reports the resulting exchange.

Protocol 4.7: Justin Struggles With Mental Multi-Unit Subtraction

T: This time, let's say that a pack of gum cost—it's really good gum, you know—it costs six dollars and 53 cents. Six dollars and 53. And I give you a 10-dollar bill [handing him a 10-dollar bill].
J: [For 7 seconds he whispers to himself while making writing motions in the air as if he were doing the traditional subtraction algorithm.] 37? Four dollars and 37 cents?
T: How'd you get that so fast?
J: 'Cause I subtracted. I knew it'd be 4 dollars 'cause it was 6. And then I just subtracted 37 from 100.
T: Wow. 37 from a hundred. How would you do that in your head? I'm just interested in how you're thinking.
J: Because I know that a hundred minus 7 would be 3. Like 79…Wait. Was it 77?
T: OK [confused].
J: I subtracted a hundred from the number and I knew 70 from it would equal 30 and I knew the rest would be, like, 7.
T: [Nods]
J: 'Cause it was 3.

My interpretation of Justin's behavior is that, during the first seven seconds, he attempted to visualize writing the traditional subtraction algorithm. His unsure tone points to an awareness that he had difficulty carrying out the traditional algorithm without pencil and paper. He also starts conflating quantities immediately upon starting his explanation: He confuses the number of cents in his answer with the number of cents in the subtrahend, but then goes on to forget the subtrahend completely, except for the presence of the 7 that he was most recently working with. He does keep track of place value in his explanation. In other words, he does not just think about subtracting 7 from 0, but instead is aware that he is subtracting 7 from 100. Again, later he talks about subtracting 70 from 100 instead of just thinking about subtracting 7 from 0. However, he does not coordinate those place values in a way that would give him the correct answer; he is aware of the composite units he is working with, but does not use their relationships, such as the equivalence of one dollar and ten 10-cent units, in operation.

Despite Justin's awareness that he was having trouble keeping track of the numbers involved when subtracting 6.53 from 10.00, he did not go back to his method of counting up that he used successfully in Protocol 4.6 and most likely had used successfully in the first four tasks of the initial interview. His decision to use the traditional algorithm as opposed to counting up when moving from problems involving quantities of at most 100 cents to problems involving multiple dollars and multiple cents could indicate a foreseen discomfort in trying to keep track of these quantities if he were to try to find the answer by counting up or strategic reasoning. In other words, he had a sense that using his counting up strategy might be overly challenging in these problems that involve three levels of units because he had not yet generalized his informal counting up strategies to these situations. As with Adam, if he is not assimilating the situation with three levels of units, he would not necessarily be able to see the 10-cent structure within the dollars simultaneously with an awareness of the multiple 100-cent, or dollar, structures. He would not come up with the strategy of adding up by dollars, then 10-cent units, then single cents without assimilating the situation in terms of all three quantities simultaneously. He seems, like Adam, to work with dollars and cents separately.

An alternative hypothesis to explain why he switched to a traditional algorithm when dealing with larger quantities is that he has had subtraction problems in the last year or two of school that utilize larger quantities of money, such as $10, so that the statement of the task activated his recognition template for a situation of a *procedural scheme* (see Chapter 5, "Michelle's Fraction Addition Scheme") in which he applies the traditional subtraction algorithm. It could be that he would have the ability to reason through problems such as the one in Protocol 4.7, but he sees the proper solution method as the traditional algorithm in these cases. Nonetheless, I do not see strong indications at this point that Justin is able to assimilate with a three-levels-of-units structure, only that he can assimilate with a two-levels-of-units structure. Justin will eventually show the ability to assimilate with three levels of units, but that will not occur until the teaching episodes.

Justin's Fractional Operations

In the first partitioning task, Justin gives a fairly solid indication that he can equi-partition an unmarked length.

Protocol 4.8: Justin equi-partitions.

T: [Puts blank rectangle of paper in front of Justin.] Five people are going to share this candy bar. That's going to be a Snickers bar. I need you to mark off the share for *one* person.
J: You'd have 4/5 left.
T: You're going to have…
J: Four—
T: You're going to have five people sharing. I want you to mark off the share for one person [pointing to the paper bar].
J: [Immediately makes a mark that would be about 1/5.]
T: Now, how do you know that would be a fair share?
J: You'd have to go like this. [Measures length of the share he marked off with two fingers and uses that length to measure off the rest of the bar. After measuring off five lengths, he stops short of the end of the bar and picks up his pencil.]
T: You want to try again?
J: [Makes a slightly bigger estimate, measures it off, and still comes up a little short. Starts erasing.]
T: [Chuckling] That's kind of hard to do, isn't it?
J: [Makes a new estimate, measures it off, and gets almost exactly five lengths in the bar. Measures it off again backwards (from right to left) to double-check his work.] Five.

When Justin makes his initial estimate, he makes the mark almost immediately and relatively accurately. That is, it is closer to 1/5 of the bar than to 1/4 or 1/6. This implies that he is not equi-segmenting, sequentially estimating segments, before he draws the initial mark.

Instead, the speed of his estimate implies that, in order to make his estimate, he projected a composite unit of 5 onto the bar in which the partitions appear *simultaneously*.

Furthermore, he independently decides to iterate the length of his estimate five times and compare the length of the combined iterations to the length of the original bar. I claim that he is iterating as opposed to segmenting because he does not seem to feel the need to form the other partitions in order to compare them. This reflects foreknowledge that the continuous units he is making are all identical. I hypothesize that he is aware of his estimate as both part of the original bar and as the iterating unit for a second, imagined bar whose length he is comparing to the original. However, it is possible that he was segmenting the original bar and not iterating to form a new bar when he checks his estimate. If he is, indeed, equi-partitioning, then this would provide further confirmation for his construction of an ENS.

He gives several more indications that he has advanced partitioning operations during the initial interview. For example, he is able to solve the problem, “This bar is 7 times bigger than your bar. Draw your bar,” which generally indicates the splitting operation. However, his initial attempt is far too big, bigger than 1/5 of the original bar. Hence I am hesitant to conclude that he can use the splitting operation in that I am sure he is aware of the need for iteration of his candy bar to form the second candy bar, but I do not know if he used an equi-partitioning operation to estimate the size of his bar. In another problem, he gave an indication of having constructed distributive partitioning, which is a more advanced fragmenting operation than equi-partitioning and so would imply both the construction of an equi-partitioning operation and ENS. Specifically, when asked, “How can you share these 3 candy bars among 4 people?,” he immediately replies, “Take a fourth of each candy bar.” However, one of the candy bars was bigger than the others, which may have helped him think about working with each of the candy bars separately.

Summary of Justin's Mathematics

Based on his initial interview, I hypothesize that Justin had constructed an ENS and could assimilate with two levels of units. However, I still wanted to see if he was assimilating with three levels of units, i.e., had constructed a GNS, or if it was within his zone of potential construction (ZPC). As we will see in Chapter 5, Justin indicates the construction of a GNS during the first month of teaching episodes.

Lily's Initial Interview

I conducted the initial interview with Lily on January 19. During the interview, she gave many indications that she had constructed an explicitly nested number sequence (ENS) and a few that she had constructed a generalized number sequence (GNS). I will limit my analysis to three of the tasks that allowed me to confidently conclude that she had, indeed, constructed an ENS and possibly a GNS. The first task I will analyze is recorded in Protocol 4.9. I mainly include this protocol to serve as a comparison with the other three students' behavior on the same problem. Lily gives the strongest indications of strategic reasoning out of all the future participants.

Protocol 4.9: Lily makes change for one dollar.

T: Now we're going to pretend that you have the store and you're selling me gum. The gum costs 63 cents, and I give you a dollar [putting a dollar bill on the table]. What change would you give me?
L: [After 14s] 37 cents?
T: How did you get that?! That was pretty fast.
L: 60 plus 40 is a hundred. There's a hundred pennies in a dollar. And I just subtracted in my head, 100 minus 63.
T: You said 63 and 40 and you got 103 first? Right?
L: Well…

> T: ‘Cause I think you did it in kind of an interesting way. I know you know how to do the subtraction [miming writing on paper] on paper and pencil. Your teachers told me you’re good at all that, so I know you can do that. But how were you figuring out in your head? Were you adding 40 first, is that what you said?
> L: Like, I thought that 60 and 40 is 100, and it’s 63, so it can’t be 40. It had to be less, and if it was 63, 7 plus 3 is 10, and then you would get 37.

At first, Lily, like Adam, is vague in her explanation after noting that 100 – 60 is 40, or, in this case, that 60 + 40 is 100. Lily says, “I subtracted in my head.” Serendipitously I misunderstood Lily’s original explanation and so I pushed her to explain herself, which resulted in her statement, “It’s 63, so it can’t be 40. It had to be less, and if it was 63, 7 plus 3 is 10, and then you would get 37.” Her goal did not seem to be subtracting 60 and 3 or counting down by 60 and 3, as I concluded for Adam. Instead, she is reasoning that because 63 is greater than 60, 100 – 63 will be less than 40. When she says, “it was 63, 7 plus 3 is 10, and then you would get 37,” she might be noting that the units’ digit of 100 – 63 must be a 7 because the answer is a multiple of 10 (because the answer is 0 modulo 10). This is the most conservative interpretation I can give of her mathematics in this situation. Yet even in this interpretation she reasons about the relative size of the addends, and so is carrying out, at least partially, strategic reasoning. However, I think it is probable, based on Lily’s facility with composite units as seen in Protocol 4.9 and her facility with operating on differences as seen in Chapter 5, that she could visualize the adjustments she was making to the addends as in Figure 4.4.

In Figure 4.4, the top sketch is of a possible assimilating structure for the dollar. Note that the 100 cents are assimilated as made up of ten 10’s. In the second sketch, Lily has recalled or determined that 40 + 60 = 100. Again, based on her later behavior, I feel comfortable saying that she saw addition and subtraction as inverses, and, in particular, she would see 40 + 60 = 100 as representing the same relationships as 100 – 60 = 40. Therefore, the second sketch would represent both a sum and a difference. At this point, Lily is aware that she needs to find a different difference, 100 – 63, and she knows that it will be smaller than 40. To find it, she decomposes the 10 between 30 and 40 into 3 and 7, takes away that 3 from the 40 and appends it to 60, given her 63 and 37 as the new addends, or 37 as the new difference. The multiple subset relationships in the bottom sketch underline the fact that an ENS is necessary for this reasoning.

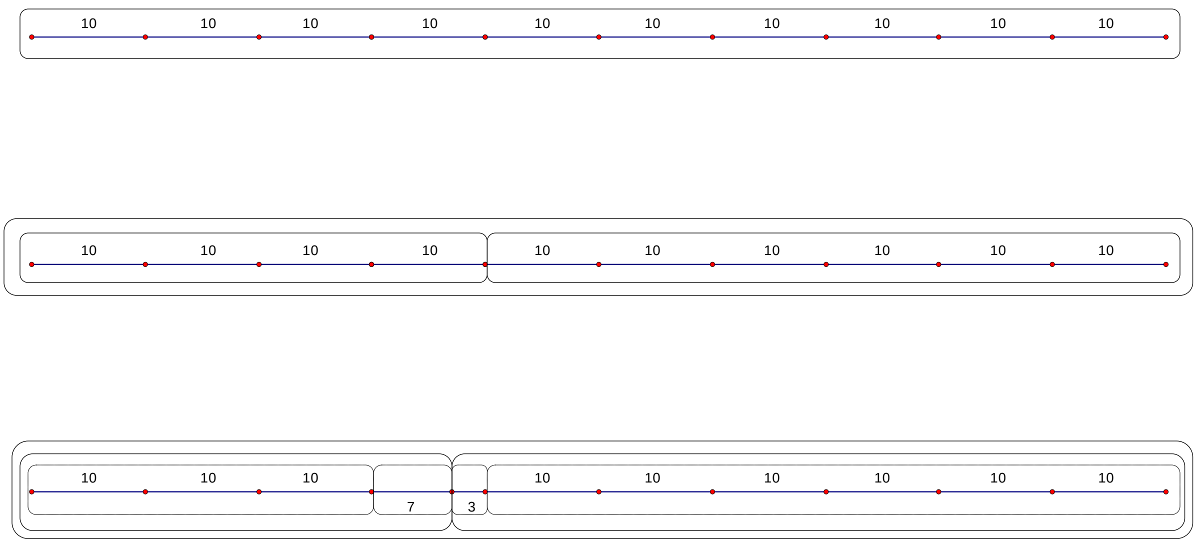


Figure 4.4. Strategic reasoning when making change for one dollar.

The task in Protocol 4.10 gives an even stronger indication that Lily had constructed an ENS. The task was meant to investigate whether she had constructed iterable composite units.

Protocol 4.10: Lily works with composite units.

T: We're putting them [pennies] into stacks of 7. I have done 6 stacks of 7. [Although there are a few empty penny rolls visible, we are just pretending that there are 6 stacks visible.] How many is that?
L: [After 4s] 42.
T: Let's say that I use two of those stacks for something else, so I've taken two of the stacks away. Now how many pennies do I have?
L: [After 2s] 28.
T: How did you get that?
L: I just multiplied. I subtracted 2 from 6 and I got 4, so 7 times 4 is 28.

In this example, Lily's spontaneous use of multiplicative language would, by itself, indicate the construction of an ENS in that multiplicative reasoning and language implies iteration of composite units. However, she gives further indication of an ENS by fluently moving back and forth between two levels of composite units, the number of stacks and the number of pennies; she has no trouble keeping track of the quantities or numbers involved. This implies that she can assimilate the situation with at least two levels of units in order to easily keep track of the two types of units (pennies and stacks of 7 pennies) and how to convert from one to another. Furthermore, the fact that she converts between the levels of units by multiplying and counting sequentially by groups of 7 implies that her units of 7 are iterable composite units and hence she has constructed an ENS.

The way I phrased my question may have implied her solution strategy in that I talk about taking away two stacks and she starts by subtracting two from six. If I knew that she had unitized the composite units of seven using her iterable ENS units, so that she was aware of the numerosity of the two groups of seven she subtracted from the six groups of seven, then I could more confidently conclude that she is assimilating the situation with three levels of units in that she would be operating on three-levels-of-unit structures, the composite units of eight groups of seven. Because she operated so quickly and saw multiplication as the dominant operation, I do think that she was aware of the groups of seven throughout. However, the context may have helped support her conceptualization of units of seven because seven pennies could be replaced with the idea of one penny roll or one stack in this particular situation. Therefore, I looked for this kind of behavior in other types of situations before hypothesizing that she can assimilate with three levels of units. The next task will also give another indication that she is assimilating with three levels of units, but I will not get definitive confirmation until the regular teaching episodes.

I did not give Lily the task of making change for $10 because she had already shown such strong indications of having constructed an ENS. In retrospect, her solution of that task may have helped me determine whether or not she was assimilating with three levels of units. However, I gave a distributive partitioning task to Lily, which I will discuss both in order to provide my strongest indication from the initial interview that Lily is assimilating with three levels of units and in order to enable comparison with the other participants. Directly before the task in Protocol 4.11, I asked Lily how she would share 3 candy bars of equal size among 5 people. She said that she would "divide by fractions" and then described an algorithm for dividing 3 by 5 that ended with multiplying 3/1 by 1/5. She did say that "3 is the whole," so I think she may have understood why dividing 3 by 5 would make sense in this context. However, I wanted to know how she would envision the partitioning of the bars that would correspond to her algorithm, so I gave her a similar distributive partitioning problem but asked her to pretend she did not know how to multiply or divide fractions when solving it.

Protocol 4.11: Lily distributively partitions.

T: Now I want you to think about this again and let's say you don't know how to do fraction multiplication and division. […] You have the three candy bars [there are 3 approximately equal rectangles drawn on a sheet of paper] and you're sharing among four people.
L: I can divide each bar into 4 pieces, and then I'll have 3 pieces.
T: OK. Then what fraction would each of them have of one candy bar?
L: 1/4.
T: Draw me what one person would get however you want to. You can divide them all up if you want to.
L: [Partitions each bar into fourths.] They would each get three of these [circling three pieces in one bar]. They would get about this much of a candy bar [draws an unpartitioned rectangle the same length as the three pieces she circled].
T: Nice. All right. What fraction would that [the unpartitioned bar] be of one candy bar?
L: 3/4?

Lily's first statement in Protocol 4.11, "I can divide each bar into 4 pieces, and then I'll have 3 pieces," implies that she is partitioning each bar into 4 parts and that she knows that 3 of these parts will make up one share. When I ask how much each person gets of one candy bar, I meant to ask what fraction the 3 parts were of one candy bar, but Lily interpreted it as asking how much of one candy bar each person gets, and she replies, “1/4.” Therefore, she seems to be thinking of taking one part from each candy bar. This demonstrates her awareness that partitioning the whole of 3 candy bars into 4 and taking one piece is equivalent to partitioning each of the 3 candy bars into 4 and putting together a share from each candy bar. Hence, by definition, she is *distributively partitioning* in this task. This is further confirmed by her knowledge that she can take the 3 pieces from just one candy bar and that the fraction of one candy bar is 3/4. Distributive partitioning is an even more advanced level of fragmenting than equi-partitioning (Steffe, 2010a, p. 69). Lily’s independent use of the three partitions to make a new unpartitioned bar that is 3/4 of one of the candy bars implies an iterative fraction scheme because she does not feel a need to work *within* a whole, but instead she can disembed her iterable unit fractions to form a new bar that can be compared to the original. This would be a stronger indication of the construction of an iterative fraction scheme if she had formed an unpartitioned bar that was longer than the original bars so that it was clear that she was not constrained in iterating her unit fractions to the whole. The construction of an iterative fraction scheme presupposes the ability to assimilate with a three-levels-of-units structure (Steffe, 2010c). Hence her ways of operating in this task imply the construction of a GNS.

Summary of Lily’s Mathematics

In her initial interview, Lily indicated the construction of a GNS. I continued to look for confirmation of her assimilating structures during the ensuing teaching episodes, and, indeed, Lily does give further indications of assimilating with a three levels of unit structure during the first month of the teaching episodes.

Michelle's Initial Interview

One of the other graduate students conducted the initial interview with Michelle on January 25, and my advisor acted as the witness. Michelle shows indications of having constructed an ENS, but no indications of being able to assimilate with three levels of units. I have included Michelle's work on two of the money tasks as a comparison with other students. In particular, her behavior on the two money tasks in Protocols 4.12 and 4.13 is similar to that of Justin and Adam in that she has trouble making change for $10, but she seems comfortable making change for $1.

Protocol 4.12: Michelle makes change for one dollar.

T: Suppose I'm buying a pack of gum from you, and it costs 63 cents, and I pay you with a dollar bill. How much change do I get?
M: 47?
T: So the gum costs 63 cents…
M: [Nods.] 63 cents…[after 14s] 37 cents!
T: Nice. 37. How did you get that?
M: I just added. I mean, not added, subtracted.
T: How?
M: I subtracted from…I went to 60 from a hundred, and then I added three.
W: But could you say more about that? I mean, how'd you do that? That's really neat!
T: From 60 to 100?
M: No, from 100 I minused it by 60--
W: How many was that?
M: 40.
W: 40? And then what'd you do?
M: I subtracted 3.
W: 3 from where?
M: From the 40.

Michelle's original answer of 47 was probably the result of using the tens' digits to estimate that the difference was around 40 and then determining that the ones' digit was 7, but not coordinating the calculations to determine that the answer was 37 instead of 47. During the rest of the teaching experiment, Michelle does not usually guess at answers, so she probably answered so quickly, and without fully thinking through her answer, due to nerves. Throughout the initial interview she gave signs of being nervous. For example, when she says, "I just added. I mean, not added, subtracted," she is speaking fairly quickly and nervously laughs at the end.

When the teacher/researcher indicates that she should try again by beginning to restate the problem, Michelle takes her time to think through the difference and comes up with the correct answer of 37. Whereas she gave the answer of 47 in an unsure tone, she says 37 quite emphatically as if she is confident about the answer. As we will see in Chapter 7, Michelle has a proclivity for using strategic reasoning to determine differences. Given that proclivity, it would not be surprising for her to attempt to determine the difference using strategic reasoning, as indicated in Figure 4.5. This would be compatible with her statement: "I went to 60 from a hundred, and then I added three." Her later statement, "I minused it by 60," would seem to contradict Figure 4.5 in that *minused* often implies taking away or counting down by 60. However, *minused* might simply imply finding a difference for Michelle in this context. What would be uncharacteristic is for Michelle to describe counting up as "adding 3" when the result would be to *decrease* the difference by 3. I think here that she is thinking about adding 3 to the 60, which would correspond to an unspoken subtraction of 3 from 40.

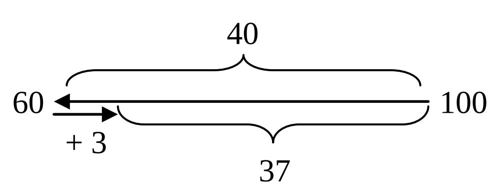


Figure 4.5. Michelle's possible strategic reasoning.

Using her second explanation, I have provided a second hypothesis for her reasoning that is illustrated in Figure 4.6. When the witness suggests that Michelle counted up, she is emphatic that she started from 100, so I have kept 100 as the starting point. If her statement, "I minused it by 60," implies that she counted down or "took away" 60 from 100, she probably had a thought process similar to the one indicated in Figure 4.6. In this case, when she said she "added 3," that would make sense if she was thinking about adding 3 to the amount she was subtracting. This is consistent with her later use of language in signed situations in which she will use the word *increasing* if she is increasing a decrease, for example (see Protocols 7.15, 7.19 and 7.22 in Chapter 7 for related usages). When she said she subtracted 3 from 40 at the end of Protocol 4.12, she would have been thinking about subtracting another 3 from 40 to get to 37.

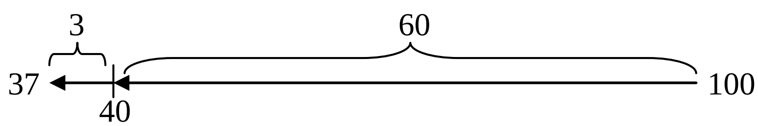


Figure 4.6. Michelle's possible counting down.

The only part of Michelle's explanation in Protocol 4.12 that would not match up well with Figure 4.6 is her original statement that she "went *to* 60 from a hundred." Justin would, at times, say he was going *to* a number, when he was adding or subtracting that number. However, Michelle never, in the entire teaching experiment, said she was going *to* a number unless she was. Therefore, I do not find Figure 4.6 to represent a viable hypothesis, which implies that Michelle was using strategic reasoning, similarly to Lily, and not counting down as were Adam and Justin. If Michelle was using strategic reasoning, then, as discussed in Lily and Adam's initial interviews, she would have already constructed an ENS.

Protocol 4.13 describes Michelle work with three levels of monetary units, 1-cent units, 10-cent units and dollars.

Protocol 4.13: Michelle makes change for ten dollars.

T: I'm buying a pack of gum from you, but right now it costs seven forty-one, seven dollars and 41 cents. I pay you with a 10-dollar bill. How much change do I get?
M: 3 dollars and 60 cents? No, 61 cents? No, no, 69 cents!
T: All right. Try one more time. The gum costs seven forty-one, seven dollars and 41, and I paid $10.
W: You're very close.
M: [Puts her hand to her head] $10 and it's seven…
T: 41
M: Seven forty-one, and you pay me $10. [After 15s] I don't know.
W: You're really close.
T: OK. Let's say the gum costs 7 dollars, and I pay you 10, how much?
M: You get 3 dollars back.
T: 3 dollars back, OK. So right now it's actually seven forty-one, and I pay you 10. Do you think I get more or less than 3 dollars?
M: Less?
T: Less. OK. So right now, think about that again. Seven forty-one and I pay you 10.
M: Two dollars and…[after 16s] sixty-…I don't know.

Like Adam, Michelle does not coordinate her calculations involving dollars and her calculations involving cents. In fact, unlike in Protocol 4.12, she does not coordinate her calculations involving 10-cent units and 1-cent units either. Similarly to Protocol 4.12, she first gives an answer, 3 dollars and 69 cents, which results from subtracting $10 – $7, 100¢ – 40¢, and 10¢ – 1¢. As in Protocol 4.12, she gives this initial answer quickly and seems flustered when she gives it. Continuing in the pattern of Protocol 4.12, the interviewer encourages her to try again, and on her second attempt she gives two indications of more focused concentration: The first is putting her hand to her head, and the second is the increased length of time she thinks before speaking. This time, though, she does not persist in her problem solving, even with the hints from the teacher, in order to figure out an answer.

Given that she had determined that there would be $2 change and that we know from Protocol 4.12 that she could figure out the number of cents needed, I think that this problem would have been in her ZPC in a different environment. In particular, I think that Michelle would have been able to solve the task in Protocol 4.13 if she were working with someone with whom she had established mathematical caring relations (Hackenberg, 2005). I will not discuss many of her teaching episodes in the first month of the teaching experiment because they mainly involve fraction tasks. However, during that month, her willingness to persist in problem solving amid uncomfortable perturbation increases noticeably. Her behavior in Protocol 4.13 is similar to behavior on earlier fraction tasks in that she does not tend to engage in independent problem solving given moderate perturbation. Once she gains confidence in both herself and her teacher/researcher (initially Dr. Steffe), she seems more willing to engage in independent problem solving. By the time she is working on additive tasks, as in Chapter 7, she is noticeably more willing to reason through problems, although I also think that the additive situations were less challenging for her than the fractional ones and hence she was not experiencing the same level of perturbation. As I will discuss in the Conclusions chapter, the construction of mathematical caring relations changes her willingness to engage problems, but I do not think that it changes her general orientation to problem solving, which seems quite low in comparison to the other participants' orientations to problem solving.

As with Adam and Justin, I hypothesize that Michelle can assimilate with two levels of units, as she seemed to do in Protocol 4.12, but not with three levels of units. If she were assimilating the task in Protocol 4.13 with three levels of units, I do not think she would experience the same level of perturbation in attempting to solve it. See the discussions for Adam and Justin in this chapter for more details on the relationship between three levels of units and the task in Protocol 4.13.

The next two tasks involve Michelle's mathematical behavior in fraction contexts. Protocol 4.14 shows Michelle's solution to a splitting task. Her behavior is consistent with the use of a splitting operation, which confirms her prior construction of an ENS.

Protocol 4.14: Michelle splits.

T: Suppose this is your bar. [Indicating a printed copy the task, including an unmarked rectangle.] My bar-- Your bar is 4 times as long as my bar. Can you draw my bar?
M: [Starts to read the printed question, which is the same.]
T: It's 4 times as long as mine.
M: [Draws three small bars in the air over part of the rectangle and then looks like she is counting visualized bars inside the big bar. Marks off an estimate that is smaller than 1/4 using a tick mark underneath the bar. Erases it and draws a bar underneath that ends where the tick mark was.] Would your bar be that long?
T: How?
M: I did it like if there was 5 squares through each bar, I mean 4 squares through each bar. Then that would be one of yours, so I did that [outlining the bar she drew] for yours.
T: OK. This is mine [pointing to smaller bar], right? And this is yours [pointing to original bar], and how did you figure out, again? So you—
M: I divided into the four. [Draws in four pieces on top bar. Sizes are not equal.] I saw that yours would be one of those [outlining the bar she drew], so…
T: It would be--, yours would be 4 times as long somehow, right? Very nice.

Unlike Adam, she quickly draws her estimate, so I know that she would not have had time to do much trial-and-error. Unlike Justin, her estimate was good, so I have no reason to doubt that she is using her equi-partitioning operation. Therefore, I conclude that Michelle assimilated the iterated bar in the question as a partition of the given bar that could be iterated to form the given bar. This implies an iterable (fractional) unit and, therefore, construction of an ENS.

Michelle had trouble, on the other hand, with the distributive partitioning task. I describe her solution in Protocol 4.15. This task does not provide additional information about her mathematical schemes and operations, but the lack of indications that Michelle had a distributive partitioning operation available to her was a factor in my decision to pair her with Adam, the only other participant who did not appear to distributively partition in this task, for the first month of the teaching experiment.

Protocol 4.15: Michelle attempts to share three candy bars among four people.

T: Here are three candy bars [putting down three identical strips of paper].
W: And share them among four people. How would you do that?
T: You want to share for four [holding up four fingers] people.
M: For four people?
T: Mm-hmm.
M: [Thinks for 70s.]
W: I think we're out of time.
T: Yes, we're out of time. What do you think?
M: I'm thinking on how I'm going to do this. I don't know if I can split all [demonstrating splitting two bars in half] these and that would be four and then just do this one [the third bar] and see how many could go evenly into those [the four halves from the first two bars]. Can I do that?
T: Yeah! You can do that.
M: [Divides two bars into halves.] There's the four they would get, and they would get…[Divides third bar into fourths.] And then they would each of them get one of those.

My interpretation of her solution is that she knew she needed to divide the bars up somehow and started by dividing them into halves. She then knew that each person would get at one half of a bar from the first two bars. This left the third bar to divide up among four people, and, hence, she partitioned it into fourths. This interpretation is also compatible with her mathematical ways of operating in the first month of the teaching episodes. When asked to do a distributive partitioning task in this first month, she again began by dividing each bar into halves.

Michelle was able to solve the problem without distributively partitioning, so Protocol 4.15 does not necessarily provide a contraindication of the ability to distributively partition. If Michelle had demonstrated the ability to distributively partition then, as with Lily, I would be able to conclude that she could assimilate the situation with three levels of units. However, her ways of operating in the initial interview only indicated an ability to assimilate with two levels of units.

Summary of Michelle's Mathematics

Based on the initial interview, I can attribute to Michelle the construction of an ENS. She did not give any strong indications that she was able to assimilate with three levels of units. At the time, I suspected that she could, and so I looked for indications of her ability to assimilate with three levels of units in the first month of the teaching experiment. However, I have not been able to find any convincing indications that she did have a three-levels-of-units assimilatory structure.

CHAPTER 5

NUMBER SCHEMES AND ADDITIVE EXPLORATIONS

After we selected participants based on their mathematical operations and their willingness to communicate about their mathematical thinking, we paired the students up and worked with them for the month of February on fraction schemes and other numerical, unsigned schemes. Based on my original analyses of the initial interviews, I found Justin and Lily to have the most sophisticated mathematical schemes and operations. In particular, I did not note the ambiguity in Justin's solutions of fraction tasks, and hence I attributed distributive partitioning to him and Lily, but not to Adam and Michelle. Distributive partitioning implies a GNS, so I had hypothesized that Justin and Lily had constructed a GNS, while Adam and Michelle had not. This resulted in the pairing of Adam with Michelle and Justin with Lily.

During the first month, based on our hypotheses about the students at the time, the teacher/researchers were looking for confirmation that Adam and Michelle had each constructed an ENS and additional data that would indicate whether each of the participants was assimilating with a two-levels-of-units or three-levels-of-units structure. For the majority of February, we used fraction tasks that were similar to tasks (cf. Steffe & Olive, 2010) I had experience with from earlier research with Dr. Steffe. For a couple of sessions in March, we worked in two signed, additive contexts—weight gain/loss and win/loss margins in a game—with the same pairings of students.

In this chapter I analyze a few tasks from the February teaching episodes to give the reader further information about the students' unsigned mathematical schemes and operations.

This analysis is a continuation of my analysis of the students' mathematics based on the initial interviews. Therefore, in this retrospective analysis, I present confirmation or contraindications of my previous hypotheses about the students, which differ from my hypotheses at the time of data collection, as well as additional indications about assimilating structures for several of the students. I will also give examples of characteristic behaviors for each of the students in the two additive contexts. At the end of the month, I decided to switch how the students were paired (see "Change of Pairings" at the end of this chapter). The bulk of my analysis will focus on the additive work done in these later pairings. Those findings are found in Chapters 6 and 7.

Adam and Michelle

Adam and Michelle were paired in this initial section of the teaching experiment. In my analysis of Adam's initial interviews in Chapter 4, I hypothesize that he has constructed a TNS. Therefore in my retrospective analysis of the February teaching episodes, I looked for indications of Adam's construction of an ENS. I was surprised that I did not find any strong indications that Adam had constructed an ENS. I was also surprised that, for the entire month, he appears to use a part-whole fraction scheme (Liss, Gammaro, & Steffe, 2012) despite his earlier partitioning behavior. He does make temporary modifications to his fraction scheme, but his ways of operating do not indicate any fractional schemes or operations that would imply the construction of an ENS. I will discuss his work on one task here to underline the fact that the data were not conclusive regarding his partitioning operations.

In my analysis of Michelle's initial interview in Chapter 4, I hypothesize that Michelle has constructed an ENS. Therefore, in my retrospective analysis of her mathematical ways of operating in February, I looked for evidence of the ability to assimilate with three levels of units. All of the possible indications of assimilating with three levels of units that I found were weak indications. I will discuss her ways of operating in what was meant to be a recursive partitioning task and in a series of tasks that involved adding fractions with unlike denominators. I selected this last series of tasks because it highlights the difficulty of interpreting her mental schemes and operations based on her fractional behavior. One reason that her mathematics was difficult to analyze is because she utilized memorized procedures in a way that obscured her quantitative understandings of mathematical situations. I also think that the use of these procedures was counter-productive to the development of her problem solving capabilities and her general mathematical growth. I will discuss this issue further in Chapter 8. A related reason that her mathematics was difficult to analyze is because she got easily frustrated when faced with a mathematical perturbation and would elicit more hints and clarifications from the teacher/researcher than the other participants. That in and of itself might not have been problematic, but she was adept at isolating useful strategies and procedures from the teacher/researcher's hints without giving strong indications that she had constructed the schemes and operations that would usually be implied by the independent development of these strategies and procedures. Hence what appears to be sophisticated fractional reasoning might be a combination of moderately sophisticated fractional understandings and memorized procedures.

An Analysis of Adam and Michelle's Unsigned Numerical Schemes and Operations

Protocol 5.1 records the discussion during Adam and Michelle's first task working in pairs. As I noted in my analysis of Adam in the initial interview, he did not show indications of the ability to recursively partition. The teacher/researcher was providing another opportunity for Adam to construct a recursive partitioning scheme. While Adam is, again, not successful in the task at hand, he does engage in partitioning behavior that I will analyze below. As in Chapter 4, in the protocols throughout this chapter, A, J, L, M, T, and W will indicate that Adam, Justin, Lily, Michelle, the teacher/researcher, or the witness is the speaker or actor, respectively. The symbol, […], indicates that dialogue has been omitted. Ellipses between lines of dialogue indicate omitted dialogue. Actions will be set apart in brackets.

Protocol 5.1: Adam and Michelle Attempt a Recursive Partitioning Task.

T: [Putting down one strip of paper] Share that into eight equal pieces. […]
M: Into eight equal pieces? […] Do it in half.
A: [Divides the bar into two equal pieces.] […]Four and four [indicating each half].
M: Then you'd do 1, 2, 3, 4 [indicating the partitions], and then those would be your other 4.
…
A: [Draws in first 4 partitions. The last one is bigger.] That's too small [the first one].
T: No, that's good. Let's pretend that they're correct. Michelle, you do the [last four] boxes.
M: [Draws in the other four partitions. Hers are not equal either.]
T: OK. Now cut off *one* piece [handing scissors to Adam].
A: [Cuts off one piece.]
T: […] Share that among […] Michelle, Adam, and [the witness]. Cut off Michelle's piece.
…
M: One, two, and three [indicating her plan]. One [drawing partitions]…one, two, three.
T: You agree, Adam?
A: Yeah.
T: Now cut off one of those pieces. […] Let Michelle cut this time.
M: [She cuts off 1/3 of the 1/8 piece.]
T: Great! […] What fraction is that [smallest] piece of the original eight pieces that we had?
M: [Looks at 7/8 piece while counting for 2s, looks up and counts 1s] I think I know, I think.
T: [After 38s, to Adam] […] Want to use all this stuff [pieces of paper] to explore around?
A: [Puts the cut-up 1/8 piece back together.]
T: You've got to put those back together, right? […] Use this [pen] some more if you want.
A: [Loosely reassembles pieces as in Figure 5.1. Points to each unmarked eighths and the cut-off pieces. Answers 93s after the question was originally asked.] Got it.
T: OK. Now, Michelle, you write your answer down. Don't let Adam see it. […]
M: [Writes "1/24".]
A: [Writes "1/4".]
T: [To Adam] How'd you think about it? You were doing some stuff over here. […]
A: I just thought it was 2/8, so I reduced it to 1/4.
T: Why would you think Michelle got 1/24?
A: I don't know.

T: Michelle, can you explain it to him?
M: Yeah. I made each of them [unmarked eighths] into 3's, and then 3 [in each unmarked eighth] plus these 3 [the cut-off pieces] is 24, and then there's 1 of them [the 1/24 piece], so it's 1/24.
T: Would that work?
A: Yeah.

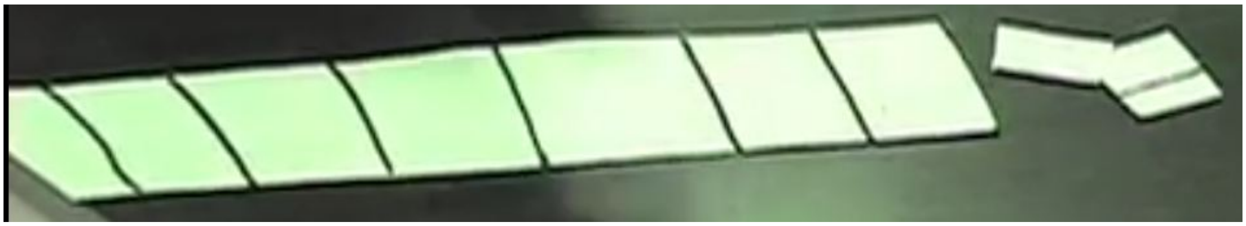


Figure 5.1. Adam reassembles three fractional pieces into a whole.

Adam's partitioning behavior. As in the initial interview, Adam's fragmenting behavior appears to be a result of partitioning and not segmenting. However, Adam's ways of operating could be explained by either an equi-partitioning operation based on the composite units of an ENS or a simultaneous partitioning operation based on the composite units of a TNS. When he draws in the first four partitions, we can see in Figure 5.1 that they were not of equal size. He independently notices that something is wrong and says, "That's too small," pointing to the first partition he drew in. On the one hand, this seems to indicate awareness that he is using the first partition to make the other partitions, because when the partitioning does not come out equally, he sees the first partition as the reason. Also, the length of the base of the first three partitions is about the same, implying that he was intending to make the first three partitions equal in size. On the other hand, there is again no indication that he was mentally disembedding the first partition and iterating it to form half of the bar. Once again, he actually draws in the partitions before deciding whether the initial estimate was too big or small, and he does not explicitly compare anything to the length of the fractional whole, the half-bar in this case. Therefore, his partitioning behavior here does not help me decide whether he has constructed an equi-partitioning operation or only a simultaneous partitioning operation.

Because the teacher/researcher was not testing whether or not Adam could equi-partition, he cuts short Adam's partitioning behavior by replying, "No, that's good. Let's pretend that they're correct." Michelle then goes on to draw more partitions that are clearly not the same size. Perhaps Adam constructs these as norms of mathematical behavior because he does not attempt to rigorously coordinate the size and number of partitions in the rest of the teaching episode. In general, neither of the students attempted to ensure equality of partitions. In Michelle's case, I think this was because, like the teacher/researcher, the drawn partitions stood in for mental equi-partitioning and hence their equality was implicit. However, additional experience with partitioning and segmenting/iterating might have been useful both for Adam to concretize/develop equi-partitioning operations and for me as a researcher to have the opportunity to see more outward indications of his partitioning behavior. The teacher/researcher would, at times, insist that Adam draw in the partitions exactly equally. However, in these situations, Adam does not succeed in drawing equal partitions, and he never develops or assimilates the strategy of measuring off the length of the estimate in order to check it rather than drawing in each partition. This indicates that his attention is on the partitions *within* the whole bar and not as separate units to be iterated or compared to the whole.

Adam's solution to the recursive partitioning portion of the task also gives indications of his fractional ways of operation. Certainly his inability to initially come up with the answer of 1/24 is a contraindication of recursive partitioning. However, the teacher/researcher makes an informative intervention in that he gives the strips of paper to Adam, encourages him to put them back together, gives him a pen, and encourages Adam to "use this some more." As we see in Figure 5.1, Adam puts the three pieces loosely together in a way that would help him revisualize the original bar. However, he does not attempt to partition any of the other eighths in order to get equal-sized pieces. In his initial interview, Adam had been given a similar task in which he had marked off 1/3 of a bar and then marked off 1/7 of the 1/3. After I asked him to find the fraction the smallest piece was of the original bar, he silently sat and thought for a long time. Eventually I asked him what he was thinking about and he replied, “I can’t really figure it out because I don’t have the rest of the pieces.” This indicates that he is aware of an incomplete partitioning, but is not sure how to get a complete partitioning, at least he is unsure of how to do it in visualized imagination. Furthermore, his initial solution of 2/8 in Protocol 5.1 implies that the eighth that is partitioned in 3 is not destroyed. That is, he is still aware of the re-partitioned eighth as a single partition.

His behavior here is consistent with his construction of non-iterable fractional units. If he considered all of the 8 partitions to be copies or iterations of his original estimate, then partitioning that estimate would open the possibility of partitioning the iterations of the estimate as well. However, if Adam does not see the 8 partitions as interchangeable, then it would make sense that his action on one partition would have no bearing on his action in other partitions. In his answer of 2/8, I think that Adam is conflating the number of (separate) pieces that make up one eighth with the number of partitions that he is giving a fractional name to. Certainly his ways of operating in Protocol 5.1 do not contraindicate my original hypotheses about Adam.

Steffe’s (2010) account of a student with a simultaneous partitioning scheme, but no equi-partitioning scheme, involves a student who had constructed an ENS. Therefore, Adam’s relatively unsophisticated fractional schemes should not be taken as evidence that he has not constructed an ENS. Nonetheless, even when working in discrete contexts, I did not find strong indications that Adam had constructed an ENS.

Michelle's recursive partitioning behavior. Protocol 5.1 provides indication that Michelle can recursively partition. She also gave an indication of having constructed a recursive partitioning scheme in her initial interview, but there she could see the original bar with both types of partitions, as Adam can in this case when he reassembles the bar. Hence in the initial interview I felt that she could have made a pseudo-empirical abstraction of drawing partitions into each original partition. In Protocol 5.1, Michelle does look at the 7/8 paper strip, but she never looks at the other pieces and she finishes up the counting in her head, using her fingers to keep track of something. Therefore, I think she was taking the re-presentation of the result of the original partitioning activity and re-partitioning it. In order for her to generate the strategy of re-partitioning the eight partitions, she would need to assimilate the bar as being made up of both an unknown number of iterations of the smallest piece and also as being comprised of eight partitions that could each be partitioned three times to form the iterations of the smallest piece. This implies assimilation with three levels of units. However, the composite unit of 8 partitions had been formed in her immediate experience. Therefore, her ability to think about this situation with the three-levels-of-units structure may have been enabled by her direct experience of forming the composite unit of eight. For that reason, her solution in Protocol 5.1 serves only as a weak indication that Michelle can assimilate with a three-levels-of-units structure. In order to attribute that assimilating structure to her, I would need confirmation when none of the levels of units were in her immediate past or present experience, which does not happen during the teaching experiment.

At the beginning of Protocol 5.1 when the students are attempting to partition the original bar into eight parts, Michelle suggests to "Do it in half." This could be taken as an indication of assimilating with a recursive partitioning scheme if I thought she was assimilating a partitioning situation with the structure of a partition of partitions. In this case, her language could imply that she was thinking of 8 as 2 groups of 4 and recognized that she could use that fact in recursively partitioning. However, I do not think that she was assimilating with a recursive partitioning scheme because she has to think for a few seconds about how to proceed once Adam has divided the bar in half. In fact, he is the one who realizes that there should be four on each side. Furthermore, other students I have worked with, including Justin, often attempt to start off a large partition by dividing a bar in half, regardless of whether the number of partitions is even or odd. Michelle herself, in Protocol 4.15, starts the distributive partitioning task by dividing each bar in half. Therefore, I think that Michelle's suggestion stemmed mainly from her realization that it would be hard to estimate 1/8 and a lucky attempt at simplifying the problem, as opposed to an anticipatory recognition that she could recursively partition to get 8 partitions.

Michelle's fraction addition scheme. As I discussed in the introduction to this section, "Adam and Michelle," most of the possible indications that Michelle was able to assimilate with three levels of units were as ambiguous as her behavior in Protocol 5.1 because, as in Protocol 5.1, there were often mitigating circumstances that may have enabled her to do demanding problems with only a two-levels-of-units assimilatory structure. Protocols 5.2, 5.3, and 5.4 give her behavior on a series of fraction tasks over the course of two consecutive days. By the end of this series of tasks, Michelle's fractional reasoning appears to be quite sophisticated. For example, she develops a fraction addition scheme. However, by considering the series of tasks together, I show that she would not necessarily require a three-levels-of units assimilating structure to develop her fraction addition scheme.

On February 16, the teacher/researcher began to work with Adam and Michelle on tasks involving recursive partitioning, common partitioning, and fraction addition. The first task he gave to both participants was to partition a bar into 20 pieces in two sets of partitions: they could not partition into 20 pieces all at once, they could only partition things into less than 20 pieces. After some clarification, both Adam and Michelle partitioned their bar into 10 vertical partitions and then made a horizontal partition that divided each tenth in half, as shown in Figure 5.2. This task was followed up partitioning into 30 without using 30, 24 without using 24, and 100 without using 100. Michelle used 15 and 2, 6 and 4, and 10 and 10 vertical and horizontal partitions, respectively, to solve these tasks.

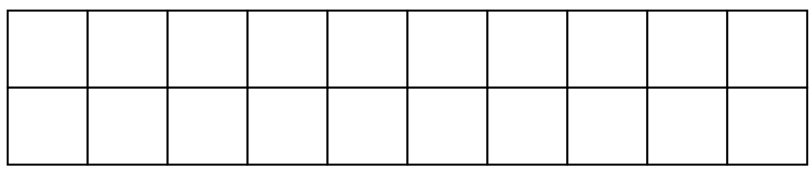


Figure 5.2. Making 20 partitions with 10 and 2.

After these recursive partitioning (from the observer's perspective) tasks, the teacher/researcher gave them a common partitioning task. Namely, he gave them each two equal-sized bars, one marked vertically into halves, and one marked vertically into thirds. He then said, "Some people come along and they want to share them so that you have an equal amount in each candy bar. Can you tell us how many people could share those candy bars?" Michelle first lined her bars up so that their horizontal edges were contiguous, as in Figure 5.3. After a brief pause, she started making vertical motions over both candy bars at approximately each sixth. She then used her pencil to divide each third into halves vertically and each half into thirds, counting to double-check the number of pieces. She looked at the bars for awhile and then made a horizontal line down the middle of each bar, forming twelfths. In her explanation, she said, "I cut these [thirds] into halves, and I got six, so I cut these [halves] into three and got six, and then I cut them through the middle." When the teacher/researcher asks her if she could do it in less than 12 pieces, she does not come up with a new answer, but she does agree with the teacher/researcher when he says that she did not need to draw the horizontal line.

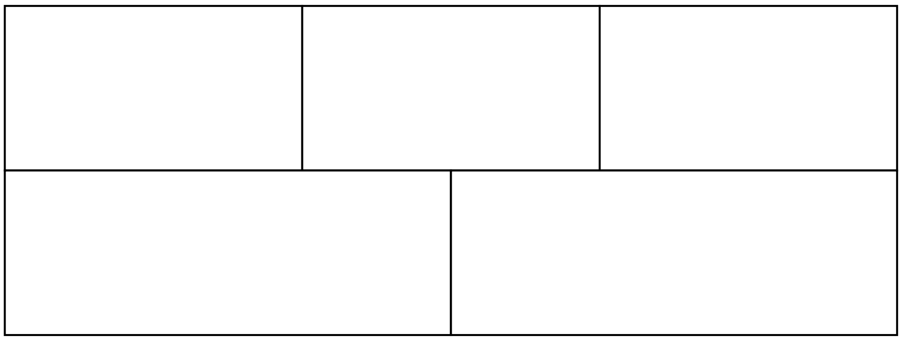


Figure 5.3. Michelle lines up halves and thirds.

Adam could not solve the problem. He divided the halves/thirds into an unequal number of pieces and/or the bars into an unequal number of pieces in his various attempts. In contrast to Adam, Michelle was recursively partitioning, i.e., partitioning an existing partition. Although she did not use fractional language consistently in this explanation, I think she was aware of the fractional relationships because she spontaneously used fractional language in similar situations in this and the next teaching episode. By partitioning a partition, she is creating three levels of units, assuming that she is not destroying the fractional relationship of the partitions to their fractional whole as she does so. Therefore, if she were using her recursive partitioning operation in assimilation of the situation, then I would attribute a three-levels-of-units assimilating structure. However, I would not attribute that construction to her if she is recursively partitioning in action only.

The central question is why she decided to divide the thirds into halves and the halves into thirds. If she understand the halves and thirds as potentially partitioned into sub-partitions and she understood that the total number of sub-partitions would be a multiple of 2 or 3 in the 2/2-bar or 3/3-bar, respectively, then I would say that she is assimilating with three levels of units. However, an equally plausible explanation is that she got the idea to divide the thirds into halves when she looked at the lined up bars and saw that the vertical line in the 2/2-bar would divide the middle third into 2 equal pieces. She seemed to estimate the first few cuts to see if they would line up with the existing partitions in both bars. In addition, she counts the six partitions in each bar before drawing the horizontal line, so I do not think that she saw the necessity of her solution giving an equal number of pieces. I think she was basing her attempt on a visual estimation. Furthermore, I think she drew in the horizontal line to make her solution look more like the solutions to the previous problems in which all of the bars ending up with crosshatching. I do not think the horizontal line was as used as part of a plan to make 12 pieces. This is both because she seemed to add the horizontal line as an afterthought and because she had to count to find out how many pieces were there when the teacher/researcher asked. In the end, her behavior was ambiguous with respect to her levels-of-units structure.

At this point, the teacher/researcher starts giving Adam and Michelle different tasks in order to stay within both students' ZPCs, while still providing challenges. Hence, I leave out all dialogue relating to Adam in the rest of this section. The next task given to Michelle is another common partitioning task.

Protocol 5.2: Michelle finds a common partitioning for fifths and sevenths.

T: Maybe you could do it with this bar, sevenths, and this bar [5/5].
M: [Starts by lining up the bars and making vertical motions over the sevenths' partitions. Then counts the partitions in each bar several times. Mimics cutting each fifth in half.] 1, *2;* 3, *4;* 5, *6;* 7, *8,*…[Mimics making horizontal partitions.]
T: [About 2.5 minutes after asking the question] It's just like before.
M: I know. [After about 30s] Can I have a hint?
T: OK. How many pieces you got here?
M: 1, 2, 3, 4, *5*.
T: How many pieces you got here?
M: 1, 2, 3, 4, 5, 6, *7*.
T: What'd you do last time? [Brings out her bars from last time.] When you found the sixth.

M: Cut them up, but I don't get how I can cut these two up to make them even.
T: In this one we had half, we had two pieces and three pieces. You cut each one of these [halves] into three and each one of these [thirds] into two.
M: Yeah!
T: You found a sixth.
M: [Counts partitions in the 7/7-bar. Partitions each fifth into sevenths. Counts partitions in the 5/5-bar. Partitions each seventh into fifths.] 35 each!
T: What'd you do?
M: I cut this [a fifth] up into five, I mean seven, and I cut this [a seventh] up into five. And then I got 35.

Protocol 5.2 lends credence to the hypothesis that Michelle did not have a plan of action based on an assimilatory recursive partitioning scheme in the initial common partitioning task. She first tried to do a visual estimate to see if the partitions would line up in some visually obvious way. When that did not work, she tried dividing each fifth in half, giving an example, as I discussed earlier, of the often-used strategy of dividing partitions in half when faced with a partitioning task. When the teacher/research intervenes and draws her attention to the number of pieces in the bar, Michelle shows no recognition that she had utilized the number of pieces in the bar in her previous solution. Therefore, I conclude that her partitioning in the initial common partitioning task was serendipitous. She saw the symmetry evident in the teacher's observation, "In this one we had…two pieces and three pieces. You cut each one of these [halves] into three and each one of these [thirds] into two," based on her exclamation, "Yeah!," and her immediate partitioning of the sevenths into fifths. However, I see no reason to hypothesize that she had a conceptual understanding of why dividing each of *a* partitions into *b* partitions and vice versa would give a common partitioning.

The task in Protocol 5.2 was Michelle's last task on February 16. The next day, the teacher/researcher started the session by asking Michelle to find a common partitioning for a 4/4-bar and a 5/5-bar. She immediately counted the partitions in each bar and partitioned each fourth into fifths and vice versa. Protocol 5.3 gives an account of her transition from that common partitioning task to a fraction addition task involving fourths and fifths.

Protocol 5.3: Michelle adds fractions with different denominators.

T: I want you to write down a fraction for 1/4 and a fraction for 1/5, but you can't use 1/4 and 1/5 [giving her a blank piece of paper].
M: Like fractions that are equivalent to 1/4?
T: [Nods.] […] You've got to use this [the bars].
M: What do you mean we have to use these?
T: You've got to use these to make a fraction for 1/4 and another fraction for 1/5.
M: [After 15s, counts the partitions in a fourth and writes, "1/4 = 5/20" and "1/5 = 4/20."
T: OK. I want you to find a fourth plus a fifth.
M: [Looking at her bars] A fourth plus a fifth. What do you mean?!
T: I want you to shade a fourth plus a fifth.
M: Like 1/4 on this one and 1/5 on this one?
T: Well, whatever you think is a fourth plus a fifth. How many of those little pieces do you get? You get twentieths, right?
M: Yeah.
T: So I want you to draw a fourth plus a fifth.
M: [Shades in 1/4 on the 4/4-bar.] I'm confused on the question.
T: I want you to find a fourth plus a fifth. How many twentieths is it?
M: There's 20 twentieths.
T: Huh?
M: [Pointing to twentieths] 1, 2, 3, 4, 5...Well, you can count.
T: You've got a fourth here and a fifth here. This is a fourth, right?
M: Mm-hmm.
T: Color in a fifth down here.
M: OK. [Shades it in.]
T: OK. Is a fourth or a fifth bigger?
M: They're the same. Wait, no, no, no! A fourth is bigger, yeah [smiling].
T: A fourth is bigger. Why?
M: 'Cause it has more pieces?
T: How many twentieths is a fourth?
M: A fourth is five. Five, yeah, five.
T: OK, so this is a fourth and a fifth, right?
M: Mm-hmm.
T: So if you put those together, how much of a whole bar do you have?
M: So if you put these two together […] [tapping the shaded in partitions]?
T: Yeah, those two. How much is that? That'd be the sum, right?
…
M: 9/20.

Michelle spoke with an exasperated tone in much of this protocol. The culminating example of her attitude towards the teacher/researcher was when she said, “1, 2, 3, 4, 5…Well, you can count.” This mildly belligerent/exasperated tone would often emerge when she was unsure of how to proceed in a problem. The teacher/researcher does an incredibly perceptive intervention when he asks, “Is a fourth or a fifth bigger?” I do not think he wanted to see if he she could answer the question because her behavior in the past had indicated that she could. Instead I think he was attempting to ask a question that (a) was a new question since the conversation was going in circles, (b) she could answer and thereby reduce her level of frustration, and (c) would reorient her to the fourths and fifths as *quantities* of the same whole. The very question presupposes the existence of a single fractional whole for both fractions, and now that she has made a common partitioning on the two bars, she can use the partitioning to quantify the difference between the two fractions. This helps her attend to the fractions as quantities with respect to the fractional whole and also attend to how the common partitioning be used to measure the two fractions. In fact, only the first two goals, of getting out of the conversational rut and raising her comfort level, were achieved. However, even after she is no longer uncomfortable in the situation, she still continues to wait for the teacher/researcher to tell her what to do: Shade in the fifth as well and tell me how many twentieths that is if you put the two shaded parts together. The other three participants in this kind of situation, would use the teacher/researcher’s questions as hints and continue to think about the problem. Michelle was unique in her ability and tendency to outwait the teacher/researcher.

As for her conceptual understanding, she seems to make sense of the results of her common partitioning procedure in that she understands she is finding equal-size pieces in both bars. As a matter of fact, the teacher/researcher asked the common partitioning task in exactly that way, “You’ve got to cut those up so you get equal shares.” However, he follows that by saying, “You did that yesterday,” so that hint could have helped her assimilate the situation as a common partitioning task. Nonetheless, the fact that she immediately begins to solve it with the procedure for common partitioning leads me to hypothesize that she is aware of a goal to make equal-size pieces. After all, she had done other types of tasks the day before, so I do not think that the hint alone would be sufficient to trigger her common partitioning scheme so quickly. She also seems to understand that the common partitioning shows the equivalent fractions she could form using her paper-and-pencil algorithm, given that she writes 4/20 and 5/20 when she is “using the bars.” However, her confusion over how to find the sum of 1/4 and 1/5 using the bars implies that she is not aware of the 4/20 and 5/20 as enumerating the magnitude of 1/5 and 1/4. She can *see* that 1/4 of the bar has 5/20 in it and 1/5 of the bar has 4/20 in it, but I do not think she equates these number since she does not seem to understand that she can use the twentieths to describe the magnitude of the sum.

Before finishing up my analysis of this task, I share a continuation of it. After Protocol 5.3, the teacher/researcher asks Michelle to find 2/5 plus 3/4. Without counting on the bars, she writes, “2/5 = 8/20” and “3/4 = 15/20.” She then shades in those quantities using squiggles, as shown in Figure 5.4. However, she does not shade in 2/5 on the 5/5-bar, but rather 8/20 on the 4/4-bar. Similarly, she shades in 15/20 on the 5/5-bar instead of 3/4 on the 4/4-bar. I say that she is shading in 8/20 and 15/20 instead of 2/5 and 3/4 because she shades in the quantities by twentieths instead of using the fourths and fifths visible on the bar. Her original shading of the fourth and fifth is very light and does not include the squiggles.

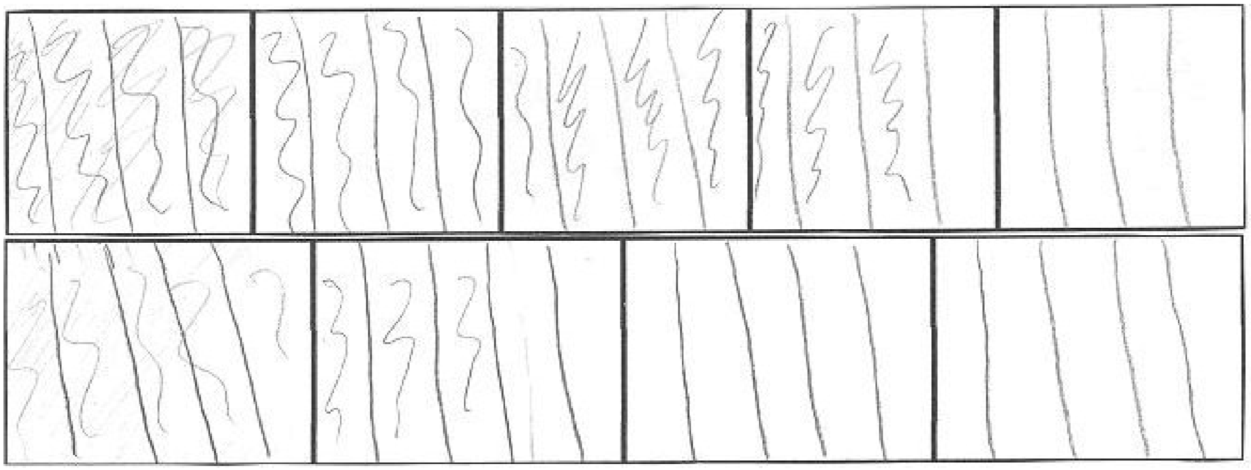


Figure 5.4. Michelle draws 2/5 and 3/4 in terms of twentieths

I interpret her behavior as indicating that she was not aware of iterating the 4/20 to get the 8/20 she wrote or iterating the 5/20 to get the 15/20 she wrote. Instead, I hypothesize that she is using a *procedural scheme*. Her use of the term *equivalent* in her earlier question, "Like fractions that are equivalent to 1/4?," implies that she has learned how to make equivalent fractions through a paper and pencil algorithm in her regular mathematics classroom. This algorithm would form the action of her procedural scheme. I call this a procedural scheme because she has a quantitative situation that activates her scheme and a quantitative goal in that she is aware of given fractional amounts of a fractional whole and the goal of forming commensurate fractions. For example, she demonstrate at least that level of quantitative understanding when she counted off given fractions in terms of twentieths using her partitioned bars in Protocol 5.3. In addition, she demonstrates that she can interpret the results of her scheme in terms of the bars in that she correctly shades 8/20 and 15/20 and gives the sum, 23/20, as her answer. However, as I have explained, I do not think she understands how the *action* of her scheme relates to her operations on the bars. For example, she most likely carried out her procedural scheme by figuring out what number she could multiply by 5 to get 20 and then multiplied the result, four, by 2 to get the numerator, 8, in 8/20. However, she does not display any awareness of how the intermediate result, four, is represented in her operations on the bars.

Her use of this procedural scheme instead of reasoning in terms of the visual information from her bars or her previous answers implies that she is not *equating* 4/20 and 1/5, for example. I think she sees them as having the same size, but as being distinct quantities. In fact, I do not see evidence in her ways of operating that 4/20 and 5/20 are iterable for her in the way that 1/5 and 1/4 are. In order for her to experience 4/20 and 1/5 as equal, she would need to be able to assimilate the twentieths with an underlying structure of five groups of four twentieths. That is, she would need to assimilate the situation with three levels of units. I do not think that constructing this structure would be sufficient to experience equality because the structure would decay and so would not be inherent in the 4/20. I theorize that the sense of experiencing two quantities as equal, as opposed to describing an equivalent amount, requires this inherent nature of the structure that implies equality.

Therefore, although Michelle does solve a fraction addition problem, which would imply assimilation with three levels of units if she were to independently solve it, the nature of quantitative understandings of the situation are obscured by her ability to form procedural schemes using algorithms generated by other people. In fact, her use of the equivalent fraction algorithm is fairly flexible, and her ability to form these procedural schemes is indicative of a fairly strong quantitative understanding of the situation. For example, her ability to construct a three levels of units structure in action allows her to form a meaning for *equivalent fraction* that involves have two fractions representing commensurate quantities.

The other participants do form procedural schemes. Lily, for example, probably forms some during her fraction work with Justin because she begins using some of Justin's fractional procedures. However, she would display an understanding of these procedures eventually. For Lily, at least two factors allowed this development of understanding. The first is that she had the mental constructions in place to spontaneously develop the underlying operations/schemes. The second is that Lily, unlike Michelle, had a tendency to try to explain her thinking and understand other peoples' thinking. (One of the other graduate students and I called her "the little teacher.") Therefore, I think she would experience perturbation if she could not explain why a borrowed procedure works or how the other person came up with it. Michelle does not seem to feel that perturbation, and so it is not clear whether she *could have* constructed some of the schemes without the borrowed procedures. This ambiguity continues in Protocol 5.4.

Michelle went on to solve 1/10 + 1/7 and 3/10 + 4/7 without partitioning the entire bar. In this case, the teacher/researcher suggests that she not partition the entire bar, and she was able to curtail her partitioning operation by only partitioning one tenth into sevenths and one seventh into tenths. She then was able to explain her answer to 3/10 + 4/7 by referring to partitions not in her visual field: "Because there's 7 pieces in each of these, so 7, 14, and 21. It's 21/70. I add the 21/70 plus 40/70 because there's 10 in each of them [sevenths] and there's 4 of them." At this point her behavior and explanation were quite sophisticated. The teacher/researcher's intervention helped her to focus on the twentieths in terms of the tenths and sevenths structures. That is, she was attending to all three levels of units as she solved the problem. However, because the sevenths and tenths were visible, she could work with those types of units intuitively. Given more opportunities to curtail her schemes, she might have made a reflective abstraction that would form a three-levels-of-units assimilating structure. However, she does not give indications of having done that by the end of February, when she starts into a focus on signed additive relationships.

An Analysis of Adam and Michelle Working With Differences

In February, I spent an entire session working with Justin and Lily on situations in which they would have to operate on differences. Out of that experience, I developed the card game, described in Chapter 3. Recall that in the card game, the students each chose a card. The player with the higher number on her/his card won the difference in the number of points between the two cards. We only worked with Adam and Michelle in the card game context for half a session on March 2 before we switch pairings. During that session, I had not yet developed the score sheet that I introduce in Chapter 3, so one focus of the session was establishing how to keep track of who won the round and who was winning the game. The second focus (for the teacher/researcher) was attempting to establish a norm of reasoning on number sequences to determine differences in the cards' values as opposed to using the traditional subtraction algorithm.

Keeping track of scores. Both Adam and Michelle went through several possibilities of how they would keep track of their scores before settling on one that both they and the teacher/researcher seemed comfortable with. The teacher/researcher asked them about who won the round and by how much, and who was winning the game and by how much, if either player was not keeping track of that information.

Michelle originally recorded her round scores by writing the amount she won or lost with a W or L behind it, respectively. Adam, on the other hand, just wrote down the undirected difference, even though he was being asked who won or lost. This is a weak indication that Michelle is assimilating these quantities as signed, and that Adam is not. The teacher/researcher, in attempting to get the students to write down their overall score, said, "Every time you have to find out who wins and who loses, and you have to find the total of the wins and losses." Michelle took this to mean that she was supposed to keep track of the wins and losses separately, so she started writing her winning and losing scores on different parts of the paper. Adam looks at her paper and then uses a similar set-up. However, in the second round, he writes down how much he is losing the game by instead of how much he lost the round by. Figure 5.5 shows how they were keeping score after the first five rounds[7]. The fact that Adam originally wrote down a 6 in his loss column after he had won by 5 points and lost by 11 points would make sense, particularly because the teacher/researcher tells him, "You write down 6 loss points," when trying to get Adam to keep track of both the round score and overall score. However, the fact that Adam then writes down 63 when he loses a round by 63 indicates that he is conflating the quantities representing the loss in a round and the overall loss margin in the game, at least notationally.

Adam	Michelle
Won	Won
5	11, 63, 80, 22
Loss	Lost
6, 63, 80, 22	5

Figure 5.5. Second attempt at making a score sheet.

After the fifth round, the teacher/researcher again attempts to get the students to keep track of their overall win/loss margin, as shown in Protocol 5.4.

Protocol 5.4: Teacher/researcher intervention in keeping score.

T: Who's the winner so far in the game?
M: Me.
A: Her.
T: By how much? So you're not keeping track of that, OK?
M: Wait. You don't want us to keep track of that? You do?

[7] In fact, Michelle had two sets of tables because she started over after the second round, but I have combined her two sets of tables in Figure 5 for easier comparison.

T: Yes, I want you to keep track as you go along, who the winner is. So let's start over. You have to keep track as we go along of who the winner of the game is. So at each point we've got to find out who's winning the game and by how much.
M: OK.
T: "Won," "lost." So you can turn this over and start over.
M: [Turns over her paper and makes *won*, *lost*, and *winner* columns.]
T: And then "winner." OK. [To Adam] So you've got to [do the same]…

Adam makes the same columns with slightly different names. However, when the students start playing the game again, they just keep track of who won the round in their *winner* column. Basically, they write down their round score twice: once in either *won* or *lost* and once in *winner*, giving the name of the person who won and their score. When the teacher/researcher realizes that they are still not keeping track of the win/loss margin, he intervenes again as shown in Protocol 5.5.

Protocol 5.5. Adam and Michelle figure out who won the game.

T: How could you find out who won the game?
A: Add all your points up.
…
T: How would we do it?
A: [Starts adding all his wins.]
M: But we lost points too.
A: All the points that you won.
M: [Starts adding her wins also.]
A: You beat me!

First, notice that the students do understand the difference between round scores and game scores, so a large part of their confusion was based on poor instructions for the game on my part. Second, Adam's comment, "Add all your points up," and his subsequent actions of adding wins indicates again that he is not thinking about the situation in terms of signed quantities. Although he is clearly differentiating between the points he won and the points Michelle won, he shows a preference for talking about points in terms of unsigned quantities, *Adam's points* and *Michelle's points*, instead of one signed quantity, *Adam's wins/losses*.

Michelle, on the other hand, seems to prefer to think of the points in reference to one quantity, her score. I discuss the differences in these ways of thinking about quantity in future chapters. For now, the main point is that my goal was for students to move away from two related, but unsigned, quantities, like Adam's quantities, to one signed quantity. Michelle was already indicating a tendency towards constructing signed quantities.

The entire game only took about 19 minutes, and the students do not come to understand that the teacher/researcher wants them to keep track of a different quantity, the win/loss margin, in that short time frame. Both students play this game in future episodes with their new partners, and are able to make sense of the new quantity given more time and more structure to their score sheet. I continue to follow their construction of this third quantity in Chapters 6 and 7.

Reasoning with number sequences. Although the teacher/researcher repeatedly tried to get the students to determine differences in the cards' values without doing the traditional subtraction algorithm, Adam was the only one to do so, and, even then, he only did it once. Protocol 5.6 describes this instance, which involves Adam reasoning about a difference. Right before the discussion in Protocol 5.6, Michelle had turned over a card worth 62 points, and Adam had turned over a card worth 39 points.

Protocol 5.6: Adam using an intuitive compensatory strategy.

A: Let's see, you beat me by 23.
T: Wait! Let's check it out. You said 22?
A&M: 23.
A: 'Cause 40 plus 22 is 62, so you subtract 1 from 40, you get 39 and so it would be twenty…
M: 21.
A: 23.
M: 23. Wait. What?!
T: What did you say again, Adam? I think that was really neat. You lost me. […]
A: OK. 40 plus 22 is 62, so if you subtract 1 from that, it's 23.
M: No, it's not. If you have to…
A: [Starts doing the traditional subtraction algorithm on his paper.]

T: I trust you, I trust you. No, don't do that…OK, you do that [laughing].
A: [Gets 23 as his answer.]
T: I like the way you're thinking, though, Adam. Can you think like he does, figuring it out that way? We call them strategies.

You can see the teacher/researcher attempting to validate Adam's type of reasoning. However, it took another session for Adam to start reasoning using his number sequence on a regular basis. Michelle took two more sessions before she started reasoning using her number sequence.

The most interesting phenomenon for me here, though, is Adam's reasoning itself. In Protocol 5.6, Adam was making motions to the left to show the decrease from 40 to 39 (or the increase by 1 of the difference), and he makes a circling motion to the right of his previous gesture to indicate the missing addend. In other problems he similarly makes hand motions as if he is picturing some kind of left-to-right sequence of numbers/units. Therefore, I hypothesize that Adam has some sort of linear visual re-presentation of his number sequence. In addition, he forms a goal of figuring out what he needs to add to 39 (his card) to get to 62 (Michelle's card), but he starts by working with 40 instead of 39. Figuring out what to add to 40 is a solvable problem for him, and he gets a missing addend of 22 as his intermediate solution. Then, he said, "so if you subtract 1 from that, it's 23." *That* referred to 40 because he was not subtracting 1 from 62 or 22; he adds 1 to 22.

For Adam, subtracting 1 from the 40, the starting addend, implies an increase of 1 in 22, the missing addend. This may not seem significant, but, as you will see in Chapter 7, neither Justin nor Michelle can consistently adjust the known addend when using strategic reasoning to find a missing addend. The fact that he is only adjusting by one might be necessary for this kind of thinking in that he might be able to adjust be one somewhat intuitively whereas adjusting by a larger number might require a more explicit sense of the additive structure he has constructed. In order to correctly make this adjustment, he either needs to have an intuitive sense of the compensatory relationship inherent in the identity, $a + b = (a + 1) + (b - 1)$, or he needs a visualization of the missing addend that maintains an awareness of the two numbers that bound it. In fact, Adam indicates this kind of visualization with larger numbers on March 22, as described in Chapter 7, but it is very difficult for him. Based on these considerations and his behavior in the next teaching session, which I analyze in Chapter 7, he was almost certainly using an intuitive compensatory strategy. Indeed, Steffe and Cobb (1988) hypothesized the same kind of reasoning in TNS students they worked with.

Justin and Lily

Justin and Lily were paired in this initial section of the teaching experiment. I was the teacher/researcher for all of Justin and Lily's teaching episodes. The witness was one of three other graduate student researchers. In Chapter 4, I confirmed that Justin can assimilate with two levels of units, and I hypothesized that he had constructed an ENS. Therefore, for Justin, I was looking for indications of the construction of a GNS. I confirmed that Lily had constructed an ENS, and I hypothesized that she had constructed a GNS. With Lily, I am looking for confirmation that she has constructed a GNS. To these ends, I analyze Justin and Lily's work on a few fraction and multiplication tasks from February. I also present Justin and Lily's work in the early additive contexts. In particular, I contrast how the two students operate with differences, including their language and notation when operating with and on differences.

An Analysis of Justin and Lily's Unsigned Numerical Schemes and Operations

Justin equi-partitions. In the first teaching episode with Justin and Lily, on February 9, I tested the hypothesis that Justin could equi-partition by putting him in a variety of partitioning situations. In Protocol 5.7, I describe the first of these tasks, in which I asked the students to partition a rectangle of paper into 8 equal pieces.

Protocol 5.7: Justin and Lily make 1/8 of a bar.

T: Your candy bar is going to be shared equally among eight people. You are going to cut off the share of one of those eight people. Actually cut it out. You can make a mark with your pencil first, but then I want you to cut off the share of one person.
J: [Makes a pretty good estimate, uses the estimate to segment the rest of the candy bar, finds out the estimate is too big, and makes a smaller estimate. He does this again. In all, he makes four different estimates before he cuts.] Eight pieces?
L: [As Justin works, she makes an estimate, repeatedly looking at what Justin is doing. Does not use her hands to measure off her estimate, but seems to do so with her eyes. Erases her original estimate and then cuts it off. Does not attempt to check this estimate.]
T: […] All right, do you all think you have a fair share? Your shares look different. Let me see yours, Justin. Put yours next to Lily's…[Justin's piece is about 1/8 and Lily's is close to 1/6.] Pretty different. How could we see which one's closer?
J: You'd go like this [picks up his piece, lines it up with the left-hand side of the bar] across it.
…
T: How would you know?
J: You'd have to count it and see if it equals it eight times.
T: Equals it eight times? Yours, that's 2, 3, 4, 5, 6 [counting as Justin moves his piece along the remaining part of the bar]. Does that work?
J: Almost. [The piece fit into the remaining bar slightly less than seven times, making it a little less than an eighth of the original bar.]
L: [Measures her piece off as well.] Mine's about 6 [referring to the piece itself and then the 5 times it fits into the remaining bar.].
T: Yours is about 6?
J: Mine is 7 plus this one [the cut-off piece].
T: 7 plus that one would equal 8. OK, good.

Justin's ability to make a good estimate before starting the segmenting process supports my hypothesis that he is projecting a composite unit of eight onto the bar in order to simultaneously be aware of the eight partitions. As in the initial interview, Justin's unprompted measuring off without making marks is an indication that he is equi-partitioning. However, in Protocol 5.7, Justin's multiplicative language, "You'd have to count it and see if *it equals it eight times*," gives me more definitive confirmation that he is equi-partitioning. In particular, when he is using his estimate to measure off more pieces, he seems to be thinking of them as iterations of the first piece, implying that he is assimilating the situation with the iterable units of his ENS. Also, the fact that both he and Lily cut off, or physically disembed, their 1/8 from the whole, but seem to be retaining the 1/8 piece mentally in the original bar, implies that they can disembed a fraction without destroying the whole. Hence I consider Justin's solution in Protocol 5.7 to confirm his construction of an equi-partitioning operation (and an ENS).

Justin and Lily assimilate with three levels of units. I began the February 10 teaching episode by determining what multiplication facts the students knew involving 11. Justin had memorized his multiplication facts up to 11x11, and Lily had memorized up to 10x11. I then posed two tasks, both described in Protocol 5.8, that were meant to elicit the articulation of iterable composite units.

Protocol 5.8: Justin and Lily operate on a three-levels-of-units structure.

T: How could you figure out 12 times 11, just using these other number facts?
L: You could do 10 times 11 and add 11.
T: Would that give you 12 times 11?
J: Yeah, you could do 10 times 11 and add 11 two times.
T: Two times. Is that what you were thinking?
L: I think you can add 11 times 2, which is 22.
T: How could you get 17 times 11?
J: You can do 10 times 11 and 7 times 11 and add that together.
T: Is there a different way?
L: You could add eleven 17 times, but that would take a long time.
T: What if I knew 5 times 11, how could I use that to help me?
J: Add 5 times 11 three times and then add 2 times 11.
T: How about if you're going to use 6 times 11?
L: You can do 10 times 11, 6 times 11 and 1.
T: Go ahead and figure it out.
J: 177.
L: [Nods.]
J: I did 10 times 11, and I added 6 times 11, and then I just added 11.
T: 10 times 11 was…
J: One hundred and ten, and I added 66 and then I added 11.
T: So 110 plus 66, what did you get?
J: 110 plus 66, I got 166.
T: [Shakes head.]
J: One hundred and *ten*…176.

T: So after you add 11?
J: 187.

Both Justin and Lily give indications in Protocol 5.8 of being able to assimilate with a three-levels-of-units units coordination structure (cf. Steffe, 2010f, pp. 90-92) when they give strategies for figuring out the product, 17x11. When Justin says, “You can do 10 times 11 and 7 times 11 and add that together,” and, similarly, when Lily says, “You can do 10 times 11, 6 times 11 and 1 [eleven],” they are assimilating the product 17x11 as 11 iterable composite units embedded in 17 iterable units. They show the iterability of the composite units of 11 by their ability to compress them into a single iterable unit that they can operate on in the form of repeated addition or multiplication. In the quotes above, Justin and Lily operate on the composite unit of 17 (units of composite units) as well in order to decompose it into 10 and 7 or 10, 6 and 1, respectively. The most impressive decomposition strategy is Justin’s strategy, “Add 5 times 11 three times and then add 2 times 11.” In this case, Justin is iterating 5x11, which is a three-levels-of-units structure. Therefore, I have confirmation that Lily has constructed a GNS, and I have a strong indication that Justin has as well.

The next two tasks are taken from the next teaching episode, on February 15, and are meant to confirm Justin’s ability to assimilate with three levels of units. In actuality, I am presenting indications that he had constructed a reversible improper fraction scheme, and then I show how this construction implies a three-levels-of-units assimilating structure by analyzing the levels of units that must necessarily be present in his assimilating structure in order to carry out his solution strategy. As the teacher/researcher, I started out the episode by giving them an unmarked bar and telling them it was 4/9 of another candy bar. I then asked them to figure out how long the other candy bar is. They both partitioned the bar into fourths and then put nine fourths (there were extra bars) together to get the answer. In Protocol 5.9, I ask the students to do a similar task given an improper fractional bar. At first they misunderstand the question, but after clarification both are able to make sense of the new problem situation.

Protocol 5.9: Justin and Lily explain how to find the original bar given 7/5 of it.

T: We'll do another one that's a little bit like that. This one's going to be harder to draw, so what I might do is let you all start it and just explain how you would finish it [putting one unmarked bar on the table]. Let's say that this piece is 7/5 of another candy bar. […] I want you to find the original candy bar, the whole candy bar.
L: So if that's 7/5….
J: So that would—
L: So…
J: You'd have to cut—
L: all of this would be shaded in and then there'd be two more.
J: Yeah [nodding], two more and another one [bar].
T: […] This already *is* 7/5 [running hand above the entire bar].
L: It's 5/5.
T: No, what I'm telling you is you have to pretend this is 7/5 already.
J: Oh!
T: Does that make sense?
J&L: [Nodding] Yeah.
…
T: Then how could you get back to your original bar?
J: You'd have to cut it all in sevenths, and you can take two parts off of that. You could put it in seven and take two off and it would equal 5/7.
T: What do you think?
L: Well, 5 is the whole then part of it can be divided into five and then the rest of the space would be the part of the other candy bar. So it would be like the candy bar was already connected, but you haven't broken it up yet.
T: Right. But how will you divide it into 5? Would you divide it equally into 5?
L: Yes [hesitantly].
T: And then what part would you take to get the original candy bar?
L: [After 23s] I don't know.
T: […] Let me give you one that's a little easier to draw, so that you can show me […] what you think you would do and then you can see if it worked.

Given that the students just finished a problem in which they were supposed to find a given fraction of a given (whole) bar, and given that I said the task in Protocol 5.9 would be similar to that task, it is not surprising that the students initially assimilated the task with the goal of finding 7/5 of the original bar. Lily's assertion that the bar is 5/5 gives credence to the idea that the students had not assimilated the problem situation as I intended. Once I clarified the problem situation, both students showed an understanding of the quantitative relationships at play: Justin does so by giving the appropriate solution strategy. Lily does so by saying, “Part of it can be divided into five and then the rest of the space would be the part of the other candy bar.” Her assimilation of the situation with an improper fraction scheme is made explicit when she goes on to say, “So it would be like the candy bar was already connected.” In order to make improper fractions of bars in the past, both students have had to connect candy bars, so Lily is referring to that past activity in her statement. When she follows this statement up with, “but you haven’t broken it up yet,” I think she is imagining a partitioning of the “already connected” bars, and understands that the first 5 of those partitions would make the whole.

My question, “How will you divide it into 5? … equally?,” confused Lily. When she says, “Yes,” I think she is referring to the fact that the five partitions of part of the candy bar would be equal. I am not sure if she is having trouble figuring out her solution strategy in the 23-second pause or whether she is unsure of what I am asking, but I think it is likely that her reversible improper fraction scheme is not anticipatory and must be carried out in action. Even though she had trouble explaining her solution strategy in Protocol 5.9, in the continuation of the discussion, given in Protocol 5.10, she is able to *carry out* the solution strategy. In Protocol 5.11, I ask her again about her solution strategy to the task in Protocol 5.9, and she wants to draw it as opposed to explaining it there as well. Therefore, I hypothesize that Lily can, at this point, only reverse her improper fraction scheme in action.

Justin’s explanation in Protocol 5.9 implies an anticipatory reversible improper fraction scheme. The only contraindication is his statement that, in the end, he would have 5/7. However, he would have 5/7 of the given bar, so his statement may just reflect his attention to the given bar as the whole at that moment. In Protocol 5.11 I will ask him about his assertion that he would have 5/7.

Protocol 5.10: Lily finds the length of a bar given 4/3 of the bar.

T: [Putting a blank bar in front of each student.] This is 4/3 of the original, so you've already found 4/3 of a candy bar and this is what you got, […] and you need to find the original candy bar. […]
J: [Immediately starts partitioning into 4. Adjusts his original estimate after measuring off with his fingers.]
L: [Thinks for 11s, then marks off about 1/4. Measures off with her fingers to check the estimate. Makes a new estimate and uses tick marks to check it. Then draws a vertical partition after the third fourth.]
T: You have your answer? Show me how long the original candy bar would be.
L: It would be from here [3/4] to here [left-side of bar].
T: […] Justin, you can see what she did, right?
J: Yeah.
T: Tell him where you said the original candy bar would be.
L: From here to here.
T: Do you agree?
J: Yeah.
T: On yours where would it be?
J: [Has all partitions drawn in.] From here to here [indicating 3/4 of the bar].

Justin's immediate partitioning behavior implies that he had a solution strategy in mind before he began to draw, which confirms that he has an anticipatory reversible improper fraction scheme. Lily is able to solve the problem and identify the quantities she is forming, which implies that she does have a reversible improper fraction scheme. However, she is hesitant in beginning and carrying the strategy, which supports my hypothesis that her reversible scheme is not anticipatory.

Protocol 5.11: Lily shows how to find the original bar given 7/5.

T: Now let's go back, and I want you to think again about the 7/5. I understand Justin's explanation completely, but for yours [Lily's] I wasn't sure what you were dividing. Think about it now in terms of the one you just did. If this is 7/5, how would you find the original candy bar back again? ... Do you want to actually draw it, would that help?
L: [Nods.]
T: Yeah. Go ahead and try to draw it.
…
L: [Partitions it into sevenths after trying several estimates.]
T: OK, so do you know where the original candy bar is?
L: This. [Covers the last 2/7 and indicates the first five partitions.]
T: That part? OK. […] What did you do? […]
L: I divided it into 7 pieces and 5 is the whole, so 5 would be a whole candy bar, and then these two [the last 2/7] would represent the rest of it.
…
L: [Shades in 5 pieces.]
T: What fraction is the shaded in part?
L: One whole.
…
T: At some point when you were talking, Justin, you said it would be 5/7. What did you mean by, "it would be 5/7"?
J: I meant that part would have been a whole [shaded in part], but then that [the other two parts] would equal 2/5 of another one.
…
T: How does that makes this [shaded] part 5/7?
J: I didn't mean to say that.
T: […] I claim that you could call this 5/7, can you see why I would say that, Lily? […]
…
L: Because this [the whole bar] is divided into 7 pieces and 5 of them are shaded in.
T: So if I call it 5/7, then what's my whole?
J: Seven.

As I discussed after Protocol 5.9, Lily hesitates when I ask her to explain her strategy in Protocol 5.11, and she immediately nods affirmatively when I ask if drawing it would help. This is consistent with her behavior in Protocols 5.9 and 5.10. In Protocol 5.11, I also follow up with Justin about his claim that the five pieces in his solution represent 5/7. Although both he and Lily could make sense of my claim that the shaded area is 5/7, Justin indicated that he was not thinking of the five pieces in that way, but as 5/5: "I meant that part would have been a whole

[5/5], but then that would equal 2/5 of another one…. I didn't mean to say that [it is 5/7]." After his clarification, I can confidently attribute a reversible improper fraction scheme to him.

I now discuss the assimilating structures required for Justin's reversible improper fraction scheme. In order to solve the first task—given 7/5 of a bar, make the original bar—Justin had to assimilate the given bar as representing the result of partitioning and iteration and then reverse the process in order to get back to the original bar. In particular, he has to project a composite unit of 7 onto the bar to form partitions that represent the 7 iterations that would have been made to form the 7/5. Similarly to the splitting operation, he is using his awareness of iteration and partitioning as related, inverse operations to be aware of a simultaneous iteration and partitioning inherent in the partitioning. However, in this case, Justin is not only aware of the iterative relationship of the seven partitions he projects to the given bar, as in the splitting operation. He is also aware of an iterative relationship to the original bar. In particular, he is aware that any of the partitions iterated five times will make up a whole. Although he only articulates the idea that five of the partitions together make up the original whole, I am assuming he is aware that any one of them could be iterated to form the whole given his iterable fractional unit. Assimilation of the problem situation with a simultaneous awareness of the imagined partitions' two different iterative relationships requires a three-levels-of-units assimilating structure: The partitions themselves are an iterable fractional unit, and they are in a multiplicative relationship with both the original bar and the given bar. Therefore, I claim that Justin's construction of a reversible improper fraction scheme implies a GNS. Because Lily had not yet formed this as an anticipatory scheme, I cannot make the same claim about her assimilating structures based on these tasks. However, given her ability to distributively partition and flexibly operate on three levels of units, as in Protocol 5.8, I feel confident in attributing a GNS to her as well.

An Analysis of Justin and Lily Working With Differences

On February 17, I did one teaching episode with Justin and Lily in which my goal was to see how they interacted with differences. For example, were they able to compare them or combine them? What kind of language and notation did they use when working in situations where differences are important quantities? Based on how they operated in this teaching episode, I planned a new context involving directed differences that I started with all four students in March.

Weight gain/loss. During the February 17 teaching episode, I gave Justin and Lily three types of problems. The first type of problem asked questions about a list of seven weekly weights of a fictional person named Jake. The main two questions were: When did Jake change weight the most? If Jake gained 6 more pounds in Week 8 than he did in Week 7, how much did he weigh at his Week 8 weigh in? The first question was meant to engender attention to differences. I hypothesized that neither Justin nor Lily would have trouble with this since they could both assimilate situations with three levels of units, and I saw assimilating a difference as at most requiring three levels of units: the difference, the two numbers being compared to form the difference, and the underlying units of 1. The next type of problem in the teaching episode was a problem that required reasoning about a difference of differences. The third type of problem was a series of number patterns that I asked the students to complete. I discuss these last two types of problems in the upcoming subsections.

The first thing that stands out when watching the teaching episode on February 17 is the contrast in how Lily and Justin used/reacted to language and notation. For example, although they agreed on the answer to the first question, Justin discusses everything in terms of a missing action, whereas Lily seems to think of the difference quantity more as a thing, a noun.

Throughout Protocol 5.12, the students are referring to Figure 5.6, and I have used bolded text to highlight key features of their language.

Initial Weight	125 – pounds
Week 1	132 – pounds
Week 2	135 – pounds
Week 3	148 pounds
Week 4	139 – pounds
Week 5	140 – pounds
Week 6	138 pounds
Week 7	140 – pounds

Figure 5.6. Weights given to Justin and Lily.

Protocol 5.12: Lily and Justin talk about differences.

T: Which week did he change weight the most?
J: [Sits back as if done after 32 seconds.]
L: [Looks up as if done after 41 seconds.]
T: Do you both have an answer?
J&L: Yes.
J: Weeks 2 and 3.
T: Between Week 2 weigh-in and Week 3 weigh-in?
J: Yeah.
T: During Week 3. Is that what you got also?
L: Yes.
T: OK, so how did you all get it?
J: I added, I subtracted, **I added** [to get from] that [Initial Weight] to that [Week 1]. That was only 7. **I added 5** to that to make it 40. **I added 8**, I knew that would be 13. Then 148 minus 139 is only 9. And then 239 plus 1 is 40. Then **minus 2** is 140. So that one [Week 3] was when **he increased** the most.
…

T: Is that the same way you were thinking about it, Lily?
L: Kind of. I was looking at it and some of them weren't very **far apart by pounds**, like maybe 2 or 3 pounds. But this one has **the biggest change**. So that's kind of how I got my answer.

I will come back to Justin's tendency to refer to quantities in terms of actions and Lily's tendency to refer to quantities as entities in the subsection, "Number Patterns," which appears later in this chapter. Following the discussion in Protocol 5.12, I gave them the second question: If Jake gained 6 more pounds in Week 8 than he did in Week 7, how much did he weigh at his Week 8 weigh in? The language of this kind of question can be confusing, but Thompson (1993) found that persistent confusions about the quantities being referred to in these kinds of problems seem to indicate difficulty in constructing differences as quantities in and of themselves that could then be compared. Surprisingly to me, Justin had quite a bit of trouble understanding the situation. Lily understood very quickly that we were comparing the differences between weigh-ins. In fact, they both gave the (incorrect) answer of 146 and a half pounds at first, which I asked them to write down. Protocol 5.13 shows my attempts to clarify the question.

Protocol 5.13: Justin's difficulty quantifying comparisons between differences.

T: Let's just double check everything here. I said that he gained 6 *more* pounds in Week 8 than Week 7, so how many pounds did he gain in Week 7?
…
J: 2 and a half.
…
T: He gained 2 and a half pounds in Week 7 and he gained 6 more pounds in Week 8. How many pounds did he gain the way you guys have done it?
L: He gained 6 pounds. So it had to be 6 *more*. So really he gained eight and a half pounds.
T: Explain what you said to Justin.
L: If he gained 2 pounds here, he gained 6 *more* pounds. Or 2 and a half pounds, so 6 plus 2 and a half is 8 and a half.
T: Why are you doing the 6 pounds plus 2 and a half?
L: Because it's 6 *more* pounds. This is how much he gained, 2 and a half pounds, but we just added 6 pounds to this [140 and a half, the Week 7 weight] instead of adding the 2 pounds, or how much he gained, *plus* the 6 pounds.
T: Hmm. What do you think, Justin?
J: I don't know.

T: OK. When I say he gained 6 *more* pounds in Week 8 than in Week 7, what am I saying 6 more than? What am I comparing?
J: 6 pounds and 140 and a half.
T: If I wrote, “Jake gained 6 pounds in Week 8,” if that’s what I said, then you both agree that your answer would be 146 and a half, right?
J: Mm-hmm.
J&L: [Nod.]
T: Here what I’m trying to say is that Jake gained 6 more pounds in Week 8 than the number of pounds he gained in Week 7.
J: OK.
…
T: I’m comparing the amount he gained, not the amount he actually weighed, so these two [questions] should have different answers. Want me to give you another one?
J: Yeah.
T: Well, first of all, does Lily’s reasoning make sense there?
J: Yeah [uncertainly].
T: Kind of?
J: Kind of.

A difference of differences. After Protocol 5.13, I still had no indication of whether Justin could differentiate between the two quantities Lily and I were talking about, the change in weight and the comparison of changes in weight. In order to investigate Justin’s quantitative reasoning with regards to a difference of differences, I gave them a different problem that would again give them the difference of differences and ask them to find one of the constituting quantities for the first differences. I decided to do this problem during the weight task. It was not in my plans for the day, but it was based on the “Sister and Brother problem” used in Sowder, Sowder, and Nickerson’s (2009) textbook as well as that and other similar problems in Thompson’s (1993) research on the construction of complex additive relationships. Although Justin ends up with the correct answer to the task in Protocol 5.14, he continues to struggle to differentiate between a difference and a difference of differences.

Protocol 5.14: Justin compares differences between basketball scores.

T: Panthers and the Dragons are playing, and then the Bulldogs are playing Tigers. […] The Panthers won by 5 more points than the Tigers won by. The Panthers ended the game with 50 points, and the Dragons ended the game with 43 points. And let’s say that the

Bulldogs ended the game with 60 points. Figure out what the Tigers' score was. [See Figure 5.7 for a summary of the problem.]

Panthers	50	?	
Dragons	43		
			5
Tigers	?	?	
Bulldogs	60		

Figure 5.7. Summary of the given quantities in Protocol 5.14.

J: They scored 5 more than the Bulldogs?
T: The Panthers won by 5 more points than the Tigers won by. I'll write that down. I said, "Panthers won by 5 more points than the Tigers won by." Go ahead and think about that. When you get an answer don't say it. You can write it down. Also try to write something down to convince me and convince each other.
J: Oh, so the Panthers got 5 more than the Tigers?
T: The Panthers *won* by 5 more points than the Tigers won by.
J: Like when they first subtracted?
T: The Panthers beat the Dragons by 5 more points than the Tigers beat the Bulldogs.
L: [Writes, "Panthers won to the dragons by 7 pts. 7 – 5 = 2 so the tigers won by 2 points. Simple subtraction problem."]
J: [Writes, "Panthers score against the Dragons was 7 points, so the Tigers beat the Bulldogs by." Looks at Lily's paper several times.]
T: Tell me why you subtracted.
L: Because they won by 2 more points than the Dragons did.
T: How do you know, though? How did you know to subtract 5 to get that?
J: [Writes: 2 points.]
…
T: Can you see what she has here? "The Panthers won by 7 points." You had that down, right?
J: [Looks at Lily's paper and then his paper.] Yeah.
T: […] Can you explain why she subtracted the 5?
J: Because they beat the…they got 5 more points, so you would have to subtract the 7 from the 5 and that would show that the Tigers only won by 2.
L: […] If they won by 5 more points than the Tigers then you could kind of just subtract this [the scores of the Dragons and Panthers] and see […] what you're going to get, that'd be 45 points. […] So they won by 45 points at the end of the game.
…
T: What'd you get for the Tigers' score?
J: [Writes "62" quickly.]
L: Uh…62 points.

Lily did not appear to be listening to my initial discussion with Justin, but even if she did, she definitely found the task much simpler than Justin did. She writes down her original answer quickly, and, when she explains her answer, she uses her answer of 2 as a margin of victory, not a score, although she adds it to the wrong team's score. I do not think this was a necessary error and she does correct herself on prompting. Justin on the other hand, does struggle to understand the task. His first question, "They scored 5 more than the Bulldogs?" is a standard interpretation of the situation for someone who is having trouble reifying the differences, i.e., considering the differences in scores as quantities in their own right that can be used as input for further operations. He is interpreting the difference between the differences as a difference between scores. His second question, "Oh, so the Panthers got 5 more than the Tigers?" may represent another unsuccessful attempt to determine my meaning in that he is again using the 5 as a difference between scores, except now he is considering the scores of the Panthers and Tigers instead of the Dragons and Bulldogs. However, it is also possible that he understands in some sense that 5 represents a difference of differences and he just does not have the language available to talk about that. This interpretation is supported by his follow-up question, "Like when they first subtracted?" in which he seems to be referring to the initial comparisons or subtractions that "they" the teams would naturally make in the given situation, namely, subtracting the score of the Dragons from the score of the Panthers and the score of the Bulldogs from the score of the Tigers.

Before we move on to the next protocol, notice the way Lily and Justin are using language in the last two protocols. Although Lily had a brief difficulty interpreting my question in Protocol 5.13, her language throughout her explanations was precise. She used *more* to differentiate between differences in weights and comparisons of differences of weights. There was only one place, in Protocol 5.13 when she says, "If they won by 5 *more* points than the Tigers," that she uses *more* to refer to a difference in scores instead of a difference of differences. Justin, on the other hand, has not developed a way to verbally differentiate between the differences and the difference of differences. Even when discussing differences his language is unclear. For example, he writes, “Panthers score against the Dragons was 7 points” when he was trying to communicate that the difference between their scores was 7 points.” In future tasks that involve comparing differences, Justin continues to successfully operate, but his use of language continues to be imprecise with regards to whether he is referring to differences or a related quantity. We can see an example of that in a new context in Protocol 5.15.

Number patterns. After working with weights and game scores, I moved to numerical sequences in which the differences are changing, so these are not arithmetic sequences, but the differences form a simple pattern; the first pattern I gave was quadratic and the second was exponential. Both students successfully solved the quadratic pattern and quickly gave an explanation. Justin wrote, “Two more each time,” and clarified his meaning by saying, “Subtract two each time.” Lily wrote in, “-2 -4 -6 -8” and explained, “Each time going down by multiples of 2.” While both students clearly saw a pattern in the differences, Lily’s observation that all of the differences are multiples of two allows her to avoid explicitly comparing differences. Justin, on the other hand, does communicate about the differences of differences, by writing that the second difference is always two. Notice, though, that he does not tell us *what* is changing by two each time, although the fact that he says, “two *more*,” implies that he is referring to a change in the differences. In his speech, his language is again imprecise. Although his words imply that he is always subtracting two to get from one element in the pattern to the next, he undoubtedly intended to give the sense that he has to subtract two *more* each time. I do not think that this persistent ambiguity in language is simply an issue of not finding the right words. I hypothesize that he has an awareness of the both the differences and the second difference as actions, but I do not think he has clarified their relationship. He is working very intuitively. Lily, on the other hand, explicitly writes the differences. My guess is that the (-) represents subtraction and not a negative quantity, based on her future behavior. Nonetheless, she is clearly differentiating between the elements of the sequence and their differences both in her language and her notation.

The next number sequence was exponential: 3, 4, 6, 10, 18, … The teacher/researcher told Justin and Lily the beginning of the sequence and they both wrote the sequence on their papers. Protocol 5.15 reports how they solved the task, and the final version of both students' work can be seen in Figures 5.8 and 5.9.

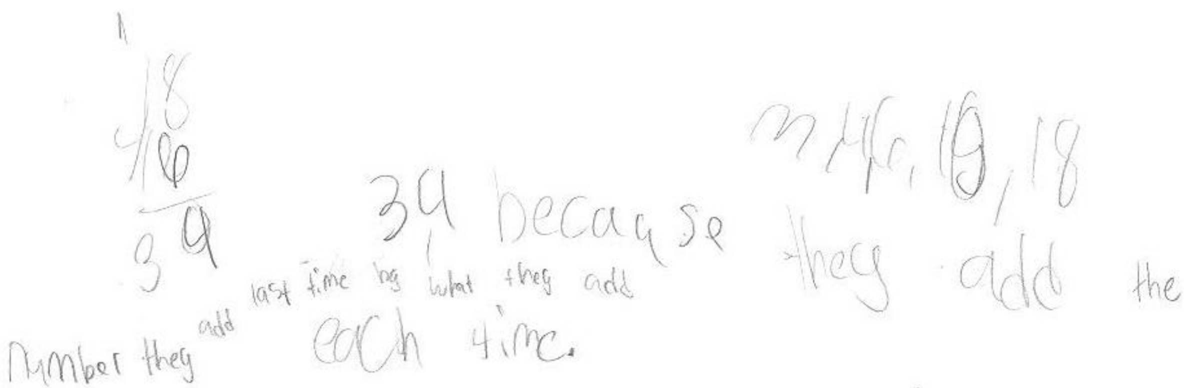


Figure 5.8. Justin's exponential sequence work.

Protocol 5.15: Justin continues an exponential number sequence.

J: [Writes, "24, because they add," puts his pencil on each number in the pattern. Adds 18 and 12 to get 30 and replaces the 24 with 30. Writes, "4 each time."]
L: [Adds 18 and 16 to get 34. Writes the differences, "+1 +2 +4 +8 +16." Underneath the differences, writes, "add the number by itself each time."]
…
T: And how did you get it?

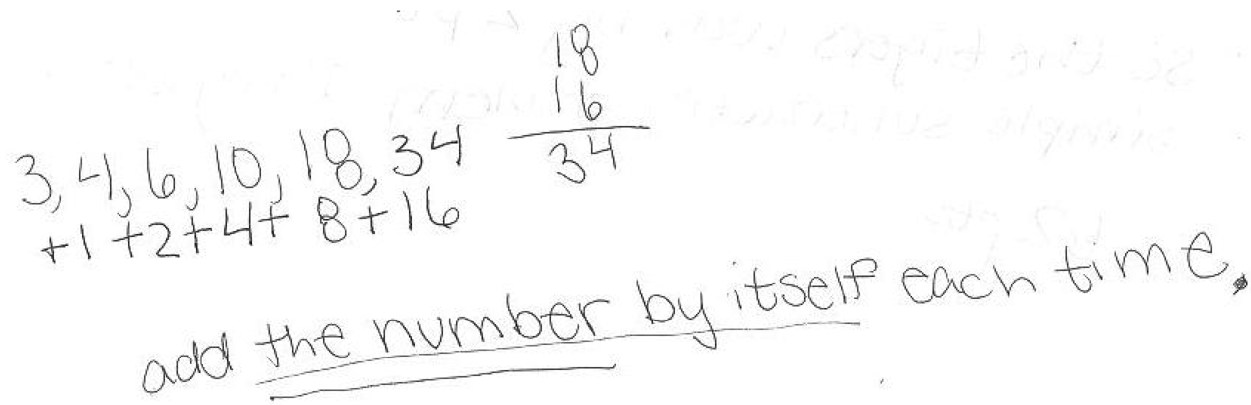


Figure 5.9. Lily's exponential sequence work.

J: Because they add and they times it and they add 1 more. They add 1 there [between 3 and 4] and then they times it [between 4 and 6] and then they add 1 more to that to equal 4 [between 6 and 10], 'cause that [between 3 and 4] would be 1, that added 1 to that and it would be 2 [between 4 and 6], and there's 4 [between 6 and 10]. And then they added another 4 there to equal 18 and so…Oh. Never mind. [Looks nervously around.]
T: Yeah, go ahead and change yours. I know you're close.
J: [Grabs eraser, replaces "4" in "4 each time" with "the number they add next time by what they add each time." Adds 18 and 16 to get 34, and puts that in place of 30.]
T: [While Justin is writing] When you wrote that, what did you mean by it?
L: On the bottom you add once each time what you had.
T: When you said, "the number," what did you mean? What is "the number"? [Underlines the phrase on Lily's paper.]
L: The numbers on the bottom, so 1 plus 1 is 2, 2 plus 2 is 4, 4 plus 4 is 8, 8 plus 8 is 16.
T: OK. What would the next number be?
L: Uh, 32.
…
L: 68
T: What do you think? We need verification.
J: [On his paper, adds 34 and 32 to get 66.] 66 because you would have to, because…I figured out that they add whatever number they use last time, like they add 4 there, so they added 8, 16.
…
J: And then 16 plus 18 equals 34. And then since it was 16 and 16 times 2 was 32, I knew I had to add 34 plus 32, would equal 66.
T: What would you have to add next time?
J: [Pauses 5 seconds.] 64.
…
T: What would you add?
J: 64 plus 66

In Protocol 5.15, Lily is still the only one notating the differences, and the notation allows her to refer to the differences as quantities more easily ("the numbers on the bottom") and allows her to see the pattern more quickly than Justin. However, Justin seems to be attending more explicitly to differences and their relationship to the given values than he has previously in the teaching episode. Here he has clearly constructed differences as quantities to be operated on: In his original explanation, he is often referring to the previous difference, or, as he is probably thinking of it, *amount added*, as "it," and then is referring to adding or multiplying "it" to get the next difference. Throughout this first explanation he is also clearly indicating the space on his paper between the numbers by using his pencil to run quickly back and forth between consecutive elements in the sequence when discussing their difference. In other words, he not only has a separate place for the differences (space between the elements), but he also is clearly referencing the constituting values of the difference through his motions. Similarly to the way in which Lily's notation seems to allow her to talk about and think about differences more explicitly, I think that having a written sequence of numbers allowed Justin to develop a way to attend to the differences and their relations to the original sequence in a way that is neither completely notated nor completely visualized. This is a hybrid of notation and visualization, perhaps parallel to a students' use of fingers to help them count their own counting acts.

Another aspect of the students' thinking exemplified in Protocol 5.15 is their attention to the differences as actions. Although they are operating on the numbers associated with these actions (how much is being added), both in Lily's notation and in the students' written explanations, they are referencing differences as actions, not as objects. Using Sfard's (1987) language, the differences are mathematical operations, not mathematical structures to both students at this point. Lily's notations of the operations might represent that she is further along in the transition from difference-as-operation to difference-as-structure. Recall that in Protocol 5.12, the first task of the episode and in the first sequence problem, Lily tended towards using nouns to refer to differences and comparisons of differences, whereas Justin tended towards verb phrases. In addition, Lily seemed to be able to attend to differences and operate on them more easily than Justin in each task.

However, this is not to imply that a student such as Justin should be pushed to notate differences at this point. His lack of notation means that he has to keep track of more quantities in his head. He carries out a remarkable coordination in Protocol 5.15, keeping track of elements of the sequence, like 66, even when they are not written, as well as the sequence of differences. This involves constructing a three-levels-of-units additive structure in which the base units are the units of one in his GNS, the second level is the original sequence of composite units, and the third level is the sequence of the differences. Note that he is operating on the differences, so he is operating on a three-levels-of-units structure. In this case, the use of notation can reduce the cognitive complexity of the problem, which can be a blessing and curse. While the notation may allow some students to solve the problem who might not otherwise be able to do so, it can also deter other students from operating on a three-level-of-unit structure of differences. Such experience with this three-level-of-unit structure may be necessary in order to reflectively abstract the addition and subtraction operations and reconstitute them as assimilating quantities.

Justin's ability to operate in these situations, but not communicate clearly about his thinking, led me to hypothesize that he does have the operations available to reify differences and operate on them, but that he needs more experience using differences as the input for further operation. This was one consideration in my development of the *card game,* which we started in March.

The card game. Recall that in the card game, the students each choose a card. The player with the higher number on her/his card wins the difference in the number of points between the two cards. Because Justin and Lily had done some good work with fractions during February, I initially planned to try to work with them not just on adding and subtracting integers, but on adding and subtracting rational numbers. Therefore, in the first iteration of the card game, on March 2, their cards contained (positive) fractions between 0 and 1 with denominators from 1 to 12. However, in about 15 minutes of playing, only five rounds were completed. It turned out that the task of finding the difference between two fractions with different denominators, while certainly solvable for both students, was a still a genuine problem solving activity in which a lot of their thought went into determining the difference. Because I was interested in getting them to a place where they were operating *on* the differences, I did not want determining the differences to take up such a large part of their time and mental energy. Therefore, moving forward, I used cards with (positive) natural number values. For the same reason, I restricted the card's value to be 100 or under. At this point in the teaching experiment, we also had decided to switch pairs. In fact, Justin and Lily were well-matched mathematically and worked well together, but the other pair of students was not working well together, so we paired Adam with Lily and Justin with Michelle.

Change of Pairings

Shortly after I began work in the additive contexts with the students, I decided to switch pairings. An unfortunate dynamic had built up between Adam and Michelle in which both students saw Michelle as more mathematically advanced. In addition, Michelle came across to me as disparaging of Adam's work at times. I was worried that this might cause Adam to withdraw from problem solving activity so that his problem solving could not be looked down upon. Just a couple of teaching episodes after I began to notice this dynamic building between Adam and Michelle, Justin and Michelle were both absent. We decided to try working with

Adam and Lily together because they had both just started a new context and so we could do similar tasks with both. Adam seemed much more enthusiastic about the mathematics when working with Lily, so we decided to keep them paired together. Lily was much more advanced than Adam, but, at the time, I felt that she and Justin had similar ways of operating mathematically. Hence I, in effect, gave up the goal of finding the limits of Lily's ZPC in order to have the chance to learn more about Adam's mathematics. The next two chapters will describe the remainder of my additive work with the new student pairings.

CHAPTER 6

JUSTIN AND MICHELLE'S ADDITIVE REASONING

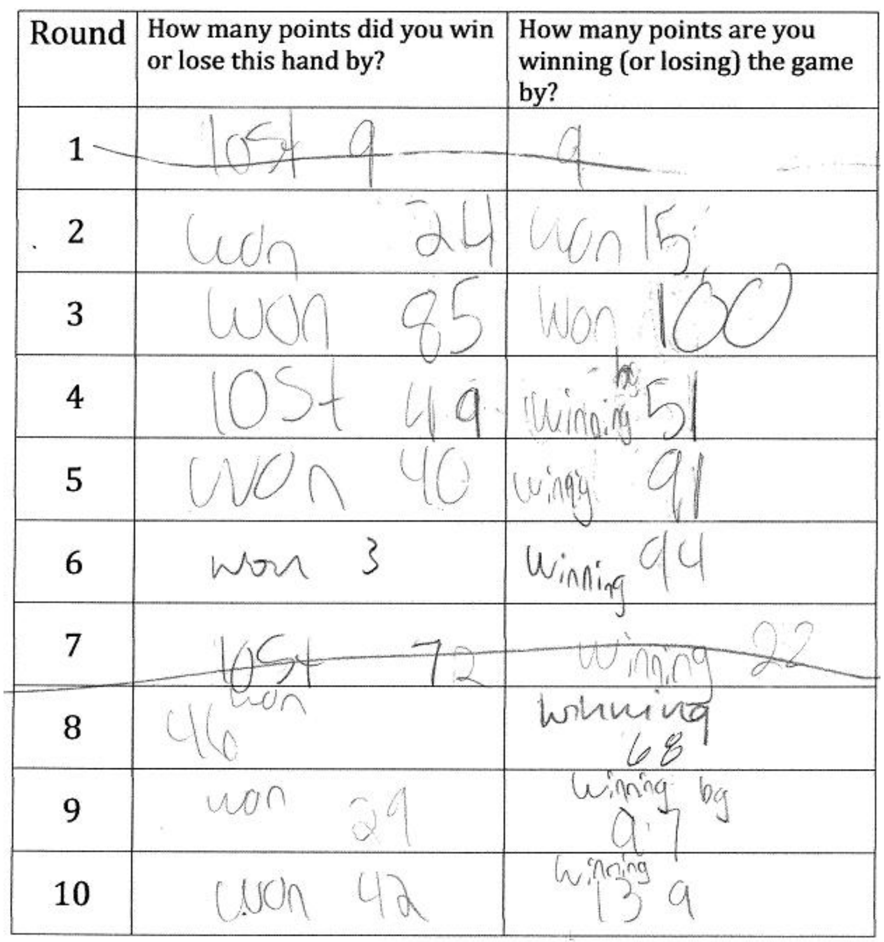
The first teaching episode when Justin and Michelle were partnered was March 9. They had a total of 12 teaching sessions together, ending on May 12. I worked with Justin individually on May 5.

The Card Game

On March 9, Justin and Michelle played the simplified version of the card game in which cards only had natural number values up to 100. In a previous episode, this version of the game had been played with Adam and Michelle and with Adam and Lily. In both instances, the students' first instincts were to keep track of their wins and losses in separate columns. Because I wanted to maximize the number of situations in which the students would have to coordinate positive and negative quantities, I decided to make a score sheet that would force the students to combine their wins and losses into a single running score (see Figure 6.1). Figure 6.2 illustrates how I would have expected both students to fill out their score sheet for the first three rounds of the game on March 9.

Mathematical Analysis of the Card Game

Before I report on the students' mathematical behavior, I will analyze the mathematics needed to fluently use this score sheet. The teacher/researchers intervened, if it was necessary, with both pairs of students to get them to specify whether specific entries represented a win or a loss. Therefore, the students were constructing signed quantities of some sort in that the values have both a magnitude (the absolute value) and an orientation (win or loss). To me, the second column represents a directed difference, A – B, in which A is the student's card's value for the student filling out the score sheet and B is the other player's card's value. However, the students never indicated that they thought of the Column 2 values in this way. The students interpreted their goal as finding out *how far apart* the two card values are, i.e., finding an undirected difference. The designation of "won" or "lost" was more like a record of who had the higher card than a sense of orientation of the difference. In other words, none of the students' words or actions implied that they were aware that the quantity represented the action they had to take to get from B to A, where a loss would mean going down the number sequence to get to A and a win would mean going up the number sequence to get to A.



Round	How many points did you win or lose this hand by?	How many points are you winning (or losing) the game by?
1	~~lost 9~~	~~9~~
2	won 24	won 15
3	won 85	won 100
4	lost 49	winning by 51
5	won 40	winning 91
6	won 3	winning 94
7	~~lost 72~~	~~winning 22~~
8	won 46	winning 68
9	won 29	winning by 97
10	won 42	winning 139

Figure 6.1. Justin's final card game score sheet on March 9.

Round		Justin's Score Sheet		Michelle's Score Sheet	
		How many points did you win or lose this hand by?	How many points are you winning (or losing) the game by?	How many points did you win or lose this hand by?	How many points are you winning (or losing) the game by?
1	Justin 91 Michelle 100	Lost 9	Losing 9	Won 9	Winning 9
2	Justin 45 Michelle 21	Won 24	Winning 15	Lost 24	Losing 15
3	Justin 87 Michelle 2	Won 85	Winning 100	Lost 85	Losing 100

Figure 6.2. Overview of the card game.

Throughout the discussion of this context, I refer to entries by their round and column placement. For example, in Figure 6.1, Justin has written "won 85" in Round 3, Column 2 (R3C2).

Column 3 is a cumulative sum of the player's relative[8] scores (Column 2 values). Therefore, from my perspective, Column 3 consists of sums of signed differences. From Round 2 on, the Column 3 value is most easily calculated by combining the previous Column 3 signed value with the current round score, also a signed value. Both the C2 and C3 values are signed numbers. Therefore, the sum of two of these values would represent a signed addition, and

[8] I use *relative* to indicate that the score is not representing the total number of points, which we were not keeping track of, and was always positive, but rather the directed difference between one player's overall score and the other player's overall score. For example, "winning 94" in Justin's R6C3 means that his overall score was 94 higher than Michelle's, although we do not know exactly what either of their overall scores was.

finding a C2 value given two consecutive C3 values would represent a signed subtraction, or a signed missing addend (MA) problem.

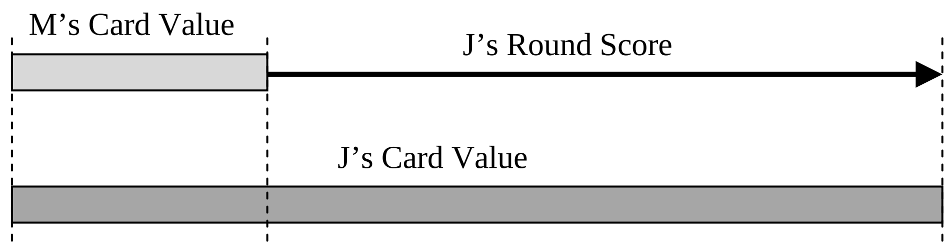


Figure 6.3. Quantities constituting C2.

To get a better sense of how I envision the relationships between the various quantities, suppose that Justin is playing against Michelle and we are filling out the score sheet from Justin’s perspective. Then Figure 6.3 shows how I would envision the relationship between the card values and the round score. In this case, Justin would have won the round. The direction and the size of the arrow would represent the sign and absolute value of the C2 score, respectively.

The additive relationships in Figure 6.3 are related to the additive relationships in an unsigned subtraction context. That is, one card value is a subset of the other and the computed quantity corresponds to the complementary subset of the larger card value. However, there is now a directionality to keep track of, which results from an ordering of the two constituting quantities. If Michelle and Justin’s bars had been switched, the same subset relationships would exist as in Figure 6.3, but the directionality of the round score would change. Therefore the round score contains a record of the order of the constituting quantities that an unsigned difference does not.

Figure 6.4 shows how the first two round scores in a game would be combined to determine the new win/loss margin, the C3 quantity. Here again, the lengths of the arrows in Figure 6.4 correspond to an additive relationship among unsigned quantities. However, unlike the unsigned sums, which contain the two as complementary subsets, a signed sum can correspond to one of the complementary subsets in the underlying unsigned additive relationship.

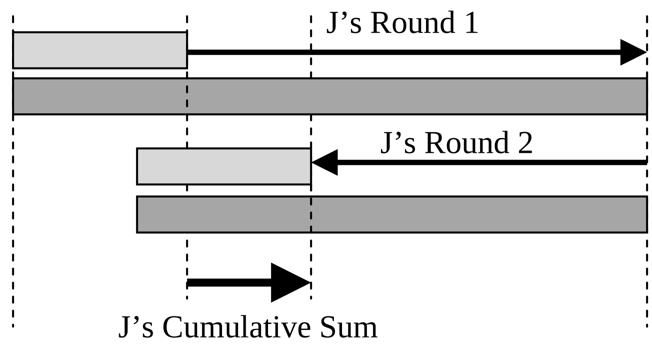


Figure 6.4. Quantities constituting C3.

Figure 6.5 illustrates all possible signed relationships that correspond to a particular unsigned additive relationship. Note that knowing which quantity corresponds to the sum is not sufficient; for each possible correspondence for the sum, there are two possibilities of signed additive relationships. In order to specify a unique signed additive relationship given the underlying unsigned additive relationship, at least two additional pieces of information are needed. For example, if I know which unsigned quantity corresponds to the sum and I know the directionality of the sum, then I can determine the signed additive relationship.

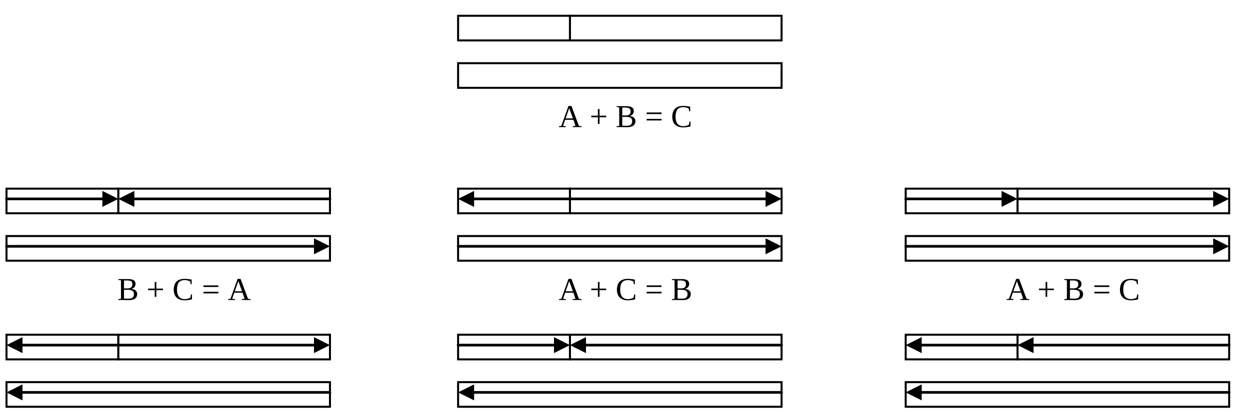


Figure 6.5. The six signed additive relationships corresponding to an unsigned one.

Analysis of the Students' Additive Operations During the Card Game

At the time of data collection, I thought of directionality as adding one extra layer of complexity to the situation that would require that students could not only construct unsigned additive relationships, but even assimilate with them. I hypothesized that, because assimilating with two levels of units is sufficient to construct unsigned additive relationships, assimilating with three levels of units, or the construction of a GNS, would be sufficient to construct signed additive relationships. As discussed in Chapter 5, Justin had constructed a GNS. Hence, I hypothesized that he would be able to operate comfortably in situations of signed addition and subtraction. Michelle, on the other, had not shown definitive indications of a GNS. Therefore, I expected her to be able to be able to find signed differences, but I thought that more sophisticated problems, such as making sense of the C3 quantity or finding a C2 quantity given a C3 quantity, might be difficult for her. In fact, Justin struggled to make sense of the card game quantities both days he spent on the card game, whereas Michelle did not appear to have any issue making sense of the quantities, although she, along with Justin, experienced difficulties when working on missing addend tasks.

The first two rounds of the card game on March 9 were uneventful. However, in Round 3, Justin deliberated at length about the value for R3C3. Figure 6.6 shows his score sheet at this point in time. He first writes, "100." He then says, "Wait," points to the 9 in R1C3, and changes his answer to 91, saying, "I put 91." Finally, he changes his answer back to 100. From the time he wrote "won 85" for R3C2 to the time he finally decided on "100" for R3C3 was about one minute. He did not interact with Michelle during this time, except to look at her initial answer, which cannot be clearly seen on the video, but is probably 105 or 109.

My interpretation of Justin's behavior is that he was trying to decide between two courses of action: (1) add the 15 and 85, or (2) add the 15 and 85 and then subtract 9. The first course of action, adding the 15 and 85, could indicate an interpretation of the situation consistent with mine. His second course of action is based on an ambiguous interpretation of the C3 values. In this interpretation he would have thought of *previous* C3 values as representing just wins or just losses, but his goal for the current C3 value was to find his margin of victory (or defeat). Under this second interpretation of the situation, he would add his current win (the round score) to the past wins (15) to get the new total wins and then combine this total (through subtraction) with the total losses to get his margin of, in this case, victory. Note that this course of action does not have an internally consistent interpretation of the C3 values because it assimilates past C3 values as points won or points lost, while it assimilates the current task with the goal of finding a comparison between points won and points lost.

Round	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
1	lost 9	9
2	won ~~24~~ 24	15
3	won 85	

Figure 6.6. Justin's score sheet in the middle of R3 on March 9.

There are other examples of situations in which Justin does not necessarily interpret C3 values as a comparison of his score and Michelle's score. Many times they seem to represent a number of points, where the orientation is positive if they are Justin's points and the orientation is negative if they are Michelle's points. In order to get the 15 in R2C3, he would have had to combine positive and negative values to get a signed sum, but the records of the negative points seem to decay for him, so that by the time he gets to the third round, he thinks of the 15 as only representing points won. My hypothesis is that Justin's positive and negative quantities are not yet unified to form a single, coherent quantity. Instead, I think that he works with two separate, but related, quantities: his (winning) points and Michelle's (losing) points. He cannot yet think about the situation in terms of a single, unifying signed quantity reflecting the directed difference between his score and Michelle's score. If he were able to assimilate with this quantity, then Column 2 scores could be assimilated as changes in this quantity, thereby unifying all of the values on the score sheet.

An alternate hypothesis is that he does not have experience with cumulative sums and hence is having trouble making sense of the C3 quantity. Similarly, he might not be sure which quantity he is supposed to be keeping track of in Column 3. Both of these hypotheses would imply that his ambiguous interpretation of the C3 quantity is not a necessary error. These alternate hypotheses do hold, at least in part. As we will see in upcoming protocols from this same March 9 teaching session, Justin went through a period of attempting to come up with a quantity that would allow him to act intersubjectively with Michelle and I; he was trying to think in a way that is internally consistent and made sense of Michelle's (and my) answers and explanations. However, the fact that within the same explanation he vacillated between interpreting C3 values as if they represent the total of just positive or just negative points and interpreting them as if they represent a cumulative sum, implies that he was not just trying out different interpretations, but that he was also conflating the possible C3 quantities. This ambiguity caused him to experience perturbation when the competing interpretations lead to conflicting outcomes, often pointed out by Michelle or I. The next few protocols illustrate some of Justin's difficulties. Note that Michelle was consistent throughout in describing a cumulative sum in reference to C3.

Protocol 6.1: Justin equivocates about C3 quantity.

M: [Draws 82.]
J: [Draws 33.] Dang it!
T: Now, before you figure out the numbers, who's going to win or lose this round?
J: She is. [Writes "lost" in R4C2.]
T: And who's going to be winning or losing the game? Can you figure that out without…
M: Me.
T: You're going to be winning the game?
M: Losing.
J: She's losing.
…
T: Losing the game and you'll be winning the game?
J: Yeah. So far.
T: How can you tell, because she won the round, so how can you tell she won't be winning the game?
M: Because I'm losing by a hundred and I only get [whispering] 7, 33…
T: Well, we don't know exactly how much you'll get. […] But we do know it's less than a hundred for sure?
M: Yeah.
J: Mm-hmm.
T: How do you know it's less than a hundred *for sure*?
J: Because it's only 80 and you have to subtract it by 33 […][whispering] 49 [points to 9 in R1C3]…57…[writes 49 in Column 2 and 58 in Column 3, then goes back and writes "won" in R2C3 and R3C3 and writes "lost" next to 58].
M: [Writes "lost" in C3.]
J: You *won* by.
M: No, I know, but I *lost* this round. I'm still losing.
J: Oh. [Erases R4C3]…[…] 49 plus 9 is 58. [Writes 58 again, does not specify winning or losing.]
…
T: Was the 58 winning or losing?
J: It was losing by 58, 'cause 9 plus [indicates R1C3 and R4C2] that would be 58.

Note that Justin did not have the correct answer at the end of Protocol 6.1. I do not realize this until later in the teaching episode. In this round, Justin worked throughout with the C3 scores as if they represent an accumulation of just positive or just negative points. In particular, because he lost by 49, he combined that loss with what he thought was a total of losses, the 9 from R1C3, giving a new total of losses, 58.

Justin experienced perturbation when Michelle noted that even though she won the round, she was still losing the game. I surmise that he erased his answer because he felt that he had not taken his wins into account, which conflicted with his sense that the C3 quantity tells whether he was winning or losing. Nonetheless, when he worked out R4C3 a second time, he came up with the same answer. At first, this was particularly surprising to me because he said he was winning the game so far before he calculated his C3 score. However, I hypothesize now that he was still teasing apart the two conflated quantities represented by his C3 values, and that he had temporarily decided that the C3 values represent either his score or Michelle's score, not how much he was winning or losing by. His C3 score of losing by 58 would not contradict the fact that he is winning overall because, given his interpretation, he had 100 points and Michelle had 58 points, so he was winning overall. The idea that the value should also represent the overall score is still lurking in his thoughts, as we will see in future rounds. Unfortunately, perhaps because I do not correct Justin's answer, Michelle eventually changed her Round 4 answers to match his, which probably alleviated any perturbation he felt at having a different answer than her. Nonetheless, the conflation that caused indecision in Round 3 caused a more explicit perturbation here. In Round 5, the perturbation would be even greater.

The full discussion of Round 5 lasted about 10.5 minutes. I edited Protocol 6.2 down to the main arguments made by Justin and Michelle, although most of the ideas are repeated multiple times over the course of the 10.5 minutes. Because the protocol is still quite long, I have added numerals to the actor's initials in order to more easily refer to specific statements. Justin started off Round 5 with the values shown in Figure 6.7.

Round	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
1	lost 9	9
2	won ~~24~~ 24	won 15
3	won 85	won ~~100~~ ~~91~~ 100
4	lost 49	~~lost 58~~ 58
5		

Figure 6.7. Justin's score sheet after Round 4 on March 9.

Protocol 6.2: Michelle and Justin solve R5 on March 9.

M1: [Draws 32.]
J1: [Draws 72.]
T1: Justin, are you winning or losing right now?
J2: [Writes "won 40" in C2 and "winning 140" in C3.] I'm winning.
…
T2: You have different answers.
J3: [Erases 140.] It's 140 minus…58. […] 100 minus 58 would be 42. [Writes "42" in R5C3.]
T3: OK. If you were winning the whole game by 100 [pointing to R3C3] and then you lost the round by 49 points [pointing to R4C2]…
J4: Oh, no. [Erases R5C3 score.] I'm winning. […] Winning by 51. [Writes, "winning 51" in R5C3.] Wouldn't I have to add that ["9" in R1C3] to that ["lost 49" in R4C2] so I could…
…
T4: You know what? I'm not sure that 58 is right for you two.
J5: I thought that because you'd have to add the 49 plus the 9.
T5: And what would that tell you?
J6: That her score's 58 and I'm winning by 42.
T6: OK. What does the 100 tell you?
J7: That's my score. I'm winning by 100.
T7: Before…

J8: Before she won by 49. [Erases “58” in R4C3.] […] [Points to “9” in R1C3 and “49” in R4C2.] Would I have to add these two together? ‘Cause if that would be 58 [R4C3], won by 40, 140 minus 58. [Erases R5C3.]
…
M2: Since I won this round […] I had to subtract 49 from 100, which is 51. So wouldn’t I just be losing by 51 in *this* round? Round 4?
T8: You’d be losing…
M3: by 51.
T9: The whole game by 51.
M4: Yeah!
T10: Yes. I think that would make sense from what you were saying.
J9: Writes “winning 51” in R4C3.] I’m winning by 51. Then I’m winning by…
…
T11: And what happened right after that?
J10: Then I won by 40. […] So would I…add the 51 [R4C3] plus 40 [R5C2] would be 91.
T12: You’d be…And would you be winning or losing?
J11: I’d be winning. […] She has fifty-something. She has 58. Would I do 140 minus 58?
T13: To find what?
J12: How much I’m winning by.
…
W: Justin, can you explain where that second line came from, how’d you know that was going to be “won 15” [R2C3]?
…
J13: ‘Cause I won by 24. Wait. What were the numbers again for that round?
T14: I’m not sure, but you know you won the round by 24 [indicating R2C2].
…
J14: I won by 24, then I subtracted 9 from the 24 and that means I’ll be winning by 15.
T15: Why did you subtract the 9 from the 24?
J15: Because she already had 9, and then if I had a 9 too, we’d be tied, and then I subtracted 24 from the 9. [See Figure 6.8 for recent changes to the score sheet.]
…

Round	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
1	lost 9	9
2	won ~~24~~ 24	won 15

3	won 85	won ~~100~~ ~~91~~ 100
4	lost 49	~~lost 58~~ ~~58~~ winning 51
5	won 40	~~winning 140 42~~ ~~winning 51~~

Figure 6.8. Justin's score sheet partway through R5 on March 9.

T16: When you figure out the 100, did you use the 9 here [R1C3]?
J16: No [perturbed]. Should be 100 minus [points to R1C3], 91. [Looks at Michelle.]
T17: You should be winning by 91? And why would you be?
J17: 'Cause if you subtract the 9 from the…if you add those [indicates "won 15" R2C3 and "won 85" R3C2] together, that'd be a hundred. Then you could subtract […][points to "won 100" in R3C3 and "9" in R1C3]…
T18: And what does the 100 tell you?
J18: That's how much I was winning by. […] And I have to subtract 9 from the 100 […] It'll give me 91.
T19: But, I mean, what does it tell you?
J19: That I'm winning by 91.
T20: I thought you just said you were winning by 100, though.
J20: I think I did that wrong. [Erases "100."] 'Cause I forgot to subtract the 9 from her score. So that was 91. [Writes "91."]
T21: Hmm. I don't know. Michelle still has her 100 there. How did you get the 100, Michelle? Can you explain it to Justin, not to me?
M5: OK. Well, since we were, since this is our score for the whole game [pointing to each entry in C3], either we're losing by 9 in the whole game or we're winning by 9 in the whole game.
J21: [Begins erasing R3C3 as Michelle continues.]
M6: And then I lost by 15 and I lost by 85 again, so I added those two numbers up, which got me a hundred. So I was losing by a hundred, which means you would be winning by a hundred during the whole game.
J22: I thought I'd be winning by 91 because your score's not, you'd have to subtract…
M7: But this R1C3 doesn't count anymore because now this R2C3 is our new score.
J23: Oh, oh yeah. [Dispirited, rewrites "100."]

Throughout the March 9 teaching episode, Justin and Michelle are attempting to communicate their different ideas of C2 and C3 quantities by differentiating their language when discussing the different quantities. The emerging distinction for these two students seems to be the use of “won” or “lost” in reference to round scores and “winning” or “losing” in reference to a cumulative sum. During this round, however, both Justin’s language—“lost by 9, won by 15, won by 100 [pointing to C3 entries]”—and Michelle’s language—“And then I lost by 15 and I lost by 85 again, so I added those two numbers up, which got me a hundred. So I was losing by a hundred, which means you would be winning by a hundred during the whole game”—is still inconsistent when referring to C3 quantities. I think this indicates that neither student has stepped back from the situation and reflected on how the C2 and C3 quantities are different and how they are related. In addition, Justin uses other language that gives us clues as to his conceptualization of the quantities at hand. In lines J6 and J7, Justin interprets his C3 value of (losing) 58 to be Michelle’s score and his C3 value of *won 100* to be his score. Furthermore, he sees the 58 as telling him both Michelle’s score and how much he is winning by, which he finds by subtracting Michelle’s score from his score. This makes explicit his current use of C3 as total wins or total losses, but not the amount he is winning or losing by. We also see that when he has been saying “winning by” in the past, he might not have meant what we thought because he said, “That’s my score. I’m winning by 100,” as if both sentences represent the same idea. Going back to the original discussion of Justin and Michelle’s ambiguous language, this shows that, not only do both misapply (from my perspective) “won” and “lost,” but Justin also misapplies “winning by” and “losing by.”

Another important section in Round 5 is lines W–J16 when the witness asks Justin, “Can you explain where that second line came from, how’d you know that was going to be “won 15” [R2C3]?” and Justin ends up reviewing his earlier calculations. The question was asked in order to point out to Justin that he has already used the 9 to calculate his C3 *won 15* score, so the 9 does not need to be added to the 49 again. When confronted with this instance where he did not give C3 in terms of total wins or losses, Justin wants to see card values again. I am assuming he is now unsure about whether he won the round by 24 or 15 because if R2C2 is supposed to represent the total wins, it should be *won 24* instead of *won 15*. Once the teacher/researcher assures him that he won the round by 24, he is able to interpret his *won 15*, which implies that he is back to a C3 quantity that is a sum of wins and losses. However, when the teacher/researcher challenges his *won 100* score by asking if he used the 9 from R1C3, he reverts to the idea that the C3 values are total wins or total losses. This shows that he is still conflating the two quantities.

An interesting aside that will be central to later discussions is Justin's nascent use of an *additive inverse* to get to a reference quantity in his explanation of how he got *won 15* in line 42. He says, "Because she already had 9, and then if I had a 9 too, we'd be tied, and then I subtracted 24 from the 9." In this case, being tied would be the reference value, both would be winning and losing by 0. He recognizes that in order to get there, he needs to use part of his score to get him back to the reference quantity and then determine how much is remaining to find his margin of victory. I will later argue that attending to the use of additive inverses to obtain reference quantities is an integral step in unifying two unsigned quantities, such as wins and losses, into a single signed quantity, such as overall score.

During Round 5, Justin seems to come up against perturbations out of his interactions with Michelle. In particular, he seems to recognize that Michelle's concept of C3 quantities and what it means to be "winning by," which I give authority to as the teacher/researcher, does not match up with his concepts. However, he does not give an indication of what sense he has made of Michelle's reasoning by the end of the round. Given his dispirited mood at the end, it is quite possible that he wrote in 100 again simply because it seemed to be the right answer in everyone else's eyes. Similarly, he seems to have written in 51 in R4C3 because he heard me tell Michelle that that was the correct answer in line 46.

In Round 6, Justin wins, so it is not clear if he thinks he is adding up wins to get R6C3 or finding out how much he is winning/losing by. Figure 6.9 shows his score sheet going into Round 7. In Round 7, he gives some indications that he is now keeping track of a winning/losing quantity instead of wins and losses. Protocol 6.3 gives the dialogue for this round.

Round	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
5	won 40	~~winning 140 42~~ ~~winning 51~~ winning 91
6	won 3	winning 94
7		

Figure 6.9. Justin's score sheet after R6 on March 9.

Protocol 6.3: Round 7 dialogue on March 9.

T: [Writes "losing 22" in Michelle's R7C3.] […]
M: Oh, that means I won! […] That means I won! [Writes "won" in R7C2.]
T: Yes, so, first of all, that means she won. […] How do you know she won?
J: [After 5 seconds.] 'Cause that means she was […] losing by 94 and now she's only losing by 22. […] [Writes "lost 72" in R7C2.]
T: Are you winning or losing the game, Justin?
J: I'm…94…I am winning [writes "winning" in C3] because she's losing.
…
T: But *also* because you were winning by 94 and how much did you lose the round by?

J: I lost by 22, no, 72.
T: Is that enough to make you lose?
J: No.
T: No, it's not enough to make you lose. OK. How much are you winning by?
J: I'm winning by 22, I think.
…
T: Go ahead and figure how much you're winning by.
J: [Whispers] 22. [Appears to be doing a calculation in his head. Writes "22" in C3. About 8 seconds have passed since the teacher spoke.] […] [Erases "22."] If I lost by 72… [Does another calculation in his head. Writes "22" again.]

In Round 7, Justin appropriately calculated to get the missing round score even though the round score was a loss and he was winning overall. Furthermore, he knows he is winning if Michelle is losing and, before calculating, he guesses that he will be winning by the same amount she is losing by, 22. Taken together, this implies that C3 is a *relative score* now. If it were not, then there would not be a necessity that there overall scores would have the same magnitude. Furthermore, is able to interpret a decrease in losing C3 scores, which again implies C3 is now a *relative score* for him. I think it is neat to see him double-check that he will also have a magnitude of 22 in C3. This indicates that he is trying out a new interpretation of C3 and is still tentatively testing it to see if it works with what Michelle and I are doing.

Despite these positive indications that Justin is making progress in constructing a cumulative signed sum as the C3 quantity, the next two protocols indicate the limitations of his ability to operate in this setting. Protocol 6.4 gives an excerpt from the Round 8 discussion, and Figure 6.10 gives the state of Justin's score sheet at the beginning of that discussion.

Round	How many points did you win or lose this hand by?	How many points are you winning (or losing) the game by?
6	won 3	winning 94
7	lost 72	winning ~~22~~ 22
8		

Figure 6.10. Justin's score sheet after R7 on March 9.

Protocol 6.4: Round 8 dialogue on March 9.

T: How about this time you just select them, don't turn them over. […] [I write "losing 68" on Michelle's R8C3 and "winning 68" on Justin's.] […] Justin, do you think you won or lost the round?
J: I think I won.
M: Because mine says, "losing." [Indicates R8C3.]
T: But the round, did he win or lose the round?
J: Oh, won.
T: How do you know that…why do we think that Justin won the round?
J: Because…
M: Because I was losing by 22 and now I'm losing by 68.
J: And I'm winning by 68 plus by 22.
T: OK. OK. What I want you to do is to come up with some possible values for the cards. This time you don't have either, so you don't know exactly what the cards are, but give me some possible values for the cards.
M: [Nods.]
J: 33 and…Wait. 35 and 33.
T: 35 and 33?!
J: 'Cause 30 and 30 would be 60 plus the 5 and 3 would be 8.
T: If you have 35 and she had 33, what would the score for the round be?
J: 35, 8, the score would be 68.
M: I'd be losing by 2!
T: Yeah.
J: Oh!
T: That probably wouldn't work, right?
M: 68 minus 22…
J: 68 plus 22…The score is 90. 68 plus [puts pencil on 22]…
T: Can you all figure out how much you won or lost the round by…without figuring out the actual card values?
M: 46.
T: 46? And did you win or lose?

M: I lost.
T: OK. Go ahead and write that in. I'm going to turn this over, and I want you all to just check my calculations, OK? [Michelle's card was 14 and Justin's was 60.]
J: 46.

When Justin and Michelle were given the C3 score, Justin incorrectly reasons that he won the round because "I'm winning by 68 plus by 22," and then he finds the round score by adding the two numbers. I think that he is conflating the winning score with the round score in this instance, and it probably indicates the lack of reversibility for his schemes of action he has developed for the card game. He then gives card values that add to 68 as the possible card values, which may indicate that, while he can assimilate with an unsigned sum structure, he cannot yet assimilate with a difference structure. Once he is given the card values, he does determine the correct round score, showing that he does continue to understand the situation.

The last two rounds yielded little information about Justin's thinking. He did come up with answers that I agreed with, but they did not represent situations that had been challenging to him earlier, such as when the C2 value and C3 value had a different directionality. At the end of the tenth round, I gave the students a final task. Protocol 6.5 reports the resulting exchange.

Protocol 6.5: Getting rid of a round.

T: If you could get rid of any round, which round would you get rid of in order to help your score?
J: The first round!
T: You'd get rid of the first round. OK. Go ahead and put a line through whichever round you get rid of.
J: [Crosses out R1.] 'Cause I won every other round.
T: Really?
J: Yep. [Points to R2-10C3 values.] Won, won, won, won, won, won, won.
M: That's sad. No, you were *winning*. You didn't win all the rounds. I won that one [R4] and that one [R7] and then the rest I lost.
J: 1, 2… [pointing to C2 scores that are losses].
T: Why did you choose that round, Justin?
J: Wait. Because that's the…the only one that I lost…Wait. [Points to 72.] 72. Ah, I would pick this one [R7].
T: Why did you pick this one [R3]?

M: Because I lost by the most.
T: You lost by the most. Why did you change your answer, Justin?
J: Because that one I only lost by 9, this one I lost by 72.
T: Oh, gotcha. OK. Now I want you to figure out, however you want to, I want you to figure out how that changes how much you won or lost the game by. And then write down your new ending score at the bottom.
M: So pretend that we never had lost, like, 85?
T: Yeah, pretend that you never had Round 3.
J: Do we add this now [R6C3 and R10C3]? [Writes 233.]
T: I do not agree with your answer.
J: [Erases.]
T: Did you get one yet?
M: Mm-mm.

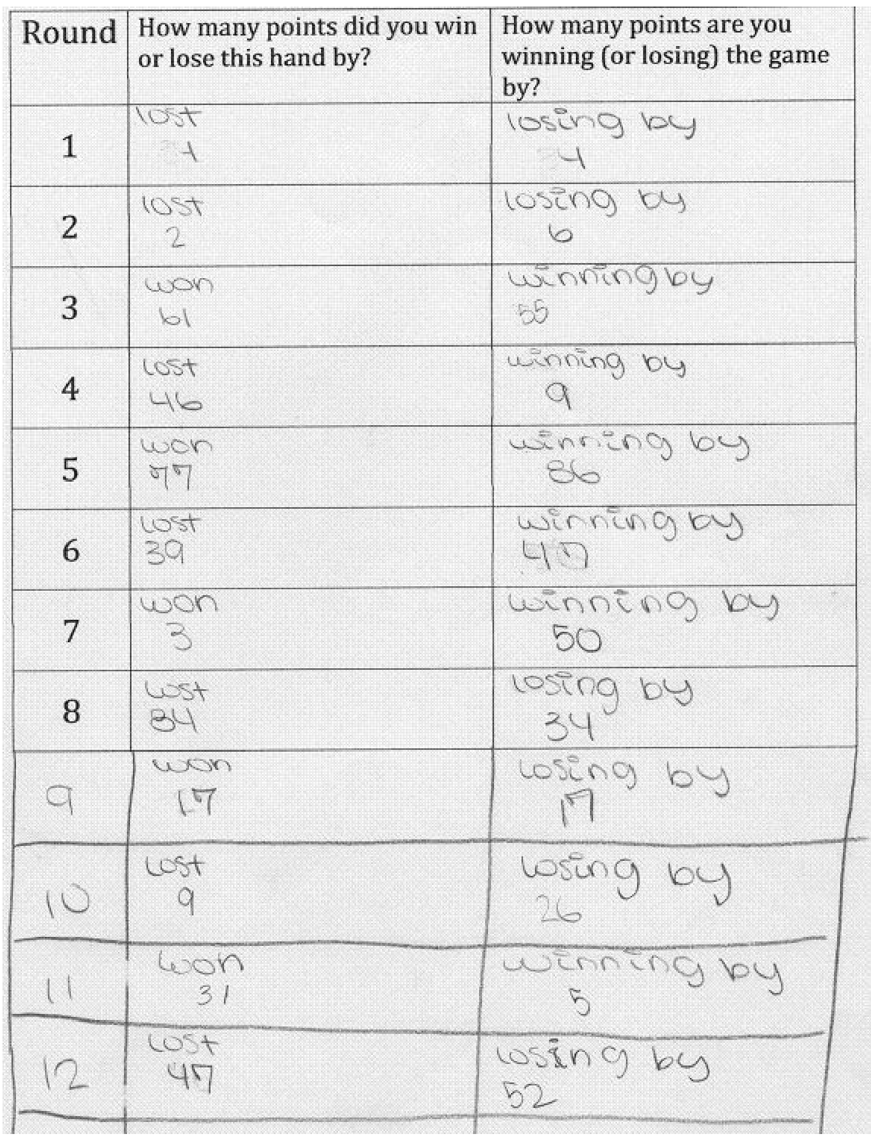
As in Protocol 6.4, Justin seems to conflate C3 and C2 values when he looks to his overall score to tell him if he won or lost the round. He does understand Michelle's logic and uses it to redo his answer. This implies that he does have an understanding of the situation that is intersubjective with Michelle's, but it is a fragile understanding in the sense that he has strong intuitions that run counter to this new understanding, probably based on his prior ways of thinking about the situation. Neither Justin nor Michelle is able to figure out how getting rid of a round would change their overall score. This probably indicates that neither of them are thinking of the C3 quantity as a sum of all the previous C2 quantities. It is interesting, though, that Justin seems aware of the cumulative nature of the C3 values in his strategy in which he uses R6C3 to stand in for all his overall score before the deleted round and R10C3 to stand in for all his overall score after the deleted round. However, he does not realize that the R7 score is one of the contributing values to the R8-10C3 scores.

Although the cumulative, recursive nature of the third column certainly caused confusion this first day, as seen in Protocol 6.5, I do not think that the cumulative, recursive nature of the third column was the fundamental issue for Justin. Instead I think the recursive nature of the quantity brought to a head the issue of working with a signed quantity as opposed to two unsigned quantities. Justin had to interpret and operate on a sum resulting from positive and negative values.

After this teaching episode, I recognized that Justin was thinking about the situation much differently than I was, but I was not sure what he was thinking. I watched the March 9 episode with one of the other researchers later the same day and we recognized that Justin was continually attempting to use the 9 from R1C3 in later calculations. However, at the time we were not able to figure out why. Because I was feeling frustrated as the teacher/researcher, I switched pairs for March 10 to allow a different teacher/researcher a chance to construct a working model of Justin's mathematics in the card game context as well as to allow the chance for alternate interventions to elicit perturbation and reflection in Justin. In retrospect, he was probably getting all the perturbation he needed from his interactions with Michelle and his quest to find an interpretation of the C2 and C3 values that would be internally consistent. Nonetheless, a new teacher/researcher worked with Justin and Michelle on March 10.

Another piece of background information for the March 10 episode is that we had two teaching goals during the card game. The first was to provide the opportunity for students to construct signed quantities, and the second was to see how well students could learn to operate in a complicated additive situation, including quantification of differences. I have not yet written about one of the ways the research team hoped to achieve these goals, which was to engender the mental strategies of counting up, counting down, and adjusting estimates. Our hypothesis was that counting up and counting down to determine differences could help the students experience the directionality inherent in the signed differences we were working with in C2. Adjusting estimates involves operating on differences, so I thought that more explicit attention to how they are using the estimates might provide an opportunity to reflect on how to operate on differences.

The teacher/researcher on March 10 is focusing on getting Justin and Michelle to use these mental strategies in place of the traditional subtraction algorithm. For that reason, many of the questions for the students focused on their procedures as opposed to the quantities involved and we do not gain too much additional information about how the students were quantitatively assimilating the situation. However, Justin clearly learns to operate in a way that is intersubjective with Michelle and the teacher/researcher.



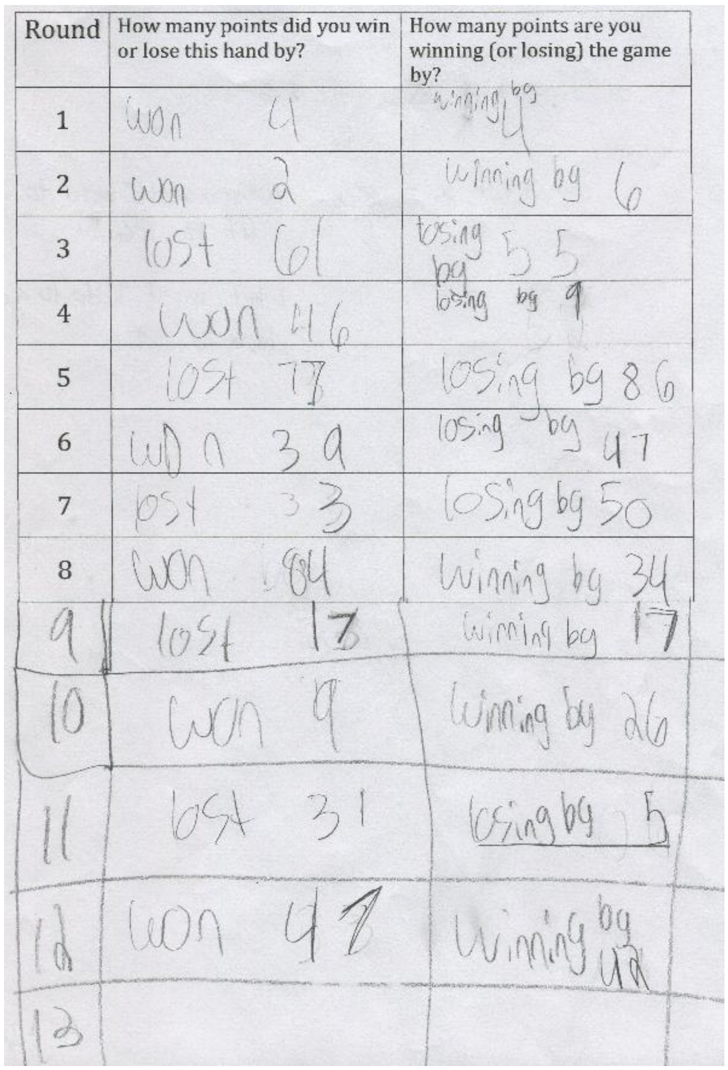
Round	How many points did you win or lose this hand by?	How many points are you winning (or losing) the game by?
1	lost 4	losing by 4
2	lost 2	losing by 6
3	won 61	winning by 55
4	lost 46	winning by 9
5	won 77	winning by 86
6	lost 39	winning by 47
7	won 3	winning by 50
8	lost 84	losing by 34
9	won 17	losing by 17
10	lost 9	losing by 26
11	won 31	winning by 5
12	lost 47	losing by 52

Figure 6.11. Michelle's score sheet on March 10.

The first striking aspect of the students' behavior in the March 10 episode is their consistent use of "won"/"lost" to describe C2 values and "winning by"/"losing by" to describe C3 values. Figures 6.11 and 6.12 show their score sheets. Both students had been using "won"/"lost" for C2 values the day before, but C3 descriptors had been inconsistent and neither had used the preposition *by*, which implies the relative nature of the C3 quantity. Michelle was the one to initiate the use of these descriptors in the first round. Justin used them from the first round on as well, but he does look at Michelle's paper before he writes "by" in R1C3, clearly imitating her. Also, I would like to note that the teacher/researcher asked only, "Who won?" after the Round 1 cards were selected, so this language did not originate from the teacher/researcher. The language is present on the score sheet, but it had been the previous day as well and neither student used it, therefore I doubt that Michelle was imitating the language of the score sheet. Both students consistently used this language in their explanations as well. The only exception is in R3 when Michelle gives her C3 value by saying, "I win by 55."

For Michelle, I hypothesize that this new language use represents a move towards seeing these values as relative scores as opposed to numbers of points. This is supported by her references to her winning and losing margins throughout instead of referring to points or Justin's score. In addition, I think the change in language indicates a more explicit awareness of the two different quantities involved.

For Justin, the meaning of "winning by" and "losing by" is more ambiguous. I say this in part because he had used these phrases to stand for several different quantities the day before. He does this again in Round 4 on March 10, which is reported in Protocol 6.6. However, after Round 4 he does refer to his values as relative values, so this might be building on his construction of a relative signed quantity from the end of the March 9 episode. In Round 6, Justin refers to a losing score for him as Michelle's winning score, allowing him to work with all "positive" or unsigned quantities. I think this is probably indicative of an underlying, continuing discomfort with signed quantities. I will discuss both Round 4 and Round 6 below. Protocol 6.6 gives the discussion for Round 4.



Round	How many points did you win or lose this hand by?	How many points are you winning (or losing) the game by?
1	won 4	winning by 4
2	won 2	winning by 6
3	lost 61	losing by 55
4	won 46	losing by 9
5	lost 77	losing by 86
6	won 39	losing by 47
7	lost 3	losing by 50
8	won 84	winning by 34
9	lost 17	winning by 17
10	won 9	winning by 26
11	lost 31	losing by 5
12	won 47	winning by 42
13		

Figure 6.12. Justin's score sheet on March 10.

Protocol 6.6: Round 4 discussion on March 10.

J: [Turns over 60.]
M: [Turns over 14. Begins doing the traditional subtraction algorithm in the air.]
T: Before you compute, I want Michelle to tell me how she might do it.
M: I'll take 10 from 60, which will give me 50, and I'll add 4. I mean take away 4.
J: Yeah.

T: Oh, that's great. Then what would you have?
M: Fifty-…46.
T: I want to know how you'll combine 46 and 55. […] You can't carry and borrow. […]
W: Would you still be losing the game or would you be winning the game?
J: I think I'd be losing. I think.
T: How're you going to do it?
J: Add 6 to 46 and if that is more than 55, then I'm winning. [Writes "52" in C3 and then adds "losing by." Looks at Michelle's paper. Begins to erase "52."]
T: Think on it awhile. Keep at it. […]
J: [Finishes erasing "52" and writes "3" instead.]
…
T: OK. Michelle, why don't you tell us how you did it?
M: Well, it was 55. I was winning by 55 so then I took the 46 and I added 10 so it would give me 56 and I just took 1 away so it was 55.
J: [Erases "3."]
T: OK. Could you get what she did?
J: I thought it would be losing by 3 because 6 plus 6 would be 12 and you'd bring the 1 up and it'd be 52 and then there'd be 3 from 55, so I'd be losing by 3.
T: OK. You're losing by 55, right?
J: Yeah.
T: And you won 46 points.
J: But wouldn't I add the 6 plus the 46?
W: […] Did you already include the 6 somewhere? Do you have to bring that back in, or no?
T: OK. Here's 55 cents. You're losing by 55 cents and you win 46 cents. How much money do you need to add to this [46 cents] to make this [55 cents]?
J: 9…9.
T: You said, "9"? How'd you figure that out?
J: 'Cause 4 plus 6 is 10. That would be 50, plus 5 is 55.
T: Give me five. You see what we're doing? You've got to figure out what we're doing…

Justin first treats the C3 value as total wins by adding the round score with his last winning C3 value to get 52. He seems to see this as a different quantity than how much he is winning or losing by since he determines that he is winning by comparing the sum (52) to 55, his last losing score. After Michelle's explanation, he seems to realize that he is supposed to put how much he is winning or losing by in C3 and so he erases his answer. However, he then thinks that the answer should be *losing by 3*, which would result from treating past C3 values as total wins or losses, not the amount he is winning or losing by. Therefore, his solution method reveals a continuing conflation of these two possible interpretations for the C3 quantity. When the teacher/researcher says that Justin won 46 points, Justin objects and wants to add in the 6 as well. In fact, he would need to add the 6 to 46 to get his total number of “winning points,” the issue lies with his interpretation of *losing by 55* in R3, which is constituted by both wins and losses. I think that Justin has trouble assimilating a quantity as being constituted by both positive and negative values in part because that involves a recognition that the magnitude of the quantity is made up of sums and differences. Just as Round 8 of March 9, I do not think Justin is assimilating numbers with difference structures inherent in them the way that he can assimilate a number as being the sum of unspecified numbers.

The interactions between the teacher/researcher and Justin in Round 4 are important for interpreting Justin’s behavior in the rest of the teaching episode. The first aspect of the interactions is the teacher/researcher’s use of the comparison of two quantities of money to model the computation of the R4C3 value. This makes clear to Justin that the teacher/researcher only wants him to use two values to do the computation. Furthermore, the teacher/researcher praises Justin enthusiastically when he correctly finds the difference by counting up. I think that Justin was slightly embarrassed that the teacher/researcher thought it was an accomplishment to find the difference of 9 because I think Justin saw that as a very easy calculation. Furthermore, after he gets the answer, the teacher/researcher asks, “Do you see what we’re doing here?” To continue to advocate his alternate method would imply that he did not understand how to compare quantities, which was not the case, so I think that the question suppressed further attempts to use his alternate methods. In answer to the teacher/researcher’s question, Justin definitely sees what the teacher/researcher and Michelle are doing because he is able to mimic it throughout the rest of the episode, but I do not think that he understands why they are using the method yet. Numerically, he can follow the computations, but quantitatively he is probably still uncertain.

In fact, Justin does not only get the same answer as the teacher/researcher for the rest of the session, but he even interprets the calculations involving wins and losses in the same way: He counts up or down to determine the distance between the numbers. This is in contrast to his use of a take-away interpretation of the calculation earlier in the session in which he finds what is left if he subtracts off one of the numbers. I hypothesize that he does not deviate from this general procedure because he does not want to get the wrong answer again. This is in contrast to Michelle, who alternates between a take-away method and a comparison method of subtraction. She uses the take-away method whenever the minuend is small (less than half of the subtrahend) and she uses the comparison method whenever the minuend is large (more than half of the subtrahend). Neither Justin nor Michelle seems to match the method of subtraction or the direction of counting for the comparison method with the quantities involved. In that sense, they are not in a position to abstract out the directionality of their differences from their subtraction method. On the other hand, they are getting experience and demonstrating that they can both conceptualize subtraction as either taking away or comparing.

Coming into Round 6, Justin was losing by 86 points and he won Round 6 by 39 points. When asked to defend his new C3 value (which does not match up with Michelle's answer) of *losing by 47*, he says, "I know she's winning by 86, so I subtracted 39 from 86. I got 47. Because that's how much I won by so you have to subtract that to see how much she would be winning by." This is the only other time Justin talks about why he is adding or subtracting in this teaching session. His explanation indicates that he is able to make sense of Michelle and the teacher/researcher's C3 solution methods: He is appropriately using his language to differentiate between the C2 and C3 quantities and he does not attempt to add in his last winning C3 score, indicating that he does interpret *losing by 86* in R5 to be a record of wins and losses, at some level. However, he also uses translates the "negative" quantity, *losing by 86*, into a positive quantity, namely, the amount Michelle is winning by. He then compares that to how much he won. Hence he still seems to be resisting a signed quantity, such as how much he is winning/losing by in favor of two unsigned quantities, the amount he is winning by versus the amount Michelle is winning by. Regardless, Justin has shown significant progress in this session in that his C3 quantities do appear to represent a cumulative sum of both wins and losses.

Summary of Justin and Michelle's ways of operating in the card game.

Michelle surprised me with her ability to operate fairly viably and fluently in the card game. I would hypothesize that she has constructed both sums and difference as reified mathematical objects. This would allow her to construct the C3 quantity, which is a combination of sums and differences of unsigned quantities. In addition, her ability to solve missing addend problems implies that she has reversible addition and subtraction schemes.

Justin, on the other hand, has not yet reified differences, which is supported by his behavior with Lily in unsigned contexts. He does seem to be forming subset relationships between addends and sum based on his intuitive use of an additive inverse in one of his explanations. However, I hypothesize that these subset relationships decay and are not available to him as assimilatory or anticipatory structures. This would explain his confusion about the nature of the C3 quantity.

Working in a Money Context

During the card game, we noticed that several of the students were relying on the traditional addition and subtraction algorithms. I was worried that this shift to a procedural scheme would inhibit the students' reasoning about the quantities in the situation. As seen in the March 10 teaching episode with Justin and Michelle, the teacher/researchers would sometimes use money to engender student-generated calculation methods. This is in line with previous research (Brenner & Moschkovich, 2002; Carraher, Carraher, & Schliemann, 1985). Therefore, we decided to move to a money context in which the students could work with actual coins in order to increase our chances of eliciting student-generated calculation strategies.

Furthermore, the C3 quantity in the card game had caused trouble for Justin and Adam, and the missing addend problems in this context were extremely difficult for Justin and Michelle. I was not sure if these difficulties had something to do with the cumulative nature of the C3 quantity or whether it represented a difficulty combining signed values. In order to clarify the difficulty, I used tasks whose quantities were less complex than those in the card game context.

I was still interested in the students working with changes or comparisons in quantity, so the general set-up for the tasks involving coins was that there was an indefinite amount of money on a plate (or in a jar) and each student either added or took away money from the plate. Their two actions, taken together, would give an overall change in the plate's value. For example, if Justin took out 35 cents and Michelle put in 23 cents, then the plate's value would have decreased by 12 cents compared to the original value of the coins on the plate. To the observer, the following signed addition expression would represent the quantitative relationships: (-35) + (+23) = (-12). There are two aspects of this set-up that are less complicated than the card game. First, the students are not constructing the signed numbers through subtraction. Second, the sum is not cumulative: We treated every pair of changes as a new situation. As before, I hypothesized that Justin would not have difficulty working in this context. Based on Michelle's ability to operate viably in the card game, I hypothesized that she would not have difficulty in this context either.

Unsigned Addition and Subtraction Results

The first day in the money context, March 15, Justin and Michelle worked with actual coins and we spent some time working on unsigned differences. For example, the teacher/researcher would have them make two piles of coins and determine how much money needed to be added to the pile with less value to make the values of the piles equal. She also asked them to come up with addition and subtraction expressions to describe such tasks, and she later gave them addition equations and asked them to come up with problem situations that would be best described by those equations.

In one of the first tasks we did, the students made a pile worth 35 cents and a pile worth 51 cents. Then both independently determined that they needed to add 16 cents to the 35-cent pile to make the values equal. After explaining their answers, they made a pile worth 16 cents and put it near the 35-cent pile. With all three piles still visible, the teacher/researcher asked them how much would need to be added to a pile of 16 cents to equal the value of a pile worth 51 cents. At first, both students began working out the problem in their heads, but Justin quickly stops and says,

> Oh! You add to 16 [touches 16-cent pile] to get to that [touches 51-cent pile], and then, which is what you start out, and this one is 35, and you have to add 35 to that [pushes the 16 and 35 pile close and then separates them] to equal 51.

During Justin's explanation, Michelle does not indicate that she noticed that the same additive relationship was being used again. The teacher/researcher asked her to explain what Justin had just said, and Michelle ran through the whole calculation again, ending up with a new pile worth 35 cents.

> Well, since it was 16 here [touches the 16-cent pile], I added 10 to it [moves a dime over], which made it 26, then 36 [moves dime], and then 46 [moves dime]. And then I added a 5, which made it 51.

The teacher/researcher then asks Michelle, “And have we seen 35 before?” Michelle looks surprised, smiles, and says, “Yeah! Right there.”

Justin’s ability and Michelle’s delay in seeing that the same underlying relationship existed in both situations is an indication that Justin creates subset relationships when solving a missing addend problem, whereas Michelle may be treating the missing addend as a transformation, not a composite unit that is contained in 51. In other words, Adam seems to have abstracted out a more structural understanding of the situation, while Michelle is focused on her counting actions and does not unitize her solution to the first problem in order to form an additive relationship between the solution and the give values. I give a more in-depth discussion of this differentiation between these two ways of understanding the problem situation in my discussion of Adam and Lily in Chapter 7. The differences in the way Adam and Lily are operating are clearer than then the differences here.

One of our goals in using coins with the students was to encourage strategic reasoning when determining unsigned sums or differences. We were successful in instigating student-generated calculations. Both Justin and Michelle reasoned strategically on March 10, the last day of the card game, but they began to reason strategically more often during this section of the teaching experiment. Even when Justin and Michelle were not strategically reasoning, they were counting up or down their number sequences in sophisticated ways.

On March 10, Justin used strategic reasoning to utilize a different minuend when strategically determining a difference: In order to find 62 – 45, he added 20 to get to 65 and then subtracted three to get to 45, giving an answer of 17. However, Justin had trouble utilizing a different subtrahend when determining difference strategically. The three times during the March 10 teaching episode and two times during this March 16 teaching episode that Justin attempted to strategically reason by adjusting the subtrahend, he was not able to adjust his calculated difference appropriately or explain his adjustments. After repeatedly getting the wrong answer with this method, he returned to counting up or down by composite units the rest of the teaching experiment. Here is the last example of Justin attempting to strategically reason.

Protocol 6.9: Justin and Michelle have trouble adjusting an estimate.

T: You removed 1588 cents, right? And then you went and added 1679 cents.
…
M: I made mine into 590, 90, OK. 600 [puts up a finger], 610, 620, 630 [putting up a finger each time], 640, 650, 660, 670, 680. That's [pointing to written "+1679"] one less than 80, so that would be 9 and then I take away 3 from that? No, I add 3 to that, so it'd be 93?
J: No, because…
M: No.
J: Because you have to take away three if you turn that into an 80 and that into a 90. You have to subtract 1 from that to make it back inside and 2 from that to make it 88.
M: So it would be 89, 87, 86.
J: Yeah.

Protocol 6.9 is interesting because of the imagery that Justin evokes in saying that, "you have to subtract 1 from that to make it back *inside*." This language implies an awareness that he is quantifying a region of the number sequence *inside*, or to the left of and including, 1679. However, he does not use the same imagery when making the bottom adjustment of changing 1590 back to 1588. Instead, he equates subtracting 2 from 1590 with subtracting 2 from the difference. My hypothesis is that either he is not attempting to reason quantitatively to determine the lower adjustment or he is not able to visualize the number sequence relationships at the bottom boundary of the difference. In either case, he is not operating on the difference with what I am calling a *difference structure*. That is, he is not holding a record of the relationship between the compared numbers and the resulting difference when he operates on the difference to adjust his estimate. He is keeping track of his actions and where his actions get him to in the number sequence. However, his is not aware of the actions as a length of the number sequence together with its ending points. He is thinking, "I added 70 to get to 109," instead of, "there are 70 numbers *between* 39 and 109." (The *between* in this context implies inclusion of the 109.) At this point, I am not sure if this is a necessary behavior based on his current ways of operating or whether he might have overcome this difficulty with increased practice.

Before moving on to my analysis of Michelle's strategic reasoning, I want to note that Justin and Michelle are engaged in joint mathematical activity in Protocol 6.9. They were the only pair to do this regularly. In this case, the joint mathematical activity is made possible by the fact that Justin was always very engaged in his partner's mathematics with both Lily and Michelle, so he could comment on Michelle's mathematical activity. However, Michelle was the one who had established the norm of working together by asking Justin questions about how he was solving a problem that she had to solve too.

As I mentioned, Michelle reasoned strategically to determine sums and differences more often than Justin. She also had trouble adjusting the subtrahend, and, as we saw in Protocol 6.9, she was not confident in her answer when she did have the right idea. However, she often adjusted the subtrahend appropriately. For example, consider her explanation of how she calculated 127 – 95: "From the 95, I just went straight to 100, and then … you'd have to add 27 to get 127. And then I just added the other 5 from 95 to 100, which got 32. She could also strategically subtract by decreasing the subtrahend. For example, to find 72 – 53, she did the following: "I went from 72, I just took 72 minus 10 is 62, and then minus another 10 is 52, and then it's 53, so I added, oh, I meant take away 1. And so I got 19." In general, she tended to have the right instinct for which way to adjust a subtrahend, but rarely felt very confident about her solution method. Eventually, she seemed to decide that Justin's method of adding up by 10's and then 1's is easier, so she sticks to that method throughout the last session using money and throughout the tasks in the number line context.

Signed Addition and Subtraction Results

Michelle continued to operate viably in the signed money context. Therefore, I only focus on the development of her explanations. I focus on the nature of Justin's explanations as well, but he also had difficulty in the first session when combining an increase and decrease to get a total change in the plate's value. Although the behavior seems quite different from his conflation with the C3 quantities in the card game, I think there is a similar explanation for his difficulties. Protocol 6.7 gives the first task involving the combination of an increase and decrease.

Protocol 6.7: Justin equivocates about the nature of a signed sum.

T: You add 94 cents to the plate and you're removing 50.
…
M: [Putting coins on the plate] 75…80…ninety-…two…94, right?
T: You add 94 and he removed 50. […] By how much did the plate's value change?
M: You took away 50…So, it was 94 and you took 50. That would be 44.
J: [Nods.]
T: […] How'd you get 44?
M: I went from 50 to 90, which gave me 40. And then I added the 4.
J: Yes, that's what I did.
T: OK. More on the plate or less on the plate?
J: Ah…there's less.
M: More.
T: What do you think, bud?
J: I thought it'd be—
T: Is there more or less on the plate?
J: It'd be less 'cause it started: 50, you took away. You took away 50 from it then you added 94.
W: But she added 94.
J: Oh.
T: What do you think?
J: But then it'd be like she only added forty-…Oh. Yeah, it's more. She added 44 to it.
M: Oh, wait. But he…
T: All together?
J: Yes.

T: She added what?
J: 94.
T: But all together…
J: The total, she would have actually been adding 44 cents.
T: Yeah, *you guys* would have been adding 44 cents with your actions.

Justin originally agrees with Michelle's solution of 44 as the magnitude, but it is not clear what Justin thinks 44 is the magnitude of. At first, the teacher/researcher assumes that he means the total change in the plate's value. However, when he says that he thinks there is less in the plate, the teacher/researcher realizes that he is thinking about the situation differently than she is. Given that he agreed with 44 as an answer, both the teacher/researcher and witness are assuming at this point that he has the types of actions mixed up, but he clearly says that 50 was taken away and 94 was added. He is aware of both quantities. The witness persists in reminding him that the 94 was added. Justin reply of, "Oh," sounds like he had a realization, which is explained when he protests that "it'd be like she only added…44 to it." He has taken the result of combining signed quantities and interpreted it in terms of one of those quantities. He knows that the 44 is tied to the idea of an increase, so he reinterprets it as another value of Michelle's increase. Even as he comes to decide that the 44 does represent "more," he continues to say that it is what Michelle added.

I think this reveals a latent conflation of the two increase quantities, and I think the underlying cause is, like with the card game, Justin's lack of a difference structure to assimilate the signed sum with. He is not aware of the relationships of the two quantities that made up the signed sum when he reflects on the signed sum. Therefore, the sum is reinterpreted in terms of the original quantities in the task, how much was removed and how much was added to the plate.

Protocol 6.8 reports the discussion of the task immediately following Protocol 6.7. Justin uses the same kind of reasoning as in Protocol 6.7, but he is more adamant about the interpretation that the sum is of the same nature as one of the increases or decreases that formed it. Protocol 6.8 also offers confirmation that Justin is using an additive inverse to get to a change of 0 when reasoning about his calculations.

Protocol 6.8: Justin reasons through 0 again while still constructing a signed quantity.

T: Justin, you're going to be adding 48 cents, and you [Michelle] are going to remove 20 cents. […] Is it going to be less or more on the plate, do you think?
J&M: More.
T: More? OK. Why more?
J: Because it's 48, you subtract 20 from that, it'd be 28. It'd be 8 more than you had.
T: 8 more cents on the plate?
J: Yeah.
T: What do you [Michelle] think? You said there's more, how much more on the plate then?
M: There would be 28 more. Wait. No. […] Because you added 28, but I don't get how you want us to think about it. Since he took, or, did you add or take?
J: I…I…
T: He added 48 […] you removed 20 cents. […] How much more's […] on the plate?
M: 28?
T: You're saying 28, it wouldn't—
J: I think it's 8 because she took away 20 and if you're adding 20, it'd be the same amount as you had and the extra 8 would be 8 more cents than you had.
T: How much did he add there then, though? Did he add…
J: 28.
M: He added only—
T: And you [Justin] were going to add how much, though?
J: 48…Since you took away…
…
M: He added 48 and I took away 20. So 48 minus 20 is 28.
J: [Acts out motions to himself as Michelle is talking.]
T: What'd you think about that, buddy? […]
J: I still think it's 8, 'cause she took away the 20 cents and then I added 48 cents [indicating both actions], so it'd be…Yeah, it'd be 28. [Smiles sheepishly.]
…
T: You want to do it with money, just to make sure we're right or you think you're good?
J: Yeah, I think I'm going to be good [reviewing actions in his head].
…
W: Why, Justin, did you think it was 8?
J: Because I forgot there was another 20 to go.

In the task immediately following Protocol 6.8, Justin added 10 cents and Michelle removed 27 cents. Justin uncertainly answered that it there would be 7 cents less on the plate and then corrected himself when the question was restated, saying, "*Twenty*-seven? Oh," as if he misheard the question. However, this was not a case of mishearing because he would have had to hear "seventeen" instead of "twenty-seven" in order to get 7 cents less, which is an unlikely mistake. In this task, he did not seem aware of where the 17 that he was operating with came from. He is surprised when, on hearing the question again, the magnitude of the decrease was 27 and not 17. I think this illustrates how quickly and unconsciously he can replace the total change with one of the constituting changes in his scheme. This is the last time he gave indications of this kind of conflation during the entire teaching experiment, although he did briefly conflate different quantities when working with missing addend situations. I think the reason he stopped conflating in this way is because he reflected on the quantities in the situation in Protocol 6.7. When he said, "I still think it's 8, 'cause she took away the 20 cents and then I added 48 cents, so it'd be…Yeah, it'd be 28," he was running through the motions again with his hands and then stopped using his hand when he paused in his argument and looked up as if he were running through the motions in his head again. Based on the fact that conflation stopped soon after this episode, I hypothesize that he attended to the 28 as resulting from both values, possibly forming a difference structure. That is, he may have still been aware of the 20 as a subset of 48 when he calculated the complementary subset of 28. Once he had a more permanent awareness of the 20 as a subset of the 28, he would not feel the need to use it again (by comparing it with the 28).

Before moving on from Protocol 6.8, note how Justin was thinking about adding 28 cents and removing 20 cents. He said, "If you're adding 20, it'd be the same amount as you had and the extra 8 would be 8 more cents than you had." His reference value, his equivalent to 0, was "the same amount as you had," and he decomposed the 28 into the amount needed to get him back to the reference value (the additive inverse of removing 20 cents) and the remainder of adding 28, which represents the total change. Although this calculation did not help him successfully solve this task, it points to a beginning of attending to directed subset relationships in this new setting.

The next day, March 17, Justin confirmed that he had a viable scheme for adding signed numbers.

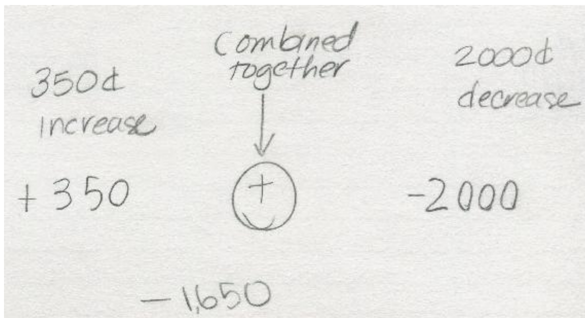


Figure 6.13. Notation used in Protocol 6.9.

Protocol 6.9: Justin finds the sign of a sum.

T: You guys are going to pick again. Keep it under 2000. [Michelle], you're going to have an increase. We'll have a decrease for you [Justin].
M: An increase of 350.
T: […] How do you think we should write that […]?
J: 350 plus my number.
T: We'll just leave it. [Michelle,] what were you going to say?
M: Plus 350 minus…
T: Yeah, just "plus 350". We're just talking about the first step. […] OK, then we're going to use this symbol [circled addition sign, see Figure 6.13] and this means "combine together." OK, Justin, what's your decrease?
J: 2000.
T: 2000 cent decrease. How would we write that number?
J: Minus 2000.
T: A negative 2000. First off, is the plate's value going to increase or decrease?
J: Decrease?
T: Why, Justin?
J: Because mine, it has more than hers. Because my number's bigger, and it's a decrease.
T: By how much is the plate's value going to decrease?
J: [Keeps track of something on his fingers.]
T: How about you explain what you did?
J: I just sub--, I pretended like that [-2000] was a positive number, so it'd be easier. So I kept adding a hundred to it, and I got 1650.
T: What do you mean you kept adding 100 to it?

J: It was…it would be, I'd add 100 to it: 450, 550, 650, then I got up to 1950, so it'd be a 50, so I got 1650. It would be a negative 1650.
T: Do you agree with his answer?
M: Yeah, I do. I just did 2000, and then I went down [by 350]. So I went to 1900, 1800, 1700, and then I went down to 1600, and then I took away the 50 [because she subtracted by 400 instead of 350] and I got 1650.

In addition to confirming a viable signed addition scheme, Justin gave an example of what I call a *holistic comparison* in the money context in order to determine the sign of the sum. A holistic comparison involves comparing sizes of a positive and negative value. In contrast, what I call a *quantitative comparison* involves forming explicit subset relationship, as Justin does when he attends to what would get him back to the 0 value. See Figure 7.5 and the surrounding discussion for further clarification between these two concepts.

Justin's holistic comparison, "Because my number's bigger, and it's a decrease," implies the equivalent of "take the sign of the larger" in the tradition rule for mixed-sign addition, "Subtract and take the sign of the larger." In the next task, Michelle gives another good example of a holistic comparison to determine sign: "Because his number's bigger than mine, and so that means that there'd be more ones being added than being taken away."

Later on March 17, we started doing situations of signed subtraction in the context of the adding and taking away money. The set-up was that both Justin and Michelle wrote down an action they would do to the plate, but hid their action from one another. Then the teacher/researcher told them what the total change to the plate's value was. Protocol 6.10 relates what happened the first time we did this with Justin and Michelle.

Protocol 6.10: First missing addend situation with Justin and Michelle.

T: You're both going to individually pick what you're going to do with the plate's value. How much of an increase or decrease, all right? But it's a secret. You're only going to show me and then what I'm going to do is tell you what the total change when we combine your actions together, the total change in the plate's value. And then you're going to try and figure out, what'd your partner do?
…

J: [Writes “increase +23”.]
M: [Writes “increase + 86”.]
T: […] We have a total change: 109 cent increase. [Writes it down.] Think about what your increase is. If you don’t remember, here it is [handing back cards]. And then think about what your partner’s action was
M: [Writes “increase +23”.]
J: [Writes “increase +86”.]
…
T: [To Michelle] How did you get your answer? […]
M: I went from 86, I added 10 ‘til I went 96, then 106, and then since it was 109, I added 3 here. And that gave me 23.
T: How did you know Justin increased his value?
M: Because I had 86 and then it went up.
T: All right. Justin?
J: I got 86 because I kept adding 10 to my 23, and then when I got to 103, that was 80, and I added 6 to that and got 86.
T: Did she have an increase in value or a decrease?
J: An increase.
T: And again, how’d you know?
J: Because if it was a decrease, it would have been a negative number.
T: Would it necessarily be a negative number?
M: [Shakes her head.]
J: No, it’d be less than the number I chose.

In Protocol 6.10, the students were dealing with a familiar kind of missing addend situation. They started off with a positive quantity, ended up with a larger positive quantity, and had to figure out how much was added (or subtracted) in between. My guess is that both assimilated this as an addition situation using their natural number addition scheme. Michelle indicates this when she described the situation: “Because I had 86 and then it went up.” Note that the 86 is not a signed number in her explanation and that the salient fact for her is that the quantity in questions “went up.” Justin, on the other hand, gave an explanation that relies on a *process of elimination*. Instead of explaining directly why Michelle had done an increase, as Michelle did with respect to Justin’s action, he explains why she could not have done a decrease: “Because if it was a decrease, it would have been a negative number.” Indirect explanations that rely on a process of elimination were common in MA tasks. It is possible that the fact that it was an increase in this case was so obvious to them that it was hard to explain. Furthermore, Justin may have been looking for a way to explain how he got his answer that was different than Michelle's explanation. Nonetheless, the prevalence of this kind of explanation was striking with many of the participants.

Justin's explanation also indicates a possible conflation, at least in language, of the second addend and the resulting change. Although it is unclear what *it* refers to in his explanation, the way that he interprets the teacher/researcher's question, "Would it necessarily be a negative number?," implies that *it* was referring to the total change: Justin answers, "No, it'd be less than the number I chose." If *it* was referring to a Michelle's action and we are assuming that Michelle's action is a decrease, as in this explanation, then it would necessarily be negative. I think it is more likely that Justin would make the conflation of thinking that the resulting change is necessarily a negative number rather than thinking that a decrease would not have to be negative. The reason I claim that he is conflating is because I think this idea that a decrease necessarily leads to a negative answer, which all secondary mathematics teachers have probably seen in some form, represents a conflation between the nature of the second addend and the result of adding, i.e., there is a conflation between the nature of an action and the result of that action. In this case, Justin is mentally going down a number sequence when he imagines a decrease and he associates this negative quality with the result of the decrease. His ability to immediately answer my question correctly implies that this conflation was not necessary, but I do think that it is indicative of an incomplete construction of an assimilating structure for these missing addend problems.

For me, the two types of situations—signed addition and signed subtract/missing addend—have the same basic assimilating structure. In other words, I have a sort of ___ + ___ =

____ sense in my head for both, which I refer to as a *generalized signed sum assimilating structure*. This is a sort of generalization of sum and difference structures. When I say that Justin did not yet have a complete assimilating structure for these problems, I mean that he was able to reflect on his actions during signed addition, but he still did not seem to assimilate the situations with this kind of additive structure as a given. Note that this is consistent with the hypothesis that he was not assimilating with a difference structure in the card game and in the signed addition situations because his behavior could imply that he is still not assimilating with a difference structure or it could simply imply that he has not generalized his sum and difference structures into a signed sum assimilating structure. The fact that he does not have an assimilating signed sum structure makes it is understandable that he would not be able to reason reversibly with his signed addition scheme to figure out the missing addend. This would explain his indirect explanations for the sign of the answer because he cannot reason from the resulting change to get the second addend, so he has to start with the first addend and imagine both increases and decreases to figure out which would make more sense for the second addend. Justin may be assimilating his own action, i.e., he may not have to mentally carry out his action to make sense of it, but he is mentally carrying out the second action when thinking through the problem. His need to carry out the actions mentally would also help explain the momentary conflation, since he would need to be actually mentally carrying out the decrease in order to make that conflation.

In the last task of the day on March 17, reported in Protocol 6.11, the teacher/researcher explicitly tells the students which kind of action to choose so that we could see them do a missing addend problem where the addends have different signs. Despite knowing that Michelle did a decrease, Justin struggles to find the right answer, but gives a weak indication of quantifying a distance between signed numbers.

Protocol 6.11: Justin and Michelle find a missing addend of a different sign with money.

T: I'm going to give a little something away because we don't want increase both times. I want you [Michelle] to choose a decrease and you [Justin] to choose an increase.
J: [Writes "+12".]
M: [Writes "-34".]
T: [Writes "22¢ decrease".]
M: [After doing a combination of what seems like a traditional addition or subtraction algorithm in the air and counting using her fingers,] 30 [putting down her the third (middle) finger on her right hand], 31, 32, 33, 34 [putting down the last two fingers on her right hand and two fingers on her left hand]. [Writes down +12.]
J: [Writes "-30", scratches it out, writes "-40", scratches it out, writes "-34". The whole time he is counting on his fingers.]
T: This is a difficult one. Can you guys tell me…
J: I had 12. I added these two together, it'd be like 2 plus 2 would be 4, I subtracted it and I got the thirty-…four. I subtracted 10 from it, 'til I got to 24, then it was 3, 4.
T: OK. OK.
M: I just counted up from 22 to 34 to see that [circles her answer].

The bell rang in between when the task was posed and when the students wrote down their initial answers, so the explanations were rushed and the teacher/researcher was not able to follow-up on her questions. However, Michelle's solution seems fairly straight forward. She said that she counted up from 22 to 34, and, indeed, she seemed to have added 8 to get to 30, so she put down her finger representing 8, the middle finger on her right hand, and then finished counting up by ones, leaving a finger pattern for 2 more than 10, or 12. I do not yet know why she knew to subtract the 34 and 22 instead of adding them. I follow up on how she decides whether to add or subtract in the next teaching session.

When looking at Justin's actions and explanations together, there is a weak indication that he was aware of quantifying a *distance between* two numbers with different signs. First, there is no indication that Justin attempted to mentally carry out a traditional algorithm for addition or subtraction. He seemed to be keeping track of quantities on his fingers the entire time. Second, this is the first time he had referred to a signed subtraction situation involving a subtrahend and minuend with different signs as a subtraction situation. Up until now, all the students would talk about this situation only in terms of addition because you would add the magnitudes of the subtrahend and minuend to get the answer. Note that he starts off with +12 and -22, yet he says he "subtracted it" to get the 34.

In particular, given his discarded answers, I hypothesize that he knew that he wanted to find the distance between +12 and -22. If these numbers had had the same sign, then a quick glance at their ones' digit would tell Justin that the difference, or distance between them, would be a multiple of 10. Given his proclivity for adding by tens and ones, I think it was probably his intention to see how many tens were between the two numbers. My guess is that the -30 came from ignoring the identical ones' digit (as you could do for unsigned differences) and finding the difference between +10 and -20. This would be in line with the first sentence of his explanation in which he deals with ones' place first and then the tens' place. However, we know he checks his answers because he tells us how he checked his final answer in the second sentence of the explanation. Therefore, I think he must have realized that (+12) + (-30) would not get him to a big enough decrease. He could have just tried the next highest multiple of 10, or he could have tried counting by tens across 0 and gotten +2, -2, -12, -22, overgeneralizing the pattern that adding by 10 always keeps the ones' digit the same. In either case, he must have also realized that -40 did not work by checking it. In fact, he probably realized that his assumption that it is a multiple of ten was not viable. In any case, according to his explanation, he ends up adding the ones' digit, but he still describes his work with the multiples of ten as subtraction. He certainly could have been grasping at straws when he added the 2's together. However, given that he seems aware that he is finding a distance between numbers, I think it is possible that, in the end, he had some visualization of the situation that allowed him to see that the ones' digits were increasing the distance on either end of the number sequence between +12 and -22. I present confirmation of this type of visualization in future tasks.

In my own visualization of Justin's task, I would see the need to add 12 and 22 because 0 partitions the difference into these two quantities. This is what I would refer to as *attending to 0 and using additive inverses* in a mixed-sign subtraction situation. Given Justin's spontaneous reasoning through 0 in signed addition situations, I was hopeful at this point in the teaching experiment that he would soon start reasoning through 0 in these new mixed-sign subtraction/missing addend situations in order to see the necessity of adding magnitudes to get the magnitude of the result.

In the first task of the next teaching episode with Justin and Michelle, on March 22, I saw a continuing evolution of their explanations with regards to signed MA situations. Protocol 6.12 describes a task where both students gave interesting explanations. Note that the students had to figure out whether the other person's action was an increase or decrease, unlike in Protocol 6.11. Like in Protocol 6.11, Justin solved a mixed-sign task, while Michelle solved a same-sign task.

Protocol 6.12: Justin and Michelle describe their signed MA solution methods.

T: You're going to choose whatever you want to do with the plate's value and then I'll tell you what the change is.
M: Do we pick…
T: Whatever you would like…
M: [Writes "+58".]
J: [Writes "-26".]
T: The plate's value increased. 32 cent increase. [Writes it.] Right now just try to figure out what your partner did. [After about 30s,] Show each other.
M: His was -26.
T: We're saying the plate's value decreases 26 cents, right? And you're saying she what?
J: She got 58, so it's +58.
T: She increased the plate's value 58 cents. How'd you know that Justin decreased the plate's value?
M: Well, since I had 58 and the increase went down, it didn't go up, so that means he had a decrease, not increase.
T: How did you know she increased the plate's value?

J: Because I know if it was less than 26, it would have been smaller than 26. Maybe even a negative number.
…
T: Why did you add the 26 and 32?
J: Because if you subtract a 26 from a number, you would get something like…you subtract the 58 from 26 and you get 32, so you have to add both of them together so you can get 58.
T: How'd you know that he decreased by 26 cents?
M: I went from 32 and then I added a 10 to it, which made it 42 and another time, which made it 52 and I just added the 8, which gave me…I mean, I added [up] to the 58 and it gave me a 6, so that'd be 26. And then to make sure I was correct, I added these two numbers and I got 58.

Michelle used an indirect explanation. Before this, she has only described how she would get the sign of the second addend in two missing addend problems, and in both of these problems both of the actions were increases. Protocol 6.10 was one of the two problems, and we saw how in that situation she directly explained why she knew Justin's action was an increase, while Justin reasoned by elimination. In Protocol 6.12, she uses that same situation of combining two increases as her point of comparison and notes that this is not a situation of combining two increases, because the starting increase decreases instead of increasing. Once she has eliminated that possibility, she knows Justin's action was a decrease. Therefore, so far we only have evidence of a direct explanation from Michelle when the situation could be assimilated with an unsigned additive structure.

Justin seemed to interpret the teacher's question, "How did you know she increased the plate's value?" as asking, "How did you know *the magnitude of her* increase?" I say this not only because that is a question he partially answered, but also because he did the same thing in the next task, reported in Protocol 6.13. Given Justin's interpretation of the question, he was correct that if the magnitude of her number is less than the magnitude of his, then the magnitude of the result would be less than his. Based on what follows in this protocol, I think his statement was drawing on his reasoning from signed addition situations with addends of different signs. In these situations, he has probably abstracted out a procedural scheme in which you subtract the smaller number from the larger number to get the magnitude of the answer. Therefore, he knows that if her increase was smaller than his decrease, then the answer would have a smaller magnitude than his action. Because the result has a larger magnitude than his action, he knows that the magnitude of her action must be larger than his.

I find it interesting that he picked up on these relationships between the magnitudes. I would have expected him to answer the question, as he interpreted it, by saying that if the result is positive, then her number must be larger, as a contrapositive to his statement in Protocol 6.9 in which he said that if his decrease is bigger than her increase, then the result will be a decrease. He also says that the resulting change would be “Maybe even negative” if her increase were smaller than his decrease. In fact, if the magnitude of her increase is less than the magnitude of his decrease, then the result would *necessarily* be negative based on the same logic he used in Protocol 6.9. He speaks slowly, as if he is thinking while he makes his statement, and looks off into space afterwards as if he is thinking. The fact that he is so tentative about the result being negative implies to me that his procedural scheme for the signed addition situations are not operating on signed quantities, but on unsigned numbers, because he had to think about whether the result of his procedure in his hypothetical situation would be positive or negative. The sign was not inherently part of the initial reasoning process.

Also note that he is not explicitly referring to the reference value of no change in the plate’s value or to the additive inverse of his action. If he were thinking of the quantities in reference to 0, then he would be more likely to use an additive inverse to get to the 0 value. This would have resulted in a determination that her magnitude would have to be 26 just to get back to no change in the plate’s value, so she needs to have a magnitude bigger than 26 to get to an increase in the plate's value. That would lead to a direct explanation that does not rely on the process of elimination. Both his and Michelle's indirect explanations imply that they were not setting up quantitative relationships that would help them determine the necessary action to get the missing addend. Instead, they seemed to try possibilities out and then check if their answer made sense by applying their signed addition scheme.

Later in Protocol 6.12, Justin explains why he added 26 to 32 to get the magnitude of Michelle's action: "Because if you subtract a 26 from a number, you would get something like…you subtract the 58 from 26 and you get 32, so you have to add both of them together so you can get 58." At the time, I thought that he was defending his answer without explaining how he got it. However, I now think that he was attempting to explain a general situation, "If you subtract 26 from *a number*, you would get something like…," but lacked the language and notation to express the generalized difference that would complete this thought. This implies that he is assimilating this situation as a difference, which is the first strong indication I have that he could assimilate with a (natural number) difference structure.

Realizing that his reasoning would be hard to explain using an unknown quantity, he illustrated his reasoning using the solution. Looking at his explanation from this perspective, he seems to be saying that he was starting with the knowledge that he would have to subtract the magnitudes of his action and Michelle's action to get the result, so he had a missing subtrahend. Based on his knowledge of unsigned addition and subtraction, he knows he can find the subtrahend by adding the minuend and difference. The idea that he knew he would have to subtract the two magnitudes of the actions to get the resulting change fits with his earlier statements. The fact that he seems to be assimilating this situation using schemes from a mixed-sign addition situation implies that he knew Michelle's action would have a different sign than his action almost immediately. The question I have is how he knew that. I presume he knew she increased the plate's value because the overall change was an increase, but his round-about explanations make me loathe to conclude that at this point.

The next task, recorded in Protocol 6.13, does not give us much more insight into how Justin is figuring out the sign, but it does give us information about how Michelle is thinking about missing addend problem.

> Protocol 6.13: Michelle starts to reason reversibly in a missing addend situation.
>
> M: [Writes "-52".]
> J: [Writes "+46".]
> T: Plate's value decreased by 6 cents. [Writes "6¢ decrease".]
> M: [After 14 seconds, writes "+58", after another 26 seconds, erases, and after another 33 seconds writes "+46" and flips her card over.]
> J: [After 14 seconds, writes "-40", after another 38 seconds, erases, and after another 21 seconds, writes "-52" and looks up as if done.]
> T: How'd you start out? How'd you know she decreased?
> J: Because I knew when she decreased, it was less than 40, so I add the 6 plus the 46 and got 52.
> T: What do you mean it was less than 40?
> J: 'Cause it de--…Wait.
> T: 'Cause you first said -40 down, right?
> J: Yeah, 'cause I thought you meant she had 40, it was 40, subtracted 40 from that, so I had 40 down. Then I didn't think…So then I added the 6 plus the other 6 and I got 52.
> …
> M: I knew his was a positive because since I decreased by 52 and then the [absolute value of the] number decreased, I thought that since my number was bigger and a decrease, I thought the answer... […] His would be the positive number. So then I did 46, and then I went up until I got 52 and it gave me 6, and I thought the answer was +46.

Both students seemed to know the sign of the other person's action and the difference between the magnitudes of their actions relatively quickly. However, I do not think either has a fully reversible mixed-sign addition scheme.

Michelle's explanation for the sign of Justin's answer does not rely on a process of elimination. She directly explains that because the size of her decrease decreased, she knew he had an increase. Because she originally did not know whose action would be bigger, indicated by her initial answer of “+58” for Justin, I do not think she used her mixed-sign addition scheme reversibly at first. However, she goes on to use her scheme reversibly to explain the sign and relative size of her answer to Justin’s. Her utterance, “His would be *the* positive number,” implies that she identifies this as a set type of problem in which there is one positive number and one negative number being combined, i.e., a situation of her mixed-sign addition scheme. Furthermore, she reasons reversibly from the fact that the resulting change is negative to figure out that the negative number in the situation, her number, must be the bigger number. And, in fact, she seems aware that her number is bigger by 6. Interestingly, she seems to interpret this as a need to find a number that, when you count up 6, would give you 52, but does not reason reversibly from the 52 to get 46. Instead she describes trying 46 and checking if it works. Given the amount of time that passed between her erasing her initial answer and writing her final answer, it is reasonable to suppose that she could have tried other magnitudes before settling on 46. Therefore, she may not be able to totally reverse her mixed-sign addition scheme, but she is able to do so at least partially.

Justin, as in Protocol 6.12, does not explain how he knows that Michelle decreased the plate’s value, but instead discusses the magnitude of her decrease. When he does so, he uses his original answer’s magnitude, 40, as the reference point. He seems to be describing a process of elimination for figuring out the magnitude in that he tried a decrease of 40 and realized that “it was less than 40,” which I interpret to mean “more than 40 less,” and so tried adding the 6 to the 46 instead of subtracting it. This is corroborated by his later explanation, “I had 40 down. Then I didn’t think…so then I added the 6 plus the other 6.” My guess is that his complete thought in the second sentence was something like, “I didn’t think that would give me the right answer.” Note that the way Justin is using “less” in his explanation is ambiguous. If he means “to the left of" or "below" in a number sequence, then he would need to say, "less than *negative* 40" to be consistent. My sense is that Justin was aware that the decrease (of 40) is making *something* less in the sense of more negative/to the left of/below and so he described doing a bigger increase as *less than* because it is making *it* more *less*. The way he expresses this fails to differentiate between the sign and size of the action. I think this is in part because he is not being explicit about what *it* is that is decreasing, i.e., the plate's value.

I had felt that Michelle had had easier missing addend problems than Justin so far because she always seemed to have same-sign missing addend problems. That is, her action and the resulting action were of the same type. I felt that these situations might be assimilated with unsigned addition and subtraction schemes. For example, if she starts with + 36 and ends with +21, then that could be assimilated as starting with 36 units and ending up with 21 units. Figuring out whether you have to add (combining with the same kinds of units) or subtract (combining with opposite kinds of units) could feel like a familiar problem. Justin, on the other hand, had had mixed-sign missing addend problems in which his action and the resulting change had different signs. I hypothesized that these kinds of missing addend problems would be more difficult for the students. In order to have more control over the types of missing addend problems each student worked on, we changed to a situation in which the two students together chose the first action and the teacher/researcher is supposed to pick a secret second action. Then the teacher/researcher tells the students what the overall change is. Therefore, the situation is the same as in the last tasks, but the students are now working on the same problem. The first missing addend situation Michelle solves where the given numbers are both negative was later in the March 22 teaching episode and is represented in Protocol 6.14.

Protocol 6.14: Michelle's first mixed-sign missing addend problem.

T: You have a 74 cent increase […] Followed by something to get us a total change in the plate's value…it decreased by 15 cents. [Writes as in Figure 6.14.]
J: Decreased by 15.
M: That means her number had to be higher than 72 and it had to be a decrease.
J: Yeah.
T: You don't mean 72, you mean what? Do you mean 72?
J: 74.
M: No, I said the number has to be…
J: Bigger than 74.
T: OK. Take your time with this one.
M: How would you work it out?
J: You could add 50 to it [the 15], it'd be 65. And then we would just add a 9 to it, so it'd be…sixty-…69.
M: 65…Then [adding] the [5 to finish out] 10 would be 70 and then the 4 would be…74. Well then, so it would be 69.
J: Yeah.
M: No, 'cause then—
J: It has to be bigger.
M: It has to be bigger than 72.
J: Seventy-…
M: Four. Wait. Then wouldn't we just add 15 to 74?
J: Yeah.
J&M: 89.
T: 89 cent…
J&M: Decrease.
T: Why would we add 74 to 15?
M: Since we decreased by 15, that meant your number had to be bigger than 74, so you just add the 15 to get your number.
T: How about you?
J: I knew it [74] was bigger that it [15], so I just started adding 10 to it, but then I knew that that was the wrong strategy, so she said that you have to add the two numbers together, so I added them together and I got 89.

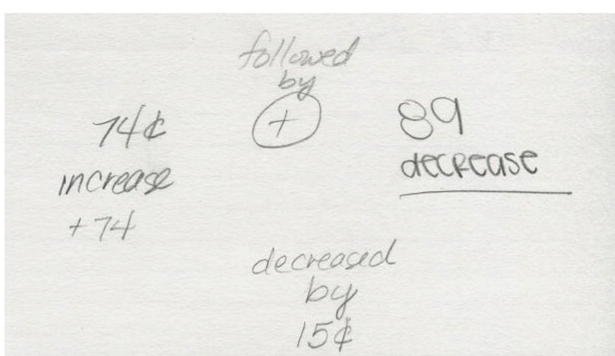


Figure 6.14. The written representation of the Protocol 6.14 problem situation.

Justin and Michelle's joint effort is a little confusing to follow, but they are initially attempting to count up from 15 to 74 and keep track of how much they are counting up by. They both end up incorrectly getting 69: Justin by adding 50, then 9, and Michelle by adding 50, then 5, then 4. Basically, they are subtracting the magnitudes when they should be adding, although they realize that the answer is wrong because of Michelle's initial estimate that the magnitude would be greater than 74. Similarly to same-sign missing addend problems where the missing addend has a different sign than the given changes, Michelle can reason about the sign and relative magnitude of the missing action, but reasons through a process of elimination to figure out how to get the actual magnitude. In her final explanation, she is still not seeing the necessity that the answer be *15* bigger than 74, just that it must be bigger than 74. There is an implied process of elimination procedure in that she seems to see adding and subtracting the magnitudes as the only two options and we know she tried to subtract them first, so is left with adding them as her only other option. Justin is more explicit about the trial-and-error nature of his solution method. He says that he added because Michelle told him to after he saw that subtracting did not give the correct answer. Notice that neither Justin nor Michelle's explanations have yet utilized the reference value of *no change* or used additive inverses in their explanations, and *all* of Justin's explanations so far have been indirect, that is, relying on a process of elimination.

At this point the teacher/researcher attempts to induce attention to the reference value and additive inverses. The resulting dialogue is shown in Protocol 6.15.

Protocol 6.15: Justin struggles with additive inverses in a missing addend situation.

T: What if this [indicating resulting change in Figure 6.14] was no change in the plate's value and we have the 74 cent increase [indicating 74 cent increase in Figure 6.14]? [Covers the resulting change in Figure 6.14.] […] I'm sorry. I'll write this. [Takes blank card and writes the problem with the usual notation.]
J: […] It'd be a decrease of 148. 148 minus 74 could be 0, like back to 74.
T: Say that again.

J: It could be 148 ‘cause 74 times 2 is 148, if that was a decrease it’d be 74 minus 148 and it’d equal 74. Yeah. So it’d be like no change, it’d be 0.
M: Wouldn’t it be a 74 decrease [leans to the left]? Since you’re going in the [leans to the right]…since…
…
T: We have a 74 cent increase [indicating it on the card], followed by *something* [indicating the unknown quantity on the card, gives you no change [indicating the resulting change of 0 on the card]--
J: Oh!
T: in the plate’s value.
J: It’d be followed by 0.
T: If you did a 74 cent increase [indicating on the card] followed by a 0 cent increase [indicating on the card], then how much would the plate’s value change?
J: It wouldn’t change by anything.
M: Wouldn’t you get 74 cents back?
T: What do you think?
J: Yeah, it’d be 74 again ‘cause you wouldn’t be adding or taking away anything.
T: Your first step is a 74 increase [indicating], so what do you need to follow that by [indicating second addend position on card] to have no change in the plate’s value? If you follow it with no change, would you have no change in the plate’s value [indicating resulting change on card]?
J: I think it wouldn’t have any change because if you decrease by 0, it’d still be 74. And if you increase by 0, it’d still be 74. So there’s no change.
M: But […] if you don’t increase or decrease by anything, then wouldn’t you get that number back? For change [indicating resulting change on card]?
J: Oh yeah.
T: Keep going with what you were thinking.
M: I didn’t really know that was correct. I just thought since a decrease of 74, since it’s an increase of 74, and you just decrease it, you won’t get anything back.
T: What do you think?
J: Yeah.
T: Explain to me how it makes sense to you now.
J: ‘Cause if you have +74 and you take out by 74, it would equal 0.

Michelle appears to understand the idea of additive inverse throughout. However, she manages to do so while only speaking about a reference quantity once: when she says, “you won’t get anything back.” She uses that same kind of language to refer to the resulting change earlier in her exchange with Justin as well. This may indicate that she is reifying the changes to think of them as signed objects that she is comparing as opposed to thinking about a change or action as the solves the problem. However, she seems to be able to flexibly move between the idea of the signed numbers as objects or changes/actions because she refers to the number you get back as “change” as well, in her question: “Wouldn’t you get that number back? For change?”

Justin, on the other hand, seems to be confused about what is meant by “no change in the plate’s value.” He is trying to use 74 as the reference value initially, and it does appear to be an unsigned 74, because he does not see the change of *74 cent increase* to *74 cent decrease* as a change in value. He then uses +74 as the reference quantity and gets 0 as the missing addend. Keep in mind that there is a card showing 0 as the resulting change this entire time, and that we have been using the phrase, “change in the plate’s value,” almost exclusively to refer to the resulting change throughout the teaching episodes involving a money context. In addition, both the teacher/researcher and Michelle have a tone as if they disagree with his answer. His tenacity in sticking to 74, or +74, as his reference value in the face of these contraindications, implies to me that he is not imagining an unknown starting value for the plate before he increases the value by 74. The only member of the written equation representing a change for him, at the moment, is the second addend. The overall change in the plate’s value is not even written as far as he is concerned because he is finding it by comparing the first addend and the sum. At the end of the protocol, he does seem to understand that the 0 is supposed to be in the position of the resulting change, but I am wondering what he means by “0” in his last statement and whether he will have issues in the future confusing reference points. In particular, in my pilot study, Brad had trouble with number line representations because the second addend could be labeled with the directd difference of the ending point with reference to 0 (giving the coordinate of the ending point) or with reference to the starting point (giving the directed length of the trip). At this point, I am curious about whether Justin will have trouble with number line representations or not. In the next teaching episode we introduce a context with just such a natural representation.

The last task from the money context with Justin and Michelle that I will share shows that even after two days of working on these missing addend situations, the solution strategies have a trial-and-error feel to them. Protocol 6.16 gives the dialogue from the last missing addend task in this context.

Protocol 6.16: Justin and Michelle describe trial-and-error solution strategies.

T: A 64 cent increase followed by something gave the plate a 24 cent decrease.
M: Would it be 88?
J: Yeah, 88.
T: How'd you guys get that so fast?
J: If you added the 24 and the 64 together it would equal 88 'cause you're decreasing it by 24, so a number minus 24 would equal 64? Or does it mean take 24 away from 64?
T: Were you asking me a question?
J: Are we supposed to subtract the 24 *from* the 64?
T: Well, n--, that's the plate's value after both actions.
J: Oh, OK, so then you'd have to add it together.
…
T: I'm still confused on why we're adding 64 and 24. And how do you know it's a decrease?
J: 'Cause if you had 88 and you subtract 24, you could get the 64 to be a de--. 'Cause it'd still be a decrease, or if you had 64 and you subtracted 24, it'd be 40.
T: How do you know it's a decrease?
M: Because our number was 64 *increase* and then we got the 24 *decrease*. That means that your number had to be a decrease because you can't just increase and increase and get a decrease.
T: Did it decrease more than 64 cents?
M: Yes.
T: How'd you know it decreased more than 64 cents?
M: […] I always add these two numbers up [first addend and sum] and then to check myself I'll do…Say when we got 88, I just did the 88 minus the 64 and got 24 decrease. But you guys get the 24 because it's a bigger number?

When Justin says, "A number minus 24 would equal 64? Or does it mean take 24 away from 64?" he seems to be certain that the -24 implies that he needs to subtract 24 from something, but he is not sure what he needs to subtract from. He is still not clear on the quantitative relationships in mixed-sign missing addend problems. He does decide on the correct interpretation, but I think this is based on teacher cues. When the teacher/researcher says, "Well, n--, that's the plate's value after both actions," I think Justin takes this as disagreement with his suggestion that 24 should be subtracted from 64. Hence, by process of elimination, he decides that the problem situation tells him that "a number minus 24 would equal 64," his only other suggested interpretation. When the teacher/researcher presses him on his interpretation, he points out that an 88 decrease works in the situation, whereas the idea that he needs to subtract 24 from 64, gives him 40, which I am assuming he is implying is the wrong answer because it is not 88, which is the correct answer. In other words, he is now using trial-and-error to back up his process of elimination by trying one of the methods and seeing if it works.

Michelle is also using process of elimination in her explanation of why the 88 is a decrease. Her reasoning is that it is a decrease because it is not an increase. The teacher/researcher attempts to induce the use of the additive inverse of a 64 increase in an explanation of why Michelle knew the missing addend would be bigger than 64 both when she asks, "Did it decrease more than 64 cents?," and when she follows up with the question, "How'd you know it decreased more than 64 cents?" However, Michelle does not seem to be able to articulate how she judges the relative size of the missing addend, although she has consistently been able to do judge the relative size in missing addend situations. This implies that she is making the estimate based on an intuitive understanding of the context, but she has not yet reflected on these types of missing addend situations enough to be aware of why she can make these judgments. As far as the way that she actually figures out the answer, she very explicitly tells us that she first tries to add up the two given magnitudes and then checks to see if the sum will work as the magnitude of the missing addend. I am guessing this is a procedural scheme that she uses when the given quantities have different signs, since she does not appear to do that in same-sign missing addend situations. At the very end, I think she is attempting to reflect on the relative magnitudes and I think she is trying to express that the larger magnitude of the 88 decrease is somehow connected to the fact that 24 is a decrease when she hypothesizes, “You guys get the 24 because it’s a bigger number?” My interpretation requires that the direction of causality be reversed in her statement, and even then, her explanation is unclear. We will have to wait for future mixed-sign missing addend situations to see if she has made progress in abstracting out the quantitative relationships at play.

Summary of Justin and Michelle’s Ways of Operating in a Money Context

Justin has several breakthroughs in this section of the teaching experiment. First, he seems to be developing the ability to reflect on a difference as the distance between two quantities, which implies the construction of a subset relationship. However, he has trouble strategically reasoning in some cases, which implies that he does not yet have an assimilatory difference structure that he can reflect upon.

With regards to signed quantities, Justin’s increased ability to reflect on difference seems to allow him to construct signed sums as viable quantities. This breakthrough happens during and after Protocol 6.8. He shows a variety of explanations including attention to additive inverses and reasoning through 0 in both addition and missing addend situations.

Michelle seems more comfortable with strategic reasoning, but she is uncertain about her answers and seems to be operating on a very intuitive level at this point. I hypothesize that this reflects the fact that she is relying on reversible unsigned addition and subtraction schemes in her strategic reasoning and not on additive subset structures. She does not reason through 0 in her explanations. When she is doing signed sums that begin with a decrease, she tends to reason indirectly, which leads me to believe that she is not able to reflect on her operations and is probably not constructing an anticipatory signed addition scheme at first. However, she later reasons reversibly with her signed addition scheme in a same-sign MA problem. Therefore, I believe that she is developing more sophisticated signed addition schemes that are both anticipatory and reversible.

Trips on a Number Line

Starting on March 24, Justin and Michelle worked in a number line context. We introduced the context by discussing how long a pole could be if it went from the surface of the Earth to the center, and then asked the students to imagine the pole going into the earth that far and also extending up out of the surface of the Earth the same distance. Both Justin and Michelle are able to solve the first task, which involves figuring out where you are from your starting point after making a series of three trips. Justin is able to solve the task more quickly and I can see him making motions up and down with his fingers in the air as I talk about the trips. Protocol 6.17 gives the discussion of the second task.

Protocol 6.17: Justin sketches trips for the first time.

T1: Write down whatever you need to remember the problem, but, just like before, don't use your paper and pencil to add or subtract. We always want you to do it in fun ways. This time we'll have Michelle, and she's going to go up 59m, and then she's going to rest, and then she goes down 88m, and then she rests again, and then she goes up 37m.
J&M: [Write down the lengths of the three trips using *up* and *down* as the descriptors.]
J1: [Moves fingers up and down at first and then keeps track of counting something on his fingers. After 40 seconds, writes "8m up".]
M1: So I'm trying to figure out how…
T2: At the very end…you do this [pointing to Michelle's paper], you rest, you do that, you rest, you do that, at the *very* end, how far you are from where you started.
M2: OK.
T3: Justin, […] would you sketch what the trips look like on the pole so we can talk about that?
J2: [Draws left image in Figure 6.15.]

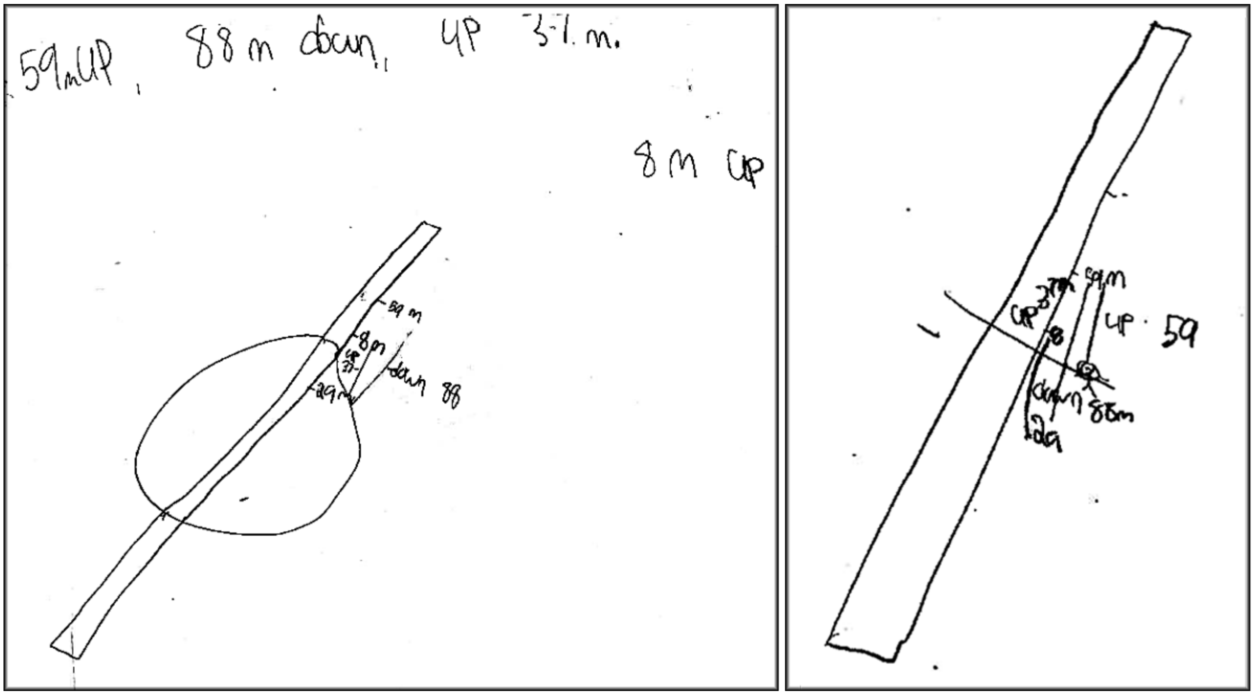


Figure 6.15. Justin's sketches in Protocol 6.17.

T4: […] Can you draw a bigger picture of just the pole and the trips?
J3: [Draws right image in Figure 6.15.]
M3: [During the 3.75 minutes since I clarified the problem, she has written and erased four answers.]
T5: Let's look at this picture first before we…Tell us what's going on here, Justin.
J4: She starts here, like she's on the ground, then she […] goes up 59ft, and then she rests, and when she goes down 88, she's inside the Earth by 29m.
T6: OK. Let's stop there because that a new number, right?
J5: Yeah.
T7: He's claiming that after you go down the 88m, you're inside the Earth 29m. Do you know how he got that? Does that make sense?
M4: Did you subtract, or…Yeah.
J6: Yeah. I subtracted 59, I took that away [indicating the distance from 59 to 0], and I added 10, I added 20 to it, so it'd be 79, then I added 9 to it and it'd be 29. So that was 29m inside the Earth. Then if you add…
T8: When you said you took the 59 away, you took it away from what?
J7: 88.
…
T9: Sorry, how'd you figure out the 29?
J8: I took 59 away from this and you're here again [pointing to the surface], and then I added 20 to it and a 9, so that was -29.
T10: How did you know you could stop when you got to the 29, though?
J9: Because if you subtract 59, it'd be 0, then 59 plus 29 is 88.
…
T11: Can you finish that up for him?

M5: OK. Yeah, so if we're down Earth 29, then I would add the 37 to the 29, which would make it […] 66 when you go *up*. So would the answer..It would be 66 *up*?! Over ground?
…
T12: I don't think that's the answer Justin had, so why don't you explain how you got yours.
J10: Yeah, I got +8 because I added +29 to that [indicates the second trip] and it made it 0 again. Then from 26 to 37 it is 8m.
T13: When you say you added the +29, where's that coming from, though?
J11: It is coming from the 37. […] I'm taking 29 from there to get back to 0. Then I added 8 to it, and it made it +8m.
T14: OK. I see what he's saying. Do you see his explanation?
M6: Yeah.
T15: […] You're going 37 *up*, so why wouldn't you add?
J12: Because it's a negative number, and if you're adding +37 it'd be back to 0 and then up 8.

Protocol 6.17 represents the first time that either of the students have drawn or seen a sketch of this number line context. Justin spontaneously labels both the length of trips and the ending position of trips separately and in distinctive ways. He labels the lengths about mid-way between the endpoints and labels the ending point at the actual point. The ability to label either clarified or represented the distinction between the two quantities, which I had been worried that he had a tendency to conflate in Protocol 6.15. Based on his behavior in Protocol 6.15, just two days before, I think that this representation probably helped him clarify the distinction. In addition, he has a visible and definite starting point, the surface of the Earth, which probably helps him keep track of the different quantities.

He uses his fingers to physically run through the actions in both of the first two tasks in this March 24 teaching episode. I think that this may signify also that he is immersed in constructing the trips and is not assimilating the tasks with a generalized sense of additively related trips as I would. The act of constructing these trips and engaging in actual actions to do would seem to be a necessary prerequisite for being able later to mentally imagine the trips and their additive relationships. Hence I was very happy that Justin was so immersed in the process of the trips at this point.

Justin also gives excellent examples here of using additive inverses to *reason through 0*. He has done this before with signed addition in a money context. He gives similar kinds of explanations in lines J6, J8, J9, and J12, although he is only pointing to the 0 and additive inverse in J6, not talking about them. However, his reasoning in J10 and J11 has a greater sense of the subset relationships between the different quantities: "I got +8 because I added +29 to that [indicates the second trip] and it made it 0 again. Then from 26 to 37 it is 8m. […] It [+29] is coming from the 37. […] I'm taking 29 from there to get back to 0" Compared to other times he has described reasoning through 0, he is more explicit in saying that he is decomposing the larger trip, +37, into the additive inverse of his current distance from 0, +29, and the remainder, +8, to get his new distance from 0. The notation could definitely help him both see and talk about these additive subset relationships more clearly, not only because it emphasizes the lengths of the number line where trips overlap, but also because 0 is always present in his visual field, helping to provide a clear reference point.

Michelle has a much harder time with this task. At one point during the explanation that is excerpted in line M5, she motions down for the -29, and motions up for *adding* the 37, but then goes on to add the 29 and 37. She then interprets the result, 66, as representing a position 66m above ground because she got it by going up. She does look at Justin's sketches, but she mainly is looking at the numbers written on her paper during his explanation. Nonetheless, her acceptance of Justin's sketch along with her hand motions implies that she does understand the situation and can picture it. What may be happening is that she is using the mentality of her trial-and-error approaches with some of the missing addend problems, where she reasons in the context to decide on the sign of her answer and the relative magnitude, in this case, a positive answer larger than 29. Then she tries adding or subtracting to get the answer. I think she tried addition first because she was thinking of +37 as adding 37, and that resulting sum would have fit with her expectations because +66 is positive and larger than 37. This solution method implies that she is not reasoning quantitatively about the situation. That is, her numerical calculations do not represent actions or quantities in the situation.

Starting with the third task, recorded in Protocol 6.18, of the teaching episode, I ask her to draw the situation. The act of drawing seems to induce her to reason about the quantities in the situations in a way compatible with Justin, although she never does independently refer to 0 in her explanations. The relationships between numbers is always more implicit for her than for Justin in this context.

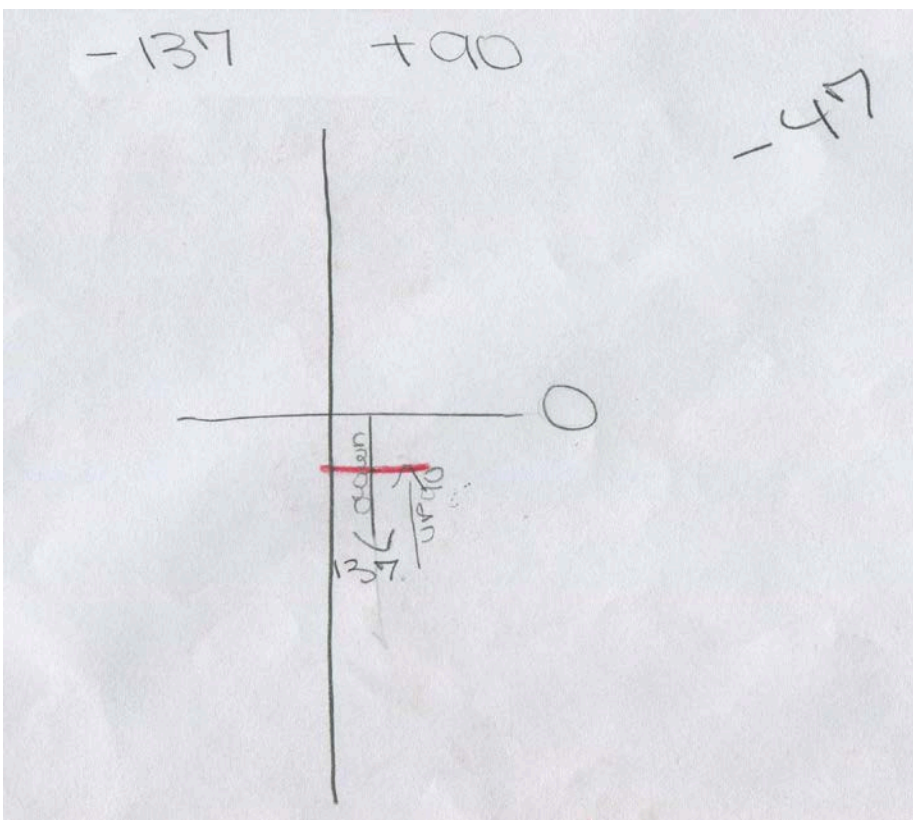


Figure 6.16. Michelle’s sketch for Protocol 6.18.

Protocol 6.18: Michelle sketches a number line problem.

T: Justin, […] you go down into the Earth 137m, and then you rest, and then you go up the pole 90m. I want to know how far you are from where you started. And see if both of you can draw the situation, kind of like Justin did.
J: [Draws a figure and gets -47 as his answer in 1.25 minutes.]
M: [Draws Figure 6.16 in the first 1.75 minutes. She takes awhile to draw the first arrow and to label the lengths. After finishing the diagram, she thinks for another 30 seconds.] Would you be below 47?
T: That's what Justin had, but actually I haven't worked it out, so, Michelle, why don't you try to convince me?
M: OK. Well, I went down 137, and since the 90 is lower than the 137 [pointing to the 90 and 137 at the top of the paper], then you're going to still be below [pointing to diagram]. So then I went up the 90, […] and then I just kept adding 10. So I went 90, 100 [putting fingers up to keep track and looking at the diagram], 110, 120, 130, which gave me 40. Then I just added the 7 [pointing to 137 in the diagram], and I got 47.
T: Make the part that's the 47 red.

When Michelle is talking about why you are still below the surface after the second trip, she seems to be referring to the size of the numbers she wrote down when I first read the problem, not the diagram. My guess is that she figured out the second arrow would end below 0 by comparing those two numbers, as she usually does to determine the sign of the answer. However, once she has drawn both arrows, she does refer to the diagram as she determines how far the second arrow's ending point is from 0. Furthermore, by pointing to the 7 in the 137 in her diagram as she counts up the final 7, she is indicating that she is aware that she is finding how much farther she needs to go to complete the 137m. Therefore, I think drawing and reasoning with the figurative material is allowing her to see how the quantities fit together additively.

Michelle and Justin are both successful in a fourth signed addition situation, but none of their explanations give us additional information about their thinking. Michelle is only asked to explain the sign. The rest of the teaching episode was spent on missing addend situations. Protocol 6.19 contains the discussion during the first of these problems.

Protocol 6.19: Justin and Michelle reason through 0 on a missing addend problem.

T: You [Justin] are going to do something, and Michelle's going to do something different, and then you're going to figure out what Justin has to do to get to where Michelle is. […]
M: [Writes "+125".]
J: I'm going to go down 154m.
M: I go up 125m.
…
T: Why don't we let A go ahead and sketch this to make sure we have the right situation.
J: [Draws the +125 and -154 trips as in Figure 6.17.]

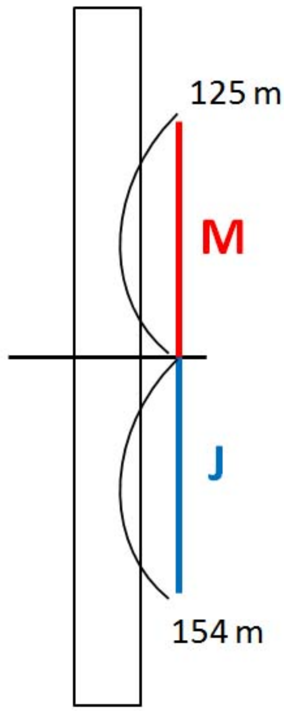


Figure 6.17. Justin reasons through 0 on a missing addend problem.

T: Where are you?
J: At 154m.
T: And what are we trying to figure out?
J: What I have to do [moving finger from -154 to +125 in the picture] to get to 125. […] I would have to get to 0 [indicates line segment from -154 to 0], which would be…
T: Don't solve it. Just show us the trip. Just draw it with a pencil.
J: I would go from here [-154] to here [0], all the way up to here [+125]. [Draws two distinct trips, as seen in Figure 6.17, but does not lift his pencil between them.]
T: All right. We're all on board?
M: On this problem, if you're at 154m, wherever Justin is, and you have to get to there to get to 0, wouldn't you just have go up one hundred and…
T: [Nods.]

M: OK. […] [Writes "+125".] So wouldn't it just be [pointing to +125] since you would have to go to 0 and go up?
T: Well, in *all* [moving finger from -154 to + 125], what does he have to do to get from here to here?
J: Ooh. Got it! [Writes "+279m".]
M: Oh. [Erases "+125" and writes "+279m".]
…
T: Do you have an interesting way? Because he got to explain last time.
M: Well, I just went from 154 to 0 and then I just went from 0 to 125, so I added those two numbers up.

This is the first time that Justin or Michelle has explicitly reasoned through 0 for a missing addend problem. My guess is that the diagram, which basically shows a pre-subdivided sum, helped the students understand why they need to add magnitudes to get the magnitude of the missing addend. Although Michelle had some difficulties, which I discuss in the following paragraph, she did make sense of Justin's use of 0 and used it herself in her final explanation. However, I would still like to see an example of her independently referring to 0 when working through a problem to confirm that she is consciously attending to 0 and additive relationships in these problems.

Michelle is a little confused about the question at first. I am not sure whether she was conflating a trip that ended at 0 with a trip that is 0 long, or whether she is so used to answering the question of how far someone is from 0, that she was answering how far Justin would have to go from 0. The following day, when sketching the signed addition situation, (+10) + (-24), she says, "You went up 10 and back down 24, so if I went up 10 and then came back down, that would get me back to 0. Then I'd go back down 24," and she ended up trying to go down 34 in all. In this case, I do not think she misunderstood the question because we were in the middle of doing several signed addition problems at the time. Therefore, it is possible that in both Protocol 6.19 and in this example from March 29, she was conflating the directed length of a trip with its endpoint.

In the next teaching episode, on March 29, we moved to using Geometers' Sketch Pad [GSP] to draw the number line trips. With this pair of students, we were hoping to start using variable quantities with them to work on additively covarying relationships such as that expressed by $x + y = 5$. We did work a little bit on these kinds of relationships with GSP, but I will not be discussing the results of that work here, instead I will be looking at their continuing evolution when reasoning about signed addition and subtraction problems with one unknown quantity. In GSP, they had several vertical number lines that were supposed to represent the pole, a horizontal line with the reference value of 0, and a centimeter grid. The tasks were already written on the GSP pages, one task per page. We intended for Justin and Michelle to come up with reasonable values for the scale of the grid depending on the magnitudes involved in each problem, but we did not explicitly tell them that. In this first teaching episode with GSP, the two students were sharing a computer in order for them to learn from each other how to use the GSP program. Justin was the first student to start drawing, and he independently decided to use the scale of 5m for each centimeter tick mark. While Justin was using the grid lines to help inform the lengths of the trips he was making, he appeared to continue to reason about the quantities in the problem as he had on the previous day. Michelle, on the other hand, tried to get the answer by estimating where the endpoint of the second trip lay in relation to the grid lines. I will refer to this behavior as *measuring* because she was, in essence, measuring how far the endpoint was from 0. I will give an example of that behavior in Protocol 6.21 from later in the teaching episode. The only task in which Michelle appeared to reason without measuring was the second task, which involved the addition of two positive trips. Protocol 6.20 describes what happened during this second task.

Protocol 6.20: Michelle reasons without measuring in GSP.

T: You climb up 35m. […]
J: [Makes Trip 1 four lengths long.] 35. Right there.
T: If this one 35, how much for the one part?
J: [Repeatedly pointing to each of the four lengths] 8… 9… or, like, 8.25.
T: You think it's true?
M: Yeah.
T: I'm not convinced yet.
…
J: [Pointing at lengths again] 8.75.
T: All right, so what would you have to do next?
J: I have to go up 15m. [Makes Trip 2 less than 2 lengths long.]
T: Is that correct? Seems correct?
M: [Nods.] Mm-hmm.
T: Now it's your turn, what do you have to do?
M: I have to go up 50. [Makes the arrow from the horizontal line to the endpoint of Trip 2.]
…
T: How did you get the 50?
M: Because he went up 35 and then he went up another 15, so I just added that.

While Michelle did not give much of an explanation here, she is clearly assimilating this as an addition situation. This is the only problem involving only positive quantities, and the only other problem she does not *measure* to find the answer could be modeled by unsigned multiplication, namely 12 – 2 – 6. Therefore, I think that she is assimilating that problem and the task in Protocol 6.20 as unsigned addition/subtraction situations, and both involve easy computations. Those two conditions seem to be what prompt Michelle not to measure to find the answer. In contrast, Protocol 6.21 gives an example of her usual measuring behavior on March 29. The problem that Michelle is modeling is a 33m trip down, followed by a 66m trip up, followed by a 45m trip down.

Protocol 6.21: Michelle measures to find the answer in GSP.

M: [Makes first arrow 3 lengths down, second arrow 6 lengths up, and third arrow a little more than 4 lengths down.]
T: Did you follow that?

J: Yeah. It look like she goes down 33, she goes back up to 66, which would be 33. Down 45, which would equal…[Moves the total trip (displacement) arrow to start at the starting point of the third trip.]
T: Wait. Where are you now?
J: Oh. I'm starting at the 33 part and going down 45. [Draws the third trip again. Pauses. Moves arrow to start at the surface of the Earth.] Would I start from here?
T: [Nods.]
J: [Makes total trip arrow.] 12.
…
T: Why?
J: 'Cause from 33 back to 0'd be 33, and you just have to subtract 12 from that [pointing to last 12 of third arrow] 'cause 33 plus 10 would be 43, plus 2 would be 12.
T: OK.
M: Since I just had to go down 33, each little square was 11, and then I just went up there, and then I just went 11, 22, 33, 44 [pointing to appropriate locations on third arrow], and then I just went down a tad since it was the extra 1, which gave me the 45 to go down. From there to there [first length of third arrow below 0] is 11, since once cube equals 11, and just go down 1 more, and it's 12.

In Protocol 6.21, Michelle does run through the actions in her explanation, but that is mainly to explain how she got the scale of 11m per cm mark and to explain that she went 1 past the -11 mark to give an ending point of -12. Note that she does not explicitly attend to the fact that the trip 45 down can be decomposed into a trip of 33 down that takes you back to the surface and a trip of 12 down below the surface to your ending point. In contrast, Justin does refer to getting "back to 0" and is explicit about trying to find how much longer the 45 down trip goes than 0 through counting up. Note that he is not using the grid to help him at this point in his reasoning. When counting up from 33 to 45, for example, he does not go from 33 to 44 and then to 45, which would be the logical way to break up the number sequence if he were using grid lines to help him estimate the answer. The reason that I think Justin's strategy is more powerful is because he is reflecting on the additive relationships while Michelle is only keeping track of number sequences. Reflecting on these additive relationships would seem essential the eventual construction of an additive assimilating structure in these signed contexts.

In the first and only missing addend problem of March 29, we can see that Justin continues to reason through 0 in missing addend situations on GSP, whereas Michelle attends to 0 only implicitly when naming the position of ending points. The problem situation the students are modeling is that Justin has climbed up 5m from the starting point, rested and then climbed again, but we don't know how far or which direction. Michelle has to climb down 11m from the starting point to reach Justin. The students' task is to figure out what Justin's second trip was.

Protocol 6.22: Missing addend problem in GSP.

T: Justin climbed 5m.
J: [Makes arrow up 1 length.]
…
T: And you have to climb down to the pole to get to 11m.
M: [Makes arrow down a tad over 2 lengths.] Now I have to find out what *his* next move is?
T: Yes. Both of you need to find out what is his second move. […] Do you know what I mean?
J: I think, go from here to here [+5 to 0], then Trip 3 from the center down to 11. Is that right?
T: Why don't you just try it? Right now we don't know what Trip 2 is, right?
J: Trip 2, I think, would be from here down to there [+5 to 0]. Then you would go from the surface down 11 to where Michelle would be. There [pointing]. That'd be Trip 3 'cause Trip 2 would be going down 5 and Trip 3 would be going down 11.
M: I thought we were only supposed to use the third one [the arrow for Trip 2].
…
J: Oh, so I'd be going down 16.
T: You think you go down 16?
M: 5, 10, 15, 16 [counting down the grid].
T: Why?
J: Because you go from 5 down to 0 is 5 and then you go down 10 and 15 plus the 1 is 16.
M: [Nods.]

In this protocol, as in Justin's drawing in Figure 6.17, he is very clear about decomposing the missing addend into the additive inverse of the first addend, needed to get back to 0, and then remainder of the missing addend that is needed to get to the ending point of the sum. In this case, he thinks at first that he needs two more trips to get to Michelle, because he wants Trip 2 to end at 0. Even when he is explaining why the *one* missing trip would be 16m down, he refers to 0 as a kind of stopping point that partitions up the trip: "Because you go from 5 down to 0 is 5 and then you go down 10 and 15 plus the 1 is 16." He is indicating the gridline for -10 in this explanation, which is why he splits up the -16 into -15 and -1, but he does not refer to the gridlines in his earlier explanations, which implies that he is reasoning more through decomposition of the missing addend then through measuring with the grid. Michelle, on the other hand, feels the need to count down using the grid before she is willing to agree with Justin's answer. Although 0 is one of her stopping points as she counts, she does not differentiate it from any of the other gridlines through her language, motions, tone of voice, or a pause.

After watching this teaching episode, I hypothesized that Michelle's additive reasoning was suppressed by the presence of the grid lines. Therefore, I did not include grid lines in the next teaching episode with GSP on April 12 and stressed that we were just sketching, not drawing, the trips. In addition, I focused on emphasizing both the end points of trips and the trips themselves. For example, if I was asking for a signed sum, I would ask both, "Where are you in relation to your starting point?" and "What would you have to do to get from the starting point to your ending point?" This was meant to induce Michelle to attend to the lengths of the trips in order to help her abstract out the subset relationships. Michelle did operate successfully without the grid, but she still did not show indications of attending to 0 in order to mentally decompose a trip into an additive inverse of an earlier trip and the remainder.

The last example I will share from Justin and Michelle's work, from April 12, is Justin's articulation of reasoning through 0 in order to *make* a diagram of the trips. In the past, he usually used the grid lines to make his diagram, but then reasoned about the lengths of the trips without attending to the grid lines. On April 12, Michelle gave him the problem, "You go up 30m. Then you go down 55." After drawing an arrow up, he starts the second trip and drags the arrowhead down to the surface, pauses, and says, “Here’s 30.” He then continues down until the part of the Trip 2 arrow below the surface is a little shorter than the part above and says, “25 down.” The fact that Justin could utilize reasoning through 0 in order to make sense of the situation implies that the decomposition of the longest trip in these problems might be an assimilating structure for him at this point. This would also be confirmed by his automatic decomposition of missing addends into two trips as discussed in Protocol 6.22.

Summary of Justin and Michelle’s Ways of Operating With Trips on the Number Line

In this last context, the ability to use diagrams seems to help subset relationships became more explicit for both Justin and Michelle. Justin’s reasoning through 0 explanations become even clearer and this seems to allow Michelle to internalize his reasoning enough to explain it to me after two tasks. This implies that Michelle does have the ability to construct subset relationships among the signed values given a visual aid. However, she does not generally do so.

On the last day, I tried to get them to notate problems of signed addition or signed MA problems with addition equations that had unknowns in the appropriate place. However, neither student was able to differentiate between the signed addition and signed MA problems. Both seemed focused on operational versus structural understandings of signed addition and subtraction.

CHAPTER 7

ADAM AND LILY'S ADDITIVE REASONING

The first teaching episode when Adam and Lily were partnered was March 3. They had a total of 11 teaching sessions together, ending on March 29. I worked with Adam individually on March 31.

The Card Game

The first four sessions were spent on the card game (see the introduction to Chapter 6), and I introduced the score sheet to them for the first time. Within the card game context, two main issues came up during analysis with respect to Adam and Lily: How do the students reason to determine sums and differences, and what does that imply about the nature of their number sequences? What is the nature of the quantities they are constructing when they fill out the second column (C2; how much the player won/lost the round by) and third column (C3; how much the player is winning/losing the game by)? My analysis is organized around those two questions.

Determining Unsigned Sums and Differences

In Chapter 5, there was only one instance in which I felt confident claiming that Adam had reasoned on his number sequences in order to determine a difference (Protocol 5.7). Based on that single incident, I could not determine the nature of his reasoning. In particular, I was not sure if he was counting down with composite units, using a compensation strategy, or *strategically reasoning* (see Chapter 2, "Explicitly Nested Number Sequence"). Within their first teaching episode together, both Adam and Lily reasoned on their number sequences to determine missing addends several times. Their explanations indicate that Adam generally used a compensation strategy, in which he can increase one addend by one and decrease the other by one in order to maintain a constant sum, and Lily generally used strategic reasoning, in which she would find an easier subtraction problem involving nearby numbers and then adjust the difference from the easier numbers appropriately to find her answer. Originally, in the March 3, I would not consider what Adam was doing to be strategic reasoning because his sense that increasing one addend and decreasing the other gives a constant sum is very intuitive at this point.

Although Adam appropriately applies his compensatory strategy several times on March 3, he also has two instances in that same episode in which he does not apply it correctly. That is, he adjusts both addends down or both addends up by 1. The first time this happened, he come up with his answer quickly, and so I thought that he might have simply made a mistake in the action of his compensatory scheme. The next time he appears to use a compensatory scheme, I challenge the answer, and he quickly changes it, even though it had been right. At this point, I began to question whether he really was using a compensatory strategy, even intuitively, or whether he was trying to imitate Lily's ways of calculating without understanding completely when she added and when she subtracted to get her final answer. Therefore, when he appears to use a compensatory strategy unsuccessfully on another task, I decide to investigate his thinking further. I describe my intervention and his responses in the following protocol. At the beginning of the protocol, he is explaining how he figured out his round score, given that Lily's card was 92 and his card was a 9. Adam eagerly volunteered his explanation and even said, "I have a cool way."

Protocol 7.1: Adam reflects on his compensation strategy.

A: 81. Because 82 plus 10, or…
T: 82?
A: Yeah, I skipped up [motioning with hand]. 82 plus 10 is 92. Or, no wait. 10 plus 82 is 92, so you just subtract…
…
T: OK. Go ahead and finish yours.
A: [Smiling] ‘Kay, so 10 plus 82 is 92, and we need 9. So 9 plus 81 is the [points to 92]. Or, yeah.
…
T: If you had a 10 instead of a 9, what would the score be?
A: I’d be losing by, well, I would have lost by eighty-…two. So you subtract 1 from that and get 81. ‘Cause you need 9. ‘Cause 81 plus 9 is 92.
T: Let’s pretend we had that [10], right? But that’s not what you have, you have this [9]. Are the numbers further away or closer when you get the 9?
A: Farther…Yeah.
T: Should you have a bigger difference or a smaller difference?
A: [Talks under his breath.] Yeah, ‘cause if it was 81, it’d be 11.

This idea of thinking about numbers being further or closer, i.e., thinking about how far apart numbers are, seemed to resonate with Adam, as I expand on in my discussion of his future strategies. However, the only indications that my intervention may have been helpful in making his compensatory strategy more explicit are that he correctly uses a compensation strategy two more times that same day. For example, when Adam is explaining how to find 62 – 33, he says, “Well. 32…Well, 62 minus 32 is…Well, 32 plus 30 is 62. So if you have 32, then, it’d be 30, but since you have 33, it’d be 29 ‘cause…” He could not come up with a further explanation on prodding., but when I ask him if the answer could be 31, he sticks with his answer this time. Therefore, I do not think that he was just guessing. Based on his explanation, I conclude that Adam is, indeed, equating two sums, 32 + 30 = 33 + 29, but he is doing so intuitively in that he cannot explain why increasing one addend and decreasing the other keeps the sum constant. In fact, he never even explicitly refers to the action of adding or subtracting one from the addends on March 3.

In contrast, Lily could explain her strategies fairly clearly, and she was able to refer to her adjustments at the beginning and end of the problem. For example, on March 3, when she is explaining why 92 – 9 is 83, she says, “Because if you were, if you said the 9 was a 10 for a second, then it would be 82, and you’d have to add the extra 1.” Lily is even more explicit in other explanations about her strategy. On March 9, when she was explaining how she added 17 and 84, she said: “I rounded the 17 to 20, and then 84 plus 20. I got 104. […] And then, since I added 3, I have to subtract 3 and this [101] is what I wanted.” That same day, she explains how she determined 90 – 26 as follows: “Um, I said that 26, and I rounded to 30. And that gives me a 60. And I have to add the 4 that I added to 26, and I got 64.” What strikes me about her explanations is that she is very aware of the relationship between the given number and the “rounded” number she works with, and so she can refer to the difference between them when she has to use it to find the final answer. Regardless of whether her “rounded number” results in a difference/sum that is too big or too small, Lily is able to figure out how to adjust it and can usually explain her strategy fairly clearly.

The next teaching episode, March 8, Adam starts out with a strategy similar to that in Protocol 7.1 in that he adjusts by 1 in the wrong direction to get his final answer . I again intervene. Based on the success of the intervention on March 3, I used a similar tactic of encouraging him to compare how far apart the pairs of numbers were. This time, though, Lily took part in the discussion, and described her thinking to Adam. Protocol 7.2 describes the interactions between the three of us. Before the protocol begins, Lily had drawn a 35 card, and Adam had drawn a 9 card. Lily came up with W26 as her round score, and Adam came up with L24 for his.

Protocol 7.2: Lily intervenes in Adam's compensation strategy.

T: Tell us how you were thinking about it.
A: 35 minus 10 is 25, then you just need to subtract 1, which is 4. 'Cause you have 9.
T: If this had been a 10 [the 9 card], that's what you're saying, kind of, right?
A: [Nods.]
T: If this had been a 10, what would the answer be?
A&L: 25.
T: You all agree on that?
L: [Nods.]
T: All right, if he had the 9 instead of a 10—
A: You do down 1.
T: Is the answer going to be bigger or smaller? Do you see what I'm saying?
L: [Nods.] Bigger.
T: Why will it be bigger, Lily?
L: Because you're subtracting 1 less than 10, so it's going to be 1 bigger.
A: [Changes his answer to L26.]

After this intervention Adam was able to not only apply a compensatory strategy of adding or subtracting 1 from one of the addends, but he was also able to generalize his strategy into what I would call strategic reasoning. First, he explains how he got 91 – 37 by saying, "I did 97 minus 37 is 60 and then I subtracted 6 because I counted up 6 to get to 97." This is the first time he has used strategic reasoning with such a large adjustment, and it may be the first time he is truly strategically reasoning: In the past, as I discussed, he was probably using an intuitive compensation strategy or counting up/down by composite units. Furthermore, his explanation refers to his adjustment of counting up by 6 and the need to counterbalance that by subtracting 6. Note, though, that the idea of counting up/down a number sequence is still prevalent in his ways of understanding the situation. He is using numbers to quantify segments of a single number sequence, partitioned up into nested subsequences. Therefore, he was still confined to working within his number sequence as opposed to disembedding a subset of the units in his number sequence in order to compare it or move it and recombine it with other subsets the way that Lily seemed to. In the above protocol, as elsewhere, his language usually refers to going up or down (to the left or right on) a number sequence: “You do down 1.” In addition, he often motioned to the left or right with his hands when he is describing how he adjusts his estimated answer, more so than any of the other participants, which supports the interpretation that the sequentiality of his number sequence is very prominent for him. Of course, all of the participants count up and down their number sequences, but Adam seems confined to that way of operating, implying that he still has not reorganized his TNS into an ENS by abstracting out iterable unit items from his arithmetic units.

Later in the session, he is trying to figure out what to add to 42 to get 120. He says, “40 plus 80 equals 120, so you subtract 2 from 80 and get 78.” This explanation is not quite as clear in that he does not explicitly refer to both adding two and subtracting two, but he has generalized his reasoning to compensating and strategically reasoning with adjustments of more than 1. This implies to me that he has made a reflective abstraction of his intuitive compensatory strategy and can operate on some kind of visualization of the addends and/or the difference. Generally, TNS students who are carrying out this kind of compensation strategy cannot generalize it to adjustments of more than 1, for example (Steffe & Cobb, 1988). Adam’s behavior is still surprising to me, and even during the episode you can hear the witness getting excited about Adam’s new forms of strategies. I would usually, a priori, attribute the construction of an explicitly nested number sequence (ENS) to a student who was able to strategically reason, but, as I have discussed Adam also had some behaviors that were more consistent with a student who is currently limited to a TNS. Furthermore, this kind of reasoning was difficult for him. For example, in the money context, after explaining a strategic reasoning strategy that involved using 1200 – 1110 to find 1199 – 1111, he said, “Wow. I actually did it….That was hard.” Therefore, my hypothesis is that he was still operating with his TNS arithmetic units, but he was explicitly aware of the subset relationships within his composite units, which allowed him to operate in a powerful way.

Another illustration that Lily seems to have a greater freedom in how she combines and decomposes addends is the contrast in how they both described an addition strategy that more or less parallels the mathematics behind the traditional addition algorithm in that they are decomposing their addends into tens and ones and adding up the place values separately. Lily's explanation took place on March 8. She was attempting to explain how she got her win/loss margin, given that she had been losing by 84 and then lost by another 36: "I took the 4 and the 6 from 30 and 80 and knew that 4 plus 6 is 10, so I added the 80 plus 30 and get 110 and then I added the other 10." I get the impression that Lily can disembed all of these composite units from her number sequence, combine them however she wants, and then put them back together. For one thing, she is simultaneously aware of two sums, the 10 and the 110, and the fact that they form a sum together. In contrast, in the next teaching episode, March 9, Adam describes a similar strategy for double-checking that 58 + 42 is 100: "'Kay. 50 plus 40 is 90. You have the 2, that's 92 and you have the 8 too." Here Adam is able to decompose the 58 and 42 into 50, 8, 40, and 2. However, he is only creating one running sum instead of creating multiple sums that can then be summed.

I hypothesize that during the two interventions on March 3 and March 9, Adam took the subsequence between two numbers, representing a difference, as an object of reflection and was able to visualize the change his initial adjustment would make to the difference and determine the inverse change that would need to be made to get his final answer. I do not think, when he counted up by 6 and then counterbalanced that by subtracting 6 at the end, that he is unitizing the 6 or the result of counting up by 6. Instead, I think he has unitized the original difference, recognized that he made the difference larger when he counting up by 6, and so had to counterbalance that by making it smaller. The fact that he uses 6 both times probably comes out of an intuitive sense that adding 6 and subtracting 6 are inverse operations. Lily, on the other hand, would be aware of the 6 as a composite unit that was being combined with or disembedded from other composite units.

Lily's ability to disembed and recombine subsets gave her great flexibility in how she could determine sums or differences, as her above quotes hopefully illustrate. She also gave an indication that she had interiorized a difference structure: On March 8, the students drew an 84 and 45 one round and were coming up with other possible card combinations that would give a difference of 42. Here and in other places, the students would find a number they could easily add or subtract the difference from and use that to generate a pair. For example, Adam gave the pair 52 and 10. Lily confused me by coming up with the pair 107 and 65. I was confused about how she thought of that combination, so I asked her how she got it. She replied, "I added 20 both of the cards." She seemed to come up with the pair easily and quickly, so I hypothesize that she was aware that she had some kind of visualization of the subset relationships that allowed her to recognize that the added 20 would "match up," or overlap, in the two numbers, so that the difference would not be affected (see Figure 7.1). She could have also reasoned that adding some amount, x, to the subtrahend would increase the difference by x and that adding x to the minuend would decrease the difference by x, thereby creating a net change of 0. However, based on how long it took her to strategically reason and the concentration she gave to that process in other instance, I do not think she reasoned in this way in this instance because she did not appear to be in concentrated thought for any amount of time.

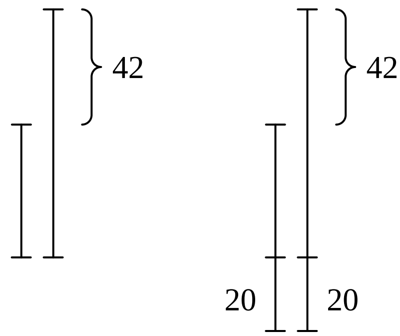


Figure 7.1. Lily's possible visualization of a difference structure.

Despite the fact that Lily's strategic reasoning was more sophisticated, it did not appear to be of a different caliber than Adam's strategic reasoning. Therefore, reasoning about unsigned sums and differences was a time when he could feel like a mathematical equal with Lily. Adam noticed by the second teaching session that Lily tended to be more articulate about her mathematics. For example, on March 8, both students were determining 120 – 42. Adam gave an explanation that involved breaking the 42 into a 40 and 2. I replied enthusiastically and gave him five. When I asked Lily for her explanation she said, "I just subtracted." Adam started laughing and said, "It's backwards now," because he was usually the one that had trouble coming up with complicated explanations.

The only other data I present about the students' unsigned schemes relates to the reversibility of their subtraction scheme. Both students were able to determine the other player's card given their card and the round score. Based on my analysis in Chapter 6, I hypothesize that this implies that they both have constructed a reversible subtraction scheme. In fact, it is a little more complicated because they had to take the sign of the round score into account when figuring out whether to add or subtract the round score from their card's value. I do not have any good indications of how the students did this, but I have the impression that they reasoned about the sign first because they would often start by saying who won, and then figure out the card value. Therefore, they probably break the problem up into two steps: first they decide who had the higher number based on the sign of the round score and then they can assimilate the situation with an unsigned difference structure in which the higher card minus the lower card gives the round score. A reversible subtraction scheme is not as sophisticated as Lily's assimilating difference structure in that Adam is not necessarily assimilating situations with the subset relationships already inherent in the situation. Instead, he can assimilate the difference as the quantity that enumerates his action in his subtraction scheme. For example, 42 would be how much he would have to count up or down to get to the other number, so then he can reverse this action and count up or down the same amount to get the missing number.

Adam and Lily's Signed Quantities

Both students are generally successful when working out additive relationships in the card game. For example, Lily can always combine her C2 value with her last C3 value to get a new C3 value, what I call a signed addition problem in this context. Adam can usually do these problems as well; during the four sessions he only has major difficulties during two tasks, which I discuss later. I also gave the students some missing addend (MA) problems in which I gave them the C3 value and they had to figure out the C2 value (and usually card values as well). Because of the cards that the students drew when I was giving them MA problems, Adam only ended up finding the difference between two positive numbers, and Lily ended up finding the difference between two negative numbers. However, both of the students had problems in which the difference was positive and problems in which the difference was negative. I am attentive to "mixed-sign" MA problems in future sessions. Adam had trouble with the first MA problem he worked on, which I will discuss, but, as with the signed addition problems, he was able to adjust his scheme(s) for MA problems to make it viable. In this section, I discuss what Adam's difficulties imply about his conceptions of the C2 and C3 quantities and his signed addition schemes, and then I discuss the kinds of explanations Lily and Adam used to explain their signed sum schemes.

Round	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
1	W86	W86
2	L14	W72
3	L10	W62
4	W12	W50
5	L69	L19
6	L83	L102

Figure 7.2. Adam's scores up to Round 6 on March 3.

Adam's signed addition. In Round 6, the first time that Adam is calculating C3 scores with both winning and losing C3 scores visible (see Figure 7.2), he, similarly to Justin, tries to not only combine the last losing score, L19, with his losing round score, L83, but also tries to combine (through subtraction) the sum with the last winning score, W50. In doing this, he treats the C3 scores as if they represent *total wins* or *total losses* instead of *total wins and losses* (actually, the total win/loss margin, but it is often ambiguous whether the students are thinking of C3 in terms of a positive/negative score or a win/loss margin). At that time, I pointed out that the last C3 value was "how much you're winning the whole game by." And he responds, "Oh yeah…That's true," and changes his answer. The fact that he reacted as if he were recognizing something about the nature of the C3 quantity implies to me that he was, indeed, thinking of the past C3 quantity as representing wins or losses, even as he was attempting to determine his new C3 value as a combination of wins and losses. He did not try to combine the last winning and losing C3 values again in the card game, so apparently my intervention was enough to help him to firm up the C3 quantity as a win/loss margin. As in Justin's case, I hypothesize that he had not yet constructed a signed sum quantity that includes records of combining both wins and losses. It appears that this was easier for him to do than for Justin. I think this relates to the fact that Adam is comfortable assimilating and operating on differences as quantities, as in his reversible subtraction scheme, so that he can more easily generalize his assimilatory concepts of differences and sums to a signed sum structure in which the signed sum consists of some combination of differences and sums.

Round	Adam's Card/ Lily's Card	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
8	(21/26)	**L5**	W73
9	**72/61**	W11	W84
10	92/**100**	~~W36~~ L8	**W76**
11	**14**/90	L76	**W0**

Figure 7.3. A selection from Adam's scores on March 8.

Adam's signed subtraction (MA). In Figure 7.3, I show some of Adam's scores from March 8. The bolded values were given to the students and the non-bolded values had to be determined. The parentheses indicate that the students did not have to figure out those exact values. Adam had trouble with the first MA problem in the card game, which happened in Round 10, but then seems to figure out a viable of operating in the situation, as evidenced both by the fact that he corrected himself and by the fact that he was able to appropriately interpret the next MA problem in Round 11.

Protocol 7.3: Adam constructs a reversible signed addition scheme in the card game.

T: All right, how did you get that?
A: Well, if I won by 76, 100 minus 36 and you get 76.
T: Now, you didn’t win the *round* by 76.
A: Oh, I’m *winning* by 76. [erasing] Which means that I won, hold on. […] [Writes, “W8,” in C2.] Hey, wait a minute. Did I win or did I lose?
T: Yeah, that’s a good first question. Did you win or did you lose?
A: To see if I’m still winning…She picked….[Changes W to L.]
T: Yeah. How can you tell just from these [C3] if you won or lost in that round [R10]? […]
A: I don’t know.
T: You don’t know? What about if you were losing by 84 and then you were losing by 76, can you tell if you won or lost that round just by looking at those two numbers [R9&10C3]?
L: I won that round because, because his won, win amount of points went down.
T: Do you agree?
A: [Shakes his head.]
T: No?
A: But I think I know what both the cards are…. I know one of them, which is 100. […] And I had 92.
T: And how do you know that?
A: Well, if I lost by 8.
T: […] Looking at these two [R9&10C3], you can’t tell if you won or lost the round?
A: Well…
T: [After about 10s] Should we try again?
A: Well, yeah, because, since, if I had 84 and I went down to 76, I lost points.
T: Exactly. Very good. All right. Let’s do another one like that. […] [Looks at cards.] […] Wow. [Writes W0 on Adam’s score sheet and L0 on Lily’s score sheet.]
A: What?! […] So we’re tied then. […] So she beat me by 76.
T: All right, what could the cards have been?
A: 76 and 0.

Adam initially treated the R10C3 score as a round score and incorrectly determined the difference, 100 – 76, to be 36, which he wrote in R10C2. Adam, by this point, is consistently using *won*/*lost* terminology for C2 values and *winning*/*losing* terminology for C3 values.

Therefore, both his calculation involving the card value and his language, “if I won by 76,” imply that he was thinking of W76 as the round score from the beginning. Of course, by this logic, he should have added to get a card value of 136, but he knew the card values did not go above 100, and Lily had just guessed that he had 100 as his card value, so, taken together, these influences may have caused him to think of the 100 as his winning card. I would not consider this to be a conflation of quantities because he is consistently treating the C3 value as a round score so far.

However, the fact that he did not seem disturbed by the idea of putting a card value in the second column and having a round score in the third column could have been made more probable by his tendency to think about both quantities as amounts of points. Lily probably does this also because, although she usually using “winning by” or “losing by” when talking about C3 quantities, we see in Protocol 7.3 that Lily calls positive C3 values, “win amount of points.” In fact, the teacher/researchers’, including myself, do it also. There is an example in the next protocol of me referring to the C3 values as a score. Thinking of C3 values as a number of “win” or “loss” points would also help explain the earlier attempt by Adam to combine the most recent positive and negative C3 values when determining the new win/loss margin. The students did not consistently refer to C2 and C3 values as a score, or a number of points. They seemed to oscillate between thinking of C2 values as a number of points and an amount they won or lost (change in points), and between thinking of C3 values as a number of points and the amount they are winning or losing by (comparison of points). In this case, thinking of both values as scores created the conditions for Adam to confuse the role of his C2 score. Elsewhere, though, he clearly is aware of the “score” really being a win/loss margin. For example, in the next protocol, I refer to the C3 value as a score, and he responds by specifying that it is how much they are winning by.

Protocol 7.3 also gives us an example of a recurring conflation for Adam: When he said, “Oh, I’m *winning* by 76, which means that I won,” he was deducing that he won the round based on the fact that he was still winning the game after the round. In other words, he conflated the nature of the increase/decrease of the C2 value with a winning/losing C3 value, respectively. There are no more clear examples where he makes this conflation in the card game context, but we will witness this conflation with Adam again in future contexts.

Notwithstanding Adam’s difficulty with this first MA problem, he successfully solved these kinds of problems throughout the card game sessions. As we saw in the last protocol, at first he was not aware that he could figure out whether he won or lost the round using only the C3 scores. He even rejected Lily’s explanation of how she figured out the sign of the round score: “I won that round because, because his won, win amount of points went down.” Nonetheless, Adam must have intuitively used the C3 scores to get the magnitude of his C2 value because Lily miscalculated and got L12 for her C2 value, so he could not have copied the answer. This intuitive calculation was probably overshadowed by his realization that he did not win based on his knowledge that Lily had drawn the highest possible card. Therefore, when I asked if he could figure out whether he won or lost from the C3 scores, he may have had the sense that he used the card values to figure out the sign of the C2 value, just as he would be aware that he usually did use the cards to come up with the C2 sign and magnitude. The second time I ask him if he can figure out whether he won or lost the round based on the C3 values, he looks at his paper and thinks for a little over 10 seconds before coming up with an explanation. My guess, based on his explanation, “If I had 84 and I went down to 76, I lost points,” is that he noticed that the C2 value was telling him *how* the C3 value changed. This appeared to be enough for him to become explicitly aware of a scheme of determining a missing C2 value from the C3 values, at least when they both are positive values.

As I said earlier, in this context, he never has to find C2 values based on C3 values with different signs. In that case, finding the difference, or change, in the C3 values would, counterintuitively, require addition. Therefore, I cannot say at this point whether his MA scheme would be viable in general. In a way, it may have fortuitous that he had an easier version of this problem the first time because although he may not have come up with a generally viable strategy, his success at coming up with an explanation undoubtedly gave him confidence to reason through future MA problems.

Even more impressive to me than Adam and Lily's ability to solve the MA problems was their ability to figure out possible card values given the two relevant C3 scores and even figure out an exact card value given the two relevant C3 scores and one of the card values. This stands in contrast to Justin and Michelle, who had trouble with these kinds of problems. In Adam's case, he seemed to very definitely break the problem of determining a missing card given one card and the C3 values into two simpler, solvable problems. The first problem was to figure out the C2 score based on the previous and current C3 scores, which he could now do, in certain situations at least. He would always write his C2 value down before thinking about the cards. Determining the card(s) given the C2 value was definitely within his capability, as I noted earlier. Lily must have solved the problem in more or less the same manner, but she did not need to write down her intermediate C2 value to find card values or explain her solution. Therefore, I think she maintained some level of awareness of all of the quantitative relationships she constructed throughout her solution process.

Notation, language, and explanations. Adam and Lily's language with reference to the signed quantities followed a similar pattern to Michelle. Sometimes they preferred to work with positive quantities. For example, Adam said, "She would get 11," instead of "I lost by 11." Lily's quote in the last protocol, "I won that round because, because his won, win amount of points went down," indicates that she is thinking about Adam's positive C3 value as decreasing instead of her negative C3 value as increasing in order to determine that she won the round. In addition, she sees the C2 value as representing a change in C3 value, but she refers to the C3 value as representing "points." Adam, on the other hand, would sometimes refer to the C2 quantities as his "points" instead of how many points he won or lost by (as in the quote, "She would get 11" instead of "She won by 11") as well as referring to C3 values as a "score" as opposed to a win/loss margin. However, they did both differentiate between the different quantities fairly clearly in their language, using "won/lost" for C2 values and "winning/losing" for C3 values, for example.

Unlike Adam's first session playing the card game (with Michelle), he began using W and L in front of his C2 scores without prompting from the first session with Lily. Lily and Justin had never kept score in the card game, and Lily did not initially use W, L, or any other notation to differentiate between values that represented a loss and values that represented a win. However, when I prompted her to do so by saying, "Did you win or did you lose?," after she wrote down her score, she immediately started using W and L for her C2 values. She also started using them for her C3 values without prompting. In addition, she described the C2 values as increases or decreases in the C3 values, so I hypothesize that she was constructing the C2 values as signed quantities from the beginning of March 3. Interestingly, Adam did not use his W/L notation for his C3 values initially. For example, after Round 4, his score sheet was as in Figure 7.4.

Round	How many points did you win or lose this hand?	How many points are you winning (or losing) the game by?
1	W86	86
2	L14	42
3	L10	32
4	W12	44

Figure 7.4. Adam's score sheet after four rounds on March 3.

He was winning the game after all four rounds, so my hypothesis is that he was thinking of the C3 value as his score, i.e., the actual number of points he had, an unsigned value. The C2 values, which he seems to be notating as signed, then, would be actions on his number of points, where W86 is adding 86 points and L14 is taking away 14 points. He did not have trouble determining the C3 value the first time he was losing the game. He simply said, "I'm losing by -21," indicating that his interpretation of the C3 quantity allows for negative values (possibly thinking of them as how many points Lily has). However, he cannot explain how he figured out the answer. Furthermore, as I discussed earlier, once he had both positive and negative C3 values, he tried to combine the most recent positive and negative values to figure out his new win/loss margin. Therefore, he does seem to be thinking of the C3 values as a number of points at times, consistent with his initial lack of signed notation.

In MA situations, we have seen explanations of the sign of C2 values from both Adam and Lily based on changes in their C3 values. In addition situations, they were also able to explain how they got the sign of their answer based on comparing the relative sizes of the wins or losses involved. Lily was able to do this right away, whereas Adam appears to appropriate Lily's explanations. Both Adam and Lily had difficulty in explaining why they were adding or subtracting in signed addition or MA problems. Adam, if he tried to explain his reasoning, would focus on how he carried out the unsigned addition or subtraction. He was not able to explain why he added or subtracted. Lily made a little more progress in this kind of explanation in that she could sometimes explain why the operation she did *not* use was inappropriate. For example, on March 8, she was losing by 23 points and then won the next round by 26 points. When I asked her why she subtracted 26 – 23 in order to determine that she was winning by 3 points, she said, "Because if I won by the 26 in that round, but he was already winning by 23, I can't be winning by 26. So you'd have to count the difference and I subtracted. The difference means subtract." In other words, she knew that she would be winning, and she knew that she would be winning by less than how much she won by in that round because Adam had more points than her going in. Therefore, she knows not to add to get her winning score because it would be too big. As you can see, her explanation is not very clear. In future contexts, we will see her further develop these kinds of explanations.

Summary of Adam and Lily's ways of operating in the card game. Adam's ways of operating during the card game seemed to evolve as he began attending more explicitly to subsets of his number sequence that represented "how far apart" numbers were. He did this when finding unsigned differences, which seemed to lead to his remarkable development of strategic reasoning within a late-stage TNS number sequence. I hypothesized that this was made possible by a more explicit aware of the nested nature of his composite units within his number sequence. However, his explanations still indicated an attention to the act of counting and the sequentiality of his number sequence, leading me to hypothesize that he had not yet reorganized his TNS into an ENS by applying his unitizing operation to his arithmetic units to form iterable unit items. As I discussed in Chapter 2, the ability to explicitly attend to the subsets relationships of a number sequence in order to strategically reason in the ways that Adam did would seem to require an abstraction of the unit item so that the student is able to condense the amount of information he or she is holding in his or her mind's eye. Based on my experience with Adam, I would hypothesize that it is possible for students to become explicitly aware of the nested nature of their number sequence at a late TNS stage, but the key constraint remaining is that they can still not disembed subsequences. For most students, that inability would be enough to make the development of strategic reasoning schemes unlikely in that they would not be able to construct a subsequence representing the difference, operate on that subsequence in order to enumerate it more easily, and then compare the two subsequences to determine how to get back to the original difference. Adam was able to finesse his inability to disembed subsequences and operate on them by applying his intuitive sense of inverse operations, such as counting up and counting down, in order to keep track of how he could undo his operations on the subsequence representing the difference.

Adam also attended to differences in a more explicit way in the MA and missing card problems. However, he only had been faced with MA tasks in which the beginning and ending (given) values were win margins. Therefore, I cannot yet conclude that he has a viable signed MA scheme.

Lily had, so far, shown no difficulty in operating in signed addition and MA situations. In unsigned situations, her explanations were very clear, referring explicitly to quantities and her operations on those quantities. In signed situations, her explanations were still developing. It was difficult for her to explain why she would add or subtract to get a signed answer. The few times she does make a non-procedural explanation, she explains why the other operation would *not* work as opposed to explaining why the appropriate operation does work.

Finally, both Adam and Lily appear to oscillate between thinking of the C3 values as an amount of points and a win/loss margin. Although I want them to be able to construct a win/loss margin as a quantity, there are situations in which it makes sense to think of the C3 values as points in order to simplify reasoning. Therefore, the only problem I see with this dual view is when the students are not able to differentiate between when it is appropriate to think of C3 values as points or not. Adam had some difficulty with this in the card game. However, both students generally seemed to have constructed viable quantitative understandings. In particular, Adam seems to have moved from a more intuitive sense of the relationship between C2 values and C3 values to a more explicit awareness that C2 values represent an action on the C3 values as he worked on the first signed MA problem.

Working in a Money Context

As I discuss in Chapter 6, we were having trouble getting Justin and Michelle to reason on their number sequences to determine differences and sums during the card game. Based on our experiences (and others' research) that students are more likely to use self-generated solution strategies when working with money, we decided to start working in a money context with them in which they were combining increases and decreases in savings to find the total change in the savings plate's value. I also simplified the additive relationships in the money context, so that there was no cumulative sum of changes, such as the C3 quantity in the card game. This was, in part, to investigate whether Justin's unexpected difficulty with the C3 quantity in the card game was based on the cumulative nature of the quantity, or whether it indicate a more general way he was assimilating signed sum situations. Although Adam and Lily were using strategies much more readily in the card game, Adam had to be constantly encouraged to do so. Furthermore, although he was not having difficulties to the extent that Justin was in the card game, I still was not sure how Adam was thinking about the MA problems in part because he had trouble explaining his solution strategies in the card game. I thought that it might be easier for both him and Lily to reflect and communicate their reasoning about the simpler signed additive relationships in the money context. We spent four teaching sessions in the money context.

During the card game, Adam had stayed fairly engaged, most likely because he was very interested in whether he was winning or losing the game: One day when they ended on a tie, he came in the next day and reminded me that they were tied. Another day when he was winning, he happily declared his good fortune made up for the previous day in which he had ended up losing by quite a bit. In the money context, he very much enjoyed working with the coins, but he started to disengage more often when we stopped acting out the tasks with actual coins. This was no doubt due in part to his difficulty in constructing the requisite quantities using only re-generated figurative material. An example of his increasing lack of engagement, as well as his positive rapport with Lily, was when Lily had just answered a question and Adam, having not heard the answer, comes up with an incorrect solution on prompting by the teacher/research. Lily says, “You know, it’s hard to believe I just said it and he wasn’t paying attention and now he didn’t know the answer.” Adam responded sheepishly, “I was staring off in space!” Lily really did seem surprised that Adam was not paying attention, indicating that this had not been characteristic behavior for him in previous teaching episodes with Lily. The fact that Adam did not get upset or flustered when she chides him is evidence of his comfort level with her. With Michelle, he would often retort with a sharp remark or attempt to defend himself.

Unsigned Addition Schemes

The first day in the money context, March 15, Adam and Lily worked with actual coins. We did the same kinds of tasks that Justin and Michelle did on their first day working with money. In particular, we mainly worked in unsigned contexts and I explored how they used addition and subtraction notation when working with unsigned quantities.

There were several indications that the two students were operating differently in addition and subtraction situations. This became particularly evident when they were working with notation. Even before we started notating, though, I asked them to make a pile of 35 cents and a pile of 51 cents and figure out what they had to do to make them worth the same amount of money. Both students added 16 cents to the pile of 35 cents and gave me multiple ways they do so using pennies, nickels, and dimes. With three piles in view from their work on that problem—a pile of 35 cents, 16 cents, and 51 cents—I reviewed our conclusions: “If I put these two together [pile of 35 and pile of 16], it’s worth the same as this [pile of 51]? You had to add 16 pennies or 16 cents.” Right after saying that, I remove the pile of 35 cents and ask, “Now, what if I gave you the problem: You have 16 cents and you have 51 cents and you need to figure out how much to add here [16] to make it worth the same as the 51 cents.” Lily immediately answers, “35 cents.” Adam asks me to repeat the question, and then he begins telling me coins to add. However, his coins, “25 cents, a dime, a nickel, and 6 pennies,” do not add up to 35 cents, and he has to start over to figure out a quarter and a dime alone would work. Furthermore, he never says, “35.” Lily, on the other hand, references the previous problem after Adam’s solution: “Well, we just took away the 35 cents, ‘cause we had to add 16 to get the 51.”

Lily knew that $C - A = B$ implies $C - B = A$, whereas Adam did not seem to recognize the implication even after Lily’s explanation. The related addition expressions would be $A + B =$

C implies B + A = C (or the converse, depending on how the student is thinking about subtraction), and if I think about why the original implication holds, I would base it on the common additive relationship underlying both quantities. Therefore, I hypothesize that Lily is aware of the common additive relationship underlying all four of these equations, whereas Adam did not assimilate the two problem situations as additively related.

I know from Adam's ways of operating in missing addend and subtraction problems that he can utilize either a sense of taking away or adding up from to solve. For example, later in this episode he explained how to solve 8 + ___ = 32: "You could subtract 32 minus 8 because to find how much [circling blank], or you can add up 8 until you get to 32." Assuming he meant "add up *from* 8," he is showing that he understands that finding the distance between 8 and 32 and finding the number 8 away from 32 give the same solution. This has an underlying relation to knowing 8 + ___ = ___ + 8. Therefore, I think Adam's failure to use a related result to solve 51 – 16 is based on the way he made sense of his solution to 51 – 35. My hypothesis is that while he can count his counting up actions to get 16, and based on his strategic reasoning elsewhere, he could attend to the 16 as enumerating a subsequence of 51, but his inability to separate his number concepts from sequential counting acts does not allow him to disembed a subset of 16 and 35, recombine them, and compare them to 51. Instead, I hypothesize that he is aware of a composite unit of 16 as the enumeration of a subsequence of counting acts, but he is not aware of it as a quantity that can stand in a structural relationship to other quantities. It is as if his quantitative relationships are based around the ability to get from one number to another. For example, 35 + 16 = 51 would imply that if you add 16 past 35, you get 51. It would not imply that the union of a set of 35 and a set of 16 has the same numerosity as a set of 51. He would know that the numerosities of the sets are the same, but it would result from his assimilation of those sets with his sequential counting sequence. Therefore, I hypothesize that he assimilated 16 as the enumeration of an action applied to 35 to get 51. I think he could have found an explanation for how to use the 51 – 35 solution to solve 51 – 16. However, the relationship would have to be constructed, it is not inherent in his addition and subtraction schemes.

Lily, on the other hand, probably constructs structural relationships in that in solving either addition or subtraction tasks, she is constructing the underlying subset relationship. In fact, I present further indications that this is the case. I call the additive structure she constructs when adding or subtracting a *generalized unsigned sum structure*: *generalized* because the same relationship is constructed in addition and subtraction situations.

When asked to notate the problem of equalizing the piles of 16 and 51, Lily writes, "16 + ___ = 51," and Adam writes, "51 – 16 = 35." Both answers support the view that Lily was solving the problem based on an additive relationship she formed in the previous problem, whereas Adam is focused on the action he used to solve each problem. Also, note that he felt compelled to write the answer. This is not an isolated incident. On the other side of the paper, I had a series of addition and subtraction problems with various terms missing, and he immediately asked if he could write all the answers in. This points to his view of addition and subtraction signs as prompts for action, not indications of an underlying additive relationship.

After Adam had filled in all the blanks in the addition and subtraction equations, I asked them to come up with word problem for the equations: "This is the opposite thing that you just did. This time I'm giving you the number sentences, and I want you to make up a situation for the number sentence. You can use coins still." The first equation was 35 + 16 = ____. Instead of writing a word problem, he first put down two groups of coins with a addition sign between them. When I ask him what question he was asking, he said, "35 cents plus 16 cents equals 51 cents," and then repeats it to Lily as, "35 plus 16." Lily, on the other hand, wrote, "Mark has 35 baseball cards, Maggie has 16 baseball cards. How many do they have all together?" After seeing what Lily did, I was hoping he would come up with a quantitative situation for the next equation, as opposed to what amounted to rewriting the equation with coins or re-stating it with "cents" added in.

The other next equation we worked on that day involved a missing addend, namely, 35 + ___ = 51. Adam had a lot of trouble with this problem. He first came up with, "Joe has a piggy bank with 51 cents. Bobby Joe, he's a farmer, has a piggy bank of 35 cents. How many would they had, have, if you subtracted them together?" Again, here, he is focusing on what he has to do to solve the problem—subtract. However, the difference does not correspond to any quantity in the problem. Going back to Adam's reversible subtraction and addition scheme, he could solve this problem given a problem situation with a difference, so he can construct an unknown quantity as representing the result of subtraction and solve for it. However, he has trouble creating a problem situation with a contextualized, unknown quantity as an addend or difference. My guess is that this is because he has to construct the additive relationship as he solves the problem. He does not come to the problem aware of possible additive relationships before he begins operating. Therefore, he does not have an assimilatory difference structure that he can use to build a problem out of.

After Adam's first attempt, I encourage him to think about it as an addition problem. He shuts down, and says, "I don't know. I'm confused," so I walk him through a problem.

Protocol 7.4: Adam comes up with his first missing addend problem.

T: All right, Bob has a piggy bank with…
A: 35
T: 35 cents in it. And then what is this going to be [pointing to blank]?
A: How much Bobby would have.

T: How much the other person has?
A: Yeah.
T: OK, and then what's this going to be [51]?
A: How much they have together.

I then have him read another problem that Lily had written in the meantime: "Bob has 35¢, Amber gave him some amount for him to have 51¢. How many ¢ did Amber give him?" Adam is impressed, saying, "It's better than mine." He then comes up with, "Daniel has a piggy bank with 35 cents in it. Billy Bob Ansley Joe is waiting on her farm for Daniel to come pick up some money so he has 51 cents. How much is she going to give him?" This problem is a hybrid of Lily's increase and his previous problem, so I do not think this indicates that he could come up with such a problem on his own.

Signed Addition Schemes

For the next three episodes, the students spent the entire time on situations in which they were constructing some form of signed quantities. In particular, we had a plate that represented a piggy bank, and the students had money in the plate they could take out and they had money of their own they could put into the plate. We first made the situation comparable to the card game in that the students would be given a single change in the plate's value, parallel to a round score in the card game, and would be asked to hypothesize about the plate's value before and after the change, parallel to determining the cards' values in the card game, which both students were adept at. This was done to form a sense of continuity to the card game, but it was also done in order to establish that we did not need to know the exact value of the plate before or after the change in order to make sense of the change as a quantity.

For the remainder of the students' work in the money context, the problems involved two people making independent changes to the plate's value, resulting in a net effect on the plate's value after both changes. Within these problems, there are two main categories: addition problems and missing addend (MA) problems. Addition problems involved combining two known changes to determine the net effect. Missing addend problems involved determining one of the original changes given the other original change and the net effect. We started with addition problems, but during the third and fourth episode in the money context, we worked on addition and MA problems.

Both students were able to combine an increase of the plate's value followed by either an increase or decrease. Both students could also combine two decreases. However, only Lily gave convincing indications that she could reason through an addition problem in which a decrease is followed by an increase. There is only one such problem that I probe Adam's thinking on in this context, although I analyze Adam's solutions to similar problems in the number line context. Protocol 7.5 gives the discussion for the first problem on March 22. Lily had decided to decrease the plate's value by 22 cents, and Adam decided to increase the plate's value by 70 cents. Although I had the coin jar with me, we did not act out this problem.

Protocol 7.5: Adam's solution to combining a decrease followed by an increase.

T: [Writes, "-22 ☐ +70."][9] What's the total change in the plate's value?

A: […] 48 cents.
T: […] You said 48, right? Increase or decrease?
A: Increase.
T: Yeah. How did you know it was an increase?
A: Well, because we didn't go past 22. The answer wasn't past 22.
T: What do you mean by "the answer wasn't past 22"?
A: 22 was a decrease. So if it was past 22 then it'd be a decrease. I don't know.
…
T: When you said the answer isn't past 22, do you mean 48?
A: 48 isn't past 28 [sic]. It'd be a 48 increase. 'Cause you subtracted 22 and you don't go past it [motioning to the left]. I don't know.
T: Past what?
A: 22.
T: OK. What would be an example where you would go past?

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[9] The actual notation we used for signed addition was an addition sign with a circle around it. I used this sign because I wanted to avoid confusion with unsigned addition.

A: 70 and 62? […] [Writes, "-62 □ +70."]
T: OK. Here you go past 62?
A: [Nods.]
T: Tell me what you mean by that in this one.
A: Because 70 minus 62 is 8, so it'd be a decrease.
T: It'd be a decrease?
A: I don't know. You make her do those. I can't do those.
T: All right. First before, I'm gonna have her explain to me the first one, but in the meantime—
A: I still don't get it.
T: Is this an increase or a decrease [-62 □ +70]?
A: An increase. […] 'Cause you don't go below 0.

The fact that I did not have the students act out this problem with coins may have contributed to Adam's confusion. He seems to know that he will need to subtract 70 – 62, and he had done many mixed-sign addition problems in the past two sessions, so he would probably have an intuitive sense that mixed-sign addition problems require subtraction either from pseudo-empirical abstraction of his previous results when acting the problems or from a sense that the actions will partly cancel each other out. Although he had never used any language such as "cancelling" and had never described the idea of an additive inverse, he did model a couple mixed-sign addition problems in which he actually matched up coins that were being put into/taken away from the plate to show the answer. In any case, he does subtract the magnitudes of the changes. However, for the sign, he compares the absolute value of the resulting difference with the absolute value of the first change in value. He has, in the past, explained how to get the sign for a mixed-sign addition by comparing the relative sizes of the *addends*, so, without the coins to work with, he might be relying on a sense of how he solved these kinds of problems: He knew that he had to subtract and compare absolute values. When he explains, "22 was a decrease. So if it was past 22 then it'd be a decrease. I don't know," he appears to experience perturbation. I think he realizes that he is using the 22 in a couple of ways, because he seems to be thinking about subtracting 22, "'Cause you subtracted 22 and you don't go past it [motioning to the left],” so 22 is enumerating a subsequence from 70 down to 48, but then he is also comparing 48 to the *position* of 22 in his number sequence. This could be related to occasional past uses of a change in quantity as a quantity, so that he was thinking of -22 as both a position and the numerosity of a subsequence. At this point, I do not have indications of what led him to recognize that 0, not 22, was the appropriate reference value for the sign his answer in the problem he made up, -62 ☐ +70.

Adam had trouble with missing addend problems in this context, although he did not have any MA problems where both given values are positive. Possibly he would have been able to do those kinds of problems in this context since he had been able to do such MA problems in the card game. I discuss Adam’s solutions to the MA tasks in the next subsection. Lily did not have trouble with any of the missing addend problems here. At this point, put together with what she had done in the card game, I feel comfortable attributing a viable MA scheme to her.

I also had an indication that Adam’s signed addition scheme is not anticipatory. On March 16, I said to him, “This time, Adam, you’re going to take out 50 cents. And Lily, you’re going to add 94 cents….Do you know what question I’m going to ask at the end, Adam?” He replied, “Nope,” but then immediately changed his mind, “Yes! How much did y’all take out toge—. No. How much did Adam take out and, when he took that out, you added that, so how much could there be?” Notice that he ran through the problem situation in order to get to the question. My guess is that he is visualizing the situation and constructing *the quantity* that he wants to solve for as he does so. A different day I ask Lily the same question and she instantly knew what I would ask. I do not take that as evidence that she has an anticipatory signed addition scheme because she could have just noticed a pattern in my language, but she did not exhibit the same contraindication. Also note the way Adam words his question, “How much could there be?" He makes it sound like he is thinking of the total change as an amount of coins somewhere. In other words, he is not thinking about the answer as a change in value, but as a pile of coins, similarly to how he would often refer to his C3 value as his "points" or "score."

Lack of reversibility. On March 17, Adam and Lily do one MA problem. In this problem, Adam was working with two given decreases and he did not give the sign of the answer, just the magnitude. March 22 is the first and last session in which Adam and Lily did mixed-sign MA problems in this context. They only did one series of related mixed-sign MA problems that day, and, in all cases, Adam subtracted the absolute value of the given values. He also always thought that the missing value would be a decrease, at least initially. I will present a series of protocols describing discussions about these problems. For the entire teaching session, we had been doing tasks in which Lily and Adam were working together to do one action and I was doing the other action. In the first protocol below we were discussing the following problem situation: Adam and Lily had decided to increase the plate's value by 52 cents, and they had to figure out what I did. I had told them that, in all, we decreased the plate's value by 12 cents, and I had written, "+52 □ K = -12," on an index card. Adam writes down the answer, "40¢ decrease." He then comes over to me and asks me if the answer is correct. I tell him to "double-check it" and, specifically, "If that's what was here [where K is], …work out that problem and make sure that's what it gives you. He then goes back to his seat, thinks for about 10 seconds, and nods.

Protocol 7.6: Adam's first mixed-sign MA solution.

T: Is my action, just looking at these numbers here [+52 □ K = -12], going to be an increase or decrease?
A: It's going to be a decrease
T: How do you know just from looking at these numbers?
A: 'Cause it's a 12 decrease so your number had to be a decrease.
T: […] And is my number going to be bigger than your number or smaller? […]

A: Smaller.
T: And why would it be smaller?
A: Because the number, it's minus 12, or 12 decrease. […] So if I was increasing it and you were decreasing it, your number would be smaller than mine 'cause mine's increasing it [pointing up] and yours is decreasing it [pointing down].
T: What if I had decreased mine by 30?
A: It would've been 22.
T: Would it be an increase of 22 or a decrease of 22?
A: In—, decrease. […]It'd be the same thing.

Because Adam wrote down an answer with the correct orientation, I assumed at the time that he realized he needed to do a decrease in order to end up with a decrease. I took his first explanation, "'Cause it's a 12 decrease so your number had to be a decrease," to be confirmation of this, although, in retrospect, this statement could reflect a number of ideas: For example, he could have thought that the second addend and the sum have to have the same sign, or he could have compared 12 and 52 and have decided that the second action decreased the value. The first possibility would fit with his behavior on March 8 when he conflated the signs of the C2 and C3 values, and the second possibility would fit with his behavior in Protocol 7.5 in which he was using the first addend as a reference point. However, the first possibility does not fit well with his other statements and ways of operating: In the card game MA problems, he did not necessarily end up with the same sign for the missing addend as he did for the sum. The second possibility would fit with all of his MA solutions so far. The question is how he was interpreting the -12. If he was just comparing absolute values, I wonder what role the negative sign played in his reasoning, if any. My guess, based on his strong association of subtraction signs with the act of taking away, decreasing the amount of something, in his unsigned subtraction schemes, is that he assimilated this task as he would the task, 52 – ___ = 12. So then the -12 could be read as signifying, "a decrease *to* 12" more so than a decrease *of* 12. I do think there was an aspect of conflating the sign of the missing addend with the sign of the sum, as in the first possibility, when he checked his answer. He would have been solving the problem, +52 ☐ -40, and I can certainly imagine him thinking, “Well, 52 – 40 is 12, so it was a decrease and it was 12.” I also think that 52 being positive here facilitated his confusion about the task because he could think of +52 as just a pile of 52 cents, an unsigned quantity. If it had been -52, I am not sure if he would have still thought about this in terms of his unsigned subtraction. Unfortunately, he does not do such problem in this context.

At this point, as the teacher/researcher, I wanted to engender additive inverse reasoning in him so that he would be aware of how the missing addend would have to compare in magnitude to the given addend. Therefore, I made the following intervention.

Protocol 7.7: Adam’s competing schemes.

T: What would I have to do to get this [-12] to be a 0?
…
A: Oh, it’d, it would be 52.
T: It would be 52. Decrease, right?
A: Decrease.
T: OK. What if I did a 51 decrease?
A: That’d be a 1 decrease.
T: If you put in 52…
A: And you put in 51.
T: [Shaking head] I took out 51.
A: You took 51, then it’d be a 1 cent increase.
T: You sure? Lily? Increase or decrease?
L: Increase.
T: Yeah, why is it an increase?
A: Can you ask her? […] My brain is smokin’.
T: Your brain, smoke coming out of your ears? All right.

There are two aspects of this protocol I will highlight. The first is that his answer of “1 decrease” fits with my interpretation of how he was thinking about the orientation of the sum. In other words, he has decreased *to* 1, so he is comfortable calling that a 1 decrease. The second aspect is that I, the teacher/researcher, switched from an abstract language of *increases* and *decreases* of an unspecified quantity to references to our past practice of putting coins in the savings plate or taking them out. I hypothesize that this shift in language prompted him to run through the situation in visualized imagination, allowing him to access his former schemes that he used when working with coins. However, this seems to be difficult to do and/or it produces a sense of perturbation at the different results of his different ways of assimilating the situation. As he says, "my brain is smokin'."

Protocol 7.8. Adam assimilates Lily's reasoning.

T: Tell us why it's an increase.
L: If we put in 52 cents, and you took away 51 cents
T: Here I'll write that down. [Writes, "+52 □ -51."] Yep.
L: Then you took away one less than we added in, so overall we would, if we put in 52 and then you subtracted 51 from what we needed, from overall, or, if we subtracted 52 minus 51, it would be 1. So…
T: Go back to your first explanation, I think that was making more sense to me.
L: If we took out, we put in more than you took out by one cent, so we increased it over all by 1 cent.
T: Yeah, that makes sense to me. Does that make sense to you, Adam? [I've written +1 under the underlined expression.]
A: Mm-hmm.
T: You sure? All right. Well then what about [writing, "+52 □ -55," and underlining]? What would that one be?
A: It'd be -3!
T: A decrease of 3? How d'you know it's going to be a decrease?
A: Because 55 is bigger than 52 and we're adding 52 in, and you're taking 55 out. So it'd be a 3 cent decrease. 'Cause we add 52, and you take out 55. You're taking out 3 cents of our money.

In this protocol, Lily continues to use language that elicits a visualization of physical actions with the coins. I steered her back to this action-oriented language and away from her comments about subtracting because, at the time, I had an intuitive sense that the first kind of language would be more helpful to Adam. In particular, I had the sense that he was getting his unsigned subtraction confused with the negative signs I was writing. Indeed, Lily's explanation does seem helpful to him, or at least it seems to have prompted him to visualize working with coins again, because he unexpectedly gives a very coherent explanation of his answer. Notice that the 3 cents is not just a decrease, but what I am “taking out … of [their] money.” He is very much picturing actions with coins and using those to assimilate the written notation at this moment.

Protocol 7.9: Adam and I discuss the sign of a missing addend.

A: Can I just go ahead and tell her the answer [for K in +52 □ K = -12]?
T: Yes.
A: [Writes, “-40,” by K.]
T: Here’s my issue.
L: That’s not what I got.
T: You guys were putting in more than I was taking out.
A: Yeah.
T: Then we would get…
A: 12.
T: An increase.
A: Yeah.
T: It’s a decrease [pointing to -12].
A: Right.
T: Something’s wrong!
A: What?!
T: Look, we’re put—, you—
A: But 52 minus 40 is 12.
T: But you put in more than you took out.
A: I’m confused.

Protocol 7.9 is a continuation of Protocol 7.8: Directly after Adam gave his explanation for -3, he asked if he could show Lily “the answer” to the original MA problem from Protocol 7.6. He clearly thinks his answer is correct both because he calls it “the answer” and because he is excited to show Lily. This implies that he did not become aware of a contradiction in the two ways he was interpreting the sign of the sum. Therefore, when he said, “My brain is smoking’,” he must be referring to trying to apply his schemes he used when working with coins to a visual re-presentation of working with coins. By the end of Protocol 7.9, he *was* confused, experiencing perturbation, but it is not clear whether he was confused about why I did not think his answer was correct or whether he was confused because he saw that his ways of operating had given him an unviable answer. Unfortunately, the bell rang at that moment, and so the discussion was cut short. This was also the last day I was planning to work with them on problems involving money because I wanted to have a chance to gather data about how they interacted with a number line-like model for signed quantities before the end of March, when both students would switch to other activities during the research time. Therefore, I did not follow up with Adam on the MA problems the next day either.

There is an indication in this protocol that further supports my view that he had competing ways of thinking about the sign of the sum. Namely, he was aware that the “total change” would be an increase in the plate’s value, but he also thought it was a decrease. Again, I hypothesize that when he was saying decrease, he meant “decrease to.” This hypothesis fits with his statement, “But 52 minus 40 is 12,” which he thought explained why his answer did represent a decrease. I conclude that the decrease represented the action he took to get *to* the (positive) 12. This way of thinking about the situation is related to conflation of the directed length of a trip and where the trip ends in relation to 0 that I saw in Brad in the pilot study in that the action that is used to get to the sum is used to name the sum. Therefore, I was very interested to see how Adam would interact with diagrams of directed trips.

On the second day of the number line context, March 24, two days after Protocols 7.5–7.9, the students did do more MA problems, which they call “the secret game.” While the teacher/researcher talked about the game in terms of trips on the vertical pole, Lily explicitly talked about “cents” at one point, and my guess is that Adam was also thinking about these problems in terms of money at first. He clearly links what he is doing on March 24 to playing “the secret game” on March 17 and 22 because he repeatedly corrects the teacher/researcher on the proper way to play. Therefore, I will present the results of my analysis of these tasks at this point also.

In the first problem, the teacher/researcher told Adam the sign for the missing addend, which involved getting from a “100,000 decrease” to an overall decrease of 99,975. Adam gets the magnitude. However, when he does not know the sign for two similar MA problems, he still gets the magnitude correct, but compares signs of the absolute values of given quantities. In the first problem, Lily tells him that his answer is only partly right, and he immediately changes the sign of his answer, implying that he is confident about the magnitude, but not the sign. In the next problem, “She decreased 100. Now then she moves again and now she’s at 50 decrease,” Adam gets 50 decrease as his answer and insists that “she went down 50.” The teacher/researcher intervenes and acts out the problem situation by moving his finger up and down a pencil. This does appear to reorient how Adam is thinking about the sign, and he agrees that she went up 50. He also gets the next MA problem correct. When he was saying, “she went down 50,” he seems to be thinking about the absolute value of her change in position, that is, “How much below the surface is she.” In his mind, that quantity is going down, getting smaller. However, two problems later, on the last MA problem in the session, he again refers to her going down when she is decreasing her distance below 0.

Adam’s solutions to these MA problems on March 24, imply to me that he is still assimilating positive and negative values as representing two related, unsigned quantities: the distance above the surface and the distance below the surface. In fact, that seems to be the way he assimilates the values for the first addend and the sum. Like in his unsigned addition and subtraction schemes, the second addend is considered to be the action or change, and so he looks to see whether he is subtracting to get the answer or adding to get the answer in order to determine the sign of the second addend.

Explanations

Lily could solve addition and MA problems, so, when analyzing her mathematics, I am more interested in how she talked about the problems, in particular, to what extent she reflected on the quantities she was constructing and their quantitative relationships. As in the card game, she continued to describe holistic comparisons to explain the sign of a sum. For example on March 16, I asked, “Is the plate worth more or less now?” after two actions, and Lily replied, “More… because he added more than I took away.” I call this kind of comparison *holistic* in order to bring attention to the possibility that the student is not thinking about a numerical relationship between values, but simply which kind of change is bigger. Note that this kind of explanation would also be possible if the students are thinking about two unsigned quantities instead of signed quantities. A holistic comparison does not focus on the sum as a quantity, and, in particular, the additive relationship between the absolute values of the addends and sum remains implicit. If a student is constrained to think about a problem in this way, the student will often not even attempt to explain why she or he added or subtracted to get the magnitude of the sum. If such an explanation is attempted, the explanation will often explain why the *other* operation is not appropriate. For example, to find the total change after a 22-cent decrease and a 70-cent increase in the plate’s value, Lily subtracts. When I ask her why she subtracted, she says,

> Because I took some out so I would add if I put that in and he put the 70 cents in, but he put the 70 cents in and I took the 22 cents out. So in all we didn’t put in, uh, 92 cents. We had to take 22 away because that’s how much I took out in the first, first, before he added it.

Therefore, she is subtracting because this is not an addition situation.

Lily's explanations for her solutions to signed addition tasks did evolve during her work in the money context. Consider her explanation in Protocol 7.8: "If we put in 52 cents, and you took away 51 cents, then you took away one less than we added in." Here she is attending to the numbers and describing the difference as a quantity: how much less I took away than they put in. I call this kind of comparison a quantitative comparison to bring attention to the students' awareness of the difference (in this case) as a quantity. Figure 7.5 illustrates how I think about the difference between the quantitative awareness implied by a holistic comparison as opposed to a quantitative comparison:

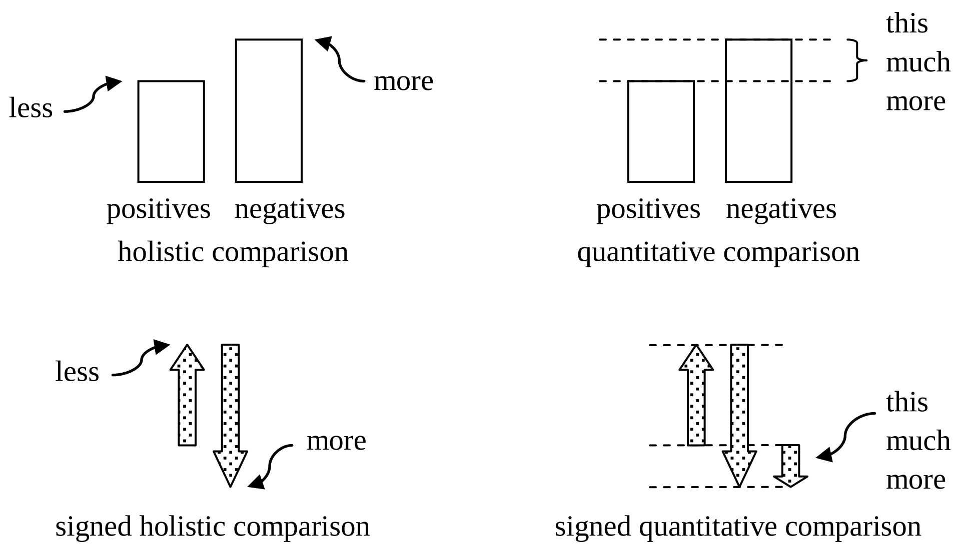

Figure 7.5. Types of quantitative awareness implied by explanations.

The same evolution happens with her explanations for signed MA tasks. For example, on March 22, I asked Lily why she added to find Adam's action when she had increased the plate's value by 32 cents and together they had decreased the plate's value by 468 cents. She said, "Like, it's [the total change is] below 1 or 0, then he's going to have more. Because if I increased it by 32, and it still wasn't enough to make it positive, then, it, my number has to be smaller than his." This is an example of a signed comparison in that she is referring directionality in order to explain the quantitative relationships. This is also an example of an intermediate step between her earlier holistic comparisons and a full-fledged quantitative comparison. She is, for example, attending to 0, but she refers only obliquely to Adam's quantity. In the very next problem, Lily decreased the plate's value by 10 cents and she knew that, along with Adam's action, the total change in value was an 8 cent increase. Lily explains, "Because it was an 8 increase, so his had to be 8 cents more than mine. 'Cause I was taking this one away." In this explanation, she is setting up a specific additive relationship between the absolute value of her answer and Adam's answer. This seems to strike a chord with Adam because he responds, "Oh, wow! Yeah. Hers had to be 8 less than mine 'cause it was an 8 cent decrease, increase, I mean." As in the card game, Adam seems drawn to focusing on how far apart numbers are, therefore, it is perhaps not surprising that he liked this explanation. Later, he mimics it when trying to help Lily find his action in one of the signed MA tasks on March 24: "If it's [the total is] blank away from your number, blank away from your number is my number."

Summary of Adam and Lily's Ways of Operating in the Money Context

Lily had constructed what I call a *generalized unsigned sum structure,* which meant that she constructed additive relationships when solving addition or subtraction problems, allowing her to be aware of the answer to any addition or subtraction problems based on the same underlying additive relationships. This construction also implies that she would be aware of the necessity of the commutative property of addition. Because this construction involves flexibly reordering and comparing quantities within a number sequence, an ENS would be necessary to develop a *generalized unsigned sum structure*. However, it does not automatically follow from the construction of an ENS because Michelle showed contraindications of having constructed such a structure. Unsurprisingly, then, Adam had not constructed a generalized unsigned sum structure. When he saw an addition or subtraction equation, it would represent an action that could be or had been carried out. I hypothesize that an addition or subtraction expression would be assimilated as a calculation to carry out and not as a yet-to-be-ennumerated quantity. This view of addition and subtraction also implied that he was not constructing a group of subsets in an additive relationship, as Lily did, instead he seemed to construct the first addend and sum as being additively related in that they were linked by an action of adding or subtracting.

Adam latched on in a couple of instances to explanations or ways of reasoning that focused on how far apart two numbers were. I hypothesize that this way of thinking about problems was exciting for him because it was within his ZPC and yet challenging. He could enumerate the number sequence to get from one number to another, but he seemed to construct that subsequence as a quantity apart from the action of counting. Adam seemed, in general, to be on the brink of reorganizing his TNS into an ENS in that he was becoming more explicitly aware of subsequences as distinct quantities from their containing sequences. However, he still appeared to be constrained to working within his TNS; he could not disembed subsequences and additively compare them back to the original sequence to form an additive structure, for example.

Adam had an easier time with the money context when he was acting out the problems with coins. Failing that, referring to quantities as actions on the coins, such as, “putting in *x* coins” or “taking out *x* coins,” seemed to facilitate his ability to construct/assimilate a viable quantity for the sum. However, using a re-presentation of working with coins appeared to be quite difficult for him. Perhaps that is why he developed a competing way of understanding the quantitative situation when he was not using coins. In particular, he assimilated the first addend and sum as unsigned quantities, and the second addend was assimilated as an action of increasing or decreasing. This is in keeping with his action-oriented idea of unsigned addition and subtraction. The tendency to assimilate the first addend and sum as unsigned quantities was also in keeping with his oscillation between thinking about changes in points and a number of points in the card game. In signed MA situations without coins, he would take the sign of the sum as indicating the type of action that was needed. It just so happened that this interpretation was consistent with the way in which the absolute values of the first addend and sum were changing in the problems he did. In other words, if A + B = C is the observer's underlying signed sum structure to the signed MA problem he was doing, then C would have a negative sign and |A| was larger than |C|, so the absolute value had decreased after the action of B.

Lily had no trouble operating in the money context. In addition, she develops her ability to explain her reasoning, which may correspond to a change in her awareness of the quantities and quantitative relationships she is constructing. In particular, she moves from holistic comparisons of the size of each kind of quantity to more quantitatively sophisticated comparisons in which she is starting to attend to the meaning of the 0 value and can refer directly to all of the quantities in the problem.

Trips on a Number Line

Neither Adam nor Michelle had developed sophisticated explanations for determining sums (or differences) by using additive inverses to decompose quantities by this point in the teaching experiment: Neither was attending to 0 or referring to the underlying subset relationships present in the signed situations. In the card game, there are not visible quantities that can be lined up in order to make subset relationships more visually salient. In the money context, we did try to engender these relationships by having both sets of students explain answers with coins, but this did not engender more explicit explanations from Adam or Michelle unless they were working with actual coins. In the number line context, students were representing linear trips, and so the overlap of directed trip lengths, which can be thought about as representing the addition of additive inverses, is visually salient. I hoped that this would provide Adam and Michelle with a catalyst for reasoning with additive inverses and a visual representation they could internalize and apply to signed situations more generally.

I introduced the new context to Adam and Lily on March 23. I asked them to imagine a pole that goes from the surface of the Earth to the center of the Earth and also extends out from the surface into space. We discussed why this scenario was not realistic, how long the pole was, how long a meter is, etc. I then explained that we would imagine the students taking trips up the pole and down the pole. Both students were actively involved in this introductory discussion and showed no indications that they were confused about the problem context. Adam and Lily had three teaching episodes together in this context, and then Adam had a final individual teaching episode in this context that I analyze in this section as well.

Adam's Conception of Length

From the first task I gave Adam and Lily, Adam had trouble visualizing or drawing the problem situations in a way that was intersubjective with Lily's and my understanding of the situation. Protocol 7.10 describes Adam and Lily's work on their first task in this context. I had already explained that the problem situation would involve Lily taking trips up and down the pole and that Lily would figure out the answer while Adam drew a picture of the situation. In fact, Adam volunteered to draw in order to avoid having to come up with a numerical answer.

Protocol 7.10: Adam and Lily's first combination of number line trips.

T: She goes 83m up the pole, she rests. She climbs 83m down the pole, she rests again, and then she climbs 1m down the pole. [Writes, "+83 ☐ -83 ☐ -1".]
L: [Smiles.]
A: What?
T: She climbs 83— You actually don't need to see it to figure out this one, huh? [Turns paper over.]

A: What are we trying to figure out?
T: We're going to figure out how far she is from where she started.
A: OK. That works.
T: Does that make sense? […]
L: I think I know.
A: Draw?! How can I draw it?
T: […] Sketch the pole and then sketch where she's going on the pole, and she'll fill in the numbers.
A: [Begins drawing the pole as in Figure 7.6.]
…
T: OK. Show me where she's starting on the pole.
A: [Draws the horizontal line that is later labeled "+83m".]
T: All right, cool. And then, do you remember what she did from there?
A: [Draws the two horizontal lines below the first one, now labeled "-83m" and "1m."]
T: What are those lines?
A: [Labels them "-83 m" and "1 m"]
T: What'd she do first? Do you remember what she did first?
A: [Writes "+83 m" by starting line.]

His incredulity at drawing the situation—" Draw?! How can I draw it?"—confused me at the time because he had been engaged and involved during our discussion of the general problem situation. For example, he gave possible lengths for the pole and even conjectured how long it would take to walk down to the center of the Earth. In retrospect, I think there were two factors at play in his confusion: his lack of a quantitative length construct that included a starting and ending point for measurement and his assimilation of at least the second trip as enumerating an action instead of a quantity of meters (being moved through). If his visualization of the trips was dominated by a sense of motion, then that would explain his confusion at drawing such a dynamic situation. I hypothesize that this alone was not the source of confusion. More important, perhaps, is the possibility that Adam was not assimilating this situation with a directed length quantity, including a reference or starting point, which, together with the length, determines the ending point.

Note that the starting point in the situation was left indefinite, so without the automatic assimilation of the situation with a length quantity that includes starting and ending values, he may have been confused about where to start drawing. However, as I discuss in the next paragraph, even after beginning to draw, his length construct causes difficulties. Although Michelle had a brief difficulty in drawing one of the number line situations, she did not struggle with this aspect of it. Adam alone seemed to have this difficulty of being aware of length, starting point, and ending point when quantifying the number line situations. However, this is akin to Brad's initial difficulties in drawing trips in the same context during my pilot study.

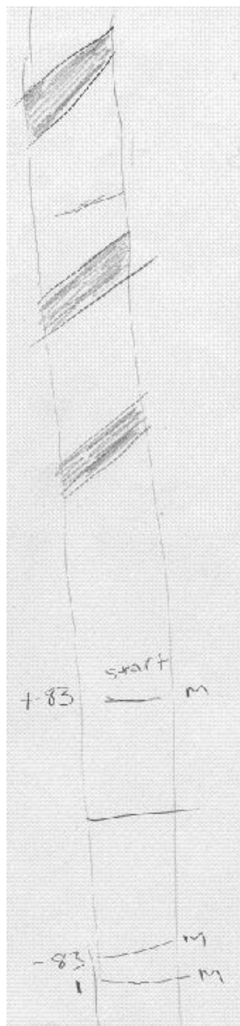

Figure 7.6. Adam's first drawing of trips on the number line.

When I gave Adam suggestions for how he can draw the situation—"Sketch the pole and then sketch where she's going on the pole"—he followed my directions exactly. That is, he drew the pole and then drew the destination of each trip, which is supported by how he labeled the lines. He had not yet labeled the "+83m" line as "start," so the first indication that he is not thinking about length in the way that Lily or I would, for example, is that he did not draw a starting point or a 0 line. Because of this, I am not sure the numbers in his labels were truly measuring a directed distance from the trips' starting points. He is the only participant I worked with in this dissertation study or pilot study who did not independently recognize these situations as representing a number line with 0 at the starting point. Even Brad automatically included a starting point; Brad was aware of the starting and ending points, he just tended to conflate them with the length of the trip.

Protocol 7.11: Adam finishes his diagram for Lily's trips.

T: OK. Show me where she's starting.
A: [Taps +83 line.]
T: OK. Write something there so we know that's where she is starting.
A: [Writes, "start".]
T: All right, and what does she do first?
A: [Points to +83.] She goes up [motioning up towards +83] 83.
T: OK. She's going to start here [start/+83 line], and she's going to go up 83.
A: [Nods.]
T: Draw me that trip.
A: [Walks up with his fingers and makes a horizontal line high on the pole.]
T: Mm-kay. And then what do you do next?
L: Then I rest and go down 83.
T: OK. Draw that trip.
A: [Draws over the start line.]
T: And then what did she do last, Adam?
A: [Makes horizontal line between start/+83 and -83.] She goes down 1.
T: How far are you from where you started?
L: Down 1m.
T: You're down 1m, and does that match up with your picture?
A: [Nods.]

Adam saw the first line he drew as being the end of the first trip, as evidenced by his hand motion that went up *to* that line when he said, "She goes up 83." Therefore, when he claimed that that line was the starting point, this confirmed earlier suspicions I had had about his first addends representing measures of states and not changes in position. In other words, +83m is how he was naming the first position. He was not aware of starting somewhere and then going up 83m until prompted: He seemed aware that the +83m line was the result of going up 83, but he did not give indications that he was aware of a starting point for the trip. Once I prompted him to draw Lily's trip of +83m, he added an new line to represent the end of the first trip, with no apparent sense of perturbation. He knew that the new line he drew was 83m up from the starting point because he knew she would go back to the starting point when she went down 83m.

In the second task of the day, Lily was supposed to draw the diagram and Adam was supposed to come up with a numerical solution to the question.

Protocol 7.12: Adam does not know where he started.

T: Adam, he's climbing around. First he climbs down the pole 70m, and then he climbs down the pole 19m, and then he climbs up the pole 33m [writing -70 □ -19 □ +33].

…

L: How many meters do you go down first?

A: 70? […] Done.

T: Done? All right, tell me how you worked through it.

A: 70 – 10 is 60, subtract that by 9, is 51.

T: Why are you subtracting? What are you subtracting?

A: So I took away 9 to get 10. 70 minus 10 is 60, subtract by 9, that's 51.

T: OK, and what are you trying to get? What does the 51 tell you?

A: What that equals [pointing to -70 □ -19].

T: I think this is the reason I probably shouldn't have written it down […] because it looks like a subtraction sign, right?

A: It *is*.

T: It's not here. Remember this isn't really an addition sign, this isn't really a subtraction sign. This is telling us you went down.

A: All right.

T: If you go down 70, you rest…

A: Right.

T: And then you go down 19.

A: Right. That's 51.

T: You've gone down 70, and then you went down further.

A: And then I went down 19. That would be 51. I've gone down 51 'cause 70 – 19 is 51.

T: Well, but don't worry about the "70 – 19 is 51." Just think about, in terms of the situation, if that answer makes sense. 'Cause you went down 70, if you went down 70 and then you went down 51, which time did you go more?

A: [Points to -70.]
T: The 70, right? You've gone down 70, then you go down even more.
A: And that's 51. Well, you're at the 51st meter, or, the 51 meter. After that 70 you go down 19 and that's 51.
T: But isn't the 51 closer to where you started than the 70 is?
A: No [shaking head].
T: How would that not be closer?
A: I don't know where I started.
T: No, you don't know where you started. That's a good point. All right.

When I was working with Adam on this task, I thought that he was simply confused by the notation. The notation does support his sense that he needs to subtract 19 from 70 to get the magnitude of the answer, as pointed out by his insistence that the negative sign is a subtraction sign. However, I think now that the problem is deeper than that. I have already indicated other instances in which he interprets a decrease as meaning a decrease in absolute value. Therefore, when I say, "you go down 19." I think he might have thought he went down in the amount he had traveled, i.e., he decreased the absolute value of his position quantity. I became very confused when he did not think that 51 was closer to his starting point than 70, but, in retrospect, he was probably perturbed by my question because he probably thought of himself as starting at -70m. My question, in that case, would be silly because clearly if he was not at -70m anymore, he was farther from where he started at 51. Again, the 70 to him does not seem to reference a signed quantity, which requires a reference point. Instead, could have represented the unsigned number of meters (possibly signifying discrete positions to him) in a trip. He did have a sense that he went down to get to the -70m, but he did not seem to have a starting point for that trip inherent in both his visualization and quantitative understanding of the situation. When he said, "I don't know where I started," I think this reflects the fact that he realized I am not referring to -70m as the starting point, and he did not have an alternative starting point in his awareness of the situation.

In order to better understand his quantitative understanding of the situation, I next asked him to draw the diagram himself, before interpreting Lily's diagram. His creation and explanation of the diagram is reported in the next protocol.

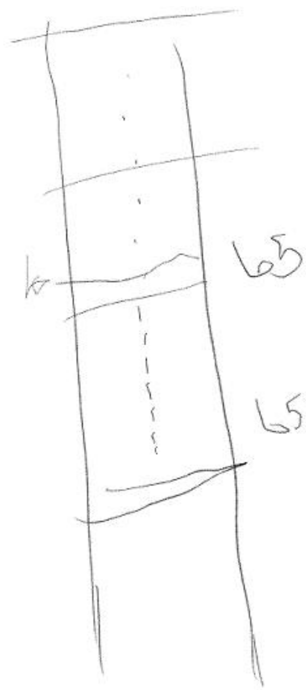

Figure 7.7. Adam sketches (-70) + (-19) and (-70) + (-5).

Protocol 7.13: Adam sketches -70 □ -19.

T: Draw me this trip [giving him a pencil]. You're starting somewhere, you're going down 70. Just draw me that part.
A: [Draws two vertical lines, seems to count down, but doesn't put a mark where he starts. Makes a horizontal line lower on the pole (fourth line down in Figure 7.7).]
T: Where do you start?
A: [Draws a horizontal line at the top of the vertical lines, which is not where he appeared to start counting down.]
T: OK. And then this is where you end after the 70? How far is it from where you started to where you ended?
A: 70m.
T: Good. We're all good. Now you rest here and then you go down 19. Show me where that would be.
A: [Draws some dashes going down as he acts out the action and then draws another horizontal line below the -70 line.]
T: All right. Where is 51 on here? If you start here [top line] and you go down 51, where's that going to be?

A: [Draws horizontal line between the start and end of his first trip.]
T: Somewhere in there, right? That [line he just draw] and that [end of second trip] are not the same, does that make sense?
A: [Nods.]
T: Yeah. How can you figure out how far you are from where you started [pointing to the end of the second trip] when you're down here? I think the way I wrote it down is confusing because it makes you just think of addition and subtraction.
A: [Nods.]
T: Try to [covering up the writing, laughing], let's say that you go down 70 and then you go down 5 more meters, where would you be?
A: [Draws horizontal line a little above the -70 line.]
T: I mean, just number-wise, where would you be? How far have you gone down?
A: [Writes 65 next to the new line.]
T: OK. You go down 70 and, when I say you go 5 more meters, I mean going further down from where you started [moving finger down the pole from the -70 line].
A: [Writes 65 next to the previously drawn line that represented -70 □ -19.]
T: Adam, come on, you're just wasting time.
A: 65! I went 5m down from 70, that's 65m.

When Adam seemed to be counting down to -70, in retrospect, he was probably just visualizing the action of going down. He seemed like he was counting because his metric for linear distance seems to be discrete. (There are more indications of this to come.) Therefore his visualization involved a series of "hops" down the pole. I do not think he visualized a specific starting point on the paper because, when I asked him to draw the starting point, he did not draw it where he had starting makes downward hops.

Adam represented, "you go down 19," with a motion going down from his -70 line, as I meant for him to do. However, when I asked him later to go down 5m from -70m, he did so, but claimed that it was 65. Therefore, and this is confirmed in the next protocol, I hypothesize that he still thought of the line 19m down from -70 as representing 51. Therefore, it is surprising that he did not experience perturbation when 51 below 0 was not at the same place as the end point for his second trip. This is further indication that the names of positions on the pole do not have a common reference point associated with them.

At the end of the protocol, I thought he was just playing around or not actually thinking about the situation. It had not occurred to me that he was not assimilating this with a (discrete) signed, number line. He seemed upset by the accusation that he was wasting time, and vigorously defended his answer. His reaction implied to me that I really did not understand his thinking, and I could tell we were both getting frustrated. Therefore, I decided to move to Lily's diagram, hoping that her alternative way of depicting the situation would make sense to him, since I could not make sense of what he was doing.

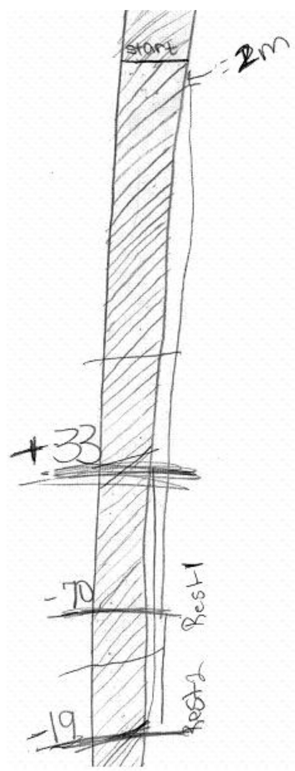

Figure 7.8. Lily's diagram for (-70) + (-19) + (+33).

Protocol 7.14: Adam interprets Lily's diagram for -70 □ -19 □ +33.

T: All right, OK. Let's look at Lily's. [See Figure 7.8.] […]
L: All right, this is where he starts. And then first he goes down 70 and each one of these lines equals 2. I did that to kind of make it more accurate. And then 70 is somewhere in this [-70] line, and then he went down another 19. And that was about here [-19 line] and then he went up 33, so I did my math and got up to here [+33 line].
T: OK, Adam, do you agree with just the general placements of things on her picture? 'Cause I think that's where the original problem is.

A: [Starts counting her diagonal lines down from the starting line.]
T: […] I mean where everything here is in relation to each other.
A: [Makes tapping motions down, pauses at +33, goes down to -70, down to “-19,” and then up again. Nods.]
T: Yeah? Starting from here [start line], can you use your pencil and reenact the trip for me?
A: [Puts eraser at the starting line and starts tapping at each diagonal line as he goes down.]
T: All right. What are you doing now?
A: Counting the lines to get to here.
T: You don’t have to count them, just show me.
A: To make sure it’s 70.
T: You don’t need to make sure it’s 70, just—
A: Because she could be wrong.
T: That’s OK because we’re just sketching right now. Show me [..] the trip […].
A: [Puts his pencil at the starting line and moves down.]
T: Where’s Adam? He’s climbing down, climbing down, climbing down. All right.
A: [Writes, “Rest 1” at -70.]
T: Awesome.
A: [Continues down to “-19 line” and writes, “Rest 2.”]
T: Awesome.
A: [Moves pencil back up to +33 and removes pencil.]
T: […] What would you have to do to get back to where you started [from +33 line]?
A: You would have to go up [starts counting up from “+33” line].
T: […] I want you to figure it out. From here to here [start to -70 line], we already talked about this, right? What’s that distance going to be?
A: 70.
T: […] We go down 19 more and we’re down here.
A: Right.
T: OK. Do you want to give me your answer or do you want me to ask Lily about it?
A: [Points to L.]

Before looking at the continuation of this discussion, I want to point out Adam’s continual attempts to count. First, I think I should have let him count. “Sketching” the situation was probably not appropriate for him, given that he was actually constructing the quantities in action and was not simply communicating a quantitative understanding. Second, he still has not shown contraindications to the hypothesis that the length is measured in discrete units for him. Lily later referred to the “spaces” that Adam had to go, indicating that she was attending to the space in between the her tick marks (see Protocol 7.20). Adam, on the other hand, never gives indications that he is quantifying a continuous quantity, even a segmented continuous quantity.

Protocol 7.15: Lily and I disagree with Adam about the where the second rest takes place.

T: Lily, where do you think we are when we get down to here [pointing to “-19” line]?
L: 89 feet from the start?
T: 89 down from the start [moving pencil from start to “-19” line]? What do you think?
A: I got 84. […] I did it like a regular problem. 70 minus 19 is 51, 51 plus 33 is 84.
T: That would be for after you’ve done all three, though, right?
A: Right.
T: And we’re only after the first two.
A: Oh, the first two? 51.
T: OK. If this is 70 [moving pencil from start to -70 line and stopping there], where would 51 be on the picture?
A: [Draws a line between -70 and the “-19” line.] No, wait. It’d be [inaudible, waving pencil above -70 line] here. [Draws a line above +33 line.]
T: […] There’s something wrong, ‘cause we’re all the way down here. We’re past 51.
A: Yeah. That means the 19’s in the wrong spot.
T: Let’s just go through this together and figure out the number, all right? We go down from here to here [start to -70]. 70 below where we started. And then we go down another 19?
A: Oh, yes.
T: You’re saying we’re 89 meters down from where we started?
L: Mm-hmm.
T: I’m certainly not convinced yet because I haven’t even heard an explanation from her. Do you think that that would be 89 [“-19” line]?
A: From the start [pointing]?
T: [Nods.] Lily, tell us why you think that’s 89, and then we’ll decide.
L: Because this is 70 feet, and we went down another 19 feet […] Because 70 plus 19 is 89.
T: OK, why are you adding 70 and 19?
L: Because right here [-70 line], the first trip, is 70 meters down, and then we went down another 19, so we had, we went down more. We need to know how much more, and we went down 19 more. So, 70 plus 19 is 89.
T: What do you think?
A: Well, um, well, I do agree that 70 plus 19 is 89. […] So what are we trying to figure out?
T: We’re trying to figure out, when we get down here, after we do, after you go another 19 down, how far are you from where you started [pointing to start and back down to “-19” line]?
A: If you went 70 down from where you were, that’s 70, and then you also go down, that’s 89! So she was right.

Adam confirms that he has competing, and in this case, contradictory ideas about the situation. He was aware that he thought -51 should be below -70, but also was aware that -51 would be above -70. In particular, Lily's depiction of the situation may have activated his recognition template for a number line in that she shows lots of tick marks and the starting point. Therefore, he is aware that if he counts down to -51, it will be before -70 because is assimilating the distance down from 0 with his number sequence at this point. This causes him to reposition 19 down. I think this represents another competing understanding of the situation I have already alluded to, between "19 down" meaning "making the amount you have gone down go 19 down," i.e., "decreasing by 19 the amount you have gone down," and "19 in the 'down direction'." Because he can make sense of the 19 down in either way, it did not perturb him to go 19m up the pole in order to go "19 down." However, almost immediately afterwards he agreed with me to go 19m further down the pole. He did not seem to recognize that he had two competing views of what "19 down" means, but he did seem to recognize that he was changing his mind multiple times about where the "19 down" trip ended. He seemed to follow Lily's explanation, which led to his realization that something was ambiguous in the situation for him: "So what are we trying to figure out?" Once he is firmly entrenched in my (and Lily's) interpretation of the 19 down of the situation and the comparison to the starting point, he agreed with our answer.

Protocol 7.16. Adam finishes determining -70 ☐ -19 ☐ +33.

T: We're down 89 and then you go up the pole 33 feet [pencil from "-19" line to "+33" line]. Now we have to figure out how far we are from where we started. […]We're down to 89, we go up 33. What if you count up? […] If we're down 89m [motioning]…
A: Yeah.
T: …and we go up 10m, where are we?
A: 99.
T: Up 10 meters. We're going back up now, so we're getting closer.
A: 79
…
T: We're down at 89 and we're going up 33, right?

…
A: 79, 69, 59, 56.
T: Mm-hmm. Oh! You got it. Are we 56 above where we started or below where we started?
A: Below [pointing down].

Protocol 7.16 shows that his reasoning within our interpretation of the situation did not invalidate his other interpretation of "up" and "down" because, after agreeing with the placement of the end of the "down 19" trip, he claims that 99 is "10 up" from 89. Although he realized something was ambiguous about his understanding of the situation, he did not realize what it was or that he was holding competing interpretation of my language. Therefore, he did not attempt to resolve the (to the observer) inconsistency in his reasoning.

In the third task of the day, I saw progress in how he was conceptualizing the quantities in the problem situation. In this task, Lily was doing three trips: First, she went up 3m and rested; second she went down 40m and rested; and third she went down another 30m and rested. I asked both students to draw a diagram of the situation before solving it because Adam had seemed to act in a way that was more subjective with Lily's and my way of acting when drawing the diagrams than when determining numerical answers. In his sketch, he started by drawing a starting line, which is the first improvement. This implies that he had some understanding of the importance of the starting line for understanding the other quantities in the problem. The second positive indication is that he immediately erased his starting line, which was low on the paper, and replaced it with a starting line high on the paper. This implies to me that he ran through a mental visualization of the trips before drawing them. Because I eventually wanted him to interiorize the quantities in his diagrams, I was happy to see that he was at least running through the situation in mental re-presentation. The third positive development is that he used arrows to draw his trips. This implies some awareness of quantifying the action, what happens *in between* the endpoints. Before this, he had never drawn anything in between the endings of trips. Nonetheless, he does not label the length of the arrow numerically, so my guess is that the arrow is primarily representing an action in his mind.

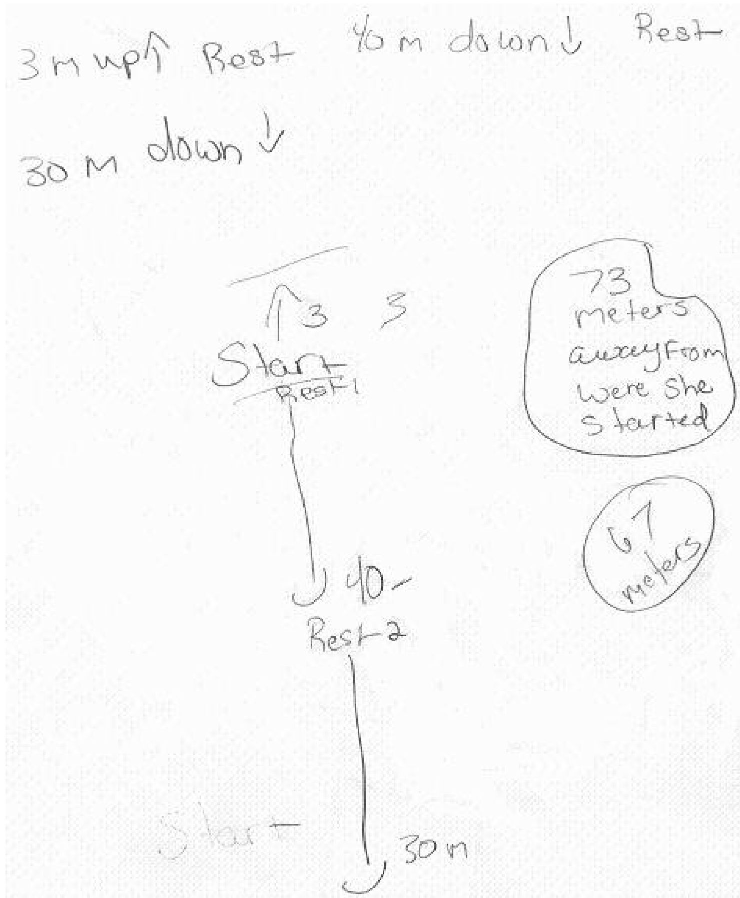

Figure 7.9. Adam anticipates the layout of a signed addition diagram.

You may notice that the diagram is not viable in that he starts the second trip from the starting line instead of the ending point for the first trip. After he drew this diagram, I asked him to explain it, and he recognized in the middle of his explanation that the second trip was starting at the wrong place, and he changed his numerical answer. Nonetheless, his original answer is revealing in that he measures the distance, using his original diagram, that she was away from the end of the first trip, not the starting line. Therefore, he was still, at this point, not differentiating clearly between reference points, possibly because he has just started to attend to them as important.

At the top of Figure 7.9, you can see what Adam wrote down to remember the problem. When I gave Adam and Lily a piece of paper, and before I told them the task, I said, "Let me just tell you this one and you all can write down whatever you want to understand the problem, OK? You can write, draw pictures, doesn't matter." I did this in lieu of notating the problem as +3 □ -40 □ -30 since I had seen, in the last problem, that Adam was interpreting the negative sign as a subtraction sign. This might have helped him interpret the "up/down" language as describing the direction of the trip, as opposed to what was happening to the absolute value. In other instances when he did look at this decrease in absolute value, the "circle-plus" notation was visible. Therefore, the notation does seem to serve as a catalyst for him to assimilate the situation in terms of unsigned addition and subtraction on absolute values of the given numbers.

However, I do not think this would be an issue for him if he could assimilate the additive relationships between the quantities in the problem. I hypothesize that his attempt to reason using notation and unsigned quantities instead of using a mental re-enactment/visualization and constructing signed quantities and quantitative relationships out of it is an indication of how difficult it was for him to construct additive relationships between signed quantities. Certainly, he was not only experiencing a confusion about what the notation meant, because we had multiple teaching sessions in which he had constructed an understanding of the notation to a problem in a way that was intersubjective with my understanding of the notation, only to revert to his unsigned interpretation of the notation later in the session or in a later session.

Multiple Reference Points

Once Adam was attending to the starting points, he seemed to become comfortable with interpreting positions on the pole with relation to the original starting point. This was probably aided by the convention we developed of referencing the surface of the Earth as the starting point. However, now that Adam was attending more explicitly to the directed distance from the starting point, he now had two quantities he needed to coordinate for each trip: the directed length of the trip and the directed distance of the end point from the starting point. In the next three protocols, I discuss difficulties that arose out of conflations in these quantities.

Although Adam did use arrows to make Figure 7.9 on March 23, he did not consistently use them on March 24. During the last session with Adam and Lily, March 29, we introduced a virtual diagram (see Figures 7.10 and 7.12) on Geometer's SketchPad (GSP). We did this in order to give Adam a situation in which he could experience a continuous quantity: He had to move the arrowhead of the arrows up or down slowly and could see the arrow getting continuously longer as he did so. Also, we hoped that the increased conformity and precision of their sketches on GSP would prompt them to notice the subset relationships among the lengths of arrows. Neither student had yet articulated that kind of quantitative comparison in their explanations in the number line context. On March 29, Adam and Lily did not get through very many tasks, and I gave them a different set of tasks, as well. The main difference was that Lily had more difficult numbers, and had to work with composite units as her scale whereas Adam's scale was units of 1m. She did not have any trouble with these tasks, and she did give a nice explanation that I discuss in the next subsection. Adam solved the tasks appropriately, but I got the impression that he was not quantitatively reasoning, but instead carrying out actions and then reading the answer off of the scale in his diagram. For that reason, on March 31, our last day with Adam, I decided to give him a diagram where he could not read the answers off the diagram because only a few tick marks were visible to guide him in making his drawing (see Figure 7.10).

Protocol 7.17 describes what took place during the first task on March 31. When the protocol begins, Adam had drawn the sketch shown in Figure 7.10 as an illustration of the following problem situation: *Adam and Katy are standing on the surface of the Earth. Katy climbs down the pole 70m, rests, and then climbs up the pole 30m. If Adam is still at the surface of the Earth, what trip should he take to get to where Katy is?*

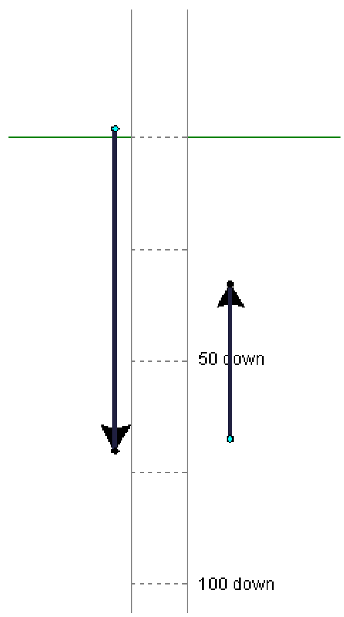

Figure 7.10. Adam's GSP sketch.

Protocol 7.17: Adam conflates the length of the second trip with its endpoint.

A: That is 25 [pointing to the horizontal mark at 25 down].
…
T: Let me draw another arrow. [Draws line segment from end of second trip to surface.]
[…] Is this part 30 [second trip] or is that part 30 [new segment]?
A: Both.
T: They're both 30?
A: [Nods.]
…

T: How far is it from the surface of the Earth all the way down to [pointing to “70 down” on the right-hand side]?
A: 70m.
T: That’s 70, right? It’s the same as that one [first trip arrow], what would that make the whole trip?
A: 70. No. 60.
T: Yeah, so there’s something a little bit off. I think one of these is not 30 and one of them is 30.
A: [Points to new segment.]
T: […] Go ahead and figure out what that is. […] You’ve got this guy going up [second trip] and how far is that?
A: 30.
T: All right. We got all that. Go ahead and finish and figure out how long your trip is going to be.

In Adam’s diagram, he had drawn the second trip as ending at 30 (down) as opposed to going 30 up. He points out to me that the first line down represented 25 down as part of his original explanation of the diagram. This implies that he meant to go to 30 (down) when drawing the second trip. This represents a conflation of how far he is from the beginning of the trip as opposed to how far he is from the surface of the Earth. If, on questioning, he had changed his mind, I might consider this an unnecessary error, in that it would not represent the result of his general ways of operating. However, he does not change his interpretation right away. I had not seen him make this kind of conflation in his first GSP session, but the numbers had been small enough that day, and the scale fine enough, that he had would have been able to actually count up 30. In this case, he is aware that “30” needs to be used to monitor his motion, but not being able to count up by 30 seems to open up the possibility of conflating 30 as representing a position in addition to the length of the trip. He also appears to be attending to unsigned quantities again in that he is not bothered by the fact that he is at 30 down, not 30 up. I assume that is because he interpreted his action as going *up* to *30*. This brings to mind similar issues in the money context in which he interpreted the sign of his sum as representing what he needed to do to get *to* that (unsigned) number as opposed to recognizing the sign as being part of the number.

The next protocol is a continuation of the discussion in Protocol 7.17, in which Adam is attempting to draw his trip, which is a trip from the surface of the Earth to where I ended up (the end point of the second arrow).

Protocol 7.18: Adam draws in overall displacement in GSP.

A: Oh. *My* trip. [Draws in an arrow from the surface down to the second trip's end.]
T: Tell me what's going on with your trip.
A: I had to go down.
T: Yes. And how far did you have to go down?
A: 28m? [Indicates that he got 28 by looking at the diagram.]
T: Yeah, 28 is a pretty good guess. How could you figure it out based on how long these arrows [first and second trip] are?
A: Wait. What?
T: […] 'Cause we don't know exactly where that [end of Adam's trip] is, right?
A: Right.
T: We're just estimating, but you know exactly how long that [first trip] is. […] You know exactly how long that [Adam's trip] is.
A: I don't know.
T: You can get it. Just think about it. […] [After about 20s] All right, what are you thinking?
A: [Shrugs.] I don't know.

In Protocol 7.18, Adam indicates that he read his answer, 28m, off of the diagram, as I suspected he had been doing during the previous session. When I attempt to get him to reason through the answer based on the other arrows, he is not sure what to do. This implies that he is not abstracting out the subset relationship implied by the diagram: the absolute value of the length of my second trip combined with the absolute value of his trip should equal the absolute value of my first trip. Given that he had done this kind of reasoning with the sketches, this points to the possibility that he had been, at least partially, depending on an imitation of Lily's way of operating at the time and was not able to independently regenerate the additive relationships in the problem. However, I do not think that possibility is likely given that his diagrams and answered often differed from Lily's, and he would not change his answer or diagram right away or even after an explanation from Lily. He seemed to be thinking through the problems on his own. Therefore, my hypothesis is that Adam did not want to engage in the problem solving here because of a combination of him not feeling well (he had the tail end of a cold that had plagued him since March 24) and his discomfort in reasoning through these kinds of problems. After all, he had had trouble in the first two sessions, with the addition problems on March 23 and with the MA problems on March 24. Therefore, he may have been avoiding problem solving activity at this time. However, this does not supercede my argument that he had not abstracted out the unsigned additive relationship. The underlying addition problem would have been trivial for him, so the issue must have been figuring out that he could or how to assimilate this situation with his unsigned addition or subtraction schemes. I strongly believe that if we were to give Adam the diagram in Figure 7.11, he would be able to tell us how long the mystery segment was. Therefore, I conjecture that his difficulty has to do with the nature of the quantities he has constructed when making the diagram.

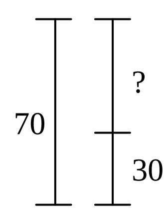

Figure 7.11. The unsigned addition relationship underlying Adam's task on March 31.

The next protocol is again a continuation of the previous discussion. In this protocol, the witness intervenes to encourage Adam to engage in problem solving activity, and, in particular, to mentally visualize the situation.

Protocol 7.19: Witness introduces a visualization of an ant walking.

W: Adam, I know you know this stuff. All right. Let's suppose there's an ant. The ant crawled down here [first trip], all the way down. How far would he crawl down there?

A: 70m.
W: Then he turns around and he crawls back how far?
A: 30.
W: How far would he have to get yet?
A: 100. Well, no.
W: How far would he have to go to get back to where he started?
A: 30. No. 40.
W: Yeah, that's it. But now how can you, if you start here [surface] and go down this way [Adam's trip], would he be going down or up? See, this way you'd be going up, right?
A: [Nods.]
W: But you started from here and you went down.
A: Right.
W: So you went down how far?
A: 28.
W: Would you go down a different distance than you go up?
A: 30.
W: 30, right. So you're going down 30, right?
A: [Nods.]
T: […] Let's say that the question had been: How far do I have to go to get back to where you are? […] I climb down 70, I climb back up 30, and then I have to go back to where I started.
A: 40.
T: 40, right? That one's easier.
A: [Nods.]

At the beginning of the protocol, Adam's answer of 100 indicates that he might have interpreted "up" as "increasing the size of," similarly to the first two problems on March 23. However, in this instance, he did experience perturbation at the mismatch between a downward trip that would result in the answer of 100 and the actual trip he has drawn. His increased experience with the vertical pseudo-number line seems to have been sufficient to reorient his awareness of position on the number line.

The intervention was successful in that Adam recognized that the witness's task represents a situation of his unsigned subtraction scheme and was able to recognize that the solution would have the same magnitude as the solution to his original task. The witness's vote of confidence undoubtedly emboldened Adam to engage in reasoning, but the task also may have been easier in that the missing quantity was of the same type and orientation as the second trip. This would make Adam more likely to consider combining the two and recognizing the equality of the length of their union with the length of the first trip. In the original task, his trip was in a different direction than my second trip, and it was a different person moving. This may have made the idea of combining their lengths less likely.

The intervention was also successful in that he solved three more tasks on GSP without incident, including another one in which he started with a trip down. This kind of task had been the most difficult for him because he had the competing interpretations of what was happening to his distance from the starting point during the second trip. In the last problem of the day, he does run into difficulties, but they are of a different sort. Interestingly, he operates in the same way that Justin did during the money context in Protocol 6.16, when Justin would find the result of two changes and then recombine it with one of the changes to get a new answer. In Adam's case, I asked him to think about the problem in his head before drawing it, to find out to what extent he had internalized the quantities in the diagrams. I then gave him the following problem, which had intentionally easy numbers to work with so that he would not be distracted from the additive relationship by his mental calculation: *If I climb down 20m, rest, and then I climb up 50m, what do you have to do to get to where I am?*

He initially answers "I have to go down 30." The only two ways I can make sense of his answer in a way that would fit with his past behavior is that he is thinking about being below +50 at the end and so says he is going down, or he is only able to visual the situation enough to recognize that the trips are not in the same direction and so he will need to subtract. All of his other problems in the number line context in which he started by going down ended below the surface. Therefore, he might have assumed he was below the surface. In either case, he did not appear to have fully internalized the quantities and their relationships that he could construct in a diagram. I then let him draw the diagram, at which point he decided the answer is “30… No, wait. 10 up ‘Cause 20’s right there [Trip 1] and then you go up 20 [Trip 2] to get back [passing 0] to the center [inaudible] 10 back up.” I think this represents, as it did with Justin, that the answer of 30 is almost immediately available to him based on his new, improved intuitive understanding of these situations. That is the good news. However, similarly to Justin, he then interprets his intuitive answer of “30 up” as my second trip. In Adam’s case, I hypothesize that the difficulty is due to the nature of his signed quantities. He has indicated in the past that positive and negative values represent two related, unsigned quantities instead of a single, signed quantity. In particular, this seems to be related to a view of the signed quantity as representing a measurement of the state of something instead of the measurement of a change in something. In the number line context, this parallels the conflation between the ending point of a trip and the directed length of the trip. On this problem, Adam reinterprets the position of the endpoint of the second trip with the directed length of the second trip. I do not think this was a necessary error in this case, but it does underscore the confusion that can arise when students begin to construct signed quantities and need to differentiate directed differences from the 0 quantity from the signed nature of changes in quantity.

Lily’s Signed Addition Schemes

As I stated the first problem, about Lily going up 83m, down 83m, down another 1m, Lily just smiled. This was my first clue that the number line context would be no harder for her than any of the others. She knew the answer to this problem right away, so I presume that she is forming a mental visualization that includes an awareness of additive relationships. Similarly, she always seemed to have a general sense of what the diagram would look like based on placement of her starting lines and her ability to come up with numerical answers without a diagram.

The only change I saw in her signed additive behavior from what she was doing in the other contexts was that she began to refer to quantities and quantitative relationships more clearly, as I hoped she would once she had the diagrams to refer to. An example is given in the following protocol. Lily has drawn a sketch on GSP for the following problem situation: *Lily and Adam started at the surface of the Earth. Lily climbed up 10m, rested, and then climbed down 24m. What does Adam need to do to get to where Lily is?* She came up with the answer 14 for the length of his trip, and I began pressing her on an explanation. Because she and Adam were doing different problems, I have left out dialogue between Adam and I in the protocol. I have also added letters to her diagram in Figure 7.12 in order to make it easier to follow her argument.

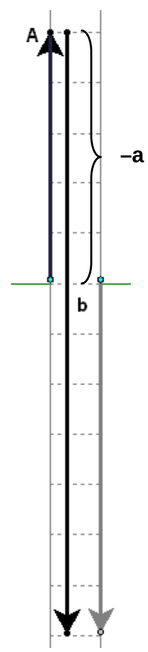

Figure 7.12. Lily's GSP sketch for Protocol 7.20.

Protocol 7.20: Lily utilizes subset relationships in her explanation.

T: All right, and how did you get that?
L: 24 minus 10 is 14.

T: […] How should I see from this picture that you should subtract 24 minus 10?
L: [After 90s in which I am working with Adam] I think I got it.
T: All right. Explain away.
L: If I'm going up 10, then I'm here [A], and this line right here for the face of the Earth to where I'm at in my first trip equals 10 also [-a]. And then I'm at the surface of the Earth and I go down. That's the amount of spaces that Adam has to go down.
T: It is [nodding]. Why would that be 24 – 10?
L: Because, in all, this line is 24, but we can't count this 10 [-a] because it's above the surface, and we're going below the surface.

In this explanation, Lily identifies part of the second trip as the additive inverse to the displacement in the first trip and notes that it has the same length as the first trip and gets her back to the surface of the Earth. She also notes that she is disembedding the 10m above the surface from the 24m trip.

In this protocol, Lily also refers to the section of the second trip that has the same directed length as Adam's trip as "the amount of spaces that Adam has to go down." In fact, she consistently attended to the spaces, or continuous segments, in her diagrams, in contrast to Adam who attended mainly to points on his diagrams. Lily does not give indications of how continuous her length concept is, but it was definitely sufficient to operate in this context when confined to working with integer quantities.

Summary of Adam and Lily's Ways of Operating With Trips on a Number Line

I mentioned earlier that I regretted not letting Adam draw more careful diagrams on the first day. In fact, when he was able to do so on the third day, using GSP, he was very engaged in the tasks. He probably was not reasoning quantitatively about the situations by trying to figure out the answer to the task without reading it off the scale on the pole, for example. Nonetheless, this experience of constructing precise diagrams was probably necessary before he could feel comfortable "sketching" the situation as we did on the first day. In addition, the time he spent drawing diagrams on GSP, did make him more intuitive about the structure of the distances from 0, as evidenced by his perturbation when he gives the incorrect answer of 100 in Protocol 7.19.

Unfortunately, I tried to force him into this kind of reasoning on the last day by greatly increasing the distance between tick marks. He was much less engaged in those problems and did experience quite a bit of perturbation in solving them. In retrospect, he would probably have been better served by spending at least two days drawing very exact diagrams at the beginning and then moving to diagrams that emphasized the quantitative relationships as opposed to the exact numerosity of the quantities involved.

Adam's extreme difficulty in making sense of the diagrams on the first day seemed to stem from the nature of the quantities, some of which he was interpreting as actions, as changing quantities that he could not draw. In addition, he was not attending to the size of actions as lengths or even to their starting point. He only attended to where the action/trip ended. This changes, and by the end of the first episode he is already notating starting lines on his own and indicating the space the trips ran through with arrows. However, he only slowly starts to develop a quantitative understanding of the lengths of trips. Even on the last day, he asks me if he is supposed to count the tick mark he starts on or not when he is measuring a trip, illustrating that he does not have a well-developed way to quantify length and he is attending to points on the diagram and not spaces in between the points.

Adam continued to have two interpretations for orientations of signed quantities. In this context, he interpreted "up" as meaning an increase in an unsigned quantity, the distance below the Earth, for example, and "down" as meaning a decrease in an unsigned quantity. Visualizing the trips happening, either by running through the trips on the diagram, as he did on March 23, or by picturing an ant walking along the diagram, as he did on March 31, helped reorient him to a meaning for the signs of quantities and language about those signs that agreed with how Lily and I were interpreting them. Whenever a diagram of a situation was not available, such as with problems on March 24 or the last problem on March 31, he interpreted the questions in terms of unsigned quantities. This behavior implies that the construction of signed quantities, as opposed to unsigned quantities, remains difficult for him.

CHAPTER 8

CONCLUSIONS

In this chapter, I discuss how my findings answer, at least in part, my research questions:

1. What difficulties do students encounter in situations of signed addition?

2. What aspects of the *mathematics of* each participant, including schemes for unsigned quantities, impede or facilitate the participant's ability to work in situations of signed addition?

3. What aspects of the social interactions among participants and the teacher/researchers impede or facilitate participants' ability to work in situations of signed addition and the researchers' ability to construct the *mathematics of* each participant in these mathematical contexts?

4. What, if any, changes are there in how the participants assimilate situations of signed addition?

In the discussion of my first research question, I discuss student difficulties and what they might imply about the mathematics of the student. In the discussion of my second research question, I summarize how each student's unsigned schemes and operations affected their construction of signed quantities and the ability to operate in additive situations with these quantities. In the discussion of my third research question, I highlight some of the social factors that affected the students' mathematical progress. In the discussion of findings in relation to my last research question, I summarize my findings related to the stages in the students' constructions of signed number addition and subtraction schemes and structures. After discussing my findings in relation to all four research questions, I link my findings back to some of the extant literature. Finally, I discuss what implications the study has for mathematics education in the middle grades.

Student Difficulties in Signed Additive Situations

In addition to answering my first research question—*What difficulties do students encounter in situations of signed addition?*—I propose explanations for these difficulties in this section. Both Lily and Michelle, and later Justin, were able to viably operate in what were, to the observer, situations of signed addition and subtraction. Therefore, many of my findings relate more to how they are able to talk about their solutions as opposed to actual difficulties. Nonetheless, there are several difficulties that my participants experienced. I have organized them based on observable behaviors.

Confusion About How to Draw a Number Line Diagram

Adam experienced an initial confusion about how to draw a series of three trips on a pseudo-number line (Protocol 7.10). This reflected three aspects of his mathematics. The first was that he did not have a static visualization that would represent a change in position. I hypothesize that he viewed the trips, in particular the second and third trips, as an action through time. Although he could use the numerical description of that action (33 m up) to monitor a reenactment or visualization of the situation, the underlying length quantity remained implicit and was not available to him as an assimilatory construct in these situations. Therefore, it was not clear to him how to make a static picture of motion. Instead he would indicate the motion of the trip by making counting motions up or down the vertical pseudo-number line and then draw only the endpoints of the trips, which he labeled with the length of the trip that ended at that position.

This initial method of drawing trips also highlights the second aspect of his mathematics that his general difficulty highlights: lack of attention to the reference point for quantities. The sense of an indefinite starting point in my original question to Adam and Lily, which did not include a reference to the surface of the Earth, may have been a source of confusion. However, it did not cause confusion in any of my other participants, and before Adam thought about how to draw the situation he indicated that the question itself was not confusing to him. I take this to mean that he could visualize the situation and understood what I was asking, which implies some awareness of a starting point. Therefore, I do not think that the inability to instantiate an indefinite starting point is the issue. Instead, in addition to using the length quantity only implicitly, Adam also used starting points only implicitly. He did not notate his starting point for the first trip in his first diagram, and once he did on my suggestion, he seemed to conflate the starting point for the first trip and second trip, both in his original labeling of the start line as the endpoint of the first trip, but also in his later statement that he did not know where he was starting.

Confusion About Increases and Decreases on a Number Line

Adam and Michelle both ran into difficulties with drawing/interpreting diagrams when they had drawn a trip down from the surface of the Earth and then were supposed to draw a second trip. In Adam's case, he was actually drawing the situation and his second trip was to go 19m down the pole. He interpreted the "down" as representing a decrease in the absolute value of his distance from 0. In Michelle's case, Justin was drawing the second trip, and it went up (Protocol 6.18). She thought she should add the absolute values of the two trips to find her distance from 0 since the trip was *going up*, which she interpreted as getting bigger or the distance going up, perhaps.

Adam has similar difficulties elsewhere, generally when a visual aid is not available, such as when he was combining changes in the money context without coins (Protocol 7.5) or solving missing addend problems without any visual aids in either the money context (Protocols 7.6–9) or number line context (see discussion after Protocol 7.9). In all of these cases, I hypothesize that the attention to absolute value implies that the students are working with signed quantities rather than a unitary signed quantity. That is, they were thinking of a number of coins moved or a distance from 0, without taking the orientation of the quantity into consideration.

Michelle did not have further difficulties with this, so this may not have been a necessary error in this case. That is, she may have been far enough along in her transition to a unitary signed quantity that she could easily alter how she interpreted language such as increase/decrease and up/down. In Adam's case, the error was recurrent, although not consistent, over several teaching sessions. Therefore, I hypothesize that he was not as far along in his transition to a unitary signed quantity and that he tended to assimilate situations in terms of unsigned quantities, only in action did he construct the relation between the two kinds of unsigned quantities.

Interpreting Signs Based on Changes in Absolute Value

Another difficulty Adam encountered that combines his attention to unsigned quantities and his lack of attention to reference points was a tendency in the same money context tasks in which he was not enacting the task with coins, to interpret the sign of the sum as representing how the absolute value of the first addend was changed by the action of the second addend. He was, again, attending to absolute value. In addition, he used the absolute value of his first addend as his reference point that he compared the absolute value of the sum with in order to determine whether he was dealing with an increase or decrease. The first time this happened, he was working on a signed addition tasks and was able to correct himself on repeated questioning by the teacher/researcher. However, the same interpretation resurfaced later in the same session when he was working in on missing addend tasks. Another aspect of Adam's mathematics that would have supported this interpretation of an increase or decrease was that he thought of the second addend, even in unsigned addition or missing addend tasks, as a transformation of the first addend into the sum. For example, 5 + 3 = 8 implies that you start at 5, count up 3, and end up at 8. The 5 and 8 are full-fledged composite units of his TNS in this scenario. The 3 is used in a more intuitive way to describe the extent of the action.

Conflating Sums and Addends

Early on in working with signed quantities, both Adam and Justin conflated an addend with the sum. Adam did this only briefly in the card game when solving his first MA problem (Protocol 7.3). Justin also did this in MA tasks in the card game. In addition, Justin interpreted the C3 quantity as a C2 quantity several times in speech, claiming that he had won every round when he had lost several rounds but was always winning the game. He also wanted to ignore the round in which he was losing the game by the most in order to improve his score, even though it was not the round that he lost by the most in. I think all of these difficulties go back to the initial need to construct a sum of changes as a quantity that includes records of both positive and negative changes. For Justin, this construction took more time because he was not using a difference as an assimilating or anticipatory concept. This meant that he had trouble constructing a signed sum that potentially represented a sum or difference of various absolute values.

Later on, in the money context, Justin conflates again by interpreting the sum as an addend and recombining it with another addend (Protocol 6.7–8). At this point, he could construct a signed sum, but he did not use it as an anticipatory or assimilating construct, so that when he very quickly and intuitively determined the sum in the money context, he did not have a pre-existent quantitative category to make sense of it with other than as one of the original actions on the plate's value by him or Michelle. Similarly, and, I think, for a similar reason, Adam makes the sum conflation in the number line context when he is solving the problem in his head (discussion after Protocol 7.19).

Finally, Adam once conflates the directed length of his second trip with the endpoint, which amounts to using the size of the trip as the size of the sum, in his case (Protocol 7.17). This was similar to Brad's difficulties in the pilot study which discuss in Chapter 1. Unlike the other difficulties in this subsection, I hypothesize that this difficulty in interpreting lengths versus endpoints is related to Adam length construct, which did not inherently attend to the starting and ending points. Both Adam and Brad had yet to construct an ENS, and I hypothesize that this hindered their ability to attend to the second addend, or the difference, in a situation as a composite unit. Furthermore, even if they did construct the second addend as a composite unit of the same type as their initial number sequences, they could not disembed subsequences to reflect on additive relationships. Directed trips are, more or less, a difference between their endpoints, at least for the observer. My argument is that Adam and Brad did not understand directed trips in that way.

Addition and Subtraction Notation

The final student difficulty I discuss here is the inability of Justin and Michelle to distinguish notationally between signed addition and signed MA problems. I hypothesize that this requires the construction of a generalized signed sum structure analogous to Lily's generalized unsigned sum structure: The generalized signed sum structure would have the added complexity of unspecified orientation as I discussed in Chapter 6 (see discussion of Figure 6.5). Justin did construct directed additive relationships as he operated in signed additive situations, as

I discuss in the next subsection, but I do not have evidence that he assimilated with them. And, even if he did assimilate with a signed addition structure, he would still need to construct an anticipatorily reversible addition scheme, which would be equivalent to constructing a generalized signed sum structure.

Although Michelle could viably operate in signed, additive situations, the fact that she never referenced 0 or additive inverses in her explanations (except for indirectly in referencing quantities represented on her diagrams) leads me to conjecture that she did not construct signed subset relationships in that way that Justin did. Instead, I think of her way of operating in these settings as analogous to how Adam operates in unsigned additive settings. In other words, she constructed quantities that would correspond to all three sets in the subset relationship, but all the quantities were not of the same type. One of the quantities would be an enumeration of a difference or of an action, but would not be constructed as a disembeddable subset that could be compared to the other sets she had constructed. Therefore, she was even further from the construction of a generalized sum structure than Justin.

In my pilot study, neither Brad nor Amanda was able to make sense of my subtraction notation, but in light of the current analysis, this is unsurprising. Neither showed evidence of a GNS, which would be helpful in constructing signed additive relationships and necessary in assimilating with them in the form of a generalized signed sum structure.

Mathematical Factors in Signed Additive Constructions

I have organized the discussion of my findings in relation to my second research question—*What aspects of the mathematics of each participant, including schemes for unsigned quantities, impede or facilitate the participant's ability to work in situations of signed addition?*—by participant.

Adam

Unlike Justin at the beginning of the card game, Adam could think of a number as being the result of adding or subtracting before action. I refer to this as an anticipatory reversible addition scheme. This scheme is probably what enabled him to operate on a hypothetical difference when strategically reasoning. It also allowed him to make sense of a signed sum as a signed quantity determined by both positive and negative quantities. The difference between him and, say, Lily in regards to their conception of a difference is that his "results of adding or subtracting," i.e., sums and differences, are experienced as a result of a transformation as opposed to a quantity in additive relationship. In other words, Adam did not form *structural* additive relationships in the sense of being aware of two addends being complementary subsets of the third. For example, if he saw the expression, 23 + 9, he would assimilate the 23 as a composite unit, made up of the first 23 arithmetic unit items in a counting sequence starting at 1 and ending at 23. The 9 by itself would imply a corresponding initial number sequence starting at 1 and ending at 9, but the presence of the addition means that the "+ 9" was *together* assimilated as a direction for how to change the 23: He was supposed to continue counting on from 23 *nine* times.

In my estimation, Adam would not assimilate the 9 as a composite unit in this setting, and would not necessarily unitize it at all. He was capable of unitizing the 9 in this context and operating on it, as he did during the action of his scheme in the later forms of his strategic reasoning (see the discussion after Protocol 7.2). However, this was not easy for him to do, as evidenced by him saying, "Wow. I actually did it….That was hard," after explaining one of his strategic reasoning strategies.

Recall that he was constrained to working within a TNS, which is made up arithmetic unit items that are still closely associated with the act of counting. If he had been able to unitize his arithmetic unit items to form iterable unit items, then that would allow him to disembed subsequences without destroying the sum in order to form additive comparisons or otherwise operate on them more easily. Adam's ability to operate on the difference during strategic reasoning is uncharacteristic for TNS students (e.g., Steffe & Cobb, 1988). Therefore, I hypothesize that Adam was on the cusp of reorganizing his TNS into an ENS. Just as monitoring seems to play a critical role, or at least by an important harbinger, of imminent construction of an INS, I conjecture that the ability to strategically reason when making adjustments of more than 1 unit to the second addend implies an imminent reorganization of the TNS. In future research, I would like to test that conjecture.

Because Adam could not form additive relationships between both addends and the sum in an unsigned situation, he could not do so in a signed situation either. I discuss in the next section that I hypothesize setting up these subset relationships is essential for moving from an assimilation of signed quantities as two unsigned but related quantities, like Adam had, to a unitary signed quantity. My basic argument is that the students need to strengthen the relation they see between their two unsigned quantities. Once there is a sense of a subset relationship, there is a sense that the students are combining these quantities, which implies that they are the same kind of quantity. If the second addend is conceived of differently from the other values, then there is not a sense that all of the quantities are the same. For example, the second addend could be constructed as a transformation while the other two values are constructed as amounts being transformed. In that scenario a mixed-sign addition would not imply the combination of the two unsigned quantities, implying that they are part of the same overarching quantity. It would imply only a possible transformation from one type of unsigned quantity to the other. Therefore, Adam was constrained in his construction of a unitary signed quantity by the nature of his number sequences.

Because Adam is dealing with two different kinds of (unsigned) quantities, he needs to keep track of not only their absolute value, but also what kind of quantity he is dealing with. He then has to construct the relation between the two quantities in order to solve an addition problem. When we worked with actual coins, the piles of coins that were being moved were placed in front of the student often closer to or farther from the savings plate, depending on whether they were being removed from the savings plate or added to it. Therefore, Adam had visual cues as to what kind of quantities each pile represented. However, without the visual aid, he had trouble constructing the signed quantities and so reverted to thinking about the situations in terms of absolute value, with sign only registering a change in absolute value.

In the number line context, he would also have a visual aid. One question that arises is why Adam did not have as much trouble in the card game with attending to changes in absolute values. My guess is that the difference laid in his motivation to attend to the different kinds of quantities. In the money situation and number line situation, there was no sense of competition. In the card game, he very much wanted to know who had won or lost a round and who was winning or losing the game. Therefore, he may have been willing to struggle with the quantities more in that situation. I do know that he *could* solve problems in the money context without coins, but his attention had to be returned to the nature of the context.

In addition to the lack of a more unified signed quantity, Adam's undeveloped sense of length as a quantity may have contributed to his difficulties operating in the number line context. As I have said, he did not attend to what was between the starting points and endpoints of trips.

When he did quantify lengths, he would do so by counting tick marks and never referred to "spaces" between the tick marks, as Lily did. In the other contexts, he was able, sometimes with difficulty, to translate the second addend into a quantity by re-presenting it to himself as a related unsigned quantity and then using the context to determine what kind of unsigned transformation to use and how to interpret the answer. For example, in the card game, he thought about a number of positive or negative points instead of a change in the number of points. In the money context, he thought about piles of coins. I hypothesize that the lack of a sophisticated length quantity was the reason that he had trouble doing the same sort of thing, at least initially, in the number line context: The underlying unsigned quantity, length, was itself problematic for him and so did not act as a way to get around the construction of a unitary signed quantity.

Justin

At first, Justin had a reversible addition scheme, but it did not seem to be anticipatory. In particular, he did not have a difference structure, which would be an assimilatory reversible addition scheme that includes potential subset relationships. Because he could operate with three levels of units, he could attend to and coordinate various number sequences to find patterns in number sequences that involved, to the observer, comparing differences. However, his lack of a difference structure meant that he did not naturally notate differences, as Lily did, and he had trouble assimilating the situation if it involved a comparison of differences, such as using a comparison of weight changes to determine a missing weight (Protocol 5.13).

Furthermore, when he got to the card game, the lack of a difference structure meant that the records of having combined positive and negative quantities through subtraction of their absolute values decayed, which made it hard for him to construct a C3 quantity that implied the combination of positive and negative quantities. He understood that the current C3 quantity required a combination of wins and losses, but it was as if the previous C3 quantities lost their sense of being a combination of sums and differences of round win/loss amounts, and so he would treat them as if they were a sum of only win amounts or only loss amounts, but not a combination (Protocols 6.1–2).

Initially in the money context, he did not readily assimilate his sum as indicating a combination of positive and negative quantities, and so he assimilated his sum as one of the given positive or negative quantities and combined it with the opposite signed quantity in order to get a new sum. This was not a necessary error, because, at this point, he was constructing a difference structure, which we see the first strong indications of in Protocol 6.13. In fact, his reflection on his solution to the tasks in Protocols 6.7–8 may have helped him construct the difference structure as an assimilatory structure, i.e., he became aware of the difference before calculating the difference of absolute values instead of forming the difference as he formed the difference of absolute values.

His lack of a difference structure was also evident in his strategic reasoning. When he attempted to reason strategically in a MA problem by adjusting the smaller (in absolute value) given quantity, he could not figure out how to adjust the estimated difference to find the actual answer. My hypothesis is that this requires comparing a hypothetical difference with a calculated difference: the difference that would result from adjusting your given values and the difference of the given values. In order to plan the solution process before action—for example, thinking, "I will subtract $(M + 3) - (N - 2)$ and then I will subtract 3 and subtract 2 to get the actual difference"—the student would actually compare two hypothetical differences. This should, for example, be in Lily's ZPC, although I do not have evidence that she ever planned her solution method before action in a strategic reasoning situation.

It was interesting that Justin started referring to additive inverses (Protocol 6.8), or their result, the 0 quantity, before he had yet finalized his construction of a difference structure, and, therefore, before his construction of a unitary signed quantity. He did not even have consistently viable signed addition/subtraction schemes at that point. This last part surprised me in retrospective analysis because I had formed a working hypothesis that students first developed signed addition and subtraction schemes, then started recognizing the importance of additive inverses to get back to a 0 value, which acted as a catalyst for developing a unitary signed quantity. In fact, when students are still working with two related, unsigned quantities as making up their signed quantities, there is a relation between the two quantities that I have referred to elsewhere as situational or intuitive. Reference to additive inverses and/or the 0 quantity can represent a more explicit awareness of this relation between the two unsigned quantities: positives and negatives. Justin's case shows that a student can develop this explicit awareness of the relation while still constructing viable signed addition and subtraction schemes.

Justin's ability to construct a difference structure and a unitary signed quantity during the course of the teaching experiment was definitely dependent on his prior construction of an ENS. As I discussed with respect to Adam, a TNS student would lack the disembedding operations necessary for construction of a difference structure. Another characteristic of Justin's mathematics that facilitated the construction of a signed quantity and, even more so, facilitated his increasing awareness of additive inverses as subsets, was his facility working with continuous length quantities.

Justin seemed to come to the study with a predisposition to think of directed quantities on a number line or number sequence. In addition, he had developed the ability to coordinate two number sequences, which allowed him to attend to multiple reference points. I hypothesize that these two characteristics of the mathematics of Justin supported an early attention to additive inverses and 0.

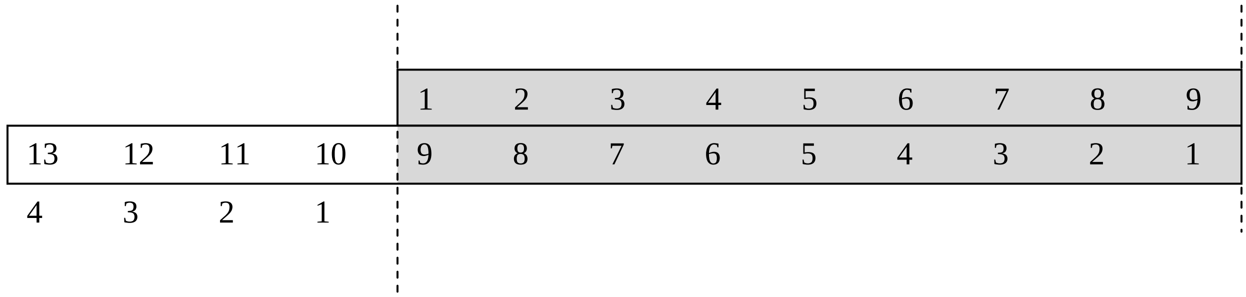

Figure 8.1. Coordinating number sequences for (+9) + (-13).

In order to attend to 0 and additive inverses at all, I hypothesize that students would need to be able to construct the kind of coordinations illustrated in Figure 8.1. Coordination of two number sequences during mixed-sign addition, by which I mean the ability to produce two directed number sequences and be aware of the positionality of numbers on the second one in relation to the reference value (0) for the first number sequence in addition to their positionality within the second number sequence: The positionality that corresponds to 0 on the first number sequence must become a new reference point on the second number sequence. The first part of this process sounds similar to double-counting when counting down a sequence. Indeed, this is basically what is going on, except that the numerosity of the double-counted portion must be the same kind of composite quantity as the quantity described by the initial number sequence. In the case of Adam, he could keep track of counting down a number sequence, but the resulting subsequence could not be disembedded to form a separate composite unit. It was a composite unit that was tied to the original number sequence.

In contrast, disembedding allows the student to count beyond 0 of the original number sequence. In particular, as in Figure 8.1, the student can use the new reference point on the second number sequence that corresponds to 0 on the original sequence in order to cut off a countable subsequence of the second number sequence representing the sum. Perhaps this ability to count down beyond 0 is the additive equivalent to the construction of an improper fraction. The double-counted subsequence must be able to go beyond the original composite whole of the first number sequence. Like the improper fraction scheme, this certainly requires the construction of three levels of units in action. However, unlike the improper fraction scheme, this probably does not strictly require assimilation with three levels of units since the subsequence of the second number sequence that extends beyond the original number sequence does not need to be compared additively or multiplicatively to the original number sequence. It is simply measured in units of one. In the case of the improper fraction scheme, the student must be aware of the multiplicative relationship of the units in the extension to the whole.

Nonetheless, in order for a student to choose to use this strategy of coordinating directed number sequences when other strategies are available, such as translating the situation into unsigned, related quantities and using the context to construct a relation between them, will not generally happen until the student can assimilate with three levels of units or has constructed a GNS. In Figure 8.1, the dotted vertical line segments represent the reference points for the two sequences. The ability to attend to the reference point corresponding to 0 in both sequences would seem to underlie the attention to 0 when solving a mixed-sign addition problem. The additive inverse to the original initial number sequence also has an “opposite” feel to it in that it is in the opposite direction of the original sequence and, because of the attention to the 0 reference point, the additive inverse is clearly marked off and unitized, making reference to the additive inverse as a quantity in problem solving more likely. Therefore, I believe that this way of coordinating number sequences triggers the development of directed additive subset relationships. The fact that a GNS supports both developments is supported by my findings that only Justin and Lily developed an attention to additive inverses in their explanations.

Lily

Lily's ability to operate in virtually all the signed, additive situations she encountered was no doubt helped by her sophisticated additive relationships with unsigned numbers. I call her assimilatory additive structure a *generalized sum structure*, and I hypothesize that this assimilatory structure allowed her to be aware of addends as complementary subsets to a sum in additive situations. I am including subtraction situations as additive situations, which is what makes this structure "generalized." She would be aware of the same underlying subset relationship in a related addition and subtraction problem, for example. She would only assimilate them differently in the sense that different quantities are unenumerated in the two problems.

In particular, her generalized sum structure supercedes and includes a difference structure, so she did not run into problems associated with the lack of a difference structure when attempting to construct signed sums and differences as combinations of both positive and negative values. I hypothesize that this allowed her to form a unitary signed quantity more quickly than other participants.

Her unsigned additive operations also ensured that she was comfortable operating on and notating differences. Because of her structural understanding of additive situations, she would not have trouble comparing two undetermined differences, as she did in the weight context, for example. In addition, her comfort in notating differences in order to extend the number sequences in February ("Number patterns" in Chapter 5) allowed her to be more confident in her solution and more explicit in her ability to describe the additive relationships at play than Justin was. Similarly, I would conjecture that her comfort in reasoning structurally about differences, that is, reasoning about the additive relationships between undetermined differences also allowed her to operate comfortably in the card game. She even had the ability, without notating intermediate steps, to determine a missing card value given two C3 values. She seemingly was aware of the C2 values as being made of differences of card values before calculating the C2 value.

Like Justin, Lily's ability to reason with additive inverses in solving signed addition problems was likely contingent on her ability to assimilate with three levels of units and her tendency to count back to determine differences. Unfortunately, I never had the opportunity to see if she could distinguish between signed addition and signed MA problems notationally, but given that she is the only student who notated unsigned MA problems as addition problems, I suspect that she may very well have been able to do so. In future research, I would like to work with a student who, like Lily, has constructed a GNS and a generalized sum structure to see if such a student can make sense of my notation for signed MA problems.

Michelle

Michelle's fairly advanced strategic reasoning (surpassed only by Lily) indicates an anticipatory reversible unsigned addition scheme. Given her construction of an ENS, she might have constructed a difference structure as an assimilatory construct. Her strategic reasoning at least indicates the ability to construct unsigned additive relationships. This seems to be enough for her to keep an awareness of a signed sum as result of unknown additions and/or subtractions, allowing her to operate smoothly in the normal course of the card game.

Like Adam, she does not seem to understand the necessity of the relationship between related addition and subtraction situations on the first day in the money context. Therefore, I do not attribute a generalized unsigned sum structure to her. This would imply that a generalized signed sum structure is not yet in her ZPC. As expected, then, she was not able to make sense of my subtraction notation.

Another factor in her struggle to make sense of my notation was her strong association of addition and subtraction signs (and equal signs) with actions she is carrying out. As with Adam, a written equation would not represent a structural relationship, but instructions for action and the result of the action. This also implies that she is not assimilating with a generalized sum structure because her sense of additive relationships is not yet structural, rooted in the constructed subset relationships that she forms in action.

The lack of indications that she had constructed a GNS fits with the finding that she did not attend to additive inverse or a 0 quantity in her explanations. Even when her comparative explanations became fairly explicit, she never gave an indication of understanding the necessity of the operation she used to determine sums or missing addends without the visual aid of a diagram. Instead, she attended to what the sign of the sum or missing addend would be through holistic comparisons (sometimes reversible holistic comparisons) and then determined how the magnitude of the solution would compare to the magnitude of the two given values. This probably reflected the fact that she did not construct signed additive relationships in the structural sense.

Social Factors

Because my third research question—*What aspects of the social interactions among participants and the teacher/researchers impede or facilitate participants' ability to work in situations of signed addition and the researchers' ability to construct the mathematics of each participant in these mathematical contexts?*—developed after data collection, my findings do not provide definitive indications of the mechanisms by which various aspects of social interactions affect the research and learning process. However, some social factors had important consequences for the study, and so I would like to discuss the relationships between these social factors and the mathematics and observable behavior of the students. At the end of the section, in Figure 8.2, I have summarized the social factors that I attended to most in data analysis and how they affected the study. I hope to follow up on the affects of some of these social factors, such as gender, in future studies that are more specifically designed for that purpose.

The two goals during data collection were to engender mathematical learning and to gather data about the students' mathematics, with the understanding that one affects the other. In particular, an abundance or dearth of opportunities for mathematical learning would both have noticeable effects on the quality of my data. In the first case, I would get indications or contraindications that a construction is within a student's ZPC. In the second case, I would not get any indications of how mathematical learning takes place. Furthermore, the more information I had about the students' mathematics, the more I could refine my model, the mathematics of students, in order to inform my interactions with the students.

Therefore, I was attentive to how cognitive and non-cognitive factors affected students' opportunities for mathematical learning and the quality of the data I was able to get about the students' mathematics. The two factors that seemed most immediately relevant to both were the students' orientation to problem solving and the quality of mathematical interactions between a pair of students and between the students and the teacher/researcher. The most striking example was Adam, who was very hard to get engaged in problem solving activity by the end of his time in the teaching experiment. The result is that I do not have very much data about how he was constructing signed quantities in the number line context. Michelle, throughout the teaching experiment, would attempt to apply procedures and look for patterns as opposed to engaging in problem solving at a deeper level. The result is very ambiguous data about her. For both students, not engaging in quantitative reasoning means that they lost opportunities for reflective abstractions and the internalization or interiorization of quantitative schemes. On the other hand, I have a lot of informative data about Justin's quantitative reasoning because he was engaged in such reasoning almost all the time he was in a teaching session.

Orientation to Quantitative Reasoning

The factors that I noticed had a direct effect on a student's tendency to engage in quantitative reasoning were four-fold: the importance to the student of conforming to teacher expectations, the students' emotional experience, the students' conceptions of what it means to do mathematics, and their current mathematical tools. Some of these factors really interacted to determine a particular students' orientation to quantitative reasoning. For example, both Lily and Michelle seemed intent on positive feedback from the teacher. Both of them came to the teaching experiment having constructed the social norm of utilizing procedures as the preferable way to solve problems, the more elaborate the procedures, the better. I assume that their application of procedures in a word problem situation would bring them accolades from their classroom teacher. Therefore, their desire to conform to teacher expectations resulted in a very procedure-centered view of doing mathematics at the start of the teaching experiment. However, I believe that Lily's desire to conform to my expectations resulted in increased quantitative reasoning both because she could reason about my tasks with limited perturbation based on her sophisticated mathematical tools and because she already had an implicit view of her mathematical procedures as dependent on underlying quantitative relationships. Therefore, my attempts to get her to quantitatively reason and explain her reasoning seemed reasonable and fit with her sense of doing mathematics. Michelle, on the other hand, experienced more perturbation when attempting to quantitatively reason. Furthermore, I think that her procedures were “black boxes” in that she had no awareness of a quantitative underpinning for her procedures, they just got her the correct answer. Because she was able to get the correct answer and explain her procedure, our requests for her to investigate why the procedures worked did not seem like important mathematics to her, and it was harder, so she resisted quantitative reasoning as long as she could get the right answer.

Attempts to conform to teacher expectations. I have already discussed some of the effects of teacher expectations with regards to Lily and Michelle’s quantitative reasoning. There are two other points that I would like to address related to teacher expectations. The first is that neither of the two boys showed the same desire to do the problems in a prescribed manner. Both seemed to come to the teaching experiment with a tendency to engage in quantitative reasoning and the teacher/researchers’ request to do did not have as big of an effect on their behavior. In fact, Adam shied away from quantitative reasoning later in the teaching experiment even though he knew and understood the kind of reasoning I was asking him to engage in. Justin almost never gave signals that he was embarrassed to get the wrong answer, and he seemed to have the attitude that anything he could do to help him solve the problem was acceptable. For example, he was not afraid to try solution strategies that might be wrong, play with the fraction pieces to solve fractional problems, count on his fingers, etc. Lily, on the other hand, was very hesitant to do or say anything that might display a lack of understanding or sophistication to the teacher. This is not to say that Justin would not ever get self-conscious. When the teacher/researcher in the money context tried to make a problem easier for him, he noticed right away and reverted to procedures. I do not think this was so much because of the teacher’s behavioral expectations, though, I think that he felt that his mathematics was being underestimated and so he imitated

Michelle's procedures until he felt that his mathematics was respected again. In contrast, in the card game, he vehemently defended his unviable solution method for determining a C3 quantity until Michelle convinced him why it did not make sense quantitatively.

This desire to conform to teacher expectations does not cut along ability lines or even the propensity for problem solving among my four participants, it cuts along gender lines. This is in line with research on the role of gender in educational experiences that has shown that girls are expected to be docile and that assertiveness is viewed negatively in the classroom (e.g., Renold, 2006). Meanwhile, boys are more encouraged to take risks. The effects of these kinds of expectations can be seen in a study in which girls are found to be less likely to answer a multiple-choice item than boys are if they are not confident about their answer (Kahle, 2004). This brings to mind the difference between Lily and Justin's behavior when thinking about solution strategies. Lily was much more worried about making a mistake.

In addition, boys' work will be praised for being unique or brilliant, while similar work from girls is less likely to be praised for those reasons and more likely to be praised for its appearance (Liu, 2006). Over time, such small indications would encourage boys to develop unique solution strategies and encourage girls to focus on doing things "the right way," which could mean imitating a teacher's procedures in a mathematics classroom. A recent study confirmed the conventional wisdom that high school mathematics teachers tend to rate boys' as having a higher mathematical ability than girls with the same test scores (Riegle-Crumb & Humphries, 2012). (Actually, the effect was only significant among White students, but the general idea holds.) In other words, teachers are more likely to attribute problem solving ability to boys and encourage problem solving activities, which are inherently error-filled endeavors, in boys. On the other hand, teachers tend to discount the sophistication of problem solving behavior in girls. This cuts close to home. One of the reasons I chose to continue with Justin instead of Lily after March was because I felt that Justin was the stronger reasoner of the two. In fact, in retrospective analysis, I have found that Lily was probably the stronger reasoner. I hypothesize that I took Justin's high level of visible problem solving activity and his willingness to make mistakes as indications of self-confidence in mathematics that was grounded in ability. In fact, he was quite capable, but Lily was too despite a lesser show of problem solving activity. I did not take gendered behavior into account when making my assessment.

I have been discussing how the desire to conform to teacher expectations can affect problem solving or quantitative reasoning behavior. However, many similar conclusions can be drawn with respect to the social interactions. As I have indicated, Lily was less likely to share developing strategies or explanations than Justin or Adam. She wanted to feel confident in her answer before volunteering information. In Michelle's case, attempts to conform to expectations were more noticeable in her individual solution strategies than in her verbal interactions. However, in both cases, the interactions between the girls and the teacher/researcher yielded less data about their mathematics. Often the best data is based on unviable solution strategies, which Michelle and Lily may have been more likely to avoid or keep to themselves even if they were facing the situation with the same mathematical tools as the Adam and Justin.

Emotional experience. When a student was feeling embarrassed or frustrated, this generally would cause the student to withdraw from engaging in difficult quantitative reasoning. On the other hand, if a student was feeling proud of a recent accomplishment, the student would be more likely to engage in difficult reasoning. The effect of emotional state on Adam was particularly pronounced. Some days he came in very tired, and I knew that I would not get any good mathematics out of him until I got him joking around again. However, I have already noted that Justin would back off from difficult reasoning if he felt embarrassed. Michelle tended to get frustrated with tasks for which she did not have a clear solution path, although she would generally re-engage fairly quickly if prompted by the teacher/researcher. Often she would do so by asking Justin or the teacher/researcher how to solve it.

Another factor I will discuss later that affects a student's emotional experience is if their ZPCs are much more limited than their partner's. This was only an issue for Adam, but it had the effect of making him feel embarrassed at times, which caused him to withdraw. When he was given different tasks than Michelle, both he and Michelle seemed to recognize that hers were harder tasks. Even when he was working with Lily, he had noticed the disparity by the second day. When he gave a sophisticated explanation and she could not, he said, "It's backwards now," indicating that he realized her explanations had been better on average than his.

Current mathematical operations and schemes. Notice that the way in which social factors play out is *mediated by mathematical ways of operating*. For instance, Lily had the same orientation to procedural explanations as Michelle originally, but her mathematical ability had given her different past mathematical experiences, which caused her to construct procedures and mathematics differently than Michelle did. More generally, the two stronger students, mathematically, seemed more inclined to try to make sense of mathematical procedures, implying that both considered there to be something to mathematics other than procedures, such as quantitative reasoning. If there is a mismatch between a students' ZPC and the mathematics underlying the procedures a student is taught to use, the potential opens up for student to interpret the procedures themselves as the important mathematics, as appeared to have happened with Michelle. Adam was an interesting case because he did enjoy problem solving and quantitative reasoning, but he did not seem to link it to his procedures. That is, he did not try to reconcile his idiosyncratic solution strategies with learned procedures. I hypothesize that understanding the mathematics behind many procedures was outside of his ZPC, hence, he would, like Michelle, have learned that there is no point in questioning the mathematics that proves the procedures work as long as they give the correct answer.

Of course, mismatches between mathematical tasks and a student's ZPC were not confined to their regular classroom. We gave students tasks outside their ZPC during the teaching experiment as well. For example, all of the students were given fractional tasks outside their ZPC at some point. Interpreting notation for signed addition was outside Adam's ZPC, and differentiating between signed addition and MA problems was outside Justin and Michelle's ZPC. When such a mismatch occurred it was usually a frustrating and defeating experience for the student, negatively affecting their emotional experience.

Finally, since one of our goals was to engender student learning, it is not surprising that some of the opportunities for mathematical learning did result in more powerful mathematical schemes or operations for the student. For example, at some point, Justin started using an anticipatory reversible addition scheme, and later a reversible signed addition scheme. Adam developed the ability to strategically reason. Michelle developed schemes for adding fractions, and Lily developed a reversible improper fraction scheme. These are just some examples of the ways in which the students' mathematics developed over the course of the teaching experiment. In the next section, "Overview of Constructions Involved in Signed Addition Situations," I discuss the development of their additive reasoning in more detail.

Social Mathematical Interactions

Mathematical interactions between students or between a teacher/researcher and the students paved the way for further opportunities for mathematical learning. This could happen through a student's attempt to make sense of another student's solution strategy, or it could happen in an attempt to explain their own solution strategy. Both encourage the student to reflect on mathematical activity and quantitative relationships. However, the quantity of interactions between students varied greatly, and the quality of the interactions of a teacher/researcher and a student also varied. I have already alluded to the impact of the student's desire to conform to teacher expectation. In addition, personal interaction styles, social groups, and disparity between the students' ZPCs could noticeably affect the frequency or quality of interactions.

Social groups. I hypothesize that *social groups* played a role in the amount of mathematical interaction between the participants. To begin with, Michelle and Justin knew each other because they were in the same "cluster" and had common classes. Similarly, Adam and Lily knew each other from their cluster.

Throughout the first month, Michelle clearly wanted to change partners. In the hall on our way to or from their classrooms, she asked several times if they were going to switch partners that day. Once we had changed partners, both pairings showed more interaction between the students than in the original pairings. In Adam and Lily's case, it was generally in the form of explaining solutions to each other. They would often react to the other person's solution also, either by praising or correcting it. Justin and Michelle also interacted more than either had with their previous partner. Michelle did not generally attend to Justin's explanations, but Justin did attend to hers. In addition, they would actually solve problems as a pair at times, as in Protocols 6.9 and 6.15. The only time Michelle did that with Adam was their very first task together (Protocol 5.1).

In the other initial pair, Justin and Lily, I conjecture that the same social divide existed. Hence, in the case of Justin and Lily, although both warmed up to the teacher/researcher, they did not interact with each other without a direct request from me. Even then, they would sometimes look at me when giving their explanation to the other person, with Lily being more likely to do this.

Student and teacher personal interaction styles. However, within the pairing of Adam and Michelle there were several mitigating factors to the difference in social group. First of all, neither student seemed to be naturally shy or quiet. Secondly, the teacher/researcher was very lively in his *style of interacting* with lots of jokes, laughter, and excitement on his part. This was often infectious and probably helped build rapport within the entire group.

In retrospective analysis I have noticed that Michelle is the only student who was involved in every instance of joint student mathematical activity. When she was working with Adam, we very quickly decided to give them different problems, so that closed off the possibility of increased mathematical interaction between them. However, when she worked with Justin, she did collaboratively solve problems with him several times. Each time she instigated the interaction by saying something like, "How are you going to solve it?" She was not just trying to get an answer from him because she would contribute to the solution activity. It is possible she was trying to use his general solution path; she got frustrated when she was not sure how to solve a problem. However, once they got started, she would sometimes instigate a change in their activity. For example, in Protocol 5.1, she challenged their choice to subtract absolute values by saying, "No, 'cause…it has to be bigger than 74. Wait. Then wouldn't we just add 15 to 74?" I worked with Michelle for several of her last sessions, and I tended to control the conversation in that I would direct my questions to specific students, removing the possibility that they would work collaboratively to answer my question. In retrospect, I should have tried to integrate more collaborative work between Justin and Michelle, as this seemed to be the interaction style she was more comfortable with and it meant that she was always engaged in the mathematical activity at hand.

Unlike with Michelle and Adam, there were not mitigating factors to overcome their membership in distinct social groups. Lily was naturally reserved, and Justin seemed naturally reserved around strangers. I base this conclusion on several observations: I had seen him interacting with friends in his regular classroom and hallway, and he was always talking and joking around in those contexts. Furthermore, he interacted much more freely with Michelle, who he knew. I even noticed that after having me as a teacher/researcher for a month, when he switched to working with a teacher/researcher he did not know as well he became slightly more reserved.

As a teacher/researcher, my interaction style is marked by a lot of long pauses in which I expect students to think about a question, perhaps because that is the way I work best when doing mathematics. Hence, my interaction style did not improve the rapport between Justin, Lily, and I. Luckily, those two students were a little like me in that they naturally stay mathematically engaged as long as there is a mathematical problem for them to work out, and they seem to do so better when they do not feel pressured or distracted. Therefore, they did do a lot of good individual mathematical thinking when working with me.

In contrast, Michelle and Adam, for different reasons, tended to withdraw from mathematical activity without interactions from the teacher/researcher. In addition, Adam was definitely a joker, and seemed to feel more comfortable in an environment where more joking was going on. Therefore, they would probably not have worked as well with me as their teacher/researcher.

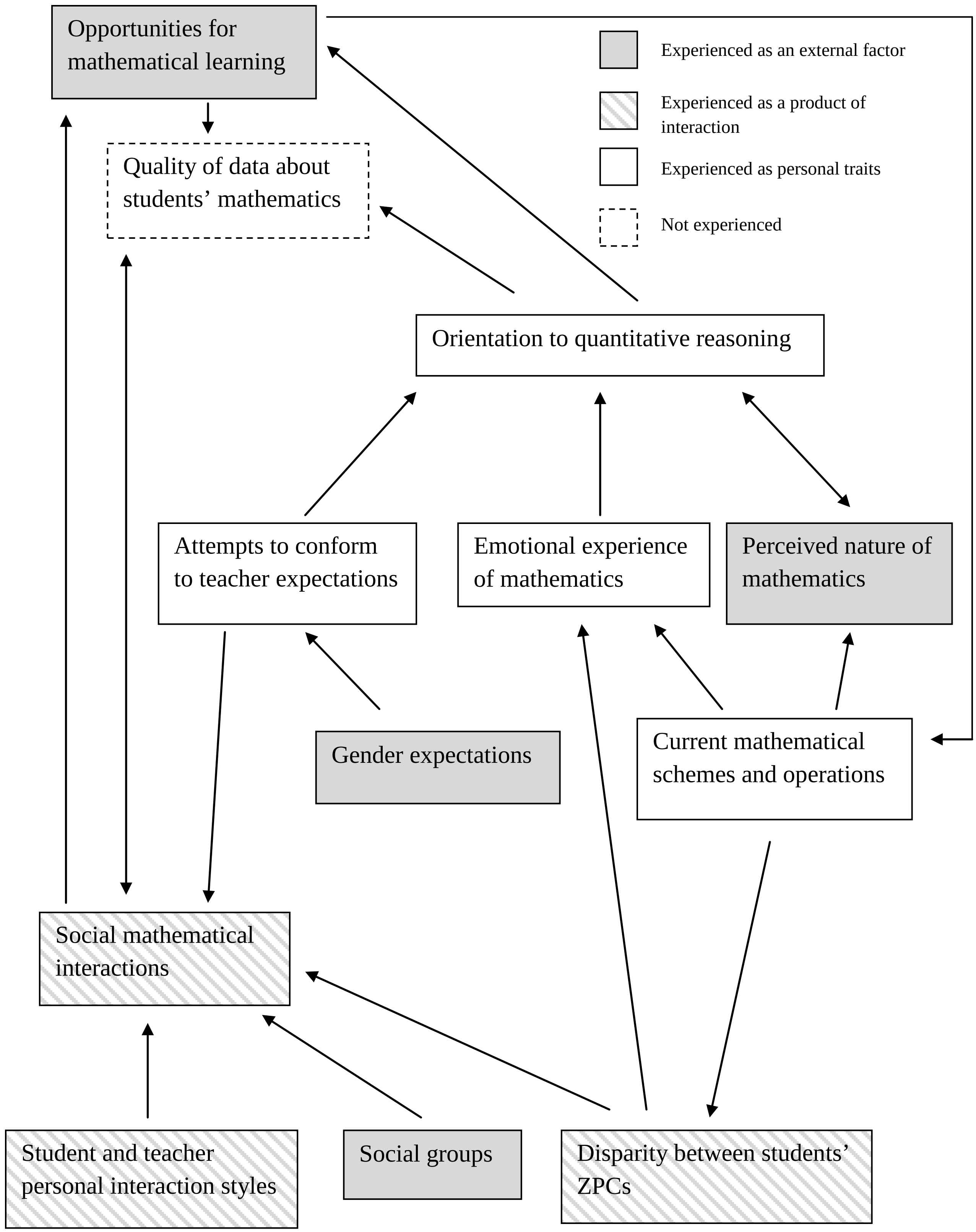

Figure 8.2. Summary of proposed social factors in my findings.

Once we switched pairings, I worked with Adam for the majority of March. He started to get stay less engaged after the card game. This was probably due, in part, to the difficulty he was having with the tasks. In addition, he had a cold for the last couple of weeks. However, he did react with more energy to other teacher/researchers who tended to put a bigger emphasis to developing rapport, so my interaction style may have played a part in his lack of engagement towards the end of March.

Disparity between students' ZPCs. Adam, while often working full-steam on problems, would withdraw mathematically when not directly given a problem. This pattern developed when he was working with Michelle and could be a result of the mismatch between his ways of operating and the questions being asked of her. Therefore, the difference in their ZPCs in a fractional setting was a factor in reducing his engagement with her mathematical activity. I noticed that in the computer environment, he paid close attention when I was showing Lily how to do something new in GSP even though she was solving different problems. Therefore, he did stay engaged when the mathematics was not overwhelming to him.

In contrast to Adam and Michelle in fractional contexts, Justin and Michelle had similar ZPCs in the additive situations. This allowed them to collaboratively solve problems in which one student was not helping or explaining to the other, but both were thinking through the problem out loud. These kinds of interactions provided rich data about their mathematical thinking.

Summary

I have summarized the above discussion in Figure 8.2. The boxes indicate aspects of the study or the students' experience that affected the study. The solid grey boxes signify those aspects of the students' experience that the students' would not necessarily feel they could affect.

The striped boxes are aspects of the students' experiences that they may feel some control over. The white boxes are aspects of the students' experience that may attribute to themselves as personal traits. The "Quantity of data about students' mathematics" box is dotted to signify that this is not part of the students' experience. The arrows in the diagram indicate an effect of some kind on the construction of another factor. The lack of an arrow does not indicate that there is no effect of one factor on another, it merely indicates that this type of effect was not salient in my analysis. For example, increased social interaction during mathematical activity could change someone's perception of mathematics from an individual pursuit to a group construction. However, my data does not provide indications of that.

Overview of Constructions Involved in Signed Addition Situations

In answer to my fourth research question—*What, if any, changes are there in how the participants assimilate situations of signed addition?*—I present in this section an overview of the various stages I saw in how students constructed signed quantities and their additive relationships. In particular, I agree with the extant literature on the initial separation between positive and negative quantities in which students assimilate them as two unsigned quantities that are related by some circumstance of the situation. For example, a student in the card game could think of positive round scores as points won and negative round scores as points the student's opponent won. The C3 quantity could represent the total number of points the student has if it is a positive value or the total number of points the opponent has if the value is negative. The relation is then that winning points decrease the opponent's score if the opponent has points and increases the student's score if the student has points. As long as such a relation is present for the students, I call these *signed quantities*.

The transition from that kind of signed quantity to the *unitary signed quantity* that I have discussed is not abrupt. It seems to consist of a gradually increasing sense that the two types of signed quantities are related until the student is able to assimilate them as, in fact, representing the same quantity. Recall that in the card game, for example, Justin, and Adam to a lesser extent, would oscillate between thinking of the C3 quantity as being part of one or the other of the C2 signed quantities and thinking of the C3 quantity as being a directed sum of the C2 signed quantities. Students first seemed to develop an awareness that sums, while still ultimately thought of as representing one of their two signed quantities, depending on the value of the sum, also contain a record that relates to both quantities. For example, in Justin's case, instead of a sum representing the total wins or the total losses in the card game, it came to represent a total of wins and losses.

The interaction of the two signed quantities through addition and later through comparisons that correspond to signed subtraction provide opportunities for students to build up this sense of relation between the quantities. The development of viable signed addition and subtraction (MA) schemes is possible at this stage, as evidenced by the ability of Adam, who was constrained to this stage throughout the teaching experiment, to consistently evaluate both signed sums and signed missing addends in the card game. In addition, explanations involving holistic comparisons will emerge at this stage.

Possible constraints to the construction of a signed sum as the first step in the transition to constructing a *unitary signed quantity* include the need to assimilate situations with a reversible addition scheme or a difference structure. Based on my findings, strategic reasoning appears to be a powerful way for students to have practice reflecting on the quantities involved in their addition and subtraction problems in order to allow the formation of a difference structure. In particular, both Adam and Lily seemed to become more aware of the subsequence/subsets they were working with. In Adam's case, he seemed to come very close to the point at which he would reorganize his TNS into an ENS when working on strategic reasoning, which would have allowed him to make more progress in his constructions during the teaching experiment. In the case of Justin and Michelle, they found an easier way to reason through the problem which allowed them to sidestep complicated strategic reasoning. However, I would conjecture that strategic reasoning in which the first and second addends are being adjusted in a problem would have been helpful for Justin in forming a difference structure. In addition, the number sequences appeared to help him reflect on differences as well.

The next step towards developing a unitary signed quantity involves assimilating both signed addends and signed sums as the same kind of quantity so that subset relationships can develop. A student could potentially do this in a situation where the signed quantities are not considered transformations of amount or position, such as in the chip model (Flores, 2008), but in this teaching experiment the tasks were meant to engender the assimilation of addends and sums as transformations. I gave examples of Adam attending to the first addend and sum in the number line context as values that did not represent transformations. This is a contraindication of this stage. In contrast, Justin, by independently labeling both the directed length of trips and their endpoints, gave indications of assimilating the given quantities as transformations in position.

This step will not be feasible in the short term if a student does not assimilate unsigned addition and subtraction situations in terms of the same quantity. For example, in Adam's case, the second addend tended to be an action or transformation, while the other addend and the sum were sets. Also note that in Adam's case, the overlapping trips in a number line model did not help because he was still constructing length as a continuous quantity.

This step might be able to be skipped if working with a chip-type model. However, this step involves the construction of transformations as quantities in the case of my participants, which is an important construction for students to make in the middle grades. Therefore, it might not be desirable to enable a student to skip this step through the use of the chip model, for example.

At this point, the opportunity is opened for students to construct subset relationships between the signed quantities. This is indicated through quantitative comparisons (as opposed to holistic comparisons) during explanations. In particular, students who are constructing subset relationships will identify a 0 state, such as “back to where we started” or “the same amount we started with.” They will also identify the additive inverse to the first addend in some way, whether by decomposing the second addend into the additive inverse and its complementary subset or by indicating how much bigger (in absolute value) the second trip needs to be to get back to 0. At this stage, I say that a student has constructed a unitary signed quantity in that their signed quantities in a problem are all of the same type and are now closed under addition and subtraction.

Although I did not run into this issue with my participants, I would like to specify that I would want students to be able to form these subset relationships even when diagrams or manipulatives are not available. This is precisely because students will need to interiorize a generalized version of these directed subset relationships eventually. In particular, if students are going to be adding or subtracting signed variable quantities, these relationships need to have been internalized. As I discuss with relation to Lily in Chapter 7, I hypothesize that the development of these subset relationships is unlikely to happen without a GNS. I would add here that, based on my analysis of Adam, I conclude that it would not be possible for a TNS student until the TNS is reorganized into an ENS. A TNS student does not set up additive subset relationships in an unsigned context because they cannot disembed their subsequences in order to disembed, combine and compare the result with the sum in the original sequence. In addition, TNS students assimilate unordered subsets with their TNS so that although they are aware of the ordering in a subset as being unimportant for the numerosity, any quantitative instantiation of the subset would have a sense of ordering to it based on the sequentiality of TNS number sequences.

The final step, which I do not have indications of in any of my participants, is the interiorization of the directed subset relationship to form an assimilatory *generalized signed sum structure*. As I mentioned in my previous subsection on Lily's additive schemes, she was probably the student with the best chance of developing or having developed this generalized signed sum structure. In future research, I hope to work with student mathematically similar to her in order to witness this final step in constructing signed, additive relationships.

I have summarized the relationship between changes in conceptions of signed quantity and addition and subtraction schemes in Figure 8.3.

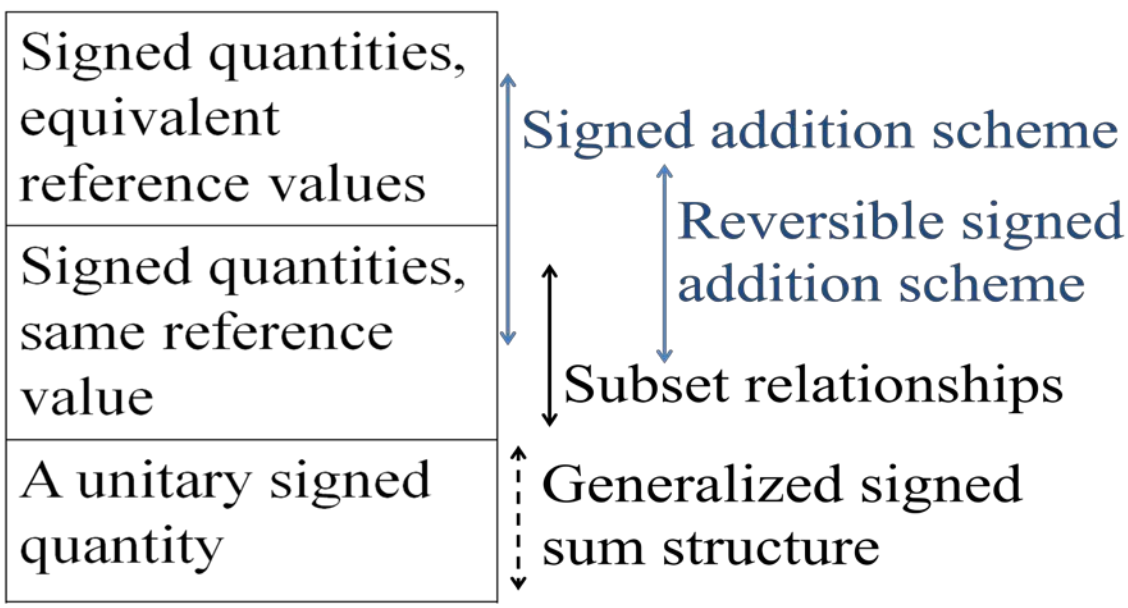

Figure 8.3. Signed additive schemes.

With reference to Figure 8.3, Adam was always in the first stage of signed quantity conceptions. Therefore, he did not construct subset relationships or a generalized signed sum structure. Michelle got to the second stage of signed quantity conceptions and could construct subset relationships with visual aids. However, she did not construct what I would call a unitary signed quantity. Lily and Justin did. I do not have evidence on whether Lily could have or had constructed a generalized signed sum structure, but I have contraindications that Justin could.

Links to Extant Literature

I have already discussed how my findings relate to literature related to gender. In this section, I discuss the relationship to literature on mathematical learning.

My findings confirmed the difficulty some students can have in working with the both addends as changes found by Vergnaud (1982). He also had found that the easiest kind of addition and subtraction situations for students involved an action as the second addend or subtrahend. This fits with my hypothesis that Adam still considered the second term in an addition or subtraction expression as representing the extent of an action, as opposed to thinking of it as a set.

My findings confirmed the existence of an early stage of signed number development in which the students think of the positive and negative quantities as more or less separate, as indicated by Peled (1991), for example. My findings add to this confirmation indications of why some students do not move on from this phase. Namely, students, like Adam, who have not yet constructed an ENS will not yet be able to construct the subset relationships necessary to transform signed quantities into a unitary signed quantity. This transformation will also not be likely to be completed by an ENS student, like Michelle.

My findings that students will sometimes conflate actions with their results, which is often equivalent to conflating a second addend and a sum, is consistent with Dreyfus and Thompson's (1985) finding that students will initially conflate directed position on a number line with the directed length of their trips on the number line. My findings also indicate that this conflation can be due to the need to develop difference as an anticipatory or assimilatory construct, as with Justin, or it can be due to a difficulty in reifying transformations, as with Adam.

Although I did not utilize a chip model (Flores, 2008), or a model in which signed quantities are related situationally by the ability to cancel out an equal value of both, my analysis leads me to believe that a student could develop a generalized sum structure in these settings, however, that construction would be dependent on the ability to form the underlying unsigned sum structure. A TNS student, like Adam, for example would not be able to do this. However, a student such as Michelle, who has constructed an ENS, may be able to develop a generalized sum structure in this setting. The idea of changes as quantities is not inherent in this situation, so the reification of transformations is not necessarily necessary to form what I would call a generalized sum structure. However, I would argue that the need to reify transformations is necessary for later work, with rates of change for example, independently of its necessity for working in additive, signed contexts.

Implications for Mathematics Education in the Middle Grades

Justin's case implies that some sixth graders of "average" mathematical ability need more practice reasoning about addends and differences. This can be done in the classroom through encouragement of strategic reasoning and through the use of unsigned complex additive situations such as those used by Thompson (1993) and colleagues or number sequences in which the obvious pattern is found by considering the differences of elements.

In fact, preparatory learning in unsigned contexts would have made the construction of a unitary signed quantity easier (or possible) for Adam and Michelle as well. For example, they both show contraindications of having constructed an unsigned generalized sum structure by not reacting to and using the underlying subset relationships in related addition and subtraction problems. This structure is an important precursor to the generalized signed sum structure and to the building of subset relationships between signed quantities. In Adam's case, strategic reasoning that involved adjustments of both addends seemed to be within his ZPD, but pushing his ability to reflect on subsequences in his TNS. This activity may be helpful for students to attend more explicitly to subset relationships. Michelle also seemed to enjoy strategic reasoning, but she probably would have benefited from encouragement to make her reasoning about adjustments more explicit.

My findings also indicate the important role that attempts to communicate mathematical thinking has in encouraging students to reflect on and internalize quantities and quantitative relationships. I believe this is shown in two ways in my study. The first is that students tried to imitate each other's solution methods, which encouraged them to think about the quantitative relationships differently. This happened, for example, when Justin tried to use Michelle's method of strategic reasoning, and when Michelle makes a quantitative comparison in imitation of Justin's previous explanation. Long-term effects were evident from Adam's attempt to strategically reason like Lily did, and Michelle's imitation of Justin's style of diagram which allowed her to make sense of all the quantities and reference points in the number line situation as well as formulate subset relationships between the directed trips. Although she did not appear to internalize the construction of these subset relationships, this is an important first step to doing so.

The second way that attempts at communication supported the mathematical reasoning of the participants was through repeated requests for clarification from a teacher/researcher. These questions generally asked why the students did something: Why did you add the two numbers? How did you know to stop counting? These questions seemed to spur students to reflect on their solution methods, which encourages mental re-presentations and reflections on mental re-presentations. I believe that this helped Adam make his subset relationships more explicit with unsigned quantities, and I believe it helped the other three make subset relationships more explicit with signed quantities.

My findings also indicate meaningful notation of addition and subtraction of signed quantities is not a reasonable expectation for students who have not yet constructed an ENS. Furthermore, the ability to carry out addition and subtraction with signed quantities does not imply that the students are aware of the additive relationships underlying the additive situations. Therefore, the ability to operate in signed contexts does not imply that the introduction of notation is desirable. With some students, such as Adam, it can lead to an attempt to operate on the symbols instead of the quantities in the situation. Finally, even students who indicate an awareness of underlying additive relationships through reference to 0 or additive inverses have not necessarily interiorized these relationships in order to form a generalized signed sum structure. This means that the students will need assistance in determining what situations would be situations of signed addition, and which would be situations of signed subtraction, notationally speaking.

At this point, I would recommend that addition and subtraction notation only be introduced to students who can construct additive signed relationships. Furthermore, if the student, like Michelle, can only produce these additive relationships with the use of a visual aid, like a diagram, then that student should not be encouraged to use addition and subtraction notation without the visual aid. Furthermore, such students may find benefit from supplementing work with transformations as signed quantities with signed quantities that do not represent transformations. However, these students will need further practice reifying transformations before moving on to high school mathematics.

My findings also have implications for mathematics education research for the middle grades. In particular, given my instructional implications, it would be helpful to have more quantitative data on the mathematical constructions of middle school students both in terms of their number sequences and how they respond to teaching interventions involving signed numbers. At the very least, policy makers should be aware of how many students would be incapable (in the short term) of meeting a learning standard involving addition and subtraction notation with signed numbers, for example.

I would also like to follow up this study with a study that looks at the mathematics students construct in other types of signed quantity tasks that do not involve transformations, and I would like to work with more students, like Lily, who could possibly construct a generalized sum structure in order to analyze that learning process. I would also like to see the difference in how students at these various stages of signed additive constructions assimilate situations involving signed variables and slope, both of which involve signed quantities and are often introduced in the middle grades.

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APPENDIX A

INITIAL INTERVIEW TASKS

1. Money issues. Ask a and b and TWO of c-e.

   a. You've started a piggy bank. So far you've saved 29 pennies. Your friend Sarah comes by and adds some more pennies to your piggy bank so that you have 41 pennies in all. How many pennies did Sarah add? *TNS.*

   b. A pack of gum costs 79 cents. How would you pay with these coins? *TNS.*

   c. I'm buying a pack of gum from you that costs 63 cents. I pay with a dollar. [Actually hand them a dollar.] What's my change? [Have coins available.] *Subtraction. ENS.*

   d. I'm buying a pack of gum from you that costs $7.41. I pay with a ten-dollar bill. [Actually hand them a ten-dollar bill.] What's my change? [Have bills and coins available.] *Subtraction. ENS.*

   e. We have 141 pennies that we are putting into penny rolls. Each roll holds 10 pennies. I've filled 8 rolls already. How many more rolls do you need to fill to finish the job? *ENS. Assimilating with 2 levels of units.*

2. Fraction problems. Bring *in strips of paper.* Ask TWO of a-c and ask d.

   a. Four people are sharing this candy bar. Cut off the share of 1 person. Can you show me that it is a fair share? *Equipartitioning if they can do it with only one mark/cut.*

   b. I only want a strip of paper that is 1/5 as long as this one. Where should I cut? How did you know? *Equipartitioning if they do it with only one mark/cut.*

c. I want a strip of paper that is 3/4 as long as this one, Where should I cut? How did you know? *Equipartitioning.*

d. Four people are sharing the candy bar. Show me the share of one person. Let's say that is my share and I want to divide it into 7 equal pieces. Show me one of those pieces. What fraction is that of the whole candy bar? *Recursive partitioning.*

3. A sweatshirt costs $36. A sweatshirt costs four times as much as a T-shirt. How much does a T-shirt cost? How many shirts could you buy for $36? *Splitting.*
4. This is your bar. It is five times longer than my bar. Show me how long my bar is. How much of your bar is my bar? *Splitting.*
5. I multiplied 50 x 34 and got 1700. What's a quick way to figure out 50 x 32? *GNS.*
6. (Optional.) Operating with differences.

a. Give the next two numbers in the sequence and explain why you chose them: 25, 23, 19, 13, … *Do students conflate the differences between the numbers in the pattern with the differences in the differences (the second differences).*

b. I leave my house and drive 16 miles east on the highway. After stopping at the store I get back in my car and drive 29 miles east. How far am I from my house? Draw a picture of the situation. How far is it from there to there? *From the store to the stopping point? Do the students conflate the distance from the house with the distance from the store?*